An Overdetermined Symmetry Problem

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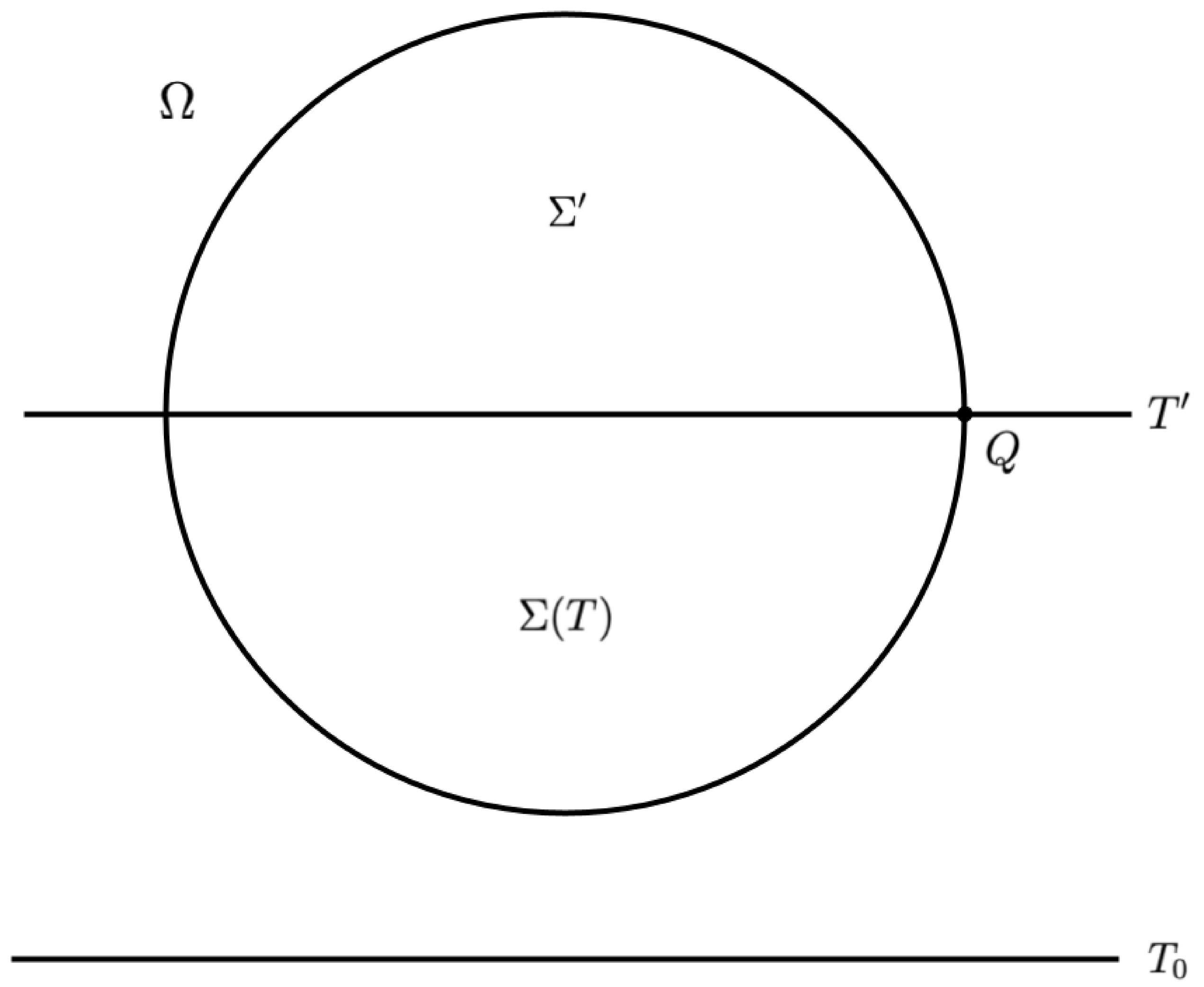
Introduction

Let us consider the following problem. Let $\Omega \subseteq \mathbb{R}^n$ be a domain that is bounded, open, and connected. Furthermore, suppose that the boundary $\partial\Omega$ is smooth. Let $u : \Omega \to \mathbb{R}$ be a C^2 function that satisfies the following conditions: $\Delta u = -1$ in Ω. and $u = 0$ and $\frac{\partial u}{\partial n} = c$ on $\partial\Omega$ for some constant c. Then, Ω must be a ball. Furthermore, we know that $u(x) = (b^2 - r^2)/2n$, where b is the ball's radius and r is the distance to its center.

First Proof

The first proof we present is from Professor James Serrin [3]. This proof utilizes the moving plane method. Let T_0 be a $n-1$ dimensional hyperplane in $\mathbb{R}^n$ that does not intersect the domain Ω. We begin to move this plane in a direction normal to itself until it intersects Ω. When this occurs, the new plane T splits Ω into two parts. The part of Ω that lies on the same side of T as our initial plane T_0 is denoted by $\Sigma(T)$. We reflect $\Sigma(T)$ in T to obtain $\Sigma' := \Sigma'(T)$. As T moves through Ω, Σ' will remain in Ω until the set Σ' meets Ω at a point P or T becomes orthogonal to Ω at some point Q. When either of these occurs, we stop moving the plane T, and we denote the resulting plane by T'. We claim that Ω is symmetric about T'. Showing this would prove the theorem. To see how, we recall that the plane T_0 was chosen arbitrarily. If Ω is symmetric about T', then Ω is symmetric in all possible directions. Since Ω is simply connected and has this strong symmetry property, it must be a ball.



To prove this, we introduce the function $v : \Sigma' \to \mathbb{R}$ defined by $v(x) = u(x')$ for $x \in \Sigma'$, where x' is the reflection of x across T'. By the maximum principle, we deduce that $u - v > 0$ or $u - v = 0$ in Σ'. Suppose that $u - v > 0$. If Σ' is internally tangent to Ω at some point P, then we may appeal to the boundary point maximum principle to deduce that $\frac{\partial}{\partial n}(u-v) > 0$ at P [1]. However, we know that $\partial u/\partial n = \partial v/\partial n = c$. If T' is orthogonal to the boundary of Ω at some point Q, then we show that u and v have the same first and second derivatives at Q. Using a modified version of the boundary point maximum principle, we can also show that $\frac{\partial}{\partial s}(u-v) > 0$ or $\frac{\partial^2}{\partial^2 s}(u-v) > 0$ for any direction s that enters Σ' non-tangentially at Q. However, this directly contradicts the fact that u and v have the same first and second derivatives at Q. We may thus conclude that Ω is symmetric about T'.

Second Proof

The second proof we present is from Weinberger [2]. To start, we first compute

$$\Delta\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial}{\partial r}(\Delta u) + 2\Delta = -2$$

where r is the distance to the origin. Using this and the fact that $\Delta u = -1$, we obtain

$$\int_\Omega \left[2u - r\frac{\partial u}{\partial r}\right] dx = \int_\Omega \left[-u\Delta\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\Delta u\right] dx$$

Using Green's identity yields

$$\int_\Omega \left[-u\Delta\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\Delta u\right] dx = \int_{\partial\Omega} \left[-u\frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\right] dS$$

By assumption, we have $u = 0$ on the boundary of Ω. Thus, we find that

$$\int_{\partial\Omega} \left[-u\frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\right] dS = \int_{\partial\Omega} r\frac{\partial r}{\partial n}\left(\frac{\partial u}{\partial n}\right)^2 dS$$

By assumption, we know that $\partial u/\partial n = c$ on the boundary of Ω. Thus, we find that

$$\int_{\partial\Omega} r\frac{\partial r}{\partial n}\left(\frac{\partial u}{\partial n}\right)^2 dS = c^2 \int_{\partial\Omega} r\frac{\partial r}{\partial n} dS$$

Appealing to the Divergence Theorem and using the fact that $\Delta\frac{1}{2}r^2 = r\Delta r$, we obtain

$$c^2\int_{\partial\Omega} r\frac{\partial r}{\partial n} dS = c^2\int_\Omega \Delta\left(\frac{1}{2}r^2\right) dx = c^2 n\int_\Omega dx = nc^2 V$$

Green's theorem also implies

$$\int_\Omega r\frac{\partial u}{\partial r} dx = -n\int_\Omega u dx$$

so that substitution yields

$$(n+2)\int_\Omega u dx = nc^2 V$$

However, we also note that

$$1 = (\Delta u)^2 \leq n\sum_{i=1}^n u_{ii}^2 \leq n\sum_{i,j} u_{ij}^2$$

by the Cauchy-Schwarz inequality. From this, we deduce that

$$\Delta\left(|\nabla u|^2 + \frac{2}{n}u\right) = 2\sum_{i,j} u_{ij}^2 - \frac{2}{n} \geq 0$$

Using this and the fact that $|\nabla u|^2 + (2/n)u = c^2$ on $\partial\Omega$, we may appeal to the maximum principle to deduce that $|\nabla u| + (2/n)u < c^2$ in Ω or $|\nabla u| + (2/n)u = c^2$ in Ω. If the former inequality held, then we could integrate over Ω to deduce that

$$(n+2)\int_\Omega u dx < nc^2 V$$

This contradiction informs us that $|\nabla u|^2 + (2/n)u = c^2$ in Ω so that

$$1 = n\sum_{i=1}^n u_{ii}^2 = \sum_{i,j} u_{ij}^2$$

which implies that $u_{ij} = -\delta_{ij}/n$. Solving the corresponding partial differential equations yields

$$u = \frac{1}{2n}(B - r^2)$$

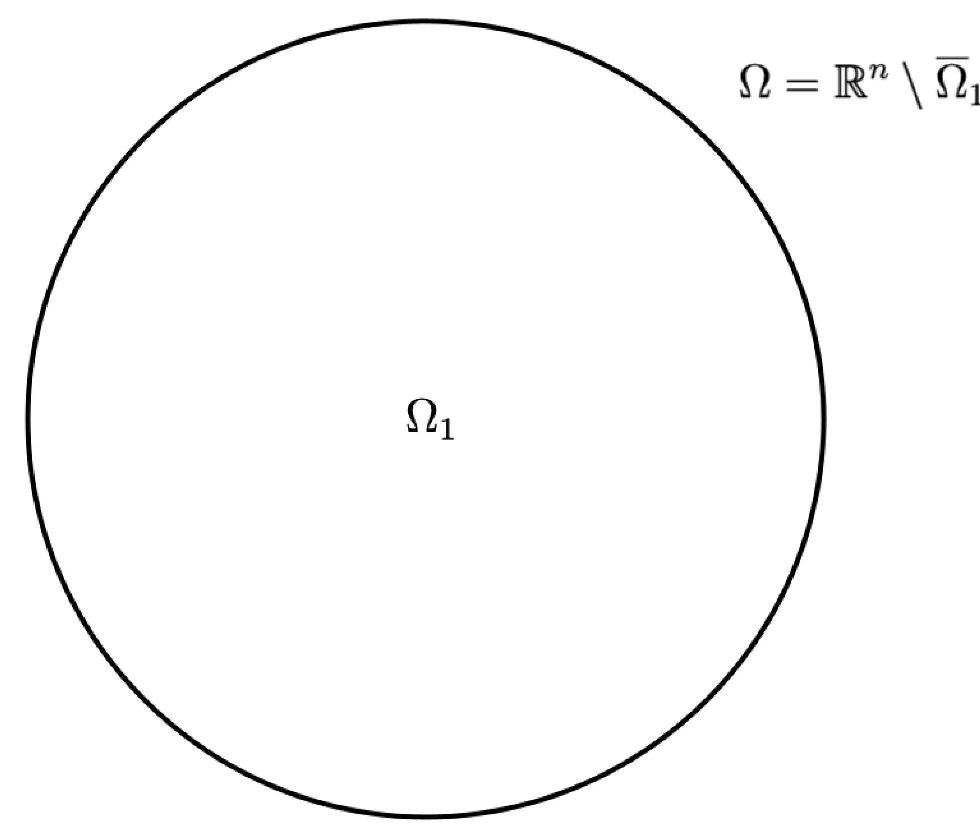
where B is a constant. Since $u = 0$ on $\partial\Omega$, B is positive and Ω is a ball of radius $B^{1/2}$.

Applications

This theorem is significant because it allows us to determine the shape of Ω from properties of u. It also has many applications in physics. For example, we may consider an incompressible viscous fluid moving through a straight pipe of cross sectional form Ω. If we fix a rectangular coordinate system with the z-axis directed along the pipe, then the velocity u depends only on x and y, and it satisfies the differential equation $\Delta u = -A$ for some constant A. Furthermore, because the fluid is viscous, we know that $u = 0$ on $\partial\Omega$; that is, there is no movement on the boundary of the pipe. Finally, we note that $\mu\partial u/\partial n$ is the tangential stress on the pipe wall, where μ is the viscosity constant. If the tangential stress is constant, then we may apply the above theorem to conclude that Ω is a circular cross section.

Generalizations

There is an interesting extension of this theorem from Wolfgang Reichel [4]. Let Ω_1 be a bounded domain with smooth boundary, and suppose that $\Omega = \mathbb{R}^n \setminus \overline{\Omega}_1$ is connected. Let u be a twice continuously differentiable function on $\overline{\Omega}$ such that $\Delta u + f(u, |\nabla u|) = 0$ in $\overline{\Omega}$, $0 \leq u < a$ in Ω, $u = a$ and $\partial u/\partial n = c \leq 0$ on $\partial\Omega_1$, and $u = \nabla u = 0$ at ∞. Furthermore, suppose that $f(p, q)$ is Lipschitz continuous in p and q and decreasing in p. Then, we may conclude that Ω_1 is a ball and that u is radially symmetric and decreasing in r. This can be proved by the moving plane method.



Acknowledgements

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