Remark on the Preceding Paper of Serrin

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Theorem 1 of the preceding paper of J. SERRIN [1] states that if the solution of the problem

$$\begin{aligned} \Delta u &= -1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

has the property that its normal derivative is a constant c on the boundary, then Ω is a ball of radius nc and $u = (n^2c^2 - r^2)/2n$ where n is the number of dimensions.

The purpose of this note is to show that this theorem (but not SERRIN's other results) can be obtained by means of more elementary arguments.

We first observe that

$$\Delta\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial}{\partial r}(\Delta u) + 2\Delta u = -2$$

where r is the distance from a fixed origin. Hence by Green's theorem

$$\begin{aligned} \int_\Omega \left[2u - r\frac{\partial u}{\partial r}\right] dx &= \int_\Omega \left[-u\Delta\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\Delta u\right] dx \\ &= \int_{\partial\Omega} \left[-u\frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\right] dS \\ &= \int_{\partial\Omega} r\frac{\partial r}{\partial n}\left(\frac{\partial u}{\partial n}\right)^2 dS = c^2\int_{\partial\Omega} r\frac{\partial r}{\partial n}\, dS = nc^2V \end{aligned}$$

where V is the volume of Ω. (At the third step we have used the fact that $u = 0$ on $\partial\Omega$.) Also by Green's theorem

$$\int_\Omega r\frac{\partial u}{\partial r}\, dx = \int_\Omega \operatorname{grad}(\tfrac{1}{2}r^2)\cdot\operatorname{grad} u\, dx = -n\int_\Omega u\, dx.$$

Thus we obtain the identity

$$(n+2)\int_\Omega u\, dx = nc^2V. \tag{2}$$

On the other hand, we observe that by Schwarz's inequality

$$1=(\Delta u)^2 \leqq n \sum_{i=1}^{n} u_{ii}^2 \leqq n \sum_{i,j=1}^{n} u_{ij}^2, \tag{3}$$

so that

$$\Delta\left(|\operatorname{grad} u|^2+\frac{2}{n}u\right)=2\sum_{i,j=1}^{n} u_{ij}^2-\frac{2}{n}\geqq 0. \tag{4}$$

Since $|\operatorname{grad} u|^2+\frac{2}{n}u=c^2$ on the boundary, we conclude from the strong maximum principle that either

$$|\operatorname{grad} u|^2+\frac{2}{n}u<c^2 \quad \text{in } \Omega \tag{5}$$

or

$$|\operatorname{grad} u|^2+\frac{2}{n}u\equiv c^2 \quad \text{in } \Omega. \tag{6}$$

If we integrate both sides of (5) over Ω and note that

$$\int_\Omega |\operatorname{grad} u|^2\,dx=-\int_\Omega u\Delta u\,dx=\int_\Omega u\,dx,$$

we obtain the inequality

$$\left(1+\frac{2}{n}\right)\int u\,dx<c^2 V,$$

which contradicts (2).

Therefore $|\operatorname{grad} u|^2+\frac{2}{u}u$ must be constant in Ω. Since its Laplacian then vanishes, we conclude that equality must hold in (4) and therefore in (3). This and the fact that $\Delta u=-1$ imply that

$$u_{ij}=-\frac{1}{n}\delta_{ij}.$$

Hence, for a suitable choice of origin we find that u is of the form

$$u=\frac{1}{2n}(A-r^2)$$

where A is a constant. Since u must vanish on $\partial\Omega$, we conclude that A is positive and that Ω is a ball of radius $A^{1/2}$.

Reference

1. SERRIN, J., A symmetry problem in potential theory. Preceding in this journal.

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