A Symmetry Problem in Potential Theory

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The following problem has been posed by Professor R. L. FOSDICK. Let Ω be a bounded open connected domain in the Euclidean space R^n having a smooth boundary $\partial\Omega$. Suppose there exists a function $u=u(x)=u(x_1, \ldots, x_n)$ satisfying the Poisson differential equation

$$\Delta u = -1 \quad \text{in } \Omega \tag{1}$$

together with the boundary conditions

$$u=0, \quad \frac{\partial u}{\partial n} = \text{constant} \quad \text{on } \partial\Omega. \tag{2}$$

Must Ω then be a ball? We shall show here that the answer is affirmative, and that u must have the specific form $(b^2-r^2)/2n$ where b is the radius of the ball and r denotes distance from its center. The precise result is as follows.

Theorem 1. *Let Ω be a domain whose boundary is of class C^2. Suppose there exists a function $u \in C^2(\bar{\Omega})$ satisfying conditions* (1) *and* (2). *Then Ω is a ball and u has the specific form noted above.*

The proof of this result is given in Section 1; in Section 3 we give various generalizations to elliptic differential equations other than (1). Before turning to the detailed arguments it will be of interest to discuss the physical motivation for the problem itself.

Consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω. If we fix rectangular coordinates in space with the z axis directed along the pipe, it is well known that the flow velocity u is then a function of x, y alone satisfying the Poisson differential equation (for $n=2$)

$$\Delta u = -A \quad \text{in } \Omega$$

where A is a constant related to the viscosity and density of the fluid and to the rate of change of pressure per unit length along the pipe. Supplementary to the differential equation one has the adherence condition

$$u=0 \quad \text{on } \partial\Omega.$$

Finally, the tangential stress per unit area on the pipe wall is given by the quantity $\mu\, \partial u/\partial n$, where μ is the viscosity. Our result states that *the tangential stress on the* *pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section.*

Exactly the same differential equation and boundary condition arise in the linear theory of torsion of a solid straight bar of cross section Ω; see [3] pp. 109–119. Theorem 1 then states that, *when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section.*

A more sophisticated example occurs in the case of a liquid rising in a straight capillary tube of cross section Ω. The function $u(x, y)$ describing the upper surface of the liquid satisfies the differential equation

$$(1+u_y^2)\,u_{xx}-2u_x u_y u_{xy}+(1+u_x^2)\,u_{yy}=\kappa u(1+u_x^2+u_y^2)^{3/2} \tag{3}$$

where κ is a certain positive constant; also the requirement that the wetting angle γ be constant leads to the boundary condition

$$\frac{\partial u}{\partial n}=-\cot\gamma=\text{constant} \quad \text{on } \partial\Omega \tag{4}$$

($\vec{n}$ being the inner normal direction). By the theorem of Section 3 it then follows that *provided the wetting angle γ is different from $\pi/2$, a liquid will rise to the same height at each point of the wall of a capillary tube if and only if the tube has circular cross section.* (When $\gamma=\pi/2$, the unique solution of (3), (4) is $u\equiv 0$ for *any* cross sectional form of the tube.)

In the final section of the paper we show that our results can be applied to somewhat more general boundary conditions than (2). A curious consequence of this generalization is the following result. Consider a viscous fluid flowing in straight streamlines through a straight pipe whose cross section is non-circular. Then there must be two points P and Q on the wall, such that the curvature of the wall is greater at P than at Q but the tangential stress is greater at Q than at P. A similar result also holds for the torsion problem.

It may be noted in conclusion that (2) constitutes Cauchy data on the boundary surface for the elliptic equation (1). It is of course well known that such data is generally overdetermined with regard to solving (1) in a given domain Ω; in light of this remark, our results provide a concrete example where the overdetermined nature of the condition can be rigorously analyzed.

1. Proof of Theorem 1

Let T_0 by a hyperplane in R^n not intersecting the domain Ω. We suppose this plane to be continuously moved normal to itself to new positions, until ultimately it begins to intersect Ω. From that moment onward, at each stage of the motion the resulting plane T will cut off from Ω a cap $\Sigma(T)$: that is, $\Sigma(T)$ will be that portion of Ω which lies on the same side of T as the original plane T_0.

For any cap $\Sigma(T)$ thus formed, we let $\Sigma'(T)$ be its reflection in T. Evidently $\Sigma'(T)$ will be contained in Ω at the beginning of the process; and indeed as T advances into Ω *the resulting cap $\Sigma'(T)$ will stay within Ω at least until one of the following two events occurs:*

(*i*) $\Sigma'(T)$ becomes internally tangent to the boundary of Ω at some point P not on T, or

(*ii*) T reaches a position where it is orthogonal to the boundary of Ω at some point Q.

We denote the plane T when it reaches either one of these positions by T'.

We now assert that Ω must be symmetric about T'. In fact, if this assertion is proved the theorem follows immediately. To see this, we observe that for *any given direction* in R^n there would then be a plane T' with normal in that direction such that Ω is symmetric about T'. Moreover, according to the construction Ω would have to be simply connected. But the only simply connected domains which have the symmetry property just noted are balls. Having thus proved that Ω is a ball we say that the function $(b^2-r^2)/2n$ is then the unique solution of the given boundary value problem ([2], pages 68–69).

In proving the assertion, we introduce a new function v defined in $\Sigma' \equiv \Sigma'(T')$ by the formula

$$v(x)=u(x') \qquad (x\in\Sigma') \tag{5}$$

where x' is the reflected value of x across T'. Evidently v satisfies the differential equation

$$\Delta v=-1 \quad \text{in } \Sigma'$$

and the boundary conditions

$$\begin{aligned} &v=u && \text{on } \partial\Sigma'\cap T' \\ v=0,\quad &\frac{\partial v}{\partial n}=\text{constant} && \text{on } \partial\Sigma'\cap \operatorname{Comp}(T'), \end{aligned}$$

the constant being the same as in (2).

Since Σ' is contained in Ω by construction, we may consider the function $u-v$ in Σ'. Evidently

$$\Delta(u-v)=0 \quad \text{in } \Sigma' \tag{6}$$

and

$$\begin{aligned} u-v&=0 \quad \text{on } \partial\Sigma'\cap T' \\ u-v&\geqq 0 \quad \text{on } \partial\Sigma'\cap \operatorname{Comp}(T'); \end{aligned} \tag{7}$$

the latter condition is a consequence of the fact that $u>0$ in Ω.* If we apply the strong version of the maximum principle ([2], page 53), it is easy to see from (6) and (7) that either

$$u-v>0 \qquad \text{at all interior points of } \Sigma' \tag{8}$$

or else $u\equiv v$ in Σ'. In the latter case it is clear that the reflected cap Σ' must coincide with that part of Ω on the same side of T' as Σ'; that is, Ω must be symmetric about T'.

* For otherwise u would have a minimum at some interior point of Ω, which is impossible since $\Delta u=-1$.

To complete the proof of the theorem it must therefore be shown that (8) is impossible. Suppose first that we are in case (*i*), that is, Σ' is internally tangent to the boundary of Ω at some point P not on T'. Then $u-v=0$ at P. Consequently, using (6), (8) and the boundary point version of the maximum principle ([2], page 65), we conclude that

$$\frac{\partial}{\partial n}(u-v)>0 \quad \text{at } P.$$

This however contradicts the fact that $\partial u/\partial n=\partial v/\partial n=$constant at P. Hence (8) is impossible in case (*i*).

In case (*ii*) the situation is more complicated, for even though $u-v=0$ at Q the boundary point version of the maximum principle does not apply (this is because Q is a right angled corner of Σ' and the requisite internally tangent ball [see [2], page 65] is not available). Consequently we must proceed in an alternate fashion. It will be shown (*a*) that $u-v$ has a zero of second order at Q and then (*b*) a contradiction will be obtained from a more delicate version of the boundary point maximum principle.

(*a*) By hypothesis the boundary of Ω is of class C^2. Consider a rectangular coordinate frame with origin at Q, the x_n axis being directed along the inward normal to $\partial\Omega$ at Q, and the x_1 axis being normal to T'. In this frame we can represent the boundary of Ω locally by the equation

$$x_n=\phi(x_1, \dots, x_{n-1}), \qquad \phi\in C^2.$$

Since u is in $C^2(\bar{\Omega})$ the condition $u=0$ on $\partial\Omega$ can then be expressed as a twice differentiable identity

$$u(x_1, \dots, x_{n-1}, \phi)\equiv 0. \tag{9}$$

Similarly, the boundary condition $\partial u/\partial n=$constant$=c$ on $\partial\Omega$ can be written as an identity,

$$\frac{\partial u}{\partial x_n}-\sum_1^{n-1}\frac{\partial u}{\partial x_k}\frac{\partial \phi}{\partial x_k}=c\left\{1+\sum_1^{n-1}\left(\frac{\partial\phi}{\partial x_k}\right)^2\right\}^{1/2} \tag{10}$$

where x_n is to be replaced throughout by $\phi(x_1, \dots, x_{n-1})$.

At this stage some simple notation will be convenient; thus

$$u_i=\frac{\partial u}{\partial x_i} \quad (i=1, \dots, n-1); \qquad u_n=\frac{\partial u}{\partial x_n}.$$

Differentiating (9) with respect to x_i, $i=1, \dots, n-1$, we now obtain

$$u_i+u_n\phi_i=0. \tag{11}$$

Evaluating this at Q, where $\phi_i=0$, we find

$$u_i=0, \quad u_n=c \quad (\text{at } Q).$$

Next differentiating (11) with respect to x_j, $j=1, \dots, n-1$, and evaluating at Q yields

$$u_{ij}+c\phi_{ij}=0 \quad (\text{at } Q)$$

where $u_{ij}=\partial^2 u/\partial x_i\,\partial x_j$. Lastly, differentiating (10) with x_l and evaluating at Q gives

$$u_{nl}=0 \quad (\text{at } Q).$$

Since also $u_{nn}=-\sum_1^{n-1} u_{ii}-1=c\Delta\varphi-1$ at Q, we have accordingly determined all the first and second derivatives of u at Q.

Since the reflected cap Σ' lies inside Ω, it is not hard to see that the second derivatives of ϕ must also satisfy

$$\phi_{1l}=0 \quad \text{at } Q, \qquad l=2,\dots,n-1.$$

Taking these relations into account, and observing that (5) implies

$$v(x_1,x_2,\dots,x_n)=u(-x_1,x_2,\dots,x_n),$$

we find that the first and second derivatives of u and v agree at Q. This completes the proof of (a).

Turning now to (b), we require the following preliminary result.

Lemma 1. *Let D^* be a domain with C^2 boundary and let T be a plane containing the normal to ∂D^* at some point Q. Let D then denote the portion of D^* lying on some particular side of T.*

Suppose that w is of class C^2 in the closure of D and satisfies

$$\Delta w \leqq 0 \quad in\ D,$$

while also $w\geqq 0$ in D and $w=0$ at Q. Let $\vec{s}$ be any direction at Q which enters D non-tangentially. Then either

$$\frac{\partial w}{\partial s}>0 \quad or \quad \frac{\partial^2 w}{\partial s^2}>0 \quad at\ Q$$

unless $w\equiv 0$.

The proof of this result will be given in Section 2. Assuming that the lemma holds, we may apply it to the function $w=u-v$ in Σ'. Since $w>0$ there, and $w=0$ at Q, this yields

$$\frac{\partial(u-v)}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2(u-v)}{\partial s^2}>0 \quad \text{at } Q$$

contradicting the fact that both u and v have the same first and second partial derivatives at Q. This completes the proof of the theorem.

2. Proof of Lemma 1

Let K_1 be a ball which is internally tangent to D^* at Q, and which touches the boundary of D^* only at Q. Such a ball exists by virtue of the fact that the boundary of D^* is of class C^2.

Construct a ball K_2 with center at Q and radius $\frac{1}{2}r_1$ where r_1 is the radius of K_1. Finally let $K'=K_1\cap K_2\cap D$. Now define the auxiliary function

$$z=z(x)=x_1\left(e^{-\alpha r^2}-e^{-\alpha r_1^2}\right)$$

where α is a positive constant to be determined (we have chosen coordinates with origin at the center of K_1 and with T being the plane $x_1=0$; moreover it can be assumed that D is on the side of T where $x_1>0$). It is clear that

$$z>0 \quad \text{in } K', \qquad z=0 \quad \text{on } \partial K_1 \text{ and on } T \tag{12}$$

(at this stage we mean by K' the *interior* of the region described, i.e., K_1, K_2 and D are taken to be open sets). We compute

$$\Delta z=2\alpha x_1 e^{-\alpha r^2}\{2\alpha r^2-(n+2)\}.$$

Now $r\geqq\frac{1}{2}r_1$ in K' so that by choosing α suitably large, say $\alpha=(n+2)\,r_1^{-2}$, we obtain $\Delta z>0$ in K'.

Now suppose w is not identically zero in D. Then by the strong maximum principle we have $w>0$ in D. We consider the part of the boundary of K' lying on ∂K_2. This set intersects the boundary of D only on the plane T. Moreover the intersection set lies at a finite distance from the corners of D. By virtue of the boundary point lemma ([2], page 65), therefore, it is not hard to see that there exists a constant $\varepsilon>0$ such that

$$w\geqq\varepsilon x_1 \quad \text{on } \partial K'\cap\partial K_2.$$

Moreover

$$w\geqq 0 \quad \text{on } \partial K'\cap\partial K_1 \text{ and } \partial K'\cap T.$$

On the other hand, it is clear that

$$z\leqq x_1 \quad \text{on } \partial K'\cap\partial K_2. \tag{13}$$

Consequently, by use of (12), the function $w-\varepsilon z$ is non-negative on the entire boundary of K', and is zero at Q. Moreover

$$\Delta(w-\varepsilon z)=\Delta w-\varepsilon\Delta z<0$$

in K'. By the maximum principle, therefore, $w-\varepsilon z>0$ in K'. Hence at Q, where $w-\varepsilon z=0$, we have either

$$\frac{\partial(w-\varepsilon z)}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2(w-\varepsilon z)}{\partial s^2}\geqq 0.$$

Since by direct calculation

$$\frac{\partial z}{\partial s}=0, \quad \frac{\partial^2 z}{\partial s^2}>0 \quad \text{at } Q,$$

this completes the proof of the lemma.

3. More General Elliptic Equations

If the proof of Theorem 1 is re-examined, one finds that the properties of the Poisson equation which were applied are the following.

(A) It is invariant to the symmetric substitution $x\to x'$.

(B) The second partial derivatives u_{nn} in an arbitrary rectangular coordinate frame can be determined in terms of the remaining second partial derivatives.

(C) The difference of two solutions obeys the strong maximum principle, and

(D) The difference of two solutions obeys both boundary point versions of the maximum principle.

Now conditions (A) and (B) are satisfied by any nonlinear *elliptic* differential equation of the form

$$a\,\Delta u + h\,u_i u_j u_{ij} = f \tag{14}$$

provided that a, f and h are functions only of u and $|p|$, where $p=(u_1, \ldots, u_n)$ denotes the gradient vector of the solution.* To see this, note that (14) is invariant under coordinate rotations as well as under substitutions $(x_1, \ldots, x_n) \to (-x_1, \ldots, x_n)$ which change the sign of a single coordinate.

On the other hand, the strong maximum principle does *not* hold for differences of solutions of (14) in the form which was applied in the proof of Theorem 1. Nevertheless, a somewhat restricted version does remain valid. Suppose in fact that the functions a, f and $h\,p_i p_j$ are continuously differentiable in u and p, and that u and v are two solutions in a domain D such that $u-v \geqq 0$ in D and $u-v=0$ at some interior point of D. Then according to a theorem of E. HOPF ([1], pages 149–150) we have $u \equiv v$.

In the proof of Theorem 1 the maximum principle was used only to show that either (8) holds or that $u \equiv v$ in Σ'. This conclusion can be reached by an alternate argument, however, using only the restricted version of the maximum principle.

Deferring this proof for a moment, it remains only to consider the two boundary point versions of the maximum principle. In this regard, the standard boundary point lemma ([2], page 65) applies immediately to the difference of solutions of (14) since this difference is zero at the point P. (The argument is the same as for the restricted version of the maximum principle above.) Finally, the required genralization of Lemma 1 needed for application to the difference of solutions of (14) will be proved in the following section.

We may thus turn our attention to showing that either (8) holds or else $u \equiv v$ in Σ'. For this purpose we shall make an additional assumption about the behavior of the solution, namely**

$$u > 0 \quad \text{in } \Omega. \tag{15}$$

(We note that this assumption applies automatically in the case of Theorem 1, as a consequence of (1), (2).) Letting c denote the constant which occurs in the boundary condition (2), we assert to begin with that either $c>0$ or else $\partial^2 u/\partial n^2 > 0$ at each point of the boundary. Thus suppose $c=0$. Then, according to the formulas given in part (a) of the previous proof, if we introduce at any fixed point P of the boundary a special coordinate frame with x_n axis directed along the inner normal to the boundary, all the first and second derivatives of u vanish at P except $\partial^2 u/\partial n^2 = u_{nn}$, and this derivative has the value

$$\frac{f(0,0)}{a(0,0)} = d$$

* From here onward we adopt the standard convention that repeated indices are to be summed from 1 to n.

** Recall that Ω is assumed to be open so that (15) is not in conflict with the given boundary condition (2).

according to (14). (Note that $a>0$ for all values of u and p because (14) is elliptic.)

Since $c=0$ by supposition, the inequality $d<0$ is obviously contradictory to (15); it remains to show that $d=0$ also cannot occur. In such a case, however, we would have $f(0,0)=0$. Then the function $\bar{u}\equiv 0$ would be a solution of (14), and correspondingly the solution u under consideration could be regarded as the *difference* of two solutions. Then in view of (15) we could apply the standard boundary point lemma to infer that $\partial u/\partial n>0$ at every point of the boundary, contradicting the original supposition that $c=0$. Hence $d>0$.

From what has just been proved it is now clear that u increases monotonically as one enters Ω along any non-tangential direction $\vec{s}$, for some positive distance so into the domain. Moreover, s_0 can be chosen independent of position on $\partial\Omega$, depending only on a bound for $\vec{s}$ away from the tangent direction.

Let us now return to the opening stages of the proof of Theorem 1. By what has been shown it is clear that immediately after T has penetrated Ω not only will $\Sigma'(T)$ be contained in Ω but also

$$(16)\qquad u>v \quad \text{at interior points of } \Sigma'(T).$$

Here we construct v in $\Sigma'(T)$ in exactly the same way as the previous construction of v in $\Sigma'(T')$.

We assert that (16) persists as T advances into Ω, for all positions of T prior to T'. Suppose in fact that there is some instant where (16) fails prior to T reaching T', Then there would be a position T'' of T such that either

$$(17)\qquad u\geqq v \quad \text{in } \Sigma'(T''), \qquad u=v \quad \text{at some interior point of } \Sigma'(T'')$$

or

$$(18)\qquad \begin{aligned} &u>v && \text{at interior points of } \Sigma'(T'') \\ &\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n} && \text{at some interior point of } T''\cap\Omega. \end{aligned}$$

If (17) holds we reach a contradiction with the restricted version of the maximum principle, and if (18) holds, with the boundary point principle. Thus the assertion is proved.

It follows by continuity that when T reaches T' we have

$$u\geqq v \quad \text{at interior points of } \Sigma'.$$

By applying the restricted maximum principle once more we see that either the strict inequality holds or else $u\equiv v$. But this is what we originally set out to demonstrate. Summing up, we have proved the following result.

Theorem 2. *Let Ω be a domain whose boundary is of class C^2. Suppose there exists a function $u\in C^2(\bar{\Omega})$ satisfying the elliptic differential equation*

$$(19)\qquad a(u,|p|)\Delta u+h(u,|p|)u_i u_j u_{ij}=f(u,|p|) \quad \text{in } \Omega$$

where a, f and $h\,p_i\,p_j$ are continuously differentiable functions of u and p (here $p=(u_1,\ldots,u_n)$ denotes the gradient vector of u). Suppose also that

$$(20)\qquad u>0 \quad \text{in } \Omega$$

and that u satisfies the boundary conditions

$$u=0, \quad \frac{\partial u}{\partial n}=\text{constant} \quad on\ \partial\Omega. \tag{2}$$

Then Ω must be a ball and u is radially symmetric.

Remarks 1. It is clear that condition (20) can be replaced by the alternate assumption that $u<0$ in Ω. Also (20) can be deleted from the statement of the theorem if it is assumed that f is never zero. To see this, note that if $f<0$, say, then u cannot take on an interior minimum value, so that the boundary condition $u=0$ on $\partial\Omega$ implies $u>0$ at interior points of Ω. In particular, one has $f=-1$ for equation (1), explaining why it was unnecessary to make the explicit assumption $u>0$ in Theorem 1.

2. Equation (3) describing the upper surface of a liquid in a capillary tube (i.e. a liquid under the combined influence of gravity and surface tension) is a special case of (19). The result stated at the end of the introduction is therefore a consequence of the following remark. *Let u be a solution of* (3), (4) *in Ω, such that $u=\alpha=$constant on $\partial\Omega$. Suppose that $\gamma\neq\pi/2$. Then $u\neq\alpha$ in Ω.* Proof. Assume first $0<\gamma<\pi/2$. Then by (4), since solutions of (3) can take on neither a positive maximum nor a negative minimum in Ω, we have $\alpha>0$ and $0\leqq u<\alpha$ in Ω. Similarly if $\gamma>\pi/2$ then $\alpha<0$ and $\alpha<u\leqq 0$ in Ω.

A further class of equations to which Theorem 2 is applicable can be obtained from regular variational problems of the form

$$\delta\int F(u,|p|)\,dx=0. \tag{21}$$

Indeed if we assume that $F(u,t)$ is three times differentiable, and that $F'(u,0)=F'''(u,0)=0$ for consistency, then the Euler-Lagrange equation for (21) can be written in the form (19) with

$$a=1, \quad h=|p|^{-2}\left(\frac{|p|F''}{F'}-1\right), \quad f=\frac{|p|}{F'}(F_u-|p|F'_u)$$

where primes here denote differentiation with respect to the second argument of F.

3. Under the hypotheses of the theorem we cannot conclude any more about the form of u then that it must be a function of the radial distance r from the center of Ω. If we write $u=u(r)$ and use primes to denote differentiation with respect to r, then one obtains the differential equation

$$a(u,|u'|)\left(u''+\frac{n-1}{r}u'\right)+h(u,|u'|)u'^2u''=f(u,|u'|), \tag{22}$$

for $0\leqq r\leqq b$ with the end conditions

$$u'(0)=0, \quad u(b)=0. \tag{23}$$

The ellipticity of (20) implies

$$a(u,|u'|)+h(u,|u'|)|u'|^2>0$$

so that in any case (22) is non-singular and can be written in the form

$$(22') \qquad u'' = F(r, u, u').$$

To make further progress it may be assumed that a and h are independent of u and that $\partial f/\partial u \geqq 0$, as is the case for the equation describing a capillary surface. Then standard arguments show that there cannot be more than one solution h. There would be some interest in pursuing the discussion of equation (22) further, as JOHNSON and PERKO have done in the case of the capillary equation (Arch. Rational Mech. Analysis, vol. 29).

4. It is of interest to observe that without some assumption of the type described in the preceeding remark, equation (22) may have more than one solution (or even infinitely many solutions) satisfying the end conditions (23). In consequence, the property of having unique solutions corresponding to given boundary data is *not* a prerequisite for Theorem 2 to hold for a given equation.

4. The Boundary Point Lemma at a Corner

Here we shall prove a generalization of Lemma 1 suitable for application to non-linear elliptic equations. We then use this result to obtain the conclusion

$$(24) \qquad \frac{\partial(u-v)}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2(u-v)}{\partial s^2} > 0 \quad \text{at } Q$$

which is needed, just as in the case of Theorem 1, for completing the proof of Theorem 2.

Lemma 2. *Let D be a domain of the type described in Lemma 1. Suppose that w is of class C^2 in the closure of D and satisfies the elliptic differential inequality*

$$Lw = a_{ij}(x)\, w_{ij} + b_i(x)\, w_i \leqq 0 \quad \text{in } D$$

where the coefficients are uniformly bounded. We assume that the matrix a_{ij} is uniformly definite

$$(25) \qquad a_{ij}(x)\, \xi_i \xi_j \geqq \kappa |\xi|^2 \quad (\kappa = \text{constant} > 0),$$

and that

$$(26) \qquad |a_{ij} \xi_i \eta_j| \leqq K(|\xi \cdot \eta| + |\xi| \cdot |d|) \quad (K = \text{constant} > 0)$$

where $\xi = (\xi_1, \ldots, \xi_n)$ is an arbitrary real vector, $\eta = (\eta_1, \ldots, \eta_n)$ is the unit normal to the plane T, and d is the distance from T.

Suppose also that $w \geqq 0$ in D and $w = 0$ at Q. Then either

$$\frac{\partial w}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial s^2} > 0 \quad \text{at } Q$$

unless $w \equiv 0$, where $\vec{s}$ is any direction at Q which enters D non-tangentially.

Proof. We proceed in the same way as in the proof of Lemma 1, except that the comparison function z and the ball K_2 must be chosen with greater care.

As in the proof of Lemma 1, we introduce the ball K_1, and then construct K_2 to have center at Q and radius θr_1, where $\theta \leqq 1/2$ is a constant to be determined.

Then in $K'=K_1\cap K_2\cap D$ we define the auxiliary function

$$z=z(x)=\{e^{-\alpha(x_1-r_1)^2}-e^{-\alpha r_1^2}\}\cdot\{e^{-\alpha r^2}-e^{-\alpha r_1^2}\}$$

where α is also to be determined. With the choice of coordinates as before, it is clear that

$$z>0 \quad \text{in } K', \qquad z=0 \quad \text{on } \partial K_1 \text{ and on } T.$$

We compute

$$\begin{aligned}Lz=&e^{-\alpha r^2}(e^{-\alpha(x_1-r_1)^2}-e^{-\alpha r_1^2})\cdot\{4\alpha^2 a_{ij}x_i x_j-2\alpha[a_{ii}+b_i x_i]\}\\&+e^{-\alpha(x_1-r_1)^2}(e^{-\alpha r^2}-e^{-\alpha r_1^2})\cdot\{4\alpha^2 a_{11}(x_1-r_1)^2-2\alpha[a_{11}+b_1(x_1-r_1)]\}\\&+8\alpha^2 e^{-\alpha r^2}e^{-\alpha(x_1-r_1)^2}\cdot(x_1-r_1)a_{1j}x_j.\end{aligned}$$

Because of the ellipticity condition (25)

$$a_{ij}x_i x_j\geqq\kappa r^2\geqq\tfrac{1}{4}\kappa r_1^2 \quad \text{in } K'$$

and, for the same reason, $a_{11}(x_1-r_1)^2\geqq\frac{1}{4}\kappa r_1^2$ in K'. Moreover by (26),

$$|a_{1j}x_j|=|a_{ij}\eta_i x_j|\leqq K(|x_1|+|x_1|)$$

since in the present case $\eta=(1,0,\ldots,0)$. Thus

$$|(x_1-r_1)a_{ij}x_j|\leqq 2x_1 r_1 K \quad \text{in } K'.$$

Finally we observe that by the mean value theorem

$$e^{-\alpha(x_1-r_1)^2}-e^{-\alpha r_1^2}\geqq 2\alpha(1-\theta)r_1 e^{-\alpha r_1^2}x_1\geqq\alpha x_1 r_1 e^{-2\alpha\theta r_1^2-\alpha(x_1-r_1)^2}.$$

Inserting these inequalities into the earlier expression for Lz, and using the fact that the terms $[a_{ii}+b_i x_i]$ and $[a_{11}+b_1(x_1-r_1)]$ are bounded, we find for large α

$$\begin{aligned}Lz\geqq&\alpha^2 x_1 r_1 e^{-\alpha(r^2+(x_1-r_1)^2)}\cdot\{(\alpha\kappa r_1^2-B)e^{-2\alpha\theta r_1^2}-16K\}\\&+\alpha e^{-\alpha(x_1-r_1)^2}(e^{-\alpha r^2}-e^{-\alpha r_1^2})\{\alpha\kappa r_1^2-B\}\end{aligned}$$

where B is an appropriate constant. By choosing $\theta=1/\alpha$ and then α suitably large it is clear that we can make the quantities in braces positive. That is, we have now constructed a function z such that $Lz>0$ in K'.

The remaining part of the proof is the same as for Lemma 1, since both the maximum principle and the boundary point lemma apply to the elliptic operator L in the same way as to the Laplace operator ([2], pages 61 and 65). The only slight difference to be noted is that (13) must now be replaced by

$$z\leqq 2\alpha r_1 x_1, \tag{13'}$$

which follows easily from the mean value theorem and the definition of z. Consequently the function $w-\varepsilon z$ of Lemma 1 must be replaced by $w-(\varepsilon/2\alpha r_1)z$ but otherwise the argument is left unchanged. This completes the demonstration of Lemma 2.

To complete the proof of Theorem 2 we must show that the final argument in the proof of Theorem 1 can be carried over to the more general case of equation (19). Thus let u and v be, respectively, the original solution and the reflected solution in the region Σ'. It has already been established that either $u-v>0$ or $u\equiv v$ in Σ'. We must show in the former case that (24) holds (the plane $T=T'$ being in the position indicated by case (*ii*) of the proof of Theorem 1). We begin by obtaining an appropriate second order differential equation for the difference function $u-v$.

Since both u and v satisfy (19) we have

$$a[u]\,\Delta u+h[u]\,u_i u_j u_{ij}=f[u]$$

$$a[v]\,\Delta v+h[v]\,v_i v_j v_{ij}=f[v]$$

where $a[u]\equiv a(u,|p|)$ and similarly for the other square brackets. Differencing these equations yields easily

$$\begin{aligned}&\{a[u]+a[v]\}\,\Delta(u-v)+\{h[u]\,u_i u_j+h[v]\,v_i v_j\}(u-v)_{ij}\\&\qquad+\{a[u]-a[v]\}\,\Delta(u+v)+\{h[u]\,u_i u_j-h[v]\,v_i v_j\}(u+v)_{ij}=2\{f[u]-f[v]\}.\end{aligned}$$

Now by the mean value theorem of multidimensional calculus

$$a[u]-a[v]=\left(\frac{\partial a}{\partial u}\right)_0(u-v)+\left(\frac{\partial a}{\partial p_i}\right)_0(u-v)_i$$

with similar expressions for $h[u]\,u_i u_j-h[v]\,v_i v_j$ and $f[u]-f[v]$. Thus if we define

$$a_{ij}(x)=\{a[u]+a[v]\}\,\delta_{ij}+\{h[u]\,u_i u_j+h[v]\,v_i v_j\}$$

it follows from the above identity that

$$a_{ij}(x)(u-v)_{ij}+b_i(x)(u-v)_i+c(x)(u-v)=0 \tag{27}$$

where $b_i(x)$ and $c(x)$ are certain bounded functions.

Here the matrix a_{ij} is uniformly definite:

$$a_{ij}\xi_i\xi_j\geqq\kappa|\xi|^2$$

since both expressions $a[u]\,\delta_{ij}+h[u]\,u_i u_j$ and $a[v]\,\delta_{ij}+h[v]\,v_i v_j$ have this property (recall that equation (19) is elliptic). Consider next the expression $a_{ij}\,\xi_i\,\eta_j$ for which the estimate (26) must be established. We have by computation

$$a_{ij}\xi_i\eta_j=\{a[u]+a[v]\}\,\xi\cdot\eta+\{h[u]\,(p\cdot\xi)(p\cdot\eta)+h[v]\,(q\cdot\xi)(q\cdot\eta)\} \tag{28}$$

where $p=(u_1,\dots,u_n)$ and $q=(v_1,\dots,v_n)$. Now v is the reflection of u in the plane $T=T'$. Hence $u=v$ on T, and moreover (in the usual rectangular coordinate frame centered at Q with x_1 axis normal to T)

$$p=(u_1,u_2,\dots,u_n),\qquad q=(-u_1,u_2,\dots,u_n)\quad\text{on } T.$$

Thus on T there holds

$$|p|=|q|,\qquad a[u]=a[v],\qquad h[u]=h[v];$$

and, since $\eta=(1,0,\ldots,0)$,

$$\begin{aligned} I &\equiv h[u](p\cdot\xi)(p\cdot\eta)+h[v](q\cdot\xi)(q\cdot\eta)\\ &=h[u]\,u_1(p-q)\cdot\xi=2h[u]\,u_1^2\,\xi_1=2h[u]\,u_1^2\,\xi\cdot\eta. \end{aligned}$$

By continuity it then follows that off T

$$|I|\leqq M(|\xi\cdot\eta|+|\xi|\cdot|x_1|) \tag{29}$$

for some constant M. Noting that also $\{a[u]+a[v]\}\leqq M'$ for some M', we have finally from (28) and (29)

$$|a_{ij}\xi_i\eta_j|\leqq K(|\xi\cdot\eta|+|\xi|\cdot|x_1|) \qquad (K=M+M')$$

as required.

The differential equation (27) for $u-v$ is not quite in the form to which Lemma 2 applies. Hence as in the proof of the restricted maximum principle we make HOPF's substitution $w=(u-v)\,e^{\beta x_1}$. Then obviously

$$a_{ij}(x)\,w_{ij}+\tilde{b}_i(x)\,w_i+\tilde{c}(x)\,w=0$$

where $\tilde{b}_i=b_i-2\beta a_{1i}$, $\tilde{c}=a_{11}\beta^2-b_1\beta+c$. Since $u-v\geqq 0$ by assumption it follows that for β sufficiently large

$$a_{ij}(x)\,w_{ij}+\tilde{b}_i(x)\,w_i\leqq 0.$$

Consequently Lemma 2 may be applied directly to w (note that w satisfies the hypotheses of Lemma 2 since $u-v=0$ at Q); we find therefore

$$\frac{\partial w}{\partial s}>0 \quad\text{or}\quad \frac{\partial^2 w}{\partial s^2}>0 \quad\text{at } Q$$

since $w\not\equiv 0$. But at Q

$$\frac{\partial(u-v)}{\partial s}=\frac{\partial w}{\partial s}, \qquad \frac{\partial^2(u-v)}{\partial s^2}=\frac{\partial^2 w}{\partial s^2}-2\beta s_1\frac{\partial w}{\partial s},$$

so that also

$$\frac{\partial(u-v)}{\partial s}>0 \quad\text{or}\quad \frac{\partial^2(u-v)}{\partial s^2}>0 \quad\text{at } Q.$$

Thus the difference $u-v$ satisfies (24), and the proof of Theorem 2 is complete.

5. A Different Boundary Condition

The previous results can be extended to more general boundary conditions than (2) without changing the basic method. We let H denote the *mean curvature* of the boundary surface $\partial\Omega$, chosen so that H is positive when the surface is "convex." Analytically, if a portion of the boundary is locally represented by the equation

$$x_n=\phi(x_1,\ldots,x_{n-1})$$

with the x_n axis directed into Ω, then on this part

$$H=\frac{(1+\phi_k\phi_k)\,\Delta\phi-\phi_i\phi_j\phi_{ij}}{(n-1)(1+\phi_k\phi_k)^{3/2}}. \tag{30}$$

We now replace (2) by the more general condition

$$u=0, \quad \frac{\partial u}{\partial n}=c(H) \quad \text{on } \partial\Omega \tag{31}$$

where c is a continuously differentiable non-decreasing function of H. Then the following result holds

Theorem 3. *The conclusions of Theorems* 1 *and* 2 *remain valid under the more general boundary condition* (31).

Proof. It will be enough to show how the proof of Theorem 1 needs to be modified to cover the more general boundary condition (31), and then to make several comments concerning the further modification required for the case of equation (19).

For the generalization of Theorem 1, then, we proceed as in the initial steps of the proof given in Section 1 to show that either (8) holds or $u \equiv v$ in Σ'. To prove then that (8) is impossible, suppose first that case (*i*) holds in the definition of Σ'. Then $u-v=0$ at P, and by the boundary point version of the maximum principle we conclude as before that

$$\frac{\partial}{\partial n}(u-v)>0 \quad \text{at } P. \tag{32}$$

On the other hand, if H is the mean curvature of $\partial\Omega$ at P and H' is the mean curvature of $\partial\Sigma'$ at P, then $H \leqq H'$ since $\Sigma' \subset \Omega$. Consequently

$$\frac{\partial u}{\partial n}=c(H) \leqq c(H')=\frac{\partial v}{\partial n}$$

which contradicts (32). Hence (8) is impossible in case (*i*).

In case (*ii*) we must further analyze the second partial derivatives of u and v at Q. For this purpose we assume for simplicity that the boundary of Ω is of class C^3, though by taking more care one can weaken this assumption. Fixing coordinates as before, we obtain then

$$u_i=0, \quad u_n=c, \quad u_{ij}+c\,\phi_{ij} \quad (\text{at } Q)$$

where i, j range from 1 to $n-1$. Moreover, differentiating (10) and evaluating at Q gives

$$u_{ni}=\frac{dc}{dH} H_i \quad (\text{at } Q)$$

since c is no longer constant. Also from (30),

$$H_i=(\Delta\varphi)_i/(n-1) \quad (\text{at } Q).$$

Since v is the reflection of u, it now follows that all the first and second partial derivatives of these functions agree at Q, with the possible exception of u_{1n} and v_{1n}, which have the values

$$u_{1n}=-v_{1n}=\frac{dc}{dH} H_1 \quad (\text{at } Q). \tag{33}$$

Now Σ' is contained in Ω; continuing the analysis of this situation (which has already yielded the relations $\phi_{1\,l}=0$ at Q), we find that the third partial derivatives of ϕ at Q must satisfy the condition

$$\varepsilon\{\phi_{lm1}\zeta_l\zeta_m+\phi_{111}\}\leqq 0 \quad (\varepsilon=\operatorname{sign}x_1 \text{ in } \Sigma')$$

where $\zeta=(\zeta_2, \ldots, \zeta_{n-1})$ is an arbitrary real vector and the indices l and m are to be summed from 2 to $n-1$. It follows that

$$\phi_{111}\leqq 0, \quad \varphi_{ll1}\leqq 0 \quad (\text{at } Q) \tag{34}$$

since Σ' may be assumed to lie in the space $x_1>0$. Clearly (34) implies

$$H_1=(\Delta\varphi)_1/(n-1)\leqq 0.$$

Thus by (33), recalling that $dc/dH\geqq 0$, we find

$$u_{1n}\leqq 0, \quad v_{1n}\geqq 0 \quad (\text{at } Q).$$

But these are the only (possibly) unequal partial derivatives of u, v at Q. Since $u>v$ in Σ', it follows that in fact $u_{1\,n}=v_{1\,n}=0$, at Q. Hence all the first and second partial derivatives of u and v agree at Q. The remainder of the proof is the same as before.

The argument for the elliptic equation (19) proceeds almost exactly as it did earlier in Section 2, though of course taking into account the modifications in the proof of Theorem 1 which were described above.

The only slight change required involves the demonstration that u increases monotonically as one enters Ω along any non-tangential direction $\vec{s}$. The difficulty is that one might have $c(H)=0$ at some points of the boundary and $c(H)>0$ at others (of course $c(H)$ can never be negative at any boundary point because $u>0$ in Ω). In any case, just as in Section 3 one can show that at any point where $\partial u/\partial n=0$ all the first and second partial derivatives of u vanish except for $\partial^2 u/\partial n^2$, and this is positive. A not too difficult compactness argument then yields the required monotonicity property of u near the boundary. This completes the proof of Theorem 3.

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