DRP Report

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Winter Break

First, let us examine the following theorem that Professor James Serrin proved. We suppose that Ω is a bounded open connected domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is smooth. Furthermore, we assume that there exists some function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\nabla u = -1$$

in Ω. Furthermore, we suppose that $u = 0$ and

$$\frac{\partial u}{\partial n} = \text{constant}$$

on $\partial\Omega$. We claim that Ω is a ball and that $u(x) = (b^2 - r^2)/2n$, where b is the ball's radius and r is the distance from the ball's center.

Week 1

To prove this, let us suppose that T_0 is an $n-1$ dimensional hyperplane in $\mathbb{R}^n$ that does not intersect the domain Ω. We suppose that this plane is moved normal to itself until it begins to intersect the domain Ω. Let us denote this new plane by T. At this point, the plane T will separate Ω into 2 subsets. Let us denote the subset that is on the same side of T as T_0 by $\Sigma(T)$, and let its reflection across T be denoted by $\Sigma'(T)$. Notice that $\Sigma'(T)$ remains inside Ω until $\Sigma'(T)$ becomes internally tangent to Ω or T is orthogonal to the boundary of Ω. When the plane T attains either of these two positions, we may denote it by T'. Next, it can be shown that reflection across T' preserves Ω. If this is true, then Ω must be a ball because it is simply connected and symmetrical in every direction and u must be of the form $(b^2 - r^2)/2n$ [1].

Week 2

To prove this, let us define the function v in Σ' as follows:

$$v(x) = u(x')$$

where $x \in \Sigma'$ and x' is obtained by reflecting x across T'. Notice that

$$\Delta v = -1$$

in Σ'. Furthermore, we have

$$v = u$$

on $\partial\Sigma' \cap T'$ and

$$v = 0, \ \frac{\partial v}{\partial n} = \text{constant}$$

on $\partial\Sigma' \cap \text{Comp}(T')$. Let us consider the function $u - v$ in Σ'. We have

$$\Delta(u - v) = 0$$

in Σ'. Furthermore, we know that

$$u - v = 0$$

on $\partial\Sigma' \cap T'$ and

$$u - v \geq 0$$

on $\partial\Sigma' \cap \text{Comp}(T')$.

Week 3

Applying the strong version of the maximum principle, we find that either $u - v > 0$ or $u - v = 0$ in Σ'. If the latter holds, then it is evident that Ω is symmetric about the plane T'. Thus, we must prove that the former case cannot happen. Before we prove this, we should discuss the Hopf lemma. The statement of this lemma is as follows: Let Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let f be a real-valued function continuous on the closure of Ω and harmonic on Ω. If x is a boundary point such that $f(x) > f(y)$ for all $y \in \Omega$ sufficiently close to x, then the normal derivative of f at x is strictly positive.

Week 4

To finish proving the theorem, we must show that $u - v > 0$ is false. First, suppose that Σ' is internally tangent to the boundary of Ω at some point P not on T'. Then $u - v = 0$ at P. Using Hopf's lemma, we may deduce that

$$\frac{\partial}{\partial n}(u - v) > 0$$

at P. This contradicts the fact that $\partial u/\partial n = \partial v/\partial n = \text{constant}$ at P. If T is orthogonal to the boundary of Ω at some point Q, Hopf's lemma does not apply. Thus, we will show that all the second derivatives of $u - v$ are 0 at Q. By our hypothesis, the bounder of Ω is of class C^2. We may consider a rectangular coordinate frame with origin at Q. Furthermore, we may suppose that the x_n axis is directed along the inward normal to $\partial\Omega$ at Q and that the x_1 axis is normal to T'. With this coordinate system, we may represent the boundary of Ω locally by the equation

$$x_n = \phi(x_1, \ldots, x_{n-1})$$

where $\phi \in C^2$. Since u is twice continuously differentiable the condition $u = 0$ on Ω can be written as follows:

$$u(x_1, \ldots, x_{n-1}, \phi) \equiv 0$$

Then, the boundary condition $\partial u / \partial n = c$ on $\partial\Omega$ may be expressed as

$$\frac{\partial u}{\partial x_n} - \sum_{k=1}^{n-1} \frac{\partial u}{\partial x_k} \frac{\partial \phi}{\partial x_k} = c \left\{ 1 + \sum_{k=1}^{n-1} \left(\frac{\partial \phi}{\partial x_k} \right)^2 \right\}^{1/2}$$

We may introduce some simple notation:

$$u_i = \frac{\partial u}{\partial x_i}$$

for every i. Differentiating $u = 0$ with respect to x_i, we find that

$$u_i + u_n \phi_i = 0$$

If we evaluate this at Q where $\phi = 0$, we find

$$u_i = 0$$

and

$$u_n = c$$

If we differentiate with respect to x_j, we find that

$$u_{ij} + c\phi_{ij} = 0$$

at Q. Furthermore, we obtain

$$u_{ni} = 0$$

at Q. Since we have

$$u_{nn} = -\sum_{i=1}^{n-1} u_{ii} - 1 = c\Delta\phi - 1$$

at Q, we have found all the first and second derivatives of u at Q. We also know that

$$\phi_{1l} = 0$$

at Q for $2 \leq l \leq n-1$. From all of this information and the fact that

$$v(x_1, x_2, \ldots, x_n) = u(-x_1, x_2, \ldots, x_n)$$

we may deduce that all the second derivatives of $u - v$ are 0 at Q.

Week 5

Next, we will prove the following lemma:

Lemma 1. *Let D^* be a domain with C^2 boundary and let T be a plane containing the normal to ∂D^* at some point Q. Let D then denote the portion of D^* lying on some particular side of T. Suppose that w is of class C^2 in the closure of D and satisfies $\Delta w \leq 0$ in D, while also $w \geq 0$ in D and $w = 0$ at Q. Let $\vec{s}$ be any direction at Q which enters D non-tangentially. Then either*

$$\frac{\partial w}{\partial s} > 0$$

or

$$\frac{\partial^2 w}{\partial s^2} > 0$$

at Q unless $w = 0$.

Let us apply this lemma to the function $w = u - v$ in Σ'. Since $w > 0$ there and $w = 0$ at Q, this yields

$$\frac{\partial (u - v)}{\partial s} > 0$$

or

$$\frac{\partial^2 (u - v)}{\partial s^2} > 0$$

which contradicts the fact that both u and v have the same first and second partial derivatives at Q. This proves the theorem.

Week 6

We will now attempt to prove this theorem. We will let K_1 be a ball internally tangent to D^* at Q and which only intersects the boundary of D^* at Q. Next, we will let K_2 be a ball with center at Q and radius $\frac{1}{2}r_1$, where r_1 is the radius of K_1. Let $K' = K_1 \cap K_2 \cap D$. We may define the following function:

$$z = z(x) = x_1(e^{-\alpha r^2} - e^{-\alpha r_1^2})$$

where α is a positive constant. We assume that the origin is the center of K_1, that T is the plane $x_1 = 0$, and that D is where $x_1 > 0$. Notice that $z > 0$ in K' and $z = 0$ in $\partial K_1 \cup T$. We then compute the laplacian of z as follows:

$$\Delta z = \sum_{i=1}^{n} \frac{\partial^2 z}{\partial {x_i}^2} = 2\alpha x_1 e^{-ar^2}(2\alpha r^2 - (n+2))$$

If we choose α to be sufficiently large, then we can ensure that $\Delta z > 0$ in K'.

Week 7

We continue the proof from the week before. We may suppose w is not equal to zero at all points in D. By the maximum principle, then, we know that $w > 0$ in D. From this, we find that $w \geq \varepsilon x_1$ on $\partial K' \cap \partial K_2$, and we know that $w \geq 0$ on $\partial K' \cap \partial K_1$ and $\partial K' \cap \partial T$ by our assumptions on W. It is also evident that $z \leq x_1$ on $\partial K' \cap \partial K_2$. We find that $w - \varepsilon z$ is non-negative on $\partial K'$ and is zero at Q. Furthermore, we have $\Delta(w - \varepsilon z) = \Delta w - \varepsilon \Delta z < 0$ in K'. The maximum principle informs us that $w - \varepsilon z > 0$ in K'. Therefore, we know that

$$\frac{\partial(w - \varepsilon z)}{\partial s} > 0$$

or

$$\frac{\partial^2(w - \varepsilon z)}{\partial s^2} \geq 0$$

We compute

$$\frac{\partial z}{\partial s} = 0$$

and

$$\frac{\partial^2 z}{\partial s^2} > 0$$

at Q, which completes the proof of the theorem.

Week 8

Next, we shall go over another proof of the same theorem. First, we compute

$$\Delta\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial}{\partial r}(\Delta u) + 2\Delta u = -2$$

Notice that

$$\int_\Omega \left[2u - r\frac{\partial u}{\partial r}\right] dx = \int_\Omega \left[-u\Delta\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\Delta u\right] dx$$

Appealing to Green's identity, we have

$$\int_{\partial\Omega} \left[-u\frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\right] dS = \int_{\partial\Omega} r\frac{\partial r}{\partial n}\left(\frac{\partial u}{\partial n}\right)^2 dS = nc^2 V$$

where V is the volume of Ω. By Green's theorem again, we have

$$\int_\Omega r\frac{\partial u}{\partial r} dx = \int_\Omega \nabla(1/2)r^2 \cdot \nabla u dx = -n\int_\Omega u dx$$

so that

$$(n+2)\int_\Omega u dx = nc^2 V$$

Week 9

By the Cauchy-Schwarz Inequality, we have

$$1 = (\Delta u)^2 \leq n \sum_{i=1}^{n} u_{ii}^2 \leq n \sum_{i,j=1}^{n} u_{ij}^2$$

Using this, we may deduce that

$$\Delta\left(|\nabla u|^2 + \frac{2}{n}u\right) = 2\sum_{i,j=1}^{n} u_{ij}^2 - \frac{2}{n} \geq 0$$

We know that

$$|\nabla u|^2 + \frac{2}{n}u = c^2$$

on $\partial\Omega$, so the strong maximum principle informs us that

$$|\nabla u|^2 + \frac{2}{n}u < c^2$$

in Ω or

$$|\nabla u|^2 + \frac{2}{n}u \equiv c^2$$

in Ω. The first case cannot happen because it would contradict the equation $(n+2)\int_\Omega u dx = nc^2V$. Thus

$$|\nabla u|^2 + \frac{2}{n}u$$

must be constant in Ω.

Week 10

This means that its Laplacian must vanish, so that

$$1 = (\Delta u)^2 \leq n \sum_{i=1}^{n} u_{ii}^2 \leq n \sum_{i,j=1}^{n} u_{ij}^2$$

This equation coupled with the fact that $\Delta u = -1$ implies that

$$u_{ij} = -\frac{\delta_{ij}}{m}$$

Thus, we may write

$$u = \frac{1}{2n}(A - r^2)$$

for some constant A. Since $u = 0$ on $\partial\Omega$, we may deduce that A is positive and that Ω is a ball [2]. Also, this relies on [**?**].

References

[1] James Serrin. A symmetry problem in potential theory. *Archive for Rational Mechanics and Analysis*, 1971.

[2] Hans Weinberger. Remark on the preceding paper of serrin. *Archive for Rational Mechanics and Analysis*, 1971.