



# A SYMMETRY PROBLEM

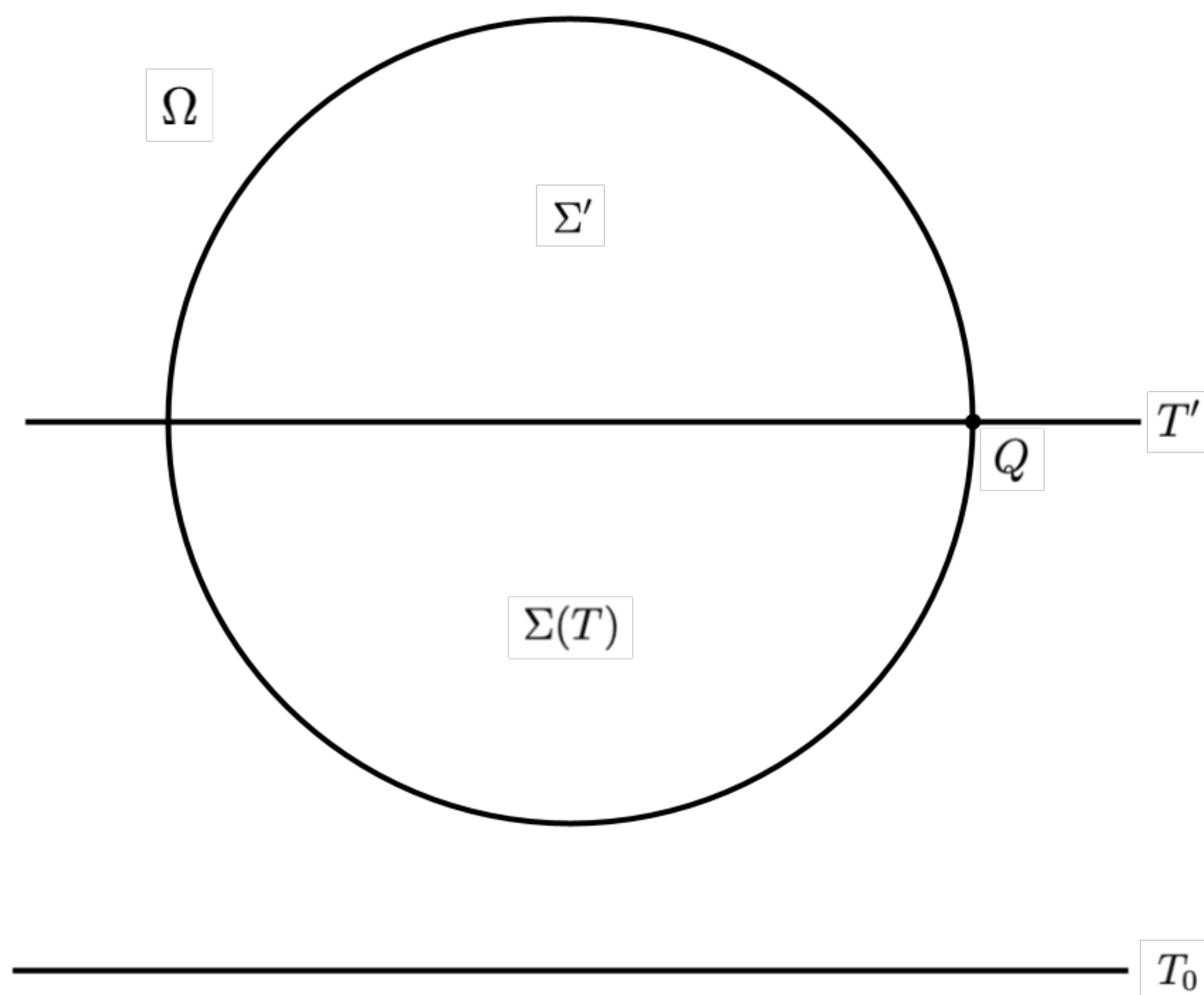
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## Introduction

Let us consider the following problem. Let  $\Omega \subseteq \mathbb{R}^n$  be a domain that is bounded, open, and connected. Furthermore, suppose that the boundary  $\partial\Omega$  is smooth. Let  $u : \Omega \rightarrow \mathbb{R}$  be a  $C^2$  function that satisfies the following conditions:  $\Delta u = -1$  in  $\Omega$ . and  $u = 0$  and  $\frac{\partial u}{\partial n} = c$  on  $\partial\Omega$  for some constant  $c$ . Then,  $\Omega$  must be a ball. Furthermore, we know that  $u(x) = (b^2 - r^2)/2n$ , where  $b$  is the ball's radius and  $r$  is the distance to its center.

## First Proof

The first proof we present is from Professor James Serrin [3]. This proof utilizes the moving plane method. Let  $T_0$  be a  $n - 1$  dimensional hyperplane in  $\mathbb{R}^n$  that does not intersect the domain  $\Omega$ . We begin to move this plane normal to itself until it intersects  $\Omega$ . When this occurs, the new plane  $T$  splits  $\Omega$  into two parts. The part of  $\Omega$  that lies on the same side of  $T$  as our initial plane  $T_0$  is denoted by  $\Sigma(T)$ . We reflect  $\Sigma(T)$  in  $T$  to obtain  $\Sigma' := \Sigma'(T)$ . As  $T$  moves through  $\Omega$ ,  $\Sigma'$  will remain in  $\Omega$  until the set  $\Sigma'$  meets  $\Omega$  at a point  $P$  or  $T$  becomes orthogonal to  $\Omega$  at some point  $Q$ . When either of these occurs, we stop moving the plane  $T$ , and we denote the resulting plane by  $T'$ . We claim that  $\Omega$  is symmetric about  $T'$ . Showing this would prove the theorem. To see how, we recall that the plane  $T_0$  was chosen arbitrarily. If  $\Omega$  is symmetric about  $T'$ , then  $\Omega$  is symmetric in all possible directions. Since  $\Omega$  is simply connected and has this strong symmetry property, it must be a ball.



To prove this, we introduce the function  $v : \Sigma' \rightarrow \mathbb{R}$  defined by  $v(x) = u(x')$  for  $x \in \Sigma'$ , where  $x'$  is the reflection of  $x$  across  $T'$ . By the maximum principle, we deduce that  $u - v > 0$  or  $u - v = 0$  in  $\Sigma'$ . Suppose that  $u - v > 0$ . If  $\Sigma'$  is internally tangent to  $\Omega$  at some point  $P$ , then we may appeal to the boundary point maximum principle to deduce that  $\frac{\partial}{\partial n}(u - v) > 0$  at  $P$  [1]. However, we know that  $\partial u / \partial n = \partial v / \partial n = c$ . If  $T'$  is orthogonal to the boundary of  $\Omega$  at some point  $Q$ , then we show that  $u$  and  $v$  have the same first and second derivatives at  $Q$ . Using a modified version of the boundary point maximum principle, we can also show that  $\frac{\partial}{\partial s}(u - v) > 0$  or  $\frac{\partial^2}{\partial s^2}(u - v) > 0$  for any direction  $s$  that enters  $\Sigma'$  non-tangentially at  $Q$ . However, this directly contradicts the fact that  $u$  and  $v$  have the same first and second derivatives at  $Q$ . We may thus conclude that  $\Omega$  is symmetric about  $T'$ .

## Second Proof

The second proof we present is from Weinberger [2]. To start, we first compute

$$\Delta \left( r \frac{\partial u}{\partial r} \right) = r \frac{\partial}{\partial r} (\Delta u) + 2\Delta = -2$$

where  $r$  is the distance to the origin. Using this and the fact that  $\Delta u = -1$ , we obtain

$$\int_{\Omega} \left[ 2u - r \frac{\partial u}{\partial r} \right] dx = \int_{\Omega} \left[ -u \Delta \left( r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \Delta u \right] dx$$

Using Green's identity yields

$$\int_{\Omega} \left[ -u \Delta \left( r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \Delta u \right] dx = \int_{\partial\Omega} \left[ -u \frac{\partial}{\partial n} \left( r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \frac{\partial u}{\partial n} \right] dS$$

By assumption, we have  $u = 0$  on  $\partial\Omega$ . Thus, we find that

$$\int_{\partial\Omega} \left[ -u \frac{\partial}{\partial n} \left( r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \frac{\partial u}{\partial n} \right] dS = \int_{\partial\Omega} r \frac{\partial r}{\partial n} \left( \frac{\partial u}{\partial n} \right)^2 dS$$

By assumption, we know that  $\partial u / \partial n = c$  on  $\partial\Omega$ . Thus, we find that

$$\int_{\partial\Omega} r \frac{\partial r}{\partial n} \left( \frac{\partial u}{\partial n} \right)^2 dS = c^2 \int_{\partial\Omega} r \frac{\partial r}{\partial n} dS$$

Appealing to the Divergence Theorem and using the fact that  $\Delta \left( \frac{1}{2} r^2 \right) = r \Delta r$ , we obtain

$$c^2 \int_{\partial\Omega} r \frac{\partial r}{\partial n} dS = c^2 \int_{\Omega} \Delta \left( \frac{1}{2} r^2 \right) dx = c^2 n \int_{\Omega} dx = nc^2 V$$

Green's theorem also implies

$$\int_{\Omega} r \frac{\partial u}{\partial r} dx = -n \int_{\Omega} u dx$$

so that substitution yields

$$(n + 2) \int_{\Omega} u dx = nc^2 V$$

However, we also note that

$$1 = (\Delta u)^2 \leq n \sum_{i=1}^n u_{ii}^2 \leq n \sum_{i,j} u_{ij}^2$$

by the Cauchy-Schwarz inequality. From this, we deduce that

$$\Delta \left( |\nabla u|^2 + \frac{2}{n} u \right) = 2 \sum_{i,j} u_{ij}^2 - \frac{2}{n} \geq 0$$

Using this and the fact that  $|\nabla u|^2 + (2/n)u = c^2$  on  $\partial\Omega$ , we may appeal to the maximum principle to deduce that  $|\nabla u| + (2/n)u < c^2$  in  $\Omega$  or  $|\nabla u| + (2/n)u = c^2$  in  $\Omega$ . If the former inequality held, then we could integrate over  $\Omega$  to deduce that

$$(n + 2) \int_{\Omega} u dx < nc^2 V$$

This contradiction informs us that  $|\nabla u|^2 + (2/n)u = c^2$  in  $\Omega$  so that

$$1 = n \sum_{i=1}^n u_{ii}^2 = \sum_{i,j} u_{ij}^2$$

which implies that  $u_{ij} = -\delta_{ij}/n$ . Solving the corresponding partial differential equations yields

$$u = \frac{1}{2n} (B - r^2)$$

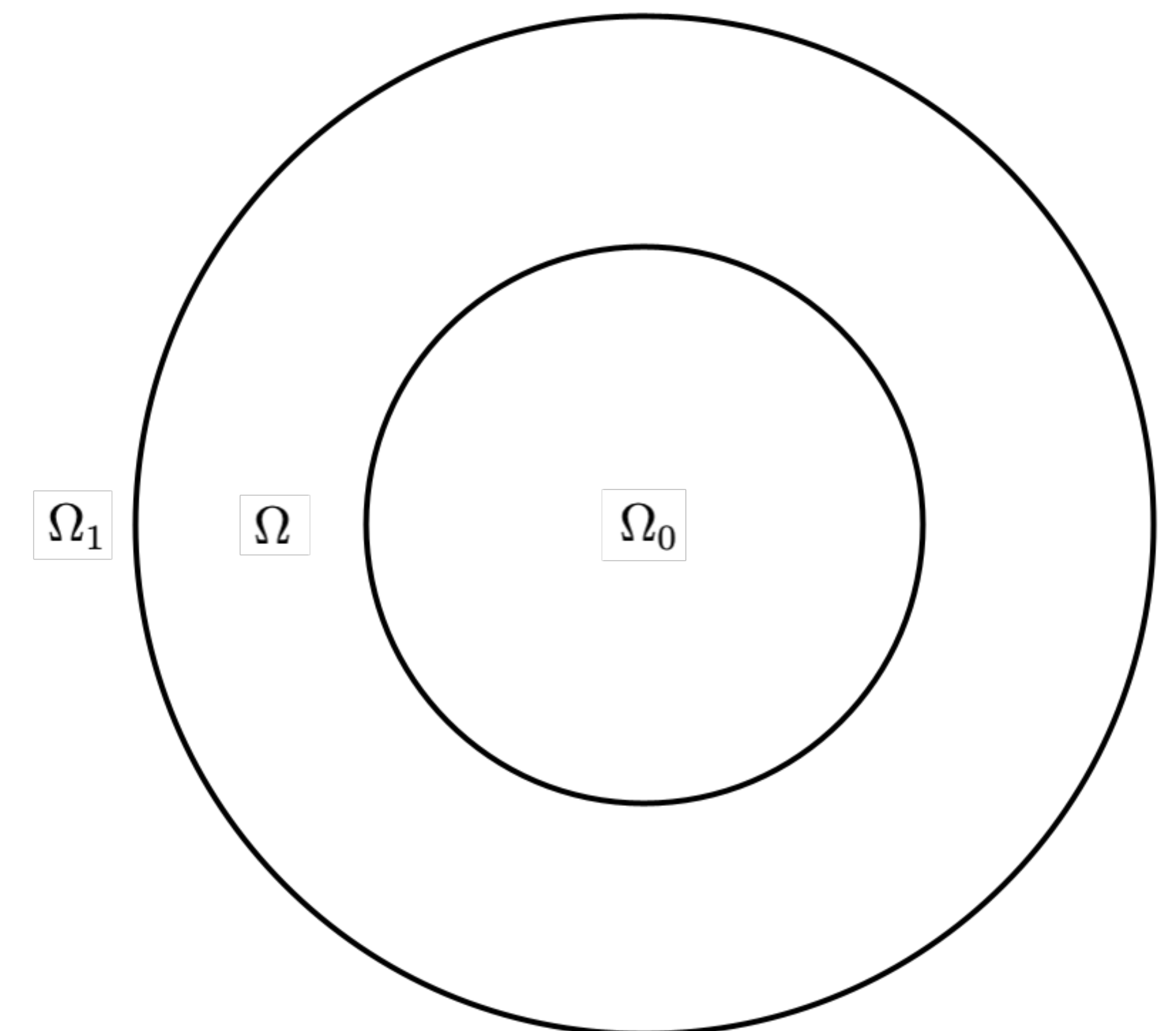
where  $B$  is a constant. Since  $u = 0$  on  $\partial\Omega$ ,  $B$  is positive and  $\Omega$  is a ball of radius  $B^{1/2}$ .

## Applications

This theorem is significant because it allows us to determine the shape of  $\Omega$  from properties of  $u$ . It also has many applications in physics. For example, we may consider an incompressible viscous fluid moving through a straight pipe of cross sectional form  $\Omega$ . If we fix a rectangular coordinate system with the  $z$ -axis directed along the pipe, then the velocity  $u$  depends only on  $x$  and  $y$ , and it satisfies the differential equation  $\Delta u = -A$  for some constant  $A$ . Furthermore, because the fluid is viscous, we know that  $u = 0$  on  $\partial\Omega$ ; that is, there is no movement on the boundary of the pipe. Finally, we note that  $\mu \partial u / \partial n$  is the tangential stress on the pipe wall, where  $\mu$  is the viscosity constant. If the tangential stress is constant, then we may apply the above theorem to conclude that  $\Omega$  is a circular cross section.

## Generalizations

There is an interesting extension of this theorem from Wolfgang Reichel [4]. Let  $\Omega_0$  and  $\Omega_1$  be smooth domains and let  $\Omega = \Omega_0 \setminus \overline{\Omega}_1$  be connected. Suppose that  $f \in C^1$  is a function satisfying  $\Delta u + f(u, |\nabla u|) = 0$  in  $\Omega$ ,  $0 < u < a$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega_0$ ,  $u = a$  on  $\partial\Omega_1$ , and  $\partial u / \partial n = c_i$  on  $\Omega_i$ . Then, we conclude that  $\Omega$  is an annulus and  $u$  is radially symmetric and decreasing in  $r$ .



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## References

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