An Overdetermined Problem in Symmetry

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Background

Let us consider the following problem. Let $\Omega \subseteq \mathbb{R}^n$ be domain that is bounded, open, and connected. Furthermore, suppose that the boundary $\partial\Omega$ of Ω is smooth. Let $u : \Omega \to \mathbb{R}$ be a function that satisfies the following conditions: $\Delta u = -1$ in Ω, $u = 0$ on $\partial\Omega$, and $\frac{\partial u}{\partial n} = c$ on Ω for some constant c. Then, Ω must be a ball. Furthermore, we know that $u(x) = (b^2 - r^2)/2n$, where b is the ball's radius and r is the distance to its center.

This theorem has many applications in physics. For example, we may consider an incompressible fluid moving through a straight pipe of cross sectional form Ω. If we fix a rectangular coordinate system with the z axis in the same direction as the pipe, then the velocity u depends only on x and y, and it satisfies the differential equation $\Delta u = -A$ for some constant A. Furthermore, because the fluid is viscous, we know that $u = 0$ on $\partial\Omega$; that is, there is no movement on the boundary of the pipe. Finally, we note that $\mu\partial\mu/\partial n$ is the tangential stress on the pipe wall, where μ is viscosity constant. If the tangential stress is constant, then we may apply the above theorem to conclude that Ω is a circular cross section.

This theorem can also be applied

First Proof

We will first prove this theorem by the moving plane method. Let T_0 be a $n-1$ dimensional hyperplane in $\mathbb{R}^n$ that does not intersect the domain Ω. We begin to move this plane until it intersects Ω. When this occurs, the new plane T splits Ω into two pieces. The piece of Ω that lies on the same side of T as our initial plane T_0 is denoted by $\Sigma(T)$. We reflect $\Sigma(T)$ in T to obtain $\Sigma'(T)$. As T is moved through Ω, it is evident that $\Sigma'(T)$ will remain in Ω unless one of the following two events occurs:

The set $\Sigma'(T)$ meets Ω at a point P

T becomes orthogonal to Ω at some point Q

When this occurs, we stop moving the plane T, and we denote the resulting plane by T'.

We claim that Ω is symmetric about T'. This is crucial because proving this would also prove the theorem by extension. To see how, we recall that the plane T was chosen arbitrarily. If we can show that Ω is symmetric about T', then we have shown that Ω is symmetric in all possible directions. Since Ω is simply connected and maintains this strong symmetry property, it must be a ball.

For convenience, let us denote $\Sigma' := \Sigma'(T)$. In order to show that this symmetry property holds, we introduce a new function $v : \Sigma' \to \mathbb{R}$ defined as follows: $v(x) = u(x')$ for $x \in \Sigma'$, where x' is obtained by reflecting x across T'. If we can show that $u = v$ in Σ', it will follow that Ω is symmetric about T'. First, we note some properties of v that can easily be obtained from the corresponding properties of u. It can easily be seen that $\Delta v = -1$ in Σ', that $v = u$ on the plane T', that $v = 0$ and $\partial v/\partial n = c$ on the boundary of Σ'. Using these facts, we deduce that $\Delta(u - v) = 0$ in Σ' and that $u - v \geq 0$ on the boundary of Σ'. By the Maximum Principle, we have $u - v > 0$ at every point in Σ' or $u - v = 0$ in Σ'. As stated above, we are trying to prove that the latter is true. Thus, we must prove that $u - v > 0$ cannot occur. For the sake of contradiction, let us suppose that $u - v > 0$ in Σ'. First, we suppose that Σ' is internally tangent to Ω at some point P. By the definitions of u and v, we have $u - v = 0$ at P. Appealing to the boundary point maximum principle, we find that $\frac{\partial}{\partial n}(u - v) > 0$ at P. However, we previously established that $\partial u/\partial n = \partial v/\partial n = c$. Thus, we have reached a contradiction. Next, we consider the case in which T' is orthogonal to the boundary of Ω at some point Q. In this case, we cannot apply the boundary point maximum principle directly because there is no ball internally tangent to Σ' at Q.

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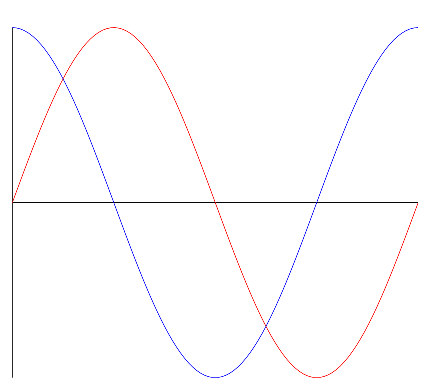

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Baz	3.14	83,742	δ
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References

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