

DRP Report

Ethan Martirosyan

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Winter Break

First, let us examine the following theorem that Professor James Serrin proved. We suppose that Ω is a bounded open connected domain in \mathbb{R}^n whose boundary $\partial\Omega$ is smooth. Furthermore, we assume that there exists some function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\nabla u = -1$$

in Ω . Furthermore, we suppose that $u = 0$ and

$$\frac{\partial u}{\partial n} = \text{constant}$$

on $\partial\Omega$. We claim that Ω is a ball and that $u(x) = (b^2 - r^2)/2n$, where b is the ball's radius and r is the distance from the ball's center.

Week 1

To prove this, let us suppose that T_0 is an $n - 1$ dimensional hyperplane in \mathbb{R}^n that does not intersect the domain Ω . We suppose that this plane is moved normal to itself until it begins to intersect the domain Ω . Let us denote this new plane by T . At this point, the plane T will separate Ω into 2 subsets. Let us denote the subset that is on the same side of T as T_0 by $\Sigma(T)$, and let its reflection across T be denoted by $\Sigma'(T)$. Notice that $\Sigma'(T)$ remains inside Ω until $\Sigma'(T)$ becomes internally tangent to Ω or T is orthogonal to the boundary of Ω . When the plane T attains either of these two positions, we may denote it by T' . Next, it can be shown that reflection across T' preserves Ω . If this is true, then Ω must be a ball because it is simply connected and symmetrical in every direction and u must be of the form $(b^2 - r^2)/2n$ [1].

Week 2

To prove this, let us define the function v in Σ' as follows:

$$v(x) = u(x')$$

where $x \in \Sigma'$ and x' is obtained by reflecting x across T' . Notice that

$$\Delta v = -1$$

in Σ' . Furthermore, we have

$$v = u$$

on $\partial\Sigma' \cap T'$ and

$$v = 0, \quad \frac{\partial v}{\partial n} = \text{constant}$$

on $\partial\Sigma' \cap \text{Comp}(T')$. Let us consider the function $u - v$ in Σ' . We have

$$\Delta(u - v) = 0$$

in Σ' . Furthermore, we know that

$$u - v = 0$$

on $\partial\Sigma' \cap T'$ and

$$u - v \geq 0$$

on $\partial\Sigma' \cap \text{Comp}(T')$.

Week 3

Applying the strong version of the maximum principle, we find that either $u - v > 0$ or $u - v = 0$ in Σ' . If the latter holds, then it is evident that Ω is symmetric about the plane T' . Thus, we must prove that the former case cannot happen. Before we prove this, we should discuss the Hopf lemma. The statement of this lemma is as follows: Let Ω is a bounded domain in \mathbb{R}^n with smooth boundary. Let f be a real-valued function continuous on the closure of Ω and harmonic on Ω . If x is a boundary point such that $f(x) > f(y)$ for all $y \in \Omega$ sufficiently close to x , then the normal derivative of f at x is strictly positive.

Week 4

To finish proving the theorem, we must show that $u - v > 0$ is false. First, suppose that Σ' is internally tangent to the boundary of Ω at some point P not on T' . Then $u - v = 0$ at P . Using Hopf's lemma, we may deduce that

$$\frac{\partial}{\partial n}(u - v) > 0$$

at P . This contradicts the fact that $\partial u / \partial n = \partial v / \partial n = \text{constant}$ at P . If T is orthogonal to the boundary of Ω at some point Q , Hopf's lemma does not apply. Thus, we will show that all the second derivatives of $u - v$ are 0 at Q . By our hypothesis, the boundary of Ω is of class C^2 . We may consider a rectangular coordinate frame with origin at Q . Furthermore, we may suppose that the x_n axis is directed along the inward normal to $\partial\Omega$ at Q and that

the x_1 axis is normal to T' . With this coordinate system, we may represent the boundary of Ω locally by the equation

$$x_n = \phi(x_1, \dots, x_{n-1})$$

where $\phi \in C^2$. Since u is twice continuously differentiable the condition $u = 0$ on Ω can be written as follows:

$$u(x_1, \dots, x_{n-1}, \phi) \equiv 0$$

Then, the boundary condition $\partial u / \partial n = c$ on $\partial\Omega$ may be expressed as

$$\frac{\partial u}{\partial x_n} - \sum_{k=1}^{n-1} \frac{\partial u}{\partial x_k} \frac{\partial \phi}{\partial x_k} = c \left\{ 1 + \sum_{k=1}^{n-1} \left(\frac{\partial \phi}{\partial x_k} \right)^2 \right\}^{1/2}$$

We may introduce some simple notation:

$$u_i = \frac{\partial u}{\partial x_i}$$

for every i . Differentiating $u = 0$ with respect to x_i , we find that

$$u_i + u_n \phi_i = 0$$

If we evaluate this at Q where $\phi = 0$, we find

$$u_i = 0$$

and

$$u_n = c$$

If we differentiate with respect to x_j , we find that

$$u_{ij} + c \phi_{ij} = 0$$

at Q . Furthermore, we obtain

$$u_{ni} = 0$$

at Q . Since we have

$$u_{nn} = - \sum_{i=1}^{n-1} u_{ii} - 1 = c \Delta \phi - 1$$

at Q , we have found all the first and second derivatives of u at Q . We also know that

$$\phi_{1l} = 0$$

at Q for $2 \leq l \leq n-1$. From all of this information and the fact that

$$v(x_1, x_2, \dots, x_n) = u(-x_1, x_2, \dots, x_n)$$

we may deduce that all the second derivatives of $u - v$ are 0 at Q .

Week 5

Next, we will prove the following lemma:

Lemma 1. *Let D^* be a domain with C^2 boundary and let T be a plane containing the normal to ∂D^* at some point Q . Let D then denote the portion of D^* lying on some particular side of T . Suppose that w is of class C^2 in the closure of D and satisfies $\Delta w \leq 0$ in D , while also $w \geq 0$ in D and $w = 0$ at Q . Let \vec{s} be any direction at Q which enters D non-tangentially. Then either*

$$\frac{\partial w}{\partial s} > 0$$

or

$$\frac{\partial^2 w}{\partial s^2} > 0$$

at Q unless $w = 0$.

Let us apply this lemma to the function $w = u - v$ in Σ' . Since $w > 0$ there and $w = 0$ at Q , this yields

$$\frac{\partial(u - v)}{\partial s} > 0$$

or

$$\frac{\partial^2(u - v)}{\partial s^2} > 0$$

which contradicts the fact that both u and v have the same first and second partial derivatives at Q . This proves the theorem.

Week 6

We will now attempt to prove this theorem. We will let K_1 be a ball internally tangent to D^* at Q and which only intersects the boundary of D^* at Q . Next, we will let K_2 be a ball with center at Q and radius $\frac{1}{2}r_1$, where r_1 is the radius of K_1 . Let $K' = K_1 \cap K_2 \cap D$. We may define the following function:

$$z = z(x) = x_1(e^{-\alpha r^2} - e^{-\alpha r_1^2})$$

where α is a positive constant. We assume that the origin is the center of K_1 , that T is the plane $x_1 = 0$, and that D is where $x_1 > 0$. Notice that $z > 0$ in K' and $z = 0$ in $\partial K_1 \cup T$. We then compute the laplacian of z as follows:

$$\Delta z = \sum_{i=1}^n \frac{\partial^2 z}{\partial x_i^2} = 2\alpha x_1 e^{-\alpha r^2} (2\alpha r^2 - (n+2))$$

If we choose α to be sufficiently large, then we can ensure that $\Delta z > 0$ in K' .

Week 7

We continue the proof from the week before. We may suppose w is not equal to zero at all points in D . By the maximum principle, then, we know that $w > 0$ in D . From this, we find that $w \geq \varepsilon x_1$ on $\partial K' \cap \partial K_2$, and we know that $w \geq 0$ on $\partial K' \cap \partial K_1$ and $\partial K' \cap \partial T$ by our assumptions on W . It is also evident that $z \leq x_1$ on $\partial K' \cap \partial K_2$. We find that $w - \varepsilon z$ is non-negative on $\partial K'$ and is zero at Q . Furthermore, we have $\Delta(w - \varepsilon z) = \Delta w - \varepsilon \Delta z < 0$ in K' . The maximum principle informs us that $w - \varepsilon z > 0$ in K' . Therefore, we know that

$$\frac{\partial(w - \varepsilon z)}{\partial s} > 0$$

or

$$\frac{\partial^2(w - \varepsilon z)}{\partial s^2} \geq 0$$

We compute

$$\frac{\partial z}{\partial s} = 0$$

and

$$\frac{\partial^2 z}{\partial s^2} > 0$$

at Q , which completes the proof of the theorem.

Week 8

Next, we shall go over another proof of the same theorem. First, we compute

$$\Delta\left(r \frac{\partial u}{\partial r}\right) = r \frac{\partial}{\partial r}(\Delta u) + 2\Delta u = -2$$

Notice that

$$\int_{\Omega} \left[2u - r \frac{\partial u}{\partial r} \right] dx = \int_{\Omega} \left[-u \Delta\left(r \frac{\partial u}{\partial r}\right) + r \frac{\partial u}{\partial r} \Delta u \right] dx$$

Appealing to Green's identity, we have

$$\int_{\partial\Omega} \left[-u \frac{\partial}{\partial n} \left(r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \frac{\partial u}{\partial n} \right] dS = \int_{\partial\Omega} r \frac{\partial r}{\partial n} \left(\frac{\partial u}{\partial n} \right)^2 dS = nc^2 V$$

where V is the volume of Ω . By Green's theorem again, we have

$$\int_{\Omega} r \frac{\partial u}{\partial r} dx = \int_{\Omega} \nabla(1/2)r^2 \cdot \nabla u dx = -n \int_{\Omega} u dx$$

so that

$$(n+2) \int_{\Omega} u dx = nc^2 V$$

Week 9

By the Cauchy-Schwarz Inequality, we have

$$1 = (\Delta u)^2 \leq n \sum_{i=1}^n u_{ii}^2 \leq n \sum_{i,j=1}^n u_{ij}^2$$

Using this, we may deduce that

$$\Delta \left(|\nabla u|^2 + \frac{2}{n} u \right) = 2 \sum_{i,j=1}^n u_{ij}^2 - \frac{2}{n} \geq 0$$

We know that

$$|\nabla u|^2 + \frac{2}{n} u = c^2$$

on $\partial\Omega$, so the strong maximum principle informs us that

$$|\nabla u|^2 + \frac{2}{n} u < c^2$$

in Ω or

$$|\nabla u|^2 + \frac{2}{n} u \equiv c^2$$

in Ω . The first case cannot happen because it would contradict the equation $(n+2) \int_{\Omega} u dx = nc^2 V$. Thus

$$|\nabla u|^2 + \frac{2}{n} u$$

must be constant in Ω .

Week 10

This means that its Laplacian must vanish, so that

$$1 = (\Delta u)^2 \leq n \sum_{i=1}^n u_{ii}^2 \leq n \sum_{i,j=1}^n u_{ij}^2$$

This equation coupled with the fact that $\Delta u = -1$ implies that

$$u_{ij} = -\frac{\delta_{ij}}{m}$$

Thus, we may write

$$u = \frac{1}{2n}(A - r^2)$$

for some constant A . Since $u = 0$ on $\partial\Omega$, we may deduce that A is positive and that Ω is a ball [2].

References

- [1] James Serrin. A symmetry problem in potential theory. *Archive for Rational Mechanics and Analysis*, 1971.
- [2] Hans Weinberger. Remark on the preceding paper of serrin. *Archive for Rational Mechanics and Analysis*, 1971.