



A SYMMETRY PROBLEM

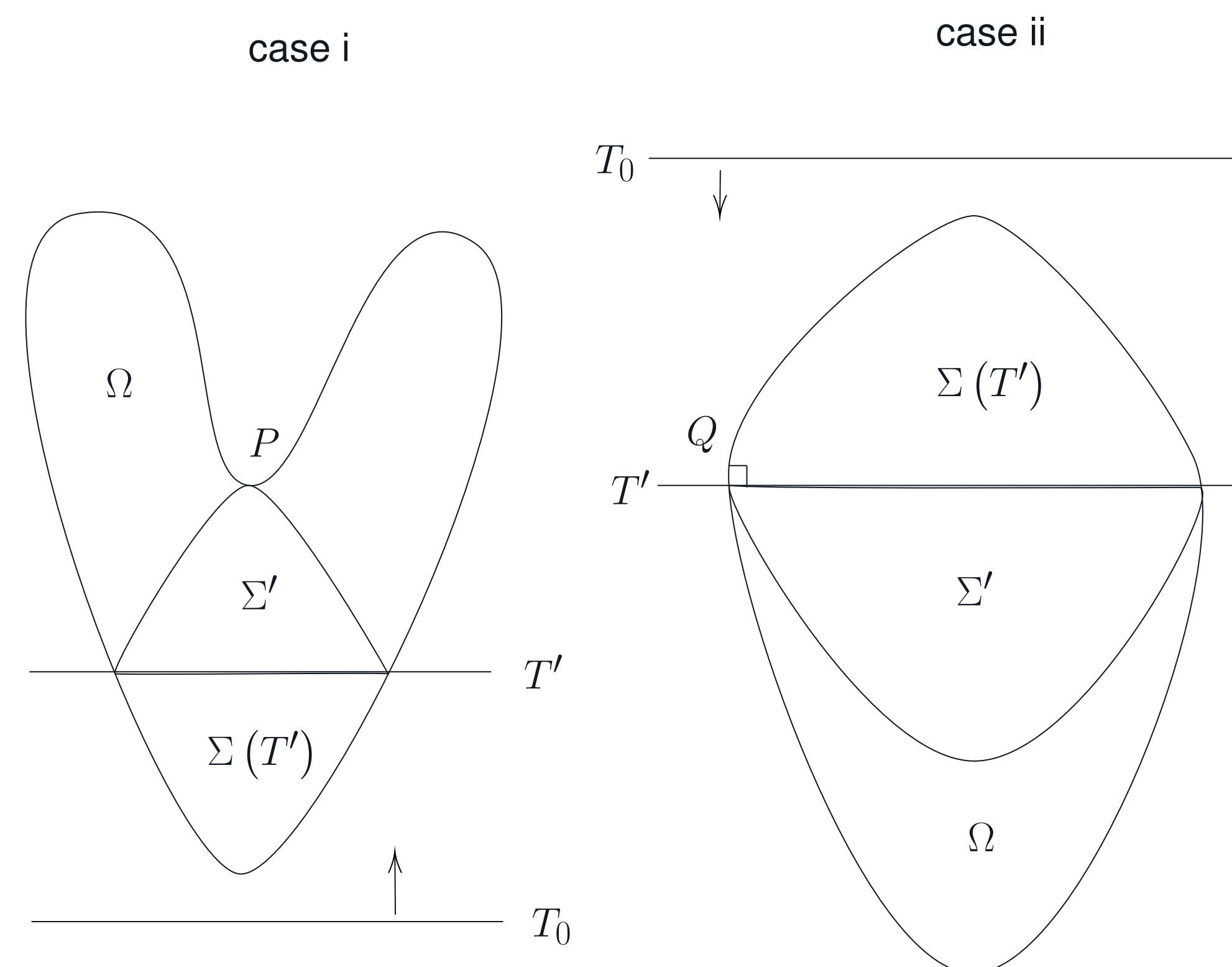
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Serrin's Overdetermined Problem

Let us consider the following problem. Let $\Omega \subseteq \mathbb{R}^n$ be a domain that is bounded, open, and connected. Furthermore, suppose that the boundary $\partial\Omega$ is smooth. Let $u : \Omega \rightarrow \mathbb{R}$ be a C^2 function that satisfies the following conditions: $\Delta u = -1$ in Ω and $u = 0$ and $\frac{\partial u}{\partial \nu} = c$ on $\partial\Omega$ for some constant c and ν is the outward normal vector to $\partial\Omega$. Then, Ω must be a ball. Furthermore, we know that $u(x) = (b^2 - r^2)/2n$, where b is the ball's radius and r is the distance to its center.

First Proof

The first proof we present is from Professor James Serrin himself [3]. This proof utilizes the **moving plane method**. Let T_0 be a $n - 1$ dimensional hyperplane in \mathbb{R}^n that does not intersect the domain Ω . We begin to move this plane normal to itself until it intersects Ω . When this occurs, the new plane T splits Ω into two parts. The part of Ω that lies on the same side of T as our initial plane T_0 is denoted by $\Sigma(T)$. We reflect $\Sigma(T)$ in T to obtain $\Sigma' := \Sigma'(T)$. As T moves through Ω , Σ' will remain in Ω until it becomes internally tangent to Ω at a point P (case i) or T becomes orthogonal to Ω at some point Q (case ii). When either of these occurs, we stop moving the plane T , and we denote the resulting plane by T' . We claim that Ω is symmetric about T' . Showing this would prove the theorem. To see how, we recall that the plane T_0 was chosen arbitrarily. If Ω is symmetric about T' , then Ω is symmetric in all possible directions. Since Ω is simply connected and has this strong symmetry property, it must be a ball.



To prove this, we introduce the function $v : \Sigma' \rightarrow \mathbb{R}$ defined by $v(x) = u(x')$ for $x \in \Sigma'$, where x' is the reflection of x across T' . By the maximum principle, we deduce that $u - v > 0$ or $u - v = 0$ in Σ' . For the sake of contradiction, suppose that $u - v > 0$. If Σ' is internally tangent to Ω at some point P , then we may appeal to the boundary point maximum principle to deduce that $\frac{\partial}{\partial \nu}(u - v) > 0$ at P [1]. However, we know that $\partial u / \partial \nu = \partial v / \partial \nu = c$. Thus we have reached a contradiction. If T' is orthogonal to the boundary of Ω at some point Q , then we show that u and v have the same first and second derivatives at Q . Using a modified version of the boundary point maximum principle, we can also show that $\frac{\partial}{\partial s}(u - v) > 0$ or $\frac{\partial^2}{\partial s^2}(u - v) > 0$ for any direction s that enters Σ' non-tangentially at Q . However, this directly contradicts the fact that u and v have the same first and second derivatives at Q . We may thus conclude that $u = v$ and that Ω is symmetric about T' .

Second Proof

The second proof we present is from Weinberger [2]. To start, we first compute

$$\Delta \left(r \frac{\partial u}{\partial r} \right) = r \frac{\partial}{\partial r} (\Delta u) + 2\Delta u = -2,$$

where r is the distance to the origin. Using this and the fact that $\Delta u = -1$, we obtain

$$\int_{\Omega} \left[2u - r \frac{\partial u}{\partial r} \right] dx = \int_{\Omega} \left[-u \Delta \left(r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \Delta u \right] dx$$

Using Green's identity yields

$$\int_{\Omega} \left[-u \Delta \left(r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \Delta u \right] dx = \int_{\partial\Omega} \left[-u \frac{\partial}{\partial \nu} \left(r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \frac{\partial u}{\partial \nu} \right] dS$$

By assumption, we have $u = 0$ on $\partial\Omega$. Thus, we find that

$$\int_{\partial\Omega} \left[-u \frac{\partial}{\partial \nu} \left(r \frac{\partial u}{\partial r} \right) + r \frac{\partial u}{\partial r} \frac{\partial u}{\partial \nu} \right] dS = \int_{\partial\Omega} r \frac{\partial r}{\partial \nu} \left(\frac{\partial u}{\partial \nu} \right)^2 dS$$

By assumption, we know that $\partial u / \partial \nu = c$ on $\partial\Omega$. Thus, we find that

$$\int_{\partial\Omega} r \frac{\partial r}{\partial \nu} \left(\frac{\partial u}{\partial \nu} \right)^2 dS = c^2 \int_{\partial\Omega} r \frac{\partial r}{\partial \nu} dS = c^2 n \int_{\Omega} dx = nc^2 V$$

where V is the volume of Ω . Green's theorem also implies

$$\int_{\Omega} r \frac{\partial u}{\partial r} dx = -n \int_{\Omega} u dx$$

so that substitution yields

$$(n + 2) \int_{\Omega} u dx = nc^2 V$$

However, we also note that

$$1 = (\Delta u)^2 \leq n \sum_{i=1}^n u_{ii}^2 \leq n \sum_{i,j} u_{ij}^2$$

by the Cauchy-Schwarz inequality. From this, we deduce that

$$\Delta \left(|\nabla u|^2 + \frac{2}{n} u \right) = 2 \sum_{i,j} u_{ij}^2 - \frac{2}{n} \geq 0$$

Using this and the fact that $|\nabla u|^2 + (2/n)u = c^2$ on $\partial\Omega$, we may appeal to the maximum principle to deduce that $|\nabla u|^2 + (2/n)u < c^2$ in Ω or $|\nabla u|^2 + (2/n)u = c^2$ in Ω . If the inequality held, then we could integrate over Ω to deduce that

$$(n + 2) \int_{\Omega} u dx < nc^2 V$$

This contradiction informs us that $|\nabla u|^2 + (2/n)u = c^2$ in Ω so that

$$\Delta \left(|\nabla u|^2 + \frac{2}{n} u \right) = 2 \sum_{i,j} u_{ij}^2 - \frac{2}{n} = 0$$

and

$$1 = n \sum_{i=1}^n u_{ii}^2 = \sum_{i,j} u_{ij}^2$$

which implies that $u_{ij} = -\delta_{ij}/n$. Solving the corresponding partial differential equations yields

$$u = \frac{1}{2n}(B - r^2)$$

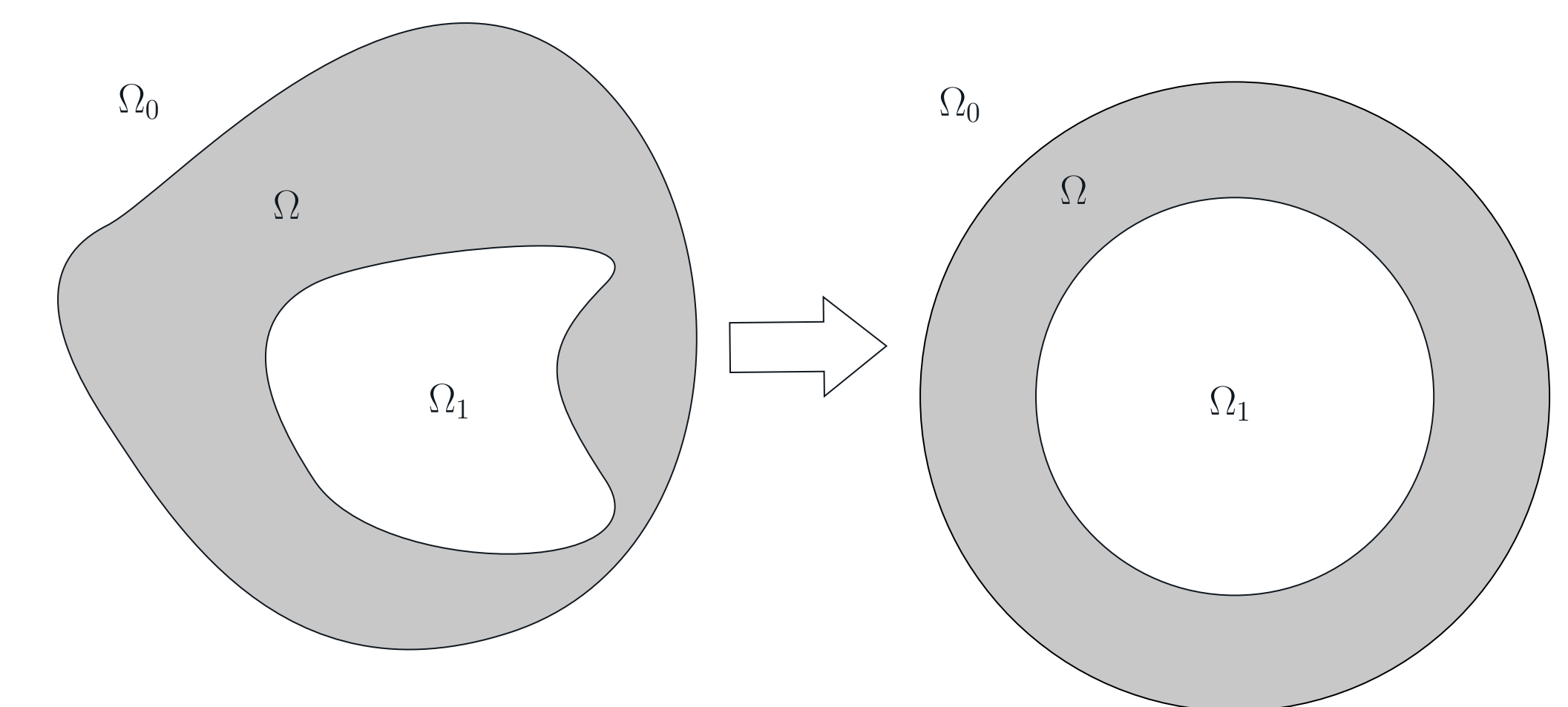
where B is a constant. Since $u = 0$ on $\partial\Omega$, B is positive and Ω is a ball of radius $B^{1/2}$.

Applications

This theorem is significant because it allows us to determine the shape of Ω from properties of u . It also has many applications in physics. For example, we may consider an incompressible viscous fluid moving through a straight pipe of cross sectional form Ω . If we fix a rectangular coordinate system with the z -axis directed along the pipe, then the velocity u depends only on x and y , and it satisfies the differential equation $\Delta u = -A$ for some constant A . Furthermore, because the fluid is viscous, we know that $u = 0$ on $\partial\Omega$; that is, there is no movement on the boundary of the pipe. Finally, we note that $\mu \partial u / \partial \nu$ is the tangential stress on the pipe wall, where μ is the viscosity constant. If the tangential stress is constant, then we may apply the above theorem to conclude that Ω is a circular cross section.

Further Results

A similar result was proved by Wolfgang Reichel [4]. Let Ω_0 and Ω_1 be smooth domains in \mathbb{R}^n and let $\Omega = \Omega_0 \setminus \Omega_1$ be connected. Suppose that $f \in C^1$ is a function satisfying $\Delta u + f(u, |\nabla u|) = 0$ in $\bar{\Omega}$, $0 < u < a$ in Ω , $u = 0$ on $\partial\Omega_0$, $u = a$ on $\partial\Omega_1$, and $\partial u / \partial \nu = c_i$ on $\partial\Omega_i$. Then, we conclude that Ω is an annulus and u is radially symmetric and decreasing in r .



The above picture demonstrates the theorem. On the left-hand side, we see our hypotheses. On the right-hand side, we see the conclusion.

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References

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