An Overdetermined Symmetry Problem

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Introduction

Let us consider the following problem. Let $\Omega \subseteq \mathbb{R}^n$ be domain that is bounded, open, and connected. Furthermore, suppose that the boundary $\partial\Omega$ is smooth. Let $u : \Omega \to \mathbb{R}$ be a function that satisfies the following conditions: $\Delta u = -1$ in Ω, $u = 0$ on $\partial\Omega$, and $\frac{\partial u}{\partial n} = c$ on $\partial\Omega$ for some constant c. Then, Ω must be a ball. Furthermore, we know that $u(x) = (b^2 - r^2)/2n$, where b is the ball's radius and r is the distance to its center.

First Proof

The first proof we present originates from Professor James Serrin [1]. We will first prove this theorem by the moving plane method. Let T_0 be a $n-1$ dimensional hyperplane in $\mathbb{R}^n$ that does not intersect the domain Ω. We begin to move this plane until it intersects Ω. When this occurs, the new plane T splits Ω into two pieces. The piece of Ω that lies on the same side of T as our initial plane T_0 is denoted by $\Sigma(T)$. We reflect $\Sigma(T)$ in T to obtain $\Sigma' := \Sigma'(T)$. As T is moved through Ω, Σ' will remain in Ω unless the set Σ' meets Ω at a point P or T becomes orthogonal to Ω at some point Q.

When either of these occurs, we stop moving the plane T, and we denote the resulting plane by T'. We claim that Ω is symmetric about T'. This is crucial because proving this would also prove the theorem by extension. To see how, we recall that the plane T was chosen arbitrarily. If we can show that Ω is symmetric about T', then we have shown that Ω is symmetric in all possible directions. Since Ω is simply connected and maintains this strong symmetry property, it must be a ball.

In order to show that this symmetry property holds, we introduce a new function $v : \Sigma' \to \mathbb{R}$ defined as follows: $v(x) = u(x')$ for $x \in \Sigma'$, where x' is obtained by reflecting x across T'. If we can show that $u = v$ in Σ', it will follow that Ω is symmetric about T'. First, we note that $\Delta v = -1$ in Σ', that $v = u$ on the plane T', that $v = 0$ and $\partial v/\partial n = c$ on the boundary of Σ'. Using these facts, we deduce that $\Delta(u - v) = 0$ in Σ' and that $u - v \geq 0$ on the boundary of Σ'. By the maximum principle, we have $u - v > 0$ at every point in Σ' or $u - v = 0$ in Σ'. As stated above, we are trying to prove that the latter is true. Thus, we must prove that $u - v > 0$ cannot occur. For the sake of contradiction, let us suppose that $u - v > 0$ in Σ'. If Σ' is internally tangent to Ω at some point P, then $u - v = 0$ at P. Appealing to the boundary point maximum principle, we find that $\frac{\partial}{\partial n}(u - v) > 0$ at P [3]. However, we previously established that $\partial u/\partial n = \partial v/\partial n = c$. Thus, we have reached a contradiction. Next, we consider the case in which T' is orthogonal to the boundary of Ω at some point Q. In this case, we cannot apply the boundary point maximum principle directly because there is no ball internally tangent to Σ' at Q. In order to circumvent this issue, we first note that

$$\frac{\partial u}{\partial x_i \partial x_j} = \frac{\partial v}{\partial x_i \partial x_j}$$

that is, u and v have the same first and second partial derivatives at Q. Finally, using a modified version of the boundary point maximum principle, we can show that

$$\frac{\partial}{\partial s}(u - v) > 0 \text{ or } \frac{\partial^2}{\partial^2 s}(u - v) > 0$$

for any direction s that enters Σ' non-tangentially at Q. However, this directly contradicts the fact that u and v have the same first and second partial derivatives at Q. Thus, we have reached a contradiction. We may conclude that Ω is symmetric about T'. Recall that T' was obtained from T_0, which was chosen arbitrarily. That is, given any direction, it is possible to find a plane with normal in that direction such that the domain Ω is symmetric about it. Since Ω is also simply connected, we may deduce that Ω is a ball.

Second Proof

The second proof we present is from Weinberger [2]. To start, we first compute

$$\Delta\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial}{\partial r}(\Delta u) + 2\Delta = -2$$

where r is the distance to the origin. Using this and the fact that $\Delta u = -1$, we have

$$\int_\Omega \left[2u - r\frac{\partial u}{\partial r}\right]dx = \int_\Omega \left[-u\Delta\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\Delta u\right]dx$$

Using Green's identity yields

$$\int_\Omega \left[-u\Delta\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\Delta u\right]dx = \int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\right]dS$$

By assumption, we have $u = 0$ on the boundary of Ω. Thus, we find that

$$\int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n}\right]dS = \int_{\partial\Omega} r\frac{\partial r}{\partial n}\left(\frac{\partial u}{\partial n}\right)^2 dS$$

By assumption, we also know that $\partial u/\partial n = c$ on the boundary of Ω. Thus, we find that

$$\int_{\partial\Omega} r\frac{\partial r}{\partial n}\left(\frac{\partial u}{\partial n}\right)^2 dS = c^2\int_{\partial\Omega} r\frac{\partial r}{\partial n}dS$$

Appealing to the divergence theorem and using the fact that $\Delta\frac{1}{2}r^2 = r\Delta r$, we obtain

$$c^2\int_{\partial\Omega} r\frac{\partial r}{\partial n}dS = c^2\int_\Omega \Delta\left(\frac{1}{2}r^2\right)dx = c^2 n\int_\Omega dx = nc^2V$$

Green's theorem also implies

$$\int_\Omega r\frac{\partial u}{\partial r}dx = -n\int_\Omega u\,dx$$

so that substitution yields

$$(n+2)\int_\Omega u\,dx = nc^2V$$

However, we also note that

$$1 = (\Delta u)^2 \leq n\sum_{i=1}^n u_{ii}^2 \leq n\sum_{i,j} u_{ij}^2$$

by the Cauchy-Schwarz inequality. From this, we deduce that

$$\Delta\left(|\nabla u|^2 + \frac{2}{n}u\right) = 2\sum_{i,j}u_{ij}^2 - \frac{2}{n} \geq 0$$

Using this and the fact that $|\nabla u|^2 + (2/n)u = c^2$ on $\partial\Omega$, we may appeal to the maximum principle to deduce that $|\nabla u| + (2/n)u < c^2$ in Ω or $|\nabla u| + (2/n)u = c^2$ in Ω. If the former inequality held, then we could integrate over Ω to deduce that

$$(n+2)\int_\Omega u\,dx < nc^2V$$

This contradiction informs us that $|\nabla u|^2 + (2/n)u = c^2$ in Ω so that

$$1 = n\sum_{i=1}^n u_{ii}^2 = \sum_{i,j}u_{ij}^2$$

which implies that $u_{ij} = -\delta_{ij}/n$. Solving the corresponding partial differential equations yields

$$u = \frac{1}{2n}(B - r^2)$$

where B is constant. Since $u = 0$ on $\partial\Omega$, B is positive and Ω is a ball of radius $B^{1/2}$.

Applications

This theorem is significant because it allows us to determine the shape of Ω from properties of u. It also has many applications in physics. For example, we may consider an incompressible fluid moving through a straight pipe of cross sectional form Ω. If we fix a rectangular coordinate system with the z axis in the same direction as the pipe, then the velocity u depends only on x and y, and it satisfies the differential equation $\Delta u = -A$ for some constant A. Furthermore, because the fluid is viscous, we know that $u = 0$ on $\partial\Omega$; that is, there is no movement on the boundary of the pipe. Finally, we note that $\mu\partial u/\partial n$ is the tangential stress on the pipe wall, where μ is the viscosity constant. If the tangential stress is constant, then we may apply the above theorem to conclude that Ω is a circular cross section.

Generalizations

References

[1] James Serrin. A Symmetry Problem in Potential Theory. *Arch. Rational Mech. Anal.* 1971.
[2] Hans Weingberger. Remark on the Preceding Paper of Serrin. *Arch. Rational Mech. Anal.* 1971.
[3] Hans Weingberger. Maximum Principles in Differential Equations. 1984.