Math 132A Homework 4

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Problem 1

Part A

First, we compute the Hessian as follows:

$$\nabla^2 f(x) = \begin{bmatrix} 12x_1^2 & -4 \\ -4 & 12x_2^2 \end{bmatrix}$$

Let us plug the point $(1, 1)$ into this Hessian to obtain

$$\begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

We claim that this matrix is positive definite. Notice that

$$\det \begin{bmatrix} 12 \end{bmatrix} = 12 > 0$$

and

$$\det \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} = 12^2 - 4^2 = 128 > 0$$

Since all the leading principal minors are positive, we find that the matrix is positive definite. By the second order sufficient conditions, we obtain that $(1, 1)$ is a local minimizer.

Next, we substitute the point $(0, 0)$ into the Hessian to obtain

$$\begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

Notice that this matrix is not positive semidefinite because

$$\det \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} = 0 - (-4)(-4) = 0 - 16 = -16 < 0$$

Since one of the principal minors is negative, this matrix is not positive semidefinite. Thus the second order necessary conditions are not satisfied, which means that the point $(0, 0)$ is not a local minimizer.

Finally, we substitute the point $(-1,-1)$ into the Hessian to obtain

$$\begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

We claim that this matrix is positive definite. Indeed, we have

$$\det \begin{bmatrix} 12 \end{bmatrix} = 12 > 0$$

and

$$\det \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} = 144 - 16 = 128 > 0$$

Since the leading principal minors are positive, this matrix is positive definite. Thus, the second order sufficient conditions are satisfied, which implies that the point $(-1,-1)$ is a local minimizer.

Part B

First, we compute the Hessian as follows:

$$\nabla^2 f(x) = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -4 \\ 0 & -4 & 10 \end{bmatrix}$$

We compute the leading principal minors. We obtain

$$\det \begin{bmatrix} 2 \end{bmatrix} = 2 > 0$$

and

$$\det \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = 8 - (-2)(-2) = 8 - 4 = 4 > 0$$

and

$$\det \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -4 \\ 0 & -4 & 10 \end{bmatrix} = 2 \begin{vmatrix} 4 & -4 \\ -4 & 10 \end{vmatrix} - (-2) \begin{vmatrix} -2 & -4 \\ 0 & 10 \end{vmatrix} = 2(40 - 16) + 2(-20) = 48 - 40 = 8 > 0$$

so we find that the matrix is positive definite. Thus, the second order sufficient conditions are satisfied, which implies that the point $(2, 2, 1)$ is a local minimizer.

Problem 2

First, we compute how many steps we will need to take in order to obtain the desired accuracy:

$$N > \frac{\ln 0.05 - \ln 1}{\ln 0.618} \approx 6.2$$

So we will need to take 7 steps in order to be within 0.05 of a minimizer x^*.

For the first step, we take

$$x_1 = 0 + \frac{3-\sqrt{5}}{2} \cdot (1-0) = \frac{3-\sqrt{5}}{2}$$

and

$$x_2 = 1 - \frac{3-\sqrt{5}}{2} \cdot (1-0) = \frac{\sqrt{5}-1}{2}$$

Now, we compute

$$f(x_1) \approx -0.579$$

and

$$f(x_2) \approx -0.484$$

Since

$$f(x_2) > f(x_1)$$

we discard the subinterval

$$\left(\frac{\sqrt{5}-1}{2}, 1 \right]$$

from our search region so that we are only looking at the interval

$$\left[0, \frac{\sqrt{5}-1}{2} \right]$$

For the second step, we take

$$x_1 = 0 + \frac{3-\sqrt{5}}{2} \cdot \left(\frac{\sqrt{5}-1}{2} - 0 \right) = \frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1)$$

and

$$x_2 = \frac{\sqrt{5}-1}{2} - \frac{3-\sqrt{5}}{2} \cdot \left(\frac{\sqrt{5}-1}{2} - 0 \right) = \frac{3-\sqrt{5}}{2}$$

(notice that x_2 from this step is equal to x_1 from the previous step). Thus, we know that

$$f(x_2) \approx -0.579$$

and we compute

$$f(x_1) \approx -0.561$$

Since

$$f(x_1) > f(x_2)$$

we discard the subinterval

$$\left[0, \frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1)\right)$$

from our search region so that we are only looking at the interval

$$\left[\frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1), \frac{\sqrt{5}-1}{2}\right]$$

For the third step, we take

$$x_1 = \frac{3-\sqrt{5}}{2}$$

(this was the value of x_2 from the previous step) and

$$x_2 = \frac{\sqrt{5}-1}{2} - \frac{3-\sqrt{5}}{2}\left(\frac{\sqrt{5}-1}{2} - \frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1)\right) = 2(\sqrt{5}-2)$$

Now, we compute

$$f(x_1) \approx -0.579$$

and

$$f(x_2) \approx -0.561$$

Since we have

$$f(x_1) < f(x_2)$$

we discard the subinterval

$$\left(2(\sqrt{5}-2), \frac{\sqrt{5}-1}{2}\right]$$

from our search region so that we are only looking at the interval

$$\left[\frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1), 2(\sqrt{5}-2)\right]$$

For the fourth step, we take

$$x_1 = \frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1) + \frac{3-\sqrt{5}}{2} \cdot \left(2(\sqrt{5}-2) - \frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1)\right) = \frac{1}{2}(7\sqrt{5}-15)$$

and

$$x_2 = \frac{3-\sqrt{5}}{2}$$

(since this was the x_1 from the previous step). Now, we compute

$$f(x_1) \approx -0.57917$$

and

$$f(x_2) \approx -0.57919$$

Since

$$f(x_1) > f(x_2)$$

we should discard the subinterval

$$\left[\frac{1}{4}(3-\sqrt{5})(\sqrt{5}-1), \frac{1}{2}(7\sqrt{5}-15)\right)$$

from our search region so that we are only looking at the interval

$$\left[\frac{1}{2}(7\sqrt{5}-15), 2(\sqrt{5}-2)\right]$$

For the fifth step, we let

$$x_1 = \frac{3-\sqrt{5}}{2}$$

(since this was the value of x_2 from the previous step) and

$$x_2 = 2(\sqrt{5}-2) - \frac{3-\sqrt{5}}{2}\left(2(\sqrt{5}-2) - \frac{1}{2}(7\sqrt{5}-15)\right) = 6\sqrt{5}-13$$

Now, we compute

$$f(x_1) \approx -0.5791951406$$

and

$$f(x_2) \approx -0.5749813593$$

Since

$$f(x_1) < f(x_2)$$

we discard the subinterval

$$(6\sqrt{5}-13, 2(\sqrt{5}-2)]$$

from our search region so that we are only looking at the interval

$$\left[\frac{1}{2}(7\sqrt{5}-15), 6\sqrt{5}-13\right]$$

For the sixth step, we let

$$x_1 = \frac{1}{2}(7\sqrt{5}-15) + \frac{3-\sqrt{5}}{2}\left(6\sqrt{5}-13-\frac{1}{2}(7\sqrt{5}-15)\right) = 10\sqrt{5}-22$$

and

$$x_2 = \frac{3-\sqrt{5}}{2}$$

Now, we compute

$$f(x_1) \approx -0.580$$

and

$$f(x_2) \approx -0.579$$

Since

$$f(x_2) > f(x_1)$$

we delete the subinterval

$$\left(\frac{3-\sqrt{5}}{2}, 6\sqrt{5}-13\right]$$

from our search region so that we are left with

$$\left[\frac{1}{2}(7\sqrt{5}-15), \frac{3-\sqrt{5}}{2}\right]$$

For the seventh (and last) step, we take

$$x_1 = \frac{1}{2}(7\sqrt{5}-15) + \frac{3-\sqrt{5}}{2}\left(\frac{3-\sqrt{5}}{2} - \frac{1}{2}(7\sqrt{5}-15)\right) = -7\sqrt{5}+16$$

and

$$x_2 = 10\sqrt{5} - 22$$

Now, we compute

$$f(x_1) \approx -0.5801763888$$

and

$$f(x_2) \approx -0.5801813331$$

Since

$$f(x_1) > f(x_2)$$

we delete the subinterval

$$\left[\frac{1}{2}(7\sqrt{5}-15), -7\sqrt{5}+16\right)$$

from our search region so that we are left with

$$\left[-7\sqrt{5}+16, \frac{3-\sqrt{5}}{2}\right]$$

The length of this interval is less than the desired tolerance $\varepsilon = 0.05$, and we know that the minimizer x^* must be in this interval.

Problem 3

First, we compute the first derivative of f:

$$f'(x) = 2x - \sin(x+2)$$

Next, we compute the second derivative of f:

$$f''(x) = 2 - \cos(x+2)$$

Now, we let $x_0 = 1$ (as given in the problem). We compute

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 1 - \frac{2 - \sin(3)}{2 - \cos(3)} \approx 0.3782994459$$

Next, we compute

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} \approx 0.378 - \frac{2(0.378) - \sin(0.378 + 2)}{2 - \cos(0.378 + 2)} \approx 0.3543168444$$

Then we compute

$$x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)} \approx 0.354 - \frac{2(0.354) - \sin(0.354 + 2)}{2 - \cos(0.354 + 2)} \approx 0.3542427589$$

Finally, we compute

$$x_4 = x_3 - \frac{f'(x_3)}{f''(x_3)} \approx 0.3542 - \frac{2(0.3542) - \sin(0.3542 + 2)}{2 - \cos(0.3542 + 2)} \approx 0.3542427582$$

This method converges very fast.