Math 132A Homework 3

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Problem 1

First, we write the Taylor polynomial as follows:

$$f(x_0 + p) \approx f(x_0) + p^T \nabla f(x_0) + \frac{1}{2} p^T \nabla^2 f(x_0) p$$

Now, we note that

$$\begin{aligned} f(x_0) = f(1, -1) &= 3(1)^4 - 2(1)^3(-1) - 4(1)^2(-1)^2 + 10(1)(-1)^3 + 2(-1)^4 \\ &= 3 + 2 - 4 - 10 + 2 = -7 \end{aligned}$$

Next, we compute

$$\nabla f(x) = \begin{bmatrix} 12x_1^3 - 6x_1^2 x_2 - 8x_1 x_2^2 + 10x_2^3 \\ -2x_1^3 - 8x_1^2 x_2 + 30x_1 x_2^2 + 8x_2^3 \end{bmatrix}$$

Now, we plug in the point $x_0 = (1, -1)$ to obtain

$$\nabla f(x_0) = \begin{bmatrix} 12(1)^3 - 6(1)^2(-1) - 8(1)(-1)^2 + 10(-1)^3 \\ -2(1)^3 - 8(1)^2(-1) + 30(1)(-1)^2 + 8(-1)^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 28 \end{bmatrix}$$

From this, we get

$$p^T \nabla f(x_0) = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 0 \\ 28 \end{bmatrix} = 0p_1 + 28p_2 = 28p_2$$

Now, we find the Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 36x_1^2 - 12x_1 x_2 - 8x_2^2 & -6x_1^2 - 16x_1 x_2 + 30x_2^2 \\ -6x_1^2 - 16x_1 x_2 + 30x_2^2 & -8x_1^2 + 60x_1 x_2 + 24x_2^2 \end{bmatrix}$$

so we get

$$\nabla^2 f(x_0) = \begin{bmatrix} 36(1)^2 - 12(1)(-1) - 8(-1)^2 & -6(1)^2 - 16(1)(-1) + 30(-1)^2 \\ -6(1)^2 - 16(1)(-1) + 30(-1)^2 & -8(1)^2 + 60(1)(-1) + 24(-1)^2 \end{bmatrix} = \begin{bmatrix} 40 & 40 \\ 40 & -44 \end{bmatrix}$$

Using this, we obtain

$$\begin{aligned} p^T \nabla^2 f(x_0) p &= \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 40 & 40 \\ 40 & -44 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 40p_1 + 40p_2 \\ 40p_1 - 44p_2 \end{bmatrix} \\ &= 40p_1^2 + 40p_1 p_2 + 40p_1 p_2 - 44p_2^2 = 40p_1^2 + 80p_1 p_2 - 44p_2^2 \end{aligned}$$

so that

$$\frac{1}{2}p^T\nabla^2 f(x_0)p = 20p_1^2 + 40p_1p_2 - 22p_2^2$$

Putting it all together, we find that

$$f(x_0) + p^T\nabla f(x_0) + \frac{1}{2}p^T\nabla^2 f(x_0)p = -7 + 28p_2 + 20p_1^2 + 40p_1p_2 - 22p_2^2$$

Now, we plug in $p = (0.1, 0.01)$ to obtain

$$f(x_0 + p) \approx -7 + 28(0.01) + 20(0.1)^2 + 40(0.1)(0.01) - 22(0.01)^2 = -6.4822$$

An exact evaluation yields

$$\begin{aligned} f(x_0 + p) = f(1.1, -0.99) &= 3(1.1)^4 - 2(1.1)^3(-0.99) - 4(1.1)^2(-0.99)^2 + 10(1.1)(-0.99)^3 + 2(-0.99)^4 \\ &= -6.4681 \end{aligned}$$

The difference between these two is

$$|-6.4822 + 6.4681| = 0.0141$$

Problem 2

The first three terms of the Taylor series are

$$f(x_0 + p) \approx f(x_0) + p^T \nabla f(x_0) + \frac{1}{2} p^T \nabla^2 f(x_0) p$$

First, we find

$$f(x_0) = f(3,4) = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Next, we compute

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} \sqrt{x_1^2 + x_2^2} \\ \frac{\partial}{\partial x_2} \sqrt{x_1^2 + x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2}} \end{bmatrix}$$

so that

$$\nabla f(x_0) = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

Thus, we obtain

$$p^T \nabla f(x_0) = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \frac{3}{5} p_1 + \frac{4}{5} p_2$$

Now, we compute the Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{\partial}{\partial x_1} \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{\partial}{\partial x_2} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{\partial}{\partial x_2} \frac{x_2}{\sqrt{x_1^2+x_2^2}} \end{bmatrix}$$

First, we compute

$$\begin{aligned} \frac{\partial}{\partial x_1} \frac{x_1}{\sqrt{x_1^2 + x_2^2}} &= \frac{\partial}{\partial x_1} x_1 (x_1^2 + x_2^2)^{-1/2} = (x_1^2 + x_2^2)^{-1/2} - x_1^2 (x_1^2 + x_2^2)^{-3/2} \\ &= (x_1^2 + x_2^2)^{-3/2} (x_1^2 + x_2^2 - x_1^2) = \frac{x_2^2}{(x_1^2 + x_2^2)^{3/2}} \end{aligned}$$

Next, we compute

$$\frac{\partial}{\partial x_1} \frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{\partial}{\partial x_1} x_2 (x_1^2 + x_2^2)^{-1/2} = -\frac{1}{2} x_2 (x_1^2 + x_2^2)^{-3/2} \cdot 2x_1 = -\frac{x_1 x_2}{(x_1^2 + x_2^2)^{3/2}}$$

and

$$\frac{\partial}{\partial x_2} \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{\partial}{\partial x_2} x_1 (x_1^2 + x_2^2)^{-1/2} = -\frac{1}{2} x_1 (x_1^2 + x_2^2)^{-3/2} \cdot 2x_2 = -\frac{x_1 x_2}{(x_1^2 + x_2^2)^{3/2}}$$

Finally, we compute

$$\begin{aligned} \frac{\partial}{\partial x_2} \frac{x_2}{\sqrt{x_1^2 + x_2^2}} &= \frac{\partial}{\partial x_2} x_2 (x_1^2 + x_2^2)^{-1/2} = (x_1^2 + x_2^2)^{-1/2} - x_2^2 (x_1^2 + x_2^2)^{-3/2} \\ &= (x_1^2 + x_2^2)^{-3/2} (x_1^2 + x_2^2 - x_2^2) = \frac{x_1^2}{(x_1^2 + x_2^2)^{3/2}} \end{aligned}$$

Now, we substitute the point $(3, 4)$ into each of these:

$$\frac{4^2}{(3^2+4^2)^{3/2}} = \frac{16}{125}$$

$$-\frac{3\cdot 4}{(3^2+4^2)^{3/2}} = -\frac{12}{125}$$

$$\frac{3^2}{(3^2+4^2)^{3/2}} = \frac{9}{125}$$

So the Hessian at the point $x_0 = (3, 4)$ is

$$\nabla^2 f(x_0) = \begin{bmatrix} 16/125 & -12/125 \\ -12/125 & 9/125 \end{bmatrix}$$

So we find that

$$\begin{aligned} p^T \nabla^2 f(x_0) p &= \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 16/125 & -12/125 \\ -12/125 & 9/125 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} 16p_1/125 - 12p_2/125 \\ -12p_1/125 + 9p_2/125 \end{bmatrix} \\ &= \frac{16}{125}p_1^2 - \frac{12}{125}p_1p_2 - \frac{12}{125}p_1p_2 + \frac{9}{125}p_2^2 = \frac{16}{125}p_1^2 - \frac{24}{125}p_1p_2 + \frac{9}{125}p_2^2 \end{aligned}$$

This gives

$$\frac{1}{2}p^T \nabla^2 f(x_0) p = \frac{8}{125}p_1^2 - \frac{12}{125}p_1p_2 + \frac{9}{250}p_2^2$$

so our Taylor polynomial is

$$5 + \frac{3}{5}p_1 + \frac{4}{5}p_2 + \frac{8}{125}p_1^2 - \frac{12}{125}p_1p_2 + \frac{9}{250}p_2^2$$

Problem 3

Part A

First, we compute

$$f_{x_1} = 4x_1^3 - 4x_2$$

and

$$f_{x_2} = 4x_2^3 - 4x_1$$

At any minimum, these two partial derivatives must be 0. So we obtain

$$4x_1^3 = 4x_2 \implies x_1^3 = x_2$$

and

$$4x_2^3 = 4x_1 \implies x_2^3 = x_1$$

This tells us that

$$(x_2^3)^3 = x_2 \implies x_2^9 - x_2 = 0 \implies x_2(x_2^8 - 1) = 0$$

This is only true when $x_2 = -1, 0, 1$. If $x_2 = -1$, then

$$x_1^3 = -1$$

so that $x_1 = -1$ (assuming that we are working over $\mathbb{R}$). If $x_2 = 0$, then

$$x_1^3 = 0$$

so that $x_1 = 0$. Finally, if $x_2 = 1$, then

$$x_1^3 = 1$$

so that $x_1 = 1$. Thus, the potential minimizers are $(-1, -1)$, $(0, 0)$, and $(1, 1)$.

Part B

We have

$$f_{x_1} = 2x_1 - 2x_2^2$$

and

$$f_{x_2} = -4x_1x_2 + 4x_2^3 - 5x_2^4$$

At any critical point, both of these partial derivatives must be 0, so we obtain

$$2x_1 - 2x_2^2 = 0$$

and

$$-4x_1x_2 + 4x_2^3 - 5x_2^4 = 0$$

From the first equation, we get

$$x_1 = x_2^2$$

Plugging this into the second equation yields

$$-4x_2^3 + 4x_2^3 - 5x_2^4 = 0 \implies x_2^4 = 0 \implies x_2 = 0 \implies x_1 = 0$$

Thus the only critical point is $(0, 0)$. So the point $(0, 0)$ is the only potential minimizer.

Part C

We note that

$$f_{x_1} = 2x_1 - 2x_2$$

and

$$f_{x_2} = 4x_2 - 2x_1 - 4x_3$$

and

$$f_{x_3} = 10x_3 - 4x_2 - 2$$

At any local minimum, all of these partial derivatives must be 0. Thus, we have

$$\begin{aligned} 2x_1 - 2x_2 &= 0 \\ 4x_2 - 2x_1 - 4x_3 &= 0 \\ 10x_3 - 4x_2 - 2 &= 0 \end{aligned}$$

Rearranging yields

$$\begin{aligned} 2x_1 - 2x_2 &= 0 \\ -2x_1 + 4x_2 - 4x_3 &= 0 \\ -4x_2 + 10x_3 &= 2 \end{aligned}$$

Now we put this into matrix form as follows:

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -4 & 0 \\ 0 & -4 & 10 & 2 \end{bmatrix}$$

We put this into row echelon form:

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -4 & 0 \\ 0 & -4 & 10 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ -2 & 4 & -4 & 0 \\ 0 & -4 & 10 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & -4 & 10 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -4 & 10 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

This tells us that

$$2x_3 = 2 \implies x_3 = 1$$

Then we get that

$$x_2 - 2x_3 = 0 \implies x_2 = 2x_3 = 2$$

Finally, we obtain

$$x_1 - x_2 = 0 \implies x_1 = x_2 = 2$$

Thus, the only potential minimizer is $(2, 2, 1)$.

Problem 4

Part A

For these problems, we will be using Sylvester's criterion. This criterion states that if a matrix is symmetric, then it is positive definite if and only if its leading principal minors are all positive. First, we note that

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^T$$

so we know that this matrix is symmetric. Next, we note that

$$\det \begin{bmatrix} 5 \end{bmatrix} = 5 > 0$$

and

$$\det \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = 25 - 16 = 9 > 0$$

so the matrix is positive definite.

Part B

First, we note that

$$\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}^T$$

so this matrix is symmetric. Next, we note that

$$\det \begin{bmatrix} 4 \end{bmatrix} = 4 > 0$$

but

$$\det \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} = 16 - 25 = -9 < 0$$

so that the matrix is not positive definite.

Part C

First, we note that

$$\begin{bmatrix} 5 & 7 & 6 \\ 7 & 10 & 8 \\ 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 6 \\ 7 & 10 & 8 \\ 6 & 8 & 10 \end{bmatrix}^T$$

so that the matrix is symmetric. Next, we must check that all of its leading principal minors are positive. We first check that

$$\det \begin{bmatrix} 5 \end{bmatrix} = 5 > 0$$

Next, we check that

$$\det \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix} = 50 - 49 = 1 > 0$$

Finally, we check that

$$\begin{aligned} \det \begin{bmatrix} 5 & 7 & 6 \\ 7 & 10 & 8 \\ 6 & 8 & 10 \end{bmatrix} &= 5 \begin{vmatrix} 10 & 8 \\ 8 & 10 \end{vmatrix} - 7 \begin{vmatrix} 7 & 8 \\ 6 & 10 \end{vmatrix} + 6 \begin{vmatrix} 7 & 10 \\ 6 & 8 \end{vmatrix} \\ &= 5(10 * 10 - 8 * 8) - 7 * (7 * 10 - 8 * 6) + 6(7 * 8 - 10 * 6) = 2 > 0 \end{aligned}$$

So we may deduce that this matrix is positive definite.