Math 132A Assignment 5
Due: Wednesday, February 19th at Midnight on Gradescope.

1. Recall from class that we discussed the general quadratic function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by
$$f(x) = \frac{1}{2}x^TQx - b^Tx,$$
for an $n \times n$ **symmetric** matrix Q and $b \in \mathbb{R}^n$.

   (a) Prove that $\nabla f(x) = Qx - b$ and $\nabla^2 f(x) = Q$.

   (b) Starting from $x^{(0)} = (1, 1.5)^T$, determine the the first three iterates in the method of steepest descent applied to such an f with
$$Q = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

2. Apply three iterations of the method of steepest descent to the function
$$f(x_1, x_2) = e^{x_1x_2} + x_1^2x_2^2$$
starting at $x^{(0)} = (0, -2)$. *To calculate the optimal step size, you can use any of the one-dimensional methods discussed in class OR ask Wolfram Alpha or equivalent program.*

3. The function
$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$
is known as *Rosenbrock's function* or the *banana function*[1]. This function is considered "nasty" and is often used to test algorithms.

   (a) Prove that $(1, 1)$ is the unique global minimizer of f.

   (b) With a starting point of $(0, 0)^T$, apply two iterations of Newton's method with step size 1

   (c) Repeat part (b) with the method of steepest descent but with fixed step size $\alpha = 0.05$.

4. Apply the Quasi-Newton method with the SR1 update formula to locate the minimizer of
$$f(x) = \frac{1}{2}x^T\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}x - x^T\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$
Start with $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $B_0 = I$ and calculate the first two iterations $(x^{(1)}, B_1)$ and $(x^{(2)}, B_2)$. At each step calculate the optimal step size[2].

5. Same as Question 1 but now use the BFGS update formula instead.

[1] To see why, try plotting $c = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$ for various positive constants c.

[2] The calculation is just finding the vertex of a parabola! No need for anything complicated here!