Math 132A Homework 2

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Problem 1

We define our variables as follows:

- b denotes the start time of task B
- f denotes the start time of task F
- e denotes the start time of task E
- p denotes the start time of task P
- d denotes the start time of task D
- l denotes the start time of task L.

We must have the nonnegativity constraints:

$$b \geq 0,\ f \geq 0,\ e \geq 0,\ p \geq 0,\ d \geq 0,\ l \geq 0$$

The first constraint tells us that F can only start after B ends. We know that B takes 3 hours, so

$$f \geq b + 3$$

The second constraint tells us that L can only start after B ends. Since B takes 3 hours, we have

$$l \geq b + 3$$

The third constraint tells us that E can only start after F ends. Since F takes 2 hours, we have

$$e \geq f + 2$$

The fourth constraint tells us that P can only start after F ends. Since F takes 2 hours, we have

$$p \geq f + 2$$

The fifth constraint tells us that D can only start after E ends. Since E takes 3 hours, we have

$$d \geq e + 3$$

The final constraint tells us that D can only start after P ends. Since P takes 4 hours, we have

$$d \geq p + 4$$

Finally, we introduce a new variable y. This variable represents the end of the final task. Thus, we require

$$y \geq b + 3,\ y \geq f + 2,\ y \geq e + 3,\ y \geq p + 4,\ y \geq d + 1, y \geq l + 2$$

We should also have $y \geq 0$ even though it is redundant. The linear program is to minimize y subject to these constraints.

Now, before we solve this with Matlab, we should put it into standard inequality form. The constraints become

$$b - f \leq -3,\ b - l \leq -3,\ f - e \leq -2,\ f - p \leq -2,\ e - d \leq -3,\ p - d \leq -4$$

and

$$b - y \leq -3,\ f - y \leq -2,\ e - y \leq -3,\ p - y \leq -4,\ d - y \leq -1,\ l - y \leq -2$$

Now, we can input this problem into Matlab as follows:

```
clear

%coefficient matrix
%(b, f, e, p, d, l, y)
%we only care about the value of y
f = [0 0 0 0 0 0 1];
%constraint matrix
A = [1 -1 0 0 0 0 0;
    1 0 0 0 0 -1 0;
    0 1 -1 0 0 0 0;
    0 1 0 -1 0 0 0;
    0 0 1 0 -1 0 0;
    0 0 0 1 -1 0 0;
    1 0 0 0 0 0 -1;
    0 1 0 0 0 0 -1;
    0 0 1 0 0 0 -1;
    0 0 0 1 0 0 -1;
    0 0 0 0 1 0 -1;
    0 0 0 0 0 1 -1;
    ];
%constraint vector
b = [-3;
    -3;
    -2;
    -2;
    -3;
    -4;
    -3;
    -2;
    -3;
    -4;
    -1;
    -2
    ];
%lower bounds
lb = [0;0;0;0;0;0;0];
ub = [inf;inf;inf;inf;inf;inf;inf];
%compute solution
x = linprog(f,A,b,[],[],lb,ub);
```

Here are the results:

	1
1	0
2	3
3	5
4	5
5	9
6	3
7	10
8	

This tells us that the optimal point is

$$(b, f, e, p, d, l, y) = (0, 3, 5, 5, 9, 3, 10)$$

In particular, this tells us that the optimal value is $y = 10$ weeks.

Problem 2

First, we must define the variables. Since this is an integer program, we introduce decision variables.

- Let p be 1 if we schedule Pedro and 0 if we do not schedule Pedro.
- Let r be 1 if we schedule Roman and 0 if we do not schedule Roman.
- Let b be 1 if we schedule Brittany and 0 if we do not schedule Brittany.
- Let m be 1 if we schedule Misha and 0 if we do not schedule Misha.
- Let y be 1 if we schedule Yian and 0 if we do not schedule Yian.
- Let a be 1 if we schedule Anasophia and 0 if we do not schedule Anasophia.
- Let t be 1 if we schedule Ty and 0 if we do not schedule Ty.

Next, we must minimize the total salary. That is, we must minimize the following expression:

$$30p + 18r + 21b + 38m + 20y + 22a + 9t$$

Now, we must find the constraints. First, we already know that

$$p, r, b, m, y, a, t \in \{0, 1\}$$

since we are assuming that this is an integer program. Furthermore, the problem tells us that at least one lifeguard must be working at all times.

From $1-2$ PM, Pedro and Roman are available, so we have

$$p + r \geq 1$$

From $2-3$ PM, Pedro and Roman are available, so we get the same constraint. From $3-4$ PM, Pedro is available, so we have

$$p \geq 1$$

From $4-5$ PM, Pedro, Brittany, and Misha are available, so we have

$$p + b + m \geq 1$$

From $5-6$ PM, Brittany, Misha, and Anasophia are available, so we have

$$b + m + a \geq 1$$

From $6-7$ PM, Brittany, Misha, Yian, and Anasophia are available, so we have

$$b + m + y + a \geq 1$$

From $7-8$ PM, Misha, Yian, and Anasophia are available, so we have

$$m + y + a \geq 1$$

From $8-9$ PM, Misha, Yian, and Ty are available, so we have

$$m + y + t \geq 1$$

Problem 3

Part A

The dual linear program is as follows:

$$\begin{aligned} \text{minimize} \quad & 4y_1 - 2y_2 \\ \text{subject to} \quad & y_1 + y_2 \geq 3 \\ & 2y_1 - y_2 \geq 1 \\ & 2y_1 + y_2 \geq 4 \\ & y_1 - y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

First, we must check that $x^* = [0, 1, 0, 2]^T$ is actually feasible. We have

$$x_1 + 2x_2 + 2x_3 + x_4 = 0 + 2(1) + 2(0) + 2 = 4 \leq 4$$

and

$$x_1 - x_2 + x_3 - x_4 = 0 - 1 + 0 - 2 = -3 < -2$$

It is easy to see that the nonnegativity conditions are satisfied.

Notice that the second and fourth entries of $x^* = [0, 1, 0, 2]$ are positive. By the complementary slackness conditions, we have

$$\begin{aligned} 2y_1 - y_2 &= 1 \\ y_1 - y_2 &= 1 \end{aligned}$$

(that is, we know that the second and fourth inequalities must be equalities). Notice that the inequality

$$x_1 - x_2 + x_3 - x_4 = 0 - 1 + 0 - 2 = -3 < -2$$

is strict. Appealing to the complementary slackness conditions again, we have $y_2 = 0$ so that the two equations become

$$\begin{aligned} 2y_1 &= 1 \\ y_1 &= 1 \end{aligned}$$

which is impossible. Thus x^* is not optimal.

Part B

First, we check to see if $x^* = [1, 0, 0, 3]^T$ is actually feasible. We have

$$x_1 + 2x_2 + 2x_3 + x_4 = 1 + 2(0) + 2(0) + 3 = 4 \leq 4$$

and

$$x_1 - x_2 + x_3 - x_4 = 1 - 0 + 0 - 3 = -2 \leq -2$$

Thus, the point x^* is feasible. Notice that the first and fourth entries of $x^* = [1, 0, 0, 3]$ are positive, so the first and fourth constraints of the dual must be equalities. This tells us that

$$y_1 + y_2 = 3$$
$$y_1 - y_2 = 1$$

Adding the two equations gives $y_1 = 2$ so that $y_2 = 1$. Thus, we obtain the point $y^* = (y_1, y_2) = (2, 1)$. We must check that this point satisfies the other conditions (namely, the second and third inequalities). Notice that

$$2y_1 - y_2 = 2(2) - 1 = 3 \geq 1$$

and

$$2y_1 + y_2 = 2(2) + 1 = 5 \geq 4$$

Also, we see that $y_1, y_2 \geq 0$ so this point is feasible for the dual. Thus the point $x^* = [1, 0, 0, 3]^T$ is optimal. We check this as follows:

$$c^T x^* = 3x_1 + x_2 + 4x_3 + x_4 = 3(1) + 0 + 4(0) + 3 = 6$$

and

$$b^T y^* = 4y_1 - 2y_2 = 4(2) - 2(1) = 6$$

Since $c^T x^* = b^T y^*$, we can be certain that x^* is optimal.

Problem 4

The dual linear program is:

$$\begin{array}{rl} \text{minimize} & 3y_1 + 2y_2 + 7y_3 + 4y_4 \\ \text{subject to} & y_1 + 2y_3 \geq 1 \\ & y_2 + 2y_4 \geq 5 \\ & y_1 + y_2 - y_3 \geq 3 \\ & 3y_1 + y_2 - y_3 + y_4 \geq 6 \\ & 3y_3 + 2y_4 \geq 6 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

Let us review the complementary slackness conditions. These conditions say that for all $j = 1, \dots, n$, we have

$$x_j^* > 0 \implies \sum_{i=1}^{m} a_{ij} y_i^* = c_j$$

and for all $i = 1, \dots, m$, we have

$$\sum_{j=1}^{n} a_{ij} x_j^* < b_i \implies y_i^* = 0$$

We can rewrite these conditions so that for all $j = 1, \dots, n$, we have

$$\sum_{i=1}^{m} a_{ij} y_i^* > c_j \implies x_j^* = 0$$

and for all $i = 1, \dots, m$, we have

$$y_i^* > 0 \implies \sum_{j=1}^{n} a_{ij} x_j^* = b_i$$

Now, let us see this in practice. We have

$$\begin{array}{l} y_1 + 2y_3 > 1 \implies x_1 = 0 \\ y_2 + 2y_4 > 5 \implies x_2 = 0 \\ y_1 + y_2 - y_3 > 3 \implies x_3 = 0 \\ 3y_1 + y_2 - y_3 + y_4 > 6 \implies x_4 = 0 \\ 3y_3 + 2y_4 > 6 \implies x_5 = 0 \end{array}$$

and

$$\begin{array}{l} y_1 > 0 \implies x_1 + x_3 + 3x_4 = 3 \\ y_2 > 0 \implies x_2 + x_3 + x_4 = 2 \\ y_3 > 0 \implies 2x_1 - x_3 - x_4 + 3x_5 = 7 \\ y_4 > 0 \implies 2x_2 + x_4 + 2x_5 = 4 \end{array}$$

First, we substitute $y^* = [1, 2, 0, 3]^T$ into the inequalities in the dual linear program:

$$y_1 + 2y_3 = 1 + 2(0) = 1$$

and

$$y_2 + 2y_4 = 2 + 2(3) = 8 > 5$$

and

$$y_1 + y_2 - y_3 = 1 + 2 - 0 = 3$$

and

$$3y_1 + y_2 - y_3 + y_4 = 3(1) + 2 - 0 + 3 = 8 > 6$$

and

$$3y_3 + 2y_4 = 3(0) + 2(3) = 6$$

Now the second and fourth inequalities are strict so that $x_2 = x_4 = 0$. Next, we know that $y_1, y_2, y_4 > 0$ so that

$$\begin{aligned} x_1 + x_3 + 3x_4 &= 3 \\ x_2 + x_3 + x_4 &= 2 \\ 2x_2 + x_4 + 2x_5 &= 4 \end{aligned}$$

Substituting $x_2 = x_4 = 0$ into these equations yields

$$\begin{aligned} x_1 + x_3 &= 3 \\ x_3 &= 2 \\ 2x_5 &= 4 \end{aligned}$$

so that $x_1 = 1$, $x_3 = 2$, and $x_5 = 2$. First, we note that

$$2x_1 - x_3 - x_4 + 3x_5 = 2(1) - 2 - 0 + 3(2) = 6 \leq 7$$

so that the third inequality is satisfied. This x^* is feasible. Thus, we find that the optimal solution $x^* = (1, 0, 2, 0, 2)$. We can check this solution as follows. Notice that

$$c^T x^* = x_1 + 5x_2 + 3x_3 + 6x_4 + 6x_5 = 1 + 5(0) + 3(2) + 6(0) + 6(2) = 1 + 6 + 12 = 19$$

and

$$b^T y^* = 3y_1 + 2y_2 + 7y_3 + 4y_4 = 3(1) + 2(2) + 7(0) + 4(3) = 3 + 4 + 12 = 19$$

Since $c^T x^* = b^T y^*$, we know that the point x^* is optimal.