Math 115A Homework 2

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Problem 1

The number of lattice points in this region can be found by counting the number of lattice points between 0 and f at each integer m between Q and R and adding these numbers. The number of lattice points between 0 and f at an arbitrary integer m is equal to the number of positive integers less than or equal to $f(m)$. By the definition of the floor function, we find that the number of lattice points between 0 and f at m is equal to $[f(m)]$. Summing this over all integers m between Q and R, we find that the number of lattice points in this region is equal to

$$\sum_{Q<n\leq R} [f(n)]$$

Problem 2

First, we note that the sum

$$\sum_{0<m<Q/2} \left[\frac{P}{Q}m\right]$$

is equal to the number of lattice points (x, y) satisfying $0 < x \leq Q/2$ and $0 < y \leq Px/Q$. That is, it is equal to the cardinality of the set

$$S_1 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 0 < x \leq Q/2,\ 0 < y \leq Px/Q\}$$

This is true because for every integer x between 0 and $Q/2$, we count $[Px/Q]$ lattice points, which is the number of positive integers y less than Px/Q. Geometrically, S_1 is the blue triangle in the image on the next page. Next, we note that the sum

$$\sum_{0<n<P/2} \left[\frac{Q}{P}n\right]$$

is the number of lattice points (x, y) satisfying $0 < y \leq P/2$ and $0 < x \leq Qy/P$. That is, it is equal to the cardinality of the set

$$S_2 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 0 < y \leq P/2,\ 0 < x \leq Qy/P\}$$

This is because for every integer y satisfying $0 < y \leq P/2$, we count $[Qy/P]$ lattice points, which is the number of positive integers x less than Qy/P. Geometrically, S_2 is the red triangle in the image on the next page. Finally,

$$\frac{P-1}{2} \cdot \frac{Q-1}{2}$$

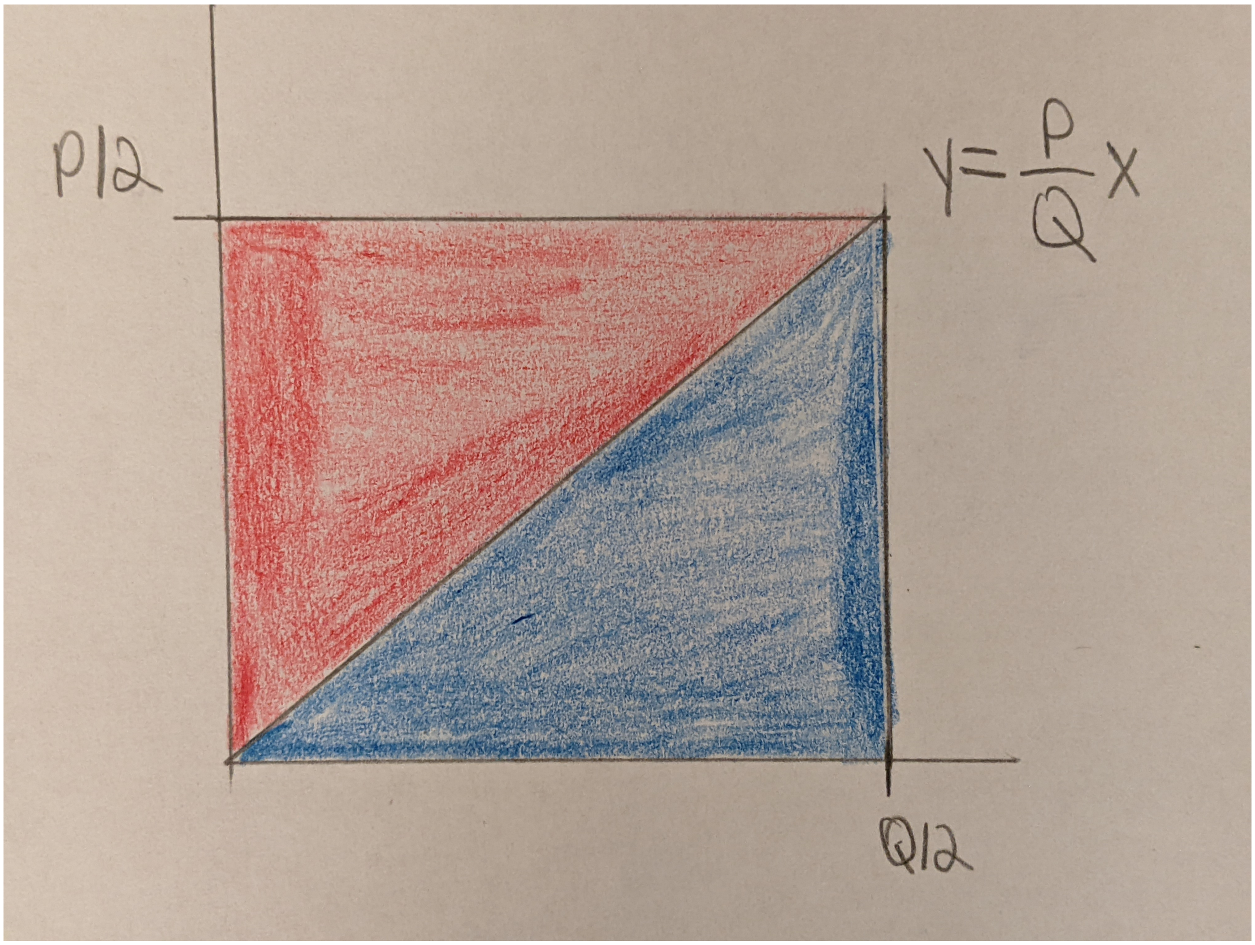
is the number of lattice points (x, y) such that $0 < x \leq Q/2$ and $0 < y \leq P/2$ or the cardinality of the set

$$S_3 = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 0 < x \leq Q/2,\ 0 < y \leq P/2\}$$

This is true because P and Q are both assumed to be odd. Geometrically, S_3 is the union of the red and blue triangles in the image on the next page, so the result is intuitively clear. Now, we must prove this rigorously. We claim that $S_3 = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$. First, we will prove that $S_3 = S_1 \cup S_2$. Let us suppose that $(x, y) \in S_3$. Then, we know that $0 < x \leq Q/2$ and $0 < y \leq P/2$. For the sake of contradiction, let us suppose that $(x, y) \notin S_1 \cup S_2$. This implies that $y > Px/Q$ and $x > Qy/P$. This yields $Qy > Px$ and $Px > Qy$ so that $Qy > Px > Qy$, a contradiction. Thus we may deduce that $(x, y) \in S_1 \cup S_2$ so that $S_3 \subseteq S_1 \cup S_2$. Next, we may suppose that $(x, y) \in S_1 \cup S_2$. Then, we know that $(x, y) \in S_1$ or $(x, y) \in S_2$. If $(x, y) \in S_1$, then we have $0 < x \leq Q/2$ and $0 < y \leq Px/Q \leq (P/Q)(Q/2) = P/2$, so $(x, y) \in S_3$. If $(x, y) \in S_2$, then we have $0 < y \leq P/2$ and $0 < x \leq Qy/P \leq (Q/P)(P/2) = Q/2$ so that $(x, y) \in S_3$. Thus, we may deduce that $S_1 \cup S_2 \subseteq S_3$ so that $S_1 \cup S_2 = S_3$. Next, we will show that $S_1 \cap S_2 = \varnothing$. For the sake of contradiction, suppose that there was some $(x, y) \in S_1 \cap S_2$. Then we would know that

$0 < y \leq Px/Q$ and $0 < x \leq Qy/P$. This yields $Qy \leq Px$ and $Px \leq Qy$ so that $Px = Qy$. Then, we find that $P \mid Qy$. Since $(P, Q) = 1$ by assumption, we realize that $P \mid y$. This is impossible since $y < P/2$. Thus, we may deduce that $S_1 \cap S_2 = \varnothing$. Now, we may write

$$\frac{P-1}{2} \cdot \frac{Q-1}{2} = |S_3| = |S_1 \cup S_2| = |S_1| + |S_2| = \sum_{0<m<Q/2} \left[\frac{P}{Q}m\right] + \sum_{0<n<P/2} \left[\frac{Q}{P}n\right]$$



Problem 3

By the division algorithm, we may write $[\alpha] = cq + r$, where $0 \le r \le c - 1$. Now, we may note that

$$\left[\frac{[\alpha]}{c}\right] = \left[\frac{cq + r}{c}\right] = \left[q + \frac{r}{c}\right] = q$$

because $r < c$. Next, we know that $\{\alpha\} = \alpha - [\alpha]$ so that $\alpha = [\alpha] + \{\alpha\}$. Thus, we find that

$$\left[\frac{\alpha}{c}\right] = \left[\frac{[\alpha] + \{\alpha\}}{c}\right] = \left[\frac{cq + r + \{\alpha\}}{c}\right] = \left[q + \frac{r + \{\alpha\}}{c}\right] = q$$

because $0 \le \{\alpha\} < 1$ and $r \le c - 1$ so that $r + \{\alpha\} < c$.

Problem 4

By Problem 3, we may suppose without loss of generality that X is an integer. In class, we proved that

$$\sum_{n \leq X} f(n) \left[\frac{X}{n} \right] = \sum_{n \leq X} g(n)$$

where f is an arithmetic function and g is the mobius transform of f. Let us take $f \equiv 1$; that is, f is identically 1. Then

$$g(n) = \sum_{d|n} f(d) = \sum_{d|n} 1 = \tau(n)$$

since τ counts the number of positive divisors of n. Substitution then yields

$$\sum_{n \leq X} \left[\frac{X}{n} \right] = \sum_{n \leq X} f(n) \left[\frac{X}{n} \right] = \sum_{n \leq X} g(n) = \sum_{n \leq X} \tau(n)$$

Problem 5

By Problem 3, we may suppose without loss of generality that X is an integer. By problem 4, we know that

$$\sum_{n \leq X} \tau(n) = \sum_{n \leq X} \left[\frac{X}{n}\right]$$

Therefore, we only have to show that

$$\sum_{n \leq X} \left[\frac{X}{n}\right] = 2 \sum_{n \leq \sqrt{X}} \left[\frac{X}{n}\right] - [\sqrt{X}]^2$$

Notice that

$$\sum_{n \leq X} \left[\frac{X}{n}\right]$$

represents the cardinality of the set

$$T = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : 0 < a \leq X,\ 0 < b \leq X/a\}$$

In the image below, T is the union of the blue, green, and red regions. Furthermore,

$$\sum_{n \leq \sqrt{X}} \left[\frac{X}{n}\right]$$

represents the cardinality of the set

$$U = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : 0 < a \leq \sqrt{X},\ 0 < b \leq X/a\}$$

In the image below, U is the union of the blue and green regions. Furthermore,

$$\sum_{n \leq \sqrt{X}} \left[\frac{X}{n}\right]$$

also represents the cardinality of the set

$$W = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : 0 < b \leq \sqrt{X},\ 0 < a \leq X/b\}$$

In the image below, W is the union of the green and red regions. Finally, we may note that $[\sqrt{X}]^2$ represents the cardinality of the set

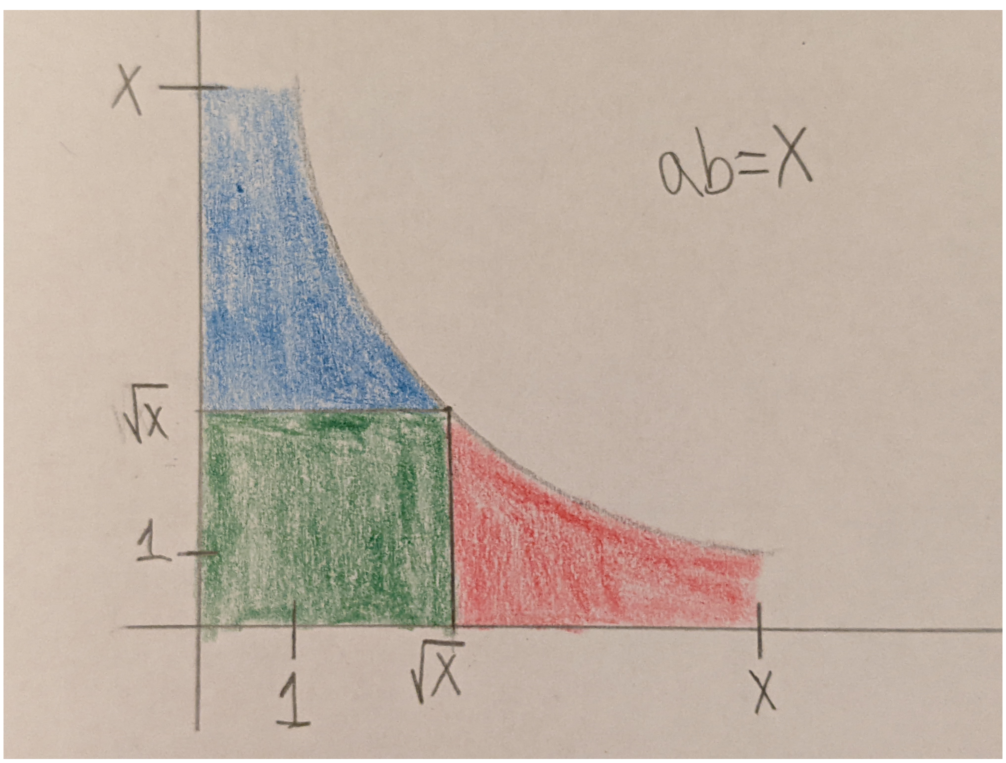
$$V = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : 0 < a \leq \sqrt{X},\ 0 < b \leq \sqrt{X}\}$$

In the image below, V is the green region. From this image, it is clear that counting all the lattice points in T is equivalent to adding all the lattice points in U and W and subtracting the points that were counted twice in V. We will now prove this rigorously. First, we will show that $T = U \cup W$. Let us suppose that $(a,b) \in T$. Then, we know that $0 < a \leq X$ and $0 < b \leq X/a$. This informs us that $0 < a \leq X/b$. For the sake of contradiction, we may suppose that $(a,b) \notin U \cup W$. Then, this means that $a > \sqrt{X}$ and $b > \sqrt{X}$. Then, we have $b \leq X/a < X/\sqrt{X} = \sqrt{X}$. This contradicts the fact that $b > \sqrt{X}$. Thus, we may deduce that $(a,b) \in U \cup W$. This shows that $T \subseteq U \cup W$. Next, we may suppose that $(a,b) \in U \cup W$. This means that $(a,b) \in U$ or $(a,b) \in W$. If $(a,b) \in U$, then we have $0 < a \leq \sqrt{X} \leq X$ and $0 < b \leq X/a$, so it is evident that $(a,b) \in T$. If $(a,b) \in W$, then we have $0 < b \leq \sqrt{X}$ and $0 < a \leq X/b$. Since $b \geq 1$, it is evident that $0 < a \leq X/b \leq X$. Furthermore, the inequality $0 < a \leq X/b$ informs us that $0 < b \leq X/a$, so we may deduce that $(a,b) \in T$. Thus, we find that $U \cup W \subseteq T$ so that $U \cup W = T$. Next, we must prove that $U \cap W = V$. First, let $(a,b) \in U \cap W$. Then, we know that $0 < a \leq \sqrt{X}$ and $0 < b \leq \sqrt{X}$. By the definition of V, we find that $(x,y) \in V$, so that $U \cap W \subseteq V$. Conversely, let us suppose that $(x,y) \in V$. Then, we know that $0 < a \leq \sqrt{X}$ and $0 < b \leq \sqrt{X}$. If $(x,y) \notin U \cap W$, then either $b > X/a$ or $a > X/b$. In the first case, we have $b > X/a \geq X/\sqrt{X} = \sqrt{X}$, which contradicts the assumption that $b \leq \sqrt{X}$. In the second case, we have $a > X/b \geq X/\sqrt{X} = \sqrt{X}$, contradicting the assumption that $a \leq \sqrt{X}$. Thus, we may deduce that $(x,y) \in U \cap W$. This implies that $V \subseteq U \cap W$ so that $V = U \cap W$. With all this information, we may deduce that $|T| = |U \cup W| = |U| + |W| - |U \cap W| = |U| + |W| - |V|$. Substituting values for $|T|$, $|U|$, $|W|$, and $|V|$, we find that

$$\sum_{n \leq X} \left[\frac{X}{n}\right] = 2 \sum_{n \leq \sqrt{X}} \left[\frac{X}{n}\right] - [\sqrt{X}]^2$$

so that

$$\sum_{n \leq X} \tau(n) = 2 \sum_{n \leq \sqrt{X}} \left[\frac{X}{n}\right] - [\sqrt{X}]^2$$



Problem 6

By Problem 3, we may suppose without loss of generality that X is an integer. In class, we proved that

$$\sum_{n \leq X} f(n) \left[\frac{X}{n}\right] = \sum_{n \leq X} g(n)$$

where f is an arithmetic function and g is its mobius transform. Let $f = \mu$, that is, we are letting f equal the mobius function μ. In this case, we have

$$g(n) = \sum_{d|n} f(d) = \sum_{d|n} \mu(d)$$

In class, we proved that $g(n) = 1$ if $n = 1$ and $g(n) = 0$ if $n > 1$. By substitution, we find that

$$\sum_{n \leq X} \mu(n) \left[\frac{X}{n}\right] = \sum_{n \leq X} f(n) \left[\frac{X}{n}\right] = \sum_{n \leq X} g(n) = 1$$

Problem 7

By Problem 3, we may suppose without loss of generality that X is an integer. In class, we proved that

$$\sum_{n \leq X} f(n) \left[\frac{X}{n} \right] = \sum_{n \leq X} g(n)$$

where f is an arithmetic function and g is its mobius transform. Let $f = \Lambda$, that is, we are letting f equal the von Mangoldt function Λ. Then, we have

$$g(n) = \sum_{d|n} f(d) = \sum_{d|n} \Lambda(d) = \log(n)$$

We proved the last equality in class. By substitution, we may deduce that

$$\sum_{n \leq X} \Lambda(n) \left[\frac{X}{n} \right] = \sum_{n \leq X} f(n) \left[\frac{X}{n} \right] = \sum_{n \leq X} g(n) = \sum_{n \leq X} \log(n)$$