Math 115A **Final Exam** **Winter 2024**

Due March 20, 2024

Name ____.

Perm Number ____.

Your Total......................

1 ...... **2** ...... **3** ...... **4** ...... **5** ...... **6** ......

1. (6 points each) Compute the Legendre symbols:

$$\left(\frac{105}{191}\right), \tag{i}$$

$$\left(\frac{56}{101}\right), \tag{ii}$$

$$\left(\frac{106}{89}\right). \tag{iii}$$

2. (6 points each) Solve each of the following congruence equations or verify that it has no solution.

$$14x \equiv 3 \pmod{31}, \tag{i}$$

$$15x \equiv 9 \pmod{35}, \tag{ii}$$

$$35x \equiv 14 \pmod{56}. \tag{iii}$$

3. Suppose $x > 2$.

(i). (10 points) Prove that

$$\sum_{n \le x} \left[\frac{x}{n}\right] = \sum_{n \le x} \tau(n),$$

where $\tau(n)$ denotes the divisor function.

(ii). (10 points) Prove that

$$\sum_{n \le x} \tau(n) = x \log x + O(x).$$

Hint: You can use (i) and the relation

$$\sum_{n\le x}\frac{1}{n}=\log x+O(1)$$

directly.

4. (10 points) Suppose a, m and b are integers, $m>1$ and $(a,m)=1$.. Prove that

$$\sum_{n=1}^{m}\left\{\frac{an+b}{m}\right\}=\frac{1}{2}(m-1).$$

In Question 5 and 6 below, p denotes a prime number >2. Write

$$\left(\frac{a}{p}\right)=0 \quad \text{if} \quad p|a.$$

5. Suppose S is a reduced system of residues $(\bmod\, p)$.

(i). (8 points) Prove that

$$\sum_{s\in S}\left(\frac{s}{p}\right)=0.$$

(ii). (8 points) Prove that

$$\sum_{s\in S}\left(\frac{1+s}{p}\right)=-1.$$

6. (18 points) Suppose $(k,p)=1$. Prove that

$$\sum_{r=1}^{p-1}\left(\frac{r(r+k)}{p}\right)=-1.$$

Hint: Let $R=\{1,2,...,p-1\}$. Verify that there is a bijection f: $R\to R$ such that

$$f(r)r\equiv 1(\bmod\, p).$$

Then use the relation

$$\left(\frac{r(r+k)}{p}\right)=\left(\frac{f(r)}{p}\right)^2\left(\frac{r(r+k)}{p}\right)$$

and Question 5 (ii).