Math 115A Winter 2024 Homework Assignment 1

Due Wednesday, January 24.

Let a, b, c,... be positive integers.

1. Suppose $(a, b) = 1$ and there is a c such that $ab = c^2$. Prove that there are c_1 and c_2 such that $a = c_1^2$ and $b = c_2^2$.

2. Suppose $x^2 + y^2 = z^2$ and $(x, y, z) = 1$.

(i). Prove that exactly one of x and y is even, the other is odd.

(ii). Suppose x is even and y is odd. Prove that

$$\left(\frac{z+y}{2}, \frac{z-y}{2}\right) = 1.$$

Then use

$$\left(\frac{x}{2}\right)^2 = \left(\frac{z+y}{2}\right)\left(\frac{z-y}{2}\right)$$

and Question 1 to conclude that there are m and n such that

$$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2.$$

3. Suppose every prime factor of N is of the form $4n + 1$. Prove that N is of the form $4M + 1$.

4. Let p_1, p_2,...,p_k are primes, each of which is > 3 and of the form $4n + 3$. Prove that the number

$$N = 4p_1p_2...p_k + 3$$

has a prime factor p which is of the form $4n + 3$ but not equal to p_1, p_2,... or p_k.

5. Conclude that there are infinitely many primes of the form $4n + 3$.

6. Prove that there are infinitely many primes of the form $6n + 5$.

7. Suppose $n \geq 2$. Prove that the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is *not* an integer.