Math 120TC Homework 3

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Exercise 4.1.1

Part A

We define the map $f : \mathbb{R}^2 \to X \setminus B^2$ as follows: $f(0) = [0]$ and $f(x) = f(re^{i\theta}) = [(r+1)e^{i\theta}]$ if $x \neq 0$. It is evident that f is bijective. Now, we must show that f and f^{-1} are continuous. First, we claim that f is continuous. Let $U \subseteq X \setminus B^2$ be open. If U does not contain $[0]$, then it is clear that $f^{-1}(U)$ is open. If $[0] \in U$, then we know that $B^2 \subseteq q^{-1}(U)$, where $q : X \to X \setminus B^2$ is the projection map. Since U is open, $q^{-1}(U)$ is open. Because B^2 is closed, we must have $B^2 \subsetneq q^{-1}(U)$. By compactness, we know that there must be some open ball D of radius strictly greater than 1 such that $D \subseteq q^{-1}(U)$. Now, we have $U = q(D) \cup U \setminus \{[0]\}$. We note that

$$f^{-1}(U) = f^{-1}(q(D)) \cup f^{-1}(U \setminus \{[0]\})$$

which is clearly open since $f^{-1}(q(D))$ and $f^{-1}(U \setminus \{[0]\})$ are open. Next, we claim that f^{-1} is continuous. We note that

$$f^{-1} \circ q(re^{i\theta}) = (\max\{0, r-1\})e^{i\theta}$$

which is continuous. Thus, if $U \subseteq \mathbb{R}^2$ is an open set, we have

$$(f^{-1} \circ q)^{-1}(U) = q^{-1} \circ f(U)$$

is open so that $f(U)$ is open. This shows that f^{-1} is continuous so that f is a homeomorphism.

Part B

Let us suppose that $f : Y \to X \setminus U$ is a homeomorphism, where Y is any metric space. Since f is bijective, there must exist y_1 and y_2 in Y such that $f(y_1) = [(0, 0)]$ and $f(y_2) = [(1, 0)]$. Because Y is a metric space, we have $d(y_1, y_2) > 0$ and let $\delta = (1/2)d(y_1, y_2)$. We consider the open ball $B(y_2; \delta)$. Since this ball is open in Y and f is a homeomorphism, we know that $f(B(y_2; \delta))$ is open so that $q^{-1}(f(B(y_2; \delta)))$ is open. Since $(1, 0) \in q^{-1}(f(B(y_2; \delta)))$, there must exist some $\varepsilon < 1$ such that $(\varepsilon, 0) \in q^{-1}(f(B(y_2; \delta)))$ so that $[(0, 0)] \in f(B(y_2; \delta))$. That is, there must exist some $y_3 \in B(y_2; \delta)$ such that $f(y_3) = [(0, 0)]$. Since $y_3 \neq y_1$, we deduce that f is not injective, which is a contradiction.