Math 120TC Homework 6

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Problem 1

First, we recall how we proved the fact that the sphere S^n is $n-1$ connected. To prove this, we first assumed that $f : S^i \to S^n$ was a continuous map and that it was not surjective. In particular, we supposed that there was some point $x \in S^n$ such that $x \notin f(S^i)$. Then, we constructed a continuous extension $\overline{f} : B^{i+1} \to S^n$ as follows: for any $v \in B^{i+1}$, we let $v = ru$, where $r \in [0,1]$ and $u \in S^i$. Then, we set

$$\overline{f}(v) = \frac{rf(u) + (1-r)(-x)}{\|rf(u) + (1-r)(-x)\|}$$

Notice that the denominator is never zero because we assumed that $f(u) \neq x$ for all $u \in S^i$. If f happened to be a space filling curve, then we constructed a new function g by linear interpolation of the images of f on the vertices of a triangulation of S^i, and we showed that g is not surjective.

Now, we will use a similar proof method to show that $S^n * F$ is n connected for any n and any finite set F. Let $f : S^i \to S^n * F$ be a continuous map for some $i \leq n$. We claim that f can be extended to a continuous map $\overline{f} : B^{i+1} \to S^n * F$. To do this, we first suppose that $f(S^i)$ does not contain any point of F. In this case, we may continuously deform the image $f(S^i)$ onto a single cone (this is only possible because we are assuming that $f(S^i)$ does not contain any point of F). We denote the vertex of this cone by x. We now extend f to B^{i+1} as follows: Let $v \in B^{i+1}$ be such that $v = ru$ for some $r \in [0,1]$ and $u \in S^i$. We draw the line from $f(u)$ to x, and we let $\overline{f}(v)$ be defined as the point $rf(u) + (1-r)x$ (notice that the origin is sent to the point x). Thus, we have demonstrated that $f : S^i \to S^n * F$ can be extended to $\overline{f} : B^{i+1} \to S^n * F$ if the image $f(S^i)$ does not contain any points of F. Next, let us suppose that the image $f(S^i)$ does contain a point of F. Then, we triangulate the sphere S^i as in the proof of the $n-1$ connectedness of S^n. We construct g by interpolating the values of f at the vertices of the triangulation. We can show that g is not surjective. Since F is a finite point set, we can perturb g if necessary to ensure that $g(S^i)$ does not contain any point of F.

Problem 2

Let D_3 denote a set with 3 distinct points. First, we claim that $(D_3)^{*(n+1)}$ is a n dimensional simplicial complex. To do this, we note that the maximum size of any simplex in $(D_3)^{*(n+1)}$ is $n+1$ since we are taking the join of a discrete set with itself $n+1$ times. By the definition of the dimension of a simplicial complex (which is defined to be the dimension of its maximal simplices), we deduce that $(D_3)^{*(n+1)}$ is an n complex. Now, we recall Sakaria's coloring/embedding theorem: Let K be a simplicial complex on n vertices, and let $\mathcal{F} = \mathcal{F}(K)$ be the system of minimal nonfaces of K. If $d \leq n - \chi(\mathrm{KG}(\mathcal{F})) - 2$, then for any continuous mapping $f : \|K\| \to \mathbb{R}^d$, the images of some two disjoint faces of K must intersect. Note that $(D_3)^{*(n+1)}$ has $3n+3$ vertices. We wish to show that it cannot be embedded in $\mathbb{R}^{2n}$. According to the above theorem, we need only show that

$$2n \leq 3n + 3 - \chi(\mathrm{KG}(\mathcal{F})) - 2$$

or

$$\chi(\mathrm{KG}(\mathcal{F})) \leq n + 1$$

Let us label the vertices of $(D_3)^{*(n+1)}$ as follows:

$$\{1', 2', 3', 1'', 2'', 3'', \ldots, 1^{(n)}, 2^{(n)}, 3^{(n)}, 1^{(n+1)}, 2^{(n+1)}, 3^{(n+1)}\}$$

The minimal nonfaces are $\{i^{(k)}, j^{(k)}\}$, where $i, j \in \{1, 2, 3\}$ and k ranges from 1 to $n+1$. Thus, these pairs are the vertices of the Kneser graph. Notice that there is an edge between the vertices $\{i^{(k)}, j^{(k)}\}$ and $\{i^{(l)}, j^{(l)}\}$ if and only if $k \neq l$. We define the coloring $c : \mathrm{KG}(\mathcal{F}) \to [n+1]$ as follows:

$$c(\{i^{(k)}, j^{(k)}\}) = k$$

It is not too difficult to see that this defines a proper coloring of the Kneser graph. Thus, we find that

$$\chi(\mathrm{KG}(\mathcal{F})) \leq n + 1$$

and that $(D_3)^{*(n+1)}$ cannot be embedded in $\mathbb{R}^{2n}$. Notice that for the case $n = 1$, we are saying that $(D_3)^{*2} = K_{3,3}$ cannot be embedded in $\mathbb{R}^2$.

Problem 3

We have the following commutative diagram.

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\psi_1} \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Thus, we have $\psi_1 \circ f = f \circ \varphi_1$. In particular, we know that

$$\psi_1(f(z)) = f(\varphi_1(z))$$

or

$$e^{2\pi i\ell/p} z^n = e^{2\pi ikn/p} z^n$$

This informs us that

$$e^{2\pi i\ell/p} = e^{2\pi ikn/p}$$

or

$$2\pi\ell/p = 2\pi kn/p + 2\pi m$$

for some integer m. Dividing by 2π gives us

$$\frac{\ell}{p} = \frac{kn}{p} + m$$

or

$$\ell = kn + pm$$

Finally, we solve for n to obtain

$$n = \frac{\ell - pm}{k}$$