Math 120TC Homework 4

Ethan Martirosyan

May 8, 2024

Exercise 4.2.3

Part A

We let $|A| = k$ and $|B| = j$. Since $U \cap V = \varnothing$, we find that $A \cap B = \varnothing$ so that $|A \cup B| = k + j$ For the sake of contradiction, let us suppose that $A \cup B$ is not affinely independent. Then, we know that the convex hull of $A \cup B$ has dimension less than $k + j - 1$. Now, let A' be chosen to have the maximum number of points such that $A'' := A \cup A'$ is affinely independent. We note that $|A'| = \dim U - |A| + 1$ so that $|A''| = \dim U + 1$. Furthermore, we let B' be chosen to have the maximum number of points such that $B'' = B \cup B'$ is affinely independent. We note that $|B'| = \dim V - |B| + 1$ so that $|B''| = \dim V + 1$. We know that $A'' \cup B''$ is not affinely independent. Thus, even though $|A'' \cup B''| = \dim U + \dim V + 2$, we know that the convex hull has dimension less than $\dim U + \dim V + 1$. However, this is equal to the dimension of the affine hull of $U \cup V$. Thus, we find that the dimension of the affine hull of $U \cup V$ is less than $\dim U + \dim V + 1$ so that U and V are not actually skew affine subspaces. This contradiction informs us that $A \cup B$ is affinely independent.

Part B

Let U denote the union of all line segments connecting a point of $\text{conv}(A)$ to a point of $\text{conv}(B)$. We claim that $U = \text{conv}(A \cup B)$. First, we let $z \in U$. Then, we may write

$$z = t \sum \alpha_i x_i + (1-t) \sum \beta_i y_i$$

where $0 \leq t, \alpha_i, \beta_i \leq 1$, $\sum \alpha_i = 1$, $\sum \beta_i = 1$, $x_i \in A$ and $y_i \in B$. We note that

$$z = \sum t\alpha_i x_i + \sum (1-t)\beta_i y_i$$

and that

$$\sum t\alpha_i + \sum (1-t)\beta_i = t \sum \alpha_i + (1-t) \sum \beta_i = t + (1-t) = 1$$

Thus, we find that $z \in \text{conv}(A \cup B)$ so that $U \subseteq \text{conv}(A \cup B)$. Next, we let $z \in \text{conv}(A \cup B)$, so we write

$$z = \sum \alpha_i x_i + \sum \beta_i y_i$$

where $x_i \in A$, $y_i \in B$, $\sum \alpha_i + \sum \beta_i = 1$. Let $K = \sum \alpha_i$, and let $J = \sum \beta_i$. Then we have $J = 1 - K$. Furthermore, we note that

$$\frac{\sum \alpha_i x_i}{K} \in \text{conv}(A)$$

and

$$\frac{\sum \beta_i y_i}{J} \in \text{conv}(B)$$

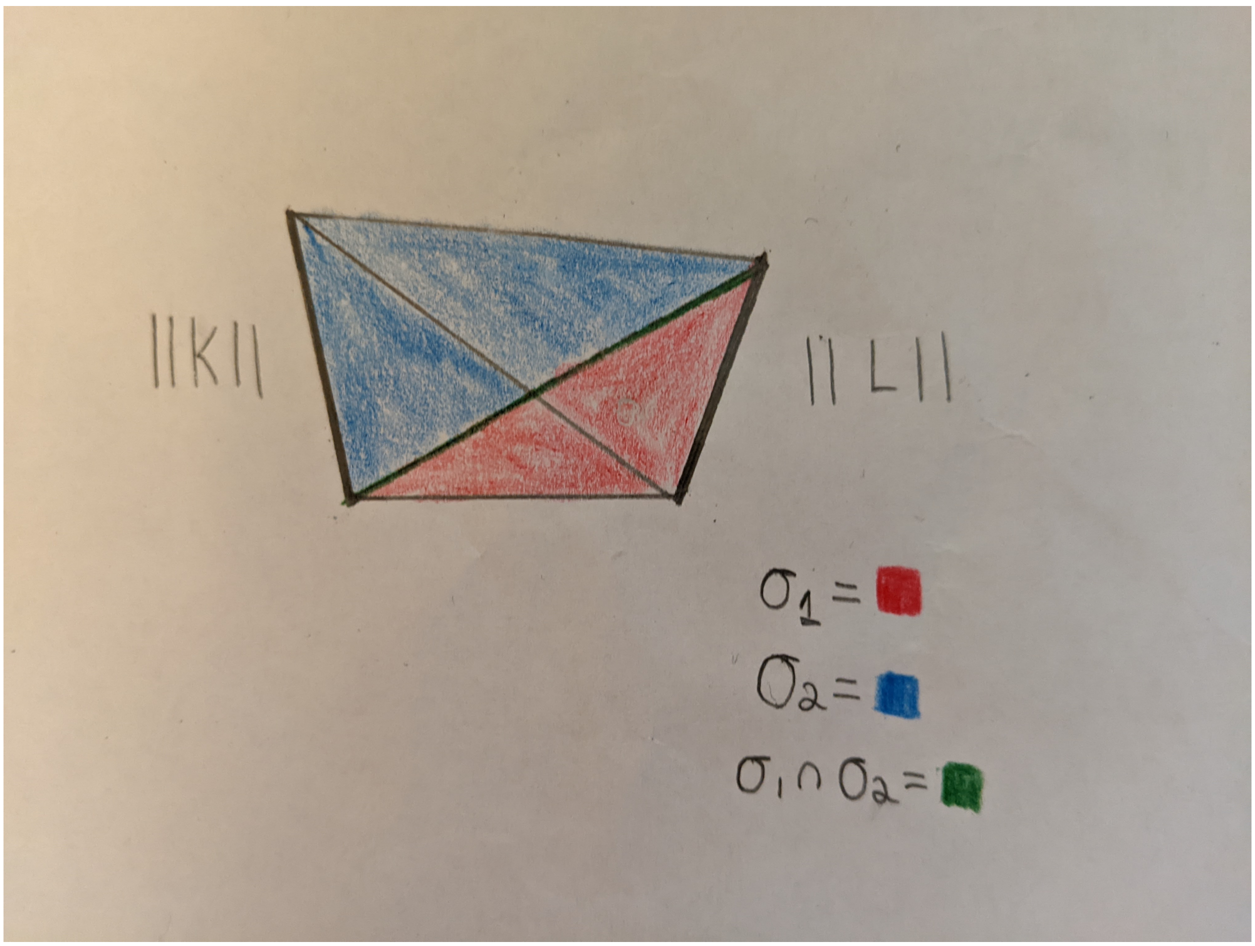
With this, we can write

$$z = K\left(\frac{\sum \alpha_i x_i}{K}\right) + J\left(\frac{\sum \beta_i y_i}{J}\right) = K\left(\frac{\sum \alpha_i x_i}{K}\right) + (1-K)\left(\frac{\sum \beta_i y_i}{J}\right)$$

so that $z \in U$ and $\text{conv}(A \cup B) \subseteq U$. Thus, we deduce that $U = \text{conv}(A \cup B)$.

Part C

First, we show that $\|\Delta^n\| * \|\Delta^m\| \cong \|\Delta^n * \Delta^m\|$. This easily follows from part B because $\|\Delta^n\| * \|\Delta^m\|$ is the union of all line segments connecting points in $\text{conv}(\Delta^n)$ to points in $\text{conv}(\Delta^m)$ and $\|\Delta^n * \Delta^m\|$ is $\text{conv}(\Delta^n \cup \Delta^m)$. Now, we claim that $\|K\| * \|L\| \cong \|K * L\|$ for arbitrary simplicial complexes. To show this, it suffices to show that $\|K\| * \|L\|$ is a geometric simplicial complex. According to the definition of simplicial complex, it must be true that every face of any simplex in $\|K\| * \|L\|$ is a simplex in $\|K\| * \|L\|$. This follows immediately from the definition of join as the union of all line segments from points in $\|K\|$ to points in $\|L\|$. Next, we must show that the intersection $\sigma_1 \cap \sigma_2$ of any two simplices is a face of both σ_1 and σ_2. We know that this is true because $\|K\| * \|L\|$ is the union of all segments connecting points of $\|K\|$ to points of $\|L\|$. The image below shows an example of this. Thus, we conclude that $\|K\| * \|L\|$ is a simplicial complex and that $\|K\| * \|L\| \cong \|K * L\|$.



Exercise 5.3.3

Part A

We construct a map $f : V_{n,2} \to S^{n-1}$ as follows:

$$f(v_1, v_2) = \frac{v_1 + v_2}{\sqrt{2}}$$

Notice that

$$f(-v_1, -v_2) = \frac{-v_1 - v_2}{\sqrt{2}} = -\frac{v_1 + v_2}{\sqrt{2}} = -f(v_1, v_2)$$

Thus f is a $\mathbb{Z}_2$-map from $V_{n,2}$ to S^{n-1} so that $\mathrm{ind}_{\mathbb{Z}_2}(V_{n,2}) \leq n - 1$.

Part B

We define a map $f : S^{n-1} \to V_{2,n}$ as follows:

$$f(x_1, \dots, x_n) = \big((x_1, \dots, x_n), (-x_2, x_1, \dots, -x_n, x_{n-1})\big)$$

(we can alternate signs in this manner because n is even). We note that

$$(x_1, \dots, x_n) \cdot (-x_2, x_1, \dots, -x_n, x_{n-1}) = x_1x_2 - x_1x_2 + \cdots - x_{n-1}x_n + x_{n-1}x_n = 0$$

so that f does indeed map S^{n-1} into $V_{2,n}$. Now, we note that

$$\begin{aligned} \nu(f(x_1, \dots, x_n)) &= \nu\big((x_1, \dots, x_n), (-x_2, x_1, \dots, -x_n, x_{n-1})\big) \\ &= \big((-x_1, \dots, -x_n), (x_2, -x_1, \dots, x_n, -x_{n-1})\big) \end{aligned}$$

and

$$f(-x_1, \dots, -x_n) = \big((-x_1, \dots, -x_n), (x_2, -x_1, \dots, x_n, -x_{n-1})\big)$$

so that f is indeed a $\mathbb{Z}_2$-map from S^{n-1} to $V_{2,n}$, thereby proving that $\mathrm{ind}_{\mathbb{Z}_2}(V_{n,2}) = n - 1$ for n even.

Part C

Suppose that n is odd. Then $S^{n-2} \subseteq \mathbb{R}^{n-1}$ and $n-1$ is even. We may define

$$f(x_1, \ldots, x_{n-1}) = \big((x_1, \ldots, x_{n-1}), (-x_2, x_1, \ldots, -x_{n-1}, x_{n-2})\big)$$

As above, we can show that

$$\nu(f(x_1, \ldots, x_{n-1})) = \big((-x_1, \ldots, -x_{n-1}), (x_2, -x_1, \ldots, x_{n-1}, -x_{n-2})\big) = f(-x_1, \ldots, -x_{n-1})$$

so that f is a $\mathbb{Z}_2$-map from S^{n-2} to $V_{2,n}$.

Exercise 5.3.4

Part A

We define $f : V_{2,n} \to V_{2,n}$ as follows:

$$f(v_1, v_2) = \left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_1 - v_2}{\sqrt{2}}\right)$$

First, we note that

$$\|v_1 + v_2\|^2 = \langle v_1 + v_2, v_1 + v_2\rangle = \langle v_1, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_1\rangle + \langle v_2, v_2\rangle = \|v_1\|^2 + \|v_2\|^2 = 2$$

and that

$$\|v_1 - v_2\|^2 = \langle v_1 - v_2, v_1 - v_2\rangle = \langle v_1, v_1\rangle - \langle v_1, v_2\rangle - \langle v_2, v_1\rangle + \langle v_2, v_2\rangle = \|v_1\|^2 + \|v_2\|^2 = 2$$

because v_1 and v_2 are orthogonal. We obtain

$$\|v_1 + v_2\| = \|v_1 - v_2\| = \sqrt{2}$$

Thus, we find that $f(v_1, v_2) \in (S^{n-1})^2$. Furthermore, we compute

$$\left\langle \frac{v_1 + v_2}{\sqrt{2}}, \frac{v_1 - v_2}{\sqrt{2}} \right\rangle = \frac{1}{2}(\langle v_1, v_1\rangle - \langle v_1, v_2\rangle + \langle v_2, v_1\rangle - \langle v_2, v_2\rangle) = 0$$

so that f does actually map $V_{2,n}$ into $V_{2,n}$. Now, we must show that $f \circ \omega_1 = \omega_2 \circ f$. Let $(v_1, v_2) \in V_{2,n}$. We compute

$$f(\omega_1(v_1, v_2)) = f(v_2, v_1) = \left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_2 - v_1}{\sqrt{2}}\right)$$

and

$$\omega_2(f(v_1, v_2)) = \omega_2\left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_1 - v_2}{\sqrt{2}}\right) = \left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_2 - v_1}{\sqrt{2}}\right)$$

so that $f \circ \omega_1 = \omega_2 \circ f$. Thus, we find that f is a $\mathbb{Z}_2$-map from $V_{2,n}$ to $V_{2,n}$. Now, we claim that $f^{-1} = f$. To see this, we compute

$$f(f(v_1, v_2)) = f\left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_2 - v_1}{\sqrt{2}}\right) = \left(\frac{\frac{v_1+v_2}{\sqrt{2}} + \frac{v_1-v_2}{\sqrt{2}}}{\sqrt{2}}, \frac{\frac{v_1+v_2}{\sqrt{2}} - \frac{v_1-v_2}{\sqrt{2}}}{\sqrt{2}}\right) = (v_1, v_2)$$

Thus, we find that f is indeed its own inverse, so we only have to show that $f : (V_{2,n}, \omega_2) \to (V_{2,n}, \omega_1)$ is a $\mathbb{Z}_2$-map. That is, we must show that $f \circ \omega_2 = \omega_1 \circ f$. Let $(v_1, v_2) \in V_{2,n}$. Then, we have

$$f(\omega_2(v_1, v_2)) = f(v_1, -v_2) = \left(\frac{v_1 - v_2}{\sqrt{2}}, \frac{v_1 + v_2}{\sqrt{2}}\right)$$

and

$$\omega_1(f(v_1, v_2)) = \omega_1\left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_1 - v_2}{\sqrt{2}}\right) = \left(\frac{v_1 - v_2}{\sqrt{2}}, \frac{v_1 + v_2}{\sqrt{2}}\right)$$

Thus we find that $f \circ \omega_2 = \omega_1 \circ f$ so that f is a $\mathbb{Z}_2$-homeomorphism from $(V_{2,n}, \omega_1)$ to $(V_{2,n}, \omega_2)$.

Part B

First, we build a map $f : (V_{n,2}, \omega_1) \to (S^{n-1}, \nu)$ as follows:

$$f(v_1, v_2) = \frac{v_1 - v_2}{\sqrt{2}}$$

We claim that $f \circ \omega_1 = \nu \circ f$. We compute

$$f(\omega_1(v_1, v_2)) = f(v_2, v_1) = \frac{v_2 - v_1}{\sqrt{2}}$$

and

$$\nu(f(v_1, v_2)) = \nu\left(\frac{v_1 - v_2}{\sqrt{2}}\right) = \frac{v_2 - v_1}{\sqrt{2}}$$

so that $f \circ \omega_1 = \nu \circ f$. Thus we deduce that $\mathrm{ind}_{\mathbb{Z}_2}(V_{n,2}) \leq n - 1$. Next, we build a map $f : S^{n-2} \to V_{2,n}$ as follows:

$$f(x_1, \ldots, x_{n-1}) = \big((0, \ldots, 0, 1), (x_1, \ldots, x_{n-1}, 0)\big)$$

Now, we claim that $\omega_2 \circ f = f \circ \nu$. Notice that

$$f(\nu(x)) = f(-x) = f(-x_1, \ldots, -x_{n-1}) = \big((0, \ldots, 0, 1), (-x_1, \ldots, -x_{n-1}, 0)\big)$$

and that

$$\omega_2(f(x)) = \omega_2\big((0, \ldots, 0, 1), (x_1, \ldots, x_{n-1}, 0)\big) = \big((0, \ldots, 0, 1), (-x_1, \ldots, -x_{n-1}, 0)\big)$$

Thus we find that $\omega_2 \circ f = f \circ \nu$ so that f is a $\mathbb{Z}_2$-map and $n - 2 \leq \mathrm{ind}_{\mathbb{Z}_2}(V_{n,2})$.