Math 120TC Homework 5

Ethan Martirosyan

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Problem 1

First, we note that

$$K_5 = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$$

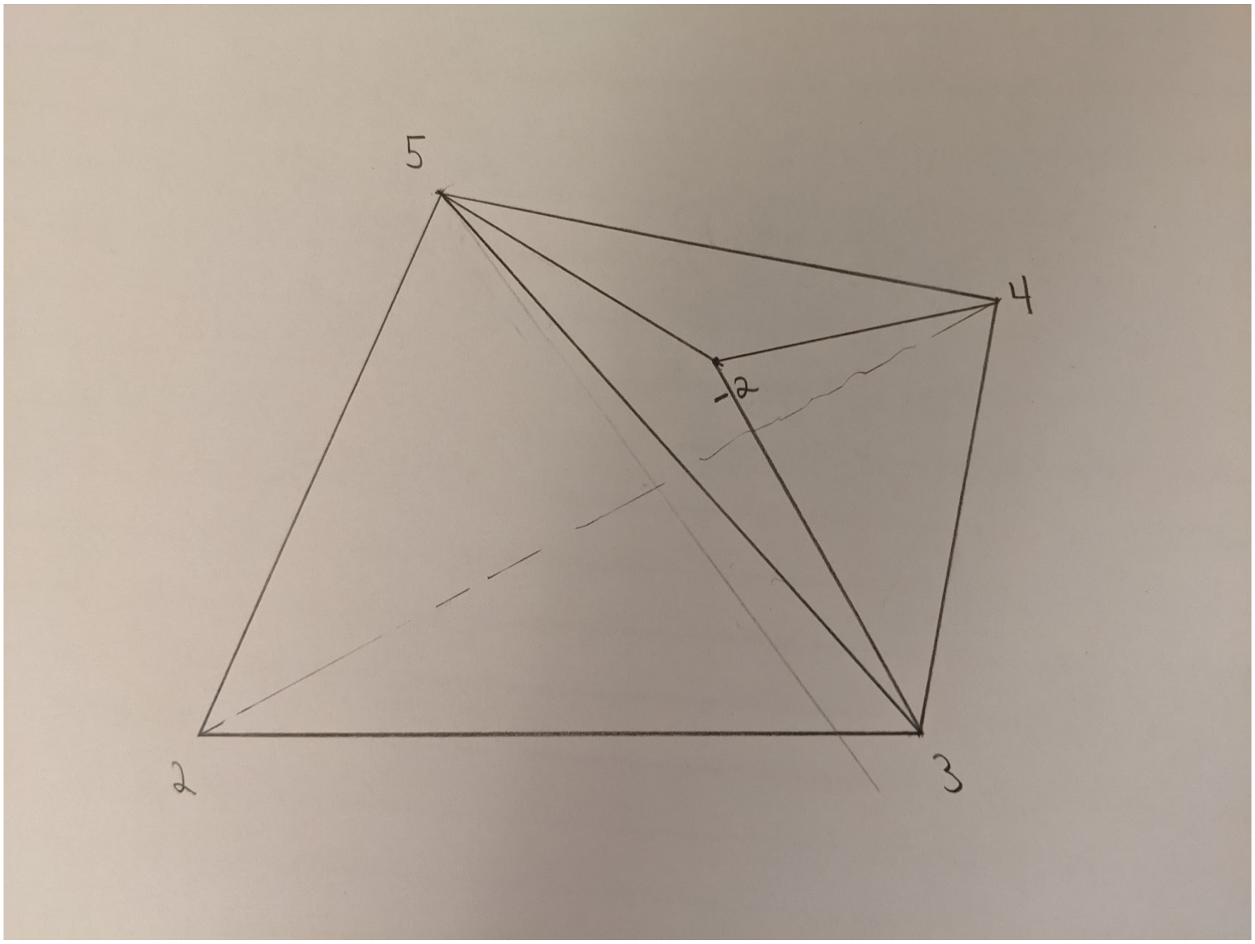
Since $V((K_5)^{*2}_{\Delta}) = V(K_5) \sqcup V(K_5)$, we know that $|V((K_5)^{*2}_{\Delta})| = 10$. We may suppose that 5 of the vertices are in $\mathbb{R}^5$ at the vectors e_1, e_2, e_3, e_4, and e_5. Then, there is a 4 dimensional hyperplane going through these five vectors, so we may consider them to be in $\mathbb{R}^4$. Then, their negatives would be the other 5 vertices. Now, we will determine the geometric nature of $(K_5)^{*2}_{\Delta}$. We claim that it is the boundary of a convex 4-polytope in $\mathbb{R}^4$. Let us write the 10 vertices as follows:

$$\{1\},\{2\},\{3\},\{4\},\{5\},\{-1\},\{-2\},\{-3\},\{-4\},\{-5\}$$

Notice that the vertex 1 is connected to 8 vertices; namely, the 8 vertices that aren't 1 or -1. Let us examine the link at 1. If we look at the vertex 2, we note that the link contains

$$\{\{2\},\{2,-3\},\{2,-4\},\{2,-5\},\{2,-3,-4\},\{2,-3,-5\},\{2,-4,-5\}\}$$

From this alone, we find that the link contains 2 complexes. We deduce that the boundary near the vertex 1 is a 3 dimensional object. Furthermore, this holds for every other vertex as well so that the deleted product $(K_5)^{*2}_{\Delta}$ must be the boundary of a convex 4 polytope. Let us examine this some more. Consider the below image:



In this picture, we have drawn the simplex corresponding to $2, 3, 4, 5$. In accordance with our notation, we have put -2 on the opposite side and connected it to $3, 4, 5$. From this picture, it is evident that the deleted join cannot be embedded into 3 space and that it must be in 4 space at least. This provides more evidence to support the assertion that the deleted join is the boundary of a 4 polytope. Just to be certain, we compute the deleted product below:

$\{1\}$ $\{1,-2\}$
$\{1,-3\}$
$\{1,-4\}$
$\{1,-5\}$
$\{1,-2,-3\}$
$\{1,-2,-4\}$
$\{1,-2,-5\}$
$\{1,-3,-4\}$
$\{1,-3,-5\}$
$\{1,-4,5\}$

$\{2\}$ $\{2,-1\}$
$\{2,-3\}$
$\{2,-4\}$
$\{2,-5\}$
$\{2,-1,-3\}$
$\{2,-1,-4\}$
$\{2,-1,-5\}$
$\{2,-3,-5\}$
$\{2,-3,-4\}$
$\{2,-4,-5\}$

$\{3\}$ $\{3,-1\}$
$\{3,-2\}$
$\{3,-4\}$
$\{3,-5\}$
$\{3,-1,-2\}$ $\{3,-1,-4\}$
$\{3,-1,-5\}$ $\{3,-2,-4\}$
$\{3,-2,-5\}$ $\{3,-4,-5\}$

$\{4\}$ $\{4,-1\}$
$\{4,-2\}$
$\{4,-3\}$
$\{4,-5\}$
$\{4,-1,-2\}$
$\{4,-1,-3\}$
$\{4,-1,-5\}$
$\{4,-2,-3\}$
$\{4,-2,-5\}$
$\{4,-3,-5\}$

$\{5\}$ $\{5,-1\}$
$\{5,-2\}$
$\{5,-3\}$
$\{5,-4\}$
$\{5,-1,-2\}$
$\{5,-1,-3\}$
$\{5,-1,-4\}$
$\{5,-2,-3\}$
$\{5,-2,-4\}$
$\{5,-3,-4\}$

$\{1,2\}$
$\{1,2,-3,-4\}$
$\{1,2,-3,-5\}$
$\{1,2,-4,-5\}$
$\{1,3\}$ $\{1,3,-4,-5\}$
$\{1,3,-2,-4\}$
$\{1,3,-2,-5\}$

$\{1,4\}$
$\{1,4,-2,-3\}$
$\{1,4,-2,-5\}$
$\{1,4,-3,-5\}$
$\{1,5\}$
$\{1,5,-2,-3\}$
$\{1,5,-2,-4\}$
$\{1,5,-3,-4\}$
$\{2,3\}$
$\{2,3,-1,-4\}$
$\{2,3,-1,-5\}$
$\{2,3,-4,-5\}$
$\{2,4\}$
$\{2,4,-1,-3\}$
$\{2,4,-1,-5\}$
$\{2,4,-3,-5\}$
$\{2,5\}$
$\{2,5,-1,-3\}$
$\{2,5,-1,-4\}$
$\{2,5,-3,-4\}$
$\{3,4\}$
$\{3,4,-1,-2\}$
$\{3,4,-1,-5\}$
$\{3,4,-2,-5\}$

$\{3,5\}$
$\{3,5,-1,-2\}$
$\{3,5,-1,-4\}$
$\{3,5,-2,-4\}$
$\{4,5\}$
$\{4,5,-1,-2\}$
$\{4,5,-1,-3\}$
$\{4,5,-2,-3\}$

Problem 2

Let $f : S^k \to S^n$ and $g : S^\ell \to S^n$ be odd maps. Then, we know that the following diagrams commute:

$$\begin{tikzcd} S^k \arrow[r, "f"] \arrow[d, "\nu"'] & S^n \arrow[d, "\nu"] \\ S^k \arrow[r, "f"] & S^n \end{tikzcd}$$

and

$$\begin{tikzcd} S^\ell \arrow[r, "g"] \arrow[d, "\nu"'] & S^n \arrow[d, "\nu"] \\ S^\ell \arrow[r, "g"] & S^n \end{tikzcd}$$

where ν represents the antipodality map on S^k, S^ℓ, or S^n (that is, we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$). Now, we have a map $f * g : S^k * S^\ell \to S^n * S^n$ given by $f * g(x, y, t) = (f(x), g(y), t)$. Let us assume that $f(S^k) \cap g(S^\ell) = \varnothing$. Then, we know that $f * g$ maps $S^k * S^\ell$ into $(S^n)^{*2}_\Delta$, that is, the deleted join of S^n considered as a space. As in Lemma 5.5.4, we can construct a $\mathbb{Z}_2$-map from $(S^n)^{*2}_\Delta \to S^n$ so that $\mathrm{ind}_{\mathbb{Z}_2}((S^n)^{*2}_\Delta) \leq n$. However, we know that $S^k * S^\ell$ is homeomorphic to $S^{k+\ell+1}$ so that $\mathrm{ind}_{\mathbb{Z}_2}(S^k * S^\ell) = k + \ell + 1 \geq n + 1$. Thus, we have reached a contradiction, and we know that $f(S^k) \cap g(S^\ell) \neq \varnothing$.