**Math 115A** **Midterm Exam** **Winter 2024**

Due February 12, 2024

Name ________________________________________.

Perm Number ____________________________________.

**Your Total**.....................

**1** ...... **2** ...... **3** ...... **4** ...... **5** ......

1. (7 points each). Find $\gcd(a, b)$ and find integers $s$ and $t$ such that

$$\gcd(a, b) = sa + tb.$$

(i). $a = 697$, $b = 391$.

(ii). $a = 484$, $b = 136$.

(iii). $a = 961$, $b = 620$.

2. (i). (8 points). Prove that for every integer $k$, $k(k + 1)$ is even.

(ii). (8 points). Prove that for every odd number $n$, $n^2 - 1$ is divisible by 8.

3. (22 points.) Suppose $X > 2$. Prove that

$$\sum_{1 \le n \le X} [\sqrt{X^2 - n^2}] = 2\left( \sum_{1 \le n \le X/\sqrt{2}} [\sqrt{X^2 - n^2}] \right) - [X/\sqrt{2}]^2.$$

Hint: Count the number of lattice points in the first quadrant region

$$x > 0, \quad y > 0, \quad x^2 + y^2 \le X^2.$$

4. (20 points). Prove that $\tau(n)$ is odd if and only if $n$ is a complete square (there is a positive integer $k$ such that $n = k^2$.)

5. (i). (11 points). Prove that if $n > 2$, then $\varphi(n)$ is an even number.

(ii). (11 points). Prove that if $n$ is divisible by $p_1 p_2$, where $p_1$ and $p_2$ are primes satisfying $3 \le p_1 < p_2$, then $n$ is *not* divisible by $\varphi(n)$.

Hint: Discuss the cases that $n$ is odd and $n$ is even respectively.