Math 122A Homework 5

Ethan Martirosyan

February 8, 2024

Chapter 3

Problem 11

Part A

First, we note that $e^z = u(x,y)+iv(x,y) = e^x \cos y + ie^x \sin y$. To verify the Cauchy-Riemann equations, we note that $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v_x$. Since we have verified that e^z satisfies the Cauchy-Riemann equations, it must be entire.

Part B

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then, we have

$$e^z = e^{x_1+x_2} \cos(y_1 + y_2) + ie^{x_1+x_2} \sin(y_1 + y_2)$$

Using the formulas for the sine and cosine of a sum as well as the fact that $e^{x_1+x_2} = e^{x_1}e^{x_2}$ (since x_1 and x_2 are real), we obtain

$$e^{x_1}e^{x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + ie^{x_1}e^{x_2}(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)$$

This is equal to

$$e^{x_1}e^{x_2} \cos y_1 \cos y_2 - e^{x_1}e^{x_2} \sin y_1 \sin y_2 + ie^{x_1}e^{x_2} \sin y_1 \cos y_2 + ie^{x_1}e^{x_2} \cos y_1 \sin y_2$$

Next, we may compute $e^{z_1}e^{z_2}$ as follows:

$$e^{z_1}e^{z_2} = (e^{x_1} \cos y_1 + ie^{x_1} \sin y_1)(e^{x_2} \cos y_2 + ie^{x_2} \sin y_2)$$

By doing some algebra, we obtain

$$e^{x_1}e^{x_2} \cos y_1 \cos y_2 - e^{x_1}e^{x_2} \sin y_1 \sin y_2 + ie^{x_1}e^{x_2} \sin y_1 \cos y_2 + ie^{x_1}e^{x_2} \cos y_1 \sin y_2$$

Thus we may conclude that

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

Problem 12

If $z = x + iy$, then we have

$$|e^z| = |e^x \cos y + ie^x \sin y| = |e^x(\cos y + i \sin y)| = |e^x|| \cos y + i \sin y| = |e^x|$$

The last equality follows because

$$| \cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

Problem 13

If $x > 0$, then $e^z = e^x e^{iy}$ approaches ∞ as $z \to \infty$ because the magnitude of e^z becomes arbitrarily large. If $x < 0$, then $e^z = e^x e^{iy}$ approaches 0 as $z \to \infty$ because the magnitude of e^z becomes arbitrarily small. If $x = 0$, then $e^z = e^{iy}$ traverses the unit circle infinitely many times. Notice that if $x \neq 0$ and $y \neq 0$, the ray from the origin is mapped to a spiral. That is, $e^z = e^x e^{iy}$ traces out a spiral as z moves along the ray.

Problem 14

Part A

Notice that $e^x e^{iy} = e^z = 1 = 1e^{i0}$. Then, we know that $e^x = 1$ and $e^{iy} = e^{i0}$, so that $x = 0$ and $y = 2k\pi$, where k is any integer. Thus, we find that $z = 2k\pi i$.

Part B

Notice that $e^x e^{iy} = e^z = i = 1e^{i\pi/2}$. Then, we know that $e^x = 1$ and $e^{iy} = e^{i\pi/2}$, so that $x = 0$ and $y = 2k\pi + \pi/2$, where k is any integer. Thus, we find that

$$z = \left(\frac{\pi}{2} + 2k\pi\right)i$$

Part C

Notice that $e^x e^{iy} = e^z = -3 = 3e^{i\pi}$. Then, we know that $e^x = 3$ and $e^{iy} = e^{i\pi}$, so that $x = \ln 3$ and $y = (2k+1)\pi$, where k is any integer. Thus, we find that

$$z = \ln 3 + (2k+1)\pi i$$

Part D

Notice that $e^x e^{iy} = e^z = 1 + i = \sqrt{2}e^{i\pi/4}$. Then, we know that $e^x = \sqrt{2}$ and $e^{iy} = e^{i\pi/4}$, so that $x = \ln 2^{1/2} = \frac{1}{2}\ln 2$ and $y = \pi/4 + 2k\pi = (2k + 1/4)\pi$. Thus, we find that

$$z = \frac{1}{2}\ln 2 + (2k + 1/4)\pi i$$

Problem 19

First, we let $w = e^z$. Then $e^w = 1$ implies that $\operatorname{Re} w = 0$ and $\operatorname{Im} w = 2\pi k$ for some integer k. Thus, we have $e^z = 2\pi k i$. Notice that k must not be zero. Then, we obtain $e^x \cos y + i e^x \sin y = 2\pi k i$. From this, we find that $e^x \cos y = 0$ and $e^x \sin y = 2\pi k$. Let us consider the first equation. Since e^x is always positive, we must have $\cos y = 0$, so that $y = \frac{\pi}{2} + r\pi$, where r is any integer. Now, we must consider whether k is positive or negative. If k is positive, then we want $\sin y$ to be positive (otherwise we won't be able to solve $e^x \sin y = 2\pi k$). Thus we may take $y = \frac{\pi}{2} + 2\pi r$. Then, we find that $e^x = 2\pi k$ so that $x = \ln(2\pi k)$. If k is negative, then we want $\sin y$ to be negative (otherwise we won't be able to solve $e^x \sin y = 2\pi k$). Thus we may take $y = -\frac{\pi}{2} + 2\pi r$ so that $x = \ln(-2\pi k)$. We may conclude that the general solution is

$$z = x + iy = \ln(2\pi|k|) + i\left(\pm \frac{\pi}{2} + 2\pi r \right)$$

where k is a nonzero integer and r is any integer.

Chapter 4

Problem 3

Notice that

$$\int_C f(z)\,\mathrm{d}z = \int_0^{2\pi} f(z(t))\dot{z}(t)\,\mathrm{d}t$$

Since $z(t) = \sin t + i\cos t$ and $f(z) = 1/z$, we find that

$$f(z(t)) = \frac{1}{z(t)} = \frac{1}{\sin t + i\cos t} = \frac{\sin t}{\sin^2 t + \cos^2 t} - i\frac{\cos t}{\sin^2 t + \cos^2 t} = \sin t - i\cos t$$

and

$$\dot{z}(t) = \cos t - i\sin t$$

Thus, we have

$$f(z(t))\dot{z}(t) = (\sin t - i\cos t)(\cos t - i\sin t) = -i(\sin^2 t + \cos^2 t) = -i$$

This informs us that

$$\int_C f(z)\,\mathrm{d}z = \int_0^{2\pi} f(z(t))\dot{z}(t)\,\mathrm{d}t = \int_0^{2\pi} -i\,\mathrm{d}t = -2\pi i$$

The result is different from that of example 2 because the curve C traverses the circle clockwise rather than counterclockwise.

Problem 6

We may write

$$\int_{|z|=1} f = Re^{i\beta}$$

for some $R \geq 0$ and $\beta \in [0, 2\pi)$. Let us parametrize the unit circle $|z| = 1$ as follows: $z(t) = e^{it}$, where $0 \leq t \leq 2\pi$. Then, we have $\dot{z}(t) = ie^{it}$ so that

$$Re^{i\beta} = \int_{|z|=1} f = \int_0^{2\pi} f(z(t))\dot{z}(t)\,\mathrm{d}t = \int_0^{2\pi} f(e^{it})ie^{it}\,\mathrm{d}t$$

Multiplying both sides of this equation by $e^{-i\beta}$ yields

$$R = \int_0^{2\pi} f(e^{it})ie^{i(t-\beta)}\,\mathrm{d}t$$

Substituting $e^{i(t-\beta)} = \cos(t-\beta) + i\sin(t-\beta)$ into the above equation, we obtain

$$R = \int_0^{2\pi} if(e^{it})\cos(t-\beta) - f(e^{it})\sin(t-\beta)\,\mathrm{d}t$$

The additivity of integration yields

$$R = i\int_0^{2\pi} f(e^{it})\cos(t-\beta)\,\mathrm{d}t - \int_0^{2\pi} f(e^{it})\sin(t-\beta)\,\mathrm{d}t$$

Since f is real-valued and R is real, we may deduce that

$$\int_0^{2\pi} f(e^{it})\cos(t-\beta)\,\mathrm{d}t = 0$$

Thus, we obtain

$$R = -\int_0^{2\pi} f(e^{it})\sin(t-\beta)\,\mathrm{d}t$$

Taking absolute values yields

$$R = |R| \leq \int_0^{2\pi} |f(e^{it})\sin(t-\beta)|\,\mathrm{d}t = \int_0^{2\pi} |f(e^{it})||\sin(t-\beta)|\,\mathrm{d}t \leq \int_0^{2\pi} |\sin(t-\beta)|\,\mathrm{d}t$$

since we assume that $|f| \leq 1$. Since sin has a period of 2π and we are integrating over an interval of length 2π, we may deduce that

$$\int_0^{2\pi} |\sin(t-\beta)|\,\mathrm{d}t = \int_0^{2\pi} |\sin t|\,\mathrm{d}t = \int_0^{\pi} \sin t\,\mathrm{d}t - \int_{\pi}^{2\pi} \sin t\,\mathrm{d}t = 4$$

since

$$\int_0^{\pi} \sin t\,\mathrm{d}t - \int_{\pi}^{2\pi} \sin t\,\mathrm{d}t = (-\cos\pi + \cos 0) - (-\cos 2\pi + \cos\pi) = 1 + 1 - (-1-1) = 4$$

Notice that

$$R = |Re^{i\beta}| = \left|\int_{|z|=1} f\right|$$

so we may conclude that

$$\left|\int_{|z|=1} f\right| \leq 4$$

Problem 8

Part A

If $k \neq -1$, then an antiderivative of z^k is

$$F(z) = \frac{1}{k+1} z^{k+1}$$

Notice that $F(z)$ is analytic on C. By Theorem 4.16, we find that

$$\int_C z^k = 0$$

Part B

Since $z = Re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we may note that

$$\int_C f(z) \, \mathrm{d}z = \int_0^{2\pi} f(z(\theta)) \dot{z}(\theta) \, \mathrm{d}\theta = \int_0^{2\pi} R^k e^{ik\theta} iRe^{i\theta} \, \mathrm{d}\theta = \int_0^{2\pi} iR^{k+1} e^{i(k+1)\theta} \, \mathrm{d}\theta$$

We have

$$e^{i(k+1)\theta} = \cos((k+1)\theta) + i \sin((k+1)\theta)$$

so substitution yields

$$\int_0^{2\pi} iR^{k+1} \cos((k+1)\theta) - R^{k+1} \sin((k+1)\theta) \, \mathrm{d}\theta$$

which equals

$$iR^{k+1} \int_0^{2\pi} \cos((k+1)\theta) \, \mathrm{d}\theta - R^{k+1} \int_0^{2\pi} \sin((k+1)\theta) \, \mathrm{d}\theta$$

Since $k \neq -1$, the above expression is equal to

$$\frac{iR^{k+1}}{k+1}(\sin(2(k+1)\pi) - \sin(0(k+1))) - \frac{R^{k+1}}{k+1}(-\cos(2(k+1)\pi) + \cos(0(k+1))) = 0$$

Problem 10

Part A

We know that e^z is entire. By Theorem 4.15, it is the derivative of some analytic function (in this case, it is the derivative of e^z). By Proposition 4.12, it is only necessary to evaluate the antiderivative at the endpoints in order to find the value of the integral. Thus, we have

$$\int_0^i e^z \, \mathrm{d}z = e^i - e^0 = \cos 1 + i \sin 1 - 1$$

Part B

We know that $\cos(2z)$ is entire. By Theorem 4.15, it is the derivative of some analytic function (in this case, it is the derivative of $\frac{1}{2}\sin(2z)$). By Proposition 4.12, it is only necessary to evaluate the antiderivative at the endpoints in order to find the value of the integral. Thus, we have

$$\int_{\pi/2}^{\pi/2+i} \cos(2z) \, \mathrm{d}z = \frac{1}{2}(\sin(\pi + 2i) - \sin(\pi)) = \frac{1}{2}\sin(\pi + 2i)$$

We may use the angle addition formula to obtain

$$\frac{1}{2}\sin(\pi + 2i) = \frac{1}{2}(\sin \pi \cos 2i + \cos \pi \sin 2i) = -\frac{1}{2}\sin(2i) = -\frac{i}{2}\sinh(2)$$

The last equality follows because $\sin(iy) = i \sinh y$ for any real y (this identity is stated in *Visual Complex Analysis*).

Problem 11

Let the curve C be defined by the function $z(t) = a + (b - a)t$, where $0 \leq t \leq 1$. Since D is convex, we know that $z(t) \in D$ for all t. Now, we may note that

$$|f(b) - f(a)| = |f(z(1)) - f(z(0))| = \left| \int_0^1 f'(z(t))\dot{z}(t)\,\mathrm{d}t \right| = \left| \int_C f'(z)\,\mathrm{d}z \right| \leq |b - a|$$

The last inequality follows by Proposition 4.10 since we know that $|f'| \leq 1$ on C and the length of the curve C is $|b - a|$.

Problem 12

First, we note that the left half-plane is convex. Therefore, we know that the curve C defined by the function $z(t) = a + (b - a)t$ for $0 \le t \le 1$ lies inside the left half-plane. Notice that the curve C is closed and bounded, so that C is compact. Furthermore, the set $R = \{z : \mathrm{Re}(z) \ge 0\}$ is closed. Thus, the distance between the curve C and the set R must be positive. This means that there must exist some $\delta < 0$ such that $\mathrm{Re}(z(t)) \le \delta$ for all t. Now, we may note that

$$|e^a - e^b| = \left| \int_0^1 \exp(z(t))\dot{z}(t)\,\mathrm{d}t \right| \le \int_0^1 |\exp(z(t))||\dot{z}(t)|\,\mathrm{d}t$$

Notice that

$$|\exp(z(t))| = \exp(\mathrm{Re}(z(t))) \le e^{\delta} < 1$$

so that

$$\int_0^1 |\exp(z(t))||\dot{z}(t)|\,\mathrm{d}t \le e^{\delta} \int_0^1 |\dot{z}(t)|\,\mathrm{d}t = e^{\delta}|a - b| < |a - b|$$

We may conclude that

$$|e^a - e^b| < |a - b|$$