Math 122A Homework 2

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Chapter 1 Problem 15

Part A

The set consisting of all points z satisfying $|z - i| \leq 1$ is the closed disk of radius 1 centered at i. This set is not a region because it is not open.

Part B

Note that

$$\left|\frac{z-1}{z+1}\right| = 1$$

if and only if

$$|z - 1| = |z + 1|$$

Geometrically, this set consists of points that have the same distance to 1 and -1. That is, it is the perpendicular bisector of the line segment connecting 1 to -1, which is to say that it is the imaginary axis. This set is not a region because it is closed.

Part C

Let $z = x + yi$. Notice that

$$|z - 2| > |z - 3|$$

if and only if

$$(x-2)^2 + y^2 > (x-3)^2 + y^2$$

if and only if

$$|x - 2| > |x - 3|$$

Comparing $|x - 2|$ and $|x - 3|$, it is evident that $|x - 2|$ exceeds $|x - 3|$ when $x > 5/2$. Therefore, this set consists of all points $z = x + iy$ where $x > 5/2$ and $y \in \mathbb{R}$. This set is a region because it is open and connected.

Part D

The set consisting of all points z such that $|z| < 1$ and $\operatorname{Im} z > 0$ is a semi-disk of radius 1 centered at 0. It is a region because it is open and connected.

Part E

Notice that

$$\frac{1}{z} = \overline{z}$$

if and only if

$$|z|^2 = z\overline{z} = 1$$

if and only if z is on the circle of radius 1 centered at 0. This set is not a region because it is not open.

Problem 25

Let T be a circle in the complex plane. We want to show that the corresponding set S on the Riemann sphere is a circle. If $(x,y) \in T$, then the corresponding point on S is

$$f(x,y) = \left(\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{x^2+y^2}{x^2+y^2+1} \right)$$

(where $f(x,y)$ is the inverse of the stereographic projection). If we can show that there exist constants A, B, C, D such that

$$A\frac{x}{x^2+y^2+1} + B\frac{y}{x^2+y^2+1} + C\frac{x^2+y^2}{x^2+y^2+1} = D$$

then we will know that $f(x,y)$ also lies on a plane, which means that the corresponding set in S is the intersection of the Riemann sphere with a plane, thus proving that S is a circle. Let us multiply the above equation by x^2+y^2+1 to obtain

$$Ax + By + C(x^2+y^2) = D(x^2+y^2+1)$$

so that

$$(C-D)(x^2+y^2) + Ax + By = D$$

Since T is a circle, we may write it as the set of points $z = x + yi$ such that $|z - \gamma| = r$, where $\gamma = a + bi$ and r is a real number. Squaring both sides, we obtain $|z-\gamma|^2 = r^2$ so that

$$(x-a)^2 + (y-b)^2 = r^2$$

Expanding this, we obtain

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

or

$$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$$

In order to put this equation into the form

$$(C-D)(x^2+y^2) + Ax + By = D$$

we let $C - D = 1$, $A = -2a$, $B = -2b$, and $D = r^2 - a^2 - b^2$. For this choice of A, B, C, D, we obtain

$$A\frac{x}{x^2+y^2+1} + B\frac{y}{x^2+y^2+1} + C\frac{x^2+y^2}{x^2+y^2+1} = D$$

By the above reasoning, S is a circle on the Riemann sphere. Next, we may suppose that T is a line. This means that for any point $(x,y) \in T$, we have $ax + by = c$, where a, b, c are real numbers. In order to put this in the form

$$(C-D)(x^2+y^2) + Ax + By = D$$

we let $C - D = 0$, $A = a$, $B = b$, and $d = c$. Then we have

$$A\frac{x}{x^2+y^2+1} + B\frac{y}{x^2+y^2+1} + C\frac{x^2+y^2}{x^2+y^2+1} = D$$

so the corresponding point in S is on a plane, which means that the set S corresponding to T under stereographic projection is a circle. Since $C = D$, we know that S goes through the point $(0,0,1)$ (this fact is mentioned in Section 1.5 of the textbook).

Problem 26

Let us write $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ where $a_n \neq 0$. We may rewrite this as

$$z^n \left(\sum_{k=0}^{n} a_k z^{k-n} \right)$$

Since $k - n < 0$ for all k such that $0 \leq k < n$, we find that

$$\lim_{z \to \infty} a_k z^{k-n} = 0$$

Thus, we may deduce that

$$\lim_{z \to \infty} \sum_{k=0}^{n-1} a_k z^{k-n} = 0$$

In particular, there must be some $M > 0$ such that $|z| > M$ implies that

$$\left| \sum_{k=0}^{n-1} a_k z^{k-n} \right| < \frac{|a_n|}{2}$$

Let us write

$$Q(z) = \sum_{k=0}^{n-1} a_k z^{k-n}$$

For $|z| > M$, we have

$$|P(z)| = \left| z^n \left(\sum_{k=0}^{n} a_k z^{k-n} \right) \right| = |z^n (a_n + Q(z))| = |z|^n |a_n + Q(z)| \geq |z|^n (|a_n| - |Q(z)|) \geq |z|^n \cdot \frac{|a_n|}{2}$$

where the first inequality holds by the reverse triangle inequality. Since $|z|^n \to \infty$ as $z \to \infty$, it is evident that $P(z) \to \infty$ as $z \to \infty$.

Problem 28

First, we will demonstrate what the function $z \mapsto 1/z$ does to points on the Riemann sphere. Let (a, b, c) be a point on the sphere. If f represents stereographic projection, then we have

$$f(a, b, c) = \left(\frac{a}{1-c}, \frac{b}{1-c} \right)$$

If $z = x + yi$, then

$$\frac{1}{z} = \frac{1}{x+yi} = \frac{1}{x+yi} \cdot \frac{x-yi}{x-yi} = \frac{x-yi}{x^2+y^2}$$

In our case, we have

$$\frac{1}{f(a,b,c)} = \frac{a/(1-c)}{(a/(1-c))^2 + (b/(1-c))^2} - \frac{b/(1-c)}{(a/(1-c))^2 + (b/(1-c))^2} i$$

Notice that

$$(a/(1-c))^2 + (b/(1-c))^2 = (a^2+b^2)/(1-c)^2 = (c-c^2)/(1-c)^2 = c(1-c)/(1-c)^2 = c/(1-c)$$

We know that $a^2 + b^2 = c - c^2$ because

$$a^2 + b^2 + (c - 1/2)^2 = 1/4$$

by assumption. By some algebra, we obtain

$$a^2 + b^2 = 1/4 - (c - 1/2)^2 = c - c^2$$

Thus, we find that

$$\frac{1}{f(a,b,c)} = \frac{a/(1-c)}{c/(1-c)} - \frac{b/(1-c)}{c/(1-c)} i = \frac{a}{c} - \frac{b}{c} i$$

From the textbook, we know that

$$f^{-1}(a/c, -b/c) = \left(\frac{a/c}{(a/c)^2 + (b/c)^2 + 1}, -\frac{b/c}{(a/c)^2 + (b/c)^2 + 1}, \frac{(a/c)^2 + (b/c)^2}{(a/c)^2 + (b/c)^2 + 1} \right)$$

Notice that

$$\frac{a/c}{(a/c)^2 + (b/c)^2 + 1} = \frac{a/c}{(a^2+b^2+c^2)/c^2} = \frac{ac}{a^2+b^2+c^2} = \frac{ac}{c-c^2+c^2} = a$$

and

$$-\frac{b/c}{(a/c)^2 + (b/c)^2 + 1} = -\frac{b/c}{(a^2+b^2+c^2)/c^2} = -\frac{bc}{a^2+b^2+c^2} = -\frac{bc}{c-c^2+c^2} = -b$$

and

$$\frac{(a/c)^2 + (b/c)^2}{(a/c)^2 + (b/c)^2 + 1} = \frac{(a^2+b^2)/c^2}{(a^2+b^2+c^2)/c^2} = \frac{a^2+b^2}{a^2+b^2+c^2} = \frac{c-c^2}{c-c^2+c^2} = \frac{c-c^2}{c} = 1 - c$$

Thus, we have

$$f^{-1}(a,b) = (a,-b,1-c)$$

This shows that for any point (a,b,c) on the Riemann sphere, the function $1/z$ takes it to the point $(a,-b,1-c)$. Since the sphere is centered at $(0,0,1/2)$, it is evident that this transformation is equivalent to reflecting across the xz plane and across the plane $z=1/2$. This is equivalent to a 180 degree rotation about the diameter that goes through the points $(1/2,0,1/2)$ and $(-1/2,0,1/2)$, which is just the x axis translated up by $1/2$ units. Now, we may note that $z \mapsto 1/z$ must take circles and lines in $\mathbb{C}$ to circles and lines in $\mathbb{C}$ because stereographic projection, its inverse, and the function $z \mapsto 1/z$ on the sphere all take circles and lines to circles and lines.

Chapter 2 Problem 1

First, we will show that if M is an analytic monomial, then $M_y = iM_x$. Suppose $M(z) = az^n = a(x+yi)^n$, where a is a complex constant. By the chain rule, we have

$$M_y = na(x+yi)^{n-1}i$$

We also have

$$M_x = na(x+yi)^{n-1}$$

so that

$$iM_x = na(x+yi)^{n-1}i$$

Thus we see that $M_y = iM_x$. Next, let us consider an analytic polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

We may write this as a sum of monomials

$$P(z) = \sum_{k=0}^{n} M^{(k)}$$

where $M^{(k)} = a_k z^k$. Taking the partial derivative of $P(z)$ with respect to y, we obtain

$$P_y = \sum_{k=0}^{n} M_y^{(k)}$$

Taking the partial derivative of $P(z)$ with respect to x and multiplying by i, we obtain

$$iP_x = \sum_{k=0}^{n} iM_x^{(k)}$$

For each k, we have $M_y^{(k)} = iM_x^{(k)}$, so that

$$P_y = \sum_{k=0}^{n} M_y^{(k)} = \sum_{k=0}^{n} iM_x^{(k)} = iP_x$$

Problem 3

Part A

For this polynomial, we have $u(x, y) = x^3 - 3xy^2 - x$ and $v(x, y) = 3x^2y - y^3 - y$. Notice that $u_x = 3x^2 - 3y^2 - 1$ and $v_y = 3x^2 - 3y^2 - 1$ so that $u_x = v_y$. We also have $u_y = -6xy$ and $v_x = 6xy$ so that $u_y = -v_x$. Thus this polynomial is analytic.

Part B

For this polynomial, we have $u(x, y) = x^2$ and $v(x, y) = y^2$. Here we have $u_x = 2x$ and $v_y = 2y$, so $u_x \neq v_y$, and this polynomial is not analytic.

Part C

For this polynomial, we have $u(x, y) = 2xy$ and $v(x, y) = y^2 - x^2$. Notice that $u_x = 2y$ and $v_y = 2y$, so $u_x = v_y$. Furthermore, we have $u_y = 2x = -v_x$, so this polynomial is analytic.

Problem 4

Suppose that $P(z) = a_0 + \cdots + a_n z^n$ is a nonconstant analytic polynomial that takes imaginary values only. Then its partial derivative with respect to y can take on imaginary values only. To prove this, consider the following expression:

$$\frac{P(x, y+h) - P(x, y)}{h}$$

It is evident that the numerator is purely imaginary. Taking the limit as $h \to 0$ along the real axis, we may deduce that P_y is purely imaginary. Similarly, we know that P_x is purely imaginary. Since P is nonconstant, we know that there must exist some point at which P_x and P_y are nonzero. At this point P_y is imaginary, and iP_x is real because P_x is imaginary.

Problem 9

Part A

To write the power series

$$\sum_{n=0}^{\infty} z^{n!}$$

in the form

$$\sum_{n=0}^{\infty} C_n z^n$$

we note that $C_n = 1$ if $n = k!$ for some integer k; otherwise $C_n = 0$. Thus we may deduce that

$$\overline{\lim}\, |C_n|^{1/n} = 1$$

so that the radius of convergence is 1.

Part B

Notice that

$$\sum_{n=0}^{\infty} (n + 2^n) z^n = \sum_{n=0}^{\infty} n z^n + \sum_{n=0}^{\infty} 2^n z^n$$

Thus, the power series

$$\sum_{n=0}^{\infty} (n + 2^n) z^n$$

must converge wherever

$$\sum_{n=0}^{\infty} n z^n$$

and

$$\sum_{n=0}^{\infty} 2^n z^n$$

converge. Notice that $n^{1/n} \to 1$ so the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n z^n$$

is 1. Furthermore, we note that $(2^n)^{1/n} = 2$ so that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n z^n$$

is 1/2. Thus, we may deduce that the power series

$$\sum_{n=0}^{\infty} (n + 2^n) z^n$$

converges when $|z| < \frac{1}{2}$. Furthermore, we know that the power series diverges when $|z| > \frac{1}{2}$ because then

$$|(n + 2^n)z^n| \geq |2z|^n - |n|$$

Notice that $|2z| > 1$ so that $|2z|^n$ increases geometrically. Since n only grows linearly, we know that $|2z|^n - |n| \to \infty$, so the power series must diverge.

Problem 10

First, we may denote the radius of convergence of the power series

$$\sum c_n z^n$$

by R and define

$$L = \lim |c_n|^{1/n}$$

Note that

$$R = \frac{1}{L}$$

Part A

We claim that the radius of convergence R' of the series

$$\sum n^p c_n z^n$$

is R. To prove this, note that

$$L' = \lim |n^p c_n|^{1/n} = \lim \sqrt[n]{n^p} \cdot \lim |c_n|^{1/n} = \lim \sqrt[n]{n^p} \cdot L$$

We claim that

$$\lim \sqrt[n]{n^p} = 1$$

To prove this, note that

$$\sqrt[n]{n^p} = \sqrt[n]{n \cdots n} = \sqrt[n]{n} \cdots \sqrt[n]{n}$$

Since $\sqrt[n]{n} \to 1$, we can apply the limit laws of multiplication to deduce that $\sqrt[n]{n^p} \to 1$. Thus $L' = L$ and

$$R' = \frac{1}{L'} = \frac{1}{L} = R$$

Part B

We claim that the radius of convergence R' of the series

$$\sum |c_n| z^n$$

is equal to R. To prove this, note that

$$L' = \lim ||c_n||^{1/n} = \lim |c_n|^{1/n} = L$$

so that

$$R' = \frac{1}{L'} = \frac{1}{L} = R$$

Part C

We claim that the radius of convergence R' of the series

$$\sum c_n^2 z^n$$

is equal to R^2. To prove this, we note that

$$L' = \lim |c_n^2|^{1/n} = \lim(|c_n|^{1/n})^2 = \left(\lim |c_n|^{1/n}\right)^2 = L^2$$

Therefore, we have

$$R' = \frac{1}{L'} = \frac{1}{L^2} = R^2$$