Math 122A Homework 3

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Chapter 2 Problem 13

Part A

Let $\beta > L$. Then, there must exist some N such that for all $n \geq N$,

$$\frac{a_{n+1}}{a_n} < \beta$$

This informs us that

$$a_{N+1} < \beta a_N$$

We also have

$$a_{N+2} < \beta a_{N+1} < \beta^2 a_N$$

By induction, we may deduce that for any positive integer p, we have

$$a_{N+p} < \beta^p a_N$$

Let $k = N + p$. Then, we have

$$a_k < \beta^k \cdot (\beta^{-N} a_N)$$

Taking the kth root of this, we obtain

$$\sqrt[k]{a_k} < \beta \cdot \sqrt[k]{\beta^{-N} a_N}$$

Notice that $\beta^{-N} a_N$ is constant, so

$$\lim_{k \to \infty} \sqrt[k]{\beta^{-N} a_N} = 1$$

Thus we may deduce that

$$\overline{\lim} \sqrt[k]{a_k} \leq \beta$$

Since this is true for every $\beta > L$, we have

$$\overline{\lim} \sqrt[k]{a_k} \leq L$$

Next, let $\alpha < L$. Then, there must exist some N such that for all $n \geq N$,

$$\frac{a_{n+1}}{a_n} > \alpha$$

Thus, we have

$$a_{N+1} > \alpha a_N$$

Notice that

$$a_{N+2} > \alpha a_{N+1} > \alpha^2 a_N$$

By induction, we know that for any integer $p > 0$, we have

$$a_{N+p} > \alpha^p a_N$$

Let $k = N + p$. Then,

$$a_k > \alpha^k \cdot (\alpha^{-N} a_N)$$

Taking the kth root yields

$$\sqrt[k]{a_k} > \alpha \sqrt[k]{\alpha^{-N} a_N}$$

Since $\alpha^{-N} a_N$ is a constant, we have

$$\lim_{k\to\infty} \sqrt[k]{\alpha^{-N} a_N} = 1$$

so that

$$\underline{\lim} \sqrt[k]{a_k} \geq \alpha$$

Since this is true for every $\alpha \leq L$, we may deduce that

$$\underline{\lim} \sqrt[k]{a_k} \geq L$$

Thus we have

$$L \leq \underline{\lim} \sqrt[k]{a_k} \leq \overline{\lim} \sqrt[k]{a_k} \leq L$$

so that

$$\lim_{k\to\infty} \sqrt[k]{a_k} = L$$

Part B

Notice that

$$\frac{(1/(n+1)!)}{(1/n!)} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

which evidently goes to zero. By Part A, we may conclude that

$$\left(\frac{1}{n!}\right)^{1/n} \to 0$$

Problem 14

Part A

Using Part A of 13, we find that

$$\frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

which goes to 0. Thus the radius of convergence is ∞.

Part B

Since the exponent of z is $2n+1$, we must be careful. Notice that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1/(2(n+1)+1)!}{1/(2n+1)!}\right| = \left|\frac{(2n+1)!}{(2n+3)!}\right| = \left|\frac{1}{(2n+2)(2n+3)}\right| = 0$$

so that

$$\overline{\lim}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

and

$$\underline{\lim}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

(the latter is true because when n is even, $a_n = 0$). Thus we may deduce that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

so the radius of convergence is ∞.

Part C

Notice that

$$\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$$

Now, we let

$$y = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

and take logarithms to obtain

$$\ln y = \lim_{n\to\infty} n \ln\frac{n}{n+1} = \lim_{n\to\infty}\frac{\ln(n/(n+1))}{1/n}$$

Since the numerator goes to $-\infty$ and the denominator goes to ∞ as $n\to\infty$, we may apply L'Hopital's rule and the chain rule to obtain

$$\frac{\frac{n+1}{n}\cdot\frac{1}{(n+1)^2}}{-\frac{1}{n^2}} = -\frac{n}{n+1}$$

which goes to -1. Thus $\ln y = -1$ and $y = e^{-1}$. Thus the radius of convergence is e.

Part D

We have

$$\frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1}$$

which goes to 0. The radius of convergence is ∞.

Chapter 3 Problem 2

Part A

The function is $f(z) = x^2 + iy^2$. Notice that $f_x = 2x$ and $f_y = 2yi$. Thus $if_x = 2xi = 2yi = f_y$ if and only if $x = y$. Thus f is differentiable at all points on the line $y = x$.

Part B

Let z be an arbitrary point on the line $y = x$. Any neighborhood of z contains points not on the line $y = x$. For these points, the Cauchy-Riemann equation $f_y = if_x$ is not satisfied since $x \neq y$. Thus f is not differentiable at these points, so there does not exist a neighborhood of z on which f is differentiable. Thus f is not analytic at any point z on the line $y = x$. It is also evident that f is not analytic if $y \neq x$ because then f is not even differentiable.

Problem 5

Notice that $f(z) = f(x, y) = u(x, y) + iv(x, y)$, where u and v are real-valued. Furthermore, we know that $f'(z) = f_x(z) = u_x + iv_x = 0$. Thus $u_x = 0$ and $v_x = 0$ at every point. By the Cauchy-Riemann equations, we have $u_y = 0$ and $v_y = 0$. Now, let (x_1, y_1) and (x_2, y_2) be two points in the region. We claim that $f(x_1, y_1) = f(x_2, y_2)$. Since we are considering a region, we know that there exists some polygonal path between these two points. If (a, b_1) and (a, b_2) are two points on this path with the same x coordinate, we may note that $u(a, b_1) - u(a, b_2) = u_y(c) \cdot (b_2 - b_1)$ for some real $c \in (b_1, b_2)$ by the Mean Value Theorem. Since $u_y = 0$, we have $u(a, b_1) = u(a, b_2)$. If (a_1, b) and (a_2, b) are two points on this path with the same y coordinate, we may note that $u(a_1, b) - u(a_2, b) = u_x(c)(a_2 - a_1)$ for some real $c \in (a_1, a_2)$ by the Mean Value Theorem. Since $u_x = 0$, we know that $u(a_1, b) = u(a_2, b)$. Thus, we know that u is constant. Similar reasoning informs us that v is constant. Because polygonal paths only contain horizontal and vertical line segments, we may deduce that $f(x_1, y_1) = u(x_1, y_1) + iv(x_1, y_1) = u(x_2, y_2) + iv(x_2, y_2) = f(x_2, y_2)$. Thus f is constant.

Problem 8

Since $u(x,y) = x^2 - y^2$, we have $u_x = 2x$ and $u_y = -2y$. Since $u_x = v_y$, we have $v_y = 2x$. Integration in y yields $v(x,y) = 2xy + c$, where $c \in \mathbb{R}$. Now we claim that f is analytic, where $f(x,y) = u(x,y) + iv(x,y) = (x^2 - y^2) + i(2xy + c)$. Notice that $u_x = 2x = v_y$ and $u_y = -2y = -v_x$. Thus f is analytic at all points. In order for f to be analytic, it is necessary and sufficient that $v(x,y) = 2xy + c$.

Problem 9

Suppose there was an analytic function $f = u + iv$ with $u(x, y) = x^2 + y^2$. Then, we have $u_x = 2x$ and $u_y = 2y$. By the Cauchy-Riemann equations, we have $v_y = u_x = 2x$ and $v_x = -u_y = -2y$. Integrating $v_y = 2x$ in y yields $v(x, y) = 2xy + c$ for some real constant c. Integrating $v_x = -2y$ in x yields $v(x, y) = -2xy + d$ for some real constant d. Then, we have $2xy + c = -2xy + d$ so that $4xy = c + d$. This is impossible since $4xy$ is not constant when x and y vary. Thus f is not analytic.

Problem 20

Notice that

$$\sin x \cosh y + i \cos x \sinh y = \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{y} + e^{-y}}{2} + i \cdot \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{y} - e^{-y}}{2}$$

is equal to

$$-i\left(\frac{e^{ix+y} - e^{-ix+y} + e^{ix-y} - e^{-ix-y}}{4}\right) + i\left(\frac{e^{ix+y} + e^{-ix+y} - e^{ix-y} - e^{-ix-y}}{4}\right)$$

which is equal to

$$i\left(\frac{e^{-ix+y} - e^{ix-y}}{2}\right)$$

This is equal to

$$\frac{e^{ix-y} - e^{-ix+y}}{2i}$$

which is equal to

$$\frac{1}{2i}(e^{i(x+yi)} - e^{-i(x+yi)}) = \sin(x + iy)$$