Math 122A Homework 6

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February 15, 2024

Chapter 5

Problem 1

To find the power series expansion of $f(z) = z^2$ around $z = 2$, we note that

$$f(z) = f(2) + f'(2)(z-2) + \frac{f''(2)}{2!}(z-2)^2 + \cdots$$

Noting that $f'(z) = 2z$ and $f''(z) = 2$, we obtain

$$f(z) = 2^2 + 2(2)(z-2) + \frac{2}{2}(z-2)^2 = 4 + 4(z-2) + (z-2)^2$$

Problem 2

To find the power series expansion for $f(z) = e^z$ about any point a, we may write

$$e^z = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$$

Since $f^{(k)}(z) = f(z)$, we find that

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k = \sum_{k=0}^{\infty} \frac{e^a}{k!}(z-a)^k = e^a \sum_{k=0}^{\infty} \frac{(z-a)^k}{k!}$$

Problem 3

Part A

In order to show that an odd entire function f has only odd terms in its power series expansion about 0, we must show that $f^{(k)}(0) = 0$ when k is even. First, notice that since f is odd, we have $f(0) = -f(-0) = -f(0)$ so that $2f(0) = 0$ and $f(0) = 0$. Thus, we may deduce that any odd function must take the value 0 at the point 0. Therefore, we must show that f^k is odd when k is even. We know that $f^{(0)} = f$ is odd. Next, we claim that $f^{(1)} = f'$ is even. Since f is odd, we have $f(z) = -f(-z)$. Taking derivatives, we obtain $f'(z) = f'(-z)$, so that f' is even. Next, we note that $f''(z) = -f''(-z)$, so that $f^{(2)} = f''$ is odd. Continuing inductively, we may deduce that f^k is odd when k is even. Therefore, we find that $f^k(0) = 0$ for even k so that f has only odd terms in its power series expansion about 0.

Part B

We claim that any even function f has only even terms in its power series expansion about 0. To prove this, we must show that $f^{(k)}(0) = 0$ when k is odd. Since f is even, we have $f(z) = f(-z)$ for all $z \in \mathbb{C}$. Taking derivatives yields $f'(z) = -f'(-z)$ so that f' is odd. As above, we know that $f^{(1)}(0) = f'(0) = 0$. Next, we find that $f''(z) = f''(-z)$ so that $f^{(2)}$ is even. Continuing inductively, we find that $f^{(k)}$ is odd for all odd k so that $f^{(k)}(0) = 0$ for all odd k. This shows that f has only even terms in its power series expansion about 0.

Problem 4

In the proof of Theorem 5.5, it was established that

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^{k+1}} \, \mathrm{d}\omega \right) z^k$$

By Corollary 2.11, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

Comparing coefficients, we find that

$$\frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^{k+1}} \, \mathrm{d}\omega = \frac{f^{(k)}(0)}{k!}$$

so that

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_C \frac{f(\omega)}{\omega^{k+1}} \, \mathrm{d}\omega$$

Problem 5

Define h as follows:

$$h(z) = f(z+a)$$

Notice that

$$h^{(k)}(z) = f^{(k)}(z+a)$$

for every positive integer k. Now, we note that

$$f^{(k)}(a) = h^{(k)}(0) = \frac{k!}{2\pi i}\int_C \frac{h(\omega)}{\omega^{k+1}}\,\mathrm{d}\omega = \frac{k!}{2\pi i}\int_C \frac{f(\omega+a)}{\omega^{k+1}}\,\mathrm{d}\omega$$

Letting $s = \omega + a$, we find that

$$\frac{k!}{2\pi i}\int_C \frac{f(\omega+a)}{\omega^{k+1}}\,\mathrm{d}\omega = \frac{k!}{2\pi i}\int_C \frac{f(s)}{(s-a)^{k+1}}\,\mathrm{d}s$$

so that

$$f^{(k)}(a) = \frac{k!}{2\pi i}\int_C \frac{f(s)}{(s-a)^{k+1}}\,\mathrm{d}s$$

Problem 6

Part A

By Problem 4, we have

$$C_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^{k+1}} \, \mathrm{d}\omega$$

Notice that the length of the path of integration is $2\pi R$ (since C is a circle with radius R) and that

$$\left| \frac{f(\omega)}{\omega^{k+1}} \right| \leq \frac{M}{R^{k+1}}$$

Thus, we realize that

$$|C_k| = \left| \frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^{k+1}} \, \mathrm{d}\omega \right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M}{R^{k+1}} = \frac{M}{R^k}$$

Part B

We suppose that an arbitrary polynomial $P(z)$ is bounded by 1 in the unit disk. Since $P(z)$ is continuous, it must also be bounded by 1 on the unit disk. Thus, we may appeal to Part A of Problem 6 with $M = 1$ and $R = 1$ to deduce that

$$|C_k| \leq \frac{M}{R^k} = \frac{1}{1^k} = 1$$

where C_k is the kth coefficient of the polynomial $P(z)$. That is, all the coefficients of the polynomial are bounded by 1.

Problem 7

Let $R > 0$ be an arbitrary positive number. On the circle $|z| = R$, we know that $f(z)$ is bounded by $A + B|z|^k = A + BR^k$. Appealing to Part A of Problem 6, we note that for any $j > k$, we have

$$|C_j| \leq \frac{A + BR^k}{R^j} = \frac{A}{R^j} + \frac{B}{R^{j-k}}$$

Since this is true for every $R > 0$, we may take the limit as R approaches ∞ in order to deduce that $C_j = 0$ for all $j > k$.

Problem 8

We claim that $C_k = 0$ for $k \geq 2$, where C_k is the kth coefficient of the power series of f centered at 0. Let $R > 0$. On the circle $|z| = R$, we know that $f(z)$ is bounded by $A + B|z|^{3/2} = A + BR^{3/2}$. By Part A of Problem 6, we have

$$|C_k| \leq \frac{A + BR^{3/2}}{R^k} = \frac{A}{R^k} + BR^{3/2-k}$$

As $R \to \infty$, it is evident that $\frac{A}{R^k} + BR^{3/2-k} \to 0$ as long as $k \geq 2$. Thus, we may deduce that f is a linear polynomial.

Problem 9

Since $|f'(z)| \leq |z|$, we may deduce from the Extended Liouville Theorem that f' is a polynomial of degree 1. Thus, f is a polynomial of degree 2. Now, we may write

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2$$

Since $|f'(0)| \leq 0$, we know that $f'(0) = 0$. Now, we note that

$$f(z) = f(0) + \frac{f''(0)}{2}z^2$$

Furthermore, we have

$$f''(0) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{z}\,\mathrm{d}z$$

where C is the unit circle. Notice that C has circumference 2π and $|f'(z)/z| \leq 1$ so that

$$|f''(0)| = \frac{1}{2\pi}\left|\int_C \frac{f'(z)}{z}\,\mathrm{d}z\right| \leq 1$$

Thus, letting $a = f(0)$ and $b = f''(0)/2$, we may conclude that

$$f(z) = a + bz^2$$

where $|b| \leq \frac{1}{2}$.