Math 122A Homework 9

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Chapter 8

Problem 1

Let $z \in \mathbb{C} \setminus S$, and let us consider the curve $\gamma(t) = tz + (1-t)\alpha$ for $t \geq 1$. Notice that $\gamma(1) = z$ and that

$$\lim_{t \to \infty} \gamma(t) = \infty$$

since $\gamma(t) = \alpha + t(z - \alpha)$ and $z \neq \alpha$. Finally, we claim that $\gamma(t) \in \mathbb{C} \setminus S$ for all $t \geq 1$. For the sake of contradiction, suppose that there exists some $t \geq 1$ such that $\omega = \gamma(t) \in S$. By the definition of S as a star-like region, we know that S contains the entire line segment between α and ω. In particular, we know that $\gamma(1) = z \in S$ (since γ is a line), contradicting our assumption that $z \in \mathbb{C} \setminus S$. Thus, we may deduce that $\gamma(t) \in \mathbb{C} \setminus S$ for all $t \geq 1$. By definition, we may conclude that S is simply connected.

Problem 2

We claim that if C is a convex region, then C is star-like. Let z_1 be an arbitrary point in C. By the definition of convexity, we know that for any other point $z \in C$, the region C contains the line-segment between z_1 and z. That is, we have found some point $z_1 \in C$ such that C contains the line segment between z_1 and z for all $z \in C$. By definition, C is a star-like region. By Problem 1, we know that C is simply connected.

Chapter 9

Problem 1

First, we show that the singularity at z_0 is not removable. For the sake of contradiction, suppose that the singularity at z_0 is removable. Then there must exist some function g analytic at z_0 such that $f(z) = g(z)$ for all z in some deleted neighborhood of z_0. Then, we find that

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = g(z_0)$$

which contradicts the fact that $f(z) \to \infty$ as $z \to z_0$. Next, we claim that the singularity at z_0 is not essential. For the sake of contradiction, suppose that the singularity at z_0 is essential. Since we are assuming that $f(z) \to \infty$ as $z \to z_0$, there must exist some $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies that $|f(z)| > 1$. However, this contradicts the Casorati-Weierstrass Theorem, which states that $f(D(z_0; \delta) \setminus \{z_0\})$ is dense in $\mathbb{C}$. Thus f has a pole at z_0.

Problem 2

Let us suppose that $|f(z)| \sim \exp(1/|z|)$ near 0. Then, we know that

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \exp(1/|z|) = \infty$$

By Problem 1, we find that f has a pole at 0. Let us suppose that this pole is of order k. Then, we know that

$$\lim_{z\to 0} z^{k+1} f(z) = 0$$

Thus, there must exist some $\delta > 0$ such that $0 < |z| < \delta$ implies that

$$|z^{k+1} f(z)| < 1$$

so that

$$|f(z)| < \frac{1}{|z|^{k+1}}$$

This implies that

$$|f(z)| \sim \frac{1}{|z|^{k+1}}$$

near 0, which contradicts our assumption that

$$|f(z)| \sim \exp(1/|z|)$$

near 0 (since $\exp(1/|z|)$ approaches ∞ much faster than $1/|z|^{k+1}$ for any positive integer k). Thus no such function f can exist.

Problem 3

First, we claim that f is a polynomial. Since f is entire, we know that it may be expressed as a power series:

$$f(z) = \sum_{k=0}^{\infty} C_k z^k$$

If all but finitely many of the C_k are zero, then f is a polynomial. Thus, we may suppose that infinitely many C_k are nonzero. Then, we have

$$g(z) := f(1/z) = \sum_{k=0}^{\infty} C_k (1/z)^k = \sum_{k=-1}^{-\infty} C_{-k} z^k$$

Since there are infinitely many non-zero terms in the principal part of the Laurent expansion of $g(z)$ about $z = 0$, we find that $z = 0$ is an essential singularity of $g(z)$. Let D be a deleted neighborhood of 0. By the Casorati-Weierstrass Theorem, we know that $g(D)$ is dense in $\mathbb{C}$. Note that $\mathbb{C} \setminus \overline{D}$ is open (since $\overline{D}$ is closed). By the Open Mapping Theorem, $g(\mathbb{C} \setminus \overline{D})$ is open. Thus $g(D) \cap g(\mathbb{C} \setminus \overline{D}) \neq \varnothing$. That is, the image of the deleted neighborhood D is dense in $\mathbb{C}$ and must intersect every open set, which includes $g(\mathbb{C} \setminus \overline{D})$. This informs us that there must exist points $z_1 \in D$ and $z_2 \in \mathbb{C} \setminus \overline{D}$ such that $g(z_1) = g(z_2)$, so that g is not injective. However, we know that g must be injective since it is the composition of the injective functions f and $1/z$. This contradiction implies that f must be a polynomial. If $\deg f \geq 2$, then f would have at least 2 roots, contradicting the injectivity of f. Thus $f(z) = az + b$ for complex constants a and b.

Problem 6

The exponential function is never zero, so the missing value is 0. Suppose that $\beta \neq 0$. We must find some z in the annulus $A = \{z \mid 0 < |z| < 1\}$ such that $e^{1/z} = \beta$. Let $\beta = Re^{i\theta}$ for some $R > 0$ and $\theta \in [0, 2\pi)$. Letting $\omega = 1/z$, we find that

$$\operatorname{Re}\omega = \log R$$

and

$$\operatorname{Im}\omega = \theta + 2\pi k$$

where k is an integer. Notice that

$$z = \frac{(\operatorname{Re}\omega)^2}{(\operatorname{Re}\omega)^2 + (\operatorname{Im}\omega)^2} - \frac{(\operatorname{Im}\omega)^2}{(\operatorname{Re}\omega)^2 + (\operatorname{Im}\omega)^2} i$$

so that z is evidently in the annulus A.

Problem 7

We may represent f and g as follows:

$$f(z) = \frac{A(z)}{(z - z_0)^m}$$

and

$$g(z) = \frac{B(z)}{(z - z_0)^n}$$

where $A(z_0), B(z_0) \neq 0$ and $A(z), B(z)$ are both analytic at z_0. First, we consider the case of $f + g$. Without loss of generality, suppose that $m \geq n$. Then we have

$$\begin{aligned} f(z) + g(z) &= \frac{A(z)}{(z - z_0)^m} + \frac{B(z)}{(z - z_0)^n} = \frac{A(z)}{(z - z_0)^m} + \frac{B(z)(z - z_0)^{m-n}}{(z - z_0)^n (z - z_0)^{m-n}} \\ &= \frac{A(z)}{(z - z_0)^m} + \frac{B(z)(z - z_0)^{m-n}}{(z - z_0)^m} = \frac{A(z) + B(z)(z - z_0)^{m-n}}{(z - z_0)^m} \end{aligned}$$

Notice that the numerator is nonzero when $z = z_0$, so that $f + g$ has a pole of order m at z_0. Similar reasoning holds if $n \geq m$. Thus we deduce that $f + g$ has a pole of order $\max\{m, n\}$ at z_0. Next, we consider the case of $f \cdot g$. We have

$$f(z) \cdot g(z) = \frac{A(z)}{(z - z_0)^m} \cdot \frac{B(z)}{(z - z_0)^n} = \frac{A(z)B(z)}{(z - z_0)^{m+n}}$$

so that $f \cdot g$ has a pole of order $m + n$ at z_0. Finally, we may consider the case of f/g. Notice that

$$\frac{f(z)}{g(z)} = \frac{A(z)}{B(z)} \frac{(z - z_0)^n}{(z - z_0)^m}$$

If $m > n$, then z_0 is a pole of order $m - n$. If $n > m$, then z_0 is a zero of order $n - m$ of f/g. If $n = m$, then z_0 is a removable singularity.

Problem 9

Part A

Notice that
$$\frac{1}{z^4+z^2} = \frac{1}{z^2(z^2+1)} = \frac{1}{z^2(z+i)(z-i)}$$
Note that 0 is a pole of order 2. This is true because
$$\lim_{z\to 0} z^2 \cdot \frac{1}{z^4+z^2} = \lim_{z\to 0} \frac{1}{(z+i)(z-i)} = 1$$
and
$$\lim_{z\to 0} z^3 \cdot \frac{1}{z^4+z^2} = \lim_{z\to 0} \frac{z}{(z+i)(z-i)} = 0$$
We claim that $\pm i$ are poles of order 1. This is true because
$$\lim_{z\to i}(z-i)\frac{1}{z^2(z+i)(z-i)} = \lim_{z\to i}\frac{1}{z^2(z+i)} = -\frac{1}{2i}$$
while
$$\lim_{z\to i}(z-i)^2\frac{1}{z^2(z+i)(z-i)} = \lim_{z\to i}\frac{(z-i)}{z^2(z+i)} = 0$$
Also, we have
$$\lim_{z\to -i}(z+i)\frac{1}{z^2(z+i)(z-i)} = \lim_{z\to -i}\frac{1}{z^2(z-i)} = \frac{1}{2i}$$
while
$$\lim_{z\to -i}(z+i)^2\frac{1}{z^2(z+i)(z-i)} = \lim_{z\to -i}\frac{(z+i)}{z^2(z-i)} = 0$$

Part B

Recall that
$$\cot z = \frac{\cos z}{\sin z}$$
The singularities are the integral multiples of π. We claim that these singularities are poles of order 1. Notice that
$$\lim_{z\to k\pi}(z-k\pi)\frac{\cos z}{\sin z} = (-1)^k \lim_{z\to k\pi}(z-k\pi)\frac{\cos z}{\sin(z-k\pi)} = (-1)^k\cos(k\pi) = (-1)^{2k} = 1$$
Thus, we have
$$\lim_{z\to k\pi}(z-k\pi)^2\frac{\cos z}{\sin z} = (-1)^k \lim_{z\to k\pi}(z-k\pi)^2\frac{\cos z}{\sin(z-k\pi)} = (-1)^k \lim_{z\to k\pi}(z-k\pi)\cos z = 0$$
This proves that the singularities are poles of order 1.

Part C

Note that

$$\csc z = \frac{1}{\sin z}$$

The singularities are again the integral multiples of π. As in Part B, we claim that they are poles of order 1. Notice that

$$\lim_{z\to k\pi}(z-k\pi)\frac{1}{\sin z} = (-1)^k \lim_{z\to k\pi}(z-k\pi)\frac{1}{\sin(z-k\pi)} = (-1)^k$$

Furthermore, we have

$$\lim_{z\to k\pi}(z-k\pi)^2\frac{1}{\sin z} = (-1)^k \lim_{z\to k\pi}(z-k\pi)^2\frac{1}{\sin(z-k\pi)} = (-1)^k \lim_{z\to k\pi}(z-k\pi) = 0$$

This shows that all the singularities are poles of order 1.

Part D

First, we claim that

$$\frac{\exp(1/z^2)}{z-1}$$

has a pole of order 1 at $z=1$. Notice that

$$\lim_{z\to 1}(z-1)\frac{\exp(1/z^2)}{z-1} = \lim_{z\to 1}\exp(1/z^2) = e$$

and that

$$\lim_{z\to 1}(z-1)^2\frac{\exp(1/z^2)}{z-1} = \lim_{z\to 1}(z-1)\exp(1/z^2) = 0$$

so that the singularity at $z=1$ is a pole of order 1. Next, we claim that the singularity at $z=0$ is essential. Notice that

$$\frac{\exp(1/z^2)}{z-1} = \sum_{k=0}^{\infty}\left(\frac{1}{z^2}\right)^k \cdot \frac{1}{k!(z-1)}$$

so that the function has infinitely many terms in the principal part of its Laurent expansion centered at 0. Thus the singularity at $z=0$ is essential.