Math 122A Homework 8

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Bak and Newman Chapter 6

Problem 10

First, we will find the minimum modulus of $z^2 - z$ in the disc $|z| \leq 1$. Notice that $0^2 - 0 = 0$, so the minimum modulus of $z^2 - z$ in the disc $|z| \leq 1$ is evidently 0. Next, we claim that the maximum modulus of $z^2 - z$ in the disc $|z| \leq 1$ is 2. Since $z^2 - z$ is analytic on the disc $|z| \leq 1$, we may apply the Maximum Modulus Theorem to deduce that the maximum of $z^2 - z$ must occur on the circle $|z| = 1$. Notice that $z^2 - z = z(z-1)$. In order to maximize this product under the constraint that $|z| = 1$, we only need to maximize $|z - 1|$ under this constraint. This maximum is achieved when $z = -1$. Thus, we may deduce that the maximum modulus of $z^2 - z$ in the disc $|z| \leq 1$ is $|z^2 - z| = |z||z-1| = |-1||-1-1| = 2$.

Problem 13

For the sake of contradiction, let us suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Let $M > 0$ be an arbitrary positive constant, and let $D_M = \{z \in \mathbb{C} : |z| \leq M\}$. By the Minimum Modulus Theorem, we know that $p(z)$ will attain its minimum on the boundary of D_M. Since $p(z)$ is a nonconstant polynomial, we know that

$$\lim_{z \to \infty} p(z) = \infty$$

This informs us that there must exist some positive constant L such that $|z| \geq L$ implies that $|p(z)| > |p(0)|$. Notice that for every $z \in \partial D_L$, we have $|p(z)| > |p(0)|$ (since $|z| = L$). This means that $p(z)$ does not attain its minimum on the boundary of D_L (because the modulus of $p(0)$ is strictly less than the modulus of $p(z)$ for every $z \in \partial D_L$), contrary to the Minimum Modulus Theorem. This contradiction informs us that there must exist some $z \in \mathbb{C}$ such that $p(z) = 0$.

Needham Chapter 7

Problem 1

For any loop L and point p such that $p \notin L$, we know that $\nu(L, p) = 0$ if and only if p is outside L and that $\nu(L, p) = \pm 1$ if and only if p is inside L. By page 341 of Needham, we know that $N(p) = |\nu(L, p)| + 2s$, where s is a non-negative integer and $N(p)$ is the number of intersection points of the ray from p with the simple loop L. From this, we obtain $|\nu(L, p)| = N(p) - 2s$. Using this equation and the fact that $|\nu(L, p)| \leq 1$, we may deduce that $N(p)$ is even if and only if $\nu(L, p) = 0$ and that $N(p)$ is odd if and only if $\nu(L, p) = \pm 1$. This informs us that if the number of intersection points is even, then p is outside L; if the number of intersection points is odd, then p is inside L.

Problem 20

If $f = g$ on Γ, then we have $f - g = 0$ on Γ. This informs us that $|f - g| = 0$ on Γ. Since $f - g$ is analytic on and inside Γ, we may apply the Maximum Modulus Theorem to deduce that $|f - g| = 0$ inside Γ so that $f = g$ throughout Γ.