# Math 122A Homework 7

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## Bak Newmann Chapter 5

### Problem 17

First, let us consider the case $n = 2$. Then, we note that

$$a_1 z_1 + a_2 z_2 = a_1 z_1 + (1 - a_1) z_2$$

which is in the convex set $S$ by the definition of convex set. Next, we may suppose that $n \geq 2$ and that

$$a_1 z_1 + \cdots + a_n z_n$$

is in $S$ whenever $a_i \geq 0$ for all $i$ and $\sum a_i = 1$. We must show that

$$a_1 z_1 + \cdots + a_{n+1} z_{n+1}$$

is in $S$ whenever $a_i \geq 0$ for all $i$ and $\sum a_i = 1$. Notice that

$$a_1 z_1 + \cdots + a_{n+1} z_{n+1} = (1 - a_{n+1}) \left( \frac{a_1}{1 - a_{n+1}} z_1 + \cdots + \frac{a_n}{1 - a_{n+1}} z_n \right) + a_{n+1} z_{n+1}$$

By our induction hypothesis, we know that

$$\frac{a_1}{1 - a_{n+1}} z_1 + \cdots + \frac{a_n}{1 - a_{n+1}} z_n$$

is in $S$. By the definition of convex set, we may deduce that

$$(1 - a_{n+1}) \left( \frac{a_1}{1 - a_{n+1}} z_1 + \cdots + \frac{a_n}{1 - a_{n+1}} z_n \right) + a_{n+1} z_{n+1}$$

is in $S$, thus completing the proof.

## Problem 19

Notice that an antiderivative of

$$P(z) = 1 + 2z + \cdots nz^{n-1}$$

is

$$Q(z) = 1 + z + \cdots + z^n$$

Using the formula for geometric sums, we obtain

$$Q(z) = 1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}$$

Notice that the zeroes of the latter expression are

$$e^{\frac{2\pi i a}{n+1}}$$

where $1 \leq a \leq n$. That is, all the zeroes of $Q(z)$ lie on the unit circle. Thus the convex hull of these solutions must lie inside the unit circle. By the Gauss-Lucas Theorem, we may deduce that all the zeroes of $P(z)$ lie inside the unit circle.

# Bak Newmann Chapter 6

## Problem 1

Let $\beta = 1 + i$. Then, we have

$$\frac{1}{z} = \frac{1}{\beta + (z - \beta)}$$

Factoring out $\beta$ yields

$$\frac{1}{\beta + (z-\beta)} = \frac{1}{\beta(1 + \frac{z-\beta}{\beta})} = \frac{1}{\beta} \cdot \frac{1}{1 - (-\frac{z-\beta}{\beta})}$$

Notice that

$$\frac{1}{\beta} \cdot \frac{1}{1 - (-\frac{z-\beta}{\beta})} = \frac{1}{\beta}\left(1 - \frac{z-\beta}{\beta} + \frac{(z-\beta)^2}{\beta^2} + \cdots\right)$$

This is equal to

$$\frac{1}{\beta}\sum_{n=0}^{\infty} \frac{(-1)^n (z-\beta)^n}{\beta^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (z-\beta)^n}{\beta^{n+1}}$$

Substituting $\beta = 1 + i$, we obtain

$$\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n (z-1-i)^n}{(1+i)^{n+1}}$$

## Problem 2

Notice that $1 - z - 2z^2 = (1-2z)(1+z)$. Now, we may write

$$\frac{1}{1-z-2z^2} = \frac{A}{1-2z} + \frac{B}{1+z}$$

where $A$ and $B$ are constants to be determined. Multiplying both sides by $1 - z - 2z^2$ yields

$$1 = A(1+z) + B(1-2z)$$

so that

$$1 = A + Az + B - 2Bz = A + B + (A - 2B)z$$

Thus, we know that $A+B=1$ and $A-2B=0$. Since $A=2B$, we obtain $2B+B=3B=1$ so that $B=1/3$ and $A=1-1/3=2/3$. We have

$$\frac{1}{1-z-2z^2} = \frac{2/3}{1-2z} + \frac{1/3}{1+z} = \frac{2}{3}\sum_{n=0}^{\infty} 2^n z^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{3}\left(\sum_{n=0}^{\infty} 2^{n+1} z^n + \sum_{n=0}^{\infty}(-1)^n z^n\right)$$

$$= \frac{1}{3}\left(\sum_{n=0}^{\infty}(2^{n+1} + (-1)^n) z^n\right) = \sum_{n=0}^{\infty} \frac{2^{n+1} + (-1)^n}{3} z^n$$

## Problem 3

Differentiating the identity

$$1 + z + z^2 + \cdots = \frac{1}{1-z}$$

yields

$$1 + 2z + \cdots = \frac{1}{(1-z)^2}$$

Multiplying both sides by $z$, we obtain

$$z + 2z^2 + \cdots = \frac{z}{(1-z)^2}$$

That is, we have established the following identity:

$$\sum nz^n = \frac{z}{(1-z)^2}$$

Differentiating this identity yields

$$\sum n^2 z^{n-1} = \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} = \frac{1-z}{(1-z)^3} + \frac{2z}{(1-z)^3} = \frac{1+z}{(1-z)^3}$$

Multiplying by $z$, we obtain

$$\sum n^2 z^n = \frac{z(1+z)}{(1-z)^3}$$

## Problem 4

For the sake of contradiction, suppose that for every positive integer $n$, we have

$$f(n) = \frac{1}{n+1} = \frac{1/n}{1/n+1}$$

Notice that the functions $f$ and $g(z) = z/(z+1)$ agree at all points in the set $S = \{1/n : n \in \mathbb{N}\}$. This set has an accumulation point at 0. By Corollary 6.10 in Bak and Newman, we may deduce that $f = g$ throughout the region $|z| < 1$. By continuity, we know that $f = g$ on $|z| = 1$ except when $z = -1$. No matter how $f(-1)$ is defined, we know that $f$ is not continuous at $-1$ since

$$\lim_{z \to -1} f(z) = \lim_{z \to -1} \frac{z}{z+1} = \infty$$

Thus $f$ is not analytic at $-1$, contrary to our assumptions about $f$.

# Needham Chapter 9

## Problem 1

Let us rewrite the integral
$$\int_0^{2\pi} \frac{\mathrm{d}t}{1+a^2-2a\cos t}$$
as a contour integral around the unit circle $C$ by introducing the substitutions $\cos t = \frac{1}{2}(z+1/z)$ and $\mathrm{d}z = iz\,\mathrm{d}t$, as the author does on page 437 of Visual Complex Analysis. Then, we find that

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1+a^2-2a\cos t} = \oint_C \frac{-i(\mathrm{d}z/z)}{1+a^2-a(z+1/z)}$$

Manipulating the integrand, we find that

$$\begin{aligned}\frac{-i(\mathrm{d}z/z)}{1+a^2-a(z+1/z)} &= \frac{i\,\mathrm{d}z}{-z-za^2+az^2+a} = \frac{i\,\mathrm{d}z}{az^2-za^2-z+a} = \frac{i\,\mathrm{d}z}{az(z-a)-(z-a)} \\ &= \frac{i\,\mathrm{d}z}{(z-a)(az-1)}\end{aligned}$$

Thus, we may conclude that

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1+a^2-2a\cos t} = \oint_C \frac{i\,\mathrm{d}z}{(z-a)(az-1)}$$

Next, we claim that
$$\oint_C \frac{i\,\mathrm{d}z}{(z-a)(az-1)} = \frac{2\pi}{1-a^2}$$
if $0 < a < 1$. Let $f(z) = i/(az-1)$. Notice that $f$ is analytic on the unit disk since its only singularity $z = \frac{1}{a}$ has modulus strictly greater than 1. Appealing to the Cauchy Integral Formula, we find that

$$\oint_C \frac{i\,\mathrm{d}z}{(z-a)(az-1)} = \oint_C \frac{f(z)}{z-a}\,\mathrm{d}z = 2\pi i \cdot f(a) = 2\pi i \cdot \frac{i}{a^2-1} = \frac{-2\pi}{a^2-1} = \frac{2\pi}{1-a^2}$$

We may deduce that
$$\int_0^{2\pi} \frac{\mathrm{d}t}{1+a^2-2a\cos t} = \frac{2\pi}{1-a^2}$$
for $0 < a < 1$.

## Problem 3

In Visual Complex Analysis, it is shown that

$$f^{(r)}(0) = \frac{r!}{2\pi i} \oint_C \frac{f(z)}{z^{r+1}} \, \mathrm{d}z$$

for any entire function $f$, any nonnegative integer $r$, and any closed loop $C$. Let $f(z) = (1+z)^n$. Then, we have

$$f^{(r)}(0) = \frac{r!}{2\pi i} \oint_C \frac{(1+z)^n}{z^{r+1}} \, \mathrm{d}z$$

By the definition of $f$, we know that

$$f'(z) = n(1+z)^{n-1}, \; f''(z) = n(n-1)(1+z)^{n-2}, \; f^{(3)}(z) = n(n-1)(n-2)(1+z)^{n-3}, \ldots$$

By induction, we may deduce that

$$f^{(r)}(z) = n(n-1)\cdots(n-r+1)(1+z)^{n-r}$$

Setting $z = 0$, we obtain

$$f^{(r)}(0) = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

We have

$$\frac{n!}{(n-r)!} = f^{(r)}(0) = \frac{r!}{2\pi i} \oint_C \frac{(1+z)^n}{z^{r+1}} \, \mathrm{d}z$$

Dividing both sides by $r!$ yields

$$\frac{n!}{r!(n-r)!} = \frac{1}{2\pi i} \oint_C \frac{(1+z)^n}{z^{r+1}} \, \mathrm{d}z$$

or

$$\binom{n}{r} = \frac{1}{2\pi i} \oint_C \frac{(1+z)^n}{z^{r+1}} \, \mathrm{d}z$$

Now, we may suppose that $C$ is the unit circle, and we will use the above formula to prove that

$$\binom{2n}{n} \leq 4^n$$

We have

$$\binom{2n}{n} = \frac{1}{2\pi i} \oint_C \frac{(1+z)^{2n}}{z^{n+1}} \, \mathrm{d}z \leq \frac{1}{2\pi} \oint_C \left| \frac{(1+z)^{2n}}{z^{n+1}} \right| \mathrm{d}z$$

Notice that

$$|(1+z)^{2n}| = |1+z|^{2n} \leq 2^{2n} = 4^n$$

and

$$|z^{n+1}| = |z|^{n+1} = 1$$

for all $z$ on the unit circle $C$. Furthermore, the circumference of $C$ is $2\pi$. Thus, using the $M$-$L$ formula from Bak and Newman, we find that

$$\frac{1}{2\pi}\oint_C \left|\frac{(1+z)^{2n}}{z^{n+1}}\right| \mathrm{d}z \leq \frac{1}{2\pi}\cdot 2\pi \cdot 4^n = 4^n$$

so that

$$\binom{2n}{n} \leq 4^n$$