Math 122B Homework 1

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Problem 1

Part A

First, we find that

$$\frac{1}{z^4+z^2} = \frac{1}{z^2(z^2+1)} = \frac{1}{z^2(z+i)(z-i)}$$

We find that $z = 0$ is a pole of order 2 and that $z = \pm i$ are simple poles. Let us compute the residue of $1/(z^4+z^2)$ at 0. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z^2 \cdot \frac{1}{z^4+z^2}\right] = \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{z^2+1}\right] = -\frac{2z}{(1+z^2)^2}$$

so that

$$\mathrm{Res}\left(\frac{1}{z^4+z^2}; 0\right) = \frac{1}{(2-1)!}\frac{\mathrm{d}}{\mathrm{d}z}\left[z^2 \cdot \frac{1}{z^4+z^2}\right]\Bigg|_{z=0} = -\frac{2\cdot 0}{(1+0^2)^2} = 0$$

Next, we will compute the residue of $1/(z^4+z^2)$ at i. Note that

$$\frac{1}{z^4+z^2} = \frac{\frac{1}{z^2(z+i)}}{z-i}$$

Letting $A(z) = 1/(z^2(z+i))$ and $B(z) = z - i$, we find that

$$\mathrm{Res}\left(\frac{1}{z^4+z^2}; i\right) = \frac{A(i)}{B'(i)} = \frac{1}{i^2(i+i)} = -\frac{1}{2i}$$

Finally, we will compute the residue of $1/(z^4+z^2)$ at $-i$. Note that

$$\frac{1}{z^4+z^2} = \frac{\frac{1}{z^2(z-i)}}{z+i}$$

Letting $A(z) = 1/(z^2(z-i))$ and $B(z) = z + i$, we find that

$$\mathrm{Res}\left(\frac{1}{z^4+z^2}; -i\right) = \frac{A(-i)}{B'(-i)} = \frac{1}{(-i)^2(-i-i)} = \frac{1}{2i}$$

Part B

Notice that

$$\cot z = \frac{\cos z}{\sin z}$$

Last quarter, we showed that $\cot z$ has a simple pole at every integral multiple of π. Letting $A(z) = \cos z$ and $B(z) = \sin z$, we find that

$$\text{Res}\bigg(\cot z; n\pi\bigg) = \frac{A(n\pi)}{B'(n\pi)} = \frac{\cos(n\pi)}{\cos(n\pi)} = 1$$

Part C

Notice that

$$\csc z = \frac{1}{\sin z}$$

Last quarter, we showed that $\csc z$ has a simple pole at every integral multiple of π. Letting $A(z) = 1$ and $B(z) = \sin z$, we find that

$$\text{Res}\bigg(\csc z; n\pi\bigg) = \frac{A(n\pi)}{B'(n\pi)} = \frac{1}{\cos(n\pi)} = \frac{1}{(-1)^n} = (-1)^n$$

Part D

Notice that the function has a simple pole at $z = 1$. Letting $A(z) = \exp(1/z^2)$ and $B(z) = z - 1$, we have

$$\text{Res}\left(\frac{\exp(1/z^2)}{z-1}; 1\right) = \frac{A(1)}{B'(1)} = \exp(1) = e$$

Furthermore, this function has an essential singularity at $z = 0$. Notice that

$$\frac{\exp(1/z^2)}{z-1} = \exp(1/z^2) \cdot -\frac{1}{1-z} = \left(1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{6z^6} + \cdots\right)(-1 - z - z^2 - z^3 - \cdots)$$

Multiplying this out, we find that the coefficient C_{-1} is

$$-1 - \frac{1}{2} - \frac{1}{6} - \cdots = -e + 1$$

so that

$$\text{Res}\left(\frac{\exp(1/z^2)}{z-1}; 0\right) = -e + 1$$

Part E

Notice that $z^2 + 3z + 2 = (z+1)(z+2)$ so that

$$\frac{1}{z^2 + 3z + 2}$$

has simple poles at -1 and -2. We may write

$$\frac{1}{z^2+3z+2} = \frac{\frac{1}{z+1}}{z+2}$$

Letting $A(z) = 1/(z+1)$ and $B(z) = z+2$, we have

$$\text{Res}\left(\frac{1}{z^2+3z+2}; -2\right) = \frac{A(-2)}{B'(-2)} = -1$$

We may also write

$$\frac{1}{z^2+3z+2} = \frac{\frac{1}{z+2}}{z+1}$$

Letting $A(z) = 1/(z+2)$ and $B(z) = z+1$, we have

$$\text{Res}\left(\frac{1}{z^2+3z+2}; -1\right) = \frac{A(-1)}{B'(-1)} = 1$$

Part F

The function $\sin(1/z)$ has an essential singularity at $z = 0$. Notice that

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \cdots$$

so that

$$\text{Res}\left(\sin\frac{1}{z}; 0\right) = 1$$

Part G

Notice that $ze^{3/z}$ has an essential singularity at $z = 0$. We have

$$ze^{3/z} = z\left(1 + \frac{3}{z} + \frac{3^2}{2z^2} + \cdots\right) = z + 3 + \frac{9}{2z} + \cdots$$

so that

$$\text{Res}\left(ze^{3/z}; 0\right) = \frac{9}{2}$$

Part H

Notice that

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

If $b^2 - 4ac \neq 0$, then the function has simple poles at

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Let

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

In particular, we may write

$$\frac{1}{az^2 + bz + c} = \frac{1}{a} \cdot \frac{1}{z - \alpha} \cdot \frac{1}{z - \beta}$$

Letting

$$A(z) = \frac{1}{a} \cdot \frac{1}{z - \alpha}$$

and

$$B(z) = z - \beta$$

we may write

$$\frac{1}{az^2 + bz + c} = \frac{A(z)}{B(z)}$$

Then, we have

$$\operatorname{Res}\left(\frac{1}{az^2 + bz + c}; \beta\right) = \frac{A(\beta)}{B'(\beta)} = \frac{1}{a} \cdot \frac{1}{\beta - \alpha} = -\frac{1}{\sqrt{b^2 - 4ac}}$$

Next, letting

$$A(z) = \frac{1}{a} \cdot \frac{1}{z - \beta}$$

and

$$B(z) = \frac{1}{z - \alpha}$$

we may write

$$\frac{1}{az^2 + bz + c} = \frac{A(z)}{B(z)}$$

Then, we have

$$\operatorname{Res}\left(\frac{1}{az^2 + bz + c}; \alpha\right) = \frac{A(\alpha)}{B'(\alpha)} = \frac{1}{a} \cdot \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{b^2 - 4ac}}$$

Next, we must consider the case in which $b^2 - 4ac = 0$. Then, there is a double pole at $z = b/2a$ and

$$\frac{1}{az^2 + bz + c} = \frac{1}{a(z - b/2a)^2}$$

The residue is

$$\frac{1}{(2-1)!} \frac{\mathrm{d}}{\mathrm{d}z} (z - b/2a)^2 \cdot \frac{1}{a(z - b/2a)^2}$$

evaluated at $z = b/2a$. However, we note that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z - b/2a)^2 \cdot \frac{1}{a(z - b/2a)^2} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{a} = 0$$

so that

$$\mathrm{Res}\left(\frac{1}{az^2 + bz + c}; \frac{b}{2a}\right) = 0$$

Problem 2

Part A

The singularities of $\cot z$ are at the integral multiples of π. The only singularity of $\cot z$ in the circle $|z| = 1$ is at 0. The residue of $\cot z$ at 0 is 1. Appealing to the Residue Theorem, we find that

$$\int_{|z|=1} \cot z \, \mathrm{d}z = 2\pi i$$

Part B

The only singularities of $1/(z-4)(z^3-1)$ in the circle $|z| = 2$ are $w_k := e^{2k\pi i/3}$, where $k = 0, 1, 2$. Notice that

$$\frac{1}{(z-4)(z^3-1)} = \frac{1}{(z-4)(z-w_0)(z-w_1)(z-w_2)}$$

so that

$$\operatorname{Res}\left(\frac{1}{(z-4)(z^3-1)}; w_0\right) = \frac{1}{(w_0-4)(w_0-w_1)(w_0-w_2)}$$

and

$$\operatorname{Res}\left(\frac{1}{(z-4)(z^3-1)}; w_1\right) = \frac{1}{(w_1-4)(w_1-w_0)(w_1-w_2)}$$

and

$$\operatorname{Res}\left(\frac{1}{(z-4)(z^3-1)}; w_2\right) = \frac{1}{(w_2-4)(w_2-w_0)(w_2-w_1)}$$

Using the Residue Theorem, we find that

$$\int_{|z|=2} \frac{\mathrm{d}z}{(z-4)(z^3-1)}$$

is equal to

$$2\pi i\left(\frac{1}{(w_0-4)(w_0-w_1)(w_0-w_2)} + \frac{1}{(w_1-4)(w_1-w_0)(w_1-w_2)} + \frac{1}{(w_2-4)(w_2-w_0)(w_2-w_1)}\right)$$

Part C

Notice that the only singularity of $\sin\frac{1}{z}$ is at 0; the residue there is 1. Appealing to the Residue Theorem, we find that

$$\int_{|z|=1} \sin\frac{1}{z} \, \mathrm{d}z = 2\pi i$$

Part D

The only singularity of $ze^{3/z}$ is at 0; the residue there is $9/2$. Appealing to the Residue Theorem, we find that

$$\int_{|z|=2} ze^{3/z} \, \mathrm{d}z = \frac{9}{2} \cdot 2\pi i = 9\pi i$$

Problem 3

Let C be the unit circle. Notice that the only singularity of $1/(1-e^{-z})^n$ in C is 0 (the singularities of $1/(1-e^{-z})^n$ are at the integral multiples of $2\pi i$). By the Residue Theorem, we find that

$$2\pi i \operatorname{Res}\left(\frac{1}{(1-e^{-z})^n};0\right) = \int_C \frac{\mathrm{d}z}{(1-e^{-z})^n}$$

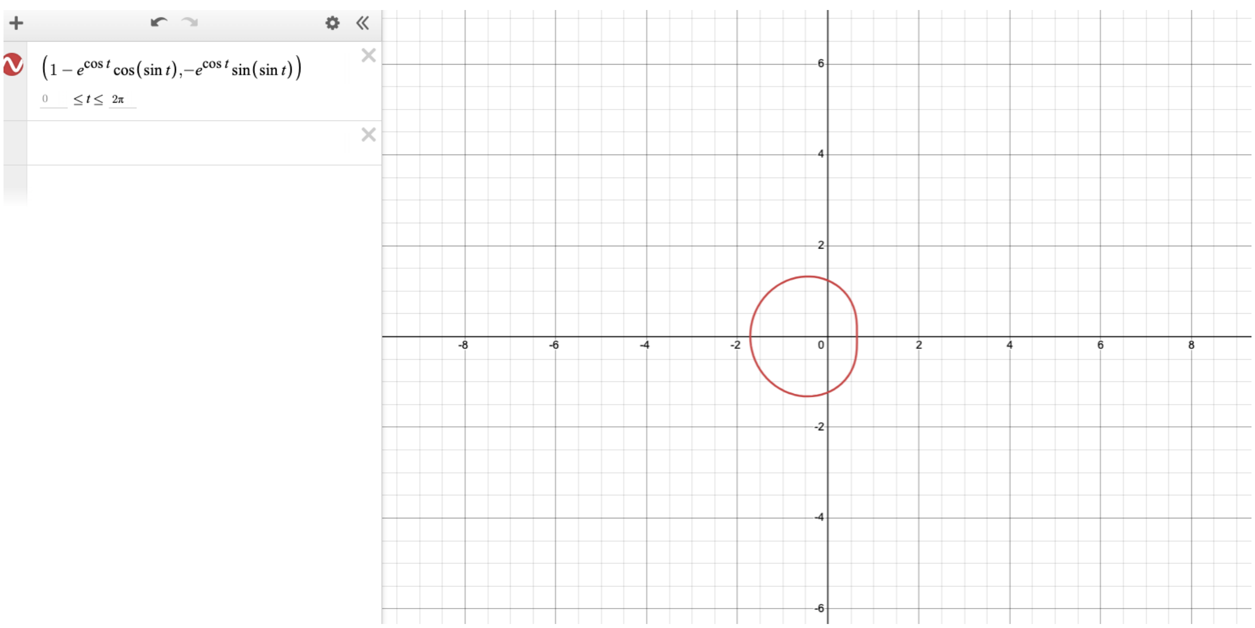
Let $w = 1-e^{-z}$. Then we have $\mathrm{d}w = e^{-z}\,\mathrm{d}z$ so that

$$\mathrm{d}z = \frac{\mathrm{d}w}{e^{-z}} = \frac{\mathrm{d}w}{1-w}$$

Let L denote the image of C under $w(z) = 1-e^{-z}$. Then, we have

$$\int_C \frac{\mathrm{d}z}{(1-e^{-z})^n} = \int_L \frac{\mathrm{d}w}{w^n(1-w)}$$

The only singularities of $1/(w^n(1-w))$ are 0 and 1. In the below image, we can see that L encloses 0 but not 1:



Thus, appealing to the Residue Theorem again, we have

$$\int_L \frac{\mathrm{d}w}{w^n(1-w)} = 2\pi i \operatorname{Res}\left(\frac{1}{w^n(1-w)};0\right)$$

Notice that

$$\frac{1}{w^n(1-w)} = \frac{1}{w^n}\cdot\frac{1}{1-w} = \frac{1}{w^n}(1+w+w^2+\cdots)$$

so that

$$\operatorname{Res}\left(\frac{1}{w^n(1-w)};0\right) = 1$$

Thus, we deduce that

$$2\pi i \mathrm{Res}\left(\frac{1}{(1-e^{-z})^n};0\right) = 2\pi i \mathrm{Res}\left(\frac{1}{w^n(1-w)};0\right) = 2\pi i$$

so that

$$\mathrm{Res}\left(\frac{1}{(1-e^{-z})^n};0\right) = 1$$

Problem 4

Notice that

$$\left(z + \frac{1}{z}\right) = \frac{z^2+1}{z}$$

Then, we have

$$\left(\frac{z^2+1}{z}\right)^{2m+1} = \frac{(z^2+1)^{2m+1}}{z^{2m+1}}$$

By the Binomial Theorem, we have

$$(z^2+1)^{2m+1} = \sum_{j=0}^{2m+1} \binom{2m+1}{j} z^{2j}$$

Then, we have

$$\frac{(z^2+1)^{2m+1}}{z^{2m+1}} = \sum_{j=0}^{2m+1} \binom{2m+1}{j} z^{2j-2m-1}$$

Thus, we have

$$\int_{|z|=1} (z+1/z)^{2m+1}\,\mathrm{d}z = \sum_{j=0}^{2m+1} \binom{2m+1}{j} \int_{|z|=1} z^{2j-2m-1}\,\mathrm{d}z$$

We know that

$$\int_{|z|=1} z^n\,\mathrm{d}z$$

is $2\pi i$ if $n=-1$ and 0 otherwise. Thus, we find that

$$\int_{|z|=1} z^{2j-2m-1}\,\mathrm{d}z$$

is 0 if $j \neq m$ and $2\pi i$ if $j=m$ so that

$$\int_{|z|=1} (z+1/z)^{2m+1}\,\mathrm{d}z = \sum_{j=0}^{2m+1} \binom{2m+1}{j} \int_{|z|=1} z^{2j-2m-1}\,\mathrm{d}z = 2\pi i \binom{2m+1}{m}$$