Math 122B Homework 5

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Problem 4

From the textbook, we know that $F(z) = z + 1/z$ maps the exterior of the unit circle onto $\mathbb{C} \setminus [-2, 2]$. Now, we claim that this mapping is unique up to an additive constant. Since $F(z) \sim z$ as $z \to \infty$, we find that

$$F(z) = z + C_0 + \frac{C_1}{z} + \frac{C_2}{z^2} + \cdots$$

where each $C_i = c_i + id_i$ for real c_i, d_i. Since we are assuming that F takes the exterior of the unit disk to the outside of a horizontal interval, we know that F takes the boundary of the unit disk to the horizontal interval. That is, we have $\operatorname{Im} f(e^{i\theta}) = c$ for all θ. Now, we compute

$$F(e^{i\theta}) = e^{i\theta} + C_0 + C_1 e^{-i\theta} + C_2 e^{-2i\theta} + \cdots$$

Notice that

$$C_k e^{-ki\theta} = (c_k + id_k)(\cos(k\theta) - i\sin(k\theta)) = c_k \cos(k\theta) + d_k \sin(k\theta) - ic_k \sin(k\theta) + id_k \cos(k\theta)$$

so that

$$\operatorname{Im} F(e^{i\theta}) = \sin\theta + d_0 - \sum_{k=1}^{\infty} c_k \sin(k\theta) + \sum_{k=1}^{\infty} d_k \cos(k\theta) = c$$

Since this must hold for all θ and c is a constant, we find that $c_1 = 1$ (so that $\sin\theta$ is cancelled from the expression), that $c_k = 0$ for all $k > 1$, and that $d_k = 0$ for all $k \geq 1$. We also obtain $d_0 = c$ so that

$$F(z) = z + C_0 + 1/z$$

as was to be shown.

Problem 5

Part A

First, we compute the image of the unit circle $|z| = 1$ under f. We find that

$$f(e^{i\theta}) = 2e^{i\theta} + \frac{1}{e^{i\theta}} = 2e^{i\theta} + e^{-i\theta} = 2(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 3\cos\theta + i\sin\theta$$

Note that

$$\frac{(3\cos\theta)^2}{9} + \sin^2\theta = \cos^2\theta + \sin^2\theta = 1$$

Thus we know that f maps the unit circle onto the ellipse

$$\frac{x^2}{9} + y^2 = 1$$

Next, we note that $f(2) = 2(2) + 1/2 = 9/2$ so that f maps the exterior of the unit circle to the exterior of the ellipse

$$\frac{x^2}{9} + y^2 = 1$$

Part B

We compute the inverse of $f(z)$ in Part A:

$$w = 2z + \frac{1}{z} \Rightarrow wz = 2z^2 + 1 \Rightarrow 2z^2 - wz + 1 = 0 \Rightarrow z = \frac{w \pm \sqrt{w^2 - 8}}{4}$$

To find out which sign we should use, we let $w = 3$. Then, we have

$$z = \frac{3 \pm \sqrt{1}}{4} = 1/2, 1$$

Thus, we choose the positive sign so that

$$f^{-1}(z) = \frac{z + \sqrt{z^2 - 8}}{4}$$

Next, we let $g(z) = z + 1/z$. Then $g \circ f^{-1}$ first takes the exterior of the ellipse to the exterior of the unit circle and then takes the exterior of the unit circle to the exterior of a real line segment.

Problem 6

Let

$$g(z) = \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)}$$

so that

$$g'(z) = \frac{f'(z)(1 - \overline{f(z_0)} f(z)) - (f(z) - f(z_0))(-f'(z))}{(1 - \overline{f(z_0)} f(z))^2}$$

which is equal to

$$\frac{f'(z) - f'(z)\overline{f(z_0)} f(z) + f(z) f'(z) - f(z_0) f'(z)}{(1 - \overline{f(z_0)} f(z))^2}$$

From this, we deduce that

$$g'(z_0) = \frac{f'(z_0) - f'(z_0)\overline{f(z_0)} f(z_0) + f(z_0) f'(z_0) - f(z_0) f'(z_0)}{(1 - \overline{f(z_0)} f(z_0))^2}$$

which is equal to

$$\frac{f'(z_0)(1 - |f(z_0)|^2)}{(1 - |f(z_0)|^2)^2} = \frac{f'(z_0)}{1 - |f(z_0)|^2}$$

Notice that

$$\arg g'(z_0) = \arg f'(z_0)$$

Set $\theta = -\arg f'(z_0)$. Let $h(z) = e^{i\theta} g(z)$. Then $h(z_0) = e^{i\theta} g(z_0) = 0$ and $h'(z_0) = e^{i\theta} g'(z_0) > 0$.

Problem 7

We define $h : R \to U$ as follows: $h(z) = \overline{f(\overline{z})}$. Notice that h is bijective since f is bijective (note that complex conjugation is a bijection on R and U since they are symmetric across the real axis). Next, we claim that h is analytic. Notice that

$$\frac{h(z+\delta)-h(z)}{\delta} = \frac{\overline{f(\overline{z}+\overline{\delta})} - \overline{f(\overline{\delta})}}{\delta} = \overline{\left(\frac{f(\overline{z}+\overline{\delta}) - f(\overline{\delta})}{\overline{\delta}}\right)}$$

Now, we have

$$h'(z) = \lim_{\delta\to 0}\frac{h(z+\delta)-h(z)}{\delta} = \lim_{\delta\to 0}\overline{\left(\frac{f(\overline{z}+\overline{\delta}) - f(\overline{\delta})}{\overline{\delta}}\right)} = \overline{\left(\lim_{\delta\to 0}\frac{f(\overline{z}+\overline{\delta}) - f(\overline{\delta})}{\overline{\delta}}\right)} = \overline{f'(\overline{z})}.$$

so that h is analytic. Thus, we have shown that h is a $1-1$ analytic mapping of R onto U. Now, we note that

$$h(z_0) = \overline{f(\overline{z_0})} = \overline{f(z_0)} = \overline{0} = 0$$

since $z_0 \in \mathbb{R}$. Furthermore, we know that

$$h'(z_0) = \overline{f'(\overline{z_0})} = \overline{f'(z_0)} = f'(z_0) > 0$$

since $z_0 \in \mathbb{R}$ and $f'(z_0) \in \mathbb{R}$. By the uniqueness aspect of the Riemann Mapping Theorem, we find that $f = h$ so that $f(z) = h(z) = \overline{f(\overline{z})}$, from which we obtain

$$\overline{f(z)} = f(\overline{z})$$

for all $z \in R$.

Problem 9

First, we consider the case in which $R = \mathbb{C}$. Then, we let $f(z) = z - z_1 + z_2$. We note that $f(z_1) = z_1 - z_1 + z_2 = z_2$. Furthermore, we know that f is bijective because f is linear. We also know that f is analytic because it is linear. Thus f is a conformal mapping of $\mathbb{C}$ onto $\mathbb{C}$ with the property that $f(z_1) = z_2$. Next, we consider the case in which $R \neq \mathbb{C}$. By the Riemann Mapping Theorem, we know that there exist $f_1 : R \to U$ and $f_2 : R \to U$ such that $f_1(z_1) = 0$ and $f_2(z_2) = 0$. Then $f = f_2^{-1} \circ f_1$ is a conformal mapping of R onto itself such that $f(z_1) = f_2^{-1} \circ f_1(z_1) = f_2^{-1}(0) = z_2$, as desired.

Problem 10

Let us suppose that $f : \mathbb{C} \to R$ was a conformal mapping. By the Riemann Mapping Theorem, there must be some conformal mapping $g : R \to U$ (this is true because $R \neq \mathbb{C}$). Let $h = g \circ f$ be the conformal mapping from $\mathbb{C}$ onto U. By Liouville's Theorem, we know that h must be constant (since h is entire and bounded). This is a contradiction. Thus, we find that there is no conformal mapping from $\mathbb{C}$ onto R.