Math 122B Homework 2

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Problem 7

By the Argument Principle, we have

$$\mathbb{Z}(f) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \frac{1}{2\pi}\Delta\arg(f(z))$$

Thus, we only have to prove that $\Delta\arg(f(z))$ can be at most 2π for any circle γ (then we can conclude that f has at most one zero in any circle so that f has at most one zero). Let γ be a circle of radius K. According to our hypothesis, $f(\gamma)$ is only real when γ is real. Notice that γ is real at K and $-K$. This means that $f(\gamma)$ cannot cross the real axis unless $\gamma(t)$ is K or $-K$. Thus f maps the upper semicircle of γ into one of the two half-planes (either upper or lower), and f maps the lower semicircle of γ into one of the two half-planes. Since the argument of $f(\gamma)$ can only increase by at most π in either of these half-planes, we deduce that $\arg(f(z)) \leq 2\pi$ so that f has at most one zero in any circle γ; thus f has at most one zero in the complex plane.

Problem 8

Part A

First, we claim that $f \neq 0$ on γ. If $f = 0$ on γ, then $|g| \leq |f|$ on γ implies that $g = 0$ on γ so that $f + g = 0$ on γ, which contradicts our assumptions. Then, as in the proof of Rouche's Theorem, we can write

$$\mathbb{Z}(f+g) = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} + \frac{1}{2\pi i}\int_\gamma \frac{(1+g/f)'}{1+g/f} = \mathbb{Z}(f) + \frac{1}{2\pi i}\int_\gamma \frac{(1+g/f)'}{1+g/f}$$

Let $\omega = 1 + g/f$. Notice that

$$|\omega - 1| = \frac{|g|}{|f|} \leq 1$$

on γ. Thus $\omega(\gamma)$ remains in the closed unit disk of radius 1 centered at 1. We claim that $\omega(\gamma)$ cannot cross 0. Suppose that $1 + g/f = 0$. Then $f + g = 0$, directly contradicting the hypothesis. Thus $\omega(\gamma)$ is a continuous curve that remains inside the unit disk centered at 1 but does not cross 0, so it does not contain 0. By Cauchy's Closed Curve Theorem, we find that

$$\frac{1}{2\pi i}\int_\gamma \frac{(1+g/f)'}{1+g/f} = 0$$

so that

$$\mathbb{Z}(f+g) = \mathbb{Z}(f)$$

Part B

Let $f(z) = 2z^4$ and $g(z) = z^5 + 1$. We know that $|f(z)| = 2 = 1 + 1 \geq |z^5 + 1| = |g(z)|$ on the unit circle. This inequality is an equality only when $z = w_k = e^{2\pi i k/5}$, where $k = 0, 1, 2, 3, 4$. We note that

$$2w_0^4 + w_0^5 + 1 = 2 + 1 + 1 = 4$$

and

$$2w_1^4 + w_1^5 + 1 = 2w_4 + 2 = 2(w_4 + 1) \neq 0$$

and

$$2w_2^4 + w_2^5 + 1 = 2w_3 + 2 = 2(w_3 + 1) \neq 0$$

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and

$$2w_4^4 + w_4^5 + 1 = 2w_1 + 2 = 2(w_1 + 1) \neq 0$$

Thus we have shown that $|f| \geq |g|$ and that $f + g \neq 0$ on the unit circle so that $\mathbb{Z}(z^5 + 2z^4 + 1) = \mathbb{Z}(f+g) = \mathbb{Z}(f) = 4$ inside $|z| = 1$.

Problem 9

Part A

Let $f_1(z) = 3e^z - z = f(z) + g(z)$ where $f(z) = 3e^z$ and $g(z) = -z$. Notice that

$$|f(z)| = |3e^z| \geq |3/e| > 1 = |g(z)|$$

for $|z| = 1$. By Rouche's Theorem, we have $\mathbb{Z}(f_1) = \mathbb{Z}(f) = 0$ inside $|z| = 1$

Part B

Let $f_2(z) = \frac{1}{3}e^z - z = g(z) + f(z)$, where $f(z) = -z$ and $g(z) = \frac{1}{3}e^z$. Notice that

$$|f(z)| = 1 > \frac{e}{3} \geq |g(z)|$$

on $|z| = 1$ so that $\mathbb{Z}(f_2) = \mathbb{Z}(f) = 1$ inside $|z| = 1$.

Part C

First we find the number of zeros of f_3 in the circle $|z| = 2$. Let $f_3(z) = z^4 - 5z + 1 = f(z) + g(z)$ where $f(z) = z^4$ and $g(z) = -5z + 1$. Notice that $|f(z)| = |z^4| = 16 > 11 \geq |-5z+1| = |g(z)|$ for $|z| = 2$ so that $\mathbb{Z}(f_3) = \mathbb{Z}(f) = 4$ inside $|z| = 2$. Next, we find the number of zeros of f_3 in the circle $|z| = 1$. Let $f_3(z) = z^4 - 5z + 1 = f(z) + g(z)$ where $f(z) = -5z$ and $g(z) = z^4 + 1$. On the circle $|z| = 1$ we have $|f(z)| = |-5z| = 5 > 2 \geq |z^4 + 1| = |g(z)|$ so that $\mathbb{Z}(f_3) = \mathbb{Z}(f) = 1$ inside $|z| = 1$. Thus the number of zeros in the annulus is $4 - 1 = 3$.

Part D

Let $f_4(z) = z^6 - 5z^4 + 3z^2 - 1 = f(z) + g(z)$, where $f(z) = -5z^4$ and $g(z) = z^6 + 3z^2 - 1$. Note that $|f(z)| = 5 \geq |z^6 + 3z^2 - 1| = |g(z)|$ on $|z| = 1$. This is only an equality when $z^6 + 3z^2 = -4$ for $|z| = 1$; that is, when $z = \pm i$. In this case, we have

$$(\pm i)^6 - 5(\pm i)^4 + 3(\pm i)^2 - 1 = -1 - 5 - 3 - 1 = -10 \neq 0$$

so that Rouches Theorem applies (by Problem 8 Part A) and we have $\mathbb{Z}(f_4) = \mathbb{Z}(f) = 4$ in the circle $|z| = 1$.

Problem 11

By Corollary 10.6, we have

$$\int_\gamma z^m \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i \sum_k \mathrm{Res}\left(\frac{z^m f'(z)}{f(z)}; z_k\right)$$

where we are summing over the zeros of f (since these are the singularities of the function we are integrating). Now, we may write $f(z) = (z - z_k)^q g(z)$ where $g(z_k) \neq 0$ and q is the order of the zero z_k. In class, we've shown that

$$\frac{f'(z)}{f(z)} = \frac{q}{z - z_k} + \frac{g'(z)}{g(z)}$$

so that

$$z^m \frac{f'(z)}{f(z)} = \frac{z^m q}{z - z_k} + z^m \frac{g'(z)}{g(z)}$$

Note that

$$2\pi i \mathrm{Res}\left(\frac{z^m f'(z)}{f(z)}; z_k\right) = \int_C z^m \frac{f'(z)}{f(z)}\,\mathrm{d}z = \int_C \frac{z^m q}{z - z_k}\,\mathrm{d}z = 2\pi i q z_k^m$$

where C is a circle containing only the singularity z_k. Notice that the second equality above holds because $z^m \frac{g'(z)}{g(z)}$ is analytic on and inside C. Thus we get

$$\mathrm{Res}\left(\frac{z^m f'(z)}{f(z)}; z_k\right) = q z_k^m$$

so that

$$\int_\gamma z^m \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i \sum_k q z_k^m$$

Problem 12

Let $R > 0$ be fixed, let $f(z) = e^z$, and let $g(z) = P_n(z) - e^z$. Notice that $P_n(z)$ converges to e^z uniformly on the closed disk of radius R, which means that $g(z)$ converges to 0 uniformly on the closed disk of radius R. Thus, there must exist some N such that $n \geq N$ implies that $|g(z)| < e^{-R} \leq |f(z)|$. By Rouches Theorem, we have

$$\mathbb{Z}(P_n(z)) = \mathbb{Z}(f + g) = \mathbb{Z}(f) = \mathbb{Z}(e^z) = 0$$

That is, $P_n(z)$ has no zeroes in the closed disk of radius R if n is sufficiently large.

Problem 13

Part A

Notice that

$$(1-z)P(z) = a_0 + a_1 z + \cdots + a_n z^n - (a_0 z + a_1 z^2 + \cdots + a_n z^{n+1})$$

which is equal to

$$a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}$$

Suppose that $P(z)$ has some root ω such that $|\omega| > 1$ and let r be such that $1 < r < |\omega|$. Let $g(z) = (1-z)P(z) + a_n z^{n+1}$ and $f(z) = -a_n z^{n+1}$. Notice that

$$\begin{aligned}|g(z)| = |(1-z)P(z) + a_n z^{n+1}| &= |a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n| \\ &< |a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1})||z|^{n+1} = |-a_n z^{n+1}| = |f(z)|\end{aligned}$$

for $|z| = r$ so that $\mathbb{Z}(f+g) = \mathbb{Z}(f)$ inside the circle of radius r. That is, the number of zeros of $(1-z)P(z)$ in this circle is the same as the number of zeros of $-a_n z^{n+1}$. Notice that the latter has $n+1$ zeros at 0. Thus $(1-z)P(z)$ must have $n+1$ zeros in the circle of radius r. Since the degree of $(1-z)P(z)$ is $n+1$, we conclude that all of the zeros of $(1-z)P(z)$ must lie in the circle of radius r. This contradicts the fact that ω is a zero of $(1-z)P(z)$. Thus we may conclude that all the zeros of $(1-z)P(z)$ lie inside the unit disk so that all the zeros of $P(z)$ also lie inside the unit disk.

Part B

Notice that

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

so that

$$\frac{1}{(1-z)^2} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{1-z} = 1 + 2z + 3z^2 + \cdots$$

Thus, we find that the sequence $P_n(z)$ converges to $q(z) = 1/(1-z)^2$ on the open unit disk $|z| < 1$. We wish to apply Rouche's Theorem to $g(z) = P_n(z) - q(z)$ and $f(z) = q(z)$ so that we can deduce that $P_n(z)$ has as many zeros as $q(z)$ in a disk of radius ρ; namely, zero. To apply Rouche's Theorem, we must show that $|P_n(z) - q(z)| < |q(z)|$ on $C(0;\rho)$. First, we find a lower bound for $q(z)$ in the unit disk. Notice that $(1-(-1))^2 = 4$, so we may deduce that $q(z) > 1/4$ in the unit disk (note that this bound also holds on the circle of radius ρ). Next, we note that $P_n(z)$ converges to $q(z)$ uniformly on the circle $C(0;\rho)$. In particular, there must exist some N such that $n > N$ and $z \in C(0;\rho)$ imply that $|P_n(z) - q(z)| < 1/4$. Thus, for sufficiently large n, we have

$$|g(z)| = |P_n(z) - q(z)| < 1/4 < |q(z)| = |f(z)|$$

so that we may apply Rouche's theorem to deduce that $f + g = P_n$ has the same number of zeros in the disk of radius ρ as $f = q(z)$, which is 0.

Problem 14

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ (we may assume without loss of generality that the leading coefficient is 1), and let C be the circle of radius $R = |a_{n-1}| + \cdots + |a_0| + 1$. Then, we have

$$\left|\sum_{k=0}^{n-1} a_k z^k\right| < 1 + \sum_{k=0}^{n-1} |a_k||z|^k \leq \left(1 + \sum_{k=0}^{n-1} |a_k|\right)|z|^{n-1} = R|z|^{n-1} = |z|^n = |z^n|$$

when z is on C. Let $g(z) = \sum_{k=0}^{n-1} a_k z^k$ and $f(z) = z^n$. Then Rouche's Theorem implies that

$$\mathbb{Z}(f+g) = \mathbb{Z}(f)$$

inside the circle C. Since f has n roots in C (0 with multiplicity n), we know that $f+g = P(z)$ must also have n roots in C.