Math 122B Homework 4

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Problem 6

First, we let the mapping f be defined as follows:

$$f(z) = \frac{-(z-1)^2}{4(z+1)^2}$$

From the textbook, we know that this mapping takes the upper semi-disk to the upper half plane. Next, we let the mapping g be defined by

$$g(z) = \frac{i-z}{i+z}$$

This mapping takes the upper half plane to the unit disk. Thus the desired mapping is $g \circ f$.

Problem 12

Let us consider the mapping $g = f_1 \circ f_2^{-1}$. This function g is an automorphism of the unit disk. Notice that

$$g(0) = f_1(f_2^{-1}(0)) = f_1(z_0) = 0$$

so that $g(z) = e^{i\theta} z$. Next, we have

$$e^{i\theta} = g'(0) = f_1'(f_2^{-1}(0))((f_2^{-1})'(0)) = f_1'(z_0) \cdot \frac{1}{f_2'(f_2^{-1}(0))} = \frac{f_1'(z_0)}{f_2'(z_0)} > 0$$

so that $e^{i\theta} = 1$ and g is the identity. From this, we deduce that $f_1 = f_2$.

Problem 13

First, we suppose that f is a conformal mapping from some half-plane H_1 to another half-plane H_2. Let g be a linear mapping that takes H_1 to the upper half-plane, and let h be a linear mapping that takes H_2 to the upper half-plane. Then $h \circ f \circ g^{-1}$ is an automorphism of the upper half-plane, so we may write

$$h \circ f \circ g^{-1} = \frac{az+b}{cz+d}$$

where a, b, c, d are real and $ad - bc > 0$. Then, we obtain

$$f = h^{-1} \circ \frac{az+b}{cz+d} \circ g$$

so that f is bilinear (since the composition of bilinear transformations is bilinear).

Next, we may suppose that f is a conformal mapping of the half-plane H to the disk D. Let g be a linear mapping from H to the upper half-plane, and let h be a linear mapping from the unit disk to D. Then $h^{-1} \circ f \circ g^{-1}$ is a conformal mapping of the upper half-plane onto U so that

$$h^{-1} \circ f \circ g^{-1} = e^{i\theta}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right)$$

for some α with $\operatorname{Im} \alpha > 0$. Thus we obtain

$$f = h \circ e^{i\theta}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right) \circ g$$

so that f is bilinear.

Next, we may suppose that f is a conformal mapping of the disk D to the half-plane H. Let g be a linear mapping of the half-plane H to the upper half-plane. Let h be a mapping of the unit disk to D. Then $h^{-1} \circ f^{-1} \circ g^{-1}$ maps the upper half-plane to the unit disk, so it is of the form

$$h^{-1} \circ f^{-1} \circ g^{-1} = e^{i\theta}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right)$$

so that

$$f^{-1} = h \circ e^{i\theta}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right) \circ g$$

Thus f^{-1} is bilinear so that f is bilinear.

Finally, we may suppose that f is a conformal mapping of the disk D_1 onto the disk D_2. Let g be a linear mapping from the unit disk to D_1, and let h be a linear mapping from D_2 to the unit disk. Then $h \circ f \circ g$ is an automorphism of the unit disk. Thus we have

$$h \circ f \circ g = e^{i\theta} z$$

so that

$$f = h^{-1} \circ e^{i\theta} z \circ g^{-1}$$

is a bilinear map.

Problem 14

First, we note that

$$f(z) = \frac{az+b}{cz+d} = \frac{az+b}{cz+d} \cdot \frac{c\overline{z}+d}{c\overline{z}+d} = \frac{ac|z|^2 + bc\overline{z} + adz + bd}{|cz+d|^2}$$

From this, we obtain

$$\operatorname{Im} f(z) = (ad-bc)\operatorname{Im} z$$

If $\operatorname{Im} z > 0$, then we know that

$$\operatorname{Im} f(z) = (ad-bc)\operatorname{Im} z < 0$$

since $ad - bc < 0$. Thus we find that f maps the upper half-plane into the lower half-plane. Next, we claim that f is onto. Let $z \in \mathbb{C}$ be such that $\operatorname{Im} z < 0$. Then, we have

$$f^{-1}(z) = \frac{dz-b}{-cz+a} = \frac{dz-b}{-cz+a} \cdot \frac{-c\overline{z}+a}{-c\overline{z}+a} = \frac{-cd|z|^2 - ab + bc\overline{z} + adz}{|-cz+a|^2}$$

We have

$$\operatorname{Im} f^{-1}(z) = (ad-bc)\operatorname{Im} z > 0$$

since $ad - bc < 0$ and $\operatorname{Im} z < 0$. Thus we find that f maps the upper half-plane onto the lower half-plane.

Problem 15

Let g be an automorphism of the first quadrant. By Lemma 13.13, we may write $g = f^{-1} \circ h \circ f$, where f is a conformal mapping of the first quadrant onto the upper half-plane and h is an automorphism of the upper half-plane. Let $f(z) = z^2$. By Theorem 13.17, we know that

$$h(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are real and $ad - bc > 0$. Thus, we find that

$$g(z) = \sqrt{\frac{az^2 + b}{cz^2 + d}}$$

Problem 18

First, we suppose that z_4 lies on the same circle or line as z_1, z_2, and z_3. Then, we claim that $[z_1, z_2, z_3, z_4]$ is real-valued. Let T be the bilinear mapping that sends z_1, z_2 and z_3 to ∞, 0, and 1. Notice that T sends the circle or line containing z_1, z_2, and z_3 to the real line (since bilinear maps take circles and lines to circles and lines). Thus z_4 is also mapped to a point on the real line so that $T(z_4) = [z_1, z_2, z_3, z_4]$ is real-valued.

Conversely, we suppose that $[z_1, z_2, z_3, z_4]$ is real-valued, and we claim that z_4 must lie on the same circle or line as z_1, z_2, and z_3. Let T be the bilinear mapping that sends z_1, z_2, and z_3 to ∞, 0, and 1. By assumption, we have $T(z_4) = [z_1, z_2, z_3, z_4] \in \mathbb{R}$. Thus, we know that $T(z_4)$ is on the real line along with $T(z_1)$, $T(z_2)$, and $T(z_3)$. Applying T^{-1}, which is bilinear, we find that z_4 is on the same circle or line as z_1, z_2, and z_3.