Math 122B Homework 3

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Chapter 11

Problem 1

Part A

Since $\deg(1+x^2)^2 - \deg x^2 \geq 2$, we have

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = 2\pi i \cdot \mathrm{Res}\left(\frac{z^2}{(1+z^2)^2}; i\right)$$

since i is the only root in the upper half plane. To find the residue, we first compute

$$(z-i)^2 \cdot \frac{z^2}{(1+z^2)^2} = (z-i)^2 \cdot \frac{z^2}{(z+i)^2(z-i)^2} = \frac{z^2}{(z+i)^2}$$

Then we differentiate:

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{(z+i)^2} = \frac{2iz}{(z+i)^3}$$

Evaluating at i, we obtain

$$\mathrm{Res}\left(\frac{z^2}{(1+z^2)^2}; i\right) = \frac{2i(i)}{(i+i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

so that

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = 2\pi i \cdot \mathrm{Res}\left(\frac{z^2}{(1+z^2)^2}; i\right) = \frac{\pi}{2}$$

Part B

Since $\deg(x^2+4)^2(x^2+9) - \deg x^2 \geq 2$, we know that this integral must converge. Furthermore, since the function is even, we have

$$\int_{0}^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,\mathrm{d}x$$

The only singularities of this function in the upper half plane are at $2i$ and $3i$. First, we compute

$$\operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};2i\right)$$

as follows: We multiply the function by $(z-2i)^2$. Then we obtain

$$(z-2i)^2\cdot\frac{z^2}{(z^2+4)^2(z^2+9)}=\frac{z^2}{(z+2i)^2(z^2+9)}$$

Then, we must differentiate:

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{(z+2i)^2(z^2+9)}=-\frac{2z(z^3-18i)}{(z+2i)^3(z^2+9)^2}$$

Substituting $z=2i$, we obtain

$$\operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};2i\right)=-\frac{2(2i)((2i)^3-18i)}{(2i+2i)^3((2i)^2+9)^2}=\frac{-104}{-1600i}=-\frac{13}{200}i$$

Next, we compute

$$\operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};3i\right)$$

as follows: Let us write

$$\frac{z^2}{(z^2+4)^2(z^2+9)}=\frac{z^2}{(z^2+4)^2(z+3i)(z-3i)}=\frac{\frac{z^2}{(z^2+4)^2(z+3i)}}{z-3i}$$

Substituting $z=3i$ into the numerator yields

$$\operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};3i\right)=\frac{(3i)^2}{((3i)^2+4)^2(3i+3i)}=\frac{-9}{(-9+4)^2(3i+3i)}=\frac{-9}{150i}=\frac{3}{50}i$$

Finally, we obtain

$$\begin{aligned}
&\int_0^\infty\frac{x^2}{(x^2+4)^2(x^2+9)}\,\mathrm{d}x=\frac{1}{2}\int_{-\infty}^\infty\frac{x^2}{(x^2+4)^2(x^2+9)}\,\mathrm{d}x\\
&=\pi i\left(\operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};2i\right)+\operatorname{Res}\left(\frac{z^2}{(z^2+4)^2(z^2+9)};3i\right)\right)\\
&=\pi i\left(-\frac{13}{200}i+\frac{3}{50}i\right)
\end{aligned}$$

Part C

First, we note that $1/(x^4+x^2+1)$ is even so that

$$\int_0^\infty\frac{\mathrm{d}x}{x^4+x^2+1}=\frac{1}{2}\int_{-\infty}^\infty\frac{\mathrm{d}x}{x^4+x^2+1}$$

We need to find the zeroes of z^4+z^2+1 in the upper half plane. Let us make the substitution $w = z^2$. Then we must solve $w^2 + w + 1 = 0$. The solutions of this are $w = e^{2\pi i/3}$ and $w = e^{4\pi i/3}$. From this, we get $z = e^{\pi i/3}$ and $z = e^{2\pi i/3}$. We now compute the residues of $1/(z^4 + z^2 + 1)$ at these two points. We have

$$\operatorname{Res}\left(\frac{1}{z^4+z^2+1}; e^{\pi i/3}\right) = \frac{1}{4(e^{\pi i/3})^3 + 2e^{\pi i/3}} = \frac{1}{-4+1+\sqrt{3}i} = \frac{1}{-3+\sqrt{3}i}$$

and

$$\operatorname{Res}\left(\frac{1}{z^4+z^2+1}; e^{2\pi i/3}\right) = \frac{1}{4(e^{2\pi i/3})^3 + 2e^{2\pi i/3}} = \frac{1}{4-1-\sqrt{3}} = \frac{1}{3+\sqrt{3}i}$$

so that

$$\int_0^\infty \frac{\mathrm{d}x}{x^4+x^2+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^4+x^2+1} = \pi i\left(\frac{1}{-3+\sqrt{3}i} + \frac{1}{3+\sqrt{3}i}\right)$$

Part E

Note that

$$\frac{\cos x}{1+x^2}$$

is an even function so that we may write

$$\int_0^\infty \frac{\cos x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\,\mathrm{d}x$$

Since the only pole of this function in the upper half plane is i, we have

$$\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\,\mathrm{d}x = \operatorname{Re}\left[2\pi i \cdot \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}; i\right)\right] = \operatorname{Re}\left(2\pi i \frac{e^{-1}}{2i}\right) = \frac{\pi}{e}$$

Thus we obtain

$$\int_0^\infty \frac{\cos x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2e}$$

Part F

First, we note that the residues are at $z = 2e^{\pi i/3}, -2, 2e^{5\pi i/3}$. We first compute

$$\operatorname{Res}\left(\frac{\log z}{z^3+8}; 2e^{\pi i/3}\right)$$

To do this, we note that

$$\frac{\log z}{(z^3+8)'} = \frac{\log z}{3z^2}$$

so that

$$\operatorname{Res}\left(\frac{\log z}{z^3+8}; 2e^{\pi i/3}\right) = \frac{\log 2e^{\pi i/3}}{3(2e^{\pi i/3})^2}$$

Similarly, we have

$$\operatorname{Res}\left(\frac{\log z}{z^3+8}; 2e^{\pi i}\right) = \frac{\log 2e^{\pi i}}{3(2e^{\pi i})^2}$$

We also have

$$\operatorname{Res}\left(\frac{\log z}{z^3+8}; 2e^{5\pi i/3}\right) = \frac{\log 2e^{5\pi i/3}}{3(2e^{5\pi i/3})^2}$$

so that the integral is just

$$-\left(\frac{\log 2e^{\pi i/3}}{3(2e^{\pi i/3})^2} + \frac{\log 2e^{\pi i}}{3(2e^{\pi i})^2} + \frac{\log 2e^{5\pi i/3}}{3(2e^{5\pi i/3})^2}\right)$$

Problem 2

Let us consider

$$e^{2iz} - 1 - 2iz = -1 - 2iz + 1 + 2iz + \frac{(2iz)^2}{2} + \cdots = \frac{(2iz)^2}{2} + \cdots$$

Thus we find that

$$\frac{e^{2iz} - 1 - 2iz}{z^2}$$

is entire. Therefore, we know that for any closed contour C_R (consisting of the line segment from $-R$ to R and the semi-circle Γ_R), we have

$$\int_{C_R} \frac{e^{2iz} - 1 - 2iz}{z^2} \, \mathrm{d}z = 0$$

Let us split this integral into the real segment and the semi-circle Γ_R. Then we have

$$\int_{-R}^{R} \frac{e^{2ix} - 1 - 2ix}{x^2} \, \mathrm{d}x + \int_{\Gamma_R} \frac{e^{2iz} - 1 - 2iz}{z^2} \, \mathrm{d}z = 0$$

Let us consider the integral

$$\int_{\Gamma_R} \frac{e^{2iz} - 1 - 2iz}{z^2} \, \mathrm{d}z = \int_{\Gamma_R} \frac{e^{2iz} - 1}{z^2} \, \mathrm{d}z - 2i \int_{\Gamma_R} \frac{1}{z} \, \mathrm{d}z$$

Using reasoning similar to that in the textbook, it can be shown that

$$\int_{\Gamma_R} \frac{e^{2iz} - 1}{z^2} \, \mathrm{d}z \to 0$$

as $R \to \infty$. Furthermore, we have

$$\int_{\Gamma_R} \frac{1}{z} \, \mathrm{d}z = \int_{-\pi/2}^{\pi/2} \frac{1}{Re^{i\theta}} \cdot iRe^{i\theta} \, \mathrm{d}\theta = i \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta = i\pi$$

Thus we obtain

$$\lim_{R \to \infty} \int_{\Gamma_R} \frac{e^{2iz} - 1 - 2iz}{z^2} \, \mathrm{d}z = 2\pi$$

so that

$$\int_{-\infty}^{\infty} \frac{e^{2ix} - 1 - 2ix}{x^2} \, \mathrm{d}x = -2\pi$$

Now, we note that

$$e^{2ix} = (e^{ix})^2 = (\cos x + i \sin x)^2 = \cos^2 x + 2i \sin x \cos x - \sin^2 x = 1 + 2i \sin x \cos x - 2\sin^2 x$$

so that

$$e^{2ix} - 1 - 2ix = -2\sin^2 x + 2i(\sin x \cos x - x)$$

From this, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{2ix} - 1 - 2ix}{x^2} \, \mathrm{d}x = -2 \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, \mathrm{d}x + 2i \int_{-\infty}^{\infty} \frac{\sin x \cos x - x}{x^2} \, \mathrm{d}x = -2\pi$$

Finally, we deduce that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, \mathrm{d}x = \pi$$

Problem 3

We first write

$$\int_{C_R} \frac{\mathrm{d}z}{1+z^n} = \int_0^R \frac{\mathrm{d}x}{1+x^n} + \int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^n} - e^{2\pi i/n} \int_0^R \frac{\mathrm{d}x}{1+x^n}$$

where the three integrals represent integration along the separate components of C_R. Next, we note that

$$\int_{C_R} \frac{\mathrm{d}z}{1+z^n} = 2\pi i \cdot \mathrm{Res}\left(\frac{1}{1+z^n}; e^{\pi i/n}\right) = \frac{2\pi i}{ne^{(n-1)\pi i/n}}$$

Furthermore, it is evident that

$$\int_{\Gamma_R} \frac{\mathrm{d}z}{1+z^n} \to 0$$

as R tends to infinity. Thus, we find that

$$\frac{2\pi i}{ne^{(n-1)\pi i/n}} = \int_0^\infty \frac{\mathrm{d}x}{1+x^n} - e^{2\pi i/n} \int_0^\infty \frac{\mathrm{d}x}{1+x^n}$$

so that

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^n} = \frac{2\pi i}{(ne^{(n-1)\pi i/n})(1-e^{2\pi i/n})}$$

Problem 8

First, we note that

$$\int_{C_R} \frac{P(z)}{Q(z)} = 2\pi i \sum_i \operatorname{Res}\left(\frac{P}{Q}; z_i\right)$$

where the sum includes every zero z_i of $Q(z)$ such that z_i is in the circle of radius R centered at 0 (which we denote by C_R). Since $\deg Q - \deg P \geq 2$, we may let $R \to \infty$ to obtain

$$\lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)} = 0$$

For sufficiently large R, the circle C_R must include all the zeroes of $Q(z)$, so we may deduce that the sum of all the residues is 0.

Chapter 13

Problem 1

Let $z_0 \neq 0$ be given. We wish to show that z^k is $1-1$ at z_0. If z_1 and z_2 have different moduli, then it is obvious that $z_1^k \neq z_2^k$. Thus, we only have to consider the case in which their moduli are the same. In this case, we know that $0 < \arg z_2 - \arg z_1 < 2\pi$. Furthermore, we know that z^k multiplies the angle between z_1 and z_2 by k. That is, we have $\arg z_2^k - \arg z_1^k = k(\arg z_2 - \arg z_1)$ (up to some multiple of 2π). However, if we require $0 < \arg z_2 - \arg z_1 < 2\pi/k$, then we know that $0 < \arg z_2^k - \arg z_1^k < 2\pi$ so that $z_1^k \neq z_2^k$. Furthermore, we know that there must exist a neighborhood around z_0 with this property for all points because $z_0 \neq 0$. Thus z^k is $1-1$ at z_0.

Problem 2

First, we will consider the effect of the mapping e^z on the line $y = y_0$. Notice that $e^z = e^{x+iy_0} = e^x e^{iy_0}$. Since x can vary, we know that $|e^z| = e^x$ can vary from 0 to ∞. Since y_0 is fixed, we know that $\arg(e^z) = y_0$ is fixed. Thus, we deduce that the mapping e^z takes the line $y = y_0$ to the ray Re^{iy_0}, where $R > 0$. Next, we will consider the effect of the mapping e^z on the line $x = x_0$. Then, we have $e^z = e^{x_0} e^{iy}$. Notice that $|e^z| = e^{x_0}$ stays constant, while $\arg(e^z) = y$ can vary. Thus e^z maps the line $x = x_0$ onto the circle centered at 0 with radius e^{x_0}.

Problem 3

Part A

First, we let $f_1(z) = z + 2$. Then $f_1(S)$ is the vertical strip where $0 < x < 3$ and $y \in \mathbb{R}$. Next, we let $f_2(z) = \frac{\pi}{3}z$. Then $f_2(f_1(S))$ is the vertical strip where $0 < x < \pi$ and $y \in \mathbb{R}$. Then we let $f_3(z) = iz$. Then $f_3(f_2(f_1(S)))$ is the horizontal strip where $0 < y < \pi$ and $x \in \mathbb{R}$. Then we let $f_4(z) = e^z$ so that $f_4(f_3(f_2(f_1(S))))$ is the entire upper half plane. Finally, we let $f_5(z) = (i - z)/(i + z)$ so that $f_5(f_4(f_3(f_2(f_1(S)))))$ is the unit disk. Letting $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, we have found our conformal mapping.

Part B

First, we appeal to Theorem 13.23:

$$\frac{w}{w+1} \cdot \frac{3}{2} = \frac{z}{z+2} \cdot \frac{4}{2}$$

Multiplying both sides by 2 yields

$$\frac{3w}{w+1} = \frac{4z}{z+2}$$

so that

$$3wz + 6w = 4zw + 4z$$

and

$$-wz + 6w = 4z$$

from which we obtain

$$w(6 - z) = 4z$$

and

$$w = \frac{4z}{6 - z}$$

Notice that the mapping

$$f(z) = \frac{4z}{6 - z}$$

is bilinear and that $24 > 0$ so that this mapping takes the upper half plane to the upper half plane and is conformal. By construction, we know that $f(-2) = -1$ and $f(0) = 0$ and $f(2) = 2$.

Part C

First, we may consider the function $\log z$. Notice that $\log z = \log re^{i\theta} = \log r + \log e^{i\theta} = \log r + i\theta$. Since $r > 0$, we know that $-\infty < \log r < \infty$. Since $0 < \theta < \pi/4$, we know that the height of the resulting horizontal strip is $\pi/4$. Thus, we may consider $f(z) = \frac{4}{\pi}\log z$. Then, we have

$$f(z) = \frac{4}{\pi}\log r + \frac{4}{\pi}i\theta$$

Since $r > 0$ and $0 < \theta < \pi/4$, we may deduce that f takes the circular arc S to the horizontal strip T.

Part D

First, we let $f(z) = \sqrt{z}$. In particular, if $z = Re^{i\theta}$ for some $R > 0$ and some $\theta \in (0, 2\pi)$, we have $\sqrt{z} = \sqrt{R}e^{i\theta/2}$. Since $S = D(0;1) \setminus [0,1]$, we know that $f(S)$ is the open semi disc $D = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$. Next, we let

$$g(z) = \frac{-(z-1)^2}{4(z+1)^2}$$

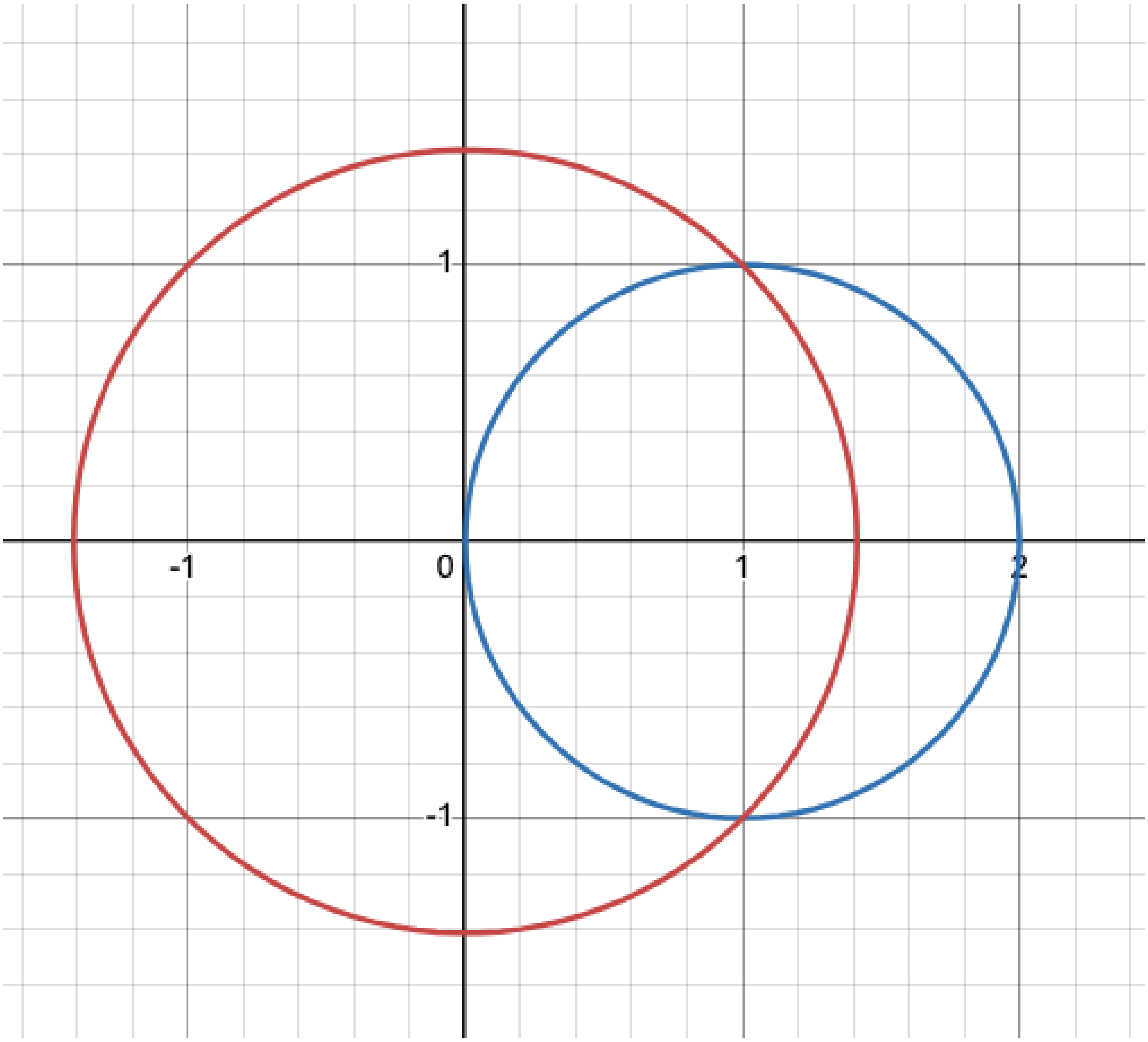
From the textbook, we know that g maps D to the upper half plane. Finally, we let

$$h(z) = \frac{z-i}{z+i}$$

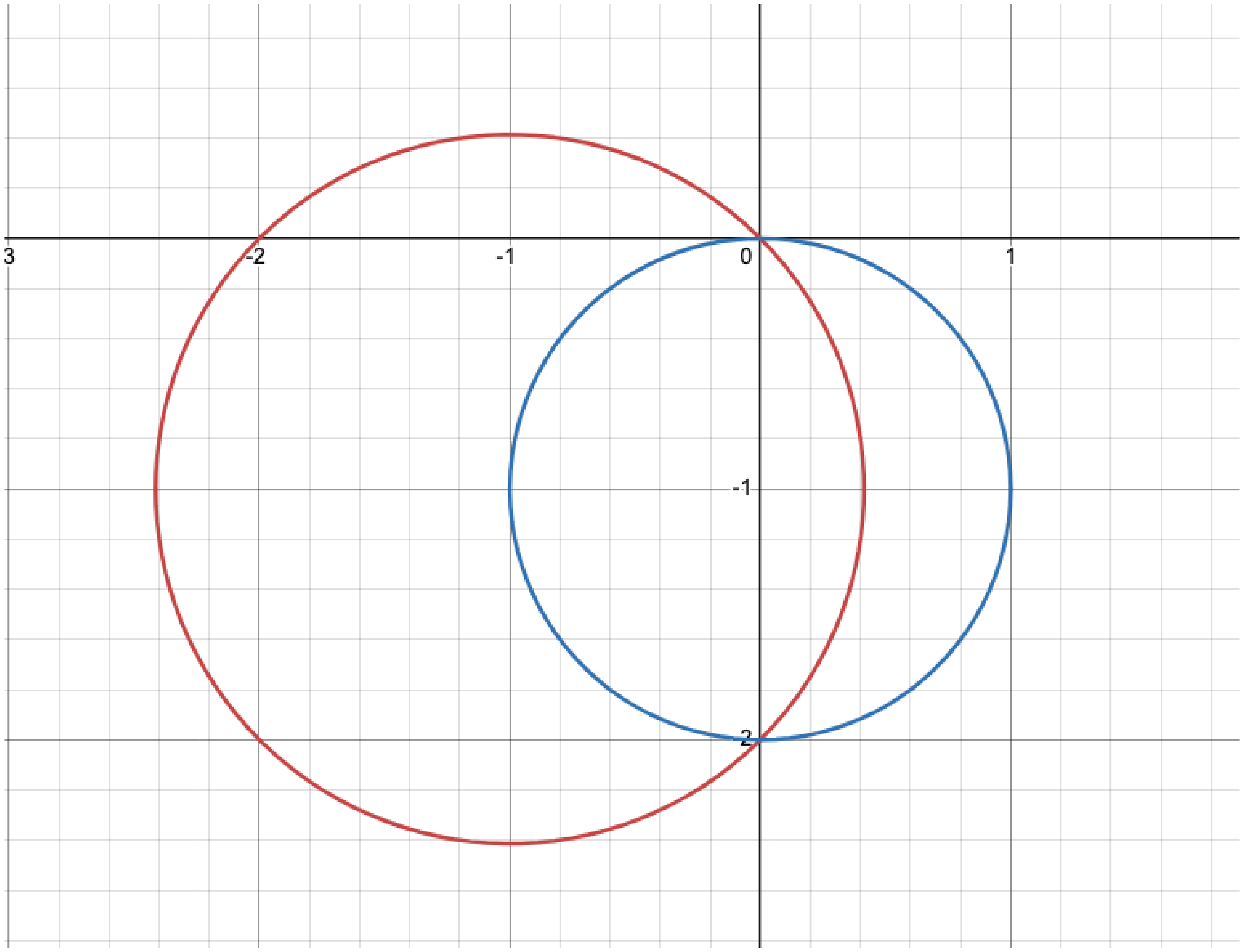
From the textbook, we know that h takes the upper half plane to the unit disk. Thus, the desired conformal mapping is $h \circ g \circ f$.

Problem 4

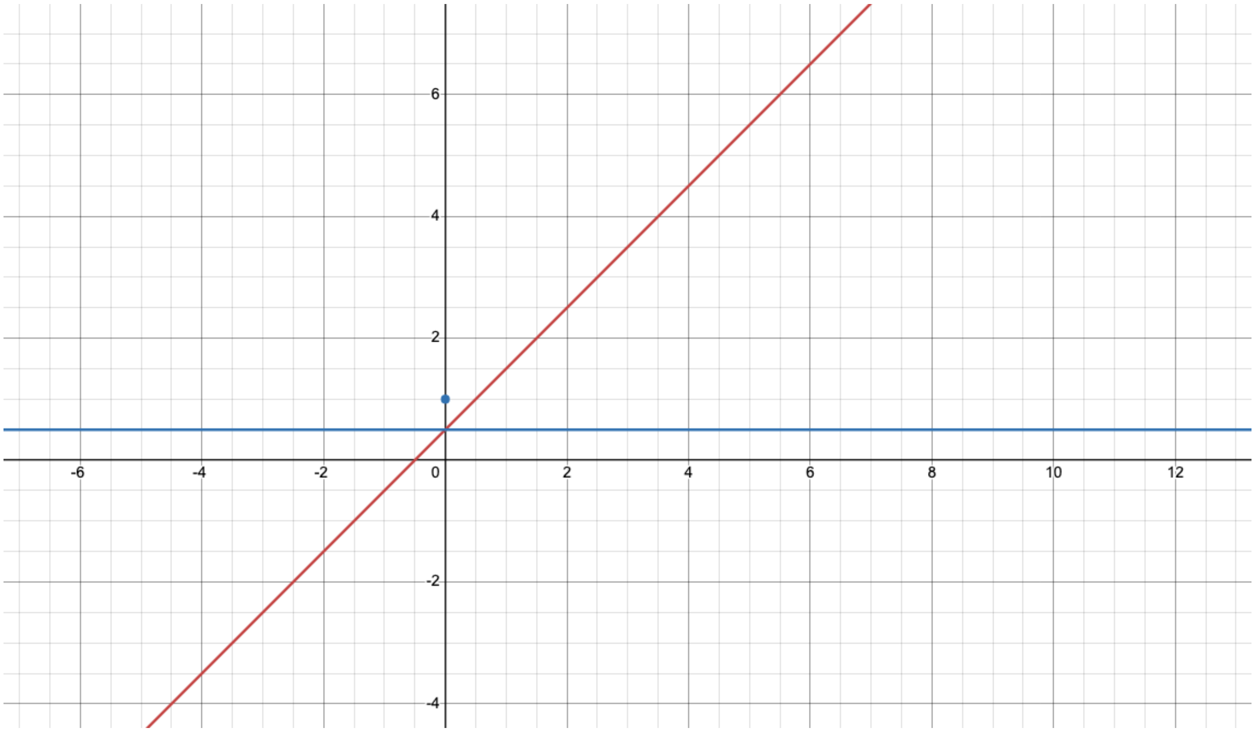
First, we draw the circles as follows:



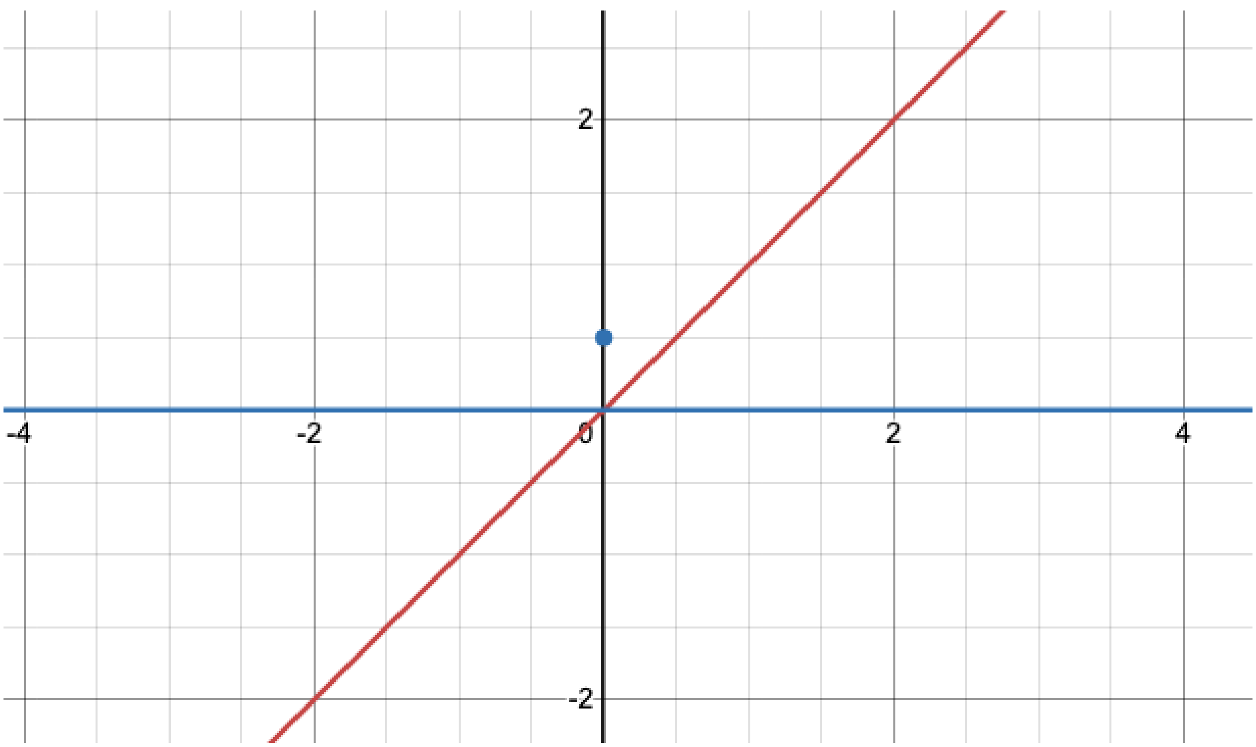
Next, we move them so that their intersection is at the origin. That is, we apply the function $f_1(z) = z - i - 1$ to both of the circles, obtaining the below image:



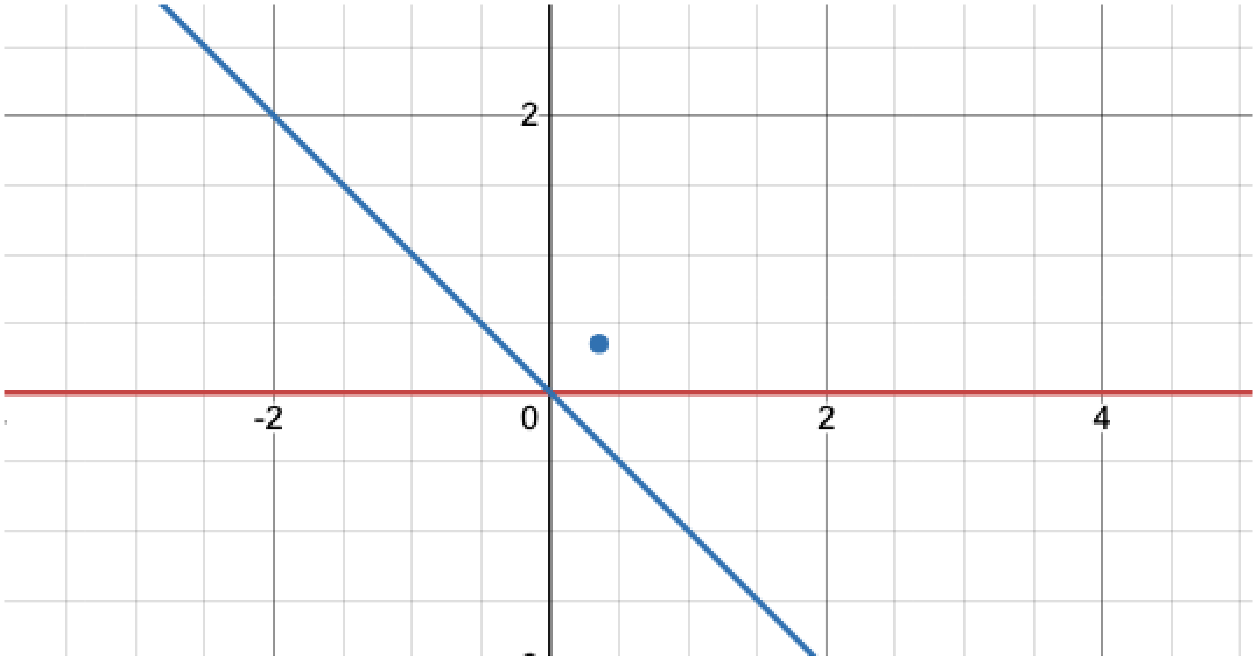
Next, we apply the mapping $f_2(z) = 1/z$. Since f_2 is a mobius transformation and $f_2(0) = \infty$, we know that f_2 sends both circles to lines. We first examine the effect of $1/z$ on the red circle. In particular, we compute $f_2(-2i) = i/2$ and $f_2(-2) = -1/2$. Computing the line between these two points yields $y = x + 1/2$. Thus the red circle is sent to the line $y = x + 1/2$. Next, we examine the effect of $1/z$ on the blue circle. In particular, we compute $f_2(1 - i) = (1 + i)/2$ and $f_2(-1 - i) = -1/2 + i/2$. The line between these two points is $y = 1/2$. Thus the blue circle is sent to the line $y = 1/2$. Finally, we would like to determine where the region between these two circles goes. We compute $f_2(-i) = i$. Thus, we obtain the following image:



Next, we apply the mapping $f_3(z) = z - i/2$ to obtain the following image:



Now, we can apply the mapping $f_4(z) = e^{-i\pi/4} z$ to obtain the following image:



Next, we can apply the mapping $f_5(z) = z^{4/3}$ so that the region with the point in it is mapped to the upper half plane. Finally, we can apply the mapping $f_6(z) = (i - z)/(i + z)$ to ensure that the upper half-plane is mapped to the unit disk. Thus we find that $f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ is the desired conformal mapping.

Problem 5

First, we apply the mapping $f_1(z) = z - \frac{1}{2}i$ to the strip $S_1 = \{z : \operatorname{Re} z > 0, 0 < \operatorname{Im} z < 1\}$ to obtain the strip $S_2 = \{z : \operatorname{Re} z > 0, -1/2 < \operatorname{Im} z < 1/2\}$. Then, we apply the mapping $f_2(z) = iz$ to the strip S_2 to obtain the strip $S_3 = \{z : -1/2 < \operatorname{Re} z < 1/2, \operatorname{Im} z > 0\}$. Then, we apply the mapping $f_3(z) = \pi z$ to the strip S_3 to obtain the strip $S_4 = \{z : -\pi/2 < \operatorname{Re} z < \pi/2, \operatorname{Im} z > 0\}$. Next, we can apply the mapping $f_4(z) = \sin z$ to S_4 to obtain the upper half plane. Finally, we can apply the mapping $f_5(z) = (z - i)/(z + i)$ to the upper half plane to obtain the unit disk.