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More sequences

First I consider the sequence $f_n(z) = \sin(nz)$. Notice that

$$\int_D \sin(nz)dA(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sin(nre^{i\theta})drd\theta = 0$$

where the equality comes from Wolfram Alpha. If we want to determine if it is bounded, we must consider the integral

$$\|\sin(nz)\|^2 = \int_D |\sin(nz)|^2 \exp\left(-\frac{1}{1-|z|^2}\right)dA(z)$$

For each n, it can be seen that the integral converges. However, we don't know if the sequence is bounded. First, let us consider the sequence without the exponential weight.

$$\|\sin(nz)\|^2 = \int_D |\sin(nz)|^2 dA(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |\sin(nre^{i\theta})|^2 drd\theta$$

We compute this for several values of n.

$$\int_0^{2\pi}\int_0^1 |\sin(re^{i\theta})|^2 drd\theta \approx 2.119$$

$$\int_0^{2\pi}\int_0^1 |\sin(2re^{i\theta})|^2 drd\theta \approx 10.0142$$

$$\int_0^{2\pi}\int_0^1 |\sin(3re^{i\theta})|^2 drd\theta \approx 39.448$$

$$\int_0^{2\pi}\int_0^1 |\sin(4re^{i\theta})|^2 drd\theta \approx 181.833$$

$$\int_0^{2\pi}\int_0^1 |\sin(5re^{i\theta})|^2 drd\theta \approx 939.957$$

Now we consider the sequence with the weight:

$$\int_D |\sin(nz)|^2\lambda(z)dA(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |\sin(nre^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right)drd\theta$$

Let us compute this integral for several values of n:

$$\int_0^{2\pi}\int_0^1 |\sin(re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.221$$

$$\int_0^{2\pi}\int_0^1 |\sin(2re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.94$$

$$\int_0^{2\pi}\int_0^1 |\sin(3re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 2.673$$

$$\int_0^{2\pi}\int_0^1 |\sin(4re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 7.868$$

$$\int_0^{2\pi}\int_0^1 |\sin(5re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 26.293$$

$$\int_0^{2\pi}\int_0^1 |\sin(6re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 97.7814$$

$$\int_0^{2\pi}\int_0^1 |\sin(7re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 393.24$$

$$\int_0^{2\pi}\int_0^1 |\sin(8re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 1677.21$$

$$\int_0^{2\pi}\int_0^1 |\sin(9re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 7488.43$$

$$\int_0^{2\pi}\int_0^1 |\sin(10re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 34685.8$$

As we can see, the norm clearly diverges as n approaches ∞. Thus, this sequence is not unbounded, so we shouldn't bother to check whether the sequence of images contains a convergent subsequence. Now I consider functions with a singularity at the point $z = 1$. First, we may try $f(z) = \frac{1}{1-z}$. We compute the unweighted integral:

$$\int_D \frac{1}{1-z} dA(z) = \int_0^1 \int_0^{2\pi} \frac{r}{1-re^{i\theta}} d\theta dr = \int_0^{2\pi}\int_0^1 \frac{r}{1-re^{i\theta}} dr d\theta = \pi$$

Let us compare the different ways of evaluating this integral. First, we can try integrating with respect to θ so that we obtain

$$\int_0^{2\pi} \frac{r}{1-re^{i\theta}} d\theta$$

but it seems extremely difficult to find a closed form expression for the result of this integral. Next, we can try integrating with respect to r so that we obtain

$$\int_0^1 \frac{r}{1-re^{i\theta}} dr = -e^{-2i\theta}(e^{i\theta} + \log(1-e^{i\theta}))$$

Then, we can integrate the result with respect to θ to obtain

$$\int_0^{2\pi} -e^{-2i\theta}(e^{i\theta} + \log(1 - e^{i\theta})d\theta = \pi$$

Now, let us compute the unweighted norm of f. We have

$$\int_D \frac{1}{|1-z|^2} dA(z) = \int_0^{2\pi} \int_0^1 \frac{r}{|1 - re^{i\theta}|^2} dr d\theta$$

Thus, we find that even without the weight, the function f can still be integrated. Next, let us try it with the weight λ.

$$\int_D \frac{1}{1-z} \lambda(z) dA(z) = \int_0^{2\pi} \int_0^1 \frac{r}{1 - re^{i\theta}} \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.4665$$

Now, we may try to compute the weighted norm of f. We obtain

$$\|f\|^2 = \int_D \frac{1}{|1-z|^2} \lambda(z) dA(z) = \int_0^{2\pi} \int_0^1 \frac{r}{|1 - re^{i\theta}|^2} \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.6892$$