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Exponential Weights

I first assume that $\phi \equiv 1$ on $\mathbb{D}$. Then we have

$$T_\phi = T_1 = P$$

Thus we are asking whether or not P is compact. An operator S is compact if and only if S takes bounded sequences to sequences with converging subsequences. Let us consider some bounded sequences. First, we have $\{z^n\}_{n\in\mathbb{N}}$. We verify that it is bounded. Note that

$$\|z^n\|^2 = \int_{\mathbb{D}} |z|^{2n} d\lambda(z) = \int_{\mathbb{D}} |z|^{2n} \exp\left(-\frac{1}{1-|z|^2}\right) dA(z) = 2\int_0^1 r^{2n+1} \exp\left(-\frac{1}{1-r^2}\right) dr$$

Let us compute this integral for some specific values of n. We have

$$\int_0^1 r \exp\left(-\frac{1}{1-r^2}\right) dr \approx 0.0742$$

$$\int_0^1 r^3 \exp\left(-\frac{1}{1-r^2}\right) dr \approx 0.0194$$

$$\int_0^1 r^5 \exp\left(-\frac{1}{1-r^2}\right) dr \approx 0.0075$$

$$\int_0^1 r^7 \exp\left(-\frac{1}{1-r^2}\right) dr \approx 0.0035$$

$$\int_0^1 r^9 \exp\left(-\frac{1}{1-r^2}\right) dr \approx 0.0018$$

From this, it is clear that the sequence $\{z^n\}_{n\in\mathbb{N}}$ is bounded and even converges. Now, we compute $P(f_m)$ as follows:

$$P(z^m) = \int_{\mathbb{D}} \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{|z^n|^2} w^m \lambda(w) dA(w) = \frac{z^m}{\|z^m\|^2} \int_{\mathbb{D}} |w|^{2m} \lambda(w) dA(w) = \frac{z^m}{\|z^m\|^2} \cdot \|z^m\|^2 = z^m$$

This is because z^m is analytic. Thus $\{P(z^n)\}_{n\in\mathbb{N}}$ is convergent.