# Compact Toeplitz Operators and Weighted Bergman Spaces

Ethan, Ryan

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- $L^2(\mathbb{D}, dA)$: space of square-integrable functions on $\mathbb{D}$
- $L_a^2$: closed subspace of analytic functions in $L^2(\mathbb{D}, dA)$
- $P : L^2(\mathbb{D}, dA) \to L_a^2$: orthogonal projection operator
- $T_u : L_a^2 \to L_a^2$: Toeplitz operator with symbol $u$. $T_u(f) = P(uf)$
- $K_z \in L_a^2$: Bergman reproducing kernel. $f(z) = \langle f, K_z \rangle$
- $k_z \in L_a^2$: normalized Bergman reproducing kernel. $k_z = K_z / \|K_z\|$
- $\tilde{S} : \mathbb{D} \to \mathbb{C}$: Berezin transform of $S$. $\tilde{S}(z) = \langle Sk_z, k_z \rangle$.
- $\varphi_z : \mathbb{D} \to \mathbb{D}$: automorphism of unit disk. $\varphi_z(w) = (z - w)/(1 - \overline{z}w)$
- $U_z : L_a^2 \to L_a^2$: $U_z f = (f \circ \varphi_z)\varphi_z'$
- $S_z : L_a^2 \to L_a^2$: $S_z = U_z S U_z$
- $H_u : L_a^2 \to (L_a^2)^\perp$: Hankel operator with symbol $u$.
  $H_u(f) = (I - P)(uf)$

## Theorem

Suppose $S$ is a finite sum of finite products of Toeplitz operators. Then the following are equivalent:

1. (i) $S$ is compact
2. (ii) $\|Sk_z\|_2 \to 0$ as $z \to \partial\mathbb{D}$
3. (iii) $\tilde{S}(z) \to 0$ as $z \to \partial\mathbb{D}$
4. (iv) $S_z 1 \to 0$ weakly in $L_a^2$ as $z \to \partial\mathbb{D}$
5. (v) $\|S_z 1\|_2 \to 0$ as $z \to \partial\mathbb{D}$
6. (vi) $\|S_z 1\|_p \to 0$ as $z \to \partial\mathbb{D}$ for all $p \in (1, \infty)$

## Fact

$U_z$ is unitary: $\langle U_z f, U_z f\rangle = \langle f, f\rangle$ for all $f \in L^2_a$.

## Proof.

$$\langle U_z f, U_z f\rangle = \int_{\mathbb{D}} |(U_z f)(w)|^2 dA(w) = \int_{\mathbb{D}} |(f \circ \varphi_z)(w)|^2 |\varphi_z'(w)|^2 dA(w)$$

Let $\lambda = \varphi_z(w)$. Then $dA(\lambda) = |\varphi_z'(w)|^2 dA(w)$ so that

$$\int_{\mathbb{D}} |(f \circ \varphi_z)(w)|^2 |\varphi_z'(w)|^2 dA(w) = \int_{\mathbb{D}} |f(\lambda)|^2 dA(\lambda) = \langle f, f\rangle$$

$\square$