Toeplitz Operators and Bergman Spaces

Polymath Jr

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- $L^2(\mathbb{D}, dA)$: space of square-integrable functions on $\mathbb{D}$
- L^2_a: closed subspace of analytic functions in $L^2(\mathbb{D}, dA)$
- $P : L^2(\mathbb{D}, dA) \to L^2_a$: orthogonal projection operator
- $T_u : L^2_a \to L^2_a$: Toeplitz operator with symbol u. $T_u(f) = P(uf)$
- $K_z \in L^2_a$: Bergman reproducing kernel. $f(z) = \langle f, K_z \rangle$
- $k_z \in L^2_a$: normalized Bergman reproducing kernel. $k_z = K_z / \|K_z\|_2$
- $\tilde{S} : \mathbb{D} \to \mathbb{C}$: Berezin transform of S. $\tilde{S}(z) = \langle S k_z, k_z \rangle$.
- $\varphi_z : \mathbb{D} \to \mathbb{D}$: automorphism of unit disk. $\varphi_z(w) = (z - w)/(1 - \bar{z} w)$
- $U_z : L^2_a \to L^2_a$: $U_z f = (f \circ \varphi_z)\varphi_z'$
- $S_z : L^2_a \to L^2_a$: $S_z = U_z S U_z$
- $H_u : L^2_a \to (L^2_a)^\perp$: Hankel operator with symbol u.
  $H_u(f) = (I - P)(uf)$

Theorem by Axler, Zheng[1]

Suppose S is a finite sum of finite products of Toeplitz operators. Then the following are equivalent:

1. (i) S is compact
2. (ii) $\|Sk_z\|_2 \to 0$ as $z \to \partial\mathbb{D}$
3. (iii) $\tilde{S}(z) \to 0$ as $z \to \partial\mathbb{D}$
4. (iv) $S_z 1 \to 0$ weakly in L_a^2 as $z \to \partial\mathbb{D}$
5. (v) $\|S_z 1\|_2 \to 0$ as $z \to \partial\mathbb{D}$
6. (vi) $\|S_z 1\|_p \to 0$ as $z \to \partial\mathbb{D}$ for all $p \in (1, \infty)$

Axler, Sheldon, and Dechao Zheng. "Compact Operators via the Berezin Transform." Indiana University Mathematics Journal, vol. 47, no. 2, 1998, pp. 387–400. JSTOR, http://www.jstor.org/stable/24899675. Accessed 30 July 2024.

We define $U_z : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ as follows: $U_z f = (f \circ \varphi_z)\varphi_z'$

Fact

U_z is unitary: $\langle U_z f, U_z f \rangle = \langle f, f \rangle$ for all $f \in A^2(\mathbb{D})$.

Proof.

$$\langle U_z f, U_z f \rangle = \int_{\mathbb{D}} |(U_z f)(w)|^2 dA(w) = \int_{\mathbb{D}} |(f \circ \varphi_z)(w)|^2 |\varphi_z'(w)|^2 dA(w)$$

Let $\lambda = \varphi_z(w)$. Then $dA(\lambda) = |\varphi_z'(w)|^2 dA(w)$ so that

$$\int_{\mathbb{D}} |(f \circ \varphi_z)(w)|^2 |\varphi_z'(w)|^2 dA(w) = \int_{\mathbb{D}} |f(\lambda)|^2 dA(\lambda) = \langle f, f \rangle$$

$\square$

Fact

$H_{\overline{f}}^* H_g = T_{fg} - T_f T_g$ for any symbols $f, g \in L^\infty(\mathbb{D}, dA)$.

Proof.

First, we compute what $H_{\overline{f}}^*$ is. Let $a \in (L_a^2)^\perp$ and $b \in L_a^2$. Then,

$$\langle H_{\overline{f}}^* a, b \rangle = \langle a, H_{\overline{f}} b \rangle = \langle a, (I - P)(\overline{f} b) \rangle = \langle a, \overline{f} b - P(\overline{f} b) \rangle$$

$$= \langle a, \overline{f} b \rangle - \langle a, P(\overline{f} b) \rangle = \langle fa, b \rangle - \langle Pa, \overline{f} b \rangle = \langle fa, b \rangle$$

Thus we deduce that $H_{\overline{f}}^* a = fa$. Now, we let $b \in L_a^2$. We obtain

$$H_{\overline{f}}^* H_g(b) = P(H_{\overline{f}}^* H_g(b)) = P(H_{\overline{f}}^*(gb - P(gb)))$$

$$= P(fgb - fP(gb)) = P(fgb) - P(fP(gb)) = T_{fg}(b) - T_f T_g(b)$$

thus proving that

$$H_{\overline{f}}^* H_g = T_{fg} - T_f T_g$$

$\square$

Lemma

Let $S : L_a^2 \to L_a^2$ be a bounded operator. Then, $\tilde{S} \circ \varphi_z = \widetilde{S_z}$

Proof.

First,

$$\tilde{S} \circ \varphi_z(w) = \tilde{S}(\varphi_z(w)) = \langle S k_{\varphi_z(w)}, k_{\varphi_z(w)} \rangle$$

and

$$\widetilde{S_z}(w) = \langle S_z k_w, k_w \rangle = \langle U_z S U_z k_w, k_w \rangle = \langle S U_z k_w, U_z k_w \rangle$$

Note that

$$U_z k_w = (k_w \circ \varphi_z) \varphi_z'$$

I appeal to the following formula[2]:

$$K_U(z, \overline{\zeta}) = \det Df(z) \overline{\det Df(\zeta)} K_V(f(z), \overline{f(\zeta)})$$

$\square$

Lebl, Jiří. Tasty Bits of Several Complex Variables. Oklahoma State University. 3 Jun 2024.

Proof.

Take $U = V = \mathbb{D}$, $f = \varphi_z$, $z = w$ and $\zeta = \varphi_z(v)$. Then

$$K_{\mathbb{D}}(w, \overline{\varphi_z(v)}) = \det D\varphi_z(w)\overline{\det D\varphi_z(\varphi_z(v))}K_{\mathbb{D}}(\varphi_z(w), \overline{v})$$

I compute

$$K_{\mathbb{D}}(w, \overline{\varphi_z(v)}) = \overline{K_w(\varphi_z(v))} = \overline{\frac{k_w(\varphi_z(v))}{1 - |w|^2}}$$

$$K_{\mathbb{D}}(\varphi_z(w), \overline{v}) = \overline{K_{\varphi_z(w)}(v)} = \overline{\frac{k_{\varphi_z(w)}(v)}{1 - |\varphi_z(w)|^2}}$$

$$\det D\varphi_z(w) = \varphi_z'(w) = \frac{|z|^2 - 1}{(1 - \overline{z}w)^2}$$

$$\overline{\det D\varphi_z(\varphi_z(v))} = \overline{\varphi_z'(\varphi_z(v))} = \overline{\frac{1}{\varphi_z'(v)}}$$

$\square$

Proof.

Putting it all together yields

$$\overline{\frac{k_w(\varphi_z(v))}{1-|w|^2}} = \frac{|z|^2-1}{(1-\bar{z}w)^2} \cdot \overline{\frac{1}{\varphi_z'(v)}} \cdot \overline{\frac{k_{\varphi_z(w)}(v)}{1-|\varphi_z(w)|^2}}$$

which simplifies to

$$k_w(\varphi_z(v))\varphi_z'(v) = k_{\varphi_z(w)}(v) \cdot -\frac{|1-z\overline{w}|^2}{(1-z\overline{w})^2}$$

With this formula, it can be shown that

$$\widetilde{S_z}(w) = \langle SU_z k_w, U_z k_w \rangle = \langle Sk_{\varphi_z(w)}, k_{\varphi_z(w)} \rangle = \tilde{S}(\varphi_z(w))$$

$\square$

Fact

ϕ vanishes on $\partial\mathbb{D} \implies T_\phi$ is compact.

Is this true on exponentially weighted spaces? Let $\lambda(z) = \exp(-1/(1-|z|^2))$ and $\phi(z) = 1 - |z|^2$. To show that $T_\phi : L^2_a(\mathbb{D}, \lambda) \to L^2_a(\mathbb{D}, \lambda)$ is not compact, it suffices to find a bounded sequence $\{f_n\}$ such that $\{T_\phi(f_n)\}$ does not contain a convergent subsequence.

Let $f_n(z) = 1/(1-z^n)$. Then

$$\|f_1\| \approx 0.468 \quad \|f_2\| \approx 0.412 \quad \|f_3\| \approx 0.396$$

$$\|f_4\| \approx 0.391 \quad \|f_5\| \approx 0.388$$

Thus $\{f_n\}$ is bounded. Furthermore,

$$\|f_1\phi\| \approx 0.331 \quad \|f_2\phi\| \approx 0.302 \quad \|f_3\phi\| \approx 0.296$$

$$\|f_4\phi\| \approx 0.294 \quad \|f_5\phi\| \approx 0.294$$

Note that

$$\|T_\phi f_n\| = \|P(\phi f_n)\| \leq \|\phi f_n\| \leq \|f_n\|$$

so $\{T_\phi(f_n)\}$ is also bounded.