Research

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Now, we let $\lambda(z) = \exp(-1/(1-|z|^2))$ and $\phi(z) = 1-|z|^2$. Let $f_n(z) = 1/(1-z^n)$. We know that $\{f_n\}$ is bounded and that $\{T_\phi(f_n)\}$ is bounded. However, we want to show that $\{T_\phi(f_n)\}$ does not contain a convergent subsequence so that we may conclude that $T_\phi : A^2(\mathbb{D}, \lambda) \to A^2(\mathbb{D}, \lambda)$ is not compact. Let us try to compute $T_\phi(f_n)$ for several values of n. Let $n = 1$. Then

$$T_\phi(f_1)(z) = P(\phi f_1)(z) = \int_{\mathbb{D}} B_\lambda(z, w) f_1(w)\phi(w) d\lambda(w) \tag{1}$$

We note that

$$f_1(w)\phi(w) = \frac{1}{1-w} \cdot (1 - w\overline{w}) = \left(\sum_{k=0}^{\infty} w^k\right)(1 - w\overline{w}) \tag{2}$$

$$= \sum_{k=0}^{\infty} w^k - \sum_{k=0}^{\infty} w^{k+1}\overline{w} = \sum_{k=0}^{\infty} w^k(1 - w\overline{w}) \tag{3}$$

and that

$$B_\lambda(z, w) = \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2} \tag{4}$$

Now, we have

$$\frac{z^0}{\|z^0\|^2}\overline{w}^0 \left(\sum_{k=0}^{\infty} w^k(1 - w\overline{w})\right) = \frac{1}{\|1\|^2}\left(\sum_{k=0}^{\infty} w^k(1 - w\overline{w})\right) \tag{5}$$

$$\frac{z^1}{\|z^1\|^2}\overline{w}^1 \left(\sum_{k=0}^{\infty} w^k(1 - w\overline{w})\right) = \frac{z}{\|z\|^2}\left(\sum_{k=0}^{\infty} w^k(\overline{w} - w\overline{w}^2)\right) \tag{6}$$

$$\frac{z^2}{\|z^2\|^2}\overline{w}^2 \left(\sum_{k=0}^{\infty} w^k(1 - w\overline{w})\right) = \frac{z^2}{\|z^2\|^2}\left(\sum_{k=0}^{\infty} w^k(\overline{w}^2 - w\overline{w}^3)\right) \tag{7}$$

$$\frac{z^3}{\|z^3\|^2}\overline{w}^3\left(\sum_{k=0}^{\infty} w^k(1-w\overline{w})\right) = \frac{z^3}{\|z^3\|^2}\left(\sum_{k=0}^{\infty} w^k(\overline{w}^3 - w\overline{w}^4)\right) \tag{8}$$

If we assume that we can interchange summation and integration, then every term with $w^a\overline{w}^b$ vanishes for $a \neq b$, so we obtain

$$\sum_{i=0}^{\infty} \frac{z^i}{\|z^i\|^2}\int_{\mathbb{D}}(w^i\overline{w}^i - w^{i+1}\overline{w}^{i+1})d\lambda(w) = \sum_{i=0}^{\infty} \frac{z^i}{\|z^i\|^2}\int_{\mathbb{D}}(|w|^{2i} - |w|^{2i+2})d\lambda(w) = \tag{9}$$

$$\sum_{i=0}^{\infty} \frac{z^i}{\|z^i\|^2}(\|z^i\|^2 - \|z^{i+1}\|^2) = \sum_{i=0}^{\infty} z^i\left(1 - \frac{\|z^{i+1}\|^2}{\|z^i\|^2}\right) \tag{10}$$

Now, we let $n = 2$. We have

$$T_\phi(f_2)(z) = P(\phi f_2)(z) = \int_{\mathbb{D}} B_\lambda(z,w)f_2(w)\phi(w)d\lambda(w) \tag{11}$$

We compute

$$f_2(w)\phi(w) = \frac{1-w\overline{w}}{1-w^2} = \left(\sum_{i=0}^{\infty} w^{2i}\right)(1-w\overline{w}) \tag{12}$$

$$= \sum_{i=0}^{\infty} w^{2i} - \sum_{i=0}^{\infty} w^{2i+1}\overline{w} = \sum_{i=0}^{\infty} w^{2i}(1-w\overline{w}) \tag{13}$$

We recall that

$$B_\lambda(z,w) = \sum_{j=0}^{\infty} \frac{z^j\overline{w}^j}{\|z^j\|^2} \tag{14}$$

Now we compute

$$\frac{z^0}{\|z^0\|^2}\overline{w}^0\left(\sum_{i=0}^{\infty} w^{2i}(1-w\overline{w})\right) = \frac{1}{\|1\|^2}\left(\sum_{i=0}^{\infty} w^{2i}(1-w\overline{w})\right) \tag{15}$$

$$\frac{z^1}{\|z^1\|^2}\overline{w}^1\left(\sum_{i=0}^{\infty} w^{2i}(1-w\overline{w})\right) = \frac{z}{\|z\|^2}\left(\sum_{i=0}^{\infty} w^{2i}(\overline{w} - w\overline{w}^2)\right) \tag{16}$$

$$\frac{z^2}{\|z^2\|^2}\overline{w}^2\left(\sum_{i=0}^{\infty} w^{2i}(1-w\overline{w})\right) = \frac{z^2}{\|z^2\|^2}\left(\sum_{i=0}^{\infty} w^{2i}(\overline{w}^2 - w\overline{w}^3)\right) \tag{17}$$

Assuming this pattern continues and interchanging integration and summation, we obtain

$$\sum_{i=0}^{\infty} \frac{z^{2i}}{\|z^{2i}\|^2}\int_{\mathbb{D}}(|w|^{4i} - |w|^{4i+2})d\lambda(w) = \sum_{i=0}^{\infty} \frac{z^{2i}}{\|z^{2i}\|^2}(\|z^{2i}\|^2 - \|z^{2i+1}\|^2) \tag{18}$$

$$= \sum_{i=0}^{\infty} z^{2i} \left(1 - \frac{\|z^{2i+1}\|^2}{\|z^{2i}\|^2}\right) \tag{19}$$

Now let $n = 3$. Note that

$$f_3(w) = \frac{1}{1 - w^3} = \sum_{k=0}^{\infty} w^{3k} \tag{20}$$

$$\phi(w) = 1 - w\overline{w} \tag{21}$$

$$B_\lambda(z, w) = \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{\|z^k\|^2} \tag{22}$$

We have

$$f_3(w)\phi(w) = \frac{1 - w\overline{w}}{1 - w^3} = \left(\sum_{k=0}^{\infty} w^{3k}\right)(1 - w\overline{w}) = \tag{23}$$

$$\sum_{k=0}^{\infty} w^{3k} - \sum_{k=0}^{\infty} w^{3k+1}\overline{w} = \sum_{k=0}^{\infty} w^{3k}(1 - w\overline{w}) \tag{24}$$

Note that

$$\frac{z^0}{\|z^0\|^2}\overline{w}^0 \left(\sum_{k=0}^{\infty} w^{3k}(1 - w\overline{w})\right) = \frac{1}{\|1\|^2}\left(\sum_{k=0}^{\infty} w^{3k}(1 - w\overline{w})\right) \tag{25}$$

$$\frac{z^1}{\|z^1\|^2}\overline{w}^1 \left(\sum_{k=0}^{\infty} w^{3k}(1 - w\overline{w})\right) = \frac{z}{\|z\|^2}\left(\sum_{k=0}^{\infty} w^{3k}(\overline{w} - w\overline{w}^2)\right) \tag{26}$$

$$\frac{z^2}{\|z^2\|^2}\overline{w}^2 \left(\sum_{k=0}^{\infty} w^{3k}(1 - w\overline{w})\right) = \frac{z^2}{\|z^2\|^2}\left(\sum_{k=0}^{\infty} w^{3k}(\overline{w}^2 - w\overline{w}^3)\right) \tag{27}$$

$$\frac{z^3}{\|z^3\|^2}\overline{w}^3 \left(\sum_{k=0}^{\infty} w^{3k}(1 - w\overline{w})\right) = \frac{z^3}{\|z^3\|^2}\left(\sum_{k=0}^{\infty} w^{3k}(\overline{w}^3 - w\overline{w}^4)\right) \tag{28}$$

Assuming the pattern continues and interchanging the integration and summation, we obtain

$$\sum_{i=0}^{\infty} \frac{z^{3i}}{\|z^{3i}\|^2} \int_{\mathbb{D}} (|w|^{6i} - |w|^{6i+2}) d\lambda(w) = \sum_{i=0}^{\infty} \frac{z^{3i}}{\|z^{3i}\|^2}(\|z^{3i}\|^2 - \|z^{3i+1}\|^2) = \tag{29}$$

$$\sum_{i=0}^{\infty} z^{3i}\left(1 - \frac{\|z^{3i+1}\|^2}{\|z^{3i}\|^2}\right) \tag{30}$$

Similarly, we find that for any $n \in \mathbb{N}$, we have

$$T_\phi(f_n)(z) = \sum_{i=0}^{\infty} z^{ni}\left(1 - \frac{\|z^{ni+1}\|^2}{\|z^{ni}\|^2}\right) \tag{31}$$

Now that we have an explicit form, we can try estimating $\|T_\phi(f_n) - T_\phi(f_m)\|$ for $n, m \in \mathbb{N}$. First, let $n = 1$ and $m = 2$. Then, we note that

$$T_\phi(f_1)(z) - T_\phi(f_2)(z) = \sum_{i=0}^{\infty} z^{i}\left(1 - \frac{\|z^{i+1}\|^2}{\|z^{i}\|^2}\right) - \sum_{i=0}^{\infty} z^{2i}\left(1 - \frac{\|z^{2i+1}\|^2}{\|z^{2i}\|^2}\right) = \tag{32}$$

$$\sum_{i=0}^{\infty} z^{2i+1}\left(1 - \frac{\|z^{2i+2}\|^2}{\|z^{2i+1}\|^2}\right) \tag{33}$$

Next, we let $n = 1$ and $m = 3$. Then

$$T_\phi(f_1)(z) - T_\phi(f_3)(z) = \sum_{i=0}^{\infty} z^{i}\left(1 - \frac{\|z^{i+1}\|^2}{\|z^{i}\|^2}\right) - \sum_{i=0}^{\infty} z^{3i}\left(1 - \frac{\|z^{3i+1}\|^2}{\|z^{3i}\|^2}\right) = \tag{34}$$

$$\sum_{i=0}^{\infty}\left(z^{3i+1}\left(1 - \frac{\|z^{3i+2}\|^2}{\|z^{3i+1}\|^2}\right) + z^{3i+2}\left(1 - \frac{\|z^{3i+3}\|^2}{\|z^{3i+2}\|^2}\right)\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{1} z^{3i+1+j}\left(1 - \frac{\|z^{3i+2+j}\|^2}{\|z^{3i+1+j}\|^2}\right) \tag{35}$$

Similarly, we let $n = 1$ and $m = 4$. Then

$$T_\phi(f_1)(z) - T_\phi(f_4)(z) = \sum_{i=0}^{\infty} z^{i}\left(1 - \frac{\|z^{i+1}\|^2}{\|z^{i}\|^2}\right) - \sum_{i=0}^{\infty} z^{4i}\left(1 - \frac{\|z^{4i+1}\|^2}{\|z^{4i}\|^2}\right) = \tag{36}$$

$$\sum_{i=0}^{\infty}\left(z^{4i+1}\left(1 - \frac{\|z^{4i+2}\|^2}{\|z^{4i+1}\|^2}\right) + z^{4i+2}\left(1 - \frac{\|z^{4i+3}\|^2}{\|z^{4i+2}\|^2}\right) + z^{4i+3}\left(1 - \frac{\|z^{4i+4}\|^2}{\|z^{4i+3}\|^2}\right)\right) = \tag{37}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{2} z^{4i+1+j}\left(1 - \frac{\|z^{4i+2+j}\|^2}{\|z^{4i+1+j}\|^2}\right) \tag{38}$$

By analogous reasoning, we find that

$$T_\phi(f_1) - T_\phi(f_m) = \sum_{i=0}^{\infty}\sum_{j=0}^{m-2} z^{mi+1+j}\left(1 - \frac{\|z^{mi+2+j}\|^2}{\|z^{mi+1+j}\|^2}\right) \tag{39}$$

We now compute

$$\|T_\phi(f_1) - T_\phi(f_2)\|^2 = \int_{\mathbb{D}} |T_\phi(f_1) - T_\phi(f_2)|^2 d\lambda = \tag{40}$$

$$\int_{\mathbb{D}} \sum_{i=0}^{\infty} z^{2i+1}\left(1 - \frac{\|z^{2i+2}\|^2}{\|z^{2i+1}\|^2}\right)\overline{\sum_{j=0}^{\infty} z^{2j+1}\left(1 - \frac{\|z^{2j+2}\|^2}{\|z^{2j+1}\|^2}\right)} d\lambda(z) = \tag{41}$$

$$\int_{\mathbb{D}} \sum_{i=0}^{\infty} z^{2i+1}\left(1-\frac{\|z^{2i+2}\|^2}{\|z^{2i+1}\|^2}\right) \sum_{j=0}^{\infty} \overline{z^{2j+1}}\left(1-\frac{\|z^{2j+2}\|^2}{\|z^{2j+1}\|^2}\right) d\lambda(z) \tag{42}$$

If we multiply out the series and assume that we can interchange summation and integration, then we obtain

$$\sum_{i=0}^{\infty}\left(1-\frac{\|z^{2i+2}\|^2}{\|z^{2i+1}\|^2}\right)^2 \int_{\mathbb{D}} |z|^{4i+2} d\lambda(z) = \sum_{i=0}^{\infty}\left(1-\frac{\|z^{2i+2}\|^2}{\|z^{2i+1}\|^2}\right)^2 \|z^{2i+1}\|^2 \tag{43}$$

Let us estimate the difference between f_n and f_m for different values of n and m. Set $n=1$. Then

$$\|f_1-f_2\|^2 = \int_{\mathbb{D}} |f_1-f_2|^2 d\lambda = \int_{\mathbb{D}} \left|\frac{1}{1-z}-\frac{1}{1-z^2}\right|^2 \lambda(z) dA(z) = \tag{44}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|\frac{1}{1-re^{i\theta}}-\frac{1}{1-(re^{i\theta})^2}\right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0496239 \tag{45}$$

$$\|f_1-f_3\|^2 = \int_{\mathbb{D}} |f_1-f_3|^2 d\lambda = \int_{\mathbb{D}} \left|\frac{1}{1-z}-\frac{1}{1-z^3}\right|^2 \lambda(z) dA(z) = \tag{46}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|\frac{1}{1-re^{i\theta}}-\frac{1}{1-(re^{i\theta})^3}\right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0618934 \tag{47}$$

$$\|f_1-f_4\|^2 = \int_{\mathbb{D}} |f_1-f_4|^2 d\lambda = \int_{\mathbb{D}} \left|\frac{1}{1-z}-\frac{1}{1-z^4}\right|^2 \lambda(z) dA(z) = \tag{48}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|\frac{1}{1-re^{i\theta}}-\frac{1}{1-(re^{i\theta})^4}\right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0664208 \tag{49}$$

$$\|f_1-f_5\|^2 = \int_{\mathbb{D}} |f_1-f_5|^2 d\lambda = \int_{\mathbb{D}} \left|\frac{1}{1-z}-\frac{1}{1-z^5}\right|^2 \lambda(z) dA(z) = \tag{50}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|\frac{1}{1-re^{i\theta}}-\frac{1}{1-(re^{i\theta})^5}\right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0684414 \tag{51}$$

$$\|f_1-f_6\|^2 = \int_{\mathbb{D}} |f_1-f_6|^2 d\lambda = \int_{\mathbb{D}} \left|\frac{1}{1-z}-\frac{1}{1-z^6}\right|^2 \lambda(z) dA(z) = \tag{52}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|\frac{1}{1-re^{i\theta}}-\frac{1}{1-(re^{i\theta})^6}\right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0694552 \tag{53}$$

$$\|f_1-f_7\|^2 = \int_{\mathbb{D}} |f_1-f_7|^2 d\lambda = \int_{\mathbb{D}} \left|\frac{1}{1-z}-\frac{1}{1-z^7}\right|^2 \lambda(z) dA(z) = \tag{54}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|\frac{1}{1-re^{i\theta}}-\frac{1}{1-(re^{i\theta})^7}\right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.070059 \tag{55}$$

$$\|f_1-f_8\|^2 = \int_{\mathbb{D}} |f_1-f_8|^2 d\lambda = \int_{\mathbb{D}} \left|\frac{1}{1-z}-\frac{1}{1-z^8}\right|^2 \lambda(z) dA(z) = \tag{56}$$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|\frac{1}{1-re^{i\theta}}-\frac{1}{1-(re^{i\theta})^8}\right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0703236 \tag{57}$$

$$\|f_1 - f_9\|^2 = \int_{\mathbb{D}} |f_1 - f_9|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z} - \frac{1}{1-z^9} \right|^2 \lambda(z) dA(z) = \tag{58}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-re^{i\theta}} - \frac{1}{1-(re^{i\theta})^9} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0705155 \tag{59}$$

$$\|f_1 - f_{10}\|^2 = \int_{\mathbb{D}} |f_1 - f_{10}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z} - \frac{1}{1-z^{10}} \right|^2 \lambda(z) dA(z) = \tag{60}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-re^{i\theta}} - \frac{1}{1-(re^{i\theta})^{10}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0706358 \tag{61}$$

Next, we let $n = 2$. Then we obtain

$$\|f_2 - f_3\|^2 = \int_{\mathbb{D}} |f_2 - f_3|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^3} \right|^2 \lambda(z) dA(z) = \tag{62}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^3} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0273926 \tag{63}$$

$$\|f_2 - f_4\|^2 = \int_{\mathbb{D}} |f_2 - f_4|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^4} \right|^2 \lambda(z) dA(z) = \tag{64}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^4} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0167966 \tag{65}$$

$$\|f_2 - f_5\|^2 = \int_{\mathbb{D}} |f_2 - f_5|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^5} \right|^2 \lambda(z) dA(z) = \tag{66}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^5} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0232063 \tag{67}$$

$$\|f_2 - f_6\|^2 = \int_{\mathbb{D}} |f_2 - f_6|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^6} \right|^2 \lambda(z) dA(z) = \tag{68}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^6} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0198311 \tag{69}$$

$$\|f_2 - f_7\|^2 = \int_{\mathbb{D}} |f_2 - f_7|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^7} \right|^2 \lambda(z) dA(z) = \tag{70}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^7} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0220185 \tag{71}$$

$$\|f_2 - f_8\|^2 = \int_{\mathbb{D}} |f_2 - f_8|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^8} \right|^2 \lambda(z) dA(z) = \tag{72}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^8} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0206997 \tag{73}$$

$$\|f_2 - f_9\|^2 = \int_{\mathbb{D}} |f_2 - f_9|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^9} \right|^2 \lambda(z) dA(z) = \tag{74}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^9} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0215970 \tag{75}$$

$$\|f_2 - f_{10}\|^2 = \int_{\mathbb{D}} |f_2 - f_{10}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^2} - \frac{1}{1-z^{10}} \right|^2 \lambda(z) dA(z) = \tag{76}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^2} - \frac{1}{1-(re^{i\theta})^{10}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0210119 \tag{77}$$

Now, we let $n = 3$. Then

$$\|f_3 - f_4\|^2 = \int_{\mathbb{D}} |f_3 - f_4|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^4} \right|^2 \lambda(z) dA(z) = \tag{78}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^4} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0132161 \tag{79}$$

$$\|f_3 - f_5\|^2 = \int_{\mathbb{D}} |f_3 - f_5|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^5} \right|^2 \lambda(z) dA(z) = \tag{80}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^5} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.0113472 \tag{81}$$

$$\|f_3 - f_6\|^2 = \int_{\mathbb{D}} |f_3 - f_6|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^6} \right|^2 \lambda(z) dA(z) = \tag{82}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^6} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00756152 \tag{83}$$

$$\|f_3 - f_7\|^2 = \int_{\mathbb{D}} |f_3 - f_7|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^7} \right|^2 \lambda(z) dA(z) = \tag{84}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^7} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00985885 \tag{85}$$

$$\|f_3 - f_8\|^2 = \int_{\mathbb{D}} |f_3 - f_8|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^8} \right|^2 \lambda(z) dA(z) = \tag{86}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^8} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00955057 \tag{87}$$

$$\|f_3 - f_9\|^2 = \int_{\mathbb{D}} |f_3 - f_9|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^9} \right|^2 \lambda(z) dA(z) = \tag{88}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^9} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00862206 \tag{89}$$

$$\|f_3 - f_{10}\|^2 = \int_{\mathbb{D}} |f_3 - f_{10}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{10}} \right|^2 \lambda(z) dA(z) = \tag{90}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{10}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00924493 \tag{91}$$

$$\|f_3 - f_{11}\|^2 = \int_{\mathbb{D}} |f_3 - f_{11}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{11}} \right|^2 \lambda(z) dA(z) = \tag{92}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{11}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00916831 \tag{93}$$

$$\|f_3 - f_{12}\|^2 = \int_{\mathbb{D}} |f_3 - f_{12}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{12}} \right|^2 \lambda(z) dA(z) = \tag{94}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{12}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00887161 \tag{95}$$

$$\|f_3 - f_{13}\|^2 = \int_{\mathbb{D}} |f_3 - f_{13}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{13}} \right|^2 \lambda(z) dA(z) = \tag{96}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{13}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00908275 \tag{97}$$

$$\|f_3 - f_{14}\|^2 = \int_{\mathbb{D}} |f_3 - f_{14}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{14}} \right|^2 \lambda(z) dA(z) = \tag{98}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{14}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00905881 \tag{99}$$

$$\|f_3 - f_{15}\|^2 = \int_{\mathbb{D}} |f_3 - f_{15}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{15}} \right|^2 \lambda(z) dA(z) = \tag{100}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{15}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00894760 \tag{101}$$

$$\|f_3 - f_{16}\|^2 = \int_{\mathbb{D}} |f_3 - f_{16}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{16}} \right|^2 \lambda(z) dA(z) = \tag{102}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{16}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00902994 \tag{103}$$

$$\|f_3 - f_{17}\|^2 = \int_{\mathbb{D}} |f_3 - f_{17}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{17}} \right|^2 \lambda(z) dA(z) = \tag{104}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{17}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00902125 \tag{105}$$

$$\|f_3 - f_{18}\|^2 = \int_{\mathbb{D}} |f_3 - f_{18}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{18}} \right|^2 \lambda(z) dA(z) = \tag{106}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{18}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00897478 \tag{107}$$

$$\|f_3 - f_{19}\|^2 = \int_{\mathbb{D}} |f_3 - f_{19}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{19}} \right|^2 \lambda(z) dA(z) = \tag{108}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{19}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00901021 \tag{109}$$

$$\|f_3 - f_{20}\|^2 = \int_{\mathbb{D}} |f_3 - f_{20}|^2 d\lambda = \int_{\mathbb{D}} \left| \frac{1}{1-z^3} - \frac{1}{1-z^{20}} \right|^2 \lambda(z) dA(z) = \tag{110}$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| \frac{1}{1-(re^{i\theta})^3} - \frac{1}{1-(re^{i\theta})^{20}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2} \right) dr d\theta \approx 0.00900671 \tag{111}$$

$$\tag{112}$$