Research

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During the first week, the mentors introduced the projects to us, and we were asked to rank the projects according to our preferences. I chose the projects regarding asymmetric colorings of graphs, ribbon knots, and Toeplitz operators on Bergman spaces, and I began reading the corresponding materials for each of these projects. I spent the second week reading Axler and Zheng's paper. To discuss their main result, I must first introduce some notation. The Bergman space of the unit disk (which we denote L^2_a) is defined to be the set of all analytic functions in $L^2(\mathbb{D}, dA)$ (which is the set of all square-integrable functions on the unit disk). It can be shown that L^2_a is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$. As such, there exists an orthogonal projection operator $P : L^2(\mathbb{D}, dA) \to L^2_a$. Now, we define the Toeplitz operator $T_\phi : L^2_a \to L^2_a$ with symbol $\phi \in L^\infty(\mathbb{D}, dA)$ as follows: $T_\phi(f) = P(\phi f)$. Next, we define the Bergman reproducing kernel: for any $z \in \mathbb{D}$, the corresponding Bergman reproducing kernel is the function $K_z \in L^2_a$ such that $f(z) = \langle f, K_z \rangle$ for all $f \in L^2_a$. The normalized Bergman reproducing kernel is the function $k_z := K_z/\|K_z\|$. Finally, for any bounded operator S on L^2_a, we define its Berezin transform $\tilde{S}$ as follows: $\tilde{S}(z) = \langle Sk_z, k_z \rangle$. Now that we have the definitions, we may state the main result of Axler and Zheng's paper: a Toeplitz operator T_u is compact if and only if its Berezin transform vanishes on the boundary of the unit disk. I spent a great deal of time reading this paper and proving all the results to myself, so I will now describe this: First they define the function $\varphi_z : D \to D$ as follows:

$$\varphi_z(w) = \frac{z - w}{1 - \overline{z}w}$$

Then they claim that $\varphi_z \circ \varphi_z$ is the identity. To show this, we compute

$$\varphi_z(\varphi_z(w)) = \frac{z - \varphi_z(w)}{1 - \overline{z}\varphi_z(w)} = \frac{z - \frac{z-w}{1-\overline{z}w}}{1 - \overline{z}\frac{z-w}{1-\overline{z}w}} = \frac{z(1 - \overline{z}w) - (z - w)}{1 - \overline{z}w - \overline{z}(z - w)} = \frac{z - |z|^2 w - z + w}{1 - \overline{z}w - |z|^2 + \overline{z}w}$$

$$= \frac{w - |z|^2 w}{1 - |z|^2} = \frac{w(1 - |z|^2)}{1 - |z|^2} = w$$

Next, they define the operator $U_z : L^2_a \to L^2_a$ by $U_z f = (f \circ \varphi_z)\varphi_z'$. They claim that this operator is unitary. We proceed to verify this:

$$\langle U_z f, U_z f \rangle = \int_D U_z f(w) \cdot \overline{U_z f(w)} dA(w) = \int_D f(\varphi_z(w))\varphi_z'(w)\overline{f(\varphi_z(w))\varphi_z'(w)} dA(w)$$

$$= \int_D |f(\varphi_z(w))|^2 |\varphi_z'(w)|^2 dA(w)$$

We make the substitution $\lambda = \varphi_z(w)$. The domain remains the same because φ_z is an automorphism of the unit disk. We also have

$$dA(\lambda) = |\varphi_z'(w)|^2 dA(w)$$

so that we obtain

$$\int_D |f(\varphi_z(w))|^2 |\varphi_z'(w)|^2 dA(w) = \int_D |f(\lambda)|^2 dA(\lambda) = \int_D f(\lambda) \cdot \overline{f(\lambda)} dA(\lambda) = \langle f, f \rangle$$

Thus, we deduce that U_z is a unitary operator (so $U_z^* = U_z^{-1}$). Next, they claim that $U_z^{-1} = U_z$. We have

$$U_z(U_z(f)) = U_z((f \circ \varphi_z)\varphi_z') = (((f \circ \varphi_z)\varphi_z') \circ \varphi_z)\varphi_z'$$

Now, we compute

$$(((f \circ \varphi_z)\varphi_z') \circ \varphi_z(w))\varphi_z'(w) = (f(\varphi_z(\varphi_z(w)))\varphi_z'(\varphi_z(w)))(\varphi_z'(w)) = f(w) \cdot \frac{1}{\varphi_z'(w)} \cdot \varphi_z'(w) = f(w)$$

Thus we deduce that $U_z U_z f = f$ for all f so that $U_z^* = U_z^{-1} = U_z$ and U_z is actually self-adjoint. The authors define the Berezin transform of a symbol u to be the Berezin transform of its Toeplitz operator T_u; that is, we have $\tilde{u} = \widetilde{T_u}$. For any orthogonal projection P, we have that $P = P^*$. Thus, we deduce that

$$\tilde{u}(z) = \widetilde{T_u}(z) = \langle T_u k_z, k_z \rangle = \langle P(uk_z), k_z \rangle = \langle uk_z, P^*(k_z) \rangle = \langle uk_z, P(k_z) \rangle = \langle uk_z, k_z \rangle$$

It can be shown that

$$K_z(\lambda) = \frac{1}{(1 - \overline{z}\lambda)^2}$$

Furthermore, we note that

$$\|K_z\|^2 = \langle K_z, K_z \rangle = K_z(z) = \frac{1}{(1 - |z|^2)^2}$$

so that

$$\|K_z\| = \frac{1}{1 - |z|^2}$$

Thus, we have

$$k_z = \frac{K_z}{\|K_z\|} = (1 - |z|^2)K_z$$

or

$$k_z(\lambda) = (1 - |z|^2)K_z(\lambda) = \frac{1 - |z|^2}{(1 - \overline{z}\lambda)^2}$$

Next, we claim that

$$|\varphi_z'(\lambda)| = |k_z(\lambda)|$$

To show this, we note that

$$\varphi_z'(\lambda) = \frac{-(1 - \overline{z}\lambda) - (z - \lambda)(-\overline{z})}{(1 - \overline{z}\lambda)^2} = \frac{|z|^2 - 1}{(1 - \overline{z}\lambda)^2}$$

so that

$$|\varphi_z'(\lambda)| = \left|\frac{|z|^2-1}{(1-\overline{z}\lambda)^2}\right| = \frac{1-|z|^2}{|1-\overline{z}\lambda|^2} = |k_z(\lambda)|$$

Now we note that

$$\tilde{u}(z) = \langle uk_z, k_z\rangle = \int_D u(w)|k_z(w)|^2 dA(w)$$

In the paper, they make the change of variables $\lambda = \varphi_z(w)$. Taking derivatives yields $dA(\lambda) = |\varphi_z'(w)|^2 dA(w) = |k_z(w)|^2 dA(w)$ so that

$$dA(w) = \frac{dA(\lambda)}{|k_z(w)|^2}$$

Since $\varphi_z \circ \varphi_z$ is the identity, we have $w = \varphi_z(\lambda)$. Thus we find that

$$\tilde{u}(z) = \int_D u(w)|k_z(w)|^2 dA(w) = \int_D u(\varphi_z(\lambda)) dA(\lambda)$$

The paper then discusses how the theorem can be used to prove several previously known results. First, we define Hankel operators. The Hankel operator H_u is defined as follows:

$$H_u f = (I-P)(uf)$$

The authors state that the main result of the paper can be used to show that H_u is compact iff $\|H_u k_z\|_2 \to 0$ as $z \to \partial D$ iff $\|u\circ\varphi_z - P(u\circ\varphi_z)\|_2 \to 0$ as $z \to \partial D$. Before proving this, we show that

$$H_{\overline{f}}^* H_g = T_{fg} - T_f T_g$$

First, we compute what $H_{\overline{f}}^*$ is. Let $a \in (L_a^2)^\perp$ and $b \in L_a^2$. Then, we have

$$\langle H_{\overline{f}}^* a, b\rangle = \langle a, H_{\overline{f}} b\rangle = \langle a, (I-P)(\overline{f}b)\rangle = \langle a, \overline{f}b - P(\overline{f}b)\rangle = \langle a, \overline{f}b\rangle - \langle a, P(\overline{f}b)\rangle$$

$$= \langle fa, b\rangle - \langle Pa, \overline{f}b\rangle = \langle fa, b\rangle$$

Thus we deduce that $H_{\overline{f}}^* a = fa$. Now, we let $b \in L_a^2$. We obtain

$$H_{\overline{f}}^* H_g(b) = P(H_{\overline{f}}^* H_g(b)) = P(H_{\overline{f}}^*(gb - P(gb))) = P(fgb - fP(gb)) = P(fgb) - P(fP(gb))$$

$$= T_{fg}(b) - T_f \circ T_g(b)$$

thus proving that

$$H_{\overline{f}}^* H_g = T_{fg} - T_f T_g$$

so that we obtain

$$H_u^* H_u = H_{\overline{\overline{u}}}^* H_u = T_{\overline{u}u} - T_{\overline{u}} T_u = T_{|u|^2} - T_{\overline{u}} T_u$$

Then, the authors state that H_u being compact is equivalent to $H_u^* H_u$ being compact and that this occurs if and only if

$$\widetilde{H_u^* H_u}(z) \to 0$$

as $z \to \partial D$ (by the theorem they will prove). Notice that

$$\widetilde{H_u^* H_u}(z) = \langle H_u^* H_u k_z, k_z\rangle = \langle H_u k_z, H_u k_z\rangle = \|H_u k_z\|^2$$

so H_u is compact if and only if $\|H_u k_z\| \to 0$ as $z \to \partial D$. Next, we note that

$$S_z = U_z S U_z = U_z (H_u^* H_u) U_z = U_z (T_{|u|^2} - T_{\overline{u}} T_u) U_z = U_z T_{|u|^2} U_z - U_z T_{\overline{u}} U_z U_z T_u U_z$$

$$= T_{|u|^2 \circ \varphi_z} - T_{\overline{u} \circ \varphi_z} T_{u \circ \varphi_z} = T_{|u \circ \varphi_z|^2} - T_{\overline{u \circ \varphi_z}} T_{u \circ \varphi_z} = H_{u \circ \varphi_z}^* H_{u \circ \varphi_z}$$

Thus we find that H_u is compact iff $\|H_{u \circ \varphi_z}^* H_{u \circ \varphi_z} 1\| \to 0$ as $z \to \partial D$. Now, we should note that

$$\|H_{u\circ\varphi_z} 1\|^2 = \langle H_{u\circ\varphi_z} 1, H_{u\circ\varphi_z} 1\rangle = \langle H_{u\circ\varphi_z}^* H_{u\circ\varphi_z} 1, 1\rangle \le \|H_{u\circ\varphi_z}^* H_{u\circ\varphi_z} 1\| \|1\| = \|H_{u\circ\varphi_z}^* H_{u\circ\varphi_z} 1\| \le$$

$$\|H_{u\circ\varphi_z}^*\| \|H_{u\circ\varphi_z} 1\|$$

This inequality shows that if

$$\|H_{u\circ\varphi_z}^* H_{u\circ\varphi_z} 1\|$$

goes to 0, then

$$\|H_{u\circ\varphi_z} 1\|$$

also goes to 0. It also shows that if

$$\|H_{u\circ\varphi_z} 1\|$$

goes to 0, then

$$\|H_{u\circ\varphi_z}^* H_{u\circ\varphi_z} 1\|$$

goes to 0. However, we note that

$$H_{u\circ\varphi_z} 1 = (I - P)(u \circ \varphi_z 1) = u \circ \varphi_z - P(u \circ \varphi_z)$$

Thus we find that H_u is compact if and only if $\|u \circ \varphi_z - P(u \circ \varphi_z)\| \to 0$ as $z \to \partial D$. Now we compute a power series for the normalized Bergman reproducing kernel k_z. Note that

$$k_z(w) = \frac{1 - |z|^2}{(1 - \overline{z} w)^2}$$

Since

$$\frac{1}{1 - \overline{z} w} = 1 + \overline{z} w + (\overline{z} w)^2 + (\overline{z} w)^3 + \cdots$$

we have

$$\frac{1}{(1 - \overline{z} w)^2} = \frac{1}{1 - \overline{z} w} \cdot \frac{1}{1 - \overline{z} w} = (1 + \overline{z} w + (\overline{z} w)^2 + (\overline{z} w)^3 + \cdots)(1 + \overline{z} w + (\overline{z} w)^2 + (\overline{z} w)^3 + \cdots)$$

$$= 1 + 2\overline{z} w + 3(\overline{z} w)^2 + 4(\overline{z} w)^3 + \cdots$$

so that

$$k_z(w) = (1 - |z|^2) \sum_{m=0}^{\infty} (m + 1) \overline{z}^m w^m$$

Applying S to both sides of this equation yields

$$Sk_z(w) = (1-|z|^2)\sum_{m=0}^{\infty}(m+1)\overline{z}^m Sw^m$$

Taking the inner product of Sk_z and k_z yields

$$\langle Sk_z, k_z\rangle = (1-|z|^2)^2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+1)(n+1)\langle Sw^m, w^n\rangle \overline{z}^m z^n$$

Now we can consider some operators. The first operator that the authors bring up in the paper is the operator S defined by

$$S\left(\sum_{n=0}^{\infty} a_n w^n\right) = \sum_{n=0}^{\infty} a_{2^n} w^{2^n}$$

or

$$S(a_0w^0 + a_1w^1 + a_2w^2 + a_3w^3 + a_4w^4 + \cdots) = a_1w^1 + a_2w^2 + a_4w^4 + \cdots$$

We claim that S is self-adjoint. To see this, we note that

$$\begin{aligned}
&\langle S(a_0w^0 + a_1w^1 + a_2w^2 + a_3w^3 + a_4w^4 + \cdots), b_0w^0 + b_1w^1 + b_2w^2 + b_3w^3 + b_4w^4 + \cdots\rangle \\
&= \langle a_1w^1 + a_2w^2 + a_4w^4 + \cdots, b_0w^0 + b_1w^1 + b_2w^2 + b_3w^3 + b_4w^4 + \cdots\rangle \\
&= \frac{1}{n+1}(a_1\overline{b_1} + a_2\overline{b_2} + a_4\overline{b_4} + \cdots)
\end{aligned}$$

and that

$$\begin{aligned}
&\langle a_0w^0 + a_1w^1 + a_2w^2 + a_3w^3 + a_4w^4 + \cdots, S(b_0w^0 + b_1w^1 + b_2w^2 + b_3w^3 + b_4w^4 + \cdots)\rangle \\
&= \langle a_0w^0 + a_1w^1 + a_2w^2 + a_3w^3 + a_4w^4 + \cdots, b_1w^1 + b_2w^2 + b_4w^4 + \cdots\rangle \\
&= \frac{1}{n+1}(a_1\overline{b_1} + a_2\overline{b_2} + a_4\overline{b_4} + \cdots)
\end{aligned}$$

Thus we deduce that $S = S^*$. Notice that S is also a projection. We note that

$$\|Sk_z\|^2 = \langle Sk_z, Sk_z\rangle = \langle S^*Sk_z, k_z\rangle = \langle SSk_z, k_z\rangle = \langle Sk_z, k_z\rangle = \tilde{S}(z)$$

Next, we note a lemma that they state in the paper, which essentially says that

$$\tilde{S}\circ\varphi_z = \widetilde{S_z}$$

First, I note that

$$\tilde{S}\circ\varphi_z(w) = \tilde{S}(\varphi_z(w)) = \langle Sk_{\varphi_z(w)}, k_{\varphi_z(w)}\rangle$$

and that

$$\widetilde{S_z}(w) = \langle S_z k_w, k_w\rangle = \langle U_zSU_zk_w, k_w\rangle = \langle SU_zk_w, U_zk_w\rangle$$

We note that

$$U_z k_w = (k_w \circ \varphi_z)\varphi_z'$$

To show this, I appeal to the formula from office hours:

$$K_U(z, \overline{\zeta}) = \det Df(z)\overline{\det Df(\zeta)} K_V(f(z), \overline{f(\zeta)})$$

Here I take $U = V = \mathbb{D}$, $f = \varphi_z$, $z = w$ and $\zeta = \varphi_z(v)$. Thus the above formula becomes

$$K_{\mathbb{D}}(w, \overline{\varphi_z(v)}) = \det D\varphi_z(w)\overline{\det D\varphi_z(\varphi_z(v))} K_{\mathbb{D}}(\varphi_z(w), \overline{v})$$

Then, I compute

$$K_{\mathbb{D}}(w, \overline{\varphi_z(v)}) = \overline{K_w(\varphi_z(v))} = \frac{\overline{k_w(\varphi_z(v))}}{1 - |w|^2}$$

$$K_{\mathbb{D}}(\varphi_z(w), \overline{v}) = \overline{K_{\varphi_z(w)}(v)} = \overline{\frac{k_{\varphi_z(w)}(v)}{1 - |\varphi_z(w)|^2}}$$

$$\det D\varphi_z(w) = \varphi_z'(w) = \frac{|z|^2 - 1}{(1 - \overline{z}w)^2}$$

and finally

$$\overline{\det D\varphi_z(\varphi_z(v))} = \overline{\varphi_z'(\varphi_z(v))} = \overline{\frac{1}{\varphi_z'(v)}}$$

Putting it all together, I obtain

$$\frac{\overline{k_w(\varphi_z(v))}}{1 - |w|^2} = \frac{|z|^2 - 1}{(1 - \overline{z}w)^2} \cdot \overline{\frac{1}{\varphi_z'(v)}} \cdot \overline{\frac{k_{\varphi_z(w)}(v)}{1 - |\varphi_z(w)|^2}}$$

I take the complex conjugate to obtain

$$\frac{k_w(\varphi_z(v))}{1 - |w|^2} = \frac{|z|^2 - 1}{(1 - z\overline{w})^2} \cdot \frac{1}{\varphi_z'(v)} \cdot \frac{k_{\varphi_z(w)}(v)}{1 - |\varphi_z(w)|^2}$$

Using the fact that

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\overline{w}|^2}$$

the above equation becomes

$$\frac{k_w(\varphi_z(v))}{1 - |w|^2} = \frac{|z|^2 - 1}{(1 - z\overline{w})^2} \cdot \frac{1}{\varphi_z'(v)} \cdot \frac{k_{\varphi_z(w)}(v)}{\frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}}$$

Simplification yields

$$k_w(\varphi_z(v))\varphi_z'(v) = k_{\varphi_z(w)}(v) \cdot -\frac{|1 - z\overline{w}|^2}{(1 - z\overline{w})^2}$$

Now, we note that

$$\langle SU_z k_w, U_z k_w \rangle = \int_{\mathbb{D}} S(U_z k_w(v))\overline{U_z k_w(v)} dA(v) = \int_{\mathbb{D}} S(k_w(\varphi_z(v))\varphi_z'(v)) \cdot \overline{k_w(\varphi_z(v))\varphi_z'(v)} dA(v)$$

$$= \int_{\mathbb{D}} Sk_{\varphi_z(w)}(v) \cdot -\frac{|1-z\overline{w}|^2}{(1-z\overline{w})^2} \cdot \overline{k_{\varphi_z(w)}(v)} \cdot \overline{-\frac{|1-z\overline{w}|^2}{(1-z\overline{w})^2}} dA(v) = \int_{\mathbb{D}} Sk_{\varphi_z(w)}(v)\overline{k_{\varphi_z(w)}(v)} dA(v)$$

$$= \langle Sk_{\varphi_z(w)}, k_{\varphi_z(w)} \rangle$$

Let the symbol ϕ be defined by $\phi(z) = z^a \overline{z}^b$, and let the function f be defined by $f(z) = z^n \overline{z}^m$. We compute

$$T_\phi(f) = P(\phi f) = \int_{\mathbb{D}} B_\lambda(z,w) f(w)\phi(w)\lambda(w) dA(w)$$

where

$$\lambda(w) = \exp\left(-\frac{1}{1-|w|^2}\right)$$

We write

$$B_\lambda(z,w) = \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2}$$

so that we obtain

$$\int_{\mathbb{D}} B_\lambda(z,w) f(w)\phi(w)\lambda(w) dA(w) = \sum_{j=0}^{\infty} \frac{z^j}{\|z^j\|^2} \int_{\mathbb{D}} w^{j+a+n}\overline{w}^{b+m}\lambda(w) dA(w)$$

Because the weight λ is radial, we note that the integral is only nonzero when $j+a+n = b+m$, or when $j = b+m-(a+n)$. In order for this to occur, we note that $b+m-(a+n)$ must be nonnegative. So we have

$$\frac{z^{b+m-a-n}}{\|z^{b+m-a-n}\|^2} \int_{\mathbb{D}} w^{b+m}\overline{w}^{b+m}\lambda(w) dA(w) = z^{b+m-a-n} \frac{\|z^{b+m}\|^2}{\|z^{b+m-a-n}\|^2}$$

where the norm is taken in the weighted Bergman space. I first assume that $\phi \equiv 1$ on $\mathbb{D}$. Then we have

$$T_\phi = T_1 = P$$

Thus we are asking whether or not P is compact. An operator S is compact if and only if S takes bounded sequences to sequences with converging subsequences. Let us consider some bounded sequences. First, we have $\{z^n\}_{n\in\mathbb{N}}$. We verify that it is bounded. Note that

$$\|z^n\|^2 = \int_{\mathbb{D}} |z|^{2n} d\lambda(z) = \int_{\mathbb{D}} |z|^{2n} \exp\left(-\frac{1}{1-|z|^2}\right) dA(z) = 2\int_0^1 r^{2n+1} \exp\left(-\frac{1}{1-r^2}\right) dr$$

Let us compute this integral for some specific values of n. We have

$$\int_0^1 r \exp\left(-\frac{1}{1-r^2}\right) dr \approx 0.0742$$

$$\int_0^1 r^3 \exp\left(-\frac{1}{1-r^2}\right) dr \approx 0.0194$$

$$\int_0^1 r^5 \exp\left(- \frac{1}{1-r^2}\right) dr \approx 0.0075$$
$$\int_0^1 r^7 \exp\left(- \frac{1}{1-r^2}\right) dr \approx 0.0035$$
$$\int_0^1 r^9 \exp\left(- \frac{1}{1-r^2}\right) dr \approx 0.0018$$

From this, it is clear that the sequence $\{z^n\}_{n\in\mathbb{N}}$ is bounded and even converges. Now, we compute $P(f_m)$ as follows:

$$P(z^m) = \int_{\mathbb{D}} \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{|z^n|^2} w^m \lambda(w) dA(w) = \frac{z^m}{\|z^m\|^2} \int_{\mathbb{D}} |w|^{2m} \lambda(w) dA(w) = \frac{z^m}{\|z^m\|^2} \cdot \|z^m\|^2 = z^m$$

This is because z^m is analytic. Thus $\{P(z^n)\}_{n\in\mathbb{N}}$ is convergent. Next I try $\phi(z) = 1 - |z|^2$. I first consider the sequence $\{z^n\}_{n\in\mathbb{N}}$. I note that

$$T_\phi(z^m) = P(\phi z^m) = \int_D \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{\|z^n\|^2} w^m (1-|w|^2) \lambda(w) dA(w) = \frac{z^m}{\|z^m\|^2} \int_D (|w|^{2m} - |w|^{2m+2}) \lambda(w) dA(w)$$

Now we compute

$$\int_D (|w|^{2m} - |w|^{2m+2}) \lambda(w) dA(w) = 2\int_0^1 (r^{2m+1} - r^{2m+3}) \exp\left(- \frac{1}{1-r^2}\right) dr$$

Let us compute this integral for several values of m.

$$2\int_0^1 (r^1 - r^3) \exp\left(- \frac{1}{1-r^2}\right) dr \approx 0.109$$
$$2\int_0^1 (r^3 - r^5) \exp\left(- \frac{1}{1-r^2}\right) dr \approx 0.0236$$
$$2\int_0^1 (r^5 - r^7) \exp\left(- \frac{1}{1-r^2}\right) dr \approx 0.008$$

In any event, it is clear that these values are less than the corresponding norms of z^m. Thus, we find that $\|T_\phi(z^m)\| \le \|z^m\|$ for all $m \in \mathbb{N}$; that is the sequence $\{T_\phi(z^n)\}_{n\in\mathbb{N}}$ converges to the function 0.

Next I consider the sequence $\{f_n\}$ where

$$f_n(w) = \frac{1}{1-|w|^n}$$

We note that

$$\|f_n\|^2 = 2\int_0^1 \frac{r}{(1-r^n)^2} \exp\left(\frac{-1}{1-r^2}\right) dr$$

Computing several values of n, we do see that this sequence is bounded. Next we compute

$$T_\phi(f_n) = P(\phi f_n) = \int_D B_\lambda(z,w) f_n(w)\phi(w)dA(w)$$

Notice that

$$f_n(w) = \frac{1}{1-|w|^n} = 1 + |w|^n + |w|^{2n} + \cdots$$

$$\phi(w) = (1-|w|^2)$$

$$B_\lambda(z,w) = \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2}$$

Thus we find that

$$T_\phi(f_n) = 1$$

for all n so that the sequence $\{T_\phi(f_n)\}$ is convergent.

Now we consider the sequence $f_m(z) = \overline{z}^m z^a$. Here we have

$$T_\phi(z^m) = P(\phi z^m) = \int_{\mathbb{D}} \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2} \overline{w}^m w^a (1 - w\overline{w})\lambda(w)dA(w)$$

Notice that this is nonzero only when $j+m=a$. That is, it is nonzero when $j = a-m$. In order for this to be true, we must have $m \leq a$. Thus it is nonzero for only finitely many m, so it too converges

Next I try the sequence $\{z^n \overline{z}^a\}$. We compute

$$T_\phi(f_n)(z) = P(\phi f_n)(z) = \int_D \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2} w^n \overline{w}^a (1 - w\overline{w})\lambda(w)dA(w)$$

We note that the integral is nonzero only when $n = a+j$. That is, we have $j = n-a \geq 0$. So we obtain

$$\frac{z^{n-a}}{\|z^{n-a}\|^2}(\|z^n\|^2 - \|z^{n+1}\|^2)$$

First I consider the sequence $f_n(z) = \sin(nz)$. Notice that

$$\int_D \sin(nz)dA(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \sin(nre^{i\theta})drd\theta = 0$$

where the equality comes from Wolfram Alpha. If we want to determine if it is bounded, we must consider the integral

$$\|\sin(nz)\|^2 = \int_D |\sin(nz)|^2 \exp\left(-\frac{1}{1-|z|^2}\right)dA(z)$$

For each n, it can be seen that the integral converges. However, we don't know if the sequence is bounded. First, let us consider the sequence without the exponential weight.

$$\|\sin(nz)\|^2 = \int_D |\sin(nz)|^2 dA(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |\sin(nre^{i\theta})|^2 drd\theta$$

We compute this for several values of n.

$$\int_0^{2\pi}\int_0^1 |\sin(re^{i\theta})|^2 drd\theta \approx 2.119$$

$$\int_0^{2\pi}\int_0^1 |\sin(2re^{i\theta})|^2 drd\theta \approx 10.0142$$

$$\int_0^{2\pi}\int_0^1 |\sin(3re^{i\theta})|^2 drd\theta \approx 39.448$$

$$\int_0^{2\pi}\int_0^1 |\sin(4re^{i\theta})|^2 drd\theta \approx 181.833$$

$$\int_0^{2\pi}\int_0^1 |\sin(5re^{i\theta})|^2 drd\theta \approx 939.957$$

Now we consider the sequence with the weight:

$$\int_D |\sin(nz)|^2 \lambda(z) dA(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 |\sin(nre^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta$$

Let us compute this integral for several values of n:

$$\int_0^{2\pi}\int_0^1 |\sin(re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta \approx 0.221$$

$$\int_0^{2\pi}\int_0^1 |\sin(2re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta \approx 0.94$$

$$\int_0^{2\pi}\int_0^1 |\sin(3re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta \approx 2.673$$

$$\int_0^{2\pi}\int_0^1 |\sin(4re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta \approx 7.868$$

$$\int_0^{2\pi}\int_0^1 |\sin(5re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta \approx 26.293$$

$$\int_0^{2\pi}\int_0^1 |\sin(6re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta \approx 97.7814$$

$$\int_0^{2\pi}\int_0^1 |\sin(7re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right) drd\theta \approx 393.24$$

$$\int_0^{2\pi}\int_0^1 |\sin(8re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 1677.21$$
$$\int_0^{2\pi}\int_0^1 |\sin(9re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 7488.43$$
$$\int_0^{2\pi}\int_0^1 |\sin(10re^{i\theta})|^2 \exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 34685.8$$

As we can see, the norm clearly diverges as n approaches ∞. Thus, this sequence is not unbounded, so we shouldn't bother to check whether the sequence of images contains a convergent subsequence. Now I consider functions with a singularity at the point $z = 1$. First, we may try $f(z) = \frac{1}{1-z}$. We compute the unweighted integral:

$$\int_D \frac{1}{1-z}dA(z) = \int_0^1\int_0^{2\pi} \frac{r}{1-re^{i\theta}}d\theta dr = \int_0^{2\pi}\int_0^1 \frac{r}{1-re^{i\theta}}drd\theta = \pi$$

Let us compare the different ways of evaluating this integral. First, we can try integrating with respect to θ so that we obtain

$$\int_0^{2\pi} \frac{r}{1-re^{i\theta}}d\theta$$

but it seems extremely difficult to find a closed form expression for the result of this integral. Next, we can try integrating with respect to r so that we obtain

$$\int_0^1 \frac{r}{1-re^{i\theta}}dr = -e^{-2i\theta}(e^{i\theta} + \log(1-e^{i\theta}))$$

Then, we can integrate the result with respect to θ to obtain

$$\int_0^{2\pi} -e^{-2i\theta}(e^{i\theta} + \log(1-e^{i\theta})d\theta = \pi$$

Now, let us compute the unweighted norm of f. We have

$$\|f\|^2 = \int_D \frac{1}{|1-z|^2}dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^2}drd\theta$$

We note that

$$1-re^{i\theta} = 1-r(\cos\theta + i\sin\theta) = (1-r\cos\theta) + i(r\sin\theta)$$

so that

$$|1-re^{i\theta}|^2 = (1-r\cos\theta)^2+(r\sin\theta)^2 = 1-2r\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta = 1-2r\cos\theta+r^2(\sin^2\theta+\cos^2\theta)$$

which equals

$$r^2 + 1 - 2r\cos\theta$$

Thus we deduce that

$$\int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^2}drd\theta = \int_0^{2\pi}\int_0^1 \frac{r}{r^2+1-2r\cos\theta}drd\theta$$

This integral appears to be extremely difficult to compute, so we may set $r = 1$. In this case, we obtain

$$\int_0^{2\pi} \frac{1}{1^2+1-2\cdot 1\cdot\cos\theta}d\theta = \frac{1}{2}\int_0^{2\pi}\frac{1}{1-\cos\theta}d\theta$$

which is infinite. Thus, we may deduce that

$$\|f\|^2 = +\infty$$

The function f is integrable, but its norm is infinite in the unweighted case. Next, let us try with the weight λ.

$$\int_D \frac{1}{1-z}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{1-re^{i\theta}}\cdot\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 0.4665$$

Now, we may try to compute the weighted norm of f. We obtain

$$\|f\|^2 = \int_D \frac{1}{|1-z|^2}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^2}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 0.6892$$

Thus, we notice significant differences between the weighted and unweighted cases. Of course, we have only computed the norms with the function

$$f(z) = \frac{1}{1-z}$$

Now, we may let

$$f_n(z) = \frac{1}{(1-z)^n}$$

so that $\{f_n\}$ is a sequence of functions that are analytic in the unit disk. We may try to compute the weighted norm of these functions as follows:

$$\|f_2\|^2 = \int_D \frac{1}{|1-z|^4}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^4}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 3.467$$

$$\|f_3\|^2 = \int_D \frac{1}{|1-z|^6}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^6}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 78.589$$

$$\|f_4\|^2 = \int_D \frac{1}{|1-z|^8}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^8}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 5497.79$$

$$\|f_5\|^2 = \int_D \frac{1}{|1-z|^{10}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^{10}}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 824121$$

$$\|f_6\|^2 = \int_D \frac{1}{|1-z|^{12}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^{12}}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 215887000$$

Here we can see that the sequence of norms is again increasing and that it appears to be unbounded. Thus, we don't need to consider the images of the functions. I am now considering trying the following sequence. First, we let

$$s_n = \sum_{j=1}^{n} \frac{1}{j^2}$$

and then we let

$$f_n(z) = \frac{1}{(1-z)^{s_n}}$$

First, we let $n = 1$ so that $s_n = 1$. Then, we have

$$f_1(z) = \frac{1}{1-z}$$

so that

$$\|f_1\|^2 = \int_D \frac{1}{|1-z|^2}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^2}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 0.6892$$

Next, we let $n = 2$ so that $s_n = 5/4$. Then we have

$$f_2(z) = \frac{1}{(1-z)^{5/4}}$$

so that

$$\|f_2\|^2 = \int_D \frac{1}{|1-z|^{5/2}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^{5/2}}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 0.897387$$

Now we let $s_3 = 1 + 1/4 + 1/9 = 49/36$. Thus, we have

$$f_3(z) = \frac{1}{(1-z)^{49/36}}$$

so that

$$\|f_3\|^2 = \int_D \frac{1}{|1-z|^{49/18}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^{49/18}}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 1.03767$$

Next, we note that $s_4 = 1 + 1/4 + 1/9 + 1/16 = 205/144$. Then, we have

$$\|f_4\|^2 = \int_D \frac{1}{|1-z|^{205/72}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^{205/72}}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 1.13509$$

We note that $s_5 = 1 + 1/4 + 1/9 + 1/16 + 1/25 = 5269/3600$. That is, we have

$$f_5(z) = \frac{1}{(1-z)^{5269/3600}}$$

so that

$$\|f_5\|^2 = \int_D \frac{1}{|1-z|^{5269/1800}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^{5269/1800}} \exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 1.20593$$

Since the sequence s_n converges, the sequence of functions f_n also probably converges and thus it is bounded. Since f_n is analytic in the unit disk D for all n, we find that $P(f_n) = f_n$ and that $P(f_n)$ also converges. Thus, we should consider the Toeplitz operator T_ϕ with symbol ϕ defined by

$$\phi(z) = 1 - |z|^2$$

Of course, the Toeplitz operator with symbol ϕ is simply defined as

$$T_\phi(f) = P(\phi f)$$

Thus, we should consider the product of ϕ and f. First, we note that

$$f(w)\phi(w) = (1+w+w^2+w^3+\cdots)(1-w\overline{w}) = (1+w+w^2+w^3+\cdots)-(w\overline{w}+w^2\overline{w}+w^3\overline{w}+w^4\overline{w}+\cdots)$$

When we integrate, all the terms will vanish except for 1 and $-w\overline{w}$. That is, we have

$$\int_D f\phi dA = \int_D (1-|z|^2)dA(z) = 2\int_0^1 r - r^3 dr = \frac{1}{2}$$

Next, we integrate with the exponential weight. That is, we have

$$\int_D f\phi d\lambda = \int_D (1-|z|^2)\lambda(z)dA(z) = 2\int_0^1 (r-r^3)\exp\left(-\frac{1}{1-r^2}\right)dr \approx 0.1096$$

Notice that

$$\int_D f\phi d\lambda = \int_D (1-|z|^2)\lambda(z)dA(z) \approx 0.1096 \le 0.4665 \approx \int_D \frac{1}{1-z}\lambda(z)dA(z) = \int_D f d\lambda$$

Furthermore, we may compute

$$\|f\phi\|^2 = \int_D |f\phi|^2 d\lambda = \int_0^{2\pi}\int_0^1 \frac{r(1-r^2)^2}{|1-re^{i\theta}|^2}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 0.344607$$

In contrast, we have

$$\|f\|^2 = \int_D |f|^2 d\lambda = \int_0^{2\pi}\int_0^1 \frac{r}{|1-re^{i\theta}|^2}\exp\left(-\frac{1}{1-r^2}\right)drd\theta \approx 0.6892$$

so that

$$\|f\phi\|^2 = \int_D |f\phi|^2 d\lambda \le \int_D |f|^2 d\lambda = \|f\|^2$$

This intuitively makes sense. To see why, we compare the functions

$$f(z) = \frac{1}{1-z}$$

and

$$f(z)\phi(z) = \frac{1-|z|^2}{1-z}$$

It is fairly easy to see that

$$|f(z)\phi(z)| \leq |f(z)|$$

or that

$$\frac{1-|z|^2}{|1-z|} \leq \frac{1}{|1-z|}$$

which can easily be deduced from the fact that

$$1-|z|^2 \leq 1$$

for all $z \in \mathbb{D}$, which is evidently true. Now, let us once again consider the sequence

$$s_n = \sum_{j=1}^{n} \frac{1}{j^2}$$

and the sequence of functions

$$f_n(z) = \frac{1}{(1-z)^{s_n}}$$

Again we see that

$$|f_n(z)\phi(z)| \leq |f_n(z)|$$

for all z in the unit disk D. To see why we note that

$$\frac{1-|z|^2}{|1-z|^{s_n}} \leq \frac{1}{|1-z|^{s_n}}$$

Multiplying this whole equation by $|1-z|^{s_n}$ yields

$$1-|z|^2 \leq 1$$

which is true for all $z \in D$. All this is to say that

$$\|f_n\phi\|^2 = \int_D |f_n\phi|^2 d\lambda \leq \int_D |f|^2 d\lambda = \|f_n\|^2$$

which informs us that

$$\|f_n\phi\| \leq \|f_n\|$$

Now, let us consider the Toeplitz operator T_ϕ again. By definition, we have

$$T_\phi(f) = P(\phi f)$$

where ϕ is the symbol and f is in the space of analytic functions that are also square integrable. Furthermore, we note that the the projection operator P is an orthogonal projection. For any function $g \in L_a^2$, we thus have

$$\|g\|^2 = \|Pg + (I-P)g\|^2 = \|Pg\|^2 + \|(I-P)g\|^2$$

which informs us that

$$\|Pg\|^2 \leq \|g\|^2$$

and thus

$$\|Pg\| \leq \|g\|$$

Applying this knowledge to the Toeplitz operator T_ϕ, we have

$$\|T_\phi(f)\| \leq \|P(\phi f)\| \leq \|\phi f\| \leq \|f\|$$

Now, let us recall the sequence of functions f_n, which we defined by

$$f_n(z) = \frac{1}{(1-z)^{s_n}}$$

The above inequality also applies to this sequence, so we find that

$$\|T_\phi(f_n)\| \leq \|f_n\|$$

for all $n \in \mathbb{N}$. Since we have already shown that the sequence $\{f_n\}$ is bounded, we may also deduce that the sequence $\{T_\phi(f_n)\}$ is bounded. Let us ascertain whether the sequence $\{f_n\phi\}$ is increasing or decreasing. First, we let $n = 1$. We note that

$$\|f_1\phi\|^2 = \int_D \frac{(1-|z|^2)^2}{|1-z|^2}\lambda(z)dA(z) \approx 0.344607$$

If we let $n = 2$, we have

$$\|f_2\phi\|^2 = \int_D \frac{(1-|z|^2)^2}{|1-z|^{5/2}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{(1-r^2)^2 r}{|1-re^{i\theta}|^{5/2}} \exp\left(-\frac{1}{1-r^2}\right) \approx 0.403613$$

Next let $n = 3$. Then we obtain

$$\|f_3\phi\|^2 = \int_D \frac{(1-|z|^2)^2}{|1-z|^{49/18}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{(1-r^2)^2 r}{|1-re^{i\theta}|^{49/18}} \exp\left(-\frac{1}{1-r^2}\right) \approx 0.440048$$

Letting $n = 4$ yields

$$\|f_4\phi\|^2 = \int_D \frac{(1-|z|^2)^2}{|1-z|^{205/72}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{(1-r^2)^2 r}{|1-re^{i\theta}|^{205/72}} \exp\left(-\frac{1}{1-r^2}\right) \approx 0.464163$$

Finally, if $n = 5$, then we obtain

$$\|f_5\phi\|^2 = \int_D \frac{(1-|z|^2)^2}{|1-z|^{5269/1800}}\lambda(z)dA(z) = \int_0^{2\pi}\int_0^1 \frac{(1-r^2)^2 r}{|1-re^{i\theta}|^{5269/1800}} \exp\left(-\frac{1}{1-r^2}\right) \approx 0.481181$$

Thus, we can numerically verify our observation that

$$\|f_n\phi\| \leq \|f_n\|$$

for certain small values of n. This also tells us that $\|f_n\phi\|$ is an increasing sequence, just as the sequence $\|f_n\|$ is increasing. I can't tell if the sequence $\|T_\phi(f_n)\|$ is increasing or decreasing from this, but I do know that it is bounded. Let us now compare some different ways of estimating the distance between f_n and f_m for $n \neq m$. By the reverse triangle inequality, we have

$$\|f_n - f_m\| \geq |\|f_n\| - \|f_m\||$$

Let us compute these approximations for several functions. First, we note that

$$\|f_1\|^2 = 0.689215$$

so that

$$\|f_1\| = 0.830190$$

We also have

$$\|f_2\|^2 = 0.897387$$

so that

$$\|f_2\| = 0.947305$$

$$\|f_1 - f_2\| \geq |\|f_1\| - \|f_2\|| \approx |0.830190 - 0.947305| = 0.117115$$

Next, we compute

$$\|f_1 - f_2\|^2 = \int_D |f_1 - f_2|^2 d\lambda = \int_0^{2\pi} \int_0^1 \left| \frac{1}{1 - re^{i\theta}} - \frac{1}{(1 - re^{i\theta})^{5/4}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0393333$$

so that

$$\|f_1 - f_2\| \approx 0.198326$$

We thus verify that

$$\|f_1 - f_2\| \approx 0.198326 \geq 0.117115 \approx |\|f_1\| - \|f_2\||$$

though the discrepancy between the actual value and the lower bound is small. We again recall that

$$\|f_2\| = 0.947305$$

and

$$\|f_3\| = \sqrt{1.03767} = 1.01866$$

so that

$$\|f_3 - f_2\| \geq |\|f_3\| - \|f_2\|| = |0.947305 - 1.01866| = 0.071355$$

and

$$\|f_3 - f_2\|^2 = \int_D |f_3 - f_2|^2 d\lambda = \int_0^{2\pi} \int_0^1 \left| \frac{1}{(1 - re^{i\theta})^{5/4}} - \frac{1}{(1 - re^{i\theta})^{49/36}} \right|^2 r \exp\left(-\frac{1}{1-r^2}\right) dr d\theta \approx 0.0117415$$

so that

$$\|f_3 - f_2\| \approx 0.108358$$

Thus, we again verify that

$$\|f_3 - f_2\| \approx 0.108358 \geq 0.071355 \approx |\|f_3\| - \|f_2\||$$

Next, we consider the functions f_3 and f_4. We note that

$$\|f_3\| = 1.01866$$

and

$$\|f_4\| = \sqrt{1.13509} = 1.06541$$

so that

$$\|f_4 - f_3\| \geq |\|f_4\| - \|f_3\|| \approx 0.04675$$

Next, we directly compute

$$\|f_4 - f_3\|^2 = \int_D |f_4 - f_3|^2 d\lambda = \int_0^{2\pi} \int_0^1 \left| \frac{1}{(1 - re^{i\theta})^{\frac{49}{36}}} - \frac{1}{(1 - re^{i\theta})^{\frac{205}{144}}} \right|^2 \cdot r \cdot \exp\left(-\frac{1}{1 - r^2}\right) dr d\theta \approx 0.00461021$$

so that

$$\|f_4 - f_3\| = \sqrt{0.00461021} \approx 0.0678985$$

verifying that

$$\|f_4 - f_3\| \approx 0.0678985 \geq 0.04675 \approx |\|f_4\| - \|f_3\||$$

However, we notice that there is not really a significant difference between the lower bound for $\|f_4 - f_3\|$ and its actual value. Finally, we consider the difference between f_4 and f_5. First, we note that

$$\|f_4\| \approx 1.06541$$

and

$$\|f_5\| \approx \sqrt{1.20593} \approx 1.09815$$

so that

$$\|f_4 - f_5\| \geq |\|f_4\| - \|f_5\|| = |1.06541 - 1.09815| = 0.03274$$

Next, I attempted to compute

$$\|f_4 - f_5\|^2 = \int_D |f_4 - f_5|^2 d\lambda$$

but it took to long to compute. Overall, however, we can see that the differences between terms do decrease rather rapidly and that the sequence does indeed converge.