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Exponential Weights

Next I try $\phi(z) = 1 - |z|^2$. I first consider the sequence $\{z^n\}_{n\in\mathbb{N}}$. I note that

$$T_\phi(z^m) = P(\phi z^m) = \int_D \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{\|z^n\|^2} w^m (1-|w|^2)\lambda(w)dA(w) = \frac{z^m}{\|z^m\|^2}\int_D (|w|^{2m}-|w|^{2m+2})\lambda(w)dA(w)$$

Now we compute

$$\int_D (|w|^{2m} - |w|^{2m+2})\lambda(w)dA(w) = 2\int_0^1 (r^{2m+1} - r^{2m+3}) \exp\left(-\frac{1}{1-r^2}\right)dr$$

Let us compute this integral for several values of m.

$$2\int_0^1 (r^1 - r^3) \exp\left(-\frac{1}{1-r^2}\right)dr \approx 0.109$$

$$2\int_0^1 (r^3 - r^5) \exp\left(-\frac{1}{1-r^2}\right)dr \approx 0.0236$$

$$2\int_0^1 (r^5 - r^7) \exp\left(-\frac{1}{1-r^2}\right)dr \approx 0.008$$

In any event, it is clear that these values are less than the corresponding norms of z^m. Thus, we find that $\|T_\phi(z^m)\| \leq \|z^m\|$ for all $m \in \mathbb{N}$; that is the sequence $\{T_\phi(z^n)\}_{n\in\mathbb{N}}$ converges to the function 0.

Next I consider the sequence $\{f_n\}$ where

$$f_n(w) = \frac{1}{1-|w|^n}$$

We note that

$$\|f_n\|^2 = 2\int_0^1 \frac{r}{(1-r^n)^2}\exp\left(\frac{-1}{1-r^2}\right)dr$$

Computing several values of n, we do see that this sequence is bounded. Next we compute

$$T_\phi(f_n) = P(\phi f_n) = \int_D B_\lambda(z,w) f_n(w)\phi(w)dA(w)$$

Notice that

$$f_n(w) = \frac{1}{1-|w|^n} = 1 + |w|^n + |w|^{2n} + \cdots$$
$$\phi(w) = (1-|w|^2)$$
$$B_\lambda(z,w) = \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2}$$

Thus we find that

$$T_\phi(f_n) = 1$$

for all n so that the sequence $\{T_\phi(f_n)\}$ is convergent.

Now we consider the sequence $f_m(z) = \overline{z}^m z^a$. Here we have

$$T_\phi(z^m) = P(\phi z^m) = \int_{\mathbb{D}} \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2} \overline{w}^m w^a (1 - w\overline{w})\lambda(w) dA(w)$$

Notice that this is nonzero only when $j + m = a$. That is, it is nonzero when $j = a - m$. In order for this to be true, we must have $m \leq a$. Thus it is nonzero for only finitely many m, so it too converges

Next I try the sequence $\{z^n \overline{z}^a\}$. We compute

$$T_\phi(f_n)(z) = P(\phi f_n)(z) = \int_D \sum_{j=0}^{\infty} \frac{z^j \overline{w}^j}{\|z^j\|^2} w^n \overline{w}^a (1 - w\overline{w})\lambda(w) dA(w)$$

We note that the integral is nonzero only when $n = a + j$. That is, we have $j = n - a \geq 0$. So we obtain

$$\frac{z^{n-a}}{\|z^{n-a}\|^2}(\|z^n\|^2 - \|z^{n+1}\|^2)$$