Toeplitz Operators on Bergman Spaces

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General Definitions

$L^p(\Omega, \lambda)$

The space of functions f on Ω such that $\int_\Omega |f|^p \, d\lambda < +\infty$

Bergman Spaces $A^p(\Omega, \lambda)$

Closed subspace of holomorphic functions in $L^p(\Omega, \lambda)$.

(Unweighted) Bergman Projection

The Bergman projection on Ω maps any function in L^p-space into the Bergman space A^p. It is defined $\mathbf{B}_\Omega(f) := \int_\Omega B(z, w) f(w) \, dV(w)$, and we call $B(z, w)$ the Bergman reproducing kernel. The explicit representation of the Bergman kernel depends on the domain Ω.

Toeplitz Operator

The Toeplitz operator T_ϕ with symbol function ϕ is an integral operator closely related to the Bergman projection. It is defined $T_\phi := \mathbf{B}_\Omega(\phi f)$.

Project 1: Boundedness of Toeplitz Operators on the Symmeterized Bidisc

Bounded Operator

An operator K is said to be bounded on Ω if, for some function f, $\|Kf\| \lesssim \|f\|$.

Toeplitz Operator (More Explicitly)

The Toeplitz operator T_ϕ, over the domain Ω and with symbol ϕ, is defined as

$$T_\phi f(z) = \int_\Omega B(z,w)\phi(w)f(w)\,dV(w),$$

where $B(z,w)$ is the Bergman kernel.

Our goal was to discern the range of r such that, for some symbol ϕ, T_ϕ is bounded from L^p to L^r. On the disc, we used Schur's test and Forelli-Rudin estimates. On the symmetrized bidisc, we adapted our strategies from the disc and also used Bell's transformation formula.

The Problem on the Disc

We let our operator be

$$T_\phi f(z) := \int_{\mathbb{D}} \frac{(1-|w|^2)^a}{(1-z\overline{w})^2} f(w)\, dV(w).$$

We obtain this by using the definition of the Bergman kernel on the unit disc $\mathbb{D}$ and by letting the symbol be $\phi(w) = (1-|w|^2)^a$. Given arbitrary values of a, p, the goal was to find the range of r such that $f \in L^p \Rightarrow Tf \in L^r$.

Strategy

Then if we define $O(f)$ to be the integral operator with kernel $\frac{(1-|w|^2)^a}{|1-z\bar{w}|^2}$, as $\|T_\phi f\| \leq \|O(|f|)\|$, it suffices to show O is a bounded operator from L^p to L^r. We have two technical tools: the Schur's Test and the Forelli-Rudin Estimates.

Schur's Test [CMO 6]

Let $\Omega \subset \mathbb{C}^N$ be a smoothly bounded domain defined by the real-valued function r, i.e. $\Omega = \{z : r(z) < 0\}$ and $dr \neq 0$ when $r = 0$.
Let $1 < p < \infty$ be given and let q be the conjugate exponent of p, $\frac{1}{p} + \frac{1}{q} = 1$. Also, let $1 < s < \infty$. Suppose that K is a measurable function on $\Omega \times \Omega$ and, for some $0 \leq t \leq 1$, and for all small $\epsilon > 0$

$$\int_\Omega |K(x,y)|^q |r(y)|^{qt} dy \leq C_{t,q} \cdot |r(x)|^{-\epsilon}$$

and

$$\int_\Omega |K(x,y)|^{(1-t)s} |r(x)|^{-\epsilon} dx \leq C_{t,s} \cdot |r(y)|^{-\epsilon}.$$

Then, for any $f \in L^p(\Omega)$, the function

$$Of(x) = \int_\Omega K(x,y) f(y) dy$$

belongs to $L^s(\Omega)$.

Using Forelli-Rudin to Apply Schur's Test

Forelli-Rudin Estimates [HKZ00]

For any $-1 < \alpha < \infty$ and any $\beta \in \mathbb{R}$.

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\overline{w}|^{2+\alpha+\beta}} dw \sim \begin{cases} 1, & \beta < 0 \\ \log(\frac{1}{1-|z|^2}), & \beta = 0 \\ \frac{1}{(1-|z|^2)^\beta}, & \beta > 0 \end{cases}$$

as $|z| \to 1^-$.

Application

Then applying Forelli-Rudin estimates and then Schur's test for our Toeplitz operator yields that our operator is bounded from L^p to L^r if $ap \geq 2$ or if $r \leq \frac{2p}{2-ap}$.

Introducing the Problem on the Symmetrized Bidisc $\mathbb{G}^2$

Symmetrized Bidisc

We define the symmetrized bidisc $\mathbb{G}^2 := \{(z_1 + z_2, z_1 z_2) : z \in \mathbb{D}^2\}$. The mapping $\phi\colon \mathbb{D}^2 \to \mathbb{G}^2$ is given by $\phi(z) = (z_1 + z_2, z_1 z_2)$.

Bergman Kernel on the Symmetrized Bidisc

In terms of λ and μ, where λ and μ are in $\mathbb{G}^2$, we will denote the Bergman on the symmetrized bidisc by $B_{\mathbb{G}^2}(\lambda, \mu)$. For $z, w \in \mathbb{D}^2$, the Bergman kernel is defined as

$$B_{\mathbb{G}^2}(\phi(z), \phi(w)) = \frac{1}{2\pi^2 (z_1 - z_2)(\overline{w_1} - \overline{w_2})} \left[\frac{1}{(1 - z_1\overline{w_1})^2 (1 - z_2\overline{w_2})^2} - \frac{1}{(1 - z_1\overline{w_2})^2 (1 - z_2\overline{w_1})^2} \right].$$

We are still figuring out what symbol to use for our operator so that the methods we used on the disc are applicable to that on the symmetrized bidisc. We have tried to let our symbol be $\psi_k(\phi(w)) = |w_1 - w_2|^k$ and $\psi_k(\phi(w)) = |w_1 - w_2|^k (1 - |w_1|^2)^k (1 - |w_2|^2)^k$.

Important Theorems for Solving on $\mathbb{G}^2$

The theorems and lemmas here are things we believe will be useful in solving this problem on the bidisc.

Bell's Transformation Formula

Letting $\mathbf{B}_\Omega$ be the Bergman projection on the domain Ω, $J_\mathbb{C}\phi$ be the complex Jacobian of ϕ, and h some holomorphic function, Bell's transformation formula tells us

$$\mathbf{B}_{\mathbb{D}^2}(J_\mathbb{C}\phi \cdot (h \circ \phi)) = J_\mathbb{C}\phi \cdot (\mathbf{B}_{\mathbb{G}^2}(h) \circ \phi).$$

Modified Schur's Test

$$\int_\Omega |K(z,w)|^{tr} |h(w)|^{-\varepsilon}\, dV(w) \lesssim |h(z)|^{-\varepsilon} \text{ for } \varepsilon \in [a,b);$$

$$\int_\Omega |K(w,z)|^{(1-t)q} |h(z)|^{-\varepsilon}\, dV(z) \lesssim |h(w)|^{-\varepsilon} \text{ for } \varepsilon \in [a',b').$$

If $\frac{a'}{b} < \frac{q}{r} < \frac{b'}{a}$ then the integral operator $\mathcal{K}$: $L^p(\Omega) \to L^q(\Omega)$ with kernel K is bounded.

Forelli-Rudin Type Estimates

For $t > 0$, $\epsilon_1 \in [0,1)$, $\epsilon_2 \in (0, 2-\epsilon_1)$, $z, a \in \mathbb{D}$, and $\epsilon_1 > 2 - t$,

$$\int_\mathbb{D} \frac{dV(w)}{|1 - z\overline{w}|^t (1-|w|^2)^{\epsilon_1} |a-w|^{\epsilon_2}} \approx |z-a|^{-\epsilon_2}(1-|z|^2)^{2-\epsilon_1-t}.$$

Project 2: Compact Toeplitz Operators on Weighted Bergman Spaces

Definition

- $k_z : \mathbb{D} \to \mathbb{C}$: $k_z(w) = (1 - |z|^2)/(1 - \bar{z}w)^2$
- $\tilde{S}$: Berezin transform of an operator S. $\tilde{S}(z) = \langle Sk_z, k_z \rangle$

Theorem by Axler, Zheng[1]

An operator T_u with $u \in L^\infty(\mathbb{D})$ is compact if and only if $\widetilde{T_u}$ vanishes on $\partial\mathbb{D}$.

[1] Axler, Sheldon, and Dechao Zheng. "Compact Operators via the Berezin Transform." Indiana University Mathematics Journal, vol. 47, no. 2, 1998, pp. 387–400. JSTOR, http://www.jstor.org/stable/24899675. Accessed 30 July 2024.

Fact

ϕ vanishes on $\partial\mathbb{D} \implies T_\phi$ is compact.

Is this true on exponentially weighted spaces? Let $\lambda(z) = \exp(-1/(1-|z|^2))$ and $\phi(z) = 1 - |z|^2$. To show that $T_\phi : A^2(\mathbb{D}, \lambda) \to A^2(\mathbb{D}, \lambda)$ is not compact, it suffices to find a bounded sequence $\{f_n\}$ such that $\{T_\phi(f_n)\}$ does not contain a convergent subsequence.

Let $f_n(z) = 1/(1-z^n)$. Then

$$\|f_1\| \approx 0.468 \quad \|f_2\| \approx 0.412 \quad \|f_3\| \approx 0.396$$

$$\|f_4\| \approx 0.391 \quad \|f_5\| \approx 0.388$$

Thus $\{f_n\}$ is bounded. Furthermore,

$$\|f_1\phi\| \approx 0.331 \quad \|f_2\phi\| \approx 0.302 \quad \|f_3\phi\| \approx 0.296$$

$$\|f_4\phi\| \approx 0.294 \quad \|f_5\phi\| \approx 0.294$$

Note that

$$\|T_\phi f_n\| = \|P(\phi f_n)\| \leq \|\phi f_n\| \leq \|f_n\|$$

so $\{T_\phi(f_n)\}$ is also bounded. We have an explicit form for $T_\phi(f_n)$:

$$T_\phi(f_n)(z) = \sum_{i=0}^{\infty} z^{ni}\left(1 - \frac{\|z^{ni+1}\|^2}{\|z^{ni}\|^2}\right)$$

Project 3: Weak-type Regularity and the Bergman Projection

Recall that a linear operator T is said to be bounded on $L^p(X)$ if there exists a constant c such that for all $f \in L^p(X)$,

$$\|Tf\|_{L^p(X)} \leq c \|f\|_{L^p(X)} .$$

Sometimes, it is known that T is bounded on $L^p(X) \iff p \in (a, b)$. This motivates the question, can we show that T satisfies some (weaker) estimates on $L^a(X)$ or $L^b(X)$?

Definition

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A linear operator T is said to be of *weak-type* (p, p) if there exists a constant c such that for any f and $\lambda > 0$,

$$|\{x \in X : |Tf(x)| > \lambda\}| \leq c \frac{\|f\|_{L^p(X)}^p}{\lambda^p}$$

Remark

One can show that being weak-type (p, p) is a necessary, but not sufficient, condition for boundedness.

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Answer: On smooth, strongly pseudoconvex domains, yes. But ...

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- This summer, we showed that it is always that case that the Bergman Projection on Generalized Hartogs Triangles $\mathbb{H}^n_{\mathbf{p}}$ does not satisfy a weak-type estimate at the lower endpoint and are continuing to work on the positive result on the upper endpoint.

Generalized Hartog's Triangles

Definition

We define generalized Hartog's triangles as

$$\mathbb{H}^n_{\mathbf{p}} := \{z \in \mathbb{C}^n \mid |z_1|^{p_1} < |z_2|^{p_2} < \cdots < |z_n|^{p_n} < 1\}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n_+$.

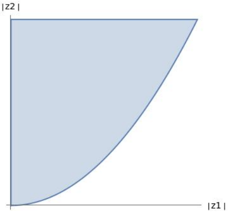

Figure: Plot of $|z_2|$ against $|z_1|$ on $\mathbb{H}^2_{(2,1)}$

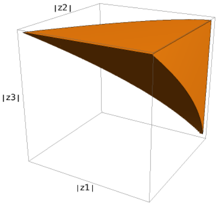

Figure: Plot of $|z_1|, |z_2|, |z_3|$ on $\mathbb{H}^3_{(3,2,3)}$

Results Obtained

Claim

For every n, the Bergman Projection on $\mathbb{H}^n_{\mathbf{p}}$ is not of weak-type (p, p) at the lower endpoint $p = \frac{2L\sum_{j=1}^{n} \frac{1}{p_n}}{L\sum_{j=1}^{n} \frac{1}{p_n} + 1}$, $L = \operatorname{lcm}(p_1, \ldots, p_n)$.

Sketch Proof

We select counterexample functions of the form

$$f_\lambda(z) = \overline{z}_1^{-a_1} |z_1|^{a_1} \overline{z}_2^{-a_2} |z_2|^{a_2} \cdots \overline{z}_{n-1}^{-a_{n-1}} |z_{n-1}|^{a_{n-1}} \overline{z}_n^{a_n} |z_n|^b$$

for some $a_1, a_2, \ldots, a_{n-1} \in \mathbb{Z}_{\geq 0}$, $a_n \in \mathbb{Z}^+$, and $b = b(\lambda) \in \mathbb{R}$ to be chosen.

Sketch Proof (cont.)

Prior Work

The Huo and Wick function was $\overline{z}_2|z_2|^{-q}$ where q is the conjugate index of $p > \frac{4}{3}$ and the Christopherson and Koenig functions were $\overline{z}_1^k|z_1|^k\overline{z}_2^a|z_2|^b$ and $\overline{z}_1^{-k}|z_1|^k\overline{z}_2^{-\ell}|z_2|^\ell\overline{z}_3^a|z_3|^b$ for some choice of k, ℓ, a, b.

We use the orthonormal basis for the Bergman kernel which is

$$\mathcal{A}_{\mathbf{p}} := \left\{ z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \mid \sum_{m=1}^{k} \frac{1+\alpha_m}{p_m} > 0, k = 1, 2, \dots, n \right\}$$

normalized. $f_\lambda(z)$ is chosen so that it (1) is in $L^p(\mathbb{H}^n_{\mathbf{p}})$ and (2) makes $|\{z \in \mathbb{H}^n_{\mathbf{p}} \mid |Pf_\lambda(z)| > \lambda\}|$ blow up compared to $\frac{\|f_\lambda\|^p_{L^p(\mathbb{H}^n_{\mathbf{p}})}}{\lambda^p}$ as $\lambda \to \infty$

Sketch Proof (cont.)

Conditions (1) and (2) essentially give upper and lower bounds on a_n in terms of the a_j, $1 \leq j \leq n-1$. We justify the existence of such an *integer* a_n via Extended Bézout's Lemma.

Conditions (1) and (2) also characterize $b = b(\lambda)$. Using this b and the newly defined set

$$\widetilde{\mathbb{H}}^n_{\mathbf{p}} := \left\{ z \in \mathbb{H}^n_{\mathbf{p}} \mid \left(\frac{1}{2}|z_n|\right)^{p_n} < |z_1|^{p_1} < |z_2|^{p_2} < \cdots < |z_n|^{p_n} < 1 \right\},$$

we can achieve a lower bound on the measure of $\{z \in \mathbb{H}^n_{\mathbf{p}} : |Pf(z)| > \lambda\}$ which demonstrates that the Bergman Projection does not satisfy a weak-type (p, p) estimate at the lower endpoint.

Questions?