Computations

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Lemma 3.1

Also, I am trying to prove Lemma 3.1 from Axler and Zheng's paper, which says that

$$\tilde{S} \circ \varphi_z = \widetilde{S_z}$$

First, I note that

$$\tilde{S} \circ \varphi_z(w) = \tilde{S}(\varphi_z(w)) = \langle S k_{\varphi_z(w)}, k_{\varphi_z(w)} \rangle$$

and that

$$\widetilde{S_z}(w) = \langle S_z k_w, k_w \rangle = \langle U_z S U_z k_w, k_w \rangle = \langle S U_z k_w, U_z k_w \rangle$$

Thus it suffices to prove that

$$U_z k_w = k_{\varphi_z(w)}$$

By the definition of U_z, I have

$$U_z k_w = (k_w \circ \varphi_z)\varphi_z'$$

so I am trying to prove that

$$(k_w \circ \varphi_z)\varphi_z' = k_{\varphi_z(w)}$$

or

$$(k_w \circ \varphi_z(v))\varphi_z'(v) = k_{\varphi_z(w)}(v)$$

To show this, I appeal to the formula from office hours:

$$K_U(z, \overline{\zeta}) = \det Df(z) \overline{\det Df(\zeta)} K_V(f(z), \overline{f(\zeta)})$$

Here I take $U = V = \mathbb{D}$, $f = \varphi_z$, $z = w$ and $\zeta = \varphi_z(v)$. Thus the above formula becomes

$$K_{\mathbb{D}}(w, \overline{\varphi_z(v)}) = \det D\varphi_z(w) \overline{\det D\varphi_z(\varphi_z(v))} K_{\mathbb{D}}(\varphi_z(w), \overline{v})$$

Then, I compute

$$K_{\mathbb{D}}(w, \overline{\varphi_z(v)}) = \overline{K_w(\varphi_z(v))} = \overline{\frac{k_w(\varphi_z(v))}{1 - |w|^2}}$$

$$K_{\mathbb{D}}(\varphi_z(w), \overline{v}) = \overline{K_{\varphi_z(w)}(v)} = \overline{\frac{k_{\varphi_z(w)}(v)}{1 - |\varphi_z(w)|^2}}$$

$$\det D\varphi_z(w) = \varphi_z'(w) = \frac{|z|^2-1}{(1-\overline{z}w)^2}$$

and finally

$$\overline{\det D\varphi_z(\varphi_z(v))} = \overline{\varphi_z'(\varphi_z(v))} = \overline{\frac{1}{\varphi_z'(v)}}$$

Putting it all together, I obtain

$$\frac{\overline{k_w(\varphi_z(v))}}{1-|w|^2} = \frac{|z|^2-1}{(1-\overline{z}w)^2} \cdot \overline{\frac{1}{\varphi_z'(v)}} \cdot \overline{\frac{k_{\varphi_z(w)}(v)}{1-|\varphi_z(w)|^2}}$$

I take the complex conjugate to obtain

$$\frac{k_w(\varphi_z(v))}{1-|w|^2} = \frac{|z|^2-1}{(1-z\overline{w})^2} \cdot \frac{1}{\varphi_z'(v)} \cdot \frac{k_{\varphi_z(w)}(v)}{1-|\varphi_z(w)|^2}$$

Using the fact that

$$1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}$$

the above equation becomes

$$\frac{k_w(\varphi_z(v))}{1-|w|^2} = \frac{|z|^2-1}{(1-z\overline{w})^2} \cdot \frac{1}{\varphi_z'(v)} \cdot \frac{k_{\varphi_z(w)}(v)}{\frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}}$$

Simplification yields

$$k_w(\varphi_z(v))\varphi_z'(v) = k_{\varphi_z(w)}(v) \cdot -\frac{|1-z\overline{w}|^2}{(1-z\overline{w})^2}$$

I can't seem to get rid of the

$$-\frac{|1-z\overline{w}|^2}{(1-z\overline{w})^2}$$

term. I would be grateful if anyone could tell me where my calculations went wrong.