Proof

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Let $\Omega \subseteq \mathbb{C}^n$ be an open bounded connected set, and let $L^2(\Omega, \lambda)$ denote the space of square integrable functions (with weight λ). Let $A^2(\Omega, \lambda)$ denote the space of holomorphic functions in $L^2(\Omega, \lambda)$. Since $A^2(\Omega, \lambda)$ is closed in $L^2(\Omega, \lambda)$, there exists an orthogonal projection operator $P : L^2(\Omega, \lambda) \to A^2(\Omega, \lambda)$. We claim that $\|P\| = 1$. First, we recall that

$$\|P\| = \sup\{\|Pf\| : f \in L^2(\Omega, \lambda), \|f\| \leq 1\}$$

We first show that $\|P\| \leq 1$. Let $f \in L^2(\Omega, \lambda)$. Since P is orthogonal, we find that

$$\|f\|^2 = \|Pf + (I - P)f\|^2 = \langle Pf + (I - P)f, Pf + (I - P)f\rangle = \tag{1}$$
$$\langle Pf, Pf\rangle + \langle Pf, (I - P)f\rangle + \langle (I - P)f, Pf\rangle + \langle (I - P)f, (I - P)f\rangle = \tag{2}$$
$$\langle Pf, Pf\rangle + \langle (I - P)f, (I - P)f\rangle = \|Pf\|^2 + \|(I - P)\|^2 \tag{3}$$

so that

$$\|Pf\|^2 = \|f\|^2 - \|(I - P)f\|^2 \leq \|f\|^2 \tag{4}$$

and

$$\|Pf\| \leq \|f\| \leq 1 \tag{5}$$

From this, we deduce that $\|P\| \leq 1$. Next, we want to show that $\|P\| \geq 1$. Let $f \in A^2(\Omega, \lambda)$ be such that $f \not\equiv 0$. Then $\|f\| > 0$ so that $f/\|f\| \in A^2(\Omega, \lambda)$. Then, we find that

$$\|P(f/\|f\|)\| = \|f/\|f\|\| = 1$$

so that $\|P\| \geq 1$. My only concern is if $A^2(\Omega, \lambda) = \{0\}$.