Topology Homework 02

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January 31, 2020

EXERCISE 1.1. *Determine all of the possible topologies on* $X = \{a, b\}$.

$\mathcal{T}_1 = \{\varnothing,\ X\}$
$\mathcal{T}_1 = \{\varnothing,\ \{a\},\ X\}$
$\mathcal{T}_1 = \{\varnothing,\ \{b\},\ X\}$
$\mathcal{T}_1 = \{\varnothing,\ \{a\},\ \{a, b\},\ X\}$
$\mathcal{T}_1 = \{\varnothing,\ \{b\},\ \{a, b\},\ X\}$
$\mathcal{T}_1 = \{\varnothing,\ \{a\},\ \{b\},\ \{a, b\},\ X\}$

EXERCISE 1.3. *Prove that a topology $\mathcal{T}$ on X is the discrete topology if and only if $\{x\} \in \mathcal{T}$ for all $x \in X$.*

Proof.
($\rightarrow$) Assume $\mathcal{T}$ is the discrete topology of X.
Since $x \in X$, $\{x\} \subseteq X$
$\therefore \{x\} \in \mathcal{T}$ *for all* $x \in X$
$\square$
($\leftarrow$) Assume $\{x\} \in \mathcal{T}$ *for all* $x \in X$
Let A a set such that $A \subseteq X$. for every point $x \in A$, we can find a set $x \in \{x\} \in \mathcal{T}$.
By the Union lemma we can say $A = \bigcup_{x \in A} \{x\}$
Since $\mathcal{T}$ is a topology, any union of open sets in $\mathcal{T}$ is an open set in $\mathcal{T}$.

This means A is an open set in $\mathcal{T}$.
Since every subset A is an open set in X, $\mathcal{T}$ is the discrete topology.
$\square$
$\therefore$ A topology $\mathcal{T}$ on X is the discrete topology if and only if $\{x\} \in \mathcal{T}$ for all $x \in X$.
$\blacksquare$

EXERCISE 1.4.

(a) *Give an example of a space where the discrete topology is the same as the finite complement topology.*

(b) *Make and prove a conjecture indicating for what class of sets the discrete and finite complement topologies coincide.*

(a) In the set $\varnothing$, both the discrete and finite complement topologies are $\{\varnothing\}$.

(b) Conjecture: The discrete and finite complement topologies for a set are equivalent if and only if the set is finite.

Proof.

($\rightarrow$) Assume the discrete and finite complement topologies for a set X are equivalent.

Consider some $A \subseteq X$
$A' \in \mathcal{T}$ since $\mathcal{T}$ is the discrete topology.
Thus, A' have a finite complement.
Thus A is finite.
Since every subset of X is finite, X is finite.
$\square$

($\leftarrow$) Let X be some finite set.

Consider some $A \subseteq X$.
Obviously, $A' \subseteq X$.
Since A' is a subset of a finite set, A' is finite.
Thus, A has a finite complement and is in the finite complement topology.
Since every $A \subseteq X$ is in the finite complement topology and the discrete topology on X, the finite complement topology and the discrete topology are equivalent for the set X.
$\square$
$\therefore$ The discrete and finite complement topologies for a set are equivalent if and only if the set is finite.
$\blacksquare$

EXERCISE 1.5. *Find three topologies on the five point set $X = \{a, b, c, d, e\}$ such that the first is strictly finer than the second and the second strictly finer than the third without using either the trivial or the discrete topology. Find a topology on X that is not comparable to each of the first three you found.*

Let $\mathcal{T}_1 = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, X\}$
Let $\mathcal{T}_2 = \{\varnothing, \{a\}, \{a, b\}, X\}$
Let $\mathcal{T}_3 = \{\varnothing, \{a\}, X\}$

It is easy to see that $\mathcal{T}_3 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_1$.
Thus, $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_2$ which is strictly finer than $\mathcal{T}_3$.

Let $\mathcal{T}_q = \{\varnothing, \{b\}, X\}$

Each $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ contain $\{a\}$, but $\{a\} \notin \mathcal{T}_q$.
Each $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ does not contain $\{b\}$, but $\{b\} \in \mathcal{T}_q$.

Thus, $\mathcal{T}_q$ does not compare to the other topologies.

EXERCISE 1.6. *Define a topology on* $\mathbb{R}$ *(by listing the open sets within it) that contain the open sets* (0,2) *and* (1,3) *and that contain as few open sets as possible.*

$\mathcal{T} = \{\varnothing, (0,2), (1,3), (1,2), (0,3), \mathbb{R}\}$

EXERCISE 1.7 *Let X be a set and assume $p \in X$. Show that the collection $\mathcal{T}$, consisting of $\varnothing$ and all subsets of X containing p, is a topology on X. This topology is called the* ***particular point topology*** *on X, and we denote it by PPX_p.*

Proof.
(1) By definition, $\varnothing \in PPX_p$.
Additionally, $p \in X$ so $X \in PPX_p$.
$\square$

(2) Let $U_1, U_2, ...U_n$ be some finite number of open sets in PPX_p.
Consider $\bigcap_{i=1}^{n} U_i$.

Each $U_i \subseteq X$ so $\bigcap_{i=1}^{n} U_i \subseteq X$
Since each U_i is an open set in PPX_p and $p \in U_i \ \forall i$ we know $p \in \bigcap_{i=1}^{n} U_i$
Thus, $\bigcap_{i=1}^{n} U_i$ is an open set in PPX_p.
$\therefore$ Any finite intersections of open sets in PPX_p is open in PPX_p.
$\square$

(3) Let $U_1, U_2, U_3...$ be some number of open sets in PPX_p.
Consider $\bigcup U_i$.

Each $U_i \subseteq X$ so $\bigcup U_i \subseteq X$.
Since U_1 is an open set in PPX_p, $p \in U_1$
Thus, $p \in \bigcup U_i$, which is open in PPX_p.
$\therefore$ Any union of open sets in PPX_p is open in PPX_p.
$\square$
By definition, PPX_p follows all three rules of being a topology.
$\therefore PPX_p$ is a topology on X.
$\blacksquare$

EXERCISE 1.8 *Let X be a set and assume $p \in X$. Show that the collection $\mathcal{T}$ consisting of X and all subsets of X that exclude p, is a topology on X. This topoogy is called the* ***excluded point topology*** *on X, and we denote it by EPX_p.*

Proof.
(1) By definition, $X \in EPX_p$.
Additionally, $p \notin \varnothing$ so $\varnothing \in EPX_p$.
$\square$

(2) Let $U_1, U_2, ...U_n$ be some finite number of open sets in EPX_p.
Consider $\bigcap_{i=1}^{n} U_i$.

Each $U_i \subseteq X$ so $\bigcap_{i=1}^{n} U_i \subseteq X$
Additionally, since $p \notin U_1$, it is certainly the case that $p \notin \bigcap_{i=1}^{n} U_i$.
Thus, $\bigcap_{i=1}^{n} U_i$ is an open set in EPX_p.
$\therefore$ Any finite intersection of open sets in EPX_p is open in EPX_p.
$\square$

(3) Let $U_1, U_2, U_3...$ be some number of open sets in EPX_p.
Consider $\bigcup U_i$.

Each $U_i \subseteq X$ so $\bigcup U_i \subseteq X$
There is no set U_i such that $p \in U_i$, so $p \notin \bigcup U_i$
Thus, $\bigcup U_i$ is an open set in EPX_p.
$\therefore$ Any union of open sets in EPX_p is open in EPX_p.
$\square$
By definition, EPX_p follows all three rules of being a topology.
$\therefore EPX_p$ is a topology on X.
$\blacksquare$