Topology Homework 03

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EXERCISE 1.16. *On the plane* $\mathbb{R}^2$ *, let*

$$\mathbb{B} = \{(a,b) \times (c,d) \subseteq \mathbb{R}^2 | a < b,\ c < d\}$$

(a) *Show that* $\mathbb{B}$ *is a basis for a topology on* $\mathbb{R}^2$.
(b) *Show that the topology* $\mathcal{T}'$ *generated by* $\mathbb{B}$ *is the standard topology on* $\mathbb{R}^2$.

(a)
(1) Consider some $(x, y) \in \mathbb{R}^2$.
Let $B' = (x-1, x+1) \times (y-1, y+1)$.
Since $x \in (x-1), x+1)$ and $y \in (y-1, y+1)$, $(x, y) \in B'$
So $\exists B' \in \mathbb{B} \ni (x,y) \in B' \forall (x,y) \in \mathbb{R}^2$
$\square$
(2) Let $B_1,\ B_2 \subseteq \mathbb{B}$
$B_1 = (a_1, b_1) \times (a_2, b_2),\ a_1 < b_1,\ a_2 < b_2$ and $B_2 = (c_1, d_1) \times (c_2, d_2),\ c_1 < c_1,\ c_2 < d_2$
Let $(x, y) \in B_1 \cap B_2$
Using the laws of algebra, we know that
$\max(a_1, c_1) < x < \min(b_1, d_1)$ and $\max(a_2, c_2) < y < \min(b_2, d_2)$

Let $B' = (\max(a_1, c_1), \min(b_1, d_1)) \times (\max(a_2, c_2), \min(b_2, d_2))$
Thus, $(x, y) \in B' \subseteq B_1 \cap B_2$
$\therefore (x,y) \in B_1 \cap B_2 \implies \exists B' \subseteq B_1 \cap B_2 \ni (x,y) \in B' \ \ \forall\ B_1, B_2 \in \mathbb{B}$
$\square$
$\therefore \mathbb{B}$ is a basis for a topology.
$\blacksquare$

(b)
Let $\mathbb{B}_1 = \{(a, b) \times (c, d)|\ a < b, c < d\}$
Let $\mathbb{B}_2 = \{B(p, r)|\ r > 0\}$
Let $\mathcal{T}_1$ be the topology generated by $\mathbb{B}_1 = \mathbb{B}$
Let $\mathcal{T}_2$ be the topology generated by $\mathbb{B}_2$ (the standard topology).

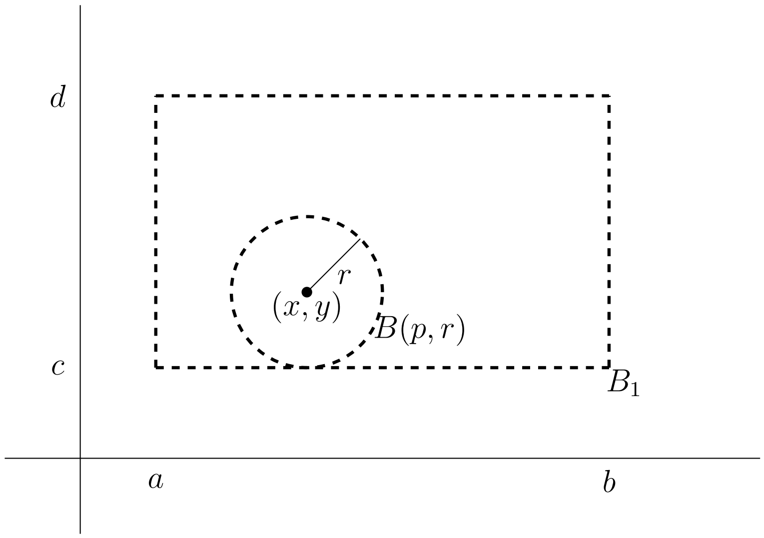

Figure 1: $\rightarrow$

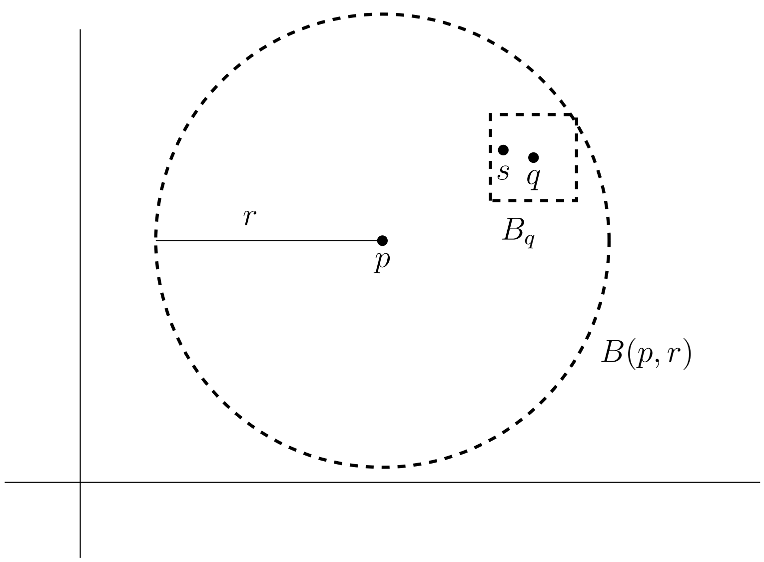

Figure 2: $\leftarrow$

$\rightarrow$ Let $B_1 \in \mathbb{B}_1$
$\exists a, b, c, d \ni B_1 = (a, b) \times (c, d)$
Let $p = (x, y) \in B_1$, so $a < x < b$ and $c < y < d$.
Let $r = min(x - a, b - x, y - c, d - y)$, so $r > 0$.

Consider $B(p,r) \in \mathbb{B}_2$
$B(p,r) \in \mathcal{T}_2$
Let $q \in B(p,r)$
$\exists m,\ \theta \ \ni 0 < m < r$ where $q = (x + m\cos\theta,\ y + m\sin\theta)$
Since $-1 \leq \sin\theta \leq 1,\ \ -r \leq m\sin\theta \leq r$
Since $-1 \leq \cos\theta \leq 1,\ \ -r \leq m\cos\theta \leq r$

$$\begin{array}{ll} a < x - r & x + r < b \\ a < x - m & x + m < b \\ \boxed{a < x + m\cos\theta} & \boxed{x + m\cos\theta < b} \end{array}$$

$$\begin{array}{ll} c < y - r & y + r < d \\ c < y - m & y + m < d \\ \boxed{c < y + m\sin\theta} & \boxed{y + m\cos\theta < d} \end{array}$$

So $q \in B_1$
Thus, $B(p,r) \subseteq B_1$
By the Union Lemma, $B_1 = \bigcup_{p \in B_1} B(p,r)$.

Let $U \in \mathcal{T}_1$.
$U = \bigcup B_k$ where $B_k \in \mathbb{B}_1$ since $\mathbb{B}_1$ generates $\mathcal{T}_1$
Thus, $U = \bigcup \bigcup_{p \in B_k} B(p,r)$, which is a union of basis elements from $\mathcal{T}_2$.
So $U \in \mathcal{T}_2$
$\therefore \mathcal{T}_1 \subseteq \mathcal{T}_2$
$\square$

$\leftarrow$ Let $B(p,r) \in \mathbb{B}_2$
Let $q = (x_q, y_q) \in B(p,r)$.
$\exists m \ni 0 < m < r$ and $d(p,q) = m$
Let $B_q = \{(x_q + \frac{m-r}{\sqrt{2}}, x_q + \frac{r-m}{\sqrt{2}}) \times (y_q + \frac{m-r}{\sqrt{2}}, y_q + \frac{r-m}{\sqrt{2}})\}$
$B_q \in \mathbb{B}_1$
Let $s \in B_q$.
By the formula for Euclidian distance

$$d(q,s) < \sqrt{\left(x_q - (x_q \pm \frac{m-r}{\sqrt{2}})\right)^2 + \left(y_q - (y_q \pm \frac{m-r}{\sqrt{2}})\right)^2}$$

$$d(q,s) < \sqrt{\left(\frac{m-r}{\sqrt{2}}\right)^2 + \left(\frac{m-r}{\sqrt{2}}\right)^2}$$

$$d(q,s) < \sqrt{(r-m)^2}$$

Since $r > m$

$$d(q,s) < r - m$$

By the triangular property
$d(p,s) < d(p,q) + d(q,s)$

$d(p, s) < m + (r - m) \quad \text{so } d(p, s) < r$

Thus, $s \in B(p, r)$.
So $B_q \subseteq B(p, r)$
By the Union lemma $\bigcup_{q \in B(p,r)} B_q = B(p, r)$

Let $U \in \mathcal{T}_2$.
$U = \bigcup B_k$ where $B_k \in \mathbb{B}_2$ since $\mathbb{B}_2$ generates $\mathcal{T}_2$.
Thus, $U = \bigcup \bigcup_{q \in B(p,r)} B_q$, which is a union of basis elements from $\mathcal{T}_1$.
So $U \in \mathcal{T}_1$
$\therefore \mathcal{T}_2 \in \mathcal{T}_1$
$\square$
$\therefore \mathcal{T}_1 = \mathcal{T}_2$
The topology $\mathcal{T}'$ generated by $\mathbb{B}$ is the standard topology on $\mathbb{R}^2$.
$\blacksquare$