Math 331 A - Probability

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HW p.129 #24,31,34,35 p.142 #62,72,76

1 p. 129 #24

If the probability density of X is given by

$$f(x) = \begin{cases} 2x^{-3} & \text{for } x > 1 \\ 0 & \text{elsewhere} \end{cases}$$

Check whether its mean and its variance exist.

By Def. 4.3

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$\mu = \int_{-\infty}^{1} 0 dx + \int_{1}^{\infty} 2x^{-2} dx$$

$$\mu = \frac{2}{-1} x^{-1} |_{1}^{\infty}$$

$$\mu = 2$$

$\boxed{\text{The mean of X exists.}}$

By Def. 4.4

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$\mu_2' = \int_{-\infty}^{1} 0 dx + \int_{1}^{\infty} 2x^{-1} dx$$

$$\mu_2' = 2ln(x)|_{1}^{\infty} = \infty$$

By Thm. 4.6

$$\sigma^2 = \mu_2' - \mu^2$$

$$\sigma^2 = \infty$$

$\boxed{\text{The variance of X does not exist.}}$

HW p.129 #24,31,34,35 p.142 #62,72,76

2 p. 129 #31

What is the smallest value k in Chebychev's theorem for which the probability that a random variable will take on a value between $\mu - k\sigma$ and $\mu + k\sigma$ is
(a) at least 0.95;
(b) at least 0.99;

(a) Let $0.95 \leq P(|x-\mu| < k\sigma)$ and $k \geq 0$
By Chebychev's Theorem

$$0.95 \leq 1 - \frac{1}{k^2}$$

$$k^2 \geq 20$$

$$k \geq \sqrt{20}$$

$\boxed{\text{Smallest value: } k = \sqrt{20}}$
(b) Let $0.99 \leq P(|x-\mu| < k\sigma)$ and $k \geq 0$
By Chebychev's Theorem

$$0.99 \leq 1 - \frac{1}{k^2}$$

$$k^2 \geq 100$$

$$k \geq \sqrt{100}$$

$\boxed{\text{Smallest value: } k = 10}$

HW p.129 #24,31,34,35 p.142 #62,72,76

3 p. 129 #34

Find the moment-generating function of the discrete random variable X whose probability density is given by

$$f(x) = 2\left(\frac{1}{3}\right)^x \text{ for } x = 1, 2, 3...$$

By Def. 4.6

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} f(x)$$

$$M_X(t) = \sum_{x \geq 0} e^{tx}\left(\frac{1}{3}\right)^x$$

$$M_X(t) = \sum_{x \geq 0} \left(\frac{1}{3}e^t\right)^x$$

By the law of convergence of geometric series

$$M_X(t) = \frac{1}{1 - \frac{1}{3}e^t}$$

$$\boxed{M_X(t) = \frac{3}{3 - e^t}}$$

HW p.129 #24,31,34,35 p.142 #62,72,76

4 p. 129 #35

If we let $R_X(t) = \ln M_x(t)$, show that $R'_X(0) = \mu$ and $R''_x(0) = \sigma^2$. Also, use these results to find the mean and the variance of a random variable X having the moment-generating function

$$M_X(t) = e^{4(e^t-1)}$$

By our definition of $R_X(t)$ we have

$$R'_X(t) = \frac{M'_X(t)}{M_X(t)}$$

By the quotient rule

$$R''_X(t) = \frac{M''_X(t)M_X(t) - M'_X(t)M'_X(t)}{(M_X(t))^2}$$

$$R'_X(0) = \frac{M'_X(0)}{M_X(0)}$$

$$R''_X(0) = \frac{M''_X(0)M_X(0) - M'_X(0)M'_X(0)}{(M_X(0))^2}$$

By Thm. 4.9

$$\frac{M'_X(0)}{M_X(0)} = \mu/1$$

$$\frac{M''_X(0)M_X(0) - M'_X(0)M'_X(0)}{(M_X(0))^2} = \frac{1 * \mu'_2 - \mu^2}{1^2} = \mu'_2 - \mu^2$$

By Thm. 4.6

$$\frac{M''_X(0)M_X(0) - M'_X(0)M'_X(0)}{(M_X(0))^2} = \sigma^2$$

$\boxed{R'_X(0) = \mu}$ $\boxed{R''_X(0) = \sigma^2}$

If we let $M_X(t) = e^{4(e^t-1)}$

$$R_X(t) = \ln M_x(t) = 4(e^t - 1)$$

$$R'_X(0) = 4(e^0) = 4$$

$$R''_X(0) = 4(e^0) = 4$$

Therefore, using our results, $\boxed{\mu = 4,\ \sigma^2 = 4}$

HW p.129 #24,31,34,35 p.142 #62,72,76 If we let $R_X(t) = \ln M_X(t)$

5 p. 142 #62

The probability that Ms. Brown will sell a piece of property at a profit of \$3000 is $\frac{3}{20}$, the probability that she will sell it at a profit of \$1,500 is $\frac{7}{20}$, the probability that she will break even is $\frac{7}{20}$, and the probability that she will lose \$1500 is $\frac{3}{20}$. What is her expected profit?

$$f(x) = \begin{cases} \frac{3}{20} & x = 3000,\ x = -1500 \\ \frac{7}{20} & x = 1500,\ x = 0 \end{cases}$$

$$E(x) = \sum_x x f(x)$$

$$E(x) = 3000\frac{3}{20} + 1500\frac{7}{20} + 0\frac{7}{20} + (-1500)\frac{3}{20}$$

$$E(x) = 750$$

$\boxed{\text{Her expected profit is \$750.}}$

HW p.129 #24,31,34,35 p.142 #62,72,76

6 p. 142 #72

With reference to Exercise 3.92 on page 107, find the mean and the variance of the random variable in question.

$$f(x) = \begin{cases} \frac{1}{288}(36 - x^2) & \text{for } -6 < x < 6 \\ 0 & \text{elsewhere} \end{cases}$$

By Def. 4.3

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$\mu = \int_{-\infty}^{-6} 0 dx + \int_{-6}^{6} \frac{x}{288}(36 - x^2) dx + \int_{6}^{\infty} 0 dx$$

$$\mu = \frac{1}{288}[18x^2 - \frac{1}{4}x^4]_{-6}^{6}$$

$$\mu = 0$$

By Def. 4.4

$$\mu_2^{'} = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$\mu_2^{'} = \int_{-\infty}^{-6} 0 dx + \int_{-6}^{6} \frac{x^2}{288}(36 - x^2) dx + \int_{6}^{\infty} 0 dx$$

$$\mu_2^{'} = \frac{1}{288}[12x^3 - \frac{1}{5}x^5]_{-6}^{6}$$

$$\mu_2^{'} = \frac{36}{5}$$

By Thm. 4.6

$$\sigma^2 = \mu_2^{'} - \mu^2$$

$$\sigma^2 = \frac{36}{5}$$

$$\boxed{\text{The R.V. X has a mean of 0 and a variance of } \frac{36}{5}}$$

HW p.129 #24,31,34,35 p.142 #62,72,76

7 p. 142 #76

A study of the nutritional value of a certain kind of bread shows that the amount of thiamine (vitamin B_1 in a slice may be looked upon as a random variable with $\mu = 0.260$ milligrams and $\sigma = 0.005$ milligrams. According to Chebychev's theorem, between what values must be the thiamine content of
(a) at least $\frac{35}{36}$ of all slices of this bread;
(b) at least $\frac{143}{144}$ of all slices of this bread;

(a) By Chebychev's Theorem, we have

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2},\ k \geq 0$$

Using the frequency interpretation of probability,

$$P(|x-\mu| < k\sigma) = \frac{35}{36}$$

$$1 - \frac{1}{k^2} \geq \frac{35}{36}$$

$$1 - \frac{1}{k^2} \geq 1 - \frac{1}{36}$$

$$k^2 \geq 36$$

Since $k \geq 0$

$$k \geq 6$$

Plugging in values for k, μ and σ, we have

$$|x - 0.26| \leq 6 * 0.005$$

The amount of thiamine (in milligrams) must be in the range [0.23,0.29] to satisfy Chebychev's Theorem.

(b) By a similar argument, without loss of generalization we have

$$k \geq 6$$

Plugging in values for k, μ and σ, we have

$$|x - 0.26| \leq 12 * 0.005$$

The amount of thiamine (in milligrams) must be in the range [0.20,0.32] to satisfy Chebychev's Theorem.