Math 331 A - Probability

Ethan Jensen

October 25th, 2019

HW p.163 #20,29,34,37 p.174 #78,80

1 p. 163 #20

Show that the moment-generating function of the geometric distribution is given by

$$M_X(t) = \frac{\theta e^t}{1 - e^t(1-\theta)}$$

By Def. 5.5, the PDF of a geometric distribution is given by

$$g(x;\theta) = \theta(1-\theta)^{x-1}$$

By Def. 4.6 we have

$$M_X(t) = \sum_x e^{tx} g(x;\theta)$$

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx}\theta(1-\theta)^{x-1}$$

$$M_X(t) = \frac{\theta}{1-\theta}\sum_{x=1}^{\infty} e^{tx}(1-\theta)^x$$

$$M_X(t) = \frac{\theta}{1-\theta}\sum_{x=1}^{\infty}(e^t(1-\theta))^x$$

By the law of convergence of geometric series we have

$$M_X(t) = \frac{\theta}{1-\theta}\frac{e^t(1-\theta)}{1-e^t(1-\theta)}$$

$$\boxed{M_X(t) = \frac{\theta e^t}{1 - e^t(1-\theta)}}$$

HW p.163 #20,29,34,37 p.174 #78,80

2 p. 163 #29

When calculating all the value in a Poisson distribution, the work can also be simplfied by first calculating $p(0; \lambda)$ and then using the recursion formula

$$p(x+1; \lambda) = \frac{\lambda}{x+1} p(x; \lambda)$$

Verify this formula.

By Def. 5.7 the PDF of a Poisson distribution is given by

$$p(x; \lambda) = \frac{\lambda^x e^- \lambda}{x!}$$

Plugging x+1 in for x, we have

$$p(x+1; \lambda) = \frac{\lambda^{x+1} e^- \lambda}{(x+1)!}$$

Using the recursive definition for the factorial function, we have

$$p(x+1; \lambda) = \frac{\lambda}{x+1} \frac{\lambda^x e^- \lambda}{x!}$$

Finally, by Def. 5.7

$$\boxed{p(x+1; \lambda) = \frac{\lambda}{x+1} p(x; \lambda)}$$

HW p.163 #20,29,34,37 p.174 #78,80

3 p. 163 #34

Show that if the limiting conditions $n \to \infty, \theta \to 0$ while $n\theta$ remains constant, are applied to the moment-generating function of the binomial distribution, we get the moment-generating function of the Poisson distribution.

By Thm. 5.4 The moment generating function for a binomial distribution is given by

$$M_X(t) = [1 + \theta(e^t - 1)]^n$$

Define a new function L(t) as

$$L(t) = \lim_{n \to \infty} [1 + \theta(e^t - 1)]^n$$

Since $n\theta$ is constant, we can define a new variable $\lambda = n\theta$. Thus, $\theta = \frac{\lambda}{n}$

$$L(t) = \lim_{n \to \infty} \left[1 + \frac{\lambda(e^t - 1)}{n}\right]^n$$

By the limit definition of e^x we have

$$L(t) = e^{\lambda(e^t - 1)}$$

By Theorem 5.9, L(t) is the moment-generating function for the Poisson distribution.

■

HW p.163 #20,29,34,37   p.174 #78,80

4 p. 163 #37

Use Theorem 5.9 to find the moment-generating function of $Y = X - \lambda$, where X is a random variable having the Poisson distribution with the parameter λ.

By Definition 5.7 we have

$$P(X = x) = P(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, 3...$$

$$P(Y = x) = P(X - \lambda = x) = P(X = x + \lambda)$$

$$P(Y = x) = \frac{\lambda^{x+\lambda} e^{-\lambda}}{(x+\lambda)!}, \quad x = \lambda, \lambda + 1, \lambda + 2, ...$$

By Def. 4.6 we have

$$M_Y(t) = \sum_{x=\lambda}^{\infty} e^{tx} \frac{\lambda^{x+\lambda} e^{-\lambda}}{(x+\lambda)!}$$

$$M_Y(t) = e^{-\lambda} \sum_{x=\lambda}^{\infty} e^{tx} \frac{\lambda^{x+\lambda}}{(x+\lambda)!}$$

Changing the bounds of summation we have

$$M_Y(t) = e^{-\lambda} \sum_{x=0}^{\infty} e^{t(x-\lambda)} \frac{\lambda^x}{x!}$$

$$M_Y(t) = e^{\lambda(-t-1)} \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!}$$

$$M_Y(t) = e^{\lambda(-t-1)} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

This summation is the Maclaurin series expansion for $e^{\lambda e^t}$.

$$M_Y(t) = e^{\lambda(-t-1)} e^{\lambda e^t}$$

$\boxed{M_Y(t) = e^{\lambda(e^t - t - 1)}}$

HW p.163 #20,29,34,37 p.174 #78,80

5 p. 174 #78

The number of monthly breakdowns of a supercomputer is a random variable having a Poisson distribution with $\lambda = 1.8$. Use the formula for the Poisson distribution to find the probabilities that this computer will function
(a) without a breakdown
(b) with only one breakdown

By Def. 5.7 The pdf of a Poisson distribution is

$$p(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

(a) Plugging in 1.8 for λ and 0 for x we have

$$p(0; 1.8) = \frac{1.8^0 e^{-1.8}}{0!} = e^{-1.8} \approx 0.1653$$

$\boxed{\text{The probability of 0 breakdowns is about 0.1653 or 16.53\%}}$

(b) Plugging in 1.8 for λ and 1 for x we have

$$p(0; 1.8) = \frac{1.8^1 e^{-1.8}}{1!} = 1.8e^{-1.8} \approx 0.2975$$

$\boxed{\text{The probability of 1 breakdown is about 0.2975 or 29.75\%}}$

HW p.163 #20,29,34,37 p.174 #78,80

6 p. 174 #80

In the inspection of a fabric produced in continuous rolls, the number of imperfections per yard is a random variable having the Poisson distribution with $\lambda = 0.25$. Find the probability that 2 yards of the fabric will have at most one imperfection using
(a) Table II;
(b) The computer printout of Figure 5.5.

$$\lambda = 0.25 \text{ imperfections/yard}$$

$$\lambda = 0.5 \text{ imperfections/2 yards}$$

Let R.V. X be a random variable that represents the distribution in the question. Since the values of X are natural numbers,

$$P(X \leq 1) = P(X = 1) + P(X = 0)$$

(a) With Table II we have

$$P(X = 0) = 0.6065, \quad P(X = 1) = 0.3033$$

$$P(X \leq 1) = 0.9098$$

$\boxed{\text{The probability of at most one imperfection is 0.9098 or 90.98\%}}$

(b) With the computer printout we have

$$P(X \leq 1) = 0.9098$$

This validates our answer in **(a)**.