Combinatorics of Integer Partitions

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February 1, 2019

1 Condition codes

Recall that $\mathbb{Z}_+^n$ is defined as

$$\mathbb{Z}_+^n = \{(x_1, x_2, ...x_n) | x_1, x_2, ...x_n \in \mathbb{Z}_+\}$$

We can define a disjoint partition on $\mathbb{Z}_+^n$ into subsets based on how elements of the tuple $(x_1, x_2, ...x_n)$ are equal to each other.

Definition 1.1. *Let S be a set of tuples. A* ***condition code*** *C on S is a tuple $(c_1, c_2, ...c_n)$ that gives S the following elementhood condition:*

> *A tuple x is an element of S if and only if there exists a tuple $t = (t_1, t_2, ...t_n) \in \mathbb{Z}_+^n$ such that for all i, x contains c_i copies of t_i.*

Definition 1.2. *A condition code C is* ***spicy*** *if it gives the following two elementhood conditions:*

> ***(a)*** *A tuple x is an element of S if and only if there exists a tuple $t = (t_1, t_2, ...t_n) \in \mathbb{Z}_+^n$ such that for all i, x contains c_i copies of t_i.*
>
> **(b)** *Each t_i is distinct.*

A condition code C that is not spicy is called **mild**

Definition 1.3. *A condition code C is* ***super spicy*** *if it gives the following two elementhood conditions:*

> ***(a)*** *A tuple x is an element of S if and only if there exists a tuple $t = (t_1, t_2, ...t_n) \in \mathbb{Z}_+^n$ such that for all i, x contains c_i copies of t_i.*
>
> **(b)** $t_i * < t_{i+1}$ if $c_i = c_{i+1}$ *and* $t_i \neq t_{i+1}$ *if* $c_i \neq c_{i+1}$ *where* $(x_n = t_i$ and $x_m = t_{i+1}) \implies n < m$
> ** Determined to be either $>$ or $<$; different for each subset*

Definition 1.4.

> **(a)** *A* ***mild subset*** *S of $\mathbb{Z}_n^+$ written $S(c_1)(c_2)...(c_k)$ is a subset of $\mathbb{Z}_n^+$ whose only restrictions on elementhood is a mild condition code.*
> **(b)** *A* ***spicy subset*** *S of $\mathbb{Z}_n^+$ written $S(c_1, c_2, ...c_k)$ is a subset of $\mathbb{Z}_n^+$*

whose only restrictions on elementhood is a spicy condition code. S can be identified as a subset of a unique mild subset S'.
(c) *A* ***super spicy subset*** *S of* $\mathbb{Z}_n^+$ *written the same as a spicy subset is a subset of* $\mathbb{Z}_n^+$ *whose only restrictions on elementhood is a super spicy condition code. S can be identified as a subset of a unique spicy subset S'.*

Example 1.1
Define and ennumerate $S(1)(1)$, all of its spicy subsets, and their super spicy subsets.
$S(1)(1)$ has two spicy subsets - namely $S(1,1)$ and $S(2)$.
$S(1,1)$ has two super spicy subsets. Label them $S_1(1,1)$ and $S_2(1,1)$.
$S(2)$ has no super spicy subsets.
From Definition 1.4 we can write
$S(1)(1) = \{(x_1, x_2) \in \mathbb{Z}_+^2\} = \mathbb{Z}_+^2$
$S(1)(1) = \{(1,1),\ (1,3),\ (4,2),\ (1,2),\ (2,3),\ (2,2),\ (3,1)...\}$

$S(1,1) = \{(x_1, x_2) \in \mathbb{Z}_+^2 | x_1 \neq x_2\}$
$S(1,1) = \{(1,2),\ (2,1),\ (1,6),\ (4,5),\ (3,2)...\}$

$S(2) = \{(x_1, x_2) \in \mathbb{Z}_+^2 | x_1 = x_2\}$
$S(2) = \{(1,1),\ (3,3),\ (7,7),\ (4,4)...\}$

$S_2(1,1) = \{(2,1),\ (4,1),\ (4,3),\ (7,5),\ (6,2)...\}$

2 A Multi-layered Partition

Note the following hierarchy:

- a mild subset is a subset of $\mathbb{Z}_+^n$.
- a spicy subset is a subset of a mild subset.
- a super spicy subset is a subset of a spicy subset.

Consider the following diagram
Lemma 1.1. *Two condition codes are equivalent if they are permutations of each other.*
Proof. Consider any two condition codes $C_1 = (a_1, a_2, ...a_n)$ and $C_2 = (b_1, b_2, ...b_n)$ which are permutations of each other.
Assume some $x \in S(C_1)$.
By Definition 1.1, a tuple x is an element in $S(C_1)$ if for all i, there exists a tuple $(t_1, t_2, ...t_n) \in \mathbb{Z}_n^+$ such that x contains a_i copies of t_i.

Applying the assumed permutation, there exists $(s_1, s_2, ...s_n) \in \mathbb{Z}_n^+$ such that for all i, x contains b_i copies of s_i.

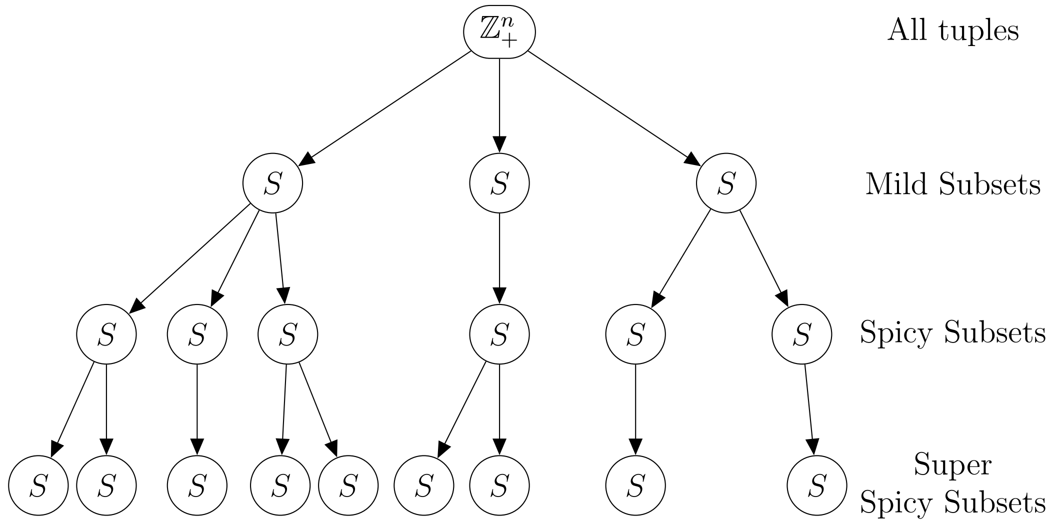

Figure 1: Partition Diagram

Thus, $x \in S(C_2)$ and $S(C_2) \subseteq S(C_1)$.
The same process can be done to show that $S(C_1) \subseteq S(C_2)$.
$\therefore S(C_1) = S(C_2)$ and the condition codes are equivalent.
■

Theorem 1.1. *The set of all super spicy subsets S with distinct condition codes C of* $\mathbb{Z}_n^+$ *forms a disjoint partition of* $\mathbb{Z}_n^+$.

Proof. Show that each distinct set is disjoint.

Consider any two distinct super spicy subsets $S(C_1)$ and $S(C_2)$, where
$C_1 = (a_1, a_2, ...a_k),\ C_2 = (b_1, b_2, ...b_k)$
Suppose $x \in S(C_1) \cap S(C_2)$.
$\exists (t_1, t_2, ...t_k) \ni$ x contains a_i copies of t_i for all i.
$\exists (s_1, s_2, ..s_k) \ni$ x contains b_i copies of s_i for all i.
This implies that $(t_1, t_2, ...t_k)$ is a permutation of $(s_1, s_2, ..s_k)$.
This further implies that $C_1 = (a_1, a_2, ...a_k),\ C_2 = (b_1, b_2, ...b_k)$ are permutations of each other and are equivalent by Lemma 1.1.
This is a contradiction.
Thus, there is no $x \in S(C_1) \cap S(C_2)$.
$S(C_1) \cap S(C_2) = \varnothing$
□
Show that $\bigcup P = \mathbb{Z}_+^n$, where P is the set of all spicy subsets.

$S \in P \implies S \subseteq \mathbb{Z}_+^n$ by definition. So $\bigcup P \subseteq \mathbb{Z}_+^n$.
Assume $x = (x_1, x_2, ...x_n) \in \mathbb{Z}_+^n$.
Let a_i be the first occurence of x_i in x.
Consider the tuple $(a_1, a_2, ...a_k)$ each of whose values are distinct values in x

and $x_n = a_i$ and $x_m = a_{i+1} \implies n < m$
There exists some spicy condition code $C = (c_1, c_2, ...c_k)$ such that x contains c_i copies of a_i for all i.
By Definition 1.2, $x \in S(C) \in P$ where P is the collection of all spicy subsets.
by the Union Lemma, $x \in \bigcup P$
$\mathbb{Z}_+^n \subseteq \bigcup P$
$\therefore \bigcup P = \mathbb{Z}_+^n$
■

Example 1.2
Show that Theorem 1.1 holds for $S(1)(1)$, and all if its super spicy subsets.

As seen in Example 1.1, the spicy subsets for $S(1)(1)$ are $S(1,1) and S(2)$, where
$S(1,1) = \{(x_1, x_2) \in \mathbb{Z}_+^2 | x_1 \neq x_2\}$

$S(2) = \{(x_1, x_2) \in \mathbb{Z}_+^2 | x_1 = x_2\}$
It is easy to see by the elementhood conditions for these two sets that they form a disjoint partition of $S(1)(1)$.

As seen in Example 1.1, thew spicy subsets for $S(1)(1)$ are
$S_1(1,1) = \{(x_1, x_2) \in \mathbb{Z}_+^2 | x_1 < x_2\}$

$S_2(1,1) = \{(x_1, x_2) \in \mathbb{Z}_+^2 | x_1 > x_2\}$
Once again, it is easy to see that these two subsets form a disjoint partition of their superset, $S(1,1)$.

This verifies Theorem 1.1 for $S(1)(1)$.

3 The D function

The fact that the spicy subsets form a disjoint partition of $\mathbb{Z}_+^n$ means that linear functions on the set $\mathbb{Z}_+^n$ can instead be applied on each of the spicy subsets instead to get the same result.

In particular, a new function can be defined called the D function, which when combined with this disjoint partition, can take a simple trigonometric identity and use it to calculate difficult infinite sums.

Definition 1.5. *Let* $x = (x_1, x_2, ...x_n) \in \mathbb{Z}_+^n$ *and S(C) be a mild subset or a super spicy subset.*
Define $D : C \to \mathbb{R}$ *by*

$$\sum_{x \in S(C)} \frac{1}{(x_1)^2} \frac{1}{(x_2)^2} \cdots \frac{1}{(x_n)^2} = \sum_{x \in S(C)} \prod_{x_i} x_i^{-2}$$

For a mild condition code we write: $D(c_1)(c_2)...(c_n)$
For a super spicy condition code we write: $D(c_1, c_2...c_n)$

Definition 1.6

Let $D_n(a)$ *represent* $\overbrace{D(a, a, ...a)}^{n}$
Let $D^n(a)$ *represent* $\overbrace{D(a)(a)...(a)}^{n}$

Theorem 1.2
$D(C_1)D(C_2) = D(C_1)(C_2)$

Proof. By Definition 1.5. , we can write

$$D(C_1)(C_2) = \sum_{x \in S(C_1)} \prod_{x_i} x_i^{-2} \sum_{y \in S(C_2)} \prod_{y_i} y_i^{-2}$$

$$D(C_1)(C_2) = \sum_{x \in S(C_1)} \sum_{y \in S(C_2)} \prod_{x_i} x_i^{-2} \prod_{y_i} y_i^{-2}$$

Let $z = (x|y)$ such that

$$z = (x_1, x_2, x_3, ...x_n, y_1, y_2, y_3...y_m)$$

Then by Definition 1.1, just combining elementhood conditions together,

$$D(C_1)D(C_2) = \sum_{z \in S(C_1)(C_2)} \prod_{Z_i} z_i^{-2}$$

$$\therefore D(C_1)D(C_2) = D(C_1)(C_2)$$

■

Theorem 1.3 $D_n(1) = \frac{\pi^{2n}}{(2n+1)!}$
Proof.
Compare the MacLaurin series of $\sinh(x)$ with the Euler product of $\sinh(x)$.

$$\sinh(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} ... = x\left(1 + \frac{x}{i\pi}\right)\left(1 - \frac{x}{i\pi}\right)\left(1 + \frac{x}{2i\pi}\right)\left(1 - \frac{x}{2i\pi}\right)...$$

Thus, by a simple calculation,

$$\frac{\sinh(\pi x)}{x} = 1 + \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + \frac{\pi^6 x^6}{7!} ... = \left(1 + \frac{x^2}{1^2}\right)\left(1 + \frac{x^2}{2^2}\right)\left(1 + \frac{x^2}{3^2}\right)...$$

When multiplying a product, to calculate the coefficient on a polynomial of the nth degree, all term combinations resulting in the nth degree of each factor must be determined and subsequently summed.

For the product expansion, the only term combinations resulting in degree 2 are when we select one x^2 term from one factor and select a 1 from each of the other factors. Comparing this to the right hand side we have

$$\frac{\pi^2}{3!} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} ... = D(1)$$

Continuing this process we find that

$$\sum_i \sum_{j<i} \frac{1}{i^2} \frac{1}{j^2} = D(1,1) = \frac{\pi^4}{5!}$$

And in general,

$$\sum_{0<x_1} \sum_{x_2<x_1} ... \sum_{x_n<x_{n-1}} \frac{1}{x_1^2} \frac{1}{x_2^2} ... \frac{1}{x_n^2} = D_n(1) = \frac{\pi^{2n}}{(2n+1)!}$$

■

Theorem 1.4 *Let C be a mild condition code and P be the collection of all super spicy condition codes C_i where all $S(C_i) \subseteq S(C)$. Then*

$$D(C) = \sum_{C_i \in P} D(C_i)$$

Proof.
$\mathbb{Z}_+^n$ has a disjoint parition into super spicy subsets by Theorem 1.2.
Since all super spicy subsets are subsets of mild subsets, every mild subset has a disjoint partion into super spicy subsets.

The sum over any set is always equal to the sum over each subset of a disjoint partition.
Let $x = (x_1, x_2, ...x_n) \in \mathbb{Z}_+^n$, C be a mild condition code, and each C_k be super spicy condition codes.

$$\sum_{x \in S(C)} \prod_{x_i} x_i^{-2} = \sum_{S(C_k) \subseteq S(C)} \sum_{x \in S(C_k)} \prod_{x_i} x_i^{-2}$$

$$\therefore D(C) = \sum_{C_i \in P} D(C_i)$$

■

4 Calculating positive even zeta values

Zeta values can be calculated using disjoint partitions. However, we must prove one more theorem to connect zeta values to the zeta function.

Theorem 1.5 $D(n) = \zeta(2n)$ *for* $n = 1, 2, 3...$

We define $\zeta(s)$ to be $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$.
By Definition 1.4 we see that any mild subset $S(C)$ where C = (n) is a set that only contains tuples of the form $(x_1, x_1, x_1, ...), x_1 \in \mathbb{Z}_+$ where each element in the tuple is the same.
By Definition 1.5 we can write

$$D(n) = \sum_{x \in S(C)} \prod_{x_i} x_i^{-2} = \sum_{x \in S(C)} x_i^{-2n}$$

The set of all x_i is just the set of positive integers. Thus,

$$D(n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \zeta(2n)$$

■

Notice that because of how the D function is defined, only positive even zeta values can be expressed in terms of the D function.

Example 1.3
From Theorem 1.5 we can express $\zeta(4)$ in terms of a D function.

$$\zeta(4) = D(2)$$

Using our results from Example 1.2 and Theorem 1.4 we can write

$$D(1)(1) = D_1(1,1) + D_2(1,1) + D(2)$$

where $D_1(1,1)$ and $D_2(1,1)$ come from the super spicy subsets $S_1(1,1)$ and $S_2(1,1)$ respectively.

Since $D_1(1,1)$ and $D_2(1,1)$ have the same value, we can write

$$D(1)(1) = 2D(1,1) + D(2)$$

From Theorem 1.3, we know that $D(1) = \frac{\pi^2}{6}$ and $D(1,1) = \frac{\pi^4}{120}$.
Plugging in these values and using Theorem 1.2 we have

$$\frac{\pi^4}{36} = \frac{\pi^4}{60} + D(2)$$

$$\therefore \zeta(4) = D(2) = \frac{\pi^4}{36} - \frac{\pi^4}{60} = \frac{\pi^4}{90}$$

5 An Example

Ex. Using disjoint partitions, calculate $\zeta(8)$
Assume $\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}$

Using Theorem 1.4 we can establish the first 4 equations.
(1) $D(1)(1)(1)(1) = 24D(1,1,1,1)+12D(1,1,2)+6D(2,2)+4D(1,3)+D(4)$
(2) $D(1)(1)(2) = 2D(1,1,2) + 2D(2,2) + 2D(1,3) + D(4)$
(3) $D(2)(2) = 2D(2,2) + D(4)$
(4) $D(1)(3) = D(1,3) + D(4)$

Plugging **(2)** into **(1)**
(5) $D(1)(1)(1)(1) = 24D(1,1,1,1)+6D(1)(1)(2)-6D(2,2)-8D(1,3)-5D(4)$
Plugging **(3)** into **(5)**
(6) $D(1)(1)(1)(1) = 24D(1,1,1,1) + 6D(1)(1)(2) - 3D(2)(2) - 8D(1,3) - 2D(4)$
Plugging **(4)** into **(6)**
(7) $D(1)(1)(1)(1) = 24D(1,1,1,1) + 6D(1)(1)(2) - 3D(2)(2) - 8D(1)(3) + 6D(4)$

Using Theorem 1.2 we can write
(8) $(D(1))^4 = 24D(1,1,1,1)+6(D(1))^2(D(2))-3(D(2))^2-8(D(1))(D(3))+6D(4)$
Using Theorem 1.3 and Theorem 1.5 we can write
(9) $\zeta(2)^4 = 24\left(\frac{\pi^8}{9!}\right) + 6\zeta(2)^2\zeta(4) - 3\zeta(4)^2 - 8\zeta(2)\zeta(6) + 6\zeta(8)$
Plugging in $\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}$ we have
(10) $\frac{\pi^8}{1296} = \frac{\pi^8}{15120} + \frac{\pi^8}{540} - \frac{\pi^8}{2700} - \frac{4\pi^8}{2835} + 6\zeta(8)$
Isolating $\zeta(8)$ we have
(11) $\zeta(8) = \frac{1}{6}\left(\frac{\pi^8}{1296} - \frac{\pi^8}{15120} - \frac{\pi^8}{540} + \frac{\pi^8}{2700} + \frac{4\pi^8}{2835}\right)$

$$\boxed{\zeta(8) = \frac{\pi^8}{9450}}$$

6 Counting spicy subsets

In the previous example, we have said that

$$D(1)(1)(2) = 2D(1,1,2) + 2D(2,2) + 2D(1,3) + D(4)$$

However, Theorem 1.4 would seem to suggest that

$$D(1)(1)(2) = D(1,1,2) + D(2,2) + D(1,3) + D(4)$$

This equation would be true if each term on the left represented a sum over a spicy subset. However, the notation we use for the D function represents a sum over super spicy subsets.

If two super spicy subsets $S(C_1)$ and $S(C_2)$ are both subsets of the same spicy subset, then $D(C_1) = D(C_2)$. Thus, all that is left to do is determine the coefficients by determining how many of each category of distinct super spicy subsets are contained in a particular mild subset.

This turns out to be a non-trivial combinatorics problem. This problem is analagous to determine all the possible ways a given set of blocks can be placed in a given set of holes.
This problem is called the **Blocks and Holes Problem**.

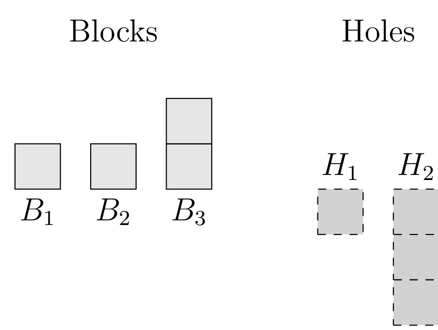

Figure 2: Blocks and Holes Scenario 1.1

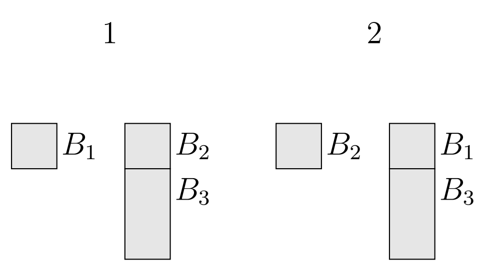

Figure 3: Blocks and Holes Solution 1.1

The above problem and solution shows that the mild subset S(1)(1)(2) contains exactly 2 super spicy subsets that are subsets of the spicy subset S(1,3).

7 References

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