Math 331 A - Probability

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HW p.183 #3,10,11,12,17,20, p.184 #23,26 p.202 #54,56,60

1 p. 183 #3

If a random variable X has a uniform density with the parameters α and β, find its distribution function.

By Defintion 6.1 the probability density of X is

$$u(x, \alpha, \beta) = \begin{cases} \frac{1}{\beta-\alpha} \text{for } \alpha < x < \beta \\ 0 \quad \text{elsewhere} \end{cases}$$

By Def. 3.5 we have the distribution function.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{\alpha}^{x} \frac{1}{\beta - \alpha} dx, \ \alpha < x < \beta$$

$$\boxed{F(x) = \frac{x - \alpha}{\beta - \alpha}, \ \alpha < x < \beta}$$

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2 p. 183 #10

Find the probabilities that a random variable will exceed 4 if it has a gamma distribution with
(a) $\alpha = 2$ and $\beta = 3$
(b) $\alpha = 3$ and $\beta = 4$

Let X be the random variable with $\alpha = 2$ and $\beta = 3$.
Let Y be the random variable with $\alpha = 3$ and $\beta = 4$.
By Definition 6.2

$$P(X = x) = g(x; 2, 3) = \begin{cases} \frac{1}{3^2\Gamma(2)}x^{2-1}e^{-x/3} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(Y = y) = g(y; 3, 4) = \begin{cases} \frac{1}{4^3\Gamma(3)}y^{3-1}e^{-y/4} & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(X > 4) = \int_4^\infty \frac{1}{3^2\Gamma(2)}x^{2-1}e^{-x/3}dx$$

$$\boxed{\textbf{(a) } P(X > 4) = \frac{7}{3}e^{-\frac{4}{3}} \approx 0.61506}$$

$$P(Y > 4) = \int_4^\infty \frac{1}{4^3\Gamma(3)}y^{3-1}e^{-y/4}dy$$

$$\boxed{\textbf{(b) } P(X > 4) = \frac{5}{2}e^{-1} \approx 0.91970}$$

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3 p. 183 #11

Show that the gamma distribution with $\alpha > 1$ has a relative maximum at $x = \beta(\alpha - 1)$. What happens when $0 < \alpha < 1$ and when $\alpha = 1$?

By Def. 6.2, for $x > 0$ the gamma distribution is defined as follows

$$g(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

$$g'(x; \alpha, \beta) = \frac{\alpha - 1 - \frac{x}{\beta}}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-2} e^{-x/\beta}$$

To find critical points, we set $g'(x; \alpha, \beta)$ equal to 0.

$$\frac{\alpha - 1 - \frac{x}{\beta}}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-2} e^{-x/\beta} = 0$$

$$(\alpha - 1 - \frac{x}{\beta}) x^{\alpha-2} = 0$$

By the zero product property, and the fact that g is non-differentiable at $x = 0$ we have

$$\alpha - 1 - \frac{x}{\beta} = 0$$

$$x = \beta(\alpha - 1)$$

These are our critical points.

$$g'(0) = 0, g'(\alpha\beta) = \frac{-1}{\beta^\alpha \Gamma(\alpha)} (\alpha\beta)^{\alpha-2} e^{-\alpha} < 0$$

Thus, $g(\beta(\alpha - 1))$ has negative concavity.

There is a local maximum at $x = \beta(\alpha - 1)$.

Note that when $0 < \alpha < 1$, the x-value for the local maximum, which is at $x = \beta(\alpha - 1)$ must be negative. But since x is never negative, there is no local maximum anywhere for the function.

When $\alpha = 1$, the x-value for the local maximum, which is at $x = \beta(1-1) = 0$.

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4 p. 183 #12

Prove Theorem 6.4, making the substitution $y = x\left(\frac{1}{\beta} - t\right)$ in the integral defining $M_X(t)$.

Theorem 6.4. The moment-generating function for the gamma distribution is given by $M_X(t) = (1 - \beta t)^{-\alpha}$
Proof.
By Def. 6.2, for $x > 0$ the gamma distribution is defined as follows

$$g(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

By Definition 4.6 we have

$$M_X(t) = \int_0^\infty e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx$$

$$M_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{tx} x^{\alpha-1} e^{-x/\beta} dx$$

$$M_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{x(t-1/\beta)} dx$$

Let $y = x\left(\frac{1}{\beta} - t\right)$. $dx = \left(\frac{\beta}{1-\beta t}\right) dy$

$$M_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty (\frac{y\beta}{1-\beta t})^{\alpha-1} e^{-y} \left(\frac{\beta}{1-\beta t}\right) dx$$

$$M_X(t) = \frac{\beta^\alpha}{(1-\beta t)^\alpha \beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y} dx$$

By the definition of the Gamma function we have

$$M_X(t) = \frac{\Gamma(\alpha)\beta^\alpha}{(1-\beta t)^\alpha \beta^\alpha \Gamma(\alpha)}$$

$$\boxed{M_X(t) = (1 - \beta t)^{-\alpha}}$$

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5 p. 183 #17

If X is a random variable having an exponential distribution with the parameter θ, use Theorems 4.10 on page 128 and 6.4 to find the moment-generating function of the random variable $Y = X - \theta$.

By Theorem 6.4, the moment-generating function for the gamma distribution is given by

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

An exponential random variable is a special case of a gamma random variable, where $\alpha = 1$ and $\beta = \theta$. Thus, the moment generating function for an exponential random variable must be given by

$$M_X(t) = (1 - \theta t)^{-1} = \frac{1}{1 - \theta t}$$

By Theorem 4.10 we have

$$M_Y(t) = M_{X-\theta}(t) = \frac{e^{-\theta t}}{1 - \theta t}$$

$$\boxed{M_Y(t) = \frac{e^{-\theta t}}{1 - \theta t}}$$

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6 p. 183 #20

A random variable X has a **Rayleigh distribution** if and only if its probability distribution is given by

$$f(x) = \begin{cases} 2\alpha x e^{-\alpha x^2} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha > 0$. Show that for this distribution
(a) $\mu = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$;
(b) $\sigma^2 = \frac{1}{\alpha}\left(1 - \frac{\pi}{4}\right)$.

By Def. 4.2

$$\mu_r' = \int_{-\infty}^{\infty} 2\alpha x^{r+1} e^{-\alpha x^2} dx$$

Let $y = \alpha x^2,\ dy = 2\alpha x dx$

$$\mu_r' = \int_0^{\infty} \left(\frac{y}{\alpha}\right)^{\frac{r}{2}} e^{-y} dy$$

$$\mu_r' = \frac{1}{\sqrt{\alpha^r}} \int_0^{\infty} y^{\frac{r}{2}} e^{-y} dy$$

By the definition of the Gamma function, we have

$$\mu_r' = \frac{1}{\sqrt{\alpha^r}} \Gamma\left(\frac{r}{2} + 1\right)$$

$$\mu_1' = \frac{1}{\sqrt{\alpha^1}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\alpha}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\alpha}}$$

$$\mu_2' = \frac{1}{\sqrt{\alpha^2}} \Gamma(2) = \frac{1}{\alpha} 1! = \frac{1}{\alpha}$$

By Def. 4.3

$\boxed{\textbf{(a) } \mu = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}}$

By Theorem 4.6 we have

$$\sigma^2 = \mu_2' - {\mu_1'}^2$$

$$\sigma^2 = \frac{1}{\alpha} - \frac{\pi}{4\alpha}$$

$\boxed{\textbf{(b) } \sigma^2 = \frac{1}{\alpha}\left(1 - \frac{\pi}{4}\right)}$

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7 p. 184 #23

A random variable X has a **Weibull distribution** if and only if its probability density is given by

$$f(x) = \begin{cases} kx^{\beta-1}e^{-\alpha x^{\beta}} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.
(a) Express k in terms of α and β.
(b) Show that $\mu = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right)$

Since f(x) is a probability distribution, integrating over its full range results in 1. (Theorem 3.5)

$$\int_0^{\infty} kx^{\beta-1}e^{-\alpha x^{\beta}}dx = 1$$

Let $y = \alpha x^{\beta} \implies dy = \alpha x^{\beta-1}$

$$\int_0^{\infty} \frac{k}{\alpha}e^{-y}dy = \frac{k}{\alpha} = 1$$

$\boxed{\textbf{(a) } k = \alpha}$

By Def. 4.2 and 4.3

$$\mu = \int_0^{\infty} x \cdot \alpha x^{\beta-1}e^{-\alpha x^{\beta}}dx = \int_0^{\infty} \alpha x^{\beta}e^{-\alpha x^{\beta}}dx$$

Let $y = \alpha x^{\beta} \implies dy = \alpha x^{\beta-1},\ x = y^{1/\beta}\alpha^{-1/\beta}$

$$\mu = \int_0^{\infty} y^{1/\beta}\alpha^{-1/\beta}e^{-y}dy = \alpha^{-1/\beta}\int_0^{\infty} y^{1/\beta}e^{-y}dy$$

By the integral definition of the Gamma function we have

$\boxed{\textbf{(b) } \mu = \alpha^{-1/\beta}\Gamma\left(1 + \frac{1}{\beta}\right)}$

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8 p. 184 #26

Show that if $\alpha > 1$ and $\beta > 1$, the beta density has a relative maximum at

$$x = \frac{\alpha - 1}{\alpha + \beta - 2}$$

By Def. 6.5 the Beta distribution is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

Taking the derivative of the Beta distribution we have

$$f'(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left((\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} - (\beta-1)x^{\alpha-1}(1-x)^{\beta-2}\right)$$

$$f'(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(((\alpha-1)(1-x) - (\beta-1)x)x^{\alpha-2}(1-x)^{\beta-2}\right)$$

$$f'(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left((\alpha-1+(2-\alpha-\beta)x)x^{\alpha-2}(1-x)^{\beta-2}\right)$$

Since $0 < x < 1$ and $\lim_{x\to 0} f(x;\alpha,\beta) = \lim_{x\to 1} f(x;\alpha,\beta) = 0$ We know that if at least one relative maximum exists for some value of x between 0 and 1.

We find said critical point by letting $f'(x;\alpha,\beta) = 0$

$$f'(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left((\alpha-1+(2-\alpha-\beta)x)x^{\alpha-2}(1-x)^{\beta-2}\right) = 0$$

Once again because $0 < x < 1$ we can safely divide out by several terms.

$$\alpha - 1 + (2-\alpha-\beta)x = 0$$

$$\boxed{x = \frac{\alpha-1}{\alpha+\beta-2} \text{ is a relative maximum.}}$$

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9 p. 202 #54

The amount of time that a watch will run without having to be reset is a random variable having an exponential distribution with θ=120 days. Find the probabilities that the watch will
(a) have to be reset in less than 24 days;
(b) not have to be reset in at least 180 days. By Def. 6.3 an exponential distribution is given by

$$f(x) = \begin{cases} \frac{1}{\theta}e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\theta > 0$.

Thus, by Def. 3.4 we have

$$P(X < 24) = \int_0^{24} \frac{1}{120}e^{-x/120}$$

$$P(X < 24) = 1 - e^{-\frac{1}{5}} \approx 0.18127$$

$\boxed{\textbf{(a)}\text{ The probability the watch will have to be reset in less than 24 days is } \approx 0.18127}$

By Def. 3.4 we have

$$P(X > 180) = \int_{180}^{\infty} \frac{1}{120}e^{-x/120}$$

$$P(X > 180) = e^{-3/2} \approx 0.22313$$

$\boxed{\textbf{(b)}\text{ The probability it will not have to be reset in at least 180 days is } \approx 0.22313}$

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10 p. 202 #56

The number of bad checks that a bank receives during a 5-hour business day is a Poisson random variable with $\lambda = 2$. What is the probability that it will not receive a bad check on any one day during the first 2 hours of business?

The exponential random variable can describe the **waiting time** of a Poisson process, with $\theta = \frac{2}{5\lambda}$ by scaling the time interval appropriately.

Let random variable X represent the time before the bank receives a bad check on any one day. From our analysis above, we have

$$P(X = x) = 2e^{-2x} \textbf{ for } x > 0$$

Thus, by Def. 3.4 we have

$$P(X > 2) = \int_{2}^{\infty} \frac{4}{5} e^{-\frac{4}{5}x}$$

$$P(X > 2) = e^{-\frac{8}{5}} \approx 0.20190$$

$$\boxed{\begin{array}{c}\text{The probability that the bank will not receive a bad check on} \\ \text{any one day during the first 2 hours of business is} \approx 0.20190\end{array}}$$

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11 p. 202 #60

If the annual proportion of new restaurants that fail in a given city may be looked upon as a random variable having a beta distribution with $\alpha = 1$ and $\beta = 4$, find

(a) the mean of this distribution, that is, the annual proportion of new restaurants that can be expected to fail in the given city;

(b) the probability that at least 25 percent of all the new restaurants will fail in the given city in any one year.

By Def. 6.5 the Beta distribution is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

Since $P(X = x) = f(x; 1, 4)$, for $x > 0$,

$$P(X = x) = \frac{\Gamma(1+4)}{\Gamma(1)\Gamma(4)} x^{1-1}(1-x)^{4-1}$$

$$P(X = x) = \frac{4 \cdot 3 \cdot 2 \cdot 1\Gamma(1)}{\Gamma(1)3 \cdot 2 \cdot 1\Gamma(1)}(1-x)^3$$

$$P(X = x) = 4(1-x)^3$$

By Def. 4.2 and 4.3 we have

$$\mu = \int_0^1 x \cdot 4(1-x)^3 = \frac{1}{5}$$

$$\boxed{\textbf{(a)}\text{ The proportion of new restaurants that can be expected to fail is } \frac{1}{5}}$$

By Def. 3.4 we have

$$P(X > \frac{1}{4}) = \int_{\frac{1}{4}}^1 4(1-x)^3 = \frac{81}{256} \approx 0.31641$$

$$\boxed{\textbf{(b)}\text{ The probability that at least 25\% of the new restaurants will fail is } \approx 0.31641}$$