Regarding Positive Even Zeta Values

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1 Condition codes

Recall that $\mathbb{Z}_+^n$ is defined as

$$\mathbb{Z}_+^n = \{(x_1, x_2, ...x_n) | x_1, x_2, ...x_n \in \mathbb{Z}_+\}$$

We can define a disjoint partition on $\mathbb{Z}_+^n$ into subsets based on how elements of the tuple $(x_1, x_2, ...x_n)$ are equal to each other.

Definition 1.1. *Let S be a set of tuples. A **condition code** C on S is a tuple $(c_1, c_2, ...c_n)$ that gives S the following elementhood condition:*

> *A tuple x is an element of S if and only if there exists a tuple $t = (t_1, t_2, ...t_n) \in \mathbb{Z}_+^n$ such that for all i, x contains c_i copies of t_i.*

Definition 1.2. *A condition code C is **spicy** if it gives the following two elementhood conditions:*

> ***(a)** A tuple x is an element of S if and only if there exists a tuple $t = (t_1, t_2, ...t_n) \in \mathbb{Z}_+^n$ such that for all i, x contains c_i copies of t_i.*
>
> **(b)** *Each t_i is distinct.*

A condition code C that is not spicy is called **mild**

Definition 1.4. *A condition code C is **super spicy** if it gives the following two elementhood conditions:*

> ***(a)** A tuple x is an element of S if and only if there exists a tuple $t = (t_1, t_2, ...t_n) \in \mathbb{Z}_+^n$ such that for all i, x contains c_i copies of t_i.*
>
> **(b)** $t_i < t_{i+1}$ if $c_i = c_{i+1}$ *and* $t_i \neq t_{i+1}$ *if* $c_i \neq c_{i+1}$

Definition 1.4.

> **(a)** *A **mild subset** S of $\mathbb{Z}_n^+$ written $S(c_1)(c_2)...(c_k)$ is a subset of $\mathbb{Z}_n^+$ whose only restrictions on elementhood is a mild condition code.*
> **(b)** *A **spicy subset** S of $\mathbb{Z}_n^+$ written $S(c_1, c_2, ...c_k)$ is a subset of $\mathbb{Z}_n^+$ whose only restrictions on elementhood is a spicy condition code. S can be identified as a subset of a unique mild subset S'.*

(c) *A **super spicy subset** S of $\mathbb{Z}_n^+$ written the same as a spicy subset is a subset of $\mathbb{Z}_n^+$ whose only restrictions on elementhood is a super spicy condition code. S can be identified as a subset of a unique spicy subset S'.*

Note the following hierarchy:

- a mild subset is a subset of $\mathbb{Z}_+^n$.
- a spicy subset is a subset of a mild subset.
- a super spicy subset is a subset of a spicy subset.

Consider the following diagram

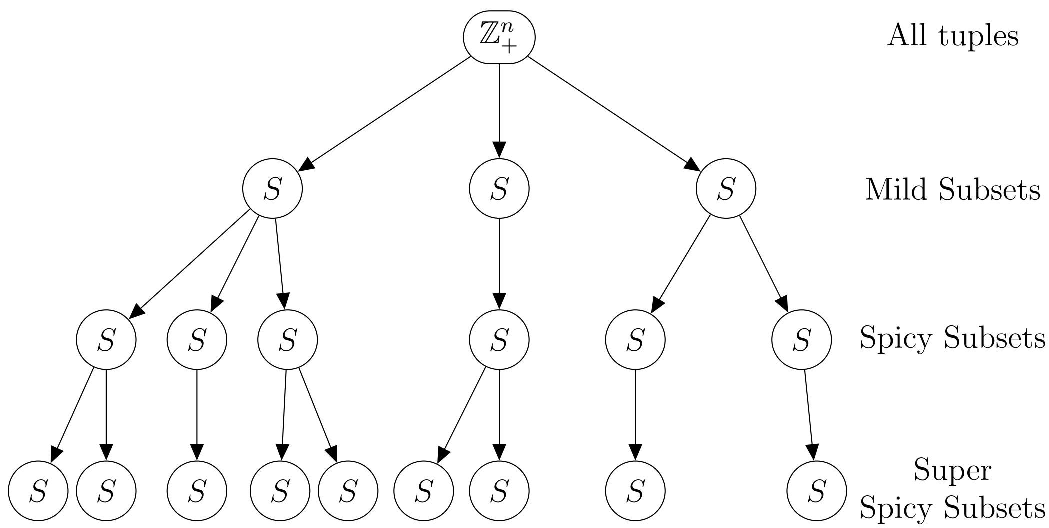

Figure 1: Partition Diagram

2 A Multi-layered Partition

Theorem 1.1. *Two condition codes are equivalent if they are permutations of each other.*
Proof. Consider any two condition codes $C_1 = (a_1, a_2, ...a_n)$ and $C_2 = (b_1, b_2, ...b_n)$ which are permutations of each other.
Assume some $x \in S(C_1)$.
By Definition 1.1, a tuple x is an element in $S(C_1)$ if for all i, there exists a tuple $(t_1, t_2, ...t_n) \in \mathbb{Z}_n^+$ such that x contains a_i copies of t_i.

Note that a permutation of said tuple $(s_1, s_2, ...s_n) \in \mathbb{Z}_n^+$ exists such that for all i, x contains b_i copies of s_i.

Thus, $x \in S(C_2)$ and $S(C_2) \subseteq S(C_1)$.
The same process can be done to show that $S(C_1) \subseteq S(C_2)$.

$\therefore S(C_1) = S(C_2)$ and the condition codes are equivalent.
$\blacksquare$

Theorem 1.2. *The set of all spicy subsets S with distinct condition codes C of $\mathbb{Z}_n^+$ forms a disjoint partition of $\mathbb{Z}_n^+$.*
Proof. Show that each distinct set is disjoint.
Consider any two distinct spicy subsets $S(C_1)$ and $S(C_2)$, where
$C_1 = (a_1, a_2, ...a_k),\ C_2 = (b_1, b_2, ...b_k)$
Suppose $x \in S_1 \cap S_2$.
$\exists(t_1, t_2, ...t_k) \ni$ x contains a_i copies of t_i for all i.
$\exists(s_1, s_2, ..s_k) \ni$ x contains b_i copies of s_i for all i.
This implies that $(t_1, t_2, ...t_k)$ is a permutation of $(s_1, s_2, ..s_k)$.
This further implies that $C_1 = (a_1, a_2, ...a_k),\ C_2 = (b_1, b_2, ...b_k)$ are permutations of each other and are equivalent by Theorem 1.1.
This is a contradiction.
Thus, there is no $x \in S_1 \cap S_2$.
$S_1 \cap S_2 = \varnothing$
$\square$
Show that $\bigcup P = \mathbb{Z}_+^n$. Where P is the set of all spicy subsets.
$S \in P \implies S \in \mathbb{Z}_+^n$ by definition. So $\bigcup P \subseteq \mathbb{Z}_+^n$.
Assume $x = (x_1, x_2, ...x_n) \in \mathbb{Z}_+^n$.
Let a_i be the first occurence of x_i in x.
Consider the tuple $(a_1, a_2, ...a_k)$ each of whose values are distinct values in x.
There exists some spicy condition code $C = (c_1, c_2, ...c_k)$ such that x contains c_i copies of a_i for all i.
By Definition 1.2, $x \in S(C) \in P$ where P is the collection of all spicy subsets.
by the Union Lemma, $x \in \bigcup P$
$\mathbb{Z}_+^n \subseteq \bigcup P$
$\therefore \bigcup P = \mathbb{Z}_+^n$
$\blacksquare$

3 The D function

Definition 1.5. *Let* $x = (x_1, x_2, ...x_n) \in \mathbb{Z}_+^n$ *and S(C) be a mild subset or a super spicy subset.*
Define $D : C \to \mathbb{R}$ *by*

$$\sum_{x \in S(C)} \frac{1}{(x_1)^2} \frac{1}{(x_2)^2} ... \frac{1}{(x_n)^2} = \sum_{x \in S(C)} \prod_{x_i} x_i^{-2}$$

For a mild condition code we write: $D(c_1)(c_2)...(c_n)$
For a super spicy condition code we write: $D(c_1, c_2...c_n)$

Theorem 1.3 *Let* C *be a mild condition code and* P *be the collection of all super spicy condition codes* C_i *which are subsets of* C. *Then*

$$D(C) = \sum_{C_i \in P} D(C_i)$$

Proof.
$\mathbb{Z}_+^n$ has a disjoint parition into super spicy subsets by Theorem 1.2.
Since all super spicy subsets are subsets of mild subsets, every spicy subset has a dijoint partion into super spicy subsets.
The sum over any set is always equal to the sum over each subset of a disjoint partition.
Let $x = (x_1, x_2, ...x_n) \in \mathbb{Z}_+^n$, C be a mild condition code, and each C_k be super spicy condition codes.

$$\sum_{x \in S(C)} \prod_{x_i} x_i^{-2} = \sum_{S(C_k) \subseteq S(C)} \sum_{x \in S(C_k)} \prod_{x_i} x_i^{-2}$$

$$\therefore D(C) = \sum_{C_i \in P} D(C_i)$$

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Definition 1.6
The shorthand for writing $D(a, a, a...a)$ *with* n as *is* $D_n(a)$.
The shorthand for writing $D(a)(a)(a)...(a)$ *with* n as *is* $D^n(a)$.

Theorem 1.5 $D(a)D(b) = D(a)(b)\ \forall a, b \in \mathbb{Z}_+$
Proof. By the

Theorem 1.6 $D_n(1) = \frac{\pi^{2n}}{(2n+1)!}$
Proof. We start by comparing the MacLaurin series of $\sinh(x)$ with the Euler product of $\sinh(x)$.

$$\sinh(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} ... = x\left(1 + \frac{x}{i\pi}\right)\left(1 - \frac{x}{i\pi}\right)\left(1 + \frac{x}{2i\pi}\right)\left(1 - \frac{x}{2i\pi}\right)...$$

Thus, by a simple calculation,

$$\frac{\sinh(\pi x)}{x} = 1 + \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + \frac{\pi^6 x^6}{7!} ... = \left(1 + \frac{x^2}{1^2}\right)\left(1 + \frac{x^2}{2^2}\right)\left(1 + \frac{x^2}{3^2}\right)...$$

When multiplying a product, to calculate the coefficient on a polynomial of the nth degree, all term combinations resulting in the nth degree of each factor must be determined and subsequently summed.

For the product expansion, the only term combinations resulting in degree 2 are when we select one x^2 term from one factor and select a 1 from each of the other factors. Comparing this to the right hand side we have

$$\frac{\pi^2}{3!} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} ... = D(1)$$

Continuing this process we find that

$$\sum_i \sum_{j<i} \frac{1}{i^2} \frac{1}{j^2} = \frac{\pi^4}{5!} = D(1, 1)$$

And in general,

$$\sum_{0<x_1} \sum_{x_2<x_1} \cdots \sum_{x_n<x_{n-1}} \frac{1}{x_1^2}\frac{1}{x_2^2} \cdots \frac{1}{x_n^2} = \frac{\pi^{2n}}{(2n+1)!} = D_n(1)$$

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4 References

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