Topology Homework 04

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EXERCISE 2.1 Determine Int(A) and Cl(A) in each case.

(a) $A = (0, 1]$ in the lower limit topology on $\mathbb{R}$.
(b) $A = \{a\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$.
(c) $A = \{a, c\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$.
(d) $A = \{b\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$.
(e) $A = (-1, 1) \cup \{2\}$ in the standard topology on $\mathbb{R}$.
(f) $A = (-1, 1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$.
(g) $A = \{(x, 0) \in \mathbb{R}^2 |\ x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology.
(h) $A = \{(0, y) \in \mathbb{R}^2 |\ y \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the vertical interval topology.
(i) $A = \{(x, 0) \in \mathbb{R}^2 |\ x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the vertical interval topology.

(a) $\boxed{\mathrm{Int(A)} = (0, 1),\ \mathrm{Cl(A)} = (0, 1]}$

(b) $\boxed{\mathrm{Int(A)} = \{a\},\ \mathrm{Cl(A)} = X}$

(c) $\boxed{\mathrm{Int(A)} = \{a\},\ \mathrm{Cl(A)} = X}$

(d) $\boxed{\mathrm{Int(A)} = \varnothing,\ \mathrm{Cl(A)} = \{c, b\}}$

(e) $\boxed{\mathrm{Int(A)} = (-1, 1),\ \mathrm{Cl(A)} = [-1, 1] \cup \{2\}}$

(f) $\boxed{\mathrm{Int(A)} = [-1, 1),\ \mathrm{Cl(A)} = [-1, 1) \cup \{2\}}$

(g) $\boxed{\mathrm{Int(A)} = \varnothing,\ \mathrm{Cl(A)} = A}$

(h) $\boxed{\mathrm{Int(A)} = A,\ \mathrm{Cl(A)} = A}$

(i) $\boxed{\mathrm{Int(A)} = \varnothing,\ \mathrm{Cl(A)} = A}$

EXERCISE 2.2 Prove Theorem 2.2, parts (ii), (iv), and (vi): Let X be a topological space and A and B be subsets of X.

(a) If C is a closed set in X and $A \subseteq C$ then Cl(A) $\subseteq C$.

(b) If $A \subseteq B$ then Cl(A) $\subseteq$ Cl(B).

(c) A is closed if and only if $A =$ Cl(A).

(a) Proof.
Let C be a closed set in X and $A \subseteq C$.
By Definition,

$$\mathrm{Cl(A)} = \bigcap_{A \subseteq C_i} C_i$$

Since $A \subseteq C$

$$\bigcap_{A \subseteq C_i} C_i \subseteq C$$

$\therefore$ Cl(A) $\subseteq C$.
■

(b) Proof.
Assume $A \subseteq B$.
Since $B \subseteq$ Cl(B), $A \subseteq$ Cl(B). By Theorem 2.2 part ii,
Cl(A) $\subseteq$ Cl(B).
■

(c) Proof.
($\rightarrow$) Assume A is closed. We know that $A \subseteq$ Cl(A).
By Definition,

$$\mathrm{Cl(A)} = \bigcap_{A \subseteq C_i} C_i$$

Since A is a closed set and $A \subseteq A$,
Cl(A) $\subseteq A$.
Thus, $A =$Cl(A).
□
($\leftarrow$) Assume $A =$Cl(A).
A = Cl(A) is an intersection of closed sets and is closed.
■

EXERCISE 2.4 Consider the particular point topology PPX_p on a set X. Determine Int(A) and Cl(A) for sets A containing p and for sets A not containing p.

If A contains p,
Int(A) $= A$, Cl(A) $= X$.

If A does not contain p,
Int(A) $= \varnothing$, Cl(A) $= A$.

EXERCISE 2.6 Prove that $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$ in the standard topology on $\mathbb{R}$.

Proof.
Let $\mathfrak{T}$ be the standard topology on $\mathbb{R}$.
Since $\mathrm{Cl}(\mathbb{Q})$ is closed, $\exists U \in \mathfrak{T} \ni \mathbb{R}-\mathrm{Cl}(\mathbb{Q}) = U$
Since $\mathbb{Q} \subseteq \mathrm{Cl}(\mathbb{Q})$, $U \subseteq \mathbb{R} - \mathbb{Q}$.

So U does not contain any rational numbers.
But, since $\mathbb{Q}$ is dense, every open interval contains rational numbers.
This means $U = \varnothing$.

$\mathbb{R}-\mathrm{Cl}(\mathbb{Q}) = \varnothing$
$\therefore \mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$.
$\blacksquare$

EXERCISE 2.10 Prove Theorem 2.5. Let X be a topological space, A be a subset of X and y be an element of X. Then $y \in$ Cl(A) if and only if every open set containing y intersects A.

Proof.
($\rightarrow$)Let $y \in$ Cl(A).
By the defintion of Cl(A),

$$y \in \bigcap_{\substack{C \subseteq A \\ \text{C closed}}} C$$

By Demorgan's Law, we have

$$y \notin \bigcup_{\substack{U \subseteq X - A \\ \text{U open}}} U$$

Thus, if $y \in U$, where U is open, then $U \nsubseteq X - A$.
$\therefore$ Every open set containing y intersects A.
$\square$
($\leftarrow$) Assume every open set containing y intersects A.
Let $U = X - \mathrm{Cl}(A)$.
Since $\mathrm{Cl}(A)$ is closed, U is open.
$y \notin U$ since U does not intersect A.
Thus, $y \in X - U$.
$\therefore y \in \mathrm{Cl}(A)$.
$\square$
$\therefore y \in$ Cl(A) if and only if every open set containing y intersects A.
$\blacksquare$

EXERCISE 2.11 Prove Theorems 2.6 parts (ii) and (iv): For sets A and B in a topological space X, the following hold:

(a) Cl$(X - A) = X-$ Int(A)

(b) Int$(A)\cap$ Int$(B) =$ Int$(A \cap B)$.

(a) Proof.

Theorem 2.6 part (i) states that for any $A \subseteq X$

$$Int(X - A) = X - Cl(A)$$

Substituting $X - A$ for A we have

$$Int(A) = X - Cl(X - A)$$

$\therefore$ Cl$(X - A) = X-$ Int(A)

$\blacksquare$

(b) Proof.

$(\rightarrow)$Let $x \in Int(A) \cap Int(B)$

$x \in Int(A)$ and $x \in Int(B)$.

Int$(A) \subseteq A$ and Int$(B) \subseteq B$

$x \in A$ and $x \in B$.

$x \in A \cap B$.

$Int(A) \cap Int(B) \subseteq Int(A) \cap B \subseteq A \cap B$.

Also, $Int(A) \cap Int(B)$ is an intersection of open sets, which is open.

So x is an element of an open subset of $A \cap B$.

Since Int$(A \cap B)$ is the union of open subsets of $A \cap B$ we can write

$x \in$ Int$(A \cap B)$

$Int(A) \cap Int(B) \subseteq$ Int$(A \cap B)$

$\square$

$(\leftarrow)$Let $x \in$ Int$(A \cap B)$.

By Definition, Int$(A \cap B)$ is an open set.

Int$(A \cap B) \subseteq A \cap B$

Int$(A \cap B) \subseteq A$ and Int$(A \cap B) \subseteq B$

So x is in an open set in A and x is in an open set in B.

This means that $X \in$ Int(A) and $x \in$ Int(B).

Thus, $x \in Int(A) \cap Int(B)$.

Int$(A \cap B) \subseteq Int(A) \cap Int(B)$

$\square$

Int$(A)\cap$ Int$(B) =$ Int$(A \cap B)$.

$\blacksquare$