Math 332 A - Mathematical Statistics

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1 p. 240 #16

Find the mean and the variance of the finite population that consists of the 10 numbers 15, 13, 18, 10, 6, 21, 7, 11, 20, and 9.

By Definition 8.5 the sample mean is defined by

$$\mu = \sum_{i=1}^{N} c_i \frac{1}{N}$$

For our population, $N = 10$. By a simple calculation,

$$\sum_{i=1}^{11} c_i = 130$$

$\boxed{\mu = 13}$

By Definition 8.5 the sample variance is defined by

$$\sigma^2 = \sum_{i=1}^{N} (c_i - \mu)^2 \frac{1}{N}$$

By a simple calculation,

$$\sum_{i=1}^{11} (c_i - \mu)^2 = 256$$

$\boxed{\sigma^2 = \frac{256}{13} \approx 19.6923}$

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2 p. 249 #21

Prove Theorem 8.10.

Theorem 8.10. If X_1 and X_2 are indepenedent random variables, X_1 has a chi-square distribution with ν_1 degrees of freedom, and $X_1 + X_2$ has a chi-square distribution with $\nu > \nu_1$ degrees of freedom, then X_2 has a chi-square distribution with $\nu - \nu_1$ degrees of freedom.
Proof.
This proof uses the moment-generating function technique.
By Theorem 7.3. We have

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t)$$

The moment generating functions of X_1 and X_2 are:

$$M_{X_1}(t) = (1-2t)^{-\nu_1/2} \text{ and } M_{X_1+X_2}(t) = (1-2t)^{-\nu/2}$$

Plugging them into the first equation, we have

$$(1-2t)^{-\nu/2} = (1-2t)^{-\nu_1/2}M_{X_2}(t)$$

$$M_{X_2}(t) = (1-2t)^{-\nu/2}(1-2t)^{\nu_1/2}$$
$$M_{X_2}(t) = (1-2t)^{-(\nu-\nu_1)/2}$$

We recognise this as the moment generating function for a chi-square random variable with $\nu - \nu_1$ degrees of freedom.

Random variables can be uniquely identified by their moment-generating functions.
$\therefore X_2$ is a chi-square random variable with $\nu - \nu_1$ degrees of freedom.
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3 p. 249 #23

Use Theorem 8.11 to show that, for random samples of size n from a normal population with the variance σ^2, the sampling distribution of S^2 has the mean σ^2 and the variance $\frac{2\sigma^4}{n-1}$.

By Theorem 8.11

$X = \dfrac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with $n-1$ degrees of freedom.

$\dfrac{(n-1)S^2}{\sigma^2}$ has the moment-generating function $M_X(t) = (1-2t)^{(n-1)/2}$

Note that $S^2 = \frac{\sigma^2}{n-1}X$. By Theorem 4.10,

$$S^2 \text{ has the moment generating function } M_{S^2}(t) = \left(1 - \frac{2\sigma^2}{n-1}t\right)^{-(n-1)/2}$$

$$\frac{d}{dt}\left(1 - \frac{2\sigma^2}{n-1}t\right)^{-(n-1)/2} = \sigma^2\left(1 - \frac{2\sigma^2}{n-1}t\right)^{-(n+1)/2}$$

$$\frac{d^2}{dt^2} = \left(1 - \frac{2\sigma^2}{n-1}t\right)^{-(n-1)/2} = \frac{\sigma^2(n+1)}{n-1}\left(1 - \frac{2\sigma^2}{n-1}t\right)^{-(n+3)/2}$$

By the definition of the moment-generating function,

$$\mu_{S^2} = \mu_1' = M'(0) = \sigma^2, \mu_2' = M''(0) = \frac{\sigma^2(n+1)}{n-1}$$

By Theorem 4.6,

$$\sigma_{S^2}^2 = \mu_2' - {\mu_1'}^2$$

$$\sigma_{S^2}^2 = \frac{\sigma^2(n+1)}{n-1} - \sigma^4 = \frac{2\sigma^4}{n-1}$$

$$\boxed{\mu_{S^2} = \sigma^2, \quad \sigma_{S^2}^2 = \frac{2\sigma^4}{n-1}}$$

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4 p. 256 #60

How many different samples of size $n = 3$ can be drawn from a finite population of size
(a) $N = 12$; **(b)** $N = 20$; **(c)** $N = 50$?

The number of samples that can be drawn is simply $\binom{N}{n}$.

$$\binom{12}{3} = 220, \ \binom{20}{3} = 1540, \ \binom{50}{3} = 22100;$$

$\boxed{\textbf{(a) } N = 12 : 220, \quad \textbf{(b) } N = 20 : 1540, \quad \textbf{(c) } N = 50 : 22100}$

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5 p. 256 #61

What is the probability of each possible sample if
(a) a random sample of size $n = 4$ is to be drawn from a finite population of size $N = 12$;
(b) a random sample of size $n = 5$ is to be drawn from a finite population of size $N = 22$?

(a) In this instance, there are $\binom{12}{4} = 495$ possible choices for the random sample. The probability of picking any particular sample satisfying those conditions is therefore 1/495.

$$\boxed{\textbf{(a)}\text{ The probability of that sample is } \frac{1}{495}}$$

(b) In this instance, there are $\binom{22}{5} = 26334$ possible choices for the random sample. The probability of picking any particular sample satisfying those conditions is therefore 1/26334.

$$\boxed{\textbf{(b)}\text{ The probability of that sample is } \frac{1}{26334}}$$

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6 p. 256 #62

If a random sample of size $n = 3$ is drawn from a finite population of size $N = 50$, what is the probability that a particular element of the population will be included in the sample?

$$\boxed{\text{The probability that a particular element will be included in the sample is } \tfrac{3}{50}.}$$

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7 p. 257 #78

The claim that the variance of a normal population is $\sigma^2 = 25$ is to be rejected if the variance of a random sample of size 16 exceeds 54.668 or is less that 12.102. What is the probability that this claim will be rejected even though $\sigma^2 = 25$?

In mathematical terms, the question is asking for the value of $E = 1 - P(12.102 \leq S^2 \leq 54.668)$.
Note that $\frac{n-1}{\sigma^2} = \frac{15}{25} = \frac{3}{5}$.

$$P(12.102 \leq S^2 \leq 54.668) = P(7.2612 \leq \frac{n-1}{\sigma^2}S^2 \leq 32.8008)$$

By Theorem 8.1, $\frac{n-1}{\sigma^2}S^2$ has a chi-square distribution with $\nu = n - 1 = 15$. Thus,

$$P(7.2612 \leq \frac{n-1}{\sigma^2}S^2 \leq 32.8008) = \int_{7.2612}^{32.8008} \frac{1}{2^{7.5}\Gamma(7.5)} x^{6.5} e^{-x/2} dx$$

This integral can be computed by hand by doing integration by parts, u-substitutions and converting to a polar integral. By Wolfram Alpha,

$$P(7.2612 \leq \frac{n-1}{\sigma^2}S^2 \leq 32.8008) \approx 0.944991$$

$$1 - P(12.102 \leq S^2 \leq 54.668) \approx 1 - 0.944991 = 0.055009$$

$\boxed{\text{The probability the claim will be rejected is about 0.055009 or 5.5009\%}}$