Topology Homework

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1 Exercise 1.27 a

The **infinite comb** C is the subset of the plane illustrated in Figure 1.17 and defined by

$$C = \{(x, 0) \mid 0 \leq x \leq 1\} \ \cup \ \{(\frac{1}{2^n}, y) \mid n = 0, 1, 2, ... \text{ and } 0 \leq y \leq 1\}$$

(a) Prove that C is not closed in the standard topology on $\mathbb{R}^2$.

(a) Proof.
Suppose C is closed in the standard topology on $\mathbb{R}^2$.
By Def. 1.14, C' is open in the standard topology on $\mathbb{R}^2$.
Since $(0, \frac{1}{2}) \notin C$, $(0, \frac{1}{2}) \in C'$. Thus, there must exist an open ball $B((x, y), r + \epsilon)$ containing $(0, \frac{1}{2})$, where $r = d((x, y), (0, \frac{1}{2}))$, $\epsilon > 0$.
For all $\epsilon \in \mathbb{R}, \exists n \in \mathbb{Z}_+$ such that $0 < \frac{1}{2^n} < \epsilon$

$$\frac{2}{2^n} < 2\epsilon, \quad \frac{1}{2^{2n}} < \epsilon$$

$$x^2 + (y - \frac{1}{2})^2 + \frac{2}{2^n}\sqrt{x^2 + (y - \frac{1}{2})^2} + \frac{1}{2^{2n}} < x^2 + (y - \frac{1}{2})^2 + 2\epsilon\sqrt{x^2 + (y - \frac{1}{2})^2} + \epsilon^2$$

$$x^2 + \frac{2}{2^n}x + \frac{1}{2^{2n}} + (y - \frac{1}{2})^2 < (r + \epsilon)^2$$

$$d\left((x, y), (\frac{1}{2^n}, \frac{1}{2})\right)^2 < (r + \epsilon)^2$$

$$d\left((x, y), (\frac{1}{2^n}, \frac{1}{2})\right) < r + \epsilon$$

Thus, the point $(\frac{1}{2^n}, \frac{1}{2})$ is contained in B.
Thus, $(\frac{1}{2^n}, \frac{1}{2})$ is contained in C'. But it is also contained in C by definition.
This is a contradiction.
$\therefore$ C is not closed in the standard topology on $\mathbb{R}^2$.
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