Regarding Positive Even Zeta Values

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1 Analyzing Positive Even Zeta Values

The Riemann Zeta function is a complex-valued function that accepts a complex number as an argument. This paper will focus entirely on the Zeta function evaluated at positive even integer values. Positive even Zeta values have an exact form in terms of π and can be calculated recursively several ways. This paper will explain two different ways of calculating said values.

The Riemann Zeta and Dirichlet Eta functions are defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

Trivially, we can show that

$$\eta(s) = (1 - 2^{1-s})\zeta(s)$$

Proof.

$$\eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} ...$$

$$\eta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} ... - 2\left[\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} ...\right]$$

$$\eta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} ... - \frac{2}{2^s}\left[\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} ...\right]$$

$$\eta(s) = \zeta(s) - \frac{2}{2^s}\zeta(s) = (1 - 2^{1-s})\zeta(s)$$

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2 Infinite Polynomials

Theorem 1.1.

$$1+\frac{\pi^2x^2}{3!}+\frac{\pi^4x^4}{5!}+\frac{\pi^6x^6}{7!}...=\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{2^2}\right)\left(1+\frac{x^2}{3^2}\right)...$$

Proof.
We start by comparing the MacLaurin series of $\sinh(x)$ with the product expansion of $\sinh(x)$. Note that the roots of $\sinh(x)$ are $k\pi i$ by Euler's formula.

$$\sinh(x)=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}...=x\left(1+\frac{x}{i\pi}\right)\left(1-\frac{x}{i\pi}\right)\left(1+\frac{x}{2i\pi}\right)\left(1-\frac{x}{2i\pi}\right)\left(1+\frac{x}{3i\pi}\right)...$$

$$\frac{\sinh(x)}{x}=\frac{1}{1!}+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}...=\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)...$$

$$\frac{\sinh(\pi x)}{x}=1+\frac{\pi^2x^2}{3!}+\frac{\pi^4x^4}{5!}+\frac{\pi^6x^6}{7!}...=\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{2^2}\right)\left(1+\frac{x^2}{3^2}\right)...$$

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When multiplying a product, to calculate the coefficient on a polynomial of the nth degree, all term combinations resulting in the nth degree of each factor must be determined and subsequently summed.

For the product expansion, the only term combinations resulting in degree 2 are when we select one x^2 term from one factor at a time and select a 1 from the other factors. Comparing this to the right hand side we have

$$\frac{\pi^2}{3!}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}...$$

Thus, using our definition of the Zeta function we have

$$\zeta(2)=\frac{\pi^2}{6}.$$

This result is incredible in and of itself, but Theorem 1.1 is the basis for computing all positive even Zeta values.

To find subsequent even Zeta values, coefficients of higher degree must be calculated for the infinite product and compared to the coefficient on the corresponding term in the infinite sum.

For example, Theorem 1.1 implies that

$$\sum_i\sum_{j<i}\frac{1}{i^2}\frac{1}{j^2}=\frac{\pi^4}{5!}$$

3 Motivation for Using Disjoint Partitions

It is often useful to use geometry to aid in understanding a concept in smaller dimensions, and then using algebra to generalize to higher dimensions. Consider the following infinitely big "Addition Square"

$$\begin{array}{ccccccc}
\frac{1}{1^2}\cdot\frac{1}{1^2} & + & \frac{1}{1^2}\cdot\frac{1}{2^2} & + & \frac{1}{1^2}\cdot\frac{1}{3^2} & + & \frac{1}{1^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{2^2}\cdot\frac{1}{1^2} & + & \frac{1}{2^2}\cdot\frac{1}{2^2} & + & \frac{1}{2^2}\cdot\frac{1}{3^2} & + & \frac{1}{2^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{3^2}\cdot\frac{1}{1^2} & + & \frac{1}{3^2}\cdot\frac{1}{2^2} & + & \frac{1}{3^2}\cdot\frac{1}{3^2} & + & \frac{1}{3^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{4^2}\cdot\frac{1}{1^2} & + & \frac{1}{4^2}\cdot\frac{1}{2^2} & + & \frac{1}{4^2}\cdot\frac{1}{3^2} & + & \frac{1}{4^2}\cdot\frac{1}{4^2} & \cdots \\
\vdots & & & & & & \vdots & \ddots
\end{array}$$

Figure 1: Addition Square

By the distributive property, the "Addition Square" can be written as

$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}...\right)^2=\zeta(2)^2=\frac{\pi^4}{36}$$

The Addition Square can also be split up into three smaller summations.

$$\begin{array}{ccccccc}
\boxed{\frac{1}{1^2}\cdot\frac{1}{1^2}} & + & \boxed{\frac{1}{1^2}\cdot\frac{1}{2^2}} & + & \boxed{\frac{1}{1^2}\cdot\frac{1}{3^2}} & + & \boxed{\frac{1}{1^2}\cdot\frac{1}{4^2}} & \cdots \\
\boxed{\frac{1}{2^2}\cdot\frac{1}{1^2}} & + & \boxed{\frac{1}{2^2}\cdot\frac{1}{2^2}} & + & \boxed{\frac{1}{2^2}\cdot\frac{1}{3^2}} & + & \boxed{\frac{1}{2^2}\cdot\frac{1}{4^2}} & \cdots \\
\boxed{\frac{1}{3^2}\cdot\frac{1}{1^2}} & + & \boxed{\frac{1}{3^2}\cdot\frac{1}{2^2}} & + & \boxed{\frac{1}{3^2}\cdot\frac{1}{3^2}} & + & \boxed{\frac{1}{3^2}\cdot\frac{1}{4^2}} & \cdots \\
\boxed{\frac{1}{4^2}\cdot\frac{1}{1^2}} & + & \boxed{\frac{1}{4^2}\cdot\frac{1}{2^2}} & + & \boxed{\frac{1}{4^2}\cdot\frac{1}{3^2}} & + & \boxed{\frac{1}{4^2}\cdot\frac{1}{4^2}} & \cdots \\
\vdots & & & & & & \vdots & \ddots
\end{array}$$

Figure 2: Addition Partition

The Diagonal Summation is equal to $\zeta(4)$. As for the Triangle Summations, we make use of Theorem 1.1.

$$1+\frac{\pi^2x^2}{3!}+\frac{\pi^4x^4}{5!}+\frac{\pi^6x^6}{7!}...=\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{2^2}\right)\left(1+\frac{x^2}{3^2}\right)...$$

Each Triangle Summation is the term sum of all term combinations of two factors in the infinite product and thus equal to the corresponding series expansion coefficient $\frac{\pi^4}{5!}$. Thus,

$$\frac{\pi^4}{36}=2\frac{\pi^4}{120}+\zeta(4)\implies\zeta(4)=\frac{\pi^4}{90}$$

4 Condition Codes for Disjoint Partitions

One useful way of constructing a disjoint partition of a multidimensional set is to separate sets by grouping. More specifically, separating sets based on the sizes of groups in which elements in particular dimensions are equal to each other. For the purposes of this paper, the Universal set for each element $x_1, x_2, ...$ will be $\mathbb{Z}^+$.

This partitioning means that sets can be given a unique condition code that describes the amount of groups of a certain size contained in said set.

For example, the set $\{(x_1, x_2, x_3) : x_1 = x_2 \neq x_3\}$ can be given the condition code (1,2) signifying exactly 1 group of size 1 (x_3), and one group of size 2 ($x_1 = x_2$).

We can then describe our universal set ${\mathbb{Z}^+}^n$ as the union of all the subsets where the condition codes are of the form $(a_1, a_2, a_3 ... a_k)$ where $a_1 + a_2 + a_3 ... + a_k = n$.

Example 1.1
Express ${\mathbb{Z}^+}^4$ as a disjoint partition of sets.

$${\mathbb{Z}^+}^4 = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$$

where the condition codes for the sets A_1 through A_5 are as follows.

$$A_1 : (1,1,1,1); \quad A_2 : (1,1,2); \quad A_3 : (1,3); \quad A_4 : (2,2); \quad A_5 : (4)$$

In set-builder notation, each of the sets A_1 through A_5 are themselves a union of smaller subsets with similar equality and inequality conditions. For example A_3 is the union of the four following smaller subsets which will be referred to as **spicy subsets**.

$$A_3 = \{(x_1, x_2, x_3, x_4) : x_1 = x_2 = x_3 \neq x_4\} \cup \{(x_1, x_2, x_3, x_4) : x_1 = x_2 = x_4 \neq x_3\}$$

$$\cup \{(x_1, x_2, x_3, x_4) : x_1 = x_3 = x_4 \neq x_2\} \cup \{(x_1, x_2, x_3, x_4) : x_2 = x_3 = x_4 \neq x_1\}$$

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This partition is complete because it ranges over the full possibilities for the condition codes. It is disjoint because the conditions of each set are unique.

The exact number of spicy subsets a for each A_1 through A_5 is very important and can be determined using an analogy called "Blocks and Holes" which will be discussed later.

5 The D Function

Definition 1.1.

$$D(a_1, a_2, a_3...) = \sum_{a \in A} \frac{1}{(x_1)^2} \frac{1}{(x_2)^2} ... \frac{1}{(x_n)^2}, \text{ } A \text{ has condition code } (a_1, a_2, a_3...)$$

where a is a previously mentioned spicy subset of A.
For example,

$$D(1,2) = \frac{1}{1^2}\frac{1}{2^4} + \frac{1}{1^2}\frac{1}{3^4} + \frac{1}{1^2}\frac{1}{4^4} + \frac{1}{2^2}\frac{1}{1^4} + \frac{1}{2^2}\frac{1}{2^4} + \frac{1}{2^2}\frac{1}{3^4} + \frac{1}{2^2}\frac{1}{4^4} + \frac{1}{3^2}\frac{1}{1^4}...$$

Theorem 1.2. $D(n) = \zeta(2n), n \in \mathbb{Z}^+$
Proof.
By Definition 1.1

$$D(n) = \sum_{a \in A} \frac{1}{(x_1)^2} \frac{1}{(x_2)^2} ... \frac{1}{(x_3)^2}, \quad A = \mathbb{Z}^{+^n}$$

by the transitive property of equality $\exists! a \in A$ which implies that $a = \mathbb{Z}^{+^n}$

$$D(n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \zeta(2n)$$

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Definition 1.2.

$$D^k(a_1, a_2, ...a_n) = (D(a_1, a_2, ...a_n))^k$$

$$D_k(a_1) = D(a_1, a_1, a_1...a_1), \text{ which has k arguments}$$

For example,

$$D^3(2) = D(2)D(2)D(2)$$

$$D_3(1) = D(1,1,1)$$

Theorem 1.3. $D_n(1) = \frac{\pi^{2n}}{(2n+1)!}$
Proof.
By Theorem 1.1 The product form must match the series form of $\frac{\sinh(x)}{x}$. It's coefficient is calculated by the sum of n term combinations of coefficients of x^{2n}. Thus,

$$\frac{1}{(2n+1)!} = \sum_{a} \frac{1}{\pi^2(x_1)^2} \frac{1}{\pi^2(x_2)^2} ... \frac{1}{\pi^2(x_n)^2}, \text{ } a = \{(x_1, x_2, ...x_n) : x_1 < x_2... < x_n\}$$

$$\frac{\pi^{2n}}{(2n+1)!} = \sum_{a \in A} \frac{1}{(x_1)^2} \frac{1}{(x_2)^2} ... \frac{1}{(x_n)^2}, \quad A = (1,1,1,1,1...) \text{ with n terms.}$$

$$\frac{\pi^{2n}}{(2n+1)!} = D_n(1)$$

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6 Algebra with the D function

Via **Theorem 1.2** and **Theorem 1.3** we have a connection between the Zeta function and π. To do the algebra, we need one final Theorem.
Theorem 1.4
$D(C)$ can always be expressed as a sum $p_1D(C_1) + p_2D(C_2) + p_3D(C_3)... + p_kD(C_k)$, where p_i is the number of spicy subsets for the set A_i with the corresponding condition code C_i.
Example 1.2
Calculate $\zeta(6)$, where $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$
By Theorem 1.4 we have the equations

$$D(1)(1)(1) = 6D(1,1,1) + 3D(1,2) + D(3) \tag{1}$$

$$D(1)(2) = D(1,2) + D(3) \tag{2}$$

From (2)

$$D(1,2) = D(1)(2) - D(3) \tag{3}$$

Substituting (3) into (1)

$$D(1)(1)(1) = 6D(1,1,1) + 3D(1)(2) - 2D(3) \tag{4}$$

By Theorem 1.2 and 1.3

$$\zeta(2)\zeta(2)\zeta(2) = 6\frac{\pi^6}{7!} + 3\zeta(2)\zeta(4) - 2\zeta(6) \tag{5}$$

Plugging in values for $\zeta(2)$ and $\zeta(4)$ we have

$$\left(\frac{\pi^2}{6}\right)^3 = 6\frac{\pi^6}{7!} + 3\left(\frac{\pi^2}{6}\right)\left(\frac{\pi^4}{90}\right) - 2\zeta(6) \tag{6}$$

Finally, we solve for $\zeta(6)$.

$$\boxed{\zeta(6) = \frac{\pi^6}{945}}$$

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In equation 1, the coefficients 6,3, and 1, can be calculated by enumeration.

$$\begin{aligned}
\{(x_1,x_2,x_3)\} = &\{(x_1,x_2,x_3) : x_1 < x_2 < x_3\} \cup \{(x_2,x_1,x_3) : x_2 \neq x_1 = x_3\} \cup \\
&\{(x_1,x_2,x_3) : x_1 < x_2 < x_3\} \cup \{(x_1,x_2,x_3) : x_1 \neq x_2 = x_3\} \cup \\
&\{(x_1,x_2,x_3) : x_1 < x_2 < x_3\} \cup \{(x_1,x_2,x_3) : x_1 = x_2 \neq x_3\} \cup \\
&\{(x_1,x_2,x_3) : x_1 < x_2 < x_3\} \cup \\
&\{(x_1,x_2,x_3) : x_1 < x_2 < x_3\} \cup \\
&\{(x_1,x_2,x_3) : x_1 < x_2 < x_3\} \cup \ \{(x_2,x_1,x_3) : x_1 = x_2 = x_3\}
\end{aligned}$$

Figure 3: Spicy Subset Enumeration

7 Blocks and Holes

We can use an analogy to calculate the number of spicy subsets.

Consider the following scenario. The scenario consists of a collection of labeled blocks of varying sizes and a collection of labeled holes of varying sizes. The goal is to count the number of ways of distributing the blocks into the holes perfectly.

- All blocks and holes are rectangles 1 unit wide. Their lengths are a whole number of units.
- The total area of the blocks equals the total area of the holes.
- The different ordering of blocks in a hole does not count as a separate arrangement.

Example 1.3
Determine the number of ways of distributing the collection of blocks [1,1,2] into the holes [1,3].

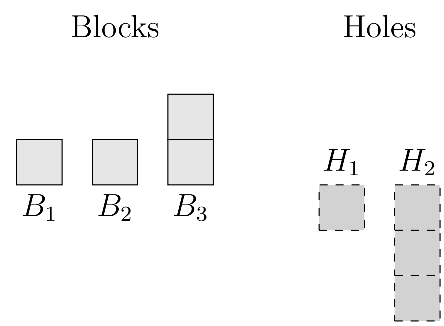

Figure 4: Blocks and Holes Scenario 1.1

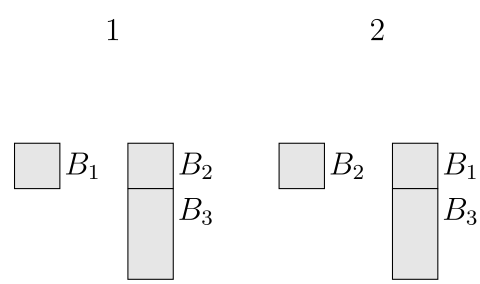

Figure 5: Blocks and Holes Solution 1.1

Thus, by enumeration, the number of arrangements of the Blocks [1,1,2] into the Holes [1,3] is 2.

8 Blocks and Holes cont.

Example 1.4 Determine the number of ways of distributing the collection of Blocks[1,1,1] into the Holes [1,1,1].

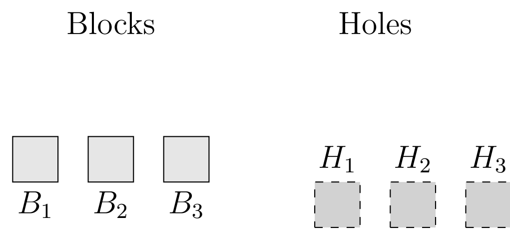

Figure 6: Blocks and Holes Scenario 1.2

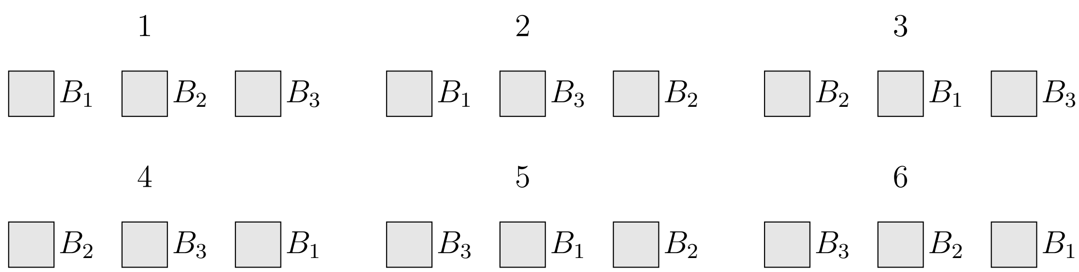

Figure 7: Blocks and Holes Solution 1.2

Thus, by enumeration, the number of Blocks[1,1,1] into the Holes[1,1,1] is 6.

9 Blocks and Holes cont.

Example 1.5
Determine the number of ways of distributing the collection of Blocks [1,1,1,2,3] into the Holes [1,3,4].

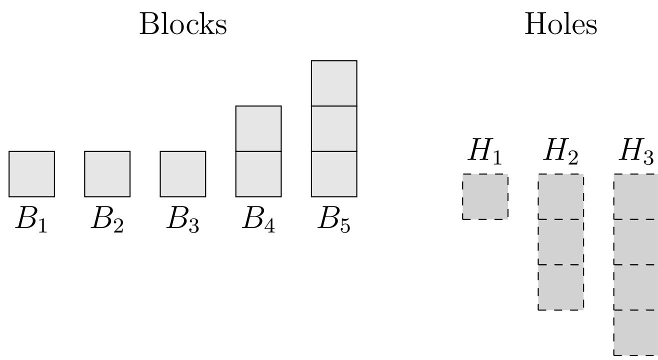

Figure 8: Blocks and Holes Scenario 1.3

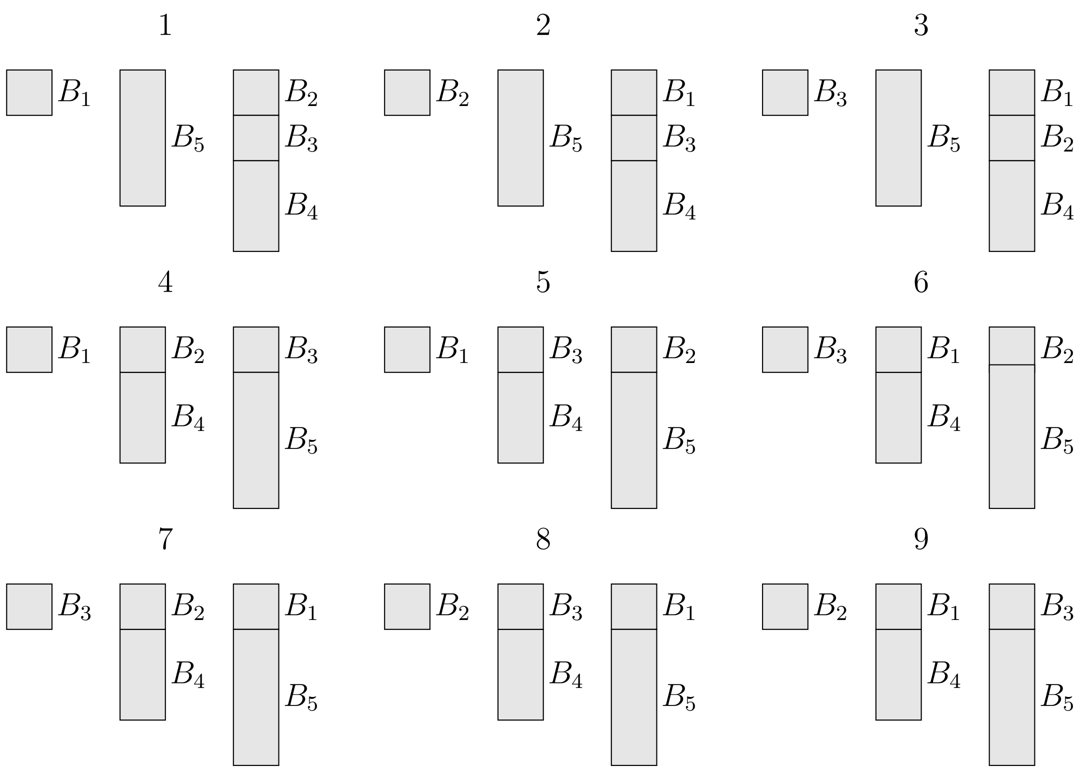

Figure 9: Blocks and Holes Solution 1.3

Thus, by enumeration the number of arrangements of the Blocks [1,1,1,2,3] into the Holes [1,3,4] is 9.

10 The Complete Algorithm

TODO: add class diagrams

11 References

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