Topology Homework 08

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EXCERCISE 4.23
Find three different topologies on the tree-point set $X = \{a, b, c\}$, each consisting of five open sets (including X and $\varnothing$), such that two of the topologies are homeomorphic to each other, but the third is not homeomorphic to the other two.

$\mathcal{T}_1 = \{\varnothing, \{a\}, \{c\}, \{a, c\}, X\}$
$\mathcal{T}_2 = \{\varnothing, \{b\}, \{c\}, \{b, c\}, X\}$
$\mathcal{T}_3 = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$

Where $f : X \to X$ defined by $f(x) = \begin{cases} b, & x = a \\ a, & x = b \\ c, & x = c \end{cases}$ is a homeomorphism.

$\mathcal{T}_1$ and $\mathcal{T}_2$ are homeomorphic, since the homeomorphism f exists. However, there is no homeomorphism between $\mathcal{T}_1$ and $\mathcal{T}_3$ or $\mathcal{T}_2$ and $\mathcal{T}_3$.

EXERCISE 4.24
Prove that a bijection $f : X \to Y$ is a homeomorphism if and only if f and f^{-1} map closed sets to closed sets.

Proof.
$(\to)$ Let $f : X \to Y$ be a homeomorphism.
Consider some closed set $C \in X$.
$C = X - U$ for some open set U in X.
$f(C) = f(X - U) = f(X) - f(U)$ since f is a bijection.
Since f is a homeomorphism, $f(U) = V$, where V is an open set.
So $f(C) = Y - V$, which is closed in Y.

Consider some closed set $D \in Y$.
$D = Y - V$ for some open set V in Y.
$f^{-1}(D) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V)$ since f is a bijection.
Since f is a homeomorphism, $f^{-1}(V) = U$, where U is an open set.
So $f^{-1}(V) = X - U$, which is closed in X.
Thus, f and f^{-1} map closed sets to closed sets.
$\square$
$(\leftarrow)$ Assume that f and f^{-1} map closed sets to closed sets.
Consider some closed set $C \in X$.
$C = X - U$ for some open set U in X.
$f(C) = f(X - U) = f(X) - f(U)$ since f is a bijection.
$f(C) = Y - f(U)$.
$f(C)$ is closed in Y.
So $f(U)$ is open in Y.
Consider some closed set $D \in Y$.
$D = Y - V$ for some open set V in Y.
$f^{-1}(D) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V)$ since f is a bijection.
$f^{-1}(D)$ is closed in X.
So $f^{-1}(V)$ is open in X.
Thus, f is a homeomorphism.
$\square$
$\therefore$ A bijection $f : X \to Y$ is a homeomorphism if and only if f and f^{-1} map closed sets to closed sets.
$\blacksquare$

EXERCISE 4.25
(a) Provide an example of a homeomorphism between $\mathbb{R}$ and the interval $(-\infty, a)$.
(b) Provide a formula for a homeomorphism between $\mathbb{R}$ and the interval (a, b), with $a < b$.
(c) Given the homeomorphism in Example 4.12 and the first two parts of this exercise, prove that if I_1 and I_2 are in collection (i) in Example 4.12, then I_1 and I_2 are topologically equivalent.

(a) Let $f : \mathbb{R} \to (-\infty, a)$ where $f(x) = a - e^{-x}$.
$\square$
(b) Let $f : \mathbb{R} \to (a, b)$ where $f(x) = a + \frac{b-a}{1+be^{-x}}$.
$\square$
(c) Let I_1 and I_2 be in the collection (i) consisting of open intervals: $(a, b), (-\infty, a), (a, \infty), \mathbb{R}$. So there are four cases for I_1 and I_2.

Case 1: $I_1 = (a, b)$ for some $a, b \in \mathbb{R}$ and $a < b$.
I_1 is homeomorphic to $\mathbb{R}$ using $f : \mathbb{R} \to I_1$ where $f(x) = a + \frac{b-a}{1+be^{-x}}$.

Case 2: $I_1 = (-\infty, a)$ for some $a \in \mathbb{R}$.
I_1 is homeomorphic to $\mathbb{R}$ using $f : \mathbb{R} \to I_1$ where $f(x) = a - e^{-x}$.

Case 3: $I_1 = (a, \infty)$ for some $a \in \mathbb{R}$.
I_1 is homeomorphic to $\mathbb{R}$ using $f : \mathbb{R} \to I_1$ where $f(x) = a + e^{x}$.

Case 4: $I_1 = \mathbb{R}$.
I_1 is homeomorphic to $\mathbb{R}$ using $f : \mathbb{R} \to I_1$ where $f(x) = x$.

In every case, we can find homeomorphisms $f : \mathbb{R} \to I_1$ and $g : \mathbb{R} \to I_2$

The function $g \circ f^{-1}(x) : I_1 \to I_2$ is a homeomorphism, since continuity is maintained through composition.
$\therefore I_1$ and I_2 are topologically equivalent.
$\blacksquare$

EXERCISE 4.26

(a) Provide a formula for a homeomorphism between the intervals $[0, \infty)$ and $[a, b)$, with $a < b$.

(b) Provide a formula for a homeomorphism between the intervals $(-\infty, 0]$ and $(a, b]$, with $a < b$.

(c) Given the homeomorphisms in Example 4.12 and the first two parts of this exercise, prove that if I_1 and I_2 are intervals in the collection (iii) in Example 4.12, then I_1 and I_2 are topologically equivalent.

(a) Let $f : [0, \infty) \to [a, b)$ where $f(x) = a + (b - a)(1 - e^{-x})$.

$\square$

(b) Let $f : (-\infty, 0] \to (a, b]$ where $f(x) = b + (a - b)(1 - e^{x})$.

$\square$

(c) Let I_1 and I_2 be in the collection (iii) consisting of open intervals: $[a, b), (a, b], (-\infty, a], [a, \infty)$.

So there are four cases for I_1 and I_2.

Case 1: $I_1 = [a, b)$ for some $a, b \in \mathbb{R}$ where $a < b$.
I_1 is homeomorphic to $[a, b)$ using
$f : I_1 \to [a, b)$ where $f(x) = b + a - x$.

Case 2: $I_1 = (a, b]$ for some $a, b \in \mathbb{R}$ where $a < b$.
I_1 is homeomorphic to $[a, b)$ using
$f : I_1 \to [a, b)$ where $f(x) = x$.

Case 3: $I_1 = (-\infty, a]$ for some $a \in \mathbb{R}$.
I_1 is homeomorphic to $[a, b)$ using
$f : I_1 \to (a, b]$ where $f(x) = a + (b - a)(1 - e^{x-a})$.

Case 4: $I_1 = [a, \infty)$ for some $a \in \mathbb{R}$.
I_1 is homeomorphic to $[a, b)$ using
$f : I_1 \to [a, b)$ where $f(x) = a + (b - a)(1 - e^{a-x})$.

In every case, we can find homeomorphisms $f : I_1 \to [a, b)$ and $g : I_2 \to [a, b)$

The function $g^{-1} \circ f(x) : I_1 \to I_2$ is a homeomorphism, since continuity is maintained through composition.

$\therefore I_1$ and I_2 are topologically equivalent.

EXERCISE 4.32
Show that homeomorphism preserves interior, closure, and boundary as indicated in the following implications:
(a) If $f : X \to Y$ is a homeomorphism, then $f(\mathrm{Int}(A)) = \mathrm{Int}(f(A))$ for every $A \subseteq X$.
(b) If $f : X \to Y$ is a homeomorphism, then $f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$ for every $A \subseteq X$.
(c) If $f : X \to Y$ is a homeomorphism, then $f(\partial(A)) = \partial(f(A))$ for every $A \subseteq X$.

(a) Since f is a homeomorphism, open sets in X map to open sets in Y. Since homeomorphisms are injective, we know that for all $U, V \subseteq X$,
$U \subseteq V \implies f(U) \subseteq f(V)$
So the biggest open subset of A maps to the biggest open subset of $f(A)$.
$\therefore f(\mathrm{Int}(A)) = \mathrm{Int}(f(A))$
$\square$

(b) Since f is a homeomorphism, closed sets in X map to closed sets in Y. Since homeomorphisms are injective, we know that for all $U, V \subseteq X$,
$U \subseteq V \implies f(U) \subseteq f(V)$
So the smallest closed superset of A maps to the smallest closed superset of $f(A)$.
$\therefore f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$
$\square$

(c) $f(\partial A) = f(\mathrm{Cl}(A) - \mathrm{Int}(A))$
Since f is a homeomorphism, and homeomorphisms are injective, we know that for all $U, V \subseteq X$, $f(U - V) = f(U) - f(V)$.
$f(\partial(A)) = f(\mathrm{Cl}(A)) - f(\mathrm{Int}(A))$
From **(a)** and **(b)**, $f(\partial(A)) = \mathrm{Cl} f((A)) - \mathrm{Int} f((A))$
$\therefore f(\partial(A)) = \partial(f(A))$
$\blacksquare$