Math 332 A - Mathematical Statistics

Ethan Jensen

February 28, 2020

HW p.287 #16,17 p.302 #64

1 p. 287 #16

If $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are independent unbiased estimators of a given parameter θ and $\text{var}(\hat{\Theta}_1) = 3\text{var}(\hat{\Theta}_2)$, find the constants a_1 and a_2 such that $a_1\hat{\Theta}_1 + a_2\hat{\Theta}_2$ in an unbiased estimator with minimum variance for such a linear combination.

Let $\lambda = \text{var}\hat{\Theta}_2$ and $C = a_1\hat{\Theta}_1 + a_2\hat{\Theta}_2$
From Theorem 4.14 we have

$$\text{var}(C) = a_1^2\text{var}(\hat{\Theta}_1) + a_2^2\text{var}(\hat{\Theta}_2) + a_1a_2\text{cov}(\hat{\Theta}_1, \hat{\Theta}_2)$$

Since the estimators are independent, their covariance is 0.

$$\text{var}(C) = a_1^2\text{var}(\hat{\Theta}_1) + a_2^2\text{var}(\hat{\Theta}_2)$$

plugging in λ for $\text{var}\hat{\Theta}_2$ we have

$$\text{var}(C) = 3a_1^2\lambda + a_2^2\lambda = (3a_1^2 + a_2^2)\lambda$$

For the linear combination to be remain unbiased, $a_1 + a_2 = 1$.
So now we can find the minimum variance.

$$\text{var}(C) = \lambda(3a_1^2 + (1 - a_1)^2) = \lambda(4a_1^2 - 2a_1 + 1)$$

We take the derivative of $\text{var}(C)$ with respect to a_1 and set that equal to 0 to find critical points.

$$\frac{d}{da_1}\text{var}(C) = 8a_1 - 2 = 0$$

So there is only one critical point with $a_1 = \frac{1}{4}$. Since there is only one critical point, we know it must be a local minimum.
Calculating for a_2 we have $a_2 = \frac{3}{4}$.

$$\boxed{a_1 = \frac{1}{4} \quad a_2 = \frac{3}{4}}$$

HW p.287 #16,17 p.302 #64

2 p. 287 #17

Show that the mean of a random variable of size n from an exponential distribution is a minimum variance estimator of the parameter θ.

We know that the distribution of an exponential random variable with paramter θ is given by

$$f(x) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0$$

$$\ln(f(x)) = -\ln(\theta) - \frac{x}{\theta}$$

$$\frac{\partial}{\partial\theta}\ln(f(x)) = -\frac{1}{\theta} + \frac{x}{\theta^2}$$

$$\left(\frac{\partial}{\partial\theta}\ln(f(x))\right) = \frac{1}{\theta^2} - \frac{2x}{\theta^3} + \frac{x^2}{\theta^4}$$

Because Expected value is a linear operator,

$$E\left(\left(\frac{\partial}{\partial\theta}\ln(f(x))\right)\right) = E(\frac{1}{\theta^2}) - E(\frac{2x}{\theta^3}) + E(\frac{x^2}{\theta^4})$$

$$E\left(\left(\frac{\partial}{\partial\theta}\ln(f(x))\right)\right) = \frac{1}{\theta^2} - \frac{2}{\theta^3}E(x) + \frac{1}{\theta^4}E(x^2)$$

With Theorem 4.6 and Corollary 6.1 we have

$$E\left(\left(\frac{\partial}{\partial\theta}\ln(f(x))\right)\right) = \frac{1}{\theta^2} - \frac{2}{\theta^2} + \frac{2}{\theta^2} = \frac{1}{\theta^2}$$

In conclusion, it is easy to see that.

$$\frac{1}{nE\left(\left(\frac{\partial}{\partial\theta}\ln(f(x))\right)\right)} = \frac{\theta^2}{n}$$

The variance of the sample mean of an exponential population is also $\frac{\theta^2}{n}$.

Therefore, by the Cramer Rao inequality, the mean of a random sample of size n from an exponential distribution is a minimum variance estimator of the parameter θ.

HW p.287 #16,17 p.302 #64

3 p. 302 #64

Given a random sample of size n from a Rayleigh population, find an estimator for its parameter α by the method of maximum likelihood.

A random variable X has a **Rayleigh distribution** if and only if its probability density is given by

$$f(x) = \begin{cases} 2\alpha x e^{-\alpha x^2} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha > 0$.
Our samples are independent, so our likelihood function is thus

$$L(\alpha) = \prod_{i=1}^{n} 2\alpha x_i e^{-\alpha x_i^2} \quad \text{for } x > 0$$

Maximizing the likeliood function is the same as maximizing the logarithm of the likelihood function.

$$\ln(L(\alpha)) = \sum_{i=1}^{n} \ln(2\alpha x_i e^{-\alpha x_i^2}) = \sum_{i=1}^{n} \left(\ln(2\alpha) + \ln(x_i) - \alpha x_i^2\right)$$

Next, we differentiate the function with respect to α and set it to 0 to find critical points.

$$\frac{d}{d\alpha} \ln(L(\alpha)) = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^2 = 0$$

$$\therefore \alpha = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

$$\boxed{\hat{\Theta} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i^2}}$$