Topology Homework 07

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EXERCISE 4.1
(a) Let X have the discrete topology and Y be an arbitrary topological space.
Show that every function $f : X \to Y$ is continuous.
(b) Let Y have the trivial topology and Y be an arbitrary topological space.
Show that every function $f : X \to Y$ is continuous.

(a) Let X have the discrete topology.
Consider some surjective function $f : X \to Y$.
Consider some open set $U \subseteq Y$.

$f^{-1}(U) \subseteq X$, which is open, since X has the discrete topology.
So for every open set U in Y, $f^{-1}(U)$ is open in X.
$\therefore$ Every function $f : X \to Y$ is continuous.

(b) Let Y have the trivial topology.
Consider some surjective function $f : X \to Y$.
The open sets in Y consist of $\varnothing$ and Y.

$f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$
$\varnothing$ and X are open sets in X.
So for every open set U in Y, $f^{-1}(U)$ is open in X.
$\therefore$ Every function $f : X \to Y$ is continuous.

EXERCISE 4.2

Prove Theorem 4.8: Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subseteq Y$.

Proof.

($\rightarrow$) Assume $f : X \to Y$ is continuous.
Consider some closed set $C \subseteq Y$.
Then $Y - C$ is open in Y.
Since f is continuous, $f^{-1}(Y - C)$ is open in X.

By Theorem 0.22, $f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C)$
So $f^{-1}(C)$ is the complement of an open set.
So $f^{-1}(C)$ is closed in X.
Thus, $f^{-1}(C)$ is closed in X for every closed set $C \subseteq Y$.
□
($\leftarrow$) Assume $f^{-1}(C)$ is closed in X for every closed set $C \subseteq Y$.
Consider some open set $U \subseteq Y$.
$U = Y - C$ for some closed set C in Y.

$f^{-1}(U) = f^{-1}(Y - C) = f^{-1}(Y) - f^{-1}(C)$ By Theorem 0.22.
$f^{-1}(U) = X - f^{-1}(C)$, where $f^{-1}(C)$ is closed in X.
So $f^{-1}(U)$ the complement of a closed set in X, which is open in X for every set U that is open in Y.
Thus, $f : X \to Y$ is continuous.
□
$\therefore$ A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set $C \subseteq Y$.
■

EXERCISE 4.7
Suppose X is a space with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $id(x) = x$, and assume that the domain X has the topology $\mathcal{T}_1$ and that the range of X has the topology $\mathcal{T}_2$. Show that id is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

Proof.
($\rightarrow$) Assume id is continuous.
Consider some open set in $U \in \mathcal{T}_2$, where $U \subseteq Y$.
$id^{-1}(U) = U$, so $U \in \mathcal{T}_1$ since id is continuous.
So $U \in \mathcal{T}_2 \implies U \in \mathcal{T}_1$.
Thus, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.
$\square$
($\leftarrow$) Assume $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.
Consider some open set $U \in \mathcal{T}_2$, where $U \subseteq Y$.
Since $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, $id^{-1}(U) = U \in \mathcal{T}_2$.
In other words, $id^{-1}(U)$ is also open in X.
Thus, id is continuous.
$\square$
$\therefore$ id is continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.
$\blacksquare$

EXERCISE 4.9
Let $f, g : X \to Y$ be continuous functions. Assume that Y is Hausdorff and that there exists a dense subset D of X such that $f(x) = g(x)$ for all $x \in D$. Prove that $f(x) = g(x)$ for all $x \in X$.

Proof.
Consider some point $x \in X$.
Suppose $f(x) \neq g(x)$.
Since Y is Hausdorff, there exist open sets $U, V \subseteq Y$ such that $f(x) \in U,\ g(x) \in V, U \cap V = \varnothing$.

Since f and g are continuous functions, there exist two sets $U_x, V_x \subseteq X \ni f(U_x) = U,\ g(V_x) = V$, where U_x and V_x are open in X.

$U_x$andV_x are both open sets that contain the point x.
So $U_x \cap V_x$ is a non-empty open set.
Since D is a dense subset of X, $\exists y \in D \ni y \in U_x \cap V_x$.
So $y \in U_x$ and $y \in V_x$.
So $f(y) \in U$ and $g(y) \in V$.
Since $y \in D$, $f(y) = g(y)$, and $f(y) \in U \cap V$.

This is a contradiction since we said $U \cap V = \varnothing$.
$\therefore f(x) = g(x)$ for all $x \in X$.
■

EXERCISE 4.13
(a) Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be continuous functions. Show that $h : X \to Y_1 \times Y_2$, defined by $h(x) = (f_1(x), f_2(x))$, is continuous as well.
(b) Extend the result of (a) to n functions, for $n > 2$.

(a) Let $U \times V$ be open in $Y_1 \times Y_2$
$h^{-1}(U \times V) = \{x| \ f_1(x) \in U \text{ and } f_2(x) \in V\}$
$h^{-1}(U \times V) = \{x| \ f_1(x) \in U\} \cap \{x| \ f_2(x) \in V\}$
$h^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Since f_1, f_2 are continuous, $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are both open in X.
So $f_1^{-1}(U) \cap f_2^{-1}(V) = h^{-1}(U \times V)$ is open in X.
$\therefore$ h is continuous as well.
■
(b)
Let $f_i : X \to Y_i, \ i = 1, 2, ...n$ be continuous functions.
Let $h : X \to Y_1 \times Y_2 \times ... \times Y_n$ defined by $h(x) = (f_1(x), f_2(x), ..., f_n(x))$.

Let $U_1 \times U_2 \times ... \times U_n$ be open in $Y_1 \times Y_2 \times ... \times Y_n$
$h^{-1}(U_1 \times U_2 \times ... \times U_n) = \{x| \ f_i(x) \in U_i \ \forall i\}$
$h^{-1}(U_1 \times U_2 \times ... \times U_n) = \bigcap_{i=1}^{n} f_i^{-1}(U_i)$

Since f_i is continuous for all i, $f_i^{-1}(U_i)$ is open in X for all i.
Thus, $\bigcap_{i=1}^{n} f_i^{-1}(U_i) = h^{-1}(U_1 \times U_2 \times ... \times U_n)$ is a finite intersection of open sets in X, and thus must also be open in X.
$\therefore$ h is continuous as well.
■

EXERCISE 4.14
Show that the addition function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = x + y$, is a continuous function.

Consider an open interval $(a, b) \subseteq \mathbb{R}$.
$f^{-1}((a, b)) = \{(x, y) \in \mathbb{R}^2 |\ a < x + y < b\}$
Now consider some $p = (x, y)$ where $p \in f^{-1}((a, b))$.

Let $B_p = B(p, r)$ where $r = min\left(\frac{x+y-a}{\sqrt{2}}, \frac{b-x-y}{\sqrt{2}}\right)$.
Now consider some $q \in B(p, r)$.
Let $m = d(p, q)$, which is less than r.
So $q = (x + m\cos\theta, y + m\sin\theta)$ for some θ

$$m < r$$
$$m < \frac{x + y - a}{\sqrt{2}}$$
$$\sqrt{2}m < x + y - a$$

Since $\cos\theta + \sin\theta < \sqrt{2},\ \forall\theta$

$$m(-\cos\theta - \sin\theta)m < x + y - a$$
$$m(\cos\theta + \sin\theta)m > -x - y + a$$
$$x + y + m\cos\theta + m\sin\theta > a$$
$$(x + m\cos\theta) + (y + m\cos\theta) > a$$

Similarly, it can be shown that

$$(x + m\cos\theta) + (y + m\cos\theta) < b$$

So $q \in f^{-1}(a, b)$
So $B_p \subseteq f^{-1}((a, b))$

By the Union Lemma, $f^{-1}((a, b)) = \bigcup_{p \in f^{-1}((a,b))} B_p$.
$f^{-1}((a, b))$ is an arbitrary union of open sets in $\mathbb{R}^2$, which makes it open.
So $f^{-1}((a, b))$ is open for all basis elements $(a, b) \in \mathbb{R}$.

Consider an open set $U \in \mathbb{R}$.
$U = B_1 \cup B_2 \cup B_3 ...$ where B_i are basis elements in $\mathbb{R}$.
By Theorem 0.22 $f^{-1}(U) = f^{-1}(B_1) \cup f^{-1}(B_2) \cup f^{-1}(B_3) ...$

$f^{-1}(U)$ can thus be written as a union of open sets in $\mathbb{R}^2$.
So $f^{-1}(U)$ is open in $\mathbb{R}^2$, and indeed is open for all $U \in \mathbb{R}$

$\therefore$ The addition function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = x + y$, is continuous.
■

EXERCISE 4.15

Let f be the multiplicative function, $f(x, y) = xy$. Complete the proof of continuity of f that was outlined in Example 4.6, by doing the following:
(a) Show that if p and q are both positive, and δ is described in the example, then $(p - \delta, p + \delta) \times (q - \delta, q + \delta) \subseteq f^{-1}((a, b))$.
(b) Consider the rest of the posibilities for p and q being positive or negative, and show that $(p - \delta, p + \delta) \times (q - \delta, q + \delta) \subseteq f^{-1}((a, b))$.

(a) Proof. Consider an open interval $(a, b) \in \mathbb{R}$.
$f^{-1}((a, b)) = \{(x, y) \in \mathbb{R}^2 | \ a < xy < b\}$
Now consider some $t = (p, q)$ where $t \in f^{-1}((a, b))$.
Let $m = \min\{b - pq, pq - a\}$
Let $B_t = (p - \delta, p + \delta) \times (q - \delta, q + \delta)$
where $\delta > 0$ is chosen such that $\delta|p|, \delta|q|, \delta^2$ are all less than $\frac{m}{3}$.

Let $(x', y') \in (p - \delta, p + \delta) \times (q - \delta, q + \delta)$

$$x' < p + \delta, \ y' < q + \delta$$
$$x'y' < (p + \delta)(q + \delta)$$
$$x'y' < pq + \delta|p| + \delta|q| + \delta^2$$
$$x'y' < pq + m$$
$$x'y' < b - m + m$$
$$x'y' < b$$

$$x' > p - \delta, \ y' > q - \delta$$
$$x'y' > pq - |p|\delta - |q|\delta + \delta^2$$
$$x'y' > pq - m/3$$
$$x'y' > a + m - m/3$$
$$x'y' > a$$

$a < x'y' < b$, so $(x', y') \in f^{-1}((a, b))$.
So $B_t \subseteq f^{-1}((a, b))$.

By the Union Lemma, $f^{-1}((a, b)) = \bigcup_{t \in f^{-1}((a,b))} B_t$.
$f^{-1}((a, b))$ is an arbitrary union of open sets in $\mathbb{R}^2$, which makes it open.
So $f^{-1}((a, b))$ is open for all basis elements $(a, b) \in \mathbb{R}$.

Consider an open set $U \in \mathbb{R}$.
$U = B_1 \cup B_2 \cup B_3 ...$ where B_i are basis elements in $\mathbb{R}$.
By Theorem 0.22 $f^{-1}(U) = f^{-1}(B_1) \cup f^{-1}(B_2) \cup f^{-1}(B_3) ...$

$f^{-1}(U)$ can thus be written as a union of open sets in $\mathbb{R}^2$.

So $f^{-1}(U)$ is open in $\mathbb{R}^2$, and indeed is open for all $U \in \mathbb{R}$

$\therefore$ For positive p and positive q, the multiplicative function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = xy$, is continuous.
$\square$

(b) Proof.
We must consider the rest of the possibilities for p and q being positive or negative.

(1) If both p and q are negative, then the same argument can be given as above since the shape $f^{-1}((a, b))$ is symmetric about $y = -x$.
(2) The shape of $f^{-1}((-a, -b))$ is identical to $f^{-1}((a, b))$, but flipped about the y-axis and the x-axis, allowing one of p and q to be negative, but still allowing the same argument to be used.
Flipping and rotation are linear transformations, which are continuous, which makes this argument rigorous.

$\therefore$ in all cases, the multiplicative function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = xy$, is continuous.
$\blacksquare$

EXERCISE 4.16
Use Example 4.6, Exercises 4.13 and 4.14, and Theorem 4.9 to show that the sum and product of a finite number of continuous functions are also continuous functions. That is, assuming that $f_1, ..., f_m : \mathbb{R} \to \mathbb{R}$ are continuous, prove that $S : \mathbb{R} \to \mathbb{R}$ and $P : \mathbb{R} \to \mathbb{R}$, defined by $S(x) = f_1(x) + ... + f_m(x)$ and $P(x) = f_1(x)f_2(x)...f_m(x)$, are continuous.

Proof.
These proofs will use Mathematical Induction.
Basis Step: Let $S : \mathbb{R} \to \mathbb{R}$ be defined by $S(x) = f_1(x)$.
Since f_1 is a continuous function, S is also a continuous function.
□
Likewise, like $P : \mathbb{R} \to \mathbb{R}$ be defined by $P(x) = f_1(x)$.
Since $f_1(x)$ is a continuous function, P is also a continuous function.
□

Induction Hypothesis: Assume $S : \mathbb{R} \to \mathbb{R}$ defined by $S(x) = f_1(x) + f_2(x) + ...f_k(x)$ is continuous, for some $k \in \mathbb{Z}_+$
Likewise, Assume $P : \mathbb{R} \to \mathbb{R}$ defined by $P(x) = f_1(x)f_2(x)...f_k(x)$ is continuous, for some $k \in \mathbb{Z}_+$

Induction Step: Consider the function $S' : \mathbb{R} \to \mathbb{R}$ defined by $S'(x) = f_1(x) + f_2(x) + ...f_k(x) + f_{k+1}(x)$.
$S'(x) = S(x) + f_{k+1}(x)$
By our induction hypothesis, S is continuous.
Since S and f_{k+1} are both continuous, $A : \mathbb{R} \to \mathbb{R}^2$ defined by $A(x) = (S(x), f_{k+1}(x))$ is continuous using our result from Exercise 4.13.
The addition function $B : \mathbb{R}^2 \to \mathbb{R}$ is also continuous using Exercise 4.14.

Notice that the function $S' : \mathbb{R} \to \mathbb{R} = B \circ A$.
S' is a continuous function by Theorem 4.9.
$\therefore$ $S : \mathbb{R} \to \mathbb{R}$, defined by $S(x) = f_1(x) + ... + f_m(x)$ is continuous for all $m \in \mathbb{Z}_+$, by Mathematical Induction.
■
Consider the function $P' : \mathbb{R} \to \mathbb{R}$ defined by $P'(x) = f_1(x)f_2(x)...f_k(x)f_{k+1}(x)$.
$P'(x) = S(x)f_{k+1}(x)$
By our induction hypothesis, P is continuous.
Since P and f_{k+1} are both continuous, $A : \mathbb{R} \to \mathbb{R}^2$ defined by $A(x) = (S(x), f_{k+1}(x))$ is continuous using our result from Exercise 4.13.
The multiplicative function $B : \mathbb{R}^2 \to \mathbb{R}$ is continuous using Exercise 4.15.

Notice that the function $P' : \mathbb{R} \to \mathbb{R} = B \circ A$.
P' is a continuous function by Theorem 4.9.
$\therefore$ $P : \mathbb{R} \to \mathbb{R}$, defined by $P(x) = f_1(x)f_2(x)...f_m(x)$ is continuous for all $m \in \mathbb{Z}_+$, by Mathematical Induction.
■

EXERCISE 4.17 Use Exercise 4.16 to show that every polynomial function $p : \mathbb{R} \to \mathbb{R}$, given by $p(x) = a_n x^n + ... + a_1 x + a_0$, is continuous.

(1) Constant functions defined by $c(x) = x_0$ are continuous. (Exm. 4.2)
(2) The identity function defined by $id(x) = x$ is continuous (Exm. 4.2)
(3) Functions composed of successive addition or multiplication of continuous functions are continuous. (Exr. 4.16)

Consider the function $p : \mathbb{R} \to \mathbb{R}$, given by $p(x) = a_n x^n + ... + a_1 x + a_0$.
Now consider a particular term in the series $a_k x^k$ for some $0 \leq k \leq n$.

Define a function $f_k : \mathbb{R} \to \mathbb{R}$ where $f_k(x) = a_k(x) \overbrace{id(x)id(x)...id(x)}^{k}$.
From (1), (2), and (3), f_k is continuous for all k.

So $p(x) = f_n(x) + ... + f_1(x) + f_0(x)$, where each f_k is continuous.
Thus, by (3), p is a continuous function.

$\therefore$ every polynomial function $p : \mathbb{R} \to \mathbb{R}$, given by $p(x) = a_n x^n + ... + a_1 x + a_0$, is continuous.
■