Topology Homework 5.1

Ethan Jensen

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EXERCISE 5.1 Show that the taxicab metric on $\mathbb{R}^2$ satisfies the properties of a metric.

The taxicab metric on $\mathbb{R}^2$ is defined by

$$d_T(p,q) = |p_1 - q_1| + |p_2 - q_2|$$

(1) ($\rightarrow$) Let $d_T(p,q) = 0$.
$|p_1 - q_1| + |p_2 - q_2| = 0$
Since $|x| = 0 \implies x = 0, \quad p_1 - q_1 = 0$ and $p_2 - q_2 = 0$
So $p_1 = q_1,\ p_2 = q_2,$ and $p = q$.

($\leftarrow$) Let p = q.
So $p_1 = q_1$ and $p_2 = q_2$
So $d_T(p,q) = |p_1 - q_1| + |p_2 - q_2| = 0 + 0 = 0$
Thus, $d_T(p,q) = 0$ if and only if $p = q$
$\square$
(2) $d_T(p,q) = |p_1 - q_1| + |p_2 - q_2| = |q_1 - p_1| + |q_2 - p_2| = d_T(q,p)$
$\square$
(3) Consider $d_T(p,q) + d_T(q,r)$
$d_T(p,q) + d_T(q,r) = |p_1 - q_1| + |p_2 - q_2| + |q_1 - r_1| + |q_2 - r_2|$
$|x| + |y| \geq |x + y|\ \forall x, y$

$$d_T(p,q) + d_T(q,r) \geq |p_1 - q_1 + q_1 - r_1| + |p_2 - q_2 + q_2 - r_2|$$

$$d_T(p,q) + d_T(q,r) \geq |p_1 - r_1| + |p_2 - r_2|$$

$$d_T(p,q) + d_T(q,r) \geq d_T(p,r)$$

$\square$
$\therefore$ The taxicab metric on $\mathbb{R}^2$ satisfies the properties of a metric.
$\blacksquare$

EXERCISE 5.2
(a) Show that the max metric on $\mathbb{R}^2$ satisfies the properties of a metric.
(b) Explain why $d(p,q) = \min\{|p_1 - q_1|, |p_2 - q_2|\}$ does not define a metric on $\mathbb{R}^2$.

The max metric on $\mathbb{R}^2$ is defined by

$$d_M(p,q) = \max\{|p_1 - q_1| + |p_2 - q_2|\}$$

(a) (1) ($\rightarrow$) Let $d_M(p,q) = 0$
$\max\{|p_1 - q_1| + |p_2 - q_2|\} = 0$
So $p_1 - q_1 = 0$ and $p_2 - q_2 = 0$
So $p_1 = q_1,\ p_2 = q_2$ and $p = q$

($\leftarrow$) Let $p = q$
So $p_1 = q_1$ and $p_2 = q_2$
So $d_M(p,q) = \max\{|p_1 - q_1|, |p_2 - q_2|\} = \max\{0, 0\} = 0$
Thus, $d_M(p,q) = 0$ if and only if $p = q$.
$\square$
(2) $d_M(p,q) = \max\{|p_1 - q_1|, |p_2 - q_2|\} = \max\{|q_1 - p_1|, |q_2 - p_2|\} = d_M(q,p)$
$\square$
(3) Consider $d_M(p,q) + d_M(q,r)$.
$d_M(p,q) + d_M(q,r) = \max\{|p_1 - q_1|, |p_2 - q_2|\} + \max\{|q_1 - r_1|, |q_2 - r_2|\}$
$\max\{a,b\} + \max\{c,d\} \geq \max\{a+c, b+d\} \forall a,b,c,d$

$$d_M(p,q) + d_T(q,r) \geq \max\{|p_1 - q_1| + |q_1 - r_1|, |p_2 - q_2| + |q_2 - r_2|\}$$

$|x| + |y| \geq |x+y|\ \forall x, y$

$$d_M(p,q) + d_T(q,r) \geq \max\{|p_1 - q_1 + q_1 - r_1|, |p_2 - q_2 + q_2 - r_2|\}$$

$$d_M(p,q) + d_T(q,r) \geq \max\{|p_1 - r_1|, |p_2 - r_2|\}$$

$$d_M(p,q) + d_M(q,r) \leq d_M(p,r)$$

$\square$
$\therefore$ The max metric on $\mathbb{R}^2$ satisfies the properties of a metric.
$\blacksquare$

(b) Consider the function $d(p,q) = \min\{|p_1 - q_1|, |p_2 - q_2|\}$.

Consider the points $p = (0,0),\ q = (0,2),\ r = (1,2)$
By our definition, $d(p,q) = 0,\ d(q,r) = 0,\ d(p,r) = 1$

So it is not the case that $d(p,q) + d(q,r) \geq d(p,r)\ \forall p,q,r$

This function is not a metric because it violates the 3rd property for a metric.
$\blacksquare$

EXERCISE 5.3 For points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$ define

$$d_V(p,q) = \begin{cases} 1 & \text{if } p_1 \neq q_1 \text{ or } |p_2 - q_2| \geq 1 \\ |p_2 - q_2| & \text{if } p_1 = q_1 \text{ and } |p_2 - q_2| < 1 \end{cases}$$

(a) Show that d_V is a metric.
(b) Describe open balls in the metric d_V.

(a) (1) $(\rightarrow)$ Let $d_V(p,q) = 0$
$d_V(p,q) \neq 1$ so $d_V = |p_2 - q_2|$ and $p_1 = q_1$
So $|p_2 - q_2| = 0$ and $p_2 = q_2$
Thus, $p = q$

$(\leftarrow)$ Let p = q.
So $p_1 = q_1$ and $p_2 = q_2$
So $|p_2 - q_2| = 0 < 1$ Thus, $d_V(p,q) = |p_2 - q_2| = 0$
□
(2) Since equality and inequality is symmetric and $|-x| = |x| \; \forall x$,

$$d_V(q,p) = \begin{cases} 1 & \text{if } q_1 \neq p_1 \text{ or } |q_2 - p_2| \geq 1 \\ |q_2 - p_2| & \text{if } q_1 = p_1 \text{ and } |q_2 - p_2| < 1 \end{cases} = d_V(p,q)$$

□
(3) Case 1: Either $d_V(p,q)$ or $d_V(q,r)$ is 1.
$d_V(p,q) \leq 1 \; \forall p, q$
$d_V(p,q) + d_V(q,r) \geq d_V(p,r)$

Case 2: $p_1 = q_1$, $q_1 = r_1$, and $|p_2 - q_2| < 1$ and $|q_2 - r_2| < 1$.
This means $p_1 = r_1$, by the transitive property of equality.
$d_V(p,r) \geq |p_2 - r_2|$
$|p_2 - q_2| + |q_2 - r_2| \geq |(p_2 - q_2) + (q_2 - r_2)| = |p_2 - r_2|$
So $|p_2 - q_2| + |q_2 - r_2| \geq d_V(p,r)$
Thus, $d_V(p,q) + d_V(q,r) \geq d_V(p,r)$

In all cases, $d_V(p,q) + d_V(q,r) \geq d_V(p,r)$
□
$d_V(p,q)$ satisfies all three conditions for a metric.
■
(b) The shape of open balls centered at some point p depends on the radius.

If $r \leq 1$, then $B_V(p,r) = \{q : \; p_1 = q_1 \text{ and } |p_2 - q_2| < r\}$ looks like a vertical open interval centered at p, with a width equal to 2r.

If $r > 1$, then $B_V(p,r) = \mathbb{R}$

EXERCISE 5.9
Let (X, d) be a metric space. A set $U \subseteq X$ is open in the topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subseteq U$.

Proof.
$(\rightarrow)$ Let $U \subseteq X$ be open in the topology on X induced by d. And let $y \in U$

Since U is open in (X,d), U can be written as a union of open balls.
So, $\exists B_d(p, r) \subseteq U \ni y \in B_D(p, r)$

Let $\delta = r - d(p, y)$.
$\delta > 0$ since $y \in B_D(p, r)$.
Consider $B_d(y, \delta) = \{q : \ d(y, q) < r - d(p, y)\}$

$\forall q \in B_d(y, \delta), \ d(p, y) + d(y, q) \geq d(p, q)$ since d is a metric.
So $d(p, q) < d(p, y) + r - d(p, y)$ and thus, $d(p, q) < r$
So $q \in B_d(p, r) \ \forall q \in B_d(y, \delta)$
Thus, $B_d(y, \delta) \subseteq B_d(p, r) \subseteq U$
□
$(\leftarrow)$ Consider some open $U \subseteq X$ where for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subseteq U$.
By the Union lemma, $U = \bigcup_{y \in U} B_d(y, \delta_y)$
Each $B_d(y, \delta_y)$ is a basis element in the metric space defined by d.
So U is a union of basis elements, which makes it open in the topology induced by d.
□
$\therefore$ A set $U \subseteq X$ is open in the topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subseteq U$.
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EXERCISE 5.15
Let (X, d) be a metric space and assume that $A \subseteq X$. Prove that $x \in \text{Cl}(A)$ if and only if there exists a sequence in A converging to x.

Proof.
$(\rightarrow)$ Assume $x \in \text{Cl}(A)$.
Then, every open set containing x intersects A at a point other than x.

Consider the sequence of open balls $B_d(x, \frac{1}{n})$ where $n \in \mathbb{Z}_+$.
Then let $q_n \neq x$ be the point of intersection between $b_d(x, \frac{1}{n})$ and A.
Then define $\{q_n\}$ as the sequence of the points q_n.

Consider some open set U where $x \in U$.
Then by Theorem 5.6, $\exists B_d(x, \delta) \subseteq U$.
Since the set of rational numbers is dense, $\exists a, b \in \mathbb{Z}_+ \ni 0 < \frac{a}{b} < \delta$.
So $\exists b \in \mathbb{Z}_+ \ni 0 < \frac{1}{b} < \delta$
So $\exists b \in \mathbb{Z})_+ \ni B_d(x, \frac{1}{n}) \subseteq B_d(x, \delta) \subseteq U$ for all $n \geq b$.
Thus, $q_n \in U$ for all $n \geq b$.

So for every open set U containing x, $\exists b \in \mathbb{Z}_+ \ni q_n \in U$ for all $n \geq b$.
So the sequence $\{q_n\}$ converges to x.
$\square$
$(\leftarrow)$ Assume that there exists a sequence $\{a_n\}$ in A converging to x.
Let U be an open set containing x.
If $x \in A$, then $x \in \text{Cl}(A)$.
Otherwise, since the sequence $\{a_n\} \subseteq A$, U intersects A at one of the sequence values, all of which cannot be equal to x, since $x \notin A$.

So every open set containing X intersects A at a point other than x.
So x is a limit point of A.
Thus, $x \in \text{Cl}(A)$.
In all cases, $x \in \text{Cl}(A)$.
$\square$
$\therefore x \in \text{Cl}(A)$ if and only if there exists a sequence in A converging to x.
$\blacksquare$