Combinatorics of Integer Partitions

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February 29, 2020

Objective

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots = \frac{\pi^2}{6}$$

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \dots$$

$$\zeta(2) = \frac{\pi^2}{6}$$

What about $\zeta(4), \zeta(6), \zeta(8)$?

Current Methods

$$\frac{1}{2q+1} = \sum_{k=1}^{s} \frac{(-1)^{k+1} 2^{k+1} (1 - 2^{1-2k}) q!}{(2q - 2k + 1)(q - k)! \pi^{2k}} \zeta(2k)$$

$$\sum_{n=0}^{\infty} \zeta(2n) x^{2n} = \frac{-\pi x}{2} \cot \pi x$$

$$\zeta(2n) = \frac{B_{2n} (2\pi)^{2n} (-1)^{n+1}}{2(2n)!}$$

Is there a better way to calculate $\zeta(2n)$?

Outline

What is a Partition?

$$4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 3 = 2 + 2 = 4$$

The collection C of partitions of 4 is

$$C = \{(1,1,1,1),\ (1,1,2),\ (1,3),\ (2,2),\ (4)\}$$

$\lambda = (1,1,2)$ is a partition of 4.

Partitions of Sequences

$$(2)+(3) = (1+1)+(1+1+1) = (1+1)+(3) = (1+1)+(1+2) =$$

$$(1+1)+(3) = (2)+(1+2) = (2)+(3)$$

The collection C of partitions of $(2,3)$ is

$$C = \{(1,1,1,1,1),\ (1,1,3),\ (1,1,1,2),\ (1,1,3),\ (1,2,2),\ (2,3)\}$$

$(1,1,1,2)$ is **finer** than $(2,3)$

Recombining a Partition

"Putting Humpty Dumpty back together again"

The collection C of recombinations of $(1, 1, 3)$ is

$$C = \{(1, 1, 3),\ (2, 3),\ (1, 4),\ (5)\}$$

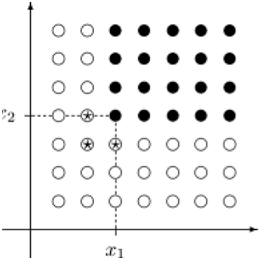
If we counted the number of ways we recombined...

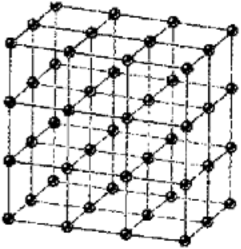
$$C = \{(1, 1, 3)_2,\ (2, 3)_1,\ (1, 4)_2,\ (5)_1\}$$

$(1, 4)$ is coarser than $(1, 1, 3)$

Partitioning Integer Lattices

Consider the Positive Integer lattices in N-Space:





We can partition sets the same way we can partition numbers and sequences.

Partition this set based on the groups of numbers in the sequence that are the same.

We want the tuple $(1, 2, 3)$ to be in a different set from $(7, 2, 2)$.

Partitions as Elementhood Conditions

Partition sets: $S(\lambda)$.

A function that takes in a partition and outputs a set.

Constructed a set using the partition as an instruction, making groups of equal elements.

Here are some elements in the set $S(2,3)$.

$(1,1,2,2,2),(4,3,4,3,4),(3,3,1,3,1),(56,2,2,2,56),(5,5,5,5,5)$

Here are some elements that are not in the set $S(2,3)$.

$$(1,2,3,4,5),(2,2,2,2,6),(3,5,3,5,7)$$

Minimal Partition Sets

Consider the set $S(1,1,1) = \mathbb{Z}_+^3$.
Possible kinds of points:

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)$$

$$(a, a, b), (a, b, a), (b, a, a)$$

$$(a, a, a)$$

For each case, we can assign a new set that only contains points of that shape.

We denote minimal partition sets as $S'(\boldsymbol{\lambda})$

Full Parititions

Consider the recombination set C of $(1, 1, 1)$.

$$C = \{(1, 1, 1)_6,\ (1, 2)_3,\ (3)_1\}$$

$$\mathbb{Z}_+^3 = S(1, 1, 1) = S'(1, 1, 1)_6 \cup S'(1, 2)_3 \cup S'(3)_1$$

How many ways are there to combine elements of a given partion to make a different coarser partition?

Blocks and Holes

Counting ways to combine elements of a partition into a coarser partition has a physical analogy.
It is called the **Blocks and Holes Problem**.

How many different ways are there to distribute the blocks into the holes?

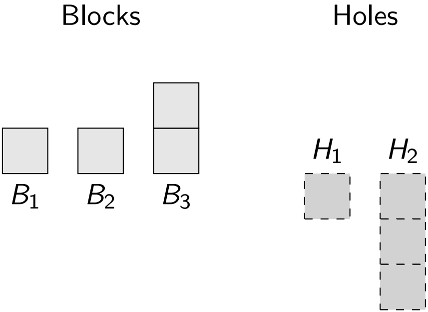


Figure: Blocks and Holes Scenario 1.1

Blocks and Holes Example

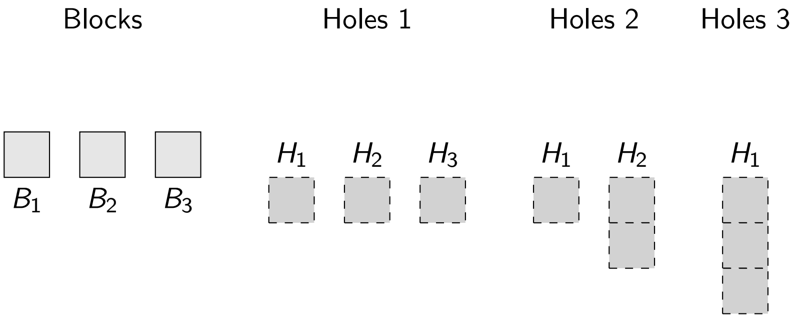


Figure: Blocks and Holes Scenario 1.2

Blocks and Holes Solution

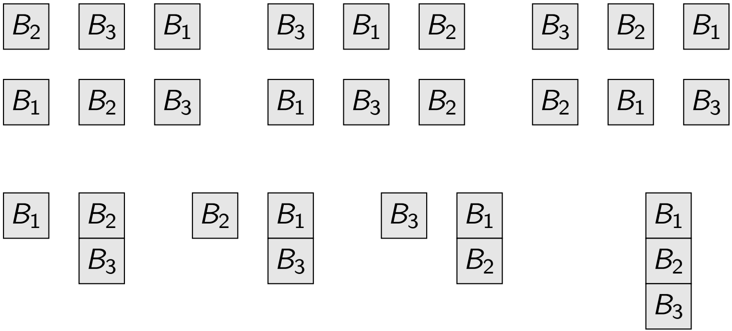


Figure: Blocks and Holes Scenario 1.2

This solution to the given blocks and holes problem demonstrates that

$$S(1,1,1) = S'(1,1,1)_6 \cup S'(1,2)_3 \cup S'(3)_1$$

Summing over a Partition

Let $\boldsymbol{\lambda}$ be a partition of an integer lattice.
Let $\mathbf{X}$ be some tuple $\mathbf{X} = (x_1, x_2, ...x_n)$
Define a new function D such that

$$D(\boldsymbol{\lambda}) = \sum_{\mathbf{X} \in S(\boldsymbol{\lambda})} \prod_{x_i \in \mathbf{X}} x_i^{-2} \quad D'(\boldsymbol{\lambda}) = \sum_{\mathbf{X} \in S'(\boldsymbol{\lambda})} \prod_{x_i \in \mathbf{X}} x_i^{-2}$$

Because D is linear **and** multiplicative!

$$D(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\lambda} \vdash \lambda_i} a_i D'(\lambda_i)$$

where a_i is the number of ways to combine $\boldsymbol{\lambda}$ into λ_i.

$$D(\lambda_a | \lambda_b) = D(\lambda_a) D(\lambda_b)$$

where $\lambda_a | \lambda_b = (\lambda_{a1}, \lambda_{a2}, ...\lambda_{ak}, \lambda_{b1}, \lambda_{b2}...\lambda_{bj})$

Solution to the Basel Problem

Comparing the Maclaurin series of $\sinh(x)$ with it's Euler product, by a simple calculation we have that

$$\frac{\sinh(\pi x)}{\pi x} = 1 + \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + \frac{\pi^6 x^6}{7!} \ldots = \left(1 + \frac{x^2}{1^2}\right)\left(1 + \frac{x^2}{2^2}\right)\left(1 + \frac{x^2}{3^2}\right)$$

By calculating the coefficient of degree 2, we must consider all the ways we can generate terms of degree 2 from the product.

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \ldots = \frac{\pi^2}{6}$$

What if we want to calculate higher power sums?

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \ldots = ?$$

Connection to Particular Zeta Values

Consider the following equation we derived earlier.

$$1 + \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + \frac{\pi^6 x^6}{7!} ... = \left(1 + \frac{x^2}{1^2}\right)\left(1 + \frac{x^2}{2^2}\right)\left(1 + \frac{x^2}{3^2}\right) ...$$

The term of coefficient of degree 4 is just $D'(1,1)$.

$$D'(1,1) = \frac{\pi^4}{5!} = \frac{\pi^4}{120}$$

We now have a way to calculate a bunch of the values for the D function.

$$D'(\overbrace{1,1,1,1...1}^{n}) = \frac{\pi^{2n}}{(2n+1)!}$$

A Quick Example

$$
\begin{array}{ccccccccc}
\frac{1}{1^2}\cdot\frac{1}{1^2} & + & \frac{1}{1^2}\cdot\frac{1}{2^2} & + & \frac{1}{1^2}\cdot\frac{1}{3^2} & + & \frac{1}{1^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{2^2}\cdot\frac{1}{1^2} & + & \frac{1}{2^2}\cdot\frac{1}{2^2} & + & \frac{1}{2^2}\cdot\frac{1}{3^2} & + & \frac{1}{2^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{3^2}\cdot\frac{1}{1^2} & + & \frac{1}{3^2}\cdot\frac{1}{2^2} & + & \frac{1}{3^2}\cdot\frac{1}{3^2} & + & \frac{1}{3^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{4^2}\cdot\frac{1}{1^2} & + & \frac{1}{4^2}\cdot\frac{1}{2^2} & + & \frac{1}{4^2}\cdot\frac{1}{3^2} & + & \frac{1}{4^2}\cdot\frac{1}{4^2} & \cdots \\
\vdots & & & & & & \vdots & \ddots
\end{array}
$$

Figure: Addition Square

Going row by row, we have that this sum is equal to

$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots\right)\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots\right)=\left(\frac{\pi^2}{6}\right)^2=\frac{\pi^4}{36}$$

A Quick Example Part

$$
\begin{array}{ccccccccc}
\frac{1}{1^2}\cdot\frac{1}{1^2} & + & \frac{1}{1^2}\cdot\frac{1}{2^2} & + & \frac{1}{1^2}\cdot\frac{1}{3^2} & + & \frac{1}{1^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{2^2}\cdot\frac{1}{1^2} & + & \frac{1}{2^2}\cdot\frac{1}{2^2} & + & \frac{1}{2^2}\cdot\frac{1}{3^2} & + & \frac{1}{2^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{3^2}\cdot\frac{1}{1^2} & + & \frac{1}{3^2}\cdot\frac{1}{2^2} & + & \frac{1}{3^2}\cdot\frac{1}{3^2} & + & \frac{1}{3^2}\cdot\frac{1}{4^2} & \cdots \\
\frac{1}{4^2}\cdot\frac{1}{1^2} & + & \frac{1}{4^2}\cdot\frac{1}{2^2} & + & \frac{1}{4^2}\cdot\frac{1}{3^2} & + & \frac{1}{4^2}\cdot\frac{1}{4^2} & \cdots \\
\vdots & & & & & & \vdots & \ddots
\end{array}
$$

Figure: Addition Square

Looking at the partition, we have that this sum is equal to

$$D'(1,1) + \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}\right)$$

Calculating Subsequent Zeta Values

The collection C of recombinations of (1,1) is

$$C = \{(1,1)_2,\ (2)_1\}$$

It follows that

$$D(1,1) = 2D'(1,1) + D'(2)$$

$$D(1)D(1) = 2D'(1,1) + D(2)$$

$$\zeta(2)^2 = 2D'(1,1) + \zeta(4)$$

$$\left(\frac{\pi^2}{6}\right)^2 = 2\frac{\pi^4}{120} + \zeta(4)$$

$$\boxed{\therefore \zeta(4) = \frac{\pi^4}{90}}$$

Calculating Subsequent Zeta Values cont.

$$(1,1,1) \to \{(1,1,1)_6 + (1,2)_3 + (3)_1\}$$

$$D(1,1,1) = 6D'(1,1,1) + 3D'(1,2) + D'(3)$$

$$(1,2) \to \{(1,2)_1 + (3)_1\}$$

$$D(1,2) = D'(1,2) + D'(3)$$

With a quick substitution, we have

$$D(1,1,1) = 6D'(1,1,1) + 3D(1,2) - 2D'(3)$$

$$(D(1))^3 = 6D'(1,1,1) + 3D(1)D(2) - 2D(3)$$

$$\frac{\pi^6}{216} = \frac{6\pi^6}{7!} + \frac{3\pi^6}{6 \cdot 90} - 2\zeta(6)$$

$$2\zeta(6) = \frac{2\pi^6}{945}$$

$$\boxed{\therefore \zeta(6) = \frac{\pi^6}{945}}$$

Solving the Blocks and Holes problem

Higher dimensions are spicy!

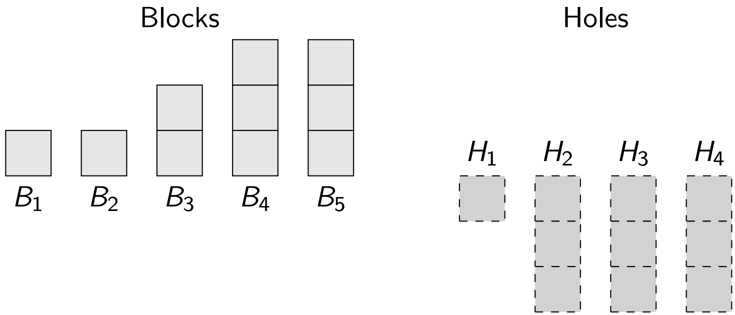


Figure: Blocks and Holes Scenario 1.1

Use a computer to virtually place blocks into holes.

The complete algorithm

We want to compute $\zeta(2n)$.

We can use the blocks and holes algorithm to build a system of equations.

After solving the system of equations we can use the master equation and recursion to solve for $\zeta(2n)$.

Conclusion

(1) Partitions

(2) Recombinations of Partitions

(3) Partitions of Integer Lattices

(4) The D function

(5) Specifying the combinatorics problem.

(5) Calculating Positive even Zeta values recursively

The importance of multiplicity of proofs

Mathematics is about methods - more so than about the Theorems themselves. Zeta values can be calculated using:

- Complex Analysis
- Bernoulli numbers
- Fourier Series

My method provides a new way to calculate positive even zeta values without using any heavy mathematical machinery.

This method is a continuagtion of the work of Leonhard Euler. Basel problem solved in (1734).

Connections to the Riemann Hypothesis

What is the hardest way to win \$1,000,000?

Proofs to difficult problems can sometimes arise from seemingly unrelated areas of mathematics.

Chaotic behavior of the primes and the Bernoulli numbers linked to the chaotic behavior of partitions.

This work provides a strong link between Combinatorics and Analytic Number Theory.

References

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