Topology Homework 03

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EXERCISE 1.10 *Show that* $\mathbb{B}$ *is a basis for a topology on* $\mathbb{R}$.

Proof.
Notice that $\bigcup_{B\in\mathbb{B}} B = \mathbb{R}$
$\therefore \forall x \in \mathbb{R} \exists B \in \mathbb{B} \ni x \in B$.
$\square$
Consider $B_1, B_2 \in \mathbb{B}$ where $B_1 = [a, b)$ and $B_2 = [c, d)$.
Let x $\in B_1 \cap B_2$.
Let $x \in B_1 \cap B_2$.
So $\max(a, c) \leq x <$min(b, d).
Thus, $x \in [\max(a, c), \min(b, d)) \in \mathbb{B}$. Call it B'.
So $\exists B' \subseteq B_1 \cap B_2 \ni x \in B' \in \mathbb{B}$.
$\blacksquare$

EXERCISE 1.11 *Show which collections of subsets of $\mathbb{R}$ are bases.*

(a) $C_1 = \{(n, n+2) \subseteq \mathbb{R} | n \in \mathbb{Z}\}$
$(0, 2)$ and $(1, 3)$ are basis elements of C_1.
By inspection, $1.5 \in (0, 2)$ and $1.5 \in (1, 3)$, so $1.5 \in (0, 2) \cap (1, 3) = (1, 2)$.
If C_1 was a basis, then there exists a basis element $B' \subseteq (1, 2)$ such that $1.5 \in B'$.
However, since all basis elements are of length 2, no basis element can be a subset of the open interval $(1, 2)$.
$\boxed{C_1 \text{ is not a basis}}$
(b) $C_2 = \{[a, b] \subseteq \mathbb{R} | a < b\}$
$[0, 1]$ and $[1, 2]$ are basis elements of C_2.
By inspection, $1 \in [0, 1] \cap [1, 2] = \{1\}$.
If C_2 was a basis, then there exists a basis element $B' \subseteq \{1\}$ such that $1 \in B'$.
However, since the upper and lower limits of the intervals in the basis set are distinct, no basis element can be a subset of $\{1\}$.
$\boxed{C_2 \text{ is not a basis}}$
(c) $C_3 = \{[a, b] \subseteq \mathbb{R} | a \leq b\}$
Consider a point $x \in \mathbb{R}$.
The basis element $[x, x] = \{x\}$ contains x.
$\square$
Consider two basis elements $[a, b]$ and $[c, d]$ with nonempty intersection, and let $x \in [a, b] \cap [c, d]$.
Let $j = \max(a, c)$ and $k = \min(b, d)$.
Since $[a, b]$ and $[c, d]$ have a nonempty intersection, we know that $j \leq k$.
So $[j, k]$ is a basis element. Call it B'.
$[a, b] \cap [c, d] = [j, k]$
Thus, $x \in [a, b] \cap [c, d] \implies x \in B'$ where B' is a basis element,
$\boxed{C_3 \text{ is a basis}}$
(d) $C_4 = \{(-x, x) \subseteq \mathbb{R} |\ x \in \mathbb{R}\}$
Consider a point $x \in \mathbb{R}$.
$x \in (x - 1, x + 1)$ where $(x - 1, x + 1)$ is a basis element.
$\square$
Consider two basis elements $(-a, a)$ and $(-b, b)$ with nonempty intersection, and let $x \in (-a, a) \cap (-b, b)$.
Either $a < b$ or $b < a$. Without loss of generality, say that $a < b$.
Since the sets are nested, $(-a, a) \cap (-b, b) = (-a, a)$, which is a basis element.
Thus, $x \in (-a, a) \cap (-b, b) \implies x \in (-a, a)$ where $(-a, a)$ is a basis element.
$\boxed{C_4 \text{ is a basis}}$
(e) $C_5 = \{(a, b) \cup \{b + 1\} \subseteq \mathbb{R} |\ a < b\}$
Each basis element in C_5 is nested and covers the full space $\mathbb{R}$, so by a similar argument to part **(d)**, C_5 is a basis for a topology on $\mathbb{R}$.
$\boxed{C_5 \text{ is a basis}}$

EXERCISE 1.13 *Consider the following six topologies defined on $\mathbb{R}$: the trivial topology, the discrete topology, the finite complement topology, the standard topoiogy, the lower limit topology, and the upper limit toology. Show how they compare to each other (finer, strictly finer, coarser, strictly coarser, noncomparable) and justify your claim.*

The **discrete topology** is the finest. It is strictly finer than all the other topologies. This is because it contains all possible subsets of $\mathbb{R}$. The other topologies have other elementhood conditions.

The **lower limit** and **upper limit** topologies are noncomparable to each other, but both are finer than the standard topology. This is because $\bigcup_{x>a}[x,b) = \bigcup_{x<b}(a,x] = (a,b)$ by the Union Lemma. All basis elements of the standard topology are open sets in said topologies.

The **standard topology** is strictly finer than the finite complement topology. By a similar idea, complements of finite sets in $\mathbb{R}$ can be expressed as a union of open intervals. However, in general, open sets in the standard topology do not have a finite complement.

The **finite complement topology** is strictly finer than the trivial topology. The trivial topology only contains $\varnothing$ and $\mathbb{R}$, which all the topologies (including the finite complement topology) contain.

The **trivial topology** is the coarsest of all topologies.

EXERCISE 1.16. *On the plane* $\mathbb{R}^2$ *, let*

$$\mathbb{B} = \{(a,b) \times (c,d) \subseteq \mathbb{R}^2 | a < b,\ c < d\}$$

(a) *Show that* $\mathbb{B}$ *is a basis for a topology on* $\mathbb{R}^2$.
(b) *Show that the topology* $\mathcal{T}'$ *generated by* $\mathbb{B}$ *is the standard topology on* $\mathbb{R}^2$.

(a)
(1) Consider some $(x,y) \in \mathbb{R}^2$.
Let $B' = (x-1, x+1) \times (y-1, y+1)$.
Since $x \in (x-1), x+1)$ and $y \in (y-1, y+1)$, $(x,y) \in B'$
So $\exists B' \in \mathbb{B} \ni (x,y) \in B' \forall (x,y) \in \mathbb{R}^2$
$\square$
(2) Let $B_1,\ B_2 \subseteq \mathbb{B}$
$B_1 = (a_1, b_1) \times (a_2, b_2),\ a_1 < b_1,\ a_2 < b_2$ and $B_2 = (c_1, d_1) \times (c_2, d_2),\ c_1 < c_1,\ c_2 < d_2$
Let $(x,y) \in B_1 \cap B_2$
Using the laws of algebra, we know that
$\max(a_1, c_1) < x < \min(b_1, d_1)$ and $\max(a_2, c_2) < y < \min(b_2, d_2)$

Let $B' = (\max(a_1, c_1), \min(b_1, d_1)) \times (\max(a_2, c_2), \min(b_2, d_2))$
Thus, $(x,y) \in B' \subseteq B_1 \cap B_2$
$\therefore (x,y) \in B_1 \cap B_2 \implies \exists B' \subseteq B_1 \cap B_2 \ni (x,y) \in B' \ \ \forall\ B_1, B_2 \in \mathbb{B}$
$\square$
$\therefore \mathbb{B}$ is a basis for a topology.
$\blacksquare$

(b)
Let $\mathbb{B}_1 = \{(a, b) \times (c, d)|\ a < b, c < d\}$
Let $\mathbb{B}_2 = \{B(p, r)|\ r > 0\}$
Let $\mathcal{T}_1$ be the topology generated by $\mathbb{B}_1 = \mathbb{B}$
Let $\mathcal{T}_2$ be the topology generated by $\mathbb{B}_2$ (the standard topology).

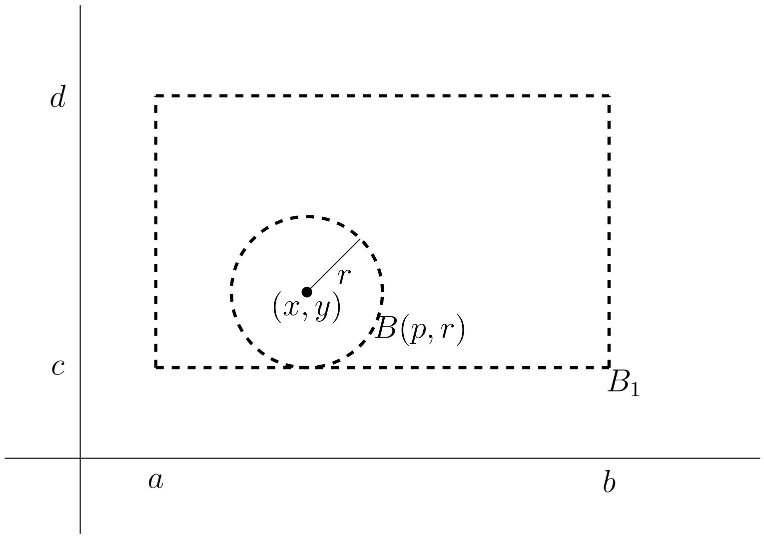

Figure 1: $\rightarrow$

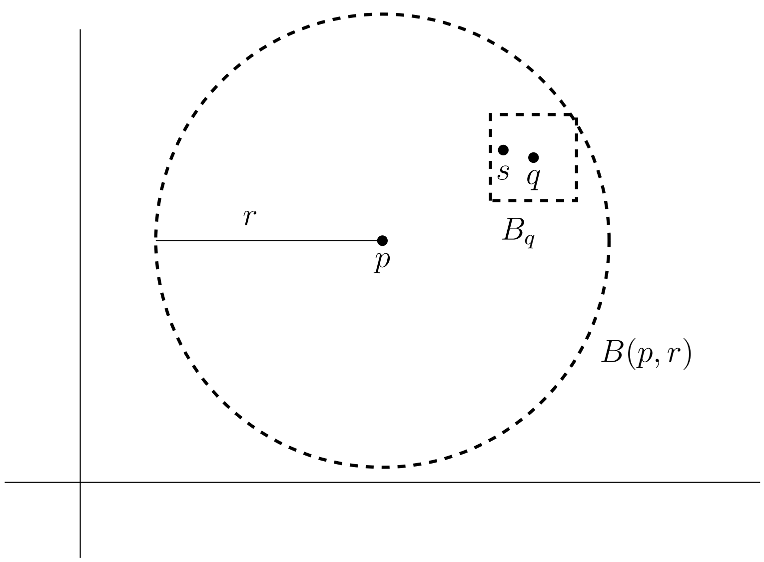

Figure 2: $\leftarrow$

$\rightarrow$ Let $B_1 \in \mathbb{B}_1$
$\exists a, b, c, d \ni B_1 = (a, b) \times (c, d)$
Let $p = (x, y) \in B_1$, so $a < x < b$ and $c < y < d$.
Let $r = min(x - a, b - x, y - c, d - y)$, so $r > 0$.

Consider $B(p,r) \in \mathbb{B}_2$
$B(p,r) \in \mathfrak{T}_2$
Let $q \in B(p,r)$
$\exists m,\ \theta\ \ni 0 < m < r$ where $q = (x + m\cos\theta,\ y + m\sin\theta)$
Since $-1 \leq \sin\theta \leq 1,\ \ -r \leq m\sin\theta \leq r$
Since $-1 \leq \cos\theta \leq 1,\ \ -r \leq m\cos\theta \leq r$

$$\begin{array}{ll} a < x - r & x + r < b \\ a < x - m & x + m < b \\ \boxed{a < x + m\cos\theta} & \boxed{x + m\cos\theta < b} \end{array}$$

$$\begin{array}{ll} c < y - r & y + r < d \\ c < y - m & y + m < d \\ \boxed{c < y + m\sin\theta} & \boxed{y + m\cos\theta < d} \end{array}$$

So $q \in B_1$
Thus, $B(p,r) \subseteq B_1$
By the Union Lemma, $B_1 = \bigcup_{p \in B_1} B(p,r)$.

Let $U \in \mathfrak{T}_1$.
$U = \bigcup B_k$ where $B_k \in \mathbb{B}_1$ since $\mathbb{B}_1$ generates $\mathfrak{T}_1$
Thus, $U = \bigcup \bigcup_{p \in B_k} B(p,r)$, which is a union of basis elements from $\mathfrak{T}_2$.
So $U \in \mathfrak{T}_2$
$\therefore \mathfrak{T}_1 \subseteq \mathfrak{T}_2$
$\square$

$\leftarrow$ Let $B(p,r) \in \mathbb{B}_2$
Let $q = (x_q, y_q) \in B(p,r)$.
$\exists m \ni 0 < m < r$ and $d(p,q) = m$
Let $B_q = \{(x_q + \frac{m-r}{\sqrt{2}}, x_q + \frac{r-m}{\sqrt{2}}) \times (y_q + \frac{m-r}{\sqrt{2}}, y_q + \frac{r-m}{\sqrt{2}})\}$
$B_q \in \mathbb{B}_1$
Let $s \in B_q$.
By the formula for Euclidian distance

$$d(q,s) < \sqrt{\left(x_q - (x_q \pm \frac{m-r}{\sqrt{2}})\right)^2 + \left(y_q - (y_q \pm \frac{m-r}{\sqrt{2}})\right)^2}$$

$$d(q,s) < \sqrt{\left(\frac{m-r}{\sqrt{2}}\right)^2 + \left(\frac{m-r}{\sqrt{2}}\right)^2}$$

$$d(q,s) < \sqrt{(r-m)^2}$$

Since $r > m$

$$d(q,s) < r - m$$

By the triangular property
$d(p,s) < d(p,q) + d(q,s)$

$d(p, s) < m + (r - m) \quad \text{so } d(p, s) < r$

Thus, $s \in B(p, r)$.
So $B_q \subseteq B(p, r)$
By the Union lemma $\bigcup_{q \in B(p,r)} B_q = B(p, r)$

Let $U \in \mathcal{T}_2$.
$U = \bigcup B_k$ where $B_k \in \mathbb{B}_2$ since $\mathbb{B}_2$ generates $\mathcal{T}_2$.
Thus, $U = \bigcup \bigcup_{q \in B(p,r)} B_q$, which is a union of basis elements from $\mathcal{T}_1$.
So $U \in \mathcal{T}_1$
$\therefore \mathcal{T}_2 \in \mathcal{T}_1$
$\square$
$\therefore \mathcal{T}_1 = \mathcal{T}_2$
The topology $\mathcal{T}'$ generated by $\mathbb{B}$ is the standard topology on $\mathbb{R}^2$.
$\blacksquare$

EXERCISE 1.17. *An* **open half plane** *is a subset of* $\mathbb{R}^2$ *in the form* $\{(x,y) \in \mathbb{R}^2 | \ Ax + By < C\}$ *for some* $A, B, C \in \mathbb{R}$ *with either* A *or* B *nonzero. Prove that open half planes are open sets in the standard topology on* $\mathbb{R}^2$.

Proof.
Let $H = \{(x,y) \in \mathbb{R}^2 | \ Ax + By < C\}$ be an open half plane and let $p = (x_p, y_p) \in H$.
Thus, $Ax_p + By_p < C$
Between any two real numbers exists another real number. Thus,

$$\exists \epsilon \ni Ax_p + By_p + \frac{\epsilon}{A+B} < C, \quad \frac{\epsilon}{A+B} > 0$$

Consider the open ball $B_p(p, \frac{\epsilon}{A+B})$
Let $q \in B_p$.

$\exists m, \ \theta \ \ni 0 < m < \frac{\epsilon}{A+B}$ where $q = (x_p + m\cos\theta, \ y_p + m\sin\theta)$
Since max(cos(θ)) and max(sin(θ)) are 1, we know
$m\cos\theta < \frac{\epsilon}{A+B}$ and $m\sin\theta < \frac{\epsilon}{A+B}$

$$A(x_p + m\cos(\theta)) + B(y_p + m\sin(\theta)) \leq A(x_p + \frac{\epsilon}{A+B}) + B(y_p + \frac{\epsilon}{A+B})$$

$$A(x_q) + B(y_q) < Ax_p + By_p + \frac{(A+B)\epsilon}{A+B} < C$$

$$A(x_q) + B(y_q) < C$$

So $q \in H$
Thus, $B_p \subseteq H$.
□
Consider $\bigcup_{p \in H} B_p$.

By the union Lemma, $\bigcup_{p \in H} B_p = H$
So H is a union of open balls.
This makes H an open set in the standard topology.

∴ Half planes are open sets in the standard topology on $\mathbb{R}^2$.
■

EXERCISE 1.25. *Prove that, in a topological space X, if U is open and C is closed, then $U - C$ is open and $C - U$ is closed.*

Proof.
Since C is closed, there exists an open set U_c such that $C = X - U_c$.
Since U is open, there exists a closed set C_u such that $U = X - C_u$

$U - C = U - (X - U_c) = U \cap U_c$
So $U - C$ is the intersection of finite open sets, and is thus open.

$C - U = C - (X - C_u) = C \cap C_u$
So $C - U$ is the intersection of closed sets, and it thus closed.
■

EXERCISE 1.26. *Prove that closed balls are closed sets on the standard topology on* $\mathbb{R}^2$.

Proof.
Consider a closed ball $\overline{B}(p,r)$.
Let q be a point outside of the closed ball $\overline{B}(p,r)$.
Thus, $d(p,q) = r + \epsilon$, where $\epsilon > 0$
Consider $B_q(q,\epsilon)$. and let some point $s \in B_q$.

$$d(s,q) < \epsilon$$

By the triangle inequality,

$$d(s,q) + d(s,r) < d(p,q)$$

$$\epsilon + d(s,r) < r + \epsilon$$
$$d(s,r) < r$$

So the point s is not in the closed ball $\overline{B}(p,r)$. It is in its complement.
So $B_q \subseteq X - \overline{B}(p,r)$.
□
Consider $\bigcup_{q \in X - \overline{B}(p,r)} B_q$

By the Union Lemma, $\bigcup_{q \in X - \overline{B}(p,r)} B_q = X - \overline{B}(p,r)$. So $X - \overline{B}(p,r)$ is a union of open balls.
This makes $X - \overline{B}(p,r)$ open in the standard topology.
So $\overline{B}(p,r)$ is closed in the standard topology.

∴ Closed balls are closed in the standard topology.
■

EXERCISE 1.28 *Which sets are closed sets in the finite complement topology on a topological space X?*

The set X and any all finite subsets of X are closed sets in the finite complement topology on X.

EXERCISE 1.29 *Which sets are closed sets in the excluded point topology EPX_p on a set X?*

All open sets in the particular point topology PPX_p are closed sets in the excluded point topology on X.

EXERCISE 1.30 *Which sets are closed sets in the particular point topology EPX_p on a set X?*

All open sets in the excluded point topology EPX_p are closed sets in the particular point topology on X.

EXERCISE 1.32 *Prove that intervals of the form $[a, b)$ are closed in the lower limit topology on $\mathbb{R}$.*

Proof.
Consider the sets $U_l = \bigcup_{c<a}[c, a)$ and $U_r = \bigcup_{d>b}[b, d)$
By the union lemma, $U_l = (-\infty, a)$ and $U_r = [b, \infty)$.

U_l and U_r are the union of open sets and are thus open.
Furthermore, $U_l \cup U_r$ must also be an open set for the same reason.

$U_l \cup U_r$ is the complement of $[a, b)$.
Since the complement of $[a, b)$ is open, $[a, b)$ is closed.

$\therefore$ Intervals of the form $[a, b)$ are closed in the lower limit topology on $\mathbb{R}$.
$\blacksquare$

EXERCISE 1.33 *Let X be a topological space*

(a) Prove that $\varnothing$ and X are closed sets.

(b) Prove that the intersection of any collection of closed sets in X is a closed set.

(c) Prove that the union of finitely many closed sets in X is a closed set.

(a) Proof.
By the definition of a topology, X and $\varnothing$ are open sets.
Their complements, $\varnothing$ and X are thus closed sets.
$\square$

(b) Proof.
Consider an intersection of closed sets $\bigcap_i C_i$.
This can be written as $X \cap \bigcap_i C_i$.

Each closed set C_i can be expressed as the complement of an open set $X - U_i$ for some U_i.
So $\bigcap_i C_i = X \cap \bigcap_i (X - U_i)$
By DeMorgan's law, we can write $\bigcap_i C_i = X - \bigcup_i U_i$

$\bigcup_i U_i$ is a union of open sets and is thus open.
Since $\bigcap_i C_i$ is the complement of an open set, $\bigcap_i C_i$ is closed.
$\square$

(c) Proof.
Consider a finite union of closed sets $\bigcup_{i=1}^n C_i$.
This can be written as $X \cap \bigcup_{i=1}^n C_i$.

Each closed set C_i can be expressed as the complement of an open set $X - U_i$ for some U_i.
So $\bigcup_{i=1}^n C_i = X \cap \bigcup_{i=1}^n (X - U_i)$
By DeMorgan's law, we can write $\bigcup_{i=1}^n C_i = X - \bigcap_{i=1}^n U_i$

$\bigcap_{i=1}^n U_i$ is an intersection of finitely many open sets and is thus open.
Since $\bigcup_{i=1}^n C_i$ is the complement of an open set, $\bigcup_{i=1}^n C_i$ is closed.
$\blacksquare$

EXERCISE 1.35 *Show that $\mathbb{R}$ in the lower limit topology is Hausdorff.*

Consider two distinct points p and q in the lower limit topology on $\mathbb{R}$.

Since p and q are distinct, either $p < q$ or $q < p$.
Without loss of generality, let p be less than q.

Consider the open sets $U_1 = [p, q)$ and $U_2 = [q, q + 1)$.
It is easy to see that U_1 and U_2 are disjoint.
Furthermore, $p \in U_1$ and $q \in U_2$.

Thus, there are disjoint neighborhoods U_1 and U_2 around p and around q respectively.

$\therefore$ $\mathbb{R}$ in the lower limit topology is Hausdorff.
$\blacksquare$

EXERCISE 1.36 *Show that $\mathbb{R}$ in the finite complement topology is not Hausdorff.*

Consider two distinct points p and q in the lower limit topology on $\mathbb{R}$.

Consider two open sets U_1 and U_2 such that $p \in U_1$ and $q \in U_2$.
U_1 and U_2 can be written as $\mathbb{R} - F_1$ and $\mathbb{R} - F_2$ respectively, where F_1 and F_2 are finite sets.

Consider $U_1 \cap U_2$.
$U_1 \cap U_2 = (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2)$
By Demorgan's Law,
$U_1 \cap U_2 = \mathbb{R} - (F_1 \cup F_2)$

Since $\mathbb{R} - (F_1 \cup F_2)$ is the difference between an uncountable set and a finite set, it is nonempty.
This means that U_1 and U_2 are not disjoint.
Thus, there are no disjoint neighborhoods around p and around q.

$\therefore$ $\mathbb{R}$ in the finite complement topology is not Hausdorff.
■