Math 331 A - Probability

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November 1st, 2019

HW p.90 #42,44,50,51 p.91 #54,62,68 p.100 #70ab,76

1 p. 90 #42

If the values of the joint probability distribution of X and Y are as shown in the table

$$\begin{array}{cc|ccc|} & & & x & \\ & & 0 & 1 & 2 \\ \hline & 0 & \frac{1}{12} & \frac{1}{6} & \frac{1}{24} \\ & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{40} \\ y & 2 & \frac{1}{8} & \frac{1}{20} & \\ & 3 & \frac{1}{120} & & \\ \hline \end{array}$$

find
(a) $P(X = 1, Y = 1)$; **(b)** $P(X = 0, 1 \leq Y < 3)$;
(c) $P(X + Y \leq 1)$; **(d)** $P(X > Y)$;

$$P(X = 1, Y = 1) = \frac{1}{4}$$

$$P(X = 0, 1 \leq Y < 3) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$$P(X + Y \leq 1) = \frac{1}{12} + \frac{1}{4} + \frac{1}{6} = \frac{1}{2}$$

$$P(X > Y) = \frac{1}{4} + \frac{1}{8} + \frac{1}{120} + \frac{1}{20} = \frac{13}{30}$$

$$\boxed{\begin{array}{ll} \textbf{(a)} P(X = 1, Y = 1) = \frac{1}{4}; & \textbf{(b)} P(X = 0, 1 \leq Y < 3) = \frac{3}{8} \\ \textbf{(c)} P(X + Y \leq 1) = \frac{1}{2}; & \textbf{(d)} P(X > Y) = \frac{13}{30} \end{array}}$$

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2 p. 90 #44

If the joint probability distribution of X and Y is given by

$$f(x,y) = c(x^2 + y^2) \quad \text{for} x = -1, 0, 1, 3; \quad y = -1, 2, 3$$

find the value of c.

By Theorem 3.7 we know

$$\sum_x \sum_y f(x,y) = 1$$

$$\sum_x \sum_y c(x^2 + y^2) = 1$$

$$c\left[3\sum_x x^2 + 4\sum_y y^2\right] = 1$$

$$c[3(1+0+1+9) + 4(1+4+9)] = 89c = 1$$

$$\boxed{c = \frac{1}{89}}$$

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3 p. 90 #50

If the joint probability density of X and Y is given by

$$f(x, y) = \begin{cases} 24xy \text{ for } 0 < x < 1, 0 < y < 1, x + y < 1 \\ 0 \qquad \text{elsewhere} \end{cases}$$

find $P(X + Y < \frac{1}{2})$.

$$x + y < \frac{1}{2} \implies y < \frac{1}{2} - x$$

By Def. 3.8 we have

$$P(X + Y < \frac{1}{2}) = \int_0^1 \int_0^{\frac{1}{2}-x} 24xydydx$$

After computing this integral we have

$$\boxed{P(X + Y < \frac{1}{2}) = \frac{1}{2}}$$

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4 p. 90 #51

If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 2 \text{ for } x > 0, y > 0, x + y < 1 \\ 0 \ \text{ elsewhere} \end{cases}$$

find
(a) $P(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$;
(b) $P(X + Y > \frac{2}{3})$;
(c) $P(X > 2Y)$.

By Def. 3.8 we have

$$P(X \leq \frac{1}{2}, Y \leq \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 2dydx = \frac{1}{2}$$

$$P(X + Y > \frac{2}{3}) = \int_0^{\frac{2}{3}} \int_0^{\frac{2}{3}-x} 2dydx = \frac{4}{9}$$

$$P(X > 2Y) = \int_0^{\frac{2}{3}} \int_{\frac{1}{2}x}^{1-x} 2dydx = \frac{2}{3}$$

$$\boxed{\begin{array}{l} \textbf{(a)} P(X \leq \frac{1}{2} Y \leq \frac{1}{2}) = \frac{1}{2} \\ \textbf{(b)} P(X + Y > \frac{2}{3}) = \frac{4}{9} \\ \textbf{(c)} P(X > 2Y) = \frac{2}{3} \end{array}}$$

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5 p. 91 #54

Find the joint probability density of the two random variables X and Y whose joint distribution function is given by

$$F(x,y) = \begin{cases} (1-e^{-x^2})(1-e^{-y^2}) & \text{for } x>0, y>0 \\ 0 & \text{elsewhere} \end{cases}$$

By Def. 3.9 we define the joint distribution function to be

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt$$

Thus,

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y}(1-e^{-x^2})(1-e^{-y^2}) \text{ for } x>0, y>0$$

$$f(x,y) = 2xe^{-x^2}2ye^{-y^2} \text{ for } x>0, y>0$$

$$\boxed{f(x,y) = \begin{cases} 4xye^{-x^2}e^{-y^2} & \text{for } x>0, y>0 \\ 0 & \text{elsewhere} \end{cases}}$$

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6 p. 91 #62

Find k if the joint probability distribution of X, Y, and Z is given by

$$f(x, y, z) = kxyz$$

for $x = 1, 2; y = 1, 2, 3; z = 1, 2.$

By Theorem 3.7 we have

$$\sum_{(x,y,z)} f(x, y, z) = 1$$

Considering all points and plugging them into our joint probability distribution and the distributive law we have

$$k(1 + 2)(1 + 2 + 3)(1 + 2) = 1$$

$$k * 54 = 1$$

$\boxed{k = \frac{1}{54}}$

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7 p. 91 #68

If the joint probability density of X,Y, and Z is given by

$$f(x,y,z) = \begin{cases} \frac{1}{3}(2x+3y+z) & \text{for } 0<x<1, 0<y<1, 0<z<1 \\ 0 & \text{elsewhere} \end{cases}$$

find
(a) $P(X=\frac{1}{2}, Y=\frac{1}{2}, Z=\frac{1}{2})$;
(b) $P(X<\frac{1}{2}, Y<\frac{1}{2}, Z<\frac{1}{2})$.

Because X,Y, and Z are continuous-values random variables, the probability X,Y or Z takes on a specific value is 0.

$$P((x,y,z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})) = 0$$

By Definition 3.8 we have

$$P(X<\frac{1}{2}, Y<\frac{1}{2}, Z<\frac{1}{2}) = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \frac{1}{3}(2x+3y+z)dxdydz$$

After computation we have

$$P(X<\frac{1}{2}, Y<\frac{1}{2}, Z<\frac{1}{2}) = \frac{1}{16}$$

$$\boxed{\begin{array}{l} \textbf{(a)} P(X=\frac{1}{2}, Y=\frac{1}{2}, Z=\frac{1}{2}) = 0 \\ \textbf{(b)} P(X<\frac{1}{2}, Y<\frac{1}{2}, Z<\frac{1}{2}) = \frac{1}{16} \end{array}}$$

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8 p. 100 #70ab

With reference to Exercise 3.42 on page 90, find
(a) the marginal distribution of X;
(b) the marginal distribution of Y;

By Definition 3.10 the marginal distributions of X and Y are given by

$$g(x) = \sum_y f(x,y), \; h(y) = \sum_x f(x,y)$$

where $g(x)$ and $h(y)$ are the marginal distributions of X and Y respectively. In Exercise 3.42, we were given a PDF in the form of a table. To calculate marginal densities using Def. 3.10, the sums of the columns and of the rows correspond to the marginal distributions of X and Y respectively.

$$\boxed{g(x) = \begin{cases} \frac{7}{15} \text{ for } x = 0 \\ \frac{7}{15} \text{ for } x = 1 \\ \frac{1}{15} \text{ for } x = 2 \\ 0 \;\; \text{elsewhere} \end{cases}}$$

$$\boxed{h(y) = \begin{cases} \frac{7}{24} \text{ for } y = 0 \\ \frac{21}{40} \text{ for } y = 1 \\ \frac{7}{40} \text{ for } y = 2 \\ \frac{1}{120} \text{ for } y = 3 \\ 0 \;\; \text{elsewhere} \end{cases}}$$

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9 p. 100 #76

If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 24y(1-x-y) & \text{for } x>0, y>0, x+y<1 \\ 0 & \text{elsewhere} \end{cases}$$

find
(a) the marginal density of X;
(b) the marginal density of Y.
Also determine whether the two random variables are independent.

By Def. 3.11

$$g(x) = \int_0^{1-x} 24y(1-x-y)dy$$

$\boxed{\textbf{(a) } g(x) = 4(1-x)^3,\ 0 < y < 1}$

By Def. 3.11

$$h(y) = \int_0^{1-y} 24y(1-x-y)dx$$

$\boxed{\textbf{(b) } h(y) = 12y(1-y)^2,\ 0 < x < 1}$

By Def. 3.14, random variables X and Y are independent if and only if $f(x,y) = g(x)h(y)$.

But $f(x,y) \neq g(x)h(y)$

$\boxed{\text{R.V.s X and Y are not independent.}}$