Topology Homework 05

Ethan Jensen, Luke Lemaitre, Kasandra Lassagne

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EXERCISE 2.13 Determine the set of limit points of A in each case.

(a) $A = (0, 1]$ in the lower limit topology on $\mathbb{R}$.
(b) $A = \{a\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$.
(c) $A = \{a, c\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$.
(d) $A = \{b\}$ in $X = \{a, b, c\}$ with topology $\{X, \varnothing, \{a\}, \{a, b\}\}$.
(e) $A = (-1, 1) \cup \{2\}$ in the standard topology on $\mathbb{R}$.
(f) $A = (-1, 1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$.
(g) $A = \{(x, 0) \in \mathbb{R}^2 |\ x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology.
(h) $A = \{(0, y) \in \mathbb{R}^2 |\ y \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the vertical interval topology.
(i) $A = \{(x, 0) \in \mathbb{R}^2 |\ x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the vertical interval topology.

Let C be the collection of limit points in each case.

(a) $\boxed{C = A}$
(b) $\boxed{C = \{b, c\}}$
(c) $\boxed{C = \{b, c\}}$
(d) $\boxed{C = \{c\}}$
(e) $\boxed{C = [-1, 1]}$
(f) $\boxed{C = [-1, 1)}$
(g) $\boxed{C = A}$
(h) $\boxed{C = A}$
(i) $\boxed{C = \varnothing}$

EXERCISE 2.15 Determine the set of limit points of $[0, 1]$ in the finite complement topology on $\mathbb{R}$.

Since open sets in the finite complement topology have finite complements and $[0, 1]$ contains infinitely many points, every open set intersects with infinitely many points in $[0, 1]$.

This means that for all points p in $\mathbb{R}$, every open set containing that point will intersect $[0, 1]$ in a point other than p. This means that every point in $\mathbb{R}$ is a limit point.

The set of limit points of $[0, 1]$ in the finite complement topology is $\mathbb{R}$.

EXERCISE 2.18 Determine the set of limit points of $A = \{\frac{1}{m} + \frac{1}{n} \in \mathbb{R} |\ m, n \in \mathbb{Z}_+\}$ in the standard topology on $\mathbb{R}$.

The set of limit points is $\{\frac{1}{m} |\ m \in \mathbb{Z}_+\} \cup \{0\}$.

EXERCISE 2.19 Show that if (x_n) is an injective sequence in $\mathbb{R}$, then (x_n) converges to every point in $\mathbb{R}$ with the finite complement topology on $\mathbb{R}$.

Consider an open set U in the finite complement topology with a complement $F = \mathbb{R} - U$.

Since (x_n) is injective in $\mathbb{R}$, values from F that show up in (x_n) will only show up once (if they show up).

Since F is finite, $\exists N \in \mathbb{Z}_+ \ni x_n \notin F$ for $n \geq N$
So $\exists N \in \mathbb{Z}_+ \ni x_n \in U$ for $n \geq N$

Therefore, all open sets in $\mathbb{R}$ will contain the sequence (x_n) after some finite number of terms.

For all open sets containing an arbitrary point p, the sequence will eventually appear in that open set after a finite number of terms, so the sequence converges to that point.

Thus, every injective sequence converges to **every point**!

EXERCISE 2.20 Prove Theorem 2.11: Let A be a subset of $\mathbb{R}^n$ in the standard topology. If x is a limit point in A, then there is a sequence of points in A that converges to x.

Proof.
Let A be a subset of $\mathbb{R}^n$ in the standard topology, and let x be a limit point of A.

Consider the sequence of open balls (B_n) defined by $B_n = B(x, \frac{1}{n})$.

Since x is a limit point, every open set around x intersects A at a point other than x.
For each open ball B_n, call this point y_n.

Then, the sequence (y_n) consisting of all of these points lives in A since each point lives in A.
Since each set is nested, for any given B_n, all values of the sequence after y_n are contained in B_n.

Now consider some open ball $B(q, r)$ that contains x.
By definition, $d(q, x) < r$.
Between every two real numbers is a rational number.
$\exists m, n \in \mathbb{Z}_+ \ni 0 \leq \frac{m}{n} \leq r - d(q, x)$
Since $m \geq 1$, $\exists n \in \mathbb{Z}_+ \ni d(q, x) + \frac{1}{n} < r$.

Consider the open ball B_n.
By the triangular property, every point in B_n is in $B(q, r)$.
So $B_n \subseteq B(q, r)$.
Thus, all values of the sequence after y_n after n terms are contained in $B(q, r)$.

Thus, any basis element containing x contains all values of the sequence (y_n) after a fixed N terms.
Every open set containing x can be written as a union of basis elements.
Therefore, every open set containing x contains all values of the sequence (y_n) after a fixed N terms.

Thus, the sequence (y_n) is a sequence of points in A that converges to x.

$\therefore$ If x is a limit point in A, then there is a sequence of points in A that converges to x.
$\blacksquare$

EXERCISE 2.21 Determine the set of limit points for the set

$$S = \{(x, \sin(\frac{1}{x})) \in \mathbb{R}^2 |\ 0 < x \leq 1\}$$

as a subset of $\mathbb{R}^2$ in the standard topology.

Let C be the set of all limit points of S.

$\boxed{C = S \cup (\{0\} \times (-1, 1))}$

EXERCISE 2.22 Consider the sequence defined by $x_n = \frac{(-1)^n}{n}$ in $\mathbb{R}$ with the standard topology.

> **(a)** Prove that every neighborhood of the point 0 contains an open interval $-\alpha, \alpha$.
> **(b)** Prove that for each open interval $(-\alpha, \alpha)$, there exists $N \in \mathbb{Z}_+$, such that $x_n \in (-\alpha, \alpha)$ for all $n \geq N$.

(a) Proof.
Consider a neighborhood U of the point 0.
In the standard topology, every neighborhood can be expressed as the union of open intervals - the basis elements for the standard topology on $\mathbb{R}$.

So $\exists B_1, B_2, B_3 ... \ni U = \bigcup_i B_i$ where each B_i is an open interval (a_i, b_i).
Since U contains 0, $\exists B_k \subseteq U \ni 0 \in B_k$, where $B_k = (a_k, b_k), a_k < 0 < b_k$.

Let $\alpha = \min\{-a_k, b_k\}$
It is easy to see that $-\alpha \geq a_k$ and $\alpha \leq b_k$.
So $(-\alpha, \alpha) \subseteq B_k \subseteq U$.
So $(-\alpha, \alpha) \subseteq U$.
$\therefore$ Every neighborhood of the point 0 contains an open interval $(-\alpha, \alpha)$.
$\square$

(b) Proof.
Consider an open interval $(-\alpha, \alpha)$.

Between any two real numbers, there exists a rational number.
$\exists m, N \in \mathbb{Z}_+ \ni 0 < \frac{m}{N} < \alpha$.
Since $m \geq 1$, $-\alpha, -\frac{1}{N} < 0 < \frac{1}{N} < \alpha$

For all $n \geq N$, $-\frac{1}{N} \geq x_n \geq \frac{1}{N}$.
Thus, for all $n \geq N$, $x_n \in (-\alpha, \alpha)$.
$\therefore$ For each open interval $(-\alpha, \alpha)$, there exists $N \in \mathbb{Z}_+$, such that $x_n \in (-\alpha, \alpha)$ for all $n \geq N$.
$\blacksquare$

EXERCISE 2.24 Determine ∂A in each case.

(a) $A = (0,1]$ in the lower limit topology on $\mathbb{R}$.
(b) $A = \{a\}$ in $X = \{a,b,c\}$ with topology $\{X, \varnothing, \{a\}, \{a,b\}\}$.
(c) $A = \{a,c\}$ in $X = \{a,b,c\}$ with topology $\{X, \varnothing, \{a\}, \{a,b\}\}$.
(d) $A = \{b\}$ in $X = \{a,b,c\}$ with topology $\{X, \varnothing, \{a\}, \{a,b\}\}$.
(e) $A = (-1,1) \cup \{2\}$ in the standard topology on $\mathbb{R}$.
(f) $A = (-1,1) \cup \{2\}$ in the lower limit topology on $\mathbb{R}$.
(g) $A = \{(x,0) \in \mathbb{R}^2 |\ x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the standard topology.
(h) $A = \{(0,y) \in \mathbb{R}^2 |\ y \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the vertical interval topology.
(i) $A = \{(x,0) \in \mathbb{R}^2 |\ x \in \mathbb{R}\}$ in $\mathbb{R}^2$ with the vertical interval topology.

(a) $\boxed{\partial A = \{1\}}$
(b) $\boxed{\partial A = \{b,c\}}$
(c) $\boxed{\partial A = \{b,c\}}$
(d) $\boxed{\partial A = \{b,c\}}$
(e) $\boxed{\partial A = \{-1,1,2\}}$
(f) $\boxed{\partial A = \{2\}}$
(g) $\boxed{\partial A = A}$
(h) $\boxed{\partial A = \varnothing}$
(i) $\boxed{\partial A = A}$

EXERCISE 2.26 Determine the boundary of each of the following subsets of $\mathbb{R}^2$ in the standard topology.

(a) $A = \{(x, x) \in \mathbb{R}^2 |\ x \in \mathbb{R}\}$
(b) $B = \{(x, y) \in \mathbb{R}^2 |\ x > 0, y \neq 0\}$
(c) $C = \{(\frac{1}{n}, 0) \in \mathbb{R}^2 |\ n \in \mathbb{Z}_+\}$
(d) $D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x^2 - y^2 < 1\}$

(a) $\boxed{\partial A = A}$

(b) $\boxed{\partial B = \{(x, y) \in \mathbb{R}^2 |\ x \geq 0, y = 0\}}$

(c) $\boxed{\partial C = C \cup \{0\}}$

(d) $\boxed{\partial D = \{(x, y) \in \mathbb{R}^2 |\ x = y \text{ or } x = -y\}}$

EXERCISE 2.28 Prove Theorem 2.15: Let A be a subset of a topological space X.

(a) ∂A is closed.

(b) $\partial A = \mathrm{Cl}(A) \cap \mathrm{Cl}(X - A)$

(c) $\partial A \cap \mathrm{Int}(A) = \varnothing$

(d) $\partial A \cup \mathrm{Int}(A) = \mathrm{Cl}(A)$

(e) $\partial A \subseteq A$ if and only if A is closed.

(f) $\partial A \cap A = \varnothing$ if and only if A is open.

(g) $\partial A = \varnothing$ if and only if A is both open and closed.

(a) Proof.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
Because $\mathrm{Int}(A)$ is open, there exists a closed set C such that $\mathrm{Int}(A) = X - C$
So $\partial A = \mathrm{Cl}(A) - (X - C) = \mathrm{Cl}(A) \cap C$.
The intersection of closed sets is a closed set.
$\therefore \partial A$ is closed.
$\square$

(b) Proof.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
By Theorem 2.6, $\partial A = \mathrm{Cl}(A) - (X - \mathrm{Cl}(A))$
$\therefore \partial A = \mathrm{Cl}(A) \cap \mathrm{Cl}(A)$
$\square$

(c) Proof.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
This means that ∂A and $\mathrm{Int}(A)$ are disjoint.
$\therefore \partial A \cap \mathrm{Int}(A) = \varnothing$.
$\square$

(d) Proof.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
$\partial A \cup \mathrm{Int}(A) = (\mathrm{Cl}(A) - \mathrm{Int}(A)) \cup \mathrm{Int}(A)$
$\therefore \partial A \cup \mathrm{Int}(A) = \mathrm{Cl}(A)$
$\square$

(e) Proof.
$(\rightarrow)$ Assume $\partial A \subseteq A$.
$\mathrm{Int}(A)$ is also a subset of A.
Thus, $\partial A \cup \mathrm{Int}(A) \subseteq A$
By part (d), we can write $\mathrm{Cl}(A) \subseteq A$.
However, we also know that $A \subseteq \mathrm{Cl}(A)$.
$\therefore A = \mathrm{Cl}(A)$ and A is closed.

$(\leftarrow)$ Assume A is closed.
This means that $\mathrm{Cl}(A) = A$.

$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
Since $\mathrm{Cl}(A) = A$, $\partial A = A - \mathrm{Int}(A)$
$\therefore \partial A \subseteq A$

$\therefore \partial A \subseteq A$ if and only if A is closed.
$\square$

(f) Proof.
$(\rightarrow)$ Assume $\partial A \cap A = \varnothing$.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
$(\mathrm{Cl}(A) - \mathrm{Int}(A)) \cap A = \varnothing$
$\mathrm{Cl}(A) \cap A - \mathrm{Int}(A) \cap A = \varnothing$
Since $\mathrm{Int}(A) \subseteq A \subseteq \mathrm{Cl}(A)$, $\quad \mathrm{Cl}(A) \cap A = A$ and $\mathrm{Int}(A) \cap A = \mathrm{Int}(A)$.
$A = \mathrm{Int}(A)$.
$\therefore A$ is open.

$(\leftarrow)$ Assume A is open.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
Since A is open, $\mathrm{Int}(A) = A$
$\partial A = \mathrm{Cl}(A) - A$.
This means ∂A and A are disjoint.
$\therefore \partial A \cap A = \varnothing$

$\therefore \partial A \cap A = \varnothing$ if and only if A is open.
$\square$

(g) Proof.
$(\rightarrow)$ Assume $\partial A = \varnothing$.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$.
$\mathrm{Cl}(A) - \mathrm{Int}(A) = \varnothing$.
Thus, $\mathrm{Cl}(A) \subseteq \mathrm{Int}(A)$.
However, $\mathrm{Int}(A) \subseteq A \subseteq \mathrm{Cl}(A)$.
Thus, $\mathrm{Int}(A) = A = \mathrm{Cl}(A)$.
Since $\mathrm{Int}(A)$ is open, and $\mathrm{Cl}(A)$ is closed,
A is both open and closed.

$(\leftarrow)$ Assume A is both open and closed.
$\partial A = \mathrm{Cl}(A) - \mathrm{Int}(A)$
Since A is both open and closed, $\mathrm{Int}(A) = A = \mathrm{Cl}(A)$
$\partial A = A - A = \varnothing$
$\square$

$\blacksquare$