Math 331 A - Probability

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1 p. 208 #3

If X has the uniform density with the parameters $\alpha = 0$ and $\beta = 1$, use the distribution function technique to find the probability density of the random variable Y = $\sqrt{X}$.

By Definition 3.3 we have

$$F_Y(y) = P(Y < y) = P(\sqrt{X} < y) = P(X < y^2)$$

By Def. 6.1 we have

$$F_Y(y) = P(X < y^2) = \int_0^{y^2} dy = y^2$$

$$f(y) = \frac{d}{dy} F_Y(y) = 2y,\ 0 < y < 1.$$

$\boxed{f(y) = 2y, \quad 0 < y < 1}$

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2 p. 208 #4

If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)} & \text{for } x>0, y>0 \\ 0 & \text{elsewhere} \end{cases}$$

and $Z = \sqrt{X^2+Y^2}$, find
(a) the distribution function of Z;
(b) the probability density of Z.

By Definition 3.3 and Definition 3.8 we have

$$F_z(z) = P(Z < z) = P(\sqrt{X^2+Y^2} < z)$$

$$F_z(z) = P(\sqrt{X^2+Y^2} < z) = \iint_A f(x,y)dA$$

Below is a drawing of region A. Converting the integral into polar we have

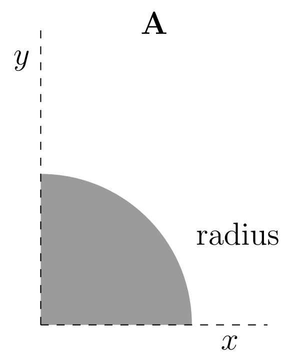

Figure 1: Region of integration

$$F_z(z) = \int_0^{\pi/2}\int_0^z 4\sin(\theta)\cos(\theta)r^2e^{-r^2}rdr\ d\theta$$

This integral can be solved using Fubini's Theorem, the double angle formula for sine, a u-substitution and integration by parts.
Then a miracle happened.

$\boxed{\textbf{(b)}\ \ F_Z(z) = 1-(1+z^2)e^{-z^2},\ 0<z<\infty}$

By Theorem 3.6
$\boxed{\textbf{(a)}\ \ f(z) = 2z^3e^{-z^2},\ 0<z<\infty}$

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3 p. 226 #43

If n independent random variables have the same gamma distribution with the parameters α and β, find the moment-generating function of their sum and, if possible, identify its distribution.

Let $Y = X_1 + X_2 + X_3 ... + X_n$ where $X_1, X_2, ...X_n$ are independent RVs with the same gamma distribution with the parameters α and β.
By Theorem 6.4 The moment-generating function of a gamma random variable X_i with α and β is given by

$$M_{X_i}(t) = (1 - \beta t)^{-\alpha}$$

By Theorem 7.3 we have

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$M_Y(t) = \prod_{i=1}^{n} (1 - \beta t)^{-\alpha}$$

$$M_Y(t) = (1 - \beta t)^{-\alpha n}$$

Variables can be uniquely identified by their moment-generating functions.

$M_Y(t)$ is a moment generating function of a gamma distribution with parameters $\alpha = \alpha n$ and $\beta = \beta$.

(a) $f(y) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{n\alpha - 1} e^{-x/\beta}$
(b) $M_Y(t) = (1 - \beta t)^{-\alpha n}$

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4 p. 226 #46

Use the result of Exercise 7.45 to show that, if n independent random variables X_i have normal distributions with the means μ_i and the standard deviations σ_i, then $Y = a_1X_1 + a_2X_2 + ...a_nX_n$ also has a normal distribution. What are the mean and the variance of this distribution?

The moment-generating function of a normal RV X_i is given by

$$M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$$

Using the result of Exercise 7.45 we have

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

$$M_Y(t) = \prod_{i=1}^{n} e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$$

$$M_Y(t) = e^{(\sum_{i=1}^{n} \mu_i)t + \frac{1}{2}(\sum_{i=1}^{n} \sigma_i^2)t^2}$$

Both $\sum_{i=1}^{n} \mu_i$ and $\sum_{i=1}^{n} \sigma_i^2$ are constants.
This means that $M_Y(t)$ is the moment-generating function of a normal distribution. In fact those summations are the mean and variance of Y respectively!

$$\boxed{Y \text{ has a normal distribution with } \mu_y = \sum_{i=1}^{n} \mu_i \text{ and } \sigma_y^2 = \sum_{i=1}^{n} \sigma_i^2}$$

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5 p. 239 #3

With reference to Exercise 8.2, show that if the two samples come from normal populations, then $\overline{X}_1 - \overline{X}_2$ is a random variable having a normal distribution with the mean $\mu_1 - \mu_2$ and the variance $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$.

The moment-generating function for a normal distribution with mean μ and variance σ^2 is given by

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

By Theorem 6.6 we get

$$M_{X_1}(t) = e^{\mu_1 t + \frac{1}{2}\left(\frac{\sigma_1^2}{n_1}\right)t^2}, \quad M_{X_2}(t) = e^{\mu_2 t + \frac{1}{2}\left(\frac{\sigma_2^2}{n_2}\right)t^2}$$

Let $\overline{Y} = \overline{X}_1 + \overline{X}_2$ be a random variable. By Theorem 4.10 we can write

$$M_Y(t) = M_{\overline{X}_1}(t) M_{\overline{X}_2}(t)$$

$$M_Y(t) = e^{\mu_1 t + \frac{1}{2}\left(\frac{\sigma_1^2}{n_1}\right)t^2} e^{\mu_2 t + \frac{1}{2}\left(\frac{\sigma_2^2}{n_2}\right)t^2} = e^{(\mu_1+\mu_2)t + \left(\frac{1}{2}\frac{\sigma_1^2}{n_1} + \frac{1}{2}\frac{\sigma_2^2}{n_2}\right)t^2}$$

This is the moment-generating function for a normal distribution.

$$\boxed{\text{$\overline{X}_1 - \overline{X}_2$ is a random variable having a normal distribution with the mean $\mu_1 - \mu_2$ and the variance $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$}}$$

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6 p. 256 #66

A random sample of size $n = 81$ is taken from an infinite population with the mean $\mu = 128$ and the standard deviation $\sigma = 6.3$. With what probability can we assert that the value we obtain for $\overline{X}$ will not fall between 126.6 and 129.4 if we use
(a) Chebychev's theorem;
(b) the central limit theorem?

We know that the standard deviation of $overline X = \frac{\sigma}{\sqrt{n}} = \frac{6.3}{9} = 0.7$.
Chebychev's theorem states that

$$P(|x - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Thus, by Chebychev's theorem

$$P(126.6 < \overline{x} < 129.4) = P(|\overline{x} - 128| < 2 \cdot 0.7) \geq 1 - \frac{1}{2^2}$$

By Theorem 2.3 we have

$\boxed{\textbf{(a)}\ P(\overline{x} < 126.6) + P(129.4 < \overline{x}) \leq 0.2500 \text{ or } 25.00\%}$

By the central limit theorem, the random variable $\overline{X}$ approaches a normal distribution with the same mean and variance.
Referencing Table III, if $z = 2$ then $P(\overline{x} < 126.6) + P(129.4 < \overline{x}) = 1 - 2(0.4772) = 0.0456$.

$\boxed{\textbf{(b)}\ P(\overline{x} < 126.6) + P(129.4 < \overline{x}) \leq 0.0456 \text{ or } 4.56\%}$

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7 p. 256 #68

A random sample of size $n = 225$ is to be taken from an exponential population with $\theta = 4$. Based on the central limit theorem, what is the probability that the mean of the sample will exceed 4.5?

By Corollary 6.1, the mean and the variance of the population are 4 and 16 respectively.
Thus, the mean and standard deviation of $\overline{X}$ are 4 and $\frac{4}{15}$ respectively.

4.5 is thus 1.875 standard deviations away from the mean.
By the central limit theorem and by referencing Table III, if $z = 1.875$ then $P(\overline{x} > 4.5) = 0.5 - 0.4696 = 0.0304$.

$\boxed{P(\overline{x} > 4.5) = 0.0304 \text{ or } 3.04\%}$

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8 p. 256 #70

A random sample of size 64 is taken from a normal population with $\mu = 51.4$ and $\sigma = 6.8$. What is the probability that the mean of the sample will
(a) exceed 52.9;
(b) fall between 50.5 and 52.3;
(c) be less than 50.6

The mean and standard deviation of $\overline{X}$ are 51.4 and 0.85 respectively.

52.9 is therefore 1.7647 standard deviations away from the mean.
Referring to Table 3 with z = 1.76 we get the value 0.4608. Thus, $P(\overline{X} > 52.9 = 0.5 - 0.4608 = 0.0392$

$\boxed{\text{(a)The probability the mean of the sample exceeds 52.9 is0.0392 or 3.92\%}}$

50.5 and 52.3 are both 1.0588 standard deviations away from the mean.
Referring to Table 3 with z = 1.06 we get the value 0.3554. Thus, $P(50.5 < \overline{X} < 52.3 = 2 * 0.3554 = 0.7108$

$\boxed{\text{(b)The probability the mean of the sample is between 50.5 and 52.3 is0.7108 or 71.08\%}}$

50.6 is 0.9412 standard deviations from the mean of the sample. Referring to Table 3 with z = 0.94 we get the value 0.3264. Thus, $P(\overline{X} < 50.6 = 0.5 - 0.3264 = 0.1736$

$\boxed{\text{(c)The probability the mean of the sample is less than 50.6 is0.1736 or 17.36\%}}$

9 Poisson-Gamma 4.3-1

Telephone calls enter a college switchboard at a mean rate of 2/3 call per minute according to a Poisson process. Let X denote the waiting time until the tenth call arrives.
(a) What is the p.d.f. of X?
(b) What are the moment-generating function, mean, and variance of X?

$$F_x(x) = P(X \leq x) = 1 - P(X > x)$$

By scaling, and by Def. 5.7 we have

$$F_x(x) = 1 - \sum_{i=0}^{k-1} \frac{()\lambda x)^i}{i!} e^{-\lambda x}$$

To get the p.d.f. of X, we differentiate both sides

$$f(x) = -\sum_{i=0}^{k-1} \frac{i(\lambda x)^{i-1} e^{-\lambda x} - (\lambda x)^i e^{-\lambda x}}{i!}$$

$$f(x) = -e^{-\lambda x} \sum_{i=0}^{k-1} \frac{\lambda^i x^{i-1}}{(i-1)!} - \frac{\lambda^{i+1} x^i}{i!}$$

This is a telescoping series! Thus, we are left with the first part evaluated at the lower bound plus the second part evaluated at the upper bound. Everything else cancels out.

$$f(x) = -e^{-\lambda x} \left(\lim_{i \to 0} \frac{\lambda^i x^{i-1}}{(i-1)!} - \frac{\lambda^k}{(k-1)!} x^{k-1} \right)$$

$$f(x) = \frac{1}{\left(\frac{1}{\lambda}\right)^k \Gamma(k)} x^{k-1} e^{\frac{-x}{\frac{1}{\lambda}}}$$

Plugging in $\frac{2}{3}$ for λ and 10 for k, we have

$$\boxed{\textbf{(a)}\ f(x) = \frac{1}{\left(\frac{3}{2}\right)^{10} \Gamma(10)} x^9 e^{\frac{-3x}{2}},\ x > 0}$$

This is the p.d.f. of a Gamma random variable with $\alpha = 10,\ \beta = \frac{3}{2}$.
By Theorem 6.3 the mean and variance of the Gamma distribution are $\mu = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$.

Plugging in 10 for α and $\frac{3}{2}$ for β we have

$$\boxed{\textbf{(b)}\ \mu_x = 15,\ \sigma_x^2 = \frac{45}{2}}$$

10 Poisson-Gamma 4.3-6

Let X denote the number of alpha particles emitted by barium-133 and observed by a Geiger counter in a fixed position. Assume X has a Poisson distribution and $\lambda = 14.7$ is the mean number of counts per second. Let W denote the waiting time to observe 100 counts.
Twenty-five independent observations of W are

6.9	7.3	6.7	6.4	6.3
5.9	7.0	7.1	6.5	7.6
7.2	7.1	6.1	7.3	7.6
7.6	6.7	6.3	5.7	6.7
7.5	5.3	5.4	7.4	6.9

Give the p.d.f., mean, and variance of W.

For this part of the problem, the table of twenty-five observations is unnecessary.
We know that W must be a Gamma random variable with $\alpha = 100$ and $\beta = 0.068$.
We can use Theorem 6.3 to find the mean and variance.

$$\boxed{\begin{array}{c} f(w) = \frac{1}{(0.068^{100})99!} w^{99} e^{-14.7w},\ w > 0 \\ \mu_w = 6.8,\ \sigma_w^2 = 680 \end{array}}$$