Topology Homework

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1 Exercise 1.34

Prove that on a finite set, the discrete topology is the only topology that is Hausdorff.

Proof. Assume that a finite set F with topology T is Hausdorff.
Consider the following recursive algorithm.

—def Split(S):
— if (S.length == 1):
— return S
— else:
— Pick two random points in S, x_1 and x_2.
— Note: The following operation is legal since S is Hausdorff.
— Pick two disjoint subsets in S, S_1 and S_2 that contain x_1 and x_2 resp.
— return Split(S_1) , Split(S_2)

The result of Split(F) is a collection C of subsets of length 1, making up a partition of F.
Since Split(S) is a combination of legal operations, $C \in T$.
All subsets of F can be represented as a union of elements in C, since they are length one.
By Definition 1.1, any union of open sets in a Topology is also an open set.
Thus, T is the discrete topology.
$\therefore$ On any finite set, the discrete topology is the only topology that is Hausdorff.
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