Topology Homework

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1 Proof of Theorem 0.9

THEOREM 0.9. *For sets A, B, and C, the following laws hold:*

Distributive Laws:

$$(i)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$(ii)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$(iii)\ A \times (B \cup C) = (A \times B) \cup (A \times C)$$
$$(iv)\ A \times (B \cap C) = (A \times B) \cap (A \times C)$$
$$(v)\ A \times (B - C) = (A \times B) - (A \times C)$$

DeMorgan's Laws:

$$(vi)\ A - (B \cup C) = (A - B) \cap (A - C)$$
$$(vii)\ A - (B \cap C) = (A - B) \cup (A - C)$$

(i) Proof.
Assume $x \in A \cap (B \cup C)$.
$x \in A$ and $x \in (B \cup C)$ by the Definition of $\cap$
$x \in A$ and ($x \in B$ or $x \in C$) by the Definition of $\cup$
($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) by the Distributive Law
$x \in A \cap B$ or $x \in A \cap C$ by the Definition of $\cap$
$x \in (A \cap B) \cup (A \cap C)$ by the Definition of $\cup$
$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
$\square$
Each step is reversible.
$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$\blacksquare$

2 Proof of Theorem 0.9 cont.

(ii) Proof.
Assume $x \in A \cup (B \cap C)$.
$x \in A$ and $x \in (B \cap C)$ by the Definition of $\cup$
$x \in A$ or $(x \in B$ and $x \in C)$ by the Definition of $\cap$
$(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$ by the Distributive Law
$x \in A \cup B$ and $x \in A \cup C$ by the Definition of $\cup$
$x \in (A \cup B) \cap (A \cup C)$ by the Definition of $\cap$
$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
$\square$
Each step is reversible.
$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$\blacksquare$

(iii) Proof.
Assume $(x, y) \in A \times (B \cup C)$.
$x \in A$ and $y \in (B \cup C)$ by the Definition of $\times$
$x \in A$ and $(y \in B$ or $y \in C)$ by the Definition of $\cup$
$(x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$ by the Distributive Law
$(x, y) \in A \times B$ or $(x, y) \in A \times C$ by the Definition of $\times$
$(x, y) \in (A \times B) \cup (A \times C)$ by the Definition of $\cup$
$\therefore A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
$\square$
Each step is reversible.
$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$
$\therefore A \times (B \cup C) = (A \times B) \cup (A \times C)$
$\blacksquare$

(iv) Proof.
Assume $(x, y) \in A \times (B \cap C)$.
$x \in A$ and $y \in (B \cap C)$ by the Definition of $\times$
$x \in A$ and $(y \in B$ and $y \in C)$ by the Definition of $\cap$
$(x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$ by the Associative Law
$(x, y) \in A \times B$ and $(x, y) \in A \times C$ by the Definition of $\times$
$(x, y) \in (A \times B) \cap (A \times C)$ by the Definition of $\cap$
$\therefore A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$
$\square$
Each step is reversible.
$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$
$\therefore A \times (B \cap C) = (A \times B) \cap (A \times C)$
$\blacksquare$

3 Proof of Theorem 0.9 cont.

(v) Proof. Assume $(x, y) \in A \times (B - C)$.
$x \in A$ and $y \in (B - C)$ by the Definition of $\times$
$x \in A$ and ($y \in B$ and $\sim y \in C$) by the Definition of $-$
($x \in A$ and $y \in B$) and $\sim (x \in A$ and $y \in C)$ by the Annullment Law.
$(x, y) \in A \times B$ and $\sim (x, y) \in A \times C$ by the Definition of $\times$
$(x, y) \in (A \times B) - (A \times C)$ by the Definition of $-$
$\therefore A \times (B - C) \subseteq (A \times B) - (A \times C)$
$\square$
Assume $(x, y) \in (A \times B) - (A \times C)$
$(x, y) \in (A \times B)$ and $\sim (x, y) \in (A \times C)$ by the Definition of $-$
$x \in A$ and $y \in B$ and $\sim (x \in A$ and $y \in C)$ by the Definition of $\times$
$x \in A$ and $y \in B$ and ($\sim x \in A$ or $\sim y \in C$) by DeMorgan's Law
$x \in A$ and $y \in B$ and $\sim y \in C$ by Elimination
$x \in A$ and $y \in (B - C)$ by the Definition of $-$
$(x, y) \in A \times (B - C)$ by the Definition of $\times$
$(A \times B) - (A \times C) \subseteq A \times (B - C)$
$\therefore A \times (B - C) = (A \times B) - (A \times C)$
$\blacksquare$
(v) Proof. Assume $x \in A - (B \cup C)$.
$x \in A$ and $\sim x \in (B \cup C)$ by the Definition of $-$
$x \in A$ and $\sim (x \in B$ or $x \in C)$
$x \in A$ and ($\sim x \in B$ and $\sim x \in C$) by Demorgan's Law
($x \in A$ and $\sim x \in B$) and ($x \in A$ and $\sim x \in C$) by the Associative law
$x \in A - B$ and $x \in A - C$ by the Definition of -
$x \in (A - B) \cap (A - C)$ by the Definition of $\cap$
$\therefore A - (B \cup C) \subseteq (A - B) \cap (A - C)$
$\square$
Each step is reversible.
$(A - B) \cap (A - C) \subseteq A - (B \cup C)$
$\therefore A - (B \cup C) = (A - B) \cap (A - C)$
$\blacksquare$
(vi) Proof. Assume $x \in A - (B \cap C)$.
$x \in A$ and $\sim x \in (B \cap C)$ by the Definition of $-$
$x \in A$ and $\sim (x \in B$ and $x \in C)$
$x \in A$ and ($\sim x \in B$ or $\sim x \in C$) by Demorgan's Law
($x \in A$ and $\sim x \in B$) or ($x \in A$ and $\sim x \in C$) by the Associative law
$x \in A - B$ or $x \in A - C$ by the Definition of -
$x \in (A - B) \cup (A - C)$ by the Definition of $\cup$
$\therefore A - (B \cap C) \subseteq (A - B) \cup (A - C)$
$\square$
Each step is reversible.
$(A - B) \cup (A - C) \subseteq A - (B \cap C)$
$\therefore A - (B \cap C) = (A - B) \cup (A - C)$
$\blacksquare$

4 Proof of Theorem 0.21

THEOREM 0.21 *If $f : X \to Y$ is a function and A and B are subsets of X, then*

$$(i)\ f(A \cup B) = f(A) \cup f(B).$$

$$(ii)\ f(A \cap B) \subseteq f(A) \cap f(B).$$

$$(iii)\ f(A) - f(B) \subseteq f(A - B)$$

(i) Proof.
Assume $y \in f(A \cup B)$
$\exists x \in A \cup B \ni y = f(x)$
$\exists x \in A$ or $\exists x \in B \ni y = f(x)$
$y \in f(A)$ or $y \in f(B)$
$y \in f(A) \cup f(B)$
$f(A \cup B) \subseteq f(A) \cup f(B)$
Assume $y \in f(A) \cup f(B)$
$\exists x \in A \ni y = f(x)$ or $\exists x \in B \ni y = f(x)$
$\exists x \in A \cup B \ni y = f(x)$ since A and B are both subsets of $A \cup B$
$y \in f(A \cup B)$
$f(A) \cup f(B) \subseteq f(A \cup B)$
$\therefore f(A \cup B) = f(A) \cup f(B)$
■

(ii) Proof.
Assume $y \in f(A \cap B)$
$\exists x \in A \cap B \ni y = f(x)$
$\exists x \in A \ni y = f(x)$ and $\exists x \in B \ni y = f(x)$ since $A \cap B$ is a subset of both A and B.
$y \in f(A)$ and $y \in f(B)$
$y \in f(A) \cap f(B)$
$\therefore f(A \cap B) \subseteq f(A) \cap f(B)$
■

(iii) Proof.
Assume $y \in f(A) - f(B)$
$y \in f(A)$ and $\sim y \in f(B)$
$\exists x \in A \ni y = f(x)$ and $\sim \exists x \in B \ni y = f(x)$
$\exists x \in A \cap B' \ni y = f(x)$ since $x \in A$, but it cannot be in B.
$y \in f(A \cap B')$
$y \in f(A - B)$, which is a different way to write the same thing.
$\therefore f(A) - f(B) \subseteq f(A - B)$
■

5 Proof of Theorem 0.22

THEOREM 0.22. *if* $f : X \to Y$ *is a function and* V *and* W *are subsets of* Y*, then*

$$(i)\ f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W).$$
$$(ii)\ f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W).$$
$$(iii)\ f^{-1}(V - W) = f^{-1}(V) - f^{-1}(W).$$

(i) Proof. Assume $x \in f^{-1}(V \cup W)$
$f(x) \in V \cup W$
$f(x) \in V$ or $f(x) \in W$
$x \in f^{-1}(V)$ or $x \in f^{-1}(W)$
$x \in f^{-1}(V) \cup f^{-1}(W)$
$f^{-1}(V) \cup f^{-1}(W) \subseteq f^{-1}(V \cup W)$
$\square$
Each step is reversible.
$f^{-1}(V \cup W) \subseteq f^{-1}(V) \cup f^{-1}(W)$
$\therefore f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$
$\blacksquare$

(ii) Proof. Assume $x \in f^{-1}(V \cap W)$
$f(x) \in V \cap W$
$f(x) \in V$ and $f(x) \in W$
$x \in f^{-1}(V)$ and $x \in f^{-1}(W)$
$x \in f^{-1}(V) \cap f^{-1}(W)$
$f^{-1}(V) \cap f^{-1}(W) \subseteq f^{-1}(V \cap W)$
$\square$
Each step is reversible.
$f^{-1}(V \cap W) \subseteq f^{-1}(V) \cap f^{-1}(W)$
$\therefore f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W)$
$\blacksquare$

(iii) Proof. Assume $x \in f^{-1}(V - W)$
$f(x) \in V - W$
$f(x) \in V$ and $\sim f(x) \in W$
$x \in f^{-1}(V)$ and $\sim x \in f^{-1}(W)$
$x \in f^{-1}(V) - f^{-1}(W)$
$f^{-1}(V - W) \subseteq f^{-1}(V) - f^{-1}(W)$
$\square$
Each step is reversible
$f^{-1}(V) - f^{-1}(W) \subseteq f^{-1}(V - W)$
$\therefore f^{-1}(V - W) = f^{-1}(V) - f^{-1}(W)$
$\blacksquare$