Topology Homework 06

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EXERCISE 3.1 Let $X = \{(x, 0) \in \mathbb{R}^2 | \ x \in \mathbb{R}\}$, the x-axis in the plane. Describe the topology that X inherits as a subspace of $\mathbb{R}^2$ with the standard topology.

X inherits the standard topology on $\mathbb{R}$.

EXERCISE 3.2 Let $Y = [-1, 1]$ has the standard topology. Which of the following sets are open in Y and which are open in $\mathbb{R}$?

$A = (-1, -1/2) \cup (1/2, 1)$
$B = (-1, -1/2] \cup [1/2, 1)$
$C = [-1, -1/2) \cup (1/2, 1]$
$D = [-1, -1/2] \cup [1/2, 1]$
$E = \bigcup_{n=1}^{\infty} (\frac{1}{1+n}, \frac{1}{n})$

$\boxed{\text{A, C, and E are open in Y}}$

$\boxed{\text{A and E are open in } \mathbb{R}}$

EXERCISE 3.3 Prove Theorem 3.4: Let X be a topological space, and let $Y \subseteq X$ have the subspace topology. Then $C \subseteq Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X.

Proof.
($\rightarrow$) Assume C is closed in Y.
Let L_Y be the set of limit points of C in Y.
Let L_X be the set of limit points of C in X.
By Theorem 2.8, Cl(C) in Y = $C \cup L_Y$ and Cl(C) in X = $C \cup L_X$.
Since $Y \subseteq X,\ L_Y = L_X \cap Y$.

Let D be Cl(C) in X. This makes D closed in X.
Notice that $D \cap Y = (C \cup L_X) \cap Y = (C \cap Y) \cup (L_X \cap Y) = C \cup L_Y$
This means $D \cap Y =$ Cl(C) in Y.
Since C is closed in Y, Cl(C) = C.

Thus, $C = D \cap Y$ for some closed set D in X.
$\square$
($\leftarrow$) Assume $C = D \cap Y$ for some closed set D in X.
Since D is closed in X, $X - D$ is open in X.
By the definition of the subspace topology, $(X - D) \cap Y$ is open in Y.
So $Y - D$ is open in Y, so its complement, $D \cap Y$, is closed in Y.
Thus, C is closed in Y.
$\square$
$\therefore C \subseteq Y$ is closed in Y if and only if $C = D \cap Y$ for some closed set D in X
$\blacksquare$

EXERCISE 3.4 Let $Y = (0, 5]$. In each case, is the set open, closed, both or neither in Y in the standard topology?

(a)$(0, 1)$ **(b)**$(0, 1]$ **(c)**$\{1\}$ **(d)**$(0, 5]$ **(e)**$(1, 2)$

(f)$[1, 2)$ **(g)**$(1, 2]$ **(h)**$[1, 2]$ **(i)**$(4, 5]$ **(j)**$[4, 5]$

(a) Open
(b) Closed
(c) Closed
(d) Open and Closed
(e) Open
(f) Neither
(g) Neither
(h) Open
(i) Open and Closed
(j) Closed

EXERCISE 3.5 Let $Y = (0, 5]$ have the subspace topology inherited from $\mathbb{R}$ with the lower limit topology. In each case, is the set open, closed, both or neither in Y in this topology?

(a) $(0, 1)$ **(b)** $(0, 1]$ **(c)** $\{1\}$ **(d)** $(0, 5]$ **(e)** $(1, 2)$

(f) $[1, 2)$ **(g)** $(1, 2]$ **(h)** $[1, 2]$ **(i)** $(4, 5]$ **(j)** $[4, 5]$

(a) Open
(b) Closed
(c) Closed
(d) Open and Closed
(e) Open
(f) Open and Closed
(g) Neither
(h) Open
(i) Open and Closed
(j) Open and Closed

EXERCISE 3.6 Let $Y = (0,4] \cup \{5\}$, In each case, is the set open, closed, both or neither in Y in the standard topology?

$$\textbf{(a)}(0,1) \quad \textbf{(b)}(0,1] \quad \textbf{(c)}\{1\} \quad \textbf{(d)}(0,4] \quad \textbf{(e)}(1,2)$$

$$\textbf{(f)}[1,4) \quad \textbf{(g)}(1,4] \quad \textbf{(h)}[1,4] \quad \textbf{(i)}\{4\} \quad \textbf{(j)}\{4,5\}$$

(a) Open
(b) Closed
(c) Closed
(d) Open and Closed
(e) Open
(f) Neither
(g) Open
(h) Closed
(i) Closed
(j) Closed

EXERCISE 3.7 Let X be a Hausdorff topological space, and Y be a subset of X. Prove that the subspace topology on Y is Hausdorff.

Proof.
Let X be a Hausdorf topological space and let Y be a subset of x.
Consider two distinct points p and q in Y.
Since X is Hausdorf, there exist two disjoint open sets U_p and U_q in X such that $p \in U_p$ and $q \in U_q$.

Let $V_p = U_p \cap Y$ and $V_q = U_q \cap Y$.
V_p and V_q are thus open sets in the subspace topology on Y.
Since $V_p \subseteq U_p$ and $V_q \subseteq U_q$, V_p and V_q must be disjoint.

Thus, for every distict points p and q, there exist two disjoint neighborhoods of p and q in the subspace topology on Y.
Thus, the subspace topology on Y is Hausdorff.

$\therefore$ If X is a Hausdorrf space and $Y \subseteq X$, then the subspace topology on Y is Hausdorff.
$\blacksquare$

EXERCISE 3.8 Let X be a topological space, and let $Y \subseteq X$ have the subspace topology.
(a) If A is open in Y, and Y is open in X, show that A is open in X.
(b) If A is closed in Y, and Y is closed in X, show that A is closed in X.

(a) Since A is open in Y, $A = B \cap Y$ for some open set $B \subseteq X$.
Since B and Y are both open in X, $B \cap Y$ is open in X.
Thus, A is open in X.
□
(b) A is closed in Y, so $A = B \cap Y$ for a closed set $B \subseteq X$ by Theorem 3.4.
Since B and Y are both closed in X, $B \cap Y$ is closed in X.
Thus, A is closed in X.
■

EXERCISE 3.9
(a) Let $K = \{\frac{1}{n} \in \mathbb{R} | \; n \in \mathbb{Z}_+\}$. Show that the standard topology on K is the discrete topology.
(b) Let $K^* = k \cup \{0\}$. Show that the standard topology on K^* is not the discrete topology.

(a) Consider a singleton set $\{\frac{1}{n}\} \subseteq K$ for some $n \in \mathbb{Z}$.

Let $a_n = \text{avg}(\frac{1}{n+1}, \frac{1}{n}) = \frac{n+1/2}{n(n+1)}$.
Let $b_n = \text{avg}(\frac{1}{n}, \frac{1}{n-1}) = \frac{n-1/2}{n(n-1)}$.
$(a_n, b_n) \cap K = \{\frac{1}{n}\}$.
So $\{\frac{1}{n}\}$ is an open set in the standard topology on K.
So every singleton set in K is an open set.

Consider a set $U \subset K$.
By the Union lemma, $U = \bigcup_{x \in U} \{x\}$.
Since U is a union of open sets, U is an open set.
So all subsets of K are open sets.
$\therefore$ The standard topology on K is the discrete topology.
□
(b) Consider $\{0\} \subseteq K^*$.
Consider a neighborhood of $(a, b) \subseteq K^*$ such that $0 \in (a, b)$.
Thus, we have $a < 0 < b$.

Between any two real numbers is a rational number.
Thus, $\exists m, n \in \mathbb{Z} \ni a < \frac{m}{n} < 0$.
So $\exists n, \in \mathbb{Z} \ni a < \frac{1}{n} < 0$.
So $\frac{1}{n} \in (a, b)$.
Thus, all open sets containing 0 have more than one element.
So $\{0\}$ is not an open set in K^*.
$\therefore$ The standard topology on K^* is not the discrete topology.
■

EXERCISE 3.10 Show that the standard topology on $\mathbb{Q}$, the set of rational numbers is not the discrete topology.

Let (a, b) be an open set containing 0.
Between any two real numbers is a rational number.
Thus, $\exists m, n \in \mathbb{Z} \ni a < \frac{m}{n} < 0$.
So $\exists n, \in \mathbb{Z} \ni a < \frac{1}{n} < 0$.
So $\frac{1}{n} \in (a, b)$.

Thus, all open sets containing 0 have more than one element.
So $\{0\}$ is not an open set in $\mathbb{Q}$.
$\therefore$ The standard topology on $\mathbb{Q}$ is not the discrete topology.
$\blacksquare$

EXERCISE 3.12 Is the finite complement topology on $\mathbb{R}^2$ the same as the product topology on $\mathbb{R}^2$ that results from taking the product $\mathbb{R}_{fc} \times \mathbb{R}_{fc}$, where $\mathbb{R}_{fc}$ is $\mathbb{R}$ in the finite complement topology? Justify your answer.

Those two topologies are not the same.

Consider the set $U = \mathbb{R} - \{0\}\mathbb{R}$. U is open in the product topology on $\mathbb{R}^2$ that results from taking the product $\mathbb{R}_{fc} \times \mathbb{R}_{fc}$ by Definition 3.6.

It is easy to see that U does not have a finite complement and thus is not open in the finite complement topology on $\mathbb{R}^2$.

Since the toplogies do not share all elements, they are not the same.

EXERCISE 3.13 Let $X = PP\mathbb{R}^2_{(0,0)}$, the particular point topology on $\mathbb{R}^2$ with the origin serving as the particular point. Is X the same as the topology that results from taking the product of $\mathbb{R}$ with itself, where each $\mathbb{R}$ has the particular point topology $PP\mathbb{R}_0$? Justify your answer.

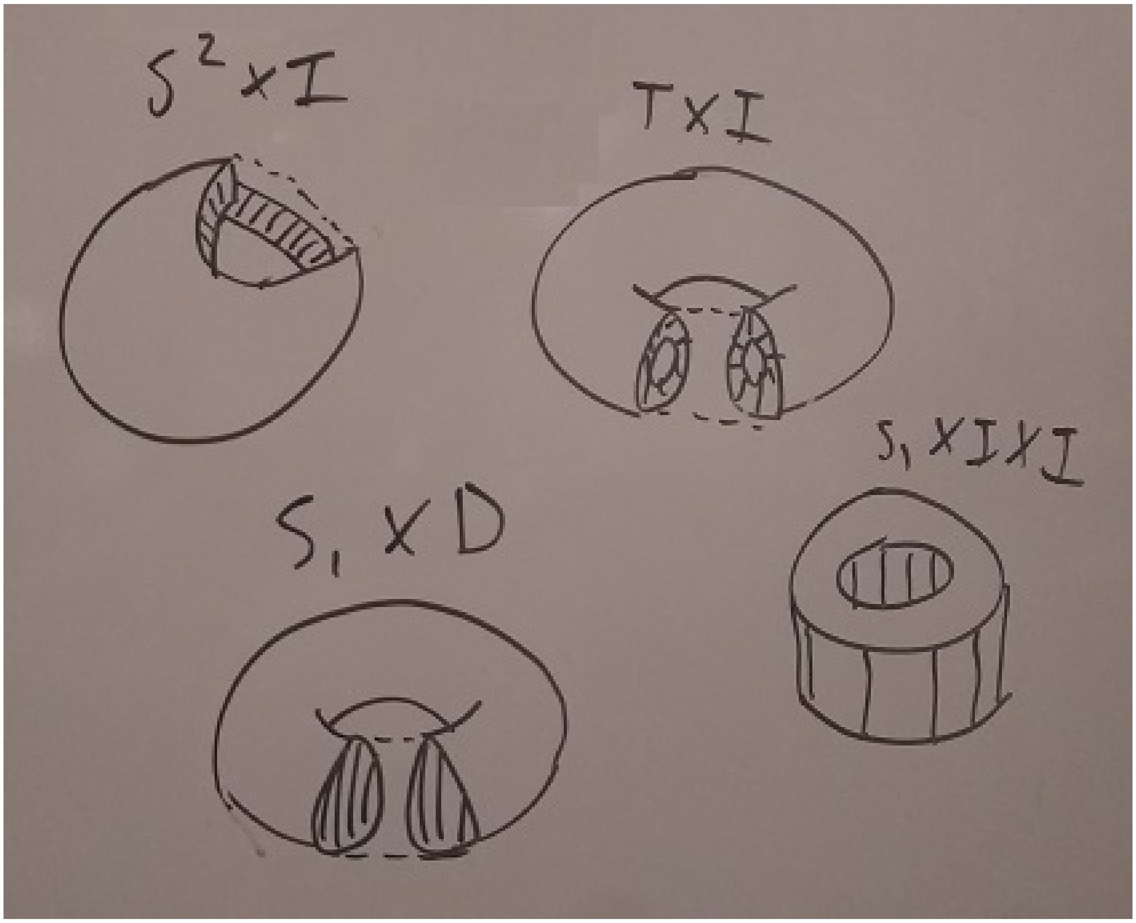
X is the same topology.

Consider an open set $U \times V$ in the product topology formed by $PP\mathbb{R}_0$ crossed with itself.
$0 \in U$ and $0 \in V$, so $(0,0) \in U \times V$.
Thus, the product topology with $PP\mathbb{R}_0$ is a subset of $PP\mathbb{R}_{(0,0)}$.

Consider an open set U in X.
For every point $(x, y) \in U$ let $V_{(x,y)} = \{x, 0\} \times \{y, 0\}$.
By the union lemma, $U = \bigcup_{(x,y)\in W} V_{(x,y)}$.
This is also a union of open sets in the product topology formed by $PP\mathbb{R}_0$ crossed with itself, which is an open set in said topology.
Thus, X is a subset of the product topology with $PP\mathbb{R}_0$.

Therefore, the product topology with $PP\mathbb{R}_0$ is the same as X.

EXERCISE 3.16 Let S^2 be the sphere, D be the disk, T be the torus, S^1 be the circle, and $I = [0, 1]$ with the standard topology. Draw pictures of the product spaces $S^2 \times I, T \times I, S^1 \times I \times I,$ **and** $S^1 \times D$.



EXERCISE 3.17 If L is a line in the plane, describe the subspace topology it inherits from $\mathbb{R}_l \times \mathbb{R}$ and from $\mathbb{R}_l \times \mathbb{R}_l$, where $\mathbb{R}_l$ is the real line in the lower limit topology. Note that the result depends on the slope of the line. In all cases, it is a familiar topology.

If L is a vertical line, it inherits the standard topology on $\mathbb{R}$ from $\mathbb{R}_l \times \mathbb{R}$. Otherwise, it inherits $\mathbb{R}_l$. Unions of open intervals as well as interals closed at the lower limit are open sets in these non-vertical lines, all of which are open in the lower limit topology.

L inherits $\mathbb{R}_l$ from $\mathbb{R}_l \times \mathbb{R}_l$ in all cases.

EXERCISE 3.18 Show that if X and Y are Hausdorff spaces, then so is the product space $X \times Y$.

Let X and Y be Hausdorff spaces.
Consider two distinct points (x_1, y_1) and (x_2, y_2) in the product space $X \times Y$.
Since the points are distinct, either $x_1 \neq x_2$ or $y_1 \neq y_2$.
Without loss of generality, let $x_1 \neq x_2$.

Since X is Hausdorff $\exists U, V \in X \ni x \in U$ and $y \in V$.
Let $U' = U \times Y$ and $V' = V \times Y$.
$(x_1, y_1) \in U'$ and $(x_2, y_2) \in V'$
Since U and V are disjoint, U' and V' are disjoint.
Thus, for any two distinct points in the product space $X \times Y$, there exist disjoint subsets that contain those points.

Therefore, the product space $X \times Y$ is Hausdorff.

EXERCISE 3.19 Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

For sake of readability, let ' denote the complement of a set.

Assume A and B are closed in X and Y respectively.
A' and B' are both open in X and Y.
So $A' \times Y$ and $X' \times B$ are open in $X \times Y$.
This means $(A' \times Y)'$ and $(X' \times B)'$ are both closed in $X \times Y$.
Thus, $(A' \times Y)' \cap (X' \times B)'$ is closed in $X \times Y$.

$$A \times B = A \times Y \cap X \times B$$

$$A \times B = (A' \times Y)' \cap (X' \times B)'$$

$\therefore A \times B$ is closed in $X \times Y$
$\blacksquare$

EXERCISE 3.20 Show that if $A \subseteq X$ and $B \subseteq Y$, then $\mathrm{Cl}(A \times B) = \mathrm{Cl}(A) \times \mathrm{Cl}(B)$.

In Exercise 3.19, we have shown that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Since $A \subseteq \mathrm{Cl}(A)$ and $B \subseteq \mathrm{Cl}(B)$ $A \times B \subseteq \mathrm{Cl}(A) \times \mathrm{Cl}(B)$.
$\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ are both closed, so $\mathrm{Cl}(A) \times \mathrm{Cl}(B)$ is closed in $X \times Y$.
Thus, $\mathrm{Cl}(A \times B) \subseteq \mathrm{Cl}(A) \times \mathrm{Cl}(B)$.
$\square$
Let $(x, y) \in \mathrm{Cl}(A) \times \mathrm{Cl}(B)$.
Let H be an open set in $X \times Y$ containing (x, y).
Since H is open, it can be written as $U \times V$, where U and V are open in X and Y respectively.
Since $x \in Cl(A)$, U has to intersect A at a point other than x, by Theorem 2.8. Call it x'. (Axiom of choice)

Thus, H intersects $A \times B$ at (x', y).
So for every open set H containing (x, y), H intersects $A \times B$ at a point other than (x, y).
That makes (x, y) a limit point of $A \times B$.
So, by Theorem 2.8, $(x, y) \in \mathrm{Cl}(A \times B)$.
Thus, $\mathrm{Cl}(A) \times \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \times B)$.
$\square$
$\therefore \mathrm{Cl}(A \times B) = \mathrm{Cl}(A) \times \mathrm{Cl}(B)$.
$\blacksquare$