Recurrence Relations in Differentiation

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Abstract. This paper shows how dynamic programming can be used to easily compute the derivatives of rational functions of polynomials of e^x. Computing these derivatives allows us to determine various interesting identities using the generating function for the Bernoulli numbers.

1 Introduction to Bernoulli numbers

Definition 0.1. The Bernoulli numbers B_i are defined recursively by

$$\sum_{i=0}^{n} \binom{n+1}{i} B_i = n+1, \quad B_0 = 1$$

The first few Bernoulli numbers are $1, \frac{1}{2}, \frac{1}{6}, 0, \frac{-1}{30}, 0, \frac{1}{42}...$
Definition 0.2.

$$\text{Define } S_k(n) = \sum_{i=0}^{n} i^k$$

which is called a power sum. Some power sum formulas are given below.

$$S_0(n) = \sum_{i=0}^{n} i^0 = n$$

$$S_1(n) = \sum_{i=1}^{n} i^1 = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$S_2(n) = \sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3(n) = \sum_{i=1}^{n} i^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$S_4(n) = \sum_{i=1}^{n} i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

These power sum formulas are used to efficiently compute sums of powers.

Theorem 0.1.

$$S_k(n) = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j x^j$$

Proof.[1]
Using telescoping series, $S_k(n+1) - S_k(n) = (n+1)^k$

$$S'_k(x+1) - S'_k(x) = k(x+1)^{k-1}$$

$$\sum_{x=0}^{n-1} S'_k(x+1) - S'_k(x) = \sum_{x=0}^{n-1} k(x+1)^{k-1}$$

Using telescoping series,

$$S'_k(n) - S'_k(0) = kS_{k-1}(n)$$

$$S'_k(x) = kS_{k-1}(x) + S'_k(0)$$

$S'_k(0)$ is just some constant. Call it a_k.
$\boxed{S'_k(x) = kS_{k-1}(x) + a_k}$
□
From this formula we have

$$S'_k(0) = kS_{k-1}(0) + a_k$$

Since $S_k(0) = 0 \forall k$

$$S'_k(0) = a_k$$

$$S''_k(0) = kS'_{k-1}(0) = ka_{k-1}$$

$$S'''(0) = kS''(k-1)(0) = k(k-1)a_{k-2}$$

And in general,
$\boxed{S^{(j)}(0) = k(k-1)(k-2)...(k-j+2)a_{k-j+1}}$
□
Using Taylor series,

$$S_k(x) = \sum_{j=0}^{k+1}\frac{S_k^{(j)}(0)}{j!}x^j$$

From the above formula we have

$$S_k(x) = \sum_{j=0}^{k+1}\frac{k!}{(k-j+1)!j!}a_{k-j+1}x^j$$

$$S_k(x) = \frac{1}{k+1}\sum_{j=0}^{k+1}\binom{k+1}{j}a_{k+1-j}x^j$$

$$S_k(x) = \frac{1}{k+1}\sum_{j=0}^{k+1}\binom{k+1}{j}a_j x^{k+1-j}$$

Now we must determine the constants a_k. Plugging in 1 for x, we have

$$S_k(1) = \frac{1}{k+1}\sum_{j=0}^{k+1}\binom{k+1}{j}a_j$$

$$k+1 = \sum_{j=0}^{k+1}\binom{k+1}{j}a_j$$

So $a_j = B_j$ are the Bernoulli numbers by the recursive definition.

$$\therefore S_k(n) = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j x^j$$

■

The power sum formula for k = 10 is

$$S_{10}(n) = \sum_{i=1}^{n} i^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

Plugging in n = 1000 we have

$$1^{10}+2^{10}+...+1000^{10} = \frac{1}{11}1000^{11}+\frac{1}{2}1000^{10}+\frac{5}{6}1000^9-1000^7+1000^5-\frac{1}{2}1000^3+\frac{5}{66}1000$$
$$= 91409924241424243424241924242500$$

Theorem 0.2.

$$B_{2n+1} = 0, \text{ for } n > 0$$

References

[1] T. Arakawa, T. Ibukiyama, M. Kaneko, and D. Zagier, *Bernoulli numbers and zeta functions.* Springer, 2014.