Math 331 A - Probability

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HW p.100 #78,82 p. 139 #42,46,49,50,54

1 p. 100 #78

With reference to Example 3.22 on page 94, find
(a) the conditional density of X_2 given $X_1 = \frac{1}{3}$ and $X_3 = 2$;
(b) the joint conditional density of X_2 and X_3 given $X_1 = \frac{1}{2}$.

From Example 3.22 we have

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2)e^{-x_3} & \text{for } 0 < x_1 < 1, 0 < x_2 < 1, x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

From Def. 3.11 the marginal densities $h(x_1, x_3)$ and $g(x_1)$ are

$$h(x_1, x_3) = \int_0^1 (x_1 + x_2)e^{-x_3} dx_2 = (x_1 + \frac{1}{2})e^{-x_3}$$

$$g(x_1) = \int_0^1 \int_0^\infty (x_1 + x_2)e^{-x_3} dx_3 \; dx_2 = x_1 + \frac{1}{2}$$

From Def. 3.13, the conditional densities $f(x_2|x_1, x_3)$ and $f(x_2, x_3|x_1)$ are

$$f(x_2|x_1, x_3) = \frac{f(x_1, x_2, x_3)}{h(x_1, x_3)}$$

$$f(x_2|x_1 = \frac{1}{3}, x_3 = 2) = \frac{f(\frac{1}{3}, x_2, 2)}{h(\frac{1}{3}, 2)} = \frac{(\frac{1}{3} + x_2)e^{-2}}{(\frac{1}{3} + \frac{1}{2})e^{-2}}$$

$$\boxed{\textbf{(a)}\ f(x_2|x_1 = \frac{1}{3}, x_3 = 2) = \frac{2 + 6x_2}{5}}$$

$$f(x_2, x_3|x_1) = \frac{f(x_1, x_2, x_3)}{g(x_1)}$$

$$f(x_2, x_3|x_1 = \frac{1}{2}) = \frac{f(\frac{1}{2}, x_2, x_3)}{g(\frac{1}{2})} = \frac{(\frac{1}{2} + x_2)e^{-x_3}}{\frac{1}{2} + \frac{1}{2}}$$

$$\boxed{\textbf{(b)}\ f(x_2, x_3|x_1 = \frac{1}{2}) = \left(\frac{1}{2} + x_2\right) e^{-x_3}}$$

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2 p. 100 #82

If the independent random variables X and Y have the marginal densities

$$f(x) = \begin{cases} \frac{1}{2} \text{ for } 0 < x < 2 \\ 0 \ \text{ elsewhere} \end{cases}$$

$$\pi(y) = \begin{cases} \frac{1}{3} \text{ for } 0 < y < 3 \\ 0 \ \text{ elsewhere} \end{cases}$$

find
(a) the joint probability density of X and Y;
(b) the value of $P(X^2 + Y^2 > 1)$

By Def. 3.14,since X and Y are independent, the joint probability density $g(x, y) = f(x)\pi(y)$.

$$\boxed{\textbf{(a)}\ g(x, y) = \frac{1}{6} \text{ for } 0 < x < 2, 0 < y < 3}$$

Finding $P(X^2 + Y^2 > 1)$ thus becomes trivial. One must only integrate the joint probability density over the correct region. Adding in the restriction $X^2 + Y^2 > 1$ along with our previous restrictions $0 < x < 2, 0 < y < 3$ we have the region of integration R.

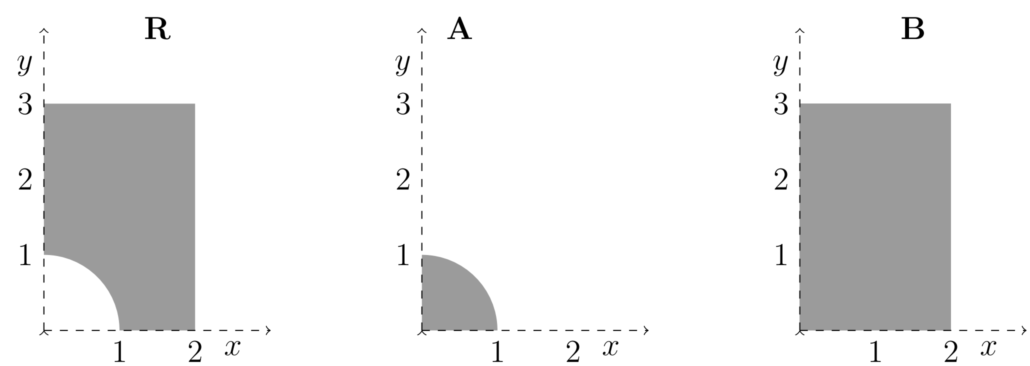

Figure 1: Regions of integration

To find the integral on region R, we can subtract the integral of $g(x, y)$ on Region A from the integral of $g(x, y)$ on Region B. Since $g(x, y)$ is a probability density, the integral of $g(x, y)$ on Region B is just 1.

$$P(X^2 + Y^2 > 1) = 1 - \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{6} r dr \, d\theta = 1 - \frac{\pi}{24}$$

$$\boxed{\textbf{(b)}\ P(X^2 + Y^2 > 1) = 1 - \frac{\pi}{24}}$$

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3 p. 139 #42

If X and Y have the joint probability distribution $f(x,y) = \frac{1}{4}$ for $(x,y) \in \{(-3,-5),(-1,-1),(1,1),(3,5)\}$, find $\text{cov}(X,Y)$.

By Theorem 4.7 we have

$$\mu'_{1,1} = \sum_{(x,y)} xy\frac{1}{4}$$

$$\mu_X = \mu_{1,0} = \sum_{(x,y)} x\frac{1}{4}$$

$$\mu_Y = \mu_{0,1} = \sum_{(x,y)} y\frac{1}{4}$$

Plugging in our points (x,y) we have

$$\mu'_{1,1} = \frac{1}{4}(15+1+1+15) = 8$$

$$\mu_X = \frac{1}{4}(-3-1+1+3) = 0$$

$$\mu_Y = \frac{1}{4}(-5-1+1+5) = 0$$

By Thm. 4.11 we have

$$\sigma_{XY} = \mu'_{1,1} - \mu_X\mu_y$$

$$\sigma_{XY} = 8 - 0 * 0$$

$\boxed{\sigma_{XY} = 8}$

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4 p. 139 #46

If X and Y have the joint probability distribution $f(-1,0)=0, f(-1,1)=\frac{1}{4}, f(0,0)=\frac{1}{6}, f(0,1)=0, f(1,0)=\frac{1}{12}, f(1,1)=\frac{1}{2}$, show that
(a) $\text{cov}(X,Y)=0$;
(b) The two random variables are not independent.

By Theorem 4.7 we have

$$\mu'_{1,1}=\sum_{(x,y)} xyf(x,y)$$

$$\mu_X=\mu_{1,0}=\sum_{(x,y)} xf(x,y)$$

$$\mu_Y=\mu_{0,1}=\sum_{(x,y)} yf(x,y)$$

Plugging in our points (x,y) we have

$$\mu'_{1,1}=0-\frac{1}{4}+0+0+\frac{1}{2}=\frac{1}{4}$$

$$\mu_X=0-\frac{1}{4}+0+0+\frac{1}{12}+\frac{1}{2}=\frac{1}{3}$$
$$\mu_Y=0+\frac{1}{4}+0+0+0+\frac{1}{2}=\frac{3}{4}$$

By Thm. 4.11 we have

$$\sigma_{XY}=\mu'_{1,1}-\mu_X\mu_Y$$
$$\sigma_{XY}=\frac{1}{4}-\frac{3}{4}\cdot\frac{1}{3}$$

$\boxed{\sigma_{XY}=0}$

By Def. 3.14 X and Y are independent if and only if f(x,y) = g(x)h(y) for every value of x and y in their ranges.

$$f(0,0)=\frac{1}{6}$$

$$g(0)=\frac{1}{6}+0=\frac{1}{6}$$
$$h(0)=0+\frac{1}{6}+\frac{1}{12}=\frac{1}{4}$$
$$f(0,0)\neq g(0)h(0)$$

$\boxed{\textbf{(b)}\text{ The two variables are not independent.}}$

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5 p. 139 #49

If X_1,X_2, and X_3 are independent and have the means 4, 9, and 3 and the variances 3,7, and 5, find the mean and the variance of
(a) $Y = 2X_1 - 3X_2 + 4X_3$;
(b) $Z = X_1 + 2X_2 - X_3$

By Theorem. 4.14

$$\mu_Y = 2\mu_{X_1} - 3\mu_{X_2} + 4\mu_{X_3}$$
$$\mu_Y = 2 \cdot 4 - 3 \cdot 9 + 4 \cdot 3 = -7$$

$$\mu_Z = \mu_{X_1} + 2\mu_{X_2} - \mu_{X_3}$$
$$\mu_Z = 4 + 2 \cdot 9 - 3 = 19$$

$\boxed{\textbf{(a) } \mu_Y = -7,\ \mu_Z = 19}$

Since X_1,X_2, and X_3 are independent we can use Corollary 4.3

$$\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2, \quad \sigma_Z^2 = \sum_{i=1}^{n} b_i^2 \sigma_{X_i}^2$$

$$\sigma_Y^2 = 2^2\sigma_{X_1}^2 + (-3)^2\sigma_{X_2}^2 + 4^2\sigma_{X_3}^2$$
$$\sigma_Y^2 = 4 \cdot 3 + 9 \cdot 7 + 16 \cdot 5 = 155$$

$$\sigma_Z^2 = 1^2\sigma_{X_1}^2 + 2^2\sigma_{X_2}^2 + (-1)^2\sigma_{X_3}^2$$
$$\sigma_Z^2 = 3 + 4 \cdot 7 + 5 = 36$$

$\boxed{\textbf{(b) } \sigma_Y^2 = 155,\ \sigma_Z^2 = 36}$

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6 p. 139 #50

Repeat both parts of Exercise 4.49, dropping the assumption of independence and using instead the information that $\text{cov}(X_1, X_2) = 1$, $\text{cov}(X_2, X_3) = -2$, $\text{cov}(X_1, X_3) = -3$.

Whether X_1,X_2, and X_3 are independent has not bearing on μ_Y and μ_Z. Thus, taking what we know from the previous problem,

$\boxed{\textbf{(a) } \mu_Y = -7,\ \mu_Z = 19}$

By Theorem 4.14

$$\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + 2 \sum_{j=1}^{n} \sum_{i<j} a_i a_j \sigma_{X_i X_j}$$

$$\sigma_Z^2 = \sum_{i=1}^{n} b_i^2 \sigma_{X_i}^2 + 2 \sum_{j=1}^{n} \sum_{i<j} b_i b_j \sigma_{X_i X_j}$$

But we know what two of the sums are from #49.

$$\sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 = 155, \quad \sum_{i=1}^{n} b_i^2 \sigma_{X_i}^2 = 36$$

Computing the other sums we have

$$2 \sum_{j=1}^{n} \sum_{i<j} a_i a_j \sigma_{X_i X_j} = 2(2 \cdot (-3) \cdot 1 + (-3) \cdot 4 \cdot -2 + 2 \cdot 4 \cdot (-3)) = -12$$

$$2 \sum_{j=1}^{n} \sum_{i<j} b_i b_j \sigma_{X_i X_j} = 2(1 \cdot (-3) \cdot 1 + 2 \cdot 4 \cdot -2 + (-1) \cdot 4 \cdot (-3)) = -14$$

So in comparison, it looks like the covariance doesn't contribute that much.

$$\sigma_Y^2 = 155 - 12, \sigma_Z^2 = 36 - 14$$

$\boxed{\textbf{(b) } \sigma_Y^2 = 143,\ \sigma_Z^2 = 22}$

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7 p. 139 #54

If var(X_1) = 5,var(X_2) = 4,var(X_3) = 7,cov(X_1, X_2) = 3,cov(X_1, X_3) = −2,and X_2 and X_3 are independent, find the covariance of $Y_1 = X_1 - 2X_2 + 3X_3$ and $Y_2 = -2X_1 + 3X_2 + 4X_3$.

First of all, since X_2 and X_3 are independent, by Thm. 4.14, cov(X_2, X_3) = 0.

By Theorem 4.15

$$\sigma_{Y_1Y_2} = \sum_{i=1}^{n} a_i b_i \sigma_{X_i}^2 + \sum_{j=1}^{n} \sum_{i<j} (a_i b_j + a_j b_i) \sigma_{X_i X_j}$$

Evaluating the first sum we have

$$\sum_{i=1}^{n} a_i b_i \sigma_{X_i} = (1 \cdot (-2) \cdot 5) + ((-2) \cdot 3 \cdot 4) + (3 \cdot 4 \cdot 7) = 50$$

Evaluating the second sum we have

$$\sum_{j=1}^{n} \sum_{i<j} (a_i b_j + a_j b_i) \sigma_{X_i X_j} = (1 \cdot 3 + (-2) \cdot (-2)) \cdot 3 + (1 \cdot 4 + (-2) \cdot 3) \cdot (-2) + ((-2) \cdot 4 + 3 \cdot 3) \cdot (0)$$

$$\sum_{j=1}^{n} \sum_{i<j} (a_i b_j + a_j b_i) \sigma_{X_i X_j} = (7) \cdot 3 + (-2) \cdot (-2) + 0 = 21 + 4 + 0 = 25$$

After adding both sums together we arrive at our final answer
$\boxed{\sigma_{Y_1Y_2} = 75}$