Math 332 A - Mathematical Statistics

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HW p.250 #38,40 p.257 #80,81,82*,83*

1 p. 250 #38

Show that for $\nu_2 > 2$ the mean of the F distribution is $\frac{\nu_2}{\nu_2-2}$, making use of the definition of F in Theorem 8.14 and the fact that for a random variable V having the chi-square distribution with ν_2 degrees of freedom,
$E\left(\frac{1}{V}\right) = \frac{1}{\nu_2-2}$.

Let U and V be independent chi-square distributions with respective ν_1 and ν_2 degrees of freedom.

With Theorem 8.14 we can write

$$F = \frac{V/\nu_1}{U/\nu_2} \implies E(F) = E\left(\frac{V/\nu_1}{U/\nu_2}\right)$$

Using Corollary 4.1, we can drag out the constants ν_1 and ν_2.

$$E(F) = \frac{\nu_2}{\nu_1}E\left(\frac{U}{V}\right)$$

In class, we determined that if U and V are independent, then certainly U and $\frac{1}{V}$ are independent.
Thus, by Thm. 4.12 we can write

$$E(F) = \frac{\nu_2}{\nu_1}E(U)E\left(\frac{1}{V}\right)$$

Since U is chi-square, we know that $E(U) = \nu_1$ by Corollary 6.1.
From the problem, we miraculously know that

$$E\left(\frac{1}{V}\right) = \frac{1}{\nu_2 - 2}$$

Combining everything together and simplifying we have

$$\mu_F = E(F) = \frac{\nu_2}{\nu_1}\frac{\nu_1}{\nu_2 - 2} = \frac{\nu_2}{\nu_2 - 2}$$

$$\boxed{\text{For } \nu_2 > 2, \text{the mean of the F distribution is } \frac{\nu_2}{\nu_2 - 2}}$$

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2 p. 250 #40

Verify that if T has a t distribution with ν degrees of freedom, the $X = T^2$ has an F distribution with $\nu_1 = 1$ and $\nu_2 = \nu$ degrees of freedom.

Let Z and Y be independent random variables with respective standard normal and chi-square distribution with ν degrees of freedom. By Theorem 8.12, X can be written as

$$X = T^2 = \left(\frac{Z}{\sqrt{Y/\nu}} \right)^2 = \frac{Z^2}{Y/\nu}$$

By Theorem 8.7, Z^2 has the chi-square distribution with 1 degree of freedom. Thus, X is a proportion of chi-square distributions, and thus has the F distribution by Theorem 8.14.
$X = \frac{Z^2/1}{Y/\nu}$

$\boxed{\text{X has the F distribution with } \nu_1 = 1 \text{ and } \nu_2 = \nu \text{ degrees of freedom.}}$

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3 p. 257 #80

A random sample of size $n = 25$ from a normal population has the mean $\overline{x} = 47$ and the standard deviation $s = 7$. If we base our decision on the statistic of Theorem 8.13, can we say that the given information supports the conjecture that the mean of the population is $\mu = 42$?

Theorem 8.13 states that $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ has the t-distribution with $n-1$ degrees of freedom.
Plugging in our values for $n, \overline{x}, s$, and μ we have $t = \frac{47-42}{7/5} = 7$.

Referring to Table IV, with $\nu = 24$, getting a value greater than 3 for t is already extremely unlikely.
$\boxed{\text{The given information does not support the conjecture.}}$

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4 p. 257 #81

A random sample of size $n = 12$ from a normal population has the mean $\overline{x} = 27.8$ and the variance $s^2 = 3.24$. If we base our decision on the statistic of Theorem 8.13, can we say that the given information supports the claim that the mean of the population $\mu = 28.5$?

Theorem 8.13 states that $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ has the t-distribution with $n-1$ degrees of freedom.
Plugging in our values for $n, \overline{x}, s$, and μ we have $t = \frac{27.8-28.5}{1.8/\sqrt{12}} \approx -1.34715$.

Referring to Table IV with $\nu = 11$, getting a value less than -1.34715 for t is 0.10. That's not that bad.
$\boxed{\text{The given information supports the conjecture.}}$

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5 p. 257 #82

If S_1 and S_2 are the standard deviations of independent random samples of size $n_1 = 61$ and $n_2 = 31$ from normal populations with $\sigma_1^2 = 12$ and $\sigma_2^2 = 18$, find $P(S_1^2/S_2^2 > 1.25)$.

By Theorem 8.15 we can say that

$$F = \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2}$$

is a random variable having an F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.
Notice that

$$P(S_1^2/S_2^2 > 1.25) = P(\frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} > 1.25\frac{\sigma_2^2}{\sigma_1^2}) = P(f > 1.875)$$

where f has an F distirbution with $\nu_1 = 30$ and $\nu_2 = 60$. Using Wolfram Alpha (not Excel), we calculate $P(f > 1.875)$.

$$\int_{1.875}^{\infty} \frac{\Gamma(45)\, 0.5^{15}\, t^{14}}{(\Gamma(15)\,\Gamma(30))\,(1+0.5\,t)^{45}}\, dt = 0.0193805$$

$$\boxed{P(f > 1.875) \approx 0.0193805}$$

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6 p. 257 #83

If S_1 and S_2 are the standard deviations of independent random samples of size $n_1 = 10$ and $n_2 = 15$ from normal populations with equal variances, find $P(S_1^2/S_2^2 < 3.05)$.

By Theorem 8.15 we can say that

$$F = \frac{S_1^2 \sigma_2^2}{S_2^2 \sigma_1^2}$$

is a random variable having an F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.
Notice that

$$P(S_1^2/S_2^2 < 3.05) = P(\frac{S_1^2 \sigma_2^2}{S_2^2 \sigma_1^2} > 1.25\frac{\sigma_2^2}{\sigma_1^2}) = P(f < 3.05)$$

where f has an F distirbution with $\nu_1 = 9$ and $\nu_2 = 14$. Using Wolfram Alpha (not Excel), we calculate $P(f > 3.05)$.

$$\int_0^{3.5} \frac{\Gamma\left(\frac{23}{2}\right)\left(\frac{9}{14}\right)^{4.5} t^{3.5} \left(1 + \frac{9t}{14}\right)^{-23/2}}{\Gamma\left(\frac{9}{2}\right)\Gamma(7)} \, dt = 0.982147$$

$\boxed{P(f > 1.875) \approx 0.982147}$