Math 332 A - Mathematical Statistics

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HW p.286 #1,5,6,13

1 p. 286 #1

If $X_1, X_2, ..., X_n$ constitute a random sample from a population with the mean μ, what condition must be imposed on the constants $a_1, a_2, ..., a_n$ so that

$$a_1X_1 + a_2X_2 + ... + a_nX_n$$

is an unbiased estimator of μ?

Since $a_1X_1 + a_2X_2 + ... + a_nX_n$ is an unbiased estimator of μ, we can write

$$E(a_1X_1 + a_2X_2 + ... + a_nX_n) = \mu$$

Expected Value is a linear operator. We can write

$$a_1E(X_1) + a_2E(X_2) + ... + a_nE(X_n) = \mu$$

Each $X_1, X_2, ..., X_n$ come from a population with mean μ we know that $E(X_1) = E(X_2) = ... = E(X_n) = \mu$ we can factor out a μ

$$\mu(a_1 + a_2 + ... + a_n) = \mu$$

$$\boxed{\sum_{k=1}^{n} a_k = 1}$$

2 p. 286 #5

Given a random sample of size n from a population that has a known mean μ and the finite variance σ^2, show that

$$\frac{1}{n} \cdot \sum_{i=1}^{n} (X_i - \mu)^2$$

is an unbiased estimator of σ^2.

By Definition 8.5, the above sum is exactly σ^2.

$$\therefore E\left(\frac{1}{n} \cdot \sum_{i=1}^{n} (X_i - \mu)^2\right) = E(\sigma^2) = \sigma^2$$

It is an unbiased estimatior of σ^2.

■

3 p. 286 #6

Use the results of Theorem 8.1 on page 233 to show that $\overline{X}^2$ is an asymptotically unbiased estimator of μ^2.

By Theorem 4.10 we know
$\mathrm{var}\bar{(}X) = E(\overline{X}^2) - (E(\overline{X}))^2$

By Theorem 8.1, $\mathrm{var}\overline{X} = \frac{\sigma^2}{n}$ and $E(\overline{X}) = \mu$
$\frac{\sigma^2}{n} = E(\overline{X}^2) - \mu^2$

Note that $E(\overline{X}^2) - \mu^2$ is the bias of the estimator $\overline{X}^2$.
As $n \to \infty$ the bias of the estimator goes to 0.
$\therefore \overline{X}^2$ is an asymptotically unbiased estimator of μ^2.
■

4 p. 286 #13

Show that if $\hat{\Theta}$ is an unbiased estimator of θ and var$(\hat{\Theta}) \neq 0$, then $\hat{\Theta}^2$ is not an unbiased estimator of θ^2.

By Definition 4.5 we can write

$$var(\hat{\Theta}) = E((\hat{\Theta} - \theta)^2) = E(\hat{\Theta}^2) - 2\theta E(\hat{\Theta}) + \theta^2$$

Since $\hat{\Theta}$ is an unbiased esitmator of θ, $E(\hat{\Theta}) = \theta$

$$var(\hat{\Theta}) = E(\hat{\Theta}^2) - 2\theta^2 + \theta^2$$

$$E(\hat{\Theta}^2) = var(\hat{\Theta}) + \theta^2$$

Since $var(\hat{\Theta})$ is non-zero, $E(\hat{\Theta}^2) \neq \theta^2$

$\therefore \hat{\Theta}^2$ is a biased estimator of θ^2.
$\blacksquare$