Topology Homework

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The following proofs were done by recalling previous Real Analysis notes, in which, similar theorems regarding countability were proved.

A complete collection of the proofs required is provided on later pages.

Lemma 1: The union of two countable sets is countable

Proof.
Let S and T be countable sets. Thm* tells us that $\exists f : \mathbb{N} \to S$ and $g : \mathbb{N} \to T \ni f$ and g are surjective.

$$\text{Consider } \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if n is odd} \\ g\left(\frac{n}{2}\right) & \text{if n is even} \end{cases}$$

so $h : \mathbb{N} \to S \cup T$ is a surjection
so, by thm *, $S \cup T$ is countable. //

Lemma 2: The product of two countable sets is countable

Proof.
Let S and T be countable sets.
thm * $\exists$ injections $f : S \to \mathbb{N}$ and $g : T \to \mathbb{N}$.
define $h(s,t) = 2^{f(s)}3^{g(t)}, h : S \times T \to \mathbb{N}$
let $h(s,t) = h(u,v)$
so $2^{f(s)}3^{g(t)} = 2^{f(u)}3^{g(v)}$
Since prime factorization is unique,
$2^{f(s)=2^{f(u)}}$ so $s = u$
and $3^{g(t)} = 3^{g(v)}$ so $t = v$
because f and g are injective.
so $s,t) = (u,v))$
$\therefore$ h is injective from $S \times T \to N$
$\therefore$ by thm * $S \times T$ is countable. //
$\blacksquare$

THEOREM 0.29. **(iv)** *A subset of a countable set is a countable set.*

Proof.
Let S be a countable set and let $T \subseteq S$
Case I: T is finite, then T is countable
Case II: T is not finite or infinite
so S must be infinite and thus, S is denumerable
So $\exists$ a bijection $f : \mathbb{N} \to S$ and we can write
$S = \{s_1, s_2, s_3, ...\}$ so $f(n) = s_n$
Let $A = \{n \in \mathbb{N} | \ s_n \in T\}$
since A is a non empty subset of $\mathbb{N}$ it has a least element. call it a_1.
so $A - \{a_1\}$ has a least element. call it a_2
so $A - \{a_1, a_2, ...a_{k-1}\}$ has a least element. call it a_k
define $g : \mathbb{N} \to \mathbb{N} \ni g(n) = a_n$
Since $a_{n+1} \notin \{a_1, a_2, ...a_n\}$ g is injective
so $f = g : \mathbb{N} \to S$ is injective
since every element of T is included in S_1
$g(\mathbb{N})$ includes all subscripts of T
so $f \circ g$ is a bijection from $\mathbb{N} \to T$
so T is denumerable and thus countable. //

THEOREM 0.29. **(v)** *A countable union of countable sets is a countable set.*

Proof.
Let $A_1, A_2, A_3, ...$ be countable sets.
Basis step: A_1 is countable by Lemma 1.
□
Induction hypothesis: Suppose $\bigcup_{i=1}^{k} A_i$ is countable for some $k \in \mathbb{Z}_+$.
Induction step: Consider some set $S = \bigcup_{i=1}^{k+1} A_i$.

S can be written as $S = A_{k+1} \cup \bigcup_{i=1}^{k} A_i$
By the induction hypothesis, $\bigcup_{i=1}^{k} A_i$ is a countable set. Thus, S is a union of two countable sets, so it is a countable set by Lemma 1.

$\therefore \bigcup_{i=1}^{n} A_i$ is a countable set for $n = 1, 2, 3...$, a countable union.
■

THEOREM 0.29. **(vi)** *A product of countable sets is a countable set.*

Proof.
Let $A_1, A_2, A_3, ...$ be countable sets.
Basis step: A_1 is countable by Lemma 2.
□
Induction hypothesis: Suppose $A_1 \times A_2 \times ... \times A_k$ is countable for some

$k \in \mathbb{Z}_+$.
Induction step: Consider some set $S = A_1 \times A_2 \times ... \times A_k \times A_{k+1}$.

S can be written as $S = A_{k+1} \times (A_1 \times A_2 \times ... \times A_k$
By the induction hypothesis, $(A_1 \times A_2 \times ... \times A_k$ is a countable set. Thus, S is a product of two countable sets, so it is a countable set by Lemma 2.

$(A_1 \times A_2 \times ... \times A_n$ is a countable set for $n = 1, 2, 3...$, a countable product.
$\blacksquare$

THEOREM: The set of real numbers is uncountable

Proof:
let $J = (0, 1)$
NTS: J is uncountable
Suppose J is countable
J can be wrriten as
$x_1 = 0.a_{11}a_{12}a_{13}a_{14}$
$x_2 = 0.a_{21}a_{22}a_{23}a_{24}$
$x_3 = 0.a_{31}a_{32}a_{33}a_{34}$
$x_4 = 0.a_{41}a_{42}a_{43}a_{44}$
$\vdots$
Where each $a_{ij} \in \{1, 2, ...9\}$
construct the number
$y = 0.b_1b_2b_3b_4...$ by

$$b_1 = \begin{cases} 3 \text{ if } a_{ii} \neq 3 \\ 7 \text{ if } a_{ii} = 3 \end{cases}$$

We know $y \in J$ but $y \neq x_i \forall i \in \mathbb{N}$ so $y \notin J \quad \rightarrow\leftarrow$
$\therefore$ J is uncountable
$\therefore$ since $J \in \mathbb{R}$, $\mathbb{R}$ is uncountable (thm) //

The following proofs is what our group came up with independent of any notes from Real Analysis.

THEOREM 0.9. *For sets A, B, and C, the following laws hold:*

Distributive Laws:

$$(i)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$(ii)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$(iii)\ A \times (B \cup C) = (A \times B) \cup (A \times C)$$
$$(iv)\ A \times (B \cap C) = (A \times B) \cap (A \times C)$$
$$(v)\ A \times (B - C) = (A \times B) - (A \times C)$$

DeMorgan's Laws:

$$(vi)\ A - (B \cup C) = (A - B) \cap (A - C)$$
$$(vii)\ A - (B \cap C) = (A - B) \cup (A - C)$$

(i) Proof.
Assume $x \in A \cap (B \cup C)$.
$x \in A$ and $x \in (B \cup C)$ by the Definition of $\cap$
$x \in A$ and ($x \in B$ or $x \in C$) by the Definition of $\cup$
($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) by the Distributive Law
$x \in A \cap B$ or $x \in A \cap C$ by the Definition of $\cap$
$x \in (A \cap B) \cup (A \cap C)$ by the Definition of $\cup$
$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
$\square$
Each step is reversible.
$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$\blacksquare$

(ii) Proof.
Assume $x \in A \cup (B \cap C)$.
$x \in A$ and $x \in (B \cap C)$ by the Definition of $\cup$
$x \in A$ or ($x \in B$ and $x \in C$) by the Definition of $\cap$
($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$) by the Distributive Law
$x \in A \cup B$ and $x \in A \cup C$ by the Definition of $\cup$
$x \in (A \cup B) \cap (A \cup C)$ by the Definition of $\cap$
$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
$\square$
Each step is reversible.
$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$\blacksquare$

(iii) Proof.
Assume $(x, y) \in A \times (B \cup C)$.
$x \in A$ and $y \in (B \cup C)$ by the Definition of $\times$
$x \in A$ and ($y \in B$ or $y \in C$) by the Definition of $\cup$
($x \in A$ and $y \in B$) or ($x \in A$ and $y \in C$) by the Distributive Law
$(x, y) \in A \times B$ or $(x, y) \in A \times C$ by the Definition of $\times$
$(x, y) \in (A \times B) \cup (A \times C)$ by the Definition of $\cup$
$\therefore A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
$\square$
Each step is reversible.
$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$
$\therefore A \times (B \cup C) = (A \times B) \cup (A \times C)$
$\blacksquare$

(iv) Proof.
Assume $(x, y) \in A \times (B \cap C)$.
$x \in A$ and $y \in (B \cap C)$ by the Definition of $\times$
$x \in A$ and ($y \in B$ and $y \in C$) by the Definition of $\cap$
($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$) by the Associative Law
$(x, y) \in A \times B$ and $(x, y) \in A \times C$ by the Definition of $\times$
$(x, y) \in (A \times B) \cap (A \times C)$ by the Definition of $\cap$
$\therefore A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$
$\square$
Each step is reversible.
$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$
$\therefore A \times (B \cap C) = (A \times B) \cap (A \times C)$
$\blacksquare$

(v) Proof. Assume $(x, y) \in A \times (B - C)$.
$x \in A$ and $y \in (B - C)$ by the Definition of $\times$
$x \in A$ and ($y \in B$ and $\sim y \in C$) by the Definition of $-$
($x \in A$ and $y \in B$) and $\sim$ ($x \in A$ and $y \in C$) by the Annullment Law.
$(x, y) \in A \times B$ and $\sim (x, y) \in A \times C$ by the Definition of $\times$
$(x, y) \in (A \times B) - (A \times C)$ by the Definition of $-$
$\therefore A \times (B - C) \subseteq (A \times B) - (A \times C)$
$\square$
Assume $(x, y) \in (A \times B) - (A \times C)$
$(x, y) \in (A \times B)$ and $\sim (x, y) \in (A \times C)$ by the Definition of $-$
$x \in A$ and $y \in B$ and $\sim$ ($x \in A$ and $y \in C$) by the Definition of $\times$
$x \in A$ and $y \in B$ and ($\sim x \in A$ or $\sim y \in C$) by DeMorgan's Law
$x \in A$ and $y \in B$ and $\sim y \in C$ by Elimination
$x \in A$ and $y \in (B - C)$ by the Definition of $-$
$(x, y) \in A \times (B - C)$ by the Definition of $\times$
$(A \times B) - (A \times C) \subseteq A \times (B - C)$

$\therefore A \times (B - C) = (A \times B) - (A \times C)$
$\blacksquare$

(v) Proof. Assume $x \in A - (B \cup C)$.
$x \in A$ and $\sim x \in (B \cup C)$ by the Definition of $-$
$x \in A$ and $\sim (x \in B$ or $x \in C)$
$x \in A$ and $(\sim x \in B$ and $\sim x \in C)$ by Demorgan's Law
$(x \in A$ and $\sim x \in B)$ and $(x \in A$ and $\sim x \in C)$ by the Associative law
$x \in A - B$ and $x \in A - C$ by the Definition of -
$x \in (A - B) \cap (A - C)$ by the Definition of $\cap$
$\therefore A - (B \cup C) \subseteq (A - B) \cap (A - C)$
$\square$
Each step is reversible.
$(A - B) \cap (A - C) \subseteq A - (B \cup C)$
$\therefore A - (B \cup C) = (A - B) \cap (A - C)$
$\blacksquare$
(vi) Proof. Assume $x \in A - (B \cap C)$.
$x \in A$ and $\sim x \in (B \cap C)$ by the Definition of $-$
$x \in A$ and $\sim (x \in B$ and $x \in C)$
$x \in A$ and $(\sim x \in B$ or $\sim x \in C)$ by Demorgan's Law
$(x \in A$ and $\sim x \in B)$ or $(x \in A$ and $\sim x \in C)$ by the Associative law
$x \in A - B$ or $x \in A - C$ by the Definition of -
$x \in (A - B) \cup (A - C)$ by the Definition of $\cup$
$\therefore A - (B \cap C) \subseteq (A - B) \cup (A - C)$
$\square$
Each step is reversible.
$(A - B) \cup (A - C) \subseteq A - (B \cap C)$
$\therefore A - (B \cap C) = (A - B) \cup (A - C)$
$\blacksquare$

THEOREM 0.21 *If $f : X \to Y$ is a function and A and B are subsets of X, then*

$$(i)\ f(A \cup B) = f(A) \cup f(B).$$
$$(ii)\ f(A \cap B) \subseteq f(A) \cap f(B).$$
$$(iii)\ f(A) - f(B) \subseteq f(A - B)$$

(i) Proof.
Assume $y \in f(A \cup B)$
$\exists x \in A \cup B \ni y = f(x)$
$\exists x \in A$ or $\exists x \in B \ni y = f(x)$
$y \in f(A)$ or $y \in f(B)$
$y \in f(A) \cup f(B)$
$f(A \cup B) \subseteq f(A) \cup f(B)$
Assume $y \in f(A) \cup f(B)$
$\exists x \in A \ni y = f(x)$ or $\exists x \in B \ni y = f(x)$
$\exists x \in A \cup B \ni y = f(x)$ since A and B are both subsets of $A \cup B$
$y \in f(A \cup B)$
$f(A) \cup f(B) \subseteq f(A \cup B)$
$\therefore f(A \cup B) = f(A) \cup f(B)$
$\blacksquare$

(ii) Proof.
Assume $y \in f(A \cap B)$
$\exists x \in A \cap B \ni y = f(x)$
$\exists x \in A \ni y = f(x)$ and $\exists x \in B \ni y = f(x)$ since $A \cap B$ is a subset of both A and B.
$y \in f(A)$ and $y \in f(B)$
$y \in f(A) \cap f(B)$
$\therefore f(A \cap B) \subseteq f(A) \cap f(B)$
$\blacksquare$

(iii) Proof.
Assume $y \in f(A) - f(B)$
$y \in f(A)$ and $\sim y \in f(B)$
$\exists x \in A \ni y = f(x)$ and $\sim \exists x \in B \ni y = f(x)$
$\exists x \in A \cap B' \ni y = f(x)$ since $x \in A$, but it cannot be in B.
$y \in f(A \cap B')$
$y \in f(A - B)$, which is a different way to write the same thing.
$\therefore f(A) - f(B) \subseteq f(A - B)$
$\blacksquare$

THEOREM 0.22. *if $f : X \to Y$ is a function and V and W are subsets of Y, then*

$$(i)\ f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W).$$
$$(ii)\ f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W).$$
$$(iii)\ f^{-1}(V - W) = f^{-1}(V) - f^{-1}(W).$$

(i) Proof. Assume $x \in f^{-1}(V \cup W)$
$f(x) \in V \cup W$
$f(x) \in V$ or $f(x) \in W$
$x \in f^{-1}(V)$ or $x \in f^{-1}(W)$
$x \in f^{-1}(V) \cup f^{-1}(W)$
$f^{-1}(V) \cup f^{-1}(W) \subseteq f^{-1}(V \cup W)$
□
Each step is reversible.
$f^{-1}(V \cup W) \subseteq f^{-1}(V) \cup f^{-1}(W)$
$\therefore f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$
■

(ii) Proof. Assume $x \in f^{-1}(V \cap W)$
$f(x) \in V \cap W$
$f(x) \in V$ and $f(x) \in W$
$x \in f^{-1}(V)$ and $x \in f^{-1}(W)$
$x \in f^{-1}(V) \cap f^{-1}(W)$
$f^{-1}(V) \cap f^{-1}(W) \subseteq f^{-1}(V \cap W)$
□
Each step is reversible.
$f^{-1}(V \cap W) \subseteq f^{-1}(V) \cap f^{-1}(W)$
$\therefore f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W)$
■

(iii) Proof. Assume $x \in f^{-1}(V - W)$
$f(x) \in V - W$
$f(x) \in V$ and $\sim f(x) \in W$
$x \in f^{-1}(V)$ and $\sim x \in f^{-1}(W)$
$x \in f^{-1}(V) - f^{-1}(W)$
$f^{-1}(V - W) \subseteq f^{-1}(V) - f^{-1}(W)$
□
Each step is reversible
$f^{-1}(V) - f^{-1}(W) \subseteq f^{-1}(V - W)$
$\therefore f^{-1}(V - W) = f^{-1}(V) - f^{-1}(W)$
■

THEOREM 0.29.

(i) A subset of a finite set is a finite set
(ii) A finite union of finite sets is a finite set
(iii) A product of finite sets is a finite set
(iv) A subset of a countable set is a countable set
(v) A countable union of countable sets is a countable set
(vi) A product of countable sets is a countable set

(i) Proof. Consider some finite set A.
A is empty or there exists a bijection $f : \{1, 2, ...n\} \to A$
If A is empty, any subset of A is empty, and is therefore finite.
If A is non empty, f gives a way to order elements in A.

Consider some subset S of A. Since S is a subset of A, elements in S can also be ordered by f.

Let $g : \{1, 2, 3, ...k\} \to S$ where $g(a)$ is the ath smallest element in $f^{-1}(S)$. Every element in S has a unique element in $\{1, 2, 3, ...k\}$ that g maps to it, so g is bijective.
Thus, S is finite.
$\therefore$ A subset of a finite set is a finite set.
$\blacksquare$

(ii) Proof. Consider the union of finite sets $\bigcup_{i=1}^{m} A_i$.
The union of any set A and the empty set is A. Thus, if any of the A_i are empty, then we can construct an equivalent finite union $\bigcup_{i=1}^{n} A_i$ such that no A_i is empty.
Next, assume all A_i are mutually disjoint.
Since each A_i are finite and nonempty, there exist bijective functions $f_1, f_2, ...f_n \ni f_i : \{1, 2, ...k_i\} \to A_i$ for some values $k_i \in \mathbb{Z}_+$

Let $P = \{2, 3, 5, ...p_n\}$ be the set of the first n primes.
Let $S = \{p_i^q | i \leq n, 1 \leq q \leq k_i\}$ be a subset of all prime powers.

Let $g : S \to \bigcup_{i=1}^{n} A_i$ where $g(p_i^q) = f_i^{-1}(q)$
By the Fundamental Theorem of Arithmetic, and the fact that all A_i are mutually disjoint, every element in $\bigcup_{i=1}^{n} A_i$ has a unique element in S that g maps to it, so g is bijective.

S is a subset of the integers, so can be ordered.
Let $h : \{1, 2, ...(k_1 + k_2 + ...k_n)\} \to S$ where $h(a)$ is the ath smallest element in S.
Every element in S gets a unique element in $\{1, 2, ...(k_1 + k_2 + ...k_n)\}$, that h maps to it, so h is bijective.

h and g are both bijective so $g \circ h : \{1, 2, ...(k_1 + k_2 + ...k_n)\} \to \bigcup_{i=1}^{n} A_i$

is also bijective.
So any finite union of mutually disjoint finite sets is finite.
If A_i are not mutually disjoint, then a union of such sets are equivalent to a finite union of mutually disjoint finite sets anyways, as shown below:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \bigcup_{i=2}^{n} \left(A_i - \bigcup_{k=1}^{i-1} A_k \right)$$

Note that each set in the above union is a subset of a finite set and therefore finite by Theorem 0.29 (i).
$\therefore$ A finite union of finite sets is finite.
■

(iii) Proof. Consider some finite sets $A_1, A_2, ...A_n$.
Let $S = A_1 \times A_2 \times ... \times A_n = \{(x_1, x_2, ...x_n) | x_i \in A_i \ \forall i\}$
If any of A_i are empty, S is empty, and therefore finite.
Otherwise, all A_i are nonempty. Since each A_i are finite and nonempty, there exist bijective functions $f_1, f_2, ...f_n \ni \ f_i : \{1, 2, ...k_i\} \to A_i$ for some values $k_i \in \mathbb{Z}_+$

Let $P = \{2, 3, 5, ...p_n\}$ be the set of the first n primes.
Let $Q = \{y \in \mathbb{Z}_+ |$y has between 1 and k_i factors of $p_i, k_m = 0$ for $m > n\}$

Let $g : Q \to S$ where $g(p_1^{x_1} p_2^{x_2} ... p_n^{x_n}) = (x_1, x_2, x_3)$
By the Fundamental Theorem of Arithmetic, each n-tuple $(x_1, x_2, ...x_n) \in S$ has a unique integer in Q that g maps to it, so g is bijective.

Q is a subset of the integers, so it is ordered.
Let $h : \{1, 2, ...(k_1 + k_2 + ...k_n)\} \to S$ where $h(a)$ is the ath smallest element in Q.
Every element in Q gets a unique element in $\{1, 2, ...(k_1 + k_2 + ...k_n)\}$, that h maps to it, so h is bijective.

h and g are both bijective so $g \circ h : \{1, 2, ...(k_1 + k_2 + ...k_n)\} \to S$ is also bijective.
Thus, S is finite.
$\therefore$ A product of finite sets is a finite set.
■

(iv) Proof. Let A be a countable set.
If A is finite, any subset of A is also finite by **(i)** and therefore countable.
If A is not finite, there exists a bijective function $f : \mathbb{Z}_+ \to A$.
f gives a way to order elements in A.
Consider some subset S of A. Since S is a subset of A, elements in S can also be ordered by f.

Let $g : \mathbb{Z}_+ \to S$ where g(a) is the ath smallest element in $f^{-1}(S)$.

Every element in S has a unique element in $\mathbb{Z}_+$ that g maps to it, so g is bijective.
$\therefore$ A subset of a countable set is a countable set.
$\blacksquare$

(v) Proof. Consider the countable sets $A_1, A_2, A_3...$
Assume all A_i are mutally disjoint.

There exist bijective functions $f_1, f_2, f_3... \ni \; f_i : \mathbb{Z}_+ \to A_i \; \forall i \in \mathbb{Z}_+$

Let $P = \{2, 3, 5...p_i...\}$ be the set of all prime numbers.
Let $S = \{2, 2^2, 2^3, 3, 3^2, ...107^{67}...\}$ be the set of all prime powers.

Let $g : S \to \bigcup_{i\geq 1} A_i$ where $g(p_i^k) = f_i^{-1}(k)$
By the Fundamental Theorem of Arithmetic, and the fact that all A_i are mutually disjoint, every element in $\bigcup_{i\geq 1} A_i$ has a unique element in S that g maps to it, so g is bijective.

S is a subset of the integers, so can be ordered.
Let $h : \{1, 2, ...(k_1 + k_2 + ...k_n)\} \to S$ where $h(a)$ is the ath smallest element in S.
Every element in S gets a unique element in $\mathbb{Z}_+$, that h maps to it, so h is bijective.

h and g are both bijective so $g \circ h : \mathbb{Z}_+ \to \bigcup_{i\geq 1} A_i$ is also bijective.
So any countable union of mutually disjoint countable sets is countable.

Using the same reasoning as **(ii)**, If A_i are not mutually disjoint, then a union of such sets is equivalent to a countable union of mutually disjoint countable sets anyways, as shown below:

$$\bigcup_{i\geq 1} A_i = A_1 \cup \bigcup_{i\geq 2} \left(A_i - \bigcup_{k=1}^{i-1} \right)$$

Note that each set in the above union is a subset of a countable set and therefore countable by Theorem 0.29 (iv).
$\therefore$ A countable union of countable sets is countable.
$\blacksquare$

(vi) Proof. Consider some countable sets $A_1, A_2, A_3....$
Let $S = A_1 \times A_2...A_n = \{(x_1, x_2, ...x_n) | x_i \in A_i \; \forall i\}$ If any of A_i are empty, S is empty, and therfore countable.
Otherwise, all A_i are nonempty. Since each A_i are countable and nonempty, there exist bijective functions $f_1, f_2, ...f_n \ni \; f_i : \mathbb{Z}_+ \to A_i \; \forall i$

Let $P = \{2, 3, 5...p_n\}$ be the set of the first n numbers.
Let $W = \{y \in \mathbb{Z}_+ | y \text{ contains no factors of } p_m \text{ if } m > n\}$

Let $g : W \to S$ where $g(p_1^{x_1} p_2^{x_2} ... p_n^{x_n}) = (x_1, x_2, ...x_n)$
By the Fundamental Theorem of Arithmetic, each n-tuple $(x_1, x_2, ...x_n) \in S$ has a unique integer in W that g maps to it, so g is bijective.

W is a subset of the integers, so it is ordered.
Let $h : \mathbb{Z}_+ \to S$ where $h(a)$ is the ath smallest element in W.
Every element in W gets a unique element in $\mathbb{Z}_+$, that h maps to it, so h is bijective.

h and g are both bijective so $g \circ h : \{\mathbb{Z}_+ \to S$ is also bijective.
Thus, S is a countable set.
$\therefore$ A product of countable sets is a countable set.
■

Prove that $\mathbb{Q}$ is countable

Proof.
Let $f : \mathbb{Z}_+ \to \mathbb{Z}_-$ where $f(a) = -a$. each element in $\mathbb{Z}_-$ get a unique element in $\mathbb{Z}_+$ that f maps to it, so f is bijective.
Thus, $\mathbb{Z}_-$ is a countable set.

$\mathbb{Z} = \{0\} \cup \mathbb{Z}_- \cup \mathbb{Z}_+$, being a countable union of countable sets is countable.

$\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$, being a product of countable sets is countable. $\{(a,b) \in \mathbb{Z}^2 | b \neq 0\} \subseteq \mathbb{Z}^2$, being a subset of a countable set is countable.

By definition, $\mathbb{Q} = \{\frac{a}{b} \in \mathbb{R}|\ (a,b) \in \mathbb{Z}^2, b \neq 0\}$
Let $g : \{(a,b) \in \mathbb{Z}^2, b \neq 0\} \to \mathbb{Q}$ where $g(a,b) = \frac{a}{b}$
each element in $\mathbb{Q}$ get a unique element in $\{(a,b) \in \mathbb{Z}^2, b \neq 0\}$ that g maps to it, so g is bijective.

Since $\{(a,b) \in \mathbb{Z}^2, b \neq 0\}$ is countable, can can define a bijection $h : \mathbb{Z}_+ \to \{(a,b) \in \mathbb{Z}^2, b \neq 0\}$

h and g are both bijective so $g \circ h : \{\mathbb{Z}_+ \to \mathbb{Q}$ is also bijective.
$\therefore$ $\mathbb{Q}$ is countable.
■

Prove that $\mathbb{R}$ is uncountable

Proof.
Recall that any real number $r \in (0,1)$ can be represented as an infinite series $r = \sum_{n=1}^{\infty} a_n 2^{-n}$, where a_n is either 0 or -1.

Suppose that the closed inteval $(0,1)$ is countable.
Then there exists a bijective function $f : \mathbb{Z}_+ \to (0,1)$, where f must have the property that $f(i) = \sum_{n=1}^{\infty} a_n^i 2^{-n}$ for some coefficients a_n^i.

Let $x = \sum_{n=1}^{\infty}(1 - a_n^n)2^{-n}$
$x \in (0,1)$ since each coefficient $1 - a_n^n$ is either a 0 or a 1.

But $\sim \exists i \ni f(i) = x$ since the coeffiecient $1 - a_n^n$ is different from a_i^i for all i. This means f is not onto. That's bad.

We assumed that f was bijective, but have shown that it is not onto. This is a contradiction.
Thus, $(0,1)$ is uncountable.
$(0,1) \subseteq \mathbb{R}$, and supersets of uncountable sets are uncountable.

$\therefore$ $\mathbb{R}$ is uncountable.
$\blacksquare$

The fact that the real numbers in $(0,1)$ can be represented as a sum of negative powers of 2 is shown in a book titled "Digital Logic Design" by Peter J. Ashenden.