LECTURE 8

Error Propagation cont.

Multiplication and Division of Counts:

$$u = xy, \frac{\partial u}{\partial x} = y, \frac{\partial u}{\partial y} = x$$
$$\sigma_u^2 = y^2\sigma_x^2 + x^2\sigma_y^2$$

Divide by $u^2 = x^2y^2$:

$\left(\frac{\sigma_u}{u}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$, i.e. the error is the quadrature sum of the fractional errors.

Mean value of the multiple independent counts:

$$\Sigma = x_1 + x_2 + x_3 + ... + x_N, \quad {}^{\partial\Sigma}/_{\partial x_i} = 1 \text{ for all counts}$$
$$\sigma_\Sigma^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + ... + \sigma_{x_N}^2$$

But $\sigma_i^2 = x_i$

$$\sigma_\Sigma^2 = x_1 + x_2 + x_3 + ... + x_N = \Sigma, \qquad \sigma_\Sigma = \sqrt{\Sigma}$$

- Lets calculate a mean from Σ, $\bar{x} = {}^{\Sigma}/_{N}$

- So then the error associated with calculating the mean

$$\sigma_{\bar{x}} = {}^{\sigma_{\bar{x}}}/_{N} = \sqrt{\Sigma}\Big/N = {}^{\sqrt{N\bar{x}}}/_{N}$$
$$\sigma_x = \sqrt{{}^{\bar{x}}/_{N}}$$

- This is the basis of the argument for if we want 2x better error we are required to invest 4x the time of measurement.

Combination of Independent Measurements with Unequal Errors:

- We use this method for the case where several experiments of varying precision measuring the same value. Best way is to obtain a weighted mean for the error such that <x> (the least mean) is a superposition of weighted mean values:

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i}$$

- Define $\alpha \equiv \sum_{i=1}^{N} a_i$

Thus $\langle x \rangle = \frac{1}{\alpha} \sum_{i=1}^{N} a_i x_i$, we seek to find <x> such that the error is minimized

$$\sigma^2_{<x>} = \sum_{i=1}^{N} \left(\frac{\partial < x >}{\partial x_i} \right)^2 \sigma^2_{x_i} = \sum_{i=1}^{N} \left(\frac{a_i}{\alpha} \right)^2 \sigma^2_{x_i} = \frac{1}{\alpha^2} \sum_{i=1}^{N} \left(a_i \sigma_{x_i} \right)^2$$

so, $\sigma^2_{<x>} = \beta / \alpha^2$ where $\beta \equiv \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$

To minimize $\sigma_{<x>}$ we minimize with respect to a typical weighting factor a_j

$$0 = \frac{\partial \sigma_{<x>}}{\partial a_j} = \frac{\alpha^2 \, \partial\beta / \partial a_j - 2\alpha\beta \, \partial\alpha / \partial a_j}{\alpha^4}; \frac{\partial \alpha}{\partial a_j} = 1, \frac{\partial \beta}{\partial a_j} = 2 a_j \sigma^2_{x_j}$$

$$= \frac{1}{\alpha^4} \left(2\alpha^2 a_j \sigma^2_{x_j} - 2\alpha\beta \right) = 0$$

$a_j = \frac{\beta}{\alpha} . \frac{1}{\sigma^2_{x_j}}$, if we choose to normalize, $\sum_{i=1}^{N} a_j = 1 = \alpha$

$$\beta = \sum_{i=1}^{N} \left(\frac{\beta}{\alpha} . \frac{1}{\sigma^2_{x_i}} \right)^2 . \sigma^2_{x_i} = \left(\sum_{i=1}^{N} \frac{1}{\sigma^2_{x_i}} \right)^{-1}$$

$a_j = \frac{1}{\sigma^2_{x_j}} \left(\sum_{j=1}^{N} \frac{1}{\sigma^2_{x_j}} \right)^{-1}$, so each data point should be weighted inversely as the square of its own error.

Thus for a minimum error in <x>, $\sigma^2_{<x>} = \beta : (\alpha = 1)$ and $\frac{1}{\sigma^2_{<x>}} = \sum_{i=1}^{N} \frac{1}{\sigma^2_{x_i}}$

Optimization of Counting Experiments

- Define S as sample counting rate without background and B as counting rate due to background
- N_B are number of background counts in time T_B and N_{S+B} are the number of source + background counts in time T_{S+B}

- Then $S = \dfrac{N_{S+B}}{T_{S+B}} - \dfrac{N_B}{T_B}$, and error propagation gives us

$$\sigma_S = \left[\left(\frac{\sigma_{N_{S+B}}}{T_{S+B}}\right)^2 + \left(\frac{\sigma_B}{T_B}\right)^2\right]^{1/2} = \left(\frac{N_{S+B}}{T_{S+B}} + \frac{N_B}{T_B}\right)^{1/2}$$

$$\sigma_S = \left(\frac{S+B}{T_{S+B}} + \frac{B}{T_B}\right)^{1/2}$$

- Since T is a constant ($T = T_{S+B} + T_B$) we can also find optimal fraction of time devoted to T_{S+B} or T_B to minimize error, square and differentiate error formula.

$2\sigma_S d\sigma_S = -\dfrac{S+B}{T_{S+B}^2} dT_{S+B} - \dfrac{B}{T_B^2} dT_B$ and set $d\sigma_S = 0$, and since T = const.; $dT_{S+B} + dT_B = 0$

$$\left.\frac{T_{S+B}}{T_B}\right|_{opt} = \sqrt{\frac{S+B}{B}}$$

- We can relate to a figure of merit in relation to the total time T or 1/T (this is the FOM)

  $\dfrac{1}{T} = \left(\dfrac{\sigma_S}{S}\right)^2 \dfrac{S^2}{\left(\sqrt{S+B} + \sqrt{B}\right)^2}$, which is in terms of the fractional error of S

Now look at the extreme cases:

S>>B: $\dfrac{1}{T} \approx \left(\dfrac{\sigma_S}{S}\right)^2 S$, in this case the statistical variation of the background is negligible and reduction in the error of S reduces the error.

S<<B: $\dfrac{1}{T} \approx \left(\dfrac{\sigma_S}{S}\right)^2 \dfrac{S^2}{4B}$, now choosing conditions that minimize S^2/B minimize the error.

<u>Limits of Detectibility</u>:

- Define the number of counts fro background in time t, N_B; and the number of counts for a sample in time t; N_S and the total counts in time t, N_T where:

$$N_S = N_T - N_B$$

- We assume a Gaussian distribution:

$$\sigma_{N_S}^2 = \sigma_{N_T}^2 + \sigma_{N_B}^2$$

Case 1: No activity present: This implies the mean values of N_T and N_B are equal and the mean of $N_S = 0$ which implies the variances of N_T and N_B are also equal.

$$\sigma_{N_S}^2 = \sigma_{N_T}^2 + \sigma_{N_B}^2 = 2\sigma_{N_B}^2$$

$$\sigma_{N_S} = \sqrt{2}\sigma_{N_B}$$

- We now want to assure that we set minimum critical level (L_c) high enough that the probability of a particular measurement N_s exceeds L_c is acceptably small.
- Given a Gaussian distribution, the probability that a random sample is within $\pm 1.645\sigma$ is 90%, and since we interpret the negative side (of a mean of 0) to determine there is no source present (removing the -5%). We can set $L_c = 1.645\sigma_{N_S} = 1.645\sqrt{2}\sigma_{N_B} = 2.326\sigma_{N_B}$ and be sure that a random sample will lie below the L_c with a 95% confidence, this has become a contemporary standard.

Case 2: Real Activity Present:

- We introduce our minimum requirement $N_D = L_c + 1.645\sigma_{N_D}$ which ensures that 95% of the area under the N_S distribution lies above L_c

$$N_D = L_c + 2.326\sigma_{N_B} = 4.653\sigma_{N_B}$$

Now the error in N_D:

$$\sigma_{N_D}^2 = \sigma_{N_T}^2 + \sigma_{N_B}^2$$
$$\sigma_{N_D}^2 = (N_D + N_B) + N_B$$

- So what is our MDA (minimum detectable activity)?
- We define MDA to be strength of a source to produce a mean value of N_S that is high enough to reduce the false rate to an acceptable level.
- As stated previously we want a 5% false negative probability we need:

$$N_D = L_c + 1.645\sigma_{N_D} \text{ where } L_c = 2.326\sigma_{N_B}$$

- We expand for detectable counts much smaller than background:

$$\sigma_{N_D}^2 = \sigma_{N_T}^2 + \sigma_{N_B}^2 = \sqrt{2}\sigma_{N_B} + 1.645$$

which gives us $N_D = 4.654\sigma_{N_B} + 2.706$, also called the Currie Equation.

- So what is the MDA $(\alpha) = {}^{N_D}\!/_{f\varepsilon T}$, where we must also account for a factor in the radiation yield per disintegration (*f*) and the efficiency of the detector (ε) and *T* is the counting time per sample.

Distribution of Time Intervals

Intervals between successive events:

- After an event, how long till the next one?
- We define the probability until the next event in the next small time, *d*t, after the delay *t*, as the product of the probability of no event (0,*t*) and the probability of the event (*t,dt*):

$I_1(t)dt = P(0) \times rdt$, where *r* is the average rate of events.

- Using the Poisson distribution, $P(0) = (rt)^0 e^{-rt} / 0!$

$$P(0) = e^{-rt}$$

$I_1(t)dt = re^{-rt}dt$, an exponential shape.

- Note that the most probable interval is 0 and the average interval length is given by

$$\bar{t}_s = \frac{\int_0^\infty tI_1(t)dt}{\int_0^\infty I_1(t)dt} = \frac{\int_0^\infty tre^{-rt}dt}{1} = \frac{1}{r}$$

Time - to - next - event :

- We now able when will the next event occur only now we don't start from the last event
- We assume a bias for long intervals 9 since we would be asking this during those intervals), this is accompanied with a weighting factor of ${}^{t}\!/_{\bar{t}} = rt$, which weights longer intervals proportional to their length.

$$I_s(t) = rtI_1(t) = r^2te^{-rt}$$

which is the product of the probability of the time interval and the probability of the event.

- the average time to next event is then:

$$\bar{t}_s = \frac{\int_0^\infty t I_S(t)dt}{\int_0^\infty I_S(t)dt} = \frac{2}{r}$$