Lecture 6

Counting Statistics and Error Prediction

Characterization of Data

Sum: $\Sigma = \sum_{i=1}^{N} X_i$

Experimental mean: $\overline{X}_e \equiv \Sigma / N$

Frequency distribution: F(x) : This is the relative frequency with which a number appear in a collection of data.

F(x) ≡ # of occurrences of X/# of measurements (N)

- The distribution is automatically normalized
$$\sum_{X=0}^{\infty} F(x) = 1$$
- As long as order doesn't matter, all the information in a data set is contained in F(x)
- One can further compute the experimental mean $\overline{X}_e$
$$\overline{X}_e = \sum_{x=0}^{\infty} xF(x)$$
- We can also compute the deviation from the true mean value :
$$\varepsilon_i \equiv X_i - \overline{X}$$
- We define the sample variance (S^2):
$$S^2 \equiv \overline{\varepsilon}^2 = 1/N \sum_{i=1}^{N} (X_i - \overline{X}_e)^2$$
which describes the internal scatter of the data.
- To describe the variance in relation to the experimental mean some changes are made (justifications are in Appendix B)
$$S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \overline{X}_e)^2$$
- This is also described through a distribution representation
$$S^2 = \sum_{x=0}^{\infty} (X - \overline{X})^2 F(x)$$
which can be shown to also give
$$S^2 = \overline{X}^2 - (\overline{X})^2$$

We note two important conclusions:

1. Any data set can be completely described by its frequency distribution function, F(x)
2. two properties of F(x) are of interest, the experimental mean and the sample variance

Statistical Models

- Show derivation of probability of decay (a binary process) for a time t, produces a probability of success of: $p = 1- e^{-\lambda t}$

A trial is defined as the observation of a given radioactive nucleus for a given time *t*. a success is when the nucleus decays during the observation and *N* is the number of radioactive nuclei, thus the number of successes (decays in the interval) will

$$dN = -\lambda N dt' = -\lambda N_0 e^{-\lambda t'} dt'$$

$$\int_0^t dN = -\lambda N_0 \int_0^t e^{-\lambda t'} dt' = \lambda N_0 \frac{1}{\lambda}[e^{-\lambda t'}]_0^t = N_0[e^{-\lambda t} - 1]$$

$$\int_0^t dN = N_0[e^{-\lambda t} - 1]$$

$$\frac{1}{N_0}\int_0^t dN = e^{-\lambda t} - 1$$

This is the probability for having no decays or failing to observe a decay (we have based this calculation on the number of nuclei remaining), so to get the number of decays we take the 1-p(failure)=p(success) or $1-\left(e^{-\lambda t} - 1\right) = 1 - e^{-\lambda t}$

- We have three distributions for binary processes

1. Binomial – has a constant probability of success
2. Poisson - probability is small and constant
3. Gaussian or Normal – average number of successes is relatively large (20-30)

Binomial Distribution:

The probability of counting exactly “x” successes in “n” trials

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

where *p* is the probability of success for each trial

We assume trials are independent (no history) so successive probabilities multiply so for x successes and n-x failures: $p^x (1-p)^{n-x}$

- The total number of trials is calculated as $(\hat{x}) \equiv \dfrac{n!}{(n-x)!x!}$ which is the number of x successes out of a trials
- From these two portions we get the Binomial distribution $\dfrac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$
- Note that in table 3.3 they display for p = 4/6, n = 10 and that $\sum_{x=0}^{n} P(x) = 1$ (it is normalized)
- We can find the mean $\bar{x} = \sum_x xP(x)$, which if we replace our P(x) we get $\bar{x} = pn$.
- Our predicted sample variance is : $\sigma^2 \equiv \sum_{x=0}^{N} (x-\bar{x})^2 P(x)$ where if we replace P(x) with the binomial value we get $\sigma^2 = pn(1-p)$ or $\bar{x}(1-p)$
- The standard deviation is just $\sigma = \sqrt{\bar{x}(1-p)}$

Poisson distribution:

- If we take a limit of small probability (still constant) the binomial distribution reduces to the Poisson Distribution

$$P(x) = \frac{(pn)^x e^{-pn}}{x!}$$

- For example: a source emits a particle per unit time with a probability of μ. We assume a δt so that at most 1 particle is detected then μδt is probability of success and 1- μδt is probability of no particle during δt

- Probability of "0" in interval (0,t+ δt) = probability of "0" (0,t) x probability of "0" in δt

$$P_0(t+\delta t) = P_0(t)(1-\mu\delta t)$$

$$\frac{P_0(t+\delta t) - P_0(t)}{\delta t} = -\mu P_0(t)$$

take the limit as δt →0

$$\frac{dP_0}{dt} = -\mu P_0(t) \quad ; \qquad P_0(t) = e^{-\mu t} \qquad \text{with } P_0(0)=1$$

for x=1 (an event) we have two possibilities, it happens in (0,t) or (t, δt)

$$P_1(t+\delta t) = P_1(t)(1-\mu\delta t) + P_0(t)(\mu\delta t)$$

$$\frac{P_1(t+\delta t) - P_1(t)}{\delta t} = -\mu P_1(t) + \mu P_0(t)$$

$$\frac{dP_1(t)}{dt} = -\mu P_1(t) + \mu e^{-\mu t}$$

$$\Rightarrow P_1(t) = \mu t e^{-\mu t}$$

- In general, $\frac{dP_x(t)}{dt} = -\mu P_x(t) + \mu P_{x-1}(t)$

For x ≥ 1 : $\frac{dP_x(t)}{dt} = \frac{(\mu t)^x}{x!} e^{-\mu t}$;(note: the Poisson distribution)

Remember the $pn = \bar{x}$

$$P(x) = \frac{(\bar{x})^x e^{-\bar{x}}}{x!}, \bar{x} = \sum_{x=0}^{n} xP(x) = pn$$

Sample variance: $\sigma^2 \equiv \sum_{x=0}^{N} (x-\bar{x})^2 P(x) = pn = \bar{x}; \sigma = \sqrt{\bar{x}}$

So the standard deviation is equal to the square root of the mean in nuclear decay detection.

Gaussian / Normal Distribution:

An approximation for p << 1 and a large mean value, the distribution becomes

$$P(x) = \frac{1}{\sqrt{2\pi\bar{x}}} e^{-(\frac{(x-\bar{x})^2}{2\bar{x}})}$$

The cumulative distribution function (sigmoid curve)

$$F(t) = \frac{1}{\sqrt{2\pi\bar{x}}} \int_{-\infty}^{\infty} e^{-(\frac{(x-\bar{x})^2}{2\bar{x}})} dx$$

The Gaussian distribution shares the following with the Poisson distribution

1. Normalized : $\sum_{x=0}^{n} P(x) = 1$
2. it is characterized by a single parameter $\bar{x} = pn$
3. The predicted variance σ^2 is equal to the mean, $\bar{x}$

The expected value of function h(x) of a random variable x:

$$E[h(x)] = \sum_{i}^{n} h(x_i)P(x_i) \qquad \text{(discrete distribution)}$$

$$E[h(x)] = \int_{-\infty}^{\infty} h(x)f(x) \qquad \text{(continuous density function)}$$

Example expected values:

Mean: $\bar{x} = \sum_{i}^{n} x_i P(x_i); \bar{x} = \int_{-\infty}^{\infty} xf(x)dx$

Variance: $\sigma^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)dx$

<u>Skewness</u>: $\gamma_1 = \frac{E[(x-\bar{x})^3]}{\sigma^3}$, skewness will be negative if f(x) has a long tail to the left of x, positive for a long tail to the right and zero if the distribution is symmetric.

Sample mean:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \ ,$$

$$E[\bar{x}] = \frac{1}{n}E[\sum_{i=1}^{n} x_i] = \frac{1}{n}\sum_{i=1}^{n} E[x_i]$$

$E[\bar{x}] = E[x_i] = \mu$, this is an unbiased estimate of the "population" mean μ

$$\mathrm{var}[\bar{x}] = \mathrm{var}\left[\frac{1}{n}\sum x_i\right] = \frac{1}{n^2}\mathrm{var}\left[\sum x_i\right]$$

$$= \frac{1}{n^2}\sum \mathrm{var}[x_i] = \frac{1}{n}\mathrm{var}[x_i]$$

$$\mathrm{var}[\bar{x}] = \frac{\sigma^2}{n}$$

- then the standard deviation of the sample mean

$$SD(\bar{x}) = \frac{SD(x_i)}{\sqrt{n}} = \frac{\sigma}{\sqrt{n}}$$

- The Central Limit Theorem: a variable x has a mean μ and a variance σ^2. If σ^2 is finite, then the distribution of the sample mean approaches a normal distribution with mean μ and variance σ^2/n as *n* tends to infinity.
- In terms of deviation, ε, the distribution becomes a density function (continuous & slow varying)

$$G(\varepsilon) = \sqrt{\frac{2}{\pi\bar{x}}}e^{-\frac{\varepsilon^2}{2\bar{x}}}$$

where one now described observing a differential probability dε about ε.

- Note a factor of two has entered since there are 2 values of x for every value of the deviation of ε.
- Since we have now moved to a continuous function we no longer discuss values, but areas under the curve, and can further generalize the function by defining $t \equiv \frac{\varepsilon}{\sigma}$

$$G(t) = \sqrt{\frac{2}{\pi}} e^{-t^2/2},$$

where t is just the observed deviation $\varepsilon \equiv |x - \bar{x}|$ normalized by σ

- The probability that a typical normalized deviation t predicted by a Gaussian distribution will be less than a specific value t_0 is given by the interval:

$$\int_0^{t_0} G(t)dt \equiv f(t_0),$$ where $f(t_0)$ is defined in table 3.4

- $f(t_0)$ is the probability of occurrence of given deviations predicted by the Gaussian Distribution.