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Radiation Detection and Measurement

Lecture 6

Chapter 3: Counting statistics and error prediction

$\bar{x}_e \equiv \Sigma / N$

Characterization of data

- Sum: $$\Sigma = \sum_{i=1}^{N} x_i$$
- Experimental mean: $$\bar{x}_e \equiv \Sigma / N$$
- Frequency distribution: F(x) : This is the relative frequency with which a number appear in a collection of data.

  F(x) ≡ (# of occurrences of X)/(# of measurements (N))

Characterization of data

- The distribution is automatically normalized:

$$\sum_{X=0}^{\infty} F(x) = 1$$

- As long as order doesn't matter, all the information in a data set is contained in F(x)
- One can further compute the experimental mean

$$\bar{x}_e = \sum_{x=0}^{\infty} xF(x)$$

Distribution function

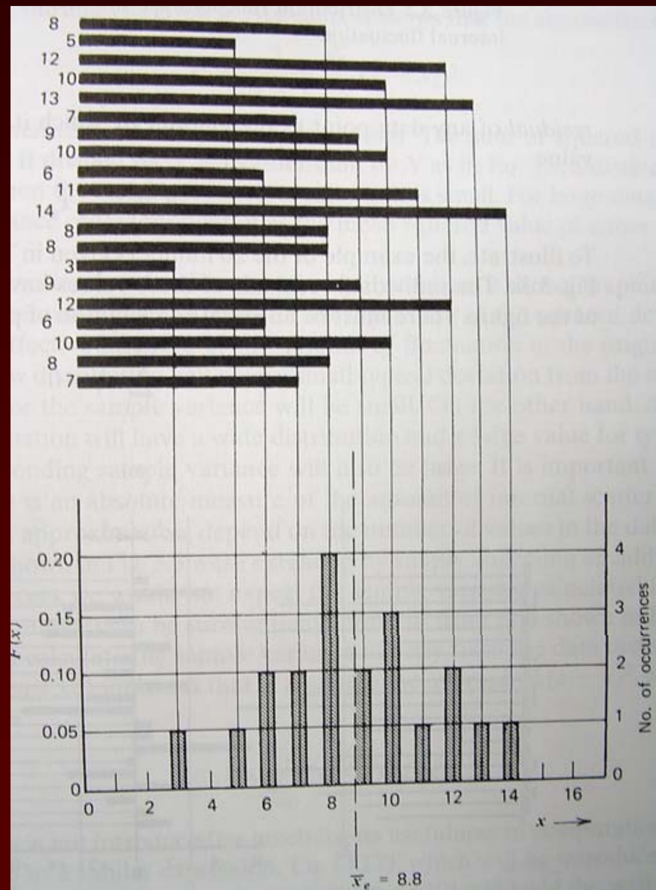

Figure 3.1 Distribution function for the data given in Table 3.1.

Characterization of data

- We can also compute the deviation from the true mean value :

$$\varepsilon_i \equiv x_i - \overline{x}$$

- We define the sample variance (S^2):

$$S^2 \equiv \overline{\varepsilon}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x}_e)^2$$

  – which describes the internal scatter of the data.

Distribution function with differing amounts of internal fluctuation

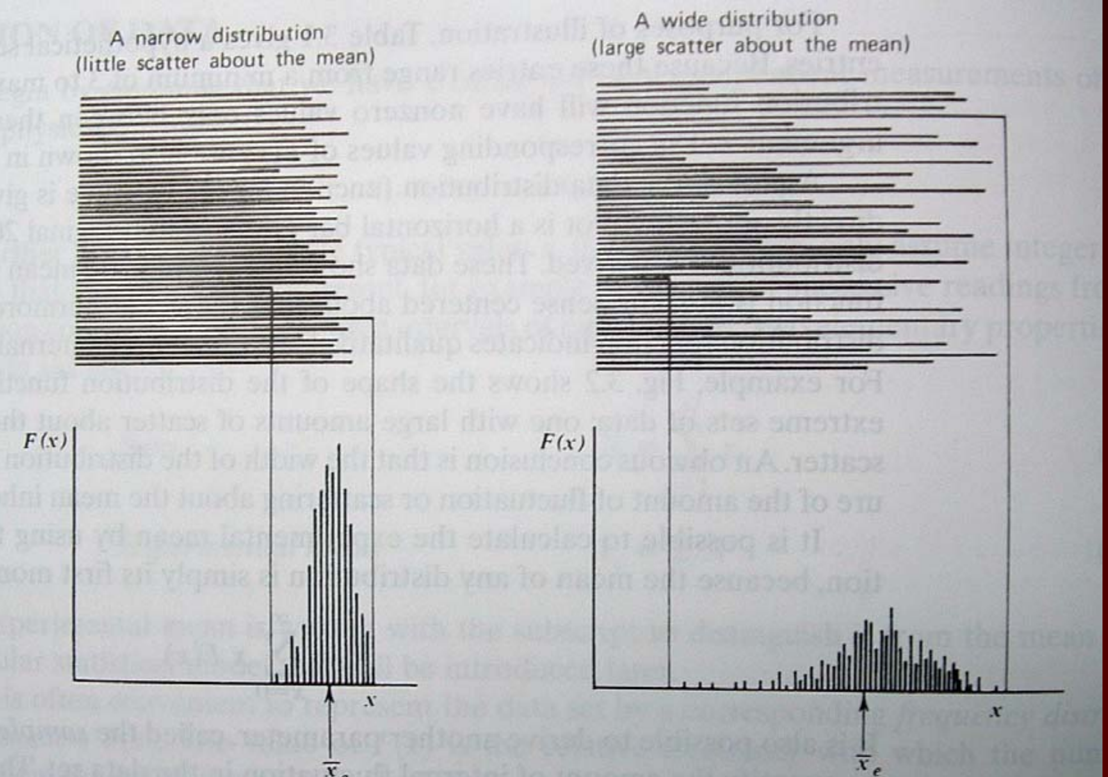

Figure 3.2 Distribution functions for two sets of data with differing amounts of internal fluctuation.

Characterization of data

- To describe the variance in relation to the experimental mean some changes are made (justifications are in Appendix B):

$$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x}_e)^2$$

- This is also described through a distribution representation:

$$S^2 = \sum_{x=0}^{\infty}(x - \overline{x})^2 F(x)$$

- which can be shown to also give:

$$S^2 = \overline{x}^2 - (\overline{x})^2$$

Characterization of data

We note two important conclusions:

- Any data set can be completely described by its frequency distribution function, F(x).

- Two properties of F(x) are of interest, the experimental mean and the sample variance.

Statistical models

- Exercise: Show derivation of probability of decay (a binary process) for a time t, produces a probability of success of:

$$p = 1 - e^{-\lambda t}$$

- A trial is defined as the observation of a given radioactive nucleus for a given time *t*. and a success is when the nucleus decays during the observation and *N* is the number of radioactive nuclei

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Exercise

$$dN = -\lambda N dt' = -\lambda N_0 e^{-\lambda t'} dt'$$

$$\int_0^t dN = -\lambda N_0 \int_0^t e^{-\lambda t'} dt' = -\lambda N_0 \frac{1}{\lambda}[e^{-\lambda t'}]_0^t = -N_0[e^{-\lambda t} - 1]$$

$$\int_0^t dN = N_0[1 - e^{-\lambda t}]$$

$$\frac{1}{N_0}\int_0^t dN = 1 - e^{-\lambda t}$$

- This is the probability for having no decays or failing to observe a decay (we have based this calculation on the number of nuclei remaining), so to get the number of decays we take the 1-p(failure) = p(success) or

$$1 - \left(e^{-\lambda t} - 1\right) = 1 - e^{-\lambda t}$$

Distributions for binomial processes

- Binomial – has a constant probability of success
- Poisson - probability is small and constant
- Gaussian or Normal – average number of successes is relatively large (20-30)

Binomial distribution

- The probability of counting exactly “x” successes in “n” trials
- where *p* is the probability of success for each trial

$$P(x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

- We assume trials are independent (no history) so successive probabilities multiply so for x successes and n-x failures:

$$p^x (1-p)^{n-x}$$

Binomial distribution

- The total number of trials is calculated as which is the number of x successes out of a trials

$$(\hat{x}) \equiv \frac{n!}{(n-x)!\,x!}$$

- From these two portions we get the Binomial distribution

$$\frac{n!}{(n-x)!\,x!}\,p^{x}(1-p)^{n-x}$$

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Binomial calculation

- p = 4/6, n = 10
- Note: $\sum_{x=0}^{n} P(x) = 1$, a normalized distribution

Table 3.3 Values of the Binomial Distribution for the Parameters $p = \frac{4}{6}$ or $\frac{2}{3}$, $n = 10$

x	$P(x)$
0	0.00002
1	0.00034
2	0.00305
3	0.01626
4	0.05690
5	0.13656
6	0.22761
7	0.26012
8	0.19509
9	0.08671
10	0.01734
	$\sum_{x=0}^{10} P(x) = 1.00000$

Binomial distribution

- We can find the mean $\bar{x} = \sum_x xP(x)$, which if we replace our P(x) we get $\bar{x} = pn$.
- Our predicted sample variance is: $\sigma^2 \equiv \sum_{x=0}^{N} (x - \bar{x})^2 P(x)$ where if we replace P(x) with the binomial value we get $\sigma^2 = pn(1 - p)$ or $\bar{x}(1 - p)$
- The standard deviation is just

$$\sigma = \sqrt{\bar{x}(1 - p)}$$

Poisson distribution

- If we take a limit of small probability (still constant) the binomial distribution reduces to the Poisson Distribution

$$P(x) = \frac{(pn)^x e^{-pn}}{x!}$$

- For example: a source emits a particle per unit time with a probability of μ. We assume a δt so that at most 1 particle is detected then $\mu\delta$t is probability of success and 1- $\mu\delta$t is probability of no particle during δt

$$\frac{P_0(t+\delta t)-P_0(t)}{\delta t}=-\mu P_0(t)$$

Poisson distribution: example

- Probability of “0” in interval
  - (0,t+ δt) = probability of “0” on (0,t) x probability of “0” in δt

$$P_0\left(t+\delta t\right)=P_0\left(t\right)\left(1-\mu\delta t\right)$$

$$\frac{P_0(t+\delta t)-P_0(t)}{\delta t}=-\mu P_0(t)$$

Poisson distribution: example

- take the limit as $\delta t \rightarrow 0$

$$\frac{dP_0}{dt} = -\mu P_0(t)$$

$$P_0(t) = e^{-\mu t}$$

- with $P_0(0)=1$

Poisson distribution: example

- for x = 1 (an event) we have two possibilities, it happens in (0,t) or (t, δt)

$$P_1(t+\delta t) = P_1(t)(1-\mu\delta t) + P_0(t)(\mu\delta t)$$

$$\frac{P_1(t+\delta t) - P_1(t)}{\delta t} = -\mu P_1(t) + \mu P_0(t)$$

$$\frac{dP_1(t)}{dt} = -\mu P_1(t) + \mu e^{-\mu t}$$

$$\Rightarrow P_1(t) = \mu t e^{-\mu t}$$

Poisson distribution: example

- In general,

$$\frac{dP_x(t)}{dt} = -\mu P_x(t) + \mu P_{x-1}(t)$$

  - For x ≥ 1: $\frac{dP_x(t)}{dt} = \frac{(\mu t)^x}{x!} e^{-\mu t}$
  - (note: the Poisson distribution)

Poisson distribution: example

- Remember the $pn = \bar{x}$

$$P(x) = \frac{(\bar{x})^x e^{-\bar{x}}}{x!}, \bar{x} = \sum_{x=0}^{n} xP(x) = pn$$

- Sample variance:

$$\sigma^2 \equiv \sum_{x=0}^{N} (x - \bar{x})^2 P(x) = pn = \bar{x}; \sigma = \sqrt{\bar{x}}$$

- So the standard deviation is equal to the square root of the mean in nuclear decay detection.

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Gaussian/Normal distribution

- An approximation for p << 1 and a large mean value, the distribution becomes:

$$P(x) = \frac{1}{\sqrt{2\pi\overline{x}}} e^{-(\frac{(x-\overline{x})^2}{2\overline{x}})}$$

- The cumulative distribution function (sigmoid curve):

$$F(t) = \frac{1}{\sqrt{2\pi\overline{x}}} \int_{-\infty}^{\infty} e^{-(\frac{(x-\overline{x})^2}{2\overline{x}})} dx$$

Discrete and continuous Gaussians

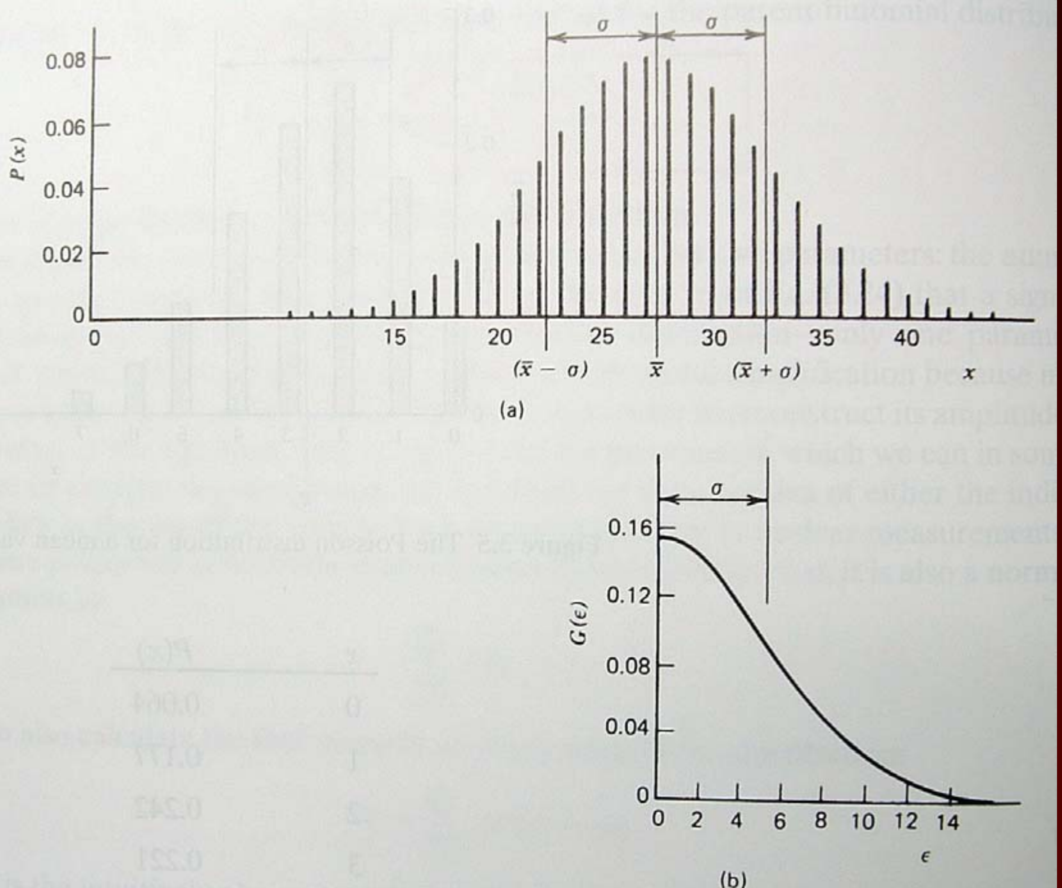

Figure 3.6 (*a*) Discrete Gaussian distribution for a mean value $x = 27.4$. (*b*) Plot of the corresponding continuous form of the Gaussian.

Gaussian vs. Poisson

- The Gaussian distribution shares the following with the Poisson distribution
  - Normalized : $$\sum_{x=0}^{n} P(x) = 1$$
  - It is characterized by a single parameter, $$\bar{x} = pn$$
  - The predicted variance σ^2 is equal to the mean, $\bar{x}$

Functions of random variables: expectation values

$$E[h(x)] = \sum_{i}^{n} h(x_i)P(x_i)$$

- Discrete variable

$$E[h(x)] = \int_{-\infty}^{\infty} h(x)f(x)$$

- Continuous variable

Examples

- Mean:

$$\bar{x} = \sum_{i}^{n} x_i P(x_i); \bar{x} = \int_{-\infty}^{\infty} x f(x) dx$$

- Variance:

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) dx$$

More examples: skewness

- <u>Skewness</u>:

$$\gamma_1 = \frac{E[(x-\bar{x})^3]}{\sigma^3}$$

  – skewness will be negative if f(x) has a long tail to the left of x, positive for a long tail to the right and zero if the distribution is symmetric.

More examples: sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$E\left[\bar{x}\right] = \frac{1}{n}E[\sum_{i=1}^{n} x_i] = \frac{1}{n}\sum_{i=1}^{n} E[x_i]$$

$$E[\bar{x}] = E[x_i] = \mu$$

– The sample mean is an unbiased estimate of the “population” mean μ

Variance of the sample mean

$$\mathrm{var}[\bar{x}] = \mathrm{var}\left[\frac{1}{n}\sum x_i\right] = \frac{1}{n^2}\mathrm{var}\left[\sum x_i\right]$$

$$= \frac{1}{n^2}\sum \mathrm{var}[x_i] = \frac{1}{n}\mathrm{var}[x_i]$$

$$\mathrm{var}[\bar{x}] = \frac{\sigma^2}{n}$$

– then the standard deviation of the sample mean is

$$SD(\bar{x}) = \frac{SD(x_i)}{\sqrt{n}} = \frac{\sigma}{\sqrt{n}}$$

The Central Limit Theorem

- A variable x has a mean μ and a variance σ^2. If σ^2 is finite, then the distribution of the sample mean approaches a normal distribution with mean μ and variance σ^2/n as n tends to infinity.

Back to the Gaussian

- In terms of deviation, ε, the distribution becomes a density function (continuous & slow varying)

$$G(\varepsilon) = \sqrt{\frac{2}{\pi \bar{x}}} e^{-\frac{\varepsilon^2}{2\bar{x}}}$$

  - where one now described observing a differential probability dε about ε
  - Note a factor of two has entered since there are 2 values of x for every value of the deviation ε.

The continuous Gaussian

- Since we have now moved to a continuous function we no longer discuss values, but areas under the curve, and can further generalize the function by defining : $t \equiv \frac{\varepsilon}{\sigma}$

$$G(t) = \sqrt{\frac{2}{\pi}} e^{-t^2/2}$$

- where t is just the observed deviation $\varepsilon \equiv |x - \bar{x}|$ normalized by σ

The continuous Gaussian cont.

- The probability that a typical normalized deviation t predicted by a Gaussian distribution will be less than a specific value t_0 is given by the interval:

$$\int_0^{t_0} G(t)\,dt \equiv f(t_0)$$

  - where $f(t_0)$ is defined in table 3.4
- $f(t_0)$ is the probability of occurrence of given deviations predicted by the Gaussian Distribution.

Discrete and continuous Gaussian: comparison

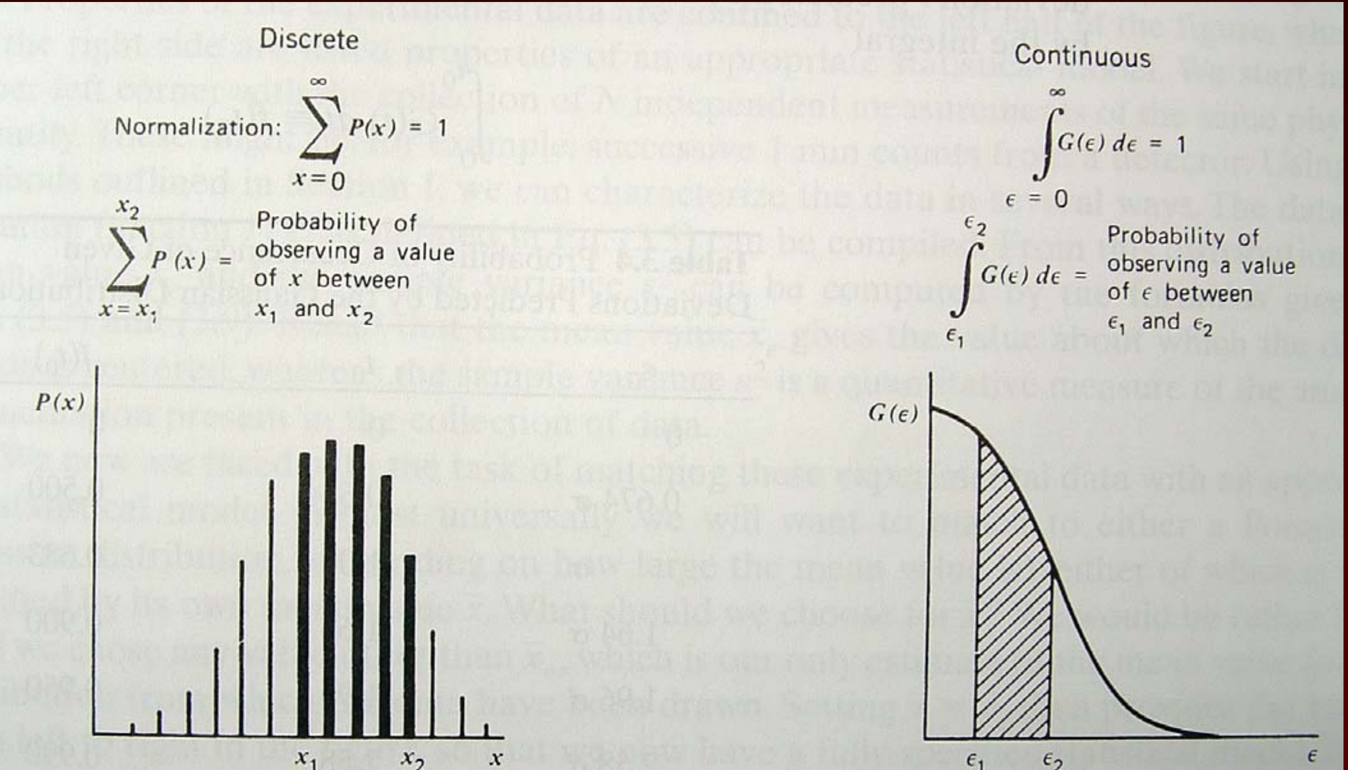

Figure 3.7 Comparison of the discrete and continuous forms of the Gaussian distribution .

The general continuous Gaussian curve

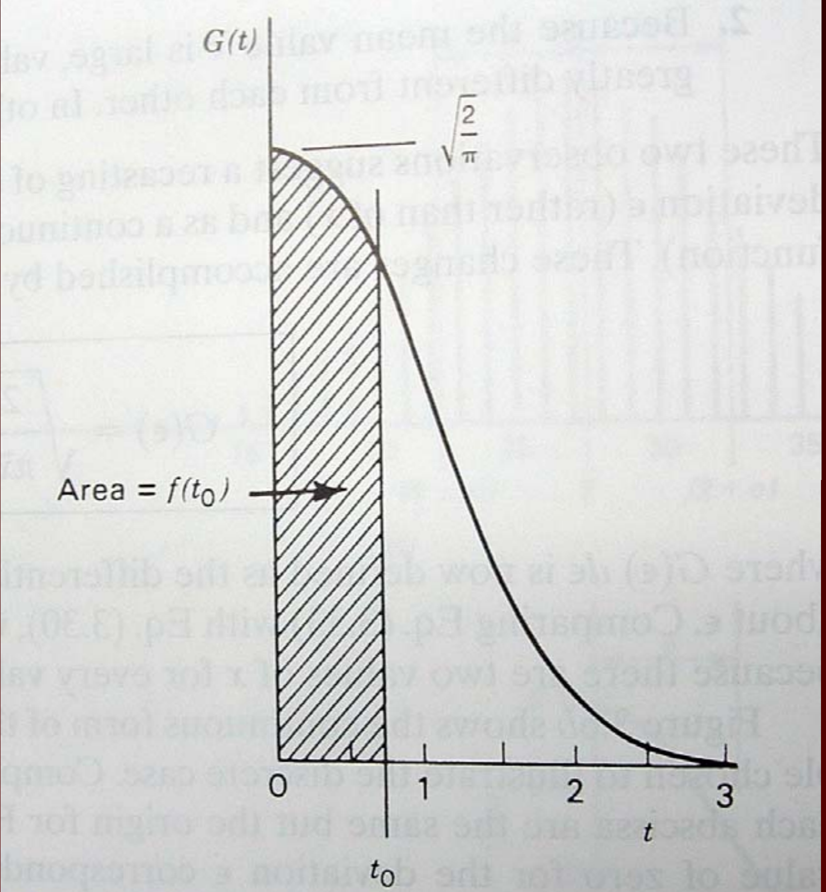

Figure 3.8 A plot of the general Gaussian curve.

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