LECTURE 7

Statistical Behavior of Counting Data for Variable Mean Value

- The Poisson distribution holds for processes that are strictly characterized by a constant mean value $\bar{x}$.
- For the case where the mean can be expect to change over the course of a measurement (i.e., short lived sources or an accelerator beam where intensity changes over measurement time)
- Modify μ to be time dependent rate (μ(t))
- The average μ or $\bar{\mu}$ is given simply by

$$\bar{\mu} = \frac{1}{T}\int_0^T \mu(t)dt$$

- It can then be shown that there is a generalized Poisson distribution whose mean is now given as a average over the interval, $\bar{\mu}T$

$$P(x) = \frac{(\bar{\mu}T)^x e^{-\bar{\mu}T}}{x!}$$

- The formulation will be accurate if a single count is performed over the time interval (0,T), and will give the same variation over repeated experiments where the parameters are the same
- When several short measurement are performed over the interval with a varying mean, we use the following Modified Poisson Distribution
- Dividing the time period (0,T) into n interval with a measurement time τ (all equal)

$$P(x) = \frac{1}{n}\sum_{j=1}^{n} P_j(x)$$

- Where $P_j(x)$ is the probability of observing x events in an interval τ
- The mean value for the interval is then $\mu_j\tau$
- The probability for observing a given value x form the collection of all measurements is then

$$P(x) = \frac{1}{n}\sum_{j=1}^{n}\frac{(\mu_j\tau)^x}{x!}e^{-\mu_j\tau}$$

- For exponentially decaying source without background we substitute $\mu(t) = \mu_0 e^{-\lambda t}$, and assume a large number of variables n, the sum goes to an integral:

$$P(x) = \frac{(\mu_0\tau)^x}{x!}e^{-\mu_0\tau}.\frac{1}{T}\int_0^T \exp[-x\lambda t + \mu_0\tau(1-e^{-\lambda t})]dt$$

- P(x) is of the form $P_0(x).C(x)$, where $P_0(x)$ is the probability over the first interval & C(x) is a “correction factor”.
- Note: Fig c.2 shows the comparison of theory with experiment.

Application of Statistical Models

Checkout of a counting system:

- Use the fluctuation to determine if statistical fluctuations are the only influence on a system
- In general we will check against the Gaussian or Poisson distribution, depending on the mean value
- We compare by calculating the experimental mean and variance of the sample set, equating the experimental mean with the real mean (by approximation) and determine the distribution variance (sqrt. of the mean)
- We compare the distributions and the means/variances
- A quantitative comparison of the distributions can also be performed:

Chi-squared (χ^2) test:

$$\chi^2 \equiv \frac{1}{\overline{X}_e}\sum_{i=1}^{N}(X_i - \overline{X}_e)^2$$

The sample variance (s^2) and χ^2 are related by:

$$\chi^2 = \frac{(N-1)s^2}{\overline{X}_e}$$

- Note that if the data is closely modeled by the Poisson distribution $s^2 \cong \sigma^2$, and for Poisson $\sigma^2 = \overline{X}$; since we have chosen $\overline{X} = \overline{X}_e$ the amount to which $s^2 / \overline{X}_e$ varies from unity is a direct measure of how far the observed sample variance differs from the predicted variance.
- As the other component of χ^2 vs. s^2 relates, as the χ^2 varies from (N-1) we see a departure from the likelihood the data is described by Poisson or Normal distribution
- (N-1) is the degrees of freedom and for various degrees of freedom we can calculate a predicted distribution of χ^2 values called a probability value or p-value
- note that the degrees of freedom is always one less than the independent measurements
- a exact fit of data to the Poisson or Normal distributions would produce a p=0.5
- Fig 3.11 shows the normalized χ^2 distribution (χ^2 / ν) where ν is the degree of freedom
- Note that the p-values are giving the probability that a random sample set of N numbers from a true Poisson set would have a larger value than y.

Precision of a Single measurement:

- Wish to assign a certain degree of uncertainty to a measurement
- Assume a single measurement of x and further that $X = \overline{X}$
- Choose a model (Poisson, normal) with $X = \overline{X}$ which allows the calculation of P(x) and σ^2
- The predicted sample variance $s^2 \geq \sigma^2$

- This limits s for a measurement and provides an estimate for s of a repeated set of imaginary measurements, but also represents the expected results of a single measurement repeated several times
- Note that we generally represent the error in terms of multiples of the standard deviation (σ)
- Table 3.6 shows the σ intervals and confidence that a value used to predict the interval will include the true mean
- Fractional standard deviation σ/X is a measure of the scatter of an experimental over the number of counts. This measure allows for a calculation of the required number of events to produce a desired error percentage, (i.e, $\sigma = \sqrt{X} \Rightarrow \sqrt{X}/X$, 100 counts produces 10% error, 10000 counts => 1%, etc)
- Note that this analysis only applies to experiments relating directly to the number of counts, not rates or sums or differences of counts, averages of counts, or any derived quantities from the counts.

Error Propagation

- The general formula for error propagation (finding how an error should change based on its functional dependence)

$$\sigma_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + ...$$

for $u= u(x, y, z...)$ and $x, y, z,\ldots$ are independent variables

Sums & Differences:

$$u = x + y \text{ or } u = x - y\,;\ \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \pm 1$$

$$\sigma_u^2 = (1)^2 \sigma_x^2 + (\pm 1)^2 \sigma_y^2$$

$$\sigma_u = (\sigma_x^2 + \sigma_y^2)^{1/2}$$

where this procedure of summing the squares of the errors and taking the square root is often called combining in quadrature or the quadrature sum.

Note that the error becomes larger for subtraction since the numbers get smaller, while the error stays the same.

Multiplying by a constant:

$$u = Ax\,,\ \frac{\partial u}{\partial x} = A$$

$$\sigma_u^2 = (A)^2 \sigma_x^2\,;\quad \sigma_u = A\sigma_x$$

$$\text{or, } v = \frac{x}{B}\,;\ \frac{\partial v}{\partial x} = \frac{1}{B}$$

$$\sigma_v^2 = \left(\frac{1}{B}\right)^2 \sigma_x\ ;\ \sigma_v = \frac{\sigma_x}{B}$$

This is an important method for count rates.

$$r = \frac{x}{t}$$

Example:

$$x = 1120 \text{ counts; } t = 5 \text{ s}$$
$$r = 1120/5 \text{ s} = 224 \text{ s}^{-1}$$
$$\sigma_r = \sigma_x / t = (1120)^{1/2}/5 \text{ sec} = 6.7 \text{ s}^{-1}$$
$$r \pm \sigma_r = 224 \pm 6.7 \ (\text{s}^{-1})$$