Manifesto on spacetime symmetries, diffeomorphisms, and conformal transformations

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At the time of writing I found myself very confused by mathematically imprecise statements made (all by physicists) regarding the role of diffeomorphisms and conformal transformations in physics. The abuse of math terminology is itself not a big problem, provided that one abuses terminology in a consistent way which doesn't interfere with physical understanding. In my experience though this isn't always the case, and so in this note we will try to set things straight and provide an unambiguous definition for all the geometric terms that usually get thrown about haphazardly: diffeomorphisms, conformal transformations, isometries, and so on. The goal is to strike a tone that provides enough precision to do away with the ambiguities present in most physics discussions, while at the same time staying away from being overly fussy about notation. This was mostly written during TASI 2019; thanks to the participants for discussions.

1 Mathematical preliminaries

Let us first recall some terminology. For us, the word "manifold" will mean a smooth, differentiable manifold. So, for us, a manifold M is a Hausdorff space[1] which is locally homeomorphic[2] to $\mathbb{R}^n$. The identification with $\mathbb{R}^n$ is accomplished locally by the coordinate functions x^μ_a, which map open subsets U_a of M (we will always refine the cover so that the U_a are simply connected) into $\mathbb{R}^n$ via $x^\mu_a : U_a \to \mathbb{R}^n$. The epithet "smooth, differentiable" just means that the transition functions $x^\mu_a \circ (x^\mu_b)^{-1}$, defined on the image of $x^\mu_a(U_a \cap U_b)$, are $\mathcal{C}^\infty$ functions.

1.1 Diffeomorphisms

A diffeomorphism f is a smooth map between manifolds that is invertible and whose Jacobian matrix is invertible everywhere. That is, a diffeomorphism between two same-dimensional

[1] A topological space is basically a set X (a bunch of points) together with a collection of all neighborhoods $N(x)$ of all points $x \in X$, with obvious consistency conditions ($x \in N(x)$ for all x, $N(x)$; if $N(x), N'(x)$ are two neighborhoods of x then $N \cap N'$ is also a neighborhood of x; every neighborhood $N(x)$ includes a sub-neighborhood $N'(x)$ such that $N(x)$ is a neighborhood of all points $y \in N'(x)$). A Hausdorff space is a topological space which is such that for any two distinct points $x, y \in N(x)$, there are always neighborhoods of x and y which are disjoint.

[2] A homeomorphism is a continuous + invertible map between two topological spaces. It is defined without reference to calculus, so it does not have to be "smooth" in any sense.

manifolds M and N is an invertible map f which assigns points in M to points in N, and which is such that if $x^\mu_{N/M}$ are coordinate maps, then the function $x^\mu_N \circ f(x^\mu_M)^{-1} : \mathbb{R}^{\dim M} \to \mathbb{R}^{\dim M}$ is differentible and has invertible Jacobian everywhere.[3] Basically, generically any "nice" thing one might imagine doing to a manifold is a diffeomorphism.[4]

The condition on the invertibility of the Jacobian ensures that the tangent spaces on M and N get mapped to eachother in a bijective way. Therefore e.g. $x \mapsto x^2$ is not a diffeomorphism from $\mathbb{R}$ to itself since it isn't invertible, and $x \mapsto x^3$ isn't a diffeomorphism from $\mathbb{R}$ to itself since although it is differentiable and invertible, the Jacobian fails to be invertible at zero.If we want practice with the terminology, we could say that f being a diffeomorphism means that the pushforward f_*, which maps $TM \to TN$, has zero kernel everywhere. Indeed, recall that vectors in T_xM are pushed forward via

$$(f_*V)^\alpha(y) = \frac{\partial y^\alpha}{\partial x^\mu}\Big|_x V^\mu(f^{-1}(y)), \qquad V \in T_xM, \tag{1}$$

which has a kernel if Df isn't invertible.

Requiring that Df be invertible everywhere is nice because it means we can use both pushforwards and pullbacks to compare both contra- and co-variant tensors on the tangent spaces of M and $f(M)$. We would be unable to do this if f failed to have an intertible Jacobian, since then we can no longer pull back vectors in T_yN[5] to compare them with vectors in $T_{f^{-1}(y)}M$, since if $f(x) = y$ then we need to write

$$(f^*V)^\mu(x) = \frac{\partial x^\mu}{\partial y^\alpha}\Big|_y V^\alpha(f(x)), \qquad V \in T_yN, \tag{2}$$

which doesn't make sense when the Jacobian of f isn't invertible.[6]

So, a diffeomorphism $f : M \to N$ is essentially a smooth invertible map between points on the two manifolds, which is such that it extends to a smooth invertible map between T_pM and $T_{f(p)}N$ for all p. Note that the definition of a diffeomorphism does *not* refer to any metric structure! A manifold is defined by a collection points and a collection of coordinate

[3] We don't want to keep track of all the notational baggage that comes along with explicitly mentioning the coordinate functions of M and N, and so in what follows we will just write e.g. $f(x) = y$ for the diffeomorphism, where x, y are coordinates on M and N (images of points in M, N under coordinate maps), respectively.

[4] Some potentially "not nice" things are also diffeomorphims. For example, the Dehn twist on one of the cycles of a T^2 is a diffeomorphism. It is a large diffeomorphism since it cannot be built out of smaller ones that are connected to the identity map: cutting the torus and rotating one of the cut boundaries by any angle $\theta \notin 2\pi\mathbb{Z}$ does not map neighboring points to neighboring points, and violates the smoothness condition.

[5] Annoying comment: we are using coordinates as stand-ins for points on manifolds, since it probably won't cause confusion. So e.g. we could also write T_yN more anal-retentively as T_qN, with $y^\mu(q) = y$.

[6] Given any smooth map $f : M \to N$, we can always pull back fields on T^*N to fields on T^*M: for every $y = f(x)$, we can write

$$(f^*V)_\mu(y) = \frac{\partial y^\alpha}{\partial x^\mu}\Big|_x V_\alpha(x), \qquad V \in T^*N. \tag{3}$$

By contrast, vector fields can only be pulled back when we can "swap upper and lower indicies" in the above, that is, when Df is invertible—i.e. when f is a diffeomorphism. Likewise, for general smooth f, we can always push forward vector fields in TM to vectors fields in TN, but we cannot push forward fields in T^*M, unless f is a diffeomorphism.

charts, and does not come equipped with any notion of distance. Indeed, let (M,g), (N,h) be two Riemannian manifolds. Then a diffeomorphism $f : M \to N$ is defined in exactly the same way as above, without reference to a metric (note that we write $f : M \to N$ and not $f : (M,g) \to (N,h)$!). For example, the unit disk with the flat Euclidean metric is diffeomorphic to the Poincare disk. This is a pretty trivial diffeomorphism, with a point on the flat disk mapped to the same point on the Poincare disk. This diffeomorphism doesn't care that it maps two points that are close together in the flat metric to two points that are (perhaps very) far apart on the Poincare disk—diffeomorphisms are smooth associations of points on one manifold to points on another, nothing more. In particular, diffeomorphisms do not generically preserve any sort of curvature. Therefore the notion of a diffeomorphism is a very broad one. Two manifolds that are diffeomorphic are isomorphic with regards to differential topology: if one only cares about the smooth structure, then diffeomorphic manifolds are the same. Basically, all reasonable things that a physicist might imagine doing to a manifold are diffeomorphisms.

Sometimes one hears a big fuss being made about active and passive diffeomorphisms. In reality, they're just different ways of keeping track of what a diffeomorphism $f : M \to M$ does. The distinction between the two is the following: to perform an active transformation, we map points on M as $M \ni p \mapsto f(p)$, but keep the coordinate maps x_a^μ on each chart fixed. Therefore the coordinates of a point p map under f as

$$x^\mu(p) \mapsto x^\mu(f(p)). \tag{4}$$

On the other hand, to perform a passive transformation, we keep the points on M fixed, but change the coordinate chart maps as $x^\mu \mapsto f(x^\mu)$. Passive transformations can therefore alternatively called "coordinate transformations." The distinction between these two points of view is shown in figure 1. I don't really see the big deal of distinguishing between these two ways of thinking—sometimes passive diffeomorphisms are viewed as trivial "coordinate transformations" while active diffeomorphisms are viewed as something more deep that only GR possesses invariance under (see e.g. stuff by Rovelli). I don't see the point in distinguishing between them, since separating a manifold into a topological space ("a background spacetime") and "a coordinate system" and treating these two as if they were separate things is not useful—spacetime is the background topological space *and* the coordinate charts; both are needed to define a spacetime.

Anyway, what would it mean for a theory to be "diffeomorphism invariant", according to the math definition of a diffeomoprhism? Such a theory would have to be completely insensitive to smooth changes in the manifold on which it is paced. Since diffeomorphisms do not care about distances, such a theory would need to be completely insensitive to sizes, shapes, and curvature—such a theory would therefore need to be topological. For example, gravity is most emphatically not diffeomorphism invariant in this sense—if this were true then gravity would be topological, which (at least in $d \geq 4$) is totally untrue + antithetical to the spirit of GR.

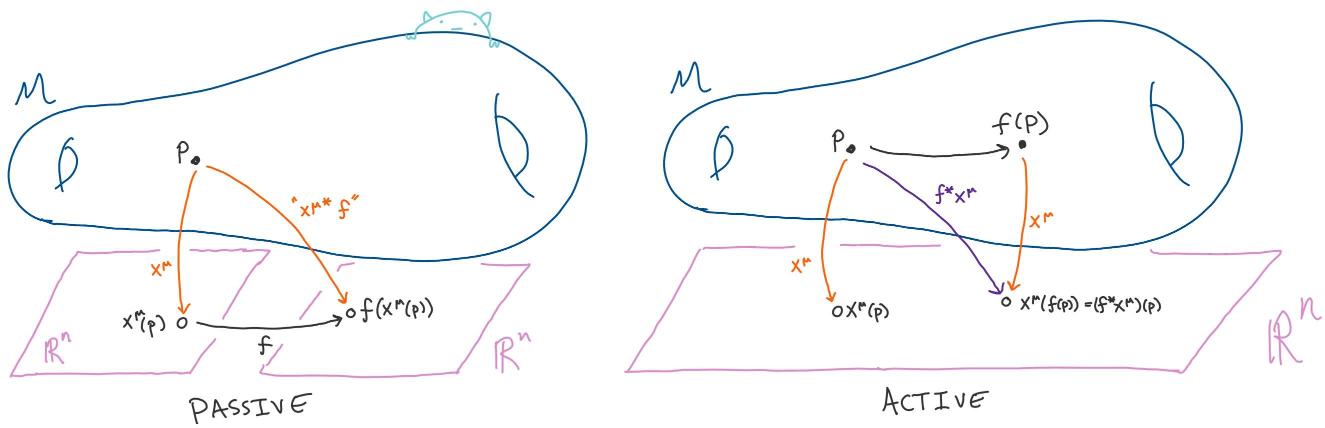

Figure 1: An illustration of the different ways to think about a diffeomoprhism $f : M \to M$. Left: an passive transformation, where we keep the points on the manifold M fixed, but change the coordinate maps $x^\mu \mapsto f(x^\mu)$. Right: an active transformation, where the coordinate charts x^μ are held fixed but the points on M are moved. Here we are letting p and $f(p)$ be within the same coordinate patch, for simplicity.

1.2 Isometries

Now we move on to Riemannian manifolds, where the presence of a metric gives us more math vocab words to use. In particular, when we have a metric, we have the notion of an isometry. An isometry is what most physicists call a diffeomorphism. That is, an isometry is a diffeomorphism between Riemannian manifolds $f : (M, g) \to (N, h)$ such that distances are preserved, viz. for any two vector fields V, W on TM, we have[7]

$$g(V, W) = h(f_* V, f_* W), \qquad \forall\, V, W \in TM, \tag{5}$$

where in our notation, we are writing a metric g as a section of $T^*M \times T^*M$, i.e. an assignment of a (0,2)-type tensor to each point of M. A simpler way of writing the above is to say that the metrics g, h are related by pullback along f:

$$g = f^* h. \tag{6}$$

Just to make the notation vis-a-vis the metric totally clear, in components we would write

$$g(V, W) = g_{\mu\nu} V^\mu W^\nu, \qquad h(f_* V, f_* W) = h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} V^\mu W^\nu, \tag{7}$$

where the metric components are e.g.

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu). \tag{8}$$

In components, this means that under an isometry, the new metric h is determined from the old metric g by

$$h_{\alpha\beta}(y) = \left(\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\alpha} \right) \bigg|_x g_{\mu\nu}(x). \tag{9}$$

[7]Note that since f is a diffeomorphism, $f_* V$ never vanishes unless V does, and so positive-definiteness of the metric is preserved.

Hence isometries are diffeomorphisms between Riemannian manfiolds such that the metric is mapped as $f : (M, g) \to (N, f_* g)$.[8] Note that for f to be an isometry, M and N do not have to be the same; they only need to be related by a diffeomorphism, with their metrics related as above. In particular, isometries are *not* necessarily transformations that preserve the components of the metric—they preserve distances, and calculating distances is done with more than just the components of the metric. A very simple example: consider the rescaling diffeomorphism of $\mathbb{R}^n$ to itself given by $y = \lambda x$, and suppose that the metric is mapped as $f : (\mathbb{R}, g) \mapsto (\mathbb{R}, \lambda^{-2} g)$. This is an isometry, since while the metric components change, distances remain the same, since inner products map as $g(V, W) \mapsto \lambda^{-2} g(\lambda V, \lambda W) = g(V, W)$.

When most physicists say "isometry", they usually actually mean "autoisometry". An autoisometry is an isometry of a Riemanninan manifold to itself, $f : (M, g) \to (M, g)$. Note that the metric on both sides of the map is the same, which means that if f is an auto-isometry, then

$$f^* g = g \implies \mathcal{L}_f g = 0, \tag{10}$$

where $\mathcal{L}_f$ is the Lie derivative along the flow of f. When we have an autoisometry, we can consider mapping the points on M with f but *not* mapping the metric components—doing this would still give us an isometry.

Recapitulating, isometries generically change both the coordinates and the metric, but are essentially geometrically trivial, since they do not change distances or curvatures. Just as a diffeomorphism is an isomorphism in the sense of differential topology, an isometry is an isomorphism in the sense of Riemannian geometry.

In my experience, physicists usually refer to isometries as either "diffeomorphisms" or "coordinate transformations", and a theory which is invariant under isometries is often referred to as "covariant" or "diffeomorphism invariant" or "diffeomorphism covariant". It's a mess. When people say "physics can't depend on a change of coordinates $\implies$ everything is diffeomoprhism invariant", they are correct only as long as they tacitly mean "provided that the metric changes accordingly with the change in coordinates in such a way as to produce an isometry". Just changing the coordinates without touching the metric components is generically *not* a physically trivial operation, since if we don't touch the metric components, then changing coordinates can change distances, curvature, etc. However, changing both the coordinates and the metric in such a way as to produce an isometry really is trivial, and it is the invariance of physical theories under this combined procedure that people generally refer to as "coordinate independence".

I actually think the smallest sin committed by physicists is to conflate isometries with diffeomorphisms, for the following reason: given a diffeomorphism $f : M \to N$, we can always construct an isometry from f, since we can always choose the metrics on the two manifolds to be related as $f : (M, g) \to (M, f_* g)$. When physicists say "diffeomorphism", they often mean this specific diffeomorphism, where the pullback of the metric on the target is the same as the original metric. To be mathematically correct, one should call this an isometry. Unfortunately, physicists (the same physicists, often) also use "diffeomorphism" to mean a spacetime symmetry generated by a diffeomorphism which is altogether distinct from an isometry. Indeed, virtually no physicists I know actually use the word diffeomorphism correctly, where in

[8] We could also equivalently have $f : (M, f^* g) \to (N, g)$.

this context ”correctly” is defined with respect to the mathematical convention. This means that in order to not come off as a total mathematical pedant and to speak like a physicist, I will have to abuse the word diffeomorphism somewhat. In what follows, I will do my best to make clear the meaning on a context-by-context basis.

2 Spacetime symmetries in field theory

Now that we know what diffeomorphisms and isometries are, what then is a “spacetime symmetry” in physics parlance? In physics, *symmetries act only on the dynamical fields.* They do not act on background fields, and nor do they act on the coordinates (which are just integration variables), and so the only thing that a symmetry action does is to replace the value of a field ϕ at a point x with the value of a different field ϕ' at *the same point* x; $\phi(x) \mapsto \phi'(x)$. Just like with any other type of symmetry, spacetime symmetries manifest themselves as ways of transforming fields and operators such that correlation functions are left invariant, and so we will have a symmetry if the fields $\phi'(x)$ have the same correlation functions has the fields $\phi(x)$.

Now as we said, symmetries do *not* act on any background fields present. To say that a theory is symmetric, one needs to make a comparison before and after acting with a symmetry transformation, and comparing two theories is only possible if they live in the same background. In non-gravitational theories, the metric is a background field—hence in field theory, symmetries do not touch the metric. This fact led to a lot of confusion for me, since many books often present a spacetime symmetry as an action on the metric. We also are often in the situation where we'd like to compare the partition functions for theories with two different metrics, e.g. when we want to compare partition functions of a theory on spheres of different radii. But really, different choices of background metric lead to different theories in some sense: this is not so for symmetries, which always relate one theory to itself.

To define a putative spacetime symmetry, one starts with a diffeomorphism $f : M \to M$ (leaving how the metric is mapped unspecified), and uses f to define an action on the fields of the theory. When spacetime symmetries act on fields (by “fields”, we mean “dynamical fields”), the most common way for fields to transform is by pushforward, viz

$$\phi(x) \mapsto (f_*\phi)(x). \tag{11}$$

For example, if $\phi(x)$ is a scalar, then $\phi(x) \mapsto \phi(f^{-1}(x))$. Since f is invertible, this definition makes sense for any kind of tensor field $\phi(x)$. Using the pullback instead is also viable, but in order to match conventions with the majority of physics literature, I will stick with the pushforward.

We will often need to look at the infinitesimal version of transformations like these. To this end, let f be an infinitesimal diffeomorphism, and let ϵ be infinitesimal. We will define the vector field ξ associated to f through

$$f(x)^\mu = x^\mu - \epsilon\xi^\mu. \tag{12}$$

The minus sign here is not a mistake, and will be explained shortly. With this definition, we see that the change in ϕ under the infinitesimal symmetry action is, assuming that ϕ

transforms by pushforward as above,

$$\delta_\xi \phi \equiv \frac{f_* \phi - \phi}{\epsilon} = \mathcal{L}_\xi \phi. \tag{13}$$

This is the reason for the sign in the definition of ξ: it allows us to write $\delta_\xi \phi$ as a Lie derivative.[9] Normally the Lie derivative is associated with the difference between a field and its pullback—here we want to both use the pushforward and get a Lie derivative as the variation (since Lie derivatives are nice to calculate with), and hence must choose the sign of ξ as above.

Of course, the pushforward is not the only way that fields can transform—for example, this transformation rule does not quite hold for conformal transformations, which we will discuss separately. But for now, until further notice, we will assume that all fields transform according to the above rule, with their variations given by the Lie derivative. If correlation functions / the action are invariant under this transformation, then we have a spacetime symmetry.

To talk more precisely about how spacetime symmetries act, in my opinion it's conceptually helpful to define vector fields on the space of field variations, and so we will now pause to develop this technology. A vector field on variational space is defined as (fancy letters will denote things in variational space)

$$\mathscr{V} = \int d^d x \, \mathscr{V}^a(x) \frac{\delta}{\delta \phi^a(x)}, \tag{15}$$

where ϕ^a stand for all the fields in the theory (the repeated a index means that we are to sum over all fields). Here the $\delta / \delta \phi^a(x)$ are basis vectors in the tangent space of field configurations, while the vector components of $\mathscr{V}$ are functions of the $\phi^a(x)$. For a $(0, n)$ form Φ (the notation is (de-Rham degree, variational degree)) defined as

$$\Phi = \int \prod_{i=1}^{n} d^d x_i \, \Phi_{a_1 \dots a_n}(x_1, \dots, x_n) \delta \phi^{a_1}(x_1) \wedge \cdots \wedge \delta \phi^{a_n}(x_n), \tag{16}$$

the variational derivative is defined as

$$\delta \Phi = \int \prod_{i=0}^{n} d^d x_i \, \frac{\delta \Phi_{a_1 \dots a_n}(x_1, \dots, x_n)}{\delta \phi^{a_0}(x_0)} \delta \phi^{a_0}(x_0) \wedge \delta \phi^{a_1}(x_1) \wedge \cdots \wedge \delta \phi^{a_n}(x_n), \tag{17}$$

while the interior product is defined as

$$i_{\mathscr{V}} \Phi = \int \prod_{i=1}^{n-1} d^d x_i \, \mathscr{V}^{a_1} \Phi_{a_1 \dots a_n}(x_1, \dots, x_n) \delta \phi^{a_2}(x_2) \wedge \cdots \wedge \delta \phi^{a_n}(x_n). \tag{18}$$

[9] For example, if ϕ is a sclar then

$$f_* \phi - \phi = f^{-1}(x)^\mu \partial_\mu \phi - \phi = \epsilon \nabla_\xi \phi, \tag{14}$$

which indeed produces the Lie derivative.

The above formulae are just meant to clarify that the variational and de-Rham differential complexes are really precisely analogous; we won't actually need them, since in what follows we'll never have occasion to work with any variational forms of degree bigger than 1.

The benefit of this notation is that it lets us define variational Lie derivatives. They are defined by Cartan's formula as usual: for a variational vector field $\mathscr{V}$, we have $\mathscr{L}_{\mathscr{V}} = \delta i_{\mathscr{V}} + i_{\mathscr{V}}\delta$. The Lie derivative of a variational scalar (i.e. just a functional of the fields, like the Lagrangian) is therefore

$$\mathcal{L}_{\mathscr{V}}\Phi = \int d^d x\, \mathscr{V}^a(x) \frac{\delta \Phi}{\delta \phi^a(x)}. \tag{19}$$

For thinking about spacetime symmetries, the relevant vector components are the variations of the fields under the symmetry: if the spacetime symmetry is generated by a vector field ξ, so that $\delta_\xi \phi^a = \mathcal{L}_\xi \phi^a$ (the ordinary Lie derivative, now), then a function Φ of the fields (for example, the Lagrangian density) transforms as[10]

$$\delta_\xi \Phi = \mathscr{L}_\xi \Phi \equiv \mathscr{L}_{\mathcal{L}_\xi \phi} \Phi = \int d^d x\, \mathcal{L}_\xi \phi^a \frac{\delta \Phi}{\delta \phi^a}. \tag{20}$$

We now define a functional Φ of all the fields (dynamical and background) to be *covariant* with respect to a vector field ξ if it satisfies

$$\mathcal{L}_\xi \Phi = \mathscr{L}_\xi \Phi. \tag{21}$$

When the Lagrangian L of a theory, written as a d-form,[11] is covariant in the above sense with respect to ξ, then the vector field ξ generates a symmetry of the theory. Physicists often say that this means "L is invariant under the diffeomorphsim ξ", but I think this is a rather poor use of terminology. The confusion here is compounded by the fact that other physicists (sometimes the same physicists) also use "diffeomorphism invariance" to refer to the invariance of action under $\mathcal{L}_\xi$, which as we have seen is technically speaking just invariance under isometries, and is a rather trivial statement.

Anyway, why is this statement, viz. the covariance of L, the right definition of a symmetry? Here's why: since in our notation L is a d-form, Cartan's formula tells us that

$$\mathcal{L}_\xi L = (i_\xi d + d i_\xi) L = d(i_\xi L), \tag{22}$$

where i_ξ is the contraction and we used $dL = 0$ on account of L being top-dimensional. Therefore, the Lie derivative of L along any vector field is always a total derivative.[12] Therefore if L is covariant with respect to ξ, then under the action of the symmetry we have

$$\delta_\xi L = \mathscr{L}_\xi L = \mathcal{L}_\xi L = d(i_\xi L), \tag{24}$$

[10] As we stressed above, symmetries act only on the dynamical fields, *not* the background fields. Therefore in this discussion, "fields" is still taken to mean "dynamical fields"—the functional derivatives appearing here act on the dynamical fields only.

[11] So e.g. we can have $L \sim F \wedge \star F$ or $L \sim d\phi \wedge \star d\phi - \phi^4 |d^d x|$, where $|d^d x| = \star 1$ is the volume d-form.

[12] Just to clarify the notation and write things in components, suppose that $L = \star\mathcal{L}$, where $\mathcal{L}$ is a scalar Lagrangian density (sorry for the profusion of $\mathcal{L}$s). Then we can write $\mathcal{L}_\xi L$ as

$$d(i_\xi L) = d(\mathcal{L} i_\xi(\star 1)) = \star\nabla_\mu(\mathcal{L}\xi^\mu) = \partial_\mu(\mathcal{L}\xi^\mu \star 1). \tag{23}$$

Here we have used $\partial_\mu(\sqrt{g} X^\mu) = \sqrt{g}\nabla_\mu X^\mu$ and $d(i_X \star 1) = \star(\nabla_\mu X^\mu)$ which holds for any vector field X^μ; the proofs of these relations are in a separate diary entry.

and so the variation of the Lagrangian is a total derivative—just the right condition for the vector field ξ to implement a symmetry.[13]

So, in what settings do we expect to have spacetime symmetries? The most common way for L to be covariant with respect to ξ is for the background fields A to all satisfy $\mathcal{L}_\xi A = 0$; then we manifestly have $\mathscr{L}_\xi = \mathcal{L}_\xi$. If the only background field is the metric, the condition $\mathcal{L}_\xi g = 0$ means that f must be an (auto-)isometry $f : (M, g) \to (M, g)$ (distances are preserved even if we map only the coordinates or the only metric components, but not both). This is why (auto-)isometries determine the spacetime symmetries present in usual QFT settings. In components, the condition that $\mathcal{L}_\xi g = 0$ of course is determined by calculating

$$\begin{aligned} g_{ab}(x+\epsilon\xi) = g_{ab}(x) + \epsilon\big[(\xi\cdot\partial)g_{ab}(x) + \partial_a\xi^c g_{cb} + \partial_b\xi^c g_{ac}\big] + O(\varepsilon^2) \\ \implies \mathcal{L}_\xi g_{ab} = (\xi\cdot\partial)g_{ab}(x) + \partial_a\xi^c g_{cb} + \partial_b\xi^c g_{ac}. \end{aligned} \tag{25}$$

and so requiring that $\mathcal{L}_\xi g = 0$ implies, after some algebra needed to covariantize the derivatives above, that ξ satisfies the Killing equation, viz.

$$\nabla_{(\mu}\xi_{\nu)} = 0. \tag{26}$$

This is the long-winded way of saying why Killing vectors produce spacetime symmetries in ordinary QFT.

Another way for L to be covariant is for L to contain no background fields at all—then there is no difference between the action of $\mathcal{L}_\xi$ and that of $\mathscr{L}_\xi$. In this case, the theory is said to be *generally covariant*. In generally covariant theories, every diffeomorphism of spacetime generates a symmetry. The most famous example of a generally covariant theory is of course GR, which achieves general covariance by making the metric dynamical and integrating over it. Other examples of generally covariant theories are TQFTs, whose actions don't contain the metric (or any other background fields) at all.

One last rather pedantic point—if spacetime symmetries don't touch the metric, why are their charges given by integrals of $T_{\mu\nu}$, which is computed from varying S wrt $g_{\mu\nu}$? The answer is basically that the difference between computing T by varying wrt $g_{\mu\nu}$ and computing T through the usual Noether procedure is captured by the Lie derivative of the Lagrangian, which as we have shown above is always a total derivative. Therefore either way of thinking about computing T is equivalent (up to a total derivative), but we usually use the method of varying wrt $g_{\mu\nu}$ since it produces a T that is manifestly symmetric. Another diary entry contains the details of this.[14] As a brief justification of this statement, we see

[13] We are tacitly ignoring issues relating to boundary conditions—the action of ξ will of course need to preserve the (spatial) boundary conditions.

[14] Another related potential point of confusion is the following: if the metric transforms trivially under (auto-)isometries, why does $T_{\mu\nu}$ still generate (auto-)isometries, given that it is computed by varying wrt the metric? The philosophy here is basically that $T_{\mu\nu}$ only appears in the context of performing isometries only on subregions of spacetime. E.g. if the vector field ξ generates an isometry, then we can consider acting on the fields with the flow of the vector field $\theta(R)\xi$, where $\theta(R)$ is the indicator function on some subregion R of spacetime. This action, which is not the action of a symmetry, is accomplished with $\int_{\partial R} d^{d-1}x^\mu\, T_{\mu\nu}\xi^\nu$, which is the charge operator localized on ∂R. Basically, we can only see the affect of symmetries when we are allowed to break them (in this case, by acting with them only on a subregion of spacetime), and so even though g does not very under (auto-)isometries, the $T_{\mu\nu}$ computed by varying wrt g still keeps track of information about symmetries.

that we can write it as

$$\mathscr{L}_\xi S + \mathcal{L}_\xi g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}} = \int \mathcal{L}_\xi L = \int d(i_\xi L) \to 0. \tag{27}$$

Therefore, at the math level, the difference between performing an infinitesimal spacetime symmetry transformation by varying the metric or by varying the fields is a total derivative, in theories that are invariant under diffeomorphisms (in the physicist's sense—"invariant under isometries" is the correct terminology). However, I think that this leads to confusion about what the symmetry transformation actually is, so it's best to just stick to thinking about the dynamical fields being acted on.

2.1 GR

The discussions concerning GR are usually most misleading vis-a-vis the role of diffeomorphisms in physics, and even ostensibly reputable sources are pretty confused about terminology. The best discussion I ended up finding seemed to be in Carroll. Although his use of the word "diffeomorphism" is technically speaking mathematically not correct (he uses the physicist's "diffeomorphism", i.e. he really means "isometry"), if we ignore this quibble, his discussion is pretty good.

From the above discussion, we can see that the statement "GR is special because it is diffeomorphism invariant" really needs to be interpreted with care. Again, if GR were literally diffeomorphism invariant in the mathematical sense, it wouldn't be a geometric theory at all, since it would be a theory that cared only about differential topology.

As we mentioned above, the whole point of GR is that it is background-independent, and hence essentially by definition it is incompatible with the existence of background fields—there is no notion of a background reference frame / prior geometry that could be used to define such fields. This means in particular that $\mathcal{L}_\xi = \mathscr{L}_\xi$ in gravitational theories; hence in GR all diffeomorphisms (in the physicist's sense) define true symmetries. This is not a uniquely gravitational feature though, since TQFTs are also formulated without any reference to background fields.[15]

The facet of this discussion that is unique to GR is the following: the fact that the field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ are invariant under mapping the metric as $g \mapsto f^* g$ for any diffeomorphism f means that there is a huge redundancy in the metric components. This is significant since in GR, the metric components are what we're solving for—this redundancy is why GR is often referred to as a gauge theory, and is the feature of GR which forces us to define all physical quantities (mass, angular momentum, etc) as boundary integrals at infinity.

[15] One caveat here is that we've only been looking at using infinitesimal diffeomorphisms to define symmetry transformations. Diffeomorphisms that aren't connected to the identity are a separate matter—the condition that $\mathscr{L}_\xi L = \mathcal{L}_\xi L$ can't be applied for large diffeomorphisms and so a generally covariant theory that's invariant under large diffeomorphisms can generically transform in a nontrivial representation of the group $\mathrm{Diff}/\mathrm{Diff}_{\mathbf{1}}$, where $\mathrm{Diff}_{\mathbf{1}}$ is the component connected to the identity.

3 Conformal transformations

Now we come to another unmitigated termonological disaster: the exact definition of a conformal transformation in QFT. After asking O(10) people who do way more CFT than I do, I got a pretty much random smattering of answers to very basic questions about definitions, which was very frustrating. One of the takeaways is that e.g. di Francesco should be banned for all purposes other than learning about minimal models and affine Lie algebras. Anyway, here we'll try to get everything straight. Our terminology can be summarized by its distinction between three things:

- conformal maps, which exist outside the context of QFT and which when discussed in a QFT context act on both the dynamical and background fields by pushforward by a diffeomorphism,
- Weyl transformations, which are rescalings of both dynamical and background fields, and
- conformal transformations, which are defined using a conformal map but which only act on the dynamical fields.

Only the last one has a chance to be a symmetry, and it is invariance under the last one which leads to conformal symmetry in the context of CFT.

3.1 Conformal maps

The mathematical definition of a conformal map is pretty straightforward. Conformal maps are diffeomorphisms in the mathematical sense, but they are not all isometries. However, they fail to be isometries in a specific way. Namely, a conformal map is a diffeomorphism $f : (M, g) \to (N, h)$ such that the pullback of the metric on N is related to the metric on M by a rescaling factor:

$$f^* h = \Omega^2 g. \tag{28}$$

In components, this reads (for $y^\mu = f(x^\mu)$)

$$h_{\alpha\beta}(y) = \Omega^2(x) \left(\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \right) \Bigg|_x g_{\mu\nu}(x). \tag{29}$$

We will usually be interested in the situation where $f : (M, g) \to (M, g)$ (one might call this an auto-conformal map; we won't bother to). Again, this will not generically be an (auto-)isometry, since $f^* g \neq g$. The condition for this to be a conformal map is of course obtained from the above equation with $h = g$. When the metric is independent of the coordinates, the condition that we have a conformal map is

$$g_{\alpha\beta} = \Omega(x)^2 \left(\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \right) \Bigg|_x g_{\mu\nu} \implies \frac{\partial y^\alpha}{\partial x^\mu} = \Omega(y) R^\alpha_\mu(y), \tag{30}$$

where R^α_μ is an (auto-)isometry, viz. $RgR^T = g$, so that e.g. $R \in SO(p,q)$. Therefore in flat space,[16] a conformal map is a local rotation (R), plus a local rescaling (Ω).

Now we will consider what an infinitesimal conformal map looks like. To this end it is convenient to write $\Omega = e^\omega$. Then subtracting g from both sides of $f^*g = \Omega^2 g$ and restricting to an infinitesimal map by taking ξ to be the vector field generating the infinitesimal action of f, we find[17]

$$\nabla_{(\mu}\xi_{\nu)} = -2\omega g_{\mu\nu}. \tag{31}$$

Contracting both sides gives us ω, and so we see that conformal maps must satisfy

$$\nabla_{(\mu}\xi_{\nu)} = -\frac{1}{d}\nabla_\lambda \xi^\lambda g_{\mu\nu}, \tag{32}$$

which is how we will be writing the conformal Killing equation.

Already, there is a terminological caveat to make. In the GR literature, a conformal map is often taken to mean a diffeomoprhism $f : (M,g) \to (N,h)$ such that $h = \Omega g$, with Ω any function (thus this is a much more broad definition of conformal map). Also, In math, the term "conformal equivalence" is used to describe two Riemannian manifolds whose metrics are related by $h = \Omega g$, again for arbitrary Ω, with a conformal map then existing between two manifolds M and N if the pullback of the metric on the target manifold N is conformally equivalent to the metric on M. Therefore "conformal equivalence" would probably be called something like "Weyl equivalence" in our terminology.

Brief miscellaneous comment: one might wonder if conformal maps preserve curvature invariants, in the sense that e.g. $f^*R = R$ for f a conformal map. In two dimensions, we can easily make concrete statements. In two dimensions all Riemannian manifolds are conformally equivalent to flat space, and so we may write the metric as $g_{\mu\nu} = e^\phi \eta_{\mu\nu}$. In complex coordinates, only $\eta_{z\bar{z}}$ is nonzero, and so the conformal killing equation for a vector field $\xi = \xi\partial + \bar{\xi}\bar{\partial}$ reads $\partial\bar{\xi} = \bar{\partial}\xi = 0$, meaning that conformal killing vectors (anti)holomorphic functions. In a previous diary entry on the c theorem, we showed that $\sqrt{g}R \propto \partial\bar{\partial}\phi$. Therefore integrating the infinitesimal transformation induced by ξ leads to $\phi \mapsto \phi + \partial\xi + \bar{\partial}\bar{\xi}$, which leaves $\partial\bar{\partial}\phi$ invariant. Therefore in two dimensions, conformal maps preserve curvature. A simpler way to argue for this is to recall that conformal maps preserve angles, meaning that doing parallel transport around a closed loop on the surface pre- and post-conformal mapping should give the same answer, meaning that indeed, conformal maps leave R invariant. This does not seem to be true in higher dimensions, though, and the formulae quicky get messier.

3.2 Conformal transformations in field theory

Now we need to discuss the notion of a conformal transformation in physics, which is something which acts only on the dynamical fields / operators of a theory. To construct a conformal transformation, we first choose a conformal map f. Now as discussed above, symmetries only act on dynamical fields, and leave the background fields (viz. the metric, which since we are not doing GR is not dynamical) and the ambient spacetime fixed. Under a

[16] In some places, conformal transformations are only defined on flat space; not here though.

[17] The minus sign is the same minus sign mentioned when defining the action of spacetime symmetries on fields. It comes up since we want $f_*g - g$, and not $f^*g - g$, to compute $\mathcal{L}_\xi g$.

conformal transformation $\partial_\mu f^\alpha = \Omega R_\mu^\alpha$, the dynamical fields / operators in the theory are defined to transform as (we will only be dealing with conformal primaries)

$$\phi(x) \mapsto \phi'(x) = \Omega(f^{-1}(x))^{-\Delta_\phi}\mathcal{R}(f^{-1}(x)) \cdot \phi(f^{-1}(x)), \tag{33}$$

where Δ_ϕ is the scaling dimension of ϕ and where $\mathcal{R}(f^{-1}(x))$ performs the action of $R(f^{-1}(x))$ in the representation appropriate for the field ϕ. Note that in accordance with this being a symmetry transformation, the background fields (viz. the metric) and the coordinate maps are left untouched. Therefore a conformal transformation really acts completely trivially geometrically—it does not change the geometry on which the fields propagate; it only changes the fields themselves.

As a mega-simple example, consider acting with a finite dilitation on some 2-pt function $\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle$ for a theory on $\mathbb{R}^n$. If f is the diffeomorphism $f : x \mapsto \lambda x$, this does

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle \mapsto \lambda^{-2\Delta}\langle \mathcal{O}_\Delta(x/\lambda)\mathcal{O}_\Delta(0)\rangle = \lambda^{-2\Delta}\frac{1}{|x/\lambda|^{2\Delta}} = \langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle. \tag{34}$$

Note that the correlator is *in*variant under the symmetry, not covariant.

3.3 Weyl transformations and the Weyl anomaly

Another thing we should get straight is the notion of a Weyl transformation. A Weyl transformation acts on the fields[18] by

$$\phi(x) \mapsto \Sigma^{-\Delta_\phi}(x)\phi(x), \qquad g_{\mu\nu}(x) \mapsto \Sigma^2(x)g_{\mu\nu}(x), \tag{35}$$

where $\Sigma(x)$ is an arbitrary non-zero function on spacetime (we are using Σ instead of Ω to emphasize that the Ω appearing in the conformal maps is a special type of rescaling, while here Σ may be any non-zero function).

Note that *Weyl transforamtions are not symmetries* in QFT, since they act nontrivially on the background metric. When we say a theory is "Weyl invariant", we just mean that correlators of $\phi(x)$ in the metric $g_{\mu\nu}$ are the same as correlators of $\Sigma^{-\Delta}(x)\phi(x)$ in the background $\Sigma^2(x)g_{\mu\nu}$. As a check that the powers of Σ are right, consider acting on $L = d^dx\,\sqrt{g}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}$ with $\Sigma = \lambda$ a constant. Then $L \mapsto \lambda^{d-2-2\Delta_\phi}L$, which gives the expected scale-invariance condition of $\Delta_\phi = (d-2)/2$.[19]

For theories with only scalars, we can think of a conformal transformation as a conformal map composed with a Weyl transformation: the conformal map acts on all fields by the pull-back, and then the Weyl transformation adds in the $\Omega^{-\Delta}$ factors to the fields by taking $\Sigma = \Omega$ and undoes the change in the metric incurred during the conformal map: therefore, letting ϕ be a scalar,

$$\begin{aligned} &\phi \xrightarrow{\text{conformal map } f} f_*\phi \xrightarrow{\text{Weyl with } \Sigma=\Omega} \Omega^{-\Delta}(x)f_*\phi \\ &g_{\mu\nu} \xrightarrow{\text{conformal map } f} (f_*g)_{\mu\nu} = \Omega^{-2}(x)g_{\mu\nu} \xrightarrow{\text{Weyl with } \Sigma=\Omega} g_{\mu\nu}, \end{aligned} \tag{36}$$

[18]In this section, we will assume for simplicity that the metric is the only background field present.
[19]To get invariance under non-constant Σ, we'd need to add the $\sqrt{g}\,R\phi^2$ term to L.

and we see that the images on the RHS are indeed the images of the fields under the conformal transformation parametrized by f. However, one must not make the mistake of thinking that conformal transformations are the same as Weyl transformations modulo "the action of a diffeomorphism": in Weyl transformations the re-scaling factor σ can be an arbitrary non-zero function, while for conformal transformations / conformal maps the re-scaling factor is much more specific: it is the Jacobian of a diffeomorphism such that the pulled-back metric differs from the original metric only by the square of the re-scaling factor. The group of Weyl transformation is infinite dimensional, while the group of (global) conformal transformations is finite dimensional—the two operations are very different.

Anyway, in equations, the relation "conformal transformation = conformal map + Weyl transformation" is captured infinitesimally by first writing, for $\Sigma = e^\sigma$ and for ξ the vector field generating a conformal map,

$$\delta_\xi^{cm}\phi = \mathcal{L}_\xi\phi, \qquad \delta_\xi^{cm} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \frac{2}{d}\nabla_\lambda\xi^\lambda g_{\mu\nu}, \qquad \delta_\sigma^w\phi = -\sigma\Delta\phi, \qquad \delta_\sigma^w g_{\mu\nu} = 2\sigma g_{\mu\nu} \tag{37}$$

where δ_ξ^{cm} is the variation under the conformal map and δ_σ^w the variation under the Weyl transformation. Therefore if we choose $\sigma = -\frac{\nabla_\lambda\xi^\lambda}{d}$, we have

$$(\delta_\xi^{cm} + \delta_\sigma^w)\phi = \left(\mathcal{L}_\xi + \frac{\Delta}{d}\nabla_\lambda\xi^\lambda\right)\phi = \delta_\xi^c\phi, \qquad (\delta_\xi^{cm} + \delta_\sigma^w)g_{\mu\nu} = 0 = \delta_\xi^c g_{\mu\nu}, \tag{38}$$

where δ_ξ^c is the variation under the conformal transformation determined by ξ.

Now let's take a look at why Weyl invariance implies conformal symmetry, again in the case of only scalar fields. Consider acting on the action (we're choosing S just because it involves less writing; one could instead study correlation functions) with an infinitesimal Weyl transformation. The change in the action is

$$\delta_\sigma^w S = \int d^dx\, \sigma(x)\left(2g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}} - \Delta_a\phi^a\frac{\delta S}{\delta\phi^a}\right). \tag{39}$$

On the other hand, for a conformal transformation

$$\delta_\xi^c S = \int d^dx\, \left(\mathcal{L}_\xi\phi + \frac{\Delta_a}{d}\nabla_\lambda\xi^\lambda\phi^a\right)\frac{\delta S}{\delta\phi^a}. \tag{40}$$

Therefore if the theory is Weyl invariant, so that $\delta_\sigma^w S = 0$ for all σ, we may choose $\sigma = -\frac{\nabla_\lambda\xi^\lambda}{d}$, and hence conclude that

$$\delta_\xi^c S = \int d^dx\, \left(\mathcal{L}_\xi\phi^a\frac{\delta S}{\delta\phi^a} + \mathcal{L}_\xi g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}\right) = \int \mathcal{L}_\xi L = 0, \tag{41}$$

and so Weyl invariance indeed implies conformal symmetry—this is not too surprising, given that the group of Weyl transformations is huge (it's infinite-dimensional!), and so Weyl invariance is rather stringent requirement to place on a QFT.

Why did we restrict ourselves to scalar fields when discussing the above? The reason is that we assumed the variation of ϕ under a conformal transformation was $(\mathcal{L}_\xi + \Delta(\nabla\cdot\xi)/d)\phi$.

As a finite transformation, this reads $\phi \mapsto \Omega^{-\Delta} f_* \phi$. However, as far as I can tell, this is *not* how non-scalar fields transformation under conformal transformations: for example, a spin-1 primary transforms as $A_\mu \mapsto \Omega^{-\Delta} R^\nu_\mu A_\nu$, where $\det R = 1$. On the other hand, $\Omega^{-\Delta}(f_* A)_\mu = \Omega^{-\Delta} \frac{\partial x^\nu}{\partial y^\mu} A_\nu = \Omega^{-\Delta-1} R^\nu_\mu A_\nu$, which is not the same. The problem comes from the fact that primaries are assumed to transform under a product of the rescaling $\Omega^{-\Delta}$ and an orthogonal transformation, but the push-forward (or pull-back) includes extra factors of Ω in the Jacobian matrices. I'm a little perplexed about this and will come back to work it out eventually, but at least there is no contradiction with what I've found in the literature, where I've only seen "Weyl $\implies$ conformal" discussed in the context of scalars.

Now let's discuss the "Weyl / conformal anomaly". The phrase "Weyl anomaly" is kind of misleading, since as defined above, Weyl transformations are not symmetries for non-dynamical metrics. I think the best way to think about the Weyl anomaly is that it is a mixed anomaly between rescaling symmetry and diffeomorphisms. Here by "rescaling symmetry", we mean the subset of conformal transformations that do

$$D_\lambda : \phi(x) \mapsto (D_\lambda \phi)(x) = \lambda^{-\Delta} \phi(x/\lambda). \tag{42}$$

Note that the metric is unchanged, since as we said, the D_λ just generate a subset of conformal transformations, viz. the scale transformations.

Let us try to weakly gauge the subset of diffeomorphisms that are a symmetry of the theory. This means coupling our theory to a background gauge field for diffeomorphisms—that background gauge field is of course the metric. Now the metric is already there in the action, but now we modify our perspective and think of it as a background gauge field, and as such allow it to transform under symmetry actions. Consider now performing the rescaling symmetry by including a shift of the metric as $D_\lambda : g_{\mu\nu}(x) \mapsto \lambda(x)^2, \phi \mapsto \lambda(x)^{-\Delta}\phi$—notice now that after letting $g_{\mu\nu}$ be a background gauge field, the rescaling transformations become local, as λ is now a function of the coordinates. This gauged local version of the rescaling symmetry is a Weyl transformation. Now when we do the calculation for the "Weyl anomaly", we see that the rescaling symmetry, even the global part with λ constant, is broken—therefore coupling the theory to a background field for diffeomorphisms has broken the rescaling symmetry, which is our basis for saying that the Weyl anomaly is really a mixed 't Hooft anomaly between rescaling and diffeomorphisms. Backing this point of view up is the fact that the $\langle T^\mu_\mu \rangle \sim R$ answer that we get for the anomaly in e.g. 2d is exactly what we expect from a 't Hooft anomaly—the symmetry breaking only shows up when we turn on a nontrivial background field, and the extent to which the symmetry is broken is determined by the curvature of the background.

3.4 Scale invariance and conformal invariance

Finally, a brief comment on scale vs conformal invariance. Often scale invariance implies conformal invariance. A very heuristic way to argue this is the following. Consider a theory with scale invariance, and consider performing a local rescaling, which acts as a rescaling within some region R and turns off into the action of the identity away from R. By performing a global scale transformation, we can stretch out the region R, and make the distance over which the action of the local resclaing falls off be arbitrarily long. Therefore a global resclaing can be used to turn a local rescaling into one which locally looks arbitrarily close to a global rescaling. In situations when this argument works we can conclude that the theory is "Weyl invariant", which as we have seen implies conformal invariance.

More precisely, one could argue that since the theory responds to an infinitesimal scale transformation via $\delta S = \int T_{\mu\nu}\delta g^{\mu\nu} = \int T_{\mu\nu}\partial^\nu \varepsilon^\mu = \int T_{\mu\nu}\delta^{\mu\nu}$, scale invariance implies $T^\mu_\mu = 0$, which implies conformal invariance. The subtlety here is that we could also have $T^\mu_\mu = \partial_\mu V^\mu$ for some V vanishing at infinity, which would give us scale invariance but not conformal invariance (no problems with this; $T_{\mu\nu}$ isn't a conformal primary anyway). This is what happens e.g. in the theory of elasticity. That this situation is a bit contrived can be argued for in the following way: if $T^\mu_\mu = \partial_\mu V^\mu$, then since T must have dimension exactly d (spacetime dimension), V must have dimension exactly $d-1$, which is the dimension that Ward identities necessitate for a conserved current (and in unitary theories, the minimum possible dimension for a vector primary). However, V is by construction not conserved. Therefore, theories with this kind of obstruction to conformal invariance must have a vector field which has exactly zero anomalous dimension, despite not being a conserved current. This is a bit pathological since "everything not forbidden is compulsory", and in an interacting theory we expect all operators whose dimensions are not protected by a symmetry to pick up a nonzero anomalous dimension. Of course in free theories, like the theory of elasticity, this expectation doesn't hold, and indeed all the examples I know of theories which are scale but not conformal are free (pure Maxwell in $d = 3$ is another example)—hence the expectation is that scale implies conformal, except in some rather isolated and fine-tuned counterexamples.