When are CS theories spin TQFTs?

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In these notes we will answer the question in the title by working through a few representative examples. I'm sure this exists in the literature somewhere, but thought it would be worthwhile to work things out in detail. We will focus on using quantization of various topological terms, although one can also compute the spectrum of line operators and look for transparent fermions (although this is more difficult in general it seems).

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One way to examine whether a CS theory is spin or not is to carefully define the CS action by breaking up the manifold into patches and defining the action in the style of DB cohomology; see a previous diary entry on this. This approach is kind of subtle for non-Abelian gauge groups though, so we will take the bounding 4-manifold approach, which is computationally simpler.

$U(1)_k$

As usual, define the CS action on a closed 3-manifold X by integrating an $F \wedge F$ term over some 4-manifold Y with $\partial Y = X$. The exponential of the action is independent of the choice of bounding 4-manifold Y provided that

$$\frac{k}{8\pi^2} \int_X F \wedge F \equiv \frac{k}{2} I \in \mathbb{Z} \tag{1}$$

for all closed 4-manifolds M. Now, $F/2\pi \in H^2(M;\mathbb{Z})$, so we know for sure that $I \in \mathbb{Z}$ since the cup product of $F/2\pi$ with itself is then in $H^4(M;\mathbb{Z})$. Now if $k \in 2\mathbb{Z}$ then the above integral is independent of M mod $2\pi\mathbb{Z}$, regardless of whether M is spin or not. Thus if $k \in 2\mathbb{Z}$, the CS theory is insensitive to the spin structure and hence is bosonic. However, suppose $k \in 2\mathbb{Z}+1$. Then the CS action is only well-defined if $I \in 2\mathbb{Z}$. The constraint $I \in 2\mathbb{Z}\ \forall M$ can only be satisfied if we restrict our attention to M such that M is spin. If M is spin then $\omega_2(TM) = 0 \mod 2$ and the intersection form is even, meaning that I is always even. So, for odd k, the theory can only be defined using spin bounding 4-manifolds, and hence the original 3-manifold needs to come equipped with a spin structure as well. Thus odd k theories are spin TQFTs.

$SU(N)_k$

Now consider $SU(N)$. Now the relevant integral over a closed 4-manifold is

$$\frac{k}{8\pi^2}\int_M \mathrm{Tr}[F\wedge F] = k\mathrm{ch}_2(F) \in k\mathbb{Z}, \tag{2}$$

since the second Chern character is the second second Chern class for $SU(N)$ on account of the tracelessness of the $SU(N)$ generators, it is quantized on account of the second Chern class being a $\mathbb{Z}$ characteristic class (for $U(1)$, the integral is just the second Chern character, which is not a class in $\mathbb{Z}$ cohomology). Note that the quantization of the integral does not depend on whether M is spin or not: the second Chern class's integrality doesn't depend on the spin nature of M, since it does not (in general) compute an intersection form. Indeed, the minimal $\mathrm{ch}_2(F) = 1$ instantons are the "small" instantons that can exist on any manifold, regardless of its topology. They are constructed from bundles which are not tensor products of line bundles (if they were their quantization would be sensitive to $\omega_2(TM)$), and since they are "small" they can exist equally happily on spin- and non-spin manifolds. So, all the $SU(N)$ CS theories are bosonic.

$U(N)_{k,q}$

Now for $U(N)_{k,q}$, which is defined though

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathcal{A}\wedge d\mathcal{A} - \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathcal{A}]\wedge d\mathrm{Tr}[\mathcal{A}]. \tag{3}$$

As explained before, the notation is done like this because q is ($1/N$ times) the effective $U(1)$ level, while k is the effective $SU(N)$ level.

Now we use the decomposition $U(N) = [SU(N)\times U(1)]/\mathbb{Z}_N$. At the level of actions, we simply write $\mathcal{A} = A + \mathcal{A}\mathbf{1}$, where A is an $SU(N)$ field (whose transition functions may fail by Nth roots of unity; see e.g. Theta Time Reversal and Temperature), $\mathcal{A}$ is a $U(1)$ field (with transition functions failing in the inverse way). The quotient comes from the correlation of the transition functions between A and $\mathcal{A}$. In terms of these fields, we have

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right] + \frac{qN}{4\pi}\mathcal{A}\wedge d\mathcal{A}, \tag{4}$$

so that qN is indeed the "effective $U(1)$ level". The scare quotes here are because $\mathcal{A}$ isn't really a $U(1)$ field, because of the quotient: only $N\mathcal{A}$ is a legit $U(1)$ field. So the legit $U(1)$ part is really

$$S \supset \frac{2\pi q/N}{8\pi^2}\int_Y d(N\mathcal{A})\wedge d(N\mathcal{A}), \tag{5}$$

where Y is a bounding 4-manifold. This would seem to indicate that we require $q \in N\mathbb{Z}$ in order for the action to be well-defined (independent of Y). But this is not quite the case, since the term in $\mathcal{L}$ involving A also stands a chance of being ill-defined on its own, due to the $\mathbb{Z}_N$ quotient. Indeed, from our previous diary entry on instanton numbers in $PSU(N)$ gauge theory, we saw that $\frac{k}{2}\int \mathrm{Tr}(dA/2\pi\wedge dA/2\pi)$ was quantized in $k/N\mathbb{Z}$. Thus the ill-defined-ness of the A part of the action alone is captured by $k/N \mod 1$. Since the transition functions of A and $\mathcal{A}$ fail the cocycle condition in opposite senses at each triple overlap of patches, the fractional part of the instanton number for the A field is the negative of that for the $\mathcal{A}$ field. Thus the total parameter measuring the ill-defined-ness of the action is actually $(k-q)/N \mod 1$. So, for a consistent theory, we need

$$k - q \in N\mathbb{Z}. \tag{6}$$

Another way to say this is that since $\mathrm{Tr}[\mathcal{A}]$ is a well-defined $U(1)$ gauge field (but not $\mathcal{A}$ itself), the appearance of the term $(k-q)\mathrm{Tr}\mathcal{A} \wedge d\mathrm{Tr}\mathcal{A}/4\pi N$ in the action means that in order for this to be well-defined we need to have $(k-q)/N \in \mathbb{Z}$.

Yet another way to say it is that the theory needs to be invariant under simultaneous shifts in the transition functions of A and $\mathcal{A}$ by elements in $\mathbb{Z}_N$, which is realized on $\mathcal{A}$ through the shift $\delta\mathcal{A} = \frac{1}{N}d\phi$ for some 2π-periodic scalar ϕ. Since we are shifting both A and $\mathcal{A}$, $\mathcal{A}$ is invariant, and the action changes by

$$\delta S = \frac{(q-k)}{2\pi} \int d\phi \wedge F_{\mathcal{A}} \tag{7}$$

(for the derivation of the fact that the prefactor is $1/2\pi$ and not $1/4\pi$, see the previous diary entry). Now since only $N\mathcal{A}$ is a $U(1)$ gauge field, the flux of $F_{\mathcal{A}}$ is quantized in $\overline{\mathbb{Z}}/N$. Thus in order for $\delta S \in \overline{\mathbb{Z}}$, we need $(q-k) \in N\mathbb{Z}$.

Anyway, when are these theories spin? Returning to the original formulation in terms of the $\mathcal{A}$ field, the appropriate four-dimensional integral to compute is

$$I = \frac{1}{8\pi^2} \int \left(k\mathrm{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] + \frac{q-k}{N} \mathrm{Tr}[F_{\mathcal{A}}] \wedge \mathrm{Tr}[F_{\mathcal{A}}] \right). \tag{8}$$

Using the definition of the second Chern class,

$$I = 2\pi \int \mathrm{c}_2(E) + 2\pi \frac{k + (q-k)/N}{8\pi^2} \int d\mathrm{Tr}\mathcal{A} \wedge d\mathrm{Tr}\mathcal{A}, \tag{9}$$

where E is the total $U(N)$ bundle. Since $\int \mathrm{ch}_2(E) \in \mathbb{Z}$ on any closed 4-manifold (spin or not), whether or not the theory is spin is determined by the second term. In particular, we get

$$k + \frac{q-k}{N} \in \begin{cases} 2\mathbb{Z} & \implies \text{not spin} \\ (2\mathbb{Z}+1) & \implies \text{spin} \end{cases}, \tag{10}$$

where these are the only two options since as we said before, $(q-k) \in N\mathbb{Z}$.

$PSU(N)_k$

As we saw in a previous diary entry, on spin manifolds, minimal $PSU(N)$ bundles have instanton numbers that are in $\frac{1}{N}\mathbb{Z}$, and thus they are only defined when the level satisfies $k \in N\mathbb{Z}$. Since the fractional part of the instanton number came from the intersection number $\int B \wedge B$ of a 2-form $\mathbb{Z}_N$ gauge field, the fractional part of the instanton number will indeed depend on the existence of a spin structure: on non-spin manifolds we only have $I \in \frac{1}{2\mathbb{Z}}$. Thus $PSU(N)_k$ is spin if the level is an odd multiple of N ($k \in 2N\mathbb{Z} + N$), and non-spin if the level is an even multiple of N ($k \in 2N\mathbb{Z}$).

For example, take $PSU(2)_2 = SO(3)_2$: we obtain this from $SU(2)_2$ by identifying the representation 1 with the trivial representation. Now $SU(2)_2$ is the Ising theory, and 1 is the fermion. So, in order to identify 1 with 0, we need a spin structure. Thus $PSU(2)_2$ is a spin CS theory.

More generally, we know that the spin j line in $SU(2)_k$ has spin

$$\theta_j = \frac{j(j+1)}{k+2}. \tag{11}$$

When $k \in 2\mathbb{Z}$, we can take the quotient to $PSU(2)_k$. The maximal spin line with $j = k/2$ is the generator of the $\mathbb{Z}_2^{(1)}$ symmetry we need to quotient by, and from the above we see that it has spin $\theta_{k/2} = k/4$. Therefore for $k \in 4\mathbb{Z}+2$ the generator is a fermion, and so $PSU(2)_k$ is spin for $k \in 4\mathbb{Z}+2$. On the other hand, when $k \in 4\mathbb{Z}$ the generator is a boson, and so for such values of k, $PSU(2)_k$ is not spin.

$DW_{p,q}$ theory

In the notation of the diary entry on anomalies in CS theories, the $DW_{p,q}$ theory is

$$\mathcal{L} = \frac{p}{4\pi} a \wedge da + \frac{q}{2\pi} a \wedge db. \tag{12}$$

Writing the action as an integral over a bounding 4-manifold tells us that these theories are spin when p is odd, and non-spin when p is even. This matches with the discussion of the 1-form symmetries of the theory in the previous diary entry: the generator for the $\mathbb{Z}_q^{(1)}$ symmetry shifting b is a boson and not anomalous, while the generator U_a for the $\mathbb{Z}_l^{(1)}$, $l \equiv \gcd(p,q)$ symmetry shifting a has spin

$$s[U_a] = \frac{p}{2l^2} \mod 1. \tag{13}$$

This means that the spin of l copies of the charge operator is $s[U_b^l] = p/2 \mod 1$. Since l copies of the charge operator gives a line that has trivial statistics with everything, we see that if $p \in 2\mathbb{Z}$ we have no problem, while if $p \in 2\mathbb{Z}+1$ then the theory has a transparent fermion. However since the theory is spin if $p \in 2\mathbb{Z}+1$ the transparent fermion is trivial, and so U_b^l is a trivial line, as required.

$SO(N)_K$

The CS action for $SO(N)_K$ is written as

$$S = \frac{k}{8\pi} \int_M \mathrm{Tr}[F_A \wedge F_A], \tag{14}$$

where the trace is taken in the vector representation. Note the factor of $1/8\pi$ in front, which differs from the usual $1/4\pi$ we've seen so far—the reason for this is ultimately that the reality of the $SO(N)$ representations ensures a doubling of the index of the Dirac operator on M, which by the index theorem lets us relate the η invariant and the CS action with an extra factor of $1/2$ compared to the normal definition—more on this in another diary entry.

Anyway, requiring that the integral be independent of the bounding 4-manifold means that for all closed M, we need

$$2\pi k \frac{1}{2 \cdot 8\pi^2} \int_M \mathrm{Tr}[F_A \wedge F_A] = \pi k \int p_1(A), \tag{15}$$

where $p_1(A)$ is the first Pontryagin class. Now this is a legit $\mathbb{Z}$ characteristic class, but unlike the second Chern class, its quantization *does* depend on the type of manifold that it's on. In particular, the relation

$$p_1(A) = P(w_2(A)) + 2w_4(A) \mod 4 \tag{16}$$

tells us that $\int p_1(A) \in 2\mathbb{Z}$ on spin manifolds. Thus $k \in 2\mathbb{Z}$ theories make sense on any manifold and are not spin, while $k \in 2\mathbb{Z}+1$ theories are spin.

So in general, the coefficient in front of the CS Lagrangian ($k/4\pi$, $k/8\pi$, etc.) can be determined by looking at how the relevant characteristic class (Chern or Pontryagin) is quantized on different types of manifolds. We should pick it so that for all k, the 2+1D CS action is well-defined on spin manifolds (but may require special choices for k to be defined on non-spin manifolds).