RG diary

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August 12, 2020

1 *More WZW things* ✓

The goal of today's problem is to try to get more familiar with the WZW term, e.g. to see how it relates to various anomalies, to become acquainted with its symmetry properties, etc.

We will be working with WZW for $SU(2)_k$[1] :

$$S = \frac{k}{8\pi}\int_M \mathrm{Tr}(dg^\dagger \wedge \star dg) + \frac{ik}{12\pi}\int_B \mathrm{Tr}[\omega\wedge\omega\wedge\omega], \qquad \omega = g^\dagger dg. \tag{1}$$

As usual, M is some 2-manifold and B is a 3-ball bounded by M. First, we will show that the WZW part of the action transforms projectively under the global symmetry $g \mapsto gh$. We will then similarly define holomorphic and antiholomorphic currents, and show how they transform under $SU(2)_L \times SU(2)_R$.

We will then consider the conformal Ward identity, and find the OPE of the currents with themselves. After splitting up the currents with a mode expansion, we will then get the commutator algebra of the modes in the usual way (in radial quantization, using contour integrals).

We can see why the WZW term comes from cohomological thinking by seeing where it goes under a global symmetry transformation of $SU(2)_R$, with $g \mapsto gh$. Under this action, the MC form for g gets conjugated and spits out a MC form for h:

$$\omega \mapsto h^\dagger \omega h + \eta, \qquad \eta \equiv h^\dagger dh. \tag{2}$$

Putting this into S and expanding,

$$S_{WZW}[gh] = S_{WZW}[g] + S_{WZW}[h] + \frac{ik}{4\pi}\int \mathrm{Tr}[\eta\wedge\eta\wedge^h\omega + \eta\wedge^h\omega\wedge^h\omega], \qquad {}^h\omega \equiv h^\dagger\omega h. \tag{3}$$

Now $\eta\wedge\eta = -d\eta$, while

$$d^h\omega = -\eta^h\omega + h^\dagger d\omega h -^h \omega\wedge\eta = -\eta\wedge^h\omega -^h\omega\wedge\eta -^h\omega\wedge^h\omega. \tag{4}$$

Solving for ${}^h\omega\wedge^h\omega$ and putting this into our expression for $S_{WZW}[gh]$, we get

$$S_{WZW}[gh] = S_{WZW}[g] + S_{WZW}[h] + \frac{ik}{4\pi}\int d\mathrm{Tr}[\eta\wedge^h\omega]. \tag{5}$$

So, the symmetry acts linearly on the WZW action up to a boundary term. This is emblematic of SPTs / anomalies: when we consider the symmetry action on some open manifold

[1] Note that in the big yellow book, the coefficients in front of the integrals are different. This is because they use a trace with a different normalization (the trace we are using is not normalized by the Dynkin index; it is the straight-up trace).

(or submanifold), we get a linear representation of the symmetry up to a term supported on the boundary of that manifold (see earlier diary entries).

When we consider the kinetic S_k term as well, a similar computation shows

$$S_k[gh] = S_k[g] + S_k[h] - \frac{k}{4\pi}\int \mathrm{Tr}(\eta \wedge \star^h\omega). \tag{6}$$

When we add these two terms, the form that couples to η in the coboundary δS of the action, namely

$$\delta S(g,h) \equiv S[g] + S[h] - S[gh], \tag{7}$$

is exactly the same type of form defined in our RG analysis of the WZW model back in a previous diary entry.

To get something useful out of this, we need to switch to using ∂_z and $\partial_{\bar{z}}$. When we do this, $\delta S_k(g,h)$ becomes, after a bit of algebra

$$\delta S_k(g,h) = \frac{k}{2\pi}\int d^2x\, \mathrm{Tr}[\eta_z\omega_{\bar{z}} + \eta_{\bar{z}}\omega_z]. \tag{8}$$

On the other hand, the WZW part is

$$\delta S_{WZW}(g,h) = \frac{k}{2\pi}\int d^2x\, \mathrm{Tr}[\eta_z\omega_{\bar{z}} - \eta_{\bar{z}}\omega_z]. \tag{9}$$

Note that $\delta S_{WZW}(g,h)$ is real since the i in the original prefactor cancels with the i generated by going to ∂_z and $\partial_{\bar{z}}$ since $\delta S_{WZW}(g,h)$ only contains terms with one derivative in x and one in y (where $z = x + iy$). Putting these together, the total coboundary of the action is (switching from an integral over $d^2x = dx \wedge dy$ to one over $dz \wedge d\bar{z}$ at the cost of a factor of $i/2$),

$$\delta S(g,h) = \frac{ik}{2\pi}\int dz \wedge d\bar{z}\, \mathrm{Tr}[h^\dagger(\partial_z h)g^\dagger\partial_{\bar{z}}g]. \tag{10}$$

Note in particular that if h is holomorphic, then $\delta S(g,h) = 0$.

This was done for the action of $SU(2)_R$. If we instead consider the $SU(2)_L$ action $g \mapsto fg$, then we instead have

$$\omega \mapsto g^\dagger\lambda g + \omega, \qquad \lambda \equiv f^\dagger df. \tag{11}$$

Thus for the action of $SU(2)_L$, it is the Cartan form for the element doing the symmetry action (namely f) that gets conjugated, instead of ω. This ends up meaning that we end up getting the same thing for $\delta S_{WZW}(f,g)$, except that η and ${}^h\omega$ change places. Since they are both 1-forms, this gives us a minus sign, and so essentially the only thing that changes is that $\delta S_{WZW}(f,g)$ term gets a minus sign relative to $\delta S_k(f,g)$, which doesn't get a minus sign since it doesn't have wedge products. Thus we find that for the $SU(2)_L$ action,

$$\delta S(f,g) = \frac{k}{\pi}\int d^2x\, \mathrm{Tr}[f^\dagger(\partial_{\bar{z}}f)g^\dagger\partial_z g]. \tag{12}$$

Note that this vanishes if f is anti-holomorphic.

So, we reach the following conclusion: the $SU(2)_L$ symmetry is implemented anomalously (i.e. the action is only invariant up to a boundary term), unless the action is done by a holomorphic function $f(z)$. Likewise, the $SU(2)_R$ symmetry is implemented anomalously unless the action is done by an anti-holomorphic function $h(\bar{z})$. This tells us that conservation of the holomorphic current (the expression for which will be recalled shortly) comes from the $SU(2)_L$ symmetry, while conservation of the antiholomorphic current comes from the $SU(2)_R$ symmetry.

Now we turn to look at the currents. We will refer to a previous diary entry where we calculated these and found δS. We found that the holomorphic (left-moving, since it doesn't depend on $\bar{z}$) and the anti-holomorphic (right-moving, since it's independent of z) currents are[2]

$$J = -k(\partial_z g)g^\dagger, \qquad \bar{J} = kg^\dagger \partial_{\bar{z}} g. \tag{13}$$

Under infinitesimal $SU(2)_L$ transformations $g \mapsto g + \gamma g$, and infinitesimal $SU(2)_R$ transformations $g \mapsto g - g\bar{\gamma}$ (we will see why the minus sign is natural in a second), a quick calculation shows that the holomorphic current varies as

$$SU(2)_L: \ \delta J = [J, \gamma] - k\partial_z \gamma, \qquad SU(2)_R: \ \delta J = kg\partial_z \bar{\gamma} g^\dagger, \tag{14}$$

so that when the variation is holomorphic J is invariant under $SU(2)_R$. Similarly, the anti-holomorphic current transforms as

$$SU(2)_R: \ \delta \bar{J} = -[\bar{J}, \bar{\gamma}] - k\partial_{\bar{z}} \bar{\gamma}, \qquad SU(2)_L: \ \delta \bar{J} = kg^\dagger (\partial_{\bar{z}} \gamma) g. \tag{15}$$

As anticipated in our calculation of the coboundary δS, the holomorphic current is identified with the $SU(2)_L$ symmetry and the antiholomorphic one with the $SU(2)_R$ symmetry. The conservation of these two currents (which comes from the equations of motion as we saw in a previous diary entry) implies that we have the symmetry

$$g(z, \bar{z}) \mapsto \Gamma(z) g(z, \bar{z}) \bar{\Gamma}^{-1}(\bar{z}) \tag{16}$$

for any homolorphic (antiholomorphic) Γ ($\bar{\Gamma}$). We've chosen the right action to involve an inverse since it makes various formulae nicer later on and is the more natural choice. The infinitesimal version of this is, writing $\Gamma(z) = \mathbf{1} + \gamma(z)$,

$$g \mapsto g + \gamma g - g\bar{\gamma}. \tag{17}$$

Consider now the ward identity for some operator $\mathcal{O}$. To compute $\langle \delta \mathcal{O} \rangle$, we need to know the variation of the action. Fortunately this was also worked out in a previous diary entry, whose results we will steal. After going through the annoying step of switching to $\partial_z, \partial_{\bar{z}}$ we find for the variation (17)

$$\delta S = -\frac{k}{2\pi} \int d^2x \, \mathrm{Tr}[\gamma \partial_{\bar{z}} J + \bar{\gamma} \partial_z \bar{J}]. \tag{18}$$

[2]We're actually changing conventions slightly for the currents compared to the last diary entry—there we were following Altland and Simon's conventions, and here we follow more standard ones (which are conjugated by g and since we are no longer at $k = 1$ have a k in front)

We can pull the derivatives out of the trace for free, since γ and $\bar{\gamma}$ are killed by $\partial_{\bar{z}}$ and ∂_z, respectively. Then since $dx \wedge dy = (i/2) dz \wedge d\bar{z}$, we can go over to an integration over $dz \wedge d\bar{z}$ and then integrate the total derivatives to get

$$\delta S = -\frac{i}{4\pi} \oint dz \, \mathrm{Tr}[\gamma J] + \frac{i}{4\pi} \oint d\bar{z} \, \mathrm{Tr}[\bar{\gamma} \bar{J}], \tag{19}$$

where the relative minus sign comes from the fact that we're taking both of the $\oint$'s to be oriented right-handedly.

Following the usual procedure of performing a shift in integration variables in the path integral which computes $\langle \mathcal{O} \rangle$, we have, for $\gamma, \bar{\gamma}$ chosen to have compact support on some ball centered on w (in radial quantization),

$$\langle \delta \mathcal{O}(w) \rangle = -\frac{i}{4\pi} \oint dz \, \langle \mathrm{Tr}[\gamma J] \mathcal{O}(w) \rangle + \frac{i}{4\pi} \oint d\bar{z} \, \langle \mathrm{Tr}[\bar{\gamma} \bar{J}] \mathcal{O}(w) \rangle, \tag{20}$$

where the contours are taken on paths enclosing the point w.

We can get the OPEs for the currents by choosing $\mathcal{O} = J^a$, where $J = J^a t^a$ with t^a the Pauli matrices. Since $\bar{\gamma}$ is antiholomorphic J is invariant under $SU(2)_R$, and so we see from our earlier result that

$$\delta J^a = i f^{abc} J^b \gamma^c - k \partial_z \gamma^a, \tag{21}$$

where for us $f^{abc} = \epsilon^{abc}$. Note that when we put this in, the LHS of (20) will only contain γ, and so the OPE between J and $\bar{J}$ must only contain non-singular terms (since the $\bar{\gamma}$ term on the RHS needs to die). Thus (not writing the expectation value brackets and taking the trace)

$$i f^{abc} J^b(w) \gamma^c(w) - k \partial_w \gamma^a(w) = -\frac{i}{2\pi} \oint dz \, \gamma^b(z) J^b(z) J^a(w). \tag{22}$$

From this, we can read off the OPE for the holomorphic currents. the f^{abc} term on the LHS has no derivatives, so we need to pick it up with a $1/(z-w)$ pole. Since the $k\partial_w \gamma^a(w)$ term has one derivative, we need to match it with a $1/(z-w)^2 = -\partial_w (z-w)^{-1}$ term. So then taking into account the $2\pi i$'s from the residues, we deduce that the singular parts of the OPE are

$$J^a(z) J^b(w) \sim \frac{k \delta_{ab}}{(z-w)^2} + i f^{abc} \frac{J^c(z)}{z-w}. \tag{23}$$

This is the current algebra we've been looking for.

Now we can define the modes of the current by their "angular momentum", i.e. what we get when we integrate the current against z^n. Since $\partial_{\bar{z}} J^a = 0$ we can expand the current as

$$J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J_n^a. \tag{24}$$

Here we picture J_n^a as an operator acting at the origin in radial quantization that sets up an associated state. The choice of power ensures that J_n^a can be found by integrating $J^a(z)$ against z^n. The commutator of two modes is

$$[J_n^a, J_m^b] = \oint_0 dw \oint_w dz \, z^n w^m J^a(z) J^b(w). \tag{25}$$

Here the w integral is taken on a contour centered on the origin, while for a given w the z integral is taken on a contour that encloses w and the $z^n w^m$ selects out the desired components of the mode expansion. This is the usual thing one gets when writing the (radial) commutator: for each w we end up doing two contours surrounding the origin along circles or radii slightly larger / smaller than $|w|$, which we then deform into a small contour enclosing w.

Anyway, now we just insert the OPE into the integral on the LHS. Remembering that we get the residue of the second-order pole with $d_z[(z-w)^2(z^n w^m/(z-w)^2)]|_{z=w}$, we do the integrals and get

$$[J^a_n, J^b_m] = if^{abc}J^c_{n+m} + kn\delta_{a,b}\delta_{n,-m}. \tag{26}$$

The $\delta_{n,-m}$ comes since after doing the z integral we have an integral over w^{n-1-m} which sets $n = m$, with the prefactor of n coming from taking the derivative when finding the residue. Finally, that the mode J^c_{n+m} is selected out can be seen by plugging in the mode expansion for J^c and noting that the integrand $z^n w^{m-l-1}/(z-w)$ is only non-zero if $l = m+n$. This is the algebra we've been looking for. We also get a similar algebra for the antiholomorphic currents by putting $\bar{J}$ into the conformal Ward identity. Also note that since the $J\bar{J}$ OPE has to have no singular terms, the two algebras are decoupled:

$$[J^a_n, \bar{J}^b_m] = 0. \tag{27}$$

2 *Magic at the $SU(2) \times SU(2)$ radius* ✓

Today we are going to learn more about the compact scalar in two dimensions and its magical properties at special values of the compactification radius.

The action is

$$S = \frac{R^2}{4\pi}\int d^2z\, \partial X \bar{\partial} X. \tag{28}$$

From the equation of motion $\bar{\partial}\partial X = 0$ we can separate X as $X = X_L(z) + X_R(\bar{z})$ inside correlation functions and away from other operators.

Define the operators

$$J^3 = i\partial X, \qquad J^\pm =: e^{\pm i2X_L}: \tag{29}$$

where we are tacitly assuming that $J^\pm$ is well-defined. Using

$$\langle X(z)X(0)\rangle = -\frac{1}{R^2}\ln|z|, \tag{30}$$

we will find the value of R for which the vertex operators $J^\pm$ are well-defined and have scaling dimension 1. Letting

$$J^1 = \frac{1}{2}(J^+ + J^-), \qquad J^2 = \frac{1}{2i}(J^+ - J^-), \tag{31}$$

we will show that the algebra obeyed by the charge operators is the $SU(2)$ algebra.

First we find the scaling dimension of J^3 and $J^\pm$. For J^3, we just differentiate the $\ln|z|$ propagator twice and see that

$$\Delta_{J^3} = 1 \tag{32}$$

as is appropriate for a conserved current.

I think we've also found the scaling dimension for $J^\pm$ in an earlier diary entry, so I'll be somewhat brief. From

$$\langle X(z,\bar z)X(0)\rangle = -\frac{1}{2R^2}(\ln z + \ln \bar z), \tag{33}$$

we get

$$\langle X_L(z)X_L(0)\rangle = -\frac{1}{2R^2}\ln z, \qquad \langle X_R(\bar z)X_R(0)\rangle = -\frac{1}{2R^2}\ln \bar z, \qquad \langle X_L(z)X_R(0)\rangle = 0. \tag{34}$$

Now, the correlation function $\langle J^\pm(z)(J^\mp)^\dagger(0)\rangle = 0$ since we violate "charge neutrality". That is, if we have a current (which for us is $j = \sum_i \sigma_i \delta(z_i)$ for $\sigma_i \in \mathbb{Z}$) and expand $X_L = \sum_i \alpha_i X_i$ with $\int X_i X_j = \delta_{ij}$, then if we have a coupling term $\int j X_L$ in the action, the path integral over X_0 enforces $\int j = 0$, i.e. charge neutrality. Since we stick in such a $\int jX_L$ term when we are computing the scaling dimension of the vertex operators $J^\pm$, we see that the two-point function of the vertex operators is only nonzero when their charge is neutral. Then we can write

$$\langle e^{\pm i2X_L(z)}e^{\mp i2X_L(0)}\rangle = \exp\left(-\frac{1}{2}\int d^2w d^2w'\, j(w)G(w,w')j(w')\right), \tag{35}$$

where $j = 2(\pm\delta(z) \mp \delta(0))$. When we normal-order the vertex operators to compute the 2-point functions of the $J^\pm$ we kill the points in the integral with support at $w = w'$, and so using the expression for the propagator we found,

$$\langle J^\pm(z)(J^\pm)^\dagger(0)\rangle = \frac{1}{z^{2/R^2}}, \tag{36}$$

so that $\Delta_{J^\pm} = 1/R^2$. This means that the $J^\pm$ operators have scaling dimension 1 when $R = 1$, giving them a chance to be conserved currents. Note that in some places (several string theory books) it's $R = \sqrt{2}$—these differences come from different conventions for the $R^2/2\pi$ in the action. The fact that the scaling dimensions of $J^\pm$ and J^3 are all the same permits the existence of a symmetry which rotates them into one another. A hint of this extra symmetry comes from looking at the mass

$$m^2 = \frac{n^2}{R^2} + w^2 R^2 + \dots, \tag{37}$$

where $\dots$ is an oscillator contribution. n/R is the momentum (which comes in units of $1/R$ because of $x \sim x + 2\pi R$), while wR with w the winding number is the energy coming from the string tension (I believe with these conventions the string tension is $T = 1/2\pi$ so that $w2\pi RT = wR$). Sending $R \leftrightarrow 1/R$ and $n \leftrightarrow w$ is a symmetry of the spectrum, which acts as a self-duality when $R = 1$. When $R \neq 1$ we just have a $U(1) \times U(1)$ symmetry for the momentum and winding separately, but when $R = 1$ we will see that we get an $SU(2) \times SU(2)$ symmetry that rotates winding and momentum into one another.

Also note that in these conventions at $R = 1$ the 2-point function for charge-1 vertex operators is

$$\langle : e^{\pm i X_L(z)} :: e^{\mp i X_L(0)} : \rangle = \frac{1}{z^{1/2}}, \tag{38}$$

which is not single-valued. So for $R = 1$, only the charge 2 vertex operators are legit local operators.

Now let's compute OPEs between the various J's, specializing to the choice of $R = 1$ where both currents have the same scaling dimensions. The first one between two J^3's is easy since we just have to do one contraction:

$$J^3(z) J^3(w) \sim \frac{1}{2(z-w)^2}. \tag{39}$$

When we do the OPE for J^3 and $J^\pm$, we just have to contract the ∂X from J^3 with one of the X's from the expansion of the vertex operator: the contractions among the X's in the vertex operator are removed by the normal ordering. So since $\partial X(z, \bar{z}) = \partial(X_L(z) + X_R(\bar{z})) = \partial X_L(z)$, some algebra gives the OPE

$$J^3(z) J^\pm(w) = i\partial X_L(z) : e^{\pm i 2 X_L(w)} : \sim \pm \partial_z \ln(z-w) J^\pm(w) \sim \pm \frac{1}{z-w} J^\pm(w), \tag{40}$$

where as usual $\sim$ means equality up to non-singular terms (in this case, just the fully normal ordered term). This allows us to compute the OPE

$$J^3(z) J^1(w) \sim \frac{1}{z-w} \frac{J^+ - J^-}{2} \sim \frac{i J^2(w)}{z-w}. \tag{41}$$

Likewise,

$$J^3(z) J^2(w) \sim -\frac{i J^1(w)}{z-w}. \tag{42}$$

Now for the vertex operator OPEs. We find the OPE with the general prescription used to convert normal-ordering things to time-ordered things:

$$\mathcal{O}_1(z) \mathcal{O}_2(w) = \exp\left(-\frac{1}{2} \int dz' dw' \, \ln(z-w) \frac{\delta}{\delta X_L(z', 1)} \frac{\delta}{\delta X_L(w', 2)} \right) : \mathcal{O}_1(z) \mathcal{O}_2(w) :, \tag{43}$$

where we are still at $R = 1$ and where the $\mathcal{O}_i$ are functionals of X_L and the notation $X_L(z, i)$ means that the functional derivative acts only on $\mathcal{O}_i$. For example, we have

$$J^+(z)J^-(w) \sim \sum_{k=1}^{\infty} \frac{\ln^k(z-w)}{2^k k!}(-1)^k 2^{2k} : e^{2iX(z)} e^{-2iX(w)} : \sim \frac{1}{(z-w)^2} : e^{2iX(z)} e^{-2iX(w)} : \quad (44)$$

Now we can expand the $X(z)$ exponential about $X(w)$ since it's inside the normal ordering, and so

$$J^+(z)J^-(w) \sim \frac{1}{(z-w)^2} + \frac{2i\partial X(w)}{z-w}. \quad (45)$$

When we compute J^-J^+ the only difference is a minus sign when expanding the vertex operators inside the normal ordering, and so

$$J^-(z)J^+(w) \sim \frac{1}{(z-w)^2} - \frac{2i\partial X(w)}{z-w}. \quad (46)$$

When we compute the $J^\pm J^\pm$ OPEs, we get an extra $(-1)^k$ which cancels the one appearing in the J^+J^- OPE, which renders all of the terms non-singular, so that

$$J^\pm(z)J^\pm(w) \sim 0, \quad (47)$$

which we expect from the fact that the LHS is not charge-neutral. We then get

$$J^1(z)J^1(w) \sim J^2(z)J^2(w) \sim \frac{J^+(z)J^-(w) + J^-(z)J^+(w)}{4} \sim \frac{1}{2(z-w)^2}, \quad (48)$$

as well as

$$J^1(z)J^2(w) \sim -J^2(z)J^1(w) \sim \frac{J^-J^+ - J^-J^+}{4i} = -\frac{\partial X(w)}{z-w} = iJ^3(w). \quad (49)$$

Collecting these together, we get

$$J^a(z)J^b(w) = \frac{1}{2(z-w)^2}\delta_{ab} + i\epsilon^{abc}\frac{J^c}{z-w}, \quad (50)$$

for $a, b, c = 1, 2, 3$. The $SU(2)$-ness of this of course comes from the $(z-w)^{-1}$ term. Indeed, this term is responsible for making the algebra of the charges the $SU(2)$ algebra. To compute $[Q^a, Q^b]$, we do the usual trick: Q^aQ^b looks like two concentric circles in radial quantization radially separated by a small distance ϵ: we turn the two associated contour integrals to an integral like $\oint_0 dw \oint_w dz$ where the subscripts indicate the center of the contour, and since only the $1/(z-w)$ pole contributes to the integral we get

$$[Q^a, Q^b] = \oint_0 dw \int_w dz \, [J^a(w), J^b(z)] = i\epsilon^{abc} \oint_0 dw \int_w dz \frac{J^c(z)}{z-w} = i\epsilon^{abc}Q^c, \quad (51)$$

which is what we wanted.

We've only been dealing with the holomorphic part X_L, but the same story plays out for X_R. Since $X_L X_R \sim 0$, the L story and the R story are completely independent, and together they generate an $SU(2)_L \times SU(2)_R$ symmetry at the self-dual point.

This manifestation of the duality can be written in a perhaps more familiar form by writing down mode expansions for X_L and X_R and identifying the $m = 0$ term in the expansions, which are the momenta p_L, p_R of the modes. From $p \propto \int dz \partial X - \int d\bar{z} \bar{\partial} X$ (think: $\partial + \bar{\partial} \propto -i\partial_x$) and $w \propto \int dx^\mu \partial_\mu X \propto \int dz \partial X + \int d\bar{z} \bar{\partial} X$, one gets (still at $R = 1$)

$$p_L = n + w, \qquad p_R = n - w, \tag{52}$$

where we are at $\alpha' = 1$. Exchanging momentum and winding sends $p_L \to p_L, p_R \to -p_R$, so that the symmetry acts oddly on the antiholomorphic component. Therefore define the field $\widetilde{X} = X_L - X_R$ as a guess for what the image of the field X is under duality (or better, just take the definition from $\partial \widetilde{X} = \partial X, \bar{\partial} \widetilde{X} = -\bar{\partial} X$). This passes basic sanity checks since the fact that $X_L X_R \sim 0$ means that X and $\widetilde{X}$ have all the same correlation functions, the same stress-energy tensor, and so on. Then by acting on $\widetilde{X}$ with $\partial \pm \bar{\partial}$, we see that the fields are related as

$$d\widetilde{X} = \star dX, \tag{53}$$

which is exactly the type of duality we are familiar with from the particle-vortex duality approach (at $R = 1$ it is a self-duality). More on this and lots of other cool things from the string theory side to perhaps appear in future diary entries.

3 *T is not a conformal primary when $c \neq 0$, and the Hamiltonian on the cylinder* ✓

Today is quick and easy—doing a calculation that I've seen in many places but never worked out for myself.

By using our knowledge of the TT OPE, we will use the conformal Ward identity to show that under the conformal transformation ξ_μ, the stress tensor changes by

$$\delta_\xi \langle T \rangle = (\xi \partial + 2\partial \xi) T + \frac{c}{12} \partial^3 \xi, \tag{54}$$

which means that T is not a primary unless we are in the trivial case where $c = 0$ (the first two terms in parenthesis are the usual transformation rules for the holomorphic part of a two-index tensor primary).

Now let $z = e^w$ be the mapping from cylindrical coordinates $w = \sigma^0 + i\sigma^1$ (with $\sigma^1 \sim \sigma^1 + 2\pi$ the spatial coordinate—maybe not the best notation) to the plane where time increases radially. We will show that

$$T(w) = z^2 T(z) - \frac{c}{24}. \tag{55}$$

Then using

$$H = \int \frac{d\sigma^1}{2\pi} T_{00}, \tag{56}$$

we will see that the Hamiltonian on the cylinder is

$$H = L_0 + \bar{L}_0 - (c + \bar{c})/24. \tag{57}$$

The conformal Ward identity for an operator X says that

$$\delta_\xi \langle X \rangle = \int d^2\sigma\, \partial_\mu \langle T^{\mu\nu} \xi_\nu X \rangle = \oint d\sigma^\mu \langle \epsilon_{\mu\nu} T^{\nu\lambda} \xi_\lambda X \rangle. \tag{58}$$

When we go to complex coordinates we get a $-i/2$ out front from the change in the ϵ tensor, since it becomes $\epsilon = -Y/2$. We also get a factor of 4 when we lower the indices on $T^{\bar{z}\bar{z}}$ to T_{zz}, and a factor of $1/2$ when raising the index on $\xi_{\bar{z}}$ to $\xi \equiv \xi^z$. Finally, we get a $-1/2\pi$ from changing T_{zz} to $T = -2\pi T_{zz}$. So the conformal Ward identity in complex coordinates is

$$\delta_\xi \langle X(w) \rangle = \frac{1}{2\pi i} \oint dz\, \langle T \xi X(w) \rangle - \frac{1}{2\pi i} \oint d\bar{z}\, \langle \bar{T} \bar{\xi} X(w) \rangle. \tag{59}$$

Here the contour is taken to be a small circle enclosing $\mathrm{Supp}(X(w))$ (or the support of a suitably smeared version of X). Now we take $X = T$ and plug in the general form of the TT OPE (and use $T\bar{T} \sim 0$ to separate the holomorphic and antiholomorphic parts). We then expand $\xi(z)$ about $z = w$. Only the $\partial^3\xi(z-w)^3$ term is able to integrate to something nonzero with the $(c/2)/(z-w)^4$ term, while only the $\partial\xi(z-w)$ term is able to make the $2T/(z-w)^2$ term nonzero, and only the zeroth order term can make the $\partial T/(z-w)$ part nonzero. So after doing the integral we get

$$\delta_\xi \langle T \rangle = (\xi\partial + 2\partial\xi)T + \frac{c}{12}\partial^3\xi \tag{60}$$

as required.

Finding the finite version of this is a pain, but we can look it up in Polchinski:

$$T(w) = (\partial_w z)^2 T(z) + \frac{c}{12}\{z; w\}, \tag{61}$$

where $\{z; w\}$ is the Schwartzian derivative:

$$\{z; w\} = \frac{\partial_w^3 z}{\partial_w z} - \frac{3}{2}\left(\frac{\partial_w^2 z}{\partial_w z}\right)^2. \tag{62}$$

Thus the stress tensor isn't a conformal primary unless $c = 0$. Checking that this Schwartzian derivative formula works is straightforward and unilluminating so I won't write it out.

We now need to do the coordinate transformation $z = e^w$ between the plane (z) and the cylinder (w) to get $T(w)$. The Schwartzian derivative in this case is easy:

$$\{z; w\} = -1/2. \tag{63}$$

So putting this in, the cylinder stress tensor is

$$T(w) = z^2 T(z) - \frac{c}{24}, \tag{64}$$

which is what we wanted.

Now for the Hamiltonian. We write it as

$$H = \frac{1}{2\pi i} \int (dw\, T(w) - d\bar{w}\, \bar{T}(\bar{w})) = \frac{1}{2\pi i} \int (dz\, z^{-1} T(w) - d\bar{z}\, \bar{z}^{-1} \bar{T}(\bar{w})), \tag{65}$$

since $dz = z dw$. Using the transformation law for T, this is

$$H = \frac{1}{2\pi i} \int \left(dz\, z^{-1} [z^2 T(z) - c/24] - d\bar{z}\, \bar{z}^{-1} [\bar{z}^2 \bar{T}(\bar{z}) - \bar{c}/24] \right). \tag{66}$$

The integral selects out the $n = 0$ components of the Laurent expansions of $T, \bar{T}$ due to the conventions on shifting the powers by 2 in the expansion. The central charge pieces are integrated against $1/z, 1/\bar{z}$ and so they survive, and then since $\int dz/z = -\int d\bar{z}/\bar{z}$, we get

$$H = L_0 + \bar{L}_0 - (c + \bar{c})/24, \tag{67}$$

as expected. That the Hamiltonian has the $L_0 + \bar{L}_0$ part is no surprise, since this is the operator that generates dilitations: since $L_0 = \frac{1}{2\pi i} \oint dz\, zT$, for a primary X we have

$$L_0 X = \frac{1}{2\pi i} \oint dz\, z \frac{h X(w)}{(z - w)^2} = h X(w), \tag{68}$$

so that indeed, L_0 performs the dilitations, and thus belongs in H since dilitations in the plane in radial quantization are the same as time evolution. Thus on the cylinder H still does dilitations, but it is accompanied by a constant piece that keeps track of a (physically meaningful) vacuum energy.[3]

[3]Normally we don't think of keeping track of a constant term in H as being a meaningful thing to do. Here though, the point of keeping track of it is to compare the differences in vacuum energies with different boundary conditions; more on this later.

4 *Linear dilaton CFT* ✓

This is an exercise John McGreevy assigned to his QFT class. Consider the linear dilaton CFT, which is a free scalar plus a coupling to gravity:

$$S = \frac{1}{2\pi\alpha'}\int d^2x\,\sqrt{g}\,\partial X\bar{\partial}X + \frac{1}{2\pi}\int d^2x\,\sqrt{g}\,QXR, \tag{69}$$

where Q is a constant, which may be either real or imaginary (if we are thinking about strings we should be writing out the spacetime index on the X like $Q_\mu X^\mu$, but for this problem we will just think of X as a scalar) and R is the two-dimensional Ricci curvature scalar.

We will show that in flat space, while the coupling to gravity doesn't affect the equations of motion, it does change the stress tensor, which is

$$T = -\frac{1}{\alpha'} :\partial X\partial X: + Q\partial^2 X. \tag{70}$$

We will then find the central charge by computing the TT OPE, and check that the TT OPE has the right form for stress tensors in CFTs. Lastly we will compute the scaling dimension of the vertex operator $:e^{ikX}:$.

In flat space, which we will be working in, the extra term in the action vanishes, so we can use the usual $-\alpha'\ln|z-w|^2/2$ propagator in what follows. However the stress tensor *will* change, since it comes from varying the metric away from flat space. Since there is a $\sqrt{g}$ in the measure, we need to compute the variation

$$\delta(\sqrt{g}R) = \delta(\sqrt{g}g^{\nu\mu}R_{\mu\nu}) = (\delta\sqrt{g})R + \sqrt{g}R_{\mu\nu}\delta g^{\nu\mu} + \sqrt{g}g^{\nu\mu}\delta R_{\mu\nu}, \tag{71}$$

with $R_{\mu\nu}$ the Ricci tensor. The variation of $\sqrt{g}$ is the usual

$$\delta\sqrt{g} = \frac{1}{2\sqrt{g}}\delta e^{\mathrm{Tr}\ln g} = \frac{1}{2}\sqrt{g}\mathrm{Tr}[g^{-1}\delta g] = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\nu\mu}, \tag{72}$$

where in the last step we used $\delta(g^{\mu\nu}g_{\nu\lambda}) = \delta(\delta^\mu{}_\lambda) = 0$. The variation of the Ricci scalar is more heinous to derive, but luckily it can be found in Wald, in section 7.5 (the one on perturbations). It turns out that $g^{\nu\mu}\delta R_{\mu\nu}$ is a total derivative:

$$g^{\nu\mu}\delta R_{\mu\nu} = \nabla^\mu[\nabla^\nu(\delta g_{\mu\nu}) - g^{\lambda\sigma}\nabla_\mu(\delta g_{\lambda\sigma})] \equiv \nabla^\mu v_\mu. \tag{73}$$

Thus the variation of the dilaton term with respect to the metric is

$$\begin{aligned}\delta S_Q &= \frac{1}{2\pi}\int d^2x\,QX\sqrt{g}\left(\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} + \nabla^\mu v_\mu\right) \\ &= \frac{1}{2\pi}\int d^2x\,QX\sqrt{g}(G_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu v_\mu),\end{aligned} \tag{74}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{75}$$

is the Einstein tensor.

Here's where being in two dimensions helps: because of the symmetries of the Riemann curvature tensor, it has only one independent component in two dimensions. That is, since $R_{\mu\nu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$ and $R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$, the only independent non-zero component is R_{0101}. Then the Ricci tensor is

$$R_{\mu\nu} = R_{0101} \begin{pmatrix} g^{11} & -g^{01} \\ -g^{10} & g^{00} \end{pmatrix} = \frac{R_{0101}}{\det g} g_{\mu\nu}, \tag{76}$$

since the matrix is $(\det g^{-1})(g^{\mu\nu})^{-1}$. On the other hand, we can contract the Ricci tensor explicitly and see that the curvature scalar is

$$R = 2 \frac{R_{0101}}{\det g}, \tag{77}$$

so that

$$R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}, \tag{78}$$

which implies the vanishing of the Einstein tensor in two dimensions, $G_{\mu\nu} = 0$. Thus from (74) we see that if we were to ignore the scalar X so that we just had gravity, we can conclude that in two dimensions the variation of the Einstein-Hilbert action is actually a total divergence, and so vanishes on closed spacetimes. This is because we should think of the Einstein-Hilbert action in this case as being $\int F$ and measuring the topology of the spacetime (more precisely, $\sqrt{g}R$ is the Euler density, so that the Einstein-Hilbert action computes the Euler characteristic), and hence it is locally a total derivative.

Now we integrate the remaining $X \nabla^\mu v_\mu$ term by parts. We get, still in flat space,

$$\delta S_Q = \frac{1}{2\pi} \int d^2x \, Q(\partial^\mu \partial^\nu X - \partial^\lambda \partial_\lambda X g^{\mu\nu}) \delta g_{\mu\nu}, \tag{79}$$

and so with the identification $\delta S = -\frac{1}{2} \int T^{\mu\nu} \delta g_{\nu\mu}$, we see that S_Q gives a contribution to the stress tensor of

$$T_Q^{\mu\nu} = \frac{1}{\pi} Q(g^{\mu\nu} \partial_\lambda \partial^\lambda - \partial^\mu \partial^\nu) X = \frac{1}{\pi} Q \Pi_T^{\mu\nu} X. \tag{80}$$

Now we switch over to complex coordinates and use the definition $T = -2\pi T_{zz} = -(\pi/2) T^{\bar{z}\bar{z}}$ to get

$$T_{zz} = \frac{-Q}{2\pi} (\partial_0^2 - \partial_1^2 - 2i \partial_0 \partial_1) X, \tag{81}$$

so that the dilaton contribution to the holomorphic part of the stress tensor is

$$T_Q = Q \partial^2 X. \tag{82}$$

Now we want to find the central charge. We thus need to look at the $1/(z-w)^4$ term in the $T(z)T(w)$ OPE. This part only comes from terms that have been fully contracted, since we need four derivatives acting on propagators to give a $1/(z-w)^4$ term. The usual piece is

$$\begin{aligned}\frac{1}{\alpha'^2}2(\langle\partial X(z)\partial X(w)\rangle)^2 &= \frac{1}{2}\left(\partial_z\partial_w \ln|z-w|^2\right)^2 \\ &= \frac{1/2}{(z-w)^4},\end{aligned} \tag{83}$$

which gives the usual contribution of 1 to the central charge. The other piece is

$$Q^2\langle\partial_z^2 X(z)\partial_w^2 X(w)\rangle = \frac{\alpha'}{2}\frac{6}{(z-w)^4}, \tag{84}$$

so that

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + O(1/z^2), \qquad c = 1 + 6\alpha' Q^2. \tag{85}$$

Note that the central charge gets *reduced* by the introduction of the dilaton coupling if Q is imaginary.

Now we should check to make sure that the $O(1/z^2)$ in the above equation really is $O(1/z^2)$, and not $O(1/z^3)$ since otherwise we're in trouble. Recall that the form of the TT OPE needs to be

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{z^2}T(w) + \frac{1}{z}\partial T(w). \tag{86}$$

One potentially troublesome term in the linear dilaton TT OPE is

$$-\frac{1}{\alpha'}Q\langle\partial_z X(z)\partial_w^2 X(w)\rangle\partial_z X(z) = \frac{2Q}{\alpha'}\frac{\alpha'}{2}\partial_z\partial_w^2 \ln|z-w|^2\partial_z X(z) = \frac{2Q\partial_z X(z)}{(z-w)^3}. \tag{87}$$

What saves us from this term is the other contraction with $z \leftrightarrow w$, since the denominator is odd under this shift. The two cross-contractions between the Q term and the regular term thus give

$$\frac{2Q(\partial_z X(z) - \partial_w X(w))}{(z-w)^3} \to \frac{2Q\partial_z^2 X(z)}{(z-w)^2}, \tag{88}$$

which is exactly the term we need for the $2T(w)/z^2$ piece of the TT OPE. It is straightforward to check that the remainder of the $(z-w)^{-2}T(w)$ term and the $(z-w)^{-1}\partial T(w)$ term are produced as well.

Let's now find the conformal dimension of the vertex operator by taking the OPE with T. We need to only look at the leading piece of the OPE which goes as $1/(z-w)^2$. The part from the $:\partial X\partial X:$ term can be found by noting that $1/(z-w)^2$ terms only occur when both ∂X operators are contracted with the vertex operator. So after figuring out the combinatorial factors,

$$-\frac{1}{\alpha'}:\partial X\partial X::e^{ikX}:= \frac{1}{\alpha'}k^2\left(\frac{\alpha'}{2}\partial_z \ln|z-w|^2\right)^2\sum_{j=0}^{\infty}\frac{1}{(j+2)!}\binom{j+2}{2}2(ik)^j:X(w)^j:+O(1/z), \tag{89}$$

so that the relevant term is

$$-\frac{1}{\alpha'} : \partial X \partial X :: e^{ikX} : = \frac{k^2\alpha'/4}{(z-w)^2} : e^{ikX} : + O(1/z), \tag{90}$$

which gives a contribution of $k^2\alpha'/4$ to the conformal dimension. The contribution from the dilaton term is

$$ikQ \sum_{j=0}^{\infty} \frac{1}{j!} \langle \partial^2 X(z) X(w) \rangle : (iX(w))^j : = \frac{ik\alpha' Q/2}{(z-w)^2} : e^{ikX} :, \tag{91}$$

which gives a contribution of $ik\alpha' Q/2$ to the conformal dimension. Thus the conformal dimension of the vertex operator is

$$h = \frac{k^2\alpha'}{4} + \frac{ik\alpha' Q}{2}. \tag{92}$$

This gives a real conformal dimension if Q is imaginary. If Q is real the scaling dimension Δ is unaffected by the coupling to the dilaton, while the conformal dimensions and spin are rendered imaginary.

5 *Bosonization and torus partition functions* ✓

I got this problem from a pset that John McGreevy assigned to his QFT class and posted online. Update: actually, looks like a large part of this is in the big yellow book (of course).

We fill find the partition function for a free compact boson on a torus with sides given by the complex numbers ω_1, ω_2. We will then show that this is the same as the partition function for a Dirac fermion on the torus, with all spin structures taken into account.

Let the two sides of the torus be given by $\omega_1 \in \mathbb{R}$, $\omega_2 \in \mathbb{C}$, with modular parameter $\tau = \omega_2/\omega_1$. The time evolution operator along a direction parallel to ω_2 for a "time" s is

$$U(s) = \exp\left(-\frac{s}{|\omega_2|} \left[\mathrm{Im}(\omega_2) H - i\mathrm{Re}(\omega_2) P \right] \right). \tag{93}$$

The signs are the way they are since Schrodinger decided that H acts as $+i\partial_t$ (so that we translate in time with $e^{-iHt} \to e^{-\tau H}$) while P acts as $-i\partial_x$ (so that we translate with e^{+iP}).

We can use the expression we found for H two days ago, namely $H = L_0 + \bar{L}_0 - (c + \bar{c})/24$. This was derived for a cylinder of radius 1, and so if the radius is instead ω_1, then the Hamiltonian is (by dimensional analysis)

$$H = \frac{2\pi}{\omega_1}\left(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}\right). \tag{94}$$

Since we are specializing to a free compact boson, we will just write the last part as $(c + \bar{c})/24 = 1/12$ in what follows.

On a cylinder of radius 1, we get the momentum operator from

$$P = \int d\sigma\, T_{\tau\sigma} = i\frac{1}{2\pi i}\int (dw\, T(w) + d\bar{w}\, \bar{T}(w)). \tag{95}$$

Recalling from two days ago how to map $T(w)$ onto the plane, we get

$$P = i\frac{1}{2\pi i}\oint \left(z^{-1}dz\, z^2 T(z) + \bar{z}^{-1}d\bar{z}\, \bar{z}^2\bar{T}(z) + (z^{-1}dz + \bar{z}^{-1}d\bar{z})\frac{c}{24}\right). \tag{96}$$

The central charge term dies since the contour with dz is minus the one with $d\bar{z}$, while the other integrals select out the $n = 0$ component of the Laurent expansions for the stress tensors, which enter with opposite sides. When on a cylinder of radius ω_1 then,

$$P = \frac{2\pi i}{\omega_1}(L_0 - \bar{L}_0). \tag{97}$$

The partition function is then

$$Z = \mathrm{Tr}\left[e^{2\pi i(L_0+\bar{L}_0)\frac{\tau-\bar{\tau}}{2}} e^{2\pi i(L_0-\bar{L}_0)\frac{\tau+\bar{\tau}}{2}} e^{-2\pi i c\frac{\tau-\bar{\tau}}{24}}\right]. \tag{98}$$

This can be cleaned up by defining

$$q \equiv e^{2\pi i\tau}. \tag{99}$$

Then we have

$$Z = \mathrm{Tr}\left[q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}\right]. \tag{100}$$

To do the trace, we need the spectrum of L_0. The action is

$$S = \frac{1}{8\pi}\int dxdt\, [(\partial_t X)^2 - (\partial_x X)^2]. \tag{101}$$

Now we expand in Fourier modes, momentarily ignoring winding number issues,

$$X = \sum_n X_n e^{ixn/r}, \tag{102}$$

where r is the radius of the spatial circle—for us, $\omega_1 = 2\pi r$. In what follows, all sums over roman letter variables will be sums over $\mathbb{Z}$. The momentum is $\pi_n = \partial_t X_{-n} r/2$, and the Hamiltonian is

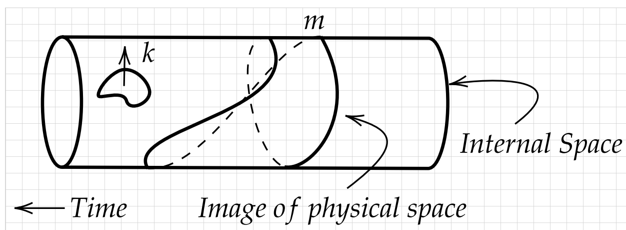
$$H = \frac{1}{r}\sum_n \left(\pi_n\pi_{-n} + \frac{n^2}{4}X_n X_{-n}\right). \tag{103}$$

This is a bunch of harmonic oscillators—from the momentum term we see that $m = 1/2$, so that the frequency of each oscillator is just $\omega_n = |n|$.

We will use this result in a bit, but first we will need to remember topological issues. Let us treat the zero mode X_0 separately from the other modes in the sum. From the commutator $[X_0, H] = 2i\pi_0/r$, we can get the time evolution of the zero mode. This further simplifies, since the compact nature of the boson forces $\pi_0 = k/R$ for $k \in \mathbb{Z}$.[4] We also need to add in a term that keeps track of the winding number of X: it is Rmx/r (with $X \sim X + 2\pi R$ defining the boson radius), which shifts as $Rmx/r \mapsto Rmx/r + 2\pi Rm$ around the spatial circle. Thus the decomposition for X

$$X(x,t) = X_0(0) + \frac{2k}{r}\tau + \frac{Rm}{r}x + \sum_{n\neq 0} X_n(t)X_{-n}(t). \tag{104}$$

As a picture, the setup is



Here the circle is the internal coordinate of the field, while the axial direction of the cylinder is time. Different images of the physical spatial circle under X are shown, one with winding number 0 and canonical momentum $\pi_0 \sim k$, and the other with winding number m (note that in a string theory context, we would replace "physical spacetime" with "worldsheet" and "internal space" with "spatial manifold".

Now we need to go to complex coordinates. There are many options, but it seems like the best choice are

$$z = e^{(\tau+ix)/r}, \qquad \bar{z} = e^{(\tau-ix)/r}. \tag{105}$$

This is kind of unpleasant since $\tau = it$ is imaginary time, but oh well—it ends up giving the answer in the form written in the problem statement. The $1/r$ in the exponents is so that r disappears from the final expression for X. With this choice of coordinates

$$x = -\frac{ir}{2}(\ln z - \ln \bar{z}), \qquad t = -\frac{ir}{2}(\ln z + \ln \bar{z}). \tag{106}$$

Putting this into $X(t)$:

$$X(z,\bar{z}) = X_0 - i(\ln z)p_L - i(\ln \bar{z})p_R + i\sum_{n\neq 0}\frac{1}{n}(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}), \tag{107}$$

where the momenta are

$$p_L = \left(\frac{k}{R} + \frac{mR}{2}\right), \qquad p_R = \left(\frac{k}{R} - \frac{mR}{2}\right). \tag{108}$$

[4]Remember that π_0 is the canonical momentum, not the physical kinematic momentum (the latter being defined through T_{01}).

We can now get expressions for $L_0, \bar{L}_0$ by using $T = -\frac{1}{2} : \partial X \partial X :$. So taking the derivative, we have

$$L_0 = \frac{1}{4\pi i} \oint dz\, z \left(p_L^2 z^{-2} + \sum_{i,j\neq 0} a_i a_j z^{-i-j-2} + 2p_L \sum_{j\neq 0} a_j z^{-j-2} \right). \tag{109}$$

The last term dies while in the second term i gets set to $-j$, so

$$L_0 = \frac{1}{2} p_L^2 + \sum_{j>0} a_{-j} a_j. \tag{110}$$

Similarly,

$$\bar{L}_0 = \frac{1}{2} p_R^2 + \sum_{j>0} \bar{a}_{-j} \bar{a}_j. \tag{111}$$

Here the sum over $j > 0$ means a sum from $\mathbb{Z} \ni j = 1$ to $j = \infty$.

We can now finally get the partition function. For each oscillator mode j, the sum over occupation numbers gives $1/(1-q^j)$. This is because as we saw earlier, the frequency of the jth mode is simply $|j|$. So the oscillator contribution to q^{L_0} is $\prod_{j>0}(1-q^j)^{-1}$. The zero modes are accounted for just by summing over all momenta k and winding numbers m, and so since the antiholomorphic oscillator contribution is the conjugate of the holomorphic contribution,

$$Z(q) = \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}, \tag{112}$$

where

$$\eta(q) \equiv q^{1/24} \prod_{j>0} (1-q^j). \tag{113}$$

Note that the theory is self-dual at the radius $R = \sqrt{2}$ (not $1/\sqrt{2}$ like earlier because of how we defined the coupling constant for the action). In the following we will consider the radius $R = 1$, which from earlier diary entries we know to be a value for which a fermion description works. Explicitly, at this radius we have

$$Z(q; R=1) = \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m/2)^2} \bar{q}^{\frac{1}{2}(k-m/2)^2}. \tag{114}$$

Our goal now is to relate this to fermions. First we break up the sum into m even and m odd:

$$Z(q; R=1) = \frac{1}{|\eta(q)|^2} \sum_{k,m} \left(q^{\frac{1}{2}(k+m)^2} \bar{q}^{\frac{1}{2}(k-m)^2} + q^{\frac{1}{2}(k+m+1/2)^2} \bar{q}^{\frac{1}{2}(k-m-1/2)^2} \right). \tag{115}$$

The first term is actually

$$\begin{aligned} \frac{1}{2|\eta(q)|^2} \left(\left| \sum_k q^{k^2/2} \right|^2 + \left| \sum_k (-1)^k q^{k^2/2} \right|^2 \right) &= \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}k^2} \bar{q}^{\frac{1}{2}m^2} \frac{1+(-1)^{k+m}}{2} \\ &= \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m)^2} \bar{q}^{\frac{1}{2}(k-m)^2}, \end{aligned} \tag{116}$$

since the term $\frac{1+(-1)^{k+m}}{2}$ projects onto configurations where $k = m$ mod 2, which is exactly fulfilled by the pair $k+m, k-m$ (we've re-labeled the summation variables—the point is that for any integers k, m, the combination $k+m, k-m$ survives the projection by $(1+(-1)^{k+m})/2)$.

We can use a similar trick for the second term in $Z(q; R=1)$: since the members of the combination $k+m, k-m$ always have the same parity, we can change the sum to run over all integers $x = k+m, y = k-m$ such that $x = y$ mod 2. We can then instead sum over *all* x, y, provided that we insert the projector $(1+(-1)^{x+y})/2$. Doing this, and then relabeling $x \to k$, $y \to m$ for consistency of notation, we have

$$\frac{1}{|\eta(q)|^2}\sum_{k,m} q^{\frac{1}{2}(k+m+1/2)^2}\bar{q}^{\frac{1}{2}(k-m-1/2)^2} = \frac{1}{2|\eta(q)|^2}\sum_{k,m} q^{\frac{1}{2}(k+1/2)^2}\bar{q}^{\frac{1}{2}(m+1/2)^2}(1+(-1)^{k+m}), \quad (117)$$

Consider the second term, proportional to $(-1)^{k+m}$. The exponential part is symmetric under the shift $k \mapsto -k-1$, but the $(-1)^{k+m}$ part picks up a minus sign, and so the second term is zero. Thus we can only keep the first term of this bit. It too factors between holomorphic and antiholomorphic contributions, and so the full partition function is

$$Z(q; R=1) = \frac{1}{2|\eta(q)|^2}\left(\left|\sum_k q^{k^2/2}\right|^2 + \left|\sum_k (-1)^k q^{k^2/2}\right|^2 + \left|\sum_k q^{\frac{1}{2}(k+1/2)^2}\right|^2\right). \quad (118)$$

This is starting to look more fermiony! The three terms are theta functions, and so we can write Z more compactly as (preserving the order of the terms as in the last equation)

$$Z(q; R=1) = \frac{1}{2|\eta(q)|^2}\left(|\theta_3(\tau)|^2 + |\theta_4(\tau)|^2 + |\theta_2(\tau)|^2\right). \quad (119)$$

The theta functions can be written as infinite products—see e.g. Polchinski. The product form for the theta functions all contain a factor that cancels the product in the $1/|\eta(q)|^2$ in the denominator, and leaves us with

$$Z(q; R=1) = \frac{1}{2|q^{1/24}|^2}\left(\prod_{j>0}|(1+q^{j-1/2})^2|^2 + \prod_{j>0}|(1-q^{j-1/2})^2|^2 + |q^{1/8}|^2\prod_{j\geq 0}|(1+q^j)^2|^2\right). \quad (120)$$

Our goal is in sight, since we are seeing the different boundary conditions for the fermions appearing.

To keep going, we will need the fermion partition functions. The Hamiltonian for the fermions is derived in the same way as the boson Hamiltonian, which we already did above. It will give us an oscillator contribution coming from the $L_0, \bar{L}_0$ operators, as well as a central charge piece that appears when we switch from the cylinder to the plane. Let's focus on a single real fermion with antiperiodic boundary conditions around the spatial circle: the central charge is $1/2$, while the oscillator expansion gives

$$L_0 = \sum_{k>0}(k-1/2)\lambda_k\lambda_{-k}, \quad (121)$$

where the λ_k's are Majorana operators and the sum is offset by 1/2 to get the boundary conditions right (note: our convention is such that the action for a single real fermion is $\frac{1}{4\pi}\int \psi\gamma^0 \partial\!\!\!/ \psi$). So, using the same logic that we used for the compact boson, the partition function for antiperiodic boundary conditions around both cycles is (NSNS / BB spin structure; B for "bounding")

$$Z_{NSNS}(q) = \mathrm{Tr}_A[q^{L_0 - 1/48}\bar{q}^{\bar{L}_0 - 1/48}] = \frac{1}{|q^{1/48}|^2}\prod_{k>0}|1+q^{k-1/2}|^2. \tag{122}$$

When we work with periodic boundary conditions around the temporal cycle, we need to insert $(-1)^F$ to implement the supertrace. This sends $q \to -q$ in the above and so

$$Z_{NSR}(q) = \mathrm{Tr}_A[(-1)^F q^{L_0 - 1/48}\bar{q}^{\bar{L}_0 - 1/48}] = \frac{1}{|q^{1/48}|^2}\prod_{k>0}|1-q^{k-1/2}|^2. \tag{123}$$

When we have periodic boundary conditions in space, the form for L_0 changes in two ways: first, the momenta live in $\mathbb{Z}$ rather than $\mathbb{Z}+\frac{1}{2}$, and second, they change by a constant since the vacuum energy on the cylinder depends on the boundary conditions (see e.g. chapter 6 of the big yellow book). In particular,

$$L_0 = \sum_{k\geq 0}\lambda_{-k}\lambda_k + \frac{1}{16}. \tag{124}$$

Thus for the RNS (periodic in space, antiperiodic in time) spin structure, the partition function is

$$Z_{RNS}(q) = \mathrm{Tr}_P[q^{L_0 - 1/48}\bar{q}^{\bar{L}_0 - 1/48}] = \frac{1}{|q^{1/48-1/16}|^2}\prod_{k\geq 0}|1+q^k|^2 = |q^{1/24}|^2\prod_{k\geq 0}|1+q^k|^2. \tag{125}$$

Finally, when we do the RR torus, we get zero because of the zero mode: when we change the thing in the product to $1-q^k$, the $k=0$ term kills the partition function. So $Z_{RR}(q)=0$.

A massless Dirac fermion for us is the same as two real fermions which are completely independent expect for the requirement that their boundary conditions match. Thus to get $Z_{\mathrm{Dirac}}(q)$ (with all spin structures counted), we just need to sum the squares of the real fermion partition functions over spin structures. This gives

$$\begin{aligned}
Z_{\mathrm{Dirac}}(q) &= \sum_{r,s\in\{NS,R\}} Z_{rs}(q)^2 \\
&= \frac{1}{2}\left(\frac{1}{|q^{2/48}|^2}\prod_{k>0}|(1+q^{k-1/2})^2|^2 + \frac{1}{|q^{2/48}|^2}\prod_{k>0}|1-q^{k-1/2}|^2 + |q^{2/24}|^2\prod_{k\geq 0}|1+q^k|^2 + 0^2\right) \\
&= \frac{1}{2|q^{1/24}|^2}\left(\prod_{j>0}|(1+q^{j-1/2})^2|^2 + \prod_{j>0}|(1-q^{j-1/2})^2|^2 + |q^{1/8}|^2\prod_{j\geq 0}|(1+q^j)^2|^2\right) \\
&= Z(q;R=1).
\end{aligned} \tag{126}$$

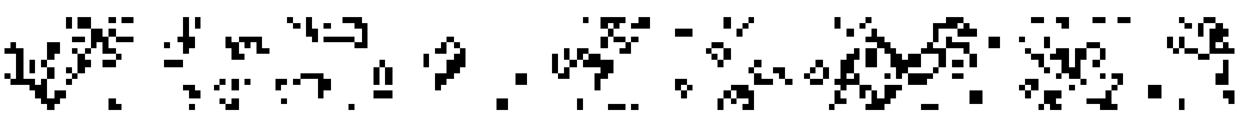
Thus we've shown that the compact boson on the torus (at $R = 1$) is the same as a Dirac fermion, which in turn is a pair of Ising models coupled together in a certain way.

Note that for this correspondence to work, we have to sum over all spin structures for the fermion (of course, in order to have modular invariance we needed to sum over all spin structures since they [except RR] are permuted into one another by modular transformations. Checking the modular invariance of the final expression can be done by looking up the transformation properties of the theta functions, see e.g. Polchinski). Note that we can see here why people talk about modular invariance giving you information about how the holomorphic and anti-holomorphic sectors talk to each other, information which isn't available on the plane. Indeed, if the two sectors didn't talk to each other then we would have $Z_{\text{Dirac}}(q) = Z_{\text{Dirac}/2}(q)Z_{\text{Dirac}/2}(\bar{q})$ for some function $Z_{\text{Dirac}/2}(q)$. This factorization property is true for a particular spin structure, but not for the whole partition function, and so the boundary conditions introduced by the torus and the requirement of modular invariance let us see the constraints on the ways that the two sectors can talk to each other.

A more suggestive way to write the sum over spin structures is to write the fermion partition function as

$$Z_{\text{Dirac}}(q) = \text{Tr}_{NS}\left[\frac{\mathbf{1} + (-1)^F}{2} q^{L_0 - 1/24}\bar{q}^{\bar{L}_0 - 1/24}\right] + \text{Tr}_R\left[\frac{\mathbf{1} + (-1)^F}{2} q^{L_0 - 1/24}\bar{q}^{\bar{L}_0 - 1/24}\right], \quad (127)$$

where the subscript on the trace indicates the spatial boundary conditions[5]. The point of writing it this way is that it takes the form of fermions coupled to a $\mathbb{Z}_2$ gauge field, with the projectors $(\mathbf{1} + (-1)^F)/2$ enforcing that all states in the Hilbert space be a singlet under the $\mathbb{Z}_2$ symmetry sending $\psi \mapsto -\psi$. That is, we can view the temporal part of the spin structure as coming from a $\mathbb{Z}_2$ gauge field (modular invariance then forces us to sum over both spatial boundary conditions and trace over the whole $\mathcal{H}_{NS} \oplus \mathcal{H}_R$ Hilbert space). The point of this remark is that if we are writing down a mapping between a bosonic theory and a fermionic one, since we never have local fermion operators in the bosonic Hilbert space—only operators which are pairs of fermions. Because of this, the sign of any fermion operators we write down must be unphysical, and so which expect there to be a $\mathbb{Z}_2$ gauge redundancy in any putative fermionic dual model we write down. In this case, we see that this thinking is correct.



[5]We could instead use $\mathbf{1} - (-1)^F$ in the second trace as well. This $\pm$ ambiguity is due to the fact that the term it appears in is the RR spin structure term, which is zero (the ambiguity relates to the two degenerate states on the RR torus which differ by whether or not the zero mode is filled)

6 *Vertex correlators* ✓

Today we have an exercise from the big yellow book, chapter 9. We will be considering a (non-compact) boson with action

$$S = \frac{1}{4\pi} \int dz d\bar{z}\, \partial\phi\bar{\partial}\phi. \tag{128}$$

Our goal will be to derive the correlation functions of vertex operators in a careful way. Here's the problem statement:

To define the vertex operator, do a mode expansion and separate out the zero mode as follows:

$$\mathcal{V}_\alpha(z,\bar{z}) = :e^{i\alpha\Phi}: V'_\alpha(z)\bar{V}'_\alpha(\bar{z}), \tag{129}$$

and show that the zero mode is

$$\Phi(z,\bar{z}) = \phi_0 - ia_0 \ln(z\bar{z}) \tag{130}$$

while the $V'_\alpha(z)$'s are

$$V'_\alpha(z) = :e^{i\alpha\phi'(z)}: = \exp\left(-\alpha \sum_{n>0} \frac{1}{n} a_{-n} z^n\right) \exp\left(\alpha \sum_{n>0} \frac{1}{n} a_n z^{-n}\right), \tag{131}$$

and likewise for the $V'_\alpha(\bar{z})$'s. Here, $\phi'(z)$ denotes the holomorphic part of ϕ with the zero mode removed.

Find the $\langle \phi'(z)\phi'(w)\rangle$ correlator, and find the n-pt correlation function of the $V'_\alpha(z)$'s. Find the n-point correlation function of the $:e^{i\alpha\phi'(z,\bar{z})}:$ zero mode vertex operators, and use these results to find the correlators for the full vertex operators $\mathcal{V}_\alpha(z,\bar{z})$.

First lets review the mode expansion. On a cylinder with circumference L, we do $\phi = \sum_n \phi_n e^{2\pi i x/L}$. Finding the Hamiltonian is straightforward:

$$H = \frac{2\pi}{L} \sum_{n\in\mathbb{Z}} (\pi_n \pi_{-n} + \frac{n^2}{4}\phi_n\phi_{-n}), \quad \pi_n = \frac{L}{4\pi}\partial_t \phi_{-n}. \tag{132}$$

We can solve the Hamiltonian by introducing oscillators a_n. We will work with the conventions in the big yellow book, so that for $n \neq 0$,

$$\phi_n = \frac{i}{n}(a_n - \bar{a}_{-n}), \qquad [a_n, a_m] = n\delta_{n+m}. \tag{133}$$

Thus ϕ is decomposed as

$$\phi(x) = \phi_0 + i \sum_{n\neq 0} \frac{1}{n}(a_n - \bar{a}_n) e^{2\pi i n x/L}. \tag{134}$$

We commutate this with the Hamiltonian to get the time dependence, which is straightforward. In particular, the time dependence of the zero mode is $\phi_0 + \frac{4\pi}{L}\pi_0 t$. As we have done in the last couple of days, we let $z = e^{2\pi(\tau - ix)/L}$, $\bar{z} = e^{2\pi(\tau+ix)/L}$ where $\tau = it$. With these conventions,

$$t = -i\frac{L}{4\pi}\ln(z\bar{z}). \tag{135}$$

Putting this in and writing a_0 for π_0, the decomposition for ϕ is

$$\phi(z,\bar{z}) = \Phi(z,\bar{z}) + i\sum_{n\neq 0}\frac{1}{n}(a_n z^{-n} + \bar{a}_n\bar{z}^{-n}), \tag{136}$$

where the zero mode part is

$$\Phi(z,\bar{z}) = \phi_0 - ia_0\ln(z\bar{z}). \tag{137}$$

Now define the field

$$\phi'(z) = \sum_{n\neq 0}\frac{1}{n}a_n z^{-n} \tag{138}$$

to be the holomorphic part of $\phi(z,\bar{z})$ with the zero mode removed. We write its two point function as

$$\langle\phi'(z)\phi'(w)\rangle = \left\langle\sum_{m<0,n>0}\frac{1}{nm}a_n a_m z^{-n}w^{-m}\right\rangle, \tag{139}$$

since a_n with $n > 0$ annihilates the vacuum (remember that a_n with $n < 0$ act as the usual creation operators). Using the commutator, we can set $m = -n$ and get

$$\langle\phi'(z)\phi'(w)\rangle = \sum_{n>0}\frac{1}{n}\frac{w^n}{z^n} = -\ln\left(1 - \frac{w}{z}\right). \tag{140}$$

Now we can get the correlator of a product of $V'_\alpha(z) =: e^{i\alpha\phi'(z)}:$ operators. To do the normal ordering we just need to move the creation operators (a_n with $n < 0$) to the left of the annihilation operators, and so

$$V'_\alpha(z) =: e^{i\alpha\phi'(z)} := \exp\left(-\alpha\sum_{n>0}\frac{1}{n}a_{-n}z^n\right)\exp\left(\alpha\sum_{n>0}\frac{1}{n}a_n z^{-n}\right), \tag{141}$$

To get the correlators involving a product of $V'_\alpha(z) =: e^{i\alpha\phi'(z)}:$ operators, we need to know how to normal-order the product. To do this, we use the CBH formula to write

$$e^A e^B = e^B e^A e^{[A,B]}, \qquad \text{if } [A,B] \in \mathbb{C}. \tag{142}$$

The commutator of the terms appearing in the exponentials in the definition of the $V'_\alpha(z)$ is a c-number, so we can use this formula. Applying this to move all the $a_{-n}, n > 0$ operators to the left in the product lets us figure out how to do the normal ordering. A bit of algebra gives (see e.g. the big yellow book, appendix 6.A)

$$\langle\prod_j V'_{\alpha_j}(z_j)\rangle = \langle:\prod_j V'_{\alpha_j}(z_j):\rangle\exp\left(-\prod_{i<k}\alpha_i\alpha_k\langle\phi'(z_i)\phi'(z_k)\rangle\right). \tag{143}$$

Using the correlator that we just derived and the fact that the fully normal-ordered term is equal to 1, we get

$$\langle \prod_j V'_{\alpha_j}(z_j) \rangle = \prod_{j<k} \left(1 - \frac{z_k}{z_j}\right)^{\alpha_j \alpha_k}. \tag{144}$$

Putting this together with the antiholomorphic piece, we have

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \left\langle \prod_j : e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle \prod_{j<k} |z_j - z_k|^{2\alpha_j \alpha_j} |z_j|^{-2\alpha_j \alpha_k}. \tag{145}$$

So, we just need the zero mode part. Remembering that $\Phi(z, \bar{z})$ is formed from a linear combination of position and momenta operators for the zero mode, we can use the same approach we used above to do the normal-ordering. We regard a_0 as the annihilation operator and ϕ_0 as the creation operator, so that to normal-order we need to shuffle the a_0's to the left. This gives

$$\begin{aligned} \left\langle \prod_j : e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle &= \left\langle : \prod_j e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle \exp\left(\sum_{i<k} [-ia_0, \phi_0] \alpha_i \alpha_k \ln(z_i \bar{z}_i) \right) \\ &= \left\langle : \prod_j e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle \prod_{i<k} |z_i|^{2\alpha_i \alpha_k}. \end{aligned} \tag{146}$$

Unlike with the non-zero-mode operators, the fully normal-ordered part actually does something. The vacua are parametrized by the value of the zero mode. So we work with the coherent vacua $|\beta\rangle$ such that $a_0|\beta\rangle = \beta|\beta\rangle$. Then we see that

$$a_0 e^{-i\alpha\phi_0}|\beta\rangle = i\frac{\delta}{\delta\phi_0} e^{-i\alpha\phi_0}|\beta\rangle = e^{-i\alpha\phi_0}(\alpha + a_0)|\beta\rangle = (\alpha + \beta) e^{-i\alpha\phi_0}|\beta\rangle, \tag{147}$$

so that $e^{-i\alpha\phi_0}|\beta\rangle = |\beta + \alpha\rangle$ shifts us between different vacua (think of electric flux operators). In the fully normal-ordered piece above, all the ϕ_0 operators stand to the left of the a_0 operators, and so we can let them act directly on the left vacuum bra. If our vacuum state is $|\beta\rangle$, then

$$\left\langle : \prod_j e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle = e^{i\sum \alpha_i \beta} \langle \beta | \gamma \rangle = \delta_{\sum_i \alpha_i, 0}, \qquad \gamma = \beta + \sum_j \alpha_j, \tag{148}$$

and so the zero mode implements the charge-neutrality condition for us.

Putting everything together, we have

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \delta_{\sum_i \alpha_i, 0} \prod_{j<k} |z_j - z_k|^{2\alpha_j \alpha_k}. \tag{149}$$

We can use this result to figure out the OPE of two vertex operators. We have

$$: e^{i\alpha\phi(z)} :: e^{i\beta\phi(w)} :=: e^{i(\alpha\phi(z) + \beta\phi(w))} : |z - w|^{2\alpha\beta}. \tag{150}$$

Note that the RHS is non-zero even when $\alpha+\beta \neq 0$. It needs to be non-zero since we need to be able to take OPEs of vertex operators that don't satisfy charge neutrality and produce a non-zero result, since e.g. the product $\mathcal{V}_1\mathcal{V}_1\mathcal{V}_{-2}$ has a non-zero vev but can be evaluated by first performing the OPE on the first two factors. Charge neutrality enters when we take the vev of the above equation, as the vev of the fully-normal-ordered part on the RHS actually vanishes when charge neutrality is not satisfied:

$$\langle : e^{i(\alpha+\beta)\phi} : \rangle = \langle e^{i\Phi(\alpha+\beta)}\rangle\langle : e^{i(\alpha+\beta)\sum_{n\neq 0}\phi_n e^{2\pi i n x/L}} : \rangle = \langle e^{i\Phi(\alpha+\beta)}\rangle = \delta_{\alpha+\beta,0}, \tag{151}$$

where we have used that the normal-ordered exponential of the non-zero modes of ϕ is equal to 1, and the fact that Φ and $\phi_{n\neq 0}$ commute.

Now, what happens when we make the boson compact? My favorite way of writing the action is to take

$$S = \frac{R^2}{4\pi}\int dz d\bar{z}\,\partial\phi\bar{\partial}\phi, \qquad \phi \sim \phi + 2\pi. \tag{152}$$

We could also change the coefficient in front of the action at the expense of changing the compactification condition on ϕ. In either case, the only real change in the above calculation is that the COM momentum π_0 takes on discrete values. Now the only change coming from compactification is in the zero mode, but changing the COM momenta to be discrete doesn't change how the zero mode implements charge-neutrality in the correlator. Therefore the correlator of the vertex operators has the same value as it does for the non-compact boson. The only aspect where the compactification radius of the boson enters is in selecting which vertex operators are allowed. In the conventions above the allowed vertex operators are $: e^{in\phi} :$ with $n \in \mathbb{Z}$; if instead $\phi \sim \phi + 2\pi\gamma$ then they are $: e^{i\lambda\phi} :$ with $\lambda \in \gamma^{-1}\mathbb{Z}$. Either way, the correlator of the vertex operators is computed using the same formula as in the non-compact case.

7 *Vertex correlators II* ✓

Today we have another quick and easy exercise from the big yellow book, chapter 9, which is the functional way of getting to the result of yesterday's diary entry.

Consider the real-space propagator

$$K(x,y) = -\ln\left(m^2[(x-y)^2 + a^2]\right), \tag{153}$$

which is the free boson correlator regulated by a mass m (long-distance cutoff) and a lattice spacing a (short-distance cutoff).

We will compute the vertex correlation function with functional methods and show that we get the result we obtained yesterday at the conformal point where $m, a \to 0$.

We use

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = Z[J] = Z[0] \exp\left(-\frac{1}{2} \int_{x,y} J(x) K(x,y) J^\dagger(y) \right), \tag{154}$$

where the relevant current for us is

$$J(x) = i \sum_j \alpha_j \delta(x - x_j). \tag{155}$$

Thus

$$Z[J] = \exp\left(\frac{1}{2} \sum_{j,k} \alpha_j \alpha_k \left(\ln(ma)^2 + \ln\left[\frac{|z_j - z_k|^2}{a^2} + 1 \right] \right) \right). \tag{156}$$

Since we are interested in sending $a^2 \to 0$, we will take $|z_j - z_k|^2 \gg a^2$ for all $j \neq k$, allowing us to get rid of the $+1$ in the ln unless $j = k$, in which case the ln vanishes. So then

$$\begin{aligned} Z[J] &= (ma)^{\left(\sum_j \alpha_j\right)^2} \prod_{i<k} \left(\frac{|z_i - z_k|^2}{a^2} \right)^{\alpha_i \alpha_k} \\ &= m^{\left(\sum_j \alpha_j\right)^2} a^{\sum_l \alpha_l^2} \prod_{i<k} |z_i - z_k|^{2\alpha_i \alpha_k}. \end{aligned} \tag{157}$$

We see that when we take $m \to 0$, $Z[J]$ vanishes unless $\sum_j \alpha_j = 0$. Even if we have charge neutrality, we still have the prefactor of $a^{\sum_l \alpha_l^2}$. This isn't a problem though, since it factors as $\prod_l a^{\alpha_l^2}$, so that if we renormalize the vertex operators with the short-distance cutoff by

$$\widetilde{\mathcal{V}}_\alpha(z, \bar{z}) \equiv a^{-\alpha^2} \mathcal{V}_\alpha(z, \bar{z}), \tag{158}$$

then we have

$$\left\langle \prod_j \widetilde{\mathcal{V}}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \delta_{\sum_i \alpha_i, 0} \prod_{j<k} |z_j - z_k|^{2\alpha_j \alpha_k}, \tag{159}$$

which is the same correlator that we found yesterday. In fact, our need to renormalize the vertex operators in the above way is not surprising, and is equivalent to the statement that the vertex operators have anomalous dimension α^2. From the 2-point function $\langle \widetilde{\mathcal{V}}_\alpha(z_1, \bar{z}_1) \widetilde{\mathcal{V}}_\alpha^\dagger(z_2, \bar{z}_2) \rangle \sim |z_1 - z_2|^{-2\Delta_\alpha}$ we see that the scaling dimension of the vertex operator $\widetilde{\mathcal{V}}_\alpha$ is $\Delta_\alpha = \alpha^2$. Thus the renormalization defined above lets us go between operators with dimensionless two-point functions ($\mathcal{V}$'s) to those with two point functions whose dimensions give the scaling dimension.

Before closing, I think it's worth elaborating a bit on the usage of the phrase "charge neutrality". The free boson has two $U(1)$ symmetries coming from the two conserved currents $J \sim i\partial\phi = i\partial\phi_+$, $\bar{J} \sim i\bar{\partial}\phi = i\bar{\partial}\phi_-$, which are conserved by virtue of the eom $\Box\phi = 0$ that allows us to split $\phi = \phi_+(z) + \phi_-(\bar{z})$. While we have focused on non-chiral vertex operators above, we can also restrict our attention to (anti)holomorphic vertex operators. Looking at e.g. the holomorphic sector, we can use the correlator $\langle \partial\phi(z) e^{i\alpha\phi_+(w)}\rangle \sim \alpha \frac{1}{z-w}$ to write

$$\left\langle J(z) \prod_j e^{i\alpha_i \phi_+(w_j)} \right\rangle \sim \sum_j \frac{\alpha_j}{z - w_j} \left\langle \prod_k e^{i\alpha_k \phi_+(w_k)} \right\rangle, \tag{160}$$

which when acted on both sides by $\oint_C dz$ for some contour C gives the usual Ward identity. Therefore the statement that the correlator on the RHS vanishes unless $\sum_j \alpha_j = 0$ is equivalent to the statement that all non-zero vertex correlators must be neutral under the two $U(1)$ symmetries.

8 *Vacuum energy and boundary conditions* ✓

Today is another quickie: deriving a statement made in Ginzbarg's lectures on CFT about vacuum energies.

Consider a free Dirac fermion, with boundary conditions on the cylinder such that

$$\psi(\sigma) = e^{2\pi i\gamma}\psi(\sigma + 2\pi), \qquad \gamma \in \mathbb{R}/\mathbb{Z}, \tag{161}$$

where σ is the spatial coordinate of the (radius 1) cylinder. We will show that the vacuum energy has a γ dependence given by

$$E_0 = \frac{1}{12} - \frac{1}{2}\gamma(1-\gamma), \tag{162}$$

which is consistent with $\gamma \in \mathbb{R}/\mathbb{Z}$ and $\gamma \sim 1 - \gamma$ (the vacuum energy only cares about the absolute value of the amount of twisting modulo 2π).

There are a few ways to do this problem. The first way is to use ζ function regularization to normal-order the L_0 operators on the cylinder. First, we note that since our complex fermion has the OPEs

$$\psi^\dagger(z)\psi^\dagger(w) \sim \psi(z)\psi(w) \sim 0, \qquad \psi^\dagger(z)\psi(w) \sim \frac{1}{z-w}, \tag{163}$$

if we write $\psi(z) = \frac{1}{\sqrt{2}}(\lambda(z) + i\eta(z))$ then we must have

$$\lambda(z)\lambda(w) \sim \eta(z)\eta(w) \sim \frac{1}{z-w}, \qquad \lambda(z)\eta(w) \sim 0, \tag{164}$$

i.e. the OPE forces the two Majoranas making up the Dirac fermion to decouple. Thus the energy momentum tensor and importantly for us the vacuum energy contribution to the dilitation generator can be obtained just by taking the answer for a single Majorana fermion and multiplying by 2. Thus on the cylinder we have

$$L_0 = \frac{1}{2}\sum_n n(:\lambda_{-n}\lambda_n: + :\eta_{-n}\eta_n:) = \sum_{n>0}(\lambda_{-n}\lambda_n + \eta_{-n}\eta_n) - \sum_{n>0} n, \quad n \in \mathbb{Z} + \gamma. \tag{165}$$

The last constant part is what shifts the vacuum energy density. We evaluate it with ζ function regularization. Letting

$$\zeta(q, r) = \sum_{n=0}^{\infty}(n+r)^q, \tag{166}$$

we see that we need to evaluate $\zeta(-1, \gamma)$. Luckily this is easily looked up:

$$\zeta(-1, \gamma) = -\frac{1}{2}(\gamma^2 - \gamma + 1/6), \tag{167}$$

and so the vacuum energy is evidently

$$E_0 = -\zeta(-1, \gamma) = \frac{1}{12} - \frac{1}{2}\gamma(1-\gamma), \tag{168}$$

as predicted.

The second, more rigorous way that doesn't use the ζ function is to find $\langle T \rangle$ directly using the mode expansion on the Dirac fermion. We decompose the Dirac fermion in a mode expansion as

$$\psi(w) = \sum_{k \in \mathbb{Z}^{>0}+\gamma} \left(\alpha_k e^{-kw} + \beta_k^\dagger e^{kw} \right), \tag{169}$$

where $w = \tau - ix$. This mode expansion comes from doing the expansion for the fermion on the circle and then getting the time dependence by commuting with the Dirac Hamiltonian. Notice that we are only summing over *positive* k in the above: here $\alpha_k = c(-k_F - k)$ destroys a left-moving particle while $\beta_k^\dagger = c(-k_F + k)$ creates a left-moving hole. The corresponding anitholomoprhic guy is the same thing but with right-moving momenta and coordinates:

$$\bar{\psi}(w) = \sum_{k \in \mathbb{Z}^{<0}+\gamma} \left(\alpha_k e^{k\bar{w}} + \beta_k^\dagger e^{-k\bar{w}} \right). \tag{170}$$

The signs in the exponents come from requiring the time dependence $e^{-\tau|k|}$, so that when $k < 0$ we need to invert the sign of $\bar{w} = \tau + ix$ in the exponent.

The modes satisfy the algebra $\{\alpha_k^\dagger, \alpha_l\} = \delta_{k,l}$; same for the β modes. Note that unlike the real fermions, there is no $\psi_0^2 = 1/2$ mode to worry about since α and $\alpha^\dagger$ are distinct. When we map these guys into the plane, we take $z = e^w$ and multiply by a factor of $(dz/dw)^{-h} = z^{-1/2}$ since the fermions have conformal dimension $h = 1/2$ to get

$$\psi(z) = \sum_{k\in\mathbb{Z}^{>0}+\gamma} \left(\alpha_k z^{-k-1/2} + \beta_k^\dagger z^{k-1/2}\right). \tag{171}$$

We can now calculate the expectation value $\langle\psi^\dagger(z)\psi(w)\rangle$:

$$\begin{aligned}\langle\psi^\dagger(z)\psi(w)\rangle &= \sum_{k,l\in\mathbb{Z}^{>0}+\gamma} z^{-k-1/2}w^{l-1/2}\langle\beta_k\beta_l^\dagger\rangle = \sum_{k\in\mathbb{Z}^{>0}+\gamma} z^{-k-1/2}w^{k-1/2} \\ &= \frac{1}{\sqrt{wz}}w^\gamma z^{-\gamma}\frac{z}{z-w},\end{aligned} \tag{172}$$

since when we complex conjugate z^x we get z^{-x} for $x \in \mathbb{R}$ since z is actually the exponential of a purely imaginary number (and so $z^* \neq \bar{z}$ with these conventions unfortunately, only when we pretend that τ is real). Also, here w is now a coordinate on the plane, and is *not* the earlier w, which was the cylinder coordinate. Sorry! Also note that regardless of γ, unlike with $\mathbb{R}$ fermions we don't have to worry about treating a zero mode separately.

Anyway, we have

$$\partial_z\langle\psi^\dagger(z)\psi(w)\rangle = (-\gamma+1/2)\frac{w^{\gamma-1/2}z^{-\gamma-1/2}}{z-w} - \frac{w^{\gamma-1/2}z^{-\gamma+1/2}}{(z-w)^2}, \tag{173}$$

and similarly for the derivative wrt w. Now the holomorphic stress tensor has the expectation value

$$T(w) = \frac{1}{2}\lim_{z\to w}\left\langle\partial_z\psi^\dagger(z)\psi(w) - \psi^\dagger(z)\partial_w\psi(w)\right\rangle + \frac{1}{\epsilon^2}, \tag{174}$$

where we've subtracted $2\cdot\frac{1}{2}\partial_z\frac{1}{z-w}$ evaluated at $z = w+\epsilon$ in line with the usual normal ordering prescription. Putting in our expressions for the derivatives,

$$\langle T(w)\rangle = \frac{1}{2}\lim_{z\to w}\left(\frac{1/2-\gamma}{z-w}(z^{-1/2-\gamma}w^{\gamma-1/2} + z^{1/2-\gamma}w^{\gamma-3/2}) - 2\frac{z^{1/2-\gamma}w^{\gamma-1/2}}{(z-w)^2}\right) + \frac{1}{\epsilon^2}. \tag{175}$$

Evaluating the term in the parenthesis for $z = w+\epsilon$ we find that it is equal to

$$\frac{1}{2}\lim_{z\to w}(\dots) = -\frac{1}{\epsilon^2} + \frac{1-4\gamma-4\gamma^2}{8w^2} + O(\epsilon). \tag{176}$$

The singular part cancels the $1/\epsilon^2$ introduced by the normal ordering, and since we are dealing with free fields there are no more singular parts leftover to be cancelled, leaving only the $1/w^2$ piece. Thus

$$\langle T(z)\rangle = \frac{1-4\gamma-4\gamma^2}{8z^2}. \tag{177}$$

Sanity check: when $\gamma = 1/2$ so that we have the normal anti-periodic boundary conditions for the fermions on the cylinder, we have $\langle T(z)\rangle = 0$ ✓.

Now let's go over to the cylinder. The holomorphic (left-moving) part of the Hamiltonian is found by (now w is the cylinder coordinate again—jeez this is awful notation)

$$H_L = \frac{1}{2\pi i}\int dw\, T(w) = \frac{1}{2\pi i}\int dz\, z^{-1}T(w) = \frac{1}{2\pi i}\int dz\, z^{-1}(z^2T(z) - 1/24), \tag{178}$$

where we used the transformation rule for T we derived earlier (with coordinates $z = e^w$) and put in the central charge $c = 1$. We see that the $n = 0$ mode of the Laurent expansion for T is selected out, which picks up the extra piece contributing to $\langle T(z)\rangle$ that we found above. So

$$E_0 = \langle L_0 \rangle + \frac{1 - 4\gamma - 4\gamma^2}{8} - \frac{1}{24} = \frac{1}{12} - \frac{1}{2}\gamma(\gamma - 1), \tag{179}$$

since $\langle L_0 \rangle = 0$ as L_0 is a sum of normal-ordered oscillators. We see that this approach gives exactly the same vacuum energy that we derived using ζ function regularization!

9 *Fermion partition functions on the torus: the functional approach* ✓

Today is a check on our understanding of fermion path integrals on the torus: the goal is to confirm some statements in Ginzbarg's CFT lectures. Here's the problem statement:

For a free Majorana fermion on a torus with a given spin structure, find $Z(q)$, where $q = e^{2\pi i\tau}$ and τ is the modular parameter. Do so using functional methods and ζ function regularization methods (the preferred regulator since it respects modular invariance) rather than using the operator approach. You only need to reproduce the q dependence; don't worry about constants and stuff.

We start from

$$Z_{XY}(q) = \mathrm{Pf}_{XY}(\partial)\mathrm{Pf}_{XY}(\bar\partial) = \sqrt{\det{}_{XY}(\partial\bar\partial)}, \tag{180}$$

where $X, Y \in \{NS, R\}$ are spin structure labels.The (un-normalized) eigenfunctions of $\partial\bar\partial$ are

$$\psi_{n,m}(\alpha,\beta) = \exp\left(\frac{2\pi i}{2i\mathrm{Im}(\tau)}\left[(n+\alpha)(z-\bar z) + (m+\beta)(\tau\bar z - \bar\tau z)\right]\right), \qquad \alpha,\beta \in \{0, 1/2\}, \tag{181}$$

where we have assumed that the spacelike edge of the torus stretches from 0 to 1 in the complex plane. One checks that

$$\begin{aligned}(z \mapsto z+1) :\ & \psi_{n,m}(\alpha,\beta) \mapsto (-1)^{2\beta}\psi_{n,m}(\alpha,\beta)\\ (z \mapsto z+\tau) :\ & \psi_{n,m}(\alpha,\beta) \mapsto (-1)^{2\alpha}\psi_{n,m}(\alpha,\beta),\end{aligned} \tag{182}$$

so that α sets the timelike boundary conditions and β sets the spacelike boundary conditions.

From the eigenvalues we get that

$$\det_{XY}(\partial\bar{\partial}) = \prod_{n,m} \frac{\pi^2}{\mathrm{Im}(\tau)^2} |n + \alpha - (m+\beta)\tau|^2. \tag{183}$$

We will ignore the $\pi^2/\mathrm{Im}(\tau)^2$ part, which by using ζ function regularization ends up contributing something with τ dependence $\sqrt{\mathrm{Im}(\tau)}$ to the partition function. This is important for maintaining modular invariance—we won't worry about it now, but will just remember that we need to re-instate this bit if we want to get something modular invariant.

First consider the case where $X, Y = R, R$, i.e. $\alpha = \beta = 0$. Then from the zero mode in the partition function, we get $Z_{RR}(q) = 0$ as expected. What about $X, Y = NS, R$, i.e. $\alpha, \beta = 1/2, 0$? This is periodic in space and anti-periodic in time. We have

$$\det_{NS,R}(\partial\bar{\partial}) \propto \prod_{n\in\mathbb{Z}} (n+1/2)^2 \prod_{n,m\in\mathbb{Z}, m\neq 0} |n + 1/2 - m\tau|^2. \tag{184}$$

Now we need

$$\prod_{n\in\mathbb{Z}} (n+x) = e^{i\pi x} - e^{-i\pi x}, \tag{185}$$

so that the first product is just a constant, and so after some algebra (combining the products for $m > 0$ and $m < 0$),

$$\det_{NS,R}(\partial\bar{\partial}) \propto \prod_{m\in\mathbb{Z}^{>0}} |q^{-m}(1+q^m)^2|^2. \tag{186}$$

Now we use

$$\prod_{m>0} q^{-m} = e^{-\zeta(-1)\ln q} = q^{1/12}, \tag{187}$$

so that, taking the square root to get the partition function,

$$Z_{NS,R}(q) \propto (q\bar{q})^{1/24} \prod_{m\in\mathbb{Z}^{\geq 0}} |1 + q^m|^2. \tag{188}$$

Now for $X, Y = R, NS$ i.e $\alpha, \beta = 0, 1/2$. Then

$$\det_{R,NS}(\partial\bar{\partial}) \propto \prod_{n\in\mathbb{Z}} |n - \tau/2|^2 \prod_{n,m\in\mathbb{Z}, m\neq 0} |n - (m+1/2)\tau|^2. \tag{189}$$

The first product gives us

$$\prod_{n\in\mathbb{Z}} |n - \tau/2|^2 = |q^{-1/4}(1 - q^{1/2})|^2. \tag{190}$$

The second product is

$$\prod_{n,m\in\mathbb{Z}, m\neq 0} |n - (m+1/2)\tau|^2 = \prod_{m\in\mathbb{Z}^{>0}} |q^{-(m+1/2)/2} q^{-(m-1/2)/2} (1 - q^{m+1/2})(1 - q^{m-1/2})|^2. \tag{191}$$

We can change the $(1-q^{m-1/2})$ to a $(1-q^{m+1/2})$ by also including a factor of $(1-q^{1/2})$, which combines with the factor in the previous product to produce an $m=0$ term in the full product. Taking everything together and taking the square root, we get

$$Z_{R,NS}(\partial\bar{\partial}) \propto (q\bar{q})^{-1/48} \prod_{m\in\mathbb{Z}^{\geq 0}} |1-q^{m+1/2}|^2, \tag{192}$$

which is exactly what we expect: the periodic boundary conditions in time give us a supertrace by sending $q \to -q$, while we have the half-odd-integer momenta needed for antiperiodic boundary conditions in space.

Finally for $\alpha = \beta = 1/2$, the $NSNS$ spin structure. We split up the product into $m=0$ and $m \neq 0$ parts as before. The $m=0$ part is

$$\prod_{n\in\mathbb{Z}} |n+1/2-\tau/2|^2 = |q^{-1/4}(1+q^{1/2})|^2. \tag{193}$$

The $m \neq 0$ part is dealt with as before: the fact that $\alpha = 1/2$ means that we get a trace instead of a supertrace, and the same sort of algebra leads to

$$Z_{NS,NS}(\partial\bar{\partial}) \propto (q\bar{q})^{-1/48} \prod_{m\in\mathbb{Z}^{\geq 0}} |1+q^{m+1/2}|^2. \tag{194}$$

10 *Orbifolding basics* ✓

Today is an elaboration on part of a problem in the big yellow book, chapter 10. We will show that on the torus,

$$Z^2_{\text{Ising}} = Z_{\text{orb}}(R=1), \tag{195}$$

where Z_{Ising} is the partition function of a single majorana fermion and $Z_{\text{orb}}(R=1)$ is the free boson at compactification radius $R=1$ orbifolded under the $\mathbb{Z}_2$ $X \mapsto -X$ symmetry.

Let's first write down Z_{Ising}, which is easy using our knowledge from the previous few diary entries. In addition to the trace of the $q^{L_0}\bar{q}^{\bar{L}_0}$ part, we need to know the vacuum energy contribution to partition function (i.e. the part that is $(q\bar{q})^{-c/24}$ if boundary conditions are not an issue). We find this contribution with the regularization

$$-\frac{1}{2}\sum_{n\in\mathbb{Z}+\alpha} = \frac{1}{24} - \frac{\alpha}{4}(1-\alpha), \tag{196}$$

which we essentially derived a few days ago (it comes from doing the normal-ordering in the oscillator sum in the Hamiltonian. This is done *on the torus*, not on the plane, and so the modding is not shifted by 1/2). If the boundary conditions in space are anitperiodic then the fermion is modded in $\mathbb{Z}+1/2$ on the torus, and hence we can take $\alpha = 1/2$ for NS spatial boundary conditions, giving a contribution of $-1/48$. If the boundary conditions are periodic, then we just set $\alpha = 0$ so that we get a $+1/24$ contribution. So then since each spin structure factors as holomorphic and antiholomorphic parts, we can just write down (remember that the the RR spin structure gives zero)

$$Z_{\text{Ising}} = \frac{1}{2}\left(\left|\frac{\theta_2}{\eta(q)}\right| + \left|\frac{\theta_3}{\eta(q)}\right| + \left|\frac{\theta_4}{\eta(q)}\right|\right), \tag{197}$$

where

$$\begin{aligned} \frac{\theta_2}{\eta(q)} &= \frac{1}{\sqrt{2}} q^{1/24} \prod_{n\in\mathbb{Z}^{\geq 0}} (1+q^n), \\ \frac{\theta_3}{\eta(q)} &= q^{-1/48} \prod_{n\in\mathbb{Z}^{\geq 0}} (1+q^{n+1/2}), \\ \frac{\theta_4}{\eta(q)} &= q^{-1/48} \prod_{n\in\mathbb{Z}^{\geq 0}} (1-q^{n+1/2}), \end{aligned} \tag{198}$$

and the η function is

$$\eta(q) = q^{1/24} \prod_{n\in\mathbb{Z}^{>0}} (1-q^n). \tag{199}$$

The $1/\sqrt{2}$ factor is the quantum dimension of the σ primary field and is inserted to have the usual expression for the modular S matrix.

Now for the orbifolded boson under the $\mathbb{Z}_2$ symmetry taking $X \to -X$. Let's recall how we arrive at the partition function. We want to "gauge" the $\mathbb{Z}_2$ symmetry (or rather, we want to do a Fourier transformation) by projecting the Hilbert space onto states which are singlets under the $\mathbb{Z}_2$, which we do with the operator $(\mathbf{1} + (-1)^X)/2$, where $(-1)^X$ is our dumb way of writing the operator which does the symmetry action on X (it inserts a $\mathbb{Z}_2$ symmetry defect that wraps around the spatial cycle of the torus and changes the temporal boundary conditions). Inserting this operator (the tube algebra Hamiltonian) ruins modular invariance since it's asymmetrical under S, and so we have also have to sum over spatial boundary conditions by inserting defects that wrap the temporal cycle. We end up summing over all (four) topologically distinct way to place $\mathbb{Z}_2$ symmetry defect lines on the torus (for us, they are added with no relative phases between them).

The torus with no symmetry defects just gives a contribution of $Z(R)/2$, the regular un-orbifolded version. The torus with a symmetry defect inserted which wraps the spatial cycle gives the holomorphic contribution

$$\frac{q^{-1/24}}{2} \mathrm{Tr}_P[(-1)^X q^{L_0}]. \tag{200}$$

Here the trace is done with periodic boundary conditions in space, and so we have to take into account zero modes. The basis states for $\mathcal{H}_0$ (the zero-mode part of the Hilbert space) that diagonalize the action of $(-1)^X$ are $|k,w\rangle \pm |-k,-w\rangle$ (k is momentum, w is winding) for $k,w \in \mathbb{Z}$, since $(-1)^X$ acts as -1 on the the momenta and winding numbers. However, recall that the k,w dependence of the spectrum of L_0 comes from the term $\frac{1}{2}p_L^2$, where $p_L = (k/R + wR/2)$ (for $\bar{L}_0$, send $w \mapsto -w$). Thus the spectrum of L_0 is unchanged under the action of $(-1)^X$, and so all the terms in the trace above with the exception of $|0,0\rangle$ die in pairs. Thus we don't even have to worry about the zero modes, and so taking the trace over the nonzero oscillator modes in the usual way, we get the holomorphic part

$$\frac{q^{-1/24}}{2} \prod_{n\in\mathbb{Z}^{>0}} \frac{1}{1+q^n}. \tag{201}$$

Note that the + sign in the denominator comes from the action of $(-1)^X$ on the left bra in the trace.

The torus with a symmetry defect inserted along the temporal cycle just changes the modding of the boson modes to be in $\mathbb{Z}+1/2$, since it changes the spatial boundary conditions to be antiperiodic. Thus there are no zero modes to worry about, and the only subtlety is the change in the vacuum energy contribution due to the altered boundary conditions. The relevant normal-ordering result for bosons is

$$\frac{1}{2}\sum_{n\in\mathbb{Z}+\alpha} n = \frac{1}{2}\left(-\frac{1}{12} + \frac{\alpha}{2}(1-\alpha)\right), \tag{202}$$

which gives us the $-1/24$ for periodic boundary conditions that we used before, and which gives $+1/48$ for antiperiodic boundary conditions (just like for the fermions, but with opposite signs). So then we get the holomorphic term

$$\frac{2q^{1/48}}{2} \prod_{n\in\mathbb{Z}^{>0}} \frac{1}{1-q^n}. \tag{203}$$

Here, the factor of 2 in the numerator comes from the fact that we have two vacua which have identical L_0 eigenvalues (± 1 eigenvalues under $(-1)^X$). The 2 in the denominator comes from the 1/2 in the projector onto $\mathbb{Z}_2$-invariant states.

Finally, the torus with defects wrapped around both cycles: this is just like the above, expect with $+q^n$ in the denominator, with the sign coming from the $(-1)^X$ acting on the oscillator modes in the trace. So we get the term

$$\frac{2q^{1/48}}{2} \prod_{n\in\mathbb{Z}^{>0}} \frac{1}{1+q^n}. \tag{204}$$

Adding everything together, remembering to square everything because of the antiholomorphic parts, and using the definitions of the θ functions and the η function, we get (note: I think the corresponding formula in the big yellow book has some incorrect factors of 2?)

$$Z_{orb}(R) = \frac{1}{2}\left(Z(R) + 2\left|\frac{\eta(q)}{\theta_2}\right| + 2\left|\frac{\eta(q)}{\theta_3}\right| + 2\left|\frac{\eta(q)}{\theta_4}\right|\right). \tag{205}$$

Note that only the first of the four terms actually depends on the compactification radius R, since it is the only term in which the presence of the zero modes contributed something nonzero to the partition function.

Specializing to $R = 1$, and recalling our result from a few days ago about the duality between Dirac fermions and the $R = 1$ boson, we have

$$Z(R=1) = Z_{\text{Dirac}} = \frac{1}{2}\left(\left|\frac{\theta_2}{\eta}\right|^2 + \left|\frac{\theta_3}{\eta}\right|^2 + \left|\frac{\theta_4}{\eta}\right|^2\right). \tag{206}$$

Now we need to make use of

$$\eta(q)^3 = \frac{1}{2}\theta_2\theta_3\theta_4, \tag{207}$$

which we use to substitute $\eta = \theta_2\theta_3\theta_4/(2\eta^2)$ in for the η's in $Z_{orb}(R = 1)$. We get

$$Z_{orb}(R=1) = \frac{1}{4}\left(\left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right|\right)^2 = Z^2_{\text{Ising}}, \tag{208}$$

which is what we wanted to show. Thus orbifolding lets us "decouple" the two majorana fermions making up a Dirac fermion from one another and allows them to have separate spin structures.

One comment on orbifolds of orbifolds (for G finite and Abelian). The orbifold of an orbifold is the original theory, essentially because the Fourier transform is an involution (well, up to a sign). To orbifold an orbifold, we sum over all ways of twisting the orbifold, where an orbifold twisted by some function $\beta : G \to U(1)$ means

$$Z^{\beta}_{orb} = \frac{1}{|G|}\sum_{g,h}\beta(h)Z^h_g, \tag{209}$$

where Z^h_g denotes the torus with a g twist in the spatial direction and an h twist in the temporal direction. To get something modular invariant, we then need to write

$$Z^{\omega}_{orb} = \frac{1}{|G|}\sum_{g,h}\omega(g,h)Z^h_g, \tag{210}$$

where $\omega(g, h) = \beta(g)\beta(h)$. To orbifold the orbifold we sum over all such functions $\omega(g, h)$, which is the dual version (in the sense of Rep $\leftrightarrow$ Vec duality) of projecting onto a G-singlet state. Thus the procedure looks like

$$Z \xrightarrow{\text{orbifold}} \frac{1}{|G|}\sum_{g,h} Z^h_g \xrightarrow{\text{orbifold the orbifold}} \frac{1}{|G|^2}\sum_{\omega\in\text{Rep}^2(G)}\sum_{g,h}\omega(g,h)Z^h_g = Z, \tag{211}$$

so that orbifolding squares to the identity (remember that we are assuming G is finite and Abelian so that $\text{Rep}(G) \cong G$). Recall that since we are summing over genuine representations here, the 2-cocycle $\omega(g, h) = \beta(g)\beta(h)$ for $\beta \in \text{Rep}(G)$ is exact. That this works is easy to explicitly verify e.g. for the $\mathbb{Z}_2$ case we've been considering, with the last equality in the above chain holding because the Fourier transform is involutive up to a sign. If $G = \mathbb{R}$ then we can telegraphically illustrate this as

$$Z(x=0) \xrightarrow{\text{orbifold}} \int dx\, Z(x) \xrightarrow{\text{orbifold the orbifold}} \int dx \int dk\, e^{ikx} Z(x) = Z(x=0), \tag{212}$$

showing that orbifolding is an involution. Here the orbifold is $Z(k=0)$ (which is a singlet under translations).

11 *The Cardy formula*

This is a basic result that I'd seen a few times but had never derived. We will show that at high energies, modular invariance implies that for a 2D CFT, the density of states satisfies

$$\ln \rho(E) = \sqrt{\frac{2\pi E c}{3}}, \tag{213}$$

where c is the central charge.

We will work on a torus with modular parameter $\tau = i\beta \in i\mathbb{R}$, with the real leg of the torus stretching from 0 to 1 as usual. We will first take (for a reason that will be clear in a sec) the "low temperature" limit of $\beta \gg 1$ so that the torus is very stretched out.

Since $\beta \gg 1$ and $\bar\tau = -\tau$, we have $q = \bar q = e^{2\pi i\tau} = q^{-2\pi\beta} \to 0$. The partition function is (assuming $c = \bar c$)

$$Z(q) = q^{-c/12}\mathrm{Tr}[q^{L_0 + \bar L_0}]. \tag{214}$$

Since $q \to 0$, we can approximate the partition function by only the terms where the modes are unoccupied, so

$$Z(q) \approx q^{-c/12}. \tag{215}$$

Now we use modular invariance—the modular S transformation inverts τ and adds a minus sign, so that $S(q) = e^{-2\pi i/\tau} = e^{-2\pi/\beta} \to 1$. Since $S(q) \to 1$, we now have a partition function dominated by *high* energy modes. Thus by considering this super-stretched torus, modular invariance means that the high and low energy parts of the partition function are linked together. Since in the S-transformed partition function the weight of higher energy modes in the sum falls off very slowly we can change the sum over states to an integral and accordingly write

$$S[Z(q)] \approx \int dE\, \rho(E) e^{-E/\beta}, \tag{216}$$

since $H = 2\pi(L_0 + \bar{L}_0 - c/12)$.

Now we make the (a priori un-justified?) assumption that we can use a saddle-point approximation. Putting the ρ in the exponential, we see that the saddle-point condition on $E = E(\beta)$ is

$$\frac{\partial \ln \rho(E)}{\partial E}\Big|_{E=E_*} = \beta^{-1}. \tag{217}$$

So then if we use this to evaluate the parition function, then by modular invariance,

$$\ln Z(q) = \ln\left(S[Z(q)]\right) \implies \frac{\pi c \beta}{6} = \ln \rho(E_*) - \beta^{-1} E_*. \tag{218}$$

Hitting both sides of the RHS with ∂_β so that we can use our saddle point condition, we get

$$\frac{\pi c}{6} = \partial_\beta E_* \partial_E \ln \rho(E)|_{E=E_*} + \frac{E_*}{\beta^2} - \beta^{-1}\partial_\beta E_* = \frac{E_*}{\beta^2} \implies \beta = \sqrt{\frac{6E_*}{\pi c}}. \tag{219}$$

We can then use the relation that we took the derivative of to get

$$\ln \rho(E_*) = \frac{\pi c \beta}{6} + \frac{E_*}{\beta} = \sqrt{\frac{2\pi E_* c}{3}}. \tag{220}$$

This provides us with a sort of a posteriori justification for the saddle point, since the at the saddle point the effective "Hamiltonian" in the parition function is $E_*/\beta - \ln \rho(E_*) \propto \beta$, which is indeed large and thus perhaps deserving of a saddle-point treatment.

Anyway, the point is that since the saddle point energy E_* is large, we see that at high energy, the (logarithm of the) density of states is controlled by the central charge in the way promised in the introduction.

12 *Alternate route to WZW central charge* ✓

This is a slightly elobarated version of yet another problem in the big yellow book, chapter 15. Here's the problem statement:

Consider the WZW theory at level k with lie algebra obtained from the group G. The current algebra is captured by the OPE

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \frac{if^{abc}J^c}{z-w}. \tag{221}$$

Show that the central charge is

$$c = \frac{k \dim G}{k+g}, \tag{222}$$

where g is the dual coxeter number (one half of the quadratic casimir of the adjoint representation), satisfying

$$g\delta_{cd} = \frac{1}{2}\sum_{a,b} f_{abc}f_{abd} = \frac{1}{2}C_2(G)\delta_{cd}. \tag{223}$$

In the expression for c, $\dim G$ is the dimension of the Lie algebra itself, not the dimension of the particular representation that the currents act on the fields with.

Show that this is the central charge *without* taking the TT OPE. You will need to find how the Virasoro algebra fits into the current algebra and will need to consider the action of various currents and Virasoro generators on a WZW primary field.

First we need to figure out the form of the energy momentum tensor (we are just trying to motivate the Sugawara construction). The action is (at the conformal fixed point)

$$S[g] = \frac{k}{8\pi}\int d^2x\, \mathrm{Tr}[\partial_\mu g^{-1}\partial^\mu g] - \frac{ik}{12\pi}\int_{M_3}\mathrm{Tr}[\omega\wedge\omega\wedge\omega], \tag{224}$$

where as usual $\omega = g^{-1}dg$. When we write this in complex coordinates the second term doesn't change since we've written it with differential forms, but the first term changes to $\frac{k}{4\pi}\int \mathrm{Tr}[\partial g^{-1}\bar{\partial}g]$ since $\partial = (\partial_0 - i\partial_1)/2$ while $|d^2x| = \frac{1}{2}|d^2z|$. Now for a scalar field we have

$$T^{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\sigma\phi g^{\sigma\nu}. \tag{225}$$

The only thing that changes for us is that we need to take the trace on the RHS. Now in our conventions $T = -2\pi T_{zz} = -\frac{\pi}{2}T^{\bar{z}\bar{z}}$ since in complex coordinates $g^{\mu\nu} = 2X$. So then for us,

$$T = -\pi\mathrm{Tr}\left[\frac{\partial\mathcal{L}}{\partial\bar{\partial}g\phi}\partial g\right]. \tag{226}$$

Now for the WZW Lagrangian, only the first kinetic term will contribute to T since the second term doesn't contain the metric (it's built out of wedge products, and so we use ϵ to contract indices rather than the metric).[6] This means we can write the classical stress tensor (i.e. the one obtained within classical field theory) as

$$T_c = -\frac{k}{2}\mathrm{Tr}[\partial g^{-1}\partial g] = \frac{1}{2k}\mathrm{Tr}[k(\partial g)g^{-1}k(\partial g)g^{-1}] = \frac{1}{2k}\sum_a J_aJ_a, \tag{227}$$

where the current is

$$J = J^at^a = -k(\partial g)g^{-1}. \tag{228}$$

[6]Alternatively, since it is linear in time derivatives it doesn't contribute to the Hamiltonian, and then by symmetry we can argue that it doesn't contribute to T.

So, this is the stress tensor we expect classically. Quantumly we try an ansatz with the same Sugawara form, except with a different coefficient, so that the ansatz for the actual stress tensor is

$$T = \frac{1}{\gamma} \sum_a (J^a J^a), \tag{229}$$

where $(\dots)$ denotes normal-ordering, which is the full taking-off-all-singular-parts normal ordering. Since the theory is interacting, the normal-ordering is more complicated than for free fields, and means that unlike for free theories, the coefficient γ will receive quantum corrects (double contractions from interactions) to the classical value of $\gamma_c = 2k$.

Now we will figure out the actual value of γ. First, some preliminary work. Since the J^a's are $(1, 0)$ currents, we define the mode expansion as

$$J^a(z) = \sum_n z^{-n-1} J^a_n. \tag{230}$$

We get the commutator of the current modes by doing the usual double contour integral with the help of the $J^a J^b$ OPE provided by the current algebra:

$$\begin{aligned}
[J^a_n, J^b_m] &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz\, z^n w^m \left(\frac{k\delta_{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w} \sum_l w^{-l-1} J^c_l \right) \\
&= nk\delta_{n+m}\delta_{ab} + \sum_l \frac{1}{2\pi i} if^{abc} \oint dw\, w^{m+n-l-1} J^c_l \\
&= nk\delta_{n+m}\delta_{ab} + if^{abc} J^c_{m+n}.
\end{aligned} \tag{231}$$

Sticking the expressions for the current modes into the stress tensor we find that the Virasoro generators (defined as usual with $T = \sum_n L_n z^{-n-2}$ so that L_n has conformal dimension n) can be written in terms of the current as

$$L_n = \frac{1}{\gamma} \sum_m : J^a_m J^a_{n-m} : \tag{232}$$

Here the normal-ordering means just what it does as if the J modes were oscillator modes of a free field: the operator with the larger mode index gets put to the right.

We also need the commutation relation between the Virasoro generators and the current modes:

$$[L_n, J^a_m] = -mJ^a_{n+m}, \tag{233}$$

which just comes from the fact that J^a, a $(1, 0)$ current, is a primary with conformal dimension $h = 1$ (again, this is easy to check from the double-contour method of computing the commutator). Note that the J^a_0 modes generate a symmetry corresponding to the Lie algebra G, as they commute with the Hamiltonian.

In the following, we will let $|\phi\rangle$ be a WZW primary state. This means the same thing as it does for primary states of the Virasoro algebra, namely

$$J^a_0|\phi\rangle = -t^a|\phi\rangle, \qquad J^a_{n>0}|\phi\rangle = 0. \tag{234}$$

Here, t^a is the representation matrix assigned to the generator a (it lives in the Lie algebra of G). We won't need to specify the exact irrep in what follows. That J_n^a annihilates $|\phi\rangle$ if $n > 0$ just means that $|\phi\rangle$ is a highest weight (or maybe better, "lowest weight" state for the current algebra. Also, note that $|\phi\rangle$ is also automatically a Virasoro primary (this isn't always the case for current algebras), since the normal-ordering in the expression of L_n in terms of the currents means that if $n > 0$, in the action of L_n on $|\phi\rangle$, the right-most current operator always has a mode number which is greater than zero.

Now on one hand, we have

$$[J_1^a, L_{-1}]|\phi\rangle = J_0^a|\phi\rangle = -t^a|\phi\rangle. \tag{235}$$

On the other hand, we have

$$[J_1^a, L_{-1}]|\phi\rangle = J_1^a L_{-1}|\phi\rangle = \frac{1}{\gamma}J_1^a(2J_{-1}^b J_0^b)|\phi\rangle = -\frac{2}{\gamma}[J_1^a, J_{-1}^b]t^b|\phi\rangle, \tag{236}$$

since only the $m = 0, m = -1$ terms in L_{-1} act nontrivially on $|\phi\rangle$. From the commutations we derived earlier, this turns into

$$-\frac{2}{\gamma}\left(k\delta_{ab}t^b + i\frac{1}{2}f_{abc}[t^c, t^b]\right)|\phi\rangle. \tag{237}$$

Using the definition of the dual coxeter number g, this becomes

$$[J_1^a, L_{-1}]|\phi\rangle = -\frac{2}{\gamma}(k+g)t^b|\phi\rangle. \tag{238}$$

Reconciling these two ways of writing the action of the commutator lets us conclude that

$$\gamma = 2(k+g), \tag{239}$$

so the correct quantum stress tensor is

$$T = \frac{1}{2(k+g)}\sum_a (J^a J^a). \tag{240}$$

Now we can get the central charge. As one does when computing unitarity constraints, we will compute the norm of the state $L_{-2}|\phi\rangle$. We will do this in two different ways: using the Virasoro algebra commutation relations and using the expression of L_{-2} in terms of the currents. We choose L_{-2} since it is the smallest weight Virasoro generator that lets us access the central charge: the central charge appears in the Virasoro algebra relations together with $\delta_{n+m}(n^3 - n)$, which gives us zero for $n = 0, 1$ but not $n = 2$.

Using the Virasoro commutation relations, we have

$$||L_{-2}|\phi\rangle||^2 = \langle\phi|[L_2, L_{-2}]|\phi\rangle = 4\langle\phi|L_0|\phi\rangle + \frac{c}{2}, \tag{241}$$

so that

$$\frac{c}{2} = -\frac{4}{\gamma}\langle\phi|t^a t^a|\phi\rangle + \langle\phi|[L_2, L_{-2}]|\phi\rangle. \tag{242}$$

Now we need to evaluate $\langle\phi|[L_2, L_{-2}]|\phi\rangle$ using the current modes, which is actually kind of gross. Keeping the terms in the mode expansion that survive, (namely $m = -2, -1, 0$) we have (repeated indices are summed)

$$\langle\phi|[L_2, L_{-2}]|\phi\rangle = \frac{1}{\gamma^2}\langle\phi|(-2t^a J_2^a + J_1^a J_1^a)(-2J_{-2}^b t^b + J_{-1}^b J_{-1}^b)|\phi\rangle. \tag{243}$$

We now must painstakingly commute the terms in the left group through until they act on the ket on the right. After doing the first series of commutations, we have

$$\begin{aligned}\langle\phi|[L_2, L_{-2}]|\phi\rangle = \gamma^{-2}\langle\phi|\Big[&4t^a(if^{abc}J_0^c + 2k\delta_{ab})t^b - 2t^a if^{abc}J_1^c J_{-1}^b - 2J_1^a if^{abc}J_{-1}^c t^b \\ &+ if^{abc}if^{acd}J_1^d J_{-1}^b - it^c f^{abc}J_1^a J_{-1}^b + k\delta_{ab}J_1^a J_{-1}^b + (k\delta_{ab} + if^{abc}J_0^c)(k\delta_{ab} + if^{abd}J_0^d)\Big]|\phi\rangle.\end{aligned} \tag{244}$$

Now we let the J_0^a's act on the primary fields and replace the remaining $J_1^a J_{-1}^b$'s with $if^{abc}J_0^c + k\delta_{ab}$. A few of the resulting terms die by the antisymmetry of the structure constants. A few comtain three t^a generators, but they always appear with some f^{abc}'s so we can antisymmetize them into two t^a's plus an extra f^{abc}. Doing this and again using the definition of the dual coexter number, we find

$$\langle\phi|[L_2, L_{-2}]|\phi\rangle = \frac{1}{\gamma^2}\left[\langle\phi|t^a t^a|\phi\rangle(8k + 8g) + (2kg + 2k^2)\dim G\right], \tag{245}$$

with $\dim G = \delta_{aa}$. Now the first bit on the RHS is $\gamma^{-2}\langle\phi|t^a t^a|\phi\rangle(8k + 8g) = 4\gamma^{-1}\langle\phi|t^a t^a|\phi\rangle$, which cancels against the $-4\gamma^{-1}\langle\phi|t^a t^a|\phi\rangle$ term in (242). Thus we have found that the central charge is given by

$$\frac{c}{2} = \frac{2kg + 2k^2}{\gamma^2}\dim G \implies c = \frac{k\dim G}{k + g}. \tag{246}$$

As examples, for $\mathfrak{su}(n)$ (for $n \geq 2$), the dual coxeter number is $g = n$, so that

$$c_{\widehat{\mathfrak{su}}(n)_k} = \frac{k(n^2 - 1)}{k + n}. \tag{247}$$

For the familiar case of $n = 2, k = 1$ we have $c = 1$, which agrees with the calculation we did earlier showing how the compact boson at the self-dual radius could be mapped onto the $\widehat{\mathfrak{su}}(2)_1$ CFT. Another interesting case is $n = k = 2$, for which

$$c_{\widehat{\mathfrak{su}}(2)_2} = \frac{3}{2}, \tag{248}$$

which hints at representations in terms of either three Majoranas or in terms of a free Majorana and a free boson (both are possible). Likewise, for $\widehat{\mathfrak{u}}(n)_k$ we have

$$c_{\widehat{\mathfrak{u}}(n)_k} = 1 + c_{\widehat{\mathfrak{su}}(n)_k}, \tag{249}$$

where the $+1$ is from the decoupled Abelian $U(1)$ factor. For example, when $k = 1$ we have

$$c_{\widehat{\mathfrak{u}}(n)_k} = n, \tag{250}$$

which is compatible with a free-fermion realization in terms of n Dirac fermions.

For $\widehat{\mathfrak{so}}(n)$ we have $g = n - 2$ so that

$$c_{\widehat{\mathfrak{so}}(n)_k} = \frac{\frac{k}{2}(n^2 - n)}{(n-2) + k}. \tag{251}$$

In particular, for $k = 1$ we have

$$c_{\widehat{\mathfrak{so}}(n)_1} = \frac{n}{2}, \tag{252}$$

which is compatible with $\widehat{\mathfrak{so}}(n)_1$ being realized by n free Majorana fermions. For $k = g = n - 2$, we have

$$c_{\widehat{\mathfrak{so}}(n)_1} = \frac{n^2 - n}{4}, \tag{253}$$

compatible with a realization in by $(n^2 - n)/2$ free Majoranas.

13 *Free fermion representation of $\widehat{\mathfrak{so}}(N)_g$ current algebra* ✓

This is a problem from the big yellow book, chapter 15. Here's the problem statement:

Consider N Majorana fermions transforming in the adjoint representation of $SO(N)$. Show how to build currents with these fermions that satisfy the $\widehat{\mathfrak{so}}(N)_g$ current algebra, where g is the dual coxeter number (see yesterday's diary entry). Compute the central charge. Answer:

$$c = \frac{1}{4}N(N-1), \tag{254}$$

which is precisely what one would expect from $\dim[\mathfrak{so}(N)] = N(N-1)/2$ (the dimension of the adjoint representation) flavors of real free fermions.

The fermions have dimension 1/2 while WZW currents have dimension $(1, 0)$, and so the J^a's we write down will need to be bilinear in the fermions. The natural choice, since we are woking in the adjoint representation, is to construct the currents with the structure constants:

$$J^a(z) = \alpha i f^{abc} \psi_b(z)\psi_c(z) \tag{255}$$

where α is some yet-to-be-determined constant. We can fix it by computing the OPE for the current. Using $\psi_a(z)\psi_b(w) \sim \delta_{ab}(z-w)^{-1}$, this is straightforward: remembering the minus signs when moving fermions around, we have

$$\begin{aligned} J^a(z)J^d(w) &\sim -\alpha^2 f^{abc} f^{def} \left(\frac{\delta_{ec}\delta_{bf} - \delta_{cf}\delta_{be}}{(z-w)^2} + \frac{1}{z-w} [\psi_b\psi_f\delta_{ce} - \psi_b\psi_e\delta_{cf} - \psi_c\psi_f\delta_{be} + \psi_c\psi_e\delta_{bf}] \right) \\ &\sim -\alpha^2 \left(-\frac{4g\delta_{ad}}{(z-w)^2} + \frac{2}{z-w} f^{abc}(\psi_b\psi_f f^{dcf} + \psi_c\psi_e f^{deb}) \right) \\ &\sim \frac{4g\alpha^2}{(z-w)^2}\delta_{ad} + i\left(\frac{4\alpha^2 i}{z-w} f^{abc} f^{dcf} \psi_b\psi_f \right). \end{aligned} \tag{256}$$

Now the anticommuting property of the fermions lets us use the identity

$$f^{abc} f^{dcf} \psi_b\psi_f = \frac{1}{2} f^{ade} f^{efg} \psi_f\psi_g. \tag{257}$$

To derive this, we use the Bianchi identity in the form

$$f^{ade} f^{efg} \psi_f\psi_g = -(f^{aeg} f^{dfe} + f^{deg} f^{fae})\psi_f\psi_g. \tag{258}$$

Plugging this in, using the antisymmetry of the ψ_a's and relabelling a bunch of variables gives the sought-for identity. Putting (257) in to the $J^a J^d$ OPE, we see that we get the OPE for an affine $\widehat{\mathfrak{so}}(N)$ algebra provided that $\beta = 1/2$ (because this choice of β fixes the $if^{ade}J^e/(z-w)$ term appearing in the OPE to have a coefficient of unity). So then recapitualating, the properly normalized currents are

$$J^a = \frac{1}{2} i f^{abc} \psi_b(z)\psi_c(w), \tag{259}$$

and they have the OPE

$$J^a(z)J^b(w) \sim \frac{g\delta_{ab}}{(z-w)^2} + i f^{abc} \frac{J^c(w)}{z-w}. \tag{260}$$

Thus the current algebra is the $\widehat{\mathfrak{so}}(N)_g$ algebra.

The stress tensor is constructed using the Sugawara strategy described in yesterday's diary entry, and we can use the formula derived yesterday to compute the central charge. Since we are at level $k = g$, we have

$$c = \frac{k \dim \mathfrak{so}(N)}{k+g} = \frac{1}{2} \dim \mathfrak{so}(N). \tag{261}$$

Now for $\mathfrak{so}(N)$ we have $\dim \mathfrak{so}(N) = \frac{1}{2}N(N-1)$,[7] meaning that

$$c = \frac{1}{4}N(N-1), \tag{263}$$

[7] Why? Consider the symmetrizer map $S : M \mapsto MM^T - \mathbf{1}$. Now $O(N) = \ker(S)$, while $\dim \operatorname{im}(S) = \sum_{i=1}^N i = \frac{1}{2}N(N+1)$ is the dimension of all symmetric matrices. So then

$$\dim O(N) = \dim GL(N) - \dim \operatorname{im}(S) = N - \frac{1}{2}N(N+1) = \frac{1}{2}N(N-1). \tag{262}$$

Then since $\dim \mathfrak{so}(N) = \dim \mathfrak{o}(N) = \dim O(N)$, $\dim \mathfrak{so}(N) = \frac{1}{2}N(N-1)$.

which is equal to 1/2 (central charge of each Majorana) times the number of fermion fields. This hints that even though from the Sugawara construction the theory doesn't look free, it actually is. This can be verified by carefully checking the equivalence of the seemingly interacting Sugawara ($J^a J^a$) stress tensor and $N(N-1)/2$ copies of $T_\psi = -\frac{1}{2}\psi\partial\psi$. Showing this is straightforward; see the big yellow book chapter 15 for hints.

14 *Sphere partition functions, the central charge, and the Weyl anomaly ✓*

Today we will show that the partition function of any 2d CFT on a sphere of radius a obeys the relation

$$\frac{d\ln Z_{S^2}}{d\ln a} = \frac{c}{3}. \tag{264}$$

To derive this, we will need to think about the trace anomaly and the central charge.

The metric on the sphere, in coordinates stereographically projected onto the plane, is $g_{\mu\nu} = \mathbf{1}_{\mu\nu}4a^2/(1+|x|^2)^2$, so that the line element is $ds^2 = \frac{4a^2}{(1+|r|^2)^2}(dx^2+dy^2)$. This means that when we vary the radius of the sphere, the variation of the metric is proportional to the metric itself: for $a \mapsto a + \delta a$ we use $g_{\mu\nu}(x) \propto a^2$ to write

$$g_{\mu\nu} \mapsto g_{\mu\nu} + 2(\delta a)a^{-1}g_{\mu\nu}. \tag{265}$$

So then applying the Ward identity (with no operator insertions), we have

$$\delta \ln Z = -\frac{1}{2}\int d^2x\, \langle T^{\mu\nu}(x)\delta g_{\mu\nu}(x)\rangle = -\int d^2x\,\sqrt{g}\langle T^{\mu\nu}(x)\rangle \delta a a^{-1} g_{\mu\nu}(x), \tag{266}$$

and so

$$\frac{d\ln Z}{d\ln a} = -\int d^2x\,\sqrt{g}\langle \mathrm{Tr}\, T_{\mu\nu}\rangle. \tag{267}$$

Because of the trace anomaly (which is relevant since we're on a sphere), this will be non-zero.

We will calculate $\langle \mathrm{Tr} T_{\mu\nu}\rangle$ by working infinitesimally and finding $\delta\langle T\rangle$, where the variation is a Weyl transformation of the metric. We can get away with doing this because we know what the form of the answer will be. Indeed, since $\langle \mathrm{Tr} T_{\mu\nu}\rangle = 0$ classically and since we are working with a CFT, the only way in which $T_{\mu\nu}$ could fail to be traceless is for there to be some of local anomaly coming from a contact term (the kind of thing which ruins the tracelessness of T in e.g. QED in four dimensions comes from scales generated during the RG flow: since we are working with a genuine CFT this sort of thing cannot happen). Since this involves UV physics and can only depend on the spacetime metric, $\langle \mathrm{Tr} T_{\mu\nu}\rangle$ must be proportional to R (the Ricci scalar), since this is the only metric-dependent, local, mass-dimension-2 scalar function that could fit the bill. We then just have to determine the coefficient β in $\langle \mathrm{Tr} T_{\mu\nu}\rangle = \beta R$, and so finding the variation of the trace of $T_{\mu\nu}$ under a Weyl transformation is good enough for our present goal.

We will choose coordinates where the metric takes the form $g_{\mu\nu} = e^{\phi(x)}\eta_{\mu\nu}$, so that an infinitesimal variation of the metric away from flat space is $\delta g_{\mu\nu}(x) = \delta\phi(x)\eta_{\mu\nu}$. The Ricci scalar can be found after some pretty heinous algebra which I'd rather not type up to be

$$R = -e^{-\phi}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi. \tag{268}$$

This implies a fact that we will need later, namely that

$$\sqrt{g}R(x) = -\Box\phi(x) = -4\partial\bar{\partial}\phi(x), \tag{269}$$

where the factor of 4 comes from our conventions where e.g. $\partial = \frac{1}{2}(\partial_0 - i\partial_1)$.

Now we return to finding the trace of T. Upon varying $g_{\mu\nu}$ away from flat space, we have (going to be using σ, σ' instead of x, y for Cartesian coordinates from now on)

$$\delta\langle T_\mu{}^\mu(\sigma)\rangle = -\frac{1}{2}\int d^2\sigma'\, \langle T_\mu{}^\mu(\sigma) T_\nu{}^\nu(\sigma')\rangle \delta\phi(\sigma'), \tag{270}$$

since the variation of the metric is proportional to $\eta_{\mu\nu}$. Now we need the OPE between the two traces of the stress tensor on the RHS of this equation.

Getting this in a precise way is a bit tricky: we will hybridize an argument in some notes by Komargodski and and an argument in Tong's CFT notes. Going back to Cartesian coordinates, we examine the two point function $\langle T_{\mu\nu}(q)T_{\alpha\beta}(p)\rangle$, where q, p are momenta. By energy conservation, this two-point function needs to be killed by $\partial_\mu, \partial_\nu, \partial_\alpha$, and ∂_β. It also needs to contain a δ function enforcing momentum conservation, and so it needs to have the form

$$\langle T_{\mu\nu}(q)T_{\alpha\beta}(p)\rangle = \delta(p+q)\left[\frac{f(q^2)}{2}(\Pi^T_{\mu\alpha}(q)\Pi^T_{\nu\beta}(q) + \Pi^T_{\mu\beta}(q)\Pi^T_{\nu\alpha}(q)) + g(q^2)\Pi^T_{\mu\nu}(q)\Pi^T_{\alpha\beta}(q)\right], \tag{271}$$

where of course the transverse projectors are $\Pi^T_{\alpha\beta}(q) = q_\alpha q_\beta - q^2\eta_{\alpha\beta}$. We can fix the form of the functions f, g by requiring scale invariance: thus they must be algebraic in q^2, and they have to go as $1/q^2$ because of requiring the two-point function to be invariant under $q \mapsto \lambda q$ for $\lambda \in \mathbb{R}$ (remember that the delta function also transforms!). So, we can write $f(q^2) = a/q^2, g(q^2) = b/q^2$. Anyway, by taking $\mu = \nu$ and summing, we get

$$\langle T_\mu{}^\mu(p)T_{\alpha\beta}(q)\rangle = \delta(p+q)\left[\frac{a}{q^2}\Pi^T_{\mu\alpha}(q)\Pi^{T\mu}_\beta(q) + \frac{b}{q^2}\Pi^{T\mu}_\mu(q)\Pi^T_{\alpha\beta}(q)\right] = -\delta(p+q)(a+b)\Pi^T_{\alpha\beta}(q). \tag{272}$$

Contracting one more time and Fourier transforming, we have

$$\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \propto \Box\delta(\sigma-\sigma'), \tag{273}$$

where the proportionality constant $(a+b)$ is something that we can't determine with this method. The point of these couple of steps is to show that the two-point function of the traces has to be proportional to a double-derivative of a δ function. To find the proportionality constant, we need another argument.

In complex coordinates, conservation of energy is $\partial T^{z\bar{z}}+\bar{\partial}T^{\bar{z}\bar{z}}=0$, or $\partial T_{z\bar{z}}-\bar{\partial}T/(2\pi)=0$. Now with our conventions $T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')=-16T_{\bar{z}z}(z)T_{\bar{w}w}(w)$. Using conservation of energy then, we have

$$\langle \partial_z T_{z\bar{z}}(z)\partial_w T_{w\bar{w}}(w)\rangle = -\frac{1}{4\pi^2}\bar{\partial}_z\bar{\partial}_w\langle T(z)T(w)\rangle. \tag{274}$$

We know what the RHS is, since we know the TT OPE. We have

$$\langle \bar{\partial}_z T_{z\bar{z}}(z)\bar{\partial}_w T_{w\bar{w}}(w)\rangle \sim \frac{1}{4\pi^2}\bar{\partial}_z\bar{\partial}_w\frac{c/2}{(z-w)^4}+\cdots = \frac{c}{48\pi^2}\partial_z^2\partial_w\bar{\partial}_z\bar{\partial}_w\frac{1}{z-w}+\ldots, \tag{275}$$

where $\ldots$ represents things that contain $\bar{\partial}_z T(z)$ and derivatives thereof. Now we need to use

$$\bar{\partial}_z\frac{1}{z-w}=2\pi\delta(z-w,\bar{z}-\bar{w}), \tag{276}$$

which one can prove by using $\int_R d^2z\bar{\partial}f(z,\bar{z})=-i\oint_{\partial R}f(z,\bar{z})$ with the function $f=1/(z-w)$ and taking R to be a region containing the point w. Thus the OPE we've been looking at is

$$\langle \bar{\partial}_z T_{z\bar{z}}(z)\bar{\partial}_w T_{w\bar{w}}(w)\rangle \sim -\partial_z\partial_w\left(\frac{c}{24\pi}\partial_z\bar{\partial}_z\delta(z-w,\bar{z}-\bar{w})\right)+\ldots, \tag{277}$$

so then using the earlier relation we found between the various stress energy tensor two-point functions, we have

$$\frac{1}{16}\partial_z\partial_w\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \sim -\partial_z\partial_w\left(\frac{c}{24\pi}\partial_z\bar{\partial}_z\delta(z-w,\bar{z}-\bar{w})\right)+\ldots \tag{278}$$

This is where the earlier analysis we did of the two-point function of T will come in handy: we know that the two point function of the traces has to be proportional to $\Box\delta(\sigma-\sigma')$. This lets us do two things: first, it lets us drop the $\ldots$ (terms that go as $\bar{\partial}_z T$; of course classically this is zero anyway) on the RHS, since we know that no operators (only c-numbers) can appear on the RHS (actually, I guess the extra terms also die because they have nonzero spin, and we have rotational invariance). It also lets us strip away the derivatives and conclude that

$$\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \sim -\frac{2c}{3\pi}\partial_z\bar{\partial}_z\delta(z-w,\bar{z}-\bar{w}). \tag{279}$$

Naively this conclusion could only be reached modulo singular terms in the kernel of $\partial_z\partial_w$, but since we know that the LHS has to be proportional to $\Box\delta(x-y)$, such terms will not appear. Now we can change the δ function and the derivatives over to σ,σ' coordinates at the cost of a factor of $1/8$ (2 from each derivative and 2 from the δ function since $dzd\bar{z}\to 2dxdy$), so that

$$\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \sim -\frac{c}{12\pi}\Box\delta(\sigma-\sigma'). \tag{280}$$

Finally, putting this into our expression for $\delta\langle T_\mu{}^\mu\rangle$, we have

$$\delta\langle T_\mu{}^\mu(\sigma)\rangle = \frac{c}{24\pi}\int d^2\sigma'\,\Box\delta(\sigma-\sigma')\delta\phi(\sigma') = \frac{c}{24\pi}\Box\delta\phi(\sigma). \tag{281}$$

Using our earlier result for the variation of the Ricci scalar, we get

$$\delta\langle T_\mu{}^\mu(\sigma)\rangle = -\frac{c}{24\pi}\delta R(\sigma). \tag{282}$$

Therefore, integrating over the variation, we can conclude that

$$\langle T_\mu{}^\mu(\sigma)\rangle = -\frac{c}{24\pi}R(\sigma), \tag{283}$$

which holds in all geometries, not just those infinitesimally close to flat space (as we discussed earlier, the form $\langle T_\mu{}^\mu(\sigma)\rangle \propto R$ is required, and the coefficient of proportionality of course is geometry-independent).

Thus using our expression for $\delta \ln Z$ under a change in a (the radius of the sphere), we finally have

$$\frac{d\ln Z}{d\ln a} = -\frac{c}{24\pi}\int d^2\sigma\, \sqrt{g}R(\sigma). \tag{284}$$

On a sphere the Ricci scalar is $2/a^2$[8], so since $\int d^2\sigma\,\sqrt{g} = 4\pi a^2$,

$$\frac{d\ln Z}{d\ln a} = \frac{c}{3}, \tag{286}$$

which is what we wanted to show.

One final comment is that this whole derivation relied on determining the contact term of $\mathrm{Tr}\, T_{\mu\nu}$ with itself. It's often the case in QFT that contact terms like this are non-universal and depend on our regularization scheme. This isn't the case here, since the contact term in question is determined by the TT OPE, i.e. by correlation functions of the stress tensor at *separated* points. This information is universal, and ensures that the contact term is universal as well. These kind of universal contact terms usually show up in the OPEs of currents and in the context of anomalies; more on this in a future diary entry.

[8] A handy formula for the Ricci scalar is $R = 2\theta/A$, where θ is the angle between a vector and the image of itself parallely-transported around the boundary of a small geodesic ball (a ball bounded by geodesics), and A is the area of the loop (I think the factor of 2 is correct; some places seem to not have it). For the sphere, parallel transporting a vector halfway around the equator and then back along the prime meridian rotates the vector by π, so that the Ricci scalar is

$$R = 2\frac{\pi}{4\pi a^2/4} = 2/a^2. \tag{285}$$

15 *The central charge and entanglement entropy* ✓

Today we're deriving another classic result that I'd seen quoted a bunch but had never seen the derivation of. We will show that, for a 2D CFT in flat space with central charge c, the entanglement entropy for an interval $[a,b]$ is

$$S \sim c \ln |a-b|. \tag{287}$$

The strategy will be to use the replica trick and think about twist operators.

We use the replica trick in the usual way, writing for $A = [a,b]$,

$$S_A = -\lim_{n\to 1} \mathrm{Tr}[\rho_A \ln \rho_A^n] = -\lim_{n\to 1} \partial_n \mathrm{Tr}[\rho_A^n] = -\lim_{\varepsilon\to 0} \frac{\mathrm{Tr}[\rho_A^{1+\varepsilon}] - \mathrm{Tr}[\rho_A]}{\varepsilon} = \lim_{n\to 1} \frac{\mathrm{Tr}[\rho_A^n]-1}{1-n}, \tag{288}$$

where $\varepsilon = n-1$ (the limit is $n \to 1^+$) and we've used $\mathrm{Tr}[\rho_A] = 1$. As usual, we calculate the nth power of the reduced density matrix by calculating the partition function on the n-sheeted replica manifold. If we let $\Sigma(a)$ denote an n-fold twist operator (which inserts the endpoint of a $z^{1/n}$ branch cut at a), then traveling around a with $\Sigma(a)$ inserted is equivalent to moving between sheets of the Riemann surface. Since each of the n sheets (each a copy of $\mathbb{C}$) has the twist operators inserted, the trace in the expression for S_A is the same as the nth power of the two-point function of the twist operators:

$$\mathrm{Tr}[\rho_A^n] = \langle \Sigma(a)\Sigma(b)\rangle \sim \frac{1}{|a-b|^{2n\Delta_\Sigma}}. \tag{289}$$

We can get the scaling dimension Δ_Σ of the twist operator by the following argument. Let w be the coordinate on the n-sheeted Riemann surface used in calculating ρ^n, and let z be a coordinate on $\mathbb{C}$. Now zoom in to one of a, b, and consider a map which locally takes the Riemann surface to $\mathbb{C}$ in a region around this point. We can map the Riemann surface to the plane by taking $z = w^{1/n}$, which "unwinds" the Riemann surface and which is single valued when acting on w (we can e.g. think about w living in a space where $e^{2\pi i n} = 1$ but $e^{2\pi i} \neq 1$). Then we can use the transformation law for the stress tensor to write

$$\langle T(w)\rangle = \left\langle \partial_w z T(z) + \frac{c}{24}\{z;w\}\right\rangle = \frac{c}{24}\{w^{1/n};w\}, \tag{290}$$

since on the plane the vev of the stress tensor vanishes (here we are thinking about the $T(w)$ insertion as occurring only on a single sheet of the replica manifold). The Schwartzian derivative is easily computed:

$$\{w^{1/n};w\} = (n^{-1}-1)(n^{-1}-2)w^{-2} - \frac{3}{2}(n^{-1}-1)^2 w^{-2} = \frac{1}{2}(1-n^{-2}), \tag{291}$$

so that

$$\langle T(w)\rangle = \frac{1}{w^2}\frac{c}{24}(1-n^{-2}). \tag{292}$$

Thus we read off the conformal dimension of the twist operators as[9]

$$h_\Sigma = \frac{c}{24}(1 - 1/n^2), \tag{294}$$

so that the scaling dimension is $\Delta_\Sigma = c(1 - n^{-2})/12$. Some sanity checks: when $n \to 1$ the twist operator becomes trivial, and $\Delta_\Sigma \to 0$ as required. Secondly, as $n \to \infty$ we have $\Delta_\Sigma \to c/12$. This makes sense because $n \to \infty$ gives a replica manifold that becomes a cylinder: the two ends of the branch cut become the two ends of the cylinder, with $n \to \infty$ meaning that wrapping around the twist operators never returns one to one's starting point (the cohomology of the cylinder is $\mathbb{Z}$, which has no elements of finite order—this is the limit $\mathbb{Z}_n \to \mathbb{Z}$ for $n \to \infty$). This agrees with $\Delta_\Sigma \to c/12$ since we know that on the cylinder, $\langle T_{cyl}(z)\rangle = cz^{-1}/24$ (see previous diary entry), giving $h_\Sigma = c/24$ and $\Delta_\Sigma = c/12$.

Anyway, using the scaling dimension we get

$$\mathrm{Tr}[\rho_A^n] = \alpha|a - b|^{-\frac{c}{6}(n - n^{-1})}, \tag{295}$$

where α is some constant that we won't be able to determine. This gives

$$S_A = -\lim_{n\to 1} \partial_n \mathrm{Tr}[\rho^n] = -\lim_{n\to 1} \partial_n e^{-\frac{\alpha c}{6}(n-n^{-1})\ln|a-b|} = \alpha\frac{c}{3}\ln|a - b|, \tag{296}$$

as required (the argument of the log will be made dimensionless with some short-distance cutoff that we haven't been writing).

16 *Basic conformal perturbation theory* ✓

Today we will look at how we can use the algebraic structure of a CFT to describe the RG flow in the vinicity of its fixed point (I read about this problem in a review on bosonization

[9] Note that we have cheated somewhat here by working locally around one of the branch cut points, essentially assuming that the branch cut runs from $z = 0$ to $z = \infty$, when in reality it runs from a to b. We can correct for this by mapping the branch cut to one stretching from 0 to ∞ by using the conformal mapping $z = (w - a)^{1/n}/(w - b)^{1/n}$ (instead of $z = w^{1/n}$). Now the resulting OPE with the stress tensor needs to have singularities at each of the termination points of the branch cuts and needs to vanish when $a = b$, and so the general form of the vev of $T(w)$ is in fact

$$\langle T(w)\rangle = \frac{c}{24}(1 - n^{-2})\frac{(a - b)^2}{(w - a)^2(w - b)^2}. \tag{293}$$

This of course gives us the same conformal dimension as derived in the non-footnoted text.

somewhere; sadly I forget which reference. This is pretty standard stuff, though). Anyway, consider perturbing a CFT by adding to the action the term

$$S = S_{CFT} + \int d^d x \, g_\alpha \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha}, \tag{297}$$

where the sum on α is implied and g_α are *dimensionless* couplings, with the $a^{-d+\Delta_\alpha}$ present to make the dimensional of the integrand correct. For studying perturbations around a fixed point, the most interesting operators to choose will be marginal ones, for which $\Delta_\alpha = d$.

Working perturbatively in the couplings g_α, we will be finding the $O(g_\alpha g_\beta)$ β functions for the g_α couplings in terms of the OPE data.

There are two ways of getting the result, which are actually rather similar: one uses dimensional analysis and block spin RG, while the other does the RG in a more symmetric way. Both methods take place in $\mathbb{R}$ space, due to the fact that we will need to use the OPEs coming from the CFT, which are awkward to formulate in momentum space.

Both methods start by writing the partition function as (here ϕ is some stand-in for an arbitrary collection of fields)

$$Z = \int \mathcal{D}\phi e^{-S_0[\phi]} \left(1 - g_\alpha \int d^d x \, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} + \frac{1}{2} g_\alpha g_\beta \int d^d x d^d y \, \mathcal{O}^\alpha(x) \mathcal{O}^\alpha(y) a^{-2d+\Delta_\alpha+\Delta_\beta} - \dots \right). \tag{298}$$

The plan will be to do some sort of real space reornmalization group step, and see how the effective coupling constants in front of each term in the expansion change.

First for the block spin method. This one is rather hueristic, so forgive the handwaving in what follows. Let's first look at the linear term, which we won't need any CFT to deal with. One one hand, we can write it as (this is not to best way of getting the first-order beta function, but it is instructive for what will follow[10])

$$g_\alpha \int d^d x \, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = \sum_I \sum_{i \in I} g_\alpha \widetilde{\mathcal{O}}^\alpha(x_i). \tag{300}$$

Here I denotes a block spin site consiting of lattice sites x_i, and $\widetilde{\mathcal{O}}^\alpha(x_i)$ is the dimensionless operator defined by $\widetilde{\mathcal{O}}^\alpha = a^{\Delta_\alpha} \mathcal{O}^\alpha$. Now let the linear size of each block be L. Then just by dimensional analysis, we have

$$g_\alpha \int d^d x \, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = \sum_I a^{-d+\Delta_\alpha} \int_{\text{block}} d^d x \, g_\alpha \mathcal{O}^\alpha = \sum_I g_\alpha (L/a)^{d-\Delta_\alpha} \widetilde{\mathcal{O}}^\alpha(x_I). \tag{301}$$

[10]Recall a simpler way: we rescale $a \mapsto a + da, g \mapsto g + dg$ and ask what dg needs to be such that the partition function is preserved to first order in the couplings. We have

$$g_\alpha \int d^d x \, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = (g_\alpha + dg_\alpha) \int d^d x \, \mathcal{O}(x) a^{-d+\Delta_\alpha} (1 + (-d + \Delta_\alpha) d \ln a) \implies dg_\alpha = d \ln a (d - \Delta_\alpha) g_\alpha, \tag{299}$$

which gives in our (high energy) convention $\beta_\alpha = (\Delta_\alpha - d) g_\alpha$ to first order in the coupling.

We could write the result of the integration as being proportional to $L^{d-\Delta_\alpha}$ since because we are perturbing around a CFT, there is no dimensionful parameter from the theory (like a correlation length) to use in place of L to obtain the required dimensionality (other than I guess a, but this can't appear since we need the beta functions to not be explicitly dependent on a). Taking the block size L to be infinitesimally larger than the lattice spacing, $L = a + da$, we get

$$g_\alpha \int d^d x\, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} \to \sum_I g_\alpha (1 + (d-\Delta_\alpha) d \ln a) \widetilde{\mathcal{O}}^\alpha(x_I). \tag{302}$$

Thus we see that we get the same thing as the original term (which was $\sum_i g_\alpha \widetilde{\mathcal{O}}^\alpha(x_i)$), just with a different coupling cosntant $g_\alpha(L)$. Then defining the β function as $\beta_\alpha = +d_{\ln a} g_\alpha$, we see that to first order in the couplings, we get the expected answer:

$$\beta_\alpha = (d - \Delta_\alpha) g_\alpha + O(g_\alpha^2). \tag{303}$$

There are several things to object to about this, some of which may be fixable. First, what do you mean "take L to be infinitessimally larger than a? You're on a lattice!". Indeed, taking $L = a + da$ is very formal. Secondly, the dimensional analysis part was kind of lame, since we actually had no way of computing any sort of geometry-dependent factors that are associated with the block spin procedure. The kind of wonky combination of mixing integrating in the continuum with the block spin aproach is also ugly. Worst of all, when we evaluated the integral with dimensional analysis, we should have obtained a sum $\sum_\beta \widetilde{\mathcal{O}}^\beta$ of different dimensionless operators that could be produced during the RG step, but we only kept the original $\beta = \alpha$ operator.

Now we look at the second order piece. We write it as

$$\begin{aligned}\frac{1}{2} g_\alpha g_\beta \int d^d x d^d y\, \mathcal{O}^\alpha(x) \mathcal{O}^\alpha(y) a^{-2d+\Delta_\alpha+\Delta_\beta} &= \frac{1}{2} \sum_{I \neq J} g_\alpha g_\beta (1 + (2d - \Delta_\alpha - \Delta_\beta)) \\ \times (1 - d \ln a) \widetilde{\mathcal{O}}^\alpha(x_I) \widetilde{\mathcal{O}}^\beta(x_J) &+ \frac{1}{2} g^\alpha g^\beta \sum_I \int_{\text{block}} d^d x d^d y\, \mathcal{O}^\alpha(x_I + x) \mathcal{O}^\beta(x_I + y) a^{-2d+\Delta_\alpha+\Delta_\beta}.\end{aligned} \tag{304}$$

The first term becomes the quadratic part of the expansion of the exponential of the action after performing the RG step, and so will not be important in what follows (plus for the case we are most interested in, where both $\mathcal{O}^\alpha, \mathcal{O}^\beta$ are marginal, this term has no affect on the β function).

Now since the distances between different sites in a single block are "below the resolution" of the theory after doing the blocking, we will take the distance $|x - y|$ to be small enough for us to profitably use the OPE. Of course this is a bit hand wavy (what about all those neighboring lattice sites whose connecting links cut a block boundary?), but we will stick with it. Using this and then doing the integral over the intra-block coordinates using dimensional analysis (again, the only dimensionful scale we have by virtute of perturbing around a CFT is L), the second term in the previous equation is,

$$\frac{1}{2}g_\alpha g_\beta \sum_I \int_{\text{block}} d^dx d^dy\, C^{\alpha\beta}_\gamma \frac{1}{|x-y|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}} a^{-2d+\Delta_\alpha+\Delta_\beta}\mathcal{O}^\gamma(x_I+x)$$
$$= \frac{1}{2}g_\alpha g_\beta \sum_I C^{\alpha\beta}_\gamma \widetilde{\mathcal{O}}^\gamma(x_I)(L/a)^{2d-\Delta_\alpha-\Delta_\beta} \tag{305}$$
$$\to \frac{1}{2}g_\alpha g_\beta \sum_I C^{\alpha\beta}_\gamma \widetilde{\mathcal{O}}^\gamma(x_I)(1+(2d-\Delta_\alpha-\Delta_\beta)d\ln a),$$

where we have taken $L \to a + da$ in the last step and assumed that $2d - \Delta_\alpha - \Delta_\beta \neq 0$. Acutally, the more relevant case (or in light of the subject, maybe I should say the more interesting case) is when all the perturbing operators are marginal. In this case dimensional analysis produces a $\ln(L/a)$, and the second term instead becomes

$$\frac{1}{2}g_\alpha g_\beta \sum_I C^{\alpha\beta}_\gamma \widetilde{\mathcal{O}}^\gamma(x_I)d\ln a. \tag{306}$$

We see that this term gives a contribution to the g_γ beta function, since it has the effect of just changing the $\widetilde{\mathcal{O}}^\gamma$ coupling constant in the block spin theory. Remembering the sign difference between this and the linear term in the expansion (298), we see that this gives

$$\beta_\gamma = -\frac{1}{2}g_\alpha g_\beta C^{\alpha\beta}_\gamma, \tag{307}$$

where we have assumed that the associated operator $\mathcal{O}_\gamma$ is marginal so that no linear part appears (again, this is the most interesting case for perturbing about a CFT fixed point).

Of course, all the gripes about the non-rigorous nature of this method that we raised when deriving the linear part of the β funciton can be raised here. In order to feel better about our result, we briefly discuss a way to make it a bit more precise by using a more symmetric approach for dealing with the second order term which I just learned about from Cardy's book (Renormalization and Scaling in Statistical Physics).

The basic idea is that we can remain in the continuum and treat the cutoff not as a lattice spacing per se, but rather just as the closest distance that operators are allowed to get from one another. So, we picture the operators as hard spheres, with the radii of the sphere being set by the cutoff. An RG step then proceeds by integrating out all terms that include operators separated a distance between a and $a + \delta a$ from one another, and then re-scaling coordinates in the usual way.[11] This makes dealing with the second order term super easy:

$$\frac{1}{2}g_\alpha g_\beta \int_{|x-y|\geq a} d^dx d^dy\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y)a^{-2d+\Delta_\alpha+\Delta_\beta} = \frac{1}{2}g_\alpha g_\beta \Big(\int_{|x-y|\geq L} d^dx d^dy\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y)a^{-2d+\Delta_\alpha+\Delta_\beta}$$
$$+ \int_{a\leq|x-y|<L} d^dx d^dy\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y)a^{-2d+\Delta_\alpha+\Delta_\beta}\Big). \tag{308}$$

[11] Only the operators in the perturbations we added behave as hard spheres; the operators in the unperturbed CFT action don't need to be dealt with in this way because their correlation functions are already scale-invariant to begin with.

Doing the renormalization step means changing the effective cutoff to L by doing the second integral on the RHS and abosrbing the result into a rescaling of the coupling constants. We do this by using the OPE: taking $L = a + da$ and using translation invariance we get

$$\begin{aligned}\frac{1}{2}g_\alpha g_\beta \int_{a\leq|x-y|<L} d^dxd^dy\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y)a^{-2d+\Delta_\alpha+\Delta_\beta}& \\ = \frac{1}{2}g_\alpha g_\beta C^{\alpha\beta}_\gamma \int d^dx\, \frac{1}{a^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}} da A(S^{d-1})a^{-d-1+\Delta_\alpha+\Delta_\beta}\mathcal{O}^\gamma(x)& \\ = \frac{1}{2}g_\alpha g_\beta C^{\alpha\beta}_\gamma A(S^{d-1})d\ln a \int d^dx\, \mathcal{O}^\gamma(x)a^{-d+\Delta_\gamma},&\end{aligned} \tag{309}$$

where $A(S^{d-1})$ is the area of the unit S^{d-1}. This gives essentially the same contribution to the β function as we got with the rather hand-waving block spin method, up to a factor of $A(S^{d-1})$. Indeed, assuming again that $\mathcal{O}^\gamma$ is marginal, we get

$$\beta_\gamma = -\frac{1}{2}A(S^{d-1})g_\alpha g_\beta C^\gamma_{\alpha\beta}. \tag{310}$$

The unasthetic factor of the sphere area can be gotten rid of by absorbing into the coupling constants.

As an example, consider e.g. $\mathfrak{su}(2)_k$. The current operators $J, \bar{J}$ for the $SU(2)_L$ and $SU(2)_R$ symmetries, respectively) have dimension 1, and so current-current interactions are marginal (they also must be dimension 2 since the stress tensor is built out of J^aJ^a terms via the Sugawara construction). Consider deforming a WZW CFT with an anisotropic "Thirring model" type current-current interaction:

$$\mathcal{L}_{int} = \sum_a \lambda_a J^a \bar{J}^a. \tag{311}$$

Since the $J^a\bar{J}^a$ terms are all marginal, the beta functions $\beta_a = -d_{\ln a}\lambda_a(L)$ (sorry for the bad notation) is determined to lowest order by the quadratic term in our expression for the beta function. Now recall that the OPE is

$$J^aJ^b \sim i\epsilon^{abc}\frac{J^c(z)}{z-w} + \frac{k\delta^{ab}}{(z-w)^2}. \tag{312}$$

Thus we have (no implicit summation)

$$(J^a\bar{J}^a)(z)\cdot(J^b\bar{J}^b)(w) \supset \sum_c \frac{1}{|z-w|^2}|\epsilon^{abc}|(J^c\bar{J}^c)(w). \tag{313}$$

We can then conclude that the $O(\lambda^2)$ β functions for the various interactions are (the numerical prefactor isn't important)

$$\beta_a = -\frac{\pi}{2}\sum_{b,c}|\epsilon^{abc}|\lambda_b\lambda_c. \tag{314}$$

As another application of this, we can do an easy check of the one-loop β function in the ϕ^4 model. We can only check up to the one-loop result since we have only kept terms quadratic in the $\mathcal{O}^\alpha$ in the expansion of the partition function. We will fix notation by

$$S = \int \left(\frac{1}{2}(\partial\phi)^2 + r\phi^2 + u\phi^4 \right). \tag{315}$$

In e.g. dimension $d = 4 - \epsilon$, the dimension of r is 2 while the dimension of u is ϵ (really 2ϵ, but we are taking $\epsilon \to 0$). This gives us the first-order terms in the β functions. Then we need the OPEs (schematic notation and writing a lot of numbers to make the combinatorics transparent)

$$\phi^2 \cdot \phi^2 \sim \frac{2 \cdot 2}{x^2}\phi^2, \qquad \phi^2 \cdot \phi^4 \sim \frac{4 \cdot 2}{x^2}\phi^4 + \frac{\frac{4!}{2 \cdot 2} \cdot 2}{r^4}\phi^2, \tag{316}$$

and

$$\phi^4 \cdot \phi^4 \sim \frac{\left(\frac{4!}{3!}\right)^2 \cdot 3!}{r^6}\phi^2 + \frac{\left(\frac{4!}{2!^2}\right)^2 \cdot 2}{r^4}\phi^4, \tag{317}$$

where we have ignored the most singular parts where all of the legs have been contracted and ingorned the ϕ^6 part in the last term. Then we can read off the OPE coefficients needed for calculating the β functions from the above formula. We will assume the coupling constants have been rescaled to get rid of the annoying factor of $A(S^{d-1})/2$ in our expression for the second-order contribution to the β functions. We get (still in the high-energy convention where we get β by differentiting wrt $\ln \Lambda$ and not $\ln a$)

$$\beta_r = 2r - 96u^2 - 24ur, \qquad \beta_u = \epsilon u - 72u^2 - 16ru. \tag{318}$$

From here one can compare to e.g. Peskin and Schroeder after traking down how the conventions differ. One can also use these to solve the for the WF fixed point, etc etc.

17 *Duality in the Ising model* ✓

Today's problem came from wanting to understand a statement in [?] about duality in the Ising model. Our goal is to explain / elaborate on the content of the mini-section on page 38 of the just-cited paper.

Let X be a Riemann surface equipped with a choice of spin structure η. Let Z_+ denote a partition function for a spin structure chosen so that $\text{Arf}(\eta) = 0$ (the spin structure can be extended to a bounding three-manifold), and Z_- a partition function for a spin structure with $\mathrm{Arf}(\eta) = 1$ (the spin structure is non-bounding). We want to examine what happens when the theory is pushed away from the self-dual conformal point by adding in a perturbation given by the energy operator $m \int \epsilon$.[12] Since $\epsilon \sim \psi\bar{\psi}$ is a fermion mass, this is indeed the right perturbation for tuning the theory away from the critical point.

Write the perturbed partition function as

$$Z_f[\eta] = \left\langle 1 - m \int_z \epsilon(z, \bar{z}) + \frac{m^2}{2} \int_{z,w} \epsilon(z, \bar{z})\epsilon(w, \bar{w}) - \dots \right\rangle_\eta , \tag{319}$$

where the expectation value is computed in the CFT (i.e. at the critical point). We claim that Z_+ is even under $m \mapsto -m$, while Z_- is odd, that is, we claim that $\langle \epsilon^k \rangle$ for k even is only nonzero when $\mathrm{Arf}(\eta) = 0$, while for k odd it is only nonzero for $\mathrm{Arf}(\eta) = 1$.

We now take a look at why this is true.[13] We will work on the torus for simplicity.

First, we note that $\langle \epsilon \rangle_\eta = 0$ if the spin structure η has antiperiodic boundary conditions in the spatial direction. This is because we are computing $\langle 0|\psi\bar{\psi}|0\rangle$ or $\langle 0|(-1)^F \psi\bar{\psi}|0\rangle$, both of which vanish since the ψs are primary fields which create states orthogonal to $|0\rangle$ when acting on $|0\rangle$. Since $\psi\psi \sim 1$, inserting an even number of ϵs results in something non-zero, while for the above reason inserting an odd number of ϵs gives zero.

Now for the spin structures with periodic boundary conditions in space. These boundary conditions are created by computing the expectation value in the states $|\sigma\rangle$ and $|\mu\rangle$, which differ by the occupation number of the fermion zero modes, and hence differ in their fermion parity (here the zero mode is a zero mode of the Hamiltonian, not a zero of the action [which doesn't exist if the time direction is antiperiodic]). More precisely, the two states differ in their $(-1)^F = (-1)^{F_L + F_R}$ eigenvalue, where $(-1)^{F_L}$ counts whether the holomorphic zero mode is filled, and $(-1)^{F_R}$ counts whether the antiholomorphic zero mode is filled. Since we are assuming the ∂ conditions for both ψ and $\bar{\psi}$ are the same, the ground states only carry a representation of the Clifford algebra associated to the total (non-chiral) zero mode algebra.

So, to do the trace, we need to sum over the two different ground states $|\sigma\rangle$ and $|\mu\rangle$, which differ in the occupation number of the zero modes. To compute e.g. $\langle \sigma|\epsilon|\sigma\rangle$, we use the operator-state correspondence to write $|\sigma\rangle = \sigma(0)|0\rangle$. Thus to get the expectation value of ϵ, we need to know the OPEs between $\psi, \bar{\psi}$, and σ, μ. Looking this up in the Big Yellow Book, we see that

$$\epsilon(z, \bar{z})\sigma(0) = i\psi(z)\bar{\psi}(\bar{z})\sigma(0) = i\psi(z)\frac{e^{-i\pi/4}}{\sqrt{2\bar{z}}}\mu(0) = \frac{1}{2|z|}\sigma(0). \tag{320}$$

Similarly,

$$\epsilon(z, \bar{z})\mu(0) = i\psi(z)\bar{\psi}(\bar{z})\mu(0) = i\psi(z)\frac{e^{i\pi/4}}{\sqrt{2\bar{z}}}\mu(0) = -\frac{1}{2|z|}\mu(0). \tag{321}$$

Note that the phase factors, which aren't always written, are very important here! Now we see that

$$\langle \epsilon(z, \bar{z}) \rangle_{RN} = \langle \sigma|\epsilon(z, \bar{z})|\sigma\rangle + \langle \mu|\epsilon(z, \bar{z})|\mu\rangle = 0. \tag{322}$$

[12] ϵ is an "energy operator" since $\epsilon \sim \psi\bar{\psi}$ where $\psi \sim X \prod Z, \bar{\psi} \sim Y \prod Z$ means that $\epsilon_j \sim \psi_j \bar{\psi}_j + \psi_j \bar{\psi}_{j+1} + h.c. \sim X_j X_{j+1} + Z_j$ contains the terms that appear in the Hamiltonian $H \sim \sum(XX + Z)$.

[13] Many thanks to Wenjie Ji for long discussions on this!

More generally, we see that $\langle \epsilon^k \rangle_{RN} = 0$ for odd k, while it is non-zero for even k. Note another rather slick way of reaching this conclusion would have been do just perform an S transformation on $\langle \epsilon \rangle_{NR} = 0$, which we already knew is true.

On the other hand, for the RR spin structure, we have

$$\begin{aligned}\langle \epsilon(z,\bar{z})\rangle_{RR} &= \langle \sigma|(-1)^F \epsilon(z,\bar{z})|\sigma\rangle + \langle \mu|(-1)^F \epsilon(z,\bar{z})|\mu\rangle = \langle \sigma|\epsilon(z,\bar{z})|\sigma\rangle - \langle \mu|\epsilon(z,\bar{z})|\mu\rangle \\ &= \frac{1}{|z|}\langle 0|\mathbf{1}|0\rangle \neq 0,\end{aligned} \tag{323}$$

where we took $|\mu\rangle$ to have an odd number of total zero modes and in the last step used $\sigma \otimes \sigma \sim \mu \otimes \mu \sim \mathbf{1} + \epsilon$ and $\langle \epsilon \rangle = 0$ (here $\langle\rangle$ without any subscript denotes the standard NN boundary conditions expectation value). So we conclude that on the RR torus, odd powers of ϵ are the ones that give nonzero expectation values. From a Lagrangian point of view, we could also argue that since $\int \mathcal{D}\psi_0 \mathbf{1} = 0$ (Grassmann integration is the same as differentiation), if there exists some ψ_0 such that $S[\psi_0] = 0$, then $Z = \int \prod_k \mathcal{D}\psi_k e^{-S} = 0$ (such a zero-action mode only exists on the RR torus). However, if we insert an ϵ into the partition function then we can do a mode expansion on it, with the term $\langle \epsilon \rangle_{RR} \supset \int \mathcal{D} \prod_k \mathcal{D}\psi_k\, \psi_0 \bar{\psi}_0 e^{-S}$ surviving and giving a non-zero expectation value.

Summing up, if $\mathrm{Arf}(\eta) = 0$ the series expansion for the parition function only includes even powers of m, while if $\mathrm{Arf}(\eta) = 1$ the series only includes odd powers. Now when we sum over spin structures, the full partition function for the bosons is

$$Z[m] = Z_+[m] \pm Z_-[m], \tag{324}$$

where the $\pm$ sign can be chosen freely (see e.g. the Big Yellow Book, chapter 11). This $\pm$ sign corresponds to projecting onto different (total) fermion parity sectors for the torus with periodic spatial ∂ conditions, which projects onto either the $|\sigma\rangle$ states or the $|\mu\rangle$ states. Since μ and σ are order-disorder duals of one another, we expect that this $\pm$ sign is switched under duality. Indeed, based on our comments above, we see that doing duality by taking $m \leftrightarrow -m$ is equivalent to flipping the $\pm$ sign, due to the evenness / oddness of the two partition functions.

We claim that doing duality, i.e. flipping the sign in the linear combination of Z_+ and Z_- is equivalent to tensoring with a Kitaev chain. Indeed, we will see that the partition function of the Kitaev chain in the topological phase is just a sign which depends on the spin structure in the right way to change the sign of Z_-.

Now let's explain this, starting with some more general comments inspired by reading [**?**] and other papers by Ryan + Kapustin. On a manifold with nontrivial topology, a bosonic theory can only be dual to a fermionic one if the duality relates a bosonic partition function to a sum over spin structures of fermionic partition functions, so that the bosonic theory has no spin structure dependence. A particular spin structure η, or a particular gauge field α, can be selected out by putting the analogue of e^{ikx} in the sum, in accordance with the relation between the two theories being related by a Fourier transform. Now recall that flat $\mathbb{Z}_2$ gauge fields are identified with elements of $H_1(X;\mathbb{Z}_2)$, so that the gauge field is defined the $\mathbb{Z}_2$ twists across each cycle α of the manifold in question (a $\mathbb{Z}_2$ twist along the cycle α

changes the boundary conditions for cycles β such that $\alpha \cap \beta = 1$). So,

$$Z_f[\eta] = \frac{1}{2}\sum_\alpha (-1)^{\eta(\alpha)} Z_b[\alpha], \tag{325}$$

and[14]

$$Z_b[\alpha] = \frac{1}{2}\sum_\eta (-1)^{\eta(\alpha)} Z_f[\eta]. \tag{327}$$

Here the action of η on $\alpha \in H_1(X;\mathbb{Z}_2)$ is given by $\eta(\alpha) = 0$ if η assigns anti-periodic (N) boundary conditions to α, and $\eta(\alpha) = 1$ if it assigns periodic (R) boundary conditions. This jives with the fact that antiperiodic boundary conditions are the "natural" ones, so that they are the identity in the $\mathbb{Z}_2$ group law. For example, if we label the nontrivial cycles on the torus as a_1, a_2, and a_1a_2, then e.g.

$$\eta_{NN}(0) = \eta_{NN}(a_1) = \eta_{NN}(a_2) = 0, \quad \eta_{NN}(a_1a_2) = 1, \tag{328}$$

while

$$\eta_{NR}(0) = \eta_{NR}(a_1) = \eta_{NR}(a_1a_2) = 0, \quad \eta_{NR}(a_2) = 1, \tag{329}$$

and

$$\eta_{RR}(0) = 0, \quad \eta_{RR}(a_1) = \eta_{RR}(a_2) = \eta_{RR}(a_1a_2) = 1. \tag{330}$$

Here the first label in the spin structure is the spatial boundary conditions and the second is the temporal ones. For us, spin structures will be identified with elements of $H^1(X;\mathbb{Z}_2)$. Here the 1-cocycleness comes from the failure of η to be a homomorphism on 1-chains:

$$\delta\eta(a,b) = \eta(a) + \eta(b) - \eta(a+b) = [a] \cap [b]. \tag{331}$$

One can check the consistency of this with the assignments of $\eta_{XY}(a)$ above.

Another way of writing $\eta(\alpha)$ for a gauge field α is by computing the integral $\eta(a) = \int_{\widehat{\eta}} \alpha$. Here, the submanifold $\eta \in H_1(X;\mathbb{Z}_2)$ is determined by the location of the branch cuts needed to determine the spin structure (each branch cut twists the boundary conditions of the fermions, with the "default" boundary condition [no branch cuts passed] being anti-periodic). Thus on T^2, the spin structure NR has a Poincare dual $\widehat{NR}$ given by the cycle which wraps the spatial cycle of the torus (so that the temporal boundary conditions are periodic).

As a sanity check, we check that the Fourier transform is involutive (it has order 2 and

[14] The normalisation factors here are specific to a torus. If we are not on the torus, we have to write the more cumbersome normalization factor

$$Z_f[\eta] = \frac{1}{\sqrt{2}^{\dim H_1(X;\mathbb{Z}_2)}}\sum_\alpha (-1)^{\eta(\alpha)} Z_b[\alpha]. \tag{326}$$

not order 4 since we're in $\mathbb{Z}_2$):

$$\begin{aligned}
Z_b[\alpha] &= \frac{1}{2}\sum_\eta (-1)^{\eta(\alpha)} Z_f[\eta] = \frac{1}{4}\sum_\eta \sum_{\alpha'} (-1)^{\eta(\alpha)+\eta(\alpha')} Z_b[\alpha'] \\
&= \frac{1}{4}\sum_{\alpha'}\sum_\eta (-1)^{\eta(\alpha+\alpha')+\int \widehat{\alpha}\cup\widehat{\alpha}'} Z_b[\alpha'] \\
&= \frac{1}{4}\sum_{\alpha'}\sum_\eta (-1)^{\eta(\alpha')+\int \widehat{\alpha}\cup\widehat{\alpha}'} Z_b[\alpha'+\alpha] \\
&= \frac{1}{2}\sum_{\alpha'} \delta(\alpha')(-1)^{\int \widehat{\alpha}\cup\widehat{\alpha}'} Z_b[\alpha'+\alpha] \\
&= Z_b[\alpha]
\end{aligned} \tag{332}$$

where we have used $\int \widehat{\alpha}\cup\widehat{\alpha} = 0$. We have also used

$$\frac{1}{2}\sum_\eta (-1)^{\eta(\alpha)} = \delta(\alpha). \tag{333}$$

This is easy to check explicitly: if $\alpha = 0$, then $(-1)^{\eta(\alpha)} = 1$ for all spin structures. If $\alpha \neq 0$, then two spin structures have $(-1)^{\eta(\alpha)} = -1$, while the other two have $(-1)^{\eta(\alpha)} = +1$, and so the sum gives zero.

We can also check that this Fourier transform formula is consistent with what we know from CFT. Recall that the Virasoro characters for the Ising CFT are given by

$$\chi_{\mathbf{1}} = \frac{1}{2}(Z_f[NN] + Z_f[NR]), \quad \chi_\epsilon = \frac{1}{2}(Z_f[NN] - Z_f[NR]) \tag{334}$$

and

$$\chi_{\sigma/\mu} = \frac{1}{2}(Z_f[RN] \mp Z_f[RR]), \tag{335}$$

where in the spin structure XY, X labels the spatial boundary conditions and Y the temporal ones. The choice of $\mp$ sign depends on what phase of the Ising model we are in (more on this later). The partition function for the Ising model with periodic boundary conditions for the spins, i.e. for a trivial gauge field $\alpha = 0$, is

$$Z_b[0] = |\chi_{\mathbf{1}}|^2 + |\chi_\epsilon|^2 + |\chi_\sigma|^2. \tag{336}$$

Now consider putting antiperiodic boundary conditions around the temporal cycle for the spin in the Ising model: this means working with a gauge field $\alpha = a_t$ which has nontrival holonomy around the temporal cycle. The energy operator ϵ and the identity are not affected, but σ gets mapped to minus itself, and so we might guess that the partition function in the presence of a_t is

$$Z_b[a_t] = |\chi_{\mathbf{1}}|^2 + |\chi_\epsilon|^2 - |\chi_\sigma|^2. \tag{337}$$

In terms of the different spin structures, this is

$$Z_b[a_t] = \frac{1}{2}(Z_f[NN] + Z_f[NR]) - \frac{1}{2}(Z_f[RN] \mp Z_f[RR]). \tag{338}$$

Since $(NN)(a_t) = (NR)(a_t) = 0$ (the Poincare dual of the former spin structure is trivial while the dual of the second one is the spatial cycle, along which a_t has trivial holonomy) while $(RN)(a_t) = (RR)(a_t) = 1$, from our Fourier transform formula we see that we should assign minus signs to the RN and RR sectors. Indeed, this is exactly what happens from flipping the sign of $|\chi\sigma|^2$ in the expression for Z_b. One can then use modular invariance of the Ising spin partition function to confirm the Fourier transform formula for the other choices of Ising spin boundary conditions.

Now let's see what the partition functions are in the different phases. In the symmetry-broken phase, nontrivial gauge field twists are not allowed: they create domain walls in Z, and since the symmetry-broken phase ground state is an eigenstate of Z, a domain wall which wraps a nontrivial cycle creates an inconcsistency in the spin configuration, and so $Z_b[\alpha] = \delta_{\alpha,0}$. This means that the fermionic partition function is just

$$Z_f[\eta] = 1. \tag{339}$$

So, the symmetry-broken phase for the bosons maps to the trivial phase for the fermions.

For the bosons, we expect that in the paramagnetic (symmetric) phase, the partition function will be insensitive to the presence of twists caused by the gauge field: in an eigenstate of X, the operator which creates a domain wall in Z acts trivially. So, we have $Z_b[\alpha] = 1$ in the symmetric phase, and thus

$$Z_f[\eta] = \frac{1}{2}\sum_\alpha (-1)^{\eta(\alpha)}. \tag{340}$$

Now if $\eta \neq RR$, $\eta(\alpha)$ is equal to 1 for only one cycle, and the sum produces $(3-1)/2 = 1$. On the other hand, if $\eta = RR$ then three of the cycles are assigned 1, and the sum produces $(1-3)/2 = -1$. So then in the symmetric phase, the fermion partition function is the Arf invariant:

$$Z_f[\eta] = (-1)^{\mathrm{Arf}(\eta)}. \tag{341}$$

This is precisely the partition function for the topological phase of the Kitaev chain. As a check that these work, we can use these fermionic partition functions to reproduce the bosonic ones. Taking $Z_f[\eta] = 1$ gives

$$Z_b[\alpha] = \frac{1}{2}\sum_\eta (-1)^{\eta(\alpha)} = \delta_{\alpha,0}, \tag{342}$$

which is the partition function for the symmetry-breaking phase[15]. Likewise, for the Kitaev chain in the topological phase,

$$Z_b[\alpha] = \frac{1}{2}\sum_\eta (-1)^{\eta(\alpha)+\mathrm{Arf}(\eta)} = 1, \tag{343}$$

[15] Note how summing over spin stuctures produces a δ function for the gauge field α (since if $\alpha \neq 0$ then two spin structures give a -1 phase and two give a $+1$), but that summing over gauge fields does not produce a δ function for the spin structure: $\frac{1}{2}\sum_\alpha (-1)^{\eta(\alpha)} \neq \delta(\eta)$ (we would have to decide what exactly we mean by $\delta(\eta)$ also). Instead, this sum gives the Arf invariant.

which in uniform in α and thus also matches the symmetric-phase partition function for the bosons (just check the last equality explicitly: for all spin structures, there is always one -1 term in the sum).

This confirms that tensoring with a Kitaev chain impliments the duality in the Ising model, since on the bosonic side interchanges the partition functions of the symmetric and symmetry-broken phases.

Now we briefly discuss how to see the different fermion parity results from a Hamiltonian viewpoint: we've seen that in the symmetric phase for the Ising spins (the topological phase for the fermions), there is a $1-(-1)^F$ projection in the path integral in the R sector, so that with periodic boundary conditions the fermion parity is odd (while it is even in the symmetry-breaking phase). The sanity check on this result is as follows. From the JW transformation, the Majorana femrions ψ satisfy $\psi_i = \prod_{j=i}^{i+L} Z_j \psi_{i+L}$ for a chain of length L (here the Hamiltonian is $H \sim -J\sum XX - h\sum Z$). The product $\prod_j Z_j = G$ is the generator of the global $\mathbb{Z}_2$ symmetry. So $\psi_i|0\rangle = G\psi_{i+L}|0\rangle = -\psi_{i+L}G|0\rangle = -\lambda_G\psi_{i+L}|0\rangle$, where λ_G is the $\mathbb{Z}_2$ charge of the ground state and we have used that the Majorana fermions anticommute with the G operator since it is a string of Z's. If the ground state has trivial $\mathbb{Z}_2$ charge then we are in the N sector and if it has -1 $\mathbb{Z}_2$ charge we are in the R sector. But G is also equal to the fermion parity operator, and so we learn that in the symmetric phase the N sector has even parity while the R sector has odd parity.

This can also be understood by actually computing H in momentum space. It is (for complex fermion operators c_k; the derivation is straightforward)

$$H = \sum_k (2h - 2J\cos(k))c_k^\dagger c_k - iJ\sin k(c_k c_{-k} + c_k^\dagger c_{-k}^\dagger). \tag{344}$$

In the N sector the fermion modes always come in pairs symmetric about $k=0$ and so $(-1)^F = 1$. In the R sector, we have a mode at $k=0$ and a mode at $k=\pi$. The latter is always un-filled since it is always at high energy. The former is filled, and hence the ground state has odd parity, provided that $J > h$, i.e. provided we are in the paramagetic phase. This provides another sanity check. A similar computation can be done for Majorana fermions γ_i, where in momentum space we have two unpaired modes, $\gamma_0^\dagger = \gamma^0$ and $\gamma_\pi^\dagger = \gamma_\pi$. Only the former is filled for the R spin structure and gives us the required odd parity. Also note that here duality does $\gamma_0 \mapsto \gamma_0, \gamma_\pi \mapsto -\gamma_\pi$, since duality in the Majorana language is equivalent to translation through half a unit cell ($i \mapsto i+1$ for the Majorana index). At the self-dual point this symmetry prevents us from hybridizing the unpaired modes with a term like $\gamma_0\gamma_\pi$. Also note that this $\mathbb{Z}_2$ duality symmetry is anomalous: it acts as a $\mathbb{Z}_2$ symmetry on the Majoranas but it actually squares to a translation, so that it cannot be implimented in an on-site way in terms of the original Ising (spin) variables. Indeed, there is no operator that we can write down in terms of the spin variables that is the charge operator for this symmetry.

Finally, we expand on why duality acts as translation by one site, and explain why this is equivalent to changing the sign of the fermion mass m. Recall that at the critical point, the Majorana chain is

$$H = iJ\sum_j^{2N} \eta_j\eta_{j+1}, \tag{345}$$

where the chain has N physical sites. Let us define the Majorana fields ξ, γ by

$$\eta_j = \frac{1}{\sqrt{2}}(\xi_j + (-1)^j \gamma_j). \tag{346}$$

The factor of $1/\sqrt{2}$ ensures that they have the usual $\{\xi_i, \xi_j\} = \{\gamma_i, \gamma_j\} = 2\delta_{ij}$ Clifford algebra relation. The $(-1)^j$ is needed since ξ, γ are the slowly varying fields which represent linearizations about the two points where the dispersion touches zero (the chemical potential for the Majoranas vanishes). Writing H in momentum space gives $H \sim \sum_{k>0} \eta_k^\dagger \eta_k \sin(k)$, so that the dispersion has zeros at $k = 0$ (the ξ mode) and at $k = \pi$ (the γ mode). Hence γ_j comes with a factor of $e^{\pi i j}$.

Anyway, putting this expansion into the Hamiltonian, we see that the $(-1)^j$ factors kill the off-diagonal terms, and so in the continuum we get a massless Majorana as expected:

$$H = \frac{iJ}{2} \sum_j (\xi_j \xi_{j+1} + \gamma_{j+1} \gamma_j) \implies S = \frac{iJ}{2} \int \bar{\Xi} \partial\!\!\!/ \Xi, \tag{347}$$

where $\Xi = (\xi, \gamma)^T$. Here we are taking $\gamma^0 = X, \gamma^1 = Y$.

Now let's add the term

$$\delta H = im \sum_j (-1)^j \eta_j \eta_{j+1}. \tag{348}$$

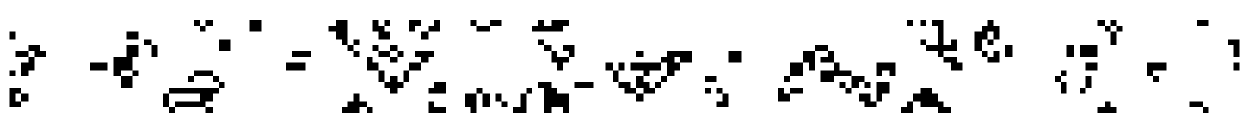
Since the coupling is alternating on each bond, the coupling strength changes within physical sites and between physical sites. Thus we expect that this term should be the one which drives us away from the critical point, where all the hopping strengths are equal. Indeed, if we put in our expansion for η, we get

$$\delta H = im \sum_j \left((-1)^j [\xi_{j+1} \xi_j - \gamma_{j+1} \gamma_j] + \xi_{j+1} \gamma_j - \gamma_{j+1} \xi_j \right). \tag{349}$$

The terms that vary as $(-1)^j$ die when we go to the continuum since they oscillate fast and cancel out, and so we have

$$\delta H = im \int dx \, (-\gamma - \partial_x \gamma) \xi + (\xi + \partial_x \xi) \gamma) \implies \delta S = 2 \int im \xi \gamma = - \int \bar{\Xi} im Z \Xi, \tag{350}$$

where the derivative terms have canceled. So δH is indeed a Majorana mass term. Now we know from above that $m \mapsto -m$ should be equivalent to doing duality. And indeed, from the definition of δH we see that sending $m \mapsto -m$ is equivalent to $j \mapsto j+1$, i.e. it is equivalent to translation by one site (half of a physical lattice constant). This is another check that duality is realized by translation through half a physical lattice site. In the continuum, it can be implemented via the transformation $\Xi \mapsto Z\Xi$, which leaves the kinetic term invariant but changes the mass term as $m \mapsto -m$. That Z is the right matrix to use here makes sense if we look at (346), where we see that duality (translation by a lattice site) should take $\xi \mapsto \xi$ and $\gamma \mapsto -\gamma$, which is exactly what Z does.



18 *Yet more on the SU(2) point of the $c = 1$ CFT* ✓

This is a fast one, and comes from a problem in a pset assigned in Ashvin Vishwanath's fall 2018 class on quantum matter. The problem statement is as follows:

Consider the fermionized description of the anisotropic XXZ chain, with Hamiltonian

$$H = \sum_i \left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + g S_i^z S_{i+1}^z \right]. \tag{351}$$

Write down an expression for $S^z(r)$ in terms of the low energy fermion fields, including components both at zero momentum and at momentum $q = \pi$.

What value of g corresponds to the self-dual $SU(2)$ point? Find out by requiring that the $q = \pi$ component of the S^z spin density have the same power law exponent as the 2-point function of the S^+ operator. We will work in conventions where the bosonized Lagrangian is written as $\mathcal{L} = (\partial_\mu \theta / 2\pi)^2/(2K)$.

First of all,

$$S_r^{\pm} \to (-1)^r (-1)^{\sum_{r'<r} n_{r'}}, \tag{352}$$

where the $(-1)^r$ is needed to cancel a minus sign from moving a string operator past a creation operator in $S_i^+ S_{i+1}^-$ in order to produce a correct-sign hopping. Therefore we have

$$S_r^z = S_r^+ S_r^- - \mathbf{1}/2 = n_r - \mathbf{1}/2. \tag{353}$$

Thus in terms of the low energy fields,

$$\begin{aligned} S_r^z = c_r^\dagger c_r - 1/2 &= (e^{-ik_F r} R^\dagger(r) + e^{ik_F r} L^\dagger(r))(e^{ik_F} R(r) + e^{-ik_F r} L(r)) - 1/2 \\ &= n_R(r) + n_L(r) + (-1)^r (L^\dagger(r) R(r) + R^\dagger(r) L(r)) - 1/2, \end{aligned} \tag{354}$$

since $k_F = \pi/2$. While $(-1)^r$ carries momentum π so does $L^\dagger(r)R(r) + h.c.$, so that $\sum_j S_j^z$ has net zero momentum as required.

The part of the S^z 2-point function that goes as $(-1)^r$ is the part involving scattering from one of the Fermi points to the other:

$$\langle S_r^z S_0^z \rangle \ni (-1)^r \langle L^\dagger(r) R(r) R^\dagger(0) L(0) \rangle \to (-1)^r \langle e^{-i\theta(r)} e^{i\theta(0)} \rangle, \tag{355}$$

since in our conventions $L^\dagger(x) \to e^{i\phi_L(x)}$, $R(x) \to e^{-i\phi_R(x)}$ and $\theta = \phi_R - \phi_L$. From the action we read off that the propagator for the θ field is

$$G_\theta(r) = -2\pi K \ln |r|, \tag{356}$$

and so

$$\langle S_r^z S_0^z \rangle \to \frac{1}{|r|^{2\pi K}}. \tag{357}$$

Now we look at the $S^\pm$ correlator. We can figure out the image of $S_r^\pm$ under bosonization from its commutation relation with Z_r, which we know maps to $n_r - 1/2 = \partial_x\theta/(2\pi) - 1/2$. Since $[Z_r, S_r^\pm]] = \pm S_r^\pm$, we guess that $S_r^\pm \to e^{\pm i\phi(r)}$. This identification is natural since we know $\phi(r)$ gets shifted by the $U(1)$ symmetry of rotations about the z axis, in the same way that S^+ does. Indeed, from the number-phase relation $[\phi(x), \partial_y\theta(y)/2\pi] = i\delta(x-y)$, we check that

$$[Z_r, S_r^\pm] \to [\partial_r\theta, e^{\pm i\phi(r)}] = \pm e^{i\pm\phi(r)} \to \pm S_r^\pm, \tag{358}$$

as required. Actually this doesn't completely fix the bosonization of $S_r^\pm$. We will actually include an explicit factor of $(-1)^r$ in its bosonization (as we wrote down above), so that

$$S_r^\pm \to (-1)^r e^{\pm i\phi(r)}. \tag{359}$$

(the reason for doing this is to get the correct sign for the fermion kinetic term).

Anyway, this means that

$$\langle S_r^+ S_0^- \rangle = (-1)^r \langle e^{i\phi(r)} e^{-i\phi(0)} \rangle = (-1)^r e^{G_\phi(r)}. \tag{360}$$

The propagator $G_\phi(r)$ for ϕ is determined by T duality: if the coefficient of the free θ action is $R^2/4\pi$, then the coefficient for the free ϕ action is $1/(4\pi R^2)$. For us $R^2 = 1/2\pi K$, and so the coefficient for the ϕ action will be $K/2$. Thus we have

$$G_\phi(r) = -\frac{1}{2\pi K}\ln|r| \implies \langle S_r^+ S_0^- \rangle \to (-1)^r \frac{1}{|r|^{1/2\pi K}}. \tag{361}$$

If we require that the power law exponents in the $(-1)^r$ parts of the $S^\pm$ and S^z two point functions match, which is a necessary requirement if the theory is to have $SU(2)$ symmetry, then we require that $2\pi K = 1/(2\pi K)$, so that we predict the $SU(2)$ point to be located at $K = 1/2\pi$. We know from above that the operator $\cos 2\theta$ becomes marginal when $K = 1/2\pi$, so that the $SU(2)$ point is characterized by the radius at which the $\cos 2\theta$ Umklapp term crosses over between relevance and irrelevance.

Hermitian conjugation and inner products for Euclidean CFTs ✓

Today we will essentially be doing a collection of several exercises in David Simmons-Duffin's class notes on CFT which tell us how to think about Hilbert space structures in Euclidean-space fomulations of CFTs.

We relate spin-1 operators in Lorentzian and Euclidean signatures via

$$\mathcal{O}_E^l(\tau, \mathbf{x}) = (-i)^{\delta_{l,0}} \mathcal{O}_L^l(t, \mathbf{x}), \tag{362}$$

where l is a vector index. The reason for the prefactor is because it is required to ensure that $O(d)$ or $O(d-1,1)$ transformations commute with the processes of continuing between signatures. We can see that this prefactor is correct by mapping a spin-1 operator to real time, doing a Lorentz tform, and then mapping back to E time:

$$\mathcal{O}_E^l(\tau, \mathbf{x}) \mapsto (-i)^{\delta_{l,0}} \mathcal{O}_L^l(t, \mathbf{x}) \mapsto \Lambda^l_m (-i)^{\delta m,0} \mathcal{O}_L^m(t, \mathbf{x}) \mapsto i^{\delta_{l,0}-\delta_{m,0}} \mathcal{O}_E^m(\tau, \mathbf{x}). \tag{363}$$

If this is to be a sensible way of going between E and L signatures, then an $O(d)$ rotation R^{lm} must be related to the $O(d-1,1)$ rotation Λ^l_m via (our use of is here is legit because we are considering the complexification of the orthogonal groups)

$$R^{lm} = \Lambda^l_m i^{\delta_{l,0}-\delta_{m,0}}. \tag{364}$$

Let's check to see that this works: if it works, then the LHS needs to preserve δ_{lm}. Being slightly callous about index placement, and working in mostly positive signature, we have

$$R^{lm} R^{ln} = \Lambda^l_m \Lambda^l_n i^{2\delta_{l,0}-\delta_{m,0}-\delta_{n,0}} = \Lambda^l_m \Lambda_{ln} i^{-\delta_{m,0}-\delta_{n,0}} = \eta_{mn} i^{-\delta_{m,0}-\delta_{n,0}} = \delta_{mn}, \tag{365}$$

and so the RHS of (364) is indeed orthogonal. A general spin operator then maps as

$$\mathcal{O}_E^{l_1\dots l_n}(\tau, \mathbf{x}) \mapsto (-i)^{\sum_i \delta_{l_i,0}} \mathcal{O}_L^{l_1\dots l_n}(t, \mathbf{x}). \tag{366}$$

Now in Euclidean signature there is no natural choice of inner product. Since we want to ground the whole formalism in real time where physics is defined, we then use the inner product in real time to construct one in Euclidean time. The Euclidean Hermitian conjugation must then reverse the Euclidean time τ—different choices of what τ is lead to different Hilbert spaces, and hence to different quantizations of the theory. In addition to reversing τ, conjugation also changes the signs of vector indices with time components, since these have an extra i associated to them in Euclidean signature, as described above. Therefore the Euclidean Hermitian conjugation acts as

$$[\mathcal{O}_E^{l_1\dots l_n}(\tau, \mathbf{x})]^\dagger = (-1)^{\sum_i \delta_{l_i,0}} \mathcal{O}_E^{l_1\dots l_n}(-\tau, \mathbf{x}). \tag{367}$$

This is actually very sensible: since $\dagger$ flips the τ coordinate, vector indices pointing in the τ direction get minus signs. An easy mistake to make is to apply this formula blindly to everything with a vector index. If do this, we would conclude that e.g. $P_1^\dagger = -P_1$ and $P_l^\dagger = P_l$ for $l > 1$ (here P_i is the physical momentum operator on e.g. the cylinder; we are not (yet) working in radial quantization, where of course the momentum is conjugated as $P^\dagger = K$). However, this is a bit distressing, since we know that P_1 is the Hamiltonian, which should be Hermitian. In fact, the transformation rule is the opposite: P_1 is Hermitian, and the rest are anti-Hermitian (which can be seen e.g. by taking the $\dagger$ of $[P_1, \phi(0)] = \partial_\tau \phi(0)$, with $\phi(0)^\dagger = \phi(0)$). Likewise, one can show that $M_{lm}^\dagger = -M_{lm}$ for $l, m > 1$, while $M_{1l}^\dagger = M_{1l}$. So, one should only apply our Hermitian conjugation rule to primaries.

When we quantize on the cylinder $\mathbb{R} \times S^{d-1}$, τ is just the z-coordinate of the cylinder, and we can use the above formula for Hermitian conjugation. However, since we will more often work in radial quantization on the plane, we need to port this definition of Hermitian conjugation into radial quantization conventions. The zero of Euclidean time is the unit sphere in $\mathbb{R}^d$, and so $\tau \mapsto -\tau$ corresponds to the inversion $x^\mu \mapsto x^\mu/x^2$. The minus sign in the mapping (367) is accounted for in radial quantization by contracting all the indices of $\mathcal{O}_E$ with the matrix $I^\mu_\nu(x) = \delta^\mu_\nu - 2x^\mu x_\nu/x^2$, taking $x^\mu \mapsto -x^\mu$. Finally, we need to remember that the cylinder and the plane have metrics differing by a Weyl rescaling. Since $r \to e^\tau$, the Weyl rescaling is determined via $r^2(dr^2/r^2 + d\Omega^2_{d-1}) \to e^{2\tau}(d\tau^2 + d\Omega^2_{d-1})$, so that the rescaling factor is $e^{2\tau}$. Now we need to remind ourselves that if we have two different coordinate systems with metrics such that $ds_1^2 = e^{2w} ds_2^2$, then an operator with scaling dimension Δ is given in the two coordinate systems by $\mathcal{O}(x_1) = e^{-w\Delta}\mathcal{O}(x_2)$. This means that $\mathcal{O}_E(x^\mu) = e^{-\tau\Delta}\mathcal{O}_E(\tau, \mathbf{x})$. Therefore we can determine how $\dagger$ acts on scalar operators in radial quantization via

$$\mathcal{O}(x^\mu)^{\dagger_r} = e^{-\Delta\tau}\mathcal{O}(\tau, \mathbf{x})^\dagger = e^{-\Delta\tau}\mathcal{O}(-\tau, \mathbf{x}) = e^{-\Delta\tau}(e^{\Delta(-\tau)}\mathcal{O}(x^\mu/x^2)) = r^{-2\Delta}\mathcal{O}(x^\mu/x^2). \quad (368)$$

For operators with spin we just add in the I matrices we mentioned above to this equation, and hence

$$[\mathcal{O}_E^{l_1\dots l_n}(x^\mu)]^{\dagger_r} = r^{-2\Delta} I^{l_1}_{m_1}(x) \cdots I^{l_n}_{m_n}(x)\mathcal{O}_E^{m_1\dots m_n}(x^\mu/x^2). \quad (369)$$

The factor of $r^{-2\Delta}$ is crucial here, since it is needed to make inner products like

$$\langle\mathcal{O}|\mathcal{O}\rangle = \langle 0|\mathcal{O}(0)^{\dagger_r}\mathcal{O}(0)|0\rangle = \lim_{r\to\infty} r^{2\Delta}\langle 0|\mathcal{O}(r)\mathcal{O}(0)|0\rangle \quad (370)$$

finite in radial quantization (and properly dimensionless).

Unitarity bounds on CFT scaling dimensions ✓

Today we'll be looking at unitarity constraints on scaling dimensions, and will essentially be doing an elaboration on an exercise in David Simmons-Duffin's class notes on CFT showing how various bounds on scaling dimensions arise from unitarity constraints.

First we will prove a bound on the scaling dimension of all primary operators with nonzero spin, namely that if $\mathcal{O}_n$ is a primary operator with spin $n > 0$, then

$$\Delta_{\mathcal{O}_n} \geq d - 2 + n. \quad (371)$$

In particular, for $n = 1$, this tells us that the minimal possible scaling dimension is $d-1$, which is the dimension that a conserved current has. In fact we will see that a spin-1 operator is a conserved current iff its scaling dimension saturates the bound: conserved currents have no anomalous dimensions, and any spin-1 operator with no anomalous dimension is a conserved current.

First we have to define what we mean by "an operator of spin n" in dimensions other than 3. In here and what follows, spin n will always refer to an operator which transforms in the n-index symmetric traceless tensor representation of $SO(d)$. While for $SO(3)$ these representations exhaust all irreps,[16] this is of course not true for $SO(d > 3)$, when irreps assigned to antisymmetric tensors become possible to construct. However, conserved currents and energy momentum tensors and the like are usually all associated to symmetric traceless representations, so we will restrict our attention to them in what follows.

The proof goes by considering the inner product between two descendants in the conformal multiplet of a spin-n conformal primary $\mathcal{O}$:

$$\left(P^\mu|\mathcal{O}^{\boldsymbol{\beta}}\rangle\right)^\dagger P_\nu|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = \langle\mathcal{O}^{\boldsymbol{\beta}}|K_\nu P^\mu|\mathcal{O}^{\boldsymbol{\alpha}}\rangle. \tag{372}$$

Here the notation is such that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ label multi-indices in accordance with the symmetric and traceless nature of $\mathcal{O}$, so that e.g. $\mathcal{O}^{\boldsymbol{\alpha}} = \mathcal{O}^{\alpha_{(1},\dots,\alpha_{n)}}$ (note the symmetrization). Note that this is not the inner product between two identical vectors and hence is not necessarily positive. Also note that we are not assuming any contraction between the μ, ν indices and those carried by $\mathcal{O}$. Using the commutator

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu}, \tag{373}$$

where $M_{\mu\nu}$ is the $SO(d)$ generator for the rotation parametrized by μ, ν in the spin-n representation, we have (since $\mathcal{O}$ is primary)

$$\left(P_\mu|\mathcal{O}^{\boldsymbol{\beta}}\rangle\right)^\dagger P_\nu|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = 2\langle\mathcal{O}^{\boldsymbol{\beta}}|\delta_{\nu\mu} - M_{\mu\nu}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle. \tag{374}$$

Now we use a clever trick: since $M_{\mu\nu} = -M_{\nu\mu}$, we can write

$$M_{\mu\nu} = \sum_{i<j=1}^{d} [A^{ij}]_{\mu\nu} M_{ij}, \tag{375}$$

where $[A^{ij}]_{\mu\nu} = \delta^i_\mu\delta^j_\nu - \delta^j_\mu\delta^i_\nu$ is the generator matrix of the rotation matrix parametrized by the tuple ij, represented in the vector representation. Let us normalize the states by

$$\langle\mathcal{O}^{\boldsymbol{\beta}}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = \delta^{\boldsymbol{\beta}\boldsymbol{\alpha}}. \tag{376}$$

Therefore we may write

$$\frac{1}{2}\left(P_\mu|\mathcal{O}^{\boldsymbol{\beta}}\rangle\right)^\dagger P_\nu|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = \delta_{\mu\nu}\delta^{\boldsymbol{\alpha}\boldsymbol{\beta}}\Delta - \sum_a (A^a \otimes M^a)^{\boldsymbol{\alpha}\boldsymbol{\beta}}_{\mu\nu}, \tag{377}$$

[16] Since any tensor with a pair of antisymmetric indices can be reduced contracting with the invariant symbol ϵ_{ijk}. This reduces the number of indices in the tensor and lets us turn antisymmetric tensors into symmetric ones. This fails for larger $SO(d)$ since contracting with the invariant ϵ symbol does not decrease the number of indices (for $d > 4$ it increases the number of indices).

where a runs over the generators of $\mathfrak{so}(d)$.

Hence we have obtained the inner product as a matrix element of a matrix constructed from a $\otimes$ of a matrix in the F irrep and one in the S_n irrep, where F is the fundamental (vector) irrep and S_n is the symmetric, traceless, n-index tensor irrep. To deal with this expression, we should reduce this $\otimes$ to its constituent irreps. This happens by writing

$$\begin{aligned}\sum_a (A^a \otimes M^a)^{\boldsymbol{\alpha\beta}}_{\mu\nu} &= \frac{1}{2}\sum_a \left[(A^a \otimes \mathbf{1} + \mathbf{1} \otimes M^a)^2 - (A^a)^2 \otimes \mathbf{1} - \mathbf{1} \otimes (M^a)^2\right) \\ &= -\frac{1}{2}\left(C_2(F \otimes S_n) - C_2(F) \otimes \mathbf{1} - \mathbf{1} \otimes C_2(S_n)\right),\end{aligned} \tag{378}$$

since the generators for the $R_1 \otimes R_2$ rep are $T^a_{12} = T^a_1 \otimes \mathbf{1} + \mathbf{1} \otimes T^a_2$. Here the minus sign is because our generators are anti-Hermitian (e.g. A^a is real and antisymmetric), and since the quadratic Casimir is positive definite and is hence $\sum_a (T^a)^\dagger T^a = -\sum_a T^a T^a$ (e.g. $L^2 = -\partial_\mu \partial^\mu$ since ∂_μ is anti-Hermitian).

The quadratic Casimir[17] for the traceless symmetric nth-rank tensor representation of $SO(d)$ is allegedly

$$C_2(S_n) = n(n + d - 2). \tag{379}$$

I tried for a stupidly long time to check this, but didn't quite figure it out. The thing I did figure out was the dimension of the representation S_n—to get C_2 we would then need the index, which I couldn't couldn't compute. The computation of the dimension is preserved for posterity's sake in the following footnote.[18]

Let's now go into a basis appropriate for the $\oplus$ decomposition of $F \otimes S_n$ into constituent irreps. In this basis, we have

$$(P|\mathcal{O}\rangle)^\dagger P|\mathcal{O}\rangle = 2\Delta\mathbf{1} - C_2(F) \otimes \mathbf{1} - \mathbf{1} \otimes C_2(S_n) + \bigoplus_{R \in F \otimes S_n} C_2(R), \tag{381}$$

[17] Just "Casimir" is imprecise terminology since for general groups there are usually many different Casimirs we can construct (involving kth powers of the generators—each k-index invariant symbol gives us such a Casimir).

[18] We want to figure out the dimension of the n-index symmetric tensor irrep of $SO(d)$. A tensor transforming under this irrep has n indices, each of which can take d values: therefore counting the number of such tensors amounts to the number of ways we can place n different objects (the indices of the tensor) in d different buckets. Only the total number of objects that gets placed in each bucket matters—the objects being placed in the buckets are all identical, on account of the symmetric property of the tensor meaning that all the indices can be freely exchanged. Recall from stat mech class that the best way to think about this is to consider counting the number of ways to arrange n objects and $d-1$ dividers between the different buckets. Since the dividers and objects are all identical, this number is $\binom{n+d-1}{n}$. Thus the number of symmetric tensors is $(n+d-1)!/(n!(d-1)!)$.

However, we need to take care of the traceless condition. This reads $T^{\mu\mu\nu_1\dots\nu_{n-2}} = 0$, where by symmetry it doesn't matter where in the index structure the two μs are. Note that the remaining indices $\nu_1, \dots, \nu_{n-2}$ are symmetric, and so the traceless condition gives us a number of constraints equal to the number of rank-$(n-2)$ symmetric tensors. By the above this is $\binom{n+d-3}{n-2}$. Therefore the dimension of S_n is

$$\dim[S_n] = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}. \tag{380}$$

For $d = 3$ we get $2n + 1$ as expected, while for e.g. $d = 4$ we get $(n+1)^2$. From the form of the quadratic Casimir, it's clear that the formula for the index $T(S_n)$ is going to be complicated.

where we are now thinking of the LHS as a matrix, rather than a matrix element. Now since the LHS is the matrix whose entries are the inner products of basis vectors for the first descendants of $\mathcal{O}$, it must be positive definite. The potential for it to have negative eigenvalues is contained within the choices for the different irreps in the $\oplus$ term: to get the strongest bound on Δ we should look for representations $R \in F \otimes S_n$ such that $C_2(R)$ is minimized. Since C_2 is larger for larger irreps, we should choose the smallest irrep appearing in the $\oplus$ decomposition. The smallest irrep is S_{n-1}, which we get from contracting the vector index of F with any of the indices of the symmetric tensors in S_n. Therefore we must have

$$2\Delta \geq (d-1) + n(n+d-2) - (n-1)(n-3+d) \implies \Delta \geq n+d-2, \tag{382}$$

as claimed. Note that this bound only applies to primary operators. For example, one shouldn't worry that the dimension of A^μ in free Maxwell theory is $(d-2)/2 < d-1$, since A^μ is not a primary (it is not even an operator in the physical Hilbert space).

Now we will prove that all scalar operators have scaling dimension bounded from below by $\Delta \geq (d-2)/2$, which is saturated for the free scalar. Consider the inner product

$$0 \leq (P_\mu P^\mu|\mathcal{O}\rangle)^\dagger P_\mu P^\mu|\mathcal{O}\rangle = \langle\mathcal{O}|K_\mu K^\mu P_\nu P^\nu|\mathcal{O}\rangle. \tag{383}$$

Getting to the bound on Δ is now just a matter of algebra. Using the commutators

$$[D, K_\mu] = -K_\mu, \qquad [M_{\mu\nu}, P_\lambda] = \delta_{\nu\lambda}P_\mu - \delta_{\mu\lambda}P_\nu, \tag{384}$$

we have, using $M_{\mu\mu} = 0$,

$$\begin{aligned}\langle\mathcal{O}|K_\mu K^\mu P_\nu P^\nu|\mathcal{O}\rangle &= \langle\mathcal{O}|K^\mu(2\delta_{\mu\nu}D - 2M_{\mu\nu} + P_\nu K_\mu)P^\nu|\mathcal{O}\rangle \\ &= \langle\mathcal{O}|2(1+\Delta)K_\nu P^\nu - 2K^\mu(dP_\mu - \delta_{\mu\nu}P_\nu - P^\nu M_{\mu\nu}) + 4(\delta_{\mu\nu}D - M_{\mu\nu})^2|\mathcal{O}\rangle \\ &= \langle\mathcal{O}|\mathcal{O}\rangle(4\Delta d(1+\Delta) - 4d^2\Delta + 4d\Delta + 4d\Delta^2) \\ &= 4d\Delta\langle\mathcal{O}|\mathcal{O}\rangle(2\Delta + 2 - d).\end{aligned} \tag{385}$$

In order for this to be positive, we need either $\Delta = 0$ (which is the case if $\mathcal{O} = \mathbf{1}$, or $\Delta \geq (d-2)/2$, proving the bound. Note that since the inner product is positive-definite, if $\mathcal{O}$ is a scalar saturating the bound, then we necessarily have $\partial^2\mathcal{O} = 0$ since in that case $\langle\partial^2\mathcal{O}|\partial^2\mathcal{O}\rangle = 0$, so that any scalar saturating the bound necessarily obeys the free-particle wave equation.

The point here is that the effects of interactions on a free scalar always increase its scaling dimension, meaning that interactions make correlations decay faster. This quasi makes sense—adding interactions means that a disturbance in a field ϕ will have a harder time propagating from x to y than it would if ϕ were free, since the disturbance can interact with fluctuations and decay faster as it propagates. This is also somewhat corroborated by imagining adding weak interactions to a given CFT, and using CPT to compute new beta functions like $\beta_g \sim y_g^{UV} g - C_{gg}^g g^2/2$—if the interaction is repulsive so that $g_* > 0$, then with positive OPE coefficients we will have $y_g^{IR} < y_g^{UV}$, meaning that interactions have increased Δ_g.

However, it is not always the case that turning on interactions increases the scaling dimensions of all operators. ethan: *to do: add the example of the $\mathbb{CP}^N$ model and the scaling dimension of $z^\dagger \sigma z$*

Inversions, reflections, and SCTs ✓

This is a short problem posed by Rychkov during his 2019 TASI lectures: show that a CFT has inversion symmetry iff it has reflection symmetry by relating the two symmetries with conjugation by an element of $SO(d+1,1)$.

Since reflections (and inversions) aren't in the identity component of $O(d+1,1)$, CFTs don't necessarily have to have them as symmetries—the only symmetry a CFT must for sure have is $SO(d+1,1)$.

Our goal is to show that

$$R_\mu = O_\mu^{-1} I O_\mu, \qquad O_\mu \in SO(d+1,1), \tag{386}$$

where $I : x^\mu \mapsto x^\mu/x^2$ is the inversion and $R_\mu : x^\nu \mapsto (-1)^{\delta_{\mu,\nu}} x^\nu$ is the reflection of the μ coordinate. If this is true, then we will have proved that reflections and inversions are continuously connected—they are in the same connected component of $O(d+1,1)$—and so a CFT which has one as a symmetry necessarily has the other as a symmetry.

The inversion exchanges zero and infinity, and since R_μ leaves both zero and infinity invariant, any candidate O_μ will need to act nontrivially at infinity. The only generator of $SO(d+1,1)$ that acts at infinity is the SCTs K_ν, and so we will need to make use of the SCTs in constructing O_μ.

First let's try to understand what the hell the SCTs actually do. A finite SCT acts on the coordinates as

$$x^\mu \mapsto \frac{x^\mu - a^\mu x^2}{1 - 2a\cdot x + a^2 x^2}. \tag{387}$$

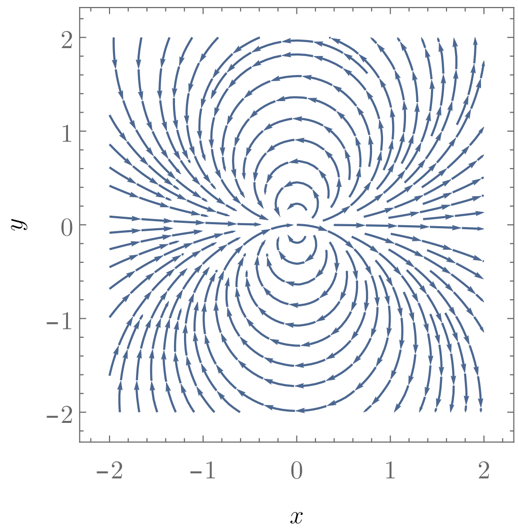
We can already see that this will move infinity, since it sends infinity to $-a^\mu/a^2$, and a^μ/a^2 to infinity. It also leaves zero invariant. In order to visualize better how it works, we can find the vector field associated with infinitesimal SCTs. Taking $|a|$ infinitesimal in the above equation, we find

$$x^\mu \mapsto 2(a\cdot x)x^\mu - a^\mu x^2 = a^\lambda(2x_\lambda x^\nu \partial_\nu - x^2\partial_\lambda)x^\mu, \tag{388}$$

and so K_μ is associated with the vector field

$$K_\mu = 2x_\mu(x\cdot\partial) - x^2\partial_\mu. \tag{389}$$

From this equation, we can see that K_μ acts by "swirling" around the origin. For example, in two dimensions, the operator K_x acts to move points along the following flow lines:



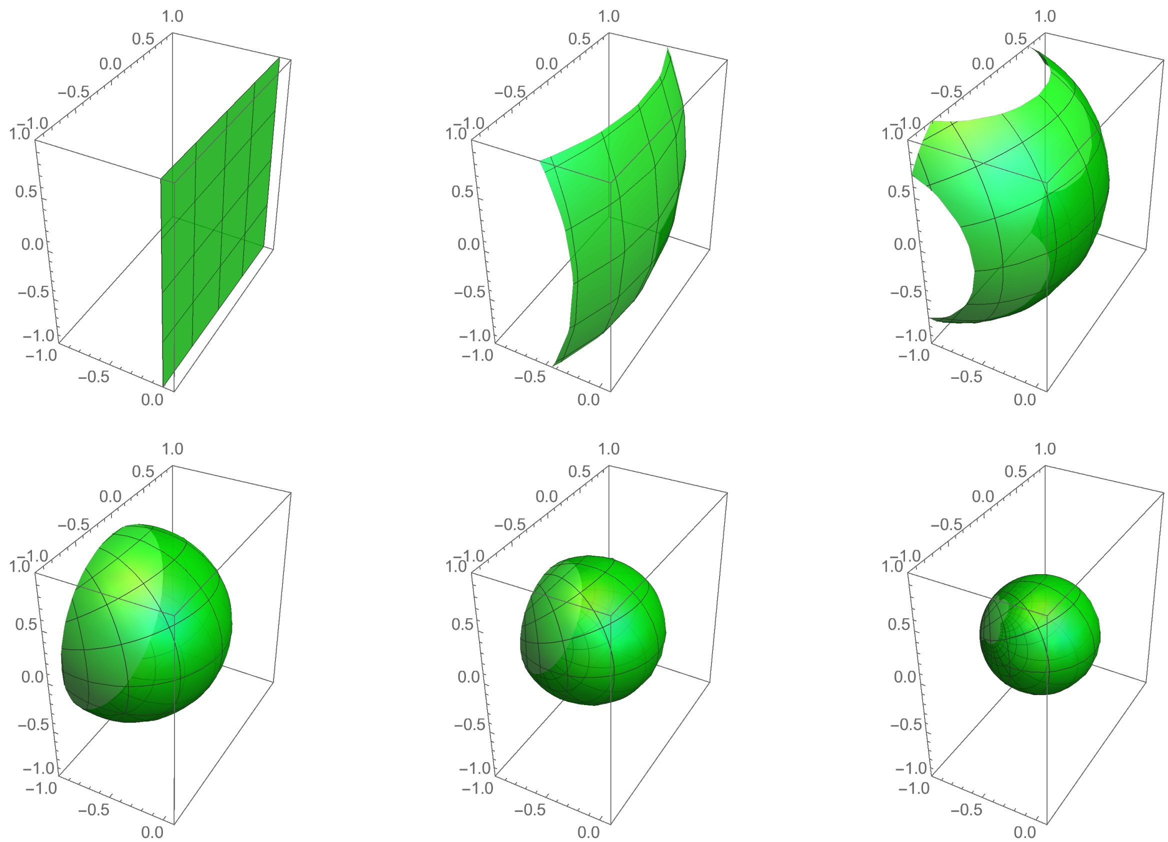
$$. \tag{390}$$

I wish this plot had been included in e.g. the big yellow book; it was only after seeing this that I finally got some intuition for what K does.

Now, in order for $O_\mu^{-1} I O_\mu$ to be a reflection, O_μ must send the $x^\mu = 0$ hyperplane to the unit sphere, since the unit sphere is what I reflects about. The O_μ^{-1} will then map the sphere back to the $x^\mu = 0$ plane, and we will have accomplished a reflection about the plane. To see why the SCT can help do this, consider e.g. the image of the y axis in $\mathbb{R}^2$ under the SCT $K_x(1)$ (here our notation is that $K_\mu(t)$ is the SCT given by (387) with $a^\mu = t\hat{x}^\mu$):

$$K_\mu(1) : (0, y) \mapsto \frac{1}{1+y^2}(-y^2, y). \tag{391}$$

One then checks that the RHS defines a circle of radius 1 centered at $(-1/2, 0)$. This also happens in higher dimensions, with $K_\mu(1)$ mapping the $x^\mu = 0$ plane to the unit sphere centered on $x^\mu = -1/2$. This is perhaps best illustrated by some graphics. In the following figure, as me move along the grid, we show the image of the yz plane under $K_x(t)$ for t ranging from 0 to 1. By the time we're done, the plane is indeed compactified to a unit sphere centered at $(-1/2, 0, 0)$:



(392)

Anyway, summing up, we see that the reflection R_μ is smoothly connected to the reflection through the homotopy

$$R_\mu(t) = [T_{-\mu}(t/2)K_\mu(t)]^{-1} I T_{-\mu}(t/2)K_\mu(t), \tag{393}$$

where $t \in [0,1]$ with $t = 1$ the reflection and $t = 0$ the inversion, and where $T_{-\mu}(t/2)$ is the translation by $t/2$ in the $-\hat{x}^\mu$ direction.

I think that this also provides us with a way to argue that any theory which is reflection-positive with respect to reflection about a plane is also reflection-positive in radial quantization (this works for primaries, but non-primary operators transform in more complicated ways under the mapping described above, though).

[redacted]

Scale invariance, conformal invariance, and algebras of charge operators ✓

Today we're going to do an elaboration on an exercise given by Slava Rychkov during his 2019 TASI lectures. The goal is to explicitly compute the algebra of the charge operators for conformal transformations by showing that

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = Q_{[\eta,\varepsilon]}(\Sigma), \tag{394}$$

where $Q_\varepsilon(\Sigma)$ is the operator which generates the conformal transformation given by the vector field ε along the submanifold Σ, and where $[\varepsilon, \eta]$ is the Lie bracket. We will be working in Euclidean spacetime of arbitrary dimension.

First, we will only assume translation and rotation symmetry. We want to calculate $Q_\varepsilon(\Sigma) \cdot T^{\mu\nu}(0)$, for Σ a surface enclosing 0, which we can do by general reasoning. The only two ϵs we have access to for translations and rotations are $\varepsilon_\mu = a_\mu$ with a_μ constant (translations) and $\epsilon_\mu = \omega_{\mu\nu}x^\nu$ with $\omega_{\mu\nu}$ constant and antisymmetric (rotations). Now since $T^{\mu\nu}$ is conserved, the Ward identity tells us that its dimension is exactly 2. Since $Q_\varepsilon(\Sigma) \cdot T$ needs to be linear in ε, dimensional analysis then tells us that each term appearing in $Q_\varepsilon(\Sigma) \cdot T$ should contain one ε, one T, and one ∂. Therefore we may write

$$Q_\varepsilon(\Sigma) \cdot T_{\mu\nu} = \epsilon^\rho \partial_\rho T_{\mu\nu} + A\partial_\mu \epsilon^\rho T_{\rho\nu} + B\partial_\nu \epsilon^\rho T_{\rho\mu} \tag{395}$$

for some constants A, B. The coefficient of the first term has been fixed to one because we want momentum to act on all operators as $P_\varepsilon = \varepsilon^\mu \partial_\mu$. By $\mu \leftrightarrow \nu$ symmetry, we have $B = A$, and taking the divergence of this then tells us that $A = 1$.[19] This means we can write

$$Q_\varepsilon(\Sigma) \cdot T_{\mu\nu} = \varepsilon^\rho \partial_\rho T_{\mu\nu} - \partial^\rho \varepsilon_\mu T_{\rho\nu} + \partial_\nu \varepsilon^\rho T_{\rho\mu}, \tag{396}$$

where we've used the Killing equation for ε in the second term to change the sign.

Now we want to ask what algebra the charges obey. The commutator of charges is computed in the usual way:

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = \int_\Sigma d^{d-1}x^\mu \int_{S_x^{d-1}} d^{d-1}y^\alpha \, T_{\mu\nu}(x)\eta^\nu(x)T_{\alpha\beta}(y)\varepsilon^\beta(y), \tag{397}$$

where S_x^{d-1} is a sphere centered on x. Shrinking down this sphere and then using (396), we have

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = \int_\Sigma d^{d-1}x^\mu \, \eta^\nu(\varepsilon^\rho \partial_\rho T_{\mu\nu} - \partial^\rho \varepsilon_\mu T_{\rho\nu} + \partial_\nu \varepsilon^\rho T_{\rho\mu}). \tag{398}$$

The annoying thing about this is that the derivatives are derivatives in the full $\mathbb{R}^d$, and not covariant derivatives for the metric restricted to the surface Σ. This makes integrating by

[19] Use $\partial_\mu \partial^\mu \varepsilon^\rho = -\partial_\mu \partial^\rho \varepsilon^\mu = -\partial^\rho(\partial \cdot \varepsilon) = 0$.

parts annoying if Σ is chosen to be some generic curved surface. However, since the charge operators are topological, we can choose any representative choice of Σ that we like—in order for integration by parts to be done easily, we will therefore choose Σ to be a hypercube. Integrations by parts can then be performed on each cube face, with the boundary terms all canceling out pairwise. Let us focus on the face F_1 of the cube with unit normal in the x^1 direction, just for concreteness. Then the first integral in (398) is

$$\int_{F_1} d^{d-1}x^1 \left[\eta^\nu \epsilon_1 \partial^1 T_{1\nu} - \epsilon_\rho \partial^\rho \eta^\nu T_{1\nu} + \epsilon_1 \partial^1 \eta^\nu T_{1\nu}\right] + \dots, \tag{399}$$

where $\dots$ is a boundary term coming from integrating by parts on F_1—these terms will end up canceling when we sum over cubes, so we will avoid writing them explicitly. The first and last terms above can be combined as

$$\int_{F_1} d^{d-1}x^1 \left[\partial^1(\eta^\nu \epsilon_1 T_{1\nu}) - (\nabla_\varepsilon \eta)^\nu T_{1\nu}\right], \tag{400}$$

since ε is Killing (here $\nabla_\varepsilon = \varepsilon^\rho \partial_\rho$).

The next term we will deal with is the second term in (398). We can integrate by parts and write this as

$$-\int_{F_1} d^{d-1}x^1 \left(\partial_1 \eta^\nu \varepsilon_1 T_{1\nu} + \eta^\nu \partial^1(\varepsilon_1 T_{1\nu}) - \partial_\rho \eta^\nu \varepsilon_1 T_{\rho\nu}\right). \tag{401}$$

By the symmetry of $T_{\mu\nu}$ and the Killing equation for η, the last term vanishes, and so this becomes

$$-\int_{F_1} d^{d-1}x^1 \, \partial_1(\eta^\nu \varepsilon_1 T_{1\nu}). \tag{402}$$

Note that this is precisely the right term to cancel the total ∂^1 derivative in (400). Therefore there are only two terms in (398) which survive, which together give

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = \int_\Sigma d^{d-1}x^\mu \, T_{\mu\nu}[(\nabla_\eta \varepsilon)^\nu - (\nabla_\varepsilon \eta)^\nu] = Q_{[\eta,\varepsilon]}(\Sigma). \tag{403}$$

Note the “backwards” order on the Lie bracket!

Now let us add the possibility that the theory also possesses scale invariance: therefore we can allow for transformations with $\partial_{(\mu}\varepsilon_{\nu)} \propto \delta_{\mu\nu}$, provided that $\mathrm{Tr}[T] = 0$. When we put this condition on T, we can allow a new term in the $Q_\varepsilon \cdot T_{\mu\nu}$ OPE, viz.

$$Q_\varepsilon(\Sigma) \cdot T_{\mu\nu} = \varepsilon^\rho \partial_\rho T_{\mu\nu} - \partial^\rho \varepsilon_\mu T_{\rho\nu} + \partial_\nu \varepsilon^\rho T_{\rho\mu} + \partial_\rho \varepsilon^\rho T_{\mu\nu}, \tag{404}$$

where the coefficient of the last term is fixed by requiring that the RHS be conserved. Now when we compute the charge commutator, we get, on each face, the integral

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] \ni \int_{F_1} d^{d-1}x^1 \, \eta^\nu[\varepsilon^\rho \partial_\rho T_{1\nu} - \partial^\rho \varepsilon_1 T_{\rho\nu} + \partial_\nu \varepsilon^\rho T_{\rho 1} + \partial_\rho \varepsilon^\rho T_{1\nu}]. \tag{405}$$

Integrating by parts and dropping the boundary terms that will cancel, the first term in the above is

$$\int_{F_1} d^{d-1}x^1 \, \left[-\nabla_\varepsilon \eta^\nu T_{1\nu} - \eta^\nu (\partial \cdot \varepsilon) T_{1\nu} + \partial_1(\eta^\nu \varepsilon^1 T_{1\nu}) \right]. \tag{406}$$

The first term contains the part of the Lie bracket that we want, while the second term cancels with the third term in (405). Therefore we get

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] \ni \int_{F_1} d^{d-1}x^1 \, \left[(\nabla_\eta \varepsilon^\nu - \nabla_\varepsilon \eta^\nu) T_{1\nu} + \partial_1(\eta^\nu \varepsilon^1 T_{1\nu}) - \eta^\nu \partial^\rho \varepsilon_1 T_{\rho\nu} \right]. \tag{407}$$

The last term is however

$$-\int_{F_1} d^{d-1}x^1 \, \left[\partial_1(\eta^\nu \varepsilon_1 T_{1\nu}) - \varepsilon_1 \partial_\rho \eta^\nu T_{\rho\nu} \right]. \tag{408}$$

The first term cancels the second term in (407), while the second term vanishes for the usual reasons of symmetry and tracelessness of T. Therefore we again get $[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = Q_{[\eta,\varepsilon]}(\Sigma)$.

ethan: *to-do: discuss why the $d = 2$ case is special, and global vs local conformal transformations*

Free Maxwell is not conformally invariant in $d \neq 4$ ✓

Today I'm going to write down some stuff I learned that was inspired during questions in one of Slava Rychkov's 2019 TASI lectures. We will see that free Maxwell theory, while scale invariant in any dimension, is only conformally invariant in $d = 4$. Update: turns out a good chunk of our discussion is contained in Rychkov's earlier paper [**?**].

There are a few ways to show that Maxwell is only conformally invariant in $d = 4$. One is through the gauge invariance of the improved stress tensor, which I won't go into. The other involve showing that the field strength F is not a primary: since F is used to create all the gauge-invariant local operators, if F is not a primary then the theory won't have any primaries, and we won't get a CFT. I've heard some people say "Maxwell can't be conformally invariant in $d \neq 4$ since only then is e^2 dimensionless", but this argument doesn't work, since it is prejudiced to a certain scaling dimension for A: for pure Maxwell the scaling dimension of A can be the scale-invariant choice of $(d-2)/2$ (no conflict with unitarity here since A isn't a primary), which gives a dimensionless e^2 (free Maxwell *is* scale invariant after all, just not conformally invariant).

F is not a primary because its 2-point functions don't have the right tensor structure

Now for our first way of showing that F isn't a primary. Since the action is free we can explicitly compute the correlator of A, and then that of F follows after some algebra. Since we just want the F correlator, we can choose any gauge fixing we like. We will make the choice where the action becomes $A \wedge \star \Box A$. With this, straightforward calculation gives (ignoring numerical prefactors of 2 and π and so on)

$$\langle \partial_\mu A_\lambda \partial_\nu A_\sigma \rangle = g_{\lambda\sigma} \partial_\mu \frac{(d-2)x_\nu}{x^d} = \frac{(d-2)g_{\lambda\sigma}}{x^d} \left(g_{\mu\nu} - d\frac{x_\mu x_\nu}{x^2} \right). \tag{409}$$

Getting the result of the correlator is then straightforward. If we let

$$\Gamma_{\mu\nu} \equiv g_{\mu\nu} - d\frac{x_\mu x_\nu}{x^2}, \tag{410}$$

then we get

$$\langle F_{\mu\nu} F_{\lambda\sigma} \rangle = \frac{d-2}{x^d} \left(\Gamma_{\mu\lambda} g_{\nu\sigma} - \Gamma_{\nu\lambda} g_{\mu\sigma} - \Gamma_{\mu\sigma} g_{\nu\lambda} + \Gamma_{\nu\sigma} g_{\mu\lambda} \right). \tag{411}$$

If we similarly define

$$\Gamma'_{\mu\nu} \equiv g_{\mu\nu} - \frac{d}{2} \frac{x_\mu x_\nu}{x^2}, \tag{412}$$

then we can write this as

$$\langle F_{\mu\nu} F_{\lambda\sigma} \rangle = \frac{2(d-2)}{x^d} (\Gamma'_{\mu\lambda} \Gamma'_{\nu\sigma} - \Gamma'_{\mu\sigma} \Gamma'_{\nu\lambda}), \tag{413}$$

since the x^μ terms with four different indices cancel and the factor of 2 out front gives the right number of terms quadratic in the metric.

Actually, the form of this correlator really follows without doing any calculations (up to the overall prefactor). Since the theory is free and F is conserved, the dimension of the expression is fixed. Since the F operators are normal-ordered, the components in the tensor structure will never pair up μ with ν or λ with σ; however, the correlator must be antisymmetric under the exchange of both pairs of indices. This basically fixes the correlator to have the structure given above (even the factor of $(d-2)$ can be motivated from our knowledge of the fact that in $d = 2$ there are no propagating degrees of freedom).

Why is this 2-point function incompatible with conformal invariance when $d \neq 4$? Because it doesn't have the right tensor structure. Indeed, the tensor structure of the 2-point function of any conformal primary transforming in some irrep of $SO(d)$ must be built entirely from the "inversion tensors" $I^{\mu\nu} = \delta^{\mu\nu} - 2x^\mu x^\nu / x^2$, which can be seen from looking at the action on the correlator by the (nontrivial) conformal transformations which fix the origin and the point x^μ—a good reference for this is DSD's class notes on CFT, section 8.2. Anyway, since we see that this is only true when $d = 4$, when Γ' becomes the inversion tensor, we can conclude that $F_{\mu\nu}$ can only be a primary in $d = 4$. $F_{\mu\nu}$ can't be a descendant since it is the "smallest" gauge-invariant field; hence free Maxwell is only a CFT in $d = 4$.

F is not a primary because it doesn't transform like one under conformal transforamtions

We now give a different "proof" for why $F_{\mu\nu}$ isn't a primary, by looking at the way it transforms under conformal transformations. This proof is premised on the assumption that A transforms in the same way as a primary would under conformal transformations, i.e. that it transforms by the pull-back of a diffeomorphism up to a conformal re-scaling factor, with scaling dimension $\Delta_A = (d-2)/2$, so that the theory is scale-invariant. Of course A really isn't a primary (its scaling dimension would violate the unitarity bound, and anyway it doesn't appear in the Hilbert space), but it also isn't gauge invariant so this doesn't bother us—we only care about F. Nevertheless, since $F = dA$, the transformation properties of F will be fixed by those of A, so in order for F to stand a chance of being a primary, A is still forced to transform in the same way that a primary would.

We will now give a general analysis of when the derivative of a vector primary is itself a primary.[20] A general CKV ξ_μ can be written as

$$\xi_\mu = a_\mu - \omega_{\mu\nu}x^\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b\cdot x. \tag{414}$$

The above terms represent translations, rotations, scale transformations, and SCTs, respectively. We calculate (in Euc. signature, for simplicity)

$$\partial_\mu \xi_\nu = \omega_{\mu\nu} + \delta_{\mu\nu}(\lambda - 2b\cdot x) + 2b_{[\mu}x_{\nu]} \equiv \delta_{\mu\nu}\sigma_\xi + \widetilde{\omega}_{\xi\mu\nu}. \tag{415}$$

Therefore we may write the transformation of A_μ under an infinitesimal conformal transformation as

$$\delta_\xi A_\mu = -\xi^\nu\partial_\nu A_\mu - \sigma_\xi \Delta_A A_\mu + \widetilde{\omega}_{\xi\mu}{}^\nu A_\nu. \tag{416}$$

This lets us figure out how $\partial_\mu A_\nu$ transforms: using the above expression for $\partial_\mu\xi_\nu$,

$$\begin{aligned}\delta_\xi(\partial_\mu A_\nu) &= -\xi^\alpha\partial_\alpha\partial_\mu A_\nu - \sigma_\xi\partial_\mu A_\nu - \Delta_A(\partial_\mu\sigma_\xi A_\nu + \sigma_\xi\partial_\mu A_\nu) - \widetilde{\omega}_{\xi\mu}{}^\alpha\partial_\alpha A_\nu - \widetilde{\omega}_{\xi\nu}{}^\alpha\partial_\mu A_\alpha + \partial_\mu\widetilde{\omega}_{\xi\nu}{}^\alpha A_\alpha \\ &= [\delta_\xi(\partial_\mu A_\nu)]_{CP} + \partial_\mu\widetilde{\omega}_{\xi\nu}{}^\alpha A_\alpha - \Delta_A\partial_\mu\sigma_\xi A_\nu,\end{aligned} \tag{417}$$

where the $[]_{CP}$ indicates the terms that would be present if $\partial_\mu A_\nu$ were a conformal primary (the σ_ξ part appears as $\sigma_\xi(\Delta_A+1)\partial_\mu A_\nu$ which is correct since $\Delta_F = \Delta_A + 1$). Therefore the remaining two terms are the potential obstruction to F being a primary. We can simplify them by noting that

$$\partial_\mu\widetilde{\omega}_{\xi\nu\alpha} = 2\delta_{\mu\alpha}b_\nu - 2\delta_{\mu\nu}b_\alpha = -\delta_{\mu\alpha}\partial_\nu\sigma_\xi + \delta_{\mu\nu}\partial_\alpha\sigma_\xi. \tag{418}$$

This means

$$\delta_\xi(\partial_\mu A_\nu) = [\delta_\xi(\partial_\mu A_\nu)]_{CP} + \partial_\lambda\sigma_\xi(\Delta_A\delta_\mu^\lambda\delta_\nu^\rho + \delta_\mu^\rho\delta_\nu^\lambda - \delta_{\mu\nu}\delta^{\lambda\rho})A_\rho. \tag{419}$$

Now consider taking the antisymmetric combination to find $\delta_\xi F_{\mu\nu}$. The antisymmetry in μ,ν means that the last term in the non-primary part dies. The first two terms in the non-primary part are related to one another by the exchange $\mu\leftrightarrow\nu$, and so they cancel after antisymmetrization if and only if $\Delta_A = 1$. Therefore we have the condition

$$F_{\mu\nu} \text{ is a conformal primary only if } \Delta_A = 1. \tag{420}$$

[20]Note that $F = dA$ does *not* imply that F is a descendant, even if A is a primary! For ϕ a scalar, $V = d\phi$ means that V is a descendant, but this isn't true if ϕ transforms in a nontrivial irrep of $SO(d)$.

This means $F_{\mu\nu}$ is a primary only if $\Delta_F = 2$, which saturates the unitarity bound on an antisymmetric tensor. Anyway, the point is that scale invariance forces us to take $\Delta_A = (d-2)/2$, and so $F_{\mu\nu}$ is only a primary in $d = 4$. Again, in $d \neq 4$ conformal invariance would then imply that $F_{\mu\nu}$ is a descendant, but there is no local field that it could possibly be the descendant of. So as before we conclude that in $d \neq 4$, free Maxwell is not a CFT.

Conservation of $T^{\mu\nu}$, different kinds of stress tensors, and useful covariant derivative identities ✓

Today we're doing a short calculation showing why $T^{\mu\nu}$ is conserved. There's a nontrivial integration by parts that is normally glossed over which we will explain. We'll also explain the relation between the two different usual ways of calculating $T_{\mu\nu}$ (by varying the fields or by varying the metric), which is annoyingly not really explained correctly in any of my books.

The covariant conservation of the stress tensor in a given theory defined on the Riemannian manifold (M, g) follows from the invariance of the action under isometries $f : (M, g) \to (M, f_* g)$. f here is an isometry since distances computed on $M = (M, g)$ match those computed in $f(M) = (f(M), f_* g)$: the line element satisfies

$$ds^2_M(x) = ds^2_{f(M)}(f(x)). \tag{421}$$

These isometries are often called diffeomorphisms by physicists, and here we will adopt this terminology as well (for better or worse).

Consider an infinitesimal flow along a diffeomorphism $f : M \to M$, where the coordinates are mapped as $x^\mu \mapsto x^\mu + \xi^\mu$. The action $\int d^d x \sqrt{g}\, \mathcal{L}$ is unchanged under both mapping the coordinates under the diffeomorphism and pulling back the fields along the diffeomorphism, since the combination of both is a reparametrization of our coordinate system (mathematically, it is an isometry), under which all theories whose Lagrangians don't explicitly depend on the coordinates (i.e. all non-pathological theories in physics) are invariant.

After mapping the fields under the diffeomorphism by replacing them with their pullbacks (but not changing the coordinates), the Lagrangian changes as

$$\delta(\sqrt{g}\mathcal{L}) = \left(\mathcal{L}_\xi \phi \frac{\delta}{\delta \phi} + (\mathcal{L}_\xi g)_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \right) (\sqrt{g}\, \mathcal{L}), \tag{422}$$

where $\mathcal{L}_\xi$ is the Lie derivative along ξ, not to be confused with the Lagrangian (sorry!). Here the fields ϕ could be any sorts of fields (well, not spinors, but just for simplicity), so that if $\phi = A_\mu$ is a vector field, $(\mathcal{L}_\xi A)_\mu = \xi^\nu \partial_\nu A + \partial_\lambda \xi^\mu A^\lambda$. Now when we translate all the fields appearing in $\mathcal{L}$ along ξ, the Lagrangian changes by a total derivative, since we are just moving the Lagrangian infinitesimally along the flow:

$$\delta(\sqrt{g}\mathcal{L}) = \partial_\mu(\xi^\mu \sqrt{g}\mathcal{L}). \tag{423}$$

As a very simple check, consider a mass term for a scalar φ. Then

$$\mathcal{L}_\xi \varphi \frac{\delta}{\delta\varphi}(\sqrt{g}\varphi^2) = \sqrt{g}\xi^\mu \partial_\mu \varphi^2, \qquad (\mathcal{L}_\xi g)_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}(\sqrt{g}\varphi^2) = \frac{1}{2}\sqrt{g}\varphi^2(g^{\mu\nu}\xi^\lambda \partial_\lambda g_{\mu\nu} + 2\partial\cdot\xi) \tag{424}$$

where we used that the Lie derivative is

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\mu\alpha}. \tag{425}$$

Now

$$\partial_\lambda \sqrt{g} = \frac{1}{2}\sqrt{g} g^{\mu\nu} \partial_\lambda g_{\mu\nu}, \tag{426}$$

and so

$$(\mathcal{L}_\xi g)_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}(\sqrt{g}\varphi^2) = \varphi^2 \partial_\mu(\sqrt{g}\xi^\mu). \tag{427}$$

Therefore the total variation is

$$\delta(\sqrt{g}\mathcal{L}) = \partial_\mu(\sqrt{g}\xi^\mu\varphi^2) \tag{428}$$

as expected.

A less trivial example is something which involves vectors and derivatives, like $\sqrt{g}A_\mu\partial^\mu\varphi$. The variations over the matter fields give

$$\left((\mathcal{L}_\xi A)_\mu \frac{\delta}{\delta A_\mu} + \mathcal{L}_\xi\varphi\frac{\delta}{\delta\varphi}\right)(\sqrt{g}A_\mu\partial^\mu\varphi) = \sqrt{g}\left(\partial^\mu\varphi(\xi^\nu\partial_\nu A_\mu + \partial_\mu\xi^\nu A_\nu) + A_\mu\partial^\mu(\xi^\nu\partial_\nu\varphi)\right), \tag{429}$$

while the variation over the metric produces, after some algebra,

$$(\mathcal{L}_\xi g)_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{g}A_\mu\partial^\mu\varphi) = (\xi^\lambda\partial_\lambda\sqrt{g} + \partial\cdot\xi)A_\mu\partial^\mu\varphi - A^\mu\partial^\nu\varphi\xi^\alpha\partial_\alpha g_{\mu\nu} - A_\lambda\partial_\alpha\xi^\lambda\partial^\alpha\varphi - A_\alpha\partial^\alpha\xi_\lambda\partial^\lambda\varphi. \tag{430}$$

Since $\partial_\alpha g_{\alpha\sigma} = -g_{\mu\nu}\partial_\alpha g^{\nu\lambda}g_{\lambda\sigma}$, the third term on the RHS is actually $+A_\mu\partial_\nu\varphi\xi^\alpha\partial_\alpha g^{\mu\nu}$. Adding up the contributions from the matter and the metric, we see that the last two terms on the LHS of the above equation cancel with two of the terms in the variation of the matter fields, again leaving us with

$$\delta(\sqrt{g}\mathcal{L}) = \partial_\mu(\sqrt{g}\xi^\mu A_\nu\partial^\nu\varphi). \tag{431}$$

Now we can see the general pattern that's at work: the variation over the metric produces derivatives of $\sqrt{g}$, ξ, and any $g_{\mu\nu}$s appearing in $\mathcal{L}$, plus some extra stuff coming from transforming the indices of the $g_{\mu\nu}$s. The variation over the matter fields produces derivatives of thins involving the matter fields, plus some extra stuff coming from the transformation of any vector indices that the matter fields have. The extra stuff from the matter fields and the extra stuff from the variation of the $g_{\mu\nu}$s cancel, since upper-index variations cancel lower-index ones. After the smoke clears, we are left with a total derivative.

Another way to see this is simply to write the Lagrangian as a d-form. If $L = \mathcal{L} \cdot \star 1$, where $\star 1$ is the volume form, then under the variation we have

$$\delta_\xi L = \mathcal{L}_\xi L = (i_\xi d + d i_\xi) L = d(i_\xi L), \tag{432}$$

where we have used Cartan's formula and that L is a top-dimensional form. Therefore the variation of the Lagrangian is indeed always a total derivative.

Since our diffeomorphism must vanish at ∂M, the upshot to the above discussion is that (no $\sqrt{g}$ in the integration measure on the LHS; it will be picked up from varying the action)

$$\int_M d^d x \left((\mathcal{L}_\xi g)_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \mathcal{L}_\xi \phi \frac{\delta}{\delta \phi} \right) S[\phi, g_{\mu\nu}] = \int_{\partial M} d^{d-1} x^\mu \sqrt{g|_{\partial M}} \xi_\mu \mathcal{L}|_{\partial M} = 0. \tag{433}$$

Now by definition, (we are working in Euclidean signature; hence the sign)

$$\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} = T^{\mu\nu}. \tag{434}$$

The conservation of currents associated to global symmetries holds on shell. So, choosing a configuration of matter fields which solves the EOM, we see that

$$\int_M d^d x \sqrt{g}\, (\mathcal{L}_\xi g)_{\mu\nu} T^{\mu\nu} = 0, \tag{435}$$

on shell.

To get a conservation law, we need to massage the Lie derivative slightly. First, add and subtract $\Gamma^\lambda_{\alpha\mu} g_{\lambda\nu} + \Gamma^\lambda_{\alpha\nu} g_{\mu\lambda}$ on the RHS of the expression (425) for the Lie derivative. The positive terms combine with the first term above to produce a covariant derivative of $g_{\mu\nu}$, which dies. The negative terms are the connection coefficients needed to turn the derivatives of ξ into covariant ones. Therefore

$$(\mathcal{L}_\xi g)_{\mu\nu} = \nabla_{(\mu} \xi_{\nu)}. \tag{436}$$

Since the stress tensor is symmetric, we then have

$$\int_M d^d x \sqrt{g}\, (\nabla_\mu \xi_\nu) T^{\mu\nu} = 0. \tag{437}$$

Now naively we would like to integrate by parts and conclude that T is covariantly conserved. This is correct, but nonzero work is required to demonstrate it. First, we need the product rule

$$\nabla_\mu (T^{\mu\nu} \xi_\nu) = (\nabla_\mu T^{\mu\nu}) \xi_\nu + T^{\mu\nu} \nabla_\mu \xi_\nu. \tag{438}$$

The first term on the RHS has two Christoffel symbol terms, which both have minus signs, while the second term on the RHS has one positive Christoffel symbol. Two of these cancel, leaving a single negative-sign Christoffel symbol, which just the right index structure to match the $\Gamma^\mu_{\mu\lambda}T^{\lambda\nu}\xi_\nu$ term on the LHS. Since this is straightforward algebra, and I won't write it out.

Now we need to argue that $\sqrt{g}\nabla_\mu(T^{\mu\nu}\xi_\nu)$ is a total derivative. This is only true because the covariant derivative is acting on a vector; if it was acting on a larger-rank tensor, it would not be a total derivative. Indeed, for any vector V^μ, we have

$$\begin{aligned}\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}V^\mu) &= \partial_\mu V^\mu + \frac{1}{2}V^\mu g^{\alpha\beta}\partial_\mu g_{\alpha\beta} \\ &= \partial_\mu V^\mu + \frac{1}{2}V^\mu g^{\alpha\beta}(\Gamma^\lambda_{\mu\alpha}g_{\lambda\beta} + \Gamma^\lambda_{\mu\beta}g_{\alpha\lambda}) \\ &= \nabla_\mu V^\mu,\end{aligned} \tag{439}$$

where in the first line we used the usual way of differentiating the determinant by writing it as $e^{\frac{1}{2}\mathrm{Tr}\ln g}$, and in the next line we used the metric compatibility of the connection to substitute in some Christoffel symbols for the derivative of the metric. Therefore we indeed have

$$\sqrt{g}\nabla_\mu(T^{\mu\nu}\xi_\nu) = \partial_\mu(\sqrt{g}\,T^{\mu\nu}\xi_\nu). \tag{440}$$

Therefore, taking ξ^μ to vanish at ∂M, we conclude that

$$\int_M d^dx\,\sqrt{g}\,\xi_\nu\nabla_\mu T^{\mu\nu} = 0, \tag{441}$$

and since this must hold for any ξ_ν, we have

$$\nabla_\mu T^{\mu\nu} = 0, \tag{442}$$

as required.

Finally, we can take a look at the relationship between the stress tensor defined here and the "canonical" stress tensor $T^{\mu\nu}_c$ obtained from the Noether procedure. Under $\phi(x^\mu) \mapsto \phi(x^\mu + \xi^\mu)$, the action changes as

$$\delta S = \int_M d^dx\,\sqrt{g}\,\nabla_\mu\xi_\nu T^{\mu\nu}_c. \tag{443}$$

Since this variation is the one produced by the second term on the LHS of (433), we have, for any matter field configuration (on-shell or off),

$$\begin{aligned}\int_M d^dx\,\sqrt{g}\,\nabla_\mu\xi_\nu T^{\mu\nu}_c &= \int_M d^dx\,\sqrt{g}\,(\mathcal{L}_\xi g)_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}S[\phi, g_{\mu\nu}] \\ &= \int_M d^dx\,\sqrt{g}\,\nabla_\mu\xi_\nu T^{\mu\nu}.\end{aligned} \tag{444}$$

Since this must hold for any ξ_ν, we may take the matter fields to be off-shell (so that $T^{\mu\nu}$ has a non-zero divergence), and conclude that

$$\nabla_\mu T^{\mu\nu}_c = \nabla_\mu T^{\mu\nu}, \tag{445}$$

which tells us that the two stress tensors agree up to something whose divergence is zero, namely the components of $d^\dagger B$, where B is a 3-form.