The Ising model on a tree

Ethan Lake

The following is a brief note discussing some aspects of the Ising model on a $k > 2$-regular tree $\mathbb{T}_k$. This was partly inspired by educational email exchanges with Vedika Khemani and Andy Lucas (the latter of whom wrote a problem exploring some aspects of this model available here).

The zeroth order thing to note about $\mathbb{T}_k$ is that a) the bounddary $\partial\mathbb{T}_k$ contains a constant fraction of the total number of sites N, and b) in addition to this, the number of sites further than a distance of $\sim \log\log(N)$ from the boundary consitute an exponentially small fraction of all sites. This makes the Ising model on $\mathbb{T}_k$ slightly sick: in Euclidean space, where the fraction of boundary sites vanishes in the TDL, the divergent susceptibility that accompanies SSB can be diagnosed by looking at how the fixed but few-in-number boundary sites exert influence on the rest of the system; on the tree, having everything so close to the boundary gives a slightly strange TDL,[1] and as we shall see gives the free energy some unusual properties.

No magnetic field, free boundary conditions

Things are simplest with free boundary conditions and in the absence of a magnetic field, although as we will see the result obtained in this limit is slightly misleading. Here the free energy can be computed exactly, by integrating out the spins from the outside in, starting with those at the boundary. For a fixed configuration $\{s_{i\in\text{bulk}}\}$ of bulk spins, we may redefine each boundary spin as $s_i \to s_i s_{\text{parent}(i)}$, where i is a boundary vertex and parent(i) is its parent vertex. This decouples the boundary spins completely, and gives

$$Z_d = Z_{d-1}(2\cosh(\beta))^{k(k-1)^{d-1}} \tag{1}$$

where Z_d is the partition function on the depth d tree and $k(k-1)^{d-1}$ is the number of leaves at depth d. Letting the total number of nodes be $N \sim (k-1)^d$, the free energy density F/N is accordingly

$$\frac{F}{\beta N} = -\ln[2\cosh(\beta)], \tag{2}$$

[1] This "problem" can be partially mitigated by working on a 2d hyperbolic manifold with periodic boundary conditions (obtained by tesselating a genus $g \gg 1$ surface M_g). However in this case g diverges in the TDL, and there exist a continuum of different length scales set by the sizes of the generators of $H_1(M_g;\mathbb{Z})$. These models are presumably much harder to analyze analytically however, so we will stick to $\mathbb{T}_k$ for now.

which is analytic at all β—there is thus apparently *no* phase transition at any finite temperature, and the system is always in the disordered phase.

This result can be understood in two ways. First, from the above strategy, we see that one can perform a change of variables which turns the model into N decoupled spins. This change of variables proceeds by taking products of spins from the outside of the tree along paths proceeding to the tree center, and will of course fail if loops were to be introduced. The thermodynamic triviality inferred from this reasoning also occurs in various glassy models, like the Newmann-Moore model, or Hamiltonians obtained from LDPC codes (see e.g. Hong, Guo, Lucas 24 or Mezard + Montanari for a contemporary discussion). Alternatively, one could decude the above result by thinking about the energy landscape on the tree. Unlike in Euclidean space, on the tree there are an extensive number of low-lying states with extensive Hamming distance from the two ground states. For example, there are $N-1$ states of energy 1, obtained by flipping all of the spins in a given subtree. This profusion of near-minima smoothen out the energy landscape and produce a thermodynamically trivial system. Both of these explanations could also be given to explain why the Ising model on a line ($\mathbb{T}_2 = \mathbb{Z}$) lacks a phase transition (although the types of barriers present in the energy landscape are very different on $\mathbb{T}_{k>2}$ and $\mathbb{Z}$).

The conclusion that there is no phase transition on $\mathbb{T}_{k>2}$ is however slightly hasty: in spite of the trivial-looking expression for F, we will momentarily see that the magnetic susceptibility diverges at a finite β, producing a non-analyticity in F in the standard limit where $N \to \infty$ before the field $h \to 0$. Thus the free energy $F(\beta, h)$ is an analytic function of β at $h = 0$, but is a non-analytic function of h at $h = 0$ for $\beta > \beta_c$. I am unaware of any other models producing a phase transition like this (which e.g. does not show up in the specific heat at all). This fact also demonstrates that despite mean field approximations working well on trees, the present model definitely yields a free energy distinct from the mean field expression.

Nonzero field / fixed boundary conditions

The near-degeneracy of the aforementioned energy landscape is lifted in the presence of a nonzero magnetic field h, or in the presence of non-free uniform boundary conditions. Consider then taking

$$H = -\sum_{\langle i,j \rangle} s_i s_j - bh \sum_{i \in \text{bulk}} s_i - h \sum_{i \in \text{boundary}} s_i, \tag{3}$$

where $b = 0$ if the field is only at the boundary and $b = 1$ if the field is uniform throughout the system. Fixing the boundary conditions to be $s_{i \in \text{boundary}} = +1$ corresponds simply to setting $b = 0, h = 1$.

After integrating out the boundary spins, we have

$$Z_d = \sum_{\{s_{\text{bulk}}\}} e^{-\beta H(\{s_{\text{bulk}}\})} \prod_{i \in S_{d-1}} [2\cosh(\beta(s_i + h))]^{k-1}, \tag{4}$$

where S_{d-1} is the set of vertices at depth $d-1$. We claim that the effect of integrating out the boundary spins can be captured entirely by renormalizing the field experienced by the spins in S_{d-1} and adjusting the partition function by an overall constant. This is true provided that we can write

$$[2\cosh(\beta(s_i + h_n))]^{k-1} e^{(k-1)f_n} = e^{\beta(h_{n-1} - bh)s_i + f_{n-1}} \tag{5}$$

for some constants f_n (with $f_d = 0$); here the $-bh$ on the RHS is included so that a spin at depth n feels an effective field of h_n. Indeed, since $s_i \in \{\pm 1\}$ we may write $e^{\beta(h_{n-1}-bh)s_i/(k-1)} = \cosh(\beta(h_{n-1} - bh)/(k-1)) + s_i \sinh(\beta(h_{n-1} - bh)/(k-1))$, while on the LHS we may use $\cosh(x + s_i y) = \cosh(x)\cosh(y) + s_i \sinh(x)\sinh(y)$. Matching the s_i on both sides gives the recursion relation (which will be familar to those who know about belief propagation—see e.g. Mezard and Montanari)

$$\tanh(\beta)\tanh(\beta h_n) = \tanh\left(\frac{\beta(h_{n-1} - bh)}{k-1}\right) \tag{6}$$

with $h_d \equiv h$, from which the f_n are determined via

$$f_{n-1} = (k-1)\left(f_n + \ln\left[\frac{2\cosh(\beta)\cosh(\beta h_n)}{\cosh(\beta(h_{n-1} - bh))}\right]\right). \tag{7}$$

From the recursion relation for Z_d we see that the free energy density on a tree with N spins is

$$F/N = -\beta^{-1} f_0/N + \cdots, \tag{8}$$

where the $\cdots$ vanish as $N \to \infty$. As a santiy check, note that when $h = b = 0$ we get $f_n/N = \ln(2\cosh(\beta)) + O(1/N)$, which agrees with the zero-field result.

The phase of the system is—at least if one defines things by associating order to bulk sensitivity to boundary fields—determined by the effective field $h_{n\to 0}$ produced deep in the bulk. Consider first the high temperature limit $\beta \ll 1$. Then

$$h_{n-1} = \beta(k-1)h_n + bh, \tag{9}$$

which is solved as

$$h_n = h\left(b\frac{1 - \widetilde{\beta}^{d-n}}{1 - \widetilde{\beta}} + \widetilde{\beta}^{d-n}\right), \qquad \widetilde{\beta} \equiv (k-1)\beta. \tag{10}$$

Thus as long as $\widetilde{\beta} \ll 1$, the effective field seen in the bulk is finite, and approaches either 0 (if $b = 0$, viz. if the field is only at the boundary), or a renormalized value of $h/(1 - \widetilde{\beta})$ (if $b = 1$, viz. if the field is everywhere).[2]

On the other hand, when $\beta \gg 1$ it is easy to see that $h_d = \varepsilon \ll 1$ does not give an h_n that converges to 0 as $n \to 0$. To find the location of the transition where the behavior switches, we can look at the stability of the recursion relation for h_n. Consider for simplicity the case with no bulk field ($b = 0$). Then expanding the recursion relation in small h_n, h_{n-1}, we have

$$\frac{h_{n-1}}{h_n} = (k-1)\tanh(\beta), \tag{11}$$

[2]In both cases, the approach to this value is exponentially fast in distance from the tree boundary.

and so the critical inverse temperature β_* satisfies $\tanh(\beta_*) = 1/(k-1)$. This can be re-written by taking $\beta_* = \ln(\lambda)$ and using $\tanh(\log(\lambda)) = (\lambda^2-1)/(\lambda^2+1)$ and then solving for λ; doing so gives

$$\beta_* = \frac{1}{2}\ln\frac{k}{k-2}. \tag{12}$$

Note as sanity checks that a) $\beta_* = \infty$ when $k=2$, recovering the absence of a transition on the line, and b) $\beta_* \to 0$ when $k \to \infty$, since the number of nearest neighbors of each spin becomes infinite in this limit. Furthermore, if we restore the Ising coupling J and take the limit $k \to \infty$ with $J\beta$ fixed, we get

$$\beta_*|_{k\to\infty} = \frac{1}{Jk}. \tag{13}$$

This limit is the one we would take when doing mean field,[3] and indeed $T_* = Jk$ matches exactly the expected mean field critical temperature (in which the magnetization m as a function of field h is $m = \tanh[\beta(kJm+h)]$). This is one sense in which mean field "works" on the tree (although there are of course distinct differences, as was seen in the behavior of $F(\beta, h=0)$).

The above result highlights why the previous conclusion about the absence of a phase transition in the zero-field case is (in some sense) too hasty: while the free energy is that of a trivial paramagnet in the absence of a field, when $\beta > \beta_*$ any nonzero boundary field will grow to a nonzero value as spins near the boundary are integrated out, leading to ordering in the center of the tree. I believe the reason the "phase transition" at β_* does not show up in e.g. the specific heat (so that the system in some respects is still thermodynamically trivial at $\beta > \beta_*$) is that it is only the susceptibility at vertices very near the center of $\mathbb{T}_k$ that diverge. Indeed, while h_n approaches h_0 exponentially fast in distance from the boundary, almost all sites are close to the bounadry, where they see a non-divergent h_n, so that $(\partial_h\langle s_i\rangle_h)|_{h=0} \to 0$ for most sites at all β.

Phase diagram

We now briefly discuss the "phase" diagram in the (β, h) plane. For us the ordered phase will be defined purely by the region in which the Gibbs measure is not unique (irrespectivity of the triviality [or not] of $F(\beta, h)$).

Consider first $\beta = \infty$. The ordered phase here in fact has a finite critical h_c, equal to

$$h_c|_{\beta=\infty} = k-2. \tag{14}$$

I believe this follows by realizing that at $h = k-2$, the energy change of flipping a line of spins (beginning and ending on $\partial\mathbb{T}_k$) vanishes when starting from a ferromagnetic state aligned against h (provided the boundary conditions at the path ends are chosen appropriately). As a partial sanity check, $h_c|_{\beta=\infty} = 0$ on the line ($k=2$). As β is decreased, we expect h_c to decrease as well, until it vanishes at β_*.

[3]Here by "mean field" we mean taking the coupling to be all-to-all and sufficiently weak to produce a well-defined TDL. It appears that sometimes people mean "infinite spatial dimension with local couplings", viz. working on $\mathbb{T}_k$, precisely what we are doing here.

The fact that at $\beta = \infty$ the different Gibbs measures at $h = k-2-\varepsilon$ are obtained by flipping spins along lines means that the number of symmetry broken minima (extremal points of the convex set of Gibbs states) is larger than two (and in fact infinite) in this regime. This comes from the fact that in this regime, the magnetization of spins in the bulk can be affected when only a vanishing fraction of the boundary spins are changed (something which is easily checked to *not* happen at $\beta = \beta_* + \varepsilon, h = 0$). Thus there must be a further transition between a doubly degenerate ferromagnetic phase—where $\langle s_{\text{center}} \rangle$ is sensitive only to the net magnetization of spins on $\partial \mathbb{T}_k$—and an extensively-degenerate phase, where $\langle s_{\text{center}} \rangle$ is sensitive to the full details of the boundary conditions.

The value of $\beta'_*(h)$ above which the system enters the extensively-degenerate phase should in fact be equal to the $\pm J$ spin glass transition temperature at bond probability $p = 1/2$ (again, at least at $h = 0$). From some results quoted in Martinelli, Sinclair and Weitz 03, at $h = 0$ this temperature is apparently

$$\beta'_*(0) = \frac{1}{2} \ln \frac{\sqrt{k+1}+1}{\sqrt{k+1}-1}. \tag{15}$$

The connection between these temperatures (at least at $h = 0$) follows from realizing that the disorder in the $\pm J$ spin glass can be completely pushed off to the boundary: the change of variables discussed above renders all couplings ferromagnetic since $\mathbb{T}_k$ has no loops, and modifies only the boundary conditions (and thus the glass problem is trivial with free boundaries), and hence it is non-homogenous boundary conditions which produce frustration (since $|\partial \mathbb{T}_k|/|\mathbb{T}_k|$ is constant, one pays a constant energy density penalty if one ignores the boundary and satisfies only the bulk interaction terms). Thus the EA order parameter at the center of $\mathbb{T}_k$, viz. $\mathbb{E}_{\text{disorder}} \langle s_{\text{center}} \rangle^2_{\text{glass}} = \mathbb{E}_{\partial\,\text{conds}} \langle s_{\text{center}} \rangle^2_{\text{clean}}$, directly measures the influence of typical boundary conditions on the tree center, thereby identifying the spin glass temperature with β'_*.

Mixing times

We now briefly comment on the mixing times t_{mix} of Glauber dynamics in the scenarios discussed above.[4] This discussion is done purely because I wanted to remember the results, and comes mostly by pulling results from the aforementioned paper by Martinelli et al.

Recall first the story in Euclidean space: for Glauber dynamics on $\mathbb{Z}^d$ with PBC, $t_{\text{mix}} = O(\log(N))$ at $\beta < \beta_*$ and $t_{\text{mix}} = \exp(\Omega(N^{1/d}))$ at $\beta > \beta_*$. The former is the smallest t_{mix} can be ($t_{\text{rel}} = O(1)$ since the spectral gap is $1/L$, and the $\log(N)$ factor comes from the fact that independently relaxing spins take $\log(N)$ to relax essentially because of rare-region effects). With fixed ferromagnetic boundary conditions, we expect $t_{\text{mix}} = O(\text{poly}(N))$ for all T, although apparently this has not actually been proven.

Things change on $\mathbb{T}_{k>2}$. For arbitrary boundary conditions, $t_{\text{mix}} = O(\log(N))$ when

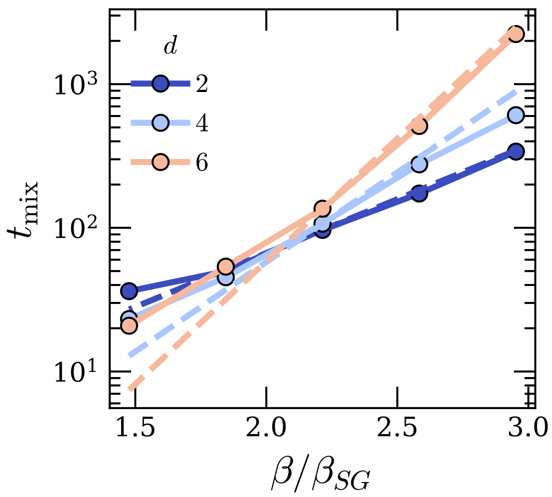
[4] Our definition of t_{mix} will be the "physicist's" definition, counting the number of time steps for Glauber dynamics to mix, with each time step involving $\Theta(N)$ updates, one applied at each site. Thus t_{mix} is shorter than its definition in the math literature by $\sim N$.

$\beta < \beta'_*$ or when $h > h_c(\beta)$, while for $\beta > \beta'_*$ and $h = 0$ one has[5]

$$t_{\rm mix} = O(N^{c\beta}) \tag{16}$$

where c is an $O(1)$ constant. With ferromagnetic boundary conditions, $t_{\rm mix} = O(\log(N))$ for all β, h; this is unsurprising since with ferromagnetic boundary conditions the mixing time on a graph G in the ordered phase should be $O(\text{poly}(\text{diam}(G)))$. I am at present not actually sure if the mixing time remains slow in the presence of a nonzero field. While the Gibbs measure is indeed not unique until a finite critical h, note that in the zero field case, it is $\beta'_*(h)$—and not $\beta_*(h)$—which is relevant for determining the regime in which mixing is slow, so that there exists a regime in the "ordered phase" (according to our definition) where mixing is nevertheless *fast* with free boundary conditions. Therefore a priori the mixing in the ordered phase at finite-field may also be fast; in fact basic reasoning about spin flips starting from the boundary make a poly(N) mixing time seem unlikely.

For this reason we will attempt to update our prior for $t_{\rm mix}$ by running some low-effore numerics. At zero field, we get



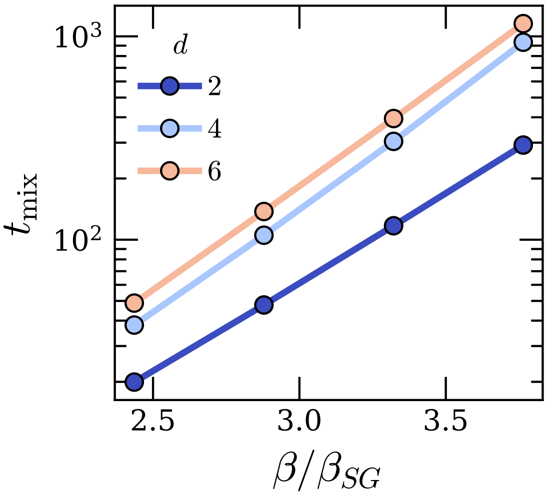
(17)

where the fits are to $t_{\rm mix} \sim N^{c\beta}$ for $c = 0.75/\beta_{SG}$ (β_{SG} is what we have been calling β'_* above). I am not sure why the transition appears to happen at $\beta = 2\beta_{SG}$ as I do not appear to be missing any obvious factors of 2.

At a finite but small magnetic field of $h = h_c(\beta = 0)/10$, and at moderately large

[5] The fact that $t_{\rm mix} = o(\exp(N^\alpha))$ for all α is immediately clear due to the fact that the energy barriers on the tree are exponential in the *log* of $|\mathbb{T}_k|$ (this is to be contrasted e.g. with the Ising model on a ER graph past the percolation threshold or on the tanner graph of a good LDPC code, which has linear confinement [barriers going as $\sim N$ rather than $\sim \log(N)$], giving gives an exponential lifetime).

values of β, we find



(18)

giving a mixing time which appears to be $t_{\mathrm{mix}} \sim \mathrm{poly}(L)e^{c\beta}$ wth c independent of L. Either the critical field for slow dynamics is extremely small or the dynamics is in fact fast at all non-zero fields (which seems more likely to me at present).