Comments on metastability in the active Ising model

Ethan Lake

It was recently appreciated in (Benvegnen,...,Tailleur 23) that the flocking phase of the active Ising model of Solon and Tailleur is in fact metastable: a large enough (and dense enough) droplet of minority spins, once nucleated, grows uncontrollably by forming a shock wave feature. In this short note we make a couple remarks on this phenomenon. Thanks to Sunghan Ro for discussions.

Mechanism and avoidance with repulsive interactions

Let us try to understand the development of the instability at a heuristic level. Consider a front where a density ρ_+ of $+$ spins collide with a density $\rho_- < \rho_+$ of $-$ spins. Let γ be the rate at which the spins on each site thermalize according to their mean-field-like Ising dynamics. If γ is much greater than the rate at which spins from the $+$ flock move into the $-$ flock, we do not expect the $+$ flock to grow, since $+$ spins that infiltrate the $-$ flock will thermalize to become $-$ before they can "overwhelm" the other $-$ spins. On the other hand, if γ is much less than this rate, the $+$ spins will convert the $-$ spins to $+$, and the flock will grow. This suggests that as long as $\rho_+ - \rho_-$ is sufficiently large (how large depends on γ), the $+$ flock will grow. If this intuition is correct, it predicts that metastability will be eliminated (in a certain region of parameter space) if an upper cutoff is placed on the number of particles per site, or if the particles are replaced by e.g. repulsively-interacting discs in an off-lattice setting.

Independence on relative flying directions

Consider more generically a situation where $+$ spins prefer to move along $\mathbf{v}^+$ and $-$ spins prefer to move along $\mathbf{v}^-$, with the usual AIM taking $\mathbf{v}^+ = -\mathbf{v}^-$. One might wonder if taking e.g. $\mathbf{v}^+ \perp \mathbf{v}^-$ could favor stability, as two opposite flocks will in then not meet "head-on", with instead one flock "ambushing" another from the side. This however turns out not to be the case, which is fairly reasonable given the discussion above.

To show this numerically in the hydrodynamic regime, we numerically integrate the hydro equations for the number densities of $+$ and $-$ spins $n^\pm$. The hydro equations are

$$\partial_t n^\pm = D\nabla^2 n^\pm + \mathbf{v}^\pm \cdot \nabla n^\pm \pm n^- e^{\beta m/\rho} \mp n^+ e^{-\beta m/\rho} \tag{1}$$

where the last terms try to align the spins on each site. Taking the symmetric and antisymmetric combinations of these equations,

$$\begin{aligned}\partial_t \rho &= D\nabla^2\rho + \mathbf{u}\cdot\nabla\rho + \mathbf{w}\cdot\nabla m \\ \partial_t m &= D\nabla^2 m + \mathbf{u}\cdot\nabla m + \mathbf{w}\cdot\nabla\rho + 2(\rho\sinh(\beta m/\rho) - m\cosh(\beta m/\rho))\end{aligned} \tag{2}$$

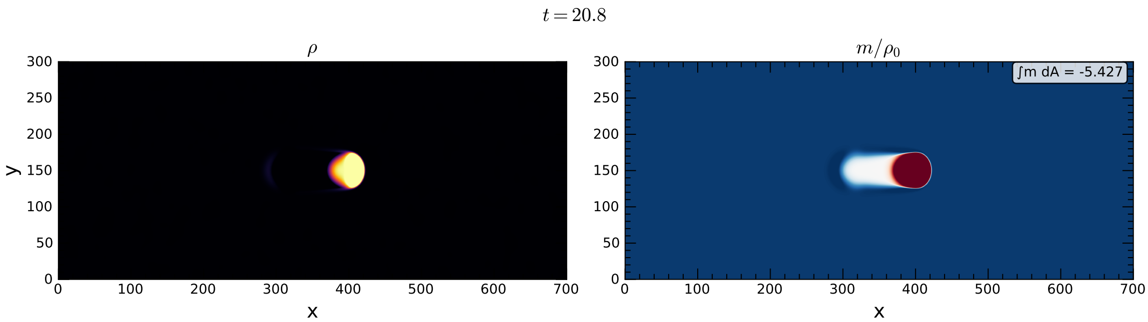
where

$$\mathbf{u} = \mathbf{v}^+ + \mathbf{v}^-, \qquad \mathbf{w} = \mathbf{v}^+ - \mathbf{v}^-. \tag{3}$$

The new feature of these equations is the addition of the terms proportional to $\mathbf{u}$, but this turns out to be innocuous.

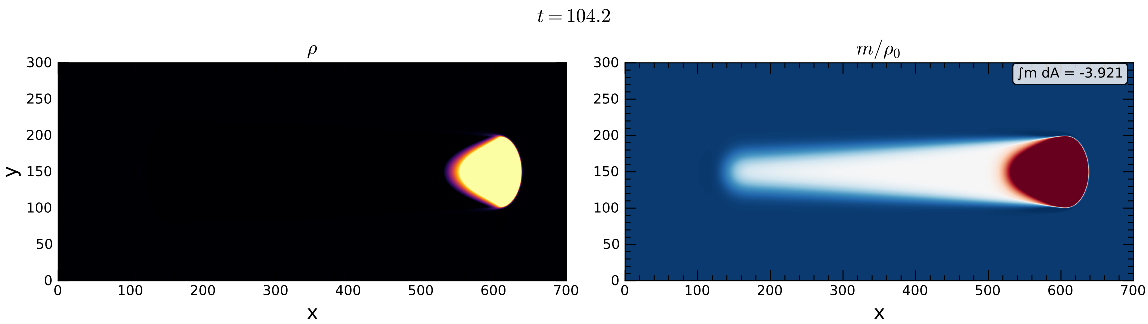
We investigate this by simulating these equations starting from a configuration with a very dense minority droplet in a homogeneous background and tracking how big the droplet gets. We will set $\beta = 2, D = 1, v = 1$, take a background density of $\rho_0 = 6$, a bubble radius of 20, a droplet density of $3\rho_0$, and set $\mathbf{v}^- = v\hat{\mathbf{x}}$.

For $\mathbf{v}^+ = -v\hat{\mathbf{x}}$, we find the same droplet profile as in the aforementioned paper: at early times,



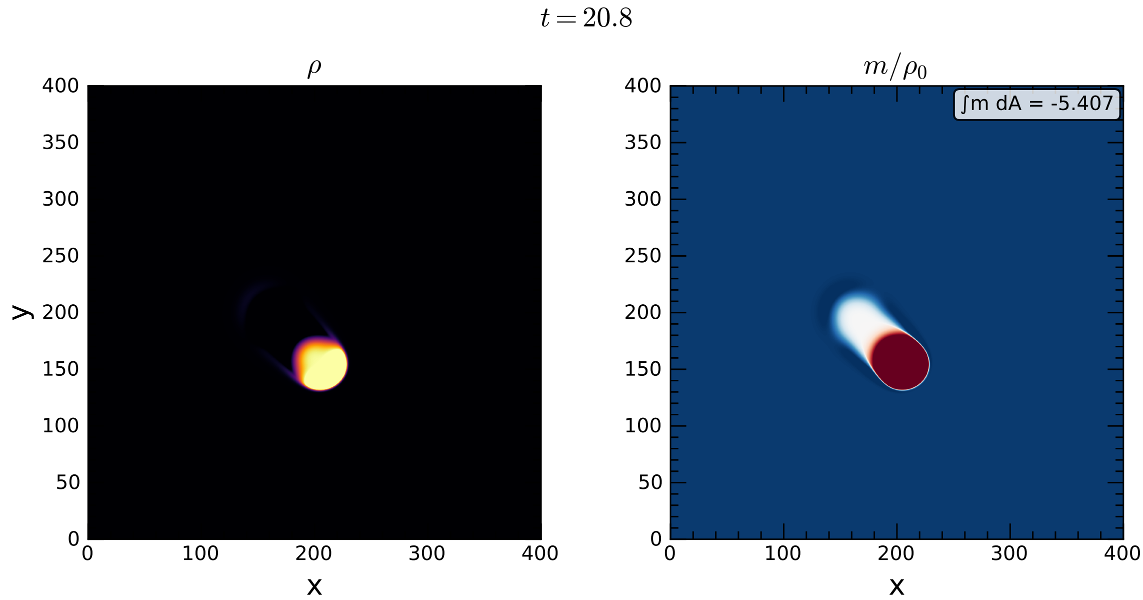
(4)

while at later times



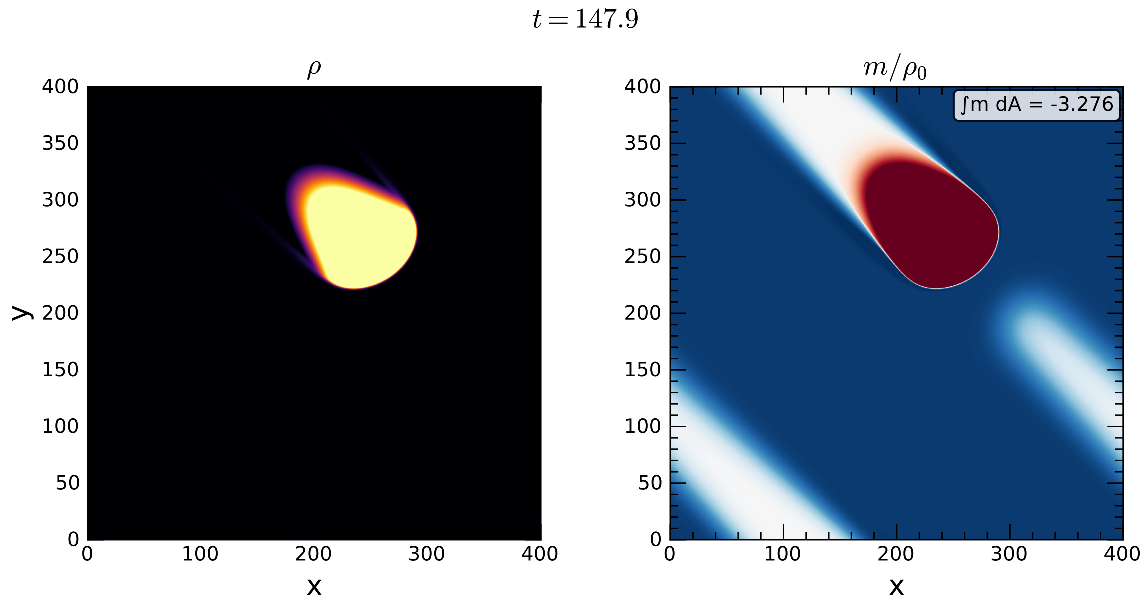
(5)

Now take $\mathbf{v}^+ = -v\hat{\mathbf{y}}$. The droplet still develops a tail and shock front; the only change is that the center of the droplet moves ballistically along $-\hat{\mathbf{y}}$. At early times:



(6)

while at later times:



(7)

A softened version

In an attempt to soften the shock front, consider a model of ferromagnetically-aligning XY spins that fly along the $\hat{\mathbf{x}}$ direction with a *speed* set by the cosine of their value. An anisotropy pinning the spins to lie in $\{0, \pi\}$ would reduce this to the AIM, but in general we instead have a softened version of the AIM (naively with the possibility to support homogeneous flocking states of different speeds).

Let us first derive the mean-field hydro equations appropriate for this scenario. Let n_θ be the number density of spins with orientation θ, and define

$$\rho = \int_\theta n_\theta, \qquad \psi = \int_\theta e^{i\theta} n_\theta \tag{8}$$

as the particle density and complex magnetization density, respectively. A reasonable noiseless Langevin equation for n_θ is

$$\partial_t n_\theta = D\nabla^2 n_\theta + v\cos(\theta)\nabla_x n_\theta - \partial_\theta \left(K n_\theta |\psi| \sin(\arg(\psi) - \theta) + \lambda \sin(2\theta) \right). \tag{9}$$

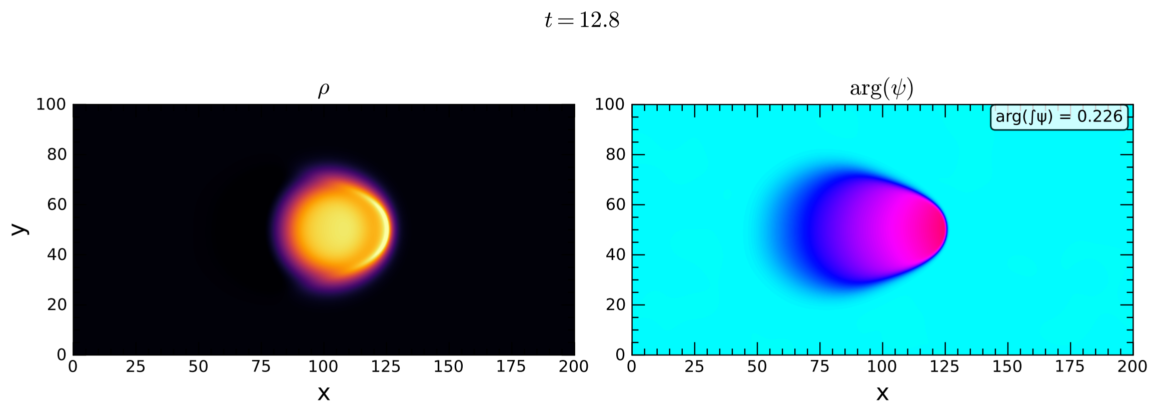
Here the K term tends to align the rotors along the direction of their magnetization, while the λ term is an anisotropy that prefers the rotors to be valued in $\{0, \pi\}$ (the ∂_θ is needed to ensure conservation of particles).

We now use this to derive Langevin equations for ρ and ψ. The equation for $\partial_t \rho$ is simple, since the ∂_θ term drops out. The equation for $\partial_t \psi$ is not closed, since it involves higher moments of the angular density (viz. integrals $\int e^{mi\theta} n_\theta$ with $|m| > 1$). For now, we will use a very simple-minded truncation by assuming that the mth moment is a product of m powers of the first moment. With this approximation, we get

$$\begin{aligned} \partial_t \rho &= D\nabla^2 \rho + v\nabla_x \frac{\psi + \psi^*}{2} \\ \partial_t \psi &= D\nabla^2 \psi + \frac{v}{2}\nabla_x(\psi^2 + \rho) + \frac{K}{2}\psi(\rho - |\psi|^2) - \frac{\lambda}{2}(\psi - \psi^*) \end{aligned} \tag{10}$$

which as required is symmetric under $\psi \mapsto -\psi, x \mapsto -x$. Note also that we essentially recover the AIM upon restricting ψ to be real.

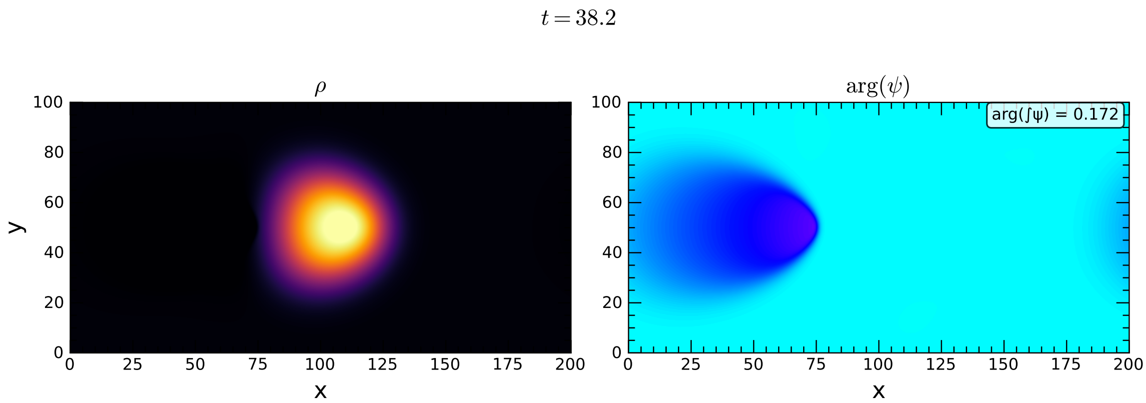
Consider taking $\lambda = 0$ and investigating the same droplet scenario as before. We set $D = v = 1$, take a backround of rotors with density $\rho_0 = 6$ and angle $\theta_0 = 0$, and place in them a droplet of radius 20, density $\rho_d = 3\rho_0$, and angle $\theta_d = \pi$.[1] A similar type of shock front begins to develop at the edge of the bubble at early times:



(11)

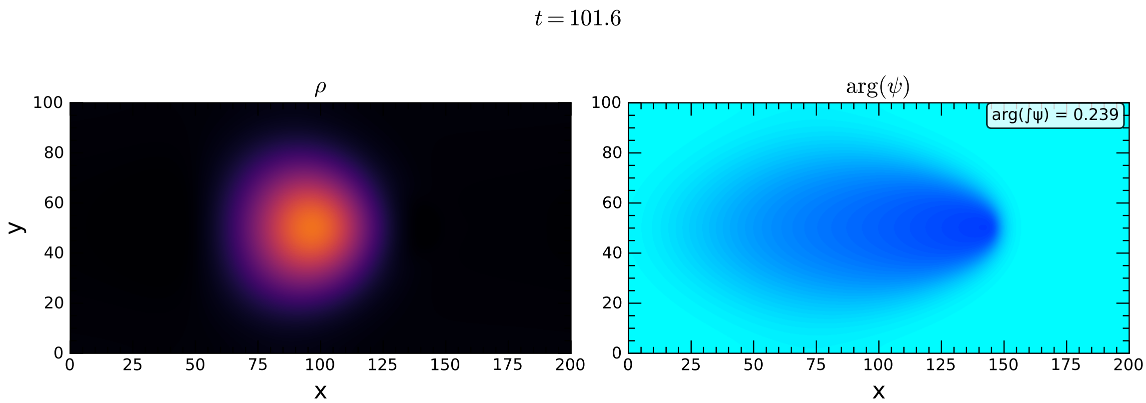
[1]We will actually shift θ_0 and θ_d slightly into the positive half-plane to break the $\theta \mapsto -\theta$ symmetry.

but it is quickly relaxed into a much more gradual domain profile:



(12)

and at long times its motion is arrested and the droplet is slowly relaxed:



(13)

Increasing the density of the droplet doesn't appear to qualitatively change this picture (increasing the velocity makes things more visually interesting but also doesn't appear to change the conclusion).