Characteristic class manipulations for Pontryagin classes

Ethan Lake

Today we review what pontryagin classes are, and prove some results about their reductions mod 2 and mod 4. These results are helpful to have when dealing with topological terms generated by integrating out massive fermions.

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First some preliminaries on the Pontryagin classes.

In what follows we will frequently need to complexify real bundles, and realify complex ones. The sequence to keep in mind for complexifying and realifying is

$$U(n) \to SO(2n) \to U(2n). \tag{1}$$

The first map is used to turn a complex $n \times n$ matrix into a real $2n \times 2n$ one, while the second map is used to complexify a real $2n \times 2n$ matrix. The second map in the sequence comes from the inclusion $\mathbb{R} \to \mathbb{C}$, while the first map comes from

$$U(n) \ni A + iB \mapsto \mathbf{1} \otimes A + J \otimes B \in SO(2n), \qquad J = -iY, \tag{2}$$

with A, B real. Here J is how we represent i in $SO(2n)$. Why is the image of $A + iB$ in $SO(2n)$? For $A + iB$ to be unitary, we need

$$(A^T - iB^T)(A + iB) = \mathbf{1} \implies A^T A + B^T B = \mathbf{1}, \quad A^T B - B^T A = 0. \tag{3}$$

Now consider $\mathbf{1} \otimes A + J \otimes B$. Then since $J^T = -J$,

$$(\mathbf{1} \otimes A^T - J \otimes B^T)(\mathbf{1} \otimes A + J \otimes B) = \mathbf{1} \otimes (A^T A + B^T B) + J \otimes (A^T B - B^T A) = \mathbf{1} \otimes \mathbf{1}, \tag{4}$$

and so $A + JB$ is indeed orthogonal.

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Now on to Pontryagin classes. For a vector bundle E (usually a real vector bundle), they are defined by

$$p_j(E) = (-1)^j c_{2j}(E \otimes \mathbb{C}), \tag{5}$$

where $E \otimes \mathbb{C}$ is the complexification of E. Pontryagin classes almost obey the same sum formula as the Chern classes. Indeed (writing $\otimes$ for $\otimes_{\mathbb{R}}$),

$$p_j(E \oplus F) = (-1)^j c_{2j}(E \otimes \mathbb{C} \oplus F \otimes \mathbb{C}) = (-1)^j [c(E \otimes \mathbb{C}) \wedge c(F \otimes \mathbb{C})]_{2j} = [p(E) \wedge p(F)]_j + \dots, \tag{6}$$

where ... are terms that involve odd Chern classes. For example,

$$p_1(E \oplus F) = p_1(E) + p_1(F) - c_1(E \otimes \mathbb{C}) \wedge c_1(F \otimes \mathbb{C}). \tag{7}$$

Now the odd Chern classes of the complexification of a real bundle are 2-torsion[1], so that the Whitney sum formula holds for Pontryagin classes only up to 2-torsion elements. Another way to say this is to realize that if L is a real line bundle, then $L \otimes L$ is trivial, since $L^* \cong L$ by the reality of L means $L \otimes L \cong L \otimes L^* \cong \mathrm{Hom}(L, L)$, which always has a global section given by the identity map. Therefore any cohomology elements that classify real line bundles must be 2-torsion, and so the appropriate cohomology for describing real line bundles is $H^1(M; \mathbb{Z}_2)$. This means that when we map $H^1(M; \mathbb{Z}_2)$ into $H^2(M; \mathbb{Z})$, which classifies complex line bundles, we should get something that's 2-torsion. As we saw above, a similar statement holds for higher degrees.

The Pontryagin classes for a complex vector bundle E, defined by the Chern classes of $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ with $E_{\mathbb{R}}$ the realification of E, can easily be computed in terms of the Chern classes of E. If E is a complex vector bundle, then the isomorphism

$$E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E} \tag{10}$$

tells us that[2]

$$c(E \otimes_{\mathbb{R}} \mathbb{C}) = c(E \oplus \bar{E}) = (1 + c_1(E) + c_2(E) + \dots)(1 - c_1(E) + c_2(E) - \dots). \tag{16}$$

[1]Proof: Let $\mathcal{L}$ be a complex line bundle. Then we claim that $c_1(\mathcal{L}) = -c_1(\bar{\mathcal{L}})$. Indeed, the first Chern class of a complex line bundle is the same as the Euler class of the underlying real bundle, which locally is expressible in terms of the logarithms of the transition functions. Since $\bar{\mathcal{L}}$ has transition functions which are conjugate to those of $\mathcal{L}$, the Euler class associated to $\mathcal{L}$ is the negative of the one associated to $\bar{\mathcal{L}}$.

Now we use the splitting principle: assume E splits as a direct sum of line bundles, so that

$$E = \bigoplus_j \mathcal{L}_j \implies c(E) = \prod_j (1 + c_1(\mathcal{L}_j)) \implies c(\bar{E}) = \prod_j (1 - c_1(\mathcal{L}_j)). \tag{8}$$

Thus

$$c_k(E) = (-1)^k c_k(\bar{E}). \tag{9}$$

Now suppose that $E = F \otimes \mathbb{C}$, for F a real line bundle. Then $E = F \otimes \mathbb{C} = F \oplus iF$ is isomorphic to $\bar{E} = F \oplus (-iF)$ (this isomorphism does *not* hold if E is a generic complex vector bundle). Therefore by the splitting principle we can conclude that $c_k(F \otimes \mathbb{C}) = -c_k(F \otimes \mathbb{C})$ for k odd, meaning that the odd Chern classes of the complexification of a real bundle are all 2-torsion.

[2]Why is this true? We just have to decipher the proof in Milnor. For any $z = x + iy \in E$, the corresponding element in $E_{\mathbb{R}}$ is obtained just by erasing the i and writing z as a tuple $(x, y) \in E_{\mathbb{R}}$. We create $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ (the fact that it's a $\otimes$ over $\mathbb{R}$ is important! We can't $\otimes$ a real bundle with something unless the tensor unit is $\mathbb{R}$) by forming the sum $E_{\mathbb{R}} \oplus E_{\mathbb{R}}$, and adding in the complex structure with the map $J : (z, w) \mapsto (w, -z)$, where z, w are both regarded as tuples of their real and imaginary parts. Now define the following two maps:

$$\mathcal{I}, \mathcal{I}^* : E \to E_{\mathbb{R}} \oplus E_{\mathbb{R}}, \quad \mathcal{I}(z) = (z, -iz), \quad \mathcal{I}^*(z) = (z, iz). \tag{11}$$

Just to be clear, if $z = x + iy$, we have $\mathcal{I}(z) = ((x, y), (y, -x)) \in E_{\mathbb{R}} \oplus E_{\mathbb{R}}$. Anyway, note that

$$J(\mathcal{I}(z)) = J(z, -iz) = (iz, -z) = \mathcal{I}(iz), \tag{12}$$

so that $\mathcal{I}$ is complex linear with respect to the complex structure on $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ (again, just to be clear, $(iz, -z) = ((-y, x), (-x, -y))$ as an element of $E_{\mathbb{R}} \oplus E_{\mathbb{R}}$). Similarly,

$$J(\mathcal{I}^*(z)) = J(z, iz) = (-iz, z) = -(iz, -z) = -\mathcal{I}^*(iz), \tag{13}$$

This relation shows that $c_{2k+1}(E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}) = 0$ since all the odd-degree parts cancel in pairs, and so the Pontryagin classes which are not of degree a multiple of 4 vanish for the realification of a complex vector bundle.

This whole song and dance of defining the Pointryagin classes in terms of the Chern classes of a complexified bundle is mainly just so that we can show that the p_k are only nonzero for $k \in 4\mathbb{Z}$, and that we can show the whitney sum formula for the p_i's. A simpler way to define them would be to use Chern-Weil and just write down the p's explicitly, but then we'd have to do invariant polynomials and stuff to see which ones could be non-zero. This is often the better way to go in terms of computing things, since the complexification is pretty trivial: we just take our real curvature form F_A, and allow ourselves to e.g. diagonalize it using complex numbers. But this approach has the disadvantage that we'd miss torsion phenomena: for example, using the expansion of $\det(\mathbf{1} + F/2\pi)$ it's easy to see that the p_i's obey a Whitney sum formula modulo torsion, but to see the torsion effects we need to work with the complexification.

One such expression is as follows. The general claim is that for an (oriented?) vector bundle, we have [1]

$$P(w_{2k}(E)) = p_i(E) + 2\sum_{j=0}^{k-1} w_{2j}(E) \cup w_{4k-2j}(E) \mod 4. \tag{17}$$

Here, the pontryagin square is a map into into $H^*(E;\mathbb{Z}_4)$; hence the mod 4 on the RHS. In particular,

$$P(w_2(E)) = p_1(E) + 2w_4(E) \mod 4. \tag{18}$$

Additionally, from the above general formula, we can conclude that

$$p_k(E) = P(w_{2k}) \mod 2. \tag{19}$$

Thus the mod 2 reduction of the Pointryagin class p_k is *not* given by w_{4k}, but rather by the square of w_{2k}.

This is easy to prove if we are dealing with the realification of a complex vector bundle. In that case, we can follow our earlier manipulations and write

$$p_k(E_\mathbb{R}) = c_{2k}(E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) = c_{2k}(E \oplus \bar{E}) = [c(E) \wedge c(\bar{E})]_{2k}, \tag{20}$$

so that $\mathcal{I}^*$ is complex anti-linear with respect to the complex structure on $E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. Note however that any $(z, w) \in E_\mathbb{R} \oplus E_\mathbb{R} \cong E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ can be written uniquely as

$$(z, w) = \frac{1}{2}\left[\mathcal{I}(z + iw) + \mathcal{I}^*(z - iw)\right]. \tag{14}$$

This means that we can take any element in $E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ and identify it with one element in the pre-image of $\mathcal{I}$, and one element in the pre-image of $\mathcal{I}^*$. The former is just E, while the latter is $\bar{E}$, since the complex structure on the pre-image of $\mathcal{I}^*$ is opposite to that of the complex structure on $E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. So finally we can conclude that

$$E_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \cong E \oplus \bar{E} \tag{15}$$

as claimed.

where the brackets instruct us to take the degree $2k$ part. Expanding out the RHS,

$$p_k(E_{\mathbb{R}}) = 2\sum_{j=1}^{k-1} c_{2k-2j}(E)\wedge c_{2j}(E) + c_k(E)\wedge c_k(E), \tag{21}$$

where all the terms involving odd Chern classes have canceled. Working mod 4, and using that the mod 2 reduction of the Chern classes for a complex vector bundle E are

$$c_k(E) = w_{2k}(E_{\mathbb{R}}) \mod 2, \tag{22}$$

we obtain (17) (we also need to use the Pontryagin square as the appropriate cohomology operation on the mod 2 reduction of $c_k(E)$).

This proof relied on using the Whitney product formula for the Chern classes of a direct sum of *complex* bundles. Thus we generically won't have $E\otimes\mathbb{C}\cong F\oplus\bar{F}$ for some complex F, unless E happens to be the realification of a complex bundle (viz. $E = F_{\mathbb{R}}$). Since in this case we can't apply the Whitney product theorem, the proof is trickier[3]. The mod 2 version of (17), however, *is* easy to prove when E is a real vector bundle. Indeed, letting E be real, we have

$$p_k(E) = c_{2k}(E\otimes_{\mathbb{R}}\mathbb{C}) = c_{2k}(E\oplus iE). \tag{23}$$

As we mentioned earlier, we can't apply the Whitney product formula on this, since E and iE are *real* vector bundles. Anyway, writing $\rho_n(\cdot)$ for the reduction of $\cdot$ mod n, we have

$$\rho_2[p_k(E)] = \rho_2[c_{2k}(E\oplus iE)] = w_{4k}((E\oplus iE)_{\mathbb{R}}) = w_{4k}(E\oplus E) = P(w_{2k}(E)) \mod 2, \tag{24}$$

where in the last step we have used the Whitney product formula mod 2 on $w_{4k}(E\oplus E)$. If we change to working mod 4 it really seems like the natural thing that appears should be the rest of the terms in (17), but as of now I don't have a full proof.

For example, consider $\mathbb{CP}^n$, which has nontrivial cohomology in even degrees[4] The Chern classes of the tangent bundle are determined by the Whitney sum formula by taking the product of $n+1$ $\mathbb{C}$ line bundles:

$$c(T\mathbb{CP}^n) = (1+z)^{\wedge(n+1)}, \tag{26}$$

where z is the generator for $H^2(\mathbb{CP}^n;\mathbb{Z})$ and we have to remember to set $c_i = 0$ if $i > n$ (i.e. if $i = n+1$). The SW classes are then obtained by taking the mod-2 reduction of this (the

[3]Actually from the Chern-Weil point of view, such a product formula kind of makes sense, even though $c(E)$ is only defined for E complex. However since we are interested in torsion phenomena, thinking about expressions like $\mathrm{Tr}[F\wedge F]$ isn't really the road we want to take.

[4]This comes from the cell decomposition of $\mathbb{CP}^n$, which we can motivate in the following way. First of all, $\mathbb{CP}^n$ consists of nonzero $n+1$ tuples $(z_0,\dots,z_n)$ modulo scaling by elements in $\mathbb{C}$. When $z_0\neq 0$, we can normalize by it and get $(1,\widetilde{z}_1,\dots,\widetilde{z}_n)$, with $\widetilde{z}_i = z_i/z_0$. This space is $\mathbb{C}^n$, and it covers all of $\mathbb{CP}^n$ except at "infinity", where $z_0 = 0$. Thus to cover $\mathbb{CP}^n$, we need to attached the space of all $(0,z_1,\dots,z_n)$ to the $\mathbb{C}^n$ space of nonzero z_0. But we still have a re-scaling freedom on the $(0,z_1,\dots,z_n)$ that we place at infinity, so $\mathbb{CP}^n$ is realized by taking $\mathbb{C}^n$ and gluing it up with a copy of $\mathbb{CP}^{n-1}$ at infinity. Iterating this process, which stops at $\mathbb{CP}^0 = \mathbb{C}^0$, we see that

$$\mathbb{CP}^n = \mathbb{C}^n\cup\mathbb{C}^{n-1}\cup\cdots\cup\mathbb{C}^0, \tag{25}$$

which gives us the cell decomposition we're familiar with.

only nonzero SW classes are even, since the odd SW classes of the realification of a complex bundle vanish). For example, take $n = 2$. Then we see that

$$w_2(T\mathbb{CP}^2) = z, \qquad w_4(T\mathbb{CP}^2) = z \cup z, \tag{27}$$

where we are implicitly working mod 2. For $n = 3$ all of the coefficients in the binomial expansion $(1, 4, 6, 4, 1)$ are even except the first and the last, and so the total SW class is (there is no contribution from the last 1 since z^4 is an 8-form, which is too big to live on $\mathbb{CP}^3$)

$$w(\mathbb{CP}^3) = 1. \tag{28}$$

Finally for $n = 4$, the binomial expansion is $(1, 5, 10, 10, 5, 1)$ and so

$$w(T\mathbb{CP}^4) = 1 + z + z^{\cup 4}. \tag{29}$$

Now we can use our characteristic classes formula to find out what the pontryagin square of w_2 is in each of these cases, using our knowledge of p_1. The Pointryagin classes are determined from the Chern classes (remember that we have to complexify the bundle first!)

$$c(T\mathbb{CP}^n \otimes_{\mathbb{R}} \mathbb{C}) = c(T\mathbb{CP}^n \oplus (T\mathbb{CP}^n)^*) = c(T\mathbb{CP}^n) \wedge c((T\mathbb{CP}^n)^*). \tag{30}$$

Since $c((T\mathbb{CP}^n)^*)$ is the same as $c(T\mathbb{CP}^n)$ but with the signs of all the odd degree terms flipped, $c((T\mathbb{CP}^n)^*) = (1-z)^{n+1}$. So then since $p_k = (-1)^k c_{2k}$, we have

$$1 - p_1 + p_2 - \cdots = (1+z)^{n+1}(1-z)^{n+1} = (1-z^2)^{n+1}, \tag{31}$$

where we remember to set $p_k = 0$ if $k \geq \lceil (n+1)/2 \rceil$ by dimensionality reasons. We see that the minus signs in the p_k with k odd on the LHS will always match a minus sign on the RHS for z^{2k}. Thus the non-zero p_k are

$$p_k(T\mathbb{CP}^n_{\mathbb{R}}) = \binom{n+1}{k} z^2, \qquad k < \lceil (n+1)/2 \rceil. \tag{32}$$

So for example,

$$p(T\mathbb{CP}^2_{\mathbb{R}}) = 1 + 3z^2, \quad p(T\mathbb{CP}^3_{\mathbb{R}}) = 1 + 4z^2, \quad p(T\mathbb{CP}^4_{\mathbb{R}}) = 1 + 5z^2 + 10z^4. \tag{33}$$

Now we can finally check our formula for p_1 mod 4 and the Pontryagin square. Our formula $P(w_2) = p_1 - 2w_4 \mod 4$ tells us that

$$P(w_2(T\mathbb{CP}^2_{\mathbb{R}})) = 3z^2 - 2z^2 \mod 4, \quad P(w_2(T\mathbb{CP}^3_{\mathbb{R}})) = 4z^2 - 2 \cdot 0 \mod 4, \tag{34}$$

and

$$P(w_2(T\mathbb{CP}^4_{\mathbb{R}})) = 5z^2 - 2 \cdot 0 \mod 4. \tag{35}$$

Thus for $\mathbb{CP}^2$ and $\mathbb{CP}^4$, since $w_2 = z$, we get $P(w_2) = w_2^2$, the usual cup square. For $\mathbb{CP}^3$ we get $P(w_2) = 0$, which we needed to get since $\mathbb{CP}^3$ is spin and has $w_2 = 0$. So, everything checks out!

References

[1] E. Thomas. On the cohomology of the real grassmann complexes and the characteristic classes of n-plane bundles. *Transactions of the American Mathematical Society*, 96(1):67–89, 1960.