Thinking about local moments in TBLG

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In what follows we will be looking at the thermodynamic properties of insulating ferromagnets. The $i\mathbb{R}$ time continuum action for such a system is

$$S = \int d^d x \, d\tau \, \left(m_0 \mathcal{L}_{WZW} + \frac{\rho}{2} |\nabla \mathbf{n}|^2 - m_0 \mathbf{H} \cdot \mathbf{n} \right), \tag{1}$$

where m_0 is the magnetization density, ρ the spin stiffness, $\mathcal{L}_{WZW}$ the WZW term endowing $\mathbf{n}$ with the right commutation relations, and where the time integral is over the thermal circle.

Consider a rescaling of space and time such that $x \mapsto \eta x, \tau \mapsto \gamma\tau$. Then correlation functions are invariant at tree-level if we map the couplings appearing in the action and T as

$$m_0 \mapsto \eta^{-d}\gamma^0 m_0, \qquad \rho \mapsto \eta^{-d+2}\gamma^{-1} m_0, \qquad H \mapsto \eta^0 \gamma^{-1} H, \qquad T \mapsto \eta^0\gamma^{-1} T. \tag{2}$$

The singular part of the free energy density at tree level thus obeys

$$f_s(m_0, T, \rho, H) = \eta^d \gamma f_s(m_0 \eta^{-d}, \rho\eta^{-d+2}\gamma^{-1}, H\gamma^{-1}, T\gamma^{-1}). \tag{3}$$

Choosing $\eta = m_0^{1/d}$ and $\gamma = T$, we have

$$f_s(m_0, T, \rho, H) = m_0 T \Phi(\rho/m_0^{1-2/d} T, H/T) = m_0 T\Phi(r, h), \tag{4}$$

where the scaling function Φ is a function of the scaling variables

$$r \equiv \rho m_0^{2/d-1} T^{-1}, \qquad h \equiv HT^{-1}. \tag{5}$$

1 Large-N mean-field

Let us first see if we can reproduce the results of [1] — this is a needed sanity check because [1] contains no details about how the calculation works.

The WZW term in (1) can be simplified in the usual way by passing to the $\mathbb{CP}^1$ representation via $\mathbf{n} = z^\dagger \boldsymbol{\sigma} z$, with $z^\dagger z = 1$. In this representation, the WZW term becomes simply $m_0 z^\dagger \partial_\tau z$.[1] The gradient term in the action becomes the Hamiltonian for the $\mathbb{CP}^1$ model.

[1] The simplification is because while there is no global coordinate system on S^2 (which makes quantization annoying), there is one on S^3, since $S^3 = SU(2)$ is a group (S^3 is parallelizable).

This contains interactions of the form $(z^\dagger \partial z)^2$, which are decoupled by integrating in a $U(1)$ gauge field $\mathbf{a}$. Enforcing $|z|^2 = 1$ with a Lagrange multiplier λ and taking the field $\mathbf{H}$ to point along the z direction wolog, we have

$$S = \int d^d x\, d\tau\, z^\dagger \left(m_0 \partial_\tau + \rho D_{\mathbf{a}}^2 + \lambda - m_0 H Z \right) z - \int d^d x\, d\tau\, \lambda. \tag{6}$$

To get a qualitative idea for what happens, we introduce a large-N limit by letting z become a $2N$-component vector, with the normalization condition $|z|^2 = N$ (sticking an N here makes things slightly nicer later on). The magnetic field is then chosen to point along the direction of the matrix $\mathbf{1}_N \oplus \mathbf{1}_{-N}$, so that we get back the spin-1 results when $N \to 1$.

With the $2N$-component generalization, integrating out the zs gives an action

$$S = N \mathrm{Tr} \ln \left(m_0 \partial_\tau - \rho D_{\mathbf{a}}^2 + \lambda - m_0 H \right) + (H \leftrightarrow -H) - 2N \int d^d x\, d\tau\, \lambda \tag{7}$$

As usual, to make progress we need to assume that λ and $\mathbf{a}$ are both uniform in space. A uniform $\mathbf{a}$ has zero field strength, and so we might as well just take $\mathbf{a} = 0$. So then in this approximation, the free energy density is

$$f/N = T \sum_{\omega \in 2\pi\mathbb{Z}} \int \frac{d^d k}{(2\pi)^d} \ln \left(-i\omega m_0 T + \rho k^2 + \lambda - m_0 H \right) + (H \leftrightarrow -H) - 2\lambda. \tag{8}$$

Mean-field solution: we then need to find the mean field solution for λ. Resolving the matsubara sum by integrating against the Bose distribution, the mean field equation is

$$1 = -\frac{T}{2} \oint \frac{dx}{2\pi} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(e^{ix} - 1)(-ixm_0 T + \rho k^2 + \lambda - m_0 H)} + (H \leftrightarrow -H), \tag{9}$$

where the contour wraps counterclockwise around the $i\mathbb{R}$ axis. Doing the integral over x, we get

$$1 = \frac{1}{2m_0} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\exp\left(\frac{1}{m_0 T} \left(\rho k^2 + \lambda - m_0 H \right) \right) - 1} + (H \leftrightarrow -H). \tag{10}$$

The remaining integral can be done in terms of polylogs. We are most interested however in the case of $d = 2$. To do the integral, we use

$$\int_0^\infty dx\, \frac{1}{Ae^x - 1} = -\ln(1 - 1/A). \tag{11}$$

After some algebra that I won't write out, we get

$$8\pi\rho/T = -\ln\left(1 - e^{-(\lambda/m_0 + H)/T}\right) - \ln\left(1 - e^{-(\lambda/m_0 - H)/T}\right). \tag{12}$$

Let us define $q \equiv e^{\lambda/m_0 T}$. Then as $r = \rho/T$ in $d = 2$, we have

$$e^{-8\pi r} q^2 = (q - e^h)(q - e^{-h}), \tag{13}$$

which is precisely the mean field equation given in [1] (up to $h \leftrightarrow h/2$ so maybe I missed a factor of 2). We can solve this to obtain the mean field solution of

$$q = \frac{\cosh(h) + \sqrt{\sinh^2(h) + e^{-8\pi r}}}{1 - e^{-8\pi r}}. \tag{14}$$

Free energy: Let us first compute the free energy density. We start with

$$f/2N = \frac{T}{2}\int \frac{d^d k}{(2\pi)^d} \ln\left(1 - \exp\left(-\frac{1}{m_0 T}(\rho k^2 + \lambda - m_0 H)\right)\right) + (H \leftrightarrow -H) - \lambda. \tag{15}$$

Specializing to $d = 2$,

$$\begin{aligned} f/2N &= \frac{m_0 T}{8\pi r}\int dx \, \ln\left(1 - e^{-x} q^{-1} e^{h}\right) + (h \leftrightarrow -h) - \lambda \\ &= -\frac{m_0 T}{8\pi r}\left(\mathrm{Li}_2(q^{-1}e^{h}) + \mathrm{Li}_2(q^{-1}e^{-h})\right) - \lambda \\ &= -\frac{m_0 T}{8\pi r}\left(\mathrm{Li}_2(q^{-1}e^{h}) + \mathrm{Li}_2(q^{-1}e^{-h}) + 8\pi r \ln q\right), \end{aligned} \tag{16}$$

which tells us the scaling function $\Phi(r, h)$.

Entropy: given that we have found the scaling function Φ above, it is easy to evaluate the entropy density via

$$s \equiv -\partial_T f/2N = m_0(r\partial_r + h\partial_h - 1)\Phi(r, h). \tag{17}$$

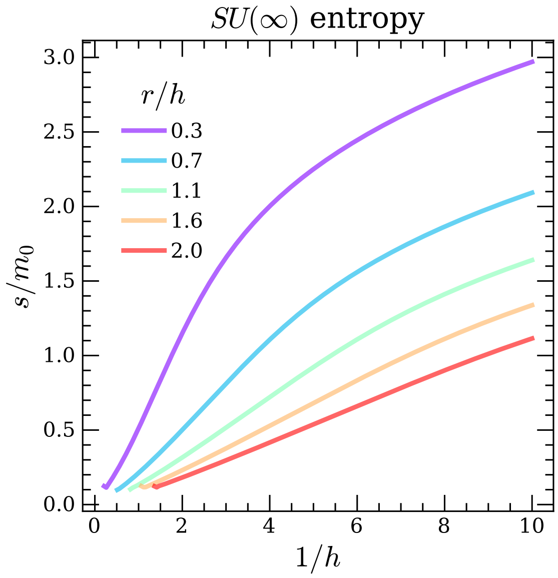
When taking derivatives we needn't worry about the r, h dependence of q, since we are working in mean-field so that $\partial_\lambda f = 0$. Using $\partial_x \mathrm{Li}_2(e^x) = -\ln(1 - e^x)$, we obtain

$$\begin{aligned} \partial_h \Phi(r, h) &= \frac{1}{8\pi r}\ln\left(\frac{q - e^{h}}{q - e^{-h}}\right) \\ \partial_r \Phi(r, h) &= \frac{1}{8\pi r^2}\left(\mathrm{Li}_2(q^{-1}e^{h}) + \mathrm{Li}_2(q^{-1}e^{-h})\right). \end{aligned} \tag{18}$$

Hence the entropy density is

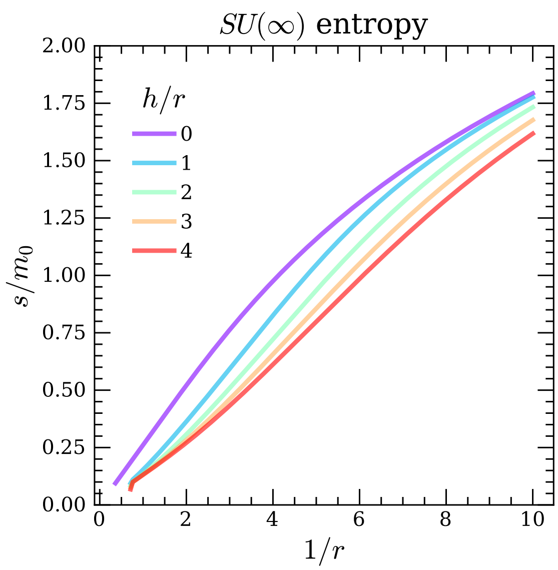
$$s = \frac{m_0}{8\pi r}\left(2\mathrm{Li}_2(q^{-1}e^{h}) + 2\mathrm{Li}_2(q^{-1}e^{-h}) + h\ln\left(\frac{q - e^{h}}{q - e^{-h}}\right) + 8\pi r \ln q\right). \tag{19}$$

Now we can make plots. For example, we can look at how s/m_0 behaves with temperature by plotting it against e.g. $1/h$ for several values of the T-independent quantity r/h:



(20)

Similarly,

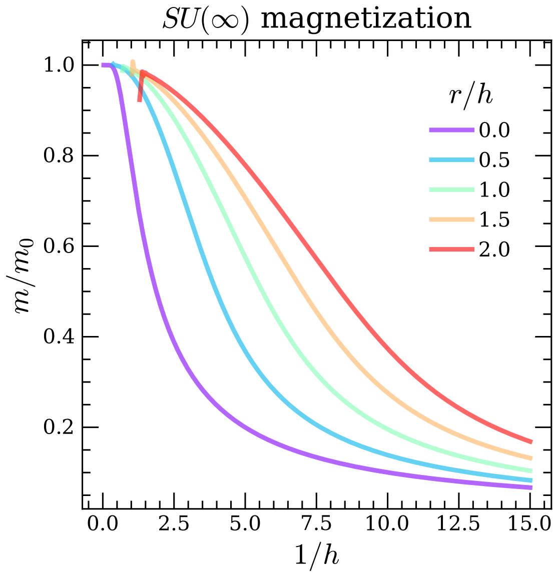


(21)

Magnetization: Let us now use this to compute the magnetization, since we can compare our answer to the result in [1]. We have in fact already done the calculation when determining the entropy, since

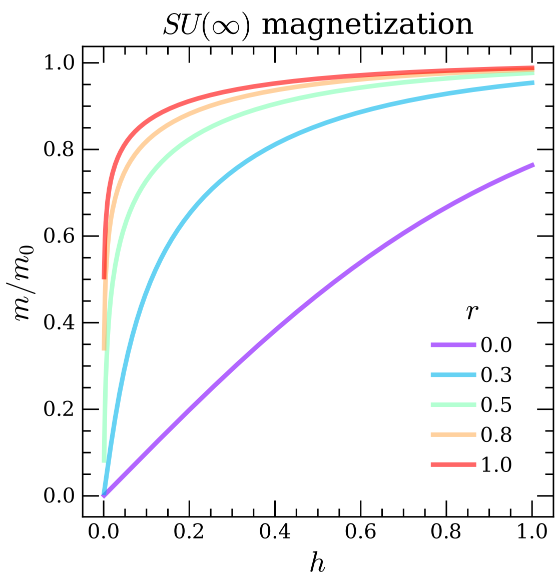
$$m \equiv -\partial_H f/2N = -\partial_h \Phi(r,h) = \frac{m_0}{8\pi r} \ln\left(\frac{q - e^{-h}}{q - e^{h}}\right). \tag{22}$$

Let us now plot the magnetization as a function of $1/h$ for a few values of r/h. The result is (there are some convergence issues with the plotting as $h \to \infty$ that I haven't bothered to fix since we know that $m = m_0$ at $h = \infty$)



(23)

which agrees with the results in [1]. Another way of plotting the results which serves as a sanity check is



(24)

where the purple line is the expected $\tanh(h)$ of the paramagnet that we get when the stiffness vanishes.

References

[1] N. Read and S. Sachdev. Continuum quantum ferromagnets at finite temperature and the quantum hall effect. *Physical review letters*, 75(19):3509, 1995.