Miscellaneous QFT problems diary

Ethan Lake

April 3, 2020

1 *Path integrals for Majorana fermions in one dimension and spin structures*

This is an elaboration on an exercise posed in Quantum Fields and Strings for Mathematicians. The problem statement is as follows:

Consider the quantum mechanics of $2n$ Majorana fermions η_i, with action

$$S = \int dt \, i \sum_j \eta_j \frac{d}{dt} \eta_j. \tag{1}$$

Suppose time is an S^1. Compute the partition function for both spin structures on this S^1. Also compute the twisted partition functions $\mathrm{Tr}_{\mathcal{H}}(g)$ and $\mathrm{sTr}_{\mathcal{H}}(g)$, where g is a matrix in the spinor representation (containing both chiralities) of $\mathrm{Spin}(2n)$ which acts to permute the fermion flavors and which can be thought of as implementing a flavor-twisted boundary condition in the path integral.[1]

Computing the Pfaffians

First we evaluate the partition functions by taking Pfaffians. There are two spin structures on S^1, which are permuted by $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$. This means there are two spin bundles on S^1, which we will write as S_+ and S_-. S_+ is the trivial $\mathbb{Z}_2$ bundle and S_- is the mobius strip. We will label the trivial bundle (the double-cover) “non-bounding” NB and the non-trivial bundle “bounding” B. This is because the nontrivial bundle bounds a disk, while the trivial

[1] The reason for choosing to twist the boundary conditions by an element of $\mathrm{Spin}(2n)$ rather than something else with the same Lie algebra (like $O(2n)$ will become clear while doing the problem. Note that the dimensions are right here since $\mathrm{Spin}(m)$ for $m \in 2\mathbb{Z}$ has dimension $2^{m/2}$ and $2n$ Majoranas have Hilbert space dimension 2^n.

bundle does not. Why? Well, when we embed the S^1 into a 2-manifold equipped with a spin structure, we need homologically trivial closed loops along which the fermion framing is parallel-transported to act as -1 in $\mathrm{Spin}(2) = U(1)$. Thus the S_- bundle is the one which can be embedded in two dimensions, while the S_+ bundle can only be embedded in two dimensions if it's embedded along a non-contractible loop. See another diary entry for more details about this. Anyway, each of the majoranas η_j, being real fields, should be viewed as sections of either the associated spinor bundle $S_+ \otimes \mathbb{R}$ or the associated spinor bundle $S_- \otimes \mathbb{R}$, depending on the spin structure.

Will we also want to compute the partition function in the presence of a background field for the global flavor symmetry $O(2n)$. Writing the twisted Dirac operator as $\not{D}_R$ for $R \in O(2n)$, we need to compute $\mathrm{Pf}(\not{D}_R) \in \mathbb{R}$ as a function of R. Unfortunately the sign is ambiguous, since we compute $\mathrm{Pf}(\not{D}_R)$ by taking a square root. Now $\mathrm{Pf}(\not{D}_R)$ can be thought of as a section of a real line bundle, the Pfaffian line bundle, over $O(2n)$. The structure group is $O(1) = \mathbb{Z}_2$, corresponding to a choice of sign in taking the square root. Unfortunately since $\pi_1(O(2n)) = \pi_1(SO(2n)) = \mathbb{Z}_2$ (the first equality from the fact that the identity component of $O(2n)$ is $SO(2n)$), this line bundle can be nontrivial, which means that the sign of the partition function may not be well defined.

However, suppose we instead twist the fields by an element in $\mathrm{Spin}(2n)$. There are no nontrivial line bundles over $\mathrm{Spin}(2n)$,[2] and so the Pfaffian line bundle must be trivial, giving us a well-defined partition function. This is why we restricted to a twist by an element of $\mathrm{Spin}(2n)$ in the problem statement. We will henceforth let $Z_\pm(R)$ denote the partition function with $\pm$ spin structure, twisted by an additional possible element $R \in \mathrm{Spin}(2n)$.

Now since the twisting by $R \in \mathrm{Spin}(2n)$ can be done by inserting R into either $\mathrm{Tr}_{\mathcal{H}}[R]$ or $\mathrm{Tr}_{\mathcal{H}}[(-1)^F R]$ (depending on the spin structure) and since $(-1)^F$ is $-\mathbf{1}$ in $\mathrm{Spin}(2n)$,[3] we have

$$Z_\pm(R) = Z_\pm(V^{-1}RV), \quad \forall\, V \in \mathrm{Spin}(2n) \tag{3}$$

so that $Z_\pm(R)$ is a class function on $\mathrm{Spin}(2n)$. This means we can, by conjugating, work in a basis where R is the exponential of an element in a maximal torus of $\mathfrak{spin}(2n)$. Letting the generators of the Clifford algebra $\mathrm{Cliff}(2n)$ be γ_i, a generic element of the chosen maximal torus can be written as

$$MT(\mathfrak{spin}(2n)) \ni -i \ln R = \sum_{j=1}^{n} r_j \gamma_{2j-1}\gamma_{2j} \tag{4}$$

for some coefficients r_j. As required, all of the $\gamma_{2j-1}\gamma_{2j}$ commute with each other, and they all square to $-\mathbf{1}$. Thus we can choose a basis in which each $\gamma_{2j-1}\gamma_{2j}$ is represented by a

[2] This is because $\pi_1(\mathrm{Spin}(2n)) = 0$. At a mathematical level, we would demonstrate this as

$$\mathrm{Line}_{\mathbb{R}}(M) = [M, BO(1)] = [M, B\mathbb{Z}_2] = H^1(M; \mathbb{Z}_2) = \mathrm{Hom}(\pi_1(M), \mathbb{Z}_2), \tag{2}$$

where M is any manifold and $\mathrm{Line}_{\mathbb{R}}(M)$ are the different real line bundles over M. Thus if $\pi_1(M) = 0$, there are no nontrivial real line bundles over M.

[3] $(-1)^F$ is the nontrivial element in the $\mathbb{Z}_2 \subset Z(\mathrm{Spin}(2n))$ that gets projected out when taking the quotient to $SO(2n)$. $Z(\mathrm{Spin}(2n))$ is either $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$, depending on whether n is odd or even, respectively. The extra $\mathbb{Z}_2 \subset Z(\mathrm{Spin}(2n))$ that remains after taking the quotient to $SO(2n)$ constitutes the center of $SO(2n)$, via $Z(SO(2n)) = \mathbb{Z}_2$.

matrix with n 2x2 blocks on the diagonal, with the matrix $J = -iY$ in the jth block and 0s everywhere else. Thus R is the exponential (no factor of i!):

$$R = \exp\left(\bigoplus_{j=1}^{n} r_j J\right), \qquad J = -iY. \tag{5}$$

To put this into the Lagrangian, we just need to construct the spin connection ω appearing in $\partial_\mu - \omega_\mu^{ij}\Sigma_{ij}$, with $\Sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$. From the above construction, normalizing so that the radius of the temporal S^1 to be 1, we see that we may take

$$\omega_t^{2j-1,2j} = r_j/2\pi. \tag{6}$$

Indeed, with this choice we have $e^{i\oint dt\, \omega_t^{ij}\Sigma_{ij}} = e^{iR}$.

We now need to compute the Pfaffian of $\not{D}_R$. Since the spin connection breaks up into a direct sum, we can consider each 2-by-2 block individually. First, consider the non-bounding partition function $Z_+(R)$. The partition function $Z_+(r_j)$ for a single block (containing a single pair of Majoranas) is[4]

$$Z_+(r_j) = \mathrm{Pf}\left(\frac{d}{d\tau} + 2\omega_t^{12}\Sigma_{12}\right) = \mathrm{Pf}\left(\frac{d}{d\tau} + \frac{r_j}{2\pi}J\right). \tag{7}$$

The relevant determinant is then (since we have chosen the S^1 to have radius 1, the eigenfunctions of $d/d\tau$ are $e^{i\omega\tau}$, $\omega \in \mathbb{Z}$)

$$\mathrm{Det}\left(\frac{d}{d\tau} + \frac{r_j}{2\pi}J\right) = \prod_{n\in\mathbb{Z}}(-n^2 + r_j^2/(2\pi)^2). \tag{8}$$

We regularize this with the ζ function:

$$\begin{aligned}\prod_{n\in\mathbb{Z}}(-n^2 + r_j^2/4\pi^2) &= \frac{r_j^2}{4\pi^2}\prod_{n\neq 0} n^2 \prod_{n\neq 0}\left(1 - \frac{r_j^2}{4\pi^2 n^2}\right)\\ &= \frac{1}{\pi^2}\left(\prod_{n=1}^{\infty} n^2\right)^2 \left(\frac{r_j}{2}\prod_{n=1}^{\infty}\left(1 - \frac{r_j^2}{4\pi^2 n^2}\right)\right)^2\\ &= 4\sin^2(r_j/2),\end{aligned} \tag{9}$$

where we used

$$\sin(x) = x\prod_{j=1}^{\infty}(1 - x^2/(\pi^2 n^2)) \tag{10}$$

to deal with the second product. To deal with the first product, we used

$$\partial_s\zeta(s) = \sum_n \partial_s e^{-s\ln n} \implies [\partial_s\zeta](0) = -\sum_n \ln n \tag{11}$$

[4]We are in Euclidean signature. In real time, the Dirac operator is $i\not{D}_R = i\partial_t + \omega_t^{ij}\Sigma_{ij}$, since $\omega_t^{ij} = -\omega_t^{ji}$ and since Σ_{ij} is anti-Hermitian (again, just to avoid confusion: ω is the spin connection for the global Spin($2n$) symmetry, not the Spin(1) symmetry of the temporal circle). When we go to Euclidean time, the Dirac operator instead becomes $\partial_\tau + \omega_t^{ij}\Sigma_{ij}$.

to write

$$e^{-2[\partial_s\zeta](0)} = e^{\sum_{n=1}^{\infty}\ln(n^2)} = \prod_{n=1}^{\infty} n^2. \tag{12}$$

Using the result $[\partial_s\zeta](0) = -\frac{1}{2}\ln 2\pi$ then tells us that

$$\prod_{n=1}^{\infty} n^2 = 2\pi, \tag{13}$$

which allows us to take care of the first product. This means that[5]

$$Z_+(R) = \mathrm{Det}^{1/2}\left(\bigoplus_j^n \left[\frac{d}{d\tau} + \frac{r_j}{2\pi}J\right]\right) = 2^n \prod_j |\sin(r_j/2)|. \tag{15}$$

Here the absolute value bars are in place because we are choosing the positive sign when taking the square root (so that $r_j \sim r_j + 2\pi$). As mentioned before, we can make a consistent choice of this sign for all r_j since $\pi_1(\mathrm{Spin}(2n)) = 0$.

Anyway, we see that if we don't twist by any gauge fields the partition function vanishes; $Z_+(\mathbf{1}) = 0$. This is to be expected, since choosing the $S_+ \otimes \mathbb{R}$ bundle gives us a zero mode of the (untwisted) Dirac operator, causing the partition function to vanish. If we twist all the fermions by a 2π rotation, which amounts to choosing $r_j = \pi$ for all j (since this is what gives $R = -\mathbf{1}$ when exponentiated), then we effectively switch to AP boundary conditions and get $Z_+(R) = 2^n$, the dimension of the Hilbert space (each of the $2n$ Majoranas gives a contribution to the dimension of $\sqrt{2}$).

We can similarly compute partition functions for the bounding bundle $S_- \otimes \mathbb{R}$ twisted by some element $R \in \mathrm{Spin}(2n)$. We just take our result for the $S_+ \otimes \mathbb{R}$ bundle, but shift $r_j \mapsto r_j + \pi$, since this adds an extra 2π twist and hence shifts the boundary conditions, as mentioned above. Thus we get

$$Z_-(R) = 2^n \prod_{j=1}^{n} |\cos(r_j/2)| \tag{16}$$

for the bounding spin bundle. As expected, this counts the dimension of the Hilbert space (and in particular is non-zero) when there is no extra $\mathrm{Spin}(2n)$ twist ($R = 0$).

We could also obtain this result by using the fact that the boundary conditions are AP

[5] If we had wanted the result in $\mathbb{R}$ time, we'd have obtained (note to self: are you sure there is no factor of i in the spin connection?)

$$Z_-^2 = \det\left(i\frac{d}{dt} + \frac{1}{2}J\right) = \prod_{n\in\mathbb{Z}} (n^2 + 1/4) = 2^{2n}\sinh^2(\pi/2). \tag{14}$$

to compute, for a single block,

$$\begin{aligned}
[Z_-(r_j)]^2 &= \prod_{n\in\mathbb{Z}-1/2} (-n^2 + r_j^2/(2\pi)^2) \\
&= \prod_{n\in\mathbb{Z}-1/2} n^2 \left[\prod_{n=1}^{\infty}\left(1 - \frac{r_j^2}{(n-1/2)^2(2\pi)^2}\right)\right]^2 \\
&= \left(\prod_{n\in\mathbb{Z}-1/2} n^2\right)\cos^2(r_j/2)
\end{aligned} \tag{17}$$

where we used the product formula

$$\cos(x) = \prod_{j=1}^{\infty}(1 - x^2/(\pi^2(n-1/2)^2)). \tag{18}$$

Now we need to deal with the remaining product. We write

$$\prod_{n\in\mathbb{Z}-1/2} n^2 = \prod_{n\in\mathbb{Z}} \frac{1}{4}(2n-1)^2 = \prod_{n\in 2\mathbb{Z}+1} n^2. \tag{19}$$

Here we have used the hilarious fact that the ζ-regularized number of integers is zero:

$$\prod_{n\in\mathbb{Z}} \alpha = e^{\sum_{n\in\mathbb{Z}}\ln\alpha} = 1. \tag{20}$$

This means that when doing ζ-function regularization,

$$\prod_{n\in\mathbb{Z}} \alpha f(n) = \prod_{n\in\mathbb{Z}} f(n) \tag{21}$$

for any non-zero constant α. We can now write the product as

$$\prod_{n\in 2\mathbb{Z}+1} n^2 = \frac{\prod_{n\in\mathbb{Z}\backslash 0} n^2}{\prod_{n\in\mathbb{Z}\backslash 0}(2n)^2} = 4. \tag{22}$$

Here we have used that the ζ-regularized number of *non-zero* integers is -1, so that

$$\frac{1}{\prod_{n\in\mathbb{Z}\backslash 0} 4} = e^{-\sum_{n\in\mathbb{Z}\backslash 0}\ln 4} = e^{\ln 4} = 4. \tag{23}$$

Putting this into our expression for $Z_-(r_j)$ and taking the product over all blocks j, we recover (**??**).

Let's briefly comment on why the two approaches (taking the momentum to be quantized in either $\mathbb{Z}$ or $\mathbb{Z}+1/2$ and not changing the $\mathrm{Spin}(2n)$ gauge field, or always taking the momentum to be in $\mathbb{Z}$ but modifying the $\mathrm{Spin}(2n)$ gauge field in the case of bounding spin structure) agree. Basically, they agree since the $\mathbb{Z}_2$ of $\mathrm{Spin}(1)$ is equal to the $(-1)^F$

of Spin($2n$), so that we can absorb the spin structure into the Spin($2n$) gauge field. At the formal level, this is just coming from the fact that we can permute spin structures by tensoring the spin bundle with the nontrivial line bundle ϵ (the mobius band) over the circle: $S_\pm \otimes \epsilon \cong S_\mp$. Since the Dirac operator is computed with a connection on the bundle $S_\pm \otimes E_{\mathrm{Spin}(2n)}$, we can write $(S_\pm \otimes \epsilon) \otimes E_{\mathrm{Spin}(2n)} = S_\pm \otimes (E_{\mathrm{Spin}(2n)} \otimes \epsilon)$, and transfer the spin structure onto the gauge connection. The point of this

$$Z_\pm(R) = Z_\mp(R_{2\pi} \cdot R), \tag{24}$$

where $R_{2\pi}$ is a 2π rotation (alias $(-1)^F$) in Spin($2n$). As we saw though, the fact that these expressions agree is not immediately obvious from the calculations.

Directly taking the trace

Let's see if we can reproduce these results by computing the trace directly. With no additional twisting going on, we know we need to get 0 for the non-bounding spin structure $S_+ \otimes \mathbb{R}$ and 2^n for the bounding spin structure $S_- \otimes \mathbb{R}$. This means that

$$Z_-(\mathbf{1}) = \mathrm{Tr}_{\mathcal{H}}(\mathbf{1}) = 2^n, \qquad Z_+(\mathbf{1}) = \mathrm{sTr}_{\mathcal{H}}(\mathbf{1}) = \mathrm{Tr}_{\mathcal{H}}[(-1)^F] = 0. \tag{25}$$

Here the supertrace sTr is a trace weighted by the matrix $(-1)^F \in \mathrm{Spin}(2n)$.

Note that the *nontrivial bundle* corresponds to the *untwisted* (regular) trace, while the *trivial* bundle corresponds to the *twisted* trace (supertrace). Again, this is because fermions "naturally" have a minus sign associated to them when traveling around closed loops (they naturally have anti-periodic boundary conditions), so that taking the regular trace with the $S_- \otimes \mathbb{R}$ bundle just leads to a sum over states (the two minus signs cancel out). However, with the $S_+ \otimes \mathbb{R}$ bundle, we have given the fermions an extra twist, and the partition function vanishes since the even and odd parts cancel.

Because $Z_\pm(R)$ is a class function on Spin($2n$), it can be written as a linear combination of the characters of Spin(n). In particular, $Z_\pm(R)$ will be a sum of characters of Spin($2n$) weighted by $(-1)^{(1\mp1)j}$, where j is a half-odd-integer or an integer according to whether the representation j of Spin($2n$) restricts to a representation of $SO(2n)$ or not. This doesn't really help (me) compute the partition function, though.

Let's start with the bounding $S_- \otimes \mathbb{R}$ bundle. We need to know how to represent R in the basis that we're taking the trace in. For some reason, I found this a bit tricky, and I think the way I do it below could be improved.

First, we know that $\dim \mathcal{H} = 2^n$, since each pair of Majoranas gives us an $SU(2)$ algebra. As we discussed earlier, R can be chosen to lie in exponential of the maximal torus of $\mathfrak{spin}(2n)$. We can then represent R on $\mathcal{H}$ as a product of factors $\otimes_{j=1}^n R_j$, where each R_j factor is a 2x2 matrix acting on the Majoranas η_{2j-1}, η_{2j}. Then we have

$$Z_-(R) = \prod_{j=1}^n \mathrm{Tr} R_j. \tag{26}$$

Therefore we just need to know how to represent the action of the spin group on a single 2x2 block. Now a single pair of Majoranas can be represented by the matrices X, Z, so that the product $XZ = J$ generates $\mathrm{Spin}(2) = U(1)$. Elements of $\mathrm{Spin}(2)$ are written as $e^{rJ/2}$, with $r = \pi$ giving $e^{\pi J/2} = J$, which anti-commutes with the fermion-odd matrices X and Z: therefore we identify it with $(-1)^F$ (note to self: the factor of $1/2$ is needed to get right results but the derivation is suspect).

With our $\otimes$ decomposition of R, which is basically a reduction $\mathrm{Spin}(2n) \to [\mathrm{Spin}(2)]^n$ (made possible by the fact that $Z_\pm(R)$ is a class function, as mentioned above), we then have

$$
\begin{aligned}
Z_-(R) = \mathrm{Tr}_{\mathcal{H}}(R) &= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} \exp\left(\frac{1}{2} r_j J\right)\right] \\
&= \prod_{j=1}^{n} \mathrm{Tr}\begin{pmatrix} \cos(r_j/2) & -\sin(r_j/2) \\ \sin(r_j/2) & \cos(r_j/2) \end{pmatrix} \\
&= 2^n \prod_{j=1}^{n} \cos(r_j/2),
\end{aligned}
\tag{27}
$$

which matches our earlier result.

Now for the non-bounding bundle $S_+ \otimes \mathbb{R}$. We have

$$
Z_+(R) = \mathrm{sTr}_{\mathcal{H}}(R) = \mathrm{Tr}_{\mathcal{H}}((-1)^F R). \tag{28}
$$

We saw above how to represent $(-1)^F$ on $\mathcal{H}$: just set $r_j = \pi$ for all j. Actually we will take $r_j = -\pi$ to match the sign conventions of the previous section; once we have fixed a choice there is no ambiguity since the Pfaffian bundle over $\mathrm{Spin}(2n)$ is trivial. With this choice,

$$
(-1)^F = \bigotimes_{j=1}^{n} e^{-\pi J/2} = (-J)^{\otimes n}. \tag{29}
$$

Now we can calculate

$$
\begin{aligned}
Z_+(R) &= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} (-J)\exp\left(\frac{1}{2} r_j J\right)\right] \\
&= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} \exp\left(\frac{1}{2} (r_j - \pi) J\right)\right] \\
&= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} \begin{pmatrix} \sin(r_j/2) & \cos(r_j/2) \\ -\cos(r_j/2) & \sin(r_j/2) \end{pmatrix}\right] \\
&= 2^n \prod_{j=1}^{n} \sin(r_j/2),
\end{aligned}
\tag{30}
$$

which reproduces our calculation using the Pfaffians.

2 *Fractional charge of fermion mass solitons in two dimensions* ✓

Today we will be doing a simple problem in the theme of charge fractionalization in 1+1D systems that I had nevertheless not previously seen a derivation of. We work in two spacetime dimensions, with the action

$$S = \int_{M_2} \left(\bar{\psi} \not{D}_A \psi + \bar{\psi} M e^{i\phi\bar{\gamma}} \psi \right), \tag{31}$$

where ϕ is a scalar field and $\bar{\gamma} = Z$ is the chirality operator (we are working in Euclidean signature). We will assume M is large, and consider a time-dependent ϕ which asymptotes to the constants $\phi_\pm$ at $x = \pm\infty$ (note that this is not the same as a Dirac fermion with a real mass that goes to $\pm m$ at $\pm\infty$, but rather a sort of complex generalization thereof). The goal will be to derive the electric charge carried by this kink.

To find the charge induced by the kink, we need to integrate out the fermions. Doing so is slightly tricky, though: we don't want to use $(\not{p} + Me^{i\phi\bar{\gamma}})^{-1}$ as a free propagator since inverting the exponential is a pain. However, we can't do the usual expansion of the $\mathrm{Tr}\ln$ while treating the mass term as a vertex, since we are interested in large M and so we won't be able to truncate the expansion. Furthermore, treating the mass term as a vertex is bad since it will turn out that the induced charge of the kink doesn't depend on M. Thus we need to find a way to include M in a propagator in a simple way such that we can do a $1/M$ expansion.

The way to most efficiently integrate out the fermions turns out to be to make use of the trick we learned in a previous diary entry: first compute the variation of the effective action and then integrate over the variation at the end. Taking the variation we get

$$\delta S = -\mathrm{Tr}\left(\delta D D^\dagger (D D^\dagger)^{-1} \right), \tag{32}$$

where this time (note that in our notation $\not{D}_A = \not{\partial} + i\not{A}$ so that $\not{D}_A^\dagger = -\not{D}_A$)

$$D = \not{D}_A + M e^{i\phi\bar{\gamma}}, \qquad D^\dagger = -\not{D}_A + M e^{-i\phi\bar{\gamma}}. \tag{33}$$

The reason for doing this is that the product $DD^\dagger$ has within it the term $G_f^{-1} = -\partial_\mu \partial^\mu + M^2$ which we can use to do an expansion in large M. If I did the algebra right, then

$$DD^\dagger = G_f^{-1} - \frac{M}{2i} \sin(\phi\bar{\gamma})(\not{\partial} + i\not{A}) + A_\mu A^\mu - i(\not{\partial}\not{A}) - 2iA^\mu \partial_\mu - iM(\not{\partial}\phi)\bar{\gamma} e^{-i\phi\bar{\gamma}}, \tag{34}$$

where we used $-i\not{\partial}\not{A} = (\not{\partial}\not{A}) - 2iA^\mu\partial_\mu + i\not{A}\not{\partial}$. This is a bit of a mess, but when we invert it the large M limit can save the day, since the largest factor of M comes in G_f^{-1} which is nice and simple. So after inverting $DD^\dagger$ in this way and dropping terms that are small in $1/M$, we get

$$\delta S = +\mathrm{Tr}\Big[(i\delta \not{A} + i\bar{\gamma}M\delta\phi e^{i\phi\bar{\gamma}})(\not{\partial} + i\not{A} - Me^{-i\phi\bar{\gamma}})G_f\Big(\mathbf{1} + G_f\Big(\frac{M}{2i}\sin(\phi\bar{\gamma})(\not{\partial} + i\not{A}) - A^2 + i\not{\partial}\not{A} - 2iA^\mu\partial_\mu + iM\not{\partial}\phi\bar{\gamma}e^{-i\phi\bar{\gamma}}\Big)\Big)\Big]. \tag{35}$$

Just so we can find our way through this mess slightly easier, let us note that

$$\int \frac{d^2k}{(2\pi)^2}G_f^2 = \int \frac{dk}{2\pi}\frac{k}{(k^2+M^2)^2} = \frac{1}{4M^2\pi}, \tag{36}$$

so that when we do the integrals we are going to be picking up factors of $1/M^2$ (the terms involving just a single G_f will die for spin trace / momentum oddness reasons).

The expression for δS above looks like a disaster, but actually a lot of things drop out under the trace or under the implicit momentum integration (by oddness). In fact, we are just interested in the charge of the kink, so we can restrict our attention to terms linear in A (since we want to find the current, by functionally differentiating wrt A and then setting $A \to 0$). If we take a critical look at δS, we see that the key surviving term is the guy coming from the $i\not{\partial}\not{A}$ piece. This part gives

$$\delta S = \mathrm{Tr}\left[\bar{\gamma}M^2\delta\phi G_f^2\not{\partial}\not{A}\right] + \dots. \tag{37}$$

Doing the trace is easy since the derivative is just acting on $\not{A}$, and so

$$\delta S = \int d^2x \int \frac{d^2k}{(2\pi)^2}\mathrm{Tr}_\sigma[\bar{\gamma}M^2\delta\phi\not{\partial}\not{A}]\frac{1}{k^2+M^2} + \dots, \tag{38}$$

where the trace is just the spin trace. The trace against $\bar{\gamma}$ selects out an antisymmetric structure for ∂ and A which gives us the field strength, and since $\mathrm{Tr}_\sigma[\mathbf{1}] = 2$ we have

$$\delta S = \frac{1}{2\pi}\int d^2x\delta\phi F + \dots, \tag{39}$$

which is a θ term controlled by the phase of the fermion mass, as expected.

Thus integrating over the variation, we see that the current is

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi, \tag{40}$$

and so the charge induced by the kink is (assuming space is a line)

$$Q = \int_{\mathbb{R}} j^0 = \int_{\mathbb{R}} \partial_x\phi\frac{1}{2\pi} = \frac{1}{2\pi}(\phi(\infty) - \phi(-\infty)). \tag{41}$$

Thus the charge is set by degree to which the phase of the fermion mass winds along the kink. The most common scenario, since we have two degenerate states at $\phi = \pi, 0$, is to have $\phi(\infty) - \phi(-\infty) = \pi$, so that the mass term goes from being $M \cdot \mathbf{1}$ at $-\infty$ to $M \cdot \mathbf{1}$ at ∞. In this case, the charge of the kink is $Q = 1/2$.

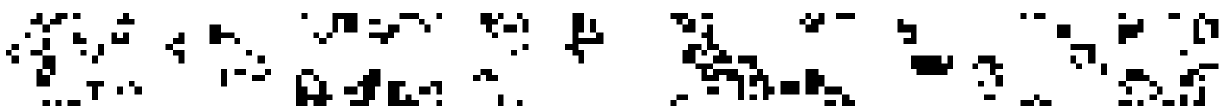
The kink also localizes a fermion zero mode. A low-tech way to see this is to just explicitly solve $H\psi = 0$ for some choice of $\phi(x)$. For example, suppose that $\phi(x) = \pi\Theta(x)$ so that the mass term is $M \cdot \mathbf{1}$ at $-\infty$ and $M \cdot \mathbf{1}$ at ∞. Then for example at $x > 0$ the equation $H\psi = 0$ reads

$$(i\partial_x - A_x)\psi_L = M\psi_R, \qquad -(i\partial_x - A_x)\psi_R = M\psi_L, \tag{42}$$

so that $-(i\partial_x - A_x)^2\psi_L = M^2\psi_L$, which we can solve handily as

$$\psi_L(x) \propto \exp\left(i\int_0^x A - M|x|\right), \tag{43}$$

which is localized around the kink and which could have been guessed from the fact that the phase of zero modes is given by the parallel transport formula (since $D_A\psi = 0$ is the requirement that ψ be parallel-transported) and their magnitude should be localized to the region where the mass changes.



3 *Large N matrix model quantum mechanics and eigenvalue distributions* ✓

This is an elaboration on a problem from a pset assigned in Hong Liu's 2018 class on AdS / CFT. We consider a matrix model with partition function

$$Z = \int \mathcal{D}M \, \exp\left(-\frac{N}{g}\mathrm{Tr}[V(M)]\right), \tag{44}$$

where $V(M)$ is a polynomial potential (so that the action is a function only of the eigenvalues of M), and the integral over M runs over all $N \times N$ Hermitian matrices. We will eventually specialize to $V(x) = x^2/2 + x^4$. We will also denote the eigenvalue density by

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i), \tag{45}$$

where $\{\lambda_i\}$ are the eigenvalues. Taking $N \to \infty$, we will assume $\rho(\lambda)$ approaches a continuous function supported on some interval $I \subset \mathbb{R}$.

Do several things:

- Find an expression for Z to leading order in the $N \to \infty$ limit.

- Define the complex function
$$F(\xi) = \int_I d\lambda\, \frac{\rho(\lambda)}{\xi - \lambda}. \tag{46}$$
Discuss the analytic properties of $F(\xi)$.

- Show that $F(\mu - i\epsilon) \in \mathbb{R}$ if $\mu \in \mathbb{R} \setminus I$, and find $F(x \to \infty)$.

- Use the previous results to determine the form of F.

- Find $\rho(\lambda)$ explicitly.

- At what value of the coupling does the free energy have a non-analytic part? Find the leading non-analytic behavior of the free energy near the critical point.

- You should find that the critical coupling is negative, $g_c = -1/48$. Does this make sense? The fact that $g_c < 0$ means that this theory has a perturbative expansion about $g = 0$ that has a finite radius of convergence—unlike what happens in normal QFT, we get more than just an asymptotic expansion. Why is this?

First, we use the Vandermonde determinant (see other diary entry) to turn the measure $\mathcal{D}M$ into an integral over the eigenvalues, weighted by an exponential involving a Coulomb eigenvalue repulsion $\ln|\lambda_i - \lambda_j|$, which contributes to the action. To get the saddle point equation, we vary the action with respect to a particular eigenvalue λ_i and set the result to zero: this gives

$$\frac{N}{g}V'(\lambda_i) = 2\sum_{j:j<i}\frac{1}{\lambda_i - \lambda_j} - 2\sum_{j:j>i}\frac{1}{\lambda_j - \lambda_i} = 2\sum_{j:j\neq i}\frac{1}{\lambda_i - \lambda_j}. \tag{47}$$

Turning the sum into an integral using the eigenvalue distribution $\rho(\lambda)$, we get

$$\frac{N}{2g}V'(\lambda) = P\int d\mu\, \frac{\rho(\mu)}{\lambda - \mu}, \tag{48}$$

where the principal value has been taken since the sum avoids terms with $i = j$ where $\lambda_i - \lambda_j = 0$.

The saddle point value for the partition function is just obtained by evaluating the matrix exponential on the saddle point distribution of eigenvalues. So the free energy is

$$\mathcal{F} = -\ln Z \approx \frac{N}{g}\sum_i V(\lambda_i) - \ln\prod_{i\neq j}|\lambda_i - \lambda_j| \to \frac{N^2}{g}\int d\lambda\, \rho(\lambda)V(\lambda) - N^2 P\int d\lambda d\mu\, \rho(\lambda)\rho(\mu)\ln|\lambda - \mu|, \tag{49}$$

where λ is a distribution of eigenvalues satisfying the saddle point equation. Here the logarithm comes from putting the vandermonde determinant in the exponential.

Now define the complex function

$$F(\xi) = \int_I d\lambda \frac{\rho(\lambda)}{\xi - \lambda}, \qquad \xi \in \mathbb{C}. \tag{50}$$

Here $I = \text{supp}(\rho) \subset \mathbb{R}$ is assumed to be a union of intervals in $\mathbb{R}$. Using the Dirac identity, we can take $\xi = \mu - i\epsilon$ for μ real and send $\epsilon \to 0$ to get

$$F(\mu - i\epsilon) = i\pi \int d\lambda\, \rho(\lambda)\delta(\mu - \lambda) + P \int d\lambda \frac{\rho(\lambda)}{\mu - \lambda}. \tag{51}$$

Here we have used the equation of motion to replace the principal part of the integral above with the derivative of $V(\mu)$. Thus we see that

$$\text{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu), \qquad \text{Re}[F(\mu - i\epsilon)] = \frac{V'(\mu)}{2g}, \qquad \mu \in I. \tag{52}$$

Note that these properties hold only for $\mu \in I$: if $\mu \in \mathbb{R} \setminus I$ then

$$\text{Im}[F(\mu - i\epsilon)] = 0, \qquad \text{Re}[F(\mu - i\epsilon)] = \int d\lambda \frac{\rho(\lambda)}{\mu - \lambda}, \qquad \mu \in \mathbb{R} \setminus I. \tag{53}$$

In particular, the real part of $F(\mu - i\epsilon)$ needn't be related to $V'(\mu)$ if $\mu \notin I$, since the saddle-point equation relating $V'(\mu)$ to the principal part of the relevant integral was derived under the assumption that $\mu \in I$.

Note that $F(\xi)$ is analytic everywhere, and on $I \subset \mathbb{R}$ it has a branch cut ($\rho(\lambda)$ is assumed to be well-behaved in the $N \to \infty$ limit). We see that across the branch cut at $\mu \in I$, $F(\xi)$ changes by $2\pi i \rho(\mu)$.

Note that if we take $\xi \in \mathbb{R}$ and send $\xi \to \infty$, we have

$$F(\xi \to \infty) = \int_I d\lambda \, \frac{\rho(\lambda)}{\xi}(1 + \lambda/\xi + \dots) = \frac{1}{\xi} + O(\xi^{-2}), \tag{54}$$

where we have used the normalization of $\rho(\lambda)$. Note that $\text{Re}[F(\mu - i\epsilon)]$ does not go to $V'(\mu)/2g$ for $\mu \notin I$ (unless V is logarithmic, which we will assume to not be the case).

We can use this information to find out what $F(\xi)$ is. In the following we will assume for simplicity that I is a single connected interval centered on zero, so that $I = [-a, a]$ for some $a \in \mathbb{R}$. This will be the case if we have a potential $V(\Lambda)$ with a unique minimum at 0, like $V(\Lambda) = \frac{1}{2}\Lambda^2 + \Lambda^4$. We will determine a self-consistently using the constraints we've derived on F.

Since we know that $F(\xi)$ is analytic but has a branch cut at $I = [-a, a]$ on the $\mathbb{R}$ axis, we expect that $\sqrt{\xi^2 - a^2} = \sqrt{\xi - a}\sqrt{\xi + a}$ will show up in $F(\xi)$ in order to give us the right branch cut structure (viz. a branch cut connecting $\pm a$), and in order to make $\text{Im}[F(\mu - i\epsilon)]$ nonzero only when $\mu \in I$. Since $F(\xi)$ is analytic, we expect $F(\xi) = g(\xi) + f(\xi)\sqrt{\xi^2 - a^2}$, where f, g are some polynomials in ξ with positive powers and real coefficients (as $\text{Im}[F(\mu - i\epsilon)] = 0$ if $\mu \notin I$).

The requirement that the real part of $F(\mu - i\epsilon)$ go to $V'(\mu)/2g$ when $\mu \in I$ tells us that $g(\xi) = V'(\xi)/2g$. We can then get $f(\xi)$ by requiring $F(\xi \to \infty) \to 1/\xi + O(1/\xi^2)$:

$$F(\xi \to \infty) \approx \frac{V'(\xi)}{2g} + f(\xi)\left(\xi - \frac{a^2}{2\xi} + O(\xi^{-3})\right). \tag{55}$$

Thus

$$f(\xi) = \frac{1}{\xi^2 - a^2/2} + \frac{\xi V'(\xi)}{2g(a^2/2 - \xi^2)}, \tag{56}$$

with a^2 to be determined by requiring $f(\xi)$ to be a $\mathbb{R}$ polynomial with positive powers. We know that $\deg(f) = \deg(V') - 1$, which again follows from our knowledge of $F(\xi \to \infty)$.

We will now specialize to the case

$$V(\lambda) = \frac{1}{2}\lambda^2 + \lambda^4. \tag{57}$$

So then $\xi V'(\xi) = \xi^2 + 4\xi^4$, and

$$f(\xi) = \frac{1}{2g(\xi^2 - a/2)}(2g - \xi^2 - 4\xi^4). \tag{58}$$

Since $V'(\lambda)$ is third order, we know that $f(\xi)$ will be second order, which allows us to stop at the leading order expansion for the square root for now. Writing $f(\xi) = A + B\xi + C\xi^2$ we see that $B = 0$, $C = -2/g$, and

$$\frac{1}{2g} + A = \frac{ca^2}{2}, \tag{59}$$

so that

$$f(\xi) = -\frac{1}{2g}(1 + 2a^2 + 4\xi^2). \tag{60}$$

This then determines the eigenvalue distribution to be, using $\mathrm{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu)$ for $\mu \in [-a, a]$,

$$\rho(\mu) = -\frac{1}{\pi}f(\mu)\sqrt{a^2 - \mu^2} = \frac{1}{2\pi g}(1 + 2a^2 + 4\mu^2)\sqrt{a^2 - \mu^2}. \tag{61}$$

As an aside, we can recover the Wigner distribution by looking at $V(\lambda) = \frac{1}{2}\lambda^2$. In this case, since we know that the degree of $f(\xi)$ is two less than the degree of $V(\xi)$, $f(\xi)$ must be a constant. Working it out and solving for a^2 in the manner described below gives

$$\rho(\mu)|_{V(\lambda)=\lambda^2/2} = \frac{1}{2\pi g}\sqrt{4g - \mu^2}, \tag{62}$$

which is the famous Wigner distribution. This is already a rather nontrivial result: if there were no interaction effects between the eigenvalues due to the Vandermonde determinant then we'd have a Gaussian distribution of eigenvalues. Turning on the interactions leads to eigenvalue repulsion and hence intuitively to a smearing out of the distribution—and yet the tails of the Gaussian are cut off to form a semicircle with finite extent, which seems to be the opposite of what we'd expect if the eigenvalues were just getting smeared out! Very interesting.

Anyway, now we return to the quartic potential (**??**). To get a, we need to get the $1/\xi$ piece of $F(\xi \to \infty)$, which requires expanding the square root to include the $-a^4/8\xi^4$ term. Setting the coefficient of the $1/\xi$ piece to 1 means that

$$3a^4 + a^2 - 4g = 0 \implies a^2 = \frac{1}{6}\left(-1 + \sqrt{1+48g}\right). \tag{63}$$

Recapitulating, we have shown that

$$F(\xi) = \frac{1}{2g}\left(\xi + 4\xi^3 - \left[4\xi^2 + \frac{2}{3} + \frac{1}{3}\sqrt{1+48g}\right]\sqrt{\xi^2 + \frac{1}{6} - \frac{1}{6}\sqrt{1+48g}}\right). \tag{64}$$

We can now get an explicit expression for the free energy

$$\mathcal{F}/N^2 \approx \frac{1}{g}\int d\lambda\, \rho(\lambda)V(\lambda) - P\int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\ln|\lambda-\mu|. \tag{65}$$

The second term in the free energy with the ln is hard to integrate, but we have another option: we can integrate the equations of motion to obtain

$$\frac{V(\lambda)-V(0)}{2g} = P\int d\mu\, \rho(\mu)\left[\ln|\lambda-\mu| - \ln|\mu|\right], \tag{66}$$

which means that (since $V(0)=0$)

$$P\int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\ln|\lambda-\mu| = \frac{1}{2g}\int d\lambda\, \rho(\lambda)V(\lambda) + P\int d\mu\, \rho(\mu)\ln|\mu|. \tag{67}$$

Putting this into the second integral in the expression for the free energy,

$$\mathcal{F}/N^2 \approx \int d\lambda\, \rho(\lambda)\left(\frac{V(\lambda)}{2g} - \ln|\lambda|\right). \tag{68}$$

The first term is

$$\frac{1}{2g}\int_{-a}^{a} d\lambda\, \rho(\lambda)(\lambda^2/2 + \lambda^4) = -\frac{a^4}{128g^2}(2 + 10a^2 + 9a^4), \tag{69}$$

while the second term is

$$-2\int_0^a d\lambda\, \rho(\lambda)\ln\lambda = \frac{a^2}{16g}\left(2 + a^2(3 + 6\ln 4) + \ln 16 - 4(1+3a^2)\ln a\right). \tag{70}$$

Now we add these two together, and carry out an expansion in small $\epsilon = g - g_c = g + 1/48$. Here the critical point $g_c = -1/48$ is the coupling at which the free energy becomes singular. This point is supposed to mark the phase transition where complicated Feynman diagrams dominate and the Feynman diagrams go over to form a continuum geometry. I think the picture is that the quantity $\Delta = g - g_c$ is a "chemical potential for triangles in the triangulation of spacetime". In the Louiville model (gravity theory with just the dilaton) that is supposed to describe the other side of the phase transition, Δ appears as a cosmological constant: writing the metric in conformal gauge as $g = e^\phi \eta$, we get an action like

$$\int \left[\frac{1}{2}(\partial\phi)^2 + \Delta e^\phi \right]. \tag{71}$$

Doing the expansion with Mathematica, we find that the free energy has an (imaginary) constant part, a term proportional to ϵ, one proportional to ϵ^2, and then one proportional to $\epsilon^{5/2}$, which is the leading singular part. So, the leading non-analytic behavior of $\mathcal{F}$ is a 5/2 power dependence on the distance from the critical point (n.b. the cancellation of the $\epsilon^{1/2}, \epsilon^{3/2}$ terms is nontrivial!).

The fact that $g_c < 0$ means that near the transition point, the potential for the eigenvalues is actually naively unstable, since $V(\pm\infty) = -\infty$. And yet, the theory we've been working with actually has a smooth well-behaved free energy for a finite range of negative g! Somehow, taking the $N \to \infty$ limit first is enough to stabilize what would otherwise be an unstable potential, allowing the $g_c < g < 0$ theory to make sense. The transition at g_c comes when the eigenvalues start "spilling over" the brim of the potential and running off towards $\pm\infty$, but this transition doesn't happen until well after the potential becomes unstable. This is actually pretty crazy—if the interactions between the eigenvalues generated by the Vandermonde determinant were positive, then one might expect that the interactions would "bind" the eigenvalues together and prevent them from spilling over the edge to the unbounded region of the potential. However, the forces between the eigenvalues are repulsive! This seems like it should make the problem even worse, with the eigenvalues getting "pushed over the edge" of the potential.

One way of understanding that the theory can be stable for $g < 0$ is to note that the perturbative diagrammatic expansion in this theory has a finite radius of convergence about $g = 0$. This is not true in typical QFTs, which have only an asymptotic expansion about the free point.[6] In QFT the number of diagrams usually grows factorially, which is the ultimate source for the vanishing of the radius of convergence for the perturbative expansion. The analytic nature of the expansion for the matrix model tells us that the number of diagrams grows more slowly—until we get to g_c the number of diagrams is controlled, and the non-analyticity doesn't hit us until the "continuum geometry" phase transition.

As a more physical take on this, let's review an old argument (due to Dyson) why QFTs should generically admit only asymptotic expansions in couplings. For example, consider QED. If the perturbative expansion in e^2 (and remember that it is e^2, and not e, which we are expanding in) has a finite radius of convergence, then QED would make sense for small negative e^2. So, take $e^2 < 0$, and consider nucleating N positron-electron pairs from the vacuum, by expending an energy $E \sim 2Nm$. Now separate the electrons and positrons into two groups, and separate the two groups from one another. The energy for this configuration is then schematically $E \sim 2Nm - e^2N^2$, which can always be made negative for N large

[6] It is important here that we are talking about the free UV fixed point, which is generically right on the boundary of allowed parameter space, as we will argue for in a second. If we had some nontrivial interacting UV fixed point then this would generically not be a worry.

enough. Therefore the vacuum is unstable, and the theory breaks down. Evidently $e^2 < 0$ doesn't make sense, and the expansion in e^2 has zero radius of convergence. The same argument can be made in ϕ^4 theory to show that the expansion in the ϕ^4 coupling cannot have a finite radius of convergence.

Large N theories, like the matrix model considered above, provide a way of getting around this argument, and they typically have a series expansion in the t' Hooft coupling gN with a finite radius of convergence. At the technical level, this is because the large-N diagrammatics allow us to resum infinitely many diagrams. Physically, we can also try running Dyson's argument: for example, consider large N YM: we nucleate N pairs of gluons from the vacuum, and then separate them in two groups. Since in the t' Hooft limit we are sending $g \to 0$, the attractive interactions between the two bunches of gluons vanishes in the large N limit, and so Dyson's argument doesn't apply.

Another subtlety needs to be dealt with: at $g \to g_c$ the saddle-point solution gives $a^2 < 0$, which seems problematic since the eigenvalues we're integrating over should always be real, due to the Hermiticity of the matrices in question. So, isn't $g < 0$ already ruled out? I guess the philosophy here is that the important thing to look at is really the singular behavior of the partition function: we used the WKB approximation to get the partition function, and while within this approximation $a^2 < 0$ strictly speaking doesn't make sense, after getting our expression for $\mathcal{F}$ we can just work with it directly, forgetting about the approximation where it came from (note to self: revisit this after I understand more).

4 *Quantization in AdS* ✓

This is a problem taken from a pset assigned in Hong Liu's holography class. The problem is as follows: consider a scalar field in AdS$_{d+1}$, with action

$$S = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2). \tag{72}$$

Here Greek indices run over time, $d-1$ "space" dimensions, and the radial coordinate z. Near the boundary at $z = 0$, we have the asymptotic expression

$$\phi(z \to 0, x) = A(x)z^{d/2-\nu} + B(x)z^{d/2+\nu}, \qquad \nu \equiv \sqrt{d^2/4 + m^2R^2}, \tag{73}$$

where R is the AdS radius, and A, B are some functions of the "spacetime coordinates". Our to-do list is as follows:

a) Define an inner product for wavefunctions and show its time independence. b) What is the condition on ν for a given ϕ to be normalizable? c) Find the stress tensor and show that it is covariantly conserved. d) Define the energy E as the integral of $\sqrt{-g}g^{tt}T_{tt}$ over a given Cauchy slice, and find the explicit form of $\partial_t E$. e) When does the energy flux at the boundary vanish? f) Show that E is finite if the chosen wavefunction is normalizable, and infinite otherwise.

a) We define the inner product as

$$\langle\phi,\psi\rangle_{\Sigma_t} = -i\int_{\Sigma_t} dz\, d\vec{x}\,\sqrt{-g}g^{tt}(\phi^*\partial_t\psi - \partial_t\phi^*\psi), \tag{74}$$

where Σ_t is any Cauchy surface, and the g^{tt} is required to make the integral invariant under rescaling of t. Taking the difference of the inner products at different times, we have

$$\langle\phi,\psi\rangle_{\Sigma_{t'}} - \langle\phi,\psi\rangle_{\Sigma_t} = -i\int_M d^dx_\perp^\mu(\phi^*\nabla_\mu\psi - \nabla_\mu\phi^*\psi), \tag{75}$$

where M is the timelike boundary of the spacetime volume bounded by the two Cauchy slices, located at $z=0$. Here we have used the fact that the integral over the bounded volume vanishes, on account of

$$\nabla_\mu(\phi^*\nabla^\mu\psi - \nabla^\mu\phi^*\psi)\sqrt{-g} = 0, \tag{76}$$

by virtue of the equations of motion, viz. $\nabla^2 = m^2$ when acting on ϕ and ψ. The integral on the RHS of (**??**) vanishes if ϕ and ψ are properly normalized at infinity, and so the inner product is time-independent.

b) ϕ has the asymptotic expansion

$$\phi(z\to 0) = A(x)z^{d/2-\nu} + B(x)z^{d/2+\nu}. \tag{77}$$

Now since $\sqrt{-g}\propto z^{-(d+1)}$ and $g^{tt}\propto z^2$, we have

$$\langle\phi,\phi\rangle_\Sigma \sim -i\int_\Sigma dz\, d\vec{x}\,(AA'z^{1-2\nu} + BB'z^{2\nu+1} + 2AB'z). \tag{78}$$

To get something finite, we need the total power of z to be greater than -1. This is always satisfied by the B mode and so the B mode is always normalizable. For the A mode to be normalizable we need $1-2\nu > -1$, so the A mode is normalizable only if

$$0 \le \nu < 1. \tag{79}$$

c) There are two terms that contribute to the stress tensor: the Lagrangian density and the $\sqrt{-g}$ in the measure. The variation of the former wrt the metric is simple, while the latter is found by using the usual

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta e^{\mathrm{Tr}\ln g} = -\frac{\sqrt{-g}}{2}\delta\mathrm{Tr}\ln g \implies \frac{\delta}{\delta g_{\mu\nu}}\sqrt{-g} = -\frac{\sqrt{-g}}{2}g^{\mu\nu}. \tag{80}$$

Since the variation of the Lagrangian density wrt the metric components is just $\sqrt{-g}\partial^\mu\phi\partial^\nu\phi$, we have

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\left(\sqrt{-g}\partial^\mu\phi\partial^\nu\phi - \frac{\sqrt{-g}g^{\mu\nu}}{2}(\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2)\right) \\ &= \nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}(\nabla_\lambda\phi\nabla^\lambda\phi + m^2\phi^2), \end{aligned} \tag{81}$$

where in the last step we have replaced ordinary derivatives with covariant derivatives since they act in the same way on ϕ, which is a scalar.

Now let's verify that T is covariantly conserved. This is straightforward since the metric is covariantly constant, meaning that we don't have to worry about the differences between raised and lowered indices, and that the covariant derivatives pass straight through the $g^{\mu\nu}$:

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla^2\phi\nabla^\nu\phi + \nabla_\mu\phi\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\nabla_\mu\nabla^\lambda\phi\nabla_\lambda\phi - m^2\phi\nabla^\nu\phi \\ &= \nabla^\nu\phi(\nabla^2\phi - m^2\phi) = 0, \end{aligned} \tag{82}$$

again by virtue of the equations of motion. Thus the stress tensor is covariantly conserved.

d) Define the energy as

$$E = \int_{\Sigma_t} dz\, d\vec{x}\,\sqrt{-g}g^{tt}T_{tt}. \tag{83}$$

The g^{tt} here is needed so that under time rescalings it cancels the rescaling of T_{tt}, so that the whole action rescales like $\sqrt{-g} \mapsto \lambda\sqrt{-g}$ under $t \mapsto \lambda^{-1}t$, which is appropriate for an energy. Another way to write $\sqrt{-g}g^{tt}$ would be $\sqrt{-g_t}n^t$, where g_t is the induced metric on Σ_t and $n^t = (0, z/R, 0, \ldots, 0)$ is the temporal unit vector in coordinates $(z, t, \vec{x})$. Under time rescalings $\sqrt{-g}g^{tt}$ transforms as a vector; as does $\sqrt{-g_t}n^t$ since $\sqrt{-g_t}$ is a scalar under time rescalings (as the metric on Σ_t involves no dt^2 piece).

We can write the time derivative of E as

$$\begin{aligned} \partial_t E &= \frac{1}{\delta t}\left(\int_{V_{\delta t}} d^dx\, dz\,\sqrt{-g}\nabla^\mu T_{t\mu} - \int_{M_{\delta t}} d^dx\,\sqrt{-g_{M_{\delta t}}}n^z T_{tz}\right) \\ &= -\int_{\Sigma_t} d\vec{x}\,\sqrt{g_{\partial\Sigma_t}}n^t n^z T_{tz}, \end{aligned} \tag{84}$$

where the various manifolds are defined straightforwardly: $V_{\delta t}$ is the spacetime volume sandwiched between Σ_t, and $\Sigma_{t+\delta t}$, and $M_{\delta t}$ is the timelike component of its boundary. In the last step, we have used the fact that the time integral in the second term on the first line just produces a factor of δt.

e) First, we need

$$n^t = (0, z/R, \vec{0}), \quad n^z = (z/R, 0, \vec{0}), \tag{85}$$

which comes from e.g. $n^z n_z = 1$. Now the induced metric on ∂_{Σ_t} has determinant $g_{\partial\Sigma_t} = +(R/z)^{2d-2}$, and so

$$F|_{z=0} = \partial_t E \sim \int_{\Sigma_t} d\vec{x}\, z^{-d+3}T_{tz} = \int_{\Sigma_t} d\vec{x}\, z^{-d+3}\nabla_t\phi\nabla_z\phi. \tag{86}$$

Now suppose $\phi \sim z^\omega$. Then

$$F|_{z=0} \sim z^{-d+3+2\omega-1}, \tag{87}$$

and so if the flux is to be zero we must have

$$F|_{z=0} = 0 \implies \omega > d/2 - 1. \tag{88}$$

Now for the Bz^Δ mode this is always true, since $\Delta = d/2 + \nu$. For the $Az^{d-\Delta}$ mode, this condition reads $\nu < 1$, which is precisely the condition that the A mode be normalizable. So we see that (non)normalizable modes have (non)zero energy flux at infinity.

f) The part of T_{tt} with the smallest power of z is the $(\nabla_t \phi)^2$ part. Again, suppose $\phi \sim z^\omega$. Then the contribution to E with the smallest power of z is

$$E \sim \int_{\Sigma_t} dz\, d\vec{x}\, z^{-d-1} z^2 z^{2\omega} \sim z^{-d+2+2\omega}. \tag{89}$$

If the energy is to be finite, we need $2(1+\omega) > d$. For the A mode we have $\omega = d/2 + \nu$, and the energy is always finite. For the B mode we have $\omega = d/2 - \nu$, and

$$E < \infty \implies \nu < 1. \tag{90}$$

Of course this is the same condition on ν for the B mode to be normalizable. So, normalizable modes, in either quantization scheme, are the ones with finite E.

5 *AdS Propagators*

This is another problem from Hong's holography class. The to-do list is:

a) How is Lorentzian AdS different from Euclidean AdS? We will use the latter spacetime in what follows. b) Let ϕ be a massive scalar field, and find the buik-to-∂ propagator $K(z, x; x')$. c) Find a relation between K and the bulk-to-bulk propagator G in terms of the limit of G as one of its arguments approaches the boundary. d) Write down a general boundary correlation function in terms of a limit of a bulk correlation function.

a) From the $1/z^2$ dependence of the metric, we see that in Euclidean AdS, the distances in the x coordinates vanish at $z = \infty$, and so $z = \infty$ is just a single point, unlike in Lorentzian AdS. Another way of seeing this is to recognize that Euclidean AdS is the Poincare disk, with $z \to \infty$ corresponding to the single point at the center of the disk.

b) We want to get the boundary-to-bulk propagator. Using the equations of motion, the propagator K at $z = \infty$ needs to satisfy

$$(\partial_M(\sqrt{-g}g^{MN}\partial_N\phi) - m^2\sqrt{-g})K = 0. \tag{91}$$

Since $z = \infty$ is a single point in the bulk, in the $z \to \infty$ limit $K(x, z; x')$ can only depend on z (not x since all x are the same at $z = \infty$, and not x' by rotational invariance of the Poincare disk). Thus, putting in the z dependence of the metric, we have

$$\left[\partial_z\left((R/z)^{d+1}(z/R)^2\partial_z\right) - m^2(R/z)^{d+1}\right]K(z \to \infty) = 0. \tag{92}$$

Assuming a power-law $K(z) \propto z^\alpha$, we have

$$(1-d)\alpha + \alpha(\alpha - 1) - m^2R^2 = 0 \implies \alpha = \frac{d}{2} \pm \sqrt{d^2/4 + m^2R^2}. \tag{93}$$

We will see later that the requirement that K go to a δ function at the $z = 0$ boundary requires us to select out the larger root (which we denote as Δ), and so

$$K(z \to \infty) = Cz^\Delta, \tag{94}$$

for some $C \in \mathbb{R}$.

Now we can use the homogeneity of AdS (despite how it looks when drawn as a Poincare disk, no point is special) to get $K(x, z; 0)$: we first perform the transformation

$$z \mapsto \frac{z}{z^2 + x^2}, \quad x^\mu \mapsto \frac{x^\mu}{z^2 + x^2} \tag{95}$$

on $K(z \to \infty)$. We then use translation invariance in the x^μ directions (rotational invariance of the Poincare disk) to get the Poisson form

$$K(x, z; x') = C\left(\frac{z}{z^2 + (x - x')^2}\right)^\Delta. \tag{96}$$

Here we see that the power of Δ gives us a δ function when $x = x'$. This power is also correct since we have

$$K(z \to 0, x; x') = z^{d-\Delta}\delta^d(x - x'). \tag{97}$$

In standard quantization, ϕ has z^Δ scaling, so that $\int d^dx'\, K(z, x; x')\phi_0(x')$ has the correct scaling.

c) We can relate the bulk-to-bulk propagator to K by using the bulk-to-boundary map and one of Greens identities, namely

$$\int_M d^{d+1}x\,\sqrt{-g}\left(\phi_1 G^{-1}\phi_2 - \phi_2 G^{-1}\phi_1\right) = \int_{\partial M} d^dx\,\sqrt{-g_\partial}\left(\phi_1 n^\mu\partial_\mu\phi_2 - \phi_2 n^\mu\partial_\mu\phi_2\right), \tag{98}$$

where g_∂ is the induced metric on the boundary, G is the bulk propagator, and n^μ is the unit normal on the boundary. The trick is then to employ this identity with $\phi_1 = K(z, x; x'), \phi_2 = G(z, x; z'', x'')$. Now since the LHS is over the bulk and since $(\nabla^2 - m^2) = G^{-1}$ annihilates K

in the bulk (the only place it doesn't annihilate K is at coincident points on the boundary), the LHS is

$$LHS = \int_M d^{d+1}x\, \sqrt{-g} K(z,x;x')(\nabla^2 - m^2)G(z,x;z'',x'') = K(x'',z'';x'), \qquad (99)$$

by definition of G. On the other hand, since $\sqrt{-g_\partial} = (R/z)^d$ and $n^\mu = z$, the RHS is

$$RHS = \int_{\partial M} d^d x\, z^{-d+1} K(z,x;x') \partial_z^{\leftrightarrow} G(z,x;z'',x''), \qquad (100)$$

where $\partial^{\leftrightarrow}$ denotes the antisymmetrized derivative. Here we have dropped the R dependence since it will cancel out in the end.

Using the asymptotic $z \to 0$ form for K as written above, we can explicitly take the derivative with respect to z and get

$$RHS = \int_{\partial M} d^d x\, z^{-d+1} \delta(x - x') \left(z^{d-\Delta} \partial_z G(z,x;z'',x'') - (d-\Delta) z^{d-\Delta-1} G(z,x;z'',x'') \right). \qquad (101)$$

Now since $G(z,x;z'',x'')$ is normalizable, we know that it has the same $z \to 0$ scaling as the bulk normalizable mode, namely z^Δ. Thus

$$RHS = z^{-d+1}(z^{d-\Delta}\Delta z^{-1} - (d-\Delta)z^{d-\Delta-1})G(z,x;z'',x''), \qquad (102)$$

where we are implicitly taking the $z \to 0$ limit. In the notation we used in class, $\Delta = d/2 + \nu$, and so

$$RHS = \lim_{z\to 0}(2\Delta - d) z^{-\Delta} G(z,x;z'',x'') = 2\nu z^{-\Delta} G(z,x;z'',x''). \qquad (103)$$

Setting this equal to LHS and moving the $2\nu z^{-\Delta}$ over to the other side and re-labeling some dummy variables, we get

$$\lim_{z\to 0} G(z,x;z',x') = \frac{z'^\Delta}{2\nu} K(z,x;x'). \qquad (104)$$

d) Let ϕ_i be the bulk scalar dual to a boundary operator $\mathcal{O}_i$. The correlation function for a product of $\mathcal{O}_i$'s at various points on the boundary can be determined by computing all Feynman diagrams in the bulk that have external legs on the boundary. Thus, a correlation function of n $\mathcal{O}_i$'s involves n K propagators (which connect the boundary $\mathcal{O}_i$'s to the part of the Feynman diagrams that live in the bulk, plus a bunch of G propagators which constitute the bulk part of the Feynman diagrams. On the other hand, we can consider the same class of Feynman diagrams, but with the external legs all made up of G propagators which terminate at points that have some small value of z. This is a bulk correlation function of ϕ_i fields. Taking the $z \to 0$ limit then gets us back to the correlation function of the $\mathcal{O}_i$'s. So, the only difference between the two correlation functions is whether we use K or G for the external legs. If we use G's, then we need to take the $z \to 0$ limit for one of G's arguments—luckily the previous part told us how to do this. So, using our result from c, we have

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{CFT} = \lim_{\{z_i\}\to 0} \prod_i (2\nu_i z_i^{-\Delta_i}) \langle \phi_1(z_1,x_1)\cdots\phi_n(z_n,x_n)\rangle. \qquad (105)$$

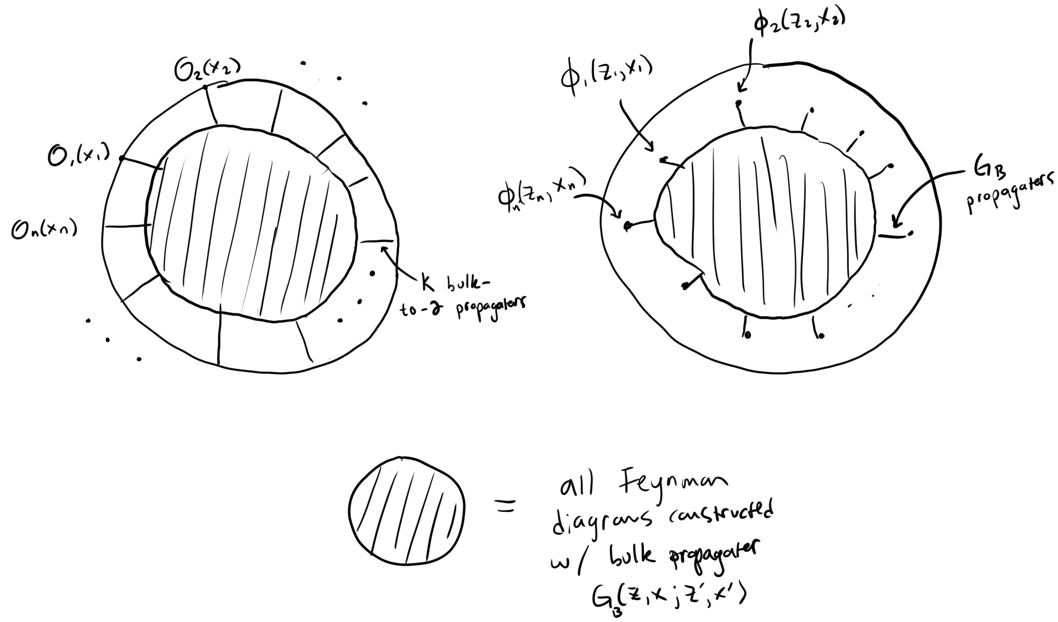
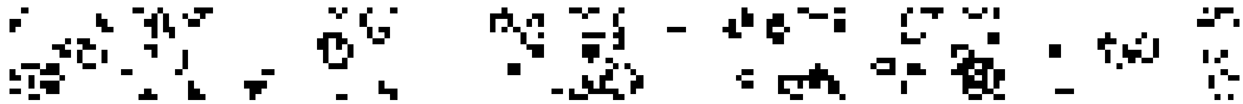


Figure 1: One of the ugliest figures I've ever made. Here, the shaded blob stands for all possible Feynman diagrams constructed from the bulk propagator.

All of this is illustrated in figure **??**.



6 *Massive vectors in AdS*

Yet another problem from Hong's holography class. Consider a massive vector field in AdS:

$$S = -\int d^{d+1}x \sqrt{-g}\left(\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}m^2 A_M A^M\right), \tag{106}$$

where M, N run over (z, x^μ). Do the following:

a) When $m^2 = 0$, find the asymptotic behavior of A_μ at the boundary $z \to 0$. b) What is the scaling dimension of the boundary current J^μ corresponding to the bulk gauge field A? c) When $m \neq 0$, what is the asymptotic behavior of A_M at the boundary? d) Now what is the scaling dimension of J^μ? e) What happens to A_z when $m \neq 0$? f) What are the differences between the massive and massless cases?

a) The equation of motion when $m = 0$ is just

$$\partial_M(\sqrt{-g}F^{MN}) = 0 \implies \partial_M(z^{-d-1}g^{ML}g^{NO}\partial_{[L}A_{O]}) = 0, \quad \forall N. \tag{107}$$

Let us assume the asymptotic behavior $A_\mu(z \to 0) \sim z^\Delta$, and work in a gauge where $A_z = 0$. The only derivative in the equations of motion we then care about in the $z \to 0$ limit is the one with $L = z$, and so setting $N = \mu$ we have

$$\partial_z(z^{-d-1+2+2}\partial_z z^\Delta) = 0 \implies (-d+3)\Delta + \Delta(\Delta - 1) = 0, \tag{108}$$

and so we have two options: $\Delta = 0$ or $\Delta = d - 2$. Thus we can write

$$A_\mu(z \to 0) = a_\mu(x) + b_\mu(x) z^{d-2}. \tag{109}$$

The a_μ is the non-renormalizable piece, while the b_μ part is renormalizable.

b) The non-renormalizable piece a_μ is the part that is relevant for computing the scaling dimension of the current in the CFT dual to A_μ, since a_μ is the part which we interpret as a change in boundary conditions. To determine the scaling dimension of J^μ, we can look at the boundary integral

$$\int_{\partial AdS} d^d x \, a_\mu J^\mu. \tag{110}$$

Now we consider performing the isometry $z \mapsto \lambda z$, $x^\mu \mapsto \lambda x^\mu$. a_μ is independent of z but it carries a covector index, so it transforms with a factor of λ. Writing $J^\mu(x/\lambda) = \lambda^{\Delta_J} J^\mu(x)$, we have

$$\int_{\partial AdS} d^d x \, a_\mu J^\mu \mapsto \lambda^{-d+1+\Delta_J} \int_{\partial AdS} d^d x \, a_\mu J^\mu. \tag{111}$$

Since we need this term to be invariant, we find that $\Delta_J = d - 1$, as expected of a current in a d-dimensional CFT.

Note that A_μ and A^μ have different scaling behaviors since they differ by e.g. $g^{\mu\mu}$, which scales as z^2. To determine the dimension of J^μ we need to integrate it against something with a covariant index, so it is the scaling of A_μ, not A^μ, which is needed.

c) When the vector field is massive, the equation of motion becomes

$$\partial_M(\sqrt{-g}F^{MN}) - \sqrt{-g}m^2 A^N = 0, \qquad \forall N. \tag{112}$$

Again, let $A_\mu \sim z^\Delta$ near the boundary. Then we have

$$\partial_z(z^{-d-1}\partial_z A_\mu g^{zz} g^{\mu\nu}) - z^{-d-1} m^2 A_\mu g^{\mu\nu} = 0, \tag{113}$$

so that

$$\Delta \partial_z (z^{-d+3+\Delta-1}) - z^{-d+1+\Delta} R^2 m^2 = 0 \implies \Delta(-d+2+\Delta) - R^2 m^2 = 0, \tag{114}$$

where the R^2 comes from the inverse metric factors. There are thus two possible choices for the scaling behavior of A_μ which are compatible with the equations of motion, and we can write

$$A_\mu = a_\mu z^{\Delta_+} + b_\mu z^{\Delta_-}, \qquad \Delta_\pm = 1 - \frac{d}{2} \pm \sqrt{(d-2)^2/4 + m^2 R^2}. \tag{115}$$

Sorry for the profusion of Δ's! It's just entrenched as a theme by this point and there's no going back.

d) In standard quantization, the non-renormalizable part will be the Δ_+ piece. Looking at the boundary term $\int_{\partial AdS} b_\mu z^{\Delta_+} J^\mu$ and performing the re-scaling of x and z tells us that $-d + 1 + \Delta_J - \Delta_+ = 0$, and so in this case J^μ has scaling dimension

$$\Delta_J = d - 1 + \Delta_+ = \frac{d}{2} + \sqrt{(d-2)^2/4 + m^2 R^2}. \tag{116}$$

Sanity check: when $m = 0$ we recover $\Delta_J = d - 1$, as required.

e) When $m \neq 0$, we can no longer use gauge invariance to fix $A_z = 0$. The z component of the equations of motion reads, focusing only on the z-dependence of A_z,

$$\partial_z(\sqrt{-g}[g^{zz}]^2 \partial_z A_z) - \sqrt{-g} m^2 g^{zz} A_z = 0. \tag{117}$$

Since all the components of the metric have the same z dependence, the z dependence of A_z is fixed in the same way as that of the A_μ.

f) In the massless case, we have gauge invariance under $A \mapsto A + d\chi$. Accordingly, the boundary J^μ operator must be divergenceless, and so it should be thought of as a conserved current. Since we integrate conserved currents over codimension 1 manifolds to get numbers, we need the dimension of J^μ to be $d-1$. By contrast, when $m \neq 0$, there is no gauge invariance, and J^μ is not a conserved current; hence its scaling dimension is not fixed at $d-1$.

7 *Wilson Loop Vevs in $\mathcal{N}=4$ SYM using AdS/CFT*

Today is another problem from a pset in Hong's holography class. This time, we're computing Wilson loops in $\mathcal{N}=4$ SYM with holography. The problem is as follows:

By evaluating the saddle-point of the NG action corresponding to a geometry in which two quarks have been inserted a distance L apart in the boundary CFT, find the potential energy $V(L)$ coming from the interaction between the two quarks (the relevant Wilson loop here is a rectangle of sides L, t, where $t \gg L$). How does the result behave in the high temperature and low-temperature limits?

We can compute Wilson loop vevs in $\mathcal{N}=4$ SYM in the limit $g_s \to 0$ (no sum over different topologies) and $\alpha' \to 0$ (when we can use the saddle-point solution to the string path integral). On the CFT side this limit is nontrivial since it corresponds to $N, \lambda \to \infty$.

We just need to compute the classical string action, since

$$\langle W(C)\rangle = Z_{str}[\partial\Sigma = C] \approx e^{iS_{cl}[\partial\Sigma=C]}, \tag{118}$$

where Σ is the string worldsheet.

At finite T (here T is temperature, not the temporal length of the Wilson loop, which we will write as t), the appropriate bulk geometry to use is an AdS-Schwarzchild black hole at temperature T. The metric is

$$ds^2 = \frac{R^2}{z^2}\left(-(1-\bar{z}^d)dt^2 + d\vec{x}^2 + \frac{1}{1-\bar{z}^d}dz^2\right), \tag{119}$$

where we've defined

$$\bar{z} \equiv z/z_0, \qquad T = \frac{d}{4\pi z_0}. \tag{120}$$

z_0 is the location of the horizon, which is closer to the $z=0$ boundary at larger temperatures.

Let us choose the contour C to run in the $x^1 - t$ plane. We can then parametrize the worldsheet with coordinates $(\tau, \sigma) = (t, x^1)$. We are interested in the energy of two quarks a distance L apart. If the temporal length T of the curve C is much larger than L, then the shape of Σ is determined by a function $z(\sigma) = z(x^1)$, with boundary conditions $z(\pm L/2) = 0$. We will use the NG action (rather than the Polyakov action) to compute S_{cl}, since we aren't ever going to need to quantize anything. The induced metric on the worldsheet is then determined by

$$ds_w^2 = \frac{R^2}{z^2}\left(-dt^2(1-\bar{z}^d) + d\sigma^2\left[1 + \frac{z'^2}{1-\bar{z}^d}\right]\right), \tag{121}$$

with $z' = \partial_\sigma z$.

The NG action is (again, assuming $t \gg L$ so that the Lagrangian on the classical solution can be treated as time-independent)

$$S_{NG} = -\frac{R^2 t}{\pi\alpha'}\int_0^{L/2}\frac{d\sigma}{z^2}\sqrt{1-\bar{z}^d + z'^2}, \tag{122}$$

where we pulled out the R^4/z^4 from the determinant of the induced metric and used the symmetry $z(\sigma) = z(-\sigma)$ that must be satisfied by the classical solution.

We can eliminate the z' inside the square root by using the equations of motion. Since $\mathcal{L}$ is independent of σ, we have

$$-z'\frac{\partial\mathcal{L}}{\partial z'} + \mathcal{L} = c, \tag{123}$$

where c is a constant. For us, this is

$$\frac{z'^2}{z^2\sqrt{1+z'^2-\bar{z}^d}} = \frac{\sqrt{1+z'^2-\bar{z}^d}}{z^2} + c, \tag{124}$$

or

$$\frac{1-\bar{z}^d}{z^2\sqrt{1+z'^2-\bar{z}^d}} = c. \tag{125}$$

We can get c by noticing that at $\sigma = 0$, $z' = 0$ by symmetry. So, let $z_* \equiv z(0)$. Then

$$c = \frac{\sqrt{1-\bar{z}_*^d}}{z_*^2}. \tag{126}$$

The energy of the Wilson line configuration is then computed as

$$E(L) = \frac{\sqrt{\lambda}}{\pi}\int_0^{z_*}\frac{dz}{z^2 z'}\sqrt{1+z'^2-\bar{z}^d}, \tag{127}$$

since $\lambda = R^4/\alpha'^2$. Solving for z' in terms of z and c, we have

$$z' = \sqrt{(1-\bar{z}^d)\left(\frac{1-\bar{z}^d}{c^2 z^4} - 1\right)}. \tag{128}$$

Putting this into the integral and doing some housekeeping, we get

$$E(L) = \frac{\sqrt{\lambda}}{\pi}\int_0^{z_*}\frac{dz}{z^2\sqrt{1-c^2z^4/(1-\bar{z}^d)}}. \tag{129}$$

Now $E(L)$ has a $z \to 0$ divergence, but this just corresponds to the diverging mass of the two quarks. Recalling that the quark mass goes as the inverse of their z-coordinates, we expect the quark mass to show up as a $1/\epsilon$ divergence if we cut the integral off below at ϵ.

Now in order to get $E(L)$, we need an expression for z_* in terms of L. We can get an integral equation which gets us part way there by solving for z' and integrating from $\sigma = -L/2$ to $\sigma = 0$:

$$\frac{L}{2} = \int_0^{z_*} dz \left[(1 - \bar{z}^d) \left(\frac{1 - \bar{z}^d}{c^2 z^4} - 1 \right) \right]^{-1/2}. \tag{130}$$

To see what's happening here more clearly, there are two limits we can take. The first is the $T \to 0$ limit (or equivalently, the small L limit). In this limit we can send $\bar{z} \to 0$ and $c^2 \to z_*^{-2}$, since when $T = 0$ the horizon is pushed to $z_0 = \infty$. In this limit, our integral equation determining z_* is

$$\frac{L}{2} = \int_0^{z_*} dz \frac{1}{\sqrt{z_*^4/z^4 - 1}} = z_* \alpha, \tag{131}$$

where $\alpha \approx 0.6$ is determined in terms of Elliptic integrals. This means that at $T = 0$, the maximal $z(\sigma)$ value on the worldsheet extends a distance into the bulk which grows linearly with L.

Now we can calculate $V(L)$ in this limit, by using $E(L) = 2M + V(L)$ and subtracting off the divergent mass piece:

$$V(L) = \frac{\sqrt{\lambda}}{\pi z_*} \left(\int_{\epsilon/z_*}^{1} \frac{dx}{x^2 \sqrt{1 - x^4}} - \frac{1}{\epsilon} \right). \tag{132}$$

The integral can be evaluated in terms of hypergeometric functions. Doing this, sending $\epsilon \to 0$, and using our expression for z_*, we get

$$V(L) = -\frac{2\alpha \sqrt{\lambda/\pi}\,\Gamma(3/4)}{L\Gamma(1/4)}. \tag{133}$$

The most important features here are the $1/L$ dependence (from scale invariance in the CFT), and the interesting $\sqrt{\lambda}$ coupling dependence.

Now we can look at what happens at higher T, or equivalently, at Wilson lines that have L large enough so that z_* approaches the horizon at z_0. By looking at (**??**), we see that z_* is monotonically increasing with L (this is a bit gross to show, but it ultimately comes down to $\partial_{z_*} c < 0$).

As we keep increasing L, there reaches a point $L_{screening}$ where (**??**) has no solution. Looking back, we see that this must mean that $c = 0$: this is when z_* “disappears” behind the black hole horizon. For $L > L_{screening}$, we no longer can have a Wilson line connecting the two quarks: the worldsheet ends up splitting apart, and terminating on the black hole. In the way we've been doing things, this corresponds to the trivial solution $V(L) = 0$, and the quarks are fully screened. The crossover between the $1/L$ dependence of the potential and the fully screened potential can be found numerically, but I'll be content with this simple understanding of the two limits.

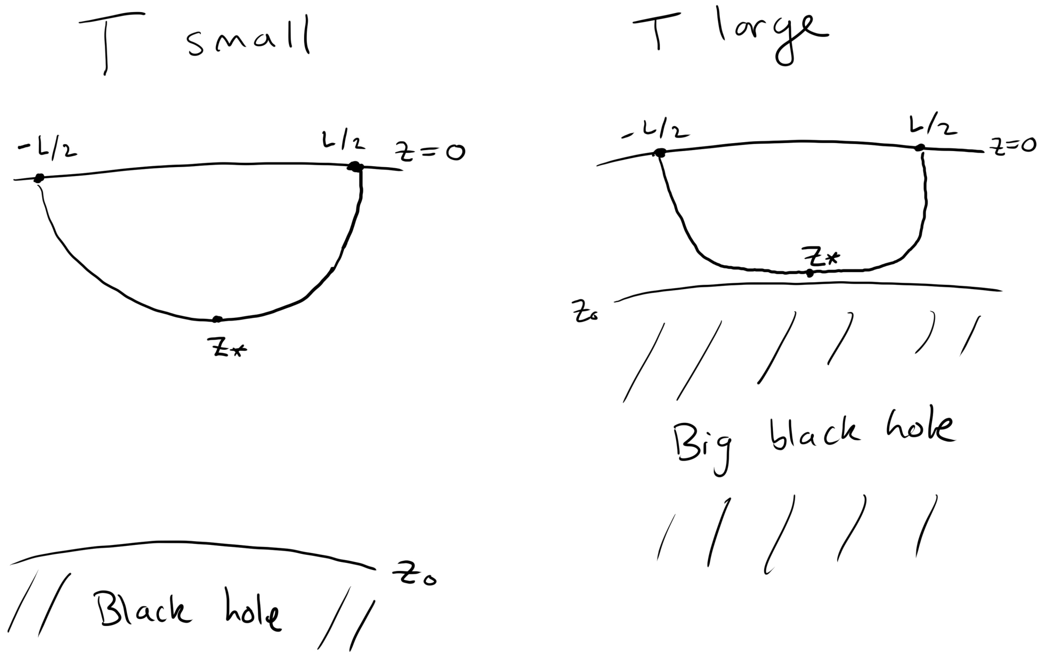


Figure 2: Another one of the ugliest figures I've ever made; explanation is in the text.

8 *Zero modes of $i\not{D}_A$ on the sphere* ✓

In today's diary entry we're going to a calculation that I've heard mentioned in a few papers, but have never actually seen worked out anywhere. We're going to explicitly construct the zero modes of the Dirac operator on a sphere, in the presence of an arbitrary $U(1)$ magnetic flux.

Spherical coordinates

In what follows, we will be using veilbeins, since that's the only method we have for dealing with fermions on curved spaces.[7] Recall that the veilbeins are found by fractionalizing the

[7]This is because fermion actions need γ^a matrices to be defined, which represent Clifford algebras. We want to represent a Clifford algebra with the relation $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ (or maybe η^{ab}), and definitely don't

metric:

$$g_{\mu\nu} = e^a_\mu e^a_\nu, \qquad e^a_\mu e^b_\nu g^{\mu\nu} = \delta^{ab}. \tag{134}$$

Since we don't want to constantly be phantoming when writing stuff out, our convention will be that, when viewing the veilbeins as a matrix, the greek (spacetime) letter will always denote the row index of the matrix, and the roman (internal space) letter will always denote the column index. When we break apart the metric like this, we pick up a gauge redundancy, since the transformation $e^a_\mu \mapsto [O]^a_b e^b_\mu$ for $O \in O(s,t)$ leaves the splitting $g_{\mu\nu} = e^a_\mu e^a_\nu$ invariant (in what follows we will only be concerned with 2+0 dimensions, so that the relevant "gauge group" is $O(2)$).

The Dirac operator is (roman indices can be raised / lowered with impunity)

$$\not{D}_A = \gamma_a e^{\mu a}(\partial_\mu + i(\omega_\mu + A_\mu)), \tag{135}$$

where ω, A are the spin and gauge connections, with

$$\omega_\mu{}^a{}_b = e^a_\nu \partial_\mu e^\nu_b + e^a_\nu \Gamma^\nu_{\mu\lambda} e^\lambda_b. \tag{136}$$

We've tried to take a sign convention that is maximally simple; ours differs from the conventions in many other places though so be careful. The spin connection is needed to ensure

want to have the anticommutator be equal to $g^{ab}(x)$; this would be a mess. Thus we need veilbeins to switch between spacetime and a frame in which the Clifford generators can be defined.

that $\not{D}_A\psi$ transforms covariantly under local $O(2)$ gauge rotations of the coordinate frames.[8]

Since the generators of Spin(d) are $-i[\gamma_a, \gamma_b]/4$, the spin connection is, quite generally,

$$\omega_\mu = \frac{1}{2}\omega_\mu^{ab}\Sigma_{ab}, \qquad \Sigma_{ab} = \frac{-i}{4}[\gamma_a, \gamma_b]. \tag{141}$$

The veilbeins for spherical coordinates on the unit S^2 are easy to write down:

$$e^a_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \sin\theta \end{pmatrix}^a_\mu, \qquad e^{\mu a} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-1}\theta \end{pmatrix}^{\mu a}. \tag{142}$$

Here the fact that the tetrads are the "square root of the metric" is made manifest. Of course, there are infinitely many other choices, related by local $O(2)$ transformations.

To get the spin connection, we will need to know that the nonzero Christoffel symbols on the sphere are

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{143}$$

Then we calculate

$$\omega^{ab}_\theta = 0, \qquad \omega^{ab}_\phi = \cos\theta J^{ab}. \tag{144}$$

[8]It is also needed to ensure that the (real-time) action is Hermitian! The added spin-connection term is actually not Hermitian, and this compensates for the non-Hermiticity of $\bar\psi(i\not\partial)\psi$ when working on a curved manifold. Indeed, under Hermitian conjugation,

$$\dagger : i\bar\psi e^{a\mu}\gamma_a\partial_\mu\psi \mapsto i\bar\psi e^{a\mu}\gamma_a\partial_\mu\psi + i\bar\psi(\partial_\mu e^{a\mu})\gamma_a\psi. \tag{137}$$

Now let's look at the spin connection part. For simplicity, we will work in Riemann normal coordinates around a certain point p, where the Christoffel symbols (but not their derivatives) can be chosen to vanish. The spin connection part of the Lagrangian density at this point is then

$$\begin{aligned} \mathcal{L} \ni -\frac{1}{2}\sum_{a,b\neq c}\psi^\dagger\gamma^0\gamma_a e^{a\mu}\omega^{bc}_\mu\Sigma_{bc}\psi &= \frac{i}{2}\sum_{a,b\neq c}\psi^\dagger\gamma^0\gamma_a e^{a\mu}(e^b_\nu\partial_\mu e^{c\nu})\gamma_b\gamma_c\psi \\ = \sum_{a\neq b\neq c}\frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_a\gamma_b\gamma_c\psi &+ \sum_{a\neq c}\frac{i}{4}\bar\psi e^{a\mu}(e^a_\nu\partial_\mu e^{\nu c} - e^c_\nu\partial_\mu e^{\nu a})\gamma_c\psi \\ = \sum_{a\neq b\neq c}\frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_a\gamma_b\gamma_c\psi &+ \frac{i}{2}\sum_c\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi \end{aligned} \tag{138}$$

The first term here is Hermitian: using $\gamma_a^\dagger\gamma_0^\dagger = \gamma_0\gamma_a$ (we are in $\mathbb{R}$ time, remember), we have

$$\dagger : \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\psi \mapsto -\frac{i}{4}\psi^\dagger e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}(\gamma_c^\dagger\gamma_b^\dagger\gamma_a^\dagger)\gamma_0^\dagger = \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_c\gamma_b\gamma_a = \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\psi, \tag{139}$$

since here $a \neq b \neq c$.

However, the second term at the end of (**??**) is actually anti-Hermitian:

$$\dagger : \frac{i}{2}\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi \mapsto -\frac{i}{2}\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi. \tag{140}$$

However, when we add in the $+i\bar\psi(\partial_\mu e^{a\mu})\gamma_a\psi$ from (**??**), we see that it combines with the RHS of the above equation to yield the LHS, giving an action that is Hermitian. Thus the second term at the end of (**??**) is a counterterm that ensures that the full action is Hermitian.

Then, if we adopt the gamma matrices $\gamma^1 = X, \gamma^2 = Y$ (this is the best choice since it makes the splitting $S = S_+ \oplus S_-$ manifest), we get

$$\omega_\theta = 0, \qquad \omega_\phi = \frac{1}{2}\cos\theta J^{ab}\Sigma_{ab} = -\frac{\cos\theta}{2}Z. \tag{145}$$

Finally, we need an expression for A_μ. We will make the usual choice for a monopole on S^2, namely (note how similar the forms of the gauge and spin connections are!)

$$A^{N/S} = n\frac{\pm 1 - \cos\theta}{2}d\phi, \tag{146}$$

which gives $\int_{S^2} F = 2\pi n$. The covariant derivatives are

$$\nabla_\theta = \partial_\theta, \qquad \nabla_\phi = \partial_\phi - \frac{iZ}{2}\cos\theta + in\frac{\pm 1 - \cos\theta}{2}. \tag{147}$$

We can now finally write down the expression for $i\not{D}_A\psi = 0$, which is

$$\not{D}_A\psi^{(N/S)} = \left[X\left(\partial_\theta + \frac{\cot\theta}{2}\right) + Y\csc\theta\left(\partial_\phi + in\left(\frac{\pm 1 - \cos\theta}{2}\right)\right)\right]\psi^{(N/S)} = 0 \tag{148}$$

or written out, (note to self: may need to come back and sort out some minus signs)

$$\begin{aligned}\left(\partial_\theta + \frac{\cot\theta}{2} - i\csc\theta\partial_\phi - n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)\right)\psi_R^{(N/S)} &= 0\\ \left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi + n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)\right)\psi_L^{(N/S)} &= 0\end{aligned} \tag{149}$$

These equations are actually very easy to solve: the ϕ dependence is $e^{il\phi}$ by symmetry, while the θ dependence is figured out by the common factor of $1/\sin\theta$ in the equations.

For $n = 0$, a solution with $l = 0$ is $\psi_{R/L} = 1/\sqrt{\sin\theta}$. However, while normalizable, this is not differentible, and therefore is not an allowed solution.[9] So there are no zero modes

[9] We are looking only for solutions in the domain of $i\not{D}_A$, which by definition are C^∞ sections of the bundle $S \otimes E$, where S is the spinor bundle and E is the gauge bundle. Indeed, the Laplacian and the Dirac operator are only defined on infinitely smooth functions, and stuff can go wrong if our functions are not infinitely differentible. For example, consider $\psi_R(\theta, \phi) = 1/\sqrt{\sin(\theta)}$. This looks like a totally fine zero mode solution for the zero-flux case $n = 0$, since $i\not{D}_0\psi = 0$ and also $\int \psi_R^\dagger\psi_R \sin\theta = 2\pi^2$ is finite, so that ψ_R is L_2 on the S^2. However, weird pathologies come up due to the non-differentibility at $0, \pi$. For example, while $\not{D}_0\psi = 0$,

$$\not{D}_0^2\psi = \left(-\csc\theta\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{4} + \frac{1}{4\sin^2\theta}\right)\psi = \sin^{-5/2}\theta(1 - \cos^2(\theta)/4) - \sin^{-1/2}(\theta)/4 \neq 0. \tag{150}$$

The problem here is that $\not{D}(\not{D}\psi) \neq (\not{D}^2)\psi$ and so the positive-definiteness arguments we made don't apply to ψ (this is why we don't want to let functions like ψ be part of the domain of definition of the Dirac operator).

Anyway, I'm still not really sure about the physicality of such a restriction. Should I really be bothered if ψ is non-differentible, provided that $\sqrt{|g|}|\psi(x)|^2$ is finite everywhere? Not sure.

when $n = 0$. This follows from the fact that there are no zero modes on spaces where the curvature scalar is nowhere negative; more on this in a separate diary entry.

Now consider $n = 1$. From the index theorem we know there should be one more L zero mode than R zero mode. And indeed, there is one L zero mode, and no R zero modes. The L zero mode is just

$$\psi_L^N(1, 1/2) = e^{i\phi/2}, \qquad \psi_L^S(1, 1/2) = e^{-i\phi/2}, \tag{151}$$

where we have adopted the notation $\psi_{L/R}^{N/S}(n, l)$. Note that the gluing transition function on the equator correctly recovers the $n = 1$ flux. Conversely, if $n = -1$ then we have an R zero mode and no L zero mode: the R zero mode is

$$\psi_R^N(-1, -1/2) = e^{-i\phi/2}, \qquad \psi_R^S(-1, -1/2) = e^{i\phi/2}. \tag{152}$$

Now take $n = \pm 2$. For $n = 2$ we expect two more L zero modes than R zero modes, and indeed, the two L zero modes are (there are no R zero modes)

$$\psi_L^N(2, 3/2) = \sin(\theta/2)e^{i3\phi/2}, \qquad \psi_L^S(2, -1/2) = \sin(\theta/2)e^{-i\phi/2} \tag{153}$$

and

$$\psi_L^N(2, 1/2) = \cos(\theta/2)e^{i\phi/2}, \qquad \psi_L^S(2, -3/2) = \cos(\theta/2)e^{-3i\phi/2}. \tag{154}$$

The situation is reversed for $n = -2$: the two zero modes are

$$\psi_R^N(-2, -3/2) = \sin(\theta/2)e^{-i3\phi/2}, \qquad \psi_R^S(-2, 3/2) = \sin(\theta/2)e^{i3\phi/2} \tag{155}$$

and

$$\psi_R^N(-2, -1/2) = \cos(\theta/2)e^{-i\phi/2}, \qquad \psi_R^S(-2, 3/2) = \cos(\theta/2)e^{3i\phi/2}. \tag{156}$$

In general, for flux n, there are n L zero modes and no R zero modes if $n > 0$, while there are n R zero modes and no L zero modes if $n < 0$. The n modes come in a series $\psi_{L/R}^N(n, l)$ with $l = n - 1/2, n - 3/2, \ldots, \pm 1/2$, with the $+$ for L and the $-$ for R. The corresponding functions on the southern patch are related by $l \mapsto l - n$, which ensures that the transition function on the equator is a large gauge transformation with winding $2\pi n$. To find out what the spins (alias angular momentum) of these zero modes are, we need to calculate the L^2 angular momentum operator, which is modified by the spin and gauge connections; we will do this in a little bit.

We've been working with a uniform field strength, but of course (by the index theorem), we know that the zero modes must persist if we take an arbitrary field configuration with the same value of $\int F$. For example, we might add the vector potential $\widetilde{A} = g(\theta)d\phi$ to the existing monopole potential, where $\widetilde{A}$ is globally defined on the sphere. Then we just have to modify our zero mode solution by $\psi_{L/R} \mapsto f_{L/R}(\theta)\psi_{L/R}$, where $f_{L/R}(\theta)$ satisfies

$$(\partial_\theta \pm \csc(\theta)g(\theta))f(\theta) = 0 \implies \psi_{L/R} = \exp\left(\mp \int_0^\theta d\theta' \, \csc(\theta')g(\theta')\right)\psi_{L/R}^0, \tag{157}$$

where $\psi_{L/R}^0$ is the solution for the homogeneous field. From this expression we see that we get a well-defined answer only if $g(\theta) \to \theta$ as $\theta \to 0, \pi$; this is the condition that $\widetilde{A}$ go to zero at the poles in this coordinate system, so that $\widetilde{A}$ is topologically trivial. For example, consider the L zero mode, and let $g(\theta) = \sin 2\theta$. Then we see that the zero-mode solution is modified by a factor of $f_L(\theta) = e^{-2\sin\theta}$, so that the weight of the wavefunction becomes concentrated near the poles, where the normalized field strength $F/\sin\theta$ becomes largest; this is because as we have seen, the L zero modes "like" positive field strength. More generally, for an arbitrary asymmetric $\widetilde{A}$, we just have to multiply our symmetric zero mode solution by a factor $f(\theta, \phi)$, where

Anyway, now some more comments on the symmetric (uniform field strength) case. we've seen that the zero mode states fall into half-odd-integer representations of $SU(2)$. This is no surprise given the symmetry of the problem (in fact, the whole spectrum of $\not{D}$ falls into $SU(2)$ representations, not just the zero modes).

To find the generators of the $SU(2)$, it is not simply enough to covariantize by making the replacement $\mathbf{L} = -i\mathbf{n} \times \partial \mapsto -i\mathbf{n} \times \nabla$, with $\mathbf{n}$ the unit vector on the sphere. Indeed, doing this leads to generators that fail to satisfy the correct $SU(2)$ commutation relations, since in the presence of background field strengths the covariant momenta $-i\nabla$ fail to commute (their commutator measures the field strength). The commutation relations are ruined not just by the gauge background field, but also by the field strength of the spin connection (the geometric curvature of the sphere). Now viewing the S^2 as living in three dimensions, the Hodge duals of the field strengths of both the gauge field and the spin connection are oriented along $\mathbf{n}$ (the magnetic field for both spin and gauge connections is radial), and so we can write $(d[\omega + A])_{\mu\nu} = \epsilon_{\mu\nu\lambda} n^\lambda B$, where in the present case $B = \mathbf{1}n/2 + Z/2$. One can check that in this case,

$$[(-i\mathbf{n} \times \nabla)^\mu, (-i\mathbf{n} \times \nabla)^\nu] = i\epsilon^{\mu\nu\lambda}((-i\mathbf{n} \times \nabla)_\lambda - Bn_\lambda). \tag{158}$$

This prompts us to take the ansatz

$$L_\mu = \epsilon_{\mu\nu\lambda} n^\nu \nabla^\lambda + Bn^\mu \tag{159}$$

for the angular momentum generators. Indeed, one can check (see the next diary for the calculation) that with this choice, the L_μ satisfy the usual $SU(2)$ commutation relations.

We can now write down the angular momentum generators explicitly. We have

$$\begin{aligned} L_z &= -i\nabla_\phi + \cos\theta B \\ L_\pm &= e^{\pm i\phi}\left(\pm\nabla_\theta + i\cot\theta\nabla_\phi + B\sin\theta\right). \end{aligned} \tag{160}$$

When simplifying this, we will use the covariant derivative $\nabla_\phi = \partial_\phi - i(Z\cos\theta + n\cos\theta)/2$, which differs from the one written above by the term $\pm in/2$, which doesn't affect the field strength and hence can be dropped without affecting the angular momentum commutation relations. However, one must use caution with this convention, since changing the convention for the gauge field *does* change the expressions for the eigenstates of $\not{D}_A$. This means that the zero mode eigenstates obtained above will *not* be appropriate eigenfunctions of the L_z and L^2 operators obtained below! Maybe someday I'll come back and redo this so that the conventions are the same. Anyway, plugging in and simplifying, we find

$$\begin{aligned} L_z &= -i\partial_\phi \\ L_\pm &= e^{\pm i\phi}\left(\pm\partial_\theta + i\cot\theta\partial_\phi + \frac{1}{2\sin\theta}(Z+n)\right). \end{aligned} \tag{161}$$

I've done a check in mathematica of the commutation relations, and they work! Yay! Note that if we had chosen to keep the factor of $\pm d\phi/2$ in the gauge connection, the eigenvalues of L_z would be shifted by $\pm 1/2$: therefore the eigenvalues of L_z are not a gauge-invariant thing to calculate, and they do not tell you about the spin of the zero mode.

As a reminder, one should not confuse the total angular momentum operator $L^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+)$ with the (negative of, depending on conventions) covariant Laplacian $-\nabla_\mu \nabla^\mu$. Indeed, the Laplacian is (we are working in conventions where the Laplacian is positive-definite)

$$\begin{aligned} -\Delta = \nabla_\mu \nabla^\mu &= \nabla_\theta^2 + \cot\theta\, \nabla_\theta + \csc^2\theta\, \nabla_\phi^2 \\ &= \partial_\theta^2 + \cot\theta\,\partial_\theta + \csc^2\theta \left(\partial_\phi^2 - i(Z+n)\cos\theta\partial_\phi - \frac{\cos^2\theta}{4}(Z+n)^2\right). \end{aligned} \tag{162}$$

In contrast, the angular momentum operator is

$$L^2 = \Delta + \frac{1}{4}(Z+n)^2. \tag{163}$$

We would like to use this result to figure out the angular momentum of the zero modes obtained previously. In our current conventions, the zero mode equation is, on the N hemisphere

$$\begin{aligned} \left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi + \frac{n}{2}\cot\theta\right)\psi_R = 0 \\ \left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi - \frac{n}{2}\cot\theta\right)\psi_L = 0 \end{aligned} \tag{164}$$

This choice for the gauge field makes finding the zero modes slightly easier. For example, if $n=1$ we see that we get a solution where $\psi_R = 1/\sqrt{4\pi}, \psi_L = 0$. The fact that the zero mode is a constant suggests that it has spin zero: to check, we act on it with L^2:

$$L^2(n=1)\psi_R = \Delta(n=1)\psi_R = 0, \tag{165}$$

as expected.

The last thing we square a lot is the Dirac operator: more on this and its relation to Δ in a future diary entry.

Stereographic projection

Now we will go through the problem again using stereographic coordinates. These coordinates are nicer since they are less singular than spherical coordinates. We usually cover the sphere in two hemispherical patches, but doing it this way means that both patches will contain a coordinate singularity for ϕ, which is no good (although for us it's okay since the zero mode solutions vanish at the poles so we can still use two patches to construct a single-valued zero mode solution). Getting a good covering of S^2 requires 4 patches, and in order to escape coordinate singularities, the patches have to have different definitions of ϕ, θ such that the $\theta = 0, \pi$ points in a given patch's coordinate system don't occur within that patch itself. Gluing together zero mode solutions like this really is a hopeless mess.

By contrast, stereographic coordinates are great! We only need two patches to cover the S^2 (not goodly, but that's okay), and within each patch we can use a metric which is perfectly singularity-free, and in fact is conformally equivalent to flat Euclidean space. Indeed, recall from a few diary entries ago that in stereographic projection of S^2 onto the plane, the metric assumes the conformally flat form (assuming the sphere has radius 1 for simplicity)

$$ds^2 = \frac{4}{(1+r^2)^2}(dx^2 + dy^2). \tag{166}$$

The tetrads are simple is this coordinate system:

$$e^a_\mu = \frac{2}{1+r^2}\mathbf{1}^a_\mu, \qquad e^{\mu a} = \frac{1+r^2}{2}\mathbf{1}^{\mu a}. \tag{167}$$

The Christoffel symbols are easily calculated to be (I won't write out the algebra)

$$\Gamma^\mu_{\nu\lambda} = \frac{2}{1+r^2}(x^\mu \delta_{\nu\lambda} - x_\nu \delta^\mu_\lambda - x_\lambda \delta^\mu_\nu). \tag{168}$$

We can then calculate

$$\omega^{ab}_\mu = \frac{2x_\mu}{1+r^2}\delta^{ab} + \delta^a_\nu \delta^{b\lambda}\Gamma^\nu_{\mu\lambda}. \tag{169}$$

Only the off-diagonal part is non-zero:

$$\omega^{12}_\mu = -\omega^{21}_\mu = \frac{2}{1+r^2}(x\delta_{\mu y} - y\delta_{x\mu}), \tag{170}$$

and so the full spin connection is

$$\omega_\mu = \frac{1}{2}\omega^{ab}_\mu \Sigma_{ab} = -\frac{i}{2}\sum_{a<b}\omega^{ab}_\mu \gamma_a\gamma_b = \frac{Z}{1+r^2}(x\delta_{\mu y} - y\delta_{\mu x}). \tag{171}$$

Therefore the equation $\not{D}_A\psi = 0$ reads

$$\left[X\left(\partial_x + iA_x - iZ\frac{y}{1+r^2}\right) + Y\left(\partial_y + iA_y + iZ\frac{x}{1+r^2}\right)\right]\psi = 0 \tag{172}$$

Decomposing this with $\psi = (\psi_L, \psi_R)^T$,

$$\begin{aligned}\left(2\partial + 2iA - \frac{\bar{z}}{1+|z|^2}\right)\psi_R = 0\\ \left(2\bar{\partial} + 2i\bar{A} - \frac{z}{1+|z|^2}\right)\psi_L = 0,\end{aligned} \tag{173}$$

where $z = x + iy$, $A = (A_x - iA_y)/2$, and $\partial = (\partial_x - i\partial_y)/2$.

Now we need to get an expression for A. We want the field strength to be proportional to the volume form, which is $4/(1+r^2)^2$. In complex coordinates,

$$\frac{4}{(1+r^2)^2}(dx^2 + dy^2) = \frac{4}{(1+|z|^2)^2}dzd\bar{z}. \tag{174}$$

Thus if we take

$$A = in\frac{\bar{z}}{2(1+|z|^2)}, \quad \bar{A} = -in\frac{z}{2(1+|z|^2)}, \tag{175}$$

then we have

$$F_{z\bar{z}} = \partial\bar{A} - \bar{\partial}A = i\frac{n}{1+|z|^2}\left(1 - \frac{|z|^2}{1+|z|^2}\right) = i\frac{n}{(1+|z|^2)^2}, \tag{176}$$

which gives

$$\int_{\mathbb{R}^2} F_{z\bar{z}}\, dz \wedge d\bar{z} = \int_{\mathbb{R}^2} \frac{n}{(1+r^2)^2} i(-2idx \wedge dy) = 2\pi n \int_0^\infty dr\, \frac{2r}{(1+r^2)^2} = 2\pi n \tag{177}$$

as desired. Thus the equations for the zero modes are (I think I missed a factor of 2 somehwere, which I am re-instating below in a very ad hoc manner: I think it is needed to get the right zero mode solutions)

$$\begin{aligned} \left(2\partial - (1/2+n)\frac{\bar{z}}{1+|z|^2}\right)\psi_R = 0 \\ \left(2\bar{\partial} - (1/2-n)\frac{z}{1+|z|^2}\right)\psi_L = 0. \end{aligned} \tag{178}$$

The solutions are then

$$\begin{aligned} \psi_R(z,\bar{z}) &= f_R(\bar{z})(1+|z|^2)^{(1/2+n)/2} \\ \psi_L(z,\bar{z}) &= f_L(z)(1+|z|^2)^{(1/2-n)/2}, \end{aligned} \tag{179}$$

where the Laurent series for $f_R(\bar{z})$ and $f_L(z)$ only involve terms of non-negative degree since we require $\psi_{R/L}$ to be finite at the origin. Now, requiring that the zero mode solutions be finite at ∞ tells us that no L zero modes exist if $n \leq 0$, while no R zero modes exist if $n \geq 0$, in agreement with what we found before. Wolog, take $n > 0$ and look at the $\psi_L(z,\bar{z})$ solutions. We can take them to be eigenstates of L_z, which in complex coordinates is

$$L_z = -i(X\partial_Y - Y\partial_X) = -i(x\partial_y - y\partial_x) = \frac{z+\bar{z}}{2}(\partial - \bar{\partial}) + \frac{z-\bar{z}}{2}(\partial + \bar{\partial}) = z\partial - \bar{z}\bar{\partial}, \tag{180}$$

where X, Y are coordinates in 3-space on the S^2 (note that the contribution from the connection to the covariant derivative has canceled with the modification of the angular momentum generators required in a magnetic field; see a subsequent diary entry for details). Note that functions only of $|z|$ have no angular momentum, as required. This means that z^α is an eigenfunction of L_z with eigenvalue $+\alpha$, while $\bar{z}^\alpha$ is an eigenfunction with eigenvalue $-\alpha$. Therefore in a basis in which L_z is diagonalized with eigenvalue $l \in \frac{1}{2}\mathbb{Z}$, the $f_L(z), f_R(\bar{z})$ will be proportional to z^l and $\bar{z}^l$, respectively. So, the zero mode eigenfunctions of L_z for $n > 0$ are of the form $z^l(1+|z|^2)^{1/4-n/2}$. At $r \to \infty$ this goes as $r^{l-n+1/2}$, so we require that $l \leq n - 1/2$ (having a constant is okay since constants have finite integrals due to the conformal factor in the metric). This recovers the situation where we have n L zero modes of angular momentum $l = n-1/2, n-3/2, \ldots, 1/2$ if $n > 0$, and n R zero modes with $l = n+1/2, \ldots, -1/2$ if $n < 0$.

Another way to derive the parity anomaly

Finally we mention a zero-mode-focused way of deriving the parity anomaly that I learned about from one of Seiberg's lectures at the Jerusalem winter school.

Consider the case of a 2+1D theory on $S^2 \times \mathbb{R}$, with a unit of flux through the spatial S^2. Then the results above tell us that the Hamiltonian (which here is just the spatial part of the Lagrangian) has a single zero mode; when we quantize we thus get two states $|0\rangle$ and $|1\rangle = \chi^\dagger|0\rangle$.

Now the charges of $|1\rangle$ and $|0\rangle$ must satisfy $q_0 = q_1 - 1$: this is just a consequence of $Q\chi^\dagger = \chi^\dagger(Q+1)$. Now, using CT symmetry,[10] we have (taking $CT|0\rangle = |1\rangle$; a possible phase factor here doesn't contribute to the discussion)

$$CTe^{iQ}|0\rangle = CTe^{iq_0}|0\rangle = e^{-iq_0}|1\rangle \tag{181}$$

but also

$$CTe^{iQ}|0\rangle = e^{iQ}CT|0\rangle = e^{q_1}|1\rangle, \tag{182}$$

so that $q_1 = -q_0$. Thus if CT really is a symmetry, we have $q_0 = -1/2, q_1 = +1/2$. This however means that both $|0\rangle$ and $|1\rangle$ are not gauge-invariant; a contradiction. Hence CT must actually be broken. Of course, this is the parity anomaly, and the way that CT gets broken is by T getting broken.

9 *Pion decay* ✓

This is essentially P&S, problem 19.2. Given the effective Lagrangian

$$\mathcal{L} = \frac{4G_F}{\sqrt{2}}(\bar{l}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L) + h.c, \tag{183}$$

find the amplitude for the decay $\pi^+ \to l^+\nu$, and compute the decay rate. Remember that the vector gauge currents are conserved, but the chiral currents are not. Parametrize them by

$$\langle 0|j^{\mu 5a}(x)|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ipx}, \tag{184}$$

where f_π is a constant.

[10] The presence of $\int dxdy\, F_{xy} = 2\pi$ means that both C and T are broken (using the historical definition of T under which magnetic fields are odd), while CT is preserved. Thus it makes sense to ask about how CT acts within the subspace of the zero modes, but not C or T individually.

We get the matrix element for the decay by writing

$$i\mathcal{M}(2\pi)^4\delta(q+k-p) = i\int d^4x\,\frac{4G_F}{\sqrt{2}}\langle l,\nu|(\bar{l}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L)|\pi^+(p)\rangle, \tag{185}$$

where k, q are the momenta of the lepton l and the neutrino. We can also write this as

$$i\mathcal{M}(2\pi)^4\delta(q+k-p) = i\bar{u}(q)\gamma^\mu\frac{1-\gamma^5}{2}v(k)\int d^4x\,e^{ix(q+k)}\frac{4G_F}{\sqrt{2}}\langle 0|(\bar{u}_L\gamma_\mu d_L)|\pi^+(p)\rangle, \tag{186}$$

where $\bar{u}$, v are the usual spinors which we use to build the plane wave solutions to the Dirac equation.

It's helpful to first rewrite the matrix element a bit by using the currents

$$j^{\mu a} = \bar{Q}\gamma^\mu\tau^a Q, \qquad j^{\mu 5a} = \bar{Q}\gamma^\mu\gamma^5\tau^a Q, \tag{187}$$

where $Q = (u,d)$ and $a = x,y,z$ is the $SU(2)$ index. First consider the combination

$$j^{\mu 1} + ij^{\mu 2}. \tag{188}$$

This choice of $SU(2)$ index structure projects onto $\bar{u}d$ currents (there is no $1/2$ needed since the generators are $\tau^a = \sigma^a/2$). We can then select out the left handed components by projecting with $(1-\gamma^5)$, so that

$$\bar{u}_L\gamma^\mu d_L = \frac{1}{2}(j^{\mu x} + ij^{\mu y} - j^{\mu 5x} - ij^{\mu 5y}). \tag{189}$$

The first two currents are conserved (they are gauge currents), so when we expand the $\bar{u}\gamma d$ term with the currents, these terms die. Using the parametrization written above for the matrix elements of the chiral currents and the fact that π^+ only has x and y $SU(2)$ indices (a π^+ is an up and an anti-down, so actually it is built as $\sigma^x - i\sigma^y$ in our basis), we get

$$i\mathcal{M}(2\pi)^4\delta(q+k-p) = i\bar{u}(q)\gamma^\mu\frac{1-\gamma^5}{2}v(k)\int d^4x\,e^{ix(q+k)}\frac{4G_F}{2}(-ip_\mu f_\pi e^{-ipx}), \tag{190}$$

and so

$$i\mathcal{M} = G_F f_\pi \bar{u}(q)\not{p}(1-\gamma^5)v(k). \tag{191}$$

Now we can get the decay rate. To square the matrix element $\mathcal{M}$, we need the spin sums

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m. \tag{192}$$

Using the Dirac equation and momentum conservation, we have

$$\bar{u}(q)\not{p} = \bar{u}(q)(\not{q} + \not{k}) \approx \bar{u}(q)\not{k}, \tag{193}$$

since the neutrino is essentially massless. Then we use the Dirac equation for the lepton to write

$$\not{k}(1-\gamma^5)v(k) = (1+\gamma^5)\not{k}v(k) = -(1+\gamma^5)m_l v(k). \tag{194}$$

So,

$$i\mathcal{M} = -G_F f_\pi m_l \bar{u}(q)(1+\gamma^5)v(k). \tag{195}$$

Squaring it and taking the trace with the help of the spin sums, this term becomes

$$G_F^2 f_\pi^2 m_l^2 \mathrm{Tr}[\not{q}(1+\gamma^5)(\not{k}-m_l)(1-\gamma^5)]. \tag{196}$$

The terms odd in γ matrices will die, so the m_l piece doesn't contribute. Then since $2\mathrm{Tr}[\not{q}\not{k}] = 2(q_\mu k_\nu)\mathrm{Tr}[g^{\mu\nu}\mathbf{1}] = 8(q\cdot k)$,

$$|\mathcal{M}|^2 = 8G_F^2 f_\pi^2 m_l^2 (E_\nu E_l + k^2), \tag{197}$$

where we went to the rest frame where $\mathbf{p} = 0$. Looking up the formula for the decay rate in P&S (and using $E_{cm} = m_\pi$), we get

$$\gamma(l^+\nu \to \pi^+) = 8G_F^2 f_\pi^2 \int \frac{d\Omega}{16\pi^2}\frac{1}{2m_\pi^2}|k| m_l^2 (E_\nu E_l + k^2). \tag{198}$$

Taking the equation $m_\pi = E_\nu + E_l$ and raising it to the fourth power gives, after some algebra,

$$k^2 = \left(\frac{m_\pi^2 - m_l^2}{2m_\pi}\right)^2 \implies E_\nu = \frac{m_\pi^2 - m_l^2}{2m_\pi}. \tag{199}$$

Then since $E_l^2 = k^2 + m_l^2$,

$$E_l = \frac{m_\pi^2 + m_l^2}{2m_\pi}. \tag{200}$$

Now we can plug these into the formula for Γ. Since $\int d\Omega = 4\pi$, this produces after some algebra,

$$\gamma(l^+\nu \to \pi^+) = \frac{m_l^2 m_\pi}{4\pi} G_F^2 f_\pi^2 (1 - m_l^2/m_\pi^2)^2. \tag{201}$$

Thus the ratio of the decay rates for $\mu^+\nu$ decay and $e^+\nu$ decay is

$$\frac{\gamma(e^+\nu \to \pi^+)}{\gamma(\mu^+\nu \to \pi^+)} = \frac{m_e^2}{m_\mu^2}\frac{(1-m_e^2/m_\pi^2)^2}{(1-m_\mu^2/m_\pi^2)^2} \ll 1. \tag{202}$$

Plugging in numbers with the Fermi constant set at $G_F = \sqrt{2}g^2/(8m_W^2)$ and $\tau_\pi \approx 2.6\times 10^{-8}$ sec, we get

$$f_\pi = \sqrt{\frac{4\pi}{\tau_\pi m_\mu^2 m_\pi}}\frac{1}{G_F(1-m_\mu^2/m_\pi^2)} \approx 100\mathrm{Mev}. \tag{203}$$

We've used just the decay rate to the muon since the contribution of the electron decay channel is suppressed by a factor of $m_e^2/m_\mu^2 \ll 1$ as we saw above.

10 *Zero modes for Dirac fermions on the torus* ✓

Today's diary entry is a simple exercise, but one which I hadn't done before: finding the spectrum of Dirac fermions on a torus in the presence of nonzero net magnetic flux.

We will work on a square torus with both side lengths set to 1 for simplicity. Suppose that over the torus, the integral of the field strength is $B \equiv \int F = 2\pi n$. We will split the torus up into two cylindrical coordinate patches. The first will be $U_1 = \{0 \leq x \leq 1/2, 0 \leq y \leq 1\}$, and the second will be $U_2 = \{1/2 \leq x \leq 1, 1/2 \leq y \leq 1\}$, with $1 \sim 0$ in both limits. The two patches overlap along the line $x = 1/2$ and along the line $x = 0 \sim 1$. We will take the transition function to be trivial on the former overlap region, and to be the exponential of a function that winds by $2\pi n$ around the y direction, viz. $g_{12} = e^{iyB}$. The natural choice for the gauge field is then $A = (0, Bx)$. This gives the correct field strength and the way it is glued between the two patches is determined correctly as

$$A^2_\mu(1, y) = g_{12}^{-1}(A^1_\mu(0, y) - i\partial_\mu)g_{12}. \tag{204}$$

The Hamiltonian $H = -i(X[\partial_x - iA_x] + Y[\partial_y - iA_y])$ is then, in this gauge,

$$H = -i \begin{pmatrix} 0 & \partial_x - i\partial_y - Bx \\ \partial_x + i\partial_y + Bx & 0 \end{pmatrix}. \tag{205}$$

Define the operator

$$\gamma \equiv \frac{1}{\sqrt{2|B|}}(\partial_x + i\partial_y + Bx), \qquad H = -i\sqrt{2|B|}\begin{pmatrix} 0 & -\gamma^\dagger \\ \gamma & 0 \end{pmatrix}. \tag{206}$$

Then

$$[\gamma, \gamma^\dagger] = \mathrm{sgn}(B). \tag{207}$$

In what follows we will assume $B > 0$, and so $\gamma, \gamma^\dagger$ obey the usual harmonic oscillator algebra.

Now if we square $H\psi = E\psi$, we get

$$2B \begin{pmatrix} \gamma^\dagger\gamma & 0 \\ 0 & \gamma^\dagger\gamma + 1 \end{pmatrix} \psi = E^2\psi. \tag{208}$$

Therefore the energy levels are

$$E_n = \pm\sqrt{2Bm}, \qquad m \in \mathbb{N}. \tag{209}$$

Now since $\gamma^\dagger\gamma$ has only non-negative eigenvalues, we see that for $\psi = (\psi_L, \psi_R)^T$, we can have ψ_L zero modes, but cannot have ψ_R zero modes. If we were to change the sign of the flux $B > 0$ by $B \mapsto -B$, then in order to maintain the right commutation relations, we would need to interchange γ and $\gamma^\dagger$. This would then give $H = 2|B|(\gamma^\dagger\gamma + 1) \oplus \gamma^\dagger\gamma$, which is the same as for $B > 0$, but with left and right components switched. Therefore for $B > 0$ we can have only L zero modes, while for $B < 0$ we can only have R zero modes. This is in agreement with the index theorem.

Anyway, to solve for the eigenspectrum, we need to find $|0\rangle$ such that $\gamma|0\rangle = 0$; we can then build up the spectrum by acting on this with creation operators. Since y doesn't appear in γ we can give the $|0\rangle$ wavefunction a y dependence of e^{iyk_y}. We then have $(\partial_x - k_y + Bx)\psi_0(x, y) = 0$, where $\langle x, y|0\rangle = \psi_0(x, y)$. This gives

$$\psi_0(x, y) = e^{iyk_y} e^{-(x-k_y/B)^2 B/2}. \tag{210}$$

If we set PBC in the y direction, then we need $k_y \in 2\pi\mathbb{Z}$. For the x direction, setting $x \sim x + 1$ means that $k_y \sim k_y + B$, Therefore $B = 2\pi n$ means that we have n different options for k_y, and so the degeneracy of the zero energy states is n. The zero modes are then $\psi = (\psi_0, 0)^T$. Note that the zero modes survive the introduction of a non-uniform perturbing flux, so long as that flux integrates to zero over the torus. For example, adding on $\widetilde{A} = \widetilde{A}_y(x)dy$ to A modifies the zero mode solution by a factor of $\exp(-\int_0^x dx'\, \widetilde{A}_y(x))$, which preserves the boundary conditions on ψ_0 provided that $\int d\widetilde{A} = 0$ (for example, we could take $\widetilde{A}_y(x) = \sin(2\pi x)$).

Excited states are constructed by acting with $(\gamma^\dagger)^n$ on $|0\rangle$ in the usual way. For example, $\gamma^\dagger|0\rangle = |1\rangle = \sqrt{2B}(x - k_y/B)\psi_0$. The state with energy $\pm\sqrt{2Bm}$, $m > 0$ is formed by taking $|m\rangle$ for the left component and $\mp i|m-1\rangle$ for the right component:

$$H\psi_{\pm m} = H\begin{pmatrix} |m\rangle \\ \mp i|m-1\rangle \end{pmatrix} = \sqrt{2B}\begin{pmatrix} 0 & i\gamma^\dagger \\ -i\gamma & 0 \end{pmatrix}\begin{pmatrix} |m\rangle \\ \mp i|m-1\rangle \end{pmatrix} = \pm\sqrt{2Bn}\begin{pmatrix} |m\rangle \\ \mp i|m-1\rangle \end{pmatrix}. \tag{211}$$

Suppose we require that the system be CT symmetric. Now in this basis, charge conjugation $C : \psi \mapsto C\psi^\dagger$ needs to satisfy $[C^\dagger(X, Y)C]^T = -(X, Y)$, and so we can take $C = Y$. If T acts as $T : \psi \mapsto Y\psi$, $i \mapsto -i$, then CT acts just as $\psi \mapsto \psi^\dagger, i \mapsto -i$. Note that $CT : \psi_{\pm m} \mapsto \psi_{\mp m}$, as expected for particle-hole symmetry.

Let $|+\rangle$ be the many-body state with the zero mode filled, and $|-\rangle$ be the many-body state with the zero mode unfilled. Since the charge operator $e^{iQ} = e^{i\int dx\, dy\, \psi^\dagger\psi}$ commutes with CT, by acting on $|\pm\rangle$ with CTe^{iQ} in two different ways (directly, or by moving the e^{iQ} to the left first), we get $e^{iq_+} = e^{-iq_-}$. Since $q_+ = q_- + 1$ (because $C : |\pm \leftrightarrow |\mp\rangle$ and because $e^{iQ}C = Ce^{-iQ}$), we then conclude that CT symmetry implies $q_\pm = \pm 1/2$. If we break C symmetry with e.g. $\delta H \propto \gamma^\dagger\gamma\mathbf{1}$, then since the charge is quantized, the charge assignment of the zero modes will remain unchanged. The fact that the preservation of T symmetry implies a charge of $\pm 1/2$ for the monopole can also be inuited from fact that T symmetry can be preserved in this model by adding "$\frac{1}{8\pi}\int AdA$" to the Lagrangian.

11 *Higgs effective potential* ✓

This is essentially a problem from Schwartz's QFT book, chapter 34. Consider a theory where a scalar field ϕ is coupled to some other fields. We want to calculate, to 1-loop, the contribution of these other fields to the effective potential for ϕ. As an example, consider the example where the other field is a fermion ψ, with the action

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi - V(\phi) + i\bar{\psi}\partial\!\!\!/\psi - Y\phi\bar{\psi}\psi. \tag{212}$$

We will find the effective potential for the ϕ field, including the contributions from the fermions and the self-coupling of the ϕ field, and will show how this generalizes when we couple ϕ to arbitrarily many different fields.

First let's recall how the effective action approach works in the usual QFT-flavored discussion. We will be working just to 1-loop order throughout.

We first include a current term $\int J\phi$ in the action, and then expand the action about the vev of ϕ, writing $\phi \mapsto \eta + \varphi$, with $\varphi = \langle\phi\rangle_J$. The generating functional of connected correlation functions is then

$$e^{-iW[J]} = \int \mathcal{D}\eta \exp\left(i\int\left[\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi+\eta] + (J_1+\delta J)(\varphi+\eta) + \frac{\delta\mathcal{L}}{\delta\phi}\Big|_\varphi \eta + \frac{1}{2}\int \eta\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi\eta\right]\right) \tag{213}$$

Here $\delta\mathcal{L}$ contains counterterms, and we've stopped the expansion of the Lagrangian at quadratic order since we will only be interested in 1-loop effects when integrating out η. As in P&S, J_1 is defined to be the current such that $\delta_\phi\mathcal{L}|_\varphi = -J_1$, so that J_1 is the current which in the classical limit gives an expectation value of φ. We then adjust the "counterterm" δJ order-by-order in perturbation theory to ensure that the expectation value of ϕ in the presence of the current really is φ, i.e. that the expectation value is not changed by a nonzero $\langle\eta\rangle$. This means that δJ is chosen to precisely cancel the tadpole diagrams involving η, and so we can write

$$e^{-iW[J]} = \int \mathcal{D}\eta \exp\left(i\int\left[\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi+\eta] + (J_1+\delta J)\varphi + \frac{1}{2}\int \eta\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi\eta\right]\right) \tag{214}$$

Doing the integral over η gives

$$-W[J] = \int\left(\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi] + J\varphi + \frac{i}{2}\mathrm{Tr}\left[\ln -\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi\right]\right), \tag{215}$$

Using the definition of the effective action as $\Gamma[\varphi] + W[J] = -\int J\varphi$, we get

$$\Gamma[\varphi] = \int\left(\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi] + \frac{i}{2}\mathrm{Tr}\left[\ln -\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi\right]\right), \tag{216}$$

which is independent of J as required (and which again is only correct to 1-loop).

Let's now consider what happens when we have a scalar coupled via a Yukawa coupling to a massless fermion, with the Lagrangian written in the problem statement. Taking $\phi \mapsto \varphi+\eta$ and then dropping the terms linear in η (since they get canceled in the effective action by the condition on the current), we see that after doing the integral over the fermions,

$$\Gamma[\varphi] = \int \left(\frac{1}{2}\varphi\Box\varphi - V(\varphi) + \delta\mathcal{L}[\varphi] + \frac{i}{2}\mathrm{Tr}\left[\ln\left(-\Box + \frac{\delta^2 V}{\delta\phi\delta\phi}\Big|_\varphi\right)\right] - i\mathrm{Tr}\left[\ln(-i\partial\!\!\!/ + Y\varphi)\right]\right). \tag{217}$$

Taking φ to be a constant lets us evaluate the traces. Let

$$m_\varphi^2 \equiv V''(\varphi) \tag{218}$$

be the effectivce mass-squared of the φ field (it may or may not be positive). Then the bosonic trace is (here V is the spacetime volume)

$$\mathrm{Tr}[\ln(-\Box + m_\phi^2)] = VI, \qquad I = \int_p \ln(-p^2 + m_\phi^2). \tag{219}$$

The integral becomes convergent if we go to Euclidean time and differentiate it three times wrt m_ϕ^2:

$$\partial^3_{m_\phi^2} I \to i\frac{2\cdot 2\pi^2}{16\pi^4}\int_0^\infty dp\,\frac{p^3}{(p^2+m_\phi^2)^3} = \frac{i}{8\pi^2}\int_0^\infty du \frac{u}{(u+m_\phi^2)^3} = \frac{1}{16\pi^2 m^2}, \tag{220}$$

where the $\to$ means that we rotated to $i\mathbb{R}$ time to do the integral. Integrating three times then gives

$$I = \frac{i}{16\pi^2}(Am_\varphi^4 + Bm_\varphi^2 + C + [m_\varphi^4/2]\ln(m_\varphi^2/\phi_R^2)), \tag{221}$$

where A, B, C are some (infinite) constants—they will be dealt with using the counterterms. ϕ_R is a dimensionful scale introduced during the renormalization process, and is fixed as one of our renormalization conditions. Since the trace we just computed appeared in the effective action with a 1/2 coefficient, the effective potential is ($V_{eff}[\varphi] = \Gamma[\varphi]/V$)

$$V_{eff}[\varphi] = C + V_R[\varphi] + \frac{1}{64\pi^2}m_\varphi^4\ln(m_\varphi^2/\phi_R^2) + \dots, \tag{222}$$

where C is a cosmological constant, $V_R[\varphi]$ is the renormalized potential (into which the terms $Am_\varphi^4 + Bm_\varphi^2$ have been absorbed), and the $\dots$ signify the contribution from the fermions (note that C and the renormalized parameters in $V_R[\varphi]$ will need to be adjusted further after we calculate the fermion contribution, which will have its own divergences as well).

Now we need to deal with the fermion contribution. We write (dropping an infinite constant)

$$\mathrm{Tr}[\ln(-i\partial\!\!\!/ + Y\varphi)] = V\int_p \mathrm{Tr}[\ln(1 - p\!\!\!/ Y\varphi/p^2)] = V\int_p \sum_{n=0}^\infty \frac{1}{n}\mathrm{Tr}[(p\!\!\!/ /p^2)^n](Y\varphi)^n. \tag{223}$$

Now the trace can be evaluated as

$$\mathrm{Tr}[(\not{p}/p^2)^n] = \begin{cases} 4p^{-n}, & n \in 2\mathbb{Z} \\ 0 & n \in 2\mathbb{Z}+1 \end{cases}. \tag{224}$$

The "proof" of this is as follows: if n is odd then we are taking a trace of an odd number of γ matrices, which vanishes since we are in four dimensions. If n is even, we have to calculate the sum

$$\sum_{\mu_1 \cdots \mu_n} \mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] p_{\mu_1} \cdots p_{\mu_n}. \tag{225}$$

Now this sum will only be nonzero if all of the μ_i group off in pairs, so that $\mu_i = \mu_j$ for some pair i, j. The trace will then produce $\pm\mathrm{Tr}[\mathbf{1}] = \pm 4$, depending on the way in which the indices get paired up. The number of ways N_n to pair up indices for a given n can be calculated inductively. For $n = 2$, there is only one way ($\mu_1 = \mu_2$), means $N_2 = 1$ and $\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}] = 4\eta^{\mu_1\mu_2}$. For $n = k+2$, the μ_1 index can be paired with $k+1$ different other indices. Once this decision is made, the remaining k indices can pair with each other in N_k ways, and so using the base case, we see that

$$N_{k+2} = (k+1)N_k \implies N_n = \prod_{i=0}^{n/2-1} (2i+1). \tag{226}$$

Now most of these N_n different pairings give different signs for the trace, and cancel out. However, since N_n is always odd, this cancellation is never complete, and always leaves behind one term. So then

$$\sum_{\mu_1 \cdots \mu_n} \mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] p_{\mu_1} \cdots p_{\mu_n} = 4\eta^{\mu_1\mu_2} \cdots \eta^{\mu_{n-1}\mu_n} p_{\mu_1} \cdots p_{\mu_n} = 4p^n, \tag{227}$$

which proves the claim.

Using this identity, we have

$$\mathrm{Tr}[\ln(-i\not{\partial}+Y\varphi)] = 4V \int_p \sum_{n \in 2\mathbb{Z}^{\geq 0}} \frac{1}{n}(Y\varphi/p)^n = 2V \int_p \ln(1-Y^2\varphi^2/p^2) = 2V \int_p \ln(-p^2+Y^2\varphi^2), \tag{228}$$

where we "un-dropped" the infinite constant from $\mathrm{Tr}\ln i\not{\partial}$ in the last step (this whole rigamarole of expanding the log and then un-expanding it was just to deal with the spin trace). Thus we are back to calculating the same integral as we did when we integrated out the bosonic field: we get

$$\mathrm{Tr}[\ln(i\not{\partial} - Y\varphi)] = \frac{1}{16\pi^2}(Y\varphi)^4 \ln\frac{(Y\varphi)^2}{\Lambda^2} + \ldots, \tag{229}$$

where the $\ldots$ are polynomials in $Y\varphi$ with infinite coefficients that will go into determining the correct counterterms to be used for determining the renormalized coupling constants. Keeping track of the sign that the functional determinant entered into $\Gamma[\varphi]$ with, we then have

$$V_{eff}[\varphi] = C' + V_R[\varphi] + \frac{1}{64\pi^2} m_\varphi^4 \ln(m_\varphi^2/\phi_R^2) - \frac{1}{16\pi^2}(Y\varphi)^4 \ln(Y^2\varphi^2/\phi_R^2). \tag{230}$$

Note how the contribution from the fermion has a numerical coefficient that is a factor of 4 greater than the boson one, and is negative. The factor of 4 ultimately comes from the fact that a Dirac fermion has 4 components; the minus sign is from the properties of fermionic functional integration. From these remarks, it is clear how to generalize the above potential to include arbitrarily many fields that couple to ϕ: let Ξ_i be a field that couples to ϕ through some interaction $\mathcal{L}_{\Xi_i\phi}$. Then after we expand about φ and drop tadpoles, the one-loop the integral over Ξ_i will produce a functional determinant, with the analogue of m_φ or $Y\varphi$ being defined as

$$m_{\Xi_i} = \frac{\delta^2\mathcal{L}_{\Xi_i\phi}}{\delta\Xi_i\delta\Xi_i}\Big|_{\phi=\varphi} \tag{231}$$

if Ξ_i is a fermion (with Yukawa coupling to ϕ), or

$$m^2_{\Xi_i} = \frac{\delta^2\mathcal{L}_{\Xi_i\phi}}{\delta\Xi_i\delta\Xi_i}\Big|_{\phi=\varphi} \tag{232}$$

if Ξ_i is a boson, with a coupling to ϕ like $\mathcal{L}_{\Xi_i\phi} \sim \phi^2\Xi_i^2 + \dots$. Then the effective potential becomes

$$V_{eff}[\varphi] = C + V_R[\varphi] + \sum_i (-1)^{2\sigma_i}\frac{n_i}{64\pi^2}m^4_{\Xi_i}\ln\frac{m^2_{\Xi_i}}{\phi_R^2}, \tag{233}$$

where σ_i is the spin of Ξ_i and n_i is the number of real dof that Ξ_i carries (e.g. 1 for a boson, 8 for an $SU(2)$ fundamental Dirac fermion, etc.).

One more helpful remark to make about the effective potential. Recall that the physical meaning of $V_{eff}[\varphi]$ is the minimum energy density state of the theory, given that the expectation value of ϕ is fixed at φ. Depending on the choice of potential $V_R[\phi]$, there is no reason that m^2_φ (going back to the case of a single scalar field) should always be positive—indeed, for the usual Mexican hat potential, it is negative near $\varphi = 0$. Then from the above, we see that the argument of the logarithm is negative, and we get an imaginary effective potential. What does this mean? Actually, the meaning is quite physical: an imaginary part means that the time evolution factor $e^{-iV_{eff}[\varphi]T}$ is exponentially damped[11], which means the state with $\langle\phi\rangle = \varphi$ is unstable. This is totally reasonable, since e.g. for the Mexican hat potential, this tells us that regions where $V''[\varphi] < 0$ are unstable: we can't have a theory where the field has zero expectation value, since such a theory is unstable. What's happening here is the ϕ^4 analogue of particle production by strong electric fields with $\langle E^2\rangle > 2m_ec^2$: forcing the vev of the electric field to be too high results in an unstable state, and screening will occur until the vev of E^2 is brought down. This is a quantum effect, so we needed to calculate the 1-loop contribution to $V_{eff}[\varphi]$ in order to see it.

As another super simple example of how this works, consider a quantum particle moving in a potential $V(x)$. Following the procedure above, we get an effective potential of (here x, like φ, is just a number, not a coordinate to be integrated over in the path integral)

$$V_{eff}(x) = V(x) + \frac{1}{2}\int dk\,\ln(k^2 + V''(x)). \tag{234}$$

[11]to check that the sign is such that it is indeed damped, we need to go back and keep track of $i\epsilon$ factors. However clearly the opposite possibility, viz. that it grows exponentially, is obviously not physical.

The integral is divergent but can be made convergent by differentiating once with respect to V''. Then integrating with respect to V'', we get (may or may not have gotten numerical factors right)

$$V_{eff} = V_R(x) + \frac{1}{16\sqrt{\pi}}\sqrt{V''(x)}, \tag{235}$$

where we absorbed a (divergent; unimportant) constant into $V(x)$ and wrote the result as $V_R(x)$. The point is that if $V''(x) < 0$, the effective potential is imaginary (keeping track of the $i\epsilon$ would inform us that the imaginary part is negative), and tells us that the particle decays with a decay rate that goes as $\sqrt{V''(x)}$, since the time evolution of the particle at this position is damped by $\sim e^{-\sqrt{V''(x)}}$ if $V''(x) < 0$. This e.g. happens for a quantum particle in the Mexican hat potential when we consider small x. Classically the particle can balance on the maximum at $x = 0$; quantum mechanically it cannot. Of course this is obvious, but here we have actually computed the precise degree to which it cannot!

12 *Freedom of the Schwinger model without bosonization*

Today we're reading the original Schwinger model paper [?] and seeing how he was able to derive the spectrum without using bosonization. The original paper is a bit abstruse in some aspects, so we will just elaborate on some things and provide details of calculations that aren't in the paper (we're also using different notation, so watch out).

The argument goes in two steps. First, we compute an exact expression for the current, and then we use this and the ward identity to derive the spectrum.

By considering the action coupled to sources as $S \ni \int(\bar{J}\psi + \bar{\psi}J)$, shifting ψ by $i\not{D}_A^{-1}J$ gives

$$Z[J] = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i\int_x \left[\bar{\psi}i\not{D}_A\psi + \frac{1}{2e^2}|F_A|^2\right] - \int_{x,y} \bar{J}(x)\mathcal{G}(x-y)J(y)\right), \tag{236}$$

where the exact Green's function for ψ is $\mathcal{G}$, with

$$\not{D}_A(x)\mathcal{G}(x-y) = \delta(x-y). \tag{237}$$

The thing that's special about 1+1D is that we can actually get a tractable expression for $\mathcal{G}$, at least at small $|x-y|$. Indeed, take the ansatz

$$\mathcal{G}(x-y) = G_0(x-y)e^{i(\phi(x)-\phi(y))}, \tag{238}$$

where $\phi(x)$ is a "Wilson line function" such that $\not{\partial}\phi = \not{A}$ (ϕ is a matrix in spin space), and $G_0(x-y)$ is the free propagator, i.e.

$$\not{\partial}_x G_0(x-y) = \delta(x-y) \implies G_0(x-y) = \int_p e^{ip\cdot(x-y)} \frac{-i\not{p}}{p^2}. \tag{239}$$

This ansatz works, since

$$(\not{\partial} - i\not{A})\left(G_0(x-y)e^{i(\phi(x)-\phi(y))}\right) = \left(\delta(x-y) + G_0(x-y)[i\not{\partial}\phi(x) - i\not{A}]\right)e^{i(\phi(x)-\phi(y))} = \delta(x-y). \tag{240}$$

Now we can get an exact expression for the current, using point splitting. We have

$$j^\mu(x) = -\lim_{\epsilon\to 0}\langle\psi_\alpha(x+\epsilon/2)\bar{\psi}_\beta(x-\epsilon/2)\rangle\gamma^\mu_{\beta\alpha}\exp\left[-i\int_{x-\epsilon/2}^{x+\epsilon/2} d\lambda^\mu\, A_\mu(\lambda)\right], \tag{241}$$

where the Wilson line has been inserted to maintain gauge invariance. So this is then

$$j^\mu(x) = \lim_{\epsilon\to 0}\left(\mathrm{Tr}[\mathcal{G}(\epsilon)\gamma^\mu](1 + i\epsilon^\mu A_\mu + \dots)\right). \tag{242}$$

Using our expression for $\mathcal{G}$, and using that

$$G_0(x) = -\not{\partial}_x \int_p \frac{e^{ip\cdot x}}{p^2} = -\frac{1}{2\pi}\not{\partial}[\ln(|x|\Lambda) + \dots] = -\frac{1}{2\pi}\frac{\not{x}}{|x|^2}, \tag{243}$$

we find, taking the trace,

$$j^\mu(x) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\frac{\epsilon_\nu}{\epsilon^2}(2g^{\mu\nu}[1 - i\epsilon^\lambda A_\lambda(x)] + i\epsilon^\lambda \mathrm{Tr}[\gamma^\mu\gamma^\nu\partial_\lambda\phi(x)]\right). \tag{244}$$

Taking the limit symmetrically, (and removing a factor of i that I must have goofed on)

$$j^\mu(x) = -\frac{1}{\pi}A^\mu(x) + \frac{1}{2\pi}\mathrm{Tr}[\partial^\mu\phi(x)]. \tag{245}$$

We can deal with the ϕ term by hitting both sides of $\not{\partial}\phi = \not{A}$ with $\not{\partial}$ and taking the trace. On one hand,[12]

$$\mathrm{Tr}[(\not{\partial})^2\phi] = \Box\mathrm{Tr}[\phi], \tag{247}$$

[12] Here $\Box = -\partial_\mu\partial^\mu$ is positive-definite. In terms of differential forms, the positive-definiteness means the sign is fixed as $\Box = +(d^\dagger d + dd^\dagger)$, which is positive definite since, letting $A = d\alpha + d^\dagger\beta + \omega$ with ω harmonic,

$$\int A\wedge\star\Box A = \int(d\alpha + d^\dagger\beta)\wedge\star(d^\dagger dd^\dagger\beta + dd^\dagger d\alpha) = ||dd^\dagger\beta||^2 + ||d^\dagger d\alpha||^2 \tag{246}$$

while on the other hand,

$$\mathrm{Tr}[\partial\!\!\!/ A\!\!\!/] = 2d^\dagger A. \tag{248}$$

Therefore

$$\mathrm{Tr}[d\phi] = 2d\Box^{-1}d^\dagger A, \tag{249}$$

and so, as differential forms,

$$j = -\frac{1}{\pi}(1-d\Box^{-1}d^\dagger)A = -\frac{1}{\pi}\Box^{-1}(\Box-dd^\dagger)A = -\frac{1}{\pi}\Box^{-1}d^\dagger dA \implies j^\mu = -\frac{1}{\pi}[\Pi_T]^{\mu\nu}A_\nu, \tag{250}$$

where Π_T is the transverse projector. Because of the presence of the projector, this result is manifestly conserved and gauge-invariant. Now in this expression, A^μ is a dynamical field, and this expression only makes sense inside of $\int \mathcal{D}A$. However, we can pull it out the path integral by coupling the gauge field to a source current J_μ and then taking $A \mapsto -ie\delta_J$.[13] Therefore we can write

$$j^\mu(x) = i\frac{e}{\pi}[\Pi_T]^{\mu\nu}\frac{\delta}{\delta J}Z[J], \tag{252}$$

where $j^\mu(x)$ is now the current in the presence of the source J, which may or may not be turned off after taking the functional derivative. Note that regardless, we will always need to have $d^\dagger J = 0$ so as to retain gauge invariance.

Anyway, while the form of the current looks simple, note that this is a *non-perturbative* result! It's also the result we'd have gotten if we computed the current using the usual 1-loop polarization bubble diagram—evidently the 1-loop result is exact, something which is made possible by a miraculous cancellation between all the other diagrams. Also, note that this result is compatible with the chiral anomaly: taking $j \mapsto \star j$ gives

$$j_A = -\frac{1}{\pi}(d^\dagger)^{-1}\star F, \tag{253}$$

giving the correct result for $d^\dagger j_A$. In fact, knowledge that the chiral anomaly is 1-loop exact, plus the above formula, would have also been a sufficient starting point to derive our expression for j^μ, since in two dimensions the vector and axial currents are just $\star$s of one another. This fact makes the cancellation of all the diagrams in the computation of j^μ less mysterious.

Now consider the consequences of the Ward identity for changing variables in the $\mathcal{D}A$ measure. The Ward identity reads (I have a sign difference from the original paper, but I think this is due to different metric signature choices)

$$\left(-i\frac{1}{e}d^\dagger d\frac{\delta}{\delta J^\mu} + (J_\mu + j_\mu)\right)Z[J] = 0, \tag{254}$$

[13]The e here appears since in our conventions the JA coupling is

$$S \ni ie^{-1}\int J\wedge\star A. \tag{251}$$

where here $j_\mu(x)$ is to be understood as a function of $-i\delta/\delta J^\mu(x)$. Now since both currents are conserved, we can freely insert the projector Π_T in front of the $J+j$ term. Plugging in our expression for the current, and writing the $d^\dagger d$ in the above equation as $\Pi_T\Box$, we have

$$[\Pi_T]^{\mu\nu}\left(-i\Box\frac{\delta}{\delta J^\nu}+i\frac{e^2}{\pi}\frac{\delta}{\delta J^\nu}+J_\nu\right)Z[J]=0, \tag{255}$$

and so, defining the massive Greens function

$$\mathcal{G}_m=\frac{1}{\Box-m^2}, \qquad m^2\equiv e^2/\pi, \tag{256}$$

we can multiply by $\mathcal{G}_m$ (since $\Box$ commutes with Π_T) and conclude that the generating functional obeys

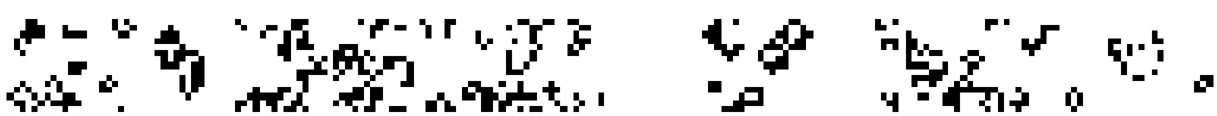
$$i[\Pi_T]^{\mu\nu}\frac{\delta}{\delta J^\nu(x)}Z[J]=\int_y[\Pi_T]^{\mu\nu}\mathcal{G}_m(x-y)J_\nu(y)Z[J]. \tag{257}$$

Solving this, we conclude that the generating functional of connected correlation functions has the *exact* expression

$$W[J]=\frac{1}{2}\int_{x,y}J_\mu(x)[\Pi_T]^{\mu\nu}\mathcal{G}_m(x-y)J_\nu(y). \tag{258}$$

Therefore the only nonzero connected correlation function in the theory is the two-point function, given by $\mathcal{G}_m(x-y)$. Therefore the model is exactly equivalent to a free massive scalar of mass m. Since $\mathcal{G}_m$ is a massive propagator, the above equation tells us that two (probe) charges stuck at positions x,y see an exponentially screened potential—this is kind of what happens after the photon is made massive through Higgsing, but of course here there was no photon to begin with, since we are in 1+1D (and likewise, there was no Goldstone boson, since although $\bar\psi\psi$ gets a vev, there is no Goldstone since the symmetry doesn't exist due to the chiral anomaly).

A natural question to ask is to what extent these features are changed when we add a mass (which ruins the soluability of the model). When we have a mass we can also have a θ term, which complicates things, but basically what happens is that the model ceases to be in a "Higgs" regime—the potential between incommensurate charges (like $J=\frac{1}{2}\widehat{C}-\frac{1}{2}\widehat{C}'$ for some curves C,C') *is* long-ranged, and leads to confinement of these charges—but also that there is screening, so that the potential between sources with integral charges is finite-ranged. More on this in a later diary entry.



13 $SO(3)$ monopoles and zero modes

Since $\pi_1(SO(N)) = \mathbb{Z}_2$, $SO(N)$ gauge theories have $\mathbb{Z}_2$ monopoles. Consider e.g. an $SO(3)$ gauge theory coupled to a Dirac fermion (in the spin 1 representation) on a spatial S^2, T^2, etc. Does the Hamiltonian generically have zero modes?

We can basically answer this question by using what we know about the index of $\not{D}_A$. In two dimensions (spatial dimensions, or 1+1D in Euclidean time), the Dirac operator $\not{D} : \Gamma(S^\pm \otimes E \to) \to \Gamma(S^\mp \otimes E)$ has a pseudoreal structure if the associated gauge bundle $E = P_G \times_\rho \mathbb{C}^{\dim \rho}$ is such that the representation ρ is either real or pseudoreal (here P_G is a principal G-bundle, with G the gauge group).

The reason for this is as follows: if ρ is real, then we can choose a connection such that $\not{A}^a T^a \mathcal{K} = -\mathcal{K} \not{A}^a T^a$, where A is the gauge connection (we are working in physicist conventions where the T^a are Hermitian). Then choosing the γ matrices to be X, Y, we see that the matrix $\mathscr{J} = \mathcal{K}(J \otimes \mathbf{1})$ (notation is spin $\otimes$ gauge) is such that

$$i\not{D}_A \mathscr{J} = \mathscr{J}(i\not{D}_A), \qquad \mathscr{J}^2 = -\mathbf{1}. \tag{259}$$

If ρ is pseudoreal, then we can find a unitary U_G such that

$$\not{A}^a T^a \mathcal{K} U_G = -\mathcal{K} U_G \not{A}^a T^a, \qquad (\mathcal{K} U_G)^2 = -\mathbf{1}. \tag{260}$$

Then we can choose the γ matrices to be the real matrices X, Z, which tells us that the operator $\mathscr{J}' = \mathcal{K}(\mathbf{1} \otimes U_G)$ satisfies

$$i\not{D}_A \mathscr{J}' = -\mathscr{J}'(i\not{D}_A), \qquad (\mathscr{J}')^2 = -\mathbf{1}. \tag{261}$$

Therefore if ρ is not complex, we can find an antilinear operator $\mathcal{J}$ that squares to $-\mathbf{1}$ and either commutes or anticommutes with $i\not{D}_A$. This means that if $\not{D}_A \psi = 0$, then $\mathscr{J}\psi$ is also a zero mode. Since $\mathscr{J}\psi$ has opposite chirality to ψ,[14] the index of $\not{D}_A$ must vanish. Therefore if the fermions transform in a representation of the gauge group that is not complex, ind $\not{D}_A = 0$, and there is nothing that protects zero modes, if they do exist, from being lifted.[15] Therefore in a generic situation, we expect no zero modes. The remainder of the diary entry is just an attempt to confirm this and to make sure we aren't missing any other symmetry that might protect the zero modes from being lifted.

[14] In the case where ρ is real, this is clear since the J tensor factor in $\mathscr{J}$ is off-diagonal, and in this basis $\bar{\gamma} = Z$. For the pseudoreal case, the choice of γ matrices means $\bar{\gamma} = Y$, so that eigenspinors of $\pm$ chirality look like $(1, \pm i)^T$. The complex conjugation in $\mathscr{J}$ exchanges these, and hence $\mathscr{J}$ anticommutes with $\bar{\gamma}$.

[15] Of course, another way to derive this would just have been to say that since in two dimensions the only gauge-invariant 2-forms for the gauge curvature that are non-vanishing are those from $U(1)$ groups, so that gauge groups like $SO(N)$ can make no contribute to ind $\not{D}_A$, by the index theorem. However the argument given in the main text doesn't depend on taking the index theorem for granted, which I think is nice.

We should also point out that this conclusion is rather special to two dimensions. This is because the γ matrices in two dimensions admit both a real and a pseudoreal structure, which meant that we could get a pseudoreal structure for the full connection on $S \otimes E$ with either a real or pseudoreal gauge connection. We also used the fact that the pseudoreal structure $\mathscr{J}$ anticommutes with $\bar{\gamma}$; in four dimensions this is not true (more on this later).

On the plane / torus

We will now specialize to $SO(3)$ gauge theory with 1 Dirac fermion in the spin 1 representation. On the plane / torus, we will work in Landau gauge, where $A_x = 0$ and A_y is a function of x only. Then $\not{D}_A \psi = 0$ reads

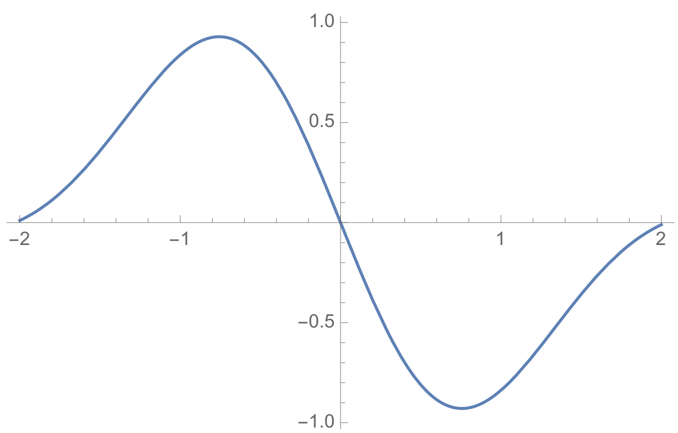
$$(\partial_x - k_y + A_y)\psi_L = 0, \qquad (\partial_x + k_y - A_y)\psi_R = 0, \tag{262}$$

where k_y is the y component of the momentum. For a uniform $U(1)$ flux, we would take $A_y = Bx$, where B is the flux density. For $SO(3)$, we let $A_y = BxT^3$.[16] After diagonalizing T^3 from $Y \oplus 0$ to $T^3 = Z \oplus 0$ and writing the spinors in flavor space as (f_1, f_2, f_3) with each f_i a two-component spinor, this gives a left-handed zero mode $(f_L, 0, 0)$ and a right-handed zero mode $(0, f_R, 0)$.

We would like to know whether these zero modes still exist after we perturb with some field strength that does not point uniformly in one direction in flavor space.[17] To do this, consider as an example adding the connection $\widetilde{A} = \widetilde{A}_y dy$, where

$$\widetilde{A}_y = \widetilde{A}_y^2 T^2 = \epsilon \cos(x) e^{-x^2/2} T^2. \tag{263}$$

This has a field strength which has zero integral over the plane[18], and so it is topologically trivial. The field strength as a function of x looks like



(264)

[16]For other gauge groups G with $\pi_1(G) \neq 0$, we can just take $A_y = A_y^a T^a = BxT^a$ for some particular generator T^a.

[17]This wording isn't very precise, since the fact that F transforms adjointly under gauge transformations means that we can perform a gauge transformation to take our uniform flux field to one in which $F_{\theta\phi}(\theta, \phi)$ has constant $\mathrm{Tr}[F \wedge \star F]$, but which has a direction in flavor space that is an arbitrary function of θ, ϕ. So we are really interested in making a perturbation that changes $\mathrm{Tr}[F \wedge \star F]$.

[18]Or torus. If we are on the torus, we take it to be big enough that the usual $e^{-(x-k_y/B)^2 B/2}$ zero mode wavefunctions have support only within width δx which is small compared to the size of the torus, so that the fact that the above wavefunctions are technically speaking not smooth over the torus doesn't really matter.

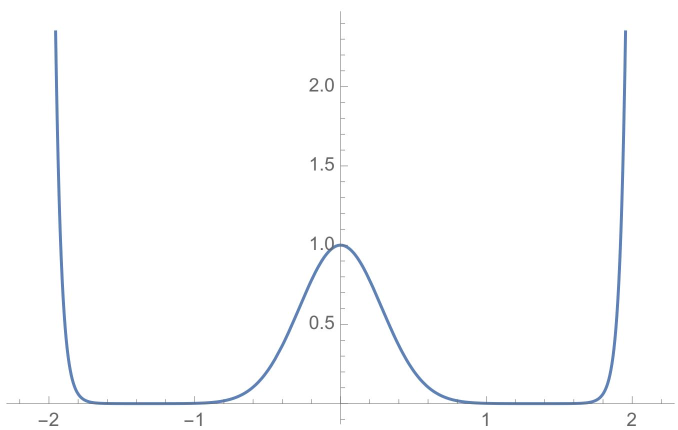
After diagonalizing T^3, the matrix T^2 in flavor space becomes

$$T^2 = \begin{pmatrix} & i \\ -i & \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} & & i \\ & & i \\ -i & -i & \end{pmatrix}. \tag{265}$$

Now since the fermions are not in a complex representation, we have ind $\not{D}_A = 0$ and we know that there will always be as many left zero modes as right zero modes.[19] Therefore to see whether adding the $\widetilde{A}_y$ term to the connection does anything to $\ker i\not{D}_A$, we can focus wolog on a certain chirality, which we will take to be L for definiteness. Therefore we are interested in whether we can find normalizable solutions to the following equations (setting $k_y = 0$ for simplicity)

$$\begin{gathered} (\partial_x + Bx)f_1 + i\frac{\widetilde{A}_y}{\sqrt{2}}f_3 = 0 \\ (\partial_x - Bx)f_2 + i\frac{\widetilde{A}_y}{\sqrt{2}}f_3 = 0 \\ \partial_x f_3 - i\frac{\widetilde{A}_y}{\sqrt{2}}(f_1 + f_2) = 0. \end{gathered} \tag{267}$$

If $\widetilde{A}_y = 0$ then we just take $f_2 = f_3 = 0$, and let f_1 be the usual harmonic oscillator solution. However, if $\widetilde{A}_y \neq 0$, this is not possible: the last equation means that either f_2 or f_3 must be nonzero if f_1 is nonzero, and the second equation then ensures that in fact $f_2 \neq 0$. Now f_2 is the mode that doesn't have a normalizable solution when $\widetilde{A}_y = 0$, and so we might expect that the $\widetilde{A}_y$ coupling ruins the normalizability of the solution. Indeed, this is what appears to happen: using the form of $\widetilde{A}_y$ above with $B = 2\pi$ and $\epsilon = .1\sqrt{2}$, a plot of the magnitude $\sum_i f_i^* f_i$ as a function of x shows a divergence:



(268)

[19] An operator that provides the (pseudo)real structure here is $\mathcal{K}(Y \otimes \mathbf{1})$, where the first tensor factor is for the spin indices and the second is for the gauge indices. Indeed, as in the previous section, using X and Y as the γ matrices, and working in a basis where the gauge generator matrices are purely imaginary and antisymmetric, we have

$$[i\not{D}_A, \mathcal{K}(Y \otimes \mathbf{1})] = 0, \tag{266}$$

and so $\mathcal{K}(Y \otimes \mathbf{1})$ provides a way to take a zero mode of a certain chirality and construct another zero mode with opposite chirality.

There of course may be something I've missed, or some tricky choice of initial conditions (the above plot was for $f_1(0) = 1, f_2(0) = f_3(0) = 0$; modifying the latter two to be nonzero makes the divergence worse) that allow this divergence to be avoided, but for now it seems to be a generic consequence of taking $\widetilde{A} \neq 0$.

On the sphere

We first need to choose a gauge connection. For a $U(1)$ monopole of flux n, the standard choice is

$$A^{N/S} = n\frac{\pm 1 - \cos\theta}{2} d\phi, \tag{269}$$

which gives $\int_{S^2} F = 2\pi n$. For a gauge group with $\pi_1(G) \neq 0$, the simplest choice for a monopole field is the above but with a T^a tacked on, where T^a is a particular (Hermitian) generator of $\mathfrak{g}$. Using results from our earlier entry on zero modes, the covariant derivatives are

$$\nabla_\theta = \partial_\theta, \qquad \nabla_\phi = \partial_\phi - \frac{iZ\otimes \mathbf{1}}{2}\cos\theta + in\frac{\pm 1 - \cos\theta}{2}\mathbf{1}\otimes T^a, \tag{270}$$

where the first tensor factor is the spin indices and the second is the gauge indices (we won't bother to explicitly write the $\otimes$ in what follows). The expression $i\not{D}_A\psi = 0$, is then, for our uniform monopole field,

$$\not{D}_A\psi^{(N/S)} = \left[X\left(\partial_\theta + \frac{\cot\theta}{2}\right) + Y\csc\theta\left(\partial_\phi + in\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\right]\psi^{(N/S)} = 0 \tag{271}$$

or written out in chiral components,

$$\begin{aligned} &\left(\partial_\theta + \frac{\cot\theta}{2} - i\csc\theta\partial_\phi + n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\psi_R^{(N/S)} = 0 \\ &\left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi - n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\psi_L^{(N/S)} = 0 \end{aligned} \tag{272}$$

In what follows, we will take $n = 1$ for concreteness. Then in the $U(1)$ case, we see that we get a single R zero mode, $\psi_R = e^{-i\phi/2}$ (the reason we get an R zero mode and not an L one is because of our sign conventions for the covariant derivative). Also note that this zero mode actually has spin zero; to see this one needs to properly calculate the angular momentum generators, which I won't go into here.

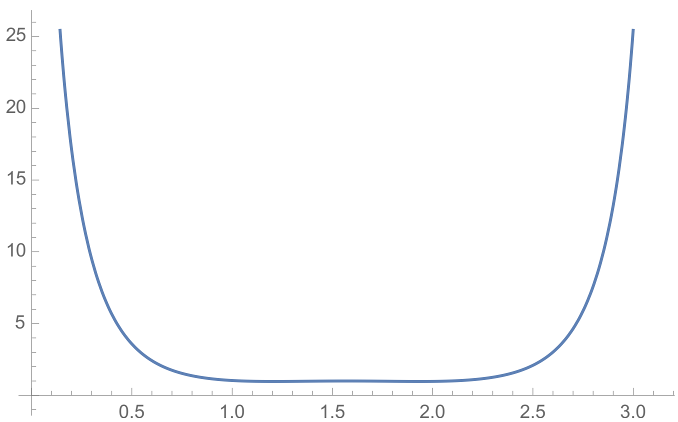
Now for $SO(3)$. If we let the field strength point in the T^3 direction, we see that we get a single L and a single R zero mode, as expected. Do these zero modes survive when a perturbation is added? Let us add the potential

$$\widetilde{A} = \widetilde{A}^2_\phi T^2\, d\phi = \epsilon\sin(2\theta)T^2\, d\phi. \tag{273}$$

This is well-defined on the sphere since $\widetilde{A}(\theta = 0, \pi) = 0$, and it is topologically trivial since $\int_{S^2} d\widetilde{A} = 0$. As mentioned before, since the zero modes for real gauge groups always come in left-right pairs, we can focus on a single handedness (we will look at R) wolog. We therefore want to find normalizable solutions to (working on the N coordinate patch)

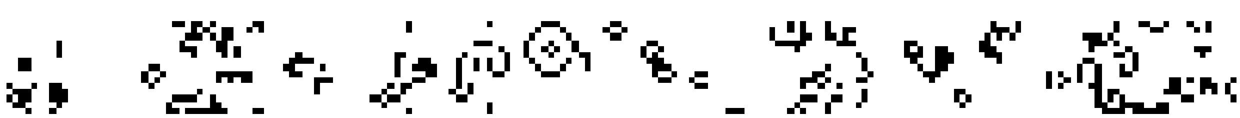
$$\begin{aligned}\left(\partial_\theta - i\csc\theta\partial_\phi + \frac{1}{2}\csc\theta\right)f_1 + i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}f_3 &= 0\\ \left(\partial_\theta + \cot\theta - i\csc\theta\partial_\phi - \frac{1}{2}\csc\theta\right)f_2 + i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}f_3 &= 0\\ \left(\partial_\theta + \frac{1}{2}\cot\theta - i\csc\theta\partial_\phi\right)f_3 - i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}(f_1+f_2) &= 0\end{aligned} \tag{274}$$

When $\epsilon = 0$ we just take $f_1 = e^{-i\phi/2}, f_2 = f_3 = 0$.[20] When $\epsilon \neq 0$ the coupling between the different modes kicks in, and as in the planar case we seem to run into normalizability problems caused by the troublesome modes f_2, f_3 being forced to be nonzero. For example, set $\epsilon = \sqrt{2}$. The natural choices for the ϕ dependence of the three modes is $f_1 \propto e^{-i\phi/2}$, $f_2 \propto e^{i\phi/2}$, and with f_3 having no ϕ dependence. With these assignments of ϕ dependence, the volume-element-normalized magnitude $\sum_i f_i^* f_i \sin\theta$ as a function of θ looks like



(277)

So, it blows up at the poles, and we don't get a legit zero mode solution. This seems to be the generic behavior for any $\epsilon \neq 0$.



[20]Here the f_3 mode has no normalizable solution when $\epsilon = 0$, since taking the ϕ dependence to be trivial means

$$f_3 \propto \frac{1}{\sqrt{\sin\theta}}, \tag{275}$$

which is not acceptable (it integrates to something finite because of the $\sin\theta$ in the measure, but it is not differentible). Similarly the f_2 mode has to be zero, since otherwise we have

$$f_2 \propto e^{i\phi/2}\csc(\theta), \tag{276}$$

which is also no good.

14 *Non relativistic limit in ϕ^4 theory UNFINISHED*

Today we have something super simple that's not usually talked about in QFT books. This is basically an exercise in Duncan's "The conceptual framework of QFT". First, we will show how to take the non-relativistic of ϕ^4 theory, with an attractive ϕ^4 potential (with a very weak ϕ^6 term added for stability reasons, which we won't keep track of). Then we will find the energy of the lowest bound state, which exists in two and three spacetime dimensions.

The action is (mostly negative signature)

$$S = \int (\phi(\Box - m^2)\phi - \lambda\phi^4/6). \tag{278}$$

Our philosophy here will be one of effective field theory. We will assume all the momentum modes of energy $k^0 > m$ have been integrated out, and that the coupling constants in the above Lagrangian are those generated during this integrating out procedure. Define the field

$$\psi(x) = \sqrt{2m}e^{imt}\int_k \theta(k^0)\phi_k e^{-ikx}, \tag{279}$$

The frequency components of ψ are all constrained to be less than m, and so when we write the action in terms of ψ, any times that do not contain an equal number of ψs and $\psi^\dagger$s will vanish, since no combination of frequencies from the Fourier modes of the ψs will be able to cancel the time dependence of the $e^{\pm imt}$ factor such a term will have, and the time integration will kill the term in question. so that

$$\phi(x) = \int_k \theta(k^0)(\phi_k e^{-ikx} + \phi^\dagger_{-k}e^{ikx}) = \frac{1}{\sqrt{2m}}(e^{-imt}\psi(x) + e^{imt}\psi^\dagger(x)). \tag{280}$$

Finally, we can drop $\psi^\dagger\partial_t^2\psi$, since it goes as k_0^2/m, which is negligible in comparison to the other terms. So then we get

$$S = \int \left(\psi^\dagger i\partial_t\psi + \frac{1}{2m}\psi^\dagger\nabla^2\psi + \lambda\psi^\dagger\psi^\dagger\psi\psi\right), \tag{281}$$

which is just what we expect for $p\dot{q} - H$.

Lets now use this action to find, QFT-style, the existence of bound states for attractive potentials, $\lambda < 0$.

15 *Self energy and particle production in fields UNFINISHED*

This is from "The conceptual framework of QFT".

We will be working in $\mathbb{R}$ time and with a mostly negative signature metric.

We write the exact propagator as

$$\mathcal{G}(p^2) = \frac{1}{p^2 - \lambda^2 + i\epsilon + (p^2 + i\epsilon)\Pi(p^2)}, \tag{282}$$

where

$$\Pi(p^2) = \int_0^\infty dm^2 \frac{s(m^2)}{p^2 - m^2 + i\epsilon}. \tag{283}$$

The goal now is to explain what $s(m^2)$ measures. To do this, note that the partition function in the presence of a current J is, for sufficiently weak J,

$$Z[J] \approx \exp\left(\frac{i}{2}\int dx\, dy\, J(x)\mathcal{G}(x-y)J(y)\right). \tag{284}$$

This reproduces the two point function correctly but $\ln Z[J]$ written in this way does not allow us to reproduce the higher-point functions; to do this we would need to go beyond quadratic order in J.

Now define

$$\Phi(x) \equiv \int dy\, \mathcal{G}^{-1}(x-y)J(y). \tag{285}$$

Then to our approximation,

16 *The non-relativistic limit of a Dirac fermion coupled to a $U(1)$ gauge field and magnetic moments*

Today we're doing an elaboration on problem 10.1 in Schwartz, which is something basic that I'd never worked through before. The goal is to take the non-relativistic limit of the Dirac equation and derive how the electron couples to EM fields in this limit. Unfortunately the method which Schwartz suggests in the problem statement doesn't work (at least, I don't think it works) when the electric field is nonzero (due to a tricky sleight of hand involving differential operators and square roots). We will follow the outline of the problem but will use what I think are more careful (but longer) methods to get the electric field dependence right.

Our sign conventions for the Dirac equation will be $(i\hbar c\not{\partial} - ec\not{A} - mc^2)\psi = 0$, so that $i\partial_t\psi = H_D\psi$ means[21]

$$H_D = -i\gamma^0\gamma^j D_j + eA^0 + \gamma^0 mc^2, \qquad D_j = \hbar c(\partial_j + i\hbar^{-1}eA_i). \tag{286}$$

Schwartz tells you to subtract off eA_0 and square, but doing this makes you liable to forget about terms like $\partial_j A_0$ which need to be retained to get e.g. the SOC term and the $\nabla^2 A_0$ term. Choosing the non-Weyl basis (so that the rest energy $\gamma^0 mc^2$ is diagonal)

$$\gamma^0 = Z\otimes \mathbf{1}, \qquad \gamma^j = J\otimes\sigma^j, \tag{287}$$

we have

$$H_D = (Z\otimes\mathbf{1})mc^2 + \phi + i(X\otimes\sigma^j)D_j, \qquad \phi\equiv eA_0. \tag{288}$$

From now on, all $\otimes$s and $\mathbf{1}$s will be omitted: X, Y, Z will be understood to live in the first $\otimes$ factor, and σ^j will be understood to live in the second.

The strategy is now to subject H to a SW transformation and systematically eliminate all the terms which are off-diagonal in the first tensor factor, so that the transformed Hamiltonian contains only things like $\mathbf{1}\otimes(\dots)$ and $Z\otimes(\dots)$. The offending off-diagonal terms will usually come in the form of $X\sigma^j\partial_j$s and $J\sigma^j\partial_j$s. Eliminating these terms will give us a Hamiltonian that we can easily find the spectrum of. We will work in the non-relativistic limit where $p^2/2m \ll mc^2$—this facilitates the diagonalization process described above because the off-diagonal terms will generally go as ratios of momenta to powers of mc^2.

To this end, rewrite the Schrodinger equation as

$$e^\Lambda i\hbar\dot{\psi} = e^\Lambda He^{-\Lambda}(e^\Lambda\psi) \implies i\hbar\partial_t\widetilde{\psi} = \left(e^\Lambda He^{-\Lambda} + i\hbar(\partial_t e^\Lambda)e^{-\Lambda}\right)\widetilde{\psi} = \widetilde{H}\widetilde{\psi}, \qquad \widetilde{\psi}\equiv e^\Lambda\psi. \tag{289}$$

Here Λ is some anti-Hermitian matrix that we will need to solve for. In tomorrow's diary entry we will prove the tools needed to show that when expanded in Λ, the Schrodinger equation for $\widetilde{\psi}$ is

$$i\hbar\partial_t\widetilde{\psi} = \sum_{k=0}^\infty\left(\frac{\mathcal{N}_k}{k!} + i\hbar\frac{\mathcal{C}_k}{(k+1)!}\right)\widetilde{\psi}, \tag{290}$$

where we have defined the nested commutators

$$\mathcal{N}_k \equiv [\Lambda, [\cdots, [\Lambda, H]\cdots]], \qquad \mathcal{C}_k\equiv[\Lambda,[\cdots,[\Lambda,\dot{\Lambda}]\cdots]], \tag{291}$$

[21]The only potentially hard-to-remember assignment of cs and $\hbar$s is the one for the vector potential. A telsa is a kg /(s C), so that eA_j is valued in kg m/s, meaning that ecA_j has units of energy.

where both terms contain k appearances of Λ.

We will now work out an expression for Λ, order-by-order in the non-relativistic limit). Because I'm feeling slightly masochistic tonight, we will go to order $(pc)^4/(mc^2)^3$. This will entail performing three SW transformations—one to push all the p terms to order p^2, one to get to p^3, and then a final one to get to p^4. Now keeping all the terms that can possibly contribute, we have

$$\begin{aligned}\widetilde{H} = H + ([\Lambda, H] + i\hbar\dot{\Lambda}) + \frac{1}{2}([\Lambda,[\Lambda,H]] + i\hbar[\Lambda,\dot{\Lambda}]) + \frac{1}{6}([\Lambda,[\Lambda,[\Lambda,H]]] + i\hbar[\Lambda,[\Lambda,\dot{\Lambda}]]) \\ + \frac{1}{24}([\Lambda,[\Lambda,[\Lambda,[\Lambda,H]]]] + i\hbar[\Lambda,[\Lambda,[\Lambda,\dot{\Lambda}]]]).\end{aligned} \tag{292}$$

Now split up the Hamiltonian as

$$H_D = H_0 + H', \qquad H_0 = \phi + Zmc^2, \qquad H' = iX\not{D}_A. \tag{293}$$

To lowest order, we should look for a Λ such that $[\Lambda, H_0] = -H' + \dots$, where $\dots$ does not contain $\not{D}_A$. Such a Λ is

$$\Lambda_1 = \frac{i}{2mc^2}J\not{D}_A = -\frac{1}{2mc^2}ZH' \implies [\Lambda_1, H_0] = \frac{1}{2}[Z, iJ\not{D}_A] + \Omega = -iX\not{D}_A + \Omega, \tag{294}$$

where we've defined

$$\Omega \equiv [\Lambda_1, \phi] = \frac{i}{2mc^2}J\hbar c\boldsymbol{\sigma}\cdot\nabla\phi. \tag{295}$$

Note that Λ_1 is properly anti-Hermitian, so that e^{Λ} implements a unitary transformation on H. We will also need

$$[\Lambda_1, H'] = \frac{1}{mc^2}Z(\not{D}_A)^2 = 4mc^2Z\Lambda_1^2 \tag{296}$$

and

$$[\Lambda_1, Z\Lambda_1^2] = \frac{iX}{4(mc^2)^3}\not{D}_A^3 = -2Z\left(iJ\frac{\not{D}_A}{2mc^2}\right)^3 = -2Z\Lambda^3. \tag{297}$$

Similarly,

$$[\Lambda_1, Z\Lambda_1^3] = -2Z\Lambda_1^4 \tag{298}$$

To save space, define the rest energy as $\alpha \equiv mc^2$. Then we have

$$\begin{aligned}[\Lambda_1,[\Lambda_1,H]] &= -4\alpha Z\Lambda_1^2 - 8\alpha Z\Lambda_1^3 + [\Lambda_1,\Omega], \\ [\Lambda_1,[\Lambda_1,[\Lambda_1,H]]] &= 8\alpha Z\Lambda_1^3 + 16\alpha Z\Lambda_1^4 + [\Lambda_1,[\Lambda_1,\Omega]] \\ [\Lambda_1,[\Lambda_1,[\Lambda_1,[\Lambda_1,H]]]] &= -16\alpha Z\Lambda_1^4 + [\Lambda_1,[\Lambda_1[\Lambda_1,\Omega]]],\end{aligned} \tag{299}$$

where in the last line we only kept terms of order Λ_1^4. To make the expansion less messy, we will work in the limit where the momenta of both the $\widetilde{\psi}$ field and the EM field are small compared to the rest energy α, so that we're basically doing an expansion in $1/\alpha$, and dropping everything that goes as $1/\alpha^{n\geq 4}$. Therefore in our order of approximation, we plug everything into $\widetilde{H}$ and get

$$\begin{aligned}\widetilde{H} &= H_0 + 2\alpha Z\left(\Lambda_1^2 - \frac{4}{3}\Lambda_1^3 + \Lambda_1^4\right) + \Omega + \frac{1}{2}[\Lambda_1, \Omega] + \frac{1}{6}[\Lambda_1, [\Lambda_1, \Omega]] \\ &\quad + i\hbar\left(\dot{\Lambda}_1 + \frac{1}{2}[\Lambda_1, \dot{\Lambda}_1] + \frac{1}{6}[\Lambda_1, [\Lambda_1, \dot{\Lambda}_1]]\right) \\ &= \widetilde{H}_0 + \widetilde{H}',\end{aligned} \tag{300}$$

where $\widetilde{H}_0$ is diagonal in the first $\otimes$ factor and $\widetilde{H}'$ is off-diagonal.

We aren't done, because there are still terms which are off-diagonal in the first $\otimes$ factor. Therefore we must perform another SW transformation to get rid of these terms. For our first transformation, note that we chose $\Lambda_1 = -\frac{1}{2\alpha}ZH'$, where H' was the term that was off-diagonal and needed to be killed. This prompts us to try a second transformation with

$$\Lambda_2 = -\frac{1}{2\alpha}Z\widetilde{H}' \approx -\frac{1}{2\alpha}Z\left(-\frac{8}{3}\alpha Z\Lambda_1^3 + \Omega + i\hbar\dot{\Lambda}_1\right), \tag{301}$$

where we dropped higher-order terms that are killed by the $1/\alpha$ in front.

To check that this works, one just needs to check that $[\Lambda_2, \widetilde{H}_0] = -\widetilde{H}' + \dots$, where $\dots$ don't involve bare $\not{D}_A$s. For example, let's check that the Λ_1^3 term in $\widetilde{H}'$ gets killed:

$$[\Lambda_2, \widetilde{H}_0] \ni [\Lambda_2, H_0] \ni \frac{4}{3}[\Lambda_1^3, Z\alpha] = \frac{8Z\alpha}{3}\Lambda_1^3, \tag{302}$$

which is exactly the right term needed to kill the $-\frac{8\alpha Z}{3}\Lambda_1^3$ appearing in $\widetilde{H}'$. The full commutator is then checked to be (still to order $1/\alpha^3$)

$$[\Lambda_2, \widetilde{H}_0] \approx -\widetilde{H}' - \frac{4}{3}[\phi, \Lambda_1^3]. \tag{303}$$

Since all the terms in Λ_2 go as at least $1/\alpha^2$, we have

$$e^{\Lambda_2}\widetilde{H}e^{-\Lambda_2} = \widetilde{H}_0 + \frac{1}{2}[\Lambda_2, \widetilde{H}'] - \frac{4}{3}[\phi, \Lambda_1^3]. \tag{304}$$

The commutator is worked out to be, to this order[22]

$$\frac{1}{2}[\Lambda_2, \widetilde{H}'] = -\frac{1}{2\alpha}[Z(\Omega + i\hbar\dot{\Lambda}_1), \Omega + i\hbar\dot{\Lambda}_1] = \frac{Z}{\alpha^3}(\hbar ce)^2(\mathbf{e}\cdot\boldsymbol{\sigma})^2 = \frac{Z}{\alpha^3}(\hbar ce)^2|\mathbf{E}|^2. \tag{306}$$

A final SW transform removes this, at the expense of generating further higher-order terms which are then thrown away. So in conclusion, to this order, the transformed Hamiltonian is

$$H_{\text{eff}} = \phi + Z\alpha + 2\alpha Z(\Lambda_1^2 + \Lambda_1^4) + \frac{1}{2}[\Lambda_1, \Omega] + \frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] + \frac{Z}{\alpha^3}(\hbar c)^2(\mathbf{e}\cdot\boldsymbol{\sigma})^2. \tag{307}$$

[22] To get this, we used

$$\dot{\Lambda}_1 = -\frac{ec}{2\alpha}\boldsymbol{\sigma}\cdot\dot{\mathbf{A}}. \tag{305}$$

Now we just need to calculate the remaining commutators. The first is

$$\frac{1}{2}[\Lambda_1, \Omega] = \frac{1}{2(2\alpha)^2}[\not{D}_A, \hbar c\boldsymbol{\sigma}\cdot\nabla\phi] = \frac{e(\hbar c)^2}{2(2\alpha)^2}\sigma^i\sigma^j\partial_i\partial_j A_0 + \frac{\hbar ce}{2(2\alpha)^2}[\sigma^i,\sigma^j]\nabla_i A_0 D_j. \tag{308}$$

The second is

$$\frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] = \frac{ec\hbar}{2(2\alpha)^2}[\not{D}_A, \boldsymbol{\sigma}\cdot\dot{\mathbf{A}}] = \frac{(\hbar c)^2 e\sigma^j\sigma^i\partial_j\dot{A}_i}{2(2\alpha)^2} + \frac{\hbar ce}{2(2\alpha)^2}[\sigma^i,\sigma^j]A_iD_j. \tag{309}$$

Adding these up and simplifying the σ commutators, we get

$$\begin{aligned}\frac{1}{2}[\Lambda_1,\Omega] + \frac{i\hbar}{2}[\Lambda_1,\dot{\Lambda}_1] &= \frac{e}{2}\left(\frac{\hbar c}{2\alpha}\right)^2\sigma^i\sigma^j\partial_j(\partial_iA_0 - \dot{A}_i) - i\frac{ec\hbar}{4\alpha^2}\sigma^k\epsilon_{ijk}(\partial_iA_0 - \dot{A}_i)D_j \\ &= \frac{e}{2}\left(\frac{\hbar c}{2\alpha}\right)^2(\nabla\cdot\mathbf{e} + i\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{e})) - i\frac{ec\hbar}{(2\alpha)^2}\boldsymbol{\sigma}\cdot(\mathbf{e}\times\mathbf{D}).\end{aligned} \tag{310}$$

These terms, which we've worked so hard for, are the ones that we miss out on if we take the approach in e.g. Schwartz.

The last thing we need to do then is to calculate Λ_1^2. It is

$$2\alpha\Lambda_1^2 = \frac{1}{2\alpha}\left[(\hbar c\partial_i + iecA_i)^2 + \frac{i[\sigma^i,\sigma^j]}{4}\hbar c^2 eF_{ij}\right] = \frac{1}{2\alpha}\left[(\hbar c\partial_i + iecA_i)^2 - 2ec^2\mathbf{S}\cdot\mathbf{B}\right], \tag{311}$$

with $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$. Therefore

$$2\alpha Z(\Lambda_1^2 + \Lambda_1^4) = -Z\left[\frac{|\boldsymbol{\pi}|^2}{2m} + 2\mu_B\mathbf{S}\cdot\mathbf{B}\right] + Z\frac{1}{8\alpha^3}(c^4|\boldsymbol{\pi}|^4 + 8\hbar^2e^2c^4|\mathbf{B}|^2), \tag{312}$$

where $\boldsymbol{\pi} = \mathbf{p} + ec\mathbf{A}$ is the canonical momentum and $\mu_B = e/(2m)$ is the Bohr magneton. Note that we have dropped the $\boldsymbol{\pi}$-$\mathbf{B}$ cross terms in the expansion above, but retained the $\mathbf{B}^2$ term: this is just because the later will fit nicely in with the $\mathbf{E}^2$ term we derived above (notice that its coefficient is $\hbar^2e^2c^2/(m^3c^4)$, which is c^2 times the coefficient of the $\mathbf{E}^2$ part). In principle we should keep the mixed term; we're just dropping it cause it's relatively high-order, and ugly.

We have finally calculated everything we need to calculate. For aesthetic purposes we will rename $Z \mapsto -Z$, just so that the $|\pi|^2$ term has a coefficient $+Z$ instead of $-Z$. Adding everything together, we get our final Hamiltonian:[23]

$$\begin{aligned}H_{\text{eff}} = Zmc^2 + eA_0 + Z\left(\frac{|\boldsymbol{\pi}|^2}{2m} - \frac{|\boldsymbol{\pi}|^4}{8m^3c^2}\right) - \frac{e\hbar}{4m^2c^2}(\hbar\nabla\cdot\mathbf{E} + i\hbar\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{E}) + \sigma\cdot(\mathbf{E}\times\boldsymbol{\pi})) \\ - 2Z\mu_B\mathbf{S}\cdot\mathbf{B} + Z\frac{\hbar^2e^2}{m^3c^2}(c^2|\mathbf{B}|^2 + |\mathbf{E}|^2).\end{aligned} \tag{313}$$

[23] I tried, but I think slightly failed, to get all the minus signs consistent in the above. I might come back and fix them later, but for now I'll just fix the signs in the answer by what we know they should be on physical grounds (of course, we can take $Z \mapsto -Z$ without affecting anything).

From this Hamiltonian, we can read off a lot. For example, we can find the gyromagnetic ratio g for the electron by choosing a rotationaly-symmetric background field and looking at the relative sizes of the spin $\mathbf{S}\cdot\mathbf{B}$ and orbital $\mathbf{L}\cdot\mathbf{B}$ interactions. The former appears just as $2\mu_B$. For the later, fix a gauge in which we have a uniform field along the z direction, viz. $\mathbf{A} = \frac{B^z}{2}(-y, x, 0)$. The lowest order term which involves $\mathbf{L}\cdot\mathbf{B}$ comes from the expansion of $|\pi|^2$, and we see that the coefficient in front of $B^z L^z$ is $e/2m = \mu_B$. Therefore the ratio of the two couplings tells us that $g = 2$.

Another thing one can do is to find the spin angular momentum of the electron, by checking that $[\mathbf{L} + a\mathbf{S}, H_{\text{eff}}] = 0$ for $a = 1$, provided that the EM potential is spherically symmetric. We can also read off the SOC interaction for a Coulomb potential; choosing $\mathbf{E} = -\partial_r A_0 \widehat{\mathbf{r}}$ the $\mathbf{E}\times\boldsymbol{\pi}$ term becomes

$$+\frac{e\hbar}{4m^2c^2}\sigma^i\left(r^{-1}\partial_r A_0[\mathbf{r}\times\boldsymbol{\pi}]_i\right) = \frac{e\hbar}{4m^2c^2} r^{-1}\partial_r A_0 \boldsymbol{\sigma}\cdot\mathbf{L}. \tag{314}$$

When we send $c \to \infty$ we get a spin $SU(2)$ symmetry that acts only on the spin indices of the spinors, but does not involve an actual action on spacetime; the $1/c^2$ effect here breaks this symmetry.

The remaining terms in H_{eff} include the relativistic $|\boldsymbol{\pi}|^4$ correction to the kinetic energy, the magnetic interaction between the electron and the magnetic field produced by $\nabla\times\mathbf{E}$, a quantum-mechanical correction to the potential energy caused by the zero-point motion of the charge density (the $\nabla\cdot\mathbf{E}$ term), and an induced kinetic term for the EM fields (the last term—I haven't seen it in any textbooks before so it may be suspect).

Added: A helpful shortcut to thinking about the structure of the terms we get is just to write the eigenvalue problem for the Dirac Hamiltonian as

$$\det\begin{pmatrix} m\mathbf{1} - \lambda\mathbf{1} & -\boldsymbol{\sigma}\cdot(\mathbf{p} + e\mathbf{A}) \\ \boldsymbol{\sigma}\cdot(\mathbf{p} + e\mathbf{A}) & -m\mathbf{1} - \lambda\mathbf{1} \end{pmatrix} = 0 \tag{315}$$

where λ is the energy. This means that in the non-relativistic limit we'll get the term $\frac{1}{2m}[\boldsymbol{\sigma}\cdot(\mathbf{p} + e\mathbf{A})]^2$. Note the $\boldsymbol{\sigma}$ here, which usually isn't written, since when $\mathbf{A} = 0$ the mometa commute with eachother and the Pauli matrices just multiply to $\mathbf{1}$s. However since now $\mathbf{p}+e\mathbf{A}$ doesn't commute with itself, the $SU(2)$ structure is important. Using the identity

$$\sigma^a v_a \sigma^b u_b = v_a u^a + i\epsilon_{abc} v^a u^b \sigma^c = \mathbf{v}\cdot\mathbf{u} + i(\mathbf{v}\times\mathbf{u})\cdot\boldsymbol{\sigma}, \tag{316}$$

where v, u are two vectors whose components may not commute with each other, we see that

$$\frac{1}{2m}[\boldsymbol{\sigma}\cdot(\mathbf{p} + e\mathbf{A})]^2 = \frac{1}{2m}((\mathbf{p} + e\mathbf{A})^2 + \hbar e\mathbf{B}\cdot\boldsymbol{\sigma}). \tag{317}$$

Writing $\mathbf{A} = \mathbf{B}\times\mathbf{r}/2$, the first term gives us the angular momentum part of the paramagnetic term ($(\mathbf{r}\times\mathbf{p})\cdot\mathbf{B}$ from $\mathbf{A}\cdot\mathbf{p}$; $\nabla\cdot\mathbf{A} = 0$ in our gauge choice) and the diamagnetic term (from the $\mathbf{A}^2$ part), and the last term is just the usual coupling of the spin to the field that makes up the other half of the paramagnetic response. As we can see from the above result for H_{eff}, getting anything beyond this requires going at least to order $1/m^2$.

17 *Current-current correlators for N scalar fields*

Today we're doing a simple problem suggested by Pufu for the attendees of one of the bootstrap schools. Nothing complicated, but I thought it would be good to have around as a reference.

Consider a theory of N free scalars. Find an expression for the 2-point function of the currents, $\langle J^\mu_{ij}(x)J^\nu_{kl}(0)\rangle$. What aspects of this 2-point function are unchanged if $O(N)$ symmetry-preserving interactions are added?

First let us recall what the currents are. Taking $\phi_i \mapsto \phi_i + \epsilon_a(x)A^a_{ij}\phi_j$ where $A^a_{ij} = |e_i\rangle\langle e_j| - |e_j\rangle\langle e_i| \in \mathfrak{so}(N)$ tells us that

$$\delta S = \int d^dx\, \partial_\mu\epsilon_a(A^a_{ij}\phi_j\partial^\mu\phi_i + Aij(\partial^\mu\phi_i)\phi_j) = \int d^dx\,(\partial_\mu\epsilon_{ij})J^\mu_{ij}, \qquad J^\mu_{ij} = \phi_i\partial^\mu\phi_j - \phi_j\partial^\mu\phi_i. \tag{318}$$

We want to compute the current-current correlators. Since we care about the coefficients, we need to remember exactly what the propagator is. We invert ∂^2 by requiring that $G_{ij}(r) = \langle\phi_i(r)\phi_j(0)\rangle$ go as $\alpha\delta_{ij}/|r|^\gamma$, where

$$\partial_\mu\frac{\alpha}{r^\gamma} = \frac{\widehat{r}^\mu}{A(S^{d-1})r^{d-1}}, \tag{319}$$

so that $\partial^2 G_{ij}(r) = \delta_{ij}\delta(r)$. Therefore

$$G_{ij}(r) = \frac{1}{(d-2)A(S^{d-1})}\frac{1}{|r|^{d-2}} \equiv Er^{2-d}. \tag{320}$$

To get the current 2pt function, we need to compute things like

$$\langle: \phi^i(x)\partial_\mu\phi^j(x) :: \phi_k(y)\partial_\nu\phi_l(y) :\rangle = (\partial^x_\mu G_{jk}(x-y))(\partial^y_\nu G_{il}(x-y)) + G_{ik}(x-y)\partial^x_\mu\partial^y_\nu G_{jl}(x-y), \tag{321}$$

where the superscripts on the derivatives just indicate which variable they are being taken with respect to. Taking the derivatives,

$$\langle: \phi^i(x)\partial_\mu\phi^j(x) :: \phi_k(y)\partial_\nu\phi_l(y) :\rangle = \delta_{ik}\delta_{jl}\frac{E^2(d-2)}{r^{2d-2}}\left(\delta_{\mu\nu} - \frac{dr_\mu r_\nu}{r^2}\right) - \delta_{il}\delta_{jk}\frac{E^2(d-2)^2}{r^{2d}}r_\mu r_\nu. \tag{322}$$

The full current-current correlator comes from taking the above and adding $-(i \leftrightarrow j) - (k \leftrightarrow l) + (i,j \leftrightarrow k,l)$. Therefore

$$\langle J^\mu_{ij}(r)J^\nu_{kl}(0)\rangle = \frac{2}{(d-2)A^2(S^{d-1})r^{2d-2}}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})\left(\delta^{\mu\nu} - 2\frac{r^\mu r^\nu}{r^2}\right). \tag{323}$$

The dimensionality is just what is needed to ensure that the current has no anomalous dimension, and the factor of 2 in the last factor is just what is needed to ensure that the RHS is divergenceless.

We can also get the dimension of the current operator with a Ward identity. This is a slightly better way of finding A since it doesn't make any assumptions about the Lagrangian. The fact that this method recovers the same result as above means that conserved currents always have an anomalous dimension of zero.

Let $\mathcal{O}$ be any operator such that is charged under the symmetry generated by J. Let us consider varying the fields as

$$\phi \mapsto \phi + \eta(x)\delta_S\phi, \tag{324}$$

where $\delta_S\phi$ is a symmetry and $\eta(x)$ is an indicator function equal to ϵ for $x \in R$ and 0 else. Letting $\mathcal{O}$ be supported at a point $y \in R$, we have (taking ϵ infinitesimal)

$$\langle \delta\mathcal{O}(y)\rangle = \int_X d^dx\,(\partial_\mu\eta(x))\langle J^\mu(x)\mathcal{O}(y)\rangle = \int_{\partial R} d^{d-1}x_\perp^\mu\,\langle J_\mu(x)\mathcal{O}(y)\rangle. \tag{325}$$

Since this equation must hold regardless of what ∂R is, on dimensional grounds we must have $[J] = d-1$, so that J must have zero anomalous dimension.

When the current is associated with a non-Abelian symmetry as in the above context, the current itself provides us with such an $\mathcal{O}$. If the structure constant $f^{abc} \neq 0$, then $\delta_a\langle J^b_\mu(x)J^c_\nu(0)\rangle \neq 0$ (here δ_a is the variation which sends $\delta_a J^b_\mu = \eta i f_{a\ c}^{\ b} J^c_\mu$, and the currents are such that $\langle J^a(x)J^b(0)\rangle \propto \delta^{ab}$), and so taking $\mathcal{O}(y) = J^b_\mu(y)J^c_\nu(0)$ gives (not keeping track of numerical factors)

$$f^a_{\ bc}\langle J^b_\mu(y)J^c(0)\rangle \sim \int_{\partial R} d^{d-1}x_\perp^\lambda\langle J^a_\lambda(x)J^b_\mu(y)J^c_\nu(0)\rangle. \tag{326}$$

Requiring this to hold for arbitrary R again tells us the dimension of J and that $\langle J(x)J(0)\rangle \sim 1/|x|^{2(d-1)}$ exactly, as long as J is conserved.

Supersymmetric localization and the geometry of phase space path integrals

Today we're doing a computation that illustrates the basic features of supersymmetric localization in a very simple geometric way. This is a slight elaboration on a problem that was assigned in David Skinner's SUSY course; I found the problem statement on his webpage.

Consider a symplectic manifold (M, ω) of dimension $2m$ equipped with a Hamiltonian map $H : M \to \mathbb{R}$ which generates a $U(1)$ action on M. Consider the zero-dimensional action (the factor of 1/2 isn't in the problem statement I found online, but I think it's necessary)

$$S = \alpha\left(H(x) + \frac{1}{2}\omega_{ab}\psi^a\psi^b\right), \tag{327}$$

where the ψ^a are a set of $2m$ Grassmann variables. Consider the supersymmetry generator

$$Q = \psi^a\frac{\partial}{\partial x^a} + V^a(x)\frac{\partial}{\partial \psi^a}, \tag{328}$$

where V is the Hamiltonian vector field associated to the action of H. Use localization techniques to exactly compute the partition function

$$Z = \frac{1}{(2\pi)^m}\int d^{2m}x\, d^{2m}\psi\, e^{-S(x,\psi)}. \tag{329}$$

First let's try to understand the SUSY operator Q better. We can get a better understanding of Q by computing its square:

$$Q^2 = V^a\frac{\partial}{\partial x^a} + \psi^a\frac{\partial V^b}{\partial x^a}\frac{\partial}{\partial \psi^b}. \tag{330}$$

In what follows, it will be helpful to think geometrically. This is done by realizing that the ψ^a variables are really just a way of writing /basis covectors dx^a. In this notation, a 1-form V is written as $V_a\psi^a$, which has all the properties of a 1-form written as $V_a dx^a$ (the weird way that fermion measures transform allows the ψ^a to behave as basis covectors). Thinking about this, we realize that $\psi^a\frac{\partial}{\partial x^a}$ is really just $d = dx^a \wedge \partial_a$, and that $V^a(x)\frac{\partial}{\partial \psi^a}$ is really just i_V, which operates by contraction: $i_V(B_a dx^a) = V^a B_a$, $i_V(C_{ab}dx^a \wedge dx^b) = V^a C_{ab} dx^b$, and so on. Therefore, in geometric language, the SUSY operator is the equivariant derivative associated with the vector field V, viz.

$$Q = d_V = d + i_V. \tag{331}$$

Now let us identify the square of Q. In differential geometry language, converting ψ^as to dx^as, we claim that

$$Q^2 = \mathcal{L}_V = i_V d + d i_V, \tag{332}$$

which is the Lie derivative along V. To show this, we can e.g. consider how Q^2 acts on a 1-form $\rho = \rho_a\psi^a = \rho_a dx^a$:

$$Q^2\rho = \left(\frac{\partial V^a}{\partial x^b}\rho_a + V^a\frac{\partial \rho_b}{\partial x^a}\right)\psi^b. \tag{333}$$

Let us then compute

$$\mathcal{L}_V\rho = i_V(d\rho) + d(i_V\rho). \tag{334}$$

The first term is

$$i_V(d\rho) = i_V\partial_a\rho_b\psi^a\psi^b = V^a\left(\frac{\partial\rho_b}{\partial x^a} - \frac{\partial\rho_a}{\partial x^b}\right)\psi^b = V^a\partial_{[a}\rho_{b]}dx^b. \tag{335}$$

The second term is

$$d(i_V\rho) = \frac{\partial V^a}{\partial x^b}\rho_a\psi^b + V^a\frac{\partial\rho_a}{\partial x^b}\psi^b. \tag{336}$$

Adding these two together, we get

$$\mathcal{L}_V\rho = \left(\frac{\partial V^a}{\partial x^b}\rho_a + V^a\frac{\partial\rho_b}{\partial x^a}\right)\psi^b = Q^2\rho, \tag{337}$$

and so Q^2 indeed acts as the Lie derivative.

Let's now show that the action is invariant under SUSY transformations. The first term transforms simply as

$$QH = \psi^a\frac{\partial H}{\partial x^a}, \tag{338}$$

which in differential geometry would read $QH = d_V H = dH$. The second term is

$$Q\left(\frac{1}{2}\omega_{ab}\psi^a\psi^b\right) = \frac{1}{2}\left(\psi^c\frac{\partial\omega_{ab}}{\partial x^c}\psi^a\psi^b + \omega_{ab}V^a\psi^b - \omega_{ab}\psi^a V^b\right) = \omega_{ab}V^a\psi^b, \tag{339}$$

where we have used the antisymmetry of ω and the fact that ω is closed as a 2-form, so that the antisymmetrization enacted by the fermions in the first term makes the first term vanish. In terms of differential geometry language, the above computation reads

$$Q\omega = d_V\omega = d\omega + i_V\omega = V^a\omega_{ab}dx^b. \tag{340}$$

Putting these two transformations together, we see that the action is indeed invariant under SUSY, since

$$QS = \psi^a\frac{\partial H}{\partial x^a} + \omega_{ab}V^a\psi^b = \psi^a\omega_{ab}V^b + \omega_{ab}V^a\psi^b = 0, \tag{341}$$

where we've used the definition of the Hamiltonian vector field V, namely that V is determined by dual to $\frac{\partial}{\partial x^a}H$ via the symplectic form, which sets up the isomorphism $\omega : TM \cong T^*M$:

$$V^a = \omega^{ab}\frac{\partial H}{\partial x^b}. \tag{342}$$

Just for fun, let's recall why this is true: a Hamiltonian vector field V is a vector field in phase space along which the symplectic form is preserved, which means that

$$0 = \mathcal{L}_V\omega = d(i_V\omega), \tag{343}$$

since $d\omega = 0$. Therefore locally $i_V\omega$ is exact, and we can write it as $i_V\omega = dH$, where H is of course the Hamiltonian.

In order to run the localization procedure, we will add a Q-exact term to the action, and then show that the path integral is unchanged by its presence. The natural Q-exact term to add is the equivariant derivative $d_V V = QV$. More formally, let g be a $U(1)$-invariant metric on M.[24] Define $\Psi = g(V, \psi) = \psi^a V_a$, and consider adding $Q\Psi$ to the action, by defining[25]

$$S_\lambda(x, \psi) \equiv S(x, \psi) + \lambda Q\Psi, \tag{344}$$

where

$$Q\Psi = V^a V_a + \psi^a \psi^b \partial_a V_b. \tag{345}$$

Now we will show that $Q^2\Psi = 0$, which will mean that $QS_\lambda = 0$. Based on the geometric interpretation of Q that we just gave, the differential-geometric way to prove this is is to show that

$$Q^2\Psi = \mathcal{L}_V V = (i_V d + d i_V)V \tag{346}$$

vanishes. Indeed, the two terms on the RHS are

$$i_V dV = i_V(\partial_a V^b dx^a \wedge dx^b) = V^a \partial_{[a} V_{b]} dx^b = (V^a \partial_a V_b - V^a \partial_b V_a) dx^b, \qquad d i_V V = 2V^a \partial_b V_a dx^b, \tag{347}$$

and so

$$\mathcal{L}_V V = V^a(\partial_a V_b + \partial_b V_a) dx^b = 0, \tag{348}$$

since V is Killing field.

Now let us consider the variation of the partition function Z_λ with respect to λ:

$$(2\pi)^m \partial_\lambda Z_\lambda = \int d^{2m}x\, d^{2m}\psi\, \partial_\lambda S_\lambda e^{-S_\lambda} = \int d^{2m}x\, d^{2m}\psi\, Q(\Psi) e^{-S_\lambda} = \int d^{2m}x\, d^{2m}\psi\, Q\left(\Psi e^{-S_\lambda}\right), \tag{349}$$

since $QS_\lambda = 0$. However, both terms that are created when Q acts on something are total derivatives—either a total ψ derivative (in which case we get zero since $\int d\psi = 0$) or a total x^a derivative (in which case we get zero if $H(x) \to \infty$ at the limits of x integration, which we assume). Therefore $\partial_\lambda Z_\lambda = 0$, and so Z_λ is actually independent of λ. This means that we can send $\lambda \to \infty$ without changing the partition function. Now the exponential of the $\lambda \psi^a \psi^b \partial_a V_b$ term, being fermionic, has a power series expansion that truncates after a finite number of terms and ensures that its contribution to the partition function occurs as λ^m. The exponential of $-\lambda V^a V_a$ will thus dominate over the λ^m term and send the integrand to zero, unless x is such that $V^a V_a(x) = 0$. Let x_* denote a point where $V^a(x_*) = 0$. Since the symplectic form is non-degenerate, this must mean that at x_* we have $(\partial_a H)(x_*) = 0$, and so the points that the integral localizes around are the critical points of the Hamiltonian.

Another way to see that the integral localizes, which doesn't rely on adding the extra term to the action, is to realize that anything which is Q-closed is also Q exact everywhere except for the critical points where $V(x_*) = 0$. Since $Qe^{-S} = 0$ the integrand in Z is Q-closed, and so this means that the integrand is a total derivative everywhere except the

[24]If we are given a metric that's not $U(1)$-invariant, we can get an invariant one just by averaging over the $U(1)$ action, so this is done wolog.

[25]I'm getting sick of the fractions, so from now on ∂_a means differentiation wrt x^a (fermionic derivatives will be expressed more verbosely).

critical points; by Stoke's theorem we then see that Z receives contributions only from the critical points.

To prove the above claim about Q-exactness, we proceed as follows. Consider the field (best thought of as an inhomogeneous differential form in $\Omega^\bullet(M)$)

$$\Gamma = V \wedge (QV)^{-1}, \tag{350}$$

which is defined on $M \setminus X_*$, where X_* is the set of critical points. Here the inverse of QV is defined as (note to self: from here until when I go back to discussing the $\lambda Q \Psi$ tactic, the operator Q is $d - i_V$ instead of $d + i_V$ as above—bleh, at least it doesn't affect the conclusions. Will cleanup later)

$$(QV)^{-1} = (dV - V^2)^{-1} = -\frac{1}{V^2}(1 - dV/V^2)^{-1} = -\frac{1}{V^2}\sum_{i=1}^{m} V^{-2k}(dV)^{\wedge k}, \tag{351}$$

where again the notation is $V = V_a\psi^a$, $dV = \partial_a V_b \psi^a \psi^b/2$. This is the proper way to take an inverse of an inhomogeneous differential form for the same reason that $\sum_j (-x)^j = (1+x)^{-1}$. In our case the sum is finite and so we get

$$(1 - dV/V^2)\sum_{j=1}^{m}(dV/V^2)^{\wedge j} = 1 - (dV/V^2)^{\wedge(m+1)}, \tag{352}$$

but the last term vanishes since its dimension is too large. Note that the inverse is only well-defined away from the critical points.

A more pedestrian approach, which convinces us that this is the right way to take the inverse, is as follows. We want to find Γ such that $Q\Gamma = 1$. Since Q is odd, we can parametrize Γ as a series of odd terms:

$$\Gamma = -\frac{A_a\psi^a}{V^2} - \frac{B_{abc}\psi^a\psi^b\psi^c}{V^2} - \dots, \tag{353}$$

where the minus signs and $1/V^2$s are just for convenience. Now the first term, when acted on by Q, is

$$Q\Gamma \ni \frac{V^a A_a}{V^2} - \frac{\partial_a A_b \psi^a\psi^b}{V^2} + \frac{2}{V^4}V^c\partial_a V_c A_b\psi^a\psi^b. \tag{354}$$

Since the first term is the only one in $Q\Gamma$ with no ψs, we need $A_a = V_a$, so that

$$Q\Gamma \ni 1 - \frac{\partial_a V_b\psi^a\psi^b}{V^2} + \frac{2}{V^4}V^c\partial_a V_c V_b\psi^a\psi^b. \tag{355}$$

The second term in Γ, when acted on by Q, contributes 2- and 4-fermion terms. If $Q\Gamma = 1$, then we need the 2-fermion terms to precisely the 2-fermion terms in the above equation. The 2-fermion terms produced by the B term in $Q\Gamma$ are

$$Q\Gamma \ni \frac{V^a}{V^2}\left(-B_{abc} + B_{bac} - B_{bca}\right)\psi^b\psi^c. \tag{356}$$

Suppose we choose $B_{abc} = V^{-2} V_a \partial_b V_c$, which is what the above formula for the inverse tells us to do. Then the above terms become

$$Q\Gamma \ni \frac{1}{V^4}\left(V^2\partial_a V_b - V_a V^c \partial_c V_b + V_a V^c \partial_b V_c\right)\psi^a\psi^b. \tag{357}$$

Therefore we have, re-naming some dummy indices and letting $\dots$ denote terms with 4 or more fermions,

$$Q\Gamma = 1 - \frac{1}{V^4}\left(V_a V^c \partial_b V_c + V_a V^c \partial_c V_b\right) + \dots = 1 - \frac{V^a V_c}{V^4}\partial_{(b}V_{c)} = 1 + \dots \tag{358}$$

since V is Killing.

One can then check that choosing the next term to be $-V^{-2}C_{abcde}\psi^a\cdots\psi^e$ with $C = (dV \wedge dV)/V^4$ cancels the 4-fermion terms in the QB_{bcd} term, and that the 6-fermion terms in the QC_{abcde} term are canceled by the next order term, and so on. This cancellation occurs up until we reach products of m fermions, at which point all further terms die by antisymmetry.

Anyway, using Γ we can note that since $Q\Gamma = 1$,

$$QS = 0 \implies Q(\Gamma S) = S, \tag{359}$$

and so S is Q-exact. Therefore all Q-closed forms are exact away from the critical points, proving our claim about localization.

Anyway, going back to the $\lambda Q\Psi$ approach which is easier to implement in practice, we expand about the critical point and then define y by $y = x\sqrt{\lambda}$ and $\widetilde{\psi}$ by $\widetilde{\psi} = \psi\sqrt{\lambda}$,[26] yielding

$$\begin{aligned} Z = Z_{\lambda\to\infty} = \frac{1}{(2\pi)^m}\sum_{x_*:V^a(x_*)=0}\int d^{2m}y\, d^{2m}\psi\, \exp\Big[\alpha\Big(&-H(x_*) \\ &-\frac{1}{2}(\partial_a\partial_b(V_cV^c))(x_*)y^a y^b - (\partial_{[a}V_{b]})(x_*)\widetilde{\psi}^a\widetilde{\psi}^b + \dots\Big), \end{aligned} \tag{360}$$

where $\dots$ are terms that vanish when $\lambda \to \infty$ and hence can be dropped. We can now do the integrals no problem:

$$Z = \sum_{x_*:V(x_*)=0} e^{-\alpha H(x_*)}\alpha^m \frac{\mathrm{Pf}[(\partial_{[a}V_{b]})(y_*)]}{\sqrt{\det[(\partial_a\partial_b V_c V^c)(y_*)]}}. \tag{361}$$

Now around each critical point, we can choose coordinates where V has the canonical form of a Hamiltonian vector field, viz.

$$V(x_*) = \sum_{i=1}^m n_i(x_*)\left(p^i\frac{\partial}{\partial q^i} - q^i\frac{\partial}{\partial p^i}\right), \tag{362}$$

where we have split up the coordinates near x_* into a set of m "coordinates" q_i and m "momenta" p_i. Here the $n_i \in \mathbb{Z}$ since H generates a $U(1)$ symmetry. With this choice of coordinates, about each critical point we have

$$\mathrm{Pf}[(\partial_{[a}V_{b]})(x_*)/2] = \prod_{i=1}^m n_i(x_*), \qquad \det[(\partial_a\partial_b V_c V^c)(x_*)/2] = \det[(\partial_a V_c \partial_b V^c)(x_*)] = \prod_{i=1}^m n_i^4(x_*), \tag{363}$$

[26] The integration measure is invariant since the fermionic and bosonic measures transform oppositely.

and so

$$Z = \sum_{y_*:V(y_*)=0} e^{-\alpha H(y_*)} \frac{\alpha^{-m}}{\prod_{i=1}^m n_i(y_*)}. \tag{364}$$

Finally, we can massage this by noting that the fermion path integral in the $\lambda = 0$ action can also be done exactly, giving

$$\int d^{2m}\psi \, e^{-\frac{1}{2}\omega_{ab}\psi^a\psi^b} = \frac{1}{m!}\omega^{\wedge m}, \tag{365}$$

which is the usual phase-space measure (on the LHS $\omega = \omega_{ab} dx^a \wedge dx^b$). Therefore the localization formula is

$$\frac{1}{m!}\int \omega^{\wedge m} \, e^{-\alpha H(x)} = \sum_{x_*:V(x_*)=0} e^{-\alpha H(x_*)} \frac{(2\pi/\alpha)^m}{\prod_{i=1}^m n_i(x_*)}. \tag{366}$$

Fun with Majorana quantum mechanics, path integrals, and traces of γ matrix exponentials

Today we're doing another problem from David Skinner's SUSY class. This one is pretty easy and shares a few features in common with a previous diary entry on Majorana path integrals, so we will be rather succinct.

Consider the QM action

$$S[\psi] = \frac{1}{2}\int_{S^1} d\tau \left(\psi_a \partial_\tau \psi^b + \omega_{ab}\psi^a\psi^b\right), \tag{367}$$

where ω is antisymmetric and $a = 1, \dots, 4$. Using the path integral obtained from this action, find expressions for

$$\mathrm{Tr}[\bar{\gamma} e^{-\omega_{ab}\gamma^a\gamma^b/4}], \qquad \mathrm{Tr}[e^{-\omega_{ab}\gamma^a\gamma^b/4}] \tag{368}$$

in terms of sinh and cosh expressions. Here $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ are the 4d Gamma matrices.

First, some preliminaries. Consider an operator $\mathcal{O}$ acting on a single two-dimensional Fock space. Then we claim that

$$\mathrm{Tr}[\mathcal{O}] = \int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} \langle -\bar{\eta}|\mathcal{O}|\eta\rangle, \tag{369}$$

where the coherent states are, in our conventions,

$$|\eta\rangle = e^{-\eta\psi^\dagger}|0\rangle, \qquad \langle\bar{\eta}| = \langle 0|e^{\bar{\eta}\psi}, \tag{370}$$

with $\psi^\dagger|0\rangle = |1\rangle$ (the ψs are operators; the ηs are Grassmann numbers). With these conventions, $\langle\bar{\eta}|\eta\rangle = e^{\bar{\eta}\eta} = 1 + \bar{\eta}\eta$. One can then check that the minus sign in the left bra in (**??**) is needed to get the trace, and that if it's replaced with a $\langle\bar{\eta}|$, one instead gets the supertrace.

When we Trotterize the above expression for the trace with $\mathcal{O} = e^{-\beta H}$ (remembering the $e^{-\bar{\eta}_t\eta_t}$ factors!), we see that we get the usual $\bar{\eta}\partial_t\eta - H$ action, provided that we can identify $\bar{\eta}_N(\eta_1 + \eta_N)/\delta\tau$ with a time derivative (where there are N steps in the Trotterization). Therefore after Trotterizing, the trace computes the path integral with antiperiodic boundary conditions. Similarly, the supertrace $\mathrm{Tr}[(-1)^F\mathcal{O}]$ computes the trace with periodic boundary conditions.

To connect with the gamma matrices, one just realizes that four majoranas, when quantized, obey the same algebra as 4-dimensional gamma matrices. More precisely, because of normalization, the identification is $\psi^a = \gamma^a/\sqrt{2}$. Since $\bar{\gamma}$ anticommutes with all of them, we have in $(-1)^F = \bar{\gamma}$ in this representation. Therefore the partition functions with the two choices of boundary conditions are

$$Z_P = \mathrm{Tr}[\bar{\gamma}e^{-\omega_{ab}\gamma^a\gamma^b/4}], \qquad Z_A = \mathrm{Tr}[e^{-\omega_{ab}\gamma^a\gamma^b/4}]. \tag{371}$$

So, to compute these traces, we can alternately compute

$$Z_{A/P} = \det\left[\frac{1}{2}\frac{d}{d\tau} - \omega/2\right]^{1/2}, \tag{372}$$

with the eigenvalues for d_τ being determined from the boundary conditions.

First consider the case when the fermions are periodic around the S^1. Then we have

$$Z_P = \det\left[\prod_{n\in\mathbb{Z}}(i\pi n + \omega/2)\right]^{1/2}, \tag{373}$$

where now the determinant is taken only in the spinor tensor factor. Therefore

$$\mathrm{Tr}[\bar{\gamma}e^{-\omega_{ab}\gamma^a\gamma^b/4}] = \det\left[\frac{\omega}{2}\prod_{n\in\mathbb{N}}(\pi^2n^2 + \omega/2)\right]^{1/2} = N\det[\sinh(\omega/2)]^{1/2}, \tag{374}$$

where we've used

$$\sinh(x) = x\prod_{n\in\mathbb{N}}(1 + x^2/(\pi n)^2), \tag{375}$$

and where N is an ω-independent constant (determined e.g. through ζ regularization).

The antiperiodic spin structure case is basically the same: we get

$$\mathrm{Tr}[e^{-\omega_{ab}\gamma^a\gamma^b/4}] = \det\left[\prod_{n\in\mathbb{N}+1/2}(\pi^2n^2+\omega/2)\right]^{1/2} = M\det[\cosh(\omega/2)]^{1/2}, \tag{376}$$

where M is another ω-independent constant. Here the relevant product formula is

$$\cosh(x) = \prod_{n\in\mathbb{N}}\left(1+\frac{x^2}{(\pi n-\pi/2)^2}\right). \tag{377}$$

18 *The effective potential and thermodynamics UNFINISHED*

I've always found the manipulations regarding the definition of the effective potential to be confusing, since (as of writing) the way the Legendre transform is usually presented doesn't make a lot of sense to me. Today, in an attempt to clarify things, we'll therefore look at analogies with thermodynamics, and how the whole construction is really just one of Lagrange multipliers.

First let's remind ourselves of how it works in thermodynamics. Our goal is to maximize the entropy, subjected in e.g. the microcanonical picture to the constraints that $\sum E_ip_i = E$ and $\sum_i p_i = 1$, where p_i is the weight of the ith state. Then we add a Lagrange multiplier we'll call $-\beta$ for the energy constraint and one we'll call $-\ln Z + 1$ for the normalization, and try to extremize

$$-\sum_i p_i\ln p_i - \beta\left(\sum_i p_iE_i - E\right) + (-\ln Z+1)\left(\sum_i p_i - 1\right). \tag{378}$$

Varying wrt p_i tells us that

$$-\ln p_i - \beta E_i - \ln Z = 0 \implies p_i = \frac{e^{-\beta E_i}}{Z}. \tag{379}$$

Okay, duh. No surprises here.

Therefore, we see that the current plays the role of a Lagrange multiplier enforcing the expectation value of ϕ be φ. Just as the temperature is an implicit function of the average energy, the current is an implicit function of the vev of the field to which it couples.

Summarizing, the correspondence is

$$
\begin{aligned}
J &\leftrightarrow \beta \\
\Gamma[\varphi] &\leftrightarrow E
\end{aligned}
\tag{380}
$$