1-form anomalies in CS theory

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In these notes we will examine the anomalies of 1-form symmetries present in various CS theories. If I remember correctly I looked at [1] when writing this, and I think in the time since this was written there have been a few more papers on the subject. I also presented this work at Daniel Harlow's group meeting; thanks to him for feedback. I have not gotten around to comparing the calculations below to the existing literature (if it now exists); any differences should assumed to be mistakes on my part.

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$U(1)_k$

Let's start with the simplest example, viz. $U(1)_k$. The 1-form symmetry acts as

$$\mathbb{Z}_k^{(1)} : A \mapsto A + \lambda, \qquad k\lambda \in 2\pi H^1(X;\mathbb{Z}). \tag{1}$$

The charge operators are of course the Wilson lines. We can see that this is a symmetry by e.g. computing the spectrum of operators in the theory, but for posterity's sake let's see how it works from the action. Since λ is flat, a naive approach tells us that $\delta S = \frac{1}{4\pi}\int (k\lambda) \wedge F_A$ under the symmetry, which is only in $\frac{1}{2}\overline{\mathbb{Z}}$ (we are using the notation $\overline{\mathbb{Z}} \equiv 2\pi\mathbb{Z}$). Note that we cannot integrate this by parts to get zero by the flatness of λ, due to A not being strictly a well-defined form[1]. As usual, the confusion can be ameliorated by writing things in terms of the field strengths by using a bounding 4-manifold M. Then

$$\delta S = \frac{k}{2\pi}\int_M F_\lambda \wedge F_A + \frac{k}{4\pi}\int_M F_\lambda \wedge F_\lambda, \tag{2}$$

where F_λ is the field strength of the extension of λ into the bulk 4-manifold M. Note that since the holonomy of λ may be nontrivial, although it is flat on the boundary, it will not be flat in M, and a priori, it will not be globally an exterior derivative, i.e. we may not have $F_\lambda = d\lambda$ globally on M.

[1] A similarly hasty use of integration by parts on the CS action leads to confusion in the usual way of showing that $k \in \mathbb{Z}$ in the CS action, namely by e.g. placing the theory on $S^1 \times S^2$ with $\int F = 2\pi$ around the S^2. In the usual story one integrates by parts to get $S = \frac{k}{2\pi}\oint A_t \int_{S^2} F_{xy} = k\oint A_t$ (there is a factor of 2 here from the IBP), which says that $k \in \mathbb{Z}$ for invariance under large gauge transformations around the S^1. But what if we first did the large gauge transformation, and then did the integration by parts? Since the field strength of the large gauge transformation vanishes, the IBP fails to pick up a factor of 2, and we conclude that the change in the action is instead $(k/4\pi)\oint \lambda \int_{S^2} F_{xy}$ for $\oint \lambda = 2\pi$, which seems to imply that $k \in 2\mathbb{Z}$ is required. So, it is best to only integrate by parts when we really know that it is legit.

Now we need to integrate by parts: we will get only boundary terms, since $dF_A = d(d\lambda) = 0$. However, doing so is slightly subtle, since λ might not be a globally well-defined form. Thus we cannot write e.g. $\int d\lambda \wedge B = \int \lambda \wedge dB + \int_{\partial M} \lambda \wedge B$ for a 2-form B (the sign is correct because of the supercommutativity of d). However, since λ is flat, we know that λ is a well-defined form on ∂M. Thus in the bulk, we may write

$$\lambda = \Lambda + B, \qquad F_B \in 2\pi H^2(M, \partial M; \mathbb{Z}), \tag{3}$$

where Λ is a $U(1)$ gauge field which is globally well-defined so that $[F_\Lambda] = [d\Lambda] = 0$ in $2\pi H^2(M;\mathbb{Z})$, and B is a non-globally-well-defined part which vanishes on ∂M since $\lambda|_{\partial M}$ is globally well-defined (thus $\lambda|_{\partial M} = \Lambda|_{\partial M}$). Thus we can write

$$\delta S = \frac{k}{2\pi}\int_M [(d\Lambda + F_B)\wedge F_A + d\Lambda \wedge F_B] + \frac{k}{4\pi}\int_M (d\Lambda \wedge d\Lambda + F_B \wedge F_B). \tag{4}$$

Assuming we choose M to be spin if k is odd, the last term vanishes modulo $\overline{\mathbb{Z}}$. Since Λ is globally well-defined, the $d\Lambda \wedge F_B$ term vanishes on account of the flatness of F_B and the fact that $F_B|_{\partial M} = 0$. Likewise the $F_B \wedge F_A$ part vanishes mod $\overline{\mathbb{Z}}$: we can see this by decomposing A in the same way that we decomposed Λ, and using that $\frac{1}{2\pi}\int F_C \wedge F_B \in \overline{\mathbb{Z}}$ for $F_C \in 2\pi H^2(M, \partial M; \mathbb{Z})$. So finally, we integrate the remaining two terms by parts and get

$$\delta S = \frac{1}{2\pi}\int (k\lambda)\wedge F_A, \tag{5}$$

since $\Lambda|_{\partial M} = \lambda|_{\partial M}$ and since $d\lambda|_{\partial M}$ is flat. But since $k\lambda$ has periods in $\overline{\mathbb{Z}}$, we see that $\delta S \in \overline{\mathbb{Z}}$, and so indeed, the 1-form transformation $\delta A = \lambda$ *is* a symmetry of the action.

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To gauge this symmetry, we want the "split" symmetry operators $U(g; C)$ (not only the full charge operators) to act trivially on the Hilbert space, where the split symmetry operators are defined on *open* submanifolds $C : \partial C \neq 0$ and implement a transformation by the group element g (here $g \in \mathbb{Z}_k$). Requiring the global (unsplit) charge operators to act trivially is equivalent to projecting onto the singlet sector of the Hilbert space, which can be done by inserting the operator

$$\Pi_{\mathbf{1}} = \sum_{C \in H_1(X;\mathbb{Z}_k)} e^{i\int_C A} \tag{6}$$

into the path integral. This is orbifolding. This is not quite how we want to think of gauging, since we haven't made the symmetry local in any way, we've just projected onto a particular (trivial) value of the charge (when we gauge we want to work with genuinely gauge-invariant states, not ones that are in some particular gauge fixing). A hint of the anomaly can be seen by noticing that the split symmetry operator $U(q, C)$, which naively includes $e^{iq\int_C A}$ since the current for the 1-form symmetry is $j = \star A$, is not gauge invariant (under both the 0-form and 1-form gauge transformations) if $\partial C \neq 0$. This is a problem since after including the 2-form gauge field we expect $U(q, C)$ to be modified by the canonical momentum for the

2-form gauge field (think of $\nabla \cdot \mathbf{E}$), which we don't expect to transform in a way that would fix this issue, and hence the gauge variance of $U(q, C)$ will continue to be problematic.

Let's now see how this plays out. We will let B be the $\mathbb{Z}_k$ 2-form gauge field. To enforce the quantization of B's periods, we will as usual add to the action the BF term (we are in $\mathbb{R}$ time, so no factors of i are included)

$$S \supset \frac{k}{2\pi} \int B \wedge d\phi, \tag{7}$$

where ϕ is a 2π-periodic scalar. Now we can fix the variance under 0-form gauge transformations of $e^{i\int_C \star j} = e^{i\int_C A}$ when $\partial C \neq 0$ by writing the operator $U(q, C)$ which implements the gauge transformation as

$$U(q, C) = e^{iq\int_C (A + d\phi)}, \tag{8}$$

provided that under $A \mapsto A + d\gamma$ we have $\phi \mapsto \phi - \gamma$ (this preserves the 2π-periodicity of ϕ, since γ is itself a 2π-periodic scalar). This makes sense, since ϕ is the canonical momentum for B, and so this is exactly what we normally do when gauging the symmetry operators: the operators which perform the gauge transformations are the original charge operators defined on open submanifolds, with the canonical momentum for the gauge field integrated along their boundaries (again as an example, the generator of gauge tforms in QED is the integral of the matter current over an open volume, together with the integral of $\star F$, the canonical momentum for the gauge field, over the boundary of the volume).

Now by design, if $D \in C_1(X; \mathbb{Z})$ is such that $C \cap D \neq 0$, then $W(D) = e^{i\int_D A}$ is not gauge invariant under the $\mathbb{Z}_k^{(1)}$ gauge transformations, since it does not commute with $U(q, C)$. Note that no matter what D is, we can always find a C such that $W(D)$ is not invariant under $U(q, C)$: this is true even when $[D] = 0$ in $H_1(X; \mathbb{Z})$, in which case $W(D)$ is actually neutral under the original 1-form global symmetry.

We can make $W(D)$ gauge invariant by attaching a surface operator built out of B to it: if $[D] = 0$ in $H_1(X, \partial X; \mathbb{Z})$ (so that D either bounds a disk, is a linear combination of nontrivial classes in $H_1(X; \mathbb{Z})$ with total "charge" zero so that it bounds some other surface, or together with a submanifold of the boundary of spacetime bounds a surface) we can find some M such that $\partial M \setminus (\partial M \cap \partial X) = D$ (here X is spacetime, and gauge transformations always vanish at ∂X). The operator

$$\widetilde{W}(M) = \exp\left(i \int_D A + i \int_M B\right) \tag{9}$$

is then gauge-invariant. Why? Because when we compute its commutation relation with $U(q, C)$ (with e.g. $C \cap D = 1$), we get one factor of $e^{2\pi i q/k}$ from the $[A, A] \sim i/k$ commutation relation, and another from the $[\phi, B] \sim i/k$ commutation relation, which occurs from the contact term between the ϕ inserted at the end of C and the B integrated over M. If $[D] \neq 0$ in $H_1(X; \mathbb{Z})$ then $W(D)$ can't be made gauge-invariant, and its vev vanishes (although this was true before gauging, since $\langle W(D) \rangle$ can then be shifted by a change in integration variables which doesn't affect the boundary conditions on A).

As we hinted at above, the anomaly is seen very simply from the fact that the operators $U(q, C)$ which perform the $\mathbb{Z}_k^{(1)}$ gauge transformations are not themselves invariant under the same $\mathbb{Z}_k^{(1)}$ transformations (although as we have seen they are at least invariant under the 0-form $U(1)$ gauge transformations on A). That is, they don't commute with themselves (because of $[A, A] \sim i/k$). Since $\partial C \neq 0$, it is impossible to attach a B surface to render $U(q, C)$ gauge invariant. Thus the $\mathbb{Z}_k^{(1)}$ symmetry can't actually be gauged.

We can also see this from the action. Basically, while F_A can be made gauge invariant by $F_A \mapsto F_A - B$, the CS term cannot be made gauge invariant since it involves more than just F_A. Indeed, let us write the variation of A under the 1-form gauge transformation as

$$\delta A = \lambda, \qquad \lambda \in \frac{1}{k} Z^1_O(X; \mathbb{Z}). \tag{10}$$

Here we have defined $Z^1_O(X;\mathbb{Z})$ as the set of 1-forms such that their spatial Poincare duals are *open* codimension-1 submanifolds of space, which have integral intersection number with every element in $C_1(X;\mathbb{Z})$ that intersects them transversely. Thus the elements in $Z^*_O(X;\mathbb{Z})$ are not closed, but they are not closed in a very specific way (we are not letting λ be an arbitrary 1-form since we are gauging a $\mathbb{Z}_k$ 1-form symmetry, and not a $U(1)$ one). Another way to say this is that

$$\lambda \in \frac{1}{k} Z^1_O(X;\mathbb{Z}) \implies \int_C \lambda \in \frac{1}{k}\mathbb{Z} \quad \forall C \in C_1(X;\mathbb{Z}), \tag{11}$$

where the value for the integral will generically depend on the exact choice of C, and not just its homotopy equivalence class. Connecting this with our earlier notation vis-a-vis the $U(q, C)$ gauge transformation operators, we would say that $U(q, C)$ shifts A by $\lambda = \frac{q}{k}\widehat{C}$, where $\widehat{C}$ is the spatial Poincare dual of $C \in C_1(X;\mathbb{Z})$.

Anyway, the CS term varies as

$$\delta \int A \wedge dA = 2\int A \wedge d\lambda + \int \lambda \wedge d\lambda. \tag{12}$$

To cancel at least the first term we can try to introduce a $\mathbb{Z}_k$ gauge field B to the action and add

$$S \supset -\frac{1}{2\pi}\int A \wedge B, \tag{13}$$

with $\delta B = d\lambda$ under the gauge transformation. However this a) cannot cancel the term in δS quadratic in λ and b) produces an extra piece linear in B. So, after adding this coupling, the total variation of S is

$$\delta S = \frac{k}{4\pi}\delta\int (A \wedge dA - 2A \wedge B) = \frac{k}{4\pi}\int (2\lambda \wedge B + \lambda \wedge d\lambda). \tag{14}$$

Of course, we can make the action gauge invariant by letting B live in four dimensions, at the price of picking up an explicit dependence on a bounding 4-manifold M. This is just because F_A can always be made gauge-invariant, and we can write the terms in our modified action involving A as

$$S \supset \frac{k}{4\pi}\int_M (F_A - B) \wedge (F_A - B), \tag{15}$$

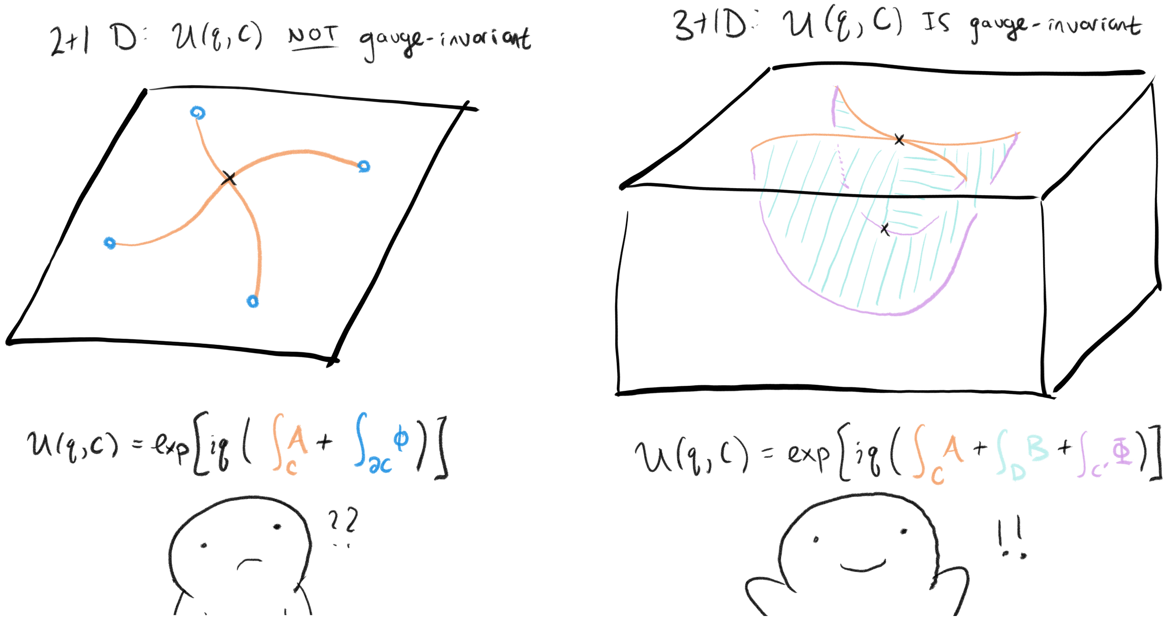

Figure 1: The generators of gauge transformations for the 1-form gauge symmetry in gauged $U(1)_k$. Contact terms that contribute to the commutator of the charge operators are marked with black x's. In 2+1D the $U(q,C)$ are charged and the symmetry can't be gauged, while in 3+1D, with B surfaces extending into the bulk, it can be.

which is manifestly gauge invariant (and still only depends on $A|_{\partial M}$). However, and this is where the anomaly comes in, it depends on the choice of M, since

$$\frac{k}{4\pi}\int_{N_4|\partial N_4=\emptyset}(F_A - B)\wedge(F_A - B) \in \frac{1}{k}\overline{\mathbb{Z}}, \tag{16}$$

which is not valued in $\overline{\mathbb{Z}}$ except in the trivial case $k=1$ where there is no symmetry to begin with (here we have used the fact that the periods of B are valued in $k^{-1}\overline{\mathbb{Z}}$ — the periods of F_A are still in $\overline{\mathbb{Z}}$ though, since the 1-form gauge transformations only change A by forms which are globally well-defined up to elements in $2\pi H^2(N_4;\mathbb{Z})$). Since k copies of this bulk action integrate to something in $\overline{\mathbb{Z}}$ over all closed 4-manifolds, we have a $\mathbb{Z}_k$ anomaly.

To write the full gauged action for the four-dimensional B, we just need to include the term which makes B into a $\mathbb{Z}_N$ gauge field. Since B lives in four dimensions, the appropriate BF term is $(k/2\pi)\int_M B\wedge F_\Phi$, where Φ is a 1-form $U(1)$ gauge field. But this term changes as $(k/2\pi)\int_M d\lambda\wedge F_\Phi = (k/2\pi)\int_{\partial M}\lambda\wedge F_\Phi$ under the 1-form gauge transformation , which is problematic. The way to get around this is to include a $-(k/2\pi)\int_{\partial M} B\wedge\Phi$ boundary term in the action. Together with the boundary term, the full part of the action involving Φ is $-(k/2\pi)\int_M F_B\wedge\Phi$, which is manifestly invariant under the 1-form gauge transformation . Recapitulating, the full action is

$$S = \frac{k}{4\pi}\int_{\partial M}(A\wedge F_A - 2A\wedge B - 2B\wedge\Phi) + \frac{k}{4\pi}\int_M(B\wedge B + 2B\wedge F_\Phi). \tag{17}$$

The full generator of gauge transformations is now

$$U(q,C) = \exp\left(iq\left[\int_C A + \int_D B + \int_{C'} \Phi\right]\right), \tag{18}$$

where D is a disk with $\partial D = C \cup C'$, $\partial C = \partial C'$, and where C' is entirely contained within the four-dimensional bulk. The $U(q,C)$'s commute with one another: the contact term between the A's on the surface is canceled between a contact term where the $\int_{C'} \Phi$ line intersects the $\int_D B$ surface. This is illustrated in Figure 1. On the left we show the $U(q,C)$ operators in the strictly 2+1D theory, which are not gauge invariant. On the right we show how, after attaching B surfaces and Φ lines to them, they become gauge invariant.

Twisted $\mathbb{Z}_N$ gauge theory

We now look at the DW theory which we will call $DW_{p,q}$, namely

$$\mathcal{L} = \frac{p}{4\pi} a \wedge da + \frac{q}{2\pi} a \wedge db. \tag{19}$$

What are the global symmetries? First, there is clearly a $\mathbb{Z}_N^{(1)}$ symmetry that shifts b by $1/N$ times a large gauge transformation. Similarly, there is also a $l \equiv \gcd(p,q)$ symmetry shifting a: this is the best we can do, as e.g. the coupling between a and b means we don't have the full $\mathbb{Z}_p^{(1)}$ symmetry of the first term, unless p divides q, and we don't have the full $\mathbb{Z}_q^{(1)}$ symmetry from shifts on a in the second term, unless q divides p.

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Let's pause for a momentum to discuss the spin and statistics of the lines in this theory. A naive reading on this would be as follows: the canonical momentum for a is $pa + qb$ and the canonical momentum for b is qa. Thus b lines commute with each other, while a lines have a self-linking phase determined by $1/p$. The mutual statistics of a and b is nonzero because they do not commute with each other, and is determined by $1/q$.

This naive reading is incorrect: even if the canonical momentum for a field ϕ as read off from $\partial\mathcal{L}/(\partial\partial\phi)$ does not involve ϕ itself, ϕ may still have nontrivial statistical interactions with itself. Indeed, the correct way to determine the commutation relations between Wilson lines is by using the inverse of the K matrix. Let's quickly remind ourselves of why: for $i \in \mathbb{Z}_{\dim K}$ and letting $\star q^\alpha \cdot J^\alpha = \star q_i^\alpha \cdot J_i^\alpha$ be the 2-form Poincare dual to a support of a particular configuration of Wilson lines $\prod_\alpha W_\alpha = \prod_\alpha e^{i\sum_j \oint_{C_\alpha} A_j}$, we have

$$\langle \prod_\alpha W_\alpha \rangle = \frac{1}{Z[J=0]} \int \prod_i \mathcal{D}A_i \exp\left(\frac{i}{4\pi}\int A_i [K]_{ij} \wedge dA_j + i\sum_{\alpha,i} q_i^\alpha \int A_i \wedge \star J_i^\alpha\right). \tag{20}$$

Shifting A to kill off the AJ coupling, we get

$$\langle \prod_\alpha W_\alpha \rangle = \exp\left(2\pi i \sum_{\alpha,\beta} q_i^\alpha q_j^\beta \int \star J_i^\alpha [K^{-1}]_{ij} \wedge d^{-1} \star J_j^\beta\right). \tag{21}$$

Taking all the Wilson loops to be supported on the boundaries of disks means that the $\star J^\alpha$ are not in $\ker(d)$, and so the above formula makes sense. Anyway, taking two linked loops, one with a unit charge for A_i and another with a unit charge for A_j (and taking the framing of each loop to be trivialized so that the diagonal in α terms in the above formula do not contribute) gives us the braiding matrix

$$[S]_{ij} = \exp\left(2\pi i [K^{-1}]_{ij}\right). \tag{22}$$

This can also be derived just by looking at $[A_i, \pi_{A_m}] = \sum_j [A_i, \bar{K}_{mj} A_j] = i\delta_{im}$. Here spacetime indices are kept implicitly, with $[A_i, A_j] = A_i \wedge A_j - A_j \wedge A_i$. Also, $\bar{K} = K/2\pi$. Anyway, multiplying by $[\bar{K}^{-1}]_{mk}$ and summing over m:

$$\sum_j [A_i, A_j]\delta_{k,j} = i \sum_m \delta_{im} [\bar{K}^{-1}]_{mk} \implies [A_i, A_j] = i[\bar{K}^{-1}]_{ij}. \tag{23}$$

Using this commutation relation to unlink any loops that are linked together in $\prod_\alpha W_\alpha$, one recovers the above expression for the S matrix (after choosing a framing).

In the present $DW_{p,q}$ example, the K matrix and its inverse are

$$K = \begin{pmatrix} p & q \\ q & 0 \end{pmatrix}, \qquad K^{-1} = \begin{pmatrix} 0 & 1/q \\ 1/q & -p/q^2 \end{pmatrix}. \tag{24}$$

Thus even though a appears in the canonical momentum for b, we see that b still fails to commute with itself. So we see that the b line is *not* a boson, despite the fact that its canonical momentum does not involve itself. In fact, it has spin $-p/2q^2$! And similarly, despite the self-CS term for a, we see that a is actually a boson! Physically, what's going on here is that b lines carry flux for a, which by the self-CS term for a have nontrivial braiding with themselves, since this term tells us that a flux also carries a charge. This allows b lines to not commute with themselves. Likewise, a lines carry a flux, which makes them seem like they would not commute with themselves. But a fluxes also carry b charge, and b charge carries a flux, and this all works out in such away that the a lines actually carry net zero a flux.

A particularly transparent example of when this happens is the case when $q = p$. In that case, we can diagonalize the K matrix by something in $SL(2, \mathbb{Z})$ via

$$K \mapsto \Lambda^T K \Lambda = qZ, \qquad \Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{25}$$

This means that in terms of the variables b and $c = a + b$, the Lagrangian is that of $U(1)_q \otimes U(1)_{-q}$. In this formulation, it is clear that b has spin $-1/(2q)$ (from the $U(1)_{-q}$ factor), while a has spin 0 mod 1, since $a + b$ has spin $+1/(2q)$ and

$$e^{2\pi i s(a)} = e^{2\pi i s(c-b)} \sqrt{[S]_{c-b,c-b}} = [S]_{c,-b} \sqrt{[S]_{c,c}[S]_{-b,-b}} = 1 \cdot \sqrt{e^{2\pi i/q} e^{-2\pi i/q}} = 1 \implies s(a) =_1 0. \tag{26}$$

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Now let's continue to look at the symmetries of the theory. The $\mathbb{Z}_q^{(1)}$ symmetry which shifts b by $d\phi/q$ is easy to identify: it is generated by the operator

$$\mathbb{Z}_q^{(1)} = \langle e^{i\oint a}\rangle, \tag{27}$$

which has the right qth root of unity phase linking with the $e^{i\oint b}$ line needed to generate the symmetry. Since $e^{i\oint a}$ has trivial self-linking, this symmetry is not anomalous.

Now for the $\mathbb{Z}_l^{(1)}$ symmetry that shifts a lines by $e^{2\pi i/l}$ (recall $l \equiv \gcd(p,q)$). Since a has trivial self-linking, the operator generating this symmetry should include $\exp(iq/l\oint b)$, since the linking of a and b lines gives a phase $e^{2\pi i/q}$. But this operator also shifts b lines, which is bad since b lines are neutral under the $\mathbb{Z}_l^{(1)}$ symmetry. If we tack on a line $e^{i\beta\oint a}$ to the symmetry generator, imposing that the generator link trivially with b tells us that

$$-\frac{pq}{q^2 l} + \frac{\beta}{q} = 0 \implies \beta = p/l. \tag{28}$$

This means that the $\mathbb{Z}_l^{(1)}$ symmetry is generated by the line

$$\mathbb{Z}_l^{(1)} = \langle \exp\left(i\frac{q}{l}\oint b + i\frac{p}{l}\oint a\right)\rangle. \tag{29}$$

What is the anomaly of this symmetry? To find out, we need the self-linking phase of the charge operator. This phase determines the anomaly as

$$\text{Anomaly} = \frac{1}{2}\left(-\frac{p}{q}\left(\frac{q}{l}\right)^2 + 2\frac{1}{q}\frac{qp}{l^2}\right) = \frac{p}{2l^2} \mod 1, \tag{30}$$

where the first term is the self-linking of b and the second is the a-b mutual phase (the factor of $1/2$ is because we want the spin of the charge operator. On spin manifolds, we should take this mod $1/2$ and not mod 1). This is indeed an anomaly appropriate for a $\mathbb{Z}_l^{(1)}$ symmetry, since it is a $\mathbb{Z}_l$ effect, in that $l(p/l^2) = p/l \in \frac{1}{2}\mathbb{Z}$ indicates that l copies of the charge operator is either trivial, or a transparent fermion. One special case that shows up often is when $p = -rq$ and the theory has two $\mathbb{Z}_q^{(1)}$ symmetries. In this case, the anomaly of the $\mathbb{Z}_q^{(1)}$ symmetry that shifts a is $-r/q$.

Finally, note that there's a mixed anomaly, of a $\mathbb{Z}_l$ character, between the two symmetries. This is just due to the fact that the generators for the $\mathbb{Z}_q^{(1)}$ and $\mathbb{Z}_l^{(1)}$ symmetries don't commute: the phase between them is $e^{2\pi i/l}$ (which is trivial if we take l copies of either generator, as it should be).

This conclusions can be corroborated by just going in and trying to gauge the symmetry directly. The symmetry that shifts b is clearly non-self-anomalous, since b only appears in the action by way of its field strength and we can just make the replacement $F_b \mapsto F_b - B_b$, where B_b is the background field for the $\mathbb{Z}_q^{(1)}$ symmetry. However, since the generator for the symmetry that shifts b carries charge under the $\mathbb{Z}_l^{(1)}$ symmetry , adding the B_b field will break the $\mathbb{Z}_l^{(1)}$ symmetry. Indeed, after adding the B_b field the action shifts by the following term under $a \mapsto a + \frac{1}{l}d\phi$:

$$\delta S = \frac{q/l}{2\pi}\int d\phi \wedge B_b \in \frac{1}{l}\overline{\mathbb{Z}}. \tag{31}$$

Thus we recover the $\mathbb{Z}_l$ mixed anomaly between the two 1-form symmetries.

Basically because of the self-CS term for a, the $\mathbb{Z}_l^{(1)}$ symmetry shifting a has a self-anomaly. To find the appropriate characterization of the anomaly, we start from the gauge-invariant bulk action (omitting the Lagrange multipliers that make B_a, B_b quantized appropriately for simplicity)

$$\begin{aligned} S &= \frac{p}{4\pi}\int_M (F_a - B_a)\wedge(F_a - B_a) + \frac{q}{2\pi}\int_M (F_a - B_a)\wedge(F_b - B_b) \\ &= S_{\partial M} + S_{bulk}, \end{aligned} \tag{32}$$

where M is some bounding 4-manifold, and

$$\begin{aligned} S_{\partial M} &= S_{DW_{p,q}} - \frac{1}{2\pi}\int_{\partial M}\left[a\wedge(pB_a + qB_b) + qb\wedge B_a\right], \\ S_{bulk} &= \frac{1}{4\pi}\int_{\partial M}\left[pB_a\wedge B_a + 2qB_a\wedge B_b\right]. \end{aligned} \tag{33}$$

The second line in the above equation parametrizes the anomaly. If we consider the dependence on the choice of M by integrating S_{bulk} over a closed 4-manifold, we see that the first term is valued in $p\overline{\mathbb{Z}}/2l^2$ on a non-spin manifold, and $p\overline{\mathbb{Z}}/2l^2$ on a spin manifold, while the second term is valued in $\overline{\mathbb{Z}}/l$. The quantization of the second term confirms the $\mathbb{Z}_l$ nature of the mixed anomaly, while the quantization of the first term confirms our result for the anomaly of the $\mathbb{Z}_l^{(1)}$ symmetry.

$U(N)_{k,q}$

Our conventions will be such that $U(N)_{k,q}$ is defined though

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathcal{A}\wedge d\mathcal{A} - \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathcal{A}]\wedge d\mathrm{Tr}[\mathcal{A}]. \tag{34}$$

The notation is done like this because q is ($1/N$ times) the effective $U(1)$ level, while k is the effective $SU(N)$ level. The reason why the effective $U(1)$ level is qN can be seen by starting with the decomposition

$$U(N)_{k,q} \cong [SU(N)_k \times U(1)_{qN}]/\mathbb{Z}_N, \tag{35}$$

where the quotient identifies the center of $SU(N)$ with the appropriate Nth roots of unity in $U(1)$. Since the quotient here says that we can freely change transition functions in the $U(1)$ bundle to make the cocycle condition fail by Nth roots of unity so long as we change the transition functions in the $SU(N)$ bundle in the opposite way, the $\mathbb{Z}_N$ quotient is equivalent to gauging the diagonal $\mathbb{Z}_N^{(1)}$ symmetry which acts on both $SU(N)$ and $U(1)$ fields; for the $U(1)$ part to be well-defined its level then needs to be in $N\mathbb{Z}$, as indicated above.

At the level of manipulating actions, we start by decomposing the $U(N)$ field as

$$\mathcal{A} = A + \boldsymbol{\mathcal{A}}\mathbf{1}, \tag{36}$$

where A is an $SU(N)$ field (whose transition functions may fail by Nth roots of unity), $\boldsymbol{\mathcal{A}}$ is a "$U(1)$ field" with transition functions failing in the inverse way—hence $N\boldsymbol{\mathcal{A}}$ is a properly-quantized $U(1)$ field, and $N\int F_{\mathcal{A}} \in \overline{\mathbb{Z}}$. The quotient comes from the correlation of the transition functions between A and $\mathcal{A}$ (more on this when we talk about $SU(N)_k$ in the next subsection). In terms of these fields, we have

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right] + \frac{qN}{4\pi}\mathcal{A}\wedge d\mathcal{A}, \tag{37}$$

so that qN is indeed the "effective $U(1)$ level" as claimed above. To get this we've used that A is traceless and that

$$\mathrm{Tr}[A\wedge A\wedge\mathcal{A}] = \mathcal{A}\wedge\mathrm{Tr}[A\wedge A] = 0 \tag{38}$$

on account of $\mathrm{Tr}[X\wedge Y] = (-1)^{|X||Y|}\mathrm{Tr}[Y\wedge X]$.

Now the $U(1)$ part started out with a $\mathbb{Z}_{qN}^{(1)}$ symmetry pre-gauging. After we gauge to perform the $\mathbb{Z}_N$ quotient though, the quantization condition on $\mathcal{A}$ is modified, so that only $NF_{\mathcal{A}}$ has periods in $\overline{\mathbb{Z}}$. Now let us shift $\mathcal{A}$ by λ, with $d\lambda = 0$. The action changes by

$$\delta S = \frac{q}{2\pi}\int \lambda\wedge(NF_{\mathcal{A}}). \tag{39}$$

Since $NF_{\mathcal{A}}$ is quantized in $\overline{\mathbb{Z}}$, we see that $\delta S\in\overline{\mathbb{Z}}$ provided that $\lambda = \frac{1}{q}d\phi$. Thus we see that the $U(N)_{k,q}$ theory has a $\mathbb{Z}_q^{(1)}$ symmetry, that acts by shifting $\mathcal{A}$.

Is it anomalous? Yes: the charge operator for the remaining $\mathbb{Z}_q^{(1)}$ symmetry is

$$U(p,C) = e^{iNp\oint_C\mathcal{A}}, \qquad p\in\mathbb{Z}_q, \tag{40}$$

with the factor of N needed to perform the shift correctly, and ensures invariance under the gauged diagonal $\mathbb{Z}_N^{(1)}$ symmetry. Computing the braiding phase of the charge operator with itself, we find a phase of $N^2/(Nq)$ since Nq is the effective $U(1)$ level. Thus the anomaly is measured by $N/q \mod 1$. This means in particular that there is no anomaly if $q = N$ (in order for the the theory to be well-defined $q = N$ means $k\in N\mathbb{Z}$). Note that the anomaly of $U(N)_{k,q}$ is the same as the anomaly of N copies of $U(1)_q$.

$SU(N)_k$

Now we look at $SU(N)_k$ CS theory. For all k, this theory has a $\mathbb{Z}_N$ 1-form symmetry, coming from the center of the gauge group. What is the anomaly of the $\mathbb{Z}_N^{(1)}$ symmetry?

Four-dimensional perspective

The easiest way of figuring this out is probably by using what we know about regular four-dimensional pure YM at various values of θ. We know that $\exp(ik\int\mathcal{L}_{CS}[A]/4\pi)$ is the operator which implements the $\theta\mapsto\theta+2\pi k$ similarity transformation in $SU(N)$ YM, where θ is 2π-periodic, and so if we know what the $\theta\mapsto\theta+2\pi k$ shift does in the $PSU(N)$ theory, where the 1-form symmetry has been gauged, we'll be able to say something about the anomaly of the gauged CS theory.

Let us now go partway towards turning the theory into a $PSU(N)$ gauge theory by adding a background $\mathbb{Z}_N$ 2-form field B (we'd get the full $PSU(N)$ theory by path integrating over

B). We went over how to do this in a previous diary entry, but I think the discussion there was a bit confused and long-winded. Here's how it works: we first write the $SU(N)$ theory as

$$\mathcal{L} = \frac{\theta}{8\pi^2}\left(\mathrm{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] - \mathrm{Tr}[F_{\mathcal{A}}] \wedge \mathrm{Tr}[F_{\mathcal{A}}]\right) + \frac{1}{2\pi} F_Y \wedge \mathrm{Tr}[\mathcal{A}]. \tag{41}$$

Here Y is a Lagrange multiplier field, and $\mathcal{A}$ is a $U(N)$ gauge field.[2]

As in the last subsection, we will find it helpful to decompose $\mathcal{A}$ as

$$\mathcal{A} = A + \boldsymbol{\mathcal{A}}\mathbf{1}. \tag{42}$$

Here A is $\mathfrak{su}(N)$-valued and $\boldsymbol{\mathcal{A}}$ is $\mathfrak{u}(1)$-valued. However, A is not a connection on a $SU(N)$ bundle, and $\boldsymbol{\mathcal{A}}$ is not a connection on a $U(1)$ bundle: rather, the transition functions g_A and $g_{\boldsymbol{\mathcal{A}}}$ satisfy

$$\delta g_A \delta g_{\boldsymbol{\mathcal{A}}} = \mathbf{1}, \qquad \delta g_A, \delta g_{\boldsymbol{\mathcal{A}}} \in \mathbf{1} e^{\frac{2\pi i}{N}\mathbb{Z}}. \tag{43}$$

In this description, we have a gauge transformation whereby the transition functions g_A and $g_{\boldsymbol{\mathcal{A}}}$ change by opposite roots of unity. Note that this means that only $NF_{\boldsymbol{\mathcal{A}}}$ is a legit $U(1)$ field strength.

Anyway, let's see why this is equivalent to the $SU(N)$ theory. We just integrate out Y: this sets $F_{\boldsymbol{\mathcal{A}}} = 0$ and the sum over $[F_Y] \in H^2(X;\mathbb{Z})$ tells us that we can set $\boldsymbol{\mathcal{A}} = \frac{1}{N} d\phi$, for $d\phi$ a large gauge transformation. The flatness constraint tells us that we will always have $\delta g_{\boldsymbol{\mathcal{A}}} = \mathbf{1}$ (since a nontrivial $\delta g_{\boldsymbol{\mathcal{A}}}$ would contribute to the 1st Chern class), and hence $\delta g_A = \mathbf{1}$ as well: now both $SU(N)$ and $U(1)$ factors are legitimate bundles. Additionally, such an $\boldsymbol{\mathcal{A}}$ can be completely absorbed into a change of the $g_{\boldsymbol{\mathcal{A}}}$ transition functions by Nth roots of unity (the transition functions change by *constants* on each double-overlap). These transition functions can then be absorbed into the g_A transition functions, and so the $\boldsymbol{\mathcal{A}}$ field completely disappears, leaving us with an $SU(N)$ action, as required.

The theory has a $\mathbb{Z}_N^{(1)}$ symmetry that comes from twisting the transition functions in the $SU(N)$ bundle by Nth roots of unity. In our $U(N)$ formulation, this is equivalent to shifting the g_A by Nth roots of unity, which in turn is equivalent to keeping the g_A transition functions fixed, but making a shift $\boldsymbol{\mathcal{A}} \mapsto \boldsymbol{\mathcal{A}} + \frac{1}{N} d\phi$. To gauge this symmetry then, we should make the replacement $F_{\mathcal{A}} \mapsto F_{\mathcal{A}} - B\mathbf{1}$. In what follows we will take B to be some fixed background field with periods in $2\pi/N$ around all closed 2-cycles. This gives us the Lagrangian

$$\mathcal{L} = \frac{\theta}{8\pi^2}\left(\mathrm{Tr}[(F_{\mathcal{A}} - B\mathbf{1})^{\wedge 2}] - (\mathrm{Tr}[F_{\mathcal{A}} - B\mathbf{1}])^{\wedge 2}\right) + \frac{1}{2\pi} Y \wedge \mathrm{Tr}[F_{\mathcal{A}} - B\mathbf{1}]. \tag{44}$$

Consider integrating out Y. This sets $F_{\boldsymbol{\mathcal{A}}} = B$, which means that $F_{\boldsymbol{\mathcal{A}}}$ becomes quantized in periods of $2\pi/N$. Because of the connection between the transition functions of the $SU(N)$ and $U(1)$ bundles, we then erase $F_{\mathcal{A}} - B\mathbf{1}$ from the action and get

$$\mathcal{L} = \frac{\theta}{8\pi^2}\mathrm{Tr}[F_A \wedge F_A], \qquad w_2(E_{PSU(N)}) = \frac{N}{2\pi} B \in H^2(X;\mathbb{Z}_N), \tag{45}$$

[2]The second term in the parenthesis ensures that, because the full term in parenthesis is the second Chern class of a $U(N)$ field, we have $\theta \sim \theta + 2\pi$ identically, without having to first integrate out Y. This is desired because $\theta \sim \theta + 2\pi$ in the $SU(N)$ theory (the $SU(N)_k$ CS theory is not spin; more on this later), while if the second term in parenthesis were absent we might not have such a periodicity.

where A is now a connection on a $PSU(N)$ bundle $E_{PSU(N)}$. Thus we see the role of B is to turn the $SU(N)$ connection into a $PSU(N)$ connection, with the topological class of the $PSU(N)$ bundle controlled by the cohomology class of B. When B gets integrated over, we perform a sum over all $PSU(N)$ bundles, and obtain a genuine $PSU(N)$ gauge theory.

In a previous diary entry we saw that the 2π periodicity in θ is lost in the $PSU(N)$ theory, and instead that changing θ by 2π induces a shift in the action given by a counterterm in B. Indeed, we can write $\mathcal{L}$ as

$$\mathcal{L} = \theta c_2(E_{U(N)}) + \frac{\theta}{4\pi}\left(-\mathrm{Tr}[F_{\mathcal{A}} \wedge B\mathbf{1}] + N\mathrm{Tr}[F_{\mathcal{A}}] \wedge B + \frac{N-N^2}{2} B \wedge B\right) + \frac{1}{2\pi} Y \wedge \mathrm{Tr}[F_{\mathcal{A}} - B\mathbf{1}]. \tag{46}$$

The advantage of writing it this way is that the $SU(N)$ part of $F_{\mathcal{A}}$ has completely disappeared into the second Chern class of the $U(N)$ bundle. Now integrating out Y, we have (using $\mathrm{Tr}[F_{\mathcal{A}} \wedge B\mathbf{1}] = 0$)

$$\mathcal{L} = \theta c_2(E_{U(N)}) + \frac{\theta}{4\pi}\left(-N + N^2 + \frac{N-N^2}{2}\right) B \wedge B, \tag{47}$$

where now the $U(N)$ bundle $E_{U(N)}$ is constrained to have first Chern class $c_1(E_{U(N)}) = \frac{N}{2\pi} B \in H^2(X;\mathbb{Z})$. Since the second Chern class is integral, under $\theta \mapsto \theta + 2\pi$ the action shifts as (modulo elements of $2\pi\mathbb{Z}$)

$$S_\theta \mapsto S_\theta + \frac{1}{4\pi}(N^2 - N) \int B \wedge B, \tag{48}$$

This is nontrivial, since $\int B \wedge B \in \mathbb{Z}/N^2$ ($\in 2\mathbb{Z}/N^2$) on generic (spin) closed 4-manifolds, and hence θ is actually not 2π periodic.

Anyway, the point is the following: consider a domain wall where θ jumps by 2π. We know that such a domain wall can be created by inserting $\exp(i\int_X \mathcal{L}_{CS}[A]/4\pi)$ into the path integral, where X is a 3-manifold defining the domain wall. By the above discussion, we know that the action differs on the two sides of the domain wall by a $B \wedge B$ counterterm in the background field. However, integrating $B \wedge B$ over an open submanifold of spacetime is not a gauge-invariant thing to do! Doing a gauge transformation on B produces an anomalous term, consisting of an integral over the codimension-1 submanifold X:

$$\delta S = \frac{i}{4\pi}(N-1) \int_X \mathrm{Tr}[2B \wedge \lambda + \lambda \wedge d\lambda], \tag{49}$$

for $\delta B = d\lambda$ (and we are tacitly writing e.g. B for $\mathbf{1}B$). Since we know that $PSU(N)$ gauge theory in four dimensions is self-consistent, this anomaly must be canceled by an anomaly of the $SU(N)_1$ CS theory.

The anomaly is determined by looking at how the shift in S_θ depends on the bounding 4-manifold. Integrating it over a closed 4-manifold tells us that $e^{i\delta S_\theta} = e^{2\pi l \frac{N-1}{2N}}$ for some $l \in \mathbb{Z}_N$. Thus we can conclude that the CS theory $SU(N)_1$ has anomaly $(N-1)/2N$ mod 1. The anomaly for $SU(N)_k$ must then be $k(N-1)/(2N) \mod 1$, since $SU(N)_k$ is the theory defined by the similarity transform on the codimension-1 slice where the $\delta\theta = 2\pi k$ domain wall happens, and the gauge-non-invariance of the bulk action in the presence of the domain wall is exactly k times the result when the θ angle jumps by 2π. So, the theory has an anomaly given by

$$\text{Anomaly} = \frac{k(N-1)}{2N} \mod 1 \quad (\text{mod}\, 1/2 \text{ if spin}). \tag{50}$$

Here the reduced anomaly for the spin case comes from the fact that the intersection form is then even, which limits the phases that δS_θ in (48) can take when integrated over closed 4-manifolds. Actually, we can do a bit better: if $k \in 2\mathbb{Z}$ then the N^2 part of (49) is trivial on all manifolds, and so we can effectively say that the anomaly is just $-k/N$ if $k \in 2\mathbb{Z}$.

Three-dimensional perspective

Now let's look at this from the three-dimensional perspective directly. One naive way to write the $SU(N)_k$ theory is to write

$$\mathcal{L} = \frac{k}{4\pi}\text{Tr}\left[\mathcal{A}\wedge d\mathcal{A} + \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right] + \frac{1}{2\pi}y\wedge d\text{Tr}[\mathcal{A}], \tag{51}$$

where y is a Lagrange multiplier that roughly speaking turns the $U(N)$ field $\mathcal{A}$ (a $\mathfrak{u}(N)$-valued form) into an $\mathfrak{su}(N)$-valued 1-form. This is not completely correct, however, since if k is odd this theory is spin, while we know that $SU(N)_k$ is non-spin for any value of k (because the $SU(N)$ instanton number is equal to $2\pi\int \mathrm{c}_2(E)$ where $\mathrm{c}_2(E)$ is the second Chern class, which is integral on all closed manifolds, spin or not).

To fix this, we will add a $U(1)_p$ term using the $U(1)$ field $\text{Tr}[\mathcal{A}]$. Note that we are free to shift the definition of the Lagrange multiplier field by

$$y \mapsto y \pm \text{Tr}[\mathcal{A}] \tag{52}$$

(since $\text{Tr}[\mathcal{A}]$ is a properly quantized $U(1)$ field), which changes p by ± 2. So, to find out how to render the theory non-spin, we just need to find out the correct parity to use for p.

Anyway, to get the answer for the correct non-spin theory, we write the full Lagrangian as

$$\mathcal{L} = \frac{k}{4\pi}\text{Tr}\left[\mathcal{A}\wedge d\mathcal{A} - \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right] + \frac{\eta_k}{4\pi}\text{Tr}[\mathcal{A}]\wedge d\text{Tr}[\mathcal{A}] + \frac{1}{2\pi}y\wedge d\text{Tr}[\mathcal{A}], \tag{53}$$

where η_k is to be determined. The integral needing to be done to check the quantization condition on η_k is

$$I = \frac{2\pi}{8\pi^2}\int_{M_4}\left(k\text{Tr}[F_\mathcal{A}\wedge F_\mathcal{A}] + \eta_k\text{Tr}[F_\mathcal{A}]\wedge\text{Tr}[F_\mathcal{A}] + 2dy\wedge d\text{Tr}[\mathcal{A}]\right) \tag{54}$$

for some closed 4-manifold M_4. The last term is always in $\overline{\mathbb{Z}}$, while the first two can be written as

$$I = 2\pi\int c_2(E_{U(N)}) + \pi(k+\eta_k)\int \text{Tr}[F_\mathcal{A}/2\pi]\wedge\text{Tr}[F_\mathcal{A}/2\pi], \tag{55}$$

where $\mathrm{c}_2(E_{U(N)})$ is the second Chern class of the $U(N)$ bundle. Since this is always an integral class regardless of the base space of the bundle, we conclude that we need $k+\eta_k$

to be even. Thus we can take e.g. $\eta_k = 0$ if $k \in 2\mathbb{Z}$, and $\eta_k = -1$ if $k \in 2\mathbb{Z}+1$. Another (simpler) choice (and the one we will adopt) is to simply set $\eta_k = -k$, which as we mentioned above is equivalent since η_k and $\eta_k \pm 2$ define equivalent theories. Adopting this choice, we have

$$\mathcal{L}_{SU(N)_k}[A] = \mathcal{L}_{U(N)_{k,k(1-N)}}[\mathcal{A}] + \frac{1}{2\pi} y \wedge d\mathrm{Tr}[\mathcal{A}]. \tag{56}$$

Thus $SU(N)_k$ is realized as a constrained version of a $U(N)_{k,q}$ theory at $q = k(1-N)$. The freedom to shift y by $\pm\mathrm{Tr}[\mathcal{A}]$ manifests itself in the equivalence $q \sim 2N$.

As we saw previously, we can split up $\mathcal{A}$ into an $SU(N)$ part A and a diagonal part $\boldsymbol{\mathcal{A}}$, provided that the cocycle conditions for the A and $\boldsymbol{\mathcal{A}}$ parts fail in canceling ways. Recall the decomposition $\mathcal{A} = A + \boldsymbol{\mathcal{A}}\mathbf{1}$. With this decomposition, in order to implement the matching-cocycle-conditions property, we require that the diagonal transformation shifting the transition functions for both A and $\boldsymbol{\mathcal{A}}$ by opposite Nth roots of unity be a gauge transformation. Note that we can do such a shift while keeping A traceless, since we are only changing the transition functions by constants: the change in transition functions is done at the level of the glueing data between patches, not at the level of the 1-forms A defined on single patches. By contrast, when we perform such a shift on $\boldsymbol{\mathcal{A}}$, we will do it by directly taking $\delta\boldsymbol{\mathcal{A}} = \frac{1}{N} d\phi$ (ϕ as usual is 2π-periodic), without changing the transition functions for the $\boldsymbol{\mathcal{A}}$ bundle. Either way we do it, the effect of this identification is to gauge a diagonal $\mathbb{Z}_N^{(1)}$ symmetry that shifts both A and $\boldsymbol{\mathcal{A}}$. The transformation acts nontrivially on A Wilson lines since they are defined by $\mathrm{Tr}[e^{i\int_{U_\alpha} A} e^{i\Lambda_{\alpha\beta}} e^{i\int_{U_\beta} A} \dots]$, with $\Lambda_{\alpha\beta}$ the transition functions between patches, and since the transformation shifts the $\Lambda_{\alpha\beta}$'s. Note that this gauge transformation, while not changing the field strength $F_{\mathcal{A}}$, *does* change the field strengths of A and $\boldsymbol{\mathcal{A}}$: if we make the cocycle condition fail by an Nth root of unity on a given triple overlap of patches, then this induces fractional flux in both A and $\boldsymbol{\mathcal{A}}$.

Now we can get a more precise understanding of what the Lagrange multiplier y is doing. Integrating out y tells us that $d\boldsymbol{\mathcal{A}} = 0$, and that $\int \boldsymbol{\mathcal{A}} \in \frac{1}{N}\overline{\mathbb{Z}}$ around all closed 1-manifolds. Thus we may write $\boldsymbol{\mathcal{A}} = \frac{1}{N} d\phi$. But we see that this is gauge-equivalent to $\boldsymbol{\mathcal{A}} = 0$ under the 1-form gauge symmetry. So, integrating out y leaves us with just the $SU(N)_k$ part of the action, which is what we want.

* * *

Returning to $\mathcal{L}$, we have

$$\mathcal{L} = \frac{k}{4\pi}\left(\mathrm{Tr}\left[A \wedge dA - \frac{2i}{3} A \wedge A \wedge A\right] + (N - N^2)\boldsymbol{\mathcal{A}} \wedge d\boldsymbol{\mathcal{A}}\right) + \frac{N}{2\pi} y \wedge d\boldsymbol{\mathcal{A}}, \tag{57}$$

again using the tracelessensss of A and the antisymmetry to kill the $A \wedge A \wedge \boldsymbol{\mathcal{A}}$ contribution. Note the $k(N-N^2)$ level of the $\boldsymbol{\mathcal{A}}$ CS term: peeking back at the analysis of the bulk gauge theory, we see that this is exactly the right number needed to cancel the bulk anomaly, and is a hint that we're on the right track.

Let's pause to figure out what the symmetry is. We started with a pure $SU(N)_k$ CS term, which as we know has a $\mathbb{Z}_N^{(1)}$ symmetry. We then wrote it in terms of a $U(N)_{k,q}$ theory plus a Lagrange multiplier, where for us we chose $q = k(1-N)$. As we saw earlier, the

$U(N)_{k,q}$ theory by itself has a $\mathbb{Z}_q^{(1)}$ global symmetry. This symmetry is generically broken by the Lagrange multiplier term, since under it we have

$$\delta S = \frac{N}{2\pi k(1-N)} \int F_y \wedge d\phi \notin \overline{\mathbb{Z}}. \tag{58}$$

So, does this mean that we have no 1-form symmetry? This would be a problem if so. But actually, the $\mathbb{Z}_N^{(1)}$ symmetry that we need to be there does exist. To see how it works, consider shifting $\mathcal{A}$ by some flat 1-form λ. The action changes as

$$\delta S = \frac{k(1-N)}{2\pi} \int \lambda \wedge (NF_{\mathcal{A}}) + \frac{N}{2\pi} \int F_y \wedge \lambda. \tag{59}$$

In order for the last term to be in $\overline{\mathbb{Z}}$, we see that we need to take $\lambda = d\phi/N$. Then the variation in S is

$$\delta S = \frac{k(1-N)/N}{2\pi} \int \lambda \wedge (NF_{\mathcal{A}}). \tag{60}$$

This is in general nontrivial, but we see that we can cancel it, if we take the symmetry transformation to involve an appropriate shift in y as well. This gives us a genuine $\mathbb{Z}_N^{(1)}$ symmetry, under which we have

$$\mathbb{Z}_N^{(1)} : \mathcal{A} \mapsto \mathcal{A} + \frac{1}{N} d\phi, \qquad y \mapsto y - \frac{k(1-N)}{N} d\phi. \tag{61}$$

If this is the right symmetry, it should shift fundamental Wilson lines by Nth roots of unity. And indeed it does:

$$\mathbb{Z}_N^{(1)} : W_f(C) = \mathrm{Tr}_f[e^{i\oint_C (A + \mathcal{A}\mathbf{1})}] \mapsto e^{i\frac{1}{N}\oint_C d\phi} W_f(C) \tag{62}$$

(note that $e^{i\oint \mathcal{A}}$ is not a gauge-invariant operator to consider the transformation properties of). Note that $e^{i\oint y}$ also shifts under the symmetry, so that it must also be electrically charged. More on this in a bit.

What is the operator which generates this symmetry? It turns out to be $\exp(i\oint y)$. This is rather surprising, since looking at the action one might be forgiven for thinking that the y line was bosonic.

To find the statistics of the y line, it is helpful to Higgs the theory down to $\mathbb{Z}_N$. In terms of the $SU(N)$ variables, the effect of the Higgsing is to leave the theory with only $\mathbb{Z}_N$ transition functions as degrees of freedom. In the continuum, it's easier to deal with this condition by writing the transition functions instead as diagonal $\mathbb{Z}_N$ 1-form matrices, with trivial transition functions. So to that end, Higgsing for us at the computational level means taking $A = 0$ and $\mathcal{A} = a$, with a a $\mathbb{Z}_N$ field. Since $A = 0$ and A has trivial transition functions, the cocycle condition will be satisfied exactly for a, and the flux of F_a will be quantized in the regular way. So, upon doing this, we get the $DW_{p,q}$ theory with $p = kN(1-N)$, $q = N$:

$$SU(N)_k \xrightarrow{Higgs} \frac{kN(1-N)}{4\pi} ada + \frac{N}{2\pi} y \wedge da. \tag{63}$$

Note that in addition to the $\mathbb{Z}^{(1)}_{\gcd(kN(1-N),N)} = \mathbb{Z}_N^{(1)}$ symmetry, we also have a symmetry that shifts y by a $\mathbb{Z}_N$ gauge field. The appearance of this magnetic symmetry is expected after we move from $SU(N)$ (which has no t'Hooft line operators since $\pi_1(SU(N)) = 0$) to $\mathbb{Z}_N$ (which does have magnetic operators since we can have $\mathbb{Z}_N$ branch cuts in the transition functions).

We've already been through this theory in lots of detail, and we learned that the mutual statistics between the a and y lines are[3]

$$[S]_{a,a} = 1, \quad [S]_{a,y} = e^{2\pi i/N}, \quad [S]_{y,y} = e^{-2\pi i k(1-N)/N}. \tag{64}$$

Recall from a ways back that we could perform a change of variables on y that shifted $k(1-N) \mapsto k(1-N) \pm 2N$. We see that this leaves the braiding phases invariant (and because of the factor of 2, it also leaves the spins invariant), and so reassuringly the shift indeed acts trivially on the modular data of the theory.

From the above entries of the S matrix, we see that the line $e^{i\oint y}$ generates the $\mathbb{Z}_N^{(1)}$ symmetry of $SU(N)_k$, since these braiding phases mean that wrapping lines with the line $e^{i\oint_C y}$ is equivalent to performing the shift (61) (where $d\phi$ is determined by the topology of C).

We can now easily figure out the anomaly: from taking the square root of $[S]_{y,y}$ to get the spin of the generating line, we read off the anomaly as $k(1-N)/2N \mod 1$. If we are on a spin manifold then having the generating line be a fermion is okay, and so in that case the anomaly is $k(1-N)/2N \mod 1/2$. Note that this is exactly the right anomaly to cancel the bulk anomaly that we derived earlier in (50)! Nice. Note that the anomaly of $SU(N)_k$ is the same as that of $[SU(N)_1]^{\otimes k}$, because of the constant k prefactor. Also note that since $N-1$ is coprime to N, we will only have a non-anomalous theory if $k \in 2N\mathbb{Z}$ (or $k \in N\mathbb{Z}$ if spin).

Can we say anything about this line in the $SU(N)$ context? Yes: under the $\mathbb{Z}_N^{(1)}$ symmetry we have

$$e^{i\oint y} \mapsto e^{2\pi i k/N} e^{i\oint y}. \tag{65}$$

Since Wilson lines in the fundamental transform with a $e^{2\pi i/N}$ phase, this tells us that the generator $e^{i\oint y}$ can be identified with a Wilson line in a k index symmetric $SU(N)$ representation. This makes sense because $e^{i\oint y}$ is the operator we get when slicing open the 2-dimensional surface operator which implements the $\mathbb{Z}_N^{(1)}$ symmetry in the 3+1 D theory. Now the $SU(N)_k$ theory lives at an interface where the bulk θ angle changes by $2\pi k$. The Witten effect means that the t'Hooft operators on both sides of the surface (which are not genuine line operators) have electric charges differing by k. This k difference in electric charges is realized by the fact that the charge operator on the interface, namely $e^{i\oint y}$, carries electric charge k.

This is a manifestation of the mixed anomaly between the $\mathbb{Z}_N^{(1)}$ symmetry and time reversal at $\theta \in \pi(2\mathbb{Z}+1)$. Indeed, consider a 2π domain wall for θ, where θ jumps from $-\pi$ to π. The operator which inserts this domain wall is the charge operator for T, since it interpolates between the two ground states (which differ by $\theta \mapsto -\theta$). The mixed anomaly comes from the fact that this domain wall operator and the surface operator which implements the $\mathbb{Z}_N^{(1)}$ symmetry don't commute: indeed, they do not commute because of a contact term, and their lack of commutativity can be seen from the fact that along their intersection is

[3] The N/N factor in the $[S]_{y,y}$ matrix element is important, since when square-rooted it contributes to the spin of the y line. However, it does not affect how y lines transform under the $\mathbb{Z}_N^{(1)}$ symmetry.

a fundamental Wilson line (since we are in four dimensions, a 3-manifold and a 2-manifold intersect at a 1-manifold).

As we have seen, if we try to gauge the $\mathbb{Z}_N^{(1)}$ symmetry in the 2+1D theory, we run in to problems since the operators which perform the gauge transformations (the fundamental Wilson lines) do not commute with each other. This can be fixed by using the same procedure as in the $U(1)_k$ case. First, we write the action for the theory as

$$\begin{aligned} S = &\frac{k}{4\pi}\int_{\partial X} \mathrm{Tr}\left[\mathcal{A}\wedge d\mathcal{A} + \frac{2i}{3}\mathcal{A}^3\right] - \frac{k}{4\pi}\int_{\partial X}\mathrm{Tr}[\mathcal{A}]\wedge \mathrm{Tr}[F_{\mathcal{A}}] + \frac{k(N-1)}{2\pi}\int_{\partial X} B\wedge \mathrm{Tr}[\mathcal{A}] \\ &+ \frac{1}{2\pi}\int_{\partial X}(y\wedge \mathrm{Tr}[F_{\mathcal{A}} - B\mathbf{1}] + N\Phi\wedge B) - \frac{N}{2\pi}\int_X F_\Phi \wedge B + \frac{k(N-N^2)}{4\pi}\int B\wedge B. \end{aligned} \tag{66}$$

To get this, we just took the four-dimensional gauged action (with Φ a 1-form Lagrange multiplier to make B a $\mathbb{Z}_N$ field), and integrated by parts. The extra $N\Phi\wedge B$ term is needed to make things gauge invariant, as we will see.

The transformation rules for the fields are as follows. First, we have a gauge transformation under which $\delta y = \delta\Phi = d\alpha$. Next, we have a $\mathbb{Z}_N^{(1)}$ gauge transformation, generated by $e^{i\oint_C y}$, which shifts

$$\mathcal{A}\mapsto \mathcal{A} + \frac{2\pi}{N}\widehat{C}, \quad B\mapsto B + \frac{2\pi}{N}d\widehat{C}, \quad y\mapsto y + \frac{2\pi k}{N}\widehat{C}, \quad \Phi\mapsto\Phi + \frac{2\pi k}{N}\widehat{C}. \tag{67}$$

Here $\widehat{C}$ is the Poincare to a possibly open curve in ∂X, with the Poincare dual having some arbitrary extension into the bulk X. Here $\widehat{C}$ is such that $\int_{C'}\widehat{C}\in\mathbb{Z}$ for all $C'\in C_1(\partial X;\mathbb{Z})$, but where the value for the integral may depend on more than just the homotopy class of C'. One can check that the action is invariant up to the term $-\frac{k}{2\pi}\int\widehat{C}\wedge F_B$, which is in $\overline{\mathbb{Z}}$ because of the quantization on F_B. Thus, the whole action is gauge-invariant.

Anyway, these transformation laws let us write down the correct, gauge invariant, generator of gauge transformations for the gauged $\mathbb{Z}_N^{(1)}$ symmetry. It is

$$U(q,C) = \exp\left(iq\left[\int_C y + \int_{C'}\Phi + k\int_D B\right]\right). \tag{68}$$

Here $C\cup C' = \partial D$, with C' only in the bulk; see Figure 1 for a similar setup. Do these operators commute with each other? Yes! $U(q,C)$ and $U(p,\widetilde{C})$, with C, $\widetilde{C}$ two intersecting curves in ∂X will have a contribution to their commutator of the form $e^{2\pi iqpk/N}$, which comes from the commutator of the two y lines. However, they will also have a compensating contribution from the commutator between the $\int_{\widetilde{C}'}\Phi$ line and the $k\int_D B$ surface (which intersect in the bulk), since Φ and B have a braiding phase of $e^{2\pi i/N}$. Thus the $U(q,C)$ are indeed legit generators of the $\mathbb{Z}_N^{(1)}$ gauge transformations.

Summary

The theories that we've looked at are

$$
\begin{aligned}
&U(1)_k : \frac{k}{4\pi} A \wedge dA \\
&DW_{p,q} : \frac{p}{4\pi} a \wedge da + \frac{q}{2\pi} a \wedge db \\
&U(N)_{k,q} : \frac{k}{4\pi}\mathrm{Tr}[\mathcal{A} \wedge d\mathcal{A} + 2i/3\mathcal{A}^3] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathcal{A}] \wedge d\mathrm{Tr}[\mathcal{A}] \qquad (k-q) \in N\mathbb{Z} \\
&SU(N)_k : U(N)_{k,k(1-N)} + \frac{1}{2\pi} y \wedge d\mathrm{Tr}[\mathcal{A}].
\end{aligned} \tag{69}
$$

The symmetries and anomalies (on general, non spin manifolds, provided that the theory is not spin) are

	1-form symmetry	Anomaly (mod 1)	Spin?
$U(1)_k$	$\mathbb{Z}_k$	$1/k$	if $k \in 2\mathbb{Z}+1$
$DW_{p,q}$	$\mathbb{Z}_q$ on b, $\mathbb{Z}_{\gcd(p,q)}$ on a	0,$p/\gcd(p,q)$,$1/\gcd(p,q)$ (mixed)	if $p \in 2\mathbb{Z}+1$
$U(N)_{k,q}$,	$\mathbb{Z}_q$	N/q	if $k+(q-k)/N \in 2\mathbb{Z}+1$
$SU(N)_k$	$\mathbb{Z}_N$	$(k-Nk)/2N$	No

(70)

Here the anomaly is determined by taking the mod 1 residue of the entry in the third column. In the last column we have indicated when the theories are spin, which will is determined in a different diary entry.

One interesting thing is to check how this is compatible with known level-rank dualities. For example, consider the duality $U(1)_N \leftrightarrow SU(N)_1$ (it is usually $U(1)_{-N}$, but in these conventions the anomalies are such that we write it as $U(1)_N$). This duality hold holds as spin TQFTs. Indeed, while they have the same $\mathbb{Z}_N^{(1)}$ symmetry, let's compare their anomalies: for $U(1)_{-N}$ we have $1/N$, while for $SU(N)_1$ we have $(1-N)/2N$. These are of course not the same. But, on a spin manifold, the anomaly of $SU(N)_k$ is actually $(k-Nk)/N \mod 1$ since the generator of the $\mathbb{Z}_N^{(1)}$ symmetry is allowed to be a fermion. Setting $k=1$ the anomaly becomes $1/N$, which matches that of the $U(1)_N$ theory.

References

[1] P.-S. Hsin, H. T. Lam, and N. Seiberg. Comments on one-form global symmetries and their gauging in 3d and 4d. *arXiv preprint arXiv:1812.04716*, 2018.