Details about thermodynamics in large-N ferromagnets

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Preface:

In these notes we will be discussing in some detail the thermodynamics of 2+1D quantum ferromagnets, treated within a large-N mean field approach. This was inspired by wanting to work out details implicitly contained in the classic paper [7], as well as to develop some technology to understand a problem in TBG (which it turned out to not be so useful for).

1 Field theory and scaling

In what follows we will be looking at the thermodynamic properties of insulating quantum Hall ferromagnets. This introductory section will essentially just be an elaboration on the arguments in [7].

The $i\mathbb{R}$ time continuum action for such a system is[1]

$$S = \int d^dx\, d\tau\, \left(m_0\mathcal{L}_{WZW} + \frac{\rho}{2}|\nabla\mathbf{n}|^2 - m_0\mathbf{H}\cdot\mathbf{n}\right), \tag{1}$$

where m_0 is the magnetization density, ρ the spin stiffness, $\mathcal{L}_{WZW}$ the WZW term endowing $\mathbf{n}$ with the right commutation relations and where the time integral is over the thermal circle. The way the WZW term works is clear in the context of quantum mechanics, but here the quantization is a bit more subtle. The well-definedness of the WZW term requires that $2m_0\int d^dx \in \mathbb{Z}$ — naively it should associate a spin 1/2 volume of size $1/m_0$, but what sort of commutation relations would be required to produce this aren't clear. Therefore one should view this action as only making sense so long as we are interested in scales much greater than $m_0^{-1/d}$.

The WZW term in (1) can be simplified in the usual way by passing to the $\mathbb{CP}^1$ representation via $\mathbf{n} = z^\dagger\boldsymbol{\sigma}z$, with the $\boldsymbol{\sigma}$ selecting out the 1 in $1/2\otimes 1/2 = 0\oplus 1$, and where $|z|^2 = 1$. As an example of how the algebra works, the kinetic term maps as

$$\frac{\rho}{2}|\nabla\mathbf{n}|^2 = \rho\left(\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ij}\delta_{kl}\right)(z_i^*\overleftrightarrow{\nabla}z_j)(z_k^*\overleftrightarrow{\nabla}z_l) = 2\rho\left(|\nabla z|^2 - |z^\dagger\nabla z|^2\right), \tag{2}$$

where we have replaced $(z^\dagger\nabla z)^2 \to -|z^\dagger\nabla z|^2$ which is valid after using $|z|^2 = 1$ and dropping total derivatives. The WZW term goes to $2m_0 z^\dagger\partial_\tau z$.[2] Decoupling the quartic interaction above with a vector field $\mathbf{a}$, we obtain

$$S = 2\int d^dx\, d\tau\, z^\dagger\left(m_0\partial_\tau - \rho D_{\mathbf{a}}^2 + \lambda - m_0 HZ/2\right)z - 2\int d^dx\, d\tau\, \lambda, \tag{6}$$

[1] We have absorbed $g\mu_B$ into the definition of H, so that H has dimensions of energy. m_0 has units of density.

[2] This is rather obvious if one simply identifies the generators of $H_2(S^2) = \mathbb{Z}$ on both sides. However I've never seen the algebra actually worked out, so here we will give an explicit proof. It first helps to write the action density of the WZW term as

$$s_{WZW} = 2m_0\int \mathrm{Tr}[\widehat{n}\, d\widehat{n}\wedge d\widehat{n}], \tag{3}$$

where $\widehat{n} \equiv \mathbf{n}\cdot\boldsymbol{\sigma}/2$ so that $n^a = \mathrm{Tr}[\sigma^a\widehat{n}]$, and the integral is over a two-manifold which bounds the thermal circle. Note that another way of writing $\widehat{n}$ is as $\widehat{n} = |z\rangle\langle z| - \mathbf{1}/2$, which follows from "$1 = 1/2\otimes 1/2 - 0$".

where λ is a Lagrange multiplier enforcing $|z|^2 = 1$ and where we have taken the field to point along the z direction without loss of generality.

Consider a rescaling of space and time such that $x \mapsto \eta x, \tau \mapsto \gamma\tau$. Then correlation functions are invariant at tree-level if we map the couplings and the temperature as

$$m_0 \mapsto \eta^{-d}\gamma^0 m_0, \qquad \rho \mapsto \eta^{-d+2}\gamma^{-1} m_0, \qquad H \mapsto \eta^0\gamma^{-1}H, \qquad T \mapsto \eta^0\gamma^{-1}T. \tag{7}$$

The singular part of the free energy density at tree level thus obeys[3]

$$f_s(m_0, T, \rho, H) = \eta^d\gamma f_s(m_0\eta^{-d}, \rho\eta^{-d+2}\gamma^{-1}, H\gamma^{-1}, T\gamma^{-1}). \tag{8}$$

Therefore choosing $\eta = m_0^{1/d}$ and $\gamma = T$ in (8), we have

$$f_s(m_0, T, \rho, H) = m_0 T\Phi(\rho/m_0^{1-2/d}T, H/T) = m_0 T\Phi(r, h), \tag{9}$$

where the scaling function Φ is a function of the scaling variables

$$r \equiv \rho m_0^{2/d-1}T^{-1}, \qquad h \equiv HT^{-1}. \tag{10}$$

The thing that is special about this type of system is that the RG flows are very strongly constrained. First, note that the constraint $m_0 \int d^dx \in \frac{1}{2}\mathbb{Z}$ implies that the scaling of m_0 above cannot receive any corrections. In fact, all of the terms in (6) flow in a way exactly determined by their scaling dimensions. Indeed, when $T = 0$, we get a theory of non-relativistic bosons in 2+1D. The fact that the z bosons are non-relativistic means that there are no diagrams in which virtual spin waves are created: this means that there is no

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Using this representation, we have

$$\begin{aligned} s_{WZW} &= m_0 \int \mathrm{Tr}\left[(2\left|z\right\rangle\left\langle z\right| - \mathbf{1})\left(\left|dz\right\rangle\left\langle z\right| + \left|z\right\rangle\left\langle dz\right|\right) \wedge \left(\left|dz\right\rangle\left\langle z\right| + \left|z\right\rangle\left\langle dz\right|\right)\right] \\ &= m_0 \int \mathrm{Tr}\left[(2\left|z\right\rangle\left\langle z|dz\right\rangle\langle z| + \left|z\right\rangle\left\langle dz\right| - \left|dz\right\rangle\left\langle z\right|) \wedge \left(\left|dz\right\rangle\left\langle z\right| + \left|z\right\rangle\left\langle dz\right|\right)\right] \\ &= m_0 \int \left(2\langle z|dz\rangle \wedge \langle z|dz\rangle + 2\langle z|dz\rangle \wedge \langle dz|z\rangle + 2\langle dz| \wedge |dz\rangle + \langle dz|z|\rangle \wedge \langle dz|z\rangle - \langle z|dz\rangle \wedge \langle z|dz\rangle\right) \\ &= 2m_0 \int \langle dz| \wedge |dz\rangle \\ &= 2im_0 \int da, \end{aligned} \tag{4}$$

where $a = -i\langle z|dz\rangle$ and we have used the supercommutativity of $\wedge$ and the fact that $\langle dz|z\rangle = -\langle z|dz\rangle$. Hence the WZW term becomes

$$S_{WZW} = 2im_0 \int d^dx\, d\tau\, a_\tau = 2m_0 \int d^dx\, d\tau\, z^\dagger \partial_\tau z, \tag{5}$$

as claimed.

[3]For notational simplicity, we will be discussing scaling with dimensionful couplings. All couplings can be rendered dimensionless using a cutoff Λ, which has dimensions of inverse length. Because we do not expect (1) to actually make sense in the continuum we will never be sending $\Lambda \to \infty$; instead we will take $\Lambda^{-d} \sim m_0^{-1}$ to be approximately the volume associated to each spin-1/2 moment.

nontrivial self-energy for the z fields, and hence no flow of the couplings in (6) beyond that of the (trivial) change of variables during the rescaling step. This argument continues to hold when $T > 0$: a nonzero T simply introduces finite-size scaling effects from the compact thermal circle, but as usual when doing finite-size scaling, the finite extent of the compact direction does not influence the RG equations (which are determined by integrating out short-distance modes).

Besides the terms in (6), are there any other couplings we need to consider? Symmetry under spatial and internal rotations together with the constraint $n^2 = 1$ mean that the most relevant terms with only spatial derivatives[4] are those with four powers of ∇, which are marginal in $d = 2$. These terms are $\nabla^2 n^a \nabla^2 n^a, (\nabla n^a \cdot \nabla n^a)^2$, and $(\nabla n^a \cdot \nabla n^b)(\nabla n^a \cdot \nabla n^b)$. The first term is never generated under RG: if it were, it would make a k^4 contribution to the self energy of the z fields, which is impossible in light of the discussion above. The latter two terms are generated though, but in fact are generated only in the combination

$$\mathcal{O}_{sw} = (\nabla n^a \cdot \nabla n^a)^2 - 2(\nabla n^a \cdot \nabla n^b)(\nabla n^a \cdot \nabla n^b), \tag{11}$$

an operator which is associated with spin wave scattering. Currently I don't have a good quick representation-theory argument for why only this particular linear combination is generated. Whatever reason there is should be formulated in term of the n^a variables only, since mapping things over into the z fields quickly gets complicated. Indeed, while on the n^a side we only need to pay attention to terms quartic in the n^as, on the z side we have $|D_{\mathbf{a}} z|^4, |z^\dagger D_{\mathbf{a}} z|^4$, and $|z^\dagger D_{\mathbf{a}} z|^2 |D_{\mathbf{a}} z|^2$, which are all different order in the z fields.

In any case, if we accept the existence of only a single marginal operator $\mathcal{O}_{sw}$, we can still derive its beta function without doing a lot of heinous algebra, provided that we are willing to work modulo constant factors. This is because in the momentum-shell RG we're doing, the cutoff dependence of each of the diagrams generating $\mathcal{O}_{sw}$ is entirely determined by a one-loop calculation. Furthermore the diagrams generating $\mathcal{O}_{sw}$ can all be summed exactly, since by the non-relativistic structure of the theory they all take the form of geometric series.

Working with the z variables, one class of diagrams which generate $\mathcal{O}_{sw}$ are those corresponding to two-body scatterings of the z fields. Because of the absence of virtual particles, the effective vertex can be determined exactly via the usual geometric series of bubble diagrams. Each three-point vertex between the z fields and $\mathbf{a}$ enters as $a_\mu \langle z_{\mathbf{k}} | z_{\mathbf{p}} \rangle (k-p)^\mu$. Therefore if the incoming momenta to the four-point vertex are $\mathbf{k}, \mathbf{k}'$ and the outgoing momenta are $\mathbf{k}+\mathbf{p}, \mathbf{k}'-\mathbf{p}$, then a single bubble is associated to the integral (setting all external legs to be at zero frequency, since the structure of the RG means that ∂_τ terms won't be generated)

$$\begin{aligned} \text{bubble} &\sim \rho^2 \int_{\mathbf{q},\omega} \frac{q^2 |\mathbf{p}-\mathbf{q}|^2}{(-i\omega m_0 + \rho|\mathbf{q}+\mathbf{k}|^2)(+i\omega m_0 + \rho|\mathbf{k}'-\mathbf{q}|^2)} \\ &= \frac{\rho}{m_0} \int_{\mathbf{q}} \frac{q^2|\mathbf{p}-\mathbf{q}|^2}{|\mathbf{k}'-\mathbf{q}|^2 + |\mathbf{q}+\mathbf{k}|^2}, \end{aligned} \tag{12}$$

where we have ignored the $\lambda \pm m_0 H/2$ terms in the propagators since they will not be relevant for determining the RG flow. To determine the RG flow for the couplings mentioned above,

[4]Anything with time derivatives will be irrelevant in $d = 2$ since the triviality of the self energy means that $\partial_\tau n^a \partial_\tau n^a$ won't be generated.

it is enough to integrate $\mathbf{q}$ over the shell $q \in [\Lambda - \delta\Lambda, \Lambda]$ and extract the term of order 4 in the external momenta and of order $\delta\Lambda/\Lambda$ in the cutoff (since we know the terms we're looking for is marginal and has four derivatives). The appropriate term is then (not writing numerical factors)

$$\delta(\text{bubble}) \supset \frac{\rho}{m_0} p^2 (k^2 + k'^2) dl, \tag{13}$$

where $dl \equiv -\delta\Lambda/\Lambda$ is the RG time. All bubbles contributing to diagrams generating $\mathcal{O}_{sw}$ are of this form. Therefore if we let λ be the coupling associated to the operator $\mathcal{O}_{sw}$, we have

$$\frac{d\lambda}{dt} = C\frac{\rho}{m_0}, \tag{14}$$

with C some positive[5] number (note that the ρ/m_0 dependence on the RHS can essentially be fixed by the scaling $x \mapsto \eta x, \tau \mapsto \gamma\tau, \lambda \mapsto \eta^2\gamma^{-1}\lambda$). This equation is furthermore exact: the only diagrams generating λ are formed from concatenations of the bubble evaluated above, and only the single-bubble diagram is relevant for determining the flow of λ.

This means that the singular part of the free energy in $d = 2$ obeys

$$f_s(m_0, T, \rho, H) = e^{-4l} f_s(e^{2l} m_0, e^{2l} T, e^{2l}\rho, e^{2l} H, Cl\rho/m_0). \tag{15}$$

Choosing $l = -\frac{1}{2}\ln T$, we have

$$f_s(m_0, T, \rho, H) = T^2 \Phi_T(m_0/T, r, h, -C\ln(T)\rho/2m_0). \tag{16}$$

Here the scaling function on the right is related to the free energy evaluated at a temperature of order Λ^2, and hence is an analytic function in its arguments. Since we expect that m_0 is of order Λ^2, $-C\ln(T)\rho/2m_0 \ll 1$. Therefore we can expand the Φ_T function in its last argument, and as a consequence the leading logarithmic correction to the scaling form of f_s looks like

$$f_s(m_0, T, \rho, H) \supset \frac{C\rho}{2m_0} T^2 |\ln T| \Phi_T'(m_0/T, r, h, 0), \tag{17}$$

when then produces contributions to the entropy with T-dependence of the form $T^{n\geq 1}|\ln T|$ (note that all the variables on the RHS are the *bare* couplings!).

2 Large N $\mathbb{CP}^N$ model

Let us first see if we can reproduce the results of [7] — this is a needed sanity check because [7] contains no details about how the calculation works.

To get analytic control of (1), we will need to consider a large-N theory which reduces to (1) in the appropriate limit. There are many large-N limits one could consider here, realized by considering sigma models where the target spaces are various Grassmannians (e.g. $U(n)/(U(p) \times U(q))$). We will mainly focus here on the case of $\mathbb{CP}^{2N-1} = S^{4N-1}/U(1)$, which is obtained from the above action simply letting z be a $2N$-component complex field,

[5]In the diagram calculated above, it is easy to show that it is positive. Plus, if it was negative we would have an unphysical instability at $\lambda = 0$.

with the results of physical interest obtained by setting $N = 1$. We will find it convenient to choose the normalization condition $|z|^2 = N$, and will take the magnetic field to point along the direction of the matrix $Z^{\oplus N}$, so that we get back the spin-1 results when $N \to 1$.[6] Integrating out the zs then gives[7]

$$S = N\mathrm{Tr}\ln\left(m_0\partial_\tau - \rho D_{\mathbf{a}}^2 + \lambda - m_0 H/2\right) + (H \leftrightarrow -H) - 2N\int d^dx\, d\tau\, \lambda \tag{18}$$

As usual, to make progress we need to assume that λ and $\mathbf{a}$ are both uniform in space. A uniform $\mathbf{a}$ has zero field strength, and so we might as well just take $\mathbf{a} = 0$. So then in this approximation, the free energy density is

$$f/N = \sum_{\omega\in 2\pi T\mathbb{Z}} \int \frac{d^dk}{(2\pi)^d} \ln\left(-i\omega m_0 + \rho k^2 + \lambda - m_0 H/2\right) + (H \leftrightarrow -H) - 2\lambda. \tag{19}$$

2.1 Mean-field solution

We then need to find the mean field solution for λ — this means working in the approximation where λ becomes merely a chemical potential, rather than an exact Lagrange multiplier.

Resolving the matsubara sum by integrating against the Bose distribution, the mean field equation is

$$1 = \frac{1}{2}\oint \frac{dz}{2\pi i}\int \frac{d^dk}{(2\pi)^d} \frac{n_B(z)}{-zm_0 + \rho k^2 + \lambda - m_0 H/2)} + (H \leftrightarrow -H), \tag{20}$$

where the contour wraps counterclockwise around the $\mathbb{R}$ axis. Doing the integral over x, we get

$$1 = \frac{1}{2m_0}\int \frac{d^dk}{(2\pi)^d} n_B((\rho k^2 + \lambda - m_0 H/2)/m_0) + (H \leftrightarrow -H). \tag{21}$$

The physical meaning of this expression is that on average, there are two z bosons per each volume $1/m_0$, with the factor of 2 coming from the fact that by quantization of the WZW term there is (roughly) a *single* spin 1/2 within each volume of size $1/m_0$.

The remaining integral can be done in terms of polylogs. We are most interested however in the case of $d = 2$. To do the integral, we use

$$\int_0^\infty dx\, \frac{1}{Ae^x - 1} = -\ln(1 - 1/A). \tag{22}$$

After some algebra that I won't write out, we get

$$8\pi\rho/T = -\ln\left(1 - e^{-(\lambda/m_0 + H/2)/T}\right) - \ln\left(1 - e^{-(\lambda/m_0 - H/2)/T}\right). \tag{23}$$

[6]The matrix $Z^{\oplus N}$ can always be chosen as a generator of $\mathfrak{su}(2N)$ (which acts naturally on $\mathbb{CP}^{2N-1}$) since the generators of $\mathfrak{su}(2N)$ form a basis for the set of $2N \times 2N$ traceless Hermitian matrices.

[7]We have taken out a factor of 2 from the argument of the log for convenience; this just modifies S by a r, h-independent constant.

Let us define the parameter

$$q \equiv e^{\lambda/m_0 T}, \tag{24}$$

which is essentially the fugacity for the z bosons, aka the Lagrange multiplier enforcing their respective expectation values.

In accordance with the scaling arguments of the previous section, let us define $r \equiv \rho/T$ and $h \equiv H/2T$ (with the factor of $1/2$ just for notational reasons). We then have

$$e^{-8\pi r} q^2 = (q - e^h)(q - e^{-h}), \tag{25}$$

which is precisely the mean field equation given in [7]. We can solve this to obtain the mean field solution of

$$q = \frac{\cosh(h) + \sqrt{\sinh^2(h) + e^{-8\pi r}}}{1 - e^{-8\pi r}}. \tag{26}$$

At zero field, this gives

$$q(h = 0) = \frac{1}{1 - e^{-4\pi r}}. \tag{27}$$

As a sanity check, consider the solution in zero field and at low $T \ll \rho$ (large r). This gives

$$q(r \gg 1, h = 0) \approx 1 + e^{-4\pi r} \implies \lambda(r \gg 1, h = 0) \approx m_0 T e^{-4\pi r}. \tag{28}$$

This is the usual result we expect for large-N.[8] Since λ is essentially the chemical potential density, it represents the change in free energy density when a single z quanta is added into the system. When a z boson is created, it spreads out over an area of $\xi_z(r)^2$, where $\xi_z(r)$ is the correlation length of the z bosons. Since in zero field the only energy cost coming from flipping spins is elastic, we then expect that the change in free energy density induced by creating a z boson is

$$\Delta f \sim \frac{\rho}{\xi_z(r)^2}. \tag{30}$$

Equating this to λ/m_0 and using the above solution for λ, we can then predict that $\xi_z(r) \sim \sqrt{r/m_0} e^{2\pi r}$. In the next subsection we will see explicitly that this is indeed the right answer.

[8] One complaint here is that technically speaking λ needs to be integrated along the imaginary axis, which is slightly disturbing since the mean-field solution for λ is real-valued. If we don't care about $1/N$ corrections this doesn't really matter: we don't actually need to integrate λ, since having λ be a constant is still good enough to get $|z|^2 = N$ inside of expectation values.

More generally, one can use a rather comical argument where one integrates over both real and imaginary parts of λ. On one hand,

$$\int \mathcal{D}\lambda_R \, \mathcal{D}\lambda_I \, e^{i\int(\lambda_R + i\lambda_I)(|z|^2 - N)} = \int \mathcal{D}l_R \, e^{i\int \lambda_R \cdot 0} \delta(|z|^2 - N) = \infty \cdot \delta(|z|^2 - N), \tag{29}$$

so that adding in the integral over the real axis only adds on a harmless factor of ∞. On the other hand, one can first minimize with respect to both λ_R and λ_I, and use the real solution in the path integral.

In any case, the fact that the saddle point is real is to be expected physically. Indeed, λ behaves as a chemical potential, and a chemical potential behaves as i times the time component of a Euclidean gauge field. Therefore in $\mathbb{R}$ time, λ acts as the time component of a gauge field, which is consistent with the fact that in the present problem the gauge field is $(\lambda, \mathbf{a})$.

On the other hand, the $r \ll 1$ high-temperature solution gives

$$q(r \ll 1, h = 0) \approx 1/4\pi r + 1/2 \implies \lambda(r \ll 1, h = 0) \approx m_0 T \ln(1/4\pi r + 1/2). \tag{31}$$

This again can be used to directly get the correlation length.

If we turn on a field, at $h \gg 1$ we have

$$q(r, h \gg 1) \approx \frac{e^h}{1 - e^{-8\pi r}} \implies \lambda(r, h \gg 1) \approx m_0 T h + m_0 T \ln\left(\frac{1}{1 - e^{-8\pi r}}\right). \tag{32}$$

The $m_0 H$ term in λ is of course just the magnetic part of the free energy density associated with flipping a spin.

2.2 Correlation Lengths

We will now compute some correlation functions, focusing at first on $h = 0$. Since we already have the solutions for λ we don't actually need to do any integrals in order to figure out what the correlation lengths are, but the integrals are still necessary given that we are also interested in knowing prefactors which multiply $e^{-x/\xi(r)}$.

Consider first the $2\pi r \ll 1$ limit, where we get a 2D stat mech model. The spatial correlation function of the z fields is

$$\langle z_i(x) z_j^\dagger(0)\rangle = \delta_{ij} T \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\rho k^2 + \lambda} = \delta_{ij} \frac{1}{4\pi r} K_0(x\sqrt{\lambda/\rho}) \overset{x\sqrt{\lambda/\rho}\gg 1}{\approx} \frac{\delta_{ij}}{\sqrt{32\pi x r^2/\xi_z(r)}} e^{-x/\xi_z(r)}, \tag{33}$$

where

$$\xi_z(2\pi r \ll 1) \approx \left(\frac{r}{m_0 \ln(1/4\pi r + 1/2)}\right)^{1/2}, \tag{34}$$

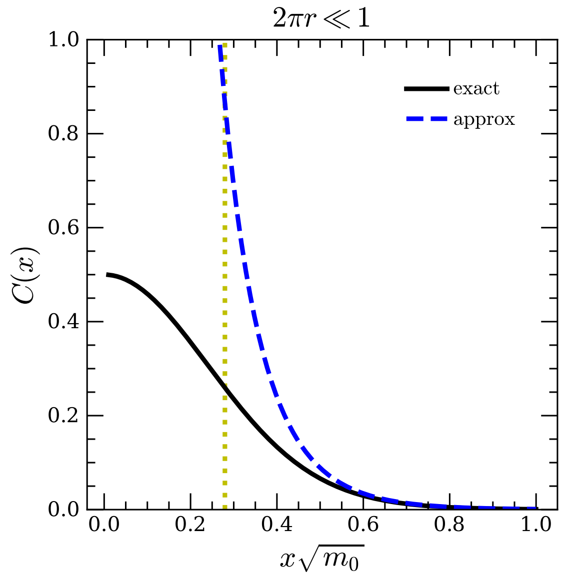
with $\xi_z(0) = 0$ as required in the present continuum model.

Of course the z fields are not physical, and what we really want is the correlation functions of the $\mathbf{n}$ fields (which are defined as above after setting $N = 1$).[9] Since the mean-field action we're working with is flavor-diagonal and quadratic in the z fields, the correlation function of the $\mathbf{n}$s is simply proportional to the square of the correlation function for the z fields:

$$\langle n^a(x) n^b(0)\rangle = \delta^{ab} C(x) \equiv \delta^{ab} \frac{\xi(r)}{8\pi r^2 x} e^{-x/\xi(r)}, \qquad \xi(r) \equiv \xi_z(r)/2. \tag{35}$$

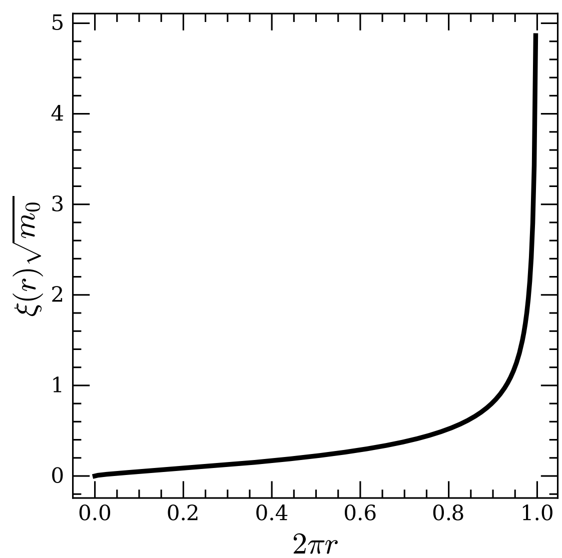
[9] What is the appropriate expression for $\mathbf{n}$ in terms of the zs at $N > 1$? If we were breaking up the coordinate on $\mathbb{CP}^{N-1}$ using $\mathbb{CP}^{N-1} = U(N)/(U(1) \times U(N-1))$ then we could write $n = VMV^\dagger$ with $M = -1 \oplus \mathbf{1}_{N-1}$. But here we're using the identification with $S^{2N-1}/U(1)$, and I'm not so sure what the best way to define the n coordinate is. The physical coordinates can be parametrized by the vector $(z_1/z_N, z_2/z_N, \ldots, 1)$, but I'm not sure how useful this is.

As a check, this approximation for the correlation function gives (at $r = 1/20$)



(36)

The function $\xi(2\pi r \ll 1)$ looks like



(37)

Note that $\xi(r) = 1/\sqrt{m_0}$ when $r \approx 0.15$. Therefore given a fixed ρ, the temperature at which the correlation length becomes $\sim 1/\sqrt{m_0}$ (viz. the size of the Moire unit cell) is $T_{UV} \approx 7\rho$. For $\rho = 1$meV this gives $T_{UV} \approx 80$K (compare this to e.g. the bandwidth / Coulomb interaction $W \sim U \sim 30$meV ~ 350 K).

Now consider the $2\pi r \gg 1$ limit; here we have to directly use the Bose function. After a change in variables, we have (omitting the flavor indices)

$$\langle z(x)z^\dagger(0)\rangle = \frac{1}{2r}\int \frac{d^2u}{(2\pi)^2}\,\frac{e^{i\mathbf{u}\cdot\mathbf{x}\sqrt{m_0/r}}}{qe^{u^2}-1} = \frac{1}{4\pi r}\int_0^\infty du\,\frac{u}{qe^{u^2}-1}J_0[ux\sqrt{m_0/r}] \tag{38}$$

At large x, only small u contributes to the integral. Therefore we can approximate[10]

$$\langle z(x)z^\dagger(0)\rangle \approx \frac{1}{4\pi r}\int_0^\infty du \frac{u}{q(1+u^2)-1} J_0[ux\sqrt{m_0/r}] \approx \frac{1}{4\pi r} K_0(x/\xi) \approx \frac{1}{\sqrt{32\pi r^2 x/\xi_z(r)}} e^{-x/\xi(r)}, \tag{40}$$

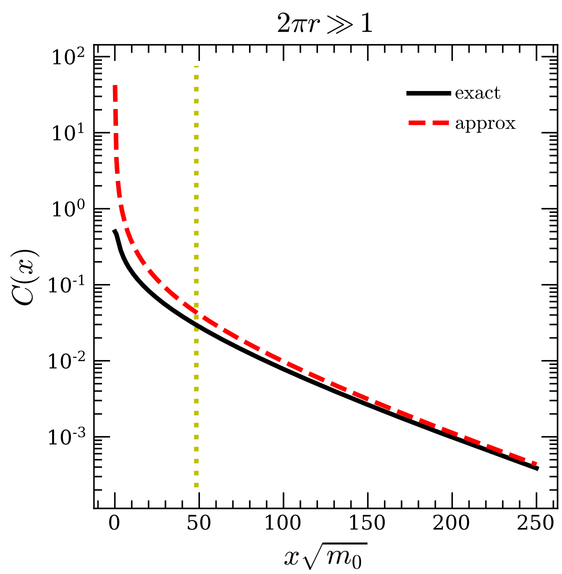
where the correlation length of the z fields is now

$$\xi_z(r) = \sqrt{\frac{r}{m_0}} e^{2\pi r}. \tag{41}$$

Therefore the correlator of the components of the $\mathbf{n}$ field at large r is

$$C(x) = \frac{\xi(r)}{8\pi r^2 x} e^{-x/\xi(r)}, \tag{42}$$

again with $\xi(r) = \xi_z(r)/2$. The correlator as a function of $x\sqrt{m_0}$ (for $r = 0.8$, and with $\xi(0.8)$ drawn as the yellow vertical line) looks like:



(43)

2.3 Free energy

Let us first compute the free energy density. We start with[11]

$$f/N = T\int \frac{d^dk}{(2\pi)^d} \ln\left(1 - \exp\left(-\frac{1}{m_0 T}(\rho k^2 + \lambda - m_0 H/2)\right)\right) + (H \leftrightarrow -H) - 2\lambda. \tag{45}$$

[10]For posterity's sake: we might think that the Bessel function imposes a soft cutoff on the integrand at the value $u_* \approx \sqrt{r/m_0}/x$, so that we could approximate the above by

$$\langle z(x)z^\dagger(0)\rangle \approx \frac{1}{2\pi r}\int_0^{u_*} \frac{u}{qe^{u^2}-1} = \frac{1}{4\pi r}\left(-\frac{r}{x^2 m_0} - \ln(q-1) + \ln(qe^{r/x^2 m_0} - 1)\right) \tag{39}$$

However, this fails at long enough distances, and does not give an exponential decay. The exponential decay comes from the oscillating Bessel function, and so one cannot break down the Bessel function with an approximation.

[11]Recall how this works formally: schematically, one can think about integrating against n_B and then deforming the contour to one which wraps the branch cut of the logarithm. Using $\int_\varepsilon^\infty dx\, \frac{1}{e^x-1} = \ln(1-e^{-\varepsilon})$

Specializing to $d = 2$,

$$\begin{aligned}
f/N &= \frac{m_0 T}{4\pi r} \int dx \, \ln\left(1 - e^{-x} q^{-1} e^{h}\right) + (h \leftrightarrow -h) - 2\lambda \\
&= -\frac{m_0 T}{4\pi r} \left(\mathrm{Li}_2(q^{-1}e^{h}) + \mathrm{Li}_2(q^{-1}e^{-h})\right) - 2\lambda \\
&= -\frac{m_0 T}{4\pi r} \left(\mathrm{Li}_2(q^{-1}e^{h}) + \mathrm{Li}_2(q^{-1}e^{-h}) + 8\pi r \ln q\right),
\end{aligned} \tag{46}$$

which tells us the scaling function $\Phi(r, h)$. Φ has the various limits

$$\begin{aligned}
\Phi(r \gg 1, h = 0) &\approx -\frac{\pi}{12r} - e^{-4\pi r}(2 - 1/2\pi) + \cdots \\
\Phi(r \ll 1, h = 0) &\approx -2 + 2\ln(4\pi r) + \cdots,
\end{aligned} \tag{47}$$

where we have used $\mathrm{Li}_2(1-x) \approx \frac{\pi^2}{6} + x(\ln(x) - 1) + \cdots$ and $\mathrm{Li}_2(x) \approx x + x^2/4 + \cdots$ for $x \to 0$. In the low-T zero-field limit we then have $f/N \sim -m_0 T^2/12\rho$, with the $-Ts$ term in $f = u - Ts$ being twice as big as the u term.

Let us also consider what happens in a large field. Here we find

$$\Phi(r, h \gg 1) \approx -\frac{1}{4\pi r}\mathrm{Li}_2\left(\frac{1}{1 - e^{-8\pi r}}\right) - 2h - 2\ln\left(\frac{1}{1 - e^{-8\pi r}}\right). \tag{48}$$

If we also take $r \gg 1$ we get (recalling that $h = H/2T$) the expected $f/N \to -m_0 H$ (as $T \to 0$, a finite field polarizes everything). However if we hold r fixed while making h large (i.e. we turn on a large field while working at constant ρ and constant T), we do *not* find that f/N reduces to $-m_0 H$. We will comment on this more when we discuss the entropy.

2.4 Entropy

Given that we have found the scaling function Φ above, it is easy to evaluate the entropy density via

$$s \equiv -\partial_T f/N = m_0(r\partial_r + h\partial_h - 1)\Phi(r, h). \tag{49}$$

---

then gives the usual stat mech result.

This is a bit glib though, since it pretends that everything is nicely behaved at infinity (which it isn't). The correct way to do things is as follows: ignoring the -2λ part and the $+(H \leftrightarrow -H)$ and writing $\varepsilon_k = \rho k^2 + \lambda - m_0 H/2$,

$$\begin{aligned}
f/N &= -\int \frac{d^d k}{(2\pi)^d} \oint \frac{dz}{2\pi i} n_B(z) \ln(z/T - \varepsilon_k/m_0 T) \\
&= \int \frac{d^d k}{(2\pi)^d} \oint \frac{dz}{2\pi i} n_B(z) \left[\int_1^{\varepsilon_k/m_0 T} dx \, \frac{1}{z/T - x} - \ln(z/T - 1)\right] \\
&\sim T \int \frac{d^d k}{(2\pi)^d} \int_1^{\varepsilon_k/m_0 T} dx \, n_B(xT) \\
&\sim T \int \frac{d^d k}{(2\pi)^d} \ln(1 - e^{-\varepsilon_k/m_0 T}),
\end{aligned} \tag{44}$$

where each $\sim$ means equality up to infinite terms which are proportional to $m_0 T$ but independent of ε_k (such terms just multiply the partition function by a constant).

When taking derivatives we needn't worry about the r, h dependence of q, since we are working in mean-field so that $\partial_\lambda f = 0$.[12]

Using $\partial_x \mathrm{Li}_2(e^x) = -\ln(1-e^x)$, we obtain

$$\begin{aligned}\partial_h \Phi(r,h) &= \frac{1}{4\pi r}\ln\left(\frac{q-e^h}{q-e^{-h}}\right)\\ \partial_r \Phi(r,h) &= \frac{1}{4\pi r^2}\left(\mathrm{Li}_2(q^{-1}e^h)+\mathrm{Li}_2(q^{-1}e^{-h})\right).\end{aligned} \tag{53}$$

Hence the entropy density is

$$s = \frac{m_0}{4\pi r}\left(2\mathrm{Li}_2(q^{-1}e^h)+2\mathrm{Li}_2(q^{-1}e^{-h})+h\ln\left(\frac{q-e^h}{q-e^{-h}}\right)+8\pi r\ln q\right). \tag{54}$$

Let us look at this function in a few different limits. At zero field and high-temperatures, we find

$$s(r \ll 1, h=0) \approx 2m_0 \ln\left(\frac{1}{4\pi r}\right) + \mathcal{O}(r^0). \tag{55}$$

In zero field and at low temperatures, we have

$$s(r \gg 1, h=0) \approx \frac{m_0 \pi}{6r} + \mathcal{O}(e^{-4\pi r}) \approx -\frac{1}{2Tm_0} f_s(r\gg 1, h=0). \tag{56}$$

The fact that we get something proportional to r^{-1} (implying $s \propto T$) is just because as $r \to \infty$ the zero-field chemical potential vanishes exponentially ($\lambda/m_0 T \sim e^{-4\pi r}$) and can be ignored, so that the problem reduces to calculating the entropy of spin waves in 2 dimensions, which indeed has a linear dependence on T.[13] Note that we might have expected s to vanish

[12] One may object to this because λ is the mean field parameter not q, and q has T-dependence that should be taken into account when computing the entropy. But in fact this contribution vanishes, and we can equally well treat q as the mean-field parameter. Indeed, since $\partial_T q = -\lambda m_0 T^{-2} q = -T^{-1} q \ln q$, we get a contribution to the entropy of

$$s/m_0 \supset q \ln q\, \partial_q \Phi. \tag{50}$$

But

$$\begin{aligned}\partial_q \Phi &= -\frac{1}{4\pi r}\left(\frac{\ln(1-e^h/q)}{q}+\frac{\ln(1-e^{-h}/q)}{q}+8\pi r/q\right)\\ &= -\frac{1}{4\pi q r}\left(\ln(q-e^h)(q-e^{-h})-2\ln q+8\pi r\right)\\ &= 0,\end{aligned} \tag{51}$$

where we have used the mean field equation in the last line and have made use of

$$\partial_x \mathrm{Li}_2(A/x) = x^{-1}\ln(1-A/x). \tag{52}$$

Hence we can treat q as fixed when computing derivatives of the free energy.

[13] Indeed, the free energy of non-relativistic bosons in d dimensions is

$$f \sim T\int \frac{d^d k}{(2\pi)^d}\ln\left(1-e^{-k^2/T}\right) \sim T^{1+d/2} \implies s \sim T^{d/2}. \tag{57}$$

exponentially, for the following reason: at low temperatures and at zero field, the spins should align themselves over areas of order $\xi^2(r) \sim e^{4\pi r}$ into large effective spins. Since the density of these large effective spins is thus proportional to $e^{-4\pi r}$, we would then expect the entropy density to vanish with the same exponential factor. However, this line of thinking neglects the fact that the degrees of freedom within each correlation area are not completely locked together, and instead host spin waves. The spin waves propagating around in the correlation regions then give the linear-in-T contribution to the entropy density.

At strong fields $h \gg 1$ we have

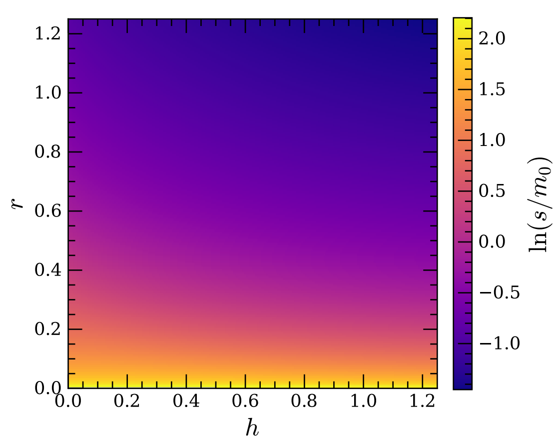
$$s(r, h \gg 1)/m_0 \approx \frac{1}{4\pi r}\left(2\mathrm{Li}_2(1-e^{-8\pi r}) + (2+h)(1-e^{-8\pi r})e^{-2h} - 8\pi r \ln(1-e^{-8\pi r})\right) + \cdots, \tag{58}$$

with the $\cdots$ vanishing as higher powers of e^{-2h}. Note that as we send $r \to 0$ in the above, we get an expression that diverges as $\ln(r)$ (this is only valid for very [unphysically] small r, viz. $8\pi r \ll 1$):

$$s(r \ll 1, h \gg 1) \approx 4 - 2\ln(8\pi) + 2(2+h)e^{-2h} - 2\ln(r). \tag{59}$$

This $\ln(r)$ divergence is unphysical (when $r \ll 1/8\pi$ the correlation length is already less than $1/\sqrt{m_0}$), and is rendered finite by the presence of a UV cutoff (the effective dimension of the Hilbert space available to the bosons in mean-field diverges as $r \to 0$ due to the integral over d^2k). Note that in these models we don't expect this theory to be sensible in the absence of a UV cutoff anyway (due to the quantization issues mentioned above), and in fact the most natural thing to do is to use $1/\sqrt{m_0}$ as a cutoff scale. This is done in Sec. 2.6. A more important issue with (59) is that $s(r \sim 1, h \gg 1)$ is non-zero even when $h \to \infty$. This is certainly unphysical in the $N = 1$ model, since as $h \to \infty$ all spin waves have an infinite gap.

Now we can make plots. A color plot which gives a qualitative feel for what's going on is

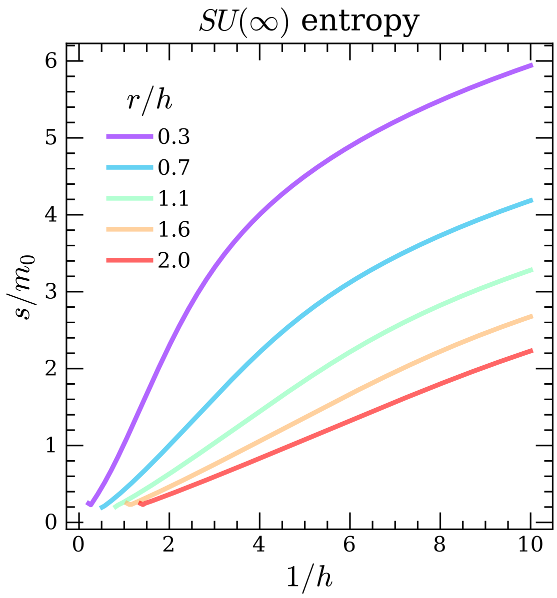


(60)

Here we've used a log scale since otherwise the (unphysical) divergence as $r \to 0$ makes the plot hard to look at.

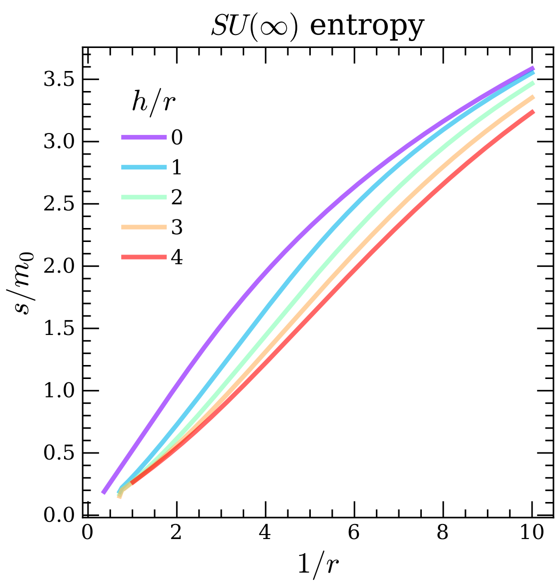
For something more quantitative, we can look for example at how s/m_0 behaves with temperature by plotting it against e.g. $1/h$ for several values of the T-independent quantity

r/h:



(61)

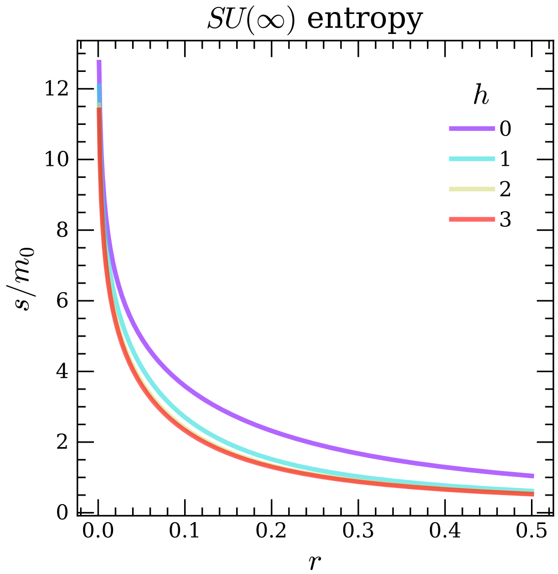
Similarly,[14]



(62)

[14]The reason why the curves have trouble going all the way down to $1/r = 0$ is mostly due to the $\ln(q - e^h)/(q - e^{-h})$ term in s; if one only wants to plot things at zero field one can ignore this term for nicer-looking plots.

The large-h behavior we mentioned above manifests itself as



(63)

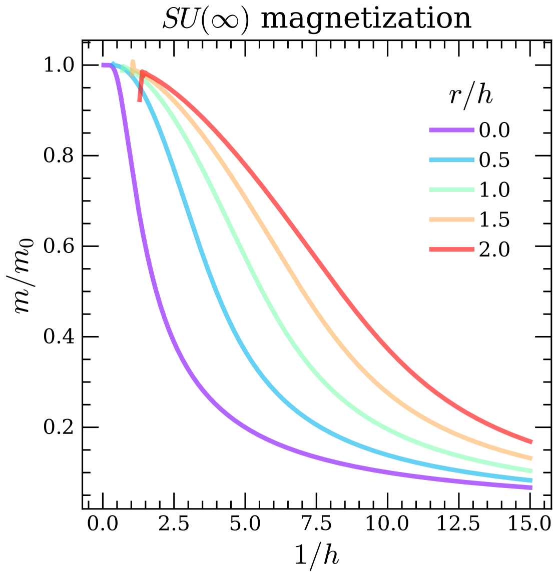
Both the $r \to 0$ divergence and a large part of the weight at $r \sim 1$ are removed by the addition of a cutoff at the scale of $1/\sqrt{m_0}$ (and as we found, we don't expect a ferromagnetic model to work when $r \lesssim 0.15$, since then the correlation length is of order $\mu_0^{-1/2}$) so the above plot should not be taken as physical — it is just given to show what the behavior of the bare continuum model looks like.

2.5 Magnetization

Let us now use this to compute the magnetization, since we can compare our answer to the result in [7]. We have in fact already done the calculation when determining the entropy, since

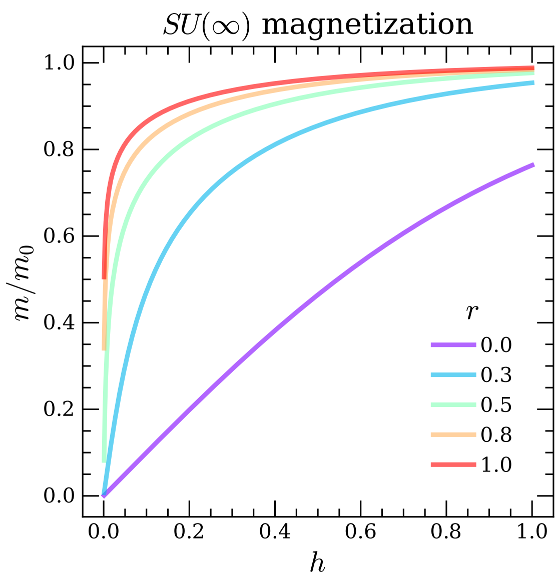
$$m \equiv -\partial_H f/N = -\frac{1}{2}\partial_h \Phi(r,h) = \frac{m_0}{8\pi r}\ln\left(\frac{q-e^{-h}}{q-e^{h}}\right). \tag{64}$$

Let us now plot the magnetization as a function of $1/h$ for a few values of r/h. The result is (there are some convergence issues with the plotting as $h \to \infty$ that I haven't bothered to fix since we know that $m = m_0$ at $h = \infty$)



(65)

which agrees with the results in [7]. Another way of plotting the results which serves as a sanity check is



(66)

where the purple line is the expected $\tanh(h)$ of the paramagnet that we get when the stiffness vanishes.

2.6 Cutting off high-T divergences

One unrealistic feature of our model so far is that the entropy density is unbounded in the high-temperature limit $r \to 0$. We can see from the plots that $s(r \to \infty) \sim -\ln r$, and indeed this is the case: not writing constants and using the $r \to 0$ mean-field solution derived above,

$$f/N \sim T\int \frac{d^2k}{(2\pi)^2}\,\ln\left(1-re^{-k^2r/m_0}\right)+m_0T\ln(r) \sim -m_0T+m_0T\ln(r) \implies s/m_0\sim 1-\ln(r). \tag{67}$$

This divergence is due to the fact that in the above treatment we are working in the continuum, with the correlation length $\xi(r)$ vanishing as in (34). In a more realistic model the $r,h\to 0$ limit would give a correlation length saturating at a length scale of order $1/\sqrt{m_0}$, with s saturating to a value of $\sim m_0\ln 2$.

The simplest way to get a correlation length which saturates to $\sim 1/\sqrt{m_0}$ at infinite temperature is simply to cut off the momenta integrals at the corresponding wavevector. Normally in QFT there is no preferred way to choose a regulator, but in the present context a short-distance cutoff is essentially part of the definition of the theory, due to the issues with quantization mentioned at the beginning of the section.

Consider then cutting the momentum integration off at the wavevector $\sqrt{Cm_0}$, where C is some order-1 number. The free energy density is

$$\begin{aligned} f/N &= T\int_0^{\sqrt{Cm_0}}\frac{dk}{2\pi}\,k\ln\left(1-q^{-1}e^{-rk^2/m_0+h}\right)+(h\leftrightarrow -h)-2\lambda \\ &= \frac{m_0T}{4\pi r}\int_0^{Cr} du\,\ln\left(1-q^{-1}e^{-u+h}\right)+(h\leftrightarrow -h)-2\lambda \\ &= -\frac{m_0T}{4\pi r}\left(\mathrm{Li}_2(q^{-1}e^{h})-\mathrm{Li}_2(q^{-1}e^{-Cr+h})+(h\leftrightarrow -h)+8\pi r\ln q\right) \end{aligned} \tag{68}$$

Therefore the cutoff manifests itself only via the extra $-\mathrm{Li}_2(q^{-1}e^{-Cr}e^{\pm h})$ terms. Taking derivatives, we find

$$\begin{aligned} s/m_0 = \frac{1}{4\pi r}\Bigg(&2\mathrm{Li}_2(q^{-1}e^{h})+2\mathrm{Li}_2(q^{-1}e^{-h})-2\mathrm{Li}_2(q^{-1}e^{-Cr+h})-2\mathrm{Li}_2(q^{-1}e^{-Cr-h}) \\ &+Cr\ln\left((1-q^{-1}e^{-Cr+h})(1-q^{-1}e^{-Cr-h})\right)+8\pi r\ln q+h\ln\left(\frac{q-e^{h}}{q-e^{-h}}\frac{q-e^{-h-Cr}}{q-e^{h-Cr}}\right)\Bigg). \end{aligned} \tag{69}$$

The magnetization is likewise determined by the last piece in the above expression, viz.

$$m/m_0=\frac{1}{4\pi r}\ln\left(\frac{q-e^{h}}{q-e^{-h}}\frac{q-e^{-h-Cr}}{q-e^{h-Cr}}\right). \tag{70}$$

Now we need to find the mean-field solution for q. The mean-field equation is

$$\begin{aligned} m_0 &= \frac{1}{4\pi}\int_0^{\sqrt{Cm_0}} dk\,k\frac{1}{qe^{rk^2/m_0+h}-1}+(h\leftrightarrow -h) \\ &= \frac{m_0}{8\pi r}\int_0^{Cr} du\,\frac{1}{qe^{u+h}-1}+(h\leftrightarrow -h) \\ &= \frac{m_0}{8\pi r}\left(-2Cr+\ln\left(\frac{1-qe^{Cr}e^{h}}{1-qe^{h}}\frac{1-qe^{Cr}e^{-h}}{1-qe^{-h}}\right)\right). \end{aligned} \tag{71}$$

In the case where $h = 0$, this gives a simple linear equation which we can solve by hand, obtaining

$$q(h=0) = \frac{e^{4\pi r} - e^{-Cr}}{e^{4\pi r} - 1}, \tag{72}$$

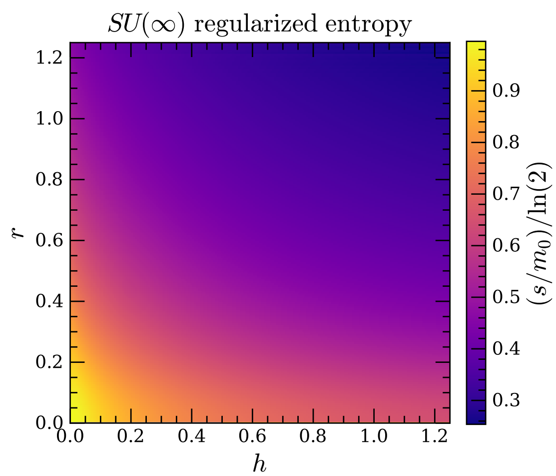
which goes to $C/4\pi + 1$ as $r \to 0$ and agrees with the result we found previously when $C \to \infty$.

The general solution for finite h is gross, and is given by the solution to the quadratic $aq^2 + bq + c = 0$, with

$$a = e^{2Cr}(e^{8\pi r} - 1), \quad b = 2\cosh(h)e^{Cr}(1 - e^{8\pi r + Cr}), \quad c = e^{8\pi r + 2Cr} - 1. \tag{73}$$

One can then examine the above expression for s/m_0 and check that it gives a (disgusting) finite expression in terms of C in the limit $r \to 0$. We will then choose C so that as $r, h \to 0$ we have $s/m_0 \to \ln 2$—the numerical value for C turns out to be $C \approx 1.31$, so that the cutoff distance is at the scale $a \approx 1/\sqrt{1.31 m_0}$, which is essentially the Moire lattice spacing. Since here $C \sim 1$ this is equivalent to imposing a cutoff in energy at the scale $\sim \rho$, as expected on physical grounds.

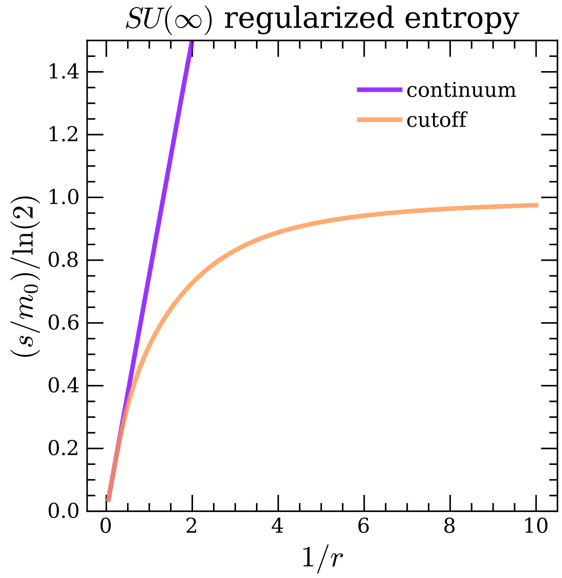
To illustrate the effects of the cutoff, we first re-plot the s/m_0 heatmap (now without a log scale!):



(74)

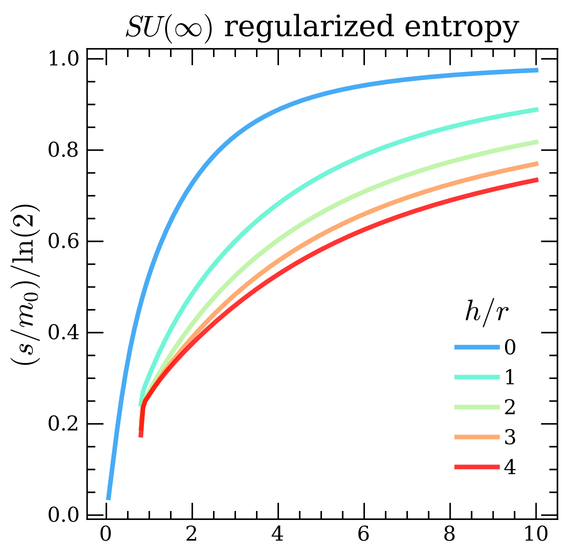
To do a more quantitative comparison, we again plot s/m_0 as a function of $1/r$ at $h = 0$:

(the upper limit of the y axis is at $\ln 2$)



(75)

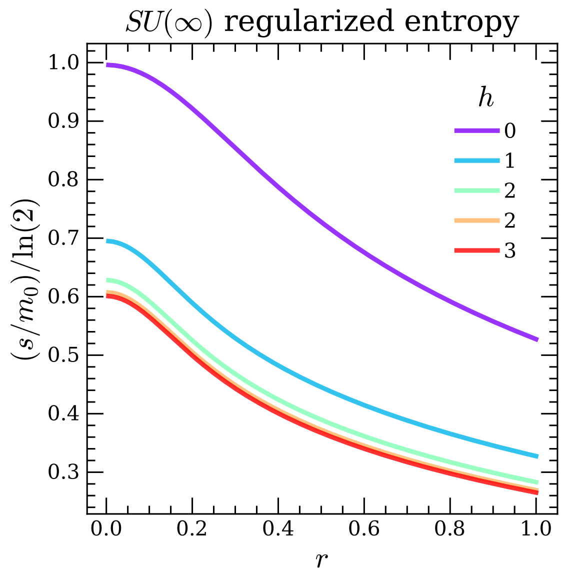
With this way of doing things we see that s/m_0, while not being close to saturated at $r \approx 1$, starts to saturate to $\ln 2$ when $r \approx 1/4$, which isn't that small. When we look at curves for various values of h/r, we obtain



(76)

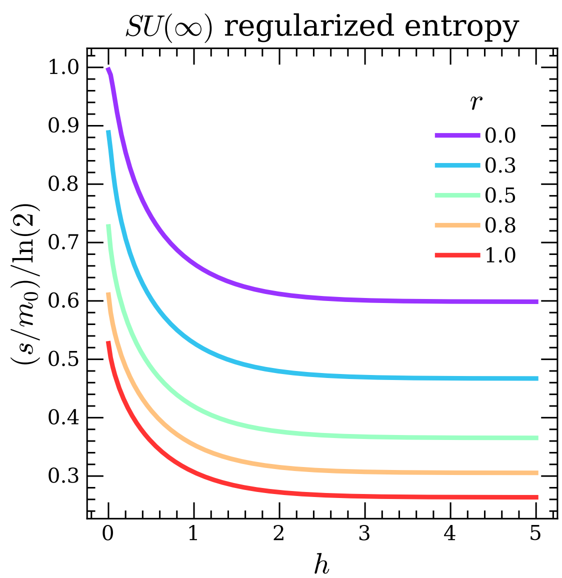
where the disappearance of the $h/r > 0$ curves at large r is due to numerical issues that I haven't bothered to fix yet. All curves properly converge to $\ln 2$ as $r \to 0$.

Finally, we see that the $\ln(r)$ divergence with $r \to 0$ at high fields seen in (63) is properly fixed by the addition of the cutoff, but that the entropy is still nonzero even as $h \to \infty$ at fixed r:



(77)

Another way of illustrating this problem is to plot this as a function of h:



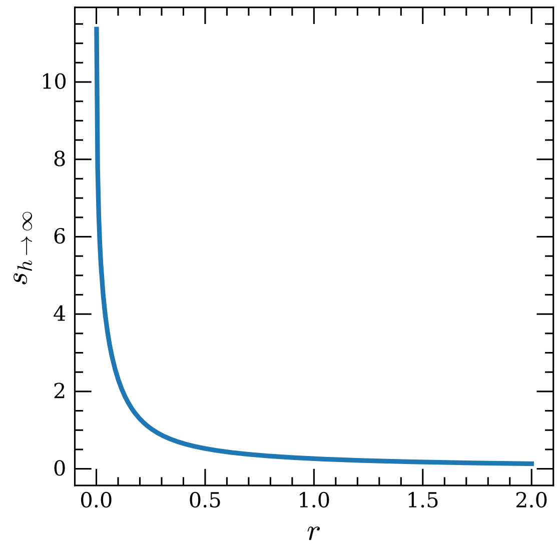
(78)

* * *

Given this, there is clearly still a problem with the mean-field limit even after regulating with a short-distance cutoff. The problem lies in the term in (58) which is independent of h, which we denote as s_{ct}:

$$s_{ct}(h \to \infty)/m_0 = \frac{1}{2\pi r}\mathrm{Li}_2(1 - e^{-8\pi r}) - 2\ln(1 - e^{-8\pi r}). \tag{79}$$

This function looks like



(80)

Physically, this arises because as $h \to \infty$, the $z_\uparrow$ boson which creates spins along the field direction has a chemical potential which is independent of h, namely $\mu_\uparrow = \ln q - h \approx -\ln(1 - e^{-8\pi r})$, with the field dependence of $\ln q$ canceling the h to prevent condensation of $z_\uparrow$. This accordingly gives an entropy of (79).

To understand where $s_{ct}(h \to \infty)$ comes from, a by now standard calculation shows that the entropy of a boson with $\langle b^\dagger b \rangle = n m_0$ and kinetic energy ρk^2 is given by

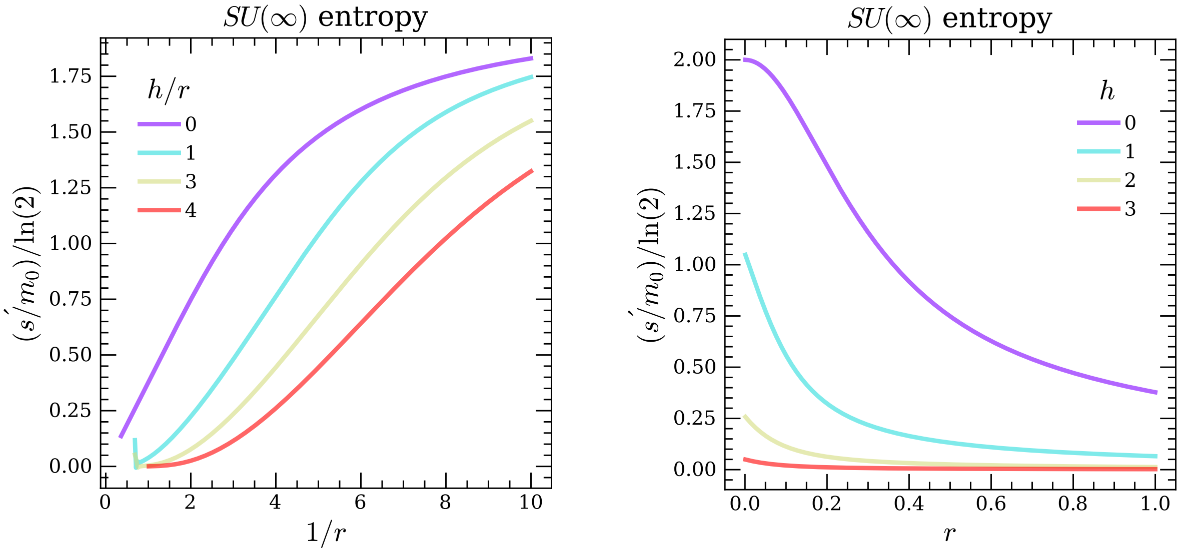
$$s_n/m_0 = -\frac{1}{2\pi r}\mathrm{Li}_2(q_n^{-1}) + n \ln q_n, \qquad q_n = \frac{1}{1 - e^{-4\pi r n}}. \tag{81}$$

When $h = \infty$ the mean-field problem gives the entropy of such a boson with $n = 2$, which plugging into the above formula we see is exactly $s_2 = s_{ct}(h \to \infty)$.

Clearly at large fields then, we must subtract off $s_{ct}(h \to \infty)$ in order to get a physically sensible result. The somewhat tricky question is what sort of subtraction to do at intermediate fields.

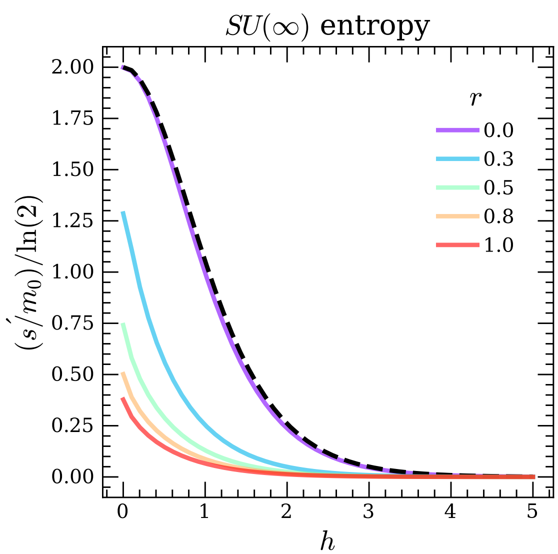
Note that $s - s_{ct}$ at $h = 0$ has a finite limit of $2m_0 \ln 2$ as $r \to 0$. We will also see shortly that when $r = 0$, $s - s_{ct}$ as a function of h is essentially exactly equal to the entropy of a paramagnet. Finally, in zero field $s - s_{ct}$ goes to $m_0\pi/12r$ when $r \gg 1$, thus while subtracting s_{ct} changes the coefficient of the $1/r$ piece by a factor of $1/2$ it does not eliminate it entirely.

We can then make a guess that $s' \equiv s - s_{ct}$ is in fact a closer approximation to the physical $N \to 1$ limit. Let us re-do a few of the previous plots for s' instead of s (*without* imposing a short distance cutoff). We get e.g.



(82)

and



, (83)

where the dashed line is the entropy of a paramagnet. These all look much closer to what we'd expect physically.

3 $1/N$ corrections

In this section we calculate the leading $1/N$ corrections to the free energy. These corrections come from the Gaussian fluctuations of the gauge field about the saddle point. As such we expect to get a contribution quadratic in T (at least at low temperatures), since this is what we get for free Maxwell theory. The only nontrivial part is figuring out the r, h dependence of the coefficient of the T^2 term.

To look at $1/N$ corrections we will write $\lambda = \lambda_* + im_0 a_0$, with λ_* the saddle-point solution (here the m_0 is needed since we want a_0 to have dimensions of ∂_τ). The free energy density is then

$$f = f_* - m_0 T \ln\left(\int \mathcal{D}a\, e^{-NS_{eff}[a]}\right), \tag{84}$$

with (where required dimensional issues are always taken care of by factors of $m_0 T$)

$$S_{eff}[a] = \sum_{s=\pm1} \mathrm{Tr}\ln\left[\mathbf{1} + G_s(\rho(-D_{\mathbf{a}}^2 + \nabla^2) + im_0 a_0)\right] - 2im_0 \int d^2x\, d\tau\, a_0\Big], \tag{85}$$

where $f_* = -NT\sum_s \mathrm{Tr}\ln G_s$ is the saddle-point free energy (the trace does not include the flavor index) and where the z propagator is

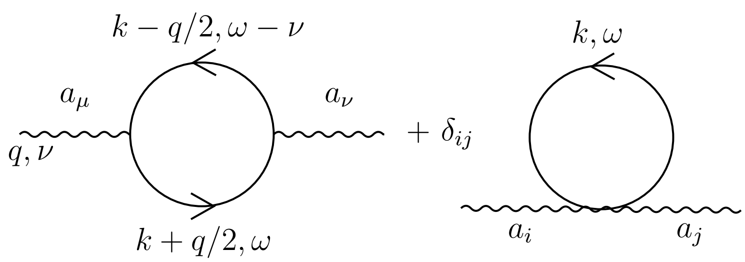
$$G_s = \frac{1}{m_0\partial_\tau - \rho\nabla^2 + \lambda_* + s m_0 H/2}. \tag{86}$$

The leading $1/N$ corrections (which are $\mathcal{O}(N^0)$) are those coming from the fluctuations of the gauge field a about its saddle point value. Hence the leading correction to the free energy is

$$\delta f/m_0T = \frac{1}{2}\mathrm{Tr}\ln\left(\left[\frac{1}{2}\frac{\delta^2 S_{eff}[a]}{\delta a_\mu \delta a_\nu}\right]^{-1}\Big|_{a_\mu=0}\right) + \delta f_{gf}/m_0T = \frac{1}{2}\mathrm{Tr}\ln\left(G_{\mu\nu}\right) + \delta f_{gf}/m_0T, \tag{87}$$

where $G_{\mu\nu}$ is the free gauge field propagator and where δf_{gf} is the contribution from the ghosts needed for gauge-fixing.

To construct $G_{\mu\nu}$, we need to evaluate the standard polarization bubbles. Diagrammatically,



$$\equiv NG_{\mu\nu}(q,\nu), \tag{88}$$

where the second diagram is relevant only if both indicies are spatial, and where the vertex factors are implicit in the diagrams.

When evaluating this this, it will be helpful to define the notation

$$\mathcal{H}_{\mathbf{k}}^s \equiv \frac{\rho}{m_0}k^2 + \lambda_*/m_0 + sH/2. \tag{89}$$

We then find for the 00 component

$$\begin{aligned} G_{00}(q,\nu) &= -\sum_s \oint \frac{dz}{2\pi i}\int \frac{d^2k}{(2\pi)^2}\frac{n_B(z)}{(-z + i\nu + \mathcal{H}_{\mathbf{k}-\mathbf{q}/2}^s)(-z + \mathcal{H}_{\mathbf{k}+\mathbf{q}/2}^s)} \\ &= \sum_s \int \frac{d^2k}{(2\pi)^2}\frac{n_B(\mathcal{H}_{\mathbf{k}+\mathbf{q}/2}^s) - n_B(\mathcal{H}_{\mathbf{k}-\mathbf{q}/2}^s)}{-i\nu + 2\rho\mathbf{k}\cdot\mathbf{q}/m_0} \\ &= \frac{1}{T}\sum_s \int \frac{d^2k}{(2\pi)^2} n_B(\mathcal{H}_{\mathbf{k}}^s)\left(\frac{1}{-i\nu/T + 2r\mathbf{k}\cdot\mathbf{q}/m_0 - rq^2/m_0} - \frac{1}{-i\nu/T + 2r\mathbf{k}\cdot\mathbf{q}/m_0 + rq^2/m_0}\right). \end{aligned} \tag{90}$$

Since to compute the free energy we are going to need to put this expression inside of a log and will therefore need to know its analytic structure, we are going to want to have an analytic approximation for the above expression. This rules out doing the $\mathbf{k}$ integral exactly (even just the component of $\mathbf{k}$ normal to $\mathbf{q}$ can't be integrated out exactly, since the resulting expression has very complicated frequency dependence). Essentially the only way forward is to expand the denominator in small $(2rk_{\parallel}q/m_0)/(-i\nu/T \pm rq^2/m_0)$ — since the Bose function suppresses the integrand exponentially in both components of $\mathbf{k}$,[15] keeping only the first few terms should be allowable. In fact we will mostly just keep only the zeroth-order term; i.e. we will proceed simply by ignoring the $\mathbf{k}\cdot\mathbf{q}$ in the denominator. The effect of first order correction on the free energy can be calculated, but the integral over q must be done numerically. For higher order corrections even the sum over ν becomes very difficult.[16]

Keeping just the zeroth-order term then, we find

$$\begin{aligned} G_{00}(q,\nu) &\sim -\frac{m_0}{T}\left(\frac{1}{m_0}\sum_s \int \frac{d^2k}{(2\pi)^2}\, n_B(\mathcal{H}^s_k)\right)\left(\frac{1}{i\nu/T - r\mathsf{q}^2} - \frac{1}{i\nu/T + r\mathsf{q}^2}\right) \\ &= -\frac{4m_0}{T}\frac{\zeta}{(i\nu/T)^2-\zeta^2}, \end{aligned} \tag{93}$$

where we have defined

$$\mathsf{q} \equiv q/\sqrt{m_0}, \qquad \zeta \equiv r\mathsf{q}^2, \tag{94}$$

and have used that the average number of bosons within an area of $1/m_0$ is 2.

[15] We are also mostly interested in low T (large r): then the bose function means that the k which contribute most to the integral are $k \sim 1/r$, which means that the $\mathbf{k}\cdot\mathbf{q}$ term is suppressed relative to the q^2 terms. This is not a very good justification, though.

[16] In case we ever want to come back to this, here is what the general series expansion looks like:

$$\begin{aligned} G_{00}(q,\nu) &\approx \frac{1}{4\pi^2 m_0}\sum_{s=\pm, n\in 2\mathbb{Z}}\int dk\, d\theta\, (\rho q\cos\theta)^n k^{n+1} n_B(\mathcal{H}^s_k)\left(\frac{1}{(i\nu m_0-\rho q^2)^{n+1}} - \frac{1}{(i\nu m_0+\rho q^2)^{n+1}}\right) \\ &= \frac{1}{\pi m_0}\sum_{s=\pm, n\in 2\mathbb{Z}}\int dk\, \frac{n!(\rho q)^n k^{n+1} n_B(\mathcal{H}^s_k)}{((n/2)!)^2 2^{n+1}}\left(\frac{1}{(i\nu m_0-\rho q^2)^{n+1}} - \frac{1}{(i\nu m_0+\rho q^2)^{n+1}}\right) \\ &= \frac{1}{\pi m_0 T}\sum_{s=\pm, n\in 2\mathbb{Z}}\int_0^\infty dx\, \frac{n!(r\mathsf{q})^n x^{n+1} n_B(\mathcal{H}^s_{x\sqrt{m_0}})}{((n/2)!)^2 2^{n+1}}\left(\frac{1}{(i\nu/T-r\mathsf{q}^2)^{n+1}} - \frac{1}{(i\nu/T+r\mathsf{q}^2)^{n+1}}\right) \\ &\equiv \sum_{n\in 2\mathbb{Z}} G_{00}(q,\nu;n), \end{aligned} \tag{91}$$

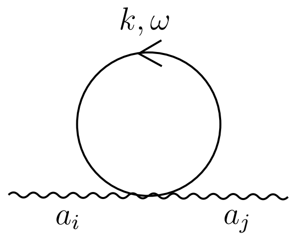
where $\mathsf{q} \equiv q/\sqrt{m_0}$. None of the expressions in the series except for the first one are particularly nice if we make no further assumptions.

Each term simplifies slightly if we are allowed to replace the BE distribution with the Maxwellian distribution: in this approximation, we have

$$\begin{aligned} G^{cl}_{00}(q,\nu) \approx \frac{1}{\pi m_0 T}\sum_{n\in 2\mathbb{Z}} &\frac{n!(r\mathsf{q})^n}{(n/2)!\cdot 2^{n+2}} r^{-n/2-1}\cosh(h) q_*^{-1} \\ &\times\left(\frac{1}{(i\nu/T-r\mathsf{q}^2)^{n+1}} - \frac{1}{(i\nu/T+r\mathsf{q}^2)^{n+1}}\right) \end{aligned} \tag{92}$$

where the 'cl' stands for 'classical approximation' and $q_* = e^{\lambda_*/m_0 T}$.

Now we need to evaluate the susceptibility for the spatial components of the gauge field. The right diagram in (88) is (remembering the symmetry factor)

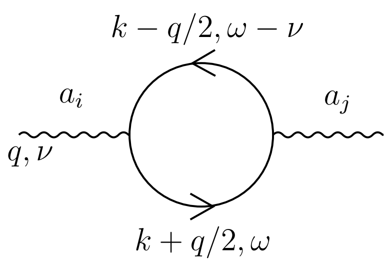
$\frac{1}{N}$ 

$$= \delta_{ij} \frac{2\rho}{m_0} \sum_s \int \frac{d^2k}{(2\pi)^2} \oint \frac{dz}{2\pi i} \frac{n_B(z)}{-z + \mathcal{H}_k^s}$$

$$= \delta_{ij} \frac{2Tr}{m_0} \sum_s \int \frac{d^2k}{(2\pi)^2} n_B(\mathcal{H}_k^s)$$

$$= \delta_{ij} 4Tr \tag{95}$$

which is the two-dimensional analogue of the usual T^2 result in three dimensions for the thermal photon mass squared. The left diagram in (88) is calculated in a similar way to G_{00}: each vertex comes with a factor of $-\rho\times$(momentum of outgoing z^* + momentum of incoming z), and so

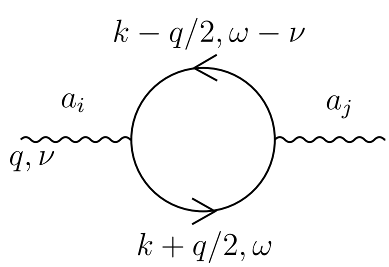
$\frac{1}{N}$ 

$$= \frac{4\rho^2}{m_0^2} \sum_s \oint \frac{dz}{2\pi i} \int \frac{d^2k}{(2\pi)^2} \frac{n_B(z) k_i k_j}{(-z + i\nu + \mathcal{H}_{\mathbf{k}-\mathbf{q}/2}^s)(-z + \mathcal{H}_{\mathbf{k}+\mathbf{q}/2}^s)}$$

$$= \frac{4Tr^2}{m_0^2} \sum_s \int \frac{d^2k}{(2\pi)^2} k_i k_j \frac{n_B(\mathcal{H}_{\mathbf{k}+\mathbf{q}/2}^s) - n_B(\mathcal{H}_{\mathbf{k}-\mathbf{q}/2}^s)}{-i\nu/T + 2r\mathbf{k}\cdot\mathbf{q}/m_0}$$

$$= \frac{4Tr^2}{m_0^2} \sum_s \int \frac{d^2k}{(2\pi)^2} n_B(\mathcal{H}_{\mathbf{k}}^s) \times \left(\frac{(k - q/2)_i (k - q/2)_j}{-i\nu/T + 2r\mathbf{k}\cdot\mathbf{q}/m_0 - \zeta} - \frac{(k + q/2)_i (k + q/2)_j}{-i\nu/T + 2r\mathbf{k}\cdot\mathbf{q}/m_0 + \zeta} \right). \tag{96}$$

As before, we can only really get nice analytic results if we drop the $\mathbf{k}\cdot\mathbf{q}$s from the denominators. In fact in order to properly compare with the other components of the polarization bubble, we will need to drop *all* ks from the term in parenthesis (this is needed if we want to preserve gauge invariance in this approximation). Therefore we have

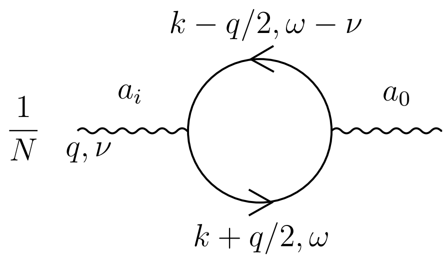
$\frac{1}{N}$ 

$$= \frac{4Tr^2}{m_0} q_i q_j \frac{\zeta}{(i\nu/T)^2 - \zeta^2} \tag{97}$$

so that

$$G_{ij}(q,\nu) = 4Tr\left(\delta_{ij} + \frac{r}{m_0}\frac{\zeta q_i q_j}{(i\nu/T)^2 - \zeta^2}\right). \tag{98}$$

Finally, we have a bubble connecting a_0 and a_i. In the same way as above, and using the same approximation of dropping k, this is (with the minus sign from the a_i vertex)



$$= -\frac{2ir}{m_0}\sum_s \int \frac{d^2k}{(2\pi)^2}\, n_B(\mathcal{H}_k^s)\left(\frac{(k-q/2)_i}{-i\nu/T - \zeta} - \frac{(k+q/2)_i}{-i\nu/T + \zeta}\right)$$

$$= \frac{4q_i\nu r}{T}\frac{1}{(i\nu/T)^2 - \zeta^2}. \tag{99}$$

Let us do a consistency check by seeing whether or not $G^{\mu\nu}$ is transverse. The timelike component of $d^\dagger G$ is[17]

$$\nu G_{00} + q^j G_{j0} = \frac{1}{(i\nu/T)^2 - \zeta^2}\left(-\frac{4\nu m_0\zeta}{T} + \frac{4\nu r q^2}{T}\right) = 0 \quad \checkmark \tag{100}$$

Likewise, the spatial component is

$$\begin{aligned} \nu G_{0i} + q^j G_{ji} &= \frac{4q^i\nu^2 r}{T}\frac{1}{(i\nu/T)^2 - \zeta^2} + 4Tr\left(q^i + q^i\frac{r\zeta q^2/m_0}{(i\nu/T)^2 - \zeta^2}\right) \\ &= \frac{q^i}{(i\nu/T)^2 - \zeta^2}\left(4\nu^2 r/T + 4Tr(i\nu/T)^2\right) = 0 \qquad \checkmark \end{aligned} \tag{101}$$

For future reference, the components of the propagator are

$$\begin{aligned} G_{00} &= -\frac{4m_0\zeta}{T}\frac{1}{(i\nu/T)^2 - \zeta^2} \\ G_{0i} &= \frac{4q_i\nu r}{T}\frac{1}{(i\nu/T)^2 - \zeta^2} \\ G_{ij} &= 4Tr\left(\delta_{ij}((i\nu/T)^2 - \zeta^2) + \frac{r\zeta q_i q_j}{m_0}\right)\frac{1}{(i\nu/T)^2 - \zeta^2} \end{aligned} \tag{102}$$

* * *

The above form of $G_{\mu\nu}$ is screaming out for us to adopt Coulomb gauge (we didn't do this earlier since the gauge-invariance of $G_{\mu\nu}$ is a check that we're doing things correctly). In this gauge, the propagator is simply

$$G^C_{\mu\nu} = -\delta_{\mu 0}\delta_{\nu 0}\frac{4m_0\zeta}{T}\frac{1}{(i\nu/T)^2 - \zeta^2} + 4\delta_{\mu i}\delta_{\nu j}\delta^{ij}Tr. \tag{103}$$

[17]Note that we've actually been Fourier transforming with *Lorentzian* signature, viz. $f(x) \sim \int d^dk\, d\nu\, e^{-i\nu t + i\mathbf{k}\cdot\mathbf{x}}$.

In Coulomb gauge, the needed gauge-fixing contribution is

$$\delta f_{gf}/m_0 T = -\ln\det(-\nabla\cdot\nabla) = -\frac{1}{Tm_0}\sum_\nu \int \frac{d^2q}{(2\pi)^2}\,\ln(\mathsf{q}^2). \tag{104}$$

This term will cancel against the q^2 in the ζ in the numerator of $\det G^C_{\mu\nu}$. Therefore

$$\delta f = \frac{1}{2}\oint \frac{dz}{2\pi i}\int \frac{d^2q}{(2\pi)^2}\,\frac{\coth(z/2T)}{2}\ln\left(\frac{64r^3T^2}{z^2-(T\zeta)^2}\right). \tag{105}$$

Note that for convergence reasons, we have resolved the sum over ν using the function $\coth(z/2T)/2$ instead of $n_B(z)$ — this happens when dealing with fields which disperse linearly.

We now need to evaluate the integral

$$I = \oint \frac{dz}{2\pi i}\int \frac{d^2q}{(2\pi)^2}\,\frac{\coth(\alpha z)}{2}\ln(z^2-y^2). \tag{106}$$

We find

$$\begin{aligned}
I &= \left(\int_{-i\infty+\eta}^{i\infty+\eta} - \int_{-i\infty-\eta}^{i\infty-\eta}\right)\frac{dz}{2\pi i}\int \frac{d^2q}{(2\pi)^2}\,\ln(z^2-y^2)\frac{\coth(\alpha z)}{2}\\
&= \int_{i\infty+\eta}^{-i\infty+\eta}\frac{dz}{2\pi i}\int\frac{d^2q}{(2\pi)^2}\,\ln(z^2-y^2)\coth(\alpha z)\\
&\sim -\int_{i\infty+\eta}^{-i\infty+\eta}\frac{dz}{2\pi i}\int\frac{d^2q}{(2\pi)^2}\int_{T^2}^{y^2}dx^2\,\frac{1}{z^2-x^2}\coth(\alpha z)\\
&= -\int\frac{d^2q}{(2\pi)^2}\int_T^y dx\,\coth(\alpha x)\\
&= -\int\frac{d^2q}{(2\pi)^2}\int_T^y dx\,(1+2n_B(2\alpha x))\\
&= -\int\frac{d^2q}{(2\pi)^2}\,(T-y-\alpha^{-1}\ln(e^{2\alpha T}-1)+\alpha^{-1}\ln(1-e^{-2\alpha y})),
\end{aligned} \tag{107}$$

where the $\sim$ means dropping an infinite constant independent of y.[18] In our application $y = T\zeta$ is independent of T, while $\alpha = 1/2T$ is independent of ρ. We will then need to drop all terms that are independent of ρ, which gives us the expected photon gas result:

$$\begin{aligned}
\delta f &\sim Tm_0\int\frac{d^2\mathsf{q}}{(2\pi)^2}\,\ln(1-e^{-r\mathsf{q}^2})\\
&= -\frac{m_0T\pi}{24r}.
\end{aligned} \tag{108}$$

This then makes a contribution to the entropy density of

$$\delta s/m_0 = \pi/12r, \tag{109}$$

[18] This can also be justified on the grounds that this term is proportional to $\sum_{n\in\mathbb{Z}} 1 = 0$ since ζ-regularized number of integers is zero.

which at $N = 1$ is half the size of the mean-field entropy.

Note that at this level of approximation, the $1/N$ corrections are independent of the field, with the gauge field a seeing only the total density of z bosons and not the magnetization. In Coulomb gauge, the leading h dependence comes from the first corrections to the propagators G_{00} and G_{ii}, viz.

$$\begin{aligned}\delta G_{00}(q,\nu) &= \frac{r^2 q^2}{T\pi m_0^2}\sum_s \int dk\, k^3 n_B(\mathcal{H}_k^s)\left(\frac{1}{(i\nu/T-\zeta)^3}-\frac{1}{(i\nu/T+\zeta)^3}\right)\\ &= \frac{q^2}{2\pi T}\sum_s \mathrm{Li}_2(e^{sh}/q_*)\left(\frac{1}{(i\nu/T-\zeta)^3}-\frac{1}{(i\nu/T+\zeta)^3}\right)\end{aligned} \tag{110}$$

and

$$\delta G_{ij}(q,\nu) = \delta_{ij}\frac{T}{2\pi}\sum_s \mathrm{Li}_2(e^{sh}/q_*)\left(\frac{1}{i\nu/T-\zeta}-\frac{1}{i\nu/T+\zeta}\right). \tag{111}$$

Including these contributions when doing the frequency sum must be done numerically, and so we'll leave it at this for now.

4 Large N $O(3N)$ models

The approach above proceeded by breaking up the $SU(2N)$-adjoint field $\mathbf{n}$ into $SU(2N)$-fundamentals. Since both $O(3)$ and $SU(2)$ act naturally on the target space of the original field theory, another approach is to break up $\mathbf{n}$ into two vectors by writing $\mathbf{n} = w^\dagger \mathbf{m} w$, for $\mathbf{m}$ the intertwiner selecting out 1 from $1\otimes 1 = 0 \oplus 1 \oplus 2$. Since the 1 on the RHS is the antisymmetric representation, the appropriate way to fractionalize $\mathbf{n}$ is by

$$n^a = w^\dagger A^a w, \tag{112}$$

where $[A^a]_{bc} = i\varepsilon^{abc}$ are the generators for the spin-1 representation of $SU(2)$ (properly normalized in accordance with $\mathrm{Tr}[T_A^a T_A^b] = N\delta^{ab}$ for T_A^a the adjoint generators of $SU(N)$). The constraints here are that $w^\dagger w = 1$ and $w^2 = 0$: indeed, with these constraints one sees that

$$\mathbf{n}\cdot\mathbf{n} = -\sum_{ij}(w_i^*)^2 w_j^2 + \left(\sum_i |w_i|^2\right)^2 = 1 \tag{113}$$

as required, and that e.g.

$$\frac{\rho}{2}|\nabla \mathbf{n}|^2 = \rho\left(|\nabla w|^2 + (w^\dagger \nabla w)^2\right), \tag{114}$$

which follows from using $w_i \nabla w^i = w_i^* \overset{\leftrightarrow}{\nabla} w_i = 0$ (note with this parametrization there is no factor of 2 as there was in the parametrization in terms of the zs). The interaction above is then decoupled in the same way as in the $\mathbb{CP}^1$ case. On can similarly check that the WZW term maps to $w^\dagger \partial_\tau w$ as in the $\mathbb{CP}^1$ case (and as with the kinetic term, there is a difference by a factor of 2).[19]

This can then be generalized by letting the ws be $3N$ component complex vectors, now with $|w|^2 = N, w^2 = 0$. As we did in the $SU(2N)$ generalization, we take the magnetic field to point along a generator of $\mathfrak{o}(3N)$ which is a $\oplus$ of N copies of the z-direction generator for $\mathfrak{o}(3)$, viz. the matrix $Y \oplus 0$, so that

$$S \supset m_0 H \int d^d x\, d\tau\, w^\dagger (Y \oplus 0)^{\oplus N} w. \tag{119}$$

Let λ be the Lagrange multiplier enforcing $|w|^2 = N$, and σ, σ^* be two Lagrange multipliers enforcing the real and imaginary parts of $w^2 = 0$. Then the action after integrating out the w fields is (it is helpful to first decompose the ws in terms of real and imaginary parts due to the form of the σ, σ^* Lagrange multipliers)

$$S = \frac{N}{2} \ln\det\big((m_0\partial_\tau - \rho\partial_x^2 + \lambda)\mathbf{1}_6 + mH_0(Y\oplus 0)\otimes \mathbf{1}_2 + \sigma_+ Z^{\oplus 3} + \sigma_- Y^{\oplus 3}\big) - N\int d^d x\, d\tau \lambda \tag{120}$$

where $\sigma_\pm = \sigma \pm \sigma^*$. The mean-field equations for $\sigma_\pm$ are easy to solve: we simply set $\sigma_\pm = 0$. Indeed, the variation with respect to $\sigma_\pm$ computes the expectation value of bilinears in w which are charged under the emergent $U(1)$ gauge field (which we have neglected to write on account of us being interested only in the mean-field solution), and so $\sigma_\pm = 0$ is the right substitution to make for a non-Higgsed mean-field solution.

[19] The strategy is the same: one writes the WZW term as an integral of $\mathrm{Tr}[\widehat{n}\, d\widehat{n} \wedge d\widehat{n}]$, where now $\widehat{n} = \frac{1}{2} n^a A^a \implies n^a = \mathrm{Tr}[\widehat{n} A^a]$ (which just selects out 1 from $0 \oplus 1 \oplus 2$). Since $\mathrm{Tr}[A^a A^b A^c] = \frac{i}{2} N f^{abc}$ for $SU(N)$ adjoints, the proportionality constant is now

$$s_{WZW} = 4m_0 \int \mathrm{Tr}[\widehat{n}\, d\widehat{n} \wedge d\widehat{n}]. \tag{115}$$

The manipluations are essentially the same as the $\mathbb{CP}^1$ case. Using $\widehat{n} = \frac{i}{2}\left(|w\rangle\langle w| - |w^*\rangle\langle w^*|\right)$ and $\langle w|w\rangle = 1$, we have

$$\begin{aligned} s_{WZW} = \frac{m_0}{2}\int \mathrm{Tr}\Big[\Big(& |w\rangle\langle w|\, dw\rangle\langle w| - |w^*\rangle\langle w^*|dw\rangle\langle w| \\ &+ |w\rangle\langle dw| - |w\rangle\langle w|\, dw^*\rangle\langle w^*| + |w^*\rangle\langle w^*|\, dw^*\rangle\langle w^*| + |w^*\rangle\langle dw^*|\Big) \wedge (\cdots)\Big], \end{aligned} \tag{116}$$

where the $\cdots$ are the expansion of $d\widehat{n}$. Then making use of $\langle w^*|w\rangle = 0$ and the anti-symmetry of the wedge product, a fair amount of algebra which won't be written out gives

$$s_{WZW} = \frac{m_0}{2}\int \left(\langle dw| \wedge |dw\rangle - \langle dw^*| \wedge |dw^*\rangle - 2\langle dw|w^*\rangle \wedge \langle dw^*|w\rangle - 2\langle w|dw^*\rangle \wedge \langle w^*|dw\rangle\right). \tag{117}$$

The last two terms cancel, and so

$$s_{WZW} = m_0 \int da, \qquad a \equiv -\frac{i}{2}\left(\langle w|dw\rangle - \langle w^*|dw^*\rangle\right). \tag{118}$$

This then becomes $m_0 \int d\tau\, w_i^* \partial_\tau w_i$, as claimed.

Since the matrix $Y \oplus 0$ has eigenvalues $-1, 0, 1$, the free energy density can be written as

$$f/N = -T \sum_{s=-1,0,1} \sum_{\omega \in 2\pi\mathbb{Z}} \int \frac{d^d k}{(2\pi)^d} \ln\left(-i\omega m_0 T + \rho k^2 + \lambda - m_0 H s\right) - \lambda. \tag{121}$$

Therefore the mean field equation for λ gives

$$1 = \frac{1}{m_0} \sum_{s=-1,0,1} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\exp\left(\frac{1}{m_0 T}\left(\rho k^2 + \lambda - m_0 H s\right)\right) - 1}. \tag{122}$$

In $d = 2$, this becomes[20]

$$1 = \frac{1}{4\pi r} \sum_{s=-1,0,1} \int du \frac{1}{e^u q e^{sh} - 1}, \tag{123}$$

so that the mean field equation for q is

$$-4\pi r = \ln\left[(1 - 1/q)(1 - e^h/q)(1 - e^{-h}/q)\right]. \tag{124}$$

Taking the exponential of both sides and multiplying through by q^3, we get

$$(q - 1)(q - e^h)(q - e^{-h}) = q^3 e^{-4\pi r}, \tag{125}$$

which is precisely the result mentioned in [7]. In zero-field, the solution is

$$q = \frac{1}{1 - e^{-4\pi r/3}} \tag{126}$$

which has the limits

$$q(r \gg 1, h = 0) \approx 1 + e^{-4\pi r/3}, \qquad q(r \gg 1, h = 0) \approx \frac{3}{4\pi r} + \frac{1}{2}, \tag{127}$$

which is the same as the $SU(N)$ case but with $r \mapsto r/3$.

The free energy is, via essentially the same calculation as before,

$$f/N = -\frac{m_0 T}{4\pi r}\left(\mathrm{Li}_2(q^{-1}e^h) + \mathrm{Li}_2(q^{-1}e^{-h}) + \mathrm{Li}_2(q^{-1}) + 4\pi r \ln q\right). \tag{128}$$

The entropy and magnetization density (both divided by N as usual) are

$$s/m_0 = \frac{1}{4\pi r}\left(2\mathrm{Li}_2(q^{-1}e^h) + 2\mathrm{Li}_2(q^{-1}e^{-h}) + 2\mathrm{Li}_2(q^{-1}) + h \ln\left(\frac{q - e^h}{q - e^{-h}}\right) + 4\pi r \ln q\right) \tag{129}$$

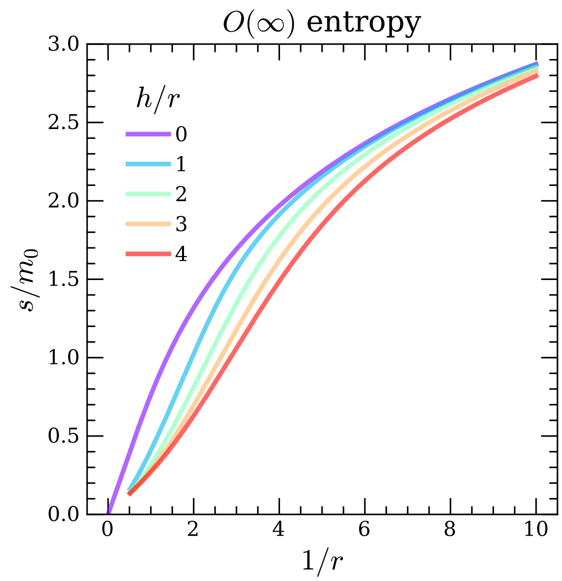
and

$$m/m_0 = \frac{1}{4\pi r} \ln\left(\frac{q - e^h}{q - e^{-h}}\right). \tag{130}$$

In particular, in zero field, the only difference between the entropy for the $SU(2N)$ and $O(3N)$ models is in the different mean-field equations for q and a 3 vs a 2 in the prefactor.

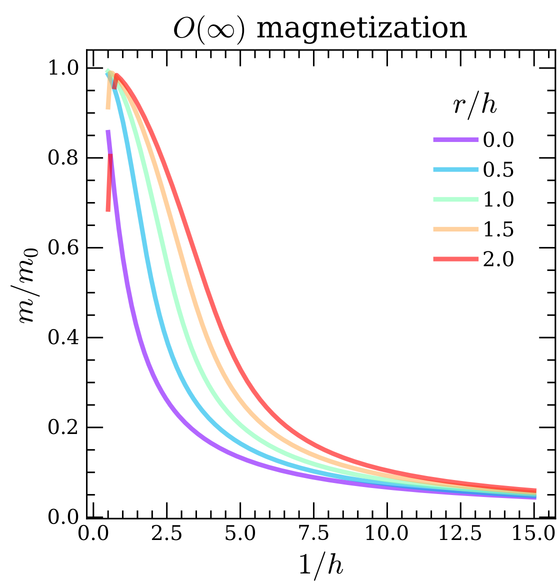
[20] In this section, we are defining $h \equiv H/T$, with no factor of 1/2.

The entropy and magnetization are qualitatively the same as the $SU(N)$ case. Plotting the entropy as a function of $1/r$ for a range of h/r, we find



(131)

Here the turn-on is quicker as a function of $1/r$, with s/m_0 saturating slightly earlier. The magnetization as a function of $1/h$ is



(132)

which again matches the plot in [7]. The magnetization is suppressed at a given $(h, r/h)$ compared to the $SU(\infty)$ case.

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