All about instantons

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These notes are a collection of things about instantons and their normalizations in different gauge groups that I thought would be handy to have around as a reference. It was inspired by reading [1] and wanting to understand + extend the results.

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1 Mathematical prelude

We begin this diary entry with definitions and useful math facts. For us, the instanton number I for a gauge bundle E over a four-dimensional spacetime X will be defined as

$$I = \int_X \mathrm{ch}_2(E), \tag{1}$$

where $\mathrm{ch}_2(E)$ is the second chern character. Recall that the Chern characters are obtained in the context of Chern-Weil theory from the expansion of $\mathrm{Tr}[e^{F/2\pi}]$ as

$$\mathrm{ch}_k(E) = \frac{1}{k!}\mathrm{Tr}\left[(F/2\pi)^{\wedge k}\right]. \tag{2}$$

Note that the Chern characters involve only a single trace, unlike the Chern classes.

We define I as the integral of $\mathrm{ch}_2(E)$ and not of $\mathrm{c}_2(E)$ (the second chern class), since it is the chern character, not the chern class, that appears in the index formula (and since we e.g. definitely want I to be nonzero when we choose the gauge group to be $U(1)$). For gauge groups like $SU(N)$ with traceless generators, $\mathrm{c}_2(E)$ and $\mathrm{ch}_2(E)$ are *almost* the same. The difference comes from torsion phenomena: the Chern classes can have torsion contributions, so that even when $\mathrm{Tr}[F]=0$, we can have e.g. $c_1 \neq 0$, provided that c_1 is pure torsion. This can happen when the gauge group is the quotient of some simply connected group, like in the case of $SU(N)/\mathbb{Z}_N$. These torsionful contributions are ignored by Chern-Weil theory, but are important to keep track of.

In contrast, the Chern characters are defined as classes in $H^*(X;\mathbb{Q})$[1], and as such *never* have any torsionful elements. They are calculated purely from the local curvature, and are only sensitive to data about the gauge group's Lie algebra (whereas the Chern classes care about the full Lie group).

An important difference between the Chern characters and the Chern classes is that Chern classes always integrate to integers. The example relevant to us is that the integral of c_2 is an integer on any manifold, spin or otherwise. In contrast, the integral of ch_2 is not generically integral on a non-spin manifold, since the intersection form on a non-spin manifold is not for sure even. Thus we should remember that the Chern classes are good $\mathbb{Z}$ characteristic classes, while the Chern characters are not.

The chern characters satisfy

$$\mathrm{ch}(E\otimes F)=\mathrm{ch}(E)\wedge\mathrm{ch}(F), \qquad \mathrm{ch}(E\oplus F)=\mathrm{ch}(E)+\mathrm{ch}(F). \tag{3}$$

The former can be seen by plugging in $F_{E\otimes F}=\mathbf{1}_G\otimes F_H+F_G\otimes\mathbf{1}_H$ into the formulae for the chern characters, while the latter is straightforward to see since the chern characters involve only a single trace. On the other hand, the Chern class of the direct sum is the wedge of the Chern classes, instead of the sum:

$$\mathrm{c}(E\oplus F)=\mathrm{c}(E)\wedge\mathrm{c}(F). \tag{4}$$

This is the Whitney sum formula and can be seen from the definition of the Chern classes in terms of the expansion of $\det(\mathbf{1}+F_A/2\pi)$, and the fact that $\det(A\oplus B)=\det(A)\det(B)$. I'm unaware of any simple formula for $\mathrm{c}(E\otimes F)$, unless $E\cong\bigoplus_i\mathcal{L}_i, F\cong\bigoplus_j\mathcal{L}'_j$ for line bundles $\mathcal{L}_i,\mathcal{L}'_j$. In that case, we have

$$\mathrm{c}(E\otimes F)=\mathrm{c}\left(\bigoplus_{i,j}\mathcal{L}_i\otimes\mathcal{L}'_j\right)=\bigwedge_{i,j}\left(1+c_1(\mathcal{L}_i\otimes\mathcal{L}'_j)\right)=\bigwedge_{i,j}(1+\mathrm{c}_1(\mathcal{L}_i)+\mathrm{c}_1(\mathcal{L}'_j)), \tag{5}$$

where we used the Whitney sum formula and the fact that $\mathrm{c}_1(\mathcal{L}\otimes\mathcal{L}')=\mathrm{c}_1(\mathcal{L})+\mathrm{c}_1(\mathcal{L}')$. This can be seen by recalling that the first Chern class can be defined by the Euler class of the underlying real bundle. Since the expression for the Euler class involves the log of the transition functions, and since the transition functions of $\mathcal{L}\otimes\mathcal{L}'$ are the product of the transition functions for $\mathcal{L}$ and $\mathcal{L}'$, the Euler class of $\mathcal{L}\otimes\mathcal{L}'$ splits as a sum of the Euler classes of each line bundle—hence $\mathrm{c}_1(\mathcal{L}\otimes\mathcal{L}')=\mathrm{c}_1(\mathcal{L})+\mathrm{c}_1(\mathcal{L}')$.

[1] The rational coefficients here are simply because the mod 1 contributions to the Chern characters come from the coefficients in the expansion of the exponential. Since the cohomology groups are isomorphic, we may equivalently just write $H^*(X;\mathbb{R})$.

Something else we sometimes need to do is to determine characteristic classes / instanton numbers for product bundles $E \otimes F$, where E is a principal G-bundle and F is a principal H-bundle. The answer for the instanton number is what you would expect: for theories not involving a $U(1)$ factor so that their Lie algebra generators are traceless, we have

$$I_{E\otimes F} = \mathrm{ch}_2(E\otimes F) = I_G \dim H + I_H \dim G, \tag{6}$$

where the dimension means the dimension of the defining representaiton of the associated Lie algebras. This follows from $\mathrm{ch}(E\otimes F) = \mathrm{ch}(E)\wedge \mathrm{ch}(F)$: taking the second order terms, we have

$$\mathrm{ch}_2(E\otimes F) = \mathrm{Tr}_G[\mathbf{1}_G]\frac{1}{8\pi^2}\mathrm{Tr}_H\left[F_H\wedge F_H\right] + \mathrm{Tr}_H[\mathbf{1}_H]\frac{1}{8\pi^2}\mathrm{Tr}_G\left[F_G\wedge F_G\right], \tag{7}$$

which gives us what we want.

2 Instantons

Normal instantons come from transitions between pure gauge field configurations in different homotopy classes of $\pi_3(G)$, where the 3 in $\pi_3(G)$ is a spatial slice (or region thereof) where the gauge fields asymptote to a constant (the elements in $\pi_3(G)$ are the glueing data for nontrivial bundles on S^4). These instantons can live in any $\mathbb{R}^4$-like region of a given 4-manifold, regardless of its topology. Furthermore they will exist for all choices of (non-Abelian) gauge groups, since $\pi_3(G) = \mathbb{Z}$ for all simple compact non-Abelian Lie groups G. These instantons are common to all gauge groups G that descend from some simply connected group $\widetilde{G}$ by quotienting by some finite Γ_G (which may be $\mathbb{Z}_1$). To show this, one uses the long exact sequence coming from $1\to\Gamma_G\to\widetilde{G}\to G\to 1$. This sequence contains

$$\ldots \to \pi_4(\widetilde{G}/G) \to \pi_3(G) \to \pi_3(\widetilde{G}) \to \pi_3(\widetilde{G}/G) \to \cdots . \tag{8}$$

Now $\pi_{k>0}(\widetilde{G}/G) = \pi_{k>0}(\Gamma_G) = 0$ since Γ_G is discrete and the homotopy groups are basepoint preserving (the basepoint is fixed to be a given element of the target space for the definition of the homotopy group, so we don't get a $|\Gamma_G|$'s worth of constant maps, we just get a $\mathbb{Z}_1$'s worth). Thus we have an isomorphism $\pi_3(G)\cong\pi_3(\widetilde{G})$, and so the "small" instantons associated with $\pi_3(G)$ have the same instanton number no matter what Γ_G is.

As in [2], we will normalize the instanton number so that the minimal "small" instanton has instanton number $I = 1$. This minimal small instanton can always be taken to be a minimal $SU(2)$ instanton, on an S^3 around which $A\sim U^\dagger dU, U\sim e^{ix^aT^a}$, for an appropriately chosen trio of generators $T^z, T^+, T^- \in \{T^a\}$, with T^z, T^+, T^- generating an $\mathfrak{su}(2)$ Lie algebra. Recall that this embedding of $\mathfrak{su}(2)$ can always be done: we pick a pair of roots T^+, T^- that are eigenvalues under the action of Ad_A where A is such that Ad_A has maximal kernel, and then from these generators we can construct a T^z in the Cartan subalgebra of $\mathfrak{g}$ that together with the $T^\pm$ generates an $\mathfrak{su}(2)$. Thus for all choices of (compact, simple) Lie group G, we can always embed an $SU(2)$ instanton through a choice of $\mathfrak{su}(2)\to\mathfrak{g}$. This induces a map $SU(2)\to G$, and the normalization of the instanton number depends on the index of this map.

One foolproof way to find the normalization for the instanton number is to compute the instanton number by requiring that for a minimal small instanton field configuration F, we have

$$1 = \frac{1}{N_{\mathfrak{g}}} \int \mathrm{Tr}_{Ad_{\mathfrak{g}}} \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right). \tag{9}$$

Here $\mathrm{Tr}_{Ad_{\mathfrak{g}}}$ is taken in the adjoint representation of $\widetilde{G}$, which is always a representation for all $G = \widetilde{G}/\Gamma_G$, and $N_{\mathfrak{g}}$ is a normalization constant that fixes the equality. For example, consider $SU(2)$. We know the bundle E with a minimal $SU(2)$ instanton is such that

$$1 = \int \mathrm{c}_1(E) = \frac{1}{2} \int \mathrm{Tr}_f \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right). \tag{10}$$

In the fundamental, we have the normalization

$$\mathrm{Tr}(T_f^a T_f^b) = \frac{\delta^{ab}}{2}. \tag{11}$$

On the other hand, in the adjoint we have

$$\mathrm{Tr}(T_{Ad}^a T_{Ad}^b) = N \delta^{ab}, \tag{12}$$

so that if we take the minimal $SU(2)$ instanton F^a but change the representation to the adjoint, the answer changes by a factor of $2N$. Thus we have

$$N_{\mathfrak{su}(N)} = 4N. \tag{13}$$

When Γ_G is nontrivial we can have "large" instantons that contribute to the instanton number I but which make rational contributions to I instead of integral contributions. This is because if the topology of spacetime is nontrivial, we can have G bundles which are not $\widetilde{G}$ bundles. This is not as contrived a scenario as it seems, since nontrivial spacetime topologies can be created by inserting t' Hooft operators with nontrivial t' Hooft flux, which exist if $\pi_1(G) = \Gamma_G$ is nontrivial. To visualize the types of processes that give nonzero "large" instanton number, we can think about the $U(1)$ case, where our intuition is aided by the fact that the integral in $\int \mathrm{ch}_2(E)$ can be interpreted as a self-intersection number of the Poincare dual of F (such an interpretation is only possible in the non-Abelian case if $E = \bigoplus_i \mathcal{L}_i$ is a direct sum of line bundles so that F is diagonal). For example, if we consider a process in which two initially separated magnetic flux loops pass through each other to form a Hopf link and then later re-separate, then the self-intersection number of $\widehat{F}/2\pi$ is 2, and we get $I = 1$.

An important comment is that the fractional part of the instanton number is due entirely to w_2, as noted in [3] (temporarily assuming that the gauge group is semisimple; $U(1)$ factors can of course give fractional contributions not associated with a torsionful class). That is, we have small instantons, which give integral contributions to the instanton number, and large instantons caused by w_2 which can be fractional, and no other types of instantons. We see this as follows: to examine different ways to construct a $G = \widetilde{G}/\Gamma_G$ bundle, we start from a trivialization over the 1-skeleton of X. This is always possible for orientable X

and connected G. Then, we try to extend this bundle over the 2-skeleton. If $\pi_1(G) = \Gamma_G$ is nontrivial, this may not be possible: if the trivialization on the 1-skeleton winds by an element of $\pi_1(G)$ along the boundary of a given 2-cell, the trivialization is not extendable into that 2-cell (this is a global obstruction if the product of all such holonomies in $\pi_1(G)$ accross all 2-cells is nontrivial). 2-cells where there is an obstruction to extending the trivialization are determined by w_2. Now we try to extend the trivialization over the 3-cells. This may be obstructed by an element in $\pi_2(G)$. However, since G is a topological group, we have the magical fact that $\pi_2(G) = \mathbb{Z}_1$, and so there is no obstruction at this level. Finally we try to extend into the 4-skeleton: this is obstructed by $\pi_3(G)$. But as we have seen $\pi_3(G) = \pi_3(\widetilde{G})$ parametrizes the "small" instantons, which are the same for both G and its universal cover, and so the contribution of $\pi_3(G)$ elements to I is always integral in our normalization. Thus the only possible contribution to the fractional part of I is w_2.

3 Examples

Now we will compute examples for a few classes of groups of interest.

As we mentioned above, if our gauge group is simply connected it can have no "large" instantons, and we will never have fractional instanton numbers. Fractional values of I ("large instantons") can arise from two things:

- $U(1)$ factors in the gauge group corresponding to $\mathbb{Z}$ factors in $\pi_1(G)$ (this is no surprise; we know that $I \in \frac{1}{2}\mathbb{Z}$ for $U(1)$ theories on general 4-manifolds)
- G bundles that cannot be lifted to $\widetilde{G}$ bundles because of a discrete obstruction, corresponding to torsion factors in $\pi_1(G)$.

In what follows we will derive expressions for the fractional part of the instanton number I for various gauge groups, written in terms of the discrete characteristic class w_2 that describes the gauge bundle (as well as the $U(1)$ part of the field strength, if it exists). We emphasize that the instanton number is never actually coming from doing a calculation with any discrete objects—the instanton number is the (integral of the) second Chern character, which never has any torsionful contributions. However, as we will see, the choice of the characteristic class w_2 puts constraints on what the instanton number can be, and so that the fractional part of I is indeed a function of w_2. This function will always turn out to be the Pontryagin square $P(w_2)$.

We will be interested in the question of whether or not $P(w_2)$ constitutes a topological action that is linearly independent from the instanton number. Recall that for $w_2 \in H^2(X;\mathbb{Z}_n)$,[2] we have $P(w_2) \in H^4(X;\mathbb{Z}_k)$, where $k = 2n$ if n is even, and $k = n$ if n is odd. We may therefore write the topological terms

$$S_{top}/2\pi \supset \frac{p}{2n}\int P(w_2), \tag{14}$$

[2]As elsewhere in the diary, we are being slightly imprecise as writing $w_2 \in H^2(X;\mathbb{Z}_n)$. In reality w_2 is a characteristic class $w_2 : b_G(X) \to H^2(X;\mathbb{Z}_n)$, i.e. a map from the isomorphism classes of G-bundles over X to the cohomology on X, in this case with $\mathbb{Z}_n$ coefficients. When we write something like $w_2 \in H^2(X;\mathbb{Z}_n)$, we are using w_2 as a standin for the cohomology class one gets when evaluating w_2 on the gauge bundle E, which is left implicit in the notation.

where $p \in \mathbb{Z}_{2n}$ if $n \in 2\mathbb{Z}$, and $p \in 2\mathbb{Z}_{2n}$ if $n \in 2\mathbb{Z}+1$.

Suppose the fractional part of the instanton number which depends on $P(w_2)$ is $(q/2n)\int P(w_2)$, so that the part of S_{top} involving w_2 is

$$S_{top} = \theta\left(\frac{q}{2n}\int P(w_2) + \dots\right) + \frac{p}{2n}\int P(w_2), \tag{15}$$

where $\dots$ are the other contributions to the instanton number. If we can equivalently write this as

$$S_{top} = (\theta + \delta\theta)\left(\frac{q}{2n}\int P(w_2) + \dots\right), \tag{16}$$

where $\delta\theta \in 2\pi\mathbb{Z}$ so as not to affect the integer part of I, then the discrete $P(w_2)$ class will not be independent from the instanton number. This is important because again, despite appearances, the stuff within the ()s in the above equation comes from integrating a local density, allowing the θ angle to take on a continuum of values.[3] We will be able to write S_{top} as (16) for any p provided that q generates all of $\mathbb{Z}_{2n}$ or $2\mathbb{Z}_{2n}$, depending on the parity of n. The condition for this to happen is that

$$\gcd(q, 2n) = \begin{cases} 1 & n \in 2\mathbb{Z} \\ 2 & n \in 2\mathbb{Z}+1 \end{cases}. \tag{17}$$

When this condition is satisfied, there is no independent torsionful characteristic class that we can add to the topological action, and hence the whole topological action will appear with a continuously tunable coefficient θ.

3.1 $SU(N)$ and $PSU(N)$

We will now specialize to the case where $G = PSU(N)$. The degree to which a given $PSU(N)$ bundle E does not lift to an $SU(N)$ bundle is determined by a class

$$w_2(E) \in H^2(X; \mathbb{Z}_N), \tag{18}$$

where X is spacetime. We can construct E by taking an $SU(2)$ bundle $\widetilde{E}$ and relaxing the cocycle condition on the transition functions to only hold modulo an Nth root of unity: $[g_{ij}g_{jk}g_{ki}]_{ab} = \delta_{ab}e^{2\pi i f_{ijk}/N}$, where the f_{ijk} are integers. The choice of f_{ijk} determines the $w_2(E)$ class, which when integrated over a given closed 2-submanifold tells us the fractional flux passing through that manifold.

A naive first guess would be that the instanton number for $PSU(N)$ bundles can be (in our normalization) an element of $\frac{1}{N}\mathbb{Z}$. This is because if E is a $PSU(N)$ bundle, then $E^{\otimes N}$ is a bundle whose transition functions those of an $SU(N)$ bundle, since the transition functions of $E^{\otimes N}$ are N-fold $\otimes$s of the transition functions for E, which ensures that the cocycle condition holds exactly in $E^{\otimes N}$ (i.e., not just up to an Nth root of unity). The instanton number for $E^{\otimes N}$ is found from

$$\mathrm{ch}(E^{\otimes N}) = \mathrm{ch}(E)^{\wedge N} = 1 + N\mathrm{ch}_2(E) + \dots, \tag{19}$$

[3] The term in ()s is well defined since a shift in $P(w_2)$ by a 4-form valued with periods in $2n\mathbb{Z}$ can be compensated for by a shift in the integer-valued part of I (the part coming from small instantons).

where we used $\mathrm{ch}_1(E) = 0$ on account of the traclessness of the $SU(N)$ generators, and so one then might think that $\int \mathrm{ch}_2(E^{\otimes N}) \in \mathbb{Z}$ on account of its transition functions satisfying the cocycle condition. This is not quite true however, and in fact $\int \mathrm{ch}_2(E^{\otimes N}) \in \frac{1}{2}\mathbb{Z}$, a situation which is possible due to the fact that for line bundles, $I \in \frac{1}{2}\mathbb{Z}$ on non-spin manifolds (we will never be restricting to spin manifolds).[4] We'll see how this works in a second.

Now let us see how such a fractional instanton number can be realized. In what follows we will basically be working out in gory detail a computation described in [3] for the case of a spin manifold. The goal is to explicitly construct a $PSU(N)$ bundle that will get us the minimal possible I of $1/2N$.

First, let us fix a class w_2. Let $\mathcal{L}$ be the line bundle over X with first Chern class reducing to w_2 mod N:

$$w_2 = \mathrm{c}_1(\mathcal{L}) \mod N. \tag{20}$$

Here the LHS is viewed as an element in $H^2(X;\mathbb{Z})$, but we will usually use the correspondence between elements of $H^*_{dR}(X;\mathbb{R})$ with quantized periods and those in $H^*(X;\mathbb{Z})$ to think of it as an actual 2-form in the de Rham sense. From $\mathcal{L}$ we can form the bundle $\mathcal{L}^{-1/N}$, defined to have transition functions which are $1/N$th roots of the transition functions of $\mathcal{L}$. In particular, the cocycle conditions in $\mathcal{L}^{-1/N}$ are only satisfied up to Nth roots of unity. We can then construct a $PSU(N)$ bundle E as follows:

$$E = \mathcal{L}^{-1/N} \otimes \left(\mathcal{L} \oplus \bigoplus_{i=1}^{N-1} T_i \right), \tag{21}$$

where T_i is a trivial line bundle. The $\mathcal{L}^{-1/N}$ means that E is not an $SU(N)$ bundle. However, the $\mathcal{L}^{-1/N}$ factor does not turn the thing in parenthesis from an $SU(N)$ bundle into a $PSU(N)$ bundle, since the thing in the parenthesis is not an $SU(N)$ bundle: it has nonzero first Chern character, which precludes it from being an $SU(N)$ bundle. Indeed, (vector bundles associated to) $SU(N)$ principal bundles always have zero first Chern character, simply because $\mathrm{Tr}(F) = 0$ (the Chern characters never have torsion; they are defined totally within the context of Chern-Weil theory). Note that this doesn't necessarily mean that the first Chern *class* must vanish though, since the Chern classes can have torsionful contributions.

Anyway, if E is to be a $PSU(N)$ bundle then since at the Lie algebra level $PSU(N)$ and $SU(N)$ are identical, E must also have a first Chern character which vanishes. This indeed is true, and is the reason for the choice of powers of $\mathcal{L}$ appearing in E: we first use

$$\mathrm{ch}(E) = \mathrm{ch}(\mathcal{L}^{-1/N}) \wedge \left(\mathrm{ch}(\mathcal{L}) + \sum_{i=1}^{N-1} \mathrm{ch}(T_i) \right). \tag{22}$$

Taking the first degree component gives

$$\mathrm{ch}_1(E) = -\frac{1}{N} c_1(\mathcal{L}) \cdot N + 1 \cdot c_1(\mathcal{L}) = 0 \tag{23}$$

[4] A better but more mathematical way to say this would be to say that we can have instanton numbers for $E^{\otimes N}$ that are in $\frac{1}{2}\mathbb{Z}$ because of the existence of the Pontryagin square operation, which lets us consistently "divide" torsionful intersection numbers by two. See the diary entry on the Pontryagin square for details.

as required.

The construction of building E from "fractional" line bundles makes it clear that it is a $PSU(N)$ bundle. If $\lambda_{ij} = e^{i2\pi g_{ij}}$ are the transition functions for $\mathcal{L}$, then the transition functions for E are the matrices

$$\Lambda_{ij} = \mathrm{diag}(e^{i2\pi g_{ij}(1-\frac{1}{N})}, e^{-i2\pi g_{ij}/N}, \dots, e^{-i2\pi g_{ij}/N}). \tag{24}$$

Note that while we still have $\det(\Lambda_{ij}) = 1$, $\delta\Lambda$ is no longer trivial:

$$(\delta\Lambda)_{ijk} = e^{-2\pi i f_{ijk}/N}\mathbf{1}, \tag{25}$$

where the $f_{ijk} \in \mathbb{Z}$ are as before determined by the class w_2. This means that the Λ_{ij} are transition functions for a $PSU(N)$ bundle, but not for an $SU(N)$ bundle[5].

Now we will compute the instanton number of E, working modulo integral classes (i.e., just focusing on the fractional part). For some reason I chose to first do this by computing the second Chern class of E, which gives the instanton number since the second Chern class and second Chern character are equal in this case (the calculation of the Chern character is a little later on). We use the Whitney sum formula to write

$$\mathrm{c}(E) = (1 + \mathrm{c}_1(\mathcal{L}^{-1/N}) + \mathrm{c}_1(\mathcal{L})) \wedge \bigwedge_{i=1}^{N-1} (1 + \mathrm{c}_1(\mathcal{L}^{-1/N})). \tag{26}$$

Taking the degree-2 part, we have

$$\mathrm{c}_2(E) = \frac{N^2 - N}{2}\mathrm{c}_1(\mathcal{L}^{-1/N}) \wedge \mathrm{c}_1(\mathcal{L}^{-1/N}) + (N-1)\mathrm{c}_1(\mathcal{L}) \wedge \mathrm{c}_1(\mathcal{L}^{-1/N}). \tag{27}$$

Now the wedge product of the chern classes is, using the Pontryagin square to take the wedge product so as to properly count the self-intersections of the w_2 surface (see the diary on the Pontryagin square for more),

$$\mathrm{c}_1(\mathcal{L}^{-1/N}) \wedge \mathrm{c}_1(\mathcal{L}^{-1/N}) = \frac{1}{N^2}P(w_2). \tag{28}$$

Then we get

$$\mathrm{c}_2(E) = \frac{P(w_2)}{2}\left(1 - \frac{1}{N} - 2 + 2\frac{1}{N}\right) = -\frac{1}{2}\left(1 - \frac{1}{N}\right)P(w_2). \tag{29}$$

We can also do the computation by computing $\mathrm{ch}_2(E)$, which should agree with $\mathrm{c}_2(E)$ since $\mathrm{c}_1(E) = 0$. The calculation goes as follows:

$$\mathrm{ch}(E) = \mathrm{ch}(\mathcal{L}^{-1/N}) \wedge (\mathrm{ch}(\mathcal{L}) + N - 1). \tag{30}$$

[5] Here it is very important that $\delta\Lambda$ is a constant Nth root of unity times $\mathbf{1}$: having different Nth roots of unity along each entry of the diagonal would be no good, since only the diagonal $\mathbb{Z}_N$ is moded out by when passing to $PSU(N)$.

Since for line bundles $\mathrm{ch}_2(\mathcal{L}) = \frac{1}{2}\mathrm{c}_1(\mathcal{L}) \wedge \mathrm{c}_1(\mathcal{L})$ we have, working modulo terms that are integral classes,[6]

$$\begin{aligned}\mathrm{ch}_2(E) &= N\mathrm{ch}_2(\mathcal{L}^{-1/N}) + \mathrm{ch}_2(\mathcal{L}) + \mathrm{ch}_1(\mathcal{L}^{-1/N}) \wedge \mathrm{ch}_1(\mathcal{L}) \\ &= P(w_2)\left(\frac{N}{2N^2} + \frac{1}{2} - \frac{1}{N}\right) \\ &= \frac{(N-1)}{2N}P(w_2).\end{aligned} \tag{31}$$

These results tell us that the instanton number is valued in $\frac{1}{N}\mathbb{Z}$ on a spin manifold, and $\frac{1}{2N}\mathbb{Z}$ on a non-spin manifold. Something that's kind of interesting here is that as mentioned above, on a non-spin manifold we can get an instanton number l such that $NI \notin \mathbb{Z}$! Again, this is interesting because from the point of view of transition functions, we might be led to expect that the fractional part of the instanton number is always valued in $\frac{1}{N}\mathbb{Z}$, given that the transition functions of an $PSU(N)$ bundle can always be made into the transition functions for an $SU(N)$ bundle by raising them to their Nth powers. This becomes a little less surprising if we consider the rather dumb (since it's just an issue of normalization) example of $U(1)/\mathbb{Z}_N$: here the instanton number in the $U(1)/\mathbb{Z}_N$ theory is valued in $\frac{1}{2N^2}\mathbb{Z}$, even though the transition functions all only fail the cocycle conditions by Nth roots of unity. Looking at this example, we see that the reason for N copies of a $PSU(N)$ bundle not necessarily giving a $I \in \mathbb{Z}$ just boils down to the fact that I is nonlinear in the field strength and that the $PSU(N)$ bundles are built from line bundles, which can have fractional instanton numbers on non-spin manifolds.[7]

To be pedantically explicit, we can do the construction of the minimal $PSU(N)$ bundle for $SU(2)$. Let A_m be the $U(1)$ gauge field for a 2π monopole whose worldline wraps some nontrivial cycle in spacetime, and let

$$A_{SO(3)} = \begin{pmatrix} A_m/2 & 0 \\ 0 & -A_m/2 \end{pmatrix}. \tag{32}$$

Note that this has zero first Chern character as required,[8] and that it is constructed as a $\oplus$ of "fractional" line bundles: the total $SO(3)$ bundle is $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$, where $\mathcal{L}$ is a legit $U(1)$ line bundle. The instanton number for this field configuration is

$$I = \frac{1}{16\pi^2}\int F_{A_m} \wedge F_{A_m} \in \frac{1}{4}\mathbb{Z} \tag{33}$$

[6] We write $\mathrm{c}_1(\mathcal{L}) = w_2 + N\alpha$, where α has integral periods. We can then throw away a term $\frac{N^2-N}{2}w_2 \wedge \alpha$ in the second step since it is an integral class, on account of $(N^2 - N)/2 \in \mathbb{Z}$ for all N.

[7] It's also kind of interesting that we can get the minimal instanton number by working with a direct sum of line bundles regardless of the spin-ness of the background manifold, since for $SU(N)$ bundles this isn't the case: consider e.g. an $SU(2)$ bundle $E = \mathcal{L} \oplus \mathcal{L}^*$. Then $\mathrm{ch}_2(E) = 2\mathrm{ch}_2(\mathcal{L})$, which on a spin manifold is in $2\mathbb{Z}$, twice the minimum allowed value. This tells us that on a spin manifold the minimal instantons for $SU(N)$ are the ones that involve twisting around more than just two axes (and hence cannot be composed into a $\oplus$ of line bundles). This is then to be contrasted with the $PSU(N)$ case, where bundles formed from direct sums of line bundles can always give the minimal instanton number.

[8] However, its first Chern class is a nontrivial element in $\mathbb{Z}_2$ cohomology—the second SW class of the $SO(3)$ bundle—which is torsionful (recall that the second SW class for a complex vector bundle is the mod-2 reduction of that bundle's first Chern class).

on general manifolds, while it is in $\frac{1}{2}\mathbb{Z}$ on spin manifolds. This is in keeping with us being able to get strength $1/2N$ instantons for $PSU(N)$ gauge theories on a generic manifold, and strength $1/N$ instantons on spin manifolds. Note how this instanton picks out a particular direction in $SU(2)$ space, namely the σ^z direction, unlike the small instantons which wrap about the entire internal $SU(2)$ space.

A final way to see all of this is to write the $PSU(N)$ field in terms of a $U(N)$ field and a $\mathbb{Z}_N$ 2-form gauge field, as was done in the previous diary entry. Using the same notation as in that diary entry, we have

$$I = \frac{1}{8\pi^2}\int \mathrm{Tr}[(F_{\mathcal{A}} - B\mathbf{1}) \wedge (F_{\mathcal{A}} - B\mathbf{1})], \tag{34}$$

where $\mathcal{A} = A_{SU(N)} + \mathcal{A}\mathbf{1}/N$ is a $U(N)$ gauge field, and $\mathcal{A}$ is a properly quantized $U(1)$ gauge field, with $F_{\mathcal{A}} = NB$ enforced through a Lagrange multiplier constraint. Then

$$I = \mathrm{c}_2(E_{U(N)}) + \frac{i}{8\pi^2}\int (F_{\mathcal{A}} \wedge F_{\mathcal{A}} + NB\wedge B - 2F\wedge B) \to \mathrm{c}_2(E_{U(N)}) + \frac{i}{8\pi^2}(N^2 - N)\int B\wedge B, \tag{35}$$

where we used the Lagrange multiplier constraint. The first term, the second Chern class of the $U(N)$ bundle, is in $\mathbb{Z}$. However, the second term is in $\frac{1}{2N}\mathbb{Z}$, since $B/2\pi$ has periods in $1/N$. Thus the instanton number for $PSU(N)$ theories is valued in $\frac{1}{2N}\mathbb{Z}$. B here is the 2-form that measures w_2 of the bundle, and so as above, we see that the fractional part of the instanton number comes from the "large" instantons (the small ones are determined by the $\mathrm{c}_2(E_{U(N)})$ factor).

Finally we ask whether the instanton number I is linearly independent from the discrete torsionful term $p/2n\int P(w_2)$. To answer this we calculate

$$\gcd(N-1, 2N) = \gcd(N-1, 2), \tag{36}$$

which according to (17) means that in this case, the discrete class $P(w_2)$ not independent from the instanton number for any N.

3.2 $SU(N)/\mathbb{Z}_M$

The computation for this case is similar to the computation for $PSU(N)$, but it's not in the literature so it's worth doing (we will be very brief, though). We can assume that $M|N$ with $1 < M < N$ wolog. We form an $SU(N)/\mathbb{Z}_M$ bundle by

$$E_{SU(N)/\mathbb{Z}_M} = \mathcal{L}^{(1-N)/M} \oplus (\mathcal{L}^{1/M})^{\oplus(N-1)}, \tag{37}$$

which has zero first Chern character and has transition functions which fail the cocycle condition by powers of $e^{2\pi i/M}\mathbf{1}_N$, as required. The second Chern character is, after a little algebra,

$$\mathrm{ch}_2[E_{SU(N)/\mathbb{Z}_M}] = \frac{N(N-1)}{2M^2}\mathrm{c}_1(\mathcal{L})^{\wedge 2}, \tag{38}$$

where $\mathrm{c}_1(\mathcal{L})$ mod M reduces to w_2 of the bundle, which is a $\mathbb{Z}_M$ valued form because $\pi_1[SU(N)/\mathbb{Z}_M] = \mathbb{Z}_M$. This result is pretty easy to understand from the $PSU(N)$ case:

the Ns in the denominator come from the number of line bundles in the direct sum (which of course doesn't change when we change the group we're quotienting by), while the N^2 in the denominator changes to M^2 because the extent to which the constituent line bundles are allowed to be fractional changes when we change the quotient group to $\mathbb{Z}_M$. Therefore the instanton number is

$$I_{E_{SU(N)/\mathbb{Z}_M}} = \frac{2\pi N(N-1)}{2M^2} \int P(w_2) + \dots, \tag{39}$$

where $\dots$ are the integer parts.

The condition for the discrete class $P(w_2)$ to not be independent from the instanton number is

$$\gcd(N(N-1)/M, 2M) = \begin{cases} 1 & M \in 2\mathbb{Z} \\ 2 & M \in 2\mathbb{Z}+1 \end{cases}. \tag{40}$$

3.3 $U(N)/\mathbb{Z}_M$

In another diary entry, we show that

$$\pi_1[U(N)/\mathbb{Z}_M] = \mathbb{Z} \times \mathbb{Z}_g, \qquad g \equiv \gcd(N, M). \tag{41}$$

Therefore $Q \equiv U(N)/\mathbb{Z}_M$ bundles will be labeled by a $\mathbb{Z}_g$ characteristic class w_2, along with the regular $\mathbb{Z}$-valued class for the "small" instantons (and, since we have a $U(1)$ factor, a fractional contribution from large instantons in the $U(1)$ part of the group).

We now need to ask how we can form Q bundles which are not $U(N)$ bundles. We claim that we can always obtain the minimal instanton number with a bundle

$$E_Q(r,q) = \mathcal{L}_d^{r/M} \otimes \begin{pmatrix} \mathcal{L}_t^{q(1-N)/g} & & & \\ & \mathcal{L}_t^{q/g} & & \\ & & \ddots & \\ & & & \mathcal{L}_t^{q/g} \end{pmatrix}, \qquad r, q \in \mathbb{Z}. \tag{42}$$

Here the line bundle $\mathcal{L}_d$ (d for "diagonal") keeps track of the fractional $U(1)$ part of the instanton number, with the transition functions in $\mathcal{L}_d^{r/M}$ failing by powers of $\zeta_M^r \mathbf{1}_N$, where $\zeta_M = e^{2\pi i/M}$. The Chern character of $\mathcal{L}_d$ does *not* reduce to anything relating to the torsionful class w_2 when modded out by some integer: the element of $\pi_1[Q]$ defined by a triple patch overlap where the transition functions of $\mathcal{L}_d^{r/M} \otimes \mathbf{1}_N$ fail the cocycle condition by $\zeta_M^r \mathbf{1}_N$ is an element of the $\mathbb{Z}$ factor in $\pi_1[Q]$ (coming from topologically nontrivial maps $\det : U(N) \to S^1$), and hence is unrelated to torsionful w_2 class.

The dependence of the instanton number on the w_2 of the bundle $E_Q(r,s)$ is instead determined by the term involving $\mathcal{L}_t$ (t for "torsion"). $\mathcal{L}_t^{q/g}$ is a line bundle that fails the cocycle conditon by $\zeta_g^q \mathbf{1}_N$ on each triple overlap; this is allowed since ζ_g is a power of ζ_M. The structure of the direct sum of the $\mathcal{L}_t$s is such that around a triple patch overlap where the cocycle condition fails, one traces out a loop in the finite $\mathbb{Z}_g$ factor of $\pi_1[Q]$ (see the diary entry on $\pi_1[U(N)/\mathbb{Z}_M]$ for details). This means that the Chern class of $\mathcal{L}_t$ can be taken to reduce to w_2 mod g. Note that the factor involving the $\mathcal{L}_t$s gives us a minimal $SU(N)/\mathbb{Z}_g$

bundle when $q = 1$, which is what we expect from general considerations of how the quotient in Q acts on the $U(1)$ and $SU(N)$ factors.

Now we should actually compute the instanton number I. The total Chern character is

$$\mathrm{ch}(E_Q(r,q)) = (N-1)\mathrm{ch}(\mathcal{L}_d^{r/M} \otimes \mathcal{L}_t^{q/g}) + \mathrm{ch}(\mathcal{L}_d^{r/M} \otimes \mathcal{L}_t^{q(1-N)/g}). \tag{43}$$

Taking the degree 2 part and simplifying modulo integer terms, we get

$$I = \int \left(\frac{Nr^2}{2M^2} \mathrm{c}_1(\mathcal{L}_d) \wedge \mathrm{c}_1(\mathcal{L}_d) + \frac{pq^2}{2g} P(w_2) \right) + \dots, \qquad p \equiv \frac{N(N-1)}{g} \in \mathbb{Z}, \tag{44}$$

again with $\dots$ representing integer contributions. Note the absence of mixed terms between the $\mathcal{L}_d$ and $\mathcal{L}_t$ factors; this is because the bundle associated with the $\mathcal{L}_t$ factors has zero first Chern class. Also note that the term $P(w_2)$ is well-defined mod $2\pi\mathbb{Z}$: its well-definedness mod $2\pi\mathbb{Z}$ for arbitrary q depends on $pg \in 2\mathbb{Z}$, but this is always the case since $pg = N(N-1) \in 2\mathbb{Z}$. The minimal instanton number is then found by choosing either r or q to be zero and the other to be 1, depending on the choices of M and N.

From the above expression, we see that the condition for the instanton number to reproduce all possible discrete $P(w_2)$ terms is that

$$\gcd\left(\frac{N(N-1)}{g}, 2g \right) = \begin{cases} 1 & g \in 2\mathbb{Z} \\ 2 & g \in 2\mathbb{Z}+1 \end{cases}. \tag{45}$$

Actually there's a small subtlety here: to "absorb" a possible discrete term into the instanton number, we have to shift θ by something in $2\pi\mathbb{Z}$. In the previous examples this hasn't been a problem, since the part of I that can be written in terms of $P(w_2)$ is the only contribution to I mod 1. However in the $U(1)$ case, we also have a fractional part of I which comes from the $U(1)$ part: the term $\mathrm{Tr}[F] \wedge \mathrm{Tr}[F] \propto \mathrm{c}_1(E)^{\wedge 2}$ appears in I, and is fractional in general. Therefore the shift of θ to cancel the discrete $P(w_2)$ term is not as innocuous as it seems, since the phase contributed by this fractional $U(1)$ part will change.

However, in the $U(N)$ case we should really be considering a more general topological action with two distinct θ angles, since in the $U(N)$ case the second Chern class and second Chern character give distinct topological terms:

$$S_{top} = \theta_1 \int \mathrm{ch}_2[E] + \theta_2 \int \frac{1}{2}\mathrm{ch}_1[E] \wedge \mathrm{ch}_1[E] + \frac{2\pi p}{2g} \int P(w_2), \tag{46}$$

where again, $\mathrm{ch}_1[E] \wedge \mathrm{ch}_1[E]$ is just a pretentious way of writing $\mathrm{Tr}[F] \wedge \mathrm{Tr}[F]/4\pi^2$. The precise statement to make is then that if (45) is satisfied, then the torsionful $P(w_2)$ term can be absorbed into the continuous θ terms by adjusting both θ_1 *and* θ_2. Just for fun, the condition that the discrete $P(w_2)$ term can be absorbed into the continuous theta term is shown as a function of N and M in figure 1.

3.4 $Sp(N)$ and $PSp(N)$

First, let's disambiguate the notation: here, by $Sp(N)$, we mean the *compact* group

$$Sp(N) \equiv U(2N) \cap Sp(2N;\mathbb{C}), \tag{47}$$

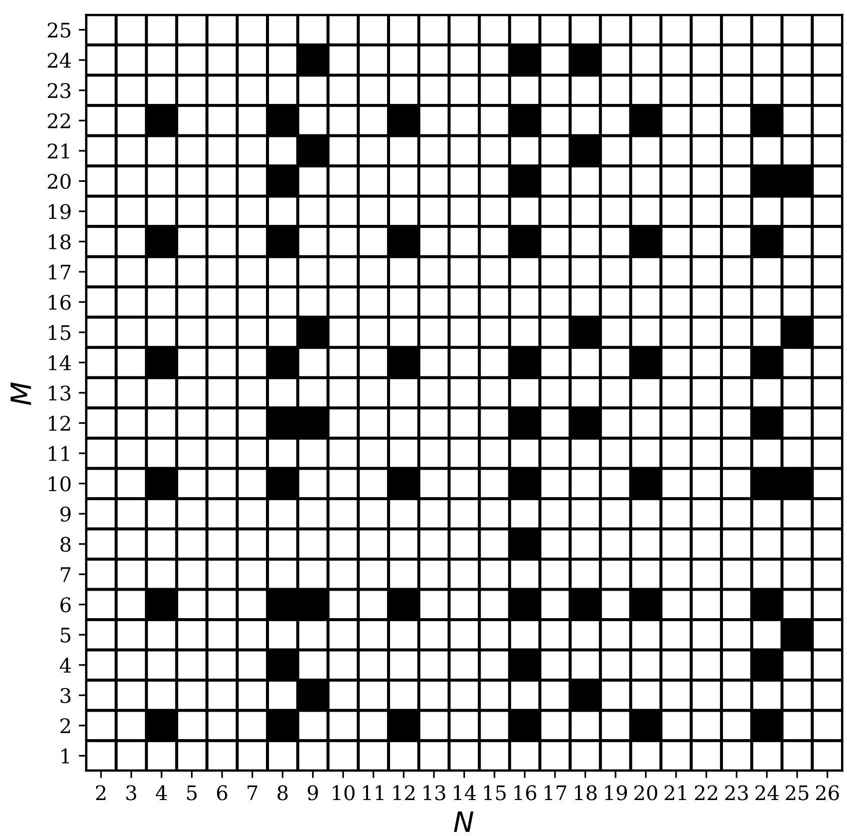

Figure 1: When the instanton number and discrete theta angle are independent for $U(N)/\mathbb{Z}_M$. A black filling means that they are independent. Going to larger values of N and M gives you something that looks like its nearly periodic, but actually isn't.

where $Sp(2N;\mathbb{C})$ is the *non-compact* group of complex $2N$-$2N$ matrices that preserve $J \otimes \mathbf{1}_{N\times N}$, in the sense that

$$U \in Sp(N) \implies U^\dagger U = 1, \qquad U^T J U = J \otimes \mathbf{1}_{N\times N}, \tag{48}$$

where

$$J \equiv (-iY) \otimes \mathbf{1}_{N\times N}, \qquad J^2 = -\mathbf{1}_{2N\times 2N} \tag{49}$$

is our choice of symplectic form. The Lie algebra for the compact symplectic group [9] can be obtained by writing a general Lie algebra element T as a linear combination

$$\mathfrak{sp}(N) \ni T = i\mathbf{1} \otimes A + X \otimes B_1 + Y \otimes B_2 + Z \otimes B_3, \tag{50}$$

where A is traceless and antisymmetric, and the B_i's are symmetric. Both A and the B_i's are real (they have to be (anti-)Hermitian in order for $e^{i\alpha T}$ to be unitary, and they have to be (anti-)symmetric in order for $e^{i\alpha T}$ to preserve the symplectic form). In this presentation we see clearly how $\mathfrak{su}(2)$ is embedded in $\mathfrak{sp}(N)$, viz. as the first factors in the $\otimes$. Additionally, we see that $Sp(1) = SU(2)$ and the center of $Sp(N)$ is $\mathbb{Z}_2$, as can be easily checked by looking for diagonal things that preserve $J \otimes \mathbf{1}$.

To get the normalization for the instanton number straight, we need to look at how $SU(2)$ embeds into $Sp(N)$. First, note that there can only be a single full $SU(2)$ factor in $Sp(N)$, since $Z(Sp(N)) = \mathbb{Z}_2 = Z(SU(2))$ means that we can't have multiple copies without having a quotient by their centers as well. We can also find such a full $SU(2)$ just by looking at matrices of the form $U \otimes \mathbf{1}_{N\times N}$, where $U \in SU(2)$. These are obviously unitary, and a quick check shows that they are also in $Sp(2N;\mathbb{C})$. Furthermore setting $U = -\mathbf{1}_{2\times 2}$ gives the center of $Sp(N)$, so we know that $Sp(N)$ really does have a full $SU(2)$ inside of it (i.e., the $SU(2)$ doesn't appear in a form where it's quotiented by $\mathbb{Z}_2$ in some way). Thus minimal $SU(2)$ instantons have instanton number 1 in $Sp(N)$, which we might write as

$$I_{Sp(N)} = \int p_1(Sp(N)). \tag{51}$$

Explicitly, we can write the gauge field $A^{Sp(N)}$ for a minimal instanton in terms of the $SU(2)$ minimal instanton gauge field $A_{SU(2)}$ as e.g.

$$A_\mu^{Sp(N)} = A_\mu^{SU(2)} \otimes E_{11}, \tag{52}$$

where E_{11} is the matrix with a 1 in the upper leftmost entry, and zeros everywhere else. Since E_{11} is symmetric, $A_\mu^{Sp(N)}$ is indeed in the Lie algebra $\mathfrak{sp}(N)$. Writing it like this, it's clear that $I_{Sp(N)} = p_1(Sp(N))$.

Now for the quotient groups $PSp(N) = Sp(N)/\mathbb{Z}_2$. How might we obtain a $PSp(N)$ bundle that's not an $Sp(N)$ bundle? We consider the bundle $E_{SO(3)} = \mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$, which is an $SO(3)$ bundle that does not lift to an $SU(2)$ bundle. Similarly to as in our discussion of $SU(N)$, $\mathcal{L}$ is the line bundle whose first Chern class reduces mod 2 to some class $w_2 \in$

[9] Why's it called symplectic? Since it preserves iY, which is the antisymmetric form used in the commutation relations for the symplectic form on phase space: if $v = (x,p)^T$, then $v^T J v = i$ is the CCR, and we can send $v \mapsto Rv$ for any $R \in Sp(N)$ preserving the CCR.

$H^2(X;\mathbb{Z}_2)$. Since roughly the transition functions fail the cocycle condition on triple overlaps by the value of w_2 on the triple overlap, the cocycle conditions of $\mathcal{L}^{1/2}$ fail by an amount controlled by $w_2/2$. We then use the diagonal embedding $SU(2) \to Sp(N)$ to use $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$ to create a $PSp(N)$ bundle that doesn't lift to an $Sp(N)$ bundle. Since the diagonal $SU(2) \to Sp(N)$ embedding sends

$$SU(2) \ni U \mapsto U \otimes \mathbf{1}_{N\times N} \in Sp(N), \tag{53}$$

the $PSp(N)$ bundle we get is a direct sum of N copies of $E_{SO(3)}$[10]:

$$E_{PSp(N)} = E_{SO(3)}^{\oplus N} = (\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2})^{\oplus N}. \tag{54}$$

Using manipulations like the ones used for looking at $PSU(N)$ bundles, we see that the instanton number mod 1 (i.e. the part of the instanton number that doesn't come from small instantons) is

$$[\mathrm{ch}_2(E_{PSp(N)})]_1 = \frac{N}{2}[\mathrm{ch}_2(\mathcal{L})]_2 = \frac{N}{2}\frac{P(w_2)}{2}, \tag{55}$$

where $P(w_2)$ is again the Pontryagin square (a $\mathbb{Z}_4$ class, since it's acting on $\mathbb{Z}_2$ cochains), and where $[]_k$ denotes the mod k reduction. Since $P(w_2)/2$ is an integer class on a spin manifold by the even-ness of the intersection form, on spin manifolds we can have fractional instantons for $PSp(N)$ if N is odd, but not if N is even. From this we see that the discrete class $P(w_2)$ is independent from I provided that $N \in 2\mathbb{Z}+1$.

Before moving on, let's just clarify why we needed to choose the diagonal embedding of $SU(2)$ into $Sp(N)$, instead of e.g. the embedding $U \mapsto E_{11} \otimes U$ used to compute the normalization of $I_{Sp(N)}$ (I'm writing the tensor product in the opposite order since I find it slightly easier to visualize). Indeed, suppose we chose this embedding for the $SO(3)$ bundle. Then we would end up with a bundle whose transition functions could fail the cocycle condition by the matrix $-\mathbf{1}_{2\times 2} \oplus \mathbf{1}_{2N-2\times 2N-2}$. In a $PSp(N)$ bundle, the transition functions are only allowed to fail the cocycle condition by the matrix $-\mathbf{1}_{2N\times 2N}$, since this is the thing that gets quotiented out by upon passing to $PSp(N)$. In contrast, if we choose the diagonal embedding $U \mapsto \mathbf{1} \otimes U$, then we get a bundle whose transition functions fail the cocycle condition by $-\mathbf{1}_{2N\times 2N}$, which is what we want. Thus, we must choose the diagonal embedding.

3.5 $SO(N)$

We will now briefly look at the normalization of the instanton number for $SO(N)$. Some time in the future I may come back and discuss $\mathrm{Spin}(N)$ and quotients of $SO(N)$.

First for $SO(3)$, which we've already mentioned above. To find the normalization, we compute the value that a minimal $SU(2)$ instanton has when lifted to the adjoint representation. This is easy: we can take the same $U^\dagger dU$ with $U \sim e^{ir^aT^a}$ type of instanton, we just have to change the T^a's. Now for $SU(N)$ we have $\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$, while for $SO(3)$ we have

[10]This is more obvious if we write the embedding as $\mathbf{1}_{N\times N} \otimes U$, and change our definition so that the elements in $Sp(N)$ preserve $\mathbf{1}_{N\times N} \otimes J$.

$\mathrm{Tr}[T^aT^b] = 2\delta^{ab}$. So $\int p_1(E_{SO(3)})$ for an $SO(3)$ bundle with a minimal $SU(2)$ instanton is $4\int p_1(E_{SU(2)})$. Thus for $SO(3)$,

$$I_{SO(3)} = \frac{1}{4}\int p_1(SO(3)). \tag{56}$$

The notation $p_1(SO(3))$ has the hopefully obvious meaning "$p_1(E)$ for some $SO(3)$ bundle E". Note that this conclusion was reached for an arbitrary manifold, spin or not spin. If we restrict ourselves to spin manifolds, the Pontryagin class is even, so that $I_{SO(3)} \in \frac{1}{2}\mathbb{Z}$ on spin manifolds. This can be proved decomposing the $SO(3)$ bundle as $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$ and realizing that the second chern character depends on the even-ness of the intersection form, or by using the relation

$$p_1(E) \mod 2 = P(w_2), \tag{57}$$

where P is the Pointryagin square. Again we see an example of the fact that the quantization of the instanton number for simply connected Lie groups doesn't depend on whether the base manifold is spin (since there the instanton number is also the Chern class, which is integral on any manifold), but that for quotients of simply connected Lie groups, the quantiation of the instanton number does depend on whether the base manifold is spin.

Now for $SO(N \geq 4)$. We use

$$SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2, \tag{58}$$

where the quotient is the diagonal $\mathbb{Z}_2$ (some people write this with $\otimes$ instead of $\times$, which I don't like: the tensor unit is $\mathbb{C}$, which means that we would already be making the $/\mathbb{Z}_2$ identification!). Note that since $Z(SU(2) \times SU(2)) = \mathbb{Z}_2^2$, taking the quotient leaves behind a factor of $\mathbb{Z}_2$ in the center, which is just right to match with $Z(SO(4)) = \mathbb{Z}_2$.

Regular (non-fractional) instantons are created in $SO(N > 3)$ through embedding a minimal $SU(2)$ instanton into one of the $SU(2)$ factors in the decomposition for the subgroup $SO(4) \subset SO(N)$. Now, we can form fractional instantons in $SO(N)$ by embedding an $SO(3)$ instanton inside of $SO(N)$. The way this embedding works is also through the $SO(4)$ subgroup (note to self: can we show there are no other ways to do the embedding?), but it is the embedding into the diagonal subgroup of $[SU(2) \times SU(2)]/\mathbb{Z}_2$. The reason that the embedding must be done though the diagonal subgroup is because $SO(3)$ has trivial center, and so we need to embed $SO(3)$ in the diagonal subgroup so that the quotient by $\mathbb{Z}_2$ gives us something without a $-\mathbf{1}$ central element. Anyway, the point of this is that the minimal fractional instanton number in $SO(N)$ will be *twice* that in $SO(3)$, since both $SU(2)$ factors contribute. So

$$I_{SO(N)} = \frac{1}{2}\int p_1(SO(N)), \qquad N \geq 4. \tag{59}$$

Again, this holds over arbitrary manifolds, be they spin or not spin. If the manifold is spin, we can conclude that $I_{SO(N)} \in \mathbb{Z}$ since in that case $p_1(SO(N))$ is an even class, as discussed earlier.

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