RG diary

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Table of contents

1 General comments on RG: the relation between the ”old-school” and Wilsonian approaches to RG, scheme-dependence, etc.

blah blah. Might want to turn into its own mini document

`gliders.pdf`

Wilsonian approach

One important point that I haven't seen in textbooks is that the procedure of integrating over the fast modes is *one-loop exact*, in that the flows of the couplings $\delta_{\ln\Lambda} g_\alpha$ in the effective action $S_{eff}[\phi_>]$ are computed only from diagrams that contain a single loop. This is simply because of phase space reasons: each independent momentum integral of the fast fields comes with a phase-space factor of $\delta\Lambda$, and so in the $\delta \ln \Lambda \to 0$ limit, only the diagrams with one momentum integration, viz. one-loop diagrams, survive. Therefore if S_I is the part of the action that mixes the high and low momentum modes, we have (in Euclidean signature)

$$S_{eff}[\phi_<] = S_0[\phi_<] + \frac{1}{2}\mathrm{Tr}\ln\left[1 + G_> \frac{\delta^2 S_I}{\delta\phi^2}\Big|_{\phi=\phi_<}\right] + \mathcal{O}((\delta\ln\Lambda)^2). \tag{1}$$

Therefore it is not that “when integrating out the fast modes, we restrict to one-loop since it's easier”, but rather “when integrating out the fast modes, one-loop is all there is!”, at least as far as getting the beta functions for the coupling constants goes.

The fact that the k-shell evolution of the coupling constants only involves digrams with a single internal line is what makes the functional RG program tractable.

Given that the differential changes in the couplings are determined by a one-loop answer, we run into a predicament: not all physical quantities show up at the 1-loop level! For example, consider the anomalous dimension in ϕ^4 theory. It is easily checked that there are no k-dependent contributions to $\Sigma(k)$ at one-loop, so that the $(\partial_x\phi)^2$ term in the action is unchanged after doing the integral over $\phi_>$. How then are we ever supposed to recover the $\eta \sim g^2$ anomalous dimension of the ϕ^4 field theory? The key is that the information about Σ is contained in *irrelevant* operators generated during the shell integration. For example, consider the two-loop answer for η, which is determined by the sunrise diagram.

The anomalous dimension is computed to lowest order by $d_{\ln\Lambda} G_>(k) = -\eta$ (recall $G_<(k) = \Lambda^{-\eta}|k|^{2-\eta}$ (we are in four dimensions for now)). The terms contributing to this at two-loop will have one loop in the high momentum shell, and one propagator running over *all* momenta. The high momentum loop can be thought of as an effective 4-point vertex for the $\phi_<$ fields, and so we can evaluate the sunrise diagram by evaluating a cactus diagram with the vertex given by the effective vertex after performing the shell integration. However, we know that a momentum-independent $\phi_<^4$ vertex doesn't give a cactus diagram with any k-dependence—therefore we need to keep the k-dependent part of the effective vertex, even though this goes as $k^{n\geq 2}\phi_<^4$, and would have naively been dropped due to irrelevance.

Schematically, write the effective action after doing the shell integration as (in what follows we will drop all numerical constants; equations should all be understood as having $\sim$s)

$$\delta S[\phi_<] \supset g^2 \delta \ln\Lambda \int_x \left[\phi_<^4 + \frac{1}{\Lambda^3}\phi_<^2(\partial\phi_<)^2\right] \tag{2}$$

Now when we compute $G_<$ using the new effective action for $\phi_<$, we see that there is now a momentum-dependent term:

$$\delta G_<(k) \supset -g^2\delta\Lambda \frac{1}{k^2}\int_q \left(G_{<,0}(q)\frac{k^2}{\Lambda^3}\right)\frac{1}{k^2} + \dots, \tag{3}$$

where the two $\frac{1}{k^2}$s are from the free propagators $G_{<,0}(k)$ on either side of the vertex and the $\dots$ are terms higher-order in k^2/Λ^2. Doing the integral (still not writing constants), we get

$$\delta G_<(k) \supset -g^2(\delta\ln\Lambda)\frac{1}{k^2} + \dots \tag{4}$$

Therefore the anomalous dimension to lowest order is[1]

$$\eta = -\frac{d\ln G}{d\ln\Lambda} \sim g^2. \tag{5}$$

The important point here is that in order to compute universal properties of the critical point like the anomalous dimension, it is *not* sufficient to do a computation which only involves an integration over a high-energy shell—one must compute diagrams where low energy modes are integrated over as well. Indeed, the RG flow of any quantity computed with a diagram at more than one loop will necessarily involve integrals over all energy ranges. This is actually very reasonable when one remembers that near critical points, modes on all scales are coupled to modes on all other scales—therefore it would be odd if one could compute universal properties of the critical point by only looking at what's going on within a small range of high energy dof. At the same time though, the RG is indeed calculating what happens when a shell of high energy dof get integrated out. The fact that the results of what happens when the high energy dof get integrated out depends on what the low energy dog are doing is a statement about how the different energy regimes are coupled to eachother. One potentially good slogan for the RG could then be "RG is the thing that tells you how different energy scales are coupled together".

[1] ethan: *explain why there's no β_g term*

Remarks on the rescaling procedure

Robert had the following nice pov: the goal in applications of QFT to most of physics is to compute correlation functions like $\langle\phi(\lambda x)\phi(\lambda y)\rangle$ in the limit $\lambda \to \infty$. The rescaling step of the RG is designed to do this by

$$\langle\phi(\lambda x)\phi(\lambda y)\rangle_{S_0} = \langle\phi(x)\phi(y)\rangle_{S_\lambda}. \tag{6}$$

That is, the RG is a way of transferring the long distance depndence of correlation functions into the form of the coupling constants in the action. Looking at the action is only useful for computing UV observables, but not IR ones. So, if we can transfer problems about IR correlators in for one action to problems about UV correlators in another action, then since the UV correlators can be gleaned from the form of the action, we can more easily read off the values for the IR correlators. (this is the rescaling, but explain why this point of view is still just a trivial change in integration variables)

Some comments on the rescaling step of the RG. Surely if we just wanted to know about the correlation functions of low energy operators in some particular model, we should just do the path integral over the high energy modes and then compute in the effective action thereby generated. This is in line with the approach of the exact RG: we get expressions for how the couplings change as a function of the cutoff, but nowhere do we talk about rescaling the cutoff back to its original value after integrating out the high energy modes. This is the type of RG procedure we usually do when working on a lattice — we decimate and get rid of some of the dof, and are then left with a new model defined on a coarse-grained version of the original lattice, with all the correlation functions in the new model on the new lattice matching the corresponding ones in the original model. This is just doing the path integral step by step. In particular by construction, all dimensionful quantities (e.g. correlation lengths) do not flow under RG. Therefore it is very misleading to say things like "the mass of the scalar flows to infinity under RG". The RG in this sense is not telling you how dimensionful quantities flow — they don't — it is rather telling you how to make simplifications in the way you describe the theory such that the dimensionful quantities computed by the theory are (approximately) unchanged. This way of doing RG is in keeping with the mantra appearing in hep, where one describes RG as "the way you change the coupling constants to keep the low energy physics fixed". Indeed, the hep approach doesn't involve any kind of rescaling like this (wavefunction renormalization is similar but not the same thing).

Why then do we usually include this extra re-scaling step when working with continuum field theories? Honestly, I'm not sure this step is actually that helpful. First, note that the rescaling step leaves all correlation functions invariant. In fact, sending $\phi(x) \mapsto \lambda^{-\Delta}\phi(\lambda x)$ is no more than a mere change of integration variables, and as such it really does nothing as far as physics is conserved — whether or not we include it is then purely a matter of whether or not doing so makes our description of the system nicer; it ultimately does not affect any physics (we could just as well do $\phi(x) \mapsto \lambda^{-42\Delta+7}\phi(\lambda x - 42)$; as long as we make this replacement everywhere no observables change[2]).

[2]Although Wilsonian RG is formulated by starting close to a fixed point, we are not using the dilitation

Instead, we do the rescaling because we want to make it easy to compare the theory before and after integrating out some high energy modes; the comparison, at least at the level of actions, is then done most naturally if the two theories have the same cutoff.

The downside of doing this is that one has to be slightly more careful when matching correlation functions. Before the re-scaling, we have[3]

$$W^n_S(k_1,\ldots,k_n) = W^n_{S'}(k_1,\ldots,k_n), \tag{7}$$

where S' is the effective action obtained after integrating out the modes with $k \in [\Lambda - \delta\Lambda, \Lambda)$ (but without re-scaling), and where the above equality holds provided all the k_i are less than $\Lambda - \delta\Lambda$. After we do the re-scaling we have to re-write this as

$$W^n_S(k_1,\ldots,k_n) = Z^{-n/2} W^n_{S'}(k_1/\lambda,\ldots,k_n/\lambda), \tag{8}$$

where on the RHS we are using the re-scaled action this time, and where the restriction on the range of k_1 is as before. The reason that this is a little weird is that if high-momenta correlators computed with the S' theory correspond to low-momenta correlators in the original theory (the one whose correlation functions we want to know)! So after doing RG in this way, we are left with a new theory, defined with the same cutoff as the original one, but which is such that its correlation functions computed at scale λk are equivalent to the correlation functions in the original theory at scale k. Now since in the deep IR we will reach a fixed point where the correlators are just power laws, the only difference between the correlators at λk and those at k will be a trivial constant factor, so as long as we're only interested in the mega-deep IR then we don't have to remember this.

Anyway, the tl;dr of this rather rambling discussion is that the re-scaling thing is not anything physical; it's just a choice made to facilitate comparing effective actions as one integrates stuff out (it is biased towards comparing actions rather than correlation functions).

Or is this just another issue of being able to interpret things actively vs. passively? In the hep approach, the input to the theory is a set of observables at low energy. The goal is then to figure out how the dimensionless parameters in the action need to change in order to keep the IR dimensionful quantities (correlation lengths, masses, etc.) fixed.

On the other hand, we could also consider having the coupling constants in the action be the input to the theory, and examining how the resulting IR dimensionful quantities change. The distinction here is whether one wants to use field theory as a way of using observables at one IR scale to make predictions for observables at a different IR scale, or whether one wants to make predictions about the IR observables associated to a particular microscopic model.

$$\phi(x) \mapsto \phi'(x) = \lambda^{\Delta}\phi(\lambda x), \qquad \lambda < 1. \tag{9}$$

Note we are always taking the active point of view, where space is fixed. This transformation then corresponds to stretching out the fields in space, plus making them smaller by scaling them by a constant.

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symmetry at the fixed point to do the rescaling. That would be e.g. acting on fields in a correlator with the dilitation, but not also acting on the fields in the action. Here we are acting on all instances of the fields, which is just a trivial change of variables.

[3] In practice, all equations like this will only be expected to hold up to powers in k_i/Λ.

Why the β function is autonomous; the scaling hypothesis

maybe talk about the scaling theory of localization

Therefore in 1D the relation is rather obvious: here $g(L)$ is a monotonic function of L, and hence we can solve for $L(g)$; therefore any explicit L-dependence in β_g can be converted into $g(L)$-dependence. However, this monotonicity property is only true when there is no critical point, and in general we cannot invert and find an expression for $L(g)$. So why does autonomy hold in general?

Hep approach

The hep approach deals with examining how the effects of fluctuations change with the scale at which measurements are made. For example, electrons in QED are screened by clouds of virtual particles. When doing scattering experiments with electrons, the strength of the interaction will depend on the energy with which they are scattered, since if the particles to be scattered have higher energies, they will peer deeper into the screening cloud, and see a larger fraction of the bare charge. The hep RG lets us calculate the effective amount of screening that occurs at different energy scales (in terms of common field theory notation, this means calculating how the couplings change as a funtion of the RG scale μ).

One more old-school way to say what the hep approach does is to say that it provides a way of telling one how to write down a set of bare couplings which reproduce a fixed set of low-energy observables given a large UV cutoff, which then in turn provides a way of writing down the limiting form of the bare couplings in the limit $\Lambda \to \infty$, allowing one to construct (at least formally, and in the renormalizable case) a truly continuum QFT.

However I think describing things in this way kind of short-sells the hep approach. It is not only useful for defining the continuum limit, but rather is useful in that it provides a powerful way of relating observables at different low-energy scales to one another.

The crucial thing about the hep approach which makes it calculationally superior to the Wilsonian approach (at least for low orders in perturbation theory) is that it only ever involves a *finite* number of coupling constants.

The input to the hep RG program is a measurement of some correlation functions at low energies. For example, suppose we measure the Coulomb force law at a momentum p^2, and get an answer V_p. Then we define the electric charge $\alpha_p \equiv p^2 V_p$. This will be some finite (and in our universe, small) number, at least for values of p^2 accessible to humanity. We then want to formulate the theory using this input parameter α_p. In particular, we want a theory that takes in α_p as input, and then tells us $\alpha_{p'}(\alpha_p)$. Computing the relationship between the α_ps at different values of p essentially the whole job that the theory is constructed to accomplish. In order for the theory to be self-consistent, it should be able to accept as input α_p and return the same values of $\alpha_{p'}$ regardless of p. This is equivalent to requiring that the parameters in the Lagrangian be independent of μ.

Put another way, the couplings are infinite because the perturbative corrections to things like e^2 involve loop integrations over an infinite energy window, and so in order for loop-corrected quantities to be finite, the bare coupling constants must correspondingly diverge. More precisely, consider e.g.

Note that irrelevant terms never show up in the hep RG, which works essentially by projecting away all irrelevant terms at the cost of making adjustments to the relevant ones. For example, consider the 4-point vertex in ϕ^4 theory in 3+1D at an energy scale p^2. The one-loop corrections to the vertex are not just proportional to $\ln(p^2/\Lambda^2)$, which diverges as $\Lambda \to \infty$. Instead, they contain additional finite contributions

How to organize fluctuations—by scale, not by number of loops!

One conceptual problem with the hep approach is that an expansion in the number of loops is *not* a physical way to organize an expansion in fluctuations. Indeed, the mere existence of autonomous RG equations (the CZ equations) for things like the n-point correlation functions $G^{(n)}$ means that the calculation of the 1-loop beta functions determines an *infinite* number of terms in the expression for $G^{(n)}$, at all orders in the couplings (i.e, at all loops).

For example, consider a marginal coupling g associated with some n-point Greens function $G_n(p)$. Let G_n be expanded as[4]

$$G_n(p) = g + \sum_{k=2} g^k \sum_{l=1}^{k-1} A_l^k t^l, \tag{10}$$

where the A_l^k are some numbers to be determined by doing Feynman diagrams, and where we have defined the RG time to the scale p as

$$t \equiv \ln(\Lambda/p), \tag{11}$$

with the sign chosen so that $t \to \infty$ in the IR. For visual clarity, we write this out longhand as

$$G_n(p) = g + g^2 A_1^1 t + g^3(A_2^2 t^2 + A_1^2 t) + g^4(A_3^3 t^3 + A_2^3 t^2 + A_1^3 t) + \dots \tag{12}$$

The terms in each group are cut off at $g^{k+1}t^k$ since a diagram with g^{k+1} vertices can have at most k totally independent momentum integrations (these diagrams are the ones of concatenated bubbles). The A_i^i terms are the leading logs, the A_{i-1}^i terms are the subleading logs, and so on.

Now write a similar expansion of the beta function as

$$\beta_g = \sum_{k=2} \beta_k g^k, \tag{13}$$

where we have started at g^2 since g is marginal. We then plug this into the equation $d_{\ln\Lambda} G_n(p) = 0$, and match coefficients in t. We find that

$$\beta_2 = -A_1^1, \qquad \beta_3 = -A_1^2, \qquad \beta_4 = -A_1^3, \qquad \dots \tag{14}$$

and

$$A_k^k = (A_1^1)^k, \qquad A_2^3 = A_1^1 A_1^2, \qquad \dots. \tag{15}$$

[4] This is the bare Greens function—we are going to be writing the β functions as the changes in the bare couplings with the cutoff, just so we don't have to deal with the anomalous dimensions from the renormalized fields in the renormalized Greens function.

Therefore the right way to organize the expansion of G_n is in fact *not* in loops (number of powers of g), but rather by the degree of subleading-ness of the log:

$$G_n(p) = g + \left(A_1^1 g^2 t + A_2^2 g^3 t^2 + A_3^3 g^4 t^3 + \ldots\right) + \left(A_1^2 g^3 t + A_2^3 g^4 t^2 + \ldots\right) + \left(A_1^3 g^4 t + \ldots\right) + \ldots, \tag{16}$$

where the first term in each group of parenthesis determines the coefficients of *all the subsequent terms* in the group. In particular, the whole n point function is determined *entirely* by the terms linear in t! This is not obvious at the level of diagrams, and provides a powerful way of summing up an infinite number of diagrams. This all really just comes from the existence of an autonomous RG equation—all the magic of the RG program lies in the existence of an appropriate CS equation! Also note that in addition, the β function for g is similarly entirely determined by the terms linear in t (as it had better be if a differential equation is to exist!). This is another explanation of why the momentum-shell RG works despite it only involving diagrams where only a single loop is in the high energy shell—these are the diagrams that produce the linear in t logs.

The above series expansions were seemingly very innocuous, and yet they let us derive constraints on the n point function valid to infinite order in perturbation theory—where was the magic? The magic was all in the fact that we wrote β as a function of g alone, and not of t. Writing $(\partial_{\ln\Lambda} + \beta_g(g,\Lambda)\partial_g)G_n(p) = 0$ is a triviality, and lets us conclude nothing. It is the fact that the β function is an *autonomous* ODE that is the crucial point. Given that β_g does not depend explicitly on $\ln\Lambda$, in the expression for $\partial_{\ln\Lambda}G_n(p)$, all the terms not linear in t will have t-dependence which must be canceled by the structure of β_g and the $\ln\Lambda$-dependence of g, and this gives rise to the dependence of all the non-linear-in-t terms on the linear ones. Now if $G_n(p)$ was a generic function of t and g, such a cancellation would not be possible. Furthermore it is *not* a priori obvious that the terms in $G_n(p)$ are related in the way we have demonstrated—that is, the explicit $\ln\Lambda$ independence of β_g is *not* obvious. The $\ln\Lambda$ independence is equivalent to renormalizability, which is generically a very non-trivial thing to prove. ethan: *but in the Wilsonian approach? Or is this just saying that the hep-th approach agrees with Wilsonian only in the case of renormalizable theories?*

Anyway, from the above relations on the coefficients in the expansion of $G_n(p)$, we see that it's really not best to organize perturbation theory by loops—rather, one should organize it by scale (i.e. by the power of $\ln(p^2/\mu^2)$ appearing in diagrams). For e.g. ϕ^4 theory this is somewhat obvious, at least for the leading logs: the fact that $A_k = A_1^k$ just comes from the fact that the leading log term t^k at a given loop order k comes from k bubbles concatenated together, so that the divergent part of the integral is simply the kth power of a single bubble. However at subleading orders and in more complicated theories these relations are less obvious.

From this point of view, should we be bothered by the fact that in theories like QED and ϕ^4 (both in four dimensions) the bare coupling constants appearing in the action are infinite? Not at all: the interactions in these theories are marginally irrelevant, and so in order for them to have a finite value in the IR, their bare values must diverge in the $\Lambda \to \infty$ limit. This is because the bare coupling constants are the effective interactions deep in the UV, at the scale Λ. Indeed, the typical one-loop relation between the bare and renormalized couplings is

$$g_B \sim g_\mu(1 - Cg_\mu \ln(\Lambda/\mu)), \tag{17}$$

where g_μ is the coupling at scale μ, i.e. the value of the four-point vertex (with the 1-loop corrections taken into account) when the external legs are set at $p^2 = \mu^2$. When we take the scale all the way to the UV by setting $\mu = \Lambda$ we get $g_B = g_\Lambda$, so that indeed the bare parameters in the Lagrangian are the UV couplings.

Note that in the hep approach, we are never integrating anything out—we are merely providing a way of relating correlation functions at different momenta.

This is why in the hep approach, it is in my opinion often conceptually clearer to avoid calculating flows by taking differentials with respect to $\ln\Lambda$. We view Λ and the bare couplings as fixed, and the effective couplings as changing as we change the scale at which they are measured.

Alternatively, if we really want to imagine changing the cutoff, then the RG provides us with a way of saying how the *bare* couplings need to change in order to keep the physical coupling at some low scale (the one we measure) fixed. The fact that we can do this is the "cutoff independence" of the theory: for a fixed input α_p at some p, the RG tells us how to construct a limit of theories with cutoff Λ and bare couplings $\alpha_B(\Lambda)$ which all produce the same effective coupling α_p at scale $p \ll \Lambda$ (again, this $p \ll \Lambda$ caveat is needed since the hep approach to RG only is able to hold α_p invariant under changes in Λ up to terms polynomial in p^2/Λ). This limit for $\Lambda \to \infty$ is what defines a continuum QFT.

One thing that is not really ever mentioned explicitly is that as we just mentioned, this whole construction works only up to order $\mathcal{O}(p^2/\Lambda^2)$. It is impossible to change Λ and adjust only the relevant / marginal terms while literally preserving correlation functions on scales below Λ. Likewise, the relation between α_p and $\alpha_{p'}$ cannot be determined as a function of the relevant / marginal couplings alone unless we permit ourselves to drop terms of order p^2/Λ^2.

In any case, powers of momenta are uninteresting in $\mathbb{R}$ space, since they are contact terms. The only things that are really relevant for $\mathbb{R}$ space physics are the logs.

ethan: *comment on IR behavior of superrenorm theories* Note that in renormalazable (but not super-renormalizable) theories, phenomena like nonzero anomalous dimensions are possible because the theory must be defined with the help of a UV regulator, since a UV fixed point doesn't exist. Thus non-universal information (Λ) shows up in things computed by the QFT, but the way in which it shows up is universal.

Relations between the two approaches; general comments

In the hep approach, you're figuring out how the coupling constants have to change in order to keep the low-energy physics fixed: the input to the RG program is IR universal quantities. In the Wilsonian approach, you're looking at how the IR physics changes as you integrate out high-energy modes: the input to the RG program is a UV Hamiltonian.

Now the hep approach is much more useful when doing calculations, since we only have a finite number of parameters to keep track of. However, the values of the coupling constants computed within the hep framework are also extremely fine-tuned: it involves projecting onto the *single* RG flow that connects the UV and IR fixed points (in the case when a UV fixed point exists). Almost all of the flows into the chosen IR fixed point will *not* lead to sensible well-defined continuum limits (only one out of an infinite number will, assuming again that a UV fixed point exists). Of course though, this is a fine thing to do as long as we only care about calculating universal quantities.

Reading qft books, one often gets the sense that things which are Λ-dependent are unphysical. This is totally untrue; things like T_c are completely physical quantities. Rather, Λ-dependent things are non-universal.

Note: we are bothered about the light mass of the Higgs, since it is relevant. Should we also be bothered about the fact that $\Lambda_{QCD}/\Lambda_{Planck} \ll 1$, or is the marginal RG flow slow enough that the "UV value of g^2" doesn't have to be "un-naturally" small?

It is important that an RG flow, in the Wilsonian sense, is not something that one can do in the lab. Instead, a Wilsonian RG flow is merely an operation one can perform on partition functions to make certain correlation functions easier to calculate.

In the hep approach, the RG flow *is* something that we can "do" in the lab—we just measure correlation functions at different energy scales.

In the Wilsonian approach, the RG is a map between different theories (it is exact when we just integrate out modes in principle, but once we re-scale the momenta we are mapping between different theories with different actions), while in the hep approach there is only ever one theory, and the RG just provides a way to relate correlation functions at different scales within that one theory.

In the Wilsonian approach, we think of the RG map as modifying coefficients of terms in the action; in the hep approach the action is always fixed—it is the same for all RG scales (the bare terms are independent of the cutoff). The things that change under RG are rather correlation functions of operators as the energy scale associated with them is changed.

Scheme-dependence

Therefore talking about "*the* IR fixed point" of a system is a bit misleading—the fixed point one gets depends on the way in which one organizes the integration over fluctuations. Changing the scheme one uses to do the RG changes where on the critical surface the fixed point is. So really universality is a property of the critical surface itself, rather than of a single point on the critical surface. ethan: *really? or is it more that changing schemes corresponds to adding in irrelevant couplings?*

Scheme (in)dependence and the physicality of the beta functions

Consider a theory with a set of dimensionless couplings g_α. We can write a series expansion for the beta function for g_α about the critical point as

$$\beta_\alpha = \sum_{n\in\mathbb{Z}^>} \beta_{n\alpha}^{\gamma_1\ldots\gamma_n} g_{\gamma_1}\cdots g_{\gamma_n}, \tag{18}$$

where the sum starts at 1 since wolog we are taking the couplings to be such that the critical point is at $g_\alpha = 0\,\forall\,\alpha$. We want to know to what extent the $\beta_{n\alpha}$ are scheme-independent, that is, to what extent they remain unchanged upon making redefinitions of the coupling constants.

From our knowledge of critical phenomena, we know that because the scaling dimensions of the fields are determined by the eigenvalues of the first derivatives of the β functions linearized about the critical point, at least $\beta_{1\alpha}$ should be scheme independent. What about the higher-order coefficients? In what follows we will show that in fact only the first non-zero $\beta_{n\alpha}$ is universal, with all higher order coefficients being scheme dependent (the vanishing of the lower-order terms with $m < n$ is also universal, e.g. a marginal coupling cannot be turned into one with a non-zero scaling dimension through a redefinition of the coupling constants).

Let f_α the coupling constants corresponding to some other way of describing the physics near the critical point (again defined so that $f_\alpha = 0 \,\forall\, \alpha$ is the critical point, so that the critical points in the f_α and g_α schemes are both at the origin of the critical manifold). For example, g_α might be the renormalized couplings one computes within dim reg, and f_α the ones computed with PV. Different choices for f_α give us different ways of parametrizing the vicinity of the critical point, and we would like to know to what extent these different parametrizations give different beta functions.

Near the critical point, both the gs and the fs will be small, and we can benefit from writing f_α in a power series

$$f_\alpha = \sum_{n\in\mathbb{Z}^>} f_{n\alpha}^{\gamma_1\cdots\gamma_n} g_{\gamma_1}\cdots g_{\gamma_n}. \tag{19}$$

Finally let $\widetilde{\beta}_\alpha$ be the beta function for f_α, and expand it too as a power series

$$\widetilde{\beta}_\alpha = \sum_{n\in\mathbb{Z}^>} \widetilde{\beta}_{n\alpha}^{\gamma_1\cdots\gamma_n} f_{\gamma_1}\cdots f_{\gamma_n}. \tag{20}$$

All of the $\beta_{n\alpha}, \widetilde{\beta}_{n\alpha}$, and $f_{n\alpha}$ will be taken to be symmetric tensors wolog.

Now a generic change of variables is generally not a useful thing to do, since we are only ever really interested in a set of variables which are scaling variables. Therefore in what follows we will take $\widetilde{\beta}_1$ and β_1 to be diagonal matrices, which following convention will be written as $\beta_{1\alpha}^{\gamma} = \delta_\alpha^\gamma y_\gamma$, $\widetilde{\beta}_{1\alpha}^{\gamma} = \delta_\alpha^\gamma \widetilde{y}_\gamma$ (no sum). Furthermore, in what follows we will wolog be working with parametrizations such that $f_\alpha = g_\alpha$ in the limit in which we approach the critical point, and hence we will set $f_1 = \mathbf{1}$ from now on.[5]

For the beta function of the reparametrized couplings, we have

$$\widetilde{\beta}_\alpha = \frac{df_\alpha}{dg_\lambda}\beta_\lambda. \tag{21}$$

We then replace everything in the above equation with their power series representations in terms of the g_αs and match coefficients. To quadratic order in the couplings, this yields the two equations

$$y_\alpha = \widetilde{y}_\alpha, \tag{22}$$

and (no sums on repeated indices)

$$y_\alpha f_{2\alpha}^{\rho\omega} + \widetilde{\beta}_{2\alpha}^{\rho\omega} = 2y_\rho f_{2\alpha}^{\rho\omega} + \beta_{2\alpha}^{\rho\omega}. \tag{23}$$

[5]Said another way, we can always perform a trivial multiplicative re-scaling of f by $f_\alpha \mapsto \frac{1}{c_\alpha} f_\alpha$ (this of course doesn't affect the scaling dims), so that $f_1 = \mathbf{1}$.

The former equation tells us that the scaling dimensions y_α are indeed scheme-independent, as expected. The latter tells us that unless the couplings are all marginal, the second-order terms in the beta functions are not scheme-independent, since the tensor $\widetilde{\beta}_{2\alpha}$ depends on the values of the $f_{2\alpha}$s. A similar analysis of the $\mathcal{O}(g^3)$ terms in the equation for $\widetilde{\beta}_\alpha$ shows that unless both the first and second-order coefficients $\beta_{2\alpha}$ vanish, the third-order coefficients are similarly scheme dependent. This then generalizes to the statement we claimed above, viz. that only the first non-zero coefficient is scheme-independent.

One consequence of the non-universality of the higher-order terms in the beta functions is that for non-marginal scaling variables, it is generically possible to make a redefinition of the coupling constants such that $\widetilde{\beta}_\alpha = y_\alpha f_\alpha$. This means that in the absence of marginal couplings, we can generically find a parametrization of the critical point such that the linear scaling law $f_\alpha \to s^{y_\alpha} f_\alpha$ holds even for f_α of order 1. For simplicity, consider the case of a single coupling with scaling dimension y. The condition that f scale linearly is, in obvious notation,

$$y(g + f_2 g^2 + f_3 g^3 + \cdots) = (1 + 2f_2 g + 3f_3 g^2 + \cdots)(yg + \beta_2 g^2 + \beta_3 g^3 + \cdots), \tag{24}$$

which means that f will scale linearly provided that we choose $f_2 = -\beta_2/y$, $f_3 = \beta_2^2/y^2 - \beta_3/2y$, and so on. Similarly, if g is marginal but $\beta_2 \neq 0$, we can work in a scheme where $\widetilde{\beta} = \beta_2 f^2$, without any higher-order terms: the condition for this at lowest order is $f_3 = -\beta_3/2\beta_2$.

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2 *Functional RG equations* ✓

Consider a scalar field theory with arbitrary interactions given by $\mathcal{L}_I[\phi]$. Impose a soft UV cutoff by modifying the action as (Euclidean signature)

$$Z[J] = \int \mathcal{D}\phi \, e^{-S + \int J\phi}, \qquad S = S_0 + S_I, \tag{25}$$

with

$$S_0 = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} \phi_p (p^2 + m^2) e^{p^2/\Lambda^2} \phi_{-p}, \tag{26}$$

and

$$S_I = \int \frac{d^4p}{(2\pi)^4} \mathcal{L}_I[\phi], \tag{27}$$

where $\mathcal{L}_I$ contains all possible interactions for ϕ_p. Here, the current J only contains momentum modes up to some scale $\mu < \Lambda$ (the energy scale below which we are interested in calculating correlation functions), and the purpose of the $e^{p^2\Lambda}$ term is so that the propagator gets smoothly cutoff at very high momentum. The action S_I explicitly depends on Λ through the dimensionality of the coupling constants appearing in S_I.

The functional RG flow works by changing the cutoff Λ while simultaneously modifying S_I in such a way that correlation functions at scales below μ are left invariant; this is an (exact) way of integrating out high-momentum modes to get an effective action for the low-momentum ones. The requirement that the correlation functions blow μ be preserved under varying Λ is that the change of the generating functional of correlation functions with respect to Λ is independent of the current:

$$\frac{d}{d\ln\Lambda}Z[J] = \text{ independent of } J, \tag{28}$$

In what follows, we will derive an expression for $d_{\ln\Lambda}S_I[\phi]$. Note: this is in Schwartz, but I think there are some typos in the problem, so don't worry about trying to derive what he tells you to derive. Polchinski's original paper may be a better reference.

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Let's first see if we can guess the answer. When we change the cutoff, what needs to happen to $\mathcal{L}_I[\phi]$ so that the partition function is unchanged? Lowering the cutoff means that certain high energy modes get integrated out. In diagrams, this means that lowering the cutoff corresponds to "collapsing" certain high energy propagator lines. There are two types of lines we can collapse: one which is an open line connecting two different vertices, and one which joins back on itself, with both ends at the same vertex. We can select out lines of the first kind by computing $\delta_{\phi_p}\mathcal{L}_I\delta_{\phi_{-p}}\mathcal{L}_I$, and we can select out the latter types of lines by doing $\delta_{\phi_p}\delta_{\phi_{-p}}\mathcal{L}_I$. After we have selected out these lines, we need to integrate over the high energy modes, which we can do by multiplying with the propagator and integrating over p. We expect there to only be one propagator since we are looking at an infinitesimal variation in $\mathcal{L}_I$, and we want to select out the lowest-order contribution in $\delta\Lambda$. Thus we expect something like

$$\frac{d}{d\ln\Lambda}\mathcal{L}_I \sim \int_p \frac{e^{-p^2/\Lambda^2}p^2/\Lambda^2}{p^2+m^2}\left(\frac{\delta\mathcal{L}_I}{\delta\phi_p}\frac{\delta\mathcal{L}_I}{\delta\phi_{-p}} + \frac{\delta^2\mathcal{L}_I}{\delta\phi_p\delta\phi_{-p}}\right). \tag{29}$$

The $e^{-p^2/\Lambda^2}p^2/\Lambda^2$ factor ensures that $\mathcal{L}_I$ only changes near the cutoff momentum, which will be corroborated soon.

To figure out what conditions $\mathcal{L}_I$ needs to satisfy, just differentiate Z with respect to Λ:

$$\frac{d}{d\ln\Lambda}Z = \int\mathcal{D}\phi\int_p\left(\frac{p^2}{\Lambda^2}e^{p^2/\Lambda^2}\phi_p(p^2+m^2)\phi_{-p} - \frac{d}{d\ln\Lambda}\mathcal{L}_I\right)e^{-S}. \tag{30}$$

This must be independent of J, which lets us figure out how $\mathcal{L}_I$ changes. Note that $\frac{d}{d\ln\Lambda}\mathcal{L}_I$ should only have support near $p^2=\Lambda^2$.

To see what we should write for $\frac{d}{d\ln\Lambda}\mathcal{L}_I$, let's calculate the functional derivatives of e^{-S}. We get

$$\frac{\delta}{\delta\phi_p}e^{-S} = -\frac{1}{(2\pi)^4}\left((p^2+m^2)\phi_{-p}e^{p^2/\Lambda^2} + \frac{\delta\mathcal{L}_I}{\delta\phi_p}\right)e^{-S}, \tag{31}$$

and

$$\frac{\delta^2}{\delta\phi_p\delta\phi_{-p}}e^{-S} = -\frac{1}{(2\pi)^4}\left((p^2+m^2)e^{p^2/\Lambda^2} + \frac{\delta^2\mathcal{L}_I}{\delta\phi_{-p}\delta\phi_p}\right)e^{-S}$$
$$+\frac{1}{(2\pi)^8}\left((p^2+m^2)\phi_{-p}e^{p^2/\Lambda^2} + \frac{\delta\mathcal{L}_I}{\delta\phi_p}\right)\left((p^2+m^2)\phi_p e^{p^2/\Lambda^2} + \frac{\delta\mathcal{L}_I}{\delta\phi_{-p}}\right)e^{-S}. \tag{32}$$

We can now make a better educated guess about the factors in $\frac{d}{d\ln\Lambda}\mathcal{L}_I$. We choose

$$\frac{d}{d\ln\Lambda}S_I = (2\pi)^4\int d^4p\,\frac{e^{-p^2/\Lambda^2}p^2/\Lambda^2}{p^2+m^2}\left(\frac{\delta S_I}{\delta\phi_p}\frac{\delta S_I}{\delta\phi_{-p}} - \frac{\delta^2 S_I}{\delta\phi_p\delta\phi_{-p}}\right). \tag{33}$$

The reason why this ends up working is the following: first, we have, for any momentum p,

$$0 = \int\mathcal{D}\phi\,\frac{\delta}{\delta\phi_p}\left[\left(\phi_p e^{p^2/\Lambda^2} + \frac{(2\pi)^4}{2}\frac{1}{p^2+m^2}\frac{\delta}{\delta\phi_{-p}}\right)e^{-S+\int J\phi}\right], \tag{34}$$

just because at the extremes of the functional integration, ϕ is infinite everywhere in spacetime, and so we get a factor of $e^{-\infty}$ from e^{-S}. Now we apply the above relation with $p > \mu$, where again μ is the scale above which the current vanishes (therefore the functional derivative is taken with respect to a high-energy mode). After some algebra that I won't write out, we evaluate the functional derivatives and get

$$\frac{e^{p^2/\Lambda^2}Z[J]}{2} = \int\mathcal{D}\phi\,\left(\phi_p\frac{p^2+m^2}{2}e^{2p^2/\Lambda^2}\phi_{-p} + \frac{1}{2(p^2+m^2)}\left(\frac{\delta^2 S_I}{\delta\phi_p\delta\phi_{-p}} - \frac{\delta S_I}{\delta\phi_p}\frac{\delta S_I}{\delta\phi_{-p}}\right)\right)e^{-S+\int J\phi}. \tag{35}$$

Next, we calculate $d_{\ln\Lambda}Z[J]$:

$$d_{\ln\Lambda}Z[J] = -\int\mathcal{D}\phi\left(\int\frac{d^4p}{(2\pi)^4}\frac{p^2+m^2}{2}\phi_p\phi_{-p}\left(-\frac{2p^2}{\Lambda^2}e^{p^2/\Lambda^2}\right) + d_{\ln\Lambda}S_I\right). \tag{36}$$

Now we plug in our ansatz for $d_{\ln\Lambda}S_I$ into this equation, and use (35) to get

$$d_{\ln\Lambda}Z[J] = \int d^4p\,e^{-p^2/\Lambda^2}\frac{p^2}{\Lambda^2}(e^{p^2/\Lambda^2}Z[J]), \tag{37}$$

which tells us that the change of the generating function under changes in the cutoff is

$$\frac{dW[J]}{d\ln\Lambda} = \int d^4p\,\frac{p^2}{\Lambda^2}, \tag{38}$$

which as required is independent of J, and so if S_I changes according to the functional differential equation provided in our ansatz, correlation functions at energy scales below Λ are all independent of Λ.

Note that all of this has been non-perturbative and exact! The equation for $d_{\ln\Lambda}S_I$ gives a *linear* differential equation for how the coupling constants in S_I change as the cutoff varies, and so the effective coupling constants at any lower scale can be found in principle only using our knowledge of the coupling constants at a fixed UV scale. Note that if S_I is only quadratic in ϕ (e.g. $S_I = \sum_k g_k \int \phi_p p^{2k} \phi_{-p}$), then S_I remains quadratic in ϕ, but if it contains any interactions, then after integrating the equation for $d_{\ln\Lambda} S_I$ down from a cutoff of Λ to a cutoff of μ, we generically will generate all possible interactions allowed by symmetry.

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3 Properties of momentum-shell propagators and details on momentum-shell RG

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$$G_>(\mathbf{r}) \equiv \int_{\Lambda-\delta\Lambda<k<\Lambda} d^d k \, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^n}. \tag{39}$$

In various dimensions up to three, this gives

$$G_>(r) = \begin{cases} \frac{1}{2\pi^2 r} \int dk \, k^{1-n} \sin(kr) & d=3 \\ \frac{1}{2\pi} \int dk \, J_0[kr], & d=2, \\ \frac{1}{\pi} \int dk \, k^{-n} \cos(kr), & d=1, \end{cases} \tag{40}$$

where the integrals are over $k \in [\Lambda - \delta\Lambda, \Lambda]$. When $\delta\Lambda/\Lambda \to 0$, this becomes

$$G_>(r) \approx \begin{cases} \Lambda^{2-n} \frac{\delta\Lambda}{2\pi^2 \Lambda r} \sin(\Lambda r) & d=3 \\ \Lambda^{1-n} \frac{\delta\Lambda}{2\pi} J_0[\Lambda r] \xrightarrow{\Lambda r \gg 1} \Lambda^{1-n} \frac{\delta\Lambda}{\sqrt{2\pi^3 \Lambda r}} \cos(\Lambda r) & d=2, \\ \Lambda^{-n} \frac{\delta\Lambda}{\pi} \cos(\Lambda r) & d=1 \end{cases} \tag{41}$$

Note how the power of r decreases by $1/2$ as we move up in dimension. Also note that in accordance with the integral being over only a finite slice of momenta, all are finite for all values of Λr.

These are the propagators for stat mech models (no time)—what about when we have a time dimension as well? We will first assume that the inverse propagator is quadratic in frequency, and that the momentum dispersion goes as k^{2l} (since k here is of order of a cutoff, we can take the dispersion to be dominated by the largest power of k^2 that appears in the action). Therefore

$$G_>(r,t) = -i \int_{\omega\in\mathbb{R}} \int_{\mathbf{k}\in\text{shell}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{-\omega^2+\alpha^2 k^{2l} - i\varepsilon}. \tag{42}$$

Here the sign of the $i\varepsilon$ is determined by remembering that proper time-ordering means that the sign of $i\varepsilon$ matches the sign of the frequency part of $G^{-1}(k,\omega)$. Doing the ω integral of course gives

$$G_>(r,t) = \theta(t)G_>(r,t) + \theta(-t)G_>(r,-t), \tag{43}$$

where

$$G_>(r,t) = \alpha^{-1} \int_{\mathbf{k}\in\text{shell}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-k^l t)}}{2k^l}. \tag{44}$$

Therefore at finite time, when the shell is narrow, we just have to multiply the above results by $e^{-i\Lambda^l\alpha|t|}$ and divide by 2:

$$G_>(r,t) \approx \begin{cases} \Lambda^{2-l}\dfrac{\delta\Lambda}{4\pi^2\Lambda\alpha r}\sin(\Lambda r)e^{-i\Lambda^l\alpha|t|} & d=3 \\ \Lambda^{1-l}\dfrac{\delta\Lambda}{4\pi\alpha}J_0[\Lambda r]e^{-i\Lambda^l\alpha|t|} \xrightarrow{\Lambda r\gg 1} \Lambda^{1-l}\dfrac{\delta\Lambda}{\sqrt{8\pi^3\Lambda r}\alpha}\cos(\Lambda r)e^{-i\Lambda^l\alpha|t|} & d=2, \\ \Lambda^{-l}\dfrac{\delta\Lambda}{2\pi\alpha}\cos(\Lambda r)e^{-i\Lambda^l\alpha|t|} & d=1 \end{cases} \tag{45}$$

In the case where the propagator is linear in frequency, we instead have

$$G_>(r,t)_{NR} = -i \int_{\omega\in\mathbb{R}} \int_{\mathbf{k}\in\text{shell}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{\omega+\alpha k^{2l}+i\varepsilon}. \tag{46}$$

We then do the ω integral and get

$$G_>(r,t)_{NR} \approx \begin{cases} \Lambda^2\dfrac{\delta\Lambda}{2\pi^2\Lambda r}\sin(\Lambda r)e^{i\Lambda^{2l}\alpha t}\theta(t) & d=3 \\ \Lambda\dfrac{\delta\Lambda}{2\pi}J_0[\Lambda r]e^{i\Lambda^{2l}\alpha t}\theta(t) \xrightarrow{\Lambda r\gg 1} \Lambda\dfrac{\delta\Lambda}{\sqrt{2\pi^3\Lambda r}}\cos(\Lambda r)e^{i\Lambda^{2l}\alpha t}\theta(t) & d=2, \\ \dfrac{\delta\Lambda}{\pi}\cos(\Lambda r)e^{i\Lambda^{2l}\alpha t}\theta(t) & d=1 \end{cases} \tag{47}$$

Note how here we have a $\theta(t)$ in the correlator: since particle-antiparticle creating processes don't happen, the vacuum has no particle-antiparticle pairs, and so acting with an annihilation operator on the vacuum gives zero. Therefore for $t>0$ we have $\langle\phi(t)\phi^\dagger(0)\rangle \neq 0$, but must have $\langle\phi^\dagger(t)\phi(0)\rangle = 0$.

We now prove some properties about $G_>(r)$. First,

$$\int d^d r\, G_>(r) = 0. \tag{48}$$

This is obvious just because $G_>$ has no Fourier components at $\mathbf{k}=0$, and can also be derived from the approximate expressions in (47). Similarly, integrating $G_>(r)$ against any even power $2m \in 2\mathbb{N}$ of r gives zero, since the power can be turned into a derivative (has to be even else we get a non-local $\sqrt{\Delta_{\mathbf{k}}^2}$)[6]:

$$\int d^d r\, r^{2m} G_>(r) = (-1)^m \int d^d k\, \frac{\nabla_{\mathbf{k}}^{2m}\delta(\mathbf{k})}{k^n} = 0. \tag{51}$$

This is basically just saying that diagrams with a single internal propagator connecting different points don't contribute to wavefunction renormalization—this is of course because they are just tree diagrams.

Using (47), one would also conclude that $\int_{\mathbf{r}} G_>(r)^s r^{2m} = 0$ for all $s, m \in \mathbb{N}$. This is not correct however, since powers of $G_>$ contain modes and zero momentum / frequency. For example,

$$\int_{\mathbf{r}} G_>(r)^2 = \int_{\mathbf{k}} G_>(\mathbf{k}) G_>(-\mathbf{k}) \propto \Lambda^{d-1-2n}\delta\Lambda. \tag{52}$$

In general, the integral of $G_>(r)^s$ will have $s-1$ sums over momenta variables, each of which lie in a shell of width $\delta\Lambda$. Therefore in general these heuristic phase space arguments tell us that

$$\int_{\mathbf{r}} G_>(r)^s \propto \Lambda^{(d-1)(s-1)-sn}(\delta\Lambda)^{s-1}. \tag{53}$$

Including even powers of r in the integrand (odd powers still vanish by presumed $SO(d)$ symmetry) gives

$$\int_{\mathbf{r}} G_>(r)^s = \int_{\mathbf{k}} \tag{54}$$

These types of integrals appear in diagrams with $s-1$ loops, so that an expansion in $\delta\Lambda/\Lambda$ is an expansion in the number of loops.

Having looked at the general properties of these correlators we will now put them to use to do momentum-shell RG. Let the action be $S_0 + S_I$, where S_I is the interaction. Write $\phi = \phi_> + \phi_<$ and let $S_e[\phi_<]$ be the effective action after integrating over $\phi_>$. Then derivatives of S_e wrt $\phi_<$ (evaluated at $\phi_< = 0$) generate connected correlation functions for $\phi_>$ in a way determined by S_I. For example,

$$-\frac{\delta^2(S_e[\phi_<] - S_0[\phi_<])}{\delta\phi_<(x)\delta\phi_<(z)}\Big|_{\phi_<=0} = \left\langle \frac{\delta^2 S_I}{\delta\phi(x)^2}\Big|_{\phi=\phi_>} \right\rangle \delta(x-y) - \left\langle \frac{\delta S_I}{\delta\phi(x)}\frac{\delta S_I}{\delta\phi(y)}\Big|_{\phi=\phi_>} \right\rangle_c. \tag{55}$$

[6]This is a bit blase, since

$$\int dx \int dy\, f(x,y) = \int dy \int dx\, f(x,y) \tag{49}$$

only if

$$\int dx\, dy\, |f(x,y)| < \infty, \tag{50}$$

which is certainly not generically the case for $f(x,y) = x^m y^{-n} e^{ikx}$ when integrated on the domain in question (since it includes all of $\mathbb{R}^d$ and the integral over $\mathbf{k}$ space is finite). However, we can always regulate the integrals so that they do converge, by adding a small imaginary momentum to make $G_>$ go as $e^{-r\varepsilon}$ at $r \to \infty$. This establishes convergence of the integral of the absolute value—we can then switch integration orders, and finally send $\varepsilon \to 0$ at the end.

Here the expectation values are wrt the fast fields, so that e.g. if $S_I = \phi^4$, $\langle \delta^2_{\phi(x)} S_I \rangle = \int \mathcal{D}\phi_> \, 12\phi^2(x) \, e^{-S_0[\phi_>]}$. A more precise way to say what $S_e[\phi_<] - S_0[\phi_<]$ is is that it's the generating functional for correlation functions of S_I. Functionally integrating this and its generalizations to higher derivatives over $\phi_<$, we get that

$$S_e[\phi_<] = S_0[\phi_<] - \sum_{k=1} (-1)^k \frac{1}{k!} \langle S_I^k \rangle_c. \tag{56}$$

Explicitly, this is

$$S_e[\phi_<] = S_0[\phi_<] - \sum_{k=1} (-1)^k \frac{1}{k!} \sum_{m_1,\dots,m_k \geq 1} \int \prod_{i=1}^k dx_i \left\langle \prod_{l=1}^k \frac{\delta^{m_l} S_I}{\delta\phi(x_l)^{m_l}} \Big|_{\phi_< = 0} \frac{\phi_<^{m_l}(x_l)}{m_l!} \right\rangle_c \tag{57}$$

A word of caution here: the cumulants appearing in this expression (the connected correlation functions) are *not* in general evaluated as $\langle (S_I - \langle S_I \rangle)^k \rangle$! Even though the second and third cumulants are indeed expectation values about the mean (so that $\langle S_I^k \rangle_c = \langle (S_I - \langle S_I \rangle)^k \rangle$ for $k = 2, 3$), this is *not* true in general. The general expression for the cumulants in terms of the moments of S_I is in general actually quite messy after $k > 3$.

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4 *Anomalous dimension at the $O(N)$ WF fixed point via the sunrise diagram and self energies* ✓

Today we're doing a rather long-winded elaboration on problem 7.2 from Sachdev's book. The goal is to compute the self energy to two-loop orderf at the WF fixed point in the $O(N)$ model in $4-\varepsilon$ dimensions in Euclidean signature, and then to use this calculation to compute the anomalous dimension of the ϕ field.

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Our conventions for the action will be that the interaction term has a $1/4!$ term in it, viz.

$$\mathcal{L} \supset \frac{r^2}{2} \sum_i \phi_i^2 + \frac{g\mu^{4-d}}{4!} \left(\sum_i \phi_i^4 + 2 \sum_{i<j} \phi_i^2 \phi_j^2 \right), \tag{58}$$

where as usual μ is some energy scale introduced to do dim reg. Sachdev uses momentum-shell RG for most of his calculations (although maybe mixes in some dim reg when doing the 2-loop computation? The details aren't given), but we will stick to dim reg.

At the critical point, the 2-point function in the $O(N)$ model will have the scale invariant form

$$\chi(k) \sim \mu^{-\eta}\frac{1}{|k|^{2-\eta}}, \tag{59}$$

where η is the anomalous dimension. At the same time, we may also write

$$\chi(k) \sim \frac{1}{k^2 - \Sigma(k)}, \tag{60}$$

where the self energy has $\Sigma(0) = 0$ at the critical point.

We would now like to compute $\Sigma(k)$ to lowest order at the critical point. The first graph that gives nontrivial momentum dependence is the sunrise diagram, which is quadratic in the coupling g. Sachdev asks you to do it directly in momentum space in the problem statement, but I think this is prohibitively hard—the calculation is done in Ramond's QFT book, but it's enough of a mess that I didn't want to go through and check that there were no typos. Seriously, it's really bad. Happily though, there is another way to do it: in real space! In real space, we need to compute[7]

$$\text{Sunrise}(x) = g^2\frac{N+2}{18}\frac{1}{((2-d)A(S^{d-1})|x|^{d-2})^3}. \tag{62}$$

To get this, we've just cubed the free propagator ($A(S^{d-1})$ is the area of the unit $(d-1)$-sphere), and tacked on the appropriate combinatorial factor. The symmetry factor comes from

$$g^2\frac{N+2}{18} = (N-1)(2g\cdot 2^2/4!)^2\cdot\frac{1}{2} + g^2\frac{1}{3!}. \tag{63}$$

In the expression for the sunrise diagram, we are interested in the value that g takes on at the fixed point, which is $O(\varepsilon)$. Therefore since we are just interested in a lowest order in ε calculation we have dropped any r-dependence in the propagators above, since the fixed point of r in this scheme is also $O(\varepsilon)$.

Getting the momentum-space result is then easy, since we may use the identity (which comes from integrating Bessel functions)

$$\int_{\mathbb{R}^d} d^dx\,|x|^{-a}e^{ix\cdot k} = \frac{(2\pi)^d\Gamma((d-a)/2)}{\pi^{d/2}2^a\Gamma(a/2)}|k|^{-(d-a)}, \tag{64}$$

and plug in $a = 3d-6$. This tells us that, letting $d = 4-\varepsilon$ so that $d-a = -2+2\varepsilon$,

$$\text{Sunrise}(k) = Cg^2(N+2)\Gamma(-1+\varepsilon)|k|^{2-2\varepsilon}, \tag{65}$$

[7] Remember that $\Sigma(k)$ contains the self-energy diagrams *with external propagators removed*, hence the sunrise diagram in $\mathbb{R}$ space only contains one integral over position. It appears in the real-space propagator via

$$G(x) = G_0(x) + \mathcal{O}(g) + g^2\int_{y,z} G_0(y)G_0^3(y-z)G_0(z-x). \tag{61}$$

Amputating the external legs $G_0(y), G_0(z-x)$ on the last term gives us the Fourier transform of the g^2 contribution to $\Sigma(k)$.

where $C > 0$ is a constant that I don't see any point in keeping track of.[8] The N dependence we'll keep, though.

Now since the fixed point value of g is at $g_* \sim \varepsilon/(N+8)$ (the exact fixed point was worked out in the diary entry on the anisotropic $O(N)$ model), working to lowest order in ε means working to $O(\varepsilon^2)$. When we send $\varepsilon \to 0$, we expand the Γ function as

$$\Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon} + \dots, \tag{66}$$

where the $\dots$ are finite. The term that cancels the pole in $1/\varepsilon$ comes from writing $|k|^{-2\varepsilon} = e^{-\varepsilon \ln k^2/\mu^2} \approx 1 - \varepsilon \ln k^2/\mu^2$, and so when $\varepsilon \to 0$ we have

$$\text{Sunrise}(k)_{\varepsilon\to 0} = Cg^2(N+2)k^2 \ln(k^2/\mu^2) + \dots, \tag{67}$$

where $\dots$ are terms that are not divergent / are polynomial in k (recall that algebraic functions of momentum become contact terms in $\mathbb{R}$ space, and so we don't care about them—it's all about the logs!). It really is worth emphasizing that this derivation via $\mathbb{R}$ space was orders of magnitude easier than the direct momentum-space computation—so keep $\mathbb{R}$ space in mind next time!

Before going on to look at the anomalous dimension, we'll look at what happens to this in $\mathbb{R}$ time. Sending $k^2 \to \mathbf{k}^2 - \omega^2$, we see that we will get an imaginary contribution when the argument of the log goes negative, and so to this order (viz. quadratic in ε), we find (still with $r = 0$)

$$\Sigma_I(0,\omega) = Cg^2(N-2)\theta(\omega)\omega^2. \tag{68}$$

Computing the above sunrise diagram would be a bit harder if we wanted to do it for $r \neq 0$, but luckily we can basically just write down the answer: since when $r \neq 0$ the threshold for particle production is $k^2 = (3r)^2$ (as there are only even-body interactions), the only possibility is to have

$$\Sigma_I(0,\omega) = Cg^2(N-2)\theta(|\omega| - 3r)(|\omega| - 3r)^2. \tag{69}$$

Now we want to get the anomalous dimension. The simple way to find η is just to solve

$$\frac{1}{k^2 - \Sigma(k)} = \frac{\mu^{-\eta}}{k^{2-\eta}} \implies \Sigma(k) = k^2(1 - e^{-\eta \ln k/\mu}) = k^2 \sum_{n=1}^{\infty} \frac{1}{n!}\eta^n(-1)^{n+1}\ln^n(k/\mu). \tag{70}$$

Let us expand η as a series,

$$\eta = \sum_{i=2}^{\infty} g^n \eta_n, \tag{71}$$

where the sum starts at $n = 2$ because the first contribution to η is the sunrise diagram. Since we have already found the $O(g^2k^2 \ln k/\mu)$ contribution, we know that

$$\eta_2 = C(N+2). \tag{72}$$

[8] From now on, "C" will be a stand-in for any positive constant that I don't want to keep track of.

Note that this immediately tells us an *infinite* number of terms appearing in the self-energy, at all orders in the coupling g: specifically, our computation of the sunrise diagram has determined the coefficients of all of the leading logs (those that go as $[\ln(k/\mu)(g^2\eta_2)]^n$) in the expansion, and therefore at a fixed order in $\ln(k/\mu)$, we know the coefficient of the lowest-order contribution in g (at order g^{2n}). Similarly, if we were brave enough to compute the subleading log term arising at three loops and going as $g^3 \ln k/\mu$, we'd also know an infinite number of terms (the subleading terms in g at a fixed order in $\ln(k/\mu)^n$, going as g^{2n+1})—this type of resummation is exactly what we're used to seeing from RG analysis.

Summarizing, we've shown that at the order of the leading logs, we have shown that the anomalous dimension at the fixed point is

$$\eta = C(N+2)g_*^2 = C'\varepsilon^2 \frac{N+2}{(N+8)^2}. \tag{73}$$

As expected from large N intuition, the anomalous dimension vanishes in the $N \to \infty$ limit.

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5 *From coupled Ising CFTs to the $O(2)$ model in three dimensions via conformal perturbation theory* ✓

Today we are doing an elaboration on an exercise given by Zohar Komargodski to the attendees of the 2017 Bootstrap school. The goal is to show that two 3d Ising models flow to the $O(2)$ fixed point when coupled through a deformation $\epsilon_1\epsilon_2$. First we'll derive some needed conformal perturbation theory results, and then we'll attack the problem both through an ϵ expansion and a direct perturbation away from the 3d Ising CFT.

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General conformal perturbation theory

Consider a CFT perturbed by a collection of operators with dimensionless couplings λ_i and scaling dimensions Δ_i:

$$\delta S = \sum_i a^{-d+\Delta_i} \lambda_i \int \mathcal{O}_i, \tag{74}$$

where a is the short-distance cutoff (from here on, summation over repeated indices is implied). We will use a derivation of the CPT beta functions inspired from reading a paper on disordered Ising models by Zohar and DSD [2].

The deformation will generically take the theory away from the fixed point. To get the β functions, we will choose an observable in the theory and require that the derivative of its expectation value with respect to the short distance cutoff vanish (again with the usual caveat that we are working modulo powers of a). There are many observables to choose from, but the one we will find most convenient is the overlap between the state $|0\rangle$ and $|\mathcal{O}_i\rangle$ in the presence of the perturbation to the action in a region R. Therefore we will need to compute[9]

$$\frac{d}{d\ln a}\langle\mathcal{O}_i|e^{-\lambda_j a^{-d+\Delta_j}\int_R \mathcal{O}_j}|0\rangle = 0. \tag{75}$$

To evaluate this, we expand the exponential to quadratic order in the couplings, and using $\langle\mathcal{O}_i| = \lim_{x\to\infty}\langle 0|x^{2\Delta_i}\mathcal{O}(x)$, we have

$$\frac{d}{d\ln a}\left\langle 1 - V_R\lambda_i a^{-y_i} + \frac{\lambda_j\lambda_k}{2}a^{-y_j-y_k}\lim_{z\to\infty}\int_R d^dy\, d^dx\, \mathcal{O}^i(z)\mathcal{O}^j(x)\mathcal{O}^k(y)\right\rangle = 0, \tag{76}$$

with $y_i \equiv d - \Delta_i$ and V_R the volume of R. Here we used the OPE to simplify the second term. If we take R to be bounded we can do the OPE between $\mathcal{O}^j$ and $\mathcal{O}^k$, and then the OPE between the resulting operator and $\mathcal{O}^i$. Since we will be doing the OPE between an operator at ∞ and one in R, it doesn't really matter where exactly in R the latter operator is located. Therefore we can always take the OPE to be taken with x located at the center of R (we'll take R to be a ball). The integral over x then produces a factor of V_R. We then have remaining an integral $S^{d-1}\int dr\, r^{\Delta_i-\Delta_j-\Delta_k+d-1}$. We will absorb the S^{d-1} factor by re-scaling all the coupling constants, and so we then have, letting $d_t = \frac{d}{d\ln a}$ be the differential for RG time,

$$y_i a^{-y_i}\lambda_i - \beta_i a^{-y_i} + a^{-(y_j+y_k)}\left(\beta_j\lambda_k - \frac{\lambda_j\lambda_k}{2}(y_j+y_k-d_t)\right)C^i_{jk}\int dr\,\frac{1}{r^{-\Delta_i+\Delta_j+\Delta_k-d+1}} = 0 \tag{77}$$

Note that to obtain this equation, we had to assume that the theory the expectation value was being taken with respect to was a CFT—otherwise the d_t pick up extra terms from the change in the action being used to construct the expectation value.

To satisfy this equation to first order in the couplings, we need $\beta_i = y_i\lambda_i + O(\lambda^2)$. In the term $\beta_j\lambda_k$, only the linear part of β_j contributes at quadratic order in the couplings, and this term cancels the $-\lambda_j\lambda_k(y_j+y_k)/2$ piece. Therefore, writing the quadratic piece of β_i as $\beta^{(2)}$, we have

$$\beta_i^{(2)}a^{-y_i} = C^i_{jk}a^{-y_j-y_k}\frac{\lambda_j\lambda_k}{2}d_t\int dr\frac{1}{r^{-\Delta_i+\Delta_j+\Delta_k-d+1}}. \tag{78}$$

The differential of the integral gives $-d_t a^{\Delta_i-\Delta_j-\Delta_k+d}/(\Delta_i-\Delta_j-\Delta_k+d) = -a^{-y_i+y_j+y_k}$, and so

$$\beta_i^{(2)} = -\frac{\lambda_j\lambda_k}{2}C^i_{jk} \implies \beta_i = y_i\lambda_i - \frac{\lambda_j\lambda_k}{2}C^i_{jk}. \tag{79}$$

[9] Other similar choices of observables, like $\langle e^{-\lambda_j a^{-d+\Delta_j}\int\mathcal{O}_j}\rangle$ or $\langle\mathcal{O}_i|e^{-\lambda_j a^{-d+\Delta_j}\int\mathcal{O}_j}|\mathcal{O}_k\rangle$, do not give us, upon differentiation, formulas which are as nice. This is because they have β functions mixed up with annoying integrals. The merits of the observable we chose to study lie mainly in the fact that the linear term in the coupling λ_i is nonzero—after differentiating this gives us a nice factor of β_i all by itself, and this makes the resulting manipulations easier.

This gives us the β functions for the couplings of the perturbations added to the action to deform the theory away from the fixed point. To see the extent to which other operators (viz. those corresponding to the scaling variables in the CFT) are modified by the perturbation, we need to make a further deformation and include these operators in the exponential. For example, we will want to compute the extent to which the perturbation modifies the scaling dimension of an operator $\mathcal{O}$ in the CFT. This is done by adding $\delta\lambda_{\mathcal{O}} \int \mathcal{O}$ to the exponential in the deformation, and then expanding the exponential to linear order in $\delta\lambda_{\mathcal{O}} = \lambda_{\mathcal{O}} - \lambda_{\mathcal{O}*}$, where $\lambda_{\mathcal{O}*}$ is the value of the coupling in the CFT. This is because in order to determine the scaling dimension of $\mathcal{O}$ in the CFT, we need to perturb away from the CFT by deforming by $\mathcal{O}$, and then examine how quickly the theory flows back to / away from the fixed point. To the extent that $\mathcal{O}$ remains a scaling variable in the presence of the $\lambda_i\mathcal{O}_i$ deformation, we then find that $\beta_{\mathcal{O}}$ is made nonzero by the term

$$\beta_{\mathcal{O}} = y_{\mathcal{O}}^{(0)}\delta\lambda_{\mathcal{O}} - (\delta\lambda_{\mathcal{O}})\lambda_j C^{\mathcal{O}}_{\mathcal{O}j}, \tag{80}$$

where $d - y_{\mathcal{O}}^{(0)}$ is the scaling dimension of $\mathcal{O}$ in the un-deformed theory. Since $\delta\lambda_{\mathcal{O}}$ vanishes at the fixed point, this means that the scaling dimension of $\mathcal{O}$ is corrected by

$$\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}}^{(0)} + \lambda_{j*}C^{\mathcal{O}}_{\mathcal{O}j}. \tag{81}$$

ϵ expansion from $d = 4$

Now we consider two Ising models, coupled though the product of their energy operators. The action in $d = 4 - \epsilon$ dimensions is

$$S = \int \left(\frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} + ta^{-2}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + ga^{-\epsilon}(\phi_1^4 + \phi_2^4) + \eta a^{-\epsilon}\phi_1^2\phi_2^2 \right). \tag{82}$$

We will evaluate the β functions around the Guassian fixed point, and use them to predict where the theory flows. The starting point of the flow is hence just free field theory, and so getting the OPE coefficients is but a simple matter of combinatorics. They are (normalizing the fields so that $\langle\phi_i(r)\phi_j(0)\rangle = \frac{1}{x^{d-2}}$)

$$C^{\eta}_{\eta\eta} = \binom{2}{1}^4 = 16, \quad C^{\eta}_{\eta g} = 2\cdot 2!\binom{4}{2} = 24, \quad C^{g}_{gg} = 2!\binom{4}{2}^2 = 72, \quad C^{g}_{\eta\eta} = 2. \tag{83}$$

The ones involving t are

$$C^t_{tt} = \binom{2}{1}^2 = 4, \quad C^t_{gg} = \binom{4}{3}^2 3! = 96, \quad C^t_{gt} = \binom{4}{2}2! = 12, \quad C^t_{\eta\eta} = \binom{2}{1}^2 2! = 8, \quad C^t_{\eta t} = 2$$
$$C^g_{tt} = 1, \quad C^\eta_{tt} = 2, \quad C^g_{tg} = \binom{4}{1}2! = 8, \quad C^\eta_{\eta t} = 2\cdot 2^2 = 8. \tag{84}$$

First let us ignore t—we tune it to zero in the UV, and then anticipate that the error in our fixed point value $t_* = 0$ will only be of order ϵ^2 (since the non-zero β function for t at

$t = 0$ comes only from η^2, g^2 terms). Then plugging the relevant OPE coefficients in to our formulae for the β functions (all of t, g, η are being treated as perturbations), we obtain

$$\beta_\eta = \epsilon\eta - 8\eta^2 - 24\eta g, \qquad \beta_g = \epsilon g - \eta^2 - 36g^2. \tag{85}$$

The flow looks like

etag_flow.pdf

(86)

There are four fixed points, only one of which is IR stable with respect to both η and g. We find that at this point,

$$\eta_* = \frac{\epsilon}{20}, \qquad g_* = \frac{\epsilon}{40}, \qquad y_\eta = -\epsilon, \qquad y_g = -\epsilon/5, \tag{87}$$

while the fixed point value of the quadratic term vanishes: $t_* = 0$. Therefore at the fixed point we may write the action, to the extent that doing so is meaningful, as

$$S = \int \left(\frac{1}{2}\partial\boldsymbol{\phi} \cdot \partial\boldsymbol{\phi} + \frac{\epsilon}{40} a^{-\epsilon} (\boldsymbol{\phi} \cdot \boldsymbol{\phi})^2 \right). \tag{88}$$

Note that the fixed-point values of η, g are such that the resulting action is $O(2)$-symmetric.

Now if we go back and include t, we have

$$\beta_t = 2t - 2t^2 - 2t\eta - 12tg - 48g^2 - 4\eta^2, \tag{89}$$

and

$$\beta_\eta = \beta_\eta(t = 0) - t^2 - 8\eta t, \qquad \beta_g = \beta_g(t = 0) - 8tg - \frac{1}{2}t^2. \tag{90}$$

When $\epsilon \ll 1$ we have $t_* \sim O(\epsilon^2)$, which can be dropped as anticipated earlier. We then get that at the fixed point, (this isn't quite correct since the presence of g^2, η^2 terms mean that t is not by itself a scaling variable at the fixed point, even to order ϵ)

$$y_t = 2 - 2\epsilon(1/20 + 6/40) + O(\epsilon^2) = 2(1 - \epsilon/5) + O(\epsilon^2). \tag{91}$$

When $\epsilon \sim 1$ the solutions for the fixed points get very complicated, and saying anything concrete gets more difficult. However, we note that setting $\epsilon = 1$ in the above equation (and yet still dropping the $O(\epsilon^2)$ terms, lol) gives a scaling dimension of $\Delta_t = 7/5 = 1.4$ for the energy operator in the $O(2)$ model. In fact the actual scaling dimension is $\Delta_t \approx 1.51$, so this laughably crude approximation actually gets the right scaling dimension within 7% error. However, the fact that it under-estimates the scaling dimension is important, since it suggests that an energy-energy deformation of two coupled $O(2)$ models is relevant, when in fact it is (barely) irrelevant.

Perturbation theory direction from the Ising CFT

Now we'll take a more direct approach. Instead of starting from the Gaussian fixed point in 4 dimensions and flowing along the WF trajectory towards the putative $O(2)$ fixed point, we will start with two coupled Ising models in three dimensions, and flow along the trajectory generated by the inter-model coupling. Since the inter-model coupling term is only very slightly relevant, the flow will likely be a lot shorter than the flow from the Gaussian fixed point in 4 dimensions, and so will hopefully provide us with a better prediction for the $O(2)$ CFT data.

The Ising models will be coupled together through their energy operators:

$$S = S_{I,1} + S_{I,2} + \eta \int \epsilon_1 \epsilon_2. \tag{92}$$

First let us recall some results for the 3d Ising model, namely [3]

$$\Delta_\epsilon \approx 1.41, \qquad \Delta_{\epsilon'} \approx 3.82, \qquad C^{\epsilon}_{\epsilon\epsilon} \approx 1.53, \qquad C^{\epsilon'}_{\epsilon\epsilon} \approx 1.54, \tag{93}$$

where ϵ is the energy operator (schematically, ϕ^2) and ϵ' is the next lightest $\mathbb{Z}_2$-even scalar in the spectrum (schematically, ϕ^4). Note that an operator like ϵ^2, with scaling dimension the relevant ≈ 2.8, doesn't appear in the spectrum. Also note that $[\epsilon_1\epsilon_2] \approx 2.82 < 3$, and so the deformation that couples the two Ising models at the decoupled fixed point $\mathcal{I}$ is slightly relevant: $y^{\mathcal{I}}_\eta \approx 0.18$. The beta function for the deformation is therefore

$$\beta_\eta = y^{\mathcal{I}}_\eta \eta - \frac{\eta^2}{2} C^{\eta}_{\eta\eta}, \qquad C^{\eta}_{\eta\eta} = (C^{\epsilon}_{\epsilon\epsilon})^2. \tag{94}$$

Therefore the IR fixed point $\mathcal{F}$ in this approximation is at

$$\eta_* = 2 y^{\mathcal{I}}_\eta (C^{\epsilon}_{\epsilon\epsilon})^{-2} \approx 0.15, \tag{95}$$

and the dimension of η at the IR fixed point $\mathcal{F}$ is, using $\beta_\eta \approx y^{\mathcal{I}}_\eta(1 - 2C^{\eta}_{\eta\eta}/(C^{\varepsilon}_{\varepsilon\varepsilon})^2\eta = -y^{\mathcal{I}}_\eta \eta$ near the IR fixed point,

$$\Delta^{\mathcal{F}}_\eta = d + y^{\mathcal{I}}_\eta \approx 3.18 \implies y^{\mathcal{F}}_\eta = -y^{\mathcal{I}}_\eta = -0.18, \tag{96}$$

Going back and calculating what we would expect for $y_\eta^{\mathcal{I}}$ at the fixed point with the ϵ expansion,[10] we obtain $y_\eta^{\mathcal{F}} = -2/5$, which is in pretty good agreement with the (presumably more accurate) result above.

We can also calculate the extent to which the coupling modifies the scaling dimension of the total energy operator $\boldsymbol{\epsilon}$, pretending that $\boldsymbol{\epsilon} \equiv \epsilon_1 + \epsilon_2$ is still an exact scaling variable at the IR fixed point. Using the results obtained in the first section, we see that the scaling dimension at the IR fixed point is approximately

$$\Delta_{\boldsymbol{\epsilon}}^{\mathcal{F}} = \Delta_\epsilon^{\mathcal{I}} + \eta_* C_{\eta\boldsymbol{\epsilon}}^{\boldsymbol{\epsilon}}. \tag{97}$$

The scaling dimension of the energy operator thus increases along the flow. Since $C_{\boldsymbol{\epsilon}\eta}^{\boldsymbol{\epsilon}} = C_{\epsilon\epsilon}^{\epsilon}$, we have, using our earlier expression for η_*,

$$\Delta_{\boldsymbol{\epsilon}}^{\mathcal{F}} = \Delta_\epsilon^{\mathcal{I}} + 2\frac{3 - \Delta_\eta^{\mathcal{I}}}{C_{\epsilon\epsilon}^{\epsilon}} \approx 1.63. \tag{98}$$

The actual scaling dimension of $\boldsymbol{\epsilon}$ in the $O(2)$ is in fact $\Delta_{\boldsymbol{\epsilon}}^{O(2)} \approx 1.51$, which eh, isn't too bad for a leading-order approximation. Unfortunately it does about the same as the more crude ϵ expansion estimate, although it over-estimates $\Delta_{\boldsymbol{\epsilon}}$ instead of under-estimating, like the ϵ expansion does (in fact, the ϵ expansion predicts $\Delta_{\boldsymbol{\epsilon}} < \Delta_\epsilon$ (by a tiny bit), which we know can't be right).

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6 *The Gross-Neveu model* ✓

Consider the Gross-Neveu model, which is a theory of N Dirac fermions in 2 spacetime dimensions, interacting through a quartic term:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - g^2(\bar{\psi}_i\psi^i)^2, \tag{99}$$

with $i = 1, \ldots, N$. Show that a dynamical mass term (which spontaneously breaks the chiral symmetry) for the fermions is generated by dimensional transmutation at one-loop order. Note that the chiral symmetry here is just a discrete $\mathbb{Z}_2$ symmetry which acts as,

$$\psi_i \mapsto \bar{\gamma}\psi_i, \tag{100}$$

and sends $\bar{\psi}_i\psi^i \mapsto -\bar{\psi}_i\psi^i$ (here $\bar{\gamma}$ has alias γ^5). In the limit $N \to \infty$, Ng^2 fixed, show that the one-loop result is exact, resulting in a mass term which is non-analytic in g^2, namely $m_{\text{eff}} = \Lambda e^{-\pi/(Ng^2)}$ (think of QCD and BCS theory).

[10] Again, because of the $g\eta$ term in β_η, η is technically speaking not a scaling variable.

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In two dimensions, we write represent the Clifford algebra with $\gamma^0 = iY, \gamma^1 = X$, so that the chirality operator is $\bar{\gamma} = \gamma^0\gamma^1 = Z$. In the absence of a $\bar{\psi}_i\psi^i$ mass term, we have the $\mathbb{Z}_2$ chiral symmetry $\psi \mapsto \bar{\gamma}\psi$, while a mass term breaks the symmetry explicitly.

To start, we decouple the fermion interaction in the usual way. Letting the decoupling HS field be called σ, the decoupled partition function is

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp\left(i\int (i\bar{\psi}\partial\!\!\!/\psi - \sigma\bar{\psi}\psi - \frac{1}{2g^2}\sigma^2)\right). \tag{101}$$

Now we integrate out the fermions to get an effective action for σ. Since there are N copies of fermions we get $\det^N(i\partial\!\!\!/ - \sigma)$, and so

$$Z = \int \mathcal{D}\sigma\, e^{iS_{\text{eff}}[\sigma]}, \tag{102}$$

where

$$S_{\text{eff}}[\sigma] = -iN\ln\det(i\partial\!\!\!/ - \sigma) - \frac{1}{2g^2}\int \sigma^2. \tag{103}$$

To evaluate the $\ln\det$, we go to momentum space in which $i\partial\!\!\!/ - \sigma$ is block-diagonalized, where each block is labeled by a different value of the momentum. One can think of the operator as becoming $\bigoplus_k(k\!\!\!/ - \sigma)$. Since $\det(\bigoplus_\alpha A_\alpha) = \det\prod_\alpha(A_\alpha \otimes \mathbf{1}_{\bar{\alpha}}) = \prod_\alpha \det A_\alpha$, we write

$$\det(i\partial\!\!\!/ - \sigma) = \prod_k \det(k\!\!\!/ - \sigma). \tag{104}$$

Evaluating the determinant with the form of the γ matrices given above gives

$$\ln\det(i\partial\!\!\!/ - \sigma) = iV\int_k \ln(\sigma^2 + k^2), \tag{105}$$

where we've rotated into Euclidean space, and V is the volume of the two-dimensional spacetime. We do the integral with the "replica trick", i.e. by using $\ln\rho = -\partial_n\rho^{-n}|_{n\to 0}$. Doing the integral (see e.g. P&S), one gets (working with the MF assumption where σ is a constant in order to do the integrals)

$$\begin{aligned} \ln\det(i\partial\!\!\!/ - \sigma) &= -iV\lim_{n\to 0}\partial_n\int_k \frac{1}{(\sigma^2 + k^2)^n} \\ &= -iV\Gamma(-d/2)\left(\frac{\sigma^2}{4\pi}\right)^{d/2}, \end{aligned} \tag{106}$$

where we need to take the $d \to 2$ limit. For $d = 2 + \epsilon$ we can use the expansion

$$\Gamma(-d/2) \approx \frac{2}{\epsilon} + \gamma. \tag{107}$$

Then taking $\epsilon \to 0$, we get[11]

$$\ln\det(i\partial\!\!\!/ - \sigma) = -iV\frac{\sigma^2}{4\pi}\left(\frac{2}{\epsilon} + \gamma - \ln(4\pi) + \ln(\sigma^2/\Lambda^2)\right) \tag{108}$$

for some renormalization scale Λ. We choose the counterterm for $1/g^2$ in the original Lagrangian to kill the $1/\epsilon$ terms and the constants. Then the effective potential is

$$\begin{aligned} V_{\text{eff}}[\sigma] &= -\frac{1}{V}\int\left[\mathcal{L} + \mathcal{L}_{ct} - iN\ln\det(i\partial\!\!\!/ - \sigma)\right] \\ &= \frac{1}{2g^2}\sigma^2 + \frac{\sigma^2}{4\pi}N\ln(\sigma^2/\Lambda^2). \end{aligned} \tag{109}$$

With this renormalization scheme we can calculate the β function, which we do by requiring that the physical effective potential be independent of the renormalization scale Λ. We get

$$\beta(g) = -\frac{dg}{d\ln\Lambda} = +\frac{N}{2\pi}g^3, \tag{110}$$

which tells us that the theory is asymptotically free: as we go to higher Λ scales, the theory becomes increasingly weakly coupled.

In any case, we can now minimize the effective potential to see if σ gets a vev in the mean-field approximation (we are ignoring IR symmetry restoration). Doing this yields

$$\sigma^2 = \Lambda^2 e^{-\frac{2\pi}{g^2 N}} \neq 0. \tag{111}$$

Thus at 1-loop order, σ gets a vev. Since $\langle\sigma\rangle \propto \langle\bar{\psi}\psi\rangle$[12] and leads to the chiral symmetry being spontaneously broken in this approximation as a result of the fermion condensate which forms. Note that the expression for the mass of the fermions has exactly the same form as the expression for the mass gap in a superconductor: it is non-analytic in the coupling, and linearly proportional to the cutoff (debeye frequency). Also note that we have two solutions for σ, consistent with the fact that the symmetry that is being spontaneously broken is a $\mathbb{Z}_2$ symmetry. This result is exact in the large N (but fixed t' Hooft coupling g^2N) limit; one can straightforwardly check that higher-loop corrections to the effective action are suppressed in powers of $1/\sqrt{N}$.

We can also check this by doing a saddle point analysis on the effective action for σ. Again in the large N limit, we expect this to be exact. Varying the effective action with respect to σ and setting the result equal to zero gives

$$\frac{\sigma(x)}{g^2} = -iN\frac{\delta}{\delta\sigma(x)}\ln\det(i\partial\!\!\!/ - \sigma). \tag{112}$$

[11] Dim reg is one way to regulate the integral. Another is to simply stay in two dimensions and do the integral as $\sim \partial_n(\sigma^2+k^2)^{1-n}|_0^\Lambda$ and take the derivative—this only gives a finite result if $n > 1$, but analytically continuing the exact result to $n \to 0$ reproduces the correct effective potential.

[12] σ appears quadratically and without derivatives, so it can be directly integrated out and we can replace σ with $\bar{\psi}\psi$ in correlation functions—of course this is exactly the point of decoupling the fermion interaction like this.

We use $\delta \ln A = A^{-1}\delta A$ and take the determinant over the spin indices explicitly to get

$$\frac{\sigma(x)}{g^2} = -2iN \int_{k=0}^{\Lambda} \langle x| \frac{\sigma}{k_\mu k^\mu + \sigma^2} |x\rangle \tag{113}$$

where Λ is a cutoff. Going to Euclidean signature to do the integral over momentum gives

$$\frac{\sigma}{g^2} = \frac{N}{2\pi} \ln\left(\frac{\Lambda^2 + \sigma^2}{\sigma^2}\right), \tag{114}$$

which when solved leads to exactly the same vev for σ that we derived using the minimal subtraction renormalization scheme.

7 *The Coleman-Weinberg potential and fluctuation-induced first order transitions* ✓

This is one of the final projects in P&S, and is basically the high-energy way of thinking about the HLM fluctuation-induced first order transitions in superconductors.

Consider scalar QED in four dimensions with a Mexican hat potential:

$$\mathcal{L} = -\frac{1}{2} F \wedge \star F + |D_A \phi|^2 + \frac{1}{2}\mu^2 |\phi|^2 - \frac{\lambda}{6}|\phi|^4, \tag{115}$$

where our sign conventions are $(D_A\phi)_\mu = (\partial_\mu - ieA_\mu)\phi$. There are several things to do:

- Compute the effective potential for ϕ to one-loop order. You should find that even at small values of $\mu^2 < 0$, where classically there is only one minimum, spontaneous symmetry breaking occurs for small e and small λ.
- Find the β functions for e and for λ.
- Use these to evaluate the ratio m_σ^2/m_A^2, where m_σ is the mass of the fluctuations about the minimum of the potential and m_A is the mass of the Higgsed photon. Show that after taking into account the RG flow, there is still a symmetry-breaking vev for ϕ even when μ^2 is negative, provided that $|\mu|$ is small.

This problem is pretty long, and it would be rather onerous to type out all the algebra, and so I'll be a bit laconic in places.

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Effective potential:

To get the effective potential, we need to integrate out the fluctuations in the ϕ field, as well as integrate out the gauge field. Following P&S we parametrize ϕ as

$$\phi = v + \frac{1}{\sqrt{2}}(\sigma + i\pi), \tag{116}$$

where σ, π are real scalars that we will integrate out.

To one-loop order, the effective action is determined by the terms second order in the fluctuations. The relevant parts of the action are

$$\begin{aligned} S = \frac{1}{2}\int \Big(&(\partial_\mu \pi \partial^\mu \pi + \partial_\mu \sigma \partial^\mu \sigma + e^2 A^2(\pi^2 + \sigma^2) + 2e^2 v^2 A^2 + 2\sqrt{2} e^2 A^2 v \sigma \\ &- (\lambda v^2/3 - \mu^2)\pi^2 - (\lambda v^2 - \mu^2)\sigma^2 \Big). \end{aligned} \tag{117}$$

The one-loop diagrams contributing to $V_{\text{eff}}(\phi)$ all have the same form: the $2n$-th order contribution to the effective potential is a graph with $2n$ external ϕ legs and a single loop connecting them (remember that the n-th order contribution to Γ are the n-point 1PI diagrams). Because of the interactions in the action, this loop can be a π loop, a σ loop, or an A loop, but there are no graphs with different types of fields running around the loop. This means that for the purpose of getting V_{eff}, we can ignore the interactions between the fields and just treat them as separate free fields.

In the transverse gauge $d^\dagger A = 0$, the gauge field contribution is

$$\frac{1}{2} A_\mu (g^{\mu\nu}\partial^2 + 2e^2 v^2) A_\nu. \tag{118}$$

This is a massive vector field, which due to the transverse constraint behaves just like three scalars, giving us a factor of

$$\int \mathcal{D}A \to [\det(\partial^2 + 2e^2 v^2)]^{-3/2}. \tag{119}$$

The σ and π contributions give us

$$\int \mathcal{D}\sigma \to [\det(\partial^2 - (\mu^2 - \lambda v^2))]^{-1/2}, \qquad \int \mathcal{D}\pi \to [\det(\partial^2 - (\mu^2 - \lambda v^2/3))]^{-1/2}. \tag{120}$$

The effective action is thus the original Mexican hat for ϕ, plus the three $\ln\det$ terms. These are calculated through dimensional regularization in the regular way, see e.g. the appendix of P&S for the integrals. We get

$$\ln\det(\partial^2 + \alpha^2) = iV \int_k \ln(k^2 + \alpha^2) = -\frac{i\alpha^2 V}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \ln\alpha^2 + \dots\right), \tag{121}$$

where the $\dots$ are constants, $d = 4 - \epsilon$, and V is the spacetime volume. We choose the counterterms in the original Lagrangian to kill off the divergent parts, using the minimal subtraction scheme. The cutoff-dependent term that each $\ln\det$ produces is

$$\ln\det(\partial^2 + \alpha^2) \to \frac{iV\alpha^4}{2(4\pi)^2}\ln(\alpha^2/\Lambda^2) \tag{122}$$

where Λ^2 is the cutoff. We choose the counterterms to cancel the $2/\epsilon + \ldots$ divergence at all scales, and to cancel the logarithmic cutoff-dependent term at the renormalization scale $M^2 < \Lambda^2$. Thus the counterterms contain logarithms of the form $-\ln(\alpha^2/M^2)$. Adding up the contributions from the three fields, and using $\Gamma[\phi]/V = -V_{\text{eff}}$ for a uniform expectation value of ϕ, we we get

$$\begin{aligned} V_{\text{eff}} = -\mu^2\phi^2 + \frac{\lambda\phi^4}{6} + \frac{1}{4(4\pi)^2}\Big(& \ln[(-\mu^2+\lambda\phi^2)/M^2](-\mu^2+\lambda\phi^2)^2 \\ & + \ln[(-\mu^2+\lambda\phi^2/3)/M^2](-\mu^2+\lambda\phi^2/3)^2 + 3\ln[2e^2\phi^2/M^2](2e^2\phi^2)^2\Big). \end{aligned} \tag{123}$$

Working at the classical critical point $\mu^2 = 0$, let us simplify the potential in the region $\lambda \sim e^4$ very small, which we will see is a region through which RG flows always pass. We get

$$V_{\text{eff}} \approx \frac{\lambda\phi^4}{6} + \frac{3e^4\phi^4}{16\pi^2}\ln\left(\frac{2e^2\phi^2}{M^2}\right). \tag{124}$$

Minimizing this, we find the following vev for ϕ^2:

$$\phi^2 = \frac{M^2}{2e^2}\exp\left(-\frac{8\pi^2\lambda}{9e^4} - \frac{1}{2}\right). \tag{125}$$

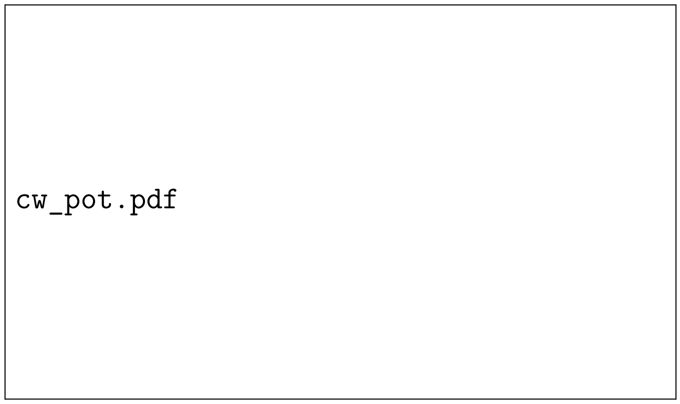
Thus the vev for ϕ is nonzero even when $\mu^2 = 0$, which could not happen classically.[13] For $\mu^2 \neq 0$, we can check by plotting the effective potential for different values of μ^2 (at fixed $\lambda \sim e^4$ and at fixed M^2), that a nontrivial minimum in V_{eff} survives even when $\mu^2 < 0$.[14] Note also that we can solve the above equation for λ as a function of e at a given renormalization scale (which we usually take to be $M^2 = 2e^2\phi^2$ for convenience). Thus we may trade two dimensionless parameters of the model (e and λ) for one dimensionless and one dimensionfull parameter — another example of dimensional transmutation.

Since the minimum of the potential becomes nonzero when the coefficient of the quadratic term is still positive, this provides us with an example of a (weakly) first order transition. This can be seen clearly by making a plot; choosing values of e^2, M^2, and λ so that the plot

[13] The fact that the Higgs transition occurs when the quadratic term is positive is rather un-intuitive in my opinion: one would think that the effects of quantum fluctuations would *decrease* the tendency to order; the naive expectation would be that the Higgs transition wouldn't occur until some negative mass at a finite distance below zero. In fact, it is the opposite situation which occurs! Crazy.

[14] When doing this, one finds regions of parameter space where for certain values of ϕ, the arguments of the logarithms become negative—this is not a pathology, and indicates that in that case, there is no field configuration with expectation value ϕ: trying to force the expectation value of the field to be ϕ leads to an unstable state. This is like if we try to force the electric field in some region of space to be greater than $\sqrt{2m_ec^2}$: such an expectation value is not stable, since the vacuum will nucleate pairs to screen the electric field. This quantum effect is exactly what the imaginary parts of the effective potential capture.

looks nice, the effective potential $V_{\text{eff}}(\phi)$ looks like



(126)

where the curves are shown for large-ish positive mass squared ($\mu^2 < 0$; blue) down to zero mass squared $\mu^2 = 0$; purple). Here we can explicitly see the first-order nature of the transition. Since the first-order nature of the transition is caused by the logarithm, which is there to account for the 1-loop fluctuations about the classical potential, this is an example of a "fluctuation-induced first-order transition". As can be seen by the dependence of the effective potential on e^2, the interesting features of this potential are entirely due to the fluctuations of the gauge field; if the gauge field is treated classically then no such first-order transition occurs.

β **functions:**

Now for the β functions. We can go back to working with A and ϕ, which is much easier since it reduces the number of diagrams we have to compute. This is allowed since we only care about β_e and β_λ. e and λ are both dimensionless at the free fixed point in this problem, and so they cannot depend on the dimensionful parameter m.

The beta function for e is easier, so we turn to that first. We get β_e by looking at one-loop corrections to the photon propagator, i.e. by examining how charge renormalization occurs.

There are two one-loop diagrams to evaluate: the polarization bubble and a diagram where a ϕ loop intersects the A propagator line at a single vertex. The latter graph is independent of the photon momentum p^2, and as such won't contribute to the beta function, since it will be completely killed by the δ_e counterterm at any RG scale M. The polarization bubble is

$$\Pi^{\mu\nu}(p^2) = -e^2 \int_q \frac{i}{q^2} \frac{i}{(q-p)^2} (2q-p)^\mu (2p-q)^\nu, \tag{127}$$

where q is the momentum flowing in the ϕ loop. The momenta in the numerator come from Fourier-transforming the vertex $eA^\mu(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger)$. We evaluate the bubble in the usual way using Feynman parameters and massaging the resulting expression into an integral that we can look up in the appendix of P&S (here $\int_x$ means $\int_0^1 dx$)

$$\begin{aligned}
\Pi^{\mu\nu}(p^2) &= e^2 \int_{q,x} \frac{(2q-p)^\mu (2q-p)^\nu}{(q^2+p^2x-2xp\cdot q)^2} \\
&= e^2 \int_{q,x} \frac{(2q+p(2x-1))^\mu (2q+p(2x-1))^\nu}{(q^2+p^2(x+x^2))^2} \\
&= e^2 \int_{q,x} \frac{g^{\mu\nu}q^2 + p^\mu p^\nu (2x-1)^2}{(q^2-\Delta)^2},
\end{aligned} \tag{128}$$

where $\Delta = (x^2+x)(-p^2)$ and where we shifted integration over q to simplify the denominator in the second step. Looking up the integrals and doing dimensional regularization, and then doing the integral over x, we get

$$\Pi^{\mu\nu}(p^2) = \frac{ie^2}{3(4\pi)^2}\left(\frac{1}{\epsilon} - \ln(-p^2/\Lambda^2) + \dots\right)(p^2 g^{\mu\nu} - p^\mu p^\nu), \tag{129}$$

where $\dots$ are irrelevant constants and the momentum dependence is required by our choice of gauge.

We then figure out the charge renormalization by setting the propagator equal to a sum over all numbers of sequential polarization bubbles in the usual way:

$$e^2(M^2=-p^2)D^{\mu\nu}(p^2) = e^2 D^{\mu\nu}(p^2) + e^2 D^{\mu\alpha}(p^2)\frac{e^2}{3(4\pi)^2}\left[\ln(-p^2/\Lambda^2)(p^2 g_{\alpha\gamma} - p_\alpha p_\gamma)\right] D^{\gamma\nu}(p^2) + \dots, \tag{130}$$

where $\dots$ contains $n > 1$ polarization bubbles, e.g. $D\Pi D\Pi D$, $D\Pi D\Pi D\Pi D$, and so on. This becomes a geometric series in the ln factor, since in our gauge $d^\dagger d\Box^{-2}d^\dagger d = d^\dagger\Box^{-1}d$, i.e. $(p^2 g_{\mu\alpha} - p_\mu p_\alpha)(g^\alpha_\nu/p^2 - p^\alpha p_\nu/p^4) = g^{\mu\nu} - p^\mu p^\nu/p^2$. Summing the geometric series, we get

$$e^2(M^2) = \frac{e^2}{1 - \frac{e^2}{3(4\pi)^2}\ln(M^2/\Lambda^2)}, \tag{131}$$

where e is as before the bare electric charge and $e(M)$ the charge at scale M. From this, we calculate the beta function

$$\beta_e = \frac{de(M)}{d\ln M} = \frac{e^3(M)}{48\pi^2}. \tag{132}$$

Now for β_λ, which is more of a pain. Since there are many 1-loop diagrams to compute if we want to get β_λ directly, we take a different approach using the CS equation. First, we find the field-strength renormalization of ϕ.

To one-loop order the ϕ propagator correction due to ϕ itself is killed by the mass counterterm (this is true because we are taking $m^2 = 0$ in the Lagrangian, see P&S chapter 12), so we only need to worry about the gauge field contribution. There are two relevant diagrams: one with a straight ϕ line and a A bubble meeting it at a single vertex, and one with a polarization bubble consisting of one A line and one ϕ line. The former diagram gives a contribution independent of the external ϕ momentum, so we can ignore it in what follows (since we'll be setting the external momentum equal to the RG scale, and then differentiating wrt that scale). The bubble diagram gives

$$\begin{aligned}\text{bubble}^{\mu\nu} &= (-ie)^2 \int_q \frac{1}{(p-q)^2}\left(\frac{g^{\mu\nu}}{p^2} + \frac{p^\mu p^\nu}{p^4}\right)(2p-q)^\mu (2p-q)^\nu \\ &= I_1 + I_2,\end{aligned} \tag{133}$$

where I_1 is the diagonal part and I_2 the $p^\mu p^\nu$ part. The former is

$$\begin{aligned} I_1 &= -e^2 \int_q \frac{(2p-q)^2}{q^2(p-q)^2} \\ &= -e^2 \int_{q,x} \frac{(p-q)^2}{(xq^2 + (1-x)(q+p)^2)^2} \\ &= -e^2 \int_{q,x} \frac{(p-q+xp)^2}{(q^2 + (x^2+x)p^2)^2} \\ &= -e^2 \int_{q,x} \frac{(1+x)^2 p^2 + q^2}{(q^2-\Delta)^2}, \qquad \Delta = -(x+x^2)p^2. \end{aligned} \tag{134}$$

Using the integrals in the appendix of P&S, and keeping only the logarithmic part, we get

$$I_1 \sim -\frac{2ie^2}{(4\pi)^2}\ln(-p^2/\Lambda^2). \tag{135}$$

For I_2, we use the Feynman trick with $(A^2B)^{-1} = -\partial_A (AB)^{-1}$ to write

$$I_2 = -e^2 \int_{q,x} \frac{2x(p^2-q^2)^2}{(x(p+q)^2 + (1-x)q^2)^3}. \tag{136}$$

We only care about divergent parts, so keeping only these, and dropping the odd-in-momentum terms that die under the integration,

$$I_2 = -e^2 \int_{q,x} \frac{2x((x^2-2)p^2q^2 + q^4)}{(q^2 + x(1+x)p^2)^3}. \tag{137}$$

Once again we turn to the P&S appendix for the integrals. Keeping only the logarithmic parts, we get

$$I_2 = \frac{ie^2}{4\pi^2} p^2 \ln(-p^2/\Lambda^2), \tag{138}$$

where the value of the numerical prefactor shouldn't be trusted up to a factor of 5 or so.

We now know the wavefunction renormalization γ, at least to one-loop order, since we now know what the δ_Z counterterm for ϕ should be. Using (see P&S chapter 12)

$$\frac{\partial \delta_Z}{\partial \ln M} = 2\gamma, \tag{139}$$

we get

$$\gamma = -\frac{3e^2}{(4\pi)^2}, \tag{140}$$

where again the numerical factor probably shouldn't be trusted since I wasn't being too careful.

Now that we have the wavefunction renormalization we can find β_λ using our knowledge of the effective potential. Just like the connected n-point functions, we can get constraints on the RG flow by evaluating $V_{\text{eff}}(\phi)$ at different renormalization scales. The effective potentials evaluated at two different scales M, M' are related by how the ϕ field scales under RG, and so taking $M' = M + \delta M$, we obtain

$$(M\partial_M + \beta_\lambda\partial_\lambda + \beta_e\partial_e - \gamma\phi\partial_\phi)V_{\text{eff}} = 0 \tag{141}$$

(see chapter 13 of P&S, the minus sign in front of γ is because the effective potential is related to inverse propagators rather than propagators like the free energy). We know the effective potential and β_e, so plugging in and doing some algebra we get

$$\beta_\lambda = \frac{1}{24\pi^2}\left(-18e^2\lambda + 54e^4 + 5\lambda^2\right). \tag{142}$$

In terms of the time $t = -\ln M$ along the RG flow, we have

$$d_t e = -\frac{e^3}{48\pi^2}, \qquad d_t\lambda = -\frac{1}{24\pi^2}\left(-18e^2\lambda + 54e^4 + 5\lambda^2\right). \tag{143}$$

Note that $d_t e$ is always negative and that $d_t\lambda$ has a negative term proportional to e^4, so that we flow to smaller values of the couplings very quickly. Also note that $d_t\lambda < -(9e^2/2 - 2\lambda)^2$, so that $d_t\lambda$ is negative definite and the flow to small λ, e is inevitable.

We already know $e(M)$ from our computation of β_e, and we can get λ for $\lambda \ll e^2$ (a regime to which the RG flow will always pass through) by solving the β function by integrating from M_0 to M:

$$\lambda(M) = e^4(M)\left(\frac{\lambda(M_0)}{e^4(M_0)} + \frac{54}{24\pi^2}\ln(M/M_0)\right). \tag{144}$$

Additionally, from the transformation

$$\phi \mapsto (1 - (\delta M/M))\phi \tag{145}$$

under RG, we get $d\phi = -d\ln M\gamma\phi$ so that

$$\phi(M) = \phi(M_0)\left(\frac{M}{M_0}\right)^{-\gamma}. \tag{146}$$

Using these expressions for e, λ, ϕ at the scale M, we can plug them into V_{eff} to figure out the effective potential at any given scale.

The mass (of the oscillatory mode about the potential minimum) is found by taking the second derivative of V_{eff} with respect to ϕ, and evaluating the result at $\langle\phi\rangle$ (which we will continue to lazily just write as ϕ). To simplify the resulting expression, we choose to evaluate it at the RG scale set by the vev, namely $M^2 = 2e^2\phi^2$. Taking the derivatives then gives

$$m_\sigma^2 = 3\phi^2\left(\frac{2\lambda}{3} - \frac{e^4}{4\pi^2}\right). \tag{147}$$

The value of ϕ is still determined by the same exponential form as in our pre-RG analysis, so that this scale for M, we have

$$\lambda = \frac{9e^4}{8\pi^2}. \tag{148}$$

This lets us write the mass more simply as

$$m_\sigma^2 = \frac{3e^4\phi^2}{2\pi^2}. \tag{149}$$

The gauge boson mass is read off from the Lagrangian as $m_A^2 = 2e^2\phi^2$, and so we determine that the ratio of the masses is

$$\frac{m_\sigma^2}{m_A^2} = \frac{3e^2}{4\pi^2}. \tag{150}$$

Finally, we look what happens if we take $\mu^2 < 0$, i.e. if we give ϕ a positive mass. Classically this would preclude symmetry breaking, but in quantum field theory this is no longer the case, because of the logarithm generated by the fluctuations. We add $m^2\phi^2$ to V_{eff}, find the second derivative of V_{eff} to get m_σ^2 as a function of ϕ, and then minimize V_{eff} to find ϕ in terms of e, λ, m, which then allows us to solve for m_σ^2. Mathematica gives the unilluminating

$$\begin{aligned}\phi &= \sqrt{\frac{2}{3}}\frac{m\pi}{e^2}\frac{2}{\sqrt{\text{ProductLog}(-16m^2\pi^2/(3e^2M^2))}} \\ &= \frac{M}{\sqrt{2}e}\exp\left(\frac{1}{2}\text{ProductLog}\left[\frac{-16m^2\pi^2}{3e^2M^2}\right]\right).\end{aligned} \tag{151}$$

One can check that this reduces to our old result when $m = 0$, and that the symmetry-breaking minimum of the potential disappears at some $m_c^2 \sim M^2e^2 > 0$, so that even with the RG analysis included, there is still symmetry breaking for $m^2 > 0$ where classically there would be none.

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8 β *function for sine-Gordon in 2+1 dimensions at finite temperature and the KT transition for* $U(1)$ *gauge theory* ✓

Today we will try to understand and elaborate on the analysis of the KT transition that occurs in $U(1)$ gauge theory in three dimensions as studied in [4]. I was originally interested in this for higher-symmetry reasons, but we won't really talk about them in what follows.

An outline of this diary entry: first, we spend a few pages motivating why we expect there to be a KT-type transition in the $U(1)$ gauge theory and explaining how we can think of the critical properties of the gauge theory in terms of the XY model in two dimensions.

We then will go through the RG calculation which shows the existence of the KT transition explicitly. This is technically interesting mostly since we will be at finite temperature.

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Confinement phase transitions in gauge theories as regular phase transitions in scalar theories

Recall that the most useful diagnostic of confinement at finite temperature is the Polyakov loop

$$P(x) = \mathrm{Tr} \exp\left(i \int dt A_0(x,t)\right), \tag{152}$$

where we are working in the convention where the gauge coupling appears in the action as $\frac{1}{2g^2} F \wedge \star F$. In what follows we will only be working with $U(1)$ theories, and so we can drop the Tr. If the vev of the Polyakov loop vanishes then the free energy of a single charge is infinite, and the theory confines. On the other hand, if the Polyakov loop is allowed to develop a vev, then we have symmetry breaking and the free energy of an external (non-dynamical) charged source is finite — this is the deconfined phase. Of course, we will be working with pure gauge theory throughout, since adding dynamical matter means that screening occurs, preventing the Polyakov loop from being a useful diagnostic of confinement (since it always has a finite vev after regularization, assuming we add a Wilson loop which is well-defined, i.e. is taken in a legit representation of $U(1)$ so that the charge of the Wilson loop is in $\mathbb{Z}$).

To come up with an effective action for the order parameter $P(x)$, we need to integrate out the other degrees of freedom, namely the spatial components of the gauge fields. We can argue that this procedure will always give us a local theory as follows: at $T \to 0$ we will assume the gauge coupling g^2 is such that the theory confines. This will be the case in the example of interest to us, since $U(1)$ gauge theory is confining at $T = 0$ in two spatial dimensions. Thus the spatial Wilson loops obey an area law, and the correlations between the spatial gauge fields are short-ranged, and so we can integrate them out while keeping the theory local.

At $T \to \infty$, we get what is effectively a zero-temperature version of the gauge theory in one less dimension, at coupling $g_{d-1}^2 = T g_d^2$, plus a field coming from the now-compactified A_0[15]. Since the lower dimensional theory is confining if the $T \to 0$ theory is (going to lower dimensions makes confinement stronger), the spatial gauge fields again have exponentially decaying correlations and can be integrated out. Note that the fact that the spatial gauge fields have area-law decaying correlations does *not* mean that the full theory (confined spatial gauge fields + compactified A_0 degree of freedom) is confining! Confinement is only probed by measuring the free energy of charged sources (captured by $P(x)$), and *not* by measuring the spatial correlations of the gauge fields! Thus at finite T, the connection between Wilson line tension and confinement is only true for certain temporally-oriented Wilson lines. Confinement is diagnosed with temporal Wilson loops in the $T = 0$ theory, which become things

[15] The coupling comes from setting a gauge in which A_0 is constant in time, and then doing the integral along the thermal circle. The relevant term in the action is then $(\beta/2g^2)\int \nabla A_0 \cdot \nabla A_0$, and so the effective coupling is $g_{\text{eff}} = \sqrt{T} g$.

related to the compactified A_0 field in the $T \to \infty$ theory—they are in principal very different objects than the spatial Wilson loops, which do not in this case carry any information about the energetics of charge sources.

Anyway, as was done in [4], we can then conjecture that at all temperatures, we can always integrate out the spatial gauge fields, producing an effective action for the Polyakov loop $P(x)$. Thus this theory can actually be described by a scalar field spin model in $d-1$ dimensions. In particular, it can be described with a global 0-form symmetry equal to the 1-form symmetry of the Polyakov loops in the gauge theory (namely, the center of the gauge group). For $U(1)$ gauge theory the dual spin model is thus an XY model, while for $\mathbb{Z}_N$ gauge theory (obtained by Higgsing a charge N scalar with the term $\lambda \cos(\partial_\mu \theta - iNA)$, taking λ to be large, and working in unitary gauge) we get an XY model with cosine-type interactions.

Let's look at the high-temperature and low-temperature limits of the gauge theory and see how they behave in the spin model. At $T \to 0$ we are in a confining phase, with a linear confining potential between free charges. Thus a two-point function of Polyakov loops goes as

$$\langle P(x) P^*(0) \rangle \sim e^{-\alpha|x|/T}, \tag{153}$$

where α is the string tension — this is an area law. This then corresponds to the disordered phase of the associated spin model, where the spin model correlation length is read off to be

$$\xi = \frac{T}{\alpha}. \tag{154}$$

So, we should think of the confining phase as the *disordered* phase.

Now we take $T \to \infty$. Generically we would expect the gauge theory to become deconfined—let's see why we would expect this. As $T \to \infty$, the thermal circle shrinks to become very small. Intuitively this means that nonzero momentum modes in the compactified direction all have very high energy, and so we can focus on field configurations that are constant in time. This means that the value of the Polyakov loop is approximately constant throughout space, meaning that the Polyakov loops have long range correlation functions, implying SSB of the 1-form symmetry and deconfinement.

More precisely, we can see this by thinking about the Wilson action for gauge theory, which looks something like

$$S \sim \sum_{\text{plaquettes}} \left(\beta_t E^2 + \beta_s B^2 \right), \tag{155}$$

where the E^2 and B^2 terms are the electric and magnetic fields at the plaquettes, realized by taking the product of Wilson lines around the edges of the (temporal and spatial, respectively) plaquettes. The ratio $\sqrt{\beta_t/\beta_s}$ represents the ratio of the spatial lattice spacing to the temporal (thermal circle) lattice spacing, and so fixing the number of lattice sites, we see that when we take T to be large, $\sqrt{\beta_t/\beta_s} \gg 1$. This means that at high T, configurations with nonzero electric flux (a definite value of the 1-form charge) are suppressed: this is deconfinement, and the electric field is "screened" (although again, we don't actually have dynamic charges).

Now consider two Polyakov loops separated by one spatial lattice spacing: $P(x)$ and $P(x + \hat{i})$. We can write

$$P(x) = \exp\left(i \int_{\text{strip}} F \right) P(x + \hat{i}), \tag{156}$$

where the integral is over the thin cylinder enclosed by the two loops. Since F on this cylinder is the electric field strength and the electric field gets frozen out at high T, we have $P(x) \approx P(x + \hat{i})$, and thus the Polyakov loop must be slowly varying in space. This means that the free energy of two external sources separated by a distance x does not grow linearly with their separation: instead, we have the scaling

$$\langle P(x)P^*(0)\rangle \sim \text{const.} \tag{157}$$

Thus at high T we are in the deconfined phase, and the system can screen charges. In the spin model, we see that this corresponds to symmetry breaking, and so the deconfined phase maps onto the ordered phase of the spin model.

It's worth emphasizing that the *high-temperature* phase of the gauge theory corresponds to the *ordered* phase of the spin model. Normally we think of going to higher temperatures as restoring symmetry, but with this example it's the opposite (and indeed, higher-form symmetries like this generically prefer to be broken at high temperature rather than low temperature). Zohar and co. have some nice recent papers emphasizing this perspective.

However, this symmetry-breaking argument is only true in large enough dimensions. Indeed, we have argued that in the case of $U(1)$ gauge theory in 2+1D at high T (which is what we will be most interested in), we can map the problem onto an XY model in two dimensions (zero temperature), which cannot actually have a symmetry-breaking phase by virtue of the Mermin-Wagner theorem. This means that at high T, this theory is actually never really deconfined at high temperature. Instead of a constant two-point function, we get

$$\langle P(x)P^*(0)\rangle \sim \frac{1}{|x|^\eta}, \tag{158}$$

where η is some constant that depends on the gauge coupling. We might call this "quasi-long-ranged-deconfinement", but we do not have a genuinely deconfined phase at high T. Neither is it deconfined at low temperature because of the effects of monopoles a la Polyakov (this is also true in the continuum without monopoles), and so actually we never have a genuine deconfined phase. However, based on our experience with the XY model, we can guess that there will be some sort of KT-type transition, where the theory always is confined but the characteristics of how it is confined change with the coupling (or with T). This is corroborated by looking at how the two-point function of the Polyakov loop scales: it goes from an exponential decay at low T to a power-law at high T, and thus there must be some kind of phase transition in between. We have good reason to expect that it will be a KT transition, because of our experience with the XY model.

Getting to the sine-Gordon action

First we want to get to a sine-Gordon action starting from the gauge theory, which will make the RG analysis easier (this is standard stuff; see e.g. Altland + Simons for the next few paragraphs). The Polyakov loops, which let us build intuition for what to expect, will actually not play a role in what follows: the dual field will still be a scalar, but it will be related to A through Hodge duality as $d\phi \sim \star F$, and not by tracing A over a circle.

We start from the usual $U(1)$ gauge theory action at coupling g, and then dualize to a compact scalar field ϕ in the usual way. We will be working on a lattice, but will use continuum notation — hopefully this will not be unduly confusing.

If we are working on a manifold with trivial first cohomology[16] and the gauge bundle is trivial, then the action in terms of the scalar ϕ is

$$S = -\frac{g^2}{2(2\pi)^2}\int d\phi \wedge \star d\phi + \int \phi \wedge q. \tag{159}$$

Here q parametrizes the locations of the instantons (which we will refer to as monopoles since we are in 2+1D): it is a three-form defined on the cubes of the spacetime lattice and is equal to

$$q = \sigma(x) d^3 x, \tag{160}$$

where $\sigma(x)$ is the monopole charge at each cube. Thus we have a situation quite similar to the XY model:

$$\text{spin waves and vortices} \sim \text{gauge fields and monopoles} \tag{161}$$

To get the partition function, we just need to know how to sum over the configurations of monopole charges. We can sum over all possible charge configurations and assume that only charge $\sigma = \pm 1$ monopoles contribute (valid in the limit where the monopole fugacity, which we have to set by hand, is small) to get

$$Z = \left\langle \sum_{N=1}^{\infty} \sum_{\{\sigma_j\}\in\mathbb{Z}_2^N} y^N \frac{1}{(N/2)!^2} \binom{N}{N/2}^{-1} \prod_{i=1}^{N} \int d^D r_i \exp\left(i \sum_{j=1}^{N} \sigma_j \phi(r_j) \right) \right\rangle \tag{162}$$

where the expectation value is taken with respect to the free ϕ action and we've defined the dimensionfull variable

$$y \equiv \frac{1}{a^D} e^{-\mu}, \tag{163}$$

with $e^{-\mu}$ the monopole fugacity and a a short-distance cutoff. Some comments are in order: first, the expectation value will vanish for any configurations that have non-zero net charge. Neutral charge configurations will have $N/2$ positive and negative charges. Since the positive charges are indistinct from one another (as are the negative charges), we are required to put in the $[(N/2)!]^{-2}$ term. Furthermore, we need to mod out by the number of ways to partition N charges into a neutral configuration, which is the reason for the $\binom{N}{N/2}$ factor. Expanding out the exponential and then re-exponentiating, we get the sine-Gordon action

$$S = \int d^D x \left(\frac{\alpha}{2} d\phi \wedge \star d\phi - 2y \cos\phi \right), \tag{164}$$

[16]This won't be true for us since we'll be on a thermal cylinder. However, the zero modes that we would have to account for actually get killed off by a choice of boundary conditions on the ends of the cylinder at spatial infinity (which we must set non-symmetrically since we are interested in SSB), and so we don't have to care about them in what follows.

where we've defined

$$\alpha \equiv (g/2\pi)^2. \tag{165}$$

Spatial RG procedure

Having obtained the sine-Gordon model, we now want to do an RG analysis and see whether we get a BKT-like RG flow. This is different from the standard analysis since we will work in arbitrary dimensions (for now), and since we're at finite temperature. Because of non-zero temperature, we will do RG in space, but not in the thermal direction. This anisotropic RG procedure is a bit awkward, but lets us capture the physics of the RG flow correctly.

We will follow [4] pretty closely for now. As usual, we split up ϕ as

$$\phi = \varphi + \chi, \tag{166}$$

where φ is the low-momentum ($p < \Lambda'$) part and χ is the high-momentum ($\Lambda' < p < \Lambda$) part. The $(\partial\phi)^2$ part splits into $(\partial\varphi)^2 + (\partial\chi)^2$ since $(\partial\varphi)^2$ is block diagonal in momentum space, and so

$$Z' = \int D\varphi \exp\left(-\frac{\alpha}{2}\int \partial_\mu\varphi\partial^\mu\varphi\right) Z'[\varphi], \tag{167}$$

where

$$Z'[\varphi] = \int D\chi \exp\left(-\frac{\alpha}{2}\int \partial_\mu\chi\partial^\mu\chi\right) \exp\left(2y\int \cos(\varphi+\chi)\right). \tag{168}$$

The strategy now is to work at weak coupling, where the monopole fugacity y is small (when the field is fluctuating slowly, adding in a monopole is costly since one has to twist the value of the field at a large number of lattice sites, while while the field is rapidly fluctuating, this twisting is sort of already going on because of the fluctuations). Now, we know that $\ln Z[J]$ is the generating function for correlation functions with disconnected pieces subtracted. For example, we schematically have

$$\left.\frac{\delta^2}{\delta J(x)\delta J(y)}\right|_{J=0} \ln Z[J] = \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle, \tag{169}$$

where $Z[J]$ includes an $\exp(\int \phi J)$ coupling. Taking n variational derivatives of $\ln Z$ produces n-point correlation functions with their disconnected pieces subtracted off. This allows us to easily write down a series expansion for $\ln Z'$ in terms of y, which appears as a current for the $\int \cos$ term. Thus

$$\ln Z' = 1 + 2y\int d^dx\langle\cos\phi\rangle + \frac{(2y)^2}{2}\int d^dx\, d^dy\ \left(\langle\cos\varphi(x)\cos\varphi(y)\rangle - \langle\cos\varphi(x)\rangle\langle\cos\varphi(y)\rangle\right) + \dots, \tag{170}$$

where the expectation values are taken with respect to the free χ action.

To compute the expectation value $\langle\cos\rangle$, we have to compute $\langle e^{\pm i\chi(x)}\rangle$. We can do this by completing the square in the action in the usual way. Since the expectation value is independent of x, we get

$$\langle\cos(\varphi+\chi)\rangle = \exp\left(-\frac{1}{2}D(0)\right)\cos\varphi. \tag{171}$$

where D is the scalar propagator for χ, $D(x-z) = \langle\chi(x)\chi(y)\rangle$. Note that in our conventions D is dimensionless, and so $D \neq \nabla^{-2}$. Instead,

$$D(x-z) = \frac{1}{2\alpha}\nabla^{-2}(x-z). \tag{172}$$

Since the mass dimension of α is $[\alpha] = D-2$, the dimension of ∇^{-2} must also be $D-2$. This works out since we want $\nabla_x^2\nabla^{-2}(x-z) = \delta(x-z)$ to be a delta function we can integrate on a D-manifold, and so the mass dimension of the delta function must be D. Also note that D is the propagator for χ, which lives at high momentum. So when defining the real-space D, we have to define it as an integral of $D(k)$ only over a momentum shell $\Lambda' < k < \Lambda$.

When we do the mixed expectation values $\langle\cos\cos\rangle$, things are only marginally more complicated: now we need to compute $\langle\exp(i[\chi(x)\pm\chi(z)])\rangle$. We still do this by completing the square, but now we get a factor like $D(x-z)$ instead of $D(0)$, so that

$$\langle\exp(i[\chi(x)\pm\chi(y)])\rangle = \exp\left(-D(0)\mp D(x-z)\right). \tag{173}$$

Putting this in and using $\cos a\cos b = (\cos(a+b)+\cos(a-b))/2$, the order y^2 term in $\ln Z'$ is

$$\int d^Dx\, d^Dz\, y^2 e^{-D(0)}\left[(e^{-D(x-z)}-1)\cos(\varphi(x)+\varphi(z)) + (e^{D(x-z)}-1)\cos(\varphi(x)-\varphi(z))\right]. \tag{174}$$

Now we need to make some approximations to make this more tractable. If we are in a big enough dimension[17], $e^{\pm D(r)}-1 \to 0$ for large r[18]. So we may be able to get away with an expansion in small $|x-z|$. So define the relative coordinate r by

$$\varphi(z) = \varphi(x) + \partial_\mu\varphi(x)r^\mu, \tag{175}$$

and substitute this in (this isn't so unreasonable—the higher-order terms are all irrelevant anyway). This generates two terms, one of which goes like $\cos(2\varphi(x)+\partial_\mu\varphi(x)r^\mu)$ and thus will be irrelevant compared to $\cos(\varphi(x))$ (and so we will drop it). The other term we expand in small r^μ, and after dropping an (infinite) φ-independent constant and integrating by parts, we get

$$\int -d^Dx\, d^Dr\, y^2 e^{-D(0)}(e^{D(x-z)}-1)(\partial_\mu\varphi(x)r^\mu)^2. \tag{176}$$

[17] For us, this big enough dimension is three: one might worry about three becoming two after compactification at high temperature, but we will be assuming that the monopole fugacity is small enough that the average separation of monopoles is much larger than the length scale set by the thermal circle, so that for the purposes of figuring out the behavior of $D(r)$, we can think of working in $\mathbb{R}^3$.

[18] $D(r)$ is proportional to $\delta\Lambda/\Lambda$, but the possible divergence we need to worry about is the one for $r\Lambda \gg \delta\Lambda/\Lambda$. In 2d we'd get a log, which would diverge at sufficiently large r.

Putting this into the full action, we obtain

$$Z = \int D\varphi \exp\left(\frac{\alpha}{2}\int d^Dx\,\varphi\partial_\mu K^{\mu\nu}\partial_\nu\varphi + 2ye^{-\frac{1}{2}D(0)}\int d^Dx\,\cos\varphi\right), \tag{177}$$

where

$$K^{\mu\nu} = \delta^{\mu\nu} + \frac{y^2e^{-D(0)}}{\alpha}\int d^Dr\,(e^{D(r)}-1)r^\mu r^\nu. \tag{178}$$

Note that K is diagonal. We should now re-scale so that the kinetic term has the same form as before we did the momentum-shell decomposition. Since $K^x \equiv K^{xx} = K^{yy} \neq K^t \equiv K^{tt}$, the scaling is anisotropic. This is of course expected since we are at finite T, and the propagator D is anisotropic. So define

$$\phi = \sqrt{K^x}\varphi \tag{179}$$

and then scale the momentum as $p' = \Lambda p/\Lambda'$ so that Λ is again made the momentum cutoff. When we scale the momentum, we are only scaling the *spatial* momentum, and leaving the temperature fixed. This produces

$$Z = \int D\phi \exp\left(\frac{\alpha}{2}\int d^Dx\,\phi\left(\partial_i\partial^i + \gamma^2\partial_t^2\right)\phi + 2ye^{-\frac{1}{2}D(0)}\int d^Dx\,\cos(\phi/\sqrt{K^x})\right), \tag{180}$$

where

$$\gamma^2 = (\Lambda/\Lambda')^2K_t/K_x. \tag{181}$$

Note that the mass dimension of y is $[y] = D$, and so the transformation of y cancels out the transformation of the integration measure in the second term. Also note that our RG step has changed the form of the $\cos\phi$ mass term, since there is now a factor of $1/\sqrt{K_x}$ inside the cosine. This means that we should go back to the original action and instead write the term as

$$\cos(\phi) \mapsto \cos(\sigma\phi), \tag{182}$$

and then examine the RG flow of the dimensionless variable σ. This is the same as looking at the RG flow for the radius of the scalar.

These transformations change the form of the χ propagator D, which is the same as the propagator for ϕ but integrated over a different range of momenta. Namely the coefficient of the time derivative part changes, since we are only scaling in space and not time. Thus we need to go back to our original action and instead write

$$L = \frac{\alpha}{2}\phi(\partial_i\partial^i + \gamma^2\partial_t\partial^t)\phi, \tag{183}$$

and then examine how γ flows under RG. This is essentially an anisotropic field strength renormalization procedure. The correct propagator for the high-momentum modes is thus (note the prefactor!)

$$D(x) = \frac{1}{2\alpha}\int_{\Lambda'}^{\Lambda}\frac{d^dp}{(2\pi)^d}T\sum_{n\in\mathbb{Z}}\frac{e^{ip\cdot x+i2\pi Tnt}}{p^2+(\gamma 2\pi Tn)^2}. \tag{184}$$

Getting the β functions

So, all told we have three things that we need to keep track of under RG: the monopole fugacity (alias y), the period of the mass term (alias σ, alias the compactification radius for the boson), and the "effective temperature" or anisotropy between space and the thermal circle (alias γ; T is held fixed).

To do the RG, we let $\Lambda' = \Lambda - \delta\Lambda$ for small $\delta\Lambda$. Then for $dD = D|_{\Lambda'=\Lambda-\delta\Lambda} - D|_{\Lambda'=\Lambda}$, we get

$$dD(x,t) = \frac{1}{2\pi}\delta\Lambda\frac{\Lambda}{2\alpha(2\pi)^2T\gamma^2}J_0(\Lambda x)\sum_n\frac{e^{i2\pi nTt}}{(\Lambda/2\pi T\gamma)^2+n^2} \tag{185}$$

in $d=2$ spatial dimensions and

$$dD(x,t) = \frac{1}{(2\pi)^2}\delta\Lambda\frac{\Lambda^2}{2\alpha(2\pi)^2T\gamma^2}\frac{\sin\Lambda x}{\Lambda x}\sum_n\frac{e^{i2\pi nTt}}{(\Lambda/2\pi T\gamma)^2+n^2} \tag{186}$$

in $d=3$.

Now we use the summation

$$\sum_{n\in\mathbb{Z}}\frac{e^{ina}}{n^2+b^2} = \frac{\pi}{b}\frac{\sinh(ab)+\sinh[(2\pi-a)b]}{\cosh(2\pi b)-1}, \tag{187}$$

which is valid for $0\leq a\leq 2\pi$ and arbitrary b (which holds for us since t is runs from 0 to $1/T$ and we are using $a=2\pi Tt$).

This gives (with $b=\Lambda/2\pi T\gamma$)

$$dD(x,t) = \frac{1}{8\pi\alpha}\frac{\delta\Lambda}{\gamma}J_0(\Lambda x)\frac{\sinh(\Lambda t/\gamma)+\sinh[(\Lambda/T-\Lambda t)/\gamma]}{\cosh(\Lambda/T\gamma)-1} \tag{188}$$

in two spatial dimensions. We can simplify this using

$$\frac{\sinh x+\sinh(y-x)}{\cosh y-1} = \frac{e^x+e^{-x+y}}{e^y-1}, \tag{189}$$

so that

$$dD(x,t) = \frac{\delta\Lambda}{8\pi\alpha\gamma}J_0(\Lambda x)\frac{e^{\Lambda t/\gamma}+e^{-\Lambda(t-1/T)/\gamma}}{e^{\Lambda/\gamma T}-1}. \tag{190}$$

Likewise in three dimensions,

$$dD(x,t) = \frac{\delta\Lambda\Lambda}{16\pi^2\alpha\gamma}\frac{\sin(\Lambda x)}{\Lambda x}\frac{e^{\Lambda t/\gamma}+e^{-\Lambda(t-1/T)/\gamma}}{e^{\Lambda/\gamma T}-1}. \tag{191}$$

To get the RG equations, we need to compute dK^x, $d\gamma$, and dy. In two dimensions, the first is (one α^{-1} from the definition of $K^{\mu\nu}$ and one from the propagator)

$$dK^x = \frac{y^2}{\alpha}\int d^2r dD(r)x^2 = \frac{\delta\Lambda}{4\alpha^2}y^2\Lambda^{-5}I_3, \qquad I_n\equiv\int dr\, r^nJ_0(r), \tag{192}$$

while the second is obtained from

$$d\gamma = \frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x). \tag{193}$$

and by using

$$dK^t = \delta\Lambda \frac{y^2\gamma^2 I_1}{4\alpha^2\Lambda^5}\left(4 + (\Lambda/T\gamma)^2 - \frac{2\Lambda}{T\gamma}\coth(\Lambda/2T\gamma)\right). \tag{194}$$

Getting dy is easy: we see that $2y$ is replaced by $2ye^{-\frac{1}{2}D(0)}$ under rescaling, so that under an infinitesimal rescaling, we have

$$dy = -\frac{1}{2}dD(0). \tag{195}$$

In three dimensions, we have the similar

$$dK^x = \frac{\delta\Lambda}{16\alpha^2}y^2\Lambda^{-5}\widetilde{I}_3, \qquad \widetilde{I}_n \equiv \int dr\, r^n \sin r \tag{196}$$

and

$$dK^t = \delta\Lambda \frac{y^2\gamma^2 \widetilde{I}_1}{4\pi\alpha^2\Lambda^5}\left(4 + (\Lambda/T\gamma)^2 - \frac{2\Lambda}{T\gamma}\coth(\Lambda/2T\gamma)\right). \tag{197}$$

$d\gamma$ is essentially the same:

$$d\gamma = \frac{3}{2}\frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x). \tag{198}$$

Of course, the I integrals are infinite, but this won't be too much of a problem (this came from the fact that we're doing RG with a hard momentum cutoff instead of something better like using the CS equation).

RG: two spatial dimensions

Our dimensionless RG parameters in two spatial dimensions are (recall that the mass dimension of y is the dimension of spacetime)

$$b \equiv \sigma\frac{T}{4\pi^2\alpha} = \frac{T}{g^2}, \qquad \tau \equiv \frac{\Lambda}{T\gamma}, \qquad m \equiv \frac{2y}{\Lambda^2 T}. \tag{199}$$

The definition of τ is used just because the ratio $\Lambda/T\gamma$ is common, and we have defined the dimensionless m (for "monopole") which is related to the instanton fugacity. The parameter $b = T/g^2$ is the coefficient that appears in front of E^2 in the gauge theory Hamiltonian and so b^{-1} gives the "temporal coupling" of the gauge theory. When b becomes very large the electric field gets frozen out and pinned to a constant value throughout space, since as the radius of the thermal circle becomes small, non-zero-mode fluctuations of the electric flux are suppressed.

Now

$$d\gamma = \frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x), \tag{200}$$

and so

$$\begin{aligned} d\ln\tau &= 2d\ln\Lambda - \frac{1}{2}(dK^t - dK^x) \\ &= d\ln\Lambda\left(2 + b^2m^2 f(\tau,\gamma)\right), \end{aligned} \tag{201}$$

where $f(\tau,\gamma)$ is a function which as $\tau \to 0$ goes to $-I_3$ (a constant).

From our expression for dK^x we also get

$$d\ln b = d\ln\Lambda\frac{y^2 I_3}{8\Lambda^4\alpha^2} = d\ln\Lambda\frac{m^2b^2\pi^4 I_3}{2}. \tag{202}$$

Finally, for dm we have

$$\begin{aligned} d\ln m &= -2d\ln\Lambda + \frac{d\Lambda}{16\pi\alpha\gamma}\coth(\Lambda/2\gamma T) \\ &= d\ln\Lambda\left(\frac{1}{4}\tau\pi b\coth(\tau/2) - 2\right), \end{aligned} \tag{203}$$

where the first term comes from dy and the second from the factor of Λ^{-2} in the definition of m and we used

$$dD(0,0) = \frac{\delta\Lambda}{8\pi\alpha\gamma}\coth(\Lambda/2\gamma T). \tag{204}$$

Absorbing the some unsightly prefactors by scaling $m \to m\pi^2\sqrt{I_3/2}$ and defining $dt = -d\ln\Lambda$ as time along the RG flow with $t = -\infty$ when no momenta have been integrated out and $t = \infty$ when all momenta have been integrated out, we can rewrite the RG equations as

$$\begin{aligned} db &= -m^2b^3dt, \\ dm &= -m\left(\frac{\pi}{4}b\tau\coth(\tau/2) - 2\right)dt \\ d\tau &= -\tau dt\left(2 - m^2b^2\widetilde{f}(\tau,\gamma)\right), \end{aligned} \tag{205}$$

where $\widetilde{f}(\tau,\gamma)$ is a rather gross function that at $\tau \to 0$ goes to 1[19]. For small enough b^2m^2, $\ln\tau$ monotonically decreases along the flow, so that $\tau \to 0$ as $t \to \infty$. This is intuitive: if we zoom out of the cylinder, the cylinder becomes "longer" (we see more of the cylinder), but since we are holding the radius of the cricle fixed, it effectively becomes "skinnier", which means high "temperature", which means low τ.

Now it is but a hop and a skip to KT! In fact, we essentially already have the KT RG equations, just with an extra parameter τ related to the fact that we're at finite temperature. However, if we are at weak enough coupling, and are just limited in the deep IR, we can take $\tau \to 0$. Heuristically, we can think that as we zoom out in real space, the aspect ratio of the cylinder changes and the thermal circle appears to shrink, thus taking us to higher temperatures (recall that $\tau \to 0$ as $T \to \infty$). That this happens is not a forgone conclusion because of the fact that γ scales under RG as well, but at weak coupling this turns out to

[19]For posterity's sake, $\widetilde{f}(\tau,\gamma) = 1 - I_3^{-1}\gamma^2(4 + \tau^2 - 2\tau\coth(\tau/2))$.

be the case. Thus we may send $\tau \to 0$ and write

$$\begin{aligned} \frac{db}{dt} &= -m^2 b^3, \\ \frac{dm}{dt} &= -m\left(\frac{\pi}{2}b - 2\right) \end{aligned} \tag{206}$$

as our RG equations. This is essentially the same as in the regular KT analysis, expect that b appears in the first equation as b^3 instead of b^2 and some of the numerical coefficients are different by various factors of π and 2. The difference in the numerical factors doesn't affect the existence of a KT transition, and the differing power of b only changes the transition quantitatively, not qualitatively (it changes the time that trajectories spend near the $y = 0$ fixed point).

To analyze our version of the KT equations, we measure the distance to the critical point with the variable x:

$$b = \frac{4}{\pi}(1 - x). \tag{207}$$

To lowest order in x, then,

$$\frac{dx}{dt} \approx \frac{16}{\pi^2}m^2, \qquad \frac{dm}{dt} \approx 2mx. \tag{208}$$

The fact that we have b^3 instead of b^2 only appears at order x^3, which is irrelevant for our discussion. We then use the above to write

$$\frac{dx^2}{dm^2} = \frac{8}{\pi^2} \implies m^2 = \frac{\pi^2}{8}(x^2 + C), \tag{209}$$

which is the familiar KT hyperbola (the only difference from regular KT is the number on the RHS). The integration constant C measures the distance from the critical trajectory, since when $C = 0$ we get the critical line. Using the RG equation for x, we see that

$$\frac{dx}{dt} = 2(x^2 + C), \tag{210}$$

which is exactly the same as the KT result. Since x is monotonically increasing, the system continues to get floppier and floppier (smaller spin stiffness) as we go to larger distances, since monopoles become more and more important at large distances. Proceeding from here, one can find the non-analyticities signifying the phase transition in the usual way. ethan: *work out the RG trajectories for 3D*

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9 β *function for scalars coupled to a non-Abelian gauge field* ✓

This is from P&S, chapter 16. Consider a non-Abelian gauge theory coupled to a scalar field:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}[F \wedge \star F] - (D_\mu\phi)^\dagger D_\mu\phi - V(\phi), \tag{211}$$

with the sign convention $D_\mu = \partial_\mu - iA^a_\mu t^a$. Here t is taken in the adjoint representation when acting on A and is taken in some representation r when acting on ϕ. We will be writing down the Feynman rules and computing the β function for g to lowest order. To speed up things, we will use the fact that for Yang-Mills with fermions, the β function is

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{n_f}{2}C(r)\right), \tag{212}$$

where $t^at^a = C_2(r)\mathbf{1}$ for the representation r (with $r = G$ indicating the adjoint) and $\mathrm{Tr}[t^at^b] = C(r)$.

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First let's get the Feynman rules. We can just do this by inspection: we have the usual 3-point and 4-point gross gauge boson vertices from the kinetic term, plus the $(D_\mu\phi)^\dagger D_\mu\phi$ term. The relevant parts of this are

$$g^2 A^a_\mu A^\mu_a \phi^\dagger\phi, \quad igA^\mu(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger). \tag{213}$$

The former term gives us a two-scalar-two-A interaction vertex. Since there are two different ways of attaching the gauge boson propagators onto the vertex, this vertex produces the Feynman rule

$$A^aA^b\phi^\dagger\phi \to ig^2\{t^a, t^b\}. \tag{214}$$

For the trivalent vertex, we fix our conventions so that the derivative acts on the incoming ϕ propagator with momentum p, so that

$$A^a_\mu\phi^\dagger\phi \to -ig(p+p')_\mu t^a. \tag{215}$$

To get β_g, we will use the CS equation on the gauge boson + two-scalars vertex (the three-point function between a gauge boson and two scalars). To do this, we need to know the counterterm δ_3 for this vertex, as well as the wavefunction renormalization counterterms δ_ϕ, δ_e for the scalar and the photon. Then varying the three-point function with respect to the RG scale, we get (to lowest order in g)

$$2\gamma_\phi + \gamma_A + \partial_{\ln M}\left(-i\delta_g - ig\sum_i(\cdots - \delta_{Z_i})\right) - i\beta_g = 0, \tag{216}$$

where the $\ldots$ are independent of the RG scale, the sum is over the external legs and comes from the external propagator corrections, and δ_g is the counterterm for the vertex we are considering. By varying the two-point functions for the gauge field and the scalar with respect to M, we get (as usual to lowest order)

$$\gamma_i = \frac{1}{2}\partial_{\ln M}\delta_{Z_i}, \tag{217}$$

so that, if we denote $g\delta_v = \delta_g$ for the vertex counterterm,

$$\beta_g = g\partial_{\ln M}(-\delta_1 + \delta_\phi + \delta_A/2). \tag{218}$$

In Abelian gauge theory the first two terms would have canceled for Ward identity reasons, but now they do not. Instead, they provide some gauge-variant divergent part which cancels with a gauge-variant divergent part of the δ_A correction for the gauge field (coming from ghosts and the gauge field self-interaction). The gauge-variant part of δ_A doesn't depend on the matter content, meaning that $-\delta_1 + \delta_\phi$ actually does not depend on the matter content either! See P&S chapter 16.5 for some more discussion about this.

This means that the only dependence of the β function on the chosen matter fields comes from their contribution to δ_A, which is very easy to compute since we only have to do a single polarization bubble integral (the $\phi^2 A^2$ vertex diagram doesn't contribute to the β function since it is momentum-independent). The diagram is (we can ignore a potential mass for the scalar since it can't contribute to the β function by dimensional analysis)

$$(\text{polarization bubble})^{\mu\nu}(q) = -\text{Tr}[t^a t^b]g^2 \int_p \frac{(2p+q)^\mu(2p+q)^\nu}{p^2(p+q)^2}. \tag{219}$$

We know the form of the bubble will have a momentum dependence dictated by being properly transverse, so we can just compute the coefficient of e.g. the q^2 piece. This gives

$$C(r)g^2\delta^{ab}\int_p \frac{4p^\mu p^\nu}{(p^2-\Delta)^2} = \frac{1}{3}\frac{g^2 C(r) g^{\mu\nu}\delta^{ab}}{(4\pi)^2} q^2 \ln(-q^2/\Lambda^2). \tag{220}$$

This thus determines the counterterm for the wavefunction renormalization of A to be

$$\delta_A = \frac{1}{3}\frac{g^2 C(r)}{(4\pi)^2}\ln(M^2/\Lambda^2). \tag{221}$$

Its contribute to the beta function is thus

$$\beta_g = \frac{1}{2}g\partial_{\ln M}\delta_A = \frac{g^3 C(r)}{3(4\pi)^2}. \tag{222}$$

Now the only difference between the β function for the theory coupled to fermions and one coupled to scalars is the different values they contribute to δ_A by way of the polarization bubble. Both bubbles have the same color trace, but the fermions have an extra trace over spin indices. This ends up contributing an extra factor of 4 (see similar calculations done earlier), and so fermions in a representation t^a contribute four times as much to β_g as scalars in the same representation do. Thus we only have to turn to P&S chapter 16 and observe that the β function for fermions coupled to a non-Abelian gauge field is

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right). \tag{223}$$

Thus we deduce that if the field is coupled to scalars instead, the beta function is

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}n_f C(r)\right). \tag{224}$$

Another method to calculate the β function is the background field method, where we integrate out fluctuations of the gauge field and express the effective action in terms of a new effective charge, which is more cmt-y. This is essentially in P&S, so I won't write it out here.

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10 *Anomalous dimension in ϕ^4 theory* ✓

Today is short and simple: we will calculate the anomalous dimension of the operator ϕ^2 in ϕ^4 theory (in four spacetime dimensions). Recall that the anomalous dimension of an operator $\mathcal{O}$ is defined by

$$d_{\ln\mu}\mathcal{O} = \gamma_{\mathcal{O}}\mathcal{O}. \tag{225}$$

We will calculate γ_{ϕ^2} to first order in λ.

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To find the anomalous dimension, we can insert the term $\int J\phi^2$ into the Lagrangian. To one loop order, J gets corrected by a ϕ^4 vertex, with the relevant graph looking like a fish. If we take J at momentum k, then the two (incoming) parts of the tail of the fish will have momenta p and $-k-p$, with the two segments of the body of the fish having momenta q and $k-q$ (we take the k momentum of the current J to be "incoming" with respect to the head of the fish). There is a symmetry factor of 2 associated with the number of ways to order the lines on the tail of the fish, and so the fish diagram is

$$\text{fish} = \frac{i^2\lambda}{2}\int_q \frac{1}{(q^2-m^2)((k-q)^2-m^2)}. \tag{226}$$

We could do Feynman parameters and dim reg, but we care only about the coefficient of the large logarithm, so we can also just directly select out the UV divergent part and do the integration: the surface area of an S^3 is $2\pi^2 R^4$, so

$$\text{fish} \sim -2\lambda\frac{\pi^2}{32\pi^4}\ln(\Lambda/\Delta), \tag{227}$$

where Δ is an invariant built out of the external momenta and the mass.

We thus renormalize the current by writing

$$J_b = J_R Z_J = J_R(1+\delta_J), \tag{228}$$

where

$$\delta_J = \frac{\lambda}{16\pi^2}\ln(\Lambda/\mu), \tag{229}$$

where μ is the RG scale at which we want the one-loop correction to vanish. Since J_b is independent of μ,

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu}(J_R(1+\delta_J)) \\ &= \frac{dJ_R}{d\ln\mu}(1+\delta_J) + J_R\frac{d}{d\ln\mu}\left(\frac{\lambda}{16\pi^2}\ln(\Lambda/\mu)\right). \end{aligned} \tag{230}$$

Which means that

$$0 = \gamma_J\left(1+\frac{\lambda}{16\pi^2}\ln(\Lambda/\mu)\right) + \frac{\beta_\lambda}{16\pi^2}\ln(\Lambda/\mu) - \frac{\lambda}{16\pi^2}. \tag{231}$$

Now β_λ is order λ^2, since it is computed by looking at the corrections to the ϕ^4 vertex which is marginal at the free fixed point. γ_J is also first order in λ, and so to first order we have $\gamma_J = \frac{\lambda}{16\pi^2}$. The anomalous dimension of ϕ^2 is the negative of this, so

$$\gamma_{\phi^2} = -\frac{\lambda}{16\pi^2}. \tag{232}$$

Alternately, we could have obtained this by calculating $G_2(\phi(x),\phi(x))$ in an RPA-like way, by summing diagrams that look like chains with all possible number of beads on them (each bead is a ϕ^4 bubble attached to the chain at a single point). At the renormalization scale μ, we would sum the series and get

$$G_2 \sim G_{2,0}\frac{1}{1-\frac{\lambda}{16\pi^2}\ln(\Lambda/\mu)}. \tag{233}$$

Taking the derivative wrt $\ln\mu$ and working to first order in λ reproduces the correct result. Thus the RG reproduces an infinite number of diagrams (subleading logs) in the usual way. In this ϕ^4 example directly summing the infinite number of diagrams is easier than calculating the beta functions, but in more complicated contexts the RG approach is much easier.

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11 *RG and the nonlinear σ model on a quotient space* ✓

This is a slight elaboration on a problem from Altland and Simons. Consider the nlσm in two dimensions, viz.

$$S = \frac{1}{\lambda}\int d^2x\, \mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}]. \tag{234}$$

We will be considering a situation of a sigma model which describes an EFT for a phase in which an $SU(2)$ symmetry is spontaneously broken to a $U(1)$ subgroup, with the fields g being valued in the quotient space $SU(2)/U(1) \sim S^2$ (which of course is not a group, despite the use of g—sorry). The goal is to compute the beta function β_λ.

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We will find the RG flow for λ by using the background field method. This proceeds by breaking up g as $g = g_s g_f$ into "slow" and "fast" degrees of freedom and integrating out g_f in the usual Polykovian way.

First let us do a bit of massaging to the action. Since working with S^2-valued fields is hard, we will instead work with $SU(2)$-valued fields and impose a local $U(1)$ gauge redundancy to recover the correct quotient space. We can then essentially guess the form of the action in this formulation in terms of $SU(2)$ fields based on the requirement of global $SU(2)$ invariance and local $U(1) \subset SU(2)$ invariance: we should get something that looks like a covariant derivative. The slow fields are being treated as background fields, so they should enter as the gauge field in the covariant derivative.

Let us write $g \in SU(2)$ by rotating away from the generator Z of the $U(1)$ unbroken subgroup as $g = g_s g_f Z g_f^{-1} g_s^{-1}$. We do this to make the fact that the action describes modes living in the coset space $SU(2)/U(1) = S^2$ manifest: Z is the generator of the unbroken $U(1)$, and $hZh^{-1} \subset U(1)$ for any h in the $U(1)$ subgroup. Let us also define

$$\Gamma = g_f Z g_f^{-1}, \qquad \Phi_\mu = g_s^{-1}\partial_\mu g_s. \tag{235}$$

Then

$$g_s^{-1}(\partial_\mu g) g_s = \Phi_\mu \Gamma + \partial_\mu \Gamma - \Gamma \Phi_\mu. \tag{236}$$

Putting this into the trace and using the cyclic property of the trace, we indeed get a Yang-Mills-y action, viz.

$$S = \frac{1}{\lambda}\int \mathrm{Tr}\left(d\Gamma + [\Phi, \Gamma]\right)^2. \tag{237}$$

It will be helpful to choose a gauge in which Φ is fixed to a particular form, which we can do using invariance of Γ under local $U(1)$ transformations. Let us take $U(1)$ to act on the left, with $g_f \mapsto h g_f$ for $h \in U(1)$. Since Z is the generator of the $U(1)$, doing this transformation doesn't affect the $[\Phi, \Gamma]$ term (essentially by construction — the form of Γ was chosen so that it lives in the coset space). However, the derivative term changes:

$$d\Gamma \mapsto dh\Gamma - h\Gamma h^{-1} dh h^{-1}. \tag{238}$$

Putting this into the action and using the cyclic property of the trace to conjugate by h, we see that the action shifts to

$$S \mapsto \frac{1}{\lambda} \int \mathrm{Tr}\left(d\Gamma + [h^{-1}(\Phi + d)h, \Gamma]\right)^2, \tag{239}$$

which is exactly what we'd expect from a gauge transformation. We can then use this freedom to fix a gauge (at least locally) in which Φ is sort of in "unitary gauge", in that it has no Z component. That is, we can fix Φ so that

$$\{\Phi_\mu, Z\} = 0. \tag{240}$$

This is accomplished I believe with the choice $h = \exp(i\psi(x)Z)$, where

$$\psi(x) = -i \int_\bullet dx^\mu \ \mathrm{Tr}[Z\Phi_\mu] + \psi(\bullet), \tag{241}$$

for some basepoint $\bullet$.

Now let us finally break apart the action into a slow component, a fast component, and a mixed term that couples slow and fast degrees of freedom. We will parametrize g_f by

$$g_f = \exp(iW), \qquad W = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \quad z \in \mathbb{C}. \tag{242}$$

Note that since W is built out of an X and a Y, this form of g_f is preserved by the action of the $U(1)$ subgroup by $h(x)$. One then sees that to second order in W,

$$g_f = Z(1 - 2iW - 2W^2). \tag{243}$$

Since the form of W means that $ZWZ = -W$, the $(d\Gamma)^2$ term in the action becomes

$$S_f = \frac{(2i)^2}{\lambda} \int \mathrm{Tr}[\partial_\mu W Z \partial^\mu W Z] = \frac{4}{\lambda} \int \mathrm{Tr}[\partial_\mu W \partial^\mu W] = \frac{8}{\lambda} \int \partial_\mu z \partial^\mu \bar{z}. \tag{244}$$

Now the commutator appearing in the action is, to second order in W,

$$[\Phi_\mu, \Gamma] = 2\Phi_\mu Z + 2iZ\{\Phi_\mu, W\} + 2Z\{\Phi_\mu, W^2\}, \tag{245}$$

where we used $Z\Phi_\mu Z = -\Phi_\mu$ by virtue of our gauge choice. Squaring this and working to quadratic order in W, we see that the slow mode part is

$$S_s = \frac{4}{\lambda} \int \mathrm{Tr}[\Phi^2], \tag{246}$$

while the slow-fast coupling part is

$$S_{sf} = -\frac{4}{\lambda} \int \mathrm{Tr}\left[Z(\Phi_\mu W + W\Phi_\mu)Z(\Phi_\mu W + W\Phi_\mu)\right] + \frac{8}{\lambda} \int \mathrm{Tr}\left[\Phi_\mu\{\Phi_\mu, W^2\}\right]. \tag{247}$$

We can massage this into

$$S_{sf} = -\frac{8}{\lambda} \int \mathrm{Tr}\left[\Phi_\mu W Z \Phi^\mu W Z\right] + \frac{8}{\lambda} \int \mathrm{Tr}\left[W^2 \Phi^2\right], \tag{248}$$

or equivalently,

$$S_{sf} = -\frac{8}{\lambda} \int \mathrm{Tr}\left[\Phi_\mu W \Phi^\mu W - W^2 \Phi^2\right]. \tag{249}$$

The mixed $d\Gamma$ and commutator cross-terms won't contribute since they vanish after we go to momentum space.

Now we take advantage of the specific form of W and Φ. Since Φ anticommutes with Z, we can write it as

$$\Phi_\mu = \begin{pmatrix} 0 & \phi_\mu \\ \bar{\phi}_\mu & 0 \end{pmatrix}, \tag{250}$$

Thus we get

$$S_{sf} = -\frac{8}{\lambda} \int \left(\phi^2 z^2 + \bar{\phi}^2 \bar{z}^2 - 2|\phi|^2 z\bar{z}\right). \tag{251}$$

To find the correction to the coupling constant induced by integrating out the fast modes, we will take $e^{-S_{sf}}$ and expand it as $1 - S_{sf} + \dots$, keeping terms of order W^2. Thus we will be computing the expectation value of the above integral with respect to the free $\partial_\mu z \partial^\mu \bar{z}$ action. Since only the $\bar{z}_p z_p$ two-point function is non-zero, the first two terms in the above expression for S_{sf} will only have a non-zero expectation value at order $z^2\bar{z}^2$, which we are dropping. Thus, after we expand the exponential, we may ignore the first two terms. Thus the relevant expectation value to compute is

$$\int \mathcal{D}z\mathcal{D}\bar{z} \left(\frac{8}{\lambda}\int_p \mathrm{Tr}[\Phi^2]\bar{z}_p z_p\right) \exp\left(-\frac{8}{\lambda}\int_p p^2 z_p \bar{z}_p\right). \tag{252}$$

We find this to be

$$\begin{aligned} \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d\ln p}{2\pi}\mathrm{Tr}[\Phi^2] &= -\frac{1}{2\pi}\ln\left(1+\frac{d\Lambda}{\Lambda}\right)\mathrm{Tr}[\Phi^2] \\ &= -\frac{\mathrm{Tr}[\Phi^2]}{2\pi} d\ln\Lambda, \end{aligned} \tag{253}$$

where $\Lambda - \delta\Lambda$ is the boundary between the fast modes and the slow modes, and where $d\Lambda = -\delta\Lambda$ is negative, since the cutoff is being decreased.

Now we re-exponentiate to find that the new slow action is

$$S_s = \frac{4}{\lambda + d\lambda}\int \mathrm{Tr}[\Phi^2], \tag{254}$$

where

$$\frac{4}{\lambda + d\lambda} = \frac{4}{\lambda} + \frac{d\ln\Lambda}{2\pi}. \tag{255}$$

We re-write this as

$$\lambda + d\lambda = \frac{\lambda}{1 + \frac{1}{8\pi}d\ln\Lambda} = \lambda - \frac{1}{8\pi}d\ln\Lambda, \tag{256}$$

and so the β function is

$$\frac{d\lambda}{d\ln\Lambda} = -\frac{\lambda^2}{8\pi}. \tag{257}$$

Thus we have asymptotic freedom, with weak coupling in the UV (as expected for a NLSM into a manifold with positive curvature, namely S^2) and with a strong-coupling disordered phase in the IR. In terms of the time flow along the RG trajectory, we can write

$$\frac{d\lambda}{dt} = \frac{\lambda^2}{8\pi}. \tag{258}$$

How special is the choice of dimension 2? To get some insight into this, we can consider doing the same calculation in dimension $d = 2+\epsilon$. The relevant integral is then schematically

$$\int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^2p}{4\pi^2} p^{-2} = \frac{1}{2\pi\epsilon}(\Lambda^{-\epsilon} - \Lambda^{-\epsilon}(1 - \epsilon d \ln \Lambda)). \tag{259}$$

The affect of the extra $\Lambda^{-\epsilon}$ is to just add a $\epsilon d \ln \Lambda$ contribution the integral, which is opposite in sign to the $d = 2$ part. Thus the β function is upgraded to

$$\frac{d\lambda}{d\ln\Lambda} = -\frac{\lambda^2}{8\pi} + \epsilon. \tag{260}$$

This tells us that while in $d = 2$ there is no phase transition and the theory just flows to a disordered state (as it must because of the CMW theorem), in $d > 2$ there is a phase transition between the ordered and disordered phases at some finite value of λ. Thus $d = 2$ is the lower critical dimension of the theory.

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12 *RG in the 1+1D $SU(N)$ WZW model* ✓

Like in yesterday's diary entry, consider the $SU(N)$ WZW model on a two-dimensional spacetime. The action is

$$S = S_{kin} + S_{wzw} = \frac{1}{\lambda}\int \mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}] - \frac{ik}{12\pi}\int_{B^3} \mathrm{Tr}[\omega \wedge \omega \wedge \omega], \tag{261}$$

where g is a map from the spacetime X (which since we will take the fields to be constant at infinity is topologically an S^2) into $SU(N)$, B^3 is a three-ball which bounds spacetime, and where ω is the Maurer-Cartan form on $SU(N)$ pulled back to B^3.

Using the background field method with an explicit momentum-shell cutoff, find the beta function for λ. Show that $\lambda = 8\pi$ is a fixed point, the existence of which is made possible by the wzw term.

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To do the RG, we split up $g = g_s g_f$ into low- and high-momentum parts. We find it helpful to parametrize g_f as $g_f = e^W$ for $W \in \mathfrak{su}(N)$. Since we are only interested in finding β_λ to one-loop order, we only need to keep terms quadratic in W (and to quadratic order, the measure $\mathcal{D}g_f$ is the same as $\mathcal{D}W$). Thus in the action we can make the replacement

$$\partial_\mu g_f \approx \partial_\mu W + \frac{1}{2}\{\partial_\mu W, W\}. \tag{262}$$

Let us focus on S_{kin} first. After some straightforward algebra, we find

$$S_{kin}[g] = S_{kin,s}[g_s] + S_{kin,f}[W] + S_{kin,sf}[g_s, W], \tag{263}$$

where

$$S_{kin,s}[g_s] = \frac{1}{\lambda}\int \mathrm{Tr}[\partial_\mu g_s \partial^\mu g_s^{-1}], \qquad S_{kin,f} = -\frac{1}{\lambda}\mathrm{Tr}[\partial_\mu W \partial^\mu W], \tag{264}$$

and

$$\begin{aligned} S_{kin,sf}[g_s, W] &= \frac{1}{2\lambda}\int \mathrm{Tr}\left[g_s^{-1}\partial_\mu g_s[\partial_\mu W, W] + (\partial_\mu g_s^{-1})g_s[W, \partial_\mu W]\right] \\ &= \frac{1}{\lambda}\int \mathrm{Tr}\left[\star\omega_s \wedge [dW, W]\right], \end{aligned} \tag{265}$$

where $\omega_s = g_s^{-1}\partial_\mu g_s \, dx^\mu$.

Now for the WZW term. The well-definedness of this term forces the coefficient to be an integer (times $i/12\pi$) which means the factor in the WZW term can't flow under RG—this does not preclude the WZW term from contributing to the RG flow of other couplings (viz. λ), however.

Let us first expand the Maurer-Cartan form ω appearing in the WZW term. Again, straightforward algebra gives

$$\omega \approx \omega_s(1 + W + W^2/2) - W\omega_s - W\omega_s W + \frac{1}{2}W^2\omega_s + \frac{1}{2}[\partial_\mu W, W] + \partial_\mu W. \tag{266}$$

Now we take the wedge product of three copies of the above expression. We only want to keep terms that have at most two derivatives in the slow fields (since they are slowly varying), and we only need to keep up to quadratic order in W. The very last $\partial_\mu W$ parts contribute a factor of $3\mathrm{Tr}[dW \wedge dW \wedge \omega_s]$. The only other term that is not cubic in ω_s comes from the commutator, and so

$$S_{top}[g_s, W] = -\frac{ik}{4\pi}\int_{B^3} \mathrm{Tr}\left[\left(\frac{1}{2}[dW, W] \wedge \omega_s + dW \wedge dW\right) \wedge \omega_s\right]. \tag{267}$$

The integrand is a total derivative:

$$d\left(-\frac{i}{8\pi}\mathrm{Tr}\left[[dW, W] \wedge \omega_s\right]\right) = -\frac{i}{8\pi}\mathrm{Tr}\left[-2dW \wedge dW \wedge \omega_s + [dW, W] \wedge \omega_s \wedge \omega_s\right], \tag{268}$$

since $d\omega_s = -\omega_s \wedge \omega_s$ (we've also used the sign rules for the supercommutativity of the wedge product). So, we have

$$S_{top}[g_s, W] = -\frac{ik}{8\pi}\int_X \mathrm{Tr}[[dW, W] \wedge \omega_s]. \tag{269}$$

To write the full slow-fast action succinctly, it is helpful to introduce

$$\widetilde{\omega} = \omega_s - \frac{i\lambda k}{8\pi} \star \omega_s, \tag{270}$$

where the $\star$ is the Hodge star on X. With this, we have

$$S = S_f[W] + S_s[g_s] + S_{sf}[g_s, W], \tag{271}$$

where

$$S_{sf}[g_s, W] = \frac{1}{\lambda} \int \mathrm{Tr}[\star\widetilde{\omega} \wedge [dW, W]]. \tag{272}$$

Thus, while the wzw term can't get renormalized through a change of its coefficient (the level), it contributes towards the renormalization of the kinetic term.

Now we expand the exponential $\exp(-S_{sf})$ in the path integral. The linear term vanishes since it contains a single fast momentum from the dW piece, and so the first relevant contribution comes from the S_{sf}^2 term. Thus the effective action for the slow fields is

$$S_{eff,s}[g_s] = S_s[g_s] - \ln\left(1 + \frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle\right), \tag{273}$$

where the expectation value is taken with respect to the $\lambda^{-1}\int \mathrm{Tr}(\partial W)^2$ action. The expectation value is

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = -\frac{4}{2\lambda^2}\int_{p,q,p',q'} p_\mu p'_\nu \langle \mathrm{Tr}[\widetilde{\omega}_q^\mu W_p W_{-q-p}]\mathrm{Tr}[\widetilde{\omega}_{q'}^\nu W_{p'} W_{-q'-p'}]\rangle, \tag{274}$$

where we've gone to momentum space and we've used $2p+q \approx 2p$ since p is a fast momentum and q is a slow momentum (corrections to this are irrelevant).

We take the expectation value by contracting the various W's. Contracting two W's in the same trace yields zero: this is because the propagator for the W fields is diagonal in the $\mathfrak{su}(N)$ generators (expand the W's in terms of T^a's in the kinetic term for W and use the orthogonality of the T^a's under the trace inner product), and so contracting two W's in the same trace produces something like $\mathrm{Tr}[\widetilde{\omega}C_2]$, where $C_2 \sim \sum_a T^a T^a$ is the quadratic Casimir, which is central in the Lie algebra. Expanding $\widetilde{\omega}$ (which lives in $\mathfrak{su}(N)$ since it is built from the Maurer-Cartan form) in terms of the traceless T^a, one sees that the trace vanishes.

So, we just need to consider the two contractions between the W's in different traces. This gives (see Altland and Simons chapter 8 for some useful identities)

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = -\frac{N}{2\lambda^2}\int_{p,q} G_p G_{p+q} p_\mu p_\nu \mathrm{Tr}[\widetilde{\omega}_q^\mu \omega_{-q}^\nu], \tag{275}$$

with the factor of N coming from the sum over the internal loop in the polarization diagram (use the double line notation to see). If we again approximate $p+q \approx p$ then we can do the integral over the fast momentum p easily:

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = -\frac{N}{8\pi}\int_q \ln\left(\frac{\Lambda}{\Lambda - \delta\Lambda}\right) \mathrm{Tr}[\widetilde{\omega}_q^\mu \widetilde{\omega}_{\mu,-q}]. \tag{276}$$

Taking $d\Lambda = -\delta\Lambda$ and expanding,

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = -d\ln\Lambda \frac{N}{8\pi}\int \mathrm{Tr}[\widetilde{\omega} \wedge \star\widetilde{\omega}], \tag{277}$$

where the integral is now in $\mathbb{R}$ space. From the definition of $\widetilde{\omega}$, one sees that this integral is

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = d\ln\Lambda \frac{N}{8\pi}\left(1 - \left(\frac{\lambda k}{8\pi}\right)^2\right)\int \mathrm{Tr}[\omega_s \wedge \star\omega_s]. \tag{278}$$

Adding this in with the $\omega_s \wedge \omega_s$ term, we see that the effective coupling $\lambda + d\lambda$ is

$$\frac{1}{\lambda + d\lambda} = \frac{1}{\lambda} + d\ln\Lambda \frac{N}{8\pi}\left(1 - \left(\frac{\lambda}{8\pi}\right)^2\right), \tag{279}$$

which gives the β function

$$\frac{d\lambda}{d\ln\Lambda} = -\frac{N\lambda^2}{8\pi}\left(1 - \left(\frac{\lambda k}{8\pi}\right)^2\right). \tag{280}$$

There are several things to note about this. First, we see that it is asymptotically free for small λ, which is what we expect based on our experience with σ models into spaces with positive curvature. However, we also see that

$$\lambda_* = 8\pi/k \tag{281}$$

constitutes a fixed point of the RG flow at 1-loop, which is made possible only by the presence of the wzw term. Such a fixed point had to occur at this value of λ for $k = 1$, since it is precisely at this value of λ that the $k = 1$ theory admits a bosonization duality to a theory of N flavors of free fermions which, being free, does not flow under RG. Thus the presence of the wzw term is essential for making the bosonization work.

Also note that the fixed point λ_* is at weak coupling in the large k limit. Therefore as $k \to \infty$ the RG distance between the free point and the nontrivial CFT goes to zero, meaning that the 1-loop expansion becomes exact in the $k \to \infty$ limit. This means that an expansion in $1/k$ is similar to an ε expansion (which is a bit hard to conceive of doing here due to the WZW term).

While we've been in two dimensions, the generalization to the theory of a vector field in S^{d+1} in d dimensions is straightforward.[20] For example, in $d = 1$ the kinetic term is irrelevant and we just get the action for a spin $k/2$ (after remembering to adjust the normalization of the WZW term as $2\pi ik/A(S^{d+1})$), while in $d = 0$ we write $\omega = (\cos\theta, \sin\theta)$ to get $S = ik\int du\,(\cos\theta\partial_u \sin\theta - \sin\theta\partial_u\cos\theta) = ik\Theta$, where we have taken "spacetime" to be an S^0 with $\theta = 0$ on one component of the S^0 and $\theta = \Theta$ on the other (this describes the action of instantons in theories with topological terms).

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[20] $\pi_{d+1}(S^{d+1}) = \mathbb{Z}$ means we always have a WZW term.

13 *Asymptotic symmetry in an RG flow* ✓

This is from P&S, chapter 12. Consider two massless copies of ϕ^4 theory interacting quadratically:

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 + (\partial\theta)^2) - \frac{\lambda}{4!}(\phi^2 + \theta^2) - \frac{2\rho}{4!}\phi^2\theta^2. \tag{282}$$

Compute the β functions for λ, ρ, and ρ/λ. Show that $\beta_{\rho/\lambda}$ has fixed points at $0, 1, 3$, and that the one at $\rho/\lambda = 1$ is stable, so that if e.g. we start with $\rho/\lambda < 3$ then we flow to a phase where $\rho = \lambda$, which is characterized by the model obtaining an $O(2)$ global symmetry in the IR. Note that this problem is a special case of that of computing the RG flow for a cubically-anisotropic $O(N)$ model which appears in another diary entry.

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We just want the β functions to leading order in the couplings. Let us find the counterterm for ρ to one-loop order by examining the one-loop corrections to a vertex with two external ϕ lines and two external θ lines. There are six diagrams, counting the original ρ term and the δ_ρ counterterm. The four other diagrams are s and t channel $2\phi \to 2\theta$ diagrams, and two u channel diagrams (warning, I may have gotten mixed up with s and u), which differ by the type of field (θ or ϕ) which propagate in the loop. The basic integral is

$$\text{some typical diagram} = -C\int_q\int_x \frac{1}{(q^2-\Delta)^2}, \tag{283}$$

where x is a Feynman parameter, C is a constant that depends on the type of diagram, and Δ is some function of the square of the external momenta (e.g. the s, t, u channels $(p_1+p_2)^2, (p_1+p_4)^2$, etc.) and x. In dim reg, this gives

$$\text{some typical diagram} = -\frac{Ci}{4\pi^2}\int_x \left(\frac{2}{\epsilon} - \ln(\Delta/\Lambda) + \dots\right), \tag{284}$$

with Λ the UV cutoff. Basically all we then need to do is compute the different Cs for the different diagrams.

The s channel diagram has two ρ vertices. Each vertex contributes a factor of $-i(2\rho/4!)\cdot 2\cdot 2 = -i\rho/3$, where the extra factors of 2 are the symmetry factor. Thus the 1-loop corrections for the s and t channels are (just writing the ln terms and putting p_1, p_2 on the "bottom" of the diagrams and p_3, p_4 on the "top")

$$-(\rho/3)^2\frac{i}{(4\pi)^2}\left(\ln\frac{-(p_1+p_2)^2}{\Lambda^2} + \ln\frac{-(p_1+p_4)^2}{\Lambda^2}\right). \tag{285}$$

The u channel diagrams have one $-i(\lambda/4!)\cdot 4!$ vertex and one $-i\rho/3$ vertex, plus a symmetry factor of $1/2$ since the two internal lines are identical. These thus add and produce

$$-(\rho/3)\lambda\frac{i}{(4\pi)^2}\ln\frac{-(p_2+p_4)^2}{\Lambda^2}. \tag{286}$$

The ρ counterterm enters into the diagrams as $-i\delta_\rho/3$, since the original ρ interaction appeared with a $1/3$ prefactor. We will fix our renormalization conditions so that the effective interaction between two θs and two ϕs is $\rho/3$ for the choice of momenta where $s = t = u = -M^2$ (the RG conditions are imposed at spacelike momenta as usual). Thus the renormalization-scale dependent part of the counterterm needs to be

$$\delta_\rho \sim \frac{1}{16\pi^2}\left(\lambda\rho + 2\rho^2/3\right)\ln(M^2/\Lambda^2). \tag{287}$$

This gives us the β function for ρ:

$$\beta_\rho = \frac{d\rho}{d\ln M} = \frac{1}{8\pi^2}\left(\lambda\rho + 2\rho^2/3\right)\ln(M^2/\Lambda^2). \tag{288}$$

Now for β_λ, which we evaluate by focusing on the correction to graphs with four external ϕ lines. In addition to the bare λ term and the $-i\delta_\lambda$ counterterm, there are six one-loop graphs. There are s,t,u channel graphs for graphs with an internal ϕ loop that go as λ^2, and likewise there are s,t,u channel graphs where the internal loop is a θ. The former three have a factor of $\lambda^2/2$ where the $1/2$ is the symmetry factor of the internal loop, and the latter three similarly have a factor of $(\rho/3)^2/2$. Thus the M-dependent part of the counterterm needed to reduce the full term to $-i\lambda$ at our RG scale is

$$\delta_\lambda \sim \frac{1}{16\pi^2}\left(3\lambda^2/2 + \rho^2/6\right)\ln(M^2/\Lambda^2), \tag{289}$$

where we've remembered to multiply by 3 since each s,t,u channel result is the same. Thus β_λ is

$$\beta_\lambda = \frac{1}{8\pi^2}\left(3\lambda^2 + \rho^2/6\right). \tag{290}$$

The β function for the ratio is

$$\begin{aligned}\beta_{\rho/\lambda} &= \frac{1}{\lambda}\left(\beta_\rho - \frac{\rho}{\lambda}\beta_\lambda\right)\\ &= \frac{1}{8\pi^2\lambda}\left(\lambda\rho + 2\rho^2/3 - 3\rho\lambda/2 - \rho^3\lambda/6\right).\end{aligned} \tag{291}$$

Note that this has fixed points at $\rho = 0$, $\rho = \lambda$, and $\rho = 3\lambda$. The $\rho = \lambda$ fixed point has an "emergent" global $O(2)$ symmetry, so we would like to know whether this fixed point is attractive or not. Indeed it is: writing $\rho = 3\lambda(1-\eta)$ for small η, one gets

$$\beta_{\rho/\lambda} \approx \frac{3\lambda\eta}{8\pi^2}, \tag{292}$$

so that ρ/λ gets smaller at long distances, approaching the $\rho/\lambda = 1$ fixed point. Likewise if $\rho = \eta\lambda$ then $\beta_{\rho/\lambda} \approx -\eta\lambda/(16\pi^2)$, so that ρ/λ increases as we flow to larger distances, again approaching the $\rho/\lambda = 1$ symmetric fixed point.

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14 ϕ^4 *theory coupled to fermions and a natural relation* ✓

This is from P&S. We consider a nlsm with symmetry group $O(2)$ coupled to fermions by using the nlsm field to create a varying chiral mass term for the fermions:

$$\mathcal{L} = \frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + i\bar{\psi}\partial\!\!\!/\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi, \tag{293}$$

where $i = 1, 2$.

Find the classical value for the fermion mass, assuming SSB occurs. Then show that this receives corrections at one-loop order, but that these corrections are *finite*.

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First, note that while the fermion mass term means that the full $O(2) \times U(1)$ symmetry is broken, the diagonal subgroup consisting of the rotations

$$\phi \mapsto R_\theta \phi, \qquad \psi \mapsto e^{-i\gamma^5\theta/2}\psi \tag{294}$$

is still a symmetry of the theory (checking this is straightforward). Secondly, assume that SSB occurs for ϕ. Wolog we can let $\langle\phi\rangle = (v, 0)^T$, so that the classical value of the fermion mass becomes

$$m_f = gv. \tag{295}$$

Now we want to find quantum corrections to this. This is most easily accomplished when we write e.g. $\phi = (v + \sigma, \pi)$, where $v = \mu/\sqrt{\lambda}$ is the classical vev of ϕ in the SSB state. σ is massive and π is massless, and the Feynman rules are derived from the relevant terms

$$\mathcal{L} \supset -ig\bar{\psi}\gamma^5\pi\psi - g\bar{\psi}\sigma\psi - \lambda v\pi^2\sigma. \tag{296}$$

There are further interactions like $-\frac{1}{2}\lambda\sigma^2\pi^2$, but these won't play a role in this problem.

To examine the corrections to m_f, we need to consider three things: the counterterm for g (which affects the mass due to $m_f = gv$) and the corrections to the propagator which come from polarization bubble diagrams with ψ and either π or σ.

The polarization bubble with σ for a ψ fermion with momentum p is (this diagram is a matrix with spinor indices for the ψ spins on the ends of the bubble)

$$\begin{aligned}\sigma \text{ bubble} &= (ig)^2 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{q\!\!\!/ + m_f}{q^2 - m_f^2} \\ &= -g^2 \int_{q,x} \frac{xp\!\!\!/ + m_f}{(q^2 - \Delta_\sigma)^2} \\ &= -g^2 \int_x (xp\!\!\!/ + m_f) \frac{i}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma\right)(1 + \epsilon \ln\sqrt{4\pi/\Delta_\sigma}),\end{aligned} \tag{297}$$

where $d = 4 - \epsilon$ and Δ_σ is built out of x, p, and m_σ. The diagram involving π is similarly

$$\begin{aligned}
\pi \text{ bubble} &= -(ig)^2 \int_q \frac{\gamma^5}{(p-q)^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \\
&= +g^2 \int_{q,x} \frac{-x\not{p} + m_f}{(q^2 - \Delta_\pi)^2} \\
&= -g^2 \int_x (x\not{p} - m_f) \frac{i}{16\pi^2} \left(\frac{2}{\epsilon} - \gamma \right) (1 + \epsilon \ln \sqrt{4\pi/\Delta_\pi}),
\end{aligned} \tag{298}$$

where Δ_π is built out of x and p. Adding these together and sending $\epsilon \to 0$ we get

$$\sigma \,\text{bubble} + \pi \,\text{bubble} = -g^2 \int_x x\not{p}(\text{divergent stuff}) - g^2 m_f \int_x \frac{i}{16\pi^2} \ln(\Delta_\pi/\Delta_\sigma). \tag{299}$$

The coefficient of the m_f term is finite, and so these diagrams lead to a finite correction to the fermion mass (the correction is non-zero since $\Delta_\sigma \neq \Delta_\pi$ due to σ being massive and π being massless). The field strength renormalization term needed because of the $x\not{p}$ term is infinite, but the mass correction is finite.

Now we need to look at the correction to the coupling g which also affects the renormalized fermion mass: if the correction to m_f is to be finite then the counterterm δ_g better be finite as well.

We will work in an RG scheme suggested in P&S, where the $\psi\psi\pi$ vertex receives no radiative corrections when the π particle carries away zero momentum. To work out δ_g, the counterterm needed to ensure that the vertex has no radiative corrections, we need to compute four diagrams (one-loop corrections to the $\psi\psi\pi$ vertex). Two of the diagrams have only one ψ propagator in the loop (the other two guys in the loop are a π and a σ) and are finite:

$$\text{two finite diagrams} \sim g^2(-\lambda v) \int_q \frac{1}{(p-q)^2} \frac{1}{(p-q)^2 - m_\sigma^2} \frac{1}{q^2 - m_f^2} (2m_f), \tag{300}$$

which goes like $\int dq\, q^3 q^{-6}$ and contains no divergences (the $\not{q}$'s in the numerator cancel since they anticommute with γ^5).

The two diagrams with two fermion propagators in the loop go as $\int dq\, q^3 q^{-4}$ and do contain a $1/\epsilon$ divergence. The one with an internal π propagator is

$$\begin{aligned}
\pi\psi\psi \,\text{loop} &= (ig)^3 i^3 \int_q \frac{1}{(p-q)^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \\
&= -g^3 \int_q \frac{1}{(p-q)^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2}.
\end{aligned} \tag{301}$$

On the other hand, the one with an internal σ propagator is

$$\begin{aligned}
\sigma\psi\psi \,\text{loop} &= g^2 (ig) i^3 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \\
&= +g^3 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2}.
\end{aligned} \tag{302}$$

These two diagrams would entirely cancel if not for the fact that the second diagram has an m_σ in one of the propagators. When we evaluate this in dimensional regularization however, the only dependence on m_σ of the second diagram comes from the $\epsilon \ln \sqrt{4\pi/\Delta_\sigma}$ term, which is not divergent. Thus all the divergent parts are independent of m_σ, and hence cancel between the two diagrams, with their sum looking something like $g^3 \int_x \ln(\Delta_\pi/\Delta_\sigma)$. So, recapitulating, the counterterm δ_g is finite, as are the polarization bubble contributions to the renormalization of m_f, so that m_f only receives finite corrections.

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15 *The ϵ expansion and β functions in the anisotropic $O(N)$ model* ✓

Today we're doing an extended version of a problem suggested by Pufu for the attendees of one of the bootstrap schools. We will be considering the $O(N)$ model deformed with an anisotropy v which reduces the symmetry to the "cubic" symmetry group $\mathbb{Z}_2^N \rtimes S_N$, acting by $\phi_i \mapsto -\phi_i, \phi_i \mapsto \phi_{\rho(i)}, \rho \in S_N$. The action is

$$S = \int d^d x \left(\frac{1}{2} \nabla_\mu \boldsymbol{\phi} \cdot \nabla^\mu \boldsymbol{\phi} + \frac{1}{2} t_0 \boldsymbol{\phi} \cdot \boldsymbol{\phi} + \frac{1}{4!} \sum_{i,j=1}^{N} (u_0 + v_0 \delta_{ij}) \phi_i^2 \phi_j^2 \right). \tag{303}$$

The numbers in the action aren't the ones given in Pufu's assignment, but I think these factors lead to the simplest manipulations.

Anyway, our goal is to study the RG properties of this model in $d = 4 - \epsilon$. We will perturb in ϵ, and let N be arbitrary. Pufu provides students with the 1-loop beta functions and asks them to analyze the RG flow. We will do this but will instead also calculate the 1-loop beta functions ourself: it's fun and this way we know that the factors we get in the beta functions are correct (the result Pufu writes down is probably correct, but one has to make some variable-re-definitions to get to his result that he doesn't mention, and which I couldn't quite figure out).

gliders.pdf

First let's define the dimensionless couplings that we will be doing RG with.[21] The dimension of ϕ_i is $(d-2)/2 = 1 - \epsilon/2$. This means that we can define a dimensionless

[21] We'll be doing dim-reg instead of conformal perturbation theory a la Cardy since the latter is much messier combinatorially.

renormalized mass through $t_0 = a^{-2} Z_t t$, where a^{-1} is the UV cutoff. Since $[\phi_i]^4 = 4 - 2\epsilon$, the bare couplings for the quartic terms are determined through dimensionless couplings u, v via

$$u_0 = u Z_u a^{-\epsilon}, \qquad v_0 = v Z_v a^{-\epsilon} \tag{304}$$

with $Z_u = 1 + \delta_u$ and likewise for Z_v (the dimensions work out since $(4 - 2\epsilon) + \epsilon = d$).

We will determine the beta functions by using the fact that the bare couplings are independent of the UV cutoff a^{-1}. Our mission is to find the appropriate counterterms δ_g, since they will carry cutoff-dependence. We will just go to 1-loop, and will work to quadratic order in all the couplings. N will be arbitrary.

The basic 4-point vertices are

threeverts.pdf

(305)

The factor of $2^3/4! = 1/3$ comes from the symmetry factor of the diagram (2!2!) and the fact that $\phi_i^2 \phi_j^2, i \neq j$ appears in the Lagrangian with coefficient $2u/4!$.

With all the outgoing legs labeled by the same index, the t-channel graphs are

alloutsame.pdf

(306)

The s and u channel processes are the same. So all the 1-loop graphs with identical external legs give us the 4-point function

$$G^{(4)}_{iiii} = -i \left((1+\delta_v) v + (1+\delta_u) u + \frac{1}{2} \left(u^2 + v^2 + 2uv + \frac{N-1}{9} u^2 \right) (V(s) + V(t) + V(u)) \right) \prod_k \frac{i}{p_k^2}, \tag{307}$$

where p_k are the external momenta and

$$V(p^2) = \int_k \frac{i^2}{k^2 (k+p)^2}. \tag{308}$$

Note that $V(p^2)$ does not include a 1/2 symmetry factor. Also note that there is no t appearing in the above propagator—we are going to be treating t as a vertex, i.e. as a (small) coupling in its own right. This is done wolog since summing up the geometric series for a straight propagator line with all possible t insertions recovers $i/(p^2 - t)$.

Anyway, the above formula for $G^{(4)}_{iiii}$ does not determine the counterterms δ_u, δ_v uniquely. To find them, we need to also renormalize the vertex with two i legs and two j legs, for $i \neq j$. The contributing diagrams are

diffloops.pdf

(309)

Adding these all up,

$$G^{(4)}_{iijj} = -i\left(\frac{1}{3}(1+\delta_u)u + \left(\frac{2}{2\cdot 3}(u^2+vu) + \frac{N-2}{2\cdot 3^2}\right)V(t) + \frac{1}{3^2}u^2(V(s)+V(t))\right)\prod_k \frac{i}{p_k^2}. \tag{310}$$

Note the absence of the 1/2 symmetry factor in the last term, since the s and t channel processes don't have internal legs with the same index. As another example of how the counting works, look at e.g. the $(u^2 + vu)$ term: the 2 in the numerator comes from taking the internal loop to be either i or j, the 2 in the denomenator is a symmetry factor, and the 3 is $2^3/4!$.

Ensuring that the divergences in $G^{(4)}_{iijj}$ are canceled lets us determine the counterterm δ_u. Setting our renormalization conditions for momenta with $s = t = u = -M^2$, some algebra tells us that the divergences in $G^{(4)}_{iijj}$ are canceled, and that the 4-point function reduces to just $-iu/(\prod p^2)$ at the scale $-M^2$, provided that

$$\delta_u = -V(-M^2)\left(\frac{N+8}{6}u + v\right). \tag{311}$$

We can now substitute this counterterm into the expression for the $G^{(4)}_{iiii}$ Greens function to solve for δ_v. Some algebra gives

$$\delta_v = -V(-M^2)\left(2u + \frac{3}{2}v\right), \tag{312}$$

which ensures that $G^{(4)}_{iiii}$ reduces to $-i(u+v)/(\prod p^2)$ at the scale $-M^2$.

Calculating $V(-M^2)$ is standard. For simplicity we will set the RG conditions to be at zero momentum. Then the integral for $V(0)$ is logarithmically divergent: when we cut it off at the UV cutoff a^{-1}, the a-dependence is

$$V(0) \supset \frac{1}{16\pi^2}\ln(a^{-2}). \tag{313}$$

We only care about the a dependence, since this is what gives the counterterms their a dependence, which is the thing that's needed to compute the beta functions.

Now in pure $\lambda\phi^4/4!$ theory (i.e. if $N=1, v=0$), a standard calculation shows that the beta function is $\beta_\lambda = \epsilon\lambda - 3\lambda^2/(16\pi^2)$. Therefore it is helpful to re-define λ by absorbing the $3/16\pi^2$ factor. We will do the same thing, by defining the new variables

$$\mathsf{u} \equiv \frac{3}{16\pi^2}u, \qquad \mathsf{v} \equiv \frac{3}{16\pi^2}v. \tag{314}$$

With these conventions, the a-dependent parts of the counterterms are, after some algebra, (note how no a^ϵs have been entering the expressions for the counterterms—both the δs and the couplings constants are dimensionless)

$$\delta_{\mathsf{v}} = \left(\frac{4}{3}\mathsf{u} + \mathsf{v}\right)\ln a, \qquad \delta_{\mathsf{u}} = \left(\frac{N+8}{9}\mathsf{u} + \frac{2}{3}\mathsf{v}\right)\ln a. \tag{315}$$

Finally, we need the mass counterterm, working to quadratic order in all the couplings. Now the corrections to the propagator that go as u, v are zero in dimensional regularization, since they are $\sim \int_k k^{-2}$ and diverge as a power law. However, since we are treating the mass as an interaction vertex, we do get a logarithmic divergence from diagrams which go as tu, tv. There are only two such 1PI diagrams where the mass appears: they are[22]

massterms.pdf, (316)

where the dot is the mass insertion. The two-point function at momentum p then contains the terms

$$G^{(2)}_{ii} = -\frac{i^2}{p^2}t(1+\delta_t)\left(\frac{v}{2} + \frac{N+2}{6}u\right)\int_k \frac{i^2}{k^4}, \tag{317}$$

[22] The δ_v and δ_u counterterms don't contribute to this order since their contributions would be third order in the couplings.

which is exactly the integral that we did above for the 1-loop calculations. Therefore the counterterm is determined as (going back to the u, v variables)

$$\delta_t = \left(\frac{1}{3}\mathsf{v} + \frac{N+2}{9}\mathsf{u}\right)\ln a. \tag{318}$$

Now we can compute the beta functions, for example for u, as follows:

$$0 = \frac{d}{d\ln a}(a^{-\epsilon}Z_{\mathsf{u}}\mathsf{u}) \implies \beta_{\mathsf{u}} = \epsilon\mathsf{u} - \frac{\mathsf{u}}{Z_{\mathsf{u}}}\frac{d\delta_{\mathsf{u}}}{d\ln a}, \tag{319}$$

and likewise for v. For t, the only change is in the first term:

$$\beta_t = 2t - \frac{t}{Z_t}\frac{d\delta_t}{d\ln a}. \tag{320}$$

To quadratic order in the coupling constants we can take $1/Z_g \to 1$ in the above expressions for each coupling g, and therefore we derive[23]

$$\begin{aligned}
\beta_{\mathsf{u}} &= \epsilon\mathsf{u} - \frac{N+8}{9}\mathsf{u}^2 - \frac{2}{3}\mathsf{u}\mathsf{v}, \\
\beta_{\mathsf{v}} &= \epsilon\mathsf{v} - \mathsf{v}^2 - \frac{4}{3}\mathsf{u}\mathsf{v}, \\
\beta_t &= 2t - \frac{1}{3}t\mathsf{v} - \frac{N+2}{9}t\mathsf{u}.
\end{aligned} \tag{321}$$

I found a reference for the above β functions in [1] (although the details of how they arrived at them aren't given), and amazingly, after converting to our normalization conventions, they agree with the above! What are the odds of that?! I can't tell you how amazed I was when I discovered this. Evidently we actually kept track of all the symmetry factors correctly—a small miracle.

Some preliminary things to notice about these beta functions: first, t only appears in the beta function for itself. This is general, and is simply because while n-valent vertices for $n > 2$ can combine to give larger-valence vertices or contract to give smaller valence vertices, 2-valent vertices can only combine to make more 2-valent vertices. This means that the mass flow won't affect where the interaction vertices flow.[24]

[23]Note that in this scheme (dimreg), all the fixed points have $t = 0$. If we were doing a different scheme, like conformal perturbation theory, there would be u^2 and v^2 terms in the beta function for t, since in conformal perturbation theory one works with the OPEs directly, which include contractions with more than 1 "loop" (in the $\mathbb{R}$ space Feynman diagrams). So, don't try to compare these β functions to the ones obtained in a different scheme; only the critical exponents are physical, and not the β functions, which are basis-dependent.

[24]This is part of a more general statement: a collection of dimensionless coupling constants g_β, with dimensionful coupling constants $g_{0\beta} = \Lambda^{[g_{0\beta}]}g_\beta$, will only appear in the beta function for a coupling g_α if

$$[g_{0\alpha}] \geq \sum_\beta [g_{0\beta}]. \tag{322}$$

In particular, this means that marginal and irrelevant operators, for which $[g_{0\alpha}] < 0$, are never renormalized

Second, the beta function for v is independent of N, roughly because the v interaction is diagonal and identical to the one in $N = 1$ ϕ^4 theory. However, the beta function for u does depend on N, which tells us that the qualitative behavior of the RG will likely depend on how big N is. Also note that the coefficients of the uv terms in the beta functions for u and v aren't the same.

Now let's analyze the beta functions. First, we will determine the fixed points. This is easy enough to do, and we will do it with slightly more generality than we have to. Let us write the generalized beta functions as

$$\beta_{g_i} = d_i g_i - \sum_j \Gamma_{ij} g_i g_j, \tag{323}$$

where the $\Gamma_{g_i g_j}$ are some numbers.[25] In the case of three couplings u, v, t where t doesn't appear in the beta functions for u, v (as above), we find four fixed points: all of them have $t_* = 0$, with the values of u and v being determined as

$$\begin{aligned}
\mathcal{G}: &\quad u_* = v_* = 0, \\
\mathcal{U}: &\quad u_* = \frac{d_u}{\Gamma_{uu}}, \quad v_* = 0, \\
\mathcal{V}: &\quad u_* = 0, \quad v_* = \frac{d_v}{\Gamma_{vv}}, \\
\mathcal{M}: &\quad u_* = \frac{d_v \Gamma_{uv} - d_u \Gamma_{vv}}{\Gamma_{uv}\Gamma_{vu} - \Gamma_{uu}\Gamma_{vv}}, \quad v_* = \frac{d_u \Gamma_{vu} - d_v \Gamma_{uu}}{\Gamma_{uv}\Gamma_{vu} - \Gamma_{uu}\Gamma_{vv}}.
\end{aligned} \tag{324}$$

$\mathcal{G}$ is the Gaussian fixed point. In our model, $\mathcal{U}$ is the nontrivial $O(N)$-symmetric fixed point, $\mathcal{V}$ is an $O(N)$-breaking fixed point where the theory splits into a sum of N decoupled ϕ^4 theories, while at the mixed fixed point $\mathcal{M}$, $O(N)$ is broken but the theory is not diagonal.

To analyze each fixed point, we need the linearized beta functions, viz.

$$\bar{\beta}_{g_i} = d_i \bar{g}_i - \sum_j \Gamma_{ij}(g_{j*}\bar{g}_i + \bar{g}_j g_{i*}), \tag{325}$$

where $\bar{g}_i \equiv g_i - g_{i*}$. Diagonalizing these equations determines the scaling variables at the appropriate fixed point.

At the Gaussian fixed point $\mathcal{G}$, we see that the (ir)relevance of the couplings are of course entirely determined by the d_g—for our $O(N)$ model this means that all of the couplings are

by relevant operators (for which $[g_{0\beta}] > 0$); hence the absence of t in the beta functions for u, v. To prove this, note that since the counterterm δ_{g_α} is dimensionless, the counterterm will appear with in diagrams in the form $\delta_{g_\alpha}\Lambda^{[g_{0\alpha}]}$. On the other hand, a correction to the g_α interaction coming from operators with couplings g_β will have a Λ dependence of $\Lambda^{\sum_\beta [g_{0\beta}]+l}$, where $l > 0$ is a cutoff dependence coming from doing loop integrals. In order for the counterterm to cancel divergences coming from these operators, we then need $[g_{0\alpha}] = \sum_\beta [g_{0\beta}] + l$, proving the claim.

[25] This is done wolog since Γ_{kj} with $j, k \neq i$ will never appear in β_{g_i} at quadratic order—this is just because all terms in the expression for β_{g_i} must contain a g_i, as we saw in the derivation above.

relevant for $\epsilon > 0$, and so $\mathcal{G}$ is very unstable. At the $\mathcal{U}$ fixed point, we have

$$\begin{aligned}
\bar{\beta}_t^{\mathcal{U}} &= \left(2 - \frac{\Gamma_{ut}}{\Gamma_{uu}} d_u\right) \bar{t}, \\
\bar{\beta}_u^{\mathcal{U}} &= -d_u \bar{u} - d_u \frac{\Gamma_{uv}}{\Gamma_{uu}} \bar{v}, \\
\bar{\beta}_v^{\mathcal{U}} &= \left(d_v - \frac{\Gamma_{vu}}{\Gamma_{uu}} d_u\right) \bar{v},
\end{aligned} \tag{326}$$

and similarly for the $\mathcal{V}$ fixed point. At the $\mathcal{M}$ fixed point, well... I won't write it out, since we are going to be specializing back to the $O(N)$ model now.

For the $O(N)$ model, the fixed points are

$$\begin{aligned}
\mathcal{G}: &\quad \mathsf{u}_* = \mathsf{v}_* = 0, \\
\mathcal{U}: &\quad \mathsf{u}_* = \frac{9\epsilon}{8+N}, \quad \mathsf{v}_* = 0, \\
\mathcal{V}: &\quad \mathsf{u}_* = 0, \quad \mathsf{v}_* = \epsilon, \\
\mathcal{M}: &\quad \mathsf{u}_* = \frac{3\epsilon}{N}, \quad \mathsf{v}_* = \frac{\epsilon(N-4)}{N}.
\end{aligned} \tag{327}$$

We are working with arbitrary N, but consider for a moment taking $N \to \infty$. We see then that the $\mathcal{U}$ fixed point merges with $\mathcal{G}$, while $\mathcal{V}$ merges with $\mathcal{M}$. Therefore the theory in the large N limit behaves as the decoupled sum of N copies of the ϕ^4 theory, up to $1/N$ corrections: this is exactly what we expect from the usual large N story, where in the $N \to \infty$ limit the different vector components decouple from one another.

Now we need to examine the stability of the different fixed points for different N. Notice that when $N < 4$, the $\mathcal{M}$ fixed point has $\mathsf{v}_* < 0$—therefore we should look at the stability of the model to make sure that this is okay. While either v, u can be negative, the potential must still be bounded from below. To find the region of stability for the potential, we look at its derivative wrt ϕ_k, where $\phi_k^2 \geq \phi_i^2 \, \forall i$. Requiring that this be positive for positive ϕ_k means that

$$\mathsf{v}\phi_k^2 + \mathsf{u}\sum_j \phi_j^2 > 0 \tag{328}$$

in the limit $\phi_k \to \infty$. If $\mathsf{v} < 0$, then we simply need $\mathsf{u} > -\mathsf{v}$. If $\mathsf{u} < 0$, then having $\mathsf{v} > -\mathsf{u}$ isn't enough: the strongest constraint comes from when all the ϕ_i^2 are equal, and tells us that in fact $\mathsf{v} > -N\mathsf{u}$. These conditions define the region of stability for our model.

Anyway, we see that $\mathsf{v}_* = \epsilon(N-4)/N$ is allowed when $\mathsf{u}_* = 3\epsilon/N$ for all $N > 1$, so that this fixed point is indeed always within the range of stability. Since the sign of v_* changes at $N = 4$, we expect that $N = 4$ might be a critical value across which the nature of the RG flow changes. Indeed, this suspicion is confirmed when we notice that at $N = 4$, the $\mathcal{U}$ and $\mathcal{M}$ fixed points become degenerate. To see exactly what happens, we linearize the beta functions, obtaining for $\mathcal{U}$ (omitting those for t since it's always relevant and hence not so interesting)

$$\bar{\beta}_{\mathsf{u}}^{\mathcal{U}} = -\epsilon\left(\bar{\mathsf{u}} + \frac{6}{N+8}\bar{\mathsf{v}}\right), \qquad \bar{\beta}_{\mathsf{v}}^{\mathcal{U}} = \epsilon\frac{N-4}{N+8}\bar{\mathsf{v}}, \tag{329}$$

n2p5ep1.pdf greyscale1.pdf

Figure 1: Left: RG flow for $N = 2, \epsilon = 1$. Right: same thing, but shown with a pretty convolution plot instead.

for $\mathcal{V}$

$$\bar{\beta}^{\mathcal{V}}_{\mathsf{u}} = \frac{\epsilon}{3}\bar{\mathsf{u}}, \qquad \bar{\beta}^{\mathcal{V}}_{\mathsf{v}} = -\epsilon\left(\bar{\mathsf{v}} + \frac{4}{3}\bar{\mathsf{u}}\right), \tag{330}$$

and for $\mathcal{M}$,

$$\bar{\beta}^{\mathcal{M}}_{\mathsf{u}} = -\frac{\epsilon}{3}\bar{\mathsf{u}}(1 + 8/N), \qquad \bar{\beta}^{\mathcal{M}}_{\mathsf{v}} = \epsilon\left(\frac{4}{N} - 1\right)\left(\bar{\mathsf{v}} - \frac{4}{3}\bar{\mathsf{u}}\right). \tag{331}$$

We could now proceed by diagonalizing these equations to get the scaling variables at each fixed point, but instead we'll just turn to pictures to better visualize things. However, we can at least see from the equations that the $\mathcal{V}$ fixed point is always unstable wrt adding u, and that the stability of the other two fixed points depends on whether N is bigger or less than 4, which in the 1-loop approximation is the critical N for which the behavior changes qualitatively.

For $N < 4$, the $\mathcal{U}$ point is the stable IR fixed point. The $\mathcal{M}$ point is located at negative v, and positive-v deviations away from it flow into $\mathcal{U}$. In figure 1, we show an example of the flow for $N = 2, \epsilon = 1$.

On the other hand, when $N > 4$, the $\mathcal{M}$ fixed point moves to positive v, u, and usurps the $\mathcal{U}$ point as the IR fixed point, with small positive v perturbations away from $\mathcal{U}$ now leading to $\mathcal{M}$. As an example, the flow with $N = 7$ is shown in figure 2.

The scaling dimensions of the various operators are easily calculated from the linearized

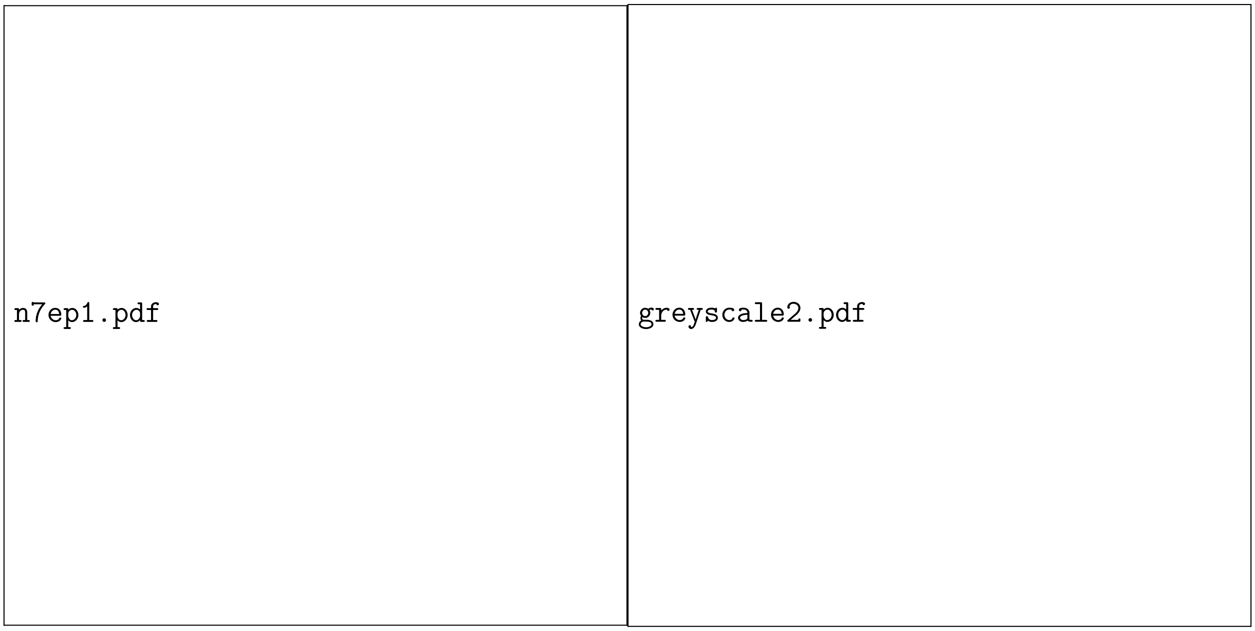

Figure 2: Same thing as the last figure, but now with $N = 7, \epsilon = 1$.

β functions: for posterity's sake, they are

$$\begin{aligned}
&\Delta^{\mathcal{G}}_{\mathsf{u}} = 2, \quad \Delta^{\mathcal{G}}_{\mathsf{v}} = \Delta^{\mathcal{G}}_{t} = 1 \\
&\Delta^{\mathcal{U}}_{\mathsf{u}} = -1, \quad \Delta^{\mathcal{U}}_{\mathsf{v}} = \frac{N-4}{N+8}, \quad \Delta^{\mathcal{U}}_{t} = \frac{N+14}{N+8}, \\
&\Delta^{\mathcal{V}}_{\mathsf{u}} = 1/3, \quad \Delta^{\mathcal{V}}_{\mathsf{v}} = -1, \quad \Delta^{\mathcal{V}}_{t} = 5/3,
\end{aligned} \tag{332}$$

while the dimensions about $\mathcal{M}$ are complicated and I don't want to write them down.

Anyway, we see that for small $N < 4$, the IR fixed point is $O(N)$ symmetric, despite the fact that the microscopic UV Lagrangian contains an interaction which breaks the $O(N)$ symmetry explicitly. This is therefore an example of an emergent symmetry. In line with our intuition from large N, this no longer occurs when N is made sufficiently large ("fluctuations become too weak at large N to ensure emergent symmetry").

Another thing to note is that if $N > 4$ and the flow starts with either $\mathsf{u}, \mathsf{v} < 0$, or if $N < 4$ and the flow starts with either $\mathsf{u} < 0$ or with v below the line connecting $\mathcal{G}$ to $\mathcal{M}$, then the flow takes us to negative values of u, v that lie outside of the region of stability for the model. When we leave the stability region this means that a previously neglected $\boldsymbol{\phi}^6$ term needs to be kept, and that a first-order transition occurs due to the shape of the potential: this then provides us with an example of a "fluctuation-induced" first order transition.

16 *Scale invariance + dimensional transmutation in quantum mechanics* ✓

Today we are mentioning a couple brief examples where dependence of various quantities on regulators occurs in scale-invariant quantum-mechanical problems. The point here is to illustrate that this is not just a phenomenon germane to QFT.

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Consider a Schrodinger equation of the form

$$\left(-\nabla^2 - \frac{\alpha}{r^2}\right)\psi = i\partial_t \psi. \tag{333}$$

This equation is left invariant under the re-scaling $r \mapsto \gamma r$, $t \mapsto \gamma^2 t$. Note that this scale invariance depends on the potential scaling as $1/r^2$, so that α is dimensionless. The scaling is generated by the operator $\mathcal{O} = \mathbf{r}\cdot\nabla + 2t\partial_t$, which does an anisotropic dilitaion in spacetime. The symmetry also gives rise to a conserved quantity, $\rho \equiv -\frac{1}{2}\mathbf{r}\cdot\nabla + t\partial_t$, since

$$[\rho, Ht] = 0, \tag{334}$$

meaning that ρ commutes with the time evolution operator e^{-iHt} and hence is conserved.

Scale invariance here manifests itself in the fact that the S-matrix is independent of E. Indeed, this can already be seen in the WKB approximation, where the phase shift in a wavepacket passing through the potential relative to the shift in the absence of the potential is

$$\delta = 2\int_0^\infty dr\, \left(\sqrt{E+\alpha/r^2} - \sqrt{E}\right) = 2\int_0^\infty dr\, \left(\sqrt{1+\alpha/r^2} - 1\right) \overset{\sim}{\to} 2\alpha \sinh^{-1}(\alpha/a), \tag{335}$$

where a is a (dimensionless, since we changed variables in the second step to a dimensionless r) short-distance cutoff. This is indeed independent of E. Furthermore, it is divergent and depends on a scale used to regulate the theory in much the same way as in examples of "dimensional transmutation" in QFT.

In two dimensions we have another option for a scale-invariant problem, since we can match the dimension of $-\nabla^2$ using a δ function potential:

$$\left(-\nabla^2 - \alpha\delta(\mathbf{r})\right)\psi = i\partial_t\psi. \tag{336}$$

Here the binding energy of the bound state depends on a regulator in the same way that e.g. the BCS gap or the mass in $1+1D$ large N σ models does: we have

$$k^2\psi_k - \alpha\psi(0) = E\psi_k \implies \frac{1}{\alpha} = \int d^2k \frac{1}{k^2 - E} = \pi\ln(\Lambda/E) \implies E = \Lambda e^{-1/\pi\alpha}, \tag{337}$$

where Λ is our UV cutoff. Thus even in quantum mechanics, divergences can manifest themselves in ways similar to the ones in QFT.

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17 *Fixed points for QCD β function* ✓

Today we're taking a very simple look at the BZ fixed point. This was suggested as an exercise at one of the bootstrap schools; I found the problem statement written online (by Komargodski).

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First, some trivial comments on why $\beta(\alpha_*) = 0$ means that α_* is an RG fixed point in a renormalizable QFT. For $\beta = \beta(\alpha, t)$ a general differential equation in terms of α and the RG time t, points where $\beta = 0$ are not in general fixed points. For example, suppose that $\beta = 2(t-1)$, so that $\alpha(t) = (t-1)^2$. Then $\beta = 0$ when $\alpha = 0$, but this is not a fixed point, since $\alpha''(t) \neq 0$ at $t = 1$. The reason why zeros of the β function are fixed points in practice is that the β function is always an autonomous DE—it never depends explicitly on the RG time (the RG flow only depends on the choice of coupling constants, not on what path was taken to get to them). Then since $\beta = \beta(\alpha)$, the root α_* with $\beta(\alpha_*) = 0$ has $\alpha(t) = \alpha_*$ as a constant solution; if β also depended on t then α_* would depend on t, and wouldn't be a fixed point. Note that the fact that we can take β to be autonomous isn't a priori obvious (and is only true up to terms that go to zero in Λ^{-1}), and is responsible for a lot of the utility of the RG.

Anyway, the beta function for non-Abelian gauge theory coupled to n fermions in a representation R is, to 2-loop, (in cond-mat conventions where $\beta > 0$ indicates that a coupling is relevant)

$$\beta(\alpha) = \frac{d\alpha}{dt} = 2(\beta_0 + \beta_1/\alpha^2), \tag{338}$$

where $t = -\ln \Lambda/\Lambda_0$ is the RG time and $\alpha = 16\pi^2/g^2$. The first term is

$$\beta_0 = 4n\frac{T_2(R)}{3} - \frac{11}{3}C_2(G). \tag{339}$$

The first order correction term is

$$\beta_1 = \left(\frac{20}{3}C_2(G) + 4C_2(R)\right) nT_2(R) - \frac{34}{3}(C_2(G))^2. \tag{340}$$

We will mainly be interested in having R be either then fundamental / anti-fundamental, or the adjoint. Note that $T_2(Ad) = C_2(G) > T_2(F)$, so that adjoint fermions push the theory closer to being IR-free (just because more fermion flavors mean more screening and hence less strongly coupled IR physics, and replacing fundamental fermions with adjoint fermions is roughly the same as increasing the number of flavors).

First take $\beta_0 < 0$. This is where the IR fixed point is stable. Mathematically, this is seen by nothing that for the fixed point to exist, $\beta_0 < 0 \implies \beta_1 > 0$. Then we see that

$$\frac{\partial \beta(\alpha)}{\partial \alpha} = -2\beta_1/\alpha^2 < 0. \tag{341}$$

The fact that the derivative of the beta function is negative at the fixed point guarantees IR stability. Anyway, we can solve for $\alpha(t)$ explicitly in terms of a productlog function, but just plotting the β function is more illuminating: for e.g. $\beta_0 = -1/2, \beta_1 = 1/2$, (this is different from a usual RG flow diagram involving two different couplings—here the y axis is α and the x axis is RG time. This is redundant since the flow of α doesn't depend on RG time, but the plot looks nice so what the hell[26]

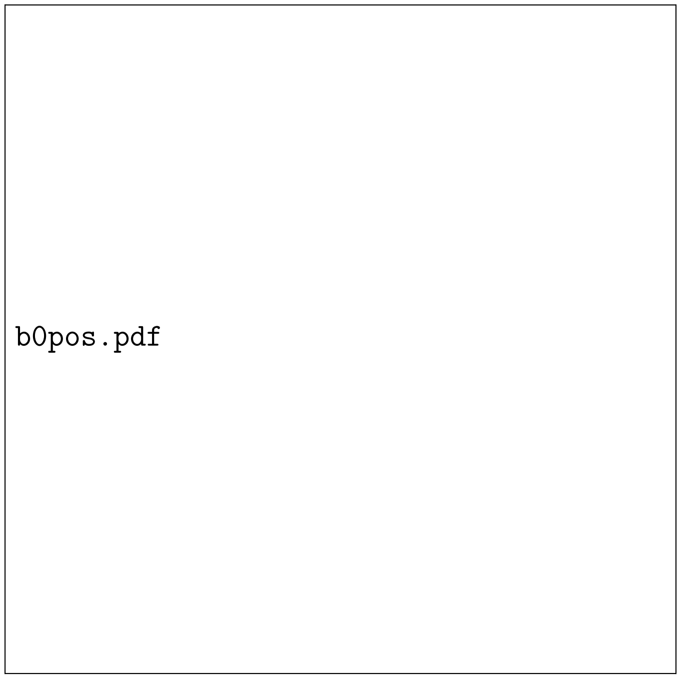
b0neg.pdf . (343)

Now suppose $\beta_0 > 0$—this is where the fixed point is IR unstable, since then we have

[26]If you prefer to have a plot of g^2 instead of α, here it is, in mini-form:

gsq.pdf (342)

$\partial_\alpha \beta(\alpha) > 0$ at the fixed point. Indeed, the RG flow is (for $\beta_0 = 1/2, \beta_1 = -1/2$)



(344)

Now we will focus on $G = SU(N)$, for which $C_2(G) = C_2(Ad) = N$. The beta function for $R = F$ the fundamental representation is

$$\beta_0^F = \frac{1}{3}\left(2n - 11N\right), \qquad \beta_1^F = -\frac{34}{3}N^2 + N\frac{13n}{3} - \frac{n}{N}, \tag{345}$$

while for the adjoint,

$$\beta_0^{Ad} = \frac{1}{3}(4Nn - 11N), \qquad \beta_1^{Ad} = N^2\left(-\frac{34}{3} + 26n\right) - 2n. \tag{346}$$

To have a fixed point, we need $\beta_1/\beta_0 < 0$. Therefore we define the two ratios

$$r_F \equiv (\beta_1/\beta_0)_F = \frac{-34N^2 + 13nN - 3n/N}{2n - 11N}, \qquad r_{Ad} \equiv (\beta_1/\beta_0)_{Ad} = \frac{(26n - 34)N - 6n/N}{4n - 11}. \tag{347}$$

We can plot the sign of the above ratios to figure out when a fixed point will occur. The rs will change sign when either β_0 or β_1 passes through zero. For the fundamental representation, we have (here the plots are just the signs of the relevant quantities; negative is blue and positive is cream)

confwindow_fund.pdf   beta0_fund.pdf

(348)

For the adjoint,

confwindow_ad.pdf   beta0_ad.pdf

(349)

Let N be fixed, let n^R_{*0} be the number of flavors at which $\beta^R_0 = 0$ (this is when the theory passes from conformal to deconfined), and let n^R_{*1} be the number of flavors at which $\beta^R_1 = 0$ (this is when the theory passes from being confined to being conformal). Then some algebra gives

$$n^F_{*0} = \frac{11N}{2}, \qquad n^F_{*1} = \frac{34N}{13 - 3/N^2} \tag{350}$$

for the fundamental and

$$n^{Ad}_{0*} = \frac{11}{4}, \qquad n^{Ad}_{*1} = \frac{34}{78 - 6/N^2} \tag{351}$$

for the adjoint. A sanity check is that $n^R_{0*} > n^R_{1*}$, since we know that when $n^R = 0$ we should have confinement.

Finally some comments on these two critical fermion numbers. The point n^R_{*0} is the exact location of the upper boundary of the conformal window, i.e. its location doesn't depend on our ignorance of the higher order terms in the expansion of β. To see this, consider n bigger than n^R_{*0}, so that $\beta_0 > 0$. Then if we start the flow at $1/\alpha$ infinitesimal, only β_0 contributes to the flow, and since here $\beta_0 > 0$, $1/\alpha$ is driven back to zero—the higher-order terms have no chance to contribute. On the other hand, suppose n is very slightly less than n^R_{*0}. If we start from $g^2 = 0$ then $\beta_0 < 0$ tells us that g^2 initially increases. This is true regardless of what the higher-order terms in β are. Now the fixed point we identified is at $(\alpha^*)^{-1} = -\beta_0/\beta_1$, which can be made arbitrarily small by tuning n arbitrarily close to n^R_{*0}. Therefore we can reach the fixed point after an arbitrarily short RG flow, which means that the fixed point can be reached, at least if we are infinitesimally close to n^R_{*0}, before any of the other terms in the β function expansion have a chance to contribute. Therefore n^R_{*0} indeed marks a sharp boundary between a region with a trivial free IR fixed point and a nontrivial (weakly) interacting one. This is similar to the WF fixed point in the ϵ expansion—we don't need to know the full expression for the β function in order to guarantee the existence of a nontrivial IR fixed point, since by tuning ϵ we can make the WF fixed point arbitrarily close to the Gaussian one.

In contrast, n^R_{*1} is not a sharp lower boundary for the conformal window. This is where $\beta_1 \to 0$, and so our expression for the fixed point is $(\alpha^*)^{-1} \to \infty$. If we start from $\alpha^{-1} = 0$ then the flow takes a long time to get to α_*, and by the time we reach it, the higher-order terms in the β function will contribute, shifting the location of the fixed point. Hence, the lower bound of the conformal window cannot be reliably calculated in perturbation theory.

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18 *Fluctuation-induced first-order transitions in $U(1)$ gauge theory a-la HLM—the cmt way of thinking about the Coleman-Weinberg potential* ✓

Today we're going to work out some of the calculations implicit in the classic HLM paper on the SCing transition being first order once the fluctuations of the gauge field are properly taken into account. This is something I'd heard about but never actually looked at before today.

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Type I SCs

The first part of the HLM paper talks about type I superconductors. Here the argument for a first-order transition is fairly straightforward. We start from the usual

$$H = \int \left(K|(\nabla - iq\mathbf{A})\psi|^2 + t|\psi|^2 + \frac{u}{2}|\psi|^4 + \frac{1}{4}F_{ij}F^{ij} \right). \tag{352}$$

The point is that type I SCs order so well that they are basically described by MFT, and the effect of fluctuations on the magnitude of the order parameter $|\psi|$ is essentially negligible—using the Ginzburg criterion (another diary entry) one can establish the irrelevance (in the colloquial sense) of fluctuations in $|\psi|$ (and hence in ψ, since as usual the HLM paper is rather caviler about issues of gauge invariance) for all reduced temperatures t expect those absurdly close to 0.

Given that we can take $\psi = \psi_0$ to be constant (and $\mathbb{R}$ wolog), all of the business is in the integral over the gauge field $\mathbf{A}$. Since $\mathbf{A}$ appears qudratically the integral is easy:

$$F[\psi_0] = \int \left(t\psi_0^2 + \frac{u}{2}\psi_0^4 \right) + V\frac{3}{2}\int_0^\Lambda d^3\mathbf{q}\, \ln[|\mathbf{q}|^2 + m^2], \qquad m^2 \equiv Kq^2\psi_0^2, \tag{353}$$

with $V = \int_{\mathbf{x}}$. Here the divvying up of the scaling dimensions in the mass is that $[\psi_0] = 0$ since we are in a regime where ψ doesn't fluctuate, and $[\sqrt{K}q] = 1$. Physically, $m = \xi_L^{-1}$, with ξ_L the London penetration depth.

Anyway, the integral gives

$$\int dq\, q^2 \ln[q^2 + m^2] = -\frac{2}{3}m^3 \arctan[q/m]\Big|_0^\Lambda + \dots, \tag{354}$$

where the $\dots$ contains a constant (diverging with Λ) and a Λm term that can be killed off with a mass counterterm for ψ. In the limit where the cutoff $\Lambda \gg m$ (here we are at t small enough that the UV cutoff length is small compared to m^{-1}, but t large enough so that ψ can be accurately treated as fluctuationless) we then get

$$F[\psi_0] = \int \left(t\psi_0^2 + \frac{u}{2}\psi_0^4 - \frac{1}{4\pi}K^{3/2}q^3\psi_0^3 \right). \tag{355}$$

The fact that the integral over $\mathbf{A}$ produces a cubic term with negative coefficient is the reason why the transition is secretly first-order. The fact that we'll get a term that goes as an odd power of ψ_0 is rather inobvious at first, since $\mathbf{A}$ only couples to ψ_0^2 in H, which means that the effective action after integrating out $\mathbf{A}$ will only contain diagrams with an even number of ψ_0s (the expansion of the $\ln[q^2 + m^2]$ term only contains terms that go as $\psi_0^{2n}, n \in \mathbb{N}$). Therefore it was important to actually do the log integral exactly (and indeed, there would have been no reason to expand the $\ln[q^2 + m^2]$ and truncate after the first few terms; we are not working at weak coupling or anything).

Type II SCs

Type II SCs are more difficult, since fluctuations in $|\psi|^2$ can no longer be ignored, forcing us to do a more careful RG analysis.

One can either break up ϕ into $\mathbb{R}$ and $i\mathbb{R}$ parts or work with $\phi, \phi^\dagger$ as separate variables. The advantage of the latter approach is the economy of notation, while its disadvantage is that I could never quite figure out the correct symmetry factors to use in diagrams involving lots of ϕs. Because of these combinatorial failures of mine, we will break up the field ψ_i as

$$\psi_j = \frac{\phi_j + i\phi_{j^*}}{\sqrt{2}}, \tag{356}$$

with ϕ_j, ϕ_{j^*} real fields and the $1/\sqrt{2}$ is there in order to get canonically normalized kinetic terms for the ϕs. We will adopt the (somewhat unusual) choice that $[\phi_j] = (d-2)/2 = 1 - \varepsilon/2$, but $[A_\mu] = 1$. This ensures that the coupling between A_μ and the bosons is exactly dimensionless, but it means that the kinetic term for A_μ, which in our conventions is where the gauge coupling g lives in the action, comes with an extra factor of $\Lambda^{-\varepsilon}$ in order that g^2 can remain dimensionless. The action is then, in $i\mathbb{R}$ time,

$$\begin{aligned} S = \int \Bigg(\sum_{j=1}^{2n} \left[\frac{1}{2} d\phi_j \wedge \star d\phi_j + \frac{1}{2} A_\mu A^\mu \phi_j^2 + \frac{u}{8} \Lambda^\varepsilon \phi_j^4 \right] + \frac{u}{4} \Lambda^\varepsilon \sum_{i<j}^{2n} \phi_i^2 \phi_j^2 \\ + \sum_{j=1}^{n} \frac{1}{2} A_\mu (\phi_j \partial^\mu \phi_{j^*} + \phi_{j^*} \partial^\mu \phi_j) + \frac{\Lambda^{-\varepsilon}}{2g^2} F_A \wedge \star F_A \Bigg) \end{aligned} \tag{357}$$

where u and g^2 are both properly dimensionless.

The Feynman rules are then (you need to actually do the Fourier tform to see how the $A\phi\phi$ vertex works)

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(358)

where we are not including symmetry factors in the displayed vertices. The momentum dependence of the $A\phi\phi$ vertex is assigned by taking all the momentum arrows to point inwards, and then taking the momentum of the ϕ_j minus the momentum of the ϕ_{j^*}.

1loopdiags.png

Figure 3: The symmetry factors and coefficients for the relevant diagrams for getting β_u; not shown are momenta and factors of $g^2\Lambda^\varepsilon$ from the gauge propagators.

The easiest beta function to get is the one for the gauge coupling. This is because the renormalization of g^2 is dictated by gauge invariance: $g^2 A_\mu A^\mu |\psi|^2$ must renormalize in the same way as $|\nabla\psi|^2$, meaning that g^2 is renormalized exactly in the opposite way as $A_\mu A^\mu$. This means the flow of g^2 is found to 1-loop just by computing the usual polarization bubble[27]:

$$\boxed{\texttt{polarizationbubble.png}} = -\frac{ni^2}{4} A^\mu_q A^\nu_q \int_k \frac{(2k+q)_\mu (2k+q)_\nu}{k^2(k+q)^2}, \tag{359}$$

which is dealt with using Feynman parameters as usual; the manipulations to get this with momentum dependence of the form $\Pi^T_{\mu\nu}(q)$ are in e.g. P&S or an earlier diary entry. The minus sign on the RHS is unfortunately crucial to keep track of[28]: if we take the lower leg of the loop to be a ϕ_j line and the upper leg to be a ϕ_{j^*} line, the the left vertex contributes a factor of $-i(2k+q)$, while the right vertex contributes $+i(2k+q)$.

I find momentum shell RG to be conceptually the clearest scheme for dealing divergences; hence the ($\mathbb{R}$ time) divergent integrals over k that we will encounter be dealt with by[29]

$$\begin{aligned}\int_{\Lambda - d\Lambda < |k| < \Lambda} d^dk\, \frac{1}{k^4} &= A(S^d) \int_{\Lambda - d\Lambda}^{\Lambda} \frac{dk}{(2\pi)^d} \frac{k^{d-1}}{k^4} = \frac{2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \frac{\Lambda^{d-4} - (\Lambda - d\Lambda)^{d-4}}{d-4} \\ &\equiv \Lambda^{-\varepsilon} \gamma(d) dt,\end{aligned} \tag{360}$$

where the RG time is $dt = d\ln\Lambda$ (here $d\Lambda > 0$ means that the cutoff is getting *smaller* in the way we've written stuff, so that indeed the sign is $dt = +d\ln\Lambda$), and we've defined

$$\gamma(d) \equiv 2\pi^{d/2}/((2\pi)^d \Gamma(d/2)). \tag{361}$$

This means that the polarization bubble gives a contribution to the photon self-energy that looks like

$$\boxed{\texttt{polarizationbubble.png}} \to -n \frac{\gamma(d)}{4 \cdot 3} \Lambda^{-\varepsilon} dt\, A^\mu_q A^\nu_q \Pi^T_{\mu\nu}(q) q^2, \tag{362}$$

[27] We don't need to consider the bubble coming from a single $A^2\phi^2$ vertex since that term is q-independent and just cancels against the q-independent part of the diagram below

[28] Although for this particular diagram the sign is fixed by unitarity, since it contributes to the A self energy.

[29] Again, we are in $i\mathbb{R}$ time. If we were in $\mathbb{R}$ time we'd need to do a Wick rotation $k^0_E = -ik^0_L$ (unfortunately the minus sign is important) in the first step.

where the internal scalars live at energies between $\Lambda - d\Lambda, \Lambda$ and the $1/3$ comes from the Feynman parameter integrals that we've not written out.

The new gauge coupling is then found to 1-loop order by putting the contributions from these diagrams into the new effective action e^{-S} (remember that $[A_q] = [A(x)] + d$)

$$\begin{aligned}\int_0^\Lambda d^dq\,\frac{\Lambda^{-\varepsilon}}{g_{t+dt}^2}A_q^\mu A_q^\nu q^2\Pi_{\mu\nu}^T(q) &= \int_0^{\Lambda-d\Lambda} d^dq\,\Lambda^{-\varepsilon}\frac{1}{g_t^2}\left(1+g_t^2 n\gamma(d)dt/12\right)A_q^\mu A_q^\nu q^2\Pi_{\mu\nu}^T(q)\\ &= \int_0^\Lambda d^dq\,\frac{\Lambda^{-\varepsilon}(1-\varepsilon dt)}{g_t^2(1-ng_t^2\gamma(d)dt/12)}A_q^\mu A_q^\nu q^2\Pi_{\mu\nu}^T(q)\\ &= \int_0^\Lambda d^dq\,\frac{\Lambda^{-\varepsilon}}{g_t^2(1+[\varepsilon-ng_t^2\gamma(d)/12]dt)}A_q^\mu A_q^\nu q^2\Pi_{\mu\nu}^T(q)\end{aligned} \tag{363}$$

From this we can read off the beta function, viz.

$$\beta_{g^2} = \frac{dg^2}{dt} = \varepsilon g^2 - \frac{n\gamma(d)}{12}g^4. \tag{364}$$

Now when we expand in ε, we can use $\gamma(4-\varepsilon) = 1/8\pi^2 + O(\varepsilon)$ and drop the $O(\varepsilon)$ terms on the assumption that the fixed point we're looking for has $g^2 \sim O(\varepsilon)$ so that εg^4 terms are negligible in our approximation. Therefore

$$\beta_{g^2} = \varepsilon g^2 - \frac{n}{4\cdot 24\pi^2}g^4. \tag{365}$$

Or, in terms of the fine structure constant $\alpha \equiv g^2/4\pi$,

$$\beta_\alpha = \varepsilon\alpha - \frac{n}{4\cdot 6\pi}\alpha^2, \tag{366}$$

with the second term $n/4$ times what it is in QED (modulo the factor of $1/4$, which is due to the $1/\sqrt{2}$ normalization in the decomposition of the complex field into its real constituent parts; the gauge coupling in bosonic QED would then usually be half of of the g^2 written here), just because we have n bosons that each act like $1/4$ of a fermion when we are (close) to four dimensions.

Now we can work on β_u. Since we already saw how the manipulations for getting the effective coupling worked in the case of g^2 (finding the term generated by integrating out the high energy modes, putting it back into the action, shifting the cutoff, etc.) we will be more terse / schematic in what follows.

We will look at the renormalization of the four-point vertex where all the flavors on the outgoing legs are identical, wolog. First, consider the series of diagrams in row a) of Figure 3 (in the second trio, $i \neq j$). All of the s, t, u channels give the same result for each choice of flavors in the loop. We see that these sum to

$$\text{diagrams in row a)} = \left(\frac{3u^2}{2}(9+(2n-1))\gamma(d)\Lambda^\varepsilon dt\right)\phi_j^4 \to \left(\Lambda^\varepsilon\frac{3u^2}{8\pi^2}(n+4)dt\right)\phi_j^4. \tag{367}$$

Since the ϕ_j^4 vertex appears in diagrams with coefficient $(u/8)4! = 3u$, we see that the above series of diagrams makes a contribution to u_{t+dt}, and hence to β_u, of

$$\beta_u \supset -\frac{n+4}{8\pi^2}u^2, \tag{368}$$

which matches with the result in the asymptotic freedom paper, a sign that our combinatorics have worked so far. As another confidence boost, this also matches the appropriate term in the beta function for the $O(2n)$ vector model, see e.g. Zinn-Justin page 649.[30]

The next diagram to tackle is b). This one is easy, since it's just the usual $\int k^{-4}$ integral:[31]

$$\text{b) diagrams } = \left(3 \cdot \frac{2}{2^2} g^4 \Lambda^\varepsilon \gamma(d) dt\right) \phi_j^4, \tag{369}$$

where we have $1/2^2$ from the vertices, 3 from the associated crossed diagrams, and $(2!)^2/2$ from the photon propagators. This will give a contribution to the u beta function like

$$\beta_u \supset -\frac{1}{2 \cdot 8\pi^2} g^4. \tag{370}$$

The other g^4 diagrams in c) are similar; they add up to give

$$\text{c) diagrams } = \frac{3}{16 \cdot 8\pi^2} g^4 \implies \beta_u \supset -\frac{1}{16 \cdot 8\pi^2} g^4, \tag{371}$$

where the 3 comes from the crossed diagrams *ethan: really? need to come back and be more vigilant about these pesky symmetry factors*

The last 1PI diagram is d). It evaluates to

$$\text{d) diagram } = -\left(6 \frac{ug^2}{4} \frac{1}{8\pi^2}\right) \phi_j^4, \tag{372}$$

where the $6 = \binom{4}{2}$ is the number of ways of putting on the photon leg, and the crucial minus sign comes from $i^2 k_\mu(-k^\mu)(-1)^3 = -1$, where the first two factors are from the photon couplings and the $(-1)^3$ comes from the fact there are three vertices and that the action appears as e^{-S} in the path integral. This then gives a contribution

$$\beta_u \supset +\frac{ug^2}{2 \cdot 8\pi^2}. \tag{373}$$

Lastly we need to do the scalar wavefunction renormalization (the diagrams in e). The scalar wavefunction renormalization is essentially the same as the photon wavefunction renormalization, since both polarization bubbles involve the same couplings (again, the $A^2\phi^2$ coupling is unimportant since it is momentum-independent and is killed by the momentum-independent part of the following diagram). Indeed, paying careful attention to signs,

scalarbubble.png

$$= +\frac{1}{4} g^2 \Lambda^\varepsilon \int \frac{(2q-k)^\mu (2q-k)_\mu}{(k-q)^2 k^2} \phi_{j;q}^2. \tag{374}$$

[30] Just to check that this wasn't an accident, we can go through the combinatorics for the case where the vertex instead has two i legs and two j legs, with $i \neq j$. The diagrams give, working from left to right along the first trio of row a), a contribution to the new u vertex of in agreement with the beta function when all the outgoing legs are identical; this also serves as a sanity check of the obvious fact that if there is no $O(2n)$ anisotropy when we start the flow, no anisotropy will be generated.

[31] We'll be working in a gauge where the photon propagator is just $\delta_{\mu\nu}/k^2$.

The integral is the same as the one we did for the photons:

$$\boxed{\texttt{scalarbubble.png}} = -\frac{\gamma(d)g^2}{4\cdot 3}dt\,\phi_{j;q}^2 q^2 \tag{375}$$

therefore the wavefunction renormalization happens by

$$\phi_{t+dt}^2 = \left(1+\frac{g^2\gamma(d)dt}{12}\right)\phi_t^2 \equiv Z_\phi \phi_t^2. \tag{376}$$

Putting this all together and accounting for the εdt contribution from the $d^d q$ measure, we have

$$\frac{4!u_{t+dt}}{8}\phi_{j;t+dt}^4 = (1+\varepsilon dt)\phi_{j;t}^4\left(\frac{4!u_t}{8} - dt\left[\frac{3u_t^2}{8\pi^2}(n+4) - \frac{g^4}{8\pi^2}(3/2+3/16) + \frac{3g^2u}{2\cdot 8\pi^2}\right]\right) \tag{377}$$

Plugging in for the wavefunction renormalization,

$$u_{t+dt} - u_t = \varepsilon u_t dt + \frac{dt}{8\pi^2}\left(-u_t^2(n+4) - g^4\frac{9}{16} + \frac{1}{3}g_t^2 u_t\right) \tag{378}$$

Therefore the beta function is

$$\beta_u = \varepsilon u - \frac{1}{8\pi^2}\left(u^2(n+4) + \frac{9}{16}g^4 - \frac{1}{3}g^2 u\right). \tag{379}$$

Note the crucial *positive* g^2u term on the RHS, which came from the diagram d (the scalar wavefunction renormalization also contributed to the g^2u with opposite sign[32], but was importantly smaller than the contribution from diagram d).

To present the beta functions together in a nice way, it is helpful to define

$$\lambda \equiv g^2/(4\cdot 8\pi^2), \qquad v \equiv u/8\pi^2. \tag{380}$$

Then the beta functions are

$$\beta_\lambda = \varepsilon\lambda - \frac{n}{3}\lambda^2, \qquad \beta_v = \varepsilon v - (n+4)v^2 + 4v\lambda/3 - 9\lambda^2. \tag{381}$$

The signs are all correct and β_λ and the first two terms in β_v are correct[33], but despite trying over and over and over to figure out where I went wrong with symmetry factors, the last two terms in β_v likely don't have the right numerical coefficients.

[32] Again, this sign is fixed by unitarity.

[33] Actually there seemed to be an annoying discrepancy of 1/2 between the $n+4$ terms in the beta functions quoted in the HLM paper and those from the asymptotic freedom paper—but comparison with e.g. the $O(2n)$ model means that the $n+4$ term in β_v is correct as written.

In any case, these algebraic quibbles don't have much bearing on the moral lessons to be taken away from this calculation. We would like to know when a second order transition is possible in three dimensions, i.e. when there exists a nontrivial fixed point with $\lambda \neq 0$ at $\varepsilon = 1$. To determine this we just solve the above β functions: besides the Gaussian and pure gauge fixed points there are two fixed points with $v \neq 0$; the one with smaller v is at

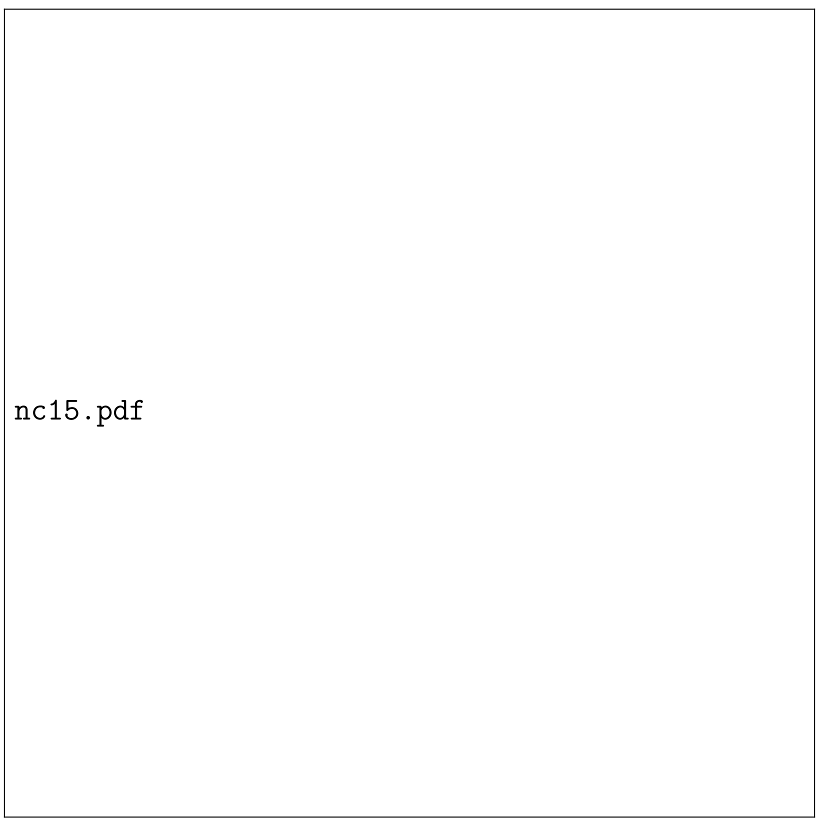
$$v_* = \frac{4n + n^2 - n\sqrt{n^2 - 316n - 1280}}{2(4n^2 + n^3)}, \tag{382}$$

which of course does not exist until the thing in the square root is positive. The condition on this turns out to be that (again this is wrong because we didn't get the exactly correct β functions, but morally you get the point)

$$n > n_* = 320. \tag{383}$$

Therefore we need quite a lot of flavors to stabilize a second order transition!

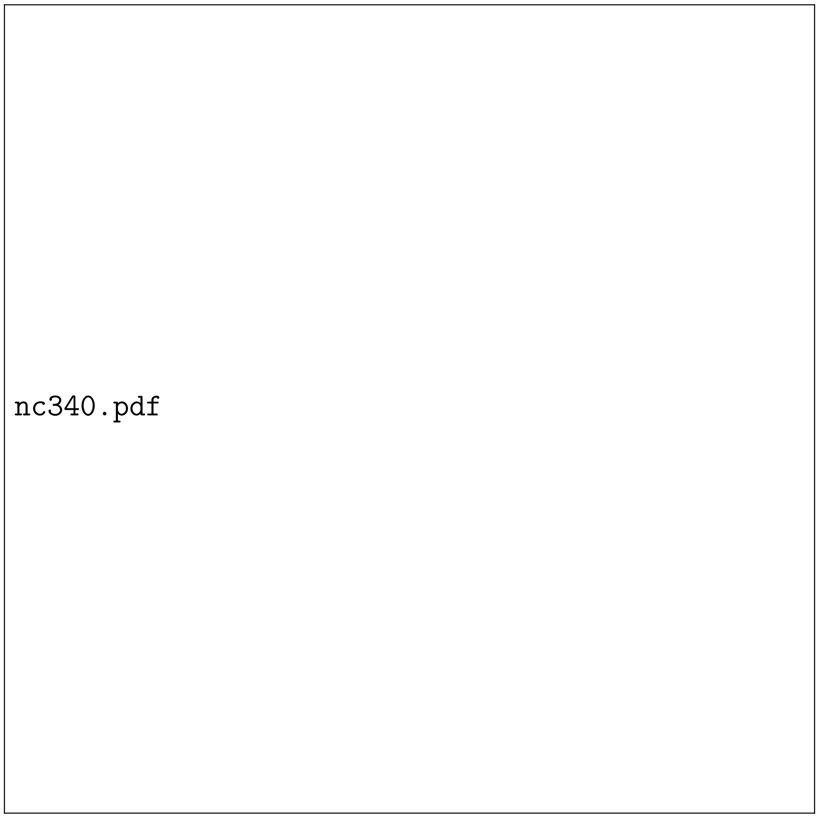
The first order transition is of course manifested in the RG flows by a runaway flow towards negative v. Working in $\varepsilon = 1$, when $n \ll n_*$ the RG flow looks like



(384)

where the y axis label is what Mathematica thinks λ is supposed to look like. Both the Gaussian fixed point and the gauge-less WF fixed point are unstable to a first-order transition. As n increases to near n_* the WF point on the $\lambda = 0$ axis gets closer and closer the Gaussian one (v_* for the gauge-less WF point goes down as $1/n$, asymptotically). When

$n \gtrsim n_*$, we instead have



(385)

which now shows that the Guassian fixed point is IR stable for a nonzero region of parameter space. The nontrivial fixed point that describes the second order transition that exists when $n > n_*$ is invisible in this picture; even at the minimal possible n of n_*, this fixed point is at $v_* = 1/640$.

Something worth noting here is that our determination of n_* strictly speaking does not work when $\varepsilon = 0$, since when $\varepsilon = 0$ there are no nontrivial fixed points (λ_* and v_* are both $\propto \varepsilon$ at every fixed point). However, it is again n_* which determines the IR stability of the Gaussian FP when $\varepsilon = 0$ (the Gaussian FP is always attractive along the $\lambda = 0$ line when $\varepsilon = 0$; we want to know when it is attractive for any non-zero value of λ). Indeed, the Gaussian FP will be IR stable provided that there exists some flow towards the origin with $\lambda = sv$ at late RG times, with $s > 0$. We plug this relation into the β functions and then solve for s: the resulting quadratic equation will have a positive solution only for certain n, and doing the calculation confirms that this n is exactly n_*. When $n > n_*$ the RG flow looks pretty much the same as it does when we had $\varepsilon = 1$; when $n < n_*$ the flow is boring: all initial points (except the v axis) just flow straight to negative v.

Note that this model is also often studied in the context of the large n expansion (with $g^2 n, un$ held fixed in a t' Hooft limit). The large n expansion is well suited for computing things like critical exponents and scaling dimensions, but extracting the above result about the first order phase transition is harder, since taking $n \to \infty$ doesn't help one to compute β functions at strong coupling. To find the first order transition in the large n expansion, one should presumably just directly calculate the free energy order by order in $1/n$, and look for a first-order transition directly in the singularity of the free energy (this is what we did in the diary entry on 2d matrix models). However, if one actually does this, it turns out (I think) that one needs to go beyond leading order in $1/n$—the crucial diagrams that led to the first-order transition were the ones in d) of Figure 3, which are suppressed in the t' Hooft limit relative to terms like the usual 1-loop all-scalar corrections to the $|\phi|^4$ vertex, which have an extra n. In the HLM paper they write down the anomalous dimension for ϕ, but I don't quite see how this gives you information about the phase transition.

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19 RG flow of $\mathbb{Z}_2^L$-symmetric coupled 2+1D Ising CFTs and gradient flow ✓

Today we will be examining the RG flow in a sum of L coupled Ising models:

$$S = \int d^{4-\varepsilon}x \left(\frac{1}{2}\sum_i \left[(\partial\phi_i)^2 + \frac{t_i\Lambda^2}{2}\phi_i^2 \right] + \frac{\Lambda^\varepsilon}{8}\sum_{i,j}\phi_i^2\phi_j^2 g_{ij} \right) \tag{386}$$

where the sums are for $i, j = 1, \ldots, L$. We will be restricting to couplings that are invariant under a $\mathbb{Z}_2^L$ action of $\phi_i \mapsto f_i\phi_i$, where each $f_i \in \pm 1$ can be chosen independently. We will show that this theory is unstable to a fluctuation-induced first order transition (at least within the ϵ expansion).

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The matrix g_{ij} can be taken to be a symmetric matrix, and for stability reasons we may take it to be positive definite. More precisely, the potential is stable as long as g_{ij} is positive semi-definite. However, if $\det g = 0$, then there exists a free direction in field space—by a change of basis, we get a decoupled flavor which does not interact. Therefore if such a g_{ij} is a fixed point it will for sure be unstable; since we are only interested in stable fixed points, degenerate g_{ij}s can be ignored.

Now enumerating all the RG fixed points for general L is essentially impossible (the symmetry group $\mathbb{Z}_2^L$ possesses too many quartic invariants). In fact even in the case of small L like $L = 4$, there are many fixed points—looking at the literature, it's actually unclear if there are any values of $N > 2$ for which the fixed points have been completely classified.

However, if we restrict our attention only to *stable* fixed points (i.e. stable with respect to all quartic perturbations respecting $\mathbb{Z}_2^L$), we will see that the situation becomes much more tractable. In what follows we will first do some direct calculations for the simplest choices of couplings, and then turn to some more powerful general statements.

First, we will assume that the g_{ij} couplings are translation-invariant and symmetric. In accordance with this, we will use the notation $g_{ij} = g_{i-j} = g_{j-i}$. Now from past experience we know that $L = 2$ critical Ising models coupled by their energy operators flows to the (relativistic) XY model, while a stack of $L = 3$ flows to the $O(3)$-symmetric fixed point; both fixed points are stable.

To find out what happens for general L, we need (at least) the 1-loop beta functions.[34] These are calculated in the usual way: the counterterms approach gives (no real need to keep track of the mass term since in dim reg t doesn't appear in the beta functions of the quartic couplings), with the notation $g_{i-j} \equiv g_{ij}$,

$$\beta_{g_j} = \varepsilon g_j - 2g_j^2 - 2g_0 g_j - \frac{1}{2}\sum_k g_k g_{j-k}. \tag{387}$$

Note how g_0 is singled out; this occurs because of the different symmetry factors for vertices that involve four identical indices.[35] The matrix determing the scaling dimensions at a given fixed point is accordingly

$$\mathcal{B}_{jl} \equiv \frac{\partial \beta_{g_j}}{\partial g_l}\Big|_{g_j = g_j^*} = \delta_{jl}(\varepsilon - 4g_j^* - 2g_0^*) - 2g_j^*\delta_{l0} - g_{j-l}^* \tag{389}$$

First, we obviously have the usual $g_0 \neq 0, g_{j\neq 0} = 0 \ \forall\ j$ Ising$^{\oplus L}$ fixed point; here $g_0^* = \frac{2}{9}\varepsilon$, g_0 is irrelevant with $y_{g_0} = -\varepsilon$, and all of the $g_{j\neq 0}$s are equally relevant with eigenvalue $y_{g_{j\neq 0}} = \varepsilon/3$.

Because of the last term in (390), it is impossible to have a fixed point with strictly finite-ranged couplings if any of the $g_{j>0}$ are non-zero, at least if we restrict to positive couplings. The solutions with maximally long-ranged coupling between the layers are those where the couplings are symmetric under the action of S_L: these are the cubic fixed points, with symmetry group $G = \mathbb{Z}_2^L \rtimes S_L$. Since G has only two quartic invariants, in this case there are only two distinct β functions to solve. Letting $h \equiv g_{j>0}$, the fixed points are determined by

$$\varepsilon g_0 = \frac{9}{2}g_0^2 + \frac{L-1}{2}h^2, \qquad \varepsilon h = 3g_0 h + \frac{L+2}{2}h^2. \tag{390}$$

The fixed points are thus

$$(g_0^*, h^*) = \begin{cases} \mathcal{C}\ : & \left(\dfrac{2\varepsilon}{9}(1-1/L), \dfrac{2\varepsilon}{3L}\right) \\ \mathcal{S}\ : & \left(\dfrac{2\varepsilon}{8+L}, \dfrac{2\varepsilon}{8+L}\right) \end{cases} \tag{391}$$

[34] It will turn out that the 1-loop answers will be good enough for assessing stability within the context of small ε for all $L \neq 4$; for $N = 4$ there's a cancellation which necessitates going to two-loop.

[35] If we prefer, in Fourier space this is

$$\beta_{g_p} = \varepsilon g_p - \frac{2}{2\pi}\int_q g_q g_{p-q} - g_p \frac{2}{2\pi}\int_q g_q - \frac{1}{2}g_p^2. \tag{388}$$

Unfortunately since the expression for β_{g_j} involves both multiplication and convolution in the layer index, Fourier transforming doesn't really help.

$\mathcal{S}$ is the usual $O(L)$-symmetric fixed point, while the cubic fixed point $\mathcal{C}$ goes over to Ising$^{\oplus L}$, up to $O(1/L)$ corrections. In this case the matrix determining the scaling dimensions is

$$\mathcal{B} = \begin{pmatrix} \varepsilon - 9g_0^* & -h^* & -h^* & \dots \\ -3h^* & \varepsilon - 4h^* - 3g_0^* & -h^* & \dots \\ -3h^* & -h^* & \varepsilon - 4h^* - 3g_0^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{392}$$

Diagonalizing this, we find three different eigenvalues (now temporarily writing $g = g_0^*$ and $h = h^*$ for notation's sake):

$$\mathrm{Spec}(\mathcal{B}) = \begin{cases} \lambda_{L-2} = \varepsilon - 3g - 3h \quad m = L-2, \\ \lambda_{\pm} = \varepsilon - 6g - \dfrac{L+2}{2}h \pm \dfrac{1}{2}\sqrt{36g^2 - (24+12L)gh + (L^2+16L-8)h^2} \quad m = 1 \end{cases} \tag{393}$$

where m denotes the multiplicity. The $\mathcal{S}$ fixed point is unstable: the eigenvalues of $\mathcal{B}$ are

$$\mathrm{Spec}(\mathcal{B})|_{\mathcal{S}} \,:\, \lambda_{L-2} = \lambda_+ = \frac{\varepsilon(L-4)}{8+L}, \qquad \lambda_- = -\varepsilon, \tag{394}$$

so that as $L \to \infty$ there are $L-1$ relevant parameters, with eigenvalues $y = +\varepsilon$. As a sanity check, these match exactly with the scaling dimensions for the $O(N)$-symmetric fixed point found in Cardy's book.

At $\mathcal{C}$, $\mathcal{B}$ has eigenvalues

$$\mathrm{Spec}(\mathcal{B})|_{\mathcal{C}} \,:\, \lambda_{L-2} = \frac{\varepsilon(L-4)}{3L}, \qquad \lambda_+ = \frac{\varepsilon(4-L)}{3L}, \qquad \lambda_- = -\varepsilon. \tag{395}$$

So this is only barely more stable than the $\mathcal{S}$ fixed point: we still have $L-2$ relevant parameters at large L, although the relevant deformations are less relevant than the ones at the $\mathcal{S}$ fixed point ($y \to \varepsilon/3$ versus $y \to \varepsilon$). Note that the scaling dimensions of the majority of the couplings have only changed with respect to the decoupled fixed point by an amount $-4\varepsilon/3L$, which vanishes as $L \to \infty$. This makes sense because $\mathcal{C}$ moves closer to the decoupled fixed point as $L \to \infty$, but it also means that we still have a long way to flow before we reach stability.

Anyway, at this point, attempting to brute-force solve the β functions and look for the stable points quickly gets out of hand. Additionally, restricting ourselves to translationally-invariant couplings is a bit fine-tuned—I don't see any reason a priori why translationally-non-invariant couplings would necessarily yield only unstable fixed points.[36] We could at-

[36] For example, a tetragonally-symmetric solution with non-zero couplings $g_{1,2} = g_{3,4} = g_{5,6} = \dots$ exists, although it turns out to be unstable (although if we cranked up these couplings and let the layers RG flow in pairs we would get a stack of XY models, which we know is stable).

tempt an analysis of the β functions in the general case, [37] but this turns out not to be necessary, as long as we are only interested in addressing the question of stability. In fact, we can use some more general results about the beta functions in ϕ^4 theories to prove that *no stable fixed points exist* for any $L > 4$.

The most concise way of going about the following is to formulate the RG flows as gradient flows on a certain space of symmetric tensors. I learned about this from Zinn-Justin's QFT and critical phenomena book—in what follows we will derive the results that are important for us. For this, let us momentarily step back to the more general case of a L-component theory with interactions g_{ijkl}, where g_{ijkl} is a symmetric tensor that satisfies $v^i v^j v^k v^l g_{ijkl} > 0$ for any real vector $v \in \mathbb{R}^L$.[38] The β function is (re-scaling away numerical factors from the loop integration as usual)

$$\beta_{ijkl} = \varepsilon g_{ijkl} - \sum_{mn} (g_{ijmn} g_{mnkl} + g_{ikmn} g_{mnjl} + g_{ilmn} g_{mnkj}). \tag{397}$$

The three quadratic terms are just the s, t, u channels (the ways of partitioning the $ijkl$ indices into unordered pairs). This simple form is why we've momentarily generalized away from the case with $\mathbb{Z}_2^L$-symmetric couplings—when we keep the couplings arbitrary the symmetry factors are much simpler to keep track of; if we worked directly with the $\mathbb{Z}_2^L$-symmetric couplings we'd have lots more δs floating around.

Anyway, the point of using this notation is that we can write

$$\beta_{ijkl} = \frac{\delta}{\delta g_{ijkl}} \mathcal{U}(g), \qquad \mathcal{U}(g) = \frac{\varepsilon}{2} \sum_{ijkl} g_{ijkl}^2 - \sum_{ijklmn} g_{ijkl} g_{klmn} g_{mnij}. \tag{398}$$

To save space, we will use the multiindex notation $\beta_I = \delta_I \mathcal{U}(g)$, where $g_{ijkl} = g_I$. One technical point that Zinn-Justin glosses over: since we are working only with symmetric tensors, we need to use the variational derivative instead of the partial derivative, since we are in a constrained space where the components of g_{ijkl} are not independent variables—it makes no sense to take the derivative with respect to g_{ijkl} while keeping g_{ijlk} fixed ($\partial_I \mathcal{U}(g) \neq \beta_I$, the difference coming in combinatorial factors from the symmetry). We can still use the partial derivative, but we have to compensate for the fact that the partial derivative is operating in a bigger space by dividing out by the symmetry factor of the tensor in question. Therefore we can write

$$\frac{\delta}{\delta g_I} = M^{IJ} \frac{\partial}{\partial g^J}, \tag{399}$$

[37]For posterity's sake, in this case the CPT approach gives (with $u = g_{ii}$)

$$\begin{aligned} \beta_u &= \varepsilon u - 36u^2 - 4 \sum_k g_{1k} g_{1k} - 4ut \\ \beta_{g_{ij}} &= \varepsilon g_{ij} - 12u g_{ij} - 8 \sum_k g_{ik} g_{kj} - 4 g_{ij} t \\ \beta_t &= 2t - 2t^2 - 6ut - \sum_j g_{1j} t - 48u^2 - \sum_j g_{1j} g_{1j}. \end{aligned} \tag{396}$$

[38]The $\mathbb{Z}_2^L$-symmetric case of interest would be $g_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl} g_{ik} + \delta_{ik}\delta_{jl} g_{ij} + \delta_{il}\delta_{jk} g_{ij})$.

where the metric is

$$M^{IJ} = \delta^{IJ}\frac{1}{N_I}, \tag{400}$$

with N_I the number of distinct permutations of I (e.g. $N_{1234} = 4!$, $N_{1223} = \binom{4}{2}$, etc.). In what follows we will use ∂s, and raise / lower indices with this metric, so that e.g. $\beta^I = M^{IJ}\partial_J \mathcal{U}(g)$. It then follows that $\mathcal{U}(g)$ is an RG monotone: letting t be RG time,

$$\frac{d\mathcal{U}(g)}{dt} = \beta^J \partial_J \mathcal{U}(g) = M^{JK}\partial_K \mathcal{U}(g)\partial_J \mathcal{U}(g) > 0, \tag{401}$$

since the metric is positive definite. We will now show that if there is a stable fixed point, it is the *unique* maximum of $\mathcal{U}(g)$ in coupling constant space. Hence if $\mathcal{U}(g) = \mathcal{U}(g')$ with $g \neq g'$, then neither g nor g' can be stable fixed points.

So, suppose g_1, g_2 are two distinct fixed points. We know that $\mathcal{U}$ is an RG monotone, and so we might be able to get an idea about the relative stability of these two fixed points by comparing $\mathcal{U}$ between them. To this end let $g(\lambda) = g_1\lambda + g_2(1-\lambda)$ be an interpolating path of couplings (this is not generically an RG flow, so $\mathcal{U}$ needn't be monotonic in λ). $\mathcal{U}$ varies along the path as

$$\partial_\lambda \mathcal{U}(g(\lambda)) = (g_1^I - g_2^I)\beta_I \equiv \Delta^I \beta_I. \tag{402}$$

Since g_1, g_2 are fixed points, $\partial_\lambda \mathcal{U}(g)|_{\lambda=0,1} = 0$. Since β is quadratic in the g_Is and vanishes at the endpoints,$\beta_I = C_I\lambda(1-\lambda)$ for some λ-independent C_I. Then another derivative wrt λ gives

$$\partial_\lambda^2 \mathcal{U}(g(\lambda)) = \Delta^I\Delta^J \partial_I\partial_J \mathcal{U}(g(\lambda)) = \Delta^I C_I(1-2\lambda). \tag{403}$$

If we evaluate this at the endpoints of the interpolation, we get

$$\Delta^I \mathcal{B}_{IJ}(g_1)\Delta^J = \Delta^I C_I, \qquad \Delta^I \mathcal{B}_{IJ}(g_2)\Delta^J = -\Delta^I C_I, \tag{404}$$

where again $\mathcal{B}_{IJ} = \partial_I\partial_J\mathcal{U}$ determines the scaling dimensions at a fixed point. Therefore

$$\Delta^I(\mathcal{B}_{IJ}(g_1) + \mathcal{B}_{IJ}(g_2))\Delta^J = 0, \tag{405}$$

which since $\Delta^I \neq 0$ means that it is impossible for $\mathcal{B}(g)$ to be positive-definite at both g_1 and g_2—hence at least one of the two fixed points must be unstable, meaning that only at most one stable fixed point exists.

A corollary of this is the following. Let $g_I = R_I^J g_J$, with R a representation of some group element of $O(L)$ acting in the vector$^{\otimes 4}$ representation. Such $O(L)$ transformations map fixed points to fixed points, since β_I transforms covariantly. Therefore from the above result, any fixed point g_I acted on non-trivially by $O(L)$ *must* be unstable, and so the only stable fixed point is the $O(L)$ symmetric one, where $g_{ijkl} \propto (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Since we know the $O(L)$-symmetric is unstable for $L > 4$ by the calculation we did above, we thus conclude that there are no stable fixed points for $L > 4$.

This result was proved by looking at the behavior of $\mathcal{U}(g)$ with respect to arbitrary variations in the space of symmetric tensors g_I. Depending on the application though, we may only be interested in variations that preserve some symmetry group $G \subset O(L)$. Making this restriction can eliminate relevant directions in coupling constant space, and result in symmetry-protected stability.[39] However, since $\lambda g_1 + (1-\lambda)g_2$ is G-invariant if both g_i are, the above result means that there is at most one G-stable fixed point for each G. Furthermore, from the previous paragraph, if g_I is a G-stable fixed point and there exists some group element in $O(L)$ which maps g_I to any G-invariant coupling distinct from g_I itself, then g_I must be unstable.

Now we can finally apply this to the problem at hand to show that there are no $\mathbb{Z}_2^L$-stable fixed points for $L > 4$. Indeed, consider the action of $\sigma \in S_L \subset O(L)$ on a given $\mathbb{Z}_2^L$-invariant coupling g_{ij}. This maps $g_{ij} \mapsto g_{\sigma(i)\sigma(j)}$, which is of course also $\mathbb{Z}_2^L$ symmetric. Therefore a necessary condition for a g_{ij} to give a $\mathbb{Z}_2^L$-stable fixed point is for $g_{\sigma(i)\sigma(j)} = g_{ij}$ for all $\sigma \in S_L$, which means that g_{ij} must have the form $g_{ij} = g_0\delta_{ij} + h$. But we have already solved this case and shown that there are no stable fixed points with this restricted class of couplings when $L > 4$. Therefore we conclude that as long as $L > 4$, the theory is unstable towards a fluctuation-induced first-order transition.[40]

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20 Scaling dimensions for a two-component compact boson ✓

Today we will calculate the scaling dimensions of the cosine operators in a general two-component compact boson model (a sigma model into T^2) in 1+1D.

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Consider a two-component compact boson theory (with both fields having 2π periodicity) given by

$$S = \frac{1}{4\pi}\int g^{\alpha\beta}\partial_\mu\theta_\alpha\partial^\mu\theta_\beta. \tag{406}$$

We can find the scaling dimensions of the cosines by decomposing the metric in veilbeins as $g_{\alpha\beta} = [E^T]_{\alpha a}\delta^{ab}E_{b\beta}$. Of course there is an $O(2)$ gauge redundancy in choosing the vielbeins; the choice we'll stick with is

$$g = \begin{pmatrix} g_1 & g' \\ g' & g_2 \end{pmatrix} \implies E_{\alpha a} = \begin{pmatrix} \sqrt{g_1} & g'/\sqrt{g_1} \\ 0 & \sqrt{g_2 - g'^2/g_1} \end{pmatrix}_{\alpha a}. \tag{407}$$

[39] A dumb example is $G = O(L)$, then the $O(L)$-symmetric fixed point is obviously stable. From the examples we worked out above, we also know that $G = \mathbb{Z}_2^L \rtimes S_N$ (where the two independent couplings were $g_{j>0}$ and g_0) is big enough to protect a stable fixed point.

[40] To build confidence, I also checked this by numerically searching for $\mathbb{Z}_2^L$-symmetric fixed points of the β functions for $L \leq 10$ (and of course found no stable ones).

Then defining new fields by $\vartheta_a \equiv E_{a\alpha}\theta^\alpha$ and letting $\boldsymbol{\theta} = (\theta_R, \theta_L)^T$, the Lagrangian is just $\mathcal{L}_0 = \frac{1}{4\pi}\partial\boldsymbol{\vartheta}^T \cdot \bar{\partial}\boldsymbol{\vartheta}$. However one must also keep in mind the periodicity of the fields, which is now modified: instead of $\boldsymbol{\vartheta} \sim \boldsymbol{\vartheta} + 2\pi\mathbf{n}$ with $\mathbf{n} \in \mathbb{Z}^2$, we instead have $\boldsymbol{\vartheta} \sim \boldsymbol{\vartheta} + 2\pi E\mathbf{n}$. This means that e.g. $e^{i\vartheta_\alpha}$ is not generically an allowed vertex operator in the theory.[41] The allowed vertex operators are instead $e^{i\mathbf{n}^T E^{-1}\boldsymbol{\vartheta}}$, and their correlation functions are computed using

$$\langle e^{i\gamma\vartheta(x)} e^{i\lambda\vartheta(y)}\rangle \sim |x-y|^{\gamma\lambda}\delta_{\gamma+\lambda} \tag{408}$$

for a field ϑ whose free term has the coefficient $R^2/4\pi$ (for us of course the ϑ fields have $R = 1$; they are at the self-dual radius).

The Lagrangian in terms of the ϑ fields is

$$\mathcal{L} = \frac{1}{4\pi}\partial\boldsymbol{\vartheta}^T \cdot \bar{\partial}\boldsymbol{\vartheta} + \sum_{\mathbf{n}\in\mathbb{Z}^2} \alpha_{\mathbf{n}} \cos\left(\mathbf{n}_\alpha E^{\alpha a}\vartheta_a\right), \tag{409}$$

where the index-up veilbein is the inverse. Using the above correlator for the ϑ vertex operators, we see that

$$\langle \cos(\mathbf{n}^T \cdot \boldsymbol{\theta}(x)) \cos(\mathbf{m}^T \cdot \boldsymbol{\theta}(0))\rangle \sim \frac{\delta_{n_\alpha E^{\alpha 1} + m_\beta E^{\beta 1}}\delta_{n_\alpha E^{\alpha 2} + m_\beta E^{\beta 2}}}{|x|^{-(n_\alpha E^{\alpha 1} m_\beta E^{\beta 1} + n_\alpha E^{\alpha 2} m_\beta E^{\beta 2})}} + (\mathbf{m} \leftrightarrow -\mathbf{m}) \tag{410}$$

Now the inverse veilbeins satisfy $E^{-1}[E^{-1}]^T = g^{-1}$, and so the exponent can be written as $n_\alpha g^{\alpha\beta} m_\beta$. The product of delta functions written out says that $(\mathbf{n}+\mathbf{m})^T E^{-1} = (0,0)$; since E is non-degenerate this means $\mathbf{n}+\mathbf{m} = (0,0)^T$. Accounting for the other possibility where the delta function sets $\mathbf{m} = +\mathbf{n}$ instead, we have

$$\langle \cos(\mathbf{n}^T \cdot \boldsymbol{\theta}(x)) \cos(\mathbf{m}^T \cdot \boldsymbol{\theta}(0))\rangle \sim \frac{\delta_{\mathbf{n}+\mathbf{m}} + \delta_{\mathbf{n}-\mathbf{m}}}{|x|^{\mathbf{n}_\alpha g^{\alpha\beta}\mathbf{n}_\beta}}, \tag{411}$$

and so the cosine (as well as the related sin) has dimension

$$\Delta_{\mathbf{n}} = \frac{1}{2} n_\alpha g^{\alpha\beta} n_\beta. \tag{412}$$

Since g (and hence g^{-1}) are positive-definite forms if the Hamiltonian is positive-definite, $\Delta_{\mathbf{n}} > 0$ as required. Sanity check: suppose that $\mathbf{n} = (1,0)^T$ and $g = R^2 \oplus 0$. Then we get a scaling dimension of $1/2R^2$, which is exactly what we expect (recall that the free fermion point is $R = 1/\sqrt{2}$, which gives a scaling dimension of 1 as required by something which fermionizes to a Dirac mass).

[41] Just to be a bit garrulous here: it'd be wrong to say that the φ fields have radius 1—the ϑ fields really don't have independent radii, unlike the θ fields: when we do the linear transformation to the ϑ fields, we change the basis vectors in the lattice that generates the torus in the sigma model, and so now instead of both fields possessing independent periodicity requirements, they have a funky periodicity relation that is mixed up between the two fields. All of this just affects what kinds of vertex operators we can write down though, and if we start from vertex operators written in terms of the θ and then map them to ones in terms of the ϑ, we can never go wrong.

The scaling dimensions of the vertex operators for the phase fields ϕ_α are computed in essentially the same way — duality inverts the metric, and so the dimension $\Delta^\vee_{\mathbf{n}}$ of the operator $\cos(\mathbf{n}\cdot\boldsymbol{\phi})$ is

$$\Delta^\vee_{\mathbf{n}} = \frac{1}{2} n_\alpha g_{\alpha\beta} n_\beta. \tag{413}$$

The same sanity checks can be done on this expression, and it passes them.

Consider first the case when $g_{LL} = g_{RR} = g$ (with $|g_{LR}| < g$ needed for stability). In this case the scaling dimensions of the θ cosines are

$$\Delta_{\mathbf{n}} = \frac{1}{2(g^2 - g_{LR}^2)}(g(n_1^2 + n_2^2) - 2g_{LR}n_1n_2). \tag{414}$$

Now it is impossible to have all of the cosines $\cos(\mathbf{n}\cdot\mathbf{t})$ and $\cos(\mathbf{n}\cdot\boldsymbol{\phi})$ be simultaneously irrelevant. However, we can find regions of parameter space without relevant deformations if we impose some symmetries.

For example, consider a theory with $U(1)^2$ symmetry, which acts by shifting the ϕ_α fields. Then no cosines of the ϕ_α fields are allowed, and we can find a stable gapless phase by finding a region of parameter space where $\Delta_{\mathbf{n}} > 2$ for all $\mathbf{n}$. The boundary of this region can be found by setting $\Delta_{\mathbf{n}} = 2$, solving for n_1 in terms of n_2, and requiring that there be no real solutions. We find $n_1 = (g/g_{LR})n_2 \pm \sqrt{(1 - g_{LR}^2/g^2)(4g - n_2^2)}$, and so since $g_{LR}^2/g^2 < 1$ by assumption, we will have no relevant cosines if $4g - n^2 < 0$ for all nonzero n_2, i.e. provided that $g < 1/4$. Therefore the stable region of parameter space is given by $g_{LR} < g < 1/4$. In the more general case when $g_{LL} \neq g_{RR}$, we instead have

$$\Delta_{\mathbf{n}} = \frac{1}{2(g_{LL}g_{RR} - g_{LR}^2)}(g_{RR}n_1^2 + g_{LL}n_2^2 - 2g_{LR}n_1n_2). \tag{415}$$

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21 *Fluctuation-induced first-order transitions in non-Abelian gauge theory*

Today we're going to look at the stability of the free fixed point in Yang-Mills coupled to n flavors of quartically interacting bosons:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}[F_A \wedge \star F_A] + |D_A\phi|^2 + m|\phi|^2 + \frac{u}{2}|\phi|^4, \tag{416}$$

where $|\phi|^2 = \langle\phi,\phi\rangle$, with $\langle\,,\,\rangle$ some invariant pairing for the representation R that ϕ transforms in. Instead of analyzing completely general choices of gauge groups and representations, we will actually just specialize to the case with gauge group $SU(N_c)$, with N_b bosons in the fundamental. viz.[42] Just as in the $U(1)$ case, we will see that at weak coupling the $m=0$ point has an instability towards negative u unless n is larger than some (often rather large) critical value (which is of course always larger than the value needed to allow the theory to be free in the IR).

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The beta function for the gauge coupling is the easiest to get, since as in the Abelian case, gauge invariance means that it can be calculated entirely in terms of the gauge field self energy. To one loop order, no quartic ϕ couplings appear in the diagrams for the A self energy; hence β_{g^2} will be a function of g^2 alone.

The first type of self-energy diagram to calculate is the one where a ϕ field flows in the loop. The only thing that makes this different from the $U(1)$ case is that there is now an extra factor of $\mathrm{Tr}[T^aT^b] = T(R)\delta^{ab}$, with $T(R)$ the index of the representation that the bosons are in.

The other contribution to the self energy comes when a gauge field flows in the loop (the "cactus diagrams" with scalar and gauge fields in the loop don't contribute since they vanish in dimensional regularization, and being momentum independent wouldn't contribute to the wavefunction renormalization anyway). The group theoretic factor comes from the f^{abc}s on the two three-point gauge boson vertices, giving $f^{abc}f^{bcd} = \mathrm{Tr}[T^a_AT^d_A] = T(A)\delta^{ad} = C_2(G)\delta^{ad}$. Since there are n bosons flowing in the polarization bubble, the beta function for the gauge coupling is therefore the standard

$$\beta_\alpha = \varepsilon\alpha + \alpha^2\left(\frac{11}{3}C_2(G) - \frac{n}{3}nT(R)\right), \tag{417}$$

where $\alpha = g^2/8\pi^2$.

The beta function for u is quite a bit more complicated but not unassailably so, since the diagrams involved don't contain any three- or four-point gauge boson vertices, which are the real source of annoyance due to the complicated numerators they bestow upon Feynman diagrams.

The term of order u^2 can be obtained following the procedure outlined in the previous diary entry on the case when $G = U(1)$; it gives us the familiar $(n+4)u^2/8\pi^2$ term.

The next easiest diagrams to compute are the ones which involve only the $\phi^\dagger A\partial\phi$ vertices. For example, consider the self energy of the scalar (needed for getting the normalization of

[42]In general, depending on the gauge group and the representation the bosons are in, there may be many different interaction terms we can write down: each term comes from an invariant symbol of the gauge group. The mass term and the quartic interaction come from $R\otimes R^* \ni \mathbf{1}$ and $(R\otimes R^*)^{\otimes 2} \ni \mathbf{1}$, but we could also have e.g. one from R^4 if $G = SU(4)$, etc. etc. Since we will be interested in the possibility of these theories describing critical points of second-order phase transitions, we will be assuming that no cubic invariant exists, so that $\mathbf{1} \notin R^{\otimes 3}, (R\otimes R^*)\otimes R$ (which rules out e.g. the fundamental of $SU(3)$ and all real representations).

the ϕ fields). The relevant diagram is (schematically)

nonabbubble.png

(418)

As of the time of writing I was not feeling like texing up the calculations; in any case they really are quite similar to those in the $U(1)$ case, which is in another diary entry. The result turns out to be

$$\beta_u = \varepsilon u - (N_b N_c + 4)u^2 + \frac{3(N_c^2-1)}{N_c}u\alpha - \frac{3(N_c-1)(N_c^2+2N_c-2)}{4N_c^2}\alpha^2, \tag{419}$$

with now $u = u/8\pi^2$ (in a computer's sense of '='). Note that N_b only appears in the first term, as this is the only one that comes from diagrams with a full ϕ loop.

In the diary entry on fluctuation-induced first-order transitions, we computed the β functions in the ϵ expansion to diagnose stability. We could do the same thing here, but for variety's sake we'll get at the answer from a different perspective.

For notation's sake, let

$$\beta_\alpha = -A\alpha, \qquad \beta_u = -au^2 + bu\alpha - c\alpha^2. \tag{420}$$

In our context, all of the above coefficients will be positive.

Suppose now that $(0,0)$ is an attractive IR fixed point, with some subset of flow lines that start at $\alpha > 0$ flowing onto $(0,0)$ (the flow lines that start on the u axis will always flow to $(0,0)$). If this is the case, then we must have a flow which flows into $(0,0)$ from a line with > 0 slope in the u-α plane. Therefore there must be a solution to the β function ODEs with $\alpha = su$ for some $s > 0$, at least for α, u very close to the origin. Close to the origin where this parametrization works, s will also be independent of RG time. Putting this into the beta functions, we see that we must have

$$-As = -a + bs - cs^2, \tag{421}$$

and since $s > 0$ we must then have

$$(A+b)^2 \geq 4ca. \tag{422}$$

N_b^* is then found by making the above an equality and solving for N_b. When we do this for e.g. $N_c = 2$, we find $N_b^* = 359$, which as with the $U(1)$ case is rather large. Not surprisingly,

N_b^* is a monotonically increasing function of N_c, e.g.

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(423)

which makes sense because the threshold for potential IR freedom increases with N_c.

Lets see this visually for the case of $N_c = 2$, where $N_b^* = 359$. For $N_b < N_b^*$, all of the flows (expect for the ones that start at $\alpha = 0$) lead to a $u < 0$ instability:

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Right at the critical number of bosons, we have

nc2nb359.pdf (425)

while when we're above N_b^*, the flows look like

nc2nb450.pdf (426)

In this plot there are lines that start at $\alpha, u > 0$ and flow onto the $\alpha = u = 0$ trivial fixed point (the separatrix can be seen as a flow line hitting the origin).

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