Astro+GR diary

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Preface:

This is a diary containing worked-out problems in astrophysics and cosmology. These are drawn from attempted solutions to problems in Weinberg's books, problems discussed in Zheng Zheng's group meetings,[1] homework problems, and from my own research. Everything is doubtless well known / in the literature, but for the most part I have avoided providing citations as looking up the answer takes away some of the fun. Finally, as I am not a licensed astrophysicst, anything that looks questionable can be assumed to be a mistake on my part.

GR and gravity

Order of magnitude problems

[1] See `here`.

Salpeter timescale

Today we are going to estimate the e-folding time τ_S (the *Salpeter time*) for the mass of an accreting black hole, accreting as fast as possible with an accretion disk of opacity ε.

We will crudely model the accreting matter as Hydrogen gas which falls isotropically into the black hole. In terms of the luminosity L of the accreting matter, the growth of the black hole mass is governed by

$$\frac{dM}{dt} = \frac{L}{c^2\varepsilon} \implies \tau_S = \frac{c^2\varepsilon M}{L}. \tag{1}$$

The placement of ε is correct since we want the luminosity to be equal to ε times the rate at which energy is deposited onto the black hole. A sanity check is that $\tau_S \to 0$ when $\varepsilon \to 0$, as in the absence of radiation pressure from the infalling matter there is nothing to restrict the accretion rate.

Now we just need to estimate L (the Eddington luminosity). This is determined by setting the outward force from radiation pressure at a point outside the black hole equal to the inward force due to the gravitational potential (make L any larger and the radiation pressure would prevent matter from accreting).

Now

$$L = \oint d^2x_\perp \, P_{rad} c, \tag{2}$$

where P_{rad} is the radiation pressure normal to the surface of a sphere of radius R enclosing the black hole. When the gravitational and radiation pressures balance, we have

$$P_{rad} = \lambda \frac{GM}{R^2}, \tag{3}$$

where λ is a constant of dimension mass / area, which we see must be $\lambda \sim m_p/\sigma$ (since the accreting matter is Hydrogen), where σ is the scattering cross section. We should probably take σ to be the electron Thompson scattering cross section σ_T since this is the dominant mechanism for the radiation to impart momentum to the infalling gas. Therefore

$$L = 4\pi c m_p GM/\sigma_T, \tag{4}$$

giving

$$\tau_S = \frac{\varepsilon c \sigma_T}{4\pi m_p G}, \tag{5}$$

which has the correct units. Numerically, taking $\varepsilon = 0.1$ gives

$$\tau_S \approx 4.7 \times 10^7 \, yr. \tag{6}$$

This means that the time for a maximally-accreting black hole to grow from $M_\odot$ to $10^9 M_\odot$ is

$$\delta t \approx 1 \, Gyr. \tag{7}$$

The interesting thing about this is of course that quasars with $M \sim 10^9 M_\odot$ have already turned on around $z \sim 7$, even though the time they have to grow is $t_q \lesssim \delta t$.

Procession of Mercury's orbit from EFT

This is a fun one from Schwartz: computing the precession of the perihelion of Mercury from basic QFT-style perturbation theory. We write the Lagrangian for GR as

$$\mathcal{L} = M_P^2 \left(-\frac{1}{2} h \Box h + h^2 \Box h \right) - hT, \tag{8}$$

where $h = h_{00}$, with $h_{\mu\nu}$ the first-order perturbation to $g_{\mu\nu}$ away from $\eta_{\mu\nu}$, and $T = T_{00}$ is (we are in $\hbar = c = 1$ units)

$$T = M\delta^3(x), \tag{9}$$

with M the mass of the sun. Looking only at T_{00} and h_{00} is valid here since we know the Schwartzchild solution only differs from $\eta_{\mu\nu}$ in the dt^2 and dr^2 parts, and we can focus only on the dt^2 part since we are interested in computing GR-induced changes to the gravitational potential set up by the sun.

Anyway, we will solve for h to $O(\lambda^4)$, where $\lambda \equiv 1/M_P = \sqrt{G_N}$ is the inverse planck mass in natural units. From the resulting gravitational potential, we will be able to find the precession of Mercury's orbit.

First re-scale h by $h \mapsto M_P h = h/\lambda$, so that

$$\mathcal{L} = -\frac{1}{2} h \Box h + \lambda h^2 \Box h - \lambda h T. \tag{10}$$

We will do perturbation theory in λ. The equations of motion are

$$\Box h = \lambda \Box (h^2) - \lambda T. \tag{11}$$

We just want to get an idea for how the calculation works — we won't be keeping too careful track of numerical factors. Now write $h = h_0 + \lambda h_1 + \dots$. The lowest order gives us $h_0 = 0$ and (all the action is taking place in space, [we are assuming a static situation] so $\Box$ is really $\partial_i \partial^i$, in mostly-negative signature)

$$h_1(x) = -\int d^3y \left[\frac{1}{\Box} \right]_{x,y} T(y) = \int d^3y \, \frac{1}{|x-y|} M\delta^3(y) = \frac{M}{x}. \tag{12}$$

When we restore units, this gives us the regular Newtonian potential. The next order in λ tells us that $h_2 = 0$, and the order after that tells us that

$$\Box h_3 = \Box(h_1^2) \implies h_3 = \left(\frac{1}{\Box}T\right)^2 = \frac{M^2}{r^2}. \tag{13}$$

Restoring h so that it is dimensionless (in natural units), we get

$$h(r) \approx \lambda^2 \frac{M}{r} + \lambda^4 \frac{M^2}{r^2}. \tag{14}$$

Restoring units gives the Newtonian potential. Thus the potential energy for a Mercury of mass m is the Newtonian one, plus a $1/r^2$ term:

$$V(r) \approx -\frac{G_N M m}{r}\left(1 + \frac{G_N M}{c^2 r}\right), \tag{15}$$

where we have restored the c^2 on dimensional grounds. Thus the correction goes as $\approx 3.3\text{km}/r$, which is clearly tiny, even for Mercury.

Now we want to find the correction this induces to the precession of Mercury's perihelion radius, which is a fun walk back through undergraduate physics. The kinetic term in the Lagrangian is $\frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$, where (r, ϕ) are the coordinates in the plane of Mercury's orbit. This of course leads to the conservation of $L = mr^2\dot{\phi}$. The eom for r is

$$m d_t^2 r = m d_t(\dot{\phi}\partial_\phi r) = m(\partial_t^2\phi\partial_\phi r + \dot{\phi}^2\partial_\phi^2 r) = \frac{L^2}{mr^3} - \partial_r V(r). \tag{16}$$

Since there are a lot of $1/r$s, we define $\rho \equiv 1/r$. We want to take the above and use angular momentum conservation to get an equation for $\partial_\phi^2 u$. We use

$$\partial_t^2\phi\partial_\phi r + \dot{\phi}^2\partial_\phi^2 r = -\frac{\partial_t^2\phi}{u^2}\partial_\phi u - \dot{\phi}^2\left(\partial_\phi^2 u\frac{1}{u^2} - \frac{\dot{\phi}^2}{u^3}(\partial_\phi u)^2\right) = -\frac{\dot{\phi}^2}{u^2}\partial_\phi^2 u - (\partial_t L)\partial_\phi u = -\frac{u^2L^2}{m^2}\partial_\phi^2 u. \tag{17}$$

This means

$$\partial_\phi^2 u = -u - \frac{m}{L^2}\partial_u V(u), \tag{18}$$

which for us gives

$$\partial_\phi^2 u = -u + \frac{m^2}{L^2}GM\left(1 + \frac{2GM}{c^2}u\right), \tag{19}$$

which we write as

$$\partial_\phi^2 u = u(\Gamma - 1) + \alpha, \qquad \Gamma = 2m^2G^2M^2/(c^2L^2), \quad \alpha = m^2GM/L^2. \tag{20}$$

Now we can solve this: $\Gamma - 1 < 0$ since $\Gamma \approx (3.3\text{km})^2c^2/(r^2v_m^2) \ll 1$ where r is Mercury's semimajor axis and v_m is Mercury's typical velocity around the sun ($r \sim 6 \times 10^7$km), and so

$$u(\phi) = A\cos\left(\phi\sqrt{1-\Gamma}\right) + \frac{\alpha}{1-\Gamma}. \tag{21}$$

We let Mercury be at perihelion when $\phi = 0$. When ϕ goes to 2π, we see that u does not exactly come back to $u(\phi = 0)$, because of the square root in the cosine. This causes the location of Mercury's perihelion to rotate. The second perihelion instead happens when

$$\phi = \frac{2\pi}{\sqrt{1-\Gamma}} \approx 2\pi(1 + \Gamma/2) \implies \delta\phi \approx \frac{2\pi m^2 G^2 M^2}{c^2 L^2}. \tag{22}$$

Putting in numbers for Mercury, this gives a precession of $\sim 26''$ per century, which is about half of what the actual answer is ($43''$ per century). Given that we weren't paying attention to constants when getting the eom, this isn't bad! Same order for magnitude, at least.

Acceleration along Killing vectors

Today is a short one, and kind of a cheat since it was part of a homework assignment in AdS/CFT class. Consider an observer moving along a Killing vector K^μ, with four-velocity U^μ such that

$$K^\mu = \alpha(x) U^\mu. \tag{23}$$

Show that the proper acceleration is

$$a^\mu = \nabla^\mu \ln \alpha. \tag{24}$$

Use this to compute the proper acceleration for $\rho =$ const observers in Rindler spacetime and $r =$const observers in Schwarzchild.

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This is straightforward; we just test the given formula and make sure it works:

$$\begin{aligned} a^\mu &= -\frac{1}{2(-K_\nu K^\nu)} \nabla^\mu (K_\lambda K^\lambda) = \frac{1}{K_\nu K^\nu} K_\lambda \nabla^\mu K^\lambda \\ &= -\frac{1}{\sqrt{-K_\nu K^\nu}} U_\lambda \nabla^\mu K^\lambda. \end{aligned} \tag{25}$$

Since $\nabla^{(\mu} K^{\nu)} = 0$ because K^μ generates an isometry,

$$a^\mu = \frac{1}{\sqrt{-K_\nu K^\nu}} U_\lambda \nabla^\lambda K^\mu = U_\lambda \nabla^\lambda U^\mu, \tag{26}$$

which is indeed the correct formula for proper acceleration. In the last step we have used

$$U_\lambda (\nabla^\lambda [-K_\nu K^\nu]^{-1/2}) U^\mu \propto U_\lambda (\nabla^\lambda K^\nu) K_\nu K^\mu \propto U_\lambda U_\nu \nabla^{(\lambda} K^{\nu)} K^\mu = 0. \tag{27}$$

Consider then Rindler spacetime $ds^2 = -\rho^2 d\eta^2 + d\rho^2$, and consider an observer at constant ρ which moves along the Killing vector $K^\mu = (\partial_\eta)^\mu = (1, 0)$. Then

$$\sqrt{-K^\mu K_\mu} = \sqrt{-g_{00}} = \rho, \tag{28}$$

so that

$$a^\mu = (0, \rho^{-1}), \tag{29}$$

since $g^{\rho\rho} = 1$. Thus the magnitude of the proper acceleration is

$$a = \sqrt{a^\mu a_\mu} = \frac{1}{\rho}. \tag{30}$$

Likewise for Schwarzchild with an observer at constant r, the Killing vector is again the timelike vector $K^\mu = (1, 0, 0, 0)$. It has magnitude $\sqrt{f(r)}$, where $f(r) = 1 - r_S/r$. Then the acceleration is purely radial:

$$a_\mu = \nabla_\mu \sqrt{f} = \delta_{\mu,r} \frac{f'(r)}{2f} dr, \tag{31}$$

so that the magnitude of the acceleration is (using $g^{rr} = f$)

$$a = \frac{f'(r)}{2\sqrt{f(r)}}. \tag{32}$$

If we want to convert this to the acceleration seen by an observer at infinity, we just need to re-scale by redshift factor $\sqrt{-K_\nu K^\nu} = \sqrt{f}$, so that the acceleration seen at infinity is

$$a_\infty = \frac{f'(r)}{2}. \tag{33}$$

In particular, the surface gravity of the black hole is found by evaluating this at $r = r_S$, for which we get

$$\kappa = a_\infty(r_S) = \frac{1}{4GM}. \tag{34}$$

Kerr-Newmann metric

A general black hole in asymptotically flat space of mass M angular momentum J and charge Q has the metric

$$ds^2 = -\frac{\rho^2 \Delta}{\Sigma} dt^2 + \frac{\Sigma}{\rho^2} \sin^2\theta (d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \tag{35}$$

where the constants are

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr, \quad a = J/M, \tag{36}$$

and

$$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \quad \omega = \frac{a}{\Sigma}(r^2 + a^2 - \Delta). \tag{37}$$

Today the to-do list is as follows:

a) Find T_H, the horizon radius, the horizon area, and the angular velocity at the black hole horizon. b) what happens when the black hole is extremal, i.e. when $M^2 = a^2 + Q^2$? c) Show that for $J = 0$, the region near the horizon of an extremal black hole is $\mathrm{AdS}_2 \times S^2$.

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a) recalling the Schwarzchild metric, the horizon radius r_+ should occur where the dr^2 part of the metric changes sign. Solving the quadratic equation $\Delta = 0$ for r and taking the larger value for r, we get

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}. \tag{38}$$

Note that this gives the correct $r_+ = 2M$ when $J = Q = 0$ (we're in units where $G_N = 1$).

To get the horizon area, we need to integrate the area element over the sphere $dr = dt = 0$ at $r = r_+$:

$$A = \int \sqrt{\det g|_{r=r_+,t=0}}\,d\phi d\theta = \int \sqrt{\Sigma}\sin\theta\, d\phi d\theta = 2\pi\int_0^\pi (r_+^2{+}a^2)\sin\theta\, d\theta = 4\pi(r_+^2{+}a^2). \tag{39}$$

To get the Hawking temperature, we consider an observer moving radially along $d\phi = d\theta = 0$ at some value of θ, ϕ (we can take $\phi = 0$ wolog since ϕ doesn't appear in the metric). Then the metric reduces to

$$ds^2 \to -\left(\frac{\rho^2\Delta}{\Sigma} - \frac{\Sigma\sin^2\theta\omega^2}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2. \tag{40}$$

Let's expand this near the horizon $r = r_+$, to first order in $r - r_+$. We will define

$$dx = \frac{1}{\sqrt{\Delta}}dr, \tag{41}$$

so as to simplify the dr^2 term in the metric. Since $\Delta = 0$ at the horizon, only terms with a $\Delta'(r_+) = 2(r_+ - M)$ will survive. Furthermore since we are interested in T_H / the surface gravity, we can set θ to be any angle we like, since we know in equilibrium the temperature will be the same over the entire horizon. We will pick $\theta = 0$, so that the metric above is approximately

$$ds^2 \approx -\frac{(\partial_r\Delta)(r_+)(r - r_+)}{r_+^2 + a^2}dt^2 + \rho^2 dx^2 = -\frac{2(r_+ - M)(r - r_+)}{r_+^2 + a^2}dt^2 + \rho^2 dx^2. \tag{42}$$

Integrating the definition for dx gives

$$x = \frac{2}{\sqrt{2(r_+ - M)}}\sqrt{r - r_+}, \tag{43}$$

so that

$$r - r_+ = \frac{r_+ - M}{2}x^2. \tag{44}$$

Putting this into the metric and defining $\rho dx = dy$, we get, after analytically continuing to $\tau = it$,

$$ds^2 \approx \frac{(r_+ - M)^2}{(r_+^2 + a^2)^2}y^2 d\tau^2 + dy^2. \tag{45}$$

Thus we are prompted to write

$$\theta \equiv \tau\frac{r_+ - M}{r_+^2 + a^2}, \tag{46}$$

and identify $\theta \sim \theta + 2\pi$ in order to get a smooth geometry. On τ this identification is

$$\tau \sim \tau + 2\pi\frac{r_+^2 + a^2}{r_+ - M} = \tau + \frac{A}{2(r_+ - M)}. \tag{47}$$

Since $\tau \sim \tau + \frac{1}{T}$ in field theory, we find that T_H is

$$T_H = \frac{2(r_+ - M)}{A}. \tag{48}$$

The angular velocity of the horizon is found by evaluating ω at $r = r_+$. There we have

$$\omega|_{r_+} = a(r_+^2 + a^2)/\Sigma = \frac{a}{r_+^2 + a^2} = \frac{4\pi a}{A}. \tag{49}$$

b) The black hole is extremal if $M^2 = a^2 + Q^2$. This means that $r_+ = M$, giving $T_H = 0$. The entropy (alias area) is finite, though:

$$S = \frac{A}{4} = \pi(M^2 + a^2). \tag{50}$$

c) For $J = 0$ we have $a = 0$, and setting $M = Q$ gives $\Delta = (r - M)^2$. Since $\Sigma \to r^4$ and $\rho \to r^2$, the metric then becomes

$$ds^2 = -(1 - M/r)^2 dt^2 + r^2 \sin^2\theta d\phi^2 + \frac{1}{(1 - M/r)^2}dr^2 + r^2 d\theta^2. \tag{51}$$

The proper distance to the horizon from any $r > r_+$ is infinite, since the integral over dr of $1/(1 - M/r)^2$ diverges as the integration limit is taken to $r \to r_+ = M$.

Near the horizon, we write $r \approx M + x$ and expand e.g. $1 - M/r \approx x/M$. Dropping the terms that go as x, x^2 in front of the ϕ, θ parts of the metric, we get

$$ds^2 \approx -\frac{x^2}{M^2}dt^2 + \frac{M^2}{x^2}dx^2 + M^2(\sin^2\theta d\phi^2 + d\theta^2), \tag{52}$$

which is the metric for the space $\mathrm{AdS}_2 \times S^2$, where the S^2 has radius M.

Baby example of RN metric and D branes

Today we'll look at a spacetime coming from $D0$ branes in four dimensions. In $G_N = 1$ units and ignoring the electric potential, the RN metric is

$$ds^2 = -f dt^2 + f^{-1} d\rho^2 + \rho^2 d\Omega_2^2, \qquad f = 1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}. \tag{53}$$

Our task is as follows:

a) For $M > Q$, find the outer event horizon. b) At the extremal point, what does the geometry reduce to? c) What happens when $Q > M$? d) Re-write the metric as

$$ds^2 = -g(r)dt^2 + h(r)(dr^2 + r^2 d\Omega_2^2). \tag{54}$$

e) Find the explicit metric in the case of $Q = M$.

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a) The horizon(s) occur where the radial and time coordinates switch between being timelike and spacelike. This occurs when $f = 0$, i.e. when

$$0 = \rho^2 - 2M\rho + Q^2 \implies \rho = M \pm \sqrt{M^2 - Q^2}. \tag{55}$$

The outer horizon is the + sign and is what we will mean by "horizon" from now on.

b) Now let the black hole be extremal, with $M = Q$. Then the horizon is at $\rho_+ = M$. Near the horizon, we expand $\rho = \rho_+ + Mz = M(1+z)$

$$f(\rho_+ + Mz) = 1 - \frac{2}{1+z} + \frac{1}{(1+z)^2} = z^2 + O(z^3). \tag{56}$$

Thus the metric goes to

$$ds^2 = -z^2 dt^2 + \frac{1}{z^2} dz^2 + M^2(1+z)^2 d\Omega_2^2 \approx \frac{1}{u^2}(du^2 + \eta_{\mu\nu} dx^\mu dx^\nu) + M^2 d\Omega_2^2, \tag{57}$$

where in the last step we have defined $u \equiv 1/z$, dropped the z-dependent pieces in the sphere part of the metric, and written $\eta_{\mu\nu} dx^\mu dx^\nu = -dt^2$ to be suggestive. The first part of the metric is AdS_2, and the second part is S^2. The two parts don't talk to eachother, and so the spacetime is of the form $AdS_2 \times S^2$. Note that the radius of the sphere is finite as $z \to 0$.

c) For $Q > M$ there are no real solutions to $f = 0$, and so if the BH is super-extremal then it doesn't have a horizon (naked singularity).

d) We want to write the metric as

$$ds^2 = -g(r)dt^2 + h(r)(dr^2 + r^2 d\Omega_2^2). \tag{58}$$

From the dt^2 term, we know that $f(\rho) = g(r)$. We also know from the S^2 part that

$$\rho(r) = r\sqrt{h(r)}. \tag{59}$$

Finally we look at the $d\rho^2$ part of the metric, which tells us that

$$\frac{d\rho}{\sqrt{f}} = \sqrt{h(r)}dr. \tag{60}$$

e) Again specify to $M = Q$, so that $f = 1 - 2M/\rho + M^2/\rho^2 = (1 - M/\rho)^2$. Then we have

$$\frac{d\rho}{1 - M/\rho} = \sqrt{h(r)}dr = \rho d\ln r \implies d\ln(\rho - M) = d\ln r, \tag{61}$$

so that we can take e.g. $r = \rho - M$. Then $g(r) = (1 - M/\rho)^2 = r^2/(r + M)^2$, while $h(r) = (1 + M/r)^2$. So the metric goes to

$$ds^2 = -\frac{r^2}{(r + M)^2}dt^2 + (1 + M/r)^2(dr^2 + r^2 d\Omega_2^2). \tag{62}$$

When we take the $r \to 0$ limit, corresponding to approaching the outer horizon $\rho \to M$, we get

$$ds^2_{r\to 0} = -\frac{r^2}{M^2}dt^2 + \frac{M^2}{r^2}dr^2 + M^2 d\Omega_2^2. \tag{63}$$

The first dt^2 and dr^2 coordinates constitute an AdS_2 with the scale M, while the last part constitutes an S^2 of *constant* radius. In particular the radius of the sphere is finite even at the horizon, which in these coordinates is at $r = 0$.

GR eom in the coordinate-free formalism

Today we will be deriving the GR eom from the EH Hilbert action within the coordinate-free formalism. This is mostly just a practice with notation.

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To compute eoms, it helps to write the EH action (in 3+1D) as

$$S \sim \int \langle e \wedge e \wedge R \rangle, \tag{64}$$

where R is the Riemann curvature tensor, viewed as a 2-form valued in $\wedge^2 V$ with V the vector bundle which the vielbeins associate to the tangent space. The brackets represent contraction of the frame indices with the ε symbol, so that in more explicit notation,

$$S \sim \int e^a_i e^b_j R_{kl}{}^{cd} \varepsilon^{ijkl} \varepsilon_{abcd}. \tag{65}$$

To see why this gives the EH action, write

$$\begin{aligned}\langle e \wedge e \wedge R\rangle &= e_i^a e_j^b R_{kl}{}^{mn} e_m^c e_n^d \varepsilon^{ijkl}\varepsilon_{abcd} \\ &= (\det e)\varepsilon_{ijmn}\varepsilon^{ijkl} R_{kl}{}^{mn} \\ &= (\det e)(\delta_k^m \delta_l^n - \delta_k^n \delta_l^m) R_{kl}{}^{mn} \\ &= 2(\det e)R \\ &= 2\sqrt{\det g}R,\end{aligned} \tag{66}$$

where we used the antisymmetry of the curvature tensor in its two bottom indices (since it is a 2-form).

Let us now determine the eoms. We will do this by treating e and ω as independent dynamical variables. e is a 1-form valued in V and ω is the connection on V, and this information fully specifies the geometry of spacetime itself, due to the fact that

$$\nabla_i e_j^a = \partial_i e_j^a + \omega \wedge e \tag{67}$$

First we will compute the eom for the connection. We start by writing R in terms of the connection ω as

$$R = d\omega + \omega \wedge \omega. \tag{68}$$

Remember that this can *not*[2] be written as $D\omega$, since ω is not a tensor — if X is a tensor with n indices, then DX needs to contain n separate $\omega \wedge X$ terms, one for each index (with signs according to whether the index is co/contra-variant). This is not true of R. However, it is true that under a variation of the connection $\delta\omega$, we have $\delta R = D\delta\omega$ (see the next diary entry. Therefore

$$\begin{aligned}\delta S &\sim \int \langle D\delta\omega \wedge e \wedge e\rangle \\ &\sim 2\int \langle \delta\omega \wedge De \wedge e\rangle,\end{aligned} \tag{69}$$

where the sign from the grading of the wedge product cancels against the antisymmetrization from the bracket. Therefore since $\delta\omega$ is arbitrary and e is generically non-degenerate, we have

$$\delta S = 0 \implies De = T = 0, \tag{70}$$

where $T = De$ is the torsion. Thus the eom tells us that the LC connection is selected out dynamically.

The eom for e is easy, and tells us that

$$\varepsilon_{abcd} R^{ab} \wedge e^c = 0 \tag{71}$$

Similar manipulations to (66) give

$$\left(R^i{}_j - \frac{1}{2}\delta^i_j R\right) e_i^a = 0, \tag{72}$$

[2]Well, it almost can — see the next diary entry.

which because of the contraction with e^a_i is actually slightly more general than the more familiar $G^{ij} = 0$.

Also note what happens in three dimensions, where we just have[3]

$$S \sim \int \langle e \wedge R \rangle. \tag{73}$$

The variation with respect to the connection again tells us that $T = 0$, but now the variation with respect to the vielbein actually just sets $R = 0$![4] This is one way of seeing why there are no gravitational waves in three dimensions, without ever introducing the metric and talking about spin 2, etc.

Poynting-Robertson effect

Collisionless dust grains orbiting a star will eventually de-orbit and be captured by the star. Today we will explain why this happens, and will calculate the characteristic timescale for this process to occur.

This effect is due to the component of the radiation pressure which acts as a force *antiparallel* to the velocity vector of the orbiting grains. This occurs because after boosting into the rest frame of the dust, the incoming light does not impact the dust in a direction normal to its velocity, but rather one which is offset from this by an angle of $\vartheta = v/c \ll 1$, where v is the velocity of the dust (think about running in the rain, and the resistance coming from the rain drops that you "run into").

The magnitude of the force on a given dust grain of radius r is

$$F = \pi r^2 \vartheta P_{rad} = \pi r^2 v \frac{L}{4\pi R^2 c^2} \tag{74}$$

where P_{rad} is the radiation pressure at the grain's orbital location (we assume the grain to be on a circular orbit of radius R), and where L is the star's luminosity. Letting m be the mass of the grain,

$$\frac{d \ln v}{dt} = \frac{r^2 L}{4 m R^2 c^2}. \tag{75}$$

[3]Note that despite the notation (and despite the fact that the 3d EH term is actually a CS term), this action is *not* chiral—when we flip a spacetime coordinate we get a minus sign from both the spatial indices and the internal indices (there is always a zero index because of the ε_{abc} in $\langle\rangle$).

[4]If we had a cosmological constant, it would set the curvature equal to this.

Now since $v = \sqrt{GM/R}$ where M is the star's mass, we have $d\ln v/dt = -(d\ln R/dt)/2$. Therefore

$$\frac{dR^2}{dt} = \frac{r^2 L}{mc^2}, \tag{76}$$

and so the time it takes the dust grain to de-orbit is

$$\tau_{insp} = \frac{R_{int}^2 mc^2}{r^2 L}, \tag{77}$$

where R_{int} is the grain's initial orbital radius. We will take $L = L_\odot$, $r = 10\mu m$, $R_{int} = 1AU$. We will estimate m by supposing that dust grains in planetary disks are as dense as I am; this gives $m \sim 5 \times 10^{-12} kg$. We then find

$$\tau_{insp} \sim 10^4\, yr, \tag{78}$$

which isn't very long!

21cm cosmology basics