Global constraints on subsystem entanglement

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In this note we will prove some rather trivial results about entanglement in systems evolving from product states under constrained dynamics. We will consider constraints coming from either a $U(1)$ conserved charge, or from a strongly-fragmenting dynamical constraint. Throughout we will consider only dynamics acting on a 1d system of size L. Thanks to Zhicheng Yang and Cheng Wang for helpful discussions.

Global $U(1)$ charge

First consider a qubit chain with a conserved $U(1)$ charge. We begin by showing a result that is painfully obvious but which will be worked out regardless for fun:

Proposition 1. *Let ρ_Q be a random density matrix with charge Q. Define $n \equiv Q/L$ and take $L \to \infty$, with n fixed. Then for any subsystem of size $A \gg 1$, with $A/L \to 0$, the entanglement of $\rho_{Q,A} \equiv Tr_{A^c}[\rho_Q]$ is almost surely (up to subleading terms scaling as $o(A)$)*

$$S(\rho_{Q,A}) = |A| H_2(n) \tag{1}$$

with H_2 the binary Shannon entropy.

In particular, when $n \neq 1/2$, A is almost surely less than maximally entangled with its complement.

Proof. Consider first the entanglement of the maximally mixed state with charge sector Q, which we write as

$$\widetilde{\mathbf{1}}_Q \equiv \frac{1}{\binom{L}{Q}} \Pi_Q \tag{2}$$

where Π_Q projects onto the charge Q sector. The reduced density matrix on A is, overloading notation by writing A instead of $|A|$:

$$\mathrm{Tr}_{A^c}(\widetilde{\mathbf{1}}_Q) = \sum_{q=0}^{A} \frac{\binom{L-A}{Q-q}}{\binom{L}{Q}} \Pi_{q,A}, \tag{3}$$

where $\Pi_{q,A}$ projects onto the Hilbert space of A having charge q. The entropy of this state is

$$S(\mathrm{Tr}_{A^c}(\widetilde{\mathbf{1}}_Q)) = -\sum_{q=0}^{A} \binom{A}{q} \frac{\binom{L-A}{Q-q}}{\binom{L}{Q}} \ln \frac{\binom{L-A}{Q-q}}{\binom{L}{Q}}. \tag{4}$$

We evaluate this by way of

$$\binom{x}{y} \approx \sqrt{\frac{x}{2\pi y(x-y)}} e^{xH_2(y/x)}. \tag{5}$$

We then use

$$H_2(x+\varepsilon) = H_2(x) + \varepsilon \ln \frac{1-x}{x} - \frac{\varepsilon^2}{2} \frac{1}{x(1-x)} \tag{6}$$

to write, in the aforementioned $L \to \infty$ limit,

$$(L-A)H_2\left(\frac{Q-q}{L-A}\right) - LH_2(n) = -A\left(H_2(n) + (q/A - n)\ln \frac{1-n}{n}\right). \tag{7}$$

This gives

$$-\binom{A}{q} \frac{\binom{L-A}{Q-q}}{\binom{L}{Q}} \ln \frac{\binom{L-A}{Q-q}}{\binom{L}{Q}} = A\sqrt{\frac{A}{2\pi q(A-q)}}\left(H_2(n) + (q/A-n)\ln\frac{1-n}{n}\right) \\ \times e^{A(H_2(q/A) - H_2(n) - (q/A-n)\ln(1-n)/n} + O(\log(A)). \tag{8}$$

Doing the integral over q and dropping all subleading terms then gives the extremely reasonable result

$$S(\mathrm{Tr}_{A^c}(\widetilde{\mathbf{1}}_Q)) = \frac{AH_2(n)}{\sqrt{2\pi An(1-n)}} \int_{\mathbb{R}} dp\, e^{-\frac{p^2}{2An(1-n)}} = AH_2(n). \tag{9}$$

We finish the proof using Page's theorem, which we fomulate as the statement

$$\mathop{\mathbb{E}}_{\rho_Q} [||\rho_{Q,A} - \mathrm{Tr}_{A^c}(\widetilde{\mathbf{1}}_Q)||_1] \leq 2^{A-L/2}. \tag{10}$$

Markov's inequality then says that with probability $1 - \Theta(\exp(-L))$, $\rho_{Q,A}$ will be exponentially close in trace distance to $\mathrm{Tr}_{A^c}(\widetilde{\mathbf{1}}_Q)$, and hence (by Fannes' inequality, if you must) the entanglement entropy of $\rho_{Q,A}$ will almost surely be exponentially close to$AH_2(n)$. $\square$

tJ_z dynamics

tJ_z dynamics is the simplest example (perhaps too simple) of exponentially fragmented dynamics with a strong Hilbert space bottleneck. We define the dynamics with an onsite Hilbert space of dimension $N+1$, $\mathcal{H}_{\text{onsite}} = \text{span}(|0\rangle, \ldots, |N\rangle)$, with the pattern formed by all of the non-zero basis states being conserved by the dynamics (the zero state $|0\rangle$ is a "free space" and may move around freely). One of the many conserved quantities of this dynamics is a $U(1)$ charge measuring the number of zeros that appear in a given string:

$$Q \equiv \sum_i |0\rangle\langle 0|_i. \tag{11}$$

As above we will let $n = Q/L$ denote the average value of this charge.

OBC: boundary subsystem

Consider an open chain of length L partitioned into a left half A and a right half A^c. While in the $U(1)$ case we could have maximally entangled subsystems at charge density $n = 1/2$, for tJ_z this is not possible: if ρ is a random state in a fixed Krylov sector, then ρ_A is less than maximally entangled with probability 1:

Proposition 2. *Let $\rho_{\mathbf{s}}$ be a random density matrix in $\mathcal{K}_{\mathbf{s}}$, the Krylov sector of states with irreducible string $\mathbf{s}$, and define $Q = |\mathbf{s}|, n = Q/L$. Then the entanglement of $\rho_{\mathbf{s},A} \equiv Tr_{A^c}[\rho_{\mathbf{s}}]$ is almost surely (up to $o(A)$ terms)*

$$S(\rho_{\mathbf{s},A}) = AH_2(n). \tag{12}$$

In particular, the maximum entropy that can be obtained is $A\ln(2) = \ln|\mathcal{K}_{\max}|$, less than the page value of $A\ln(3)$.

Proof. The proof is exactly the same as in the case for a $U(1)$ conserved charge, as in this particular setup the spin degrees of freedom contribute no entropy: given the value of the total $U(1)$ charge in A, the spin pattern in A is uniquely fixed. Thus the entropy comes entirely from the $U(1)$ charge, whose expected density is n, around which the probability distribution defined by the RDM on A concentrates. $\square$

Note that in order to maximize the entanglement entropy in this case, we need to choose a sector at half filling for the $U(1)$ charge. While such sectors are the ones of largest dimension, they are *not* typical: if we instead pick a *random* (computational basis) product state in which to initialize the dynamics, we will instead almost surely have $n = 1/(N+1)$ giving a value of $S(\rho_{\mathbf{s},A})$ even further from the Page value.

PBC

Things change when periodic boundary conditions are applied.

Proposition 3. *Let $\mathbf{s}$ be randomly chosen from the set of irriducible strings of length $|\mathbf{s}| = Q$, and let $\rho_{\mathbf{s}}$ be a random density matrix in $\mathcal{K}_{\mathbf{s}}$. Then the entanglement of $\rho_{\mathbf{s},A}$ is almost surely (up to $o(A)$ terms)*

$$S(\rho_{\mathbf{s},A}) = A\left(H_2(n) + (1-n)\ln(N)\right) \tag{13}$$

with $n = Q/L$ as before.

The physical interpretation of this result is very simple: the first term comes from the entropy of the $U(1)$ charge, and the second term comes from the entropy produced by moving $\mathbf{s}$ so that different substrings occupy A. $S(\rho_{\mathbf{s},A})$ is maximized when $n = 1/(N+1)$, the expected value of n in a random state, for which it equals the Page value.

Proof. We again consider the maximally mixed state $\widetilde{\mathbf{1}}_{\mathbf{s}}$ over the sector $\mathcal{K}_{\mathbf{s}}$. With PBC, the dimension of this sector is $\binom{L}{Q}N_\sigma$, where $N_\sigma = |\Sigma_{\mathbf{s}}|$ is the number of cyclic permutations $\sigma \in \Sigma_{\mathbf{s}}$ of $\mathbf{s}$ such that $\mathbf{s} \neq \sigma(\mathbf{s})$. We may decompose $\widetilde{\mathbf{1}}_{\mathbf{s}}$ as

$$\widetilde{\mathbf{1}}_{\mathbf{s}} = \frac{1}{N_{\sigma,\mathbf{s}}\binom{L}{Q}} \sum_{\sigma\in\Sigma_{\mathbf{s}}} \sum_{q=0}^{A} \Pi_{\sigma(\mathbf{s})_{1:A-q},A} \otimes \Pi_{\sigma(\mathbf{s})_{A-q:},L-A}, \tag{14}$$

where $\sigma(\mathbf{s})_{1:A-q}$ is the string defined by the first $A-q$ characters of $\sigma(\mathbf{s})$, and $\sigma(\mathbf{s})_{A-q:}$ is the remainder of $\mathbf{s}$. The RDM on A can then be rewritten as

$$\rho_{\mathbf{s},A} = \sum_{q=0}^{A} \sum_{\mathbf{s}_A \in \mathsf{dist}_{A-q}(\mathbf{s})} \sum_{\sigma\in\Sigma_{\mathbf{s}}\,:\,\sigma(\mathbf{s})_{1:A-q}=\mathbf{s}_A} \frac{1}{N_\sigma \binom{L}{Q}} \binom{L-A}{Q-q} \Pi_{\mathbf{s}_A,A}, \tag{15}$$

where $\mathsf{dist}_{A-q}(\mathbf{s})$ is the set of distinct substrings of $\mathbf{s}$ of length $A-q$. If $\mathbf{s}$ is chosen at random, each of these substrings will occur with equal probabilities, and so we will assume that $\mathbf{s}$ is chosen so that the emperical distribution on $\mathbf{s}_A$ is exactly uniform (and the entropy is maximized in this case). Note that in order for this to be the case, we require that $L > 2^A$.

To simplify notation, define $N_{\sigma|\mathbf{s}_A} \equiv |\{\sigma \in \Sigma_{\mathbf{s}} \,:\, \sigma(\mathbf{s})_{1:A-q} = \mathbf{s}_A\}|$. Then with the above assumption on $\mathbf{s}$,

$$\rho_{\mathbf{s},A} = \sum_{q=0}^{A} \sum_{\mathbf{s}_A \in \mathsf{dist}_{A-q}(\mathbf{s})} \frac{N_{\sigma|\mathbf{s}_A}}{N_\sigma \binom{L}{Q}} \binom{L-A}{Q-q} \Pi_{\mathbf{s}_A}. \tag{16}$$

Now again by the above assumption, $|\mathsf{dist}_{A-q}(\mathbf{s})| = N^{A-q}$, and $N_{\sigma|\mathbf{s}_A}/N_\sigma = N^{-(A-q)}$ (since the latter measures the fraction of cyclic translates of $\mathbf{s}$ that begin with $\mathbf{s}_A$, and all $\mathbf{s}_A$ occur with equal probability by assumption). Therefore the entanglement entropy is then

$$\begin{aligned} S(\rho_{\mathbf{s},A}) &= -\sum_{q=0}^{A} \frac{\binom{A}{q}\binom{L-A}{Q-q}}{\binom{L}{Q}} \left(\ln \frac{\binom{L-A}{Q-q}}{\binom{L}{Q}} + \ln N^{-(A-q)} \right) \\ &= AH_2(n) + A \ln N \sum_{q=0}^{A} (1-q/A) \frac{\binom{A}{q}\binom{L-A}{Q-q}}{\binom{L}{Q}}, \end{aligned} \tag{17}$$

where we used our earlier result for the case of a single $U(1)$ charge. The same techniques as used above can then be used to show that the remaining sum evaluates to $1-n$ in the appropriate limit, giving the desired result. □

For this result to work, we needed $L > N^A$, and it is straightforward to show that if $L = o(M^A)$ for any $M < N$, then $S(\rho_{\mathbf{s},A})$ must be less than the Page value. Thus maximally mixing a subsystem with tJ_z dynamics is possible with periodic boundary conditions, but doing so requires exponential space resources.

OBC: bulk subsystem

As long as the subsystem we consider is located deep in the bulk, this case is not fundamentally different from the PBC setup. The only difference lies in the spatial resources required to achieve a maximal contribution from the spin part of the entropy: to get a uniform distribution on the $\mathbf{s}_A$, L must be large enough that all of the $\sim N^A$ distinct choices of $\mathbf{s}_A$ are within a $\sim \sqrt{L}$ distance of A, since each A-sized substring in $\mathbf{s}$ can only diffusive over a distance of $\sim \sqrt{L}$. Therefore Page values can only be achieved when $L = \Omega(N^{2A})$, which necessitates exponentially more spatial resources than in the case with periodic boundary conditions.

Exponentially modulated symmetry

Finally we consider a spin-1 model whose dynamics conserves

$$Q = \sum_i 2^i S_i^z. \tag{18}$$

This dynamics mandates that subsystems quenched from product states always be at sub-Page entanglement for *all* boundary conditions, even when $L = \infty$.

This can be seen by considering the operator

$$Q_A = \sum_{i \in A} 2^{i - i_{l,A}} n_i, \tag{19}$$

where $i_{l,A}$ is the site at the leftmost end of A; with this definition Q_A can take on any integer between 0 and $Q_{A,\max} = 2^{|A|+1} - 2$. Suppose at $t = 0$ the system is initialized in a product state with a particular initial value $Q_A(0)$ of Q_A. Then after evolving the full system $A \cup A^c$ under constraint-preserving dynamics, the value $Q_A(t)$ of Q_A at time t must be expressible as

$$Q_A(t) = Q_A(0) + a + b2^{|A|}, \tag{20}$$

where $a, b \in \{-1, 0, 1\}$ express the distinct ways that particles can be transferred between A and A^c.

Now it turns out that at large l, the size of the largest symmetry sector on a system of size l scales as $\phi^l + \cdots$, where the $\cdots$ are subleading. Thus

$$\mathrm{rank}(\rho_A(t)) \leq 9\phi^{|A|}, \tag{21}$$

where the factor of 9 comes from the number of ways of choosing a, b. The entanglement entropy of ρ_A is accordingly upper bounded as

$$S(\rho_A(t)) \leq |A| \ln(\phi) + \mathrm{const}, \tag{22}$$

which since $\ln(\phi) < \ln(3)$ means that the coefficient of the volume law can never be made to match the scaling of a random state. It is remarkable that just the imposition of a global symmetry—albeit a rather unconventional one—can produce behavior so at odds with ETH-like expectations.