Stochastic dipolar dynamics

Ethan Lake

In these notes we numerically investiage a family of classical Markov circuits with dipole-conserving dynamics.[1] These Markov processes are designed to crudely mimic the large-hopping regime of the dipolar Bose-Hubbard model (DBHM), in which a *fractured Bose glass* occurs. In 1d, this model looks like

$$H = -\sum_i \sum_{r=0} (t_r b_i^\dagger b_{i+1} b_{i+1+r} b_{i+2+r}^\dagger + h.c.) + \sum_i \left(-\mu n_i + \frac{U}{2} n_i(n_i - 1) + \cdots \right), \quad (1)$$

where the $\cdots$ include e.g. nearest-neighbor repuslive interactions, which for the present purposes are not important. The hopping strengths are assumed to decay exponentially with r as

$$t_r = \frac{t}{\lambda} e^{-r/\lambda} \quad (2)$$

for some length scale λ and some energy t.

When t is large enough (roughly $t \gtrsim U$), the system wants to form a giant droplet constaining (at fixed filling) all the bosons in the system. This is because when acting on a region with average density n, both the kinetic and potential terms scale as n^2 — thus when $t \gtrsim U$, the energy can always be lowered by an arbitrary amount by congregating the bosons into one giant droplet. In a real system this collapse is obstructed by the kinematic constraint of dipole conservation, which freezes out particle motion after empty voids between droplets form. The natural expectation is then that a real system will spontaneously (Krylov) fracture into many disconnected droplets, which then remain as metastable states for long time scales. Dipole condensates will form in these droplets, which themselves are rather complicated quantum-mechanical objects.

As an attempt to describe the droplet formation within a more tractable model, we study a dipole-conserving family of Markov circuits. The configuration space consists of strings $\mathbf{s} \in \mathbb{N}^L$ (here $0 \in \mathbb{N}$) which are acted on at each timestep t by a Markov processes $\mathcal{M}$ which proposes the update

$$\mathcal{M}_{i,r}^{\pm} : \mathbf{s} \mapsto \mathcal{O}_{i,r}^{\pm} \mathbf{s}, \qquad \mathcal{O}_{i,r}^{\pm} \equiv b_i^{\pm} b_{i+1}^{\mp} b_{i+1+r}^{\mp} b_{i+2+r}^{\pm}, \quad (3)$$

where $b_i^{\pm}$ changes the ith entry of $\mathbf{s}$ by ± 1 (with $b_i^{\pm} b_{i+1}^{\mp} b_{i+1+r}^{\mp} b_{i+2+r}^{\pm}$ tacitly assumed to act as the identity if either of the b_j^- operators act on a location in which $s_j = 0$). Here the $\pm$ sign and site index i are chosen randomly, and the range $r \in \mathbb{N}$ is chosen with probability

$$P(r) = \frac{e^{-r/\lambda}}{\sum_{j \in \mathbb{N}} e^{-j/\lambda}}. \quad (4)$$

[1] The infinite-temperature dynamics of a similar model was studied by Morningstar, Khemani, and Huse; our diagonstics of the phase transition will be similar to theirs.

A given update (3) is accepted according to some dynamical rule which preserves detailed balance. For concreteness we will choose Glauber dynamics; thus the update is accepted with probability

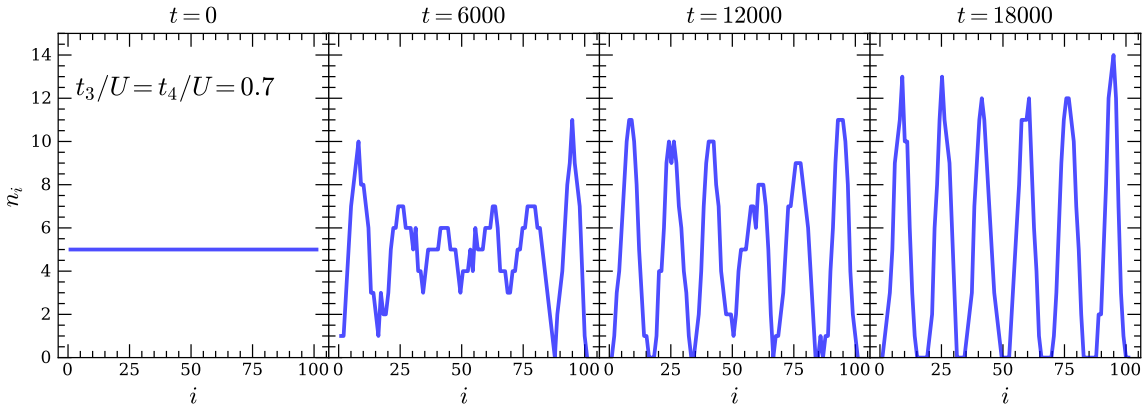
$$P_{acc} = \frac{x}{1+x}, \qquad x = \exp\left(\frac{E_{\mathbf{s}} - E_{\mathcal{O}^{\pm}_{i,r}\mathbf{s}}}{T}\right) \tag{5}$$

for some temperature T and energy function $E_{\mathbf{s}}$. $E_{\mathbf{s}}$ should be chosen in such a way so as to mimic the quantum dropleting phenomenon of the DBHM Hamiltonian. The exact details are not important for the universal phenomena we are trying to describe, and we accordingly make the choice

$$E_{\mathbf{s}} = -U\sum_i s_i^2 + \sum_{r\in\mathbb{N}} t_{r+3}\left(\frac{s_i + s_{i+1} + s_{i+r+1} + s_{i+r+2}}{4}\right)^2, \tag{6}$$

where the t_{r+3} should at least be nonzero at least for $r = 0, 1$ and decaying for $r \to \infty$, but are otherwise not particularly strongly constrained.

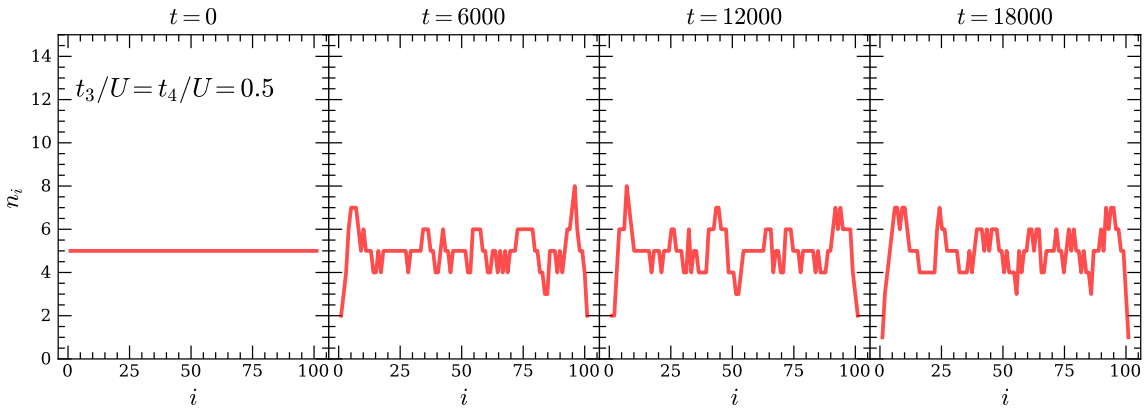
We now study the dynamics of this Markov process when initialized on a uniform configuration of density n, with $\mathbf{s}_0 = (n, \ldots, n)$. For numerical simplicity, we choose $t_3, t_4 \neq 0$ and take $t_{k>4} = 0$, with our goal being to map out the dynamical phase diagram as a function of $t_{3,4}/U, T, n$, and λ. As an example of an evolution in the fractured phase for a system of size $L = 100$, we have



(7)

Here $t_3/U = t_4/U = 0.7$, $\lambda = 1, n = 5$, $T = 0.6$ and the plot headings represent simulation time (with one time step corresponding to L Markov updates). As the simulation time $t \to \infty$, the fractured droplet configuration seen in the rightmost

panel persists. As an example of an evolution in the *un*fractured phase, we have



(8)

where the only change is that $t_3/U = t_4/U = 0.5$ (the phase boundary is thus evidently between 0.5 and 0.7).

There are several methods for detecting the fractured phase in a more quantitative way, among which are the density of zeros

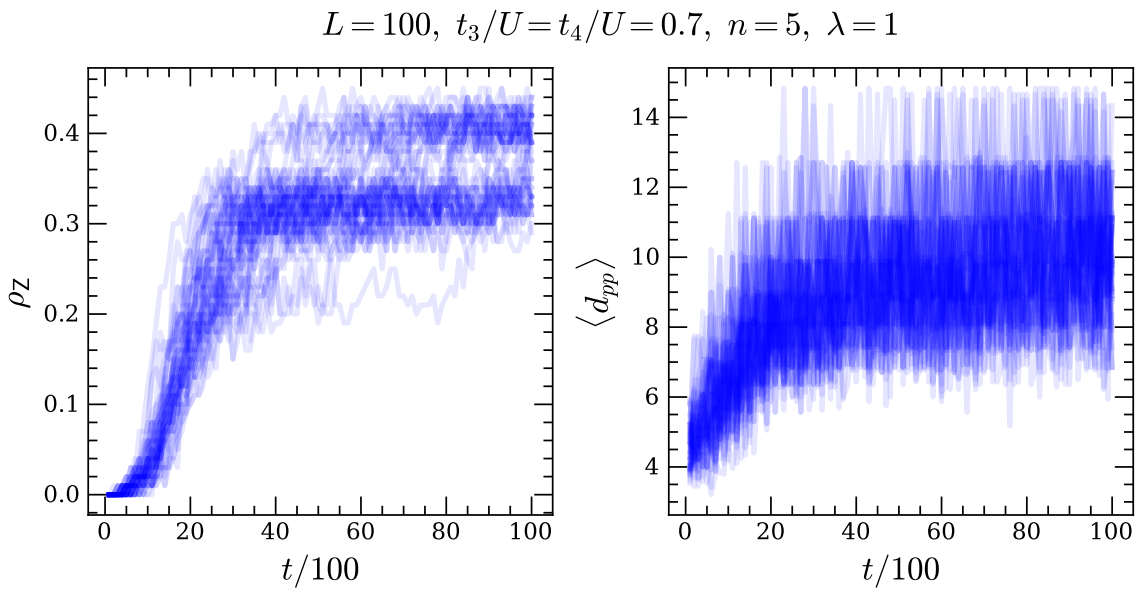
$$\rho_{\mathsf{Z}} \equiv \frac{1}{L}\sum_i \langle \delta_{s_i,0} \rangle, \tag{9}$$

the average squared density fluctuation

$$A \equiv \frac{1}{L}\sum_i \langle (s_i - n)^2 \rangle, \tag{10}$$

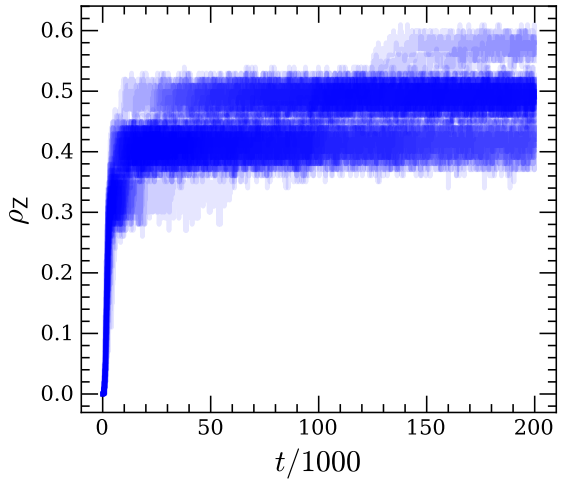
which can be thought of as a form of Edwards-Anderson order parameter for diagnosing glassiness, and the average peak-to-peak distance $\langle d_{pp} \rangle$ of $\mathbf{s}$, where a peak is defined as a local maximum in $\mathbf{s}$ as a function of position i. We can use these quantities to determine when the system has 'equilibrated' to a metastable fractured droplet configuration, although there are often ralarge sample-to-sample fluctuations (where by 'sample' we mean a particular instatiation of the chosen stochastic updates): for $t_3/U = t_4/U = 0.7$ and $\lambda = 1, n = 5, T = 1, L = 100$, the values of $\rho_{\mathsf{Z}}, \langle d_{pp} \rangle$ for 100

different realizations give



(11)

where the two bands in the left plot are due to the system being able to form configurations with different droplet numbers. If we significantly increase the time window over which we track the system, we see multiple bands corresponding to smaller droplet numbers (note the different x axis scale)

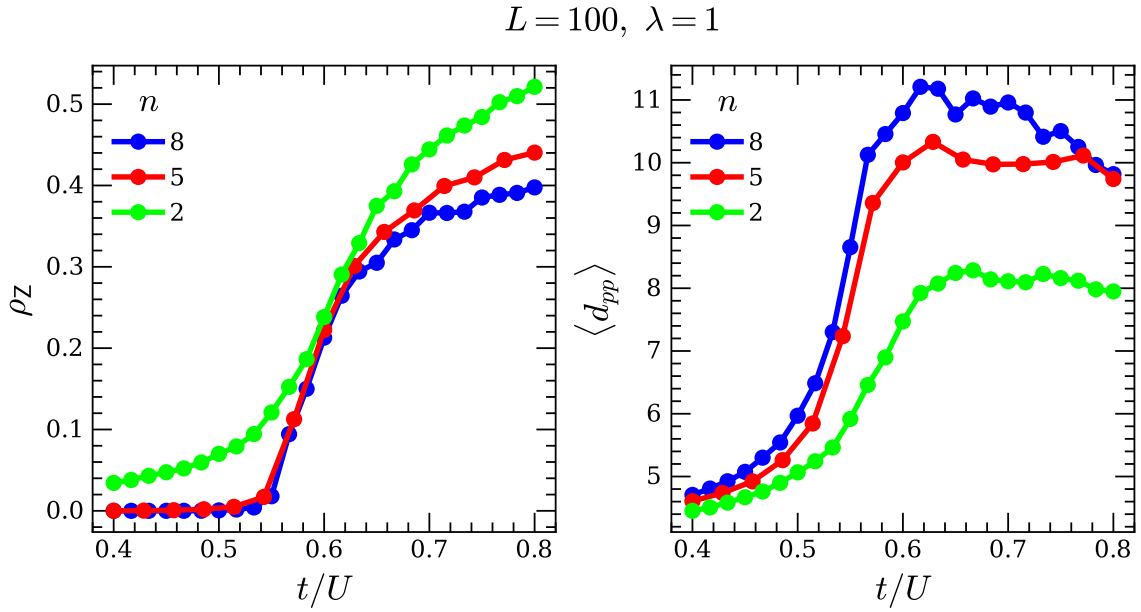


(12)

Note that the timescale for droplet annihilation increases as the droplet number decreases (the expectation is that the transition times diverge exponentially with inverse droplet number). The $t \to \infty$ limit is to have $\rho_Z = 1 - \varepsilon$.

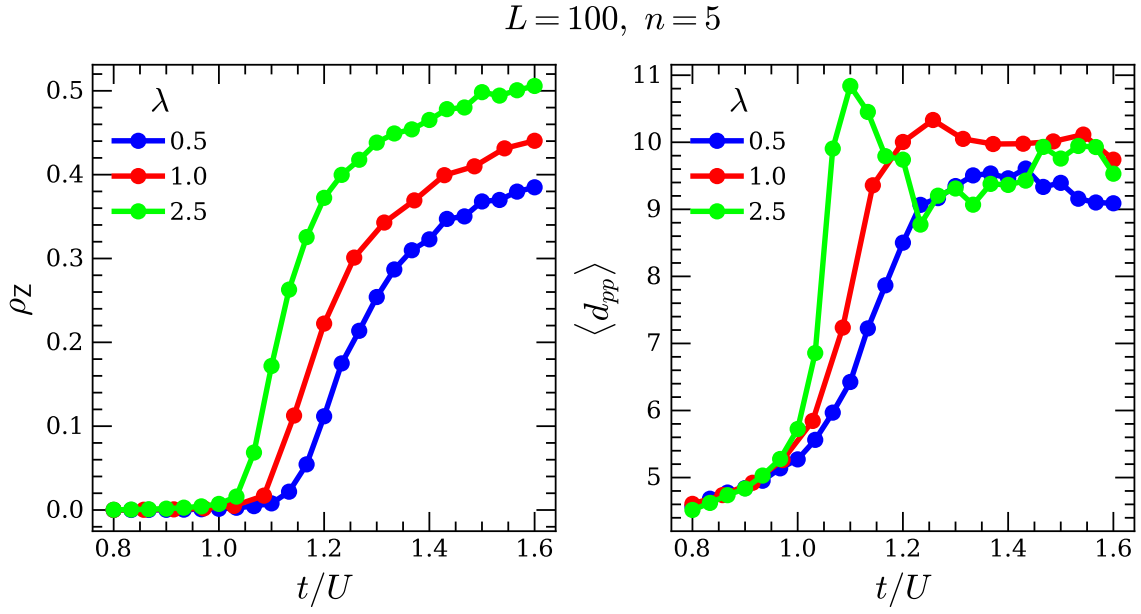
We now turn to examining for what parameters the system becomes glassy. Numerically this can be done by sampling $\rho_Z, \langle d_{pp} \rangle$, as long as we wait to collect data until the initial transient relaxation timescale has passed (this happens at $t \approx 3000$ for the choices of parameters in the above plots). First we take $t_3 = t_4 \equiv t$. Spontaneous fracturing and droplet formation is observed to occur at a critical value of t/U which becomes more sharply defined at larger densities (where ρ_Z is a sharper diagnostic of

droplet formation). Setting $t_3 = t_4 = t$ and taking $\lambda = 1$, $L = 100$, we have



(13)

The location of the transition appears to move slightly with changing λ (larger λ means that *longer*-ranged hoppings are favored in the update rules):

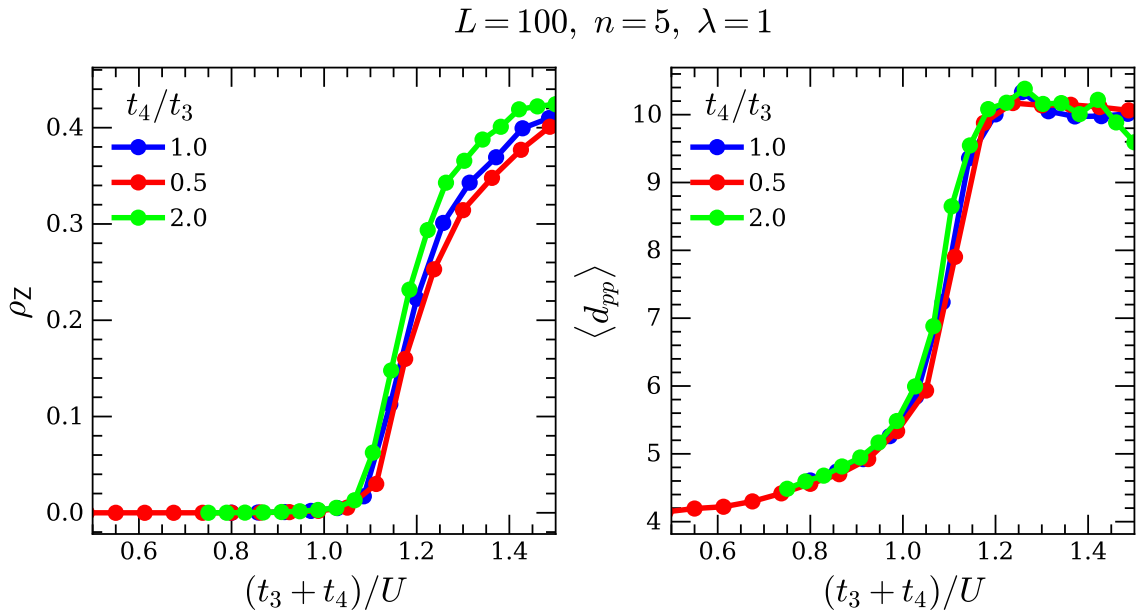


(14)

That the largest value of λ has the most noisey $\langle d_{pp} \rangle$ is not surprising, as larger hopping distances allows the droplets to more easily re-adjust their configurations.

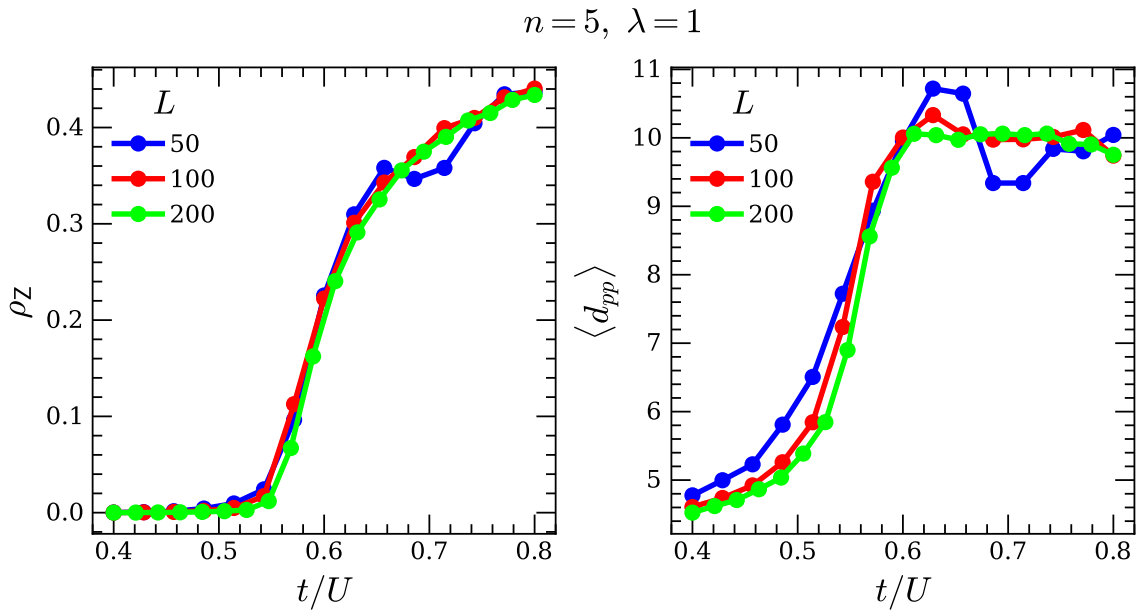
It turns out that setting $t_3 = t_4$ is done wolog, since the location of the transition

is seen to depend only on the combination $t_3 + t_4$, and not on t_3/t_4:



(15)

The transition is also not appreciably affected by changing the system size, being as it is determined by the droplet formation process, which occurs on a scale set by n, λ which is much smaller than L. Indeed,



(16)