2019 physics diary / problem list

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Preface: Disclaimer: I only know that my answers are correct for $\sim 2/3$ of the problems, and of course even correct answers are bound to have a nonzero number of typos / mistakes.

A few problems are listed in multiple section headings as appropriate.

1 Condensed matter problems

5 Quantum information problems

6 Miscellanea

7 *January 4 — Yet another way to derive the chiral anomaly in two dimensions*

Today is a short one: we'll be calculating the chiral anomaly / ABJ anomaly / mixed t' Hooft anomaly between vector and axial fields. We'll do this by looking at a Ward identity that gives the conservation of the axial current, which is a slightly different way compared to any that I've seen in books.

Solution:

Since we already know the answer and have indeed derived it several times in last year's physics diary, we won't worry too much about keeping numerical factors correct. The effective Euclidean action for the background vector field is

$$Z[A] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-\int [\bar{\psi} i \not{D} \psi + J_\mu A^\mu]\right). \tag{1}$$

In two dimensions, we have (using the gamma matrices $\gamma^0 = X, \gamma^1 = Y, \bar{\gamma} = Z$) for a fermion $\psi = (L, R)^T$

$$J_0 = n_L + n_R, J_1 = i(n_L - n_R), \qquad \mathcal{J}_0 = n_L - n_R, \mathcal{J}_1 = i(n_L + n_R), \tag{2}$$

where $\mathcal{J}$ is the axial current. This means that in two dimensions we have

$$\mathcal{J}_\mu = -i\epsilon_{\mu\nu} J^\nu \implies \star d^\dagger \mathcal{J} = -idJ. \tag{3}$$

We can use this in a ward identity as follows: suppose we shift the gauge field by $\delta A = \star d\lambda$, where λ is a 0-form (with compact support). Then

$$\int \delta(J \wedge \star A) = \int dJ \wedge \lambda = i \int \star d^\dagger \mathcal{J} \wedge \lambda, \tag{4}$$

and so evidently we have, taking λ to be infinitesimal so that we can expand to first order in λ,

$$Z[A + \star d\lambda] \approx Z[A] \left(1 - i \int \lambda \wedge \star \langle d^\dagger \mathcal{J} \rangle_A \right). \tag{5}$$

On the other hand, we can get an explicit expression for the lower orders in the expansion for $Z[A]$. Putting the $\det \not{D}_A$ in the exponent in the usual way, we get the usual representation of $Z[A]$ as a sum of bubbles with A lines sticking out of them. The first order tadpole graph gives zero, while the second gives the usual polarization bubble. One can evaluate this explicitly, or use gauge invariance to write down the answer (up to the coefficient). So, to second order,

$$Z[A] \approx \exp\left[-\frac{1}{2\pi} \int F \wedge \star \left(\frac{1}{\Box} F\right)\right]. \tag{6}$$

This is the unique gauge-invariant dimension-2 thing we can build that's quadratic in A. Another way to write it uses

$$\int F \wedge \star (\Box^{-1} F) = A \wedge \star \frac{d^\dagger d}{\Box} A \to A_\mu (g^{\mu\nu} - q^\mu q^\nu / q^2) A_\nu, \tag{7}$$

which is the usual projector onto the transverse modes. Anyway, varying this to first order in λ, gives

$$Z[A + \star d\lambda] \approx Z[A] \left(1 - \frac{1}{\pi} \int F \wedge \star \Box^{-1} d d^\dagger \star \lambda \right). \tag{8}$$

Since it is acting on an exact form, $\Box^{-1} = (dd^\dagger)^{-1} = (d^\dagger)^{-1} d^{-1}$, and so after integrating by parts,

$$Z[A + \star d\lambda] \approx Z[A] \left(1 - \frac{1}{\pi} \int \lambda \wedge F \right). \tag{9}$$

Now we can match up the two ways of calculating the partition function to obtain

$$\langle d^\dagger \mathcal{J} \rangle_A = -\frac{i}{\pi} \star F, \tag{10}$$

which is the anomaly we wanted to show. The i is from our choice of Euclidean signature, and the $1/\pi$ (instead of $1/2\pi$) ensures that $\int \star d^\dagger \mathcal{J} \in 2\mathbb{Z}$ regardless of the A background, which is consistent with overall fermion number conservation.

8 *January 4 — Free energies, contact terms, and anomalies*

Today's diary entry is a recapitulation of some things I learned about anomalies from Zohar at the 2018 / 2019 Jerusalem winter school on QFT.

Solution:

In the case of a continuous symmetry, we can relate this to the divergence of the relevant current. Here the current is defined as $J^\mu = \delta S/\delta A_\mu$, and so in particular may include the background field itself, as in the case of e.g. scalar QED, where the background field contributions to J^μ are needed to render it gauge-invariant . Consider performing an infinitesimal gauge transformation by $\delta A = d\epsilon$. Then the action shifts by $\int \epsilon \wedge \star d^\dagger J$, and so the partition function is

$$Z[A + d\epsilon] = Z[A]\left(1 + i\int \epsilon \wedge \star\langle d^\dagger J\rangle_A\right). \tag{11}$$

Taking logs to write this in terms of the free energy $\mathcal{F}[A]$,

$$i\int \epsilon \wedge \star\langle d^\dagger J\rangle_A = \delta_\epsilon \mathcal{F}[A]. \tag{12}$$

Saying that all counterterms built from background gauge fields are unphysical is going too far, though. After all, the CS term $A \wedge dA$ determines the Hall conductivity (here A is a *background* EM field, not the dynamical field which is integrated over in the field theory description of the Hall effect), which is physical and well-defined (that the CS term only affects correlation functions at coincident points can also be understood from the fact that CS theory has no radiation: the equations of motions have no derivatives, and the field strengths can be solved for as a local function of the sources). But how can the Hall conductivity be well-defined on the field theory level, if the Hall response is determined by a contact term in the background gauge fields? One way to argue (thanks to Senthil for bringing this up) is that counterterms we add to change our regularization prescription should be able to be added locally, that is, we should be able to add them with a spatially varying coefficient. Of course, adding the term $\int \alpha(x) A\wedge dA$ is not allowed because it breaks gauge invariance, and so the CS term is not a trivial change in regularization scheme in the same sense that e.g. $\int F \wedge \star F$ would be (which gives a very singular $\Box\delta(x)$ modification to the current-current correlation function).

Consider the mixed anomaly t'Hooft anomaly between $U(1)^A$ and $U(1)^V$ for fermions in even dimensions. We know that e.g. in 1+1D, a $U(1)^A$ gauge transformation $\delta A^A = d\lambda$ produces, in the usual regularization scheme, a term

$$\delta\mathcal{F}[A^A, A^V] = \frac{1}{2\pi}\int \lambda F^V = \delta_\lambda \frac{1}{2\pi}\int (d^{-1}A^A)F^V, \tag{13}$$

which is the variation of a term that is singular at zero momentum.

9 *January 6 — C, T, and fermions in three dimensions*

Today's diary entry is a careful compendium of various facts about fermions and their symmetries in three spacetime dimensions.

Solution:

In this diary entry we will be in 2+1 dimensions, in $\mathbb{R}$ time. We will use the Weyl basis for the γ matrices:

$$\gamma_0 = iY, \quad \gamma_1 = X, \quad \gamma_2 = Z. \tag{14}$$

This has the advantage that all of the γ_μ's are real, which simplifies calculations with T. From the commutation relations of the γ's, we see that this choice works provided we use mostly positive signature.

Now we'll set conventions for what we mean by C and T. In QFT, T is an antiunitary operator that sends t to $-t$. However, we have many options for what we mean by T, since we can compose T with any unitary transformation that commutes with the Lorentz group. For a given situation, some of these choices for T will be symmetries, while others will not. In the following, by T, we will mean the antiunitary operator that acts on a Dirac fermion $\psi = \psi_1 + i\psi_2$ (here both ψ_1, ψ_2 are real Majorana fermions, and $\psi_i = (\psi_{i,L},\ \psi_{i,R})^T$) as

$$T : \psi(t,x) \mapsto \gamma_0\psi(-t,x), \quad \psi_1(t,x) \mapsto \gamma_0\psi_1(-t,x), \quad \psi_2(t,x) \mapsto -\gamma_0\psi_2(-t,x). \tag{15}$$

The γ_0 here switches L and R movers, which is something we want T to do. We will often write transformations like this as e.g. $T : \psi \mapsto \gamma_0\psi$, with the reversal of the time coordinate left implicit.

We could have also chosen to not put the minus sign in the transformation of ψ_2, and then we'd get a map $T : \psi \to \psi^\dagger$. This transformation will usually be denoted by CT, since we will define

$$C : \psi_i(t,x) \mapsto (-1)^{i+1}\psi_i(t,x), \tag{16}$$

which flips the sign of the imaginary part of ψ. Finally we have parity, which we take to act as

$$P : \psi_i(t,x,y) \mapsto \gamma_1\psi_i(t,-x,y). \tag{17}$$

Note here we are being sloppy and writing x for either one spatial coordinate, or as shorthand for both spatial coordinates. We are also being sloppy in calling it parity: a better name would be reflection, since, by virtue of the fact that we are in two dimensions, the action of P as a matrix representing the Lorentz group has determinant $+1$, and not -1 (as in an even number of spacetime dimensions). P here only inverts one of the space coordinates, not all of them[1].

[1] These assignments for the actions of C, P, T are specific to our choice of signature $(-,+,+)$. Unfortunately, the representation theory of the various pin groups means that these choices do not carry over to other signatures. For example, suppose we chose the signature $(+,-,-)$, with γ matrices (X, iY, iZ). Now suppose T acted as $T = T_U K$, with K complex conjugation and T_U unitary. Then in order to preserve $\bar\psi i\partial\!\!\!/\psi$, we need to have $T_U = \gamma^2$! Note that the Hermitian term $m\bar\psi\psi$ is T-odd. Similarly, for P to preserve the kinetic term, we can again take it to act as γ^2, so that T and P only differ by complex conjugation. What a mess!

The final symmetry we'll be thinking about is the regular vector $U(1)$ symmetry. Written out explicitly, the $U(1)$ symmetry acts as a rotation on the Majorana fermions:

$$R_\alpha : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{18}$$

The current $J = \bar{\psi}\gamma_\mu\psi\, dx^\mu$ is odd under T, so that J_0 is even while J_1 is odd. For example,

$$T(\bar{\psi}\gamma_1\psi) = \bar{\psi}\gamma_0^T\gamma_0\gamma_1\gamma_0\psi = \bar{\psi}\gamma_0^2\gamma_0\gamma_1\psi = -\bar{\psi}\gamma_1\psi. \tag{19}$$

This means the charge operator $Q = \int J_0$ is even under T. Since $C(\psi) = \psi^\dagger$, J is odd under C, and so is Q. J_1 is odd under P while the other components are even, so $P(J) = J$ as a differential form. Anyway, from these definitions we see that we have the algebra

$$T^2 = (CT)^2 = \gamma_0^2 = (-1)^F, \quad C^2 = P^2 = \mathbf{1}, \quad Te^{iQ} = e^{-iQ}T, \quad Ce^{iQ} = e^{-iQ}C. \quad Pe^{iQ} = e^{iQ}P. \tag{20}$$

To summarize, the various symmetries act on a gauge field and the various components of its field strength as (the differential form dA transforms in the same way as A does)

$$\begin{aligned} T &: A \mapsto -A, \quad E^i \mapsto E^i \quad B^i \mapsto -B^i \\ C &: A \mapsto A, \quad E^i \mapsto -E^i, \quad B^i \mapsto -B^i \\ P &: A \mapsto A, \quad E^i \mapsto -E^i, \quad B^i \mapsto B^i \end{aligned} \tag{21}$$

For posterity's sake, we record the easily proved facts that for any real differential form B,

$$T[B] = (-1)^s B \implies T[dB] = (-1)^s dB, \quad T[\star B] = (-1)^{s+1} dB, \tag{22}$$

and like wise for T replaced by P (again, here P is really an inversion of one space coordinate, and is not a parity transformation in the correct sense of the word).

There are three types of masses we will consider for the fermions. They are defined as

$$\begin{aligned} \bar{\psi}M_D\psi &\equiv im_D(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) \\ \bar{\psi}M_-\psi &\equiv im_-(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2) \\ \bar{\psi}M_m\psi &\equiv im_m\bar{\psi}_1\psi_2. \end{aligned} \tag{23}$$

The second two break $U(1)$, and as we will see, are related by a $U(1)$ rotation to one another; hence they are not strictly speaking independent. We will look at each of the three masses in turn. As we will show in a second, they preserve or break the symmetries defined above in the following way:

M_D

First for the regular Dirac mass. One useful fact is that $\bar{\psi}_i\psi_i$ is even under T:

$$T(\bar{\psi}_i\psi_i) = \psi_i\gamma_0^T\gamma_0\gamma_0\psi_i = \bar{\psi}_i\psi_i. \tag{24}$$

Thus

$$T(\bar{\psi}M_D\psi) = -\bar{\psi}M_D\psi, \tag{25}$$

so that M_D is odd under T. Similarly, one shows that the Dirac mass is odd under P. On the other hand it preserves $U(1)$, since as a bilinear form for the vector (ψ_1, ψ_2), it is the identity. That it preserves $U(1)$ can also be checked explicitly, using the fact that $\bar{\psi}_1\psi_2 = \bar{\psi}_2\psi_1$ (there is no minus sign here, because of a minus sign picked up from the definition of γ_0). Finally, it also preserves C, since it is quadratic in ψ_2. Since it preserves $U(1)$ and C, there can be no anomalies involving combinations of just these two symmetries.

One perspective on why m_D breaks parity is the following. Consider solving the Dirac equation in 2+1D: writing the Dirac spinor as $\psi = (\psi_+, \psi_-)^T$ (no more Majoranas until the next subsection), we can consider going into the $\mathbf{k} = 0$ rest frame, wherein we have

$$\partial_t\psi_+ = -m_D\psi_-, \qquad \partial_t\psi_- = m_D\psi_+. \tag{26}$$

Solutions to this are $\psi_+ = e^{\pm i m_D t}$, with the $\pm$ sign free to be chosen at will. We need to fix a convention, and will choose the $+$ sign. This gives the solution $\psi = (\psi_+, \psi_-)^T = (e^{im_Dt}, ie^{im_Dt})^T$, which is a $+$ eigenvector of J. What spin does this have? Spatial rotations are implemented in Spin(3) by $i[\gamma^1, \gamma^2]/4 = J/2$, and so we see that ψ has spin 1/2. Now consider changing the sign of m_D: this is equivalent to changing our convention about which sign to choose in $e^{\pm i m_D t}$, which changes the eigenvalue of ψ under J, and means that ψ now has spin $-1/2$. Now while the choice of spin $\pm 1/2$ is a convention, after fixing a convention, the difference in spins between positive and negative m_D is not. Since a definite spin is picked out for m_D nonzero, T and P must be broken by a nonzero Dirac mass.

M_m

Now for the Majorana mass. Since $\bar{\psi}_i\psi_i$ is even under T, $\bar{\psi}_1\psi_2$ is odd. Thus

$$T(\bar{\psi}M_m\psi) = +\bar{\psi}M_m\psi. \tag{27}$$

However, since the Majoranna mass is linear in ψ_2, it is odd under C. By CPT symmetry it is thus odd under P as well (which is easily checked).

The Majorana mass also breaks $U(1)$, as is easily checked (as a bilinear form it is the matrix X, which has determinant -1 and thus can't transform in the trivial representation of $U(1)$). One also checks that under repeated applications of conjugation by the matrix representing a rotation $\pi/4$,

$$M_m \mapsto M_- \mapsto -M_m \to -M_- \to M_m. \tag{28}$$

Since M_m goes to minus itself under a $\pi/2$ rotation, M_m transforms in the charge 2 representation of $U(1)$. Thus it breaks the $U(1)$ symmetry down to the $\mathbb{Z}_2$ of $(-1)^F$ symmetry, which can never be broken since $(-1)^F$ is part of the Loretnz group ($(-1)^F$ is the generator of the center of $SU(2) = \text{Spin}(3)$).

However, saying that M_m breaks charge conjugation is a little bit hasty. As mentioned earlier, we are free to modify any of the symmetry operators by the action of a unitary operator which commutes with the Lortenz group — our usual example of such an operator will be a rotation which performs the $U(1)$ symmetry. To this end, define a new charge conjugation operator by

$$C_m \equiv Ce^{i\pi Q/2}, \quad C_m : \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \mapsto -\begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix}. \tag{29}$$

With this definition we still have $C_m^2 = \mathbf{1}$, but now C_m no longer commutes with T (and so instead of $(C_mT)^2 = T^2 = (-1)^F$, we have $(C_mT)^2 = T^2 e^{i\pi Q} C^2 = (-1)^{2F} = \mathbf{1}$). The point of doing this is that M_m is even under this charge conjugation, since

$$C_m M_m C_m = C e^{i\pi/2} M_m e^{-i\pi Q/2} C = -C M_m C = M_m, \tag{30}$$

since as a bilinear form between $\bar{\psi}$ and ψ, M_m is X while C is Z.

CPT means that we must also be able to define a P that is preserved by M_m, since it preserves T and a C as well (CPT just means that there exists *a* choice of C, P, and T such that their product acts as the identity on the terms in the Lagrangian—a generic choice of such symmetry operators will not always have a product which acts as the identity). In this case, since the charge operator commutes with P, we define

$$P_m \equiv P e^{-i\pi Q/2}, \tag{31}$$

which means that $C_m P_m T$ is a symmetry of the M_m mass. The price of realizing these symmetries is that we get more complicated relations among the symmetry generators, e.g. how now neither the parity nor the charge conjugation operators commute with T.

M_-

Finally we turn to M_m, which is related to M_m by a $\pi/4$ $U(1)$ rotation, as we just saw. (thus it also has charge 2 under the $U(1)$). It is like the reverse of M_m: it breaks T (since the $\bar{\psi}_i\psi_i$ terms are T-invariant), but not C (since it is biliear in ψ_2). Even though it is odd under T, saying that it breaks time reversal is a bit hasty. Indeed, consider the time reversal operator

$$T_- \equiv T e^{i\pi Q/2}. \tag{32}$$

It acts on the Majoranas as

$$T_- : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto -\gamma_0 \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}. \tag{33}$$

Now since conjugating with $e^{i\pi Q/2}$ sends M_- to $-M_-$ (as M_- transforms in the charge-2 representation of $U(1)$), and since M_D is odd under T, we see that M_- is preserved by T_-. It's also easy to check that $T_-^2 = T^2 = (-1)^F$, and that

$$T_- C = C T_- (-1)^F \implies (CT_-)^2 = \mathbf{1}. \tag{34}$$

As it stands M_- repsects a time reversal and a charge conjugation, but not a parity. Thus by CPT we can find some new definition of P such that P is preserved. Indeed, we take

$$P_- \equiv P e^{i\pi Q/2}, \qquad P_- : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \gamma_1 \begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix}. \tag{35}$$

Then since M_- is in the charge-2 rep of $U(1)$, conjugating it by the $e^{i\pi Q/2}$ factor (which commutes with P) gives a factor of $e^{2\pi i/2} = -1$, which cancels its oddness under P. Thus M_- is even under P_-, and so is preserved by CP_-T_-.

Thus we conclude that identifying a mass term as odd or even under "time reversal" or "charge conjugation" or "parity" is a bit subtle, since we have to specify exactly how these symmetries act. For some (legitimate) choices the symmetries may be broken, while for other (equally legitimate!) choices they may not be (nothing could be done to preserve a choice of each of C, P, and T in the case of the Dirac mass, though: since P commutes with e^{iQ} and since M_D is $U(1)$ invariant, nothing can change the fact that P is broken, and hence by CPT nothing can change the fact that at least one of C and T is broken).

10 *January 7* — **unfinished** *The parity anomaly revisited*

Today's diary entry is an elaboration on an exercise that Nati Seiberg assigned to the students at the 2018 / 2019 Jerusalem winter school on QFT. The problem was to explain why, in dualities, the T and CT symmetries are often exchanged.

Solution:

We can also include a (properly quantized) CS counterterm in the background fields, in accordance with the discussion in the previous diary entry. So the action under consideration is (in $\mathbb{R}$ time)

$$S = \int \left[\bar{\psi} i \not{D}_A \psi + \frac{k}{4\pi} A \wedge dA \right]. \tag{36}$$

We can get the anomaly by specifying to a certain gauge field configuration. We will take space to be an S^2 with $\int_{S^2} F_A = 2\pi$, and will suppose that the spatial gauge fields are time-independent. The CS term then integrates to

$$\frac{k}{2\pi} \int A_0 F_{xy} = k \int dt A_0. \tag{37}$$

Now the monopole background means the Hamiltonian for the fermions (not the Dirac operator, necessarily, and not including the counterterm) has a zero mode (see a diary entry in 2018; a monopole background of flux $2\pi n$, $n \in \mathbb{Z}$, supports n zero modes). Since the Hamiltonian for the fermions is the spatial part of $\bar{\psi} i \not{D}_A \psi$, the action for this zero mode is

$$S_0 = \int dt \, \left(\bar{\psi}(i\partial_t + A_0)\psi + kA_0 \right). \tag{38}$$

Since the zero mode on the monopole is a two-level system, this is exactly equivalent to the problem of a single free fermion in quantum mechanics, coupled to a background $U(1)$ field.

We have already analyzed this in an earlier diary entry, where we saw that it had a mixed anomaly between $U(1)$ and C (or T). There are two states, $|k\rangle, |k+1\rangle$. Time reversal doesn't exchange them, while we define C to act as

$$C : |l\rangle \mapsto |-l + 2k + 1\rangle, \tag{39}$$

so that C interchanges $|k\rangle$ and $|k+1\rangle$ Suppose we now compactify time to a circle of circumference β. We can easily compute the partition function for this system, since we just have to sum $e^{i\beta H}$ over the two states. The Hamiltonian is just $QA_0 = (\bar{\psi}\psi + k)A_0$, and so

$$Z[A] = e^{ik\oint A_0}(1 + e^{i\oint A_0}). \tag{40}$$

This is not time reversal invariant unless k is the (disallowed) value of $-1/2$[2]. Similarly, one checks that the algebra is actually (here R_α is the $U(1)$ rotation by α)

$$TR_\alpha = R_{-\alpha}Te^{-i(2k+1)}, \tag{41}$$

which is only the expected algebra if $k = -1/2$.

Now we write the partition function as

$$Z[A] = e^{ik\oint A_0}e^{i\pi\eta(A)/2}|Z[A]|, \qquad |Z[A]| = e^{i\pi\eta(A)/2} + e^{-i\pi\eta(A)/2}, \tag{42}$$

where we have defined

$$\eta(A) \equiv \left(\frac{1}{\pi}\oint A_0 + 1\right)_2 - 1, \tag{43}$$

where the subscript means mod 2. This expression is gauge invariant and well-defined under changing transition functions by 2π, and so $Z[A]$ is manifestly gauge invariant. This way of defining the eta invariant means that $\eta(A)$ takes values in $[-1,1)$, so that $\cos(\pi\eta(A)/2) = |Z[A]|$ is always positive.

Note that $\eta(A)$ has a discontinuity of 2 when the holonomy of A_0 crosses π. This means that the function $e^{i\pi\eta(A)/2}$ has a discontinuity at $\oint A_0 = \pi$, where it goes from i to $-i$. When we square $e^{i\pi\eta(A)/2}$ we get the level-1 CS term, which is always well-defined. Anyway, the point of writing things this way is that the free energy now has a singular contribution as a function of the background fields: as discussed earlier, such a singular behavior is a requirement for the existence of an anomaly.

Despite this singularity, the partition function $Z[A]$ is still continuous. This is because the singularity in the free energy happens exactly when $|Z| = 0$, due to a zero mode of the Dirac operator which occurs at the point where $\oint A_0 = \pi$. That $|Z|$ vanishes at this point is obvious from the above expression we wrote for it, but we can also see it by looking at the fermions. On the S^1 of time, we work with the usual $\psi(t) = -\psi(t+\beta)$ boundary condition. Then the frequency is modded in $\frac{2\pi}{\beta}(l - 1/2), l \in \mathbb{Z}$, and we can decompose

$$\psi(t) = \sum_{l\in\mathbb{Z}} e^{\frac{2\pi i}{\beta}(l-1/2)t}\psi_k, \tag{44}$$

and so the condition for ψ_k to be a zero mode of $i\not{D}_A$ is that $-2\pi(l-1/2) = kA_0$. This happens if $k\int A_0 = \pi$, since then we can fix a gauge in which $A_0 = \pi/k\beta$. Basically, the π flux around the S^1 that is threaded in when $\oint A_0 = \pi$ "cancels" the AP boundary conditions for fermions,

[2]We are working in the convention where $T(A) = -A$ as a differential form, so that e.g. $T(A_0) = +A_0$. This might make $Z[A]$ look T invariant as written, but remember that it came from using $\int F_A = 2\pi$. The time-reversed version of this is $T(\int F_A) = -2\pi$, which means that an extra minus sign appears in the T transformation of $i\oint A_0$.

giving them effectively periodic boundary conditions and hence a zero mode. A Ramond spin structure would be handled by inserting $(-1)^F$ into the trace in the computation of $Z[A]$, giving

$$Z_R[A] = e^{i\pi\eta(A)/2+ik\oint A_0}\sin(\pi\eta(A)/2), \tag{45}$$

and so the zero mode of $\not{D}_A$ exists if $\oint A_0 \in 2\pi\mathbb{Z}$ (which one can also see from decomposing $\psi(t)$ in frequency modes).

Anyway, let's return to three dimensions, on a general manifold. The definition of $\eta(A)$ is such that it agrees with the one above after taking the specific case of a 2π flux through a spatial S^2:

$$\eta(A) = \left(\frac{1}{4\pi^2}\int A\wedge dA + 1\right)_2 - 1, \tag{46}$$

and the partition function is analogously

$$Z[A] = |Z[A]| \exp\left(\frac{ik}{4\pi}\int A\wedge dA + \frac{\pi i}{2}\eta(A)\right). \tag{47}$$

As in the fermion quantum mechanics model, if we didn't have the 1/2 multiplying the $\eta(A)$ we would be able to choose the k counterterm such that $Z[A]$ would be real.

11 *January 9 — T, CT, and dualities*

Today's diary entry is an elaboration on an exercise that Nati Seiberg assigned to the students at the 2018 / 2019 Jerusalem winter school on QFT. The problem was to explain why, in dualities, the T and CT symmetries are often exchanged.

Solution:

In the following, we will use a notation where $\mathscr{T}$, $\mathscr{C}\mathscr{T}$ are the "duals" of T and CT under some "duality map" $\mathscr{D}$. They are defined by

$$\mathscr{D}[T\mathcal{O}] = \mathscr{T}\mathscr{D}[\mathcal{O}], \quad \mathscr{D}[CT\mathcal{O}] = \mathscr{C}\mathscr{T}\mathscr{D}[\mathcal{O}], \tag{48}$$

where $\mathcal{O}$ is any field that has an image under $\mathscr{D}$. The claim is that the usual story for dualities is $\mathscr{T} = CT, \mathscr{C}\mathscr{T} = T$.

Particle on a ring

The first, simplest possible example is that of the duality (just a Fourier transform) between p and q for a particle on a ring, with Lagrangian

$$\mathcal{L}[q] = (\partial_t q)^2 - q^2. \tag{49}$$

We define the symmetries T and C to act as

$$T : q \mapsto q, \qquad CT : q \mapsto -q. \tag{50}$$

The conjugate variable is p. When we write the path integral in the Hamiltonian formulation, we have the Berry phase term $S \supset i\int dt\,\dot{q}p$. Since $dt, i, \dot{q}$ are all odd under T, while dt, i are odd under CT, invariance of this term under the symmetries tells us that p transforms as

$$T : p \mapsto -p, \qquad CT : p \mapsto p. \tag{51}$$

Now duality here is

$$\mathscr{D} : q \mapsto p, \quad p \mapsto -q. \tag{52}$$

Here the minus sign, which says that $\mathscr{D}^2 = -\mathbf{1}$, can be seen in several ways. One is that we require the symplectic form $dq \wedge dp$ to be invariant, with the antisymmetry of the $\wedge$ product necessitating the minus sign. Another way to see this is to note that the square of the Fourier transform is an inversion. That is, letting $\mathcal{F}$ be the Fourier transform,

$$\mathcal{F}^2[f(t)] = \mathcal{F}\int dt\, e^{i\omega t} f(t) = \int d\omega \int dt\, e^{i\omega t'} e^{i\omega t} f(t) = f(-t). \tag{53}$$

This is just because of the fact that the Fourier transform performs a $\pi/2$ rotation in frequency-time space, with a π rotation then corresponding to a reversal of the time coordinate. Since dualities are often performed by a Fourier transform, and the one in the present context indeed is, we expect $\mathscr{D}^2 : q \mapsto -q$, which it does. As we will see, this holds even for more advanced kinds of dualities like Electromagnetic duality, which again is basically just a Fourier transform.

Anyway, we can make the (now obvious) fact that $\mathscr{D} : C \leftrightarrow CT$ blindingly obvious by drawing the following commutative diagram:

$$\begin{tikzcd} q \arrow[r, "\mathscr{D}"] \arrow[d, "T"] & p \arrow[d, "\mathscr{T}"] \\ q \arrow[r, "\mathscr{D}"] & p \end{tikzcd}, \tag{54}$$

which tells us that $\mathscr{T} = CT$. A similar diagram shows that $\mathscr{C}\mathscr{T} = T$.

Electromagnetic duality in four dimensions

A slightly more sophisticated example is electromagnetic duality in 3+1 dimensions. As explained two diary entries ago, we take $T, C : A \mapsto -A$. Thus the vector components E^i are even under time reversal while those of B^i are odd, with both E^i and B^i odd under C.

As we have seen several times in previous diary entries, electromagnetic duality is (up to a constant of proportionality involving the gauge coupling), implemented by Hodge duality: $\mathscr{D} : F \mapsto \star F$. Recall how this works: we implement $F = dA$ by the Lagrange multiplier term

$$S \supset \frac{i}{2\pi}\int F \wedge d\widetilde{A}, \tag{55}$$

and then integrate out F by doing a shift of F by something proportional to $\star d\widetilde{A}$. If we were to insert F into the path integral, since $\langle F \rangle = 0$ when integrating out F, after the shift to eliminate the Lagrange multiplier term we'd be left with a path integral containing just an insertion of $\star d\widetilde{A}$; hence why $\mathscr{D}$ is basically Hodge duality.

On the components of the field strength, the duality is

$$\mathscr{D} : E \mapsto B, B \mapsto -E. \tag{56}$$

The minus sign in the second map is a Lorentzian minus sign coming from lowering a time index on F, and ensures that $\mathscr{D}^2 = -\mathbf{1}$ (this is just because $\star^2 = (-1)^{1+p(D-p)}$ on p-forms in D-dimensional Minkowski space. In Euclidean signature this minus sign is picked up since the proportionality constant between $\mathscr{D}(F)$ and $\star F$ is imaginary).

Now we can draw the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\mathscr{D}"] \arrow[d, "T"] & B \arrow[d, "\mathscr{T}"] \\ E \arrow[r, "\mathscr{D}"] & B \end{tikzcd} . \tag{57}$$

Thus we conclude that $\mathscr{T}$ must act trivially on B, from which we can make the identification $\mathscr{T} = CT$. Similarly,

$$\begin{tikzcd} E \arrow[r, "\mathscr{D}"] \arrow[d, "CT"] & B \arrow[d, "\mathscr{C}\mathscr{T}"] \\ -E \arrow[r, "\mathscr{D}"] & -B \end{tikzcd} , \tag{58}$$

so that $\mathscr{C}\mathscr{T} = T$ since B is odd under $\mathscr{C}\mathscr{T}$. Thus duality exchanges T and CT. Another way to see this is to couple the theory to a background field for the $U(1)$ 1-form symmetry, and to use the transformation properties of this background field under T and CT to fix the identification of the symmetries on both sides. If H is the 2-form background field, we have (this was derived in a previous diary entry)

$$i\int ||F_A - H||^2 \leftrightarrow i\int (||F_{\widetilde{A}}||^2 + F_{\widetilde{A}} \wedge H), \tag{59}$$

where we have omitted real constants (and are in $\mathbb{R}$ time). On the LHS, we see that H transforms in the same way under T, CT as F, and hence as A, does. On the RHS, because of the i, we see that H transforms in the opposite way as $F_{\widetilde{A}}$. Thus since the field strengths are odd under T and even under CT, duality exchanges C and CT. Using this type of $iF_{\widetilde{A}} \wedge H$ term (think $ik \cdot x$) to show that $T \leftrightarrow CT$ is a typical strategy.

Yet another way of seeing the identification is to require that the Lagrange multiplier term $\frac{i}{2\pi}\int F \wedge d\widetilde{A}$ be invariant under T and CT (although this isn't totally rigorous since this term implements the constraint regardless of the sign that it appears in S with).

3d particle-vortex duality

We can also look at a slightly more complicated example, that of the duality between the 2+1 dimensional WF scalar and another 2+1 WF scalar coupled to a gauge field (and deformed away from their critical points by mass terms, with the mass terms defined to vanish at the critical point). The duality is (adding a background field A to both theories)

$$|D_A\phi|^2 + r|\phi|^2 + u|\phi|^4 \leftrightarrow |D_a\widetilde{\phi}|^2 + \widetilde{r}|\widetilde{\phi}^2| + \widetilde{u}|\widetilde{\phi}|^4 + \frac{i}{2\pi}a \wedge F_A. \tag{60}$$

Since we are choosing $T(A) = -A$, we need $T(\phi) = \phi$. Then from the T-invariance of the CS term on the RHS (this is not a functional δ function since A is not integrated over and a is not a Lagrange multiplier, so the sign is meaningful), we see that $T(a) = a$, i.e. that T acts as CT on the dual gauge field. From the invariance of $|D_a\widetilde{\phi}|^2$, we get $T(\widetilde{\phi}) = \widetilde{\phi}^\dagger$, in keeping with T acting as CT on the dual fields.

12 *January 10 — T, CT, and bosonization in 1+1D*

Today's diary entry is similar to yesterday's, and concerns the realization of T symmetry in bosonization of fermions coupled to a $U(1)$ gauge field in 1+1D.

Solution:

Since we have fermions and spacetime symmetries involved, we can already anticipate a panoply of different choices for how the symmetries act. To identify how the symmetries act, it is helpful to add a classical background field to both sides, which lets us set the standards for T, CT on both sides of the duality.

First we have to get the conventions straight for where the currents go under duality. We will work in $\mathbb{R}$ time, with conventions where $\mathscr{D}[\psi_\pm] = e^{-i\phi_\pm}$. These are the conventions in which the Dirac mass maps to $\cos(\theta)$, where $\theta = \phi_+ - \phi_-$ is the dual field to ϕ. In these conventions, the vector current for the fermions maps to the vector current for the bosons (*not* the topological current for the bosons), so that in the usual notation $\mathscr{D}[J_\pm] = \partial_\pm\phi$. Thus we have (temporarily in Euclidean signature, and keeping track of factors of 2)

$$\mathscr{D}\left[\int J \wedge \star A\right] = \mathscr{D}\left[\int (J_+A_- + J_-A_+)\right] = \frac{1}{2}\int (\partial_+\phi A_- + \partial_-\phi A_+) = \frac{1}{2}\int (\partial_+\theta A_- - \partial_-\theta A_+), \tag{61}$$

which we can integrate by parts to write as

$$\mathscr{D}\left[\int J \wedge \star A\right] = i\int \theta F_A, \tag{62}$$

where the final sign may well be wrong (note to self: the factor of i there is definitely existent, but was getting lost in calculations for some reason). Anyway, the duality map $\mathscr{D}$ thus does

$$\mathscr{D} : \frac{1}{2\pi}\bar{\psi} i \not{D}_A \psi \leftrightarrow \frac{1}{8\pi} d\phi \wedge \star d\phi + \frac{1}{2\pi}\theta F_A. \tag{63}$$

The last term is well-defined even though θ is not a legit 0-form, because of the quantization on F_A. If we had used the convention where $\mathscr{D}[\psi_\pm] = e^{\mp i\phi_\pm}$, we'd have $\frac{i}{2\pi}\int F_A\phi$ instead on the RHS. The coefficient of $1/2\pi$ is correct since a chiral transformation $\psi \mapsto e^{i\bar{\gamma}\alpha}\psi$ does $\phi_\pm \mapsto \phi_\pm \pm \alpha$ in our conventions, under which $\theta \mapsto \theta + 2\alpha$. This reproduces the shift in the action of $i(\alpha/\pi)\int F_A$ that comes from the chiral anomaly on the fermion side.

Now for the symmetries. Since $T(A) = -A$, we need (as differential forms, so that $J_\psi = \bar{\psi}\gamma_\mu\psi dx^\mu$)

$$T(\star J_\psi) = \star J_\psi \implies T(J_\psi) = -J_\psi, \tag{64}$$

so that the transformation of J_ψ correctly matches that of A, allowing $\int J_\psi \wedge \star A$ to be T-invariant. This transformation rule makes sense because T always takes $\psi^\dagger_\pm\psi_\pm \mapsto \psi^\dagger_\mp\psi_\mp$, and hence sends $J^0_\psi \mapsto J^0_\psi, J^1_\psi \mapsto -J^1_\psi$. Charge conjugation C sends $A \mapsto -A$, so that $CT(A) = A$, and likewise $CT(J_\psi) = J_\psi$. In $\mathbb{R}$ time we have $\mathscr{J}_\psi = \star J_\psi$ where $\mathscr{J}$ is the axial current, and so $T(\mathscr{J}) = \mathscr{J}, CT(\mathscr{J}) = -\mathscr{J}$.

Now we have to fix conventions for how the fermions transform under C, P, T. We will fix the signature to be $(+,-)$, with γ matrices $\gamma^0 = X, \gamma^1 = -iY = J$.[3] For time reversal, both choices $T = Y\mathcal{K}$ or $T = X\mathcal{K}$ are consistent: the former gives $T^2 = (-1)^F$ while the later gives $T^2 = \mathbf{1}$.[4] We will denote these two choices by T_- and T_+, respectively. We will take charge conjugation to be performed by $C : \psi \mapsto \bar{\psi}C^\dagger, \bar{\psi} \mapsto -C\psi$, which is a symmetry of the action if $C = Y$.[5] Component-wise, this is

$$C : \psi_\pm \mapsto \pm i\psi^\dagger_\pm. \tag{65}$$

CPT invariance forces us to choose $P = X$ if we work with T_+, and $P = J$ if we work with T_- (if we had chosen $(-,+)$ signature, the two actions of P would be reversed). More on the method to this madness will be explained in a later diary entry. Working out the action on the field components, we see that

$$(CT_+)^2 = (-1)^F, \qquad (CT_-)^2 = +\mathbf{1}, \tag{66}$$

so that the square of $CT_\pm$ is opposite to the square of $T_\pm$.

T_+: Consider first the case when fermions are Kramers singlets. Then applying T_+ to $\mathscr{D}[\psi_\pm] = e^{-i\phi_\pm}$, we get

$$\begin{tikzcd} \psi_\pm \arrow[r, "\mathscr{D}"] \arrow[d, "T_+"] & e^{-i\phi_\pm} \arrow[d, "\mathscr{T}_+"] \\ \psi_\mp \arrow[r, "\mathscr{D}"] & e^{-i\phi_\mp} \end{tikzcd} . \tag{67}$$

Thus we have[6]

$$\mathscr{T}_+ : \phi_\pm \mapsto -\phi_\mp, \qquad \phi \mapsto -\phi, \quad \theta \mapsto \theta. \tag{68}$$

[3]If we change the signature, we will be able to change whether $P^2 = \mathbf{1}$ or $-\mathbf{1}$ on fermions: since we are mostly focused on time reversal, we won't worry about exploring all these options, and will just choose $(+,-)$ signature, in which $P^2 = \pm\mathbf{1}$ is determined from the choice of $T^2 = \pm\mathbf{1}$ and CPT invariance.

[4]This is because for $T = U_T\mathcal{K}$, we need $U_T^\dagger Z U_T = -Z$.

[5]Note that the free action would also be preserved by $C : \psi_\pm \mapsto \psi^\dagger_\pm$. However, in this case, the Dirac mass $m_D\bar{\psi}\psi$ would be C-odd, which we don't want.

[6]When we look at the action of symmetries on ϕ, θ, which are related to fermion bilinears, we need to remember Klein factors, which are acted on by the γ matrices. For example, $\mathscr{D}[\psi^\dagger_+\psi_-] = \kappa_+\kappa_-e^{i\theta}, \mathscr{D}[\psi^\dagger_-\psi_+] = \kappa_-\kappa_+e^{-i\theta}$, and so $\mathscr{T}_+ : \theta \mapsto \theta$ only makes sense if $\mathscr{T}_+ : \kappa_\pm \to \kappa_\mp$.

Charge conjugation evidently acts on the boson side as $\mathscr{C} : \phi_\pm \mapsto -\phi_\pm \mp \pi/2$. However, one must be careful in deriving its action on θ and ϕ. θ and ϕ are fermion-parity even, which leaves room for a minus sign to enter in the action of charge conjugation which doesn't appear when C acts on single fermion operators. So for example,

$$C : \psi_+^\dagger \psi_- \mapsto -\psi_+\psi_- = +\psi_-^\dagger \psi_+ \implies \mathscr{C} : e^{i\theta} \mapsto e^{-i\theta}. \tag{69}$$

If we had naively applied $\mathscr{C}$ to $\phi_+ - \phi_-$, we would have obtained $\mathscr{C} : \theta \mapsto -\theta - \pi$ instead. The difference between this transformation and the correct result comes down to the fact that when we bosonize, we only apply the mosonization map $\mathscr{D}$ to things that are normal-ordered, and so we have to keep track of possible normal-ordering signs that appear. Similarly, considering the action of C on $\psi_+^\dagger \psi_-^\dagger$, one finds the action of $\mathscr{C}$ on ϕ, so that

$$\mathscr{C} : \phi \mapsto -\phi, \quad \theta \mapsto -\theta. \tag{70}$$

This means that the Dirac mass $\bar\psi\psi \to 2\cos\theta$ is $\mathscr{C}$-even while the chiral mass $i\bar\psi Z\psi \to 2\sin\theta$ is $\mathscr{C}$-odd, as expected.

Putting these together,

$$\mathscr{C}\mathscr{T}_+ : \phi_\pm \mapsto \phi_\mp \mp \pi/2, \qquad \phi \mapsto \phi, \quad \theta \mapsto -\theta. \tag{71}$$

A sanity check is that $(\mathscr{C}\mathscr{T}_+)^2 = -\mathbf{1}$ when acting on $\phi_\pm$ (we will always have $(\mathscr{C}\mathscr{T}_\pm)^2 = \mathbf{1}$ when acting on ϕ or θ since these variables are fermion-parity even, so we just need to check the action of $(\mathscr{C}\mathscr{T}_\pm)^2$ on $\phi_\pm$). Indeed, keeping in mind that in this representation the complex conjugation in $\mathscr{T}_+$ sends scalars to minus themselves,

$$(\mathscr{C}\mathscr{T}_+)^2 : \phi_\pm \to -\phi_\mp \to -(-\phi_\mp \pm \pi/2) \to -\phi_\pm \pm \pi/2 \to \phi_\pm \pm \pi. \tag{72}$$

Finally, with this choice of T_+, we have $P_+ = X$ on the fermions; thus

$$\mathscr{P}_+ : \phi_\pm \mapsto \phi_\mp, \qquad \phi \mapsto \phi, \quad \theta \mapsto -\theta. \tag{73}$$

T_-: Now consider the case when the fermions are Kramers doublets. We take $T_- = Y\mathcal{K}$, so that $T : \psi_\pm \mapsto \pm i\psi_\mp$. Then we have

$$\begin{tikzcd} \psi_\pm \arrow[r, "\mathscr{D}"] \arrow[d, "T_-"] & e^{-i\phi_\pm} \arrow[d, "\mathscr{T}_-"] \\ e^{\pm i\pi/2}\psi_\mp \arrow[r, "\mathscr{D}"] & e^{-i\phi_\mp \pm i\pi/2} \end{tikzcd} . \tag{74}$$

This identifies the bosonized action of T_- as

$$\mathscr{T}_- : \phi_\pm \mapsto -\phi_\mp \mp \pi/2, \quad \phi \mapsto -\phi, \quad \theta \mapsto \theta - \pi. \tag{75}$$

Consequently,

$$\mathscr{C}\mathscr{T}_- : \phi_\pm \mapsto \phi_\mp \pm \pi, \qquad \phi \mapsto \phi, \quad \theta \mapsto -\theta + \pi. \tag{76}$$

Lastly, we have parity, which acts as $P_- = J$ on the fermions. Thus

$$\mathscr{P}_- : \phi_\pm \mapsto \phi_\mp + (1 \mp 1)\pi/2, \qquad \phi \mapsto \phi + \pi, \quad \theta \mapsto -\theta - \pi. \tag{77}$$

$$\begin{array}{c|c|c|c|c|c|c|c} & C & P_+ & T_+ & CT_+ & P_- & T_- & CT_- \\ \hline \phi & - & + & - & + & \phi+\pi & - & + \\ \hline \theta & - & - & + & - & -\theta-\pi & \theta-\pi & -\theta+\pi \\ \hline m_D & + & + & + & + & - & - & - \\ \hline m_5 & - & - & + & - & + & - & + \\ \hline \int \theta F_A & + & + & + & + & \frac{1}{2}\int F_A & -\frac{1}{2}\int F_A & -\frac{1}{2}\int F_A \end{array} \tag{78}$$

Note that $CP_\pm T_\pm$ acts trivially on θ and $\int \theta F_A$, as required.[7]

This means that the Kramers doublet case is anomalous! Indeed, we can already get a hint of this from the fact that if time reversal acts as T_-, neither of the fermion mass terms are time-reversal invariant! Therefore we cannot do PV regularization without breaking T_- symmetry; hence the anomaly.

13 *January 26 — 1-form anomalies in CS theory*

After another break for traveling, it's time to get back to the diary. Today, the goal is to examine the anomalous nature of the 1-form symmetry that's present in various CS theories.

Solution:

$U(1)_k$

Let's start with $U(1)_k$. The 1-form symmetry acts as

$$\mathbb{Z}_k^{(1)} : A \mapsto A + \lambda, \qquad k\lambda \in 2\pi H^1(X;\mathbb{Z}). \tag{79}$$

The charge operators are of course the Wilson lines. We can see that this is a symmetry by e.g. computing the spectrum of operators in the theory, but for posterity's sake let's see how it works from the action (this is the same computation needed to show that $U(1)_1$ is invariant under large gauge transformations). Since λ is flat, a naive approach tells us that $\delta S = \frac{1}{4\pi}\int (k\lambda) \wedge F_A$ under the symmetry, which is only in $\frac{1}{2}\overline{\mathbb{Z}}$ (we are using the notation $\overline{\mathbb{Z}} \equiv 2\pi\mathbb{Z}$). Note that we cannot integrate this by parts to get zero by the flatness of λ, due

[7] CP_-T_- doesn't act trivially on ϕ but that's okay since $e^{i\phi} \sim \psi_+^\dagger \psi_-^\dagger$ won't appear in by itself in a Lorentz-invariant theory (it will only appear as $\partial\phi$ and $\bar{\partial}\phi$).

to A not being strictly a well-defined form[8]. As usual, the confusion can be ameliorated by writing things in terms of the field strengths by using a bounding 4-manifold M. Then

$$\delta S = \frac{k}{2\pi}\int_M F_\lambda \wedge F_A + \frac{k}{4\pi}\int_M F_\lambda \wedge F_\lambda, \tag{80}$$

where F_λ is the field strength of the extension of λ into the bulk 4-manifold M. Note that since the holonomy of λ is nontrivial, although it is flat on the boundary, it will not be flat in M, and a priori, it will not be globally an exterior derivative, i.e. we may not have $F_\lambda = d\lambda$ globally on M.

Warning: the following paragraph will be slightly pedantic. Now we need to integrate by parts: we will get only boundary terms, since $dF_A = d(d\lambda) = 0$. However, doing so is slightly subtle, since λ might not be a globally well-defined form. Thus we cannot write e.g. $\int d\lambda \wedge B = \int \lambda \wedge dB + \int_{\partial M} \lambda \wedge B$ for a 2-form B (the sign is correct because of the supercommutativity of d). However, since λ is flat, we know that λ is a well-defined form on ∂M. Thus in the bulk, we may write

$$\lambda = \Lambda + B, \qquad F_B \in 2\pi H^2(M, \partial M; \mathbb{Z}), \tag{81}$$

where Λ is a $U(1)$ gauge field which is globally well-defined so that $[F_\Lambda] = [d\Lambda] = 0$ in $2\pi H^2(M;\mathbb{Z})$, and B is a non-globally-well-defined part which vanishes on ∂M since $\lambda|_{\partial M}$ is globally well-defined (thus $\lambda|_{\partial M} = \Lambda|_{\partial M}$). Thus we can write

$$\delta S = \frac{k}{2\pi}\int_M [(d\Lambda + F_B)\wedge F_A + d\Lambda \wedge F_B] + \frac{k}{4\pi}\int_M (d\Lambda \wedge d\Lambda + F_B \wedge F_B). \tag{82}$$

Assuming we choose M to be spin if k is odd, the last term vanishes modulo $\overline{\mathbb{Z}}$. Since Λ is globally well-defined, the $d\Lambda \wedge F_B$ term vanishes on account of the flatness of F_B and the fact that $F_B|_{\partial M} = 0$. Likewise the $F_B \wedge F_A$ part vanishes mod $\overline{\mathbb{Z}}$: we can see this by decomposing A in the same way that we decomposed Λ, and using that $\frac{1}{2\pi}\int F_C \wedge F_B \in \overline{\mathbb{Z}}$ for $F_C \in 2\pi H^2(M, \partial M; \mathbb{Z})$. So finally, we integrate the remaining two terms by parts and get

$$\delta S = \frac{1}{2\pi}\int (k\lambda)\wedge F_A, \tag{83}$$

since $\Lambda|_{\partial M} = \lambda|_{\partial M}$ and since $d\lambda|_{\partial M}$ is flat. But since $k\lambda$ has periods in $\overline{\mathbb{Z}}$, we see that $\delta S \in \overline{\mathbb{Z}}$, and so indeed, the 1-form transformation $\delta A = \lambda$ *is* a symmetry of the action.

To gauge this symmetry, we want the "split" symmetry operators $U(g;\Sigma)$ (not only the full charge operators) to act trivially on the Hilbert space, where the split symmetry

[8] A similarly hasty use of integration by parts on the CS action leads to confusion in the usual way of showing that $k \in \mathbb{Z}$ in the CS action, namely by e.g. placing the theory on $S^1 \times S^2$ with $\int F = 2\pi$ around the S^2. In the usual story one integrates by parts to get $S = \frac{k}{2\pi}\oint A_t \int_{S^2} F_{xy} = k\oint A_t$ (there is a factor of 2 her from the IBP), which says that $k \in \mathbb{Z}$ for invariance under large gauge transformations around the S^1. But what if we first did the large gauge transformation, and then did the integration by parts? Since the field strength of the large gauge transformation vanishes, the IBP fails to pick up a factor of 2, and we conclude that the change in the action is instead $(k/4\pi)\oint \lambda \int_{S^2} F_{xy}$ for $\oint \lambda = 2\pi$, which seems to imply that $k \in 2\mathbb{Z}$ is required. So, it is best to only integrate by parts when we really know that it is legit.

operators are defined on *open* submanifolds $C : \partial C \neq 0$ and implement a transformation by the group element g (here $g \in \mathbb{Z}_k$). Requiring the charge operators to act trivially is equivalent to projecting onto the singlet sector of the Hilbert space, which can be done by inserting the operator

$$\Pi_{\mathbf{1}} = \sum_{C \in H_1(X;\mathbb{Z})} \sum_{q \in \mathbb{Z}_k} e^{iq \int_C A} \tag{84}$$

into the path integral. This is orbifolding. In my opinion though, this is not the same as gauging, since we haven't made the symmetry local in any way. The split symmetry operator $U(q, C)$ is naively $e^{iq \int_C A}$, but if $\partial C \neq 0$ then $U(q, C)$ is not gauge invariant, so gauging $\mathbb{Z}_k^{(1)}$ requires something more. Of course, doing the gauging requires the addition of a $\mathbb{Z}_k$ 2-form field. Let's see how this works.

We will let B be the $\mathbb{Z}_k$ 2-form gauge field. This means that we have to add to the action a BF term (we are in $\mathbb{R}$ time, so no factors of i are included)

$$S \supset \frac{k}{2\pi} \int B \wedge d\phi, \tag{85}$$

where ϕ is a 2π-periodic scalar. Now the naive split symmetry operator which performs the gauge transformation is $e^{i \int_C A}$, but this isn't gauge invariant (under the regular 0-form gauge symmetry) for $\partial C \neq 0$. We can fix this by writing the operator $U(q, C)$ which implements the gauge transformation as

$$U(q, C) = e^{iq \int_C (A + d\phi)}, \tag{86}$$

provided that under $A \mapsto A + d\gamma$ we have $\phi \mapsto \phi - \gamma$ (this preserves the 2π-periodicity of ϕ, since γ is itself a 2π-periodic scalar). This makes since, since ϕ is the canonical momentum for B, and so this is exactly what we normally do when gauging the symmetry operators: the operators which perform the gauge transformations are the original charge operators defined on open submanifolds, with the canonical momentum for the gauge field integrated along their boundaries (e.g. the generator of gauge tforms in QED is the integral of the matter current over an open volume, together with the integral of $\star F$, the canonical momentum for the gauge field, over the boundary of the volume).

Now by design, if $D \in C_1(X; \mathbb{Z})$ is such that $C \cap D \neq 0$, then $W(D) = e^{i \int_D A}$ is not gauge invariant under the $\mathbb{Z}_k^{(1)}$ gauge transformations, since it does not commute with $U(q, C)$. Note that no matter what D is, we can always find a C such that $W(D)$ is not invariant under $U(q, C)$: this is true even when $[D] = 0$ in $H_1(X; \mathbb{Z})$, in which case $W(D)$ is actually neutral under the original 1-form global symmetry.

We can make $W(D)$ gauge invariant by attaching a surface operator built out of B to it: if $[D] = 0$ in $H_1(X, \partial X; \mathbb{Z})$ (so that D either bounds a disk, is a linear combination of nontrivial classes in $H_1(X; \mathbb{Z})$ with total "charge" zero so that it bounds some other surface, or together with a submanifold of the boundary of spacetime bounds a surface) we can find some M such that $\partial M \setminus (\partial M \cap \partial X) = D$ (here X is spacetime, and gauge transformations always vanish at ∂X). The operator

$$\widetilde{W}(M) = \exp\left(i \int_D A + i \int_M B \right) \tag{87}$$

is then gauge-invariant. Why? Because when we compute its commutation relation with $U(q,C)$ (with e.g. $C \cap D = 1$), we get one factor of $e^{2\pi i q/k}$ from the $[A,A] \sim i/k$ commutation relation, and another from the $[\phi, B] \sim i/k$ commutation relation, which occurs from the contact term between the ϕ inserted at the end of C and the B integrated over M. If $[D] \neq 0$ in $H_1(X;\mathbb{Z})$ then $W(D)$ can't be made gauge-invariant, and its vev vanishes (although this was true before gauging, since $\langle W(D) \rangle$ can then be shifted by a change in integration variables which doesn't affect the boundary conditions on A).

The anomaly is then seen very simply from the fact that the operators $U(q,C)$ which perform the $\mathbb{Z}_k^{(1)}$ gauge transformations are not themselves invariant under the same $\mathbb{Z}_k^{(1)}$ transformations (although they are invariant under the 0-form $U(1)$ gauge transformations on A). That is, they don't commute with themselves (because of $[A,A] \sim i/k$). Since $\partial C \neq 0$, it is impossible to attach a B surface to render $U(q,C)$ gauge invariant. Thus the $\mathbb{Z}_k^{(1)}$ symmetry can't actually be gauged.

We can also see this from the action. Basically, while F_A can be made gauge invariant by $F_A \mapsto F_A - B$, the CS term cannot be made gauge invariant since it involves more than just F_A. Indeed, let us write the variation of A under the 1-form gauge transformation as

$$\delta A = \lambda, \qquad \lambda \in \frac{1}{k} Z^1_O(X;\mathbb{Z}). \tag{88}$$

Here we have defined $Z^1_O(X;\mathbb{Z})$ as the set of 1-forms such that their spatial Poincare duals are *open* codimension-1 submanifolds of space, which have integral intersection number with every element in $C_1(X;\mathbb{Z})$ that intersects them transversely. Thus the elements in $Z^*_O(X;\mathbb{Z})$ are not closed, but they are not closed in a very specific way. Another way to say this is that

$$\lambda \in \frac{1}{k} Z^1_O(X;\mathbb{Z}) \implies \int_C \lambda \in \frac{1}{k}\mathbb{Z} \quad \forall C \in C_1(X;\mathbb{Z}), \tag{89}$$

where the value for the integral will generically depend on the exact choice of C, and not just its homotopy equivalence class. Connecting this with our earlier notation vis-a-vis the $U(q,C)$ gauge transformation operators, we would say that $U(q,C)$ shifts A by $\lambda = \frac{q}{k}\widehat{C}$, where $\widehat{C}$ is the spatial Poincare dual of $C \in C_1(X;\mathbb{Z})$.

Anyway, the CS term varies as

$$\delta \int A \wedge dA = 2\int A \wedge d\lambda + \int \lambda \wedge d\lambda. \tag{90}$$

To cancel at least the first term we can try to introduce a $\mathbb{Z}_k$ gauge field B to the action and add

$$S \supset -\frac{1}{2\pi} \int A \wedge B, \tag{91}$$

with $\delta B = d\lambda$ under the gauge transformation. However this a) cannot cancel the term in δS quadratic in λ and b) produces an extra piece linear in B. So, after adding this coupling, the total variation of S is

$$\delta S = \frac{k}{4\pi}\delta \int (A \wedge dA - 2A \wedge B) = \frac{k}{4\pi} \int (2\lambda \wedge B + \lambda \wedge d\lambda). \tag{92}$$

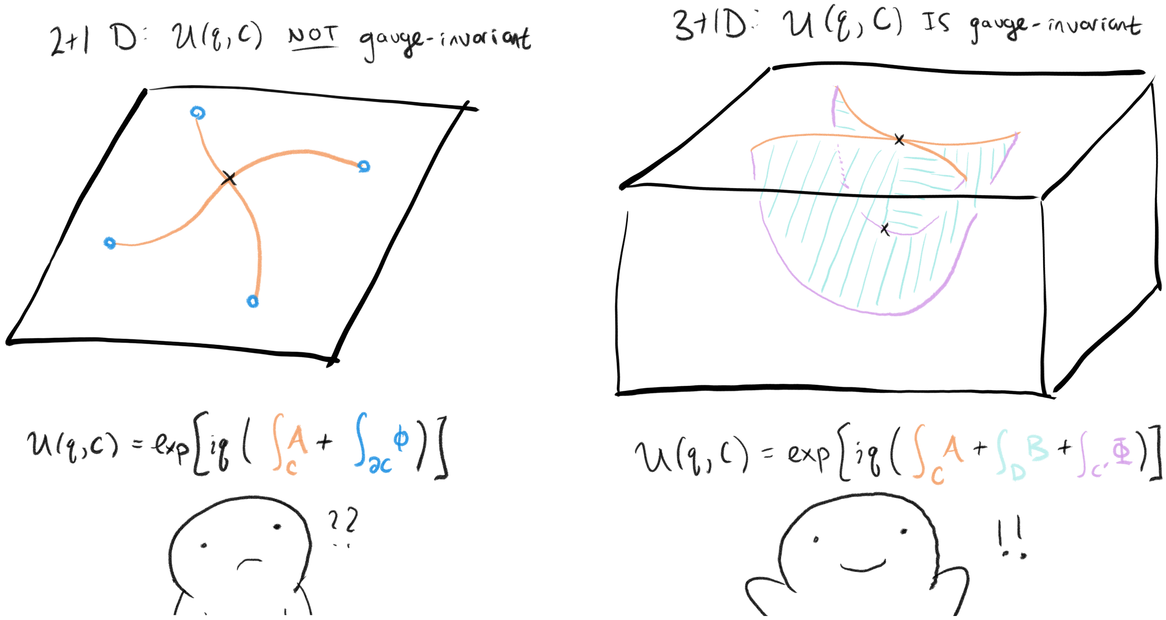

Figure 1: The generators of gauge transformations for the 1-form gauge symmetry in gauged $U(1)_k$. Contact terms that contribute to the commutator of the charge operators are marked with black x's. In 2+1D the $U(q, C)$ are charged and the symmetry can't be gauged, while in 3+1D, with B surfaces extending into the bulk, it can be.

Of course, we can make the action gauge invariant by letting B live in four dimensions, at the price of picking up an explicit dependence on a bounding 4-manifold M. This is just because F_A can always be made gauge-invariant, and we can write the terms in our modified action involving A as

$$S \supset \frac{k}{4\pi} \int_M (F_A - B) \wedge (F_A - B), \tag{93}$$

which is manifestly gauge invariant (and still only depends on $A|_{\partial M}$). However, and this is where the anomaly comes in, it depends on the choice of M, since

$$\frac{k}{4\pi} \int_{N_4 | \partial N_4 = \emptyset} (F_A - B) \wedge (F_A - B) \in \frac{1}{k}\overline{\mathbb{Z}}, \tag{94}$$

which is not valued in $\overline{\mathbb{Z}}$ except in the trivial case $k = 1$ where there is no symmetry to begin with (here we have used the fact that the periods of B are valued in $k^{-1}\overline{\mathbb{Z}}$ — the periods of F_A are still in $\overline{\mathbb{Z}}$ though, since the 1-form gauge transformations only change A by forms which are globally well-defined up to elements in $2\pi H^2(N_4; \mathbb{Z})$ (or are they always globally well-defined?)). Since k copies of this bulk action integrate to something in $\overline{\mathbb{Z}}$ over all closed 4-manifolds, we have a $\mathbb{Z}_k$ anomaly.

To write the full gauged action for the four-dimensional B, we just need to include the term which makes B into a $\mathbb{Z}_N$ gauge field. Since B lives in four dimensions, the appropriate BF term is $(k/2\pi) \int_M B \wedge F_\Phi$, where Φ is a 1-form $U(1)$ gauge field. But this term changes as $(k/2\pi) \int_M d\lambda \wedge F_\Phi = (k/2\pi) \int_{\partial M} \lambda \wedge F_\Phi$ under the 1-form gauge transformation , which is problematic. The way to get around this is to include a $-(k/2\pi)\int_{\partial M} B\wedge\Phi$ boundary term in the action. Together with the boundary term, the full part of the action involving Φ is $-(k/2\pi)\int_M F_B\wedge\Phi$, which is manifestly invariant under the 1-form gauge transformation . Recapitulating, the full action is

$$S=\frac{k}{4\pi}\int_{\partial M}(A\wedge F_A-2A\wedge B-2B\wedge\Phi)+\frac{k}{4\pi}\int_M(B\wedge B+2B\wedge F_\Phi). \tag{95}$$

The full generator of gauge transformations is now

$$U(q,C)=\exp\left(iq\left[\int_C A+\int_D B+\int_{C'}\Phi\right]\right), \tag{96}$$

where D is a disk with $\partial D=C\cup C'$, $\partial C=\partial C'$, and where C' is entirely contained within the four-dimensional bulk. The $U(q,C)$'s commute with one another: the contact term between the A's on the surface is canceled between a contact term where the $\int_{C'}\Phi$ line intersects the $\int_D B$ surface. This is illustrated in Figure 1. On the left we show the $U(q,C)$ operators in the strictly 2+1D theory, which are not gauge invariant. On the right we show how, after attaching B surfaces and Φ lines to them, they become gauge invariant.

Twisted $\mathbb{Z}_N$ gauge theory

We now look at the DW theory which we will call $DW_{p,q}$, namely

$$\mathcal{L}=\frac{p}{4\pi}a\wedge da+\frac{q}{2\pi}a\wedge db. \tag{97}$$

What are the global symmetries? First, there is clearly a $\mathbb{Z}_N^{(1)}$ symmetry that shifts b by $1/N$ times a large gauge transformation. Similarly, there is also a $l\equiv\gcd(p,q)$ symmetry shifting a: this is the best we can do, as e.g. the coupling between a and b means we don't have the full $\mathbb{Z}_p^{(1)}$ symmetry of the first term, unless p divides q, and we don't have the full $\mathbb{Z}_q^{(1)}$ symmetry from shifts on a in the second term, unless q divides p.

Let's pause for a momentum to discuss the spin and statistics of the lines in this theory. A naive reading on this would be as follows: the canonical momentum for a is $pa+qb$ and the canonical momentum for b is qa. Thus b lines commute with each other, while a lines have a self-linking phase determined by $1/p$. The mutual statistics of a and b is nonzero because they do not commute with each other, and is determined by $1/q$.

This naive reading is incorrect: even if the canonical momentum for a field does not involve that field itself, the field may still fail to commute with itself (by which I mean, certain components of that field may still fail to commute with other components of that field). Indeed, the correct way to determine the commutation relations between Wilson lines is by using the inverse of the K matrix. For $i\in\mathbb{Z}_{\dim K}$ and letting $\star q^\alpha\cdot J^\alpha=\star q_i^\alpha\cdot J_i^\alpha$ be the 2-form Poincare dual to a support of a particular configuration of Wilson lines $\prod_\alpha W_\alpha=\prod_\alpha e^{i\sum_j\oint_{C_\alpha}A_j}$, we have

$$\langle\prod_\alpha W_\alpha\rangle=\frac{1}{Z[J=0]}\int\prod_i\mathcal{D}A_i\exp\left(\frac{i}{4\pi}\int A_i[K]_{ij}\wedge dA_j+i\sum_{\alpha,i}q_i^\alpha\int A_i\wedge\star J_i^\alpha\right). \tag{98}$$

Shifting A to kill off the AJ coupling, we get

$$\langle \prod_\alpha W_\alpha \rangle = \exp\left(2\pi i \sum_{\alpha,\beta} q_i^\alpha q_j^\beta \int \star J_i^\alpha [K^{-1}]_{ij} \wedge d^{-1} \star J_j^\beta \right). \tag{99}$$

Taking all the Wilson loops to be supported on the boundaries of disks means that the $\star J^\alpha$ are not in $\ker(d)$, and so the above formula makes sense. Anyway, taking two linked loops, one with a unit charge for A_i and another with a unit charge for A_j (and taking the framing of each loop to be trivialized so that the diagonal in α terms in the above formula do not contribute) gives us the braiding matrix

$$[S]_{ij} = \exp\left(2\pi i [K^{-1}]_{ij}\right). \tag{100}$$

This can also be derived just by looking at $\sum_j [A_i, \bar{K}_{mj} A_j] = i\delta_{im}$. Here spacetime indices are kept implicitly, with $[A_i, A_j] = A_i \wedge A_j - A_j \wedge A_i$. Also, $\bar{K} = K/2\pi$. Anyway, multiplying by $[\bar{K}^{-1}]_{mk}$ and summing over m:

$$\sum_j [A_i, A_j]\delta_{k,j} = i \sum_m \delta_{im} [\bar{K}^{-1}]_{mk} \implies [A_i, A_j] = i[\bar{K}^{-1}]_{ij}. \tag{101}$$

Using this commutation relation to unlink any loops that are linked together in $\prod_\alpha W_\alpha$, one recovers the above expression for the S matrix (after choosing a framing).

In the present $DW_{p,q}$ example, the K matrix and its inverse are

$$K = \begin{pmatrix} p & q \\ q & 0 \end{pmatrix}, \qquad K^{-1} = \begin{pmatrix} 0 & 1/q \\ 1/q & -p/q^2 \end{pmatrix}. \tag{102}$$

Thus even though a appears in the canonical momentum for b, we see that b still fails to commute with itself. So we see that the b line is *not* a boson, despite the fact that its canonical momentum does not involve itself. In fact, it has spin $-p/2q^2$! And similarly, despite the self-CS term for a, we see that a is actually a boson! Physically, what's going on here is that b lines carry flux for a, which by the self-CS term for a have nontrivial braiding with themselves, since this term tells us that a flux also carries a charge. This allows b lines to not commute with themselves. Likewise, a lines carry a flux, which makes them seem like they would not commute with themselves. But a fluxes also carry b charge, and b charge carries a flux, and this all works out in such away that the a lines actually carry net zero a flux.

A particularly transparent example of when this happens is the case when $q = p$. In that case, we can diagonalize the K matrix by something in $SL(2,\mathbb{Z})$ via

$$K \mapsto \Lambda^T K \Lambda = qZ, \qquad \Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{103}$$

This means that in terms of the variables b and $c = a + b$, the Lagrangian is that of $U(1)_q \otimes U(1)_{-q}$. In this formulation, it is clear that b has spin $-1/(2q)$ (from the $U(1)_{-q}$ factor), while a has spin 0 mod 1, since $a + b$ has spin $+1/(2q)$ and

$$e^{2\pi i s(a)} = e^{2\pi i s(c-b)} \sqrt{[S]_{c-b,c-b}} = [S]_{c,-b} \sqrt{[S]_{c,c}[S]_{-b,-b}} = 1 \cdot \sqrt{e^{2\pi i/q} e^{-2\pi i/q}} = 1 \implies s(a) =_1 0. \tag{104}$$

Now let's look at the symmetries of the theory. The $\mathbb{Z}_q^{(1)}$ symmetry which shifts b by $d\phi/q$ is easy to identify: it is generated by the operator

$$\mathbb{Z}_q^{(1)} = \langle e^{i\oint a} \rangle, \tag{105}$$

which has the right qth root of unity phase linking with the $e^{i\oint b}$ line needed to generate the symmetry. Since $e^{i\oint a}$ has trivial self-linking, this symmetry is not anomalous.

Now for the $\mathbb{Z}_l^{(1)}$ symmetry that shifts a lines by $e^{2\pi i/l}$ (recall $l \equiv \gcd(p,q)$). Since a has trivial self-linking, the operator generating this symmetry should include $\exp(iq/l \oint b)$, since the linking of a and b lines gives a phase $e^{2\pi i/q}$. But this operator also shifts b lines, which is bad since b lines are neutral under the $\mathbb{Z}_l^{(1)}$ symmetry. If we tack on a line $e^{i\beta\oint a}$ to the symmetry generator, imposing that the generator link trivially with b tells us that

$$-\frac{pq}{q^2 l} + \frac{\beta}{q} = 0 \implies \beta = p/l. \tag{106}$$

This means that the $\mathbb{Z}_l^{(1)}$ symmetry is generated by the line

$$\mathbb{Z}_l^{(1)} = \langle \exp\left(i\frac{q}{l}\oint b + i\frac{p}{l}\oint a \right) \rangle. \tag{107}$$

What is the anomaly of this symmetry? To find out, we need the self-linking phase of the charge operator. This phase determines the anomaly as

$$\text{Anomaly} = \frac{1}{2}\left(-\frac{p}{q}\left(\frac{q}{l}\right)^2 + 2\frac{1}{q}\frac{qp}{l^2} \right) = \frac{p}{2l^2} \mod 1, \tag{108}$$

where the first term is the self-linking of b and the second is the a-b mutual phase (the factor of $1/2$ is because we want the spin of the charge operator. On spin manifolds, we should take this mod $1/2$ and not mod 1). This is indeed an anomaly appropriate for a $\mathbb{Z}_l^{(1)}$ symmetry, since it is a $\mathbb{Z}_l$ effect, in that $l(p/l^2) = p/l \in \frac{1}{2}\mathbb{Z}$ indicates that l copies of the charge operator is either trivial, or a transparent fermion. One special case that shows up often is when $p = -rq$ and the theory has two $\mathbb{Z}_q^{(1)}$ symmetries. In this case, the anomaly of the $\mathbb{Z}_q^{(1)}$ symmetry that shifts a is $-r/q$.

Finally, note that there's a mixed anomaly, of a $\mathbb{Z}_l$ character, between the two symmetries. This is just due to the fact that the generators for the $\mathbb{Z}_q^{(1)}$ and $\mathbb{Z}_l^{(1)}$ symmetries don't commute: the phase between them is $e^{2\pi i/l}$ (which is trivial if we take l copies of either generator, as it should be).

This conclusions can be corroborated by just going in and trying to gauge the symmetry directly. The symmetry that shifts b is clearly non-self-anomalous, since b only appears by way of its field strength and we can just make the replacement $F_b \mapsto F_b - B_b$, where B_b is the background field for the $\mathbb{Z}_q^{(1)}$ symmetry. However, since the generator for the symmetry that shifts b carries charge under the $\mathbb{Z}_l^{(1)}$ symmetry , adding the B_b field will break the $\mathbb{Z}_l^{(1)}$ symmetry. Indeed, after adding the B_b field the action shifts by the following term under $a \mapsto a + \frac{1}{l}d\phi$:

$$\delta S = \frac{q/l}{2\pi}\int d\phi \wedge B_b \in \frac{1}{l}\overline{\mathbb{Z}}. \tag{109}$$

Thus we recover the $\mathbb{Z}_l$ mixed anomaly between the two 1-form symmetries.

Basically because of the self-CS term for a, the $\mathbb{Z}_l^{(1)}$ symmetry shifting a has a self-anomaly. To find the appropriate characterization of the anomaly, we start from the gauge-invariant bulk action (ignoring the Lagrange multipliers that make B_a, B_b quantized appropriately for simplicity)

$$\begin{aligned} S &= \frac{p}{4\pi}\int_M (F_a - B_a)\wedge(F_a - B_a) + \frac{q}{2\pi}\int_M (F_a - B_a)\wedge(F_b - B_b) \\ &= S_{\partial M} + S_{bulk}, \end{aligned} \tag{110}$$

where M is some bounding 4-manifold, and

$$\begin{aligned} S_{\partial M} &= S_{DW_{p,q}} - \frac{1}{2\pi}\int_{\partial M}\left[a\wedge(pB_a + qB_b) + qb\wedge B_a\right], \\ S_{bulk} &= \frac{1}{4\pi}\int_{\partial M}\left[pB_a\wedge B_a + 2qB_a\wedge B_b\right]. \end{aligned} \tag{111}$$

The second line in the above equation parametrizes the anomaly. If we consider the dependence on the choice of M by integrating S_{bulk} over a closed 4-manifold, we see that the first term is valued in $p\overline{\mathbb{Z}}/2l^2$ on a non-spin manifold, and $p\overline{\mathbb{Z}}/2l^2$ on a spin manifold, while the second term is valued in $\overline{\mathbb{Z}}/l$. The quantization of the second term confirms the $\mathbb{Z}_l$ nature of the mixed anomaly, while the quantization of the first term confirms our result for the anomaly of the $\mathbb{Z}_l^{(1)}$ symmetry.

$U(N)_{k,q}$

Our conventions will be such that $U(N)_{k,q}$ is defined though

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathscr{A}\wedge d\mathscr{A} - \frac{2i}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathscr{A}]\wedge d\mathrm{Tr}[\mathscr{A}]. \tag{112}$$

The notation is done like this because q is ($1/N$ times) the effective $U(1)$ level, while k is the effective $SU(N)$ level. The reason why the effective $U(1)$ level is qN can be seen by starting with the decomposition

$$U(N)_{k,q} \cong [SU(N)_k \times U(1)_{qN}]/\mathbb{Z}_N, \tag{113}$$

where the quotient identifies the center of $SU(N)$ with the appropriate Nth roots of unity in $U(1)$. Since the quotient here says that we can freely change transition functions in the $U(1)$ bundle to make the cocycle condition fail by Nth roots of unity so long as we change the transition functions in the $SU(N)$ bundle in the opposite way, the $\mathbb{Z}_N$ quotient is equivalent to gauging the diagonal $\mathbb{Z}_N^{(1)}$ symmetry which acts on both $SU(N)$ and $U(1)$ fields.

At the level of actions, we simply write

$$\mathscr{A} = A + \mathcal{A}\mathbf{1}, \tag{114}$$

where A is an $SU(N)$ field (whose transition functions may fail by Nth roots of unity), $\mathcal{A}$ is a "$U(1)$ field" with transition functions failing in the inverse way—hence $N\mathcal{A}$ is a properly-quantized $U(1)$ field, and $N\int F_{\mathcal{A}} \in \overline{\mathbb{Z}}$. The quotient comes from the correlation of the transition functions between A and $\mathcal{A}$ (more on this when we talk about $SU(N)_k$ in the next subsection). In terms of these fields, we have

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right] + \frac{qN}{4\pi}\mathcal{A}\wedge d\mathcal{A}, \tag{115}$$

so that qN is indeed the "effective $U(1)$ level". To get this we've used that A is traceless and that

$$\mathrm{Tr}[A\wedge A\wedge \mathcal{A}] = \mathcal{A}\wedge \mathrm{Tr}[A\wedge A] = 0 \tag{116}$$

on account of $\mathrm{Tr}[X\wedge Y] = (-1)^{|X||Y|}\mathrm{Tr}[Y\wedge X]$.

Now the $U(1)$ part started out with a $\mathbb{Z}_{qN}^{(1)}$ symmetry pre-gauging. After we gauge to perform the $\mathbb{Z}_N$ quotient though, the quantization condition on $\mathcal{A}$ is modified, so that only $NF_{\mathcal{A}}$ has periods in $\overline{\mathbb{Z}}$. Now let us shift $\mathcal{A}$ by λ, with $d\lambda = 0$. The action changes by

$$\delta S = \frac{q}{2\pi}\int \lambda\wedge(NF_{\mathcal{A}}). \tag{117}$$

Since $NF_{\mathcal{A}}$ is quantized in $\overline{\mathbb{Z}}$, we see that $\delta S \in \overline{\mathbb{Z}}$ provided that $\lambda = \frac{1}{q}d\phi$. Thus we see that the $U(N)_{k,q}$ theory has a $\mathbb{Z}_q^{(1)}$ symmetry, that acts by shifting $\mathcal{A}$.

Is it anomalous? Yes: the charge operator for the remaining $\mathbb{Z}_q^{(1)}$ symmetry is

$$U(p,C) = e^{iNp\oint_C \mathcal{A}}, \qquad p\in\mathbb{Z}_q, \tag{118}$$

with the factor of N needed to perform the shift correctly, and ensures invariance under the gauged diagonal $\mathbb{Z}_N^{(1)}$ symmetry. Computing the braiding phase of the charge operator with itself, we find a phase of $N^2/(Nq)$ since Nq is the effective $U(1)$ level. Thus the anomaly is measured by $N/q \mod 1$. This means in particular that there is no anomaly if $q = N$ (in order for the the theory to be well-defined $q = N$ means $k \in N\mathbb{Z}$). Note that the anomaly of $U(N)_{k,q}$ is the same as the anomaly of N copies of $U(1)_q$.

$SU(N)_k$

Now we look at $SU(N)_k$ CS theory. For all k, this theory has a $\mathbb{Z}_N$ 1-form symmetry, coming from the center of the gauge group. What is the anomaly of the $\mathbb{Z}_N^{(1)}$ symmetry?

Four-dimensional perspective

The easiest way of figuring this out is probably by using what we know about regular four-dimensional pure YM at various values of θ. We know that $\exp(ik\int\mathcal{L}_{CS}[A]/4\pi)$ is the operator which implements the $\theta\mapsto\theta+2\pi k$ similarity transformation in $SU(N)$ YM, where θ is 2π-periodic, and so if we know what the $\theta\mapsto\theta+2\pi k$ shift does in the $PSU(N)$ theory, where the 1-form symmetry has been gauged, we'll be able to say something about the anomaly of the gauged CS theory.

Let us now go partway towards turning the theory into a $PSU(N)$ gauge theory by adding a background $\mathbb{Z}_N$ 2-form field B (we'd get the full $PSU(N)$ theory by path integrating over

B). We went over how to do this in last year's diary, but I think the discussion there was a bit confused and long-winded. Here's how it works: we first write the $SU(N)$ theory as

$$\mathcal{L} = \frac{\theta}{8\pi^2}\left(\mathrm{Tr}[F_{\mathscr{A}} \wedge F_{\mathscr{A}}] - \mathrm{Tr}[F_{\mathscr{A}}] \wedge \mathrm{Tr}[F_{\mathscr{A}}]\right) + \frac{1}{2\pi} F_Y \wedge \mathrm{Tr}[\mathscr{A}]. \tag{119}$$

Here Y is a Lagrange multiplier field, and $\mathscr{A}$ is a $U(N)$ gauge field.[9]

As in the last subsection, we will find it helpful to decompose $\mathscr{A}$ as

$$\mathscr{A} = A + \mathcal{A}\mathbf{1}. \tag{120}$$

Here A is $\mathfrak{su}(N)$-valued and $\mathcal{A}$ is $\mathfrak{u}(1)$-valued. However, A is not a connection on a $SU(N)$ bundle, and $\mathcal{A}$ is not a connection on a $U(1)$ bundle: rather, the transition functions g_A and $g_{\mathcal{A}}$ satisfy

$$\delta g_A \delta g_{\mathcal{A}} = \mathbf{1}, \qquad \delta g_A, \delta g_{\mathcal{A}} \in \mathbf{1} e^{\frac{2\pi i}{N}\mathbb{Z}}. \tag{121}$$

In this description, we have a gauge transformation whereby the transition functions g_A and $g_{\mathcal{A}}$ change by opposite roots of unity. Note that this means that only $NF_{\mathcal{A}}$ is a legit $U(1)$ field strength.

Anyway, let's see why this is equivalent to the $SU(N)$ theory. We just integrate out Y: this sets $F_{\mathcal{A}} = 0$ and the sum over $[F_Y] \in H^2(X;\mathbb{Z})$ tells us that we can set $\mathcal{A} = \frac{1}{N} d\phi$, for $d\phi$ a large gauge transformation. The flatness constraint tells us that we will always have $\delta g_{\mathcal{A}} = \mathbf{1}$ (since a nontrivial $\delta g_{\mathcal{A}}$ would contribute to the 1st Chern class), and hence $\delta g_A = \mathbf{1}$ as well: now both $SU(N)$ and $U(1)$ factors are legitimate bundles. Additionally, such an $\mathcal{A}$ can be completely absorbed into a change of the $g_{\mathcal{A}}$ transition functions by Nth roots of unity (the transition functions change by *constants* on each double-overlap). These transition functions can then be absorbed into the g_A transition functions, and so the $\mathcal{A}$ field completely disappears, leaving us with an $SU(N)$ action, as required.

The theory has a $\mathbb{Z}_N^{(1)}$ symmetry that comes from twisting the transition functions in the $SU(N)$ bundle by Nth roots of unity. In our $U(N)$ formulation, this is equivalent to shifting the $g_{\mathcal{A}}$ by Nth roots of unity, which in turn is equivalent to keeping the $g_{\mathcal{A}}$ transition functions fixed, but making a shift $\mathcal{A} \mapsto \mathcal{A} + \frac{1}{N} d\phi$. To gauge this symmetry then, we should make the replacement $F_{\mathcal{A}} \mapsto F_{\mathcal{A}} - B\mathbf{1}$. In what follows we will take B to be some fixed background field with periods in $2\pi/N$ around all closed 2-cycles. This gives us the Lagrangian

$$\mathcal{L} = \frac{\theta}{8\pi^2}\left(\mathrm{Tr}[(F_{\mathscr{A}} - B\mathbf{1})^{\wedge 2}] - (\mathrm{Tr}[F_{\mathscr{A}} - B\mathbf{1}])^{\wedge 2}\right) + \frac{1}{2\pi} Y \wedge \mathrm{Tr}[F_{\mathscr{A}} - B\mathbf{1}]. \tag{122}$$

Consider integrating out Y. This sets $F_{\mathcal{A}} = B$, which means that $F_{\mathcal{A}}$ becomes quantized in periods of $2\pi/N$. Because of the connection between the transition functions of the $SU(N)$ and $U(1)$ bundles, we then erase $F_{\mathcal{A}} - B\mathbf{1}$ from the action and get

$$\mathcal{L} = \frac{\theta}{8\pi^2}\mathrm{Tr}[F_A \wedge F_A], \qquad w_2(E_{PSU(N)}) = \frac{N}{2\pi} B \in H^2(X;\mathbb{Z}_N), \tag{123}$$

[9] The second term in the parenthesis ensures that, because the full term in parenthesis is the second Chern class of a $U(N)$ field, we have $\theta \sim \theta + 2\pi$ identically, without having to first integrate out Y. This is desired because $\theta \sim \theta + 2\pi$ in the $SU(N)$ theory (the $SU(N)_k$ CS theory is not spin; more on this later), while if the second term in parenthesis were absent we might not have such a periodicity.

where A is now a connection on a $PSU(N)$ bundle $E_{PSU(N)}$. Thus we see the role of B is to turn the $SU(N)$ connection into a $PSU(N)$ connection, with the topological class of the $PSU(N)$ bundle controlled by the cohomology class of B. When B gets integrated over, we perform a sum over all $PSU(N)$ bundles, and obtain a genuine $PSU(N)$ gauge theory.

In a previous diary entry we saw that the 2π periodicity in θ is lost in the $PSU(N)$ theory, and instead that changing θ by 2π induces a shift in the action given by a counterterm in B. Indeed, we can write $\mathcal{L}$ as

$$\mathcal{L} = \theta c_2(E_{U(N)}) + \frac{\theta}{4\pi}\left(-\mathrm{Tr}[F_{\mathscr{A}} \wedge B\mathbf{1}] + N\mathrm{Tr}[F_{\mathscr{A}}] \wedge B + \frac{N-N^2}{2} B \wedge B\right) + \frac{1}{2\pi} Y \wedge \mathrm{Tr}[F_{\mathscr{A}} - B\mathbf{1}]. \tag{124}$$

The advantage of writing it this way is that the $SU(N)$ part of $F_{\mathscr{A}}$ has completely disappeared into the second Chern class of the $U(N)$ bundle. Now integrating out Y, we have (using $\mathrm{Tr}[F_A \wedge B\mathbf{1}] = 0$)

$$\mathcal{L} = \theta c_2(E_{U(N)}) + \frac{\theta}{4\pi}\left(-N + N^2 + \frac{N-N^2}{2}\right) B \wedge B, \tag{125}$$

where now the $U(N)$ bundle $E_{U(N)}$ is constrained to have first Chern class $c_1(E_{U(N)}) = \frac{N}{2\pi} B \in H^2(X;\mathbb{Z})$. Since the second Chern class is integral, under $\theta \mapsto \theta + 2\pi$ the action shifts as (modulo elements of $2\pi\mathbb{Z}$)

$$S_\theta \mapsto S_\theta + \frac{1}{4\pi}(N^2 - N) \int B \wedge B, \tag{126}$$

This is nontrivial, since $\int B \wedge B \in \mathbb{Z}/N^2$ ($\in 2\mathbb{Z}/N^2$) on generic (spin) closed 4-manifolds, and hence θ is actually not 2π periodic.

Anyway, the point is the following: consider a domain wall where θ jumps by 2π. We know that such a domain wall can be created by inserting $\exp(i \int_X \mathcal{L}_{CS}[A]/4\pi)$ into the path integral, where X is a 3-manifold defining the domain wall. By the above discussion, we know that the action differs on the two sides of the domain wall by a $B \wedge B$ counterterm in the background field. However, integrating $B \wedge B$ over an open submanifold of spacetime is not a gauge-invariant thing to do! Doing a gauge transformation on B produces an anomalous term, consisting of an integral over the codimension-1 submanifold X:

$$\delta S = \frac{i}{4\pi}(N-1) \int_X \mathrm{Tr}[2B \wedge \lambda + \lambda \wedge d\lambda], \tag{127}$$

for $\delta B = d\lambda$ (and we are tacitly writing e.g. B for $\mathbf{1}B$). Since we know that $PSU(N)$ gauge theory in four dimensions is self-consistent, this anomaly must be canceled by an anomaly of the $SU(N)_1$ CS theory.

The anomaly is determined by looking at how the shift in S_θ depends on the bounding 4-manifold. Integrating it over a closed 4-manifold tells us that $e^{i\delta S_\theta} = e^{2\pi l \frac{N-1}{2N}}$ for some $l \in \mathbb{Z}_N$. Thus we can conclude that the CS theory $SU(N)_1$ has anomaly $(N-1)/2N$ mod 1. The anomaly for $SU(N)_k$ must then be $k(N-1)/(2N) \mod 1$, since $SU(N)_k$ is the theory defined by the similarity transform on the codimension-1 slice where the $\delta\theta = 2\pi k$ domain wall happens, and the gauge-non-invariance of the bulk action in the presence of the domain wall is exactly k times the result when the θ angle jumps by 2π. So, the theory has an anomaly given by

$$\text{Anomaly} = \frac{k(N-1)}{2N} \mod 1 \quad (\text{mod}\, 1/2 \text{ if spin}). \tag{128}$$

Here the reduced anomaly for the spin case comes from the fact that the intersection form is then even, which limits the phases that δS_θ in (126) can take when integrated over closed 4-manifolds. Actually, we can do a bit better: if $k \in 2\mathbb{Z}$ then the N^2 part of (127) is trivial on all manifolds, and so we can effectively say that the anomaly is just $-k/N$ if $k \in 2\mathbb{Z}$.

Three-dimensional perspective

Now let's look at this from the three-dimensional perspective directly. One naive way to write the $SU(N)_k$ theory is to write

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathscr{A} \wedge d\mathscr{A} + \frac{2i}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A}\right] + \frac{1}{2\pi} y \wedge d\mathrm{Tr}[\mathscr{A}], \tag{129}$$

where y is a Lagrange multiplier that roughly speaking turns the $U(N)$ field $\mathscr{A}$ (a $\mathfrak{u}(N)$-valued form) into an $\mathfrak{su}(N)$-valued 1-form. This is not completely correct, however, since if k is odd this theory is spin, while we know that $SU(N)_k$ is non-spin for any value of k (because the $SU(N)$ instanton number is equal to $2\pi \int \mathrm{c}_2(E)$ where $\mathrm{c}_2(E)$ is the second Chern class, which is integral on all closed manifolds, spin or not).

To fix this, we will add a $U(1)_p$ term using the $U(1)$ field $\mathrm{Tr}[\mathscr{A}]$. Note that we are free to shift the definition of the Lagrange multiplier field by

$$y \mapsto y \pm \mathrm{Tr}[\mathscr{A}] \tag{130}$$

(since $\mathrm{Tr}[\mathscr{A}]$ is a properly quantized $U(1)$ field), which changes p by ± 2. So, to find out how to render the theory non-spin, we just need to find out the correct parity to use for p.

Anyway, to get the answer for the correct non-spin theory, we write the full Lagrangian as

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathscr{A} \wedge d\mathscr{A} - \frac{2i}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A}\right] + \frac{\eta_k}{4\pi}\mathrm{Tr}[\mathscr{A}] \wedge d\mathrm{Tr}[\mathscr{A}] + \frac{1}{2\pi} y \wedge d\mathrm{Tr}[\mathscr{A}], \tag{131}$$

where η_k is to be determined. The integral needing to be done to check the quantization condition on η_k is

$$I = \frac{2\pi}{8\pi^2}\int_{M_4} \left(k\mathrm{Tr}[F_{\mathscr{A}} \wedge F_{\mathscr{A}}] + \eta_k \mathrm{Tr}[F_{\mathscr{A}}] \wedge \mathrm{Tr}[F_{\mathscr{A}}] + 2dy \wedge d\mathrm{Tr}[\mathscr{A}]\right) \tag{132}$$

for some closed 4-manifold M_4. The last term is always in $\overline{\mathbb{Z}}$, while the first two can be written as

$$I = 2\pi \int c_2(E_{U(N)}) + \pi(k + \eta_k) \int \mathrm{Tr}[F_{\mathscr{A}}/2\pi] \wedge \mathrm{Tr}[F_{\mathscr{A}}/2\pi], \tag{133}$$

where $\mathrm{c}_2(E_{U(N)})$ is the second Chern class of the $U(N)$ bundle. Since this is always an integral class regardless of the base space of the bundle, we conclude that we need $k + \eta_k$

to be even. Thus we can take e.g. $\eta_k = 0$ if $k \in 2\mathbb{Z}$, and $\eta_k = -1$ if $k \in 2\mathbb{Z}+1$. Another (simpler) choice (and the one we will adopt) is to simply set $\eta_k = -k$, which as we mentioned above is equivalent since η_k and $\eta_k \pm 2$ define equivalent theories. Adopting this choice, we have

$$\mathcal{L}_{SU(N)_k}[A] = \mathcal{L}_{U(N)_{k,k(1-N)}}[\mathscr{A}] + \frac{1}{2\pi} y \wedge d\mathrm{Tr}[\mathscr{A}]. \tag{134}$$

Thus $SU(N)_k$ is realized as a constrained version of a $U(N)_{k,q}$ theory at $q = k(1-N)$. The freedom to shift y by $\pm\mathrm{Tr}[\mathscr{A}]$ manifests itself in the equivalence $q \sim 2N$.

As we saw previously, we can split up $\mathscr{A}$ into an $SU(N)$ part A and a diagonal part $\mathcal{A}$, provided that the cocycle conditions for the A and $\mathcal{A}$ parts fail in canceling ways. Recall the decomposition $\mathscr{A} = A + \mathcal{A}\mathbf{1}$. With this decomposition, in order to implement the matching-cocycle-conditions property, we require that the diagonal transformation shifting the transition functions for both A and $\mathcal{A}$ by opposite Nth roots of unity be a gauge transformation. Note that we can do such a shift while keeping A traceless, since we are only changing the transition functions by constants: the change in transition functions is done at the level of the glueing data between patches, not at the level of the 1-forms A defined on single patches. By contrast, when we perform such a shift on $\mathcal{A}$, we will do it by directly taking $\delta\mathcal{A} = \frac{1}{N}d\phi$ (ϕ as usual is 2π-periodic), without changing the transition functions for the $\mathcal{A}$ bundle. Either way we do it, the effect of this identification is to gauge a diagonal $\mathbb{Z}_N^{(1)}$ symmetry that shifts both A and $\mathcal{A}$. The transformation acts nontrivially on A Wilson lines since they are defined by $\mathrm{Tr}[e^{i\int_{U_\alpha} A} e^{i\Lambda_{\alpha\beta}} e^{i\int_{U_\beta} A} \dots]$, with $\Lambda_{\alpha\beta}$ the transition functions between patches, and since the transformation shifts the $\Lambda_{\alpha\beta}$'s. Note that this gauge transformation, while not changing the field strength $F_{\mathscr{A}}$, *does* change the field strengths of A and $\mathcal{A}$: if we make the cocycle condition fail by an Nth root of unity on a given triple overlap of patches, then this induces fractional flux in both A and $\mathcal{A}$.

Now we can get a more precise understanding of what the Lagrange multiplier y is doing. Integrating out y tells us that $d\mathcal{A} = 0$, and that $\int \mathcal{A} \in \frac{1}{N}\overline{\mathbb{Z}}$ around all closed 1-manifolds. Thus we may write $\mathcal{A} = \frac{1}{N}d\phi$. But we see that this is gauge-equivalent to $\mathcal{A} = 0$ under the 1-form gauge symmetry. So, integrating out y leaves us with just the $SU(N)_k$ part of the action, which is what we want.

Anyway, returning to $\mathcal{L}$, we have

$$\mathcal{L} = \frac{k}{4\pi}\left(\mathrm{Tr}\left[A \wedge dA - \frac{2i}{3} A \wedge A \wedge A\right] + (N - N^2)\mathcal{A} \wedge d\mathcal{A}\right) + \frac{N}{2\pi} y \wedge d\mathcal{A}, \tag{135}$$

again using the tracelessensss of A and the antisymmetry to kill the $A \wedge A \wedge \mathcal{A}$ contribution. Note the $k(N - N^2)$ level of the $\mathcal{A}$ CS term: peeking back at the analysis of the bulk gauge theory, we see that this is exactly the right number needed to cancel the bulk anomaly, and is a hint that we're on the right track.

Let's pause to figure out what the symmetry is. We started with a pure $SU(N)_k$ CS term, which as we know has a $\mathbb{Z}_N^{(1)}$ symmetry. We then wrote it in terms of a $U(N)_{k,q}$ theory plus a Lagrange multiplier, where for us we chose $q = k(1-N)$. As we saw earlier, the $U(N)_{k,q}$ theory by itself has a $\mathbb{Z}_q^{(1)}$ global symmetry. This symmetry is generically broken by the Lagrange multiplier term, since under it we have

$$\delta S = \frac{N}{2\pi k(1-N)} \int F_y \wedge d\phi \notin \overline{\mathbb{Z}}. \tag{136}$$

So, does this mean that we have no 1-form symmetry? This would be a problem if so. But actually, the $\mathbb{Z}_N^{(1)}$ symmetry that we need to be there does exist. To see how it works, consider shifting $\mathcal{A}$ by some flat 1-form λ. The action changes as

$$\delta S = \frac{k(1-N)}{2\pi}\int \lambda \wedge (NF_{\mathcal{A}}) + \frac{N}{2\pi}\int F_y \wedge \lambda. \tag{137}$$

In order for the last term to be in $\overline{\mathbb{Z}}$, we see that we need to take $\lambda = d\phi/N$. Then the variation in S is

$$\delta S = \frac{k(1-N)/N}{2\pi}\int \lambda \wedge (NF_{\mathcal{A}}). \tag{138}$$

This is in general nontrivial, but we see that we can cancel it, if we take the symmetry transformation to involve an appropriate shift in y as well. This gives us a genuine $\mathbb{Z}_N^{(1)}$ symmetry, under which we have

$$\mathbb{Z}_N^{(1)} : \mathcal{A} \mapsto \mathcal{A} + \frac{1}{N}d\phi, \qquad y \mapsto y - \frac{k(1-N)}{N}d\phi. \tag{139}$$

If this is the right symmetry, it should shift fundamental Wilson lines by Nth roots of unity. And indeed it does:

$$\mathbb{Z}_N^{(1)} : W_f(C) = \mathrm{Tr}_f[e^{i\oint_C (A+\mathcal{A}\mathbf{1})}] \mapsto e^{i\frac{1}{N}\oint_C d\phi} W_f(C) \tag{140}$$

(note that $e^{i\oint \mathcal{A}}$ is not a gauge-invariant operator to consider the transformation properties of). Note that $e^{i\oint y}$ also shifts under the symmetry, so that it must also be electrically charged. More on this in a bit.

What is the operator which generates this symmetry? It turns out to be $\exp(i\oint y)$. This is rather surprising, since looking at the action one might be forgiven for thinking that the y line was bosonic.

To find the statistics of the y line, it is helpful to Higgs the theory down to $\mathbb{Z}_N$. In terms of the $SU(N)$ variables, the effect of the Higgsing is to leave the theory with only $\mathbb{Z}_N$ transition functions as degrees of freedom. In the continuum, it's easier to deal with this condition by writing the transition functions instead as diagonal $\mathbb{Z}_N$ 1-form matrices, with trivial transition functions. So to that end, Higgsing for us at the computational level means taking $A = 0$ and $\mathcal{A} = a$, with a a $\mathbb{Z}_N$ field. Since $A = 0$ and A has trivial transition functions, the cocycle condition will be satisfied exactly for a, and the flux of F_a will be quantized in the regular way. So, upon doing this, we get the $DW_{p,q}$ theory with $p = kN(1-N)$, $q = N$:

$$SU(N)_k \xrightarrow{Higgs} \frac{kN(1-N)}{4\pi} ada + \frac{N}{2\pi} y \wedge da. \tag{141}$$

Note that in addition to the $\mathbb{Z}^{(1)}_{\gcd(kN(1-N),N)} = \mathbb{Z}_N^{(1)}$ symmetry, we also have a symmetry that shifts y by a $\mathbb{Z}_N$ gauge field. The appearance of this magnetic symmetry is expected after we move from $SU(N)$ (which has no t'Hooft line operators since $\pi_1(SU(N)) = 0$) to $\mathbb{Z}_N$ (which does have magnetic operators since we can have $\mathbb{Z}_N$ branch cuts in the transition functions).

We've already been through this theory in lots of detail, and we learned that the mutual statistics between the a and y lines are[10]

$$[S]_{a,a} = 1, \quad [S]_{a,y} = e^{2\pi i/N}, \quad [S]_{y,y} = e^{-2\pi i k(1-N)/N}. \tag{142}$$

Recall from a ways back that we could perform a change of variables on y that shifted $k(1-N) \mapsto k(1-N) \pm 2N$. We see that this leaves the braiding phases invariant (and because of the factor of 2, it also leaves the spins invariant), and so reassuringly the shift indeed acts trivially on the modular data of the theory.

From the above entries of the S matrix, we see that the line $e^{i\oint y}$ generates the $\mathbb{Z}_N^{(1)}$ symmetry of $SU(N)_k$, since these braiding phases mean that wrapping lines with the line $e^{i\oint_C y}$ is equivalent to performing the shift (139) (where $d\phi$ is determined by the topology of C).

We can now easily figure out the anomaly: from taking the square root of $[S]_{y,y}$ to get the spin of the generating line, we read off the anomaly as $k(1-N)/2N \mod 1$. If we are on a spin manifold then having the generating line be a fermion is okay, and so in that case the anomaly is $k(1-N)/2N \mod 1/2$. Note that this is exactly the right anomaly to cancel the bulk anomaly that we derived earlier in (128)! Nice. Note that the anomaly of $SU(N)_k$ is the same as that of $[SU(N)_1]^{\otimes k}$, because of the constant k prefactor. Also note that since $N-1$ is coprime to N, we will only have a non-anomalous theory if $k \in 2N\mathbb{Z}$ (or $k \in N\mathbb{Z}$ if spin).

Can we say anything about this line in the $SU(N)$ context? Yes: under the $\mathbb{Z}_N^{(1)}$ symmetry we have

$$e^{i\oint y} \mapsto e^{2\pi i k/N} e^{i\oint y}. \tag{143}$$

Since Wilson lines in the fundamental transform with a $e^{2\pi i/N}$ phase, this tells us that the generator $e^{i\oint y}$ can be identified with a Wilson line in a k index symmetric $SU(N)$ representation. This makes sense because, as noted in [**?**], $e^{i\oint y}$ is the operator we get when slicing open the 2-dimensional surface operator which implements the $\mathbb{Z}_N^{(1)}$ symmetry in the 3+1 D theory. Now the $SU(N)_k$ theory lives at an interface where the bulk θ angle changes by $2\pi k$. The Witten effect means that the t'Hooft operators on both sides of the surface (which are not genuine line operators) have electric charges differing by k. This k difference in electric charges is realized by the fact that the charge operator on the interface, namely $e^{i\oint y}$, carries electric charge k.

This is a manifestation of the mixed anomaly between the $\mathbb{Z}_N^{(1)}$ symmetry and time reversal at $\theta \in \pi(2\mathbb{Z}+1)$. Indeed, consider a 2π domain wall for θ, where θ jumps from $-\pi$ to π. The operator which inserts this domain wall is the charge operator for T, since it interpolates between the two ground states (which differ by $\theta \mapsto -\theta$). The mixed anomaly comes from the fact that this domain wall operator and the surface operator which implements the $\mathbb{Z}_N^{(1)}$ symmetry don't commute: indeed, they do not commute because of a contact term, and their lack of commutativity can be seen from the fact that along their intersection is a fundamental Wilson line (since we are in four dimensions, a 3-manifold and a 2-manifold intersect at a 1-manifold).

[10] The N/N factor in the $[S]_{y,y}$ matrix element is important, since when square-rooted it contributes to the spin of the y line. However, it does not affect how y lines transform under the $\mathbb{Z}_N^{(1)}$ symmetry.

As we have seen, if we try to gauge the $\mathbb{Z}_N^{(1)}$ symmetry in the 2+1D theory, we run in to problems since the operators which perform the gauge transformations (the fundamental Wilson lines) do not commute with each other. This can be fixed by using the same procedure as in the $U(1)_k$ case. First, we write the action for the theory as

$$\begin{aligned}S = \frac{k}{4\pi}\int_{\partial X} \mathrm{Tr}\left[\mathscr{A}\wedge d\mathscr{A} + \frac{2i}{3}\mathscr{A}^3\right] - \frac{k}{4\pi}\int_{\partial X}\mathrm{Tr}[\mathscr{A}]\wedge\mathrm{Tr}[F_{\mathscr{A}}] + \frac{k(N-1)}{2\pi}\int_{\partial X} B\wedge\mathrm{Tr}[\mathscr{A}] \\ + \frac{1}{2\pi}\int_{\partial X}(y\wedge\mathrm{Tr}[F_{\mathscr{A}} - B\mathbf{1}] + N\Phi\wedge B) - \frac{N}{2\pi}\int_X F_\Phi\wedge B + \frac{k(N-N^2)}{4\pi}\int B\wedge B.\end{aligned} \tag{144}$$

To get this, we just took the four-dimensional gauged action (with Φ a 1-form Lagrange multiplier to make B a $\mathbb{Z}_N$ field), and integrated by parts. The extra $N\Phi\wedge B$ term is needed to make things gauge invariant, as we will see.

The transformation rules for the fields are as follows. First, we have a gauge transformation under which $\delta y = \delta\Phi = d\alpha$. Next, we have a $\mathbb{Z}_N^{(1)}$ gauge transformation, generated by $e^{i\oint_C y}$, which shifts

$$\mathscr{A}\mapsto\mathscr{A} + \frac{2\pi}{N}\widehat{C},\quad B\mapsto B + \frac{2\pi}{N}d\widehat{C},\quad y\mapsto y + \frac{2\pi k}{N}\widehat{C},\quad \Phi\mapsto\Phi + \frac{2\pi k}{N}\widehat{C}. \tag{145}$$

Here $\widehat{C}$ is the Poincare to a possibly open curve in ∂X, with the Poincare dual having some arbitrary extension into the bulk X. Here $\widehat{C}$ is such that $\int_{C'}\widehat{C}\in\mathbb{Z}$ for all $C'\in C_1(\partial X;\mathbb{Z})$, but where the value for the integral may depend on more than just the homotopy class of C'. One can check that the action is invariant up to the term $-\frac{k}{2\pi}\int\widehat{C}\wedge F_B$, which is in $\overline{\mathbb{Z}}$ because of the quantization on F_B. Thus, the whole action is gauge-invariant.

Anyway, these transformation laws let us write down the correct, gauge invariant, generator of gauge transformations for the gauged $\mathbb{Z}_N^{(1)}$ symmetry. It is

$$U(q,C) = \exp\left(iq\left[\int_C y + \int_{C'}\Phi + k\int_D B\right]\right). \tag{146}$$

Here $C\cup C' = \partial D$, with C' only in the bulk; see Figure 1 for a similar setup. Do these operators commute with each other? Yes! $U(q,C)$ and $U(p,\widetilde{C})$, with C, $\widetilde{C}$ two intersecting curves in ∂X will have a contribution to their commutator of the form $e^{2\pi iqpk/N}$, which comes from the commutator of the two y lines. However, they will also have a compensating contribution from the commutator between the $\int_{\widetilde{C}'}\Phi$ line and the $k\int_D B$ surface (which intersect in the bulk), since Φ and B have a braiding phase of $e^{2\pi i/N}$. Thus the $U(q,C)$ are indeed legit generators of the $\mathbb{Z}_N^{(1)}$ gauge transformations.

Summary

Since this diary entry has kind of exploded, let's make a summary table. The theories that we've looked at are

$$
\begin{aligned}
&U(1)_k : \frac{k}{4\pi} A \wedge dA \\
&DW_{p,q} : \frac{p}{4\pi} a \wedge da + \frac{q}{2\pi} a \wedge db \\
&U(N)_{k,q} : \frac{k}{4\pi} \mathrm{Tr}[\mathscr{A} \wedge d\mathscr{A} + 2i/3\mathscr{A}^3] + \frac{q-k}{4\pi N} \mathrm{Tr}[\mathscr{A}] \wedge d\mathrm{Tr}[\mathscr{A}] \qquad (k-q) \in N\mathbb{Z} \\
&SU(N)_k : U(N)_{k,k(1-N)} + \frac{1}{2\pi} y \wedge d\mathrm{Tr}[\mathscr{A}].
\end{aligned}
\tag{147}
$$

The symmetries and anomalies (on general, non spin manifolds, provided that the theory is not spin) are

	1-form symmetry	Anomaly (mod 1)	Spin?
$U(1)_k$	$\mathbb{Z}_k$	$1/k$	if $k \in 2\mathbb{Z}+1$
$DW_{p,q}$	$\mathbb{Z}_q$ on b, $\mathbb{Z}_{\gcd(p,q)}$ on a	0,$p/\gcd(p,q)$,$1/\gcd(p,q)$ (mixed)	if $p \in 2\mathbb{Z}+1$
$U(N)_{k,q}$,	$\mathbb{Z}_q$	N/q	if $k+(q-k)/N \in 2\mathbb{Z}+1$
$SU(N)_k$	$\mathbb{Z}_N$	$(k-Nk)/2N$	No

(148)

Here the anomaly is determined by taking the mod 1 residue of the entry in the third column. In the last column we have indicated when the theories are spin, which will be determined in the subsequent diary entry.

One interesting thing is to check how this is compatible with known level-rank dualities. For example, consider the duality $U(1)_N \leftrightarrow SU(N)_1$ (it is usually $U(1)_{-N}$, but in these conventions the anomalies are such that we write it as $U(1)_N$). This duality hold holds as spin TQFTs. Indeed, while they have the same $\mathbb{Z}_N^{(1)}$ symmetry, let's compare their anomalies: for $U(1)_{-N}$ we have $1/N$, while for $SU(N)_1$ we have $(1-N)/2N$. These are of course not the same. But, on a spin manifold, the anomaly of $SU(N)_k$ is actually $(k-Nk)/N \mod 1$ since the generator of the $\mathbb{Z}_N^{(1)}$ symmetry is allowed to be a fermion. Setting $k=1$ the anomaly becomes $1/N$, which matches that of the $U(1)_N$ theory.

14 *January 28 — When are CS theories spin TQFTs?*

Today's problem statement is straightforward: answer the question in the title, to the best of your abilities (i.e. just work through a few examples).

Solution:

One way to examine whether a CS theory is spin or not is to carefully define the CS action by breaking up the manifold into patches and defining the action in the style of

DB cohomology; see a previous diary entry on this. This approach is kind of subtle for non-Abelian gauge groups though, so we will take a different, simpler, approach.

$U(1)_k$

As usual, define the CS action on a closed 3-manifold X by integrating an $F \wedge F$ term over some 4-manifold Y with $\partial Y = X$. The exponential of the action is independent of the choice of bounding 4-manifold Y provided that

$$\frac{k}{8\pi^2} \int_X F \wedge F \equiv \frac{k}{2} I \in \mathbb{Z} \tag{149}$$

for all closed 4-manifolds M. Now, $F/2\pi \in H^2(M;\mathbb{Z})$, so we know for sure that $I \in \mathbb{Z}$ since the cup product of $F/2\pi$ with itself is then in $H^4(M;\mathbb{Z})$. Now if $k \in 2\mathbb{Z}$ then the (exponential of the) above integral is independent of M, regardless of whether M is spin or not. Thus if $k \in 2\mathbb{Z}$, the CS theory is insensitive to the spin structure and hence is bosonic. However, suppose $k \in 2\mathbb{Z}+1$. Then the CS action is only well-defined if $I \in 2\mathbb{Z}$. The constraint $I \in 2\mathbb{Z}\ \forall M$ can only be satisfied if we restrict our attention to M such that M is spin. If M is spin then $\omega_2(TM) = 0 \mod 2$ and the intersection form is even, meaning that I is always even. So, for odd k, the theory can only be defined using spin bounding 4-manifolds, and hence the original 3-manifold needs to come equipped with a spin structure as well. Thus odd k theories are spin TQFTs.

$SU(N)_k$

Now consider $SU(N)$. Now the relevant integral over a closed 4-manifold is

$$\frac{k}{8\pi^2} \int_M \mathrm{Tr}[F \wedge F] = k\mathrm{ch}_2(F) \in k\mathbb{Z}, \tag{150}$$

since the second Chern character is the second second Chern class for $SU(N)$ on account of the tracelessness of the $SU(N)$ generators, it is quantized on account of the second Chern class being a $\mathbb{Z}$ characteristic class (for $U(1)$, the integral is just the second Chern character, which is not a class in $\mathbb{Z}$ cohomology). Note that the quantization of the integral does not depend on whether M is spin or not: the second Chern class's integrality doesn't depend on the spin nature of M, since it does not (in general) compute an intersection form. Indeed, the minimal $\mathrm{ch}_2(F) = 1$ instantons are the "small" instantons that can exist on any manifold, regardless of its topology. They are constructed from bundles which are not tensor products of line bundles (if they were their quantization would be sensitive to $\omega_2(TM)$), and since they are "small" they can exist equally happily on spin- and non-spin manifolds. So, all the $SU(N)$ CS theories are bosonic.

$U(N)_{k,q}$

Now for $U(N)_{k,q}$, which is defined though

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathscr{A} \wedge d\mathscr{A} - \frac{2i}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A}\right] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathscr{A}] \wedge d\mathrm{Tr}[\mathscr{A}]. \tag{151}$$

As explained before, the notation is done like this because q is ($1/N$ times) the effective $U(1)$ level, while k is the effective $SU(N)$ level.

Now we use the decomposition $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$. At the level of actions, we simply write $\mathscr{A} = A + \mathcal{A}\mathbf{1}$, where A is an $SU(N)$ field (whose transition functions may fail by Nth roots of unity), $\mathcal{A}$ is a $U(1)$ field (with transition functions failing in the inverse way). The quotient comes from the correlation of the transition functions between A and $\mathcal{A}$. In terms of these fields, we have

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[A \wedge dA - \frac{2i}{3}A \wedge A \wedge A\right] + \frac{qN}{4\pi}\mathcal{A} \wedge d\mathcal{A}, \tag{152}$$

so that qN is indeed the "effective $U(1)$ level". The scare quotes here are because $\mathcal{A}$ isn't really a $U(1)$ field, because of the quotient: only $N\mathcal{A}$ is a legit $U(1)$ field. So the legit $U(1)$ part is really

$$S \supset \frac{2\pi q/N}{8\pi^2}\int_Y d(N\mathcal{A}) \wedge d(N\mathcal{A}), \tag{153}$$

where Y is a bounding 4-manifold. This would seem to indicate that we require $q \in N\mathbb{Z}$ in order for the action to be well-defined (independent of Y). But this is not quite the case, since the term in $\mathcal{L}$ involving A also stands a chance of being ill-defined on its own, due to the $\mathbb{Z}_N$ quotient. Indeed, from our previous diary entry on instanton numbers in $PSU(N)$ gauge theory, we saw that $\frac{k}{2}\int \mathrm{Tr}(dA/2\pi \wedge dA/2\pi)$ was quantized in $k/N\mathbb{Z}$. Thus the ill-defined-ness of the A part of the action alone is captured by $k/N \mod 1$. Since the transition functions of A and $\mathcal{A}$ fail the cocycle condition in opposite senses at each triple overlap of patches, the fractional part of the instanton number for the A field is the negative of that for the $\mathcal{A}$ field. Thus the total parameter measuring the ill-defined-ness of the action is actually $(k-q)/N \mod 1$. So, for a consistent theory, we need

$$k - q \in N\mathbb{Z}. \tag{154}$$

Another way to say this is that since $\mathrm{Tr}[\mathscr{A}]$ is a well-defined $U(1)$ gauge field (but not $\mathscr{A}$ itself), the appearance of the term $(k-q)\mathrm{Tr}\mathscr{A} \wedge d\mathrm{Tr}\mathscr{A}/4\pi N$ in the action means that in order for this to be well-defined we need to have $(k-q)/N \in \mathbb{Z}$.

Yet another way to say it is that the theory needs to be invariant under simultaneous shifts in the transition functions of A and $\mathcal{A}$ by elements in $\mathbb{Z}_N$, which is realized on $\mathcal{A}$ through the shift $\delta\mathcal{A} = \frac{1}{N}d\phi$ for some 2π-periodic scalar ϕ. Since we are shifting both A and $\mathcal{A}$, $\mathscr{A}$ is invariant, and the action changes by

$$\delta S = \frac{(q-k)}{2\pi}\int d\phi \wedge F_{\mathcal{A}} \tag{155}$$

(for the derivation of the fact that the prefactor is $1/2\pi$ and not $1/4\pi$, see the previous diary entry). Now since only $N\mathcal{A}$ is a $U(1)$ gauge field, the flux of $F_{\mathcal{A}}$ is quantized in $\overline{\mathbb{Z}}/N$. Thus in order for $\delta S \in \overline{\mathbb{Z}}$, we need $(q-k) \in N\mathbb{Z}$.

Anyway, when are these theories spin? Returning to the original formulation in terms of the $\mathscr{A}$ field, the appropriate four-dimensional integral to compute is

$$I = \frac{1}{8\pi^2}\int \left(k\mathrm{Tr}[F_{\mathscr{A}} \wedge F_{\mathscr{A}}] + \frac{q-k}{N}\mathrm{Tr}[F_{\mathscr{A}}] \wedge \mathrm{Tr}[F_{\mathscr{A}}]\right). \tag{156}$$

Using the definition of the second Chern class,

$$I = 2\pi \int \mathrm{c}_2(E) + 2\pi \frac{k + (q-k)/N}{8\pi^2} \int d\mathrm{Tr}\mathscr{A} \wedge d\mathrm{Tr}\mathscr{A}, \tag{157}$$

where E is the total $U(N)$ bundle. Since $\int \mathrm{ch}_2(E) \in \mathbb{Z}$ on any closed 4-manifold (spin or not), whether or not the theory is spin is determined by the second term. In particular, we get

$$k + \frac{q-k}{N} \in \begin{cases} 2\mathbb{Z} & \implies \text{not spin} \\ (2\mathbb{Z}+1) & \implies \text{spin} \end{cases}, \tag{158}$$

where these are the only two options since as we said before, $(q-k) \in N\mathbb{Z}$.

$PSU(N)_k$

As we saw in a previous diary entry, on spin manifolds, minimal $PSU(N)$ bundles have instanton numbers that are in $\frac{1}{N}\mathbb{Z}$, and thus they are only defined when the level satisfies $k \in N\mathbb{Z}$. Since the fractional part of the instanton number came from the intersection number $\int B \wedge B$ of a 2-form $\mathbb{Z}_N$ gauge field, the fractional part of the instanton number will indeed depend on the existence of a spin structure: on non-spin manifolds we only have $I \in \frac{1}{2\mathbb{Z}}$. Thus $PSU(N)_k$ is spin if the level is an odd multiple of N ($k \in 2N\mathbb{Z} + N$), and non-spin if the level is an even multiple of N ($k \in 2N\mathbb{Z}$).

For example, take $PSU(2)_2 = SO(3)_2$: we obtain this from $SU(2)_2$ by identifying the representation 1 with the trivial representation. Now $SU(2)_2$ is the Ising theory, and 1 is the fermion. So, in order to identify 1 with 0, we need a spin structure. Thus $PSU(2)_2$ is a spin CS theory.

More generally, we know that the spin j line in $SU(2)_k$ has spin

$$\theta_j = \frac{j(j+1)}{k+2}. \tag{159}$$

When $k \in 2\mathbb{Z}$, we can take the quotient to $PSU(2)_k$. The maximal spin line with $j = k/2$ is the generator of the $\mathbb{Z}_2^{(1)}$ symmetry we need to quotient by, and from the above we see that it has spin $\theta_{k/2} = k/4$. Therefore for $k \in 4\mathbb{Z}+2$ the generator is a fermion, and so $PSU(2)_k$ is spin for $k \in 4\mathbb{Z}+2$. On the other hand, when $k \in 4\mathbb{Z}$ the generator is a boson, and so for such values of k, $PSU(2)_k$ is not spin.

$DW_{p,q}$ theory

In the notation of last time, the $DW_{p,q}$ theory is

$$\mathcal{L} = \frac{p}{4\pi} a \wedge da + \frac{q}{2\pi} a \wedge db. \tag{160}$$

Writing the action as an integral over a bounding 4-manifold tells us that these theories are spin when p is odd, and non-spin when p is even. This matches with the discussion of the 1-form symmetries of the theory in the previous diary entry: the generator for the

$\mathbb{Z}_q^{(1)}$ symmetry shifting b is a boson and not anomalous, while the generator U_a for the $\mathbb{Z}_l^{(1)}$, $l \equiv \gcd(p,q)$ symmetry shifting a has spin

$$s[U_a] = \frac{p}{2l^2} \mod 1. \tag{161}$$

This means that the spin of l copies of the charge operator is $s[U_b^l] = p/2 \mod 1$. Since l copies of the charge operator gives a line that has trivial statistics with everything, we see that if $p \in 2\mathbb{Z}$ we have no problem, while if $p \in 2\mathbb{Z}+1$ then the theory has a transparent fermion. However since the theory is spin if $p \in 2\mathbb{Z}+1$ the transparent fermion is trivial, and so U_b^l is a trivial line, as required.

$SO(N)_K$

The CS action for $SO(N)_K$ is written as

$$S = \frac{k}{8\pi}\int_M \mathrm{Tr}[F_A \wedge F_A], \tag{162}$$

where the trace is taken in the vector representation. Note the factor of $1/8\pi$ in front, which differs from the usual $1/4\pi$ we've seen so far. Requiring that the integral be independent of the bounding 4-manifold means that for all closed M, we need

$$2\pi k \frac{1}{2\cdot 8\pi^2}\int_M \mathrm{Tr}[F_A \wedge F_A] = \pi k \int p_1(A), \tag{163}$$

where $p_1(A)$ is the first Pontryagin class. Now this is a legit $\mathbb{Z}$ characteristic class, but unlike the second Chern class, its quantization *does* depend on the type of manifold that it's on. In particular, the relation

$$p_1(A) = P(w_2(A)) + 2w_4(A) \mod 4 \tag{164}$$

tells us that $\int p_1(A) \in 2\mathbb{Z}$ on spin manifolds. Thus $k \in 2\mathbb{Z}$ theories make sense on any manifold and are not spin, while $k \in 2\mathbb{Z}+1$ theories are spin.

So in general, the coefficient in front of the CS Lagrangian ($k/4\pi$, $k/8\pi$, etc.) can be determined by looking at how the relevant characteristic class (Chern or Pontryagin) is quantized on different types of manifolds. We should pick it so that for all k, the 2+1D CS action is well-defined on spin manifolds (but may require special choices for k to be defined on non-spin manifolds).

Before wrapping up, note how we never needed to compute the spectrum of line operators to make these statements, although that's certainly one way to figure our whether they are spin or not. However, just knowing whether they are spin already tells us a nonzero amount about their spectrum: we already know that e.g. a transparent fermion cannot appear in the spectrum for $SU(N)_k$ or $U(1)_{2k}$, but that one must appear in $U(1)_{2k+1}$.

15 *January 30 — Gauge (in)variance of non-abelian CS action and building instantons*

Compute the gauge variation of the CS action, and use the result to show how one can build instantons on S^4.

Solution:

We will work in math conventions where the gauge transformation acts as

$$A \mapsto g^{-1}(A+d)g = g^{-1}Ag + \omega. \tag{165}$$

The Lagrangian in these conventions is then

$$\mathcal{L} = \frac{ik}{4\pi}\mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right). \tag{166}$$

If we had $A \mapsto g^{-1}(A+id)g$ instead, we'd need to tack an i onto the $2/3$ (which can be seen by tracking the i through the following manipulations). The gauge variation of the first part is

$$\mathrm{Tr}[A\wedge dA] \mapsto \mathrm{Tr}\left[(A^g+\omega)\wedge(-\omega A^g + (dA)^g - A^g\omega - \omega\wedge\omega)\right], \tag{167}$$

where $A^g \equiv g^{-1}Ag$ and we've used $d\omega = -\omega\wedge\omega$. Now we use

$$\mathrm{Tr}[X\wedge Y] = (-1)^{|X||Y|}\mathrm{Tr}[Y\wedge X] \tag{168}$$

to write this term as

$$\mathrm{Tr}[A\wedge dA] \mapsto \mathrm{Tr}\left[A\wedge dA - 3\omega\wedge\omega\wedge A^g - \omega^{\wedge 3} + \omega(dA)^g - 2A^g\wedge A^g\wedge\omega\right]. \tag{169}$$

Clearly, the $A^{\wedge 3}$ term is going to be needed if we want to get something gauge invariant. This term changes as

$$\frac{2}{3}\mathrm{Tr}[A^{\wedge 3}] \mapsto \frac{2}{3}\mathrm{Tr}\left[A^{\wedge 3} + \omega^{\wedge 3} + 3(A^g\wedge\omega\wedge\omega + A^g\wedge A^g\wedge\omega\right]. \tag{170}$$

Adding these two contributions, we see that

$$\delta\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[A^g\wedge\omega\wedge\omega(2-3) + \omega^{\wedge 3}(-1+2/3) + A^g\wedge A^g\wedge\omega(2-2) + \omega\wedge(dA)^g\right]. \tag{171}$$

We can collect two of the surviving terms into a total derivative, so that

$$\delta\mathcal{L} = -\frac{k}{4\pi}d\mathrm{Tr}[\omega\wedge A^g] - \frac{k}{12\pi}\mathrm{Tr}[\omega^{\wedge 3}]. \tag{172}$$

Now the first term doesn't contribute to δS, since $\omega|_{\partial X} = 0$ if g is a gauge transformation and we fix ∂ conds on A ($X =$ spacetime). Since $\omega|_{\partial X} = 0$, the second term in $\delta\mathcal{L}$ is 2π times the winding number density for a map from the (compactification of) X to the target Lie group. Together, these terms tell us what kind of WZW needs to live on ∂X in order for gauge invariance to be manifest with free boundary conditions on A. The winding number (okay, "winding" is probably best reserved for $S^1 \to S^1$ situations—maybe "wrapping" would be pedantically better) term integrates to something in $\overline{\mathbb{Z}}$ (using (168) it's straightforward to show that $\mathrm{Tr}[\omega^{\wedge 4}] = 0$, so that the winding number density term is closed. Showing that the $1/12\pi$ coefficient is the correct normalization can be done by computing the integral for a fixed example field configuration; see one of 2018's diary entries on WZW models for more detail. The winding number is $W = \frac{1}{24\pi^2}\int \mathrm{Tr}[\omega^{\wedge 3}]$.) Anyway, using this quantization on the integral of the $\omega^{\wedge 3}$ term, we see that the whole CS action is indeed gauge invariant modulo elements of $\overline{\mathbb{Z}}$.

Now we use this result to construct $SU(N)$ instantons on S^4. We will cover S^4 with two patches U_N and U_S, each homeomorphic to a 3-ball, with $U_N \cap U_S = S^3_{eq}$, the equatorial 3-sphere. We want to compute $I = \frac{1}{8\pi^2}\int \mathrm{Tr}[F \wedge F]$. Now on each patch U_N, U_S, the gauge field A is a well-defined 1-form, and so we can use Stoke's theorem. Thus

$$I = \frac{1}{8\pi^2}\left(\int_{U_S} \mathrm{Tr}[F_{A_S} \wedge F_{A_S}] + \int_{U_N} \mathrm{Tr}[F_{A_N} \wedge F_{A_N}]\right) = \frac{1}{2\pi}\int_{S^3_{eq}} (\mathcal{L}_{CS_1}[A_N] - \mathcal{L}_{CS_1}[A_S]), \tag{173}$$

where $\mathcal{L}_{CS_1}[A]$ is the CS action at level 1.

Now to create an instanton we glue up the sections of the gauge bundle on U_N to those on U_S through a large gauge transformation[11]. The existence of nontrivial gauge transformations in this case is guaranteed from $\pi_3(SU(N)) = \mathbb{Z}$. So, we choose the transition function g_{NS} such that g_{NS} is a nontrivial homotopy class in $\pi_3(SU(N))$. Then the gauge fields get glued together as $A_N = g^\dagger_{NS}(A_S + d)g_{NS}$. So

$$I = \frac{1}{2\pi}\int_{S^3_{eq}} (\mathcal{L}_{CS_1}[g^\dagger_{NS}(A_S + d)g_{NS}] - \mathcal{L}_{CS_1}[A_S]). \tag{176}$$

Now we can use our result for the gauge variation of $\mathcal{L}_{CS_1}$ to write

$$I = \frac{1}{2\pi}\int_{S^3_{eq}} \left(-\frac{1}{4\pi} d\mathrm{Tr}[\omega_{NS} \wedge g^\dagger_{NS} A_N g_{NS}] - \frac{1}{12\pi}\mathrm{Tr}[\omega^{\wedge 3}_{NS}]\right), \qquad \omega_{NS} = g^\dagger_{NS} dg_{NS}. \tag{177}$$

[11]This is exactly the same as how we build e.g. mangetic monopoles on S^2 for $U(1)$ gauge theory: we take the gauge field on the northern / southern hemispheres to be e.g.

$$A_N = \frac{1-\cos\theta}{2}d\phi, \quad A_S = \frac{-1-\cos\theta}{2}d\phi, \tag{174}$$

so that on the equator, $A_N - A_S = d\phi$, which means that on the equator, A_N and A_S differ by a large gauge transformation on the S^1 (also note how A_N is not well-defined at the south pole $\theta = \pi$, and A_S is not well-defined at the north pole $\theta = 0$). The "instanton" number is then

$$\frac{1}{2\pi}\left(\int_{U_N} F_{A_N} + \int_{U_S} F_{A_S}\right) = \frac{1}{2\pi}\int_{S^1_{eq}} (A_N - A_S) = 1, \tag{175}$$

where $U_{N/S}$ is the northern / southern hemisphere.

The first term dies, and so we get

$$I = -\frac{1}{24\pi^2}\int \mathrm{Tr}[\omega_{NS}^{\wedge 3}] = -W \in \mathbb{Z}, \tag{178}$$

which is (the negative of; sorry for the dumb sign choice) the winding number of g_{NS}.

Let's remind ourselves why the $1/24\pi^2$ coefficient is there, just for fun. We'll do the calculation for $SU(2)$ for simplicity. The winding number 1 map in $\pi_3(SU(N))$ is

$$g_{NS} = x_\mu \widetilde{\sigma}^\mu, \tag{179}$$

with $x_\mu \in S^3$ a unit vector and $\widetilde{\sigma}^\mu = (\mathbf{1}, iX, iY, iZ)$. Note that as required, $g_{NS}g_{NS}^\dagger = x_\mu x^\mu \mathbf{1} = \mathbf{1}$, and $\det g_{NS} = x_\mu x^\mu = 1$.

To evaluate the winding number integral we can either go to spherical coordinates and do lots of algebra, or use a clever trick. The clever trick is as follows: since g_{NS} is uniform on the S^3, we just need to compute the winding number density at a particular point on the 3-sphere, and then multiply the result by $2\pi^2 = \mathrm{vol}\,(S^3)$. Let us choose the north pole, where the field points in the $\mathbf{1}$ direction. Now ω is

$$\omega = (x_\mu \widetilde{\sigma}^\mu)^\dagger d(x_\mu \widetilde{\sigma}^\mu) = (x_\mu \widetilde{\sigma}^\mu)^\dagger (\sigma_\nu - x^\lambda \sigma_\lambda x_\nu)\, dx^\nu. \tag{180}$$

Evaluating this at the point $x_\nu = (1,0,0,0)$, the only derivatives that enter are ∂_i, where $i \in \{x,y,z\}$, since these are the coordinates in the tangent space at $(1,0,0,0)$. Thus ω becomes just $\sigma_i dx^i$, and the integrand is

$$\frac{i^3}{24\pi^2}\mathrm{Tr}[\sigma^i\sigma^j\sigma^k] dx^i \wedge dx^j \wedge dx^k = \frac{i^3}{4\pi^2}\mathrm{Tr}[XYZ] d^3x = \frac{1}{2\pi^2} d^3x, \tag{181}$$

which is just the pullback of the volume form on S^3. Multiplying this by the volume of S^3 we get 1, and so the $1/24\pi^2$ normalization was indeed correct.

What should we do if we want a winding number $W > 1$ map? To get winding number 1 we pulled back the volume form, so to get winding number W we should pullback W times the volume form. There are some physics books which say that for $\omega = g^{-1}dg$ we should keep the g configuration for $W = 1$ but replace g by g^W: this is wrong since then W appears in I as W^3 as I is cubic in the ω's. Instead, break up the spacetime S^3 as SS^2, where SS^{d-1} denotes the suspension of S^{d-1} by S^1. Let the coordinate on the S^1 that's doing the suspension be θ, and the coordinates on the S^2 be ϕ, ψ. Then if $g_1(\theta, \phi, \psi)$ is the winding number 1 configuration, the winding number W configuration is $g_W = g_1(W\theta, \phi, \psi)$. What's going on here is that the map completes winding number 1 during $\theta \in [0, \pi/W)$, and so the total winding number for $\theta \in [0, \pi]$ is W (we imagine composing W identity maps $S^3 \to S^3$ in a row, with the basepoint [$\theta = 0$ point] of each map being the terminal point [$\theta = \pi$ point] of the previous one). This is also clear from the integral formula: the wedge product means that the integrand contains one derivative for each of the coordinates on the S^3, so multiplying one of the coordinates by W will increase the integrand by a factor of W.

How do we make instantons for other gauge groups G? We do this by using a map $SU(2) \to G$ induced from a map $\mathfrak{su}(2) \to \mathfrak{g}$, which always exists because of roots. Specifically we can always get a winding number $W = 1$ map by taking the gauge configuration

$$g_{NS}^{SU(N)} = \mathbf{1}_{N-2} \oplus g_{NS}^{SU(2)}, \tag{182}$$

which is in $SU(N)$ as required.

We can also get winding number $W > 1$ maps by embedding the $SU(2)$ instanton inside of $SU(N)$ with a different representation. For example, for the map $SU(2) \to SU(3)$, we can choose to embed the $SU(2)$ either in the fundamental, or in the adjoint. The difference in the instanton number just comes from the difference in the trace of the generators; in this case we can get winding number $W = \pm 4$.

16 *February 3 — Topological terms from integrating out fermions in four dimensions and some characteristic class relations for vector bundles*

Today is just basically a small compendium of results about what kind of θ terms are produced when integrating out massive fermions in four dimensions.

Solution:

Complex / Dirac fermions

First we will look at the case where the fermions transform in a complex representation of the full symmetry group (involving spacetime symmetries) that we will assume to include a $U(1)$ fermion number conservation factor. In this case, there is no antisymmetric bilinear form we can use to construct a symmetric action involving a single fermion field, and so any symmetry-preserving Dirac operator appearing in the action will have to pair two independent fields $\bar{\psi}$ and ψ, with opposite charges under the $U(1)$ (this is what we mean when we say the fermion transforms in a complex representation: there is no antisymmetric bilinear form, invariant under the symmetry, that pairs a single fermion field with itself). Thus in the basis $(\psi, \bar{\psi})^T$, the Lagrangian will be purely off-diagonal:

$$\mathcal{L} = \begin{pmatrix} \psi & \bar{\psi} \end{pmatrix} \begin{pmatrix} 0 & i\not{D} \\ -i\not{D}^T & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \tag{183}$$

where $i\not{D}$ is Hermitian. The minus sign and transpose are needed since the fermions need to be paired antisymmetrically: otherwise the action vanishes, since e.g.

$$\bar{\psi}_\alpha [i\not{D}]^{\alpha\beta} \psi_\beta = -\psi_\beta [i\not{D}^T]^{\beta\alpha} \bar{\psi}_\alpha. \tag{184}$$

When doing manipulations like this, we should remember that $\partial^T = \overleftarrow{\partial} = -\partial$. So in this notation, the derivative in the matrix $[\not{D}]^{\alpha\beta}$ always acts on the second index: thus the ψ in $[\not{D}]^{\alpha\beta}\psi_\beta$ is differentiated, while the ψ in $[\not{D}]^{\beta\alpha}\psi_\beta$ is not.

Anyway, the point of this is just to note that this structure for the Lagrangian means that integrating out the fermions ψ and $\bar{\psi}$ produces a determinant (of $i\not{D}$) rather than just a

Pfaffian. Working in Euclidean signature and adding a mass m and a gauge field A produces a partition function $Z[A;m] = \det(i\not{D}_A - m)$, where m is real (in Euclidean time γ^0 is Hermitian, so the Lagrangian is $\bar{\psi}(i\not{D}_A - m)\psi$).

Since $\not{D}_A$ anticommutes with $\bar{\gamma}$ (recall that we are working in four dimensions), if ψ is an eigenspinor of $i\not{D}_A$ with non-zero eigenvalue, then $\bar{\gamma}\psi$ is a linearly independent eigenspinor with an eigenvalue of the opposite sign (they are linearly independent since they have different eigenvalues: $\langle\psi, i\not{D}_A\psi\rangle = \lambda\langle\psi,\psi\rangle \implies \langle\lambda, i\not{D}_A\bar{\gamma}\psi\rangle = -\lambda\langle\psi,\bar{\gamma}\psi\rangle = \langle\bar{\gamma}\psi, i\not{D}_A\psi\rangle = +\langle\psi,\bar{\gamma}\psi\rangle \implies \langle\psi,\bar{\gamma}\psi\rangle = 0$). Since they are linearly independent, $\psi_\pm \equiv (1\pm\bar{\gamma})\psi/2$ must be nonzero for both choices of sign: non-zero modes have support on both chirality subspaces, and so every non-zero-mode comes as a member of a positive-negative eigenvalue pair.

Now for the partition function: we have

$$\det(i\not{D}_A - m) = \left(\prod_{\lambda_j>0}(\lambda_j - m)(-\lambda_j - m)\right) m^{N_+ + N_-}, \tag{185}$$

where N_σ is the number of zero-modes with chirality σ. Note that when we say "number of zero-modes", we really mean "number of positive-charge zero modes": we are just computing the determinant of $i\not{D}_A$ as it acts on ψ, and not on $\bar{\psi}$. This number can be odd (and is the number relevant for computing the parition function), but the full number of zero modes, of both positive and negative charges, is always even. Indeed, if $\psi_\pm$ is a zero mode for the field ψ then its complex conjugate is a zero mode for the field $\bar{\psi}$[12] and so the full number of zero modes (for both the fields ψ and $\bar{\psi}$) is actually $2(N_+ + N_-)$.

The factor of $m^{N_+ + N_-}$ can also be understood from looking at how the zero modes get paired up by the mass term: each positively-charged zero mode ψ_+ appears in the path integral together with its negatively-charged partner as

$$\int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\int \psi_-^\dagger m \psi_+} = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \left(1 - \int \psi_-^\dagger m \psi_+\right) \propto m, \tag{186}$$

because of how Grassmann integration works. Thus we get a factor of m for each positively-charged zero mode.

Anyway, note how the product in the expression for $\det(i\not{D}_A - m)$ is independent of the sign of m. Thus we have

$$\frac{Z[A;m]}{Z[A;-m]} = (-1)^{N_+ + N_-} = (-1)^{N_+ - N_-} = e^{i\pi\,\mathrm{ind}(i\not{D}_A)}. \tag{187}$$

Now the index of the Dirac operator, for A a connection on a bundle E, is

$$\mathrm{ind}(i\not{D}_A) = \int \widehat{A}\wedge \mathrm{ch}(E). \tag{188}$$

Here $\mathrm{ch}(E)$ is the Chern *character* of the bundle E, *not* the Chern class. So we can write this as

$$\mathrm{ind}(i\not{D}_A) = \int \widehat{A}\wedge e^{F_A/2\pi}. \tag{189}$$

[12]The complex conjugate zero mode has opposite chirality: the associated zero mode of $\bar{\psi}$ is $\bar{\psi}_\pm = \gamma^0\psi_\pm^*$, and $\bar{\gamma}\bar{\psi}_\pm = -\gamma^0\bar{\gamma}\psi_\pm^* = \mp\bar{\psi}_\pm$. Note how here we are treating $\bar{\psi}$ as a column vector like ψ, which is a slightly more transparent thing to do since it really is an independent field.

Now the Dirac genus only involves pontryagin (spelling?! Can never remember) classes since it's a characteristic class in the real (involving traceless field strengths) tangent bundle. Thus only $4n$-dimensional classes contribute to $\widehat{A}$. For a 4-manifold M, we just need $\widehat{A} = 1 - \frac{1}{24}p_1(TM) + \dots$, and $\mathrm{ch}(E) = \mathrm{Tr}[\mathbf{1}] + \mathrm{Tr}[F_A/2\pi] + \frac{1}{2}\mathrm{Tr}[F_A/2\pi \wedge F_A/2\pi] + \dots$., with the trace taken in the fundamental representation Then

$$\mathrm{ind}(i\not{D}_A) = -\frac{\dim(E)}{24}\int p_1(TM) + \frac{1}{8\pi^2}\int \mathrm{Tr}[F_A \wedge F_A]. \tag{190}$$

Here $p_1(TM)$ is $\mathrm{Tr}[R \wedge R]$ with some normalization that I can never remember. Writing the gravitational contribution in terms of the signature with $\int \widehat{A} = \sigma/8$, we have

$$\frac{Z[A;m]}{Z[A;-m]} = \exp\left(\frac{i\pi}{8\pi^2}\int \mathrm{Tr}[F_A \wedge F_A] - i\pi\frac{\dim E}{8}\sigma\right), \tag{191}$$

where σ is the signature. Note that this is a completely non-perturbative result.[13]

On a spin manifold $\sigma \in 16\mathbb{Z}$, and so the signature part makes no contribution. On a general non-spin manifold, σ can be an arbitrary integer, and so the fact that $\mathrm{ind}(i\not{D}_A) \in \mathbb{Z}$ tells us that for a spinc connection A,

$$\frac{1}{2}\int \frac{F_A}{2\pi} \wedge \frac{F_A}{2\pi} - \frac{\sigma}{8} \in \mathbb{Z}, \tag{192}$$

which means that for a spinc connection,

$$\frac{1}{2}\int \frac{F_A}{2\pi} \wedge \frac{F_A}{2\pi} \in \frac{1}{8}\mathbb{Z}. \tag{193}$$

Of course, this makes total sense: if A is spinc then $2F_A/2\pi$ is an integer class, and so we can write the above integral as $\frac{1}{8}\int (2F_A/2\pi) \wedge (2F_A/2\pi)$, which is then manifestly in $\frac{1}{8}\mathbb{Z}$.

Pseudoreal / Majoranna fermions

So far we've seen what topological term gets generated upon integrating out a Dirac fermion. What about a Majorana fermion? Our fermion χ will be assumed to transform in a pseudo-real representation of the full symmetry group, so that there exists an antisymmetric bilinear form J invariant under the symmetry, which allows us to construct a symmetric mass term via $\chi_\alpha J^{\alpha\beta}\chi_\beta$. Since J is an invariant form then so too is $J(i\not{D}_A)$[14] and so the pairing for the kinetic term is then

$$\mathcal{L} \supset \chi^T J(i\not{D}_A)\chi. \tag{194}$$

[13]This is because locally, the integrand is a total derivative. If any Feynman diagram were to contribute to the effective action for A, it would then in momentum space contain a factor of p_{tot}, where p_{tot} is the sum of the momenta on all the external A legs attached to the diagram. Since momentum is conserved $p_{tot} = 0$, and therefore no Feynman diagram can contribute to this result. The only times when such topological terms can show up diagramatically is when there is an operator insertion (like j^μ_A) in the diagram to provide some extra momentum.

[14]Here it's best to think about $i\not{D}_A$ as being an operator rather than a bilinear form: J is used to raise / lower fermion indices, and $i\not{D}_A$ preserves the index placement. Thus J pairs two lower-index or two upper-index fermions, while $i\not{D}_A$ pairs and upper one with a lower one or vice versa.

In what follows we will take J to be real, with $J^2 = -\mathbf{1}$, so that J can be thought of as a complex structure. I think that this can be done wolog (with this convention the Hermitian mass term is $i\chi^T J\chi$). Note that

$$\chi^T(Ji\not{D}_A)\chi = -\chi^T[Ji\not{D}_A]^T\chi \implies [Ji\not{D}_A]^T = -Ji\not{D}_A \implies [i\not{D}_A]^T J = Ji\not{D}_A. \tag{195}$$

Since $J^2 = -\mathbf{1}$, we then have

$$J[i\not{D}_A]^T J = -i\not{D}_A, \tag{196}$$

which indeed is telling us that J is a kind of complex structure. Now consider the operator JK, where K is complex conjugation. We have

$$JK[i\not{D}_A]^\dagger KJ = -i\not{D}_A. \tag{197}$$

Since $(JK)^2 = -\mathbf{1}$ (this is true even if we chose J to be Hermitian instead of anti-Hermitian, since then J would be complex) and $i\not{D}_A$ is Hermitian (in Euclidean signature), we have

$$JKi\not{D}_A = i\not{D}_A JK. \tag{198}$$

Therefore pseudoreal fermions come equipped with an antiunitary action JK that commutes with the Dirac operator. Since $(JK)^2 = -\mathbf{1}$, we can then conclude that all eigenspinors of $i\not{D}_A$ come in pairs (related by JK) with identical eigenvalues: each eigenspinor χ comes with a linearly independent eigenspinor $JK\chi$, with the same eigenvalue λ, so then in the basis $(\chi,\, JK\chi)$, $i\not{D}_A$ has a block $\lambda\mathbf{1}_{2\times 2}$. Multiplying this by J, we see that a single eigenvalue λ of the Dirac operator then appears in the Lagrangian as

$$\mathcal{L} \supset \begin{pmatrix} \chi & JK\chi \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} \chi \\ JK\chi \end{pmatrix}. \tag{199}$$

Summing over all such pairs $\chi, JK\chi$, we get a big antisymmetric matrix. If we expand $e^{-\int \mathcal{L}}$ to the order at which the Grassmann integration gives something nonzero, we see that the partition function becomes $\mathrm{Pf}(Ji\not{D}_A) = \pm[\det(i\not{D}_A)]^{1/2}$, since $\det J = 1$.

One other thing we will need to know is that the doubling of the spectrum because of JK also restricts to a doubling of each eigenspinor of definite helicity. So this means that all non-zero-mode eigenspinors of $i\not{D}_A$ are qudrupled (one for each chirality, and for each chirality two modes related by JK), while the zero-modes are merely doubled. To show this, we observe that since $(JK)^2 = -\mathbf{1}$ and since $\bar\gamma$ is a product of an even number of γ matrices,

$$JK\bar\gamma JK = -\prod_j (JK\gamma_j JK) = -\prod_j(-\gamma_j(JK)^2) = -\bar\gamma \implies JK\bar\gamma = \bar\gamma JK, \tag{200}$$

and so

$$[JK, \bar\gamma] = 0. \tag{201}$$

Thus each definite-chirality mode is doubled (note to self: is this a Euclidean-time specific statement?).

Anyway, we can now compute the topological term induced by integrating out the fermions. Since the Pfaffian is the square root of the determinant, we can just naively take the square root of the partition function we found for the Dirac fermion (which did give a determinant), and write

$$\frac{Z[A;m]}{Z[A;-m]} = (-1)^{(N_+ + N_-)/2}. \tag{202}$$

Now because of the doubling of the spectrum due to the JK operator we discussed above, and because it commutes with $\bar{\gamma}$, we know that (unlike in the Dirac case), both N_+ and N_- must separately be even. Thus we have $(-1)^{N_-/2} = (-1)^{-N_-/2}$, and so

$$\frac{Z[A;m]}{Z[A;-m]} = (-1)^{\operatorname{ind}(i\not{D}_A)/2} = \exp\left(\frac{1}{2}\left[\frac{i\pi}{8\pi^2}\int \mathrm{Tr}[F_A \wedge F_A] - i\pi \frac{\dim E}{8}\sigma\right]\right). \tag{203}$$

This response is in keeping with the fact that a Majoranna is "half" a Dirac fermion.

For Majoranas, because the topological term involves $e^{\pi i p_1(B)/2}$ where B is either the gauge bundle or the tangent bundle, it evidently helps to have expressions for the Pontryagin classes mod 4. The most useful such relation is the one derived in the subsequent diary entry, namely

$$P(w_2(E)) = p_1(E) + 2w_4(E) \mod 4, \tag{204}$$

for any vector bundle (real or complex) E. For $SO(3)$ the $w_4(E)$ term is trivial, but in general it makes a contribution[15]. In any case, this means that we can write the topological response as

$$\frac{Z[A;m]}{Z[A;-m]} = \exp\left(\frac{i\pi}{2}\left[P(w_2(E)) + 2w_4(E) - \frac{\dim E}{8}\sigma\right]\right). \tag{205}$$

Here the second SW class of the gauge bundle is allowed to be non-trivial, as long as the second SW class of the tangent bundle is also non-trivial in the same way, so that $w_2(E) + w_2(TM) = 0 \mod 2$. Anyway, note that on a spin manifold, the even-ness of the intersection form means that $\int p_1(E) \in 2\mathbb{Z}$, and so the even-ness of the index of the Dirac operator means that $\frac{\dim E}{8}\sigma \in 2\mathbb{Z}$. In particular, taking just a single Majoranna fermion (not coupled to any gauge field) tells us that on a spin manifold, $\sigma \in 16\mathbb{Z}$. On the other hand, on a non-spin manifold, σ can be any integer, and this places constraints on how the Pointryagin term is quantized (although to work on a non-spin manifold, we need to be able to choose $w_2(E)$ in such a way that the full gauge- + spin-connection satisfies the cocycle condition).

[15] Recall that the kth SW class is the obstruction to finding $\mathrm{Rank}(E) - k + 1$ nowhere vanishing sections of E, and so they become trivial for $k > \mathrm{Rank}(E)$. As a consequence, any $SO(3)$ bundle has $w_4 = 0$, since the SW classes $w_k(E)$ with $k > \mathrm{Rank}(E)$ all vanish, and $\dim[SO(3)] = 3$.

An equivalent way to discuss w_k is to say that if the kth SW class is nonzero, then there is an obstruction to extending the trivialization of the bundle over the k-skeleton. But the converse is not true: there are plenty of cases where there is an obstruction to extending the trivialization, but the associated SW class vanishes. In general the obstruction to extend a G-bundle over the k-skeleton is captured by $\pi_{k-1}(G)$. This could fail to get detected by the SW classes either due to the fact that homotopy groups carry more data than cohomology groups, or because the obstructions always vanish mod 2. For example, the obstruction to extending an $SO(3)$ bundle over the 4-skeleton is non-zero as $\pi_3(SO(3)) = \mathbb{Z}$, even though $w_4 = 0$ because $4 > 3$. Moreover, no mod 2 class could detect this obstruction, since $\pi_3(SO(3))$ should really be thought of as $2\mathbb{Z}$. This is because elements in $\pi_3(SO(3))$ descend from elements in $\pi_3(S^3) = \mathbb{Z}$ from the map $S^3 \to SO(3)$, which is a double cover. Therefore a winding number 1 map in $\pi_3(S^3)$ maps onto a winding number 2 map in $\pi_3(SO(3))$; hence $\pi_3(SO(3)) = 2\mathbb{Z}$.

The typical example for Majorana fermions is when A is a connection for an $SO(n)$ associated bundle. When n is even, $-\mathbf{1} \in SO(n)$ and $-\mathbf{1} \in \mathrm{Spin}(4)$ act in the same way on fermions, and so our fermion is really coupled to a $[\mathrm{Spin}(4) \times SO(n)]/\mathbb{Z}_2$ connection. Now for $n = \dim E$ even, the gravitational term is quantized in $\frac{1}{4}\mathbb{Z}$; thus in order to maintain the integrality of the index of the Dirac operator, the Pointryagin class must also be quantized in $\frac{1}{4}\mathbb{Z}$, in such a way that the gauge and gravitational contributions add to give something in $2\mathbb{Z}$. This quantization makes sense, since when passing from $SO(n) \to SO(n)/\mathbb{Z}_2$ for n even, the quantization of the instanton number (alias $\int p_1(E)$) changes by a factor of $1/4$ on a general non-spin manifold (see a diary entry in last year's diary for a discussion of why). By contrast when n is odd, there is no $\mathbb{Z}_2$ identification between the gauge and spin connections, and in order for our fermion to be well-defined, we need to work on a spin manifold.

For example, consider $SO(3)$.

17 *February 5 — Characteristic class manipulations for Pontryagin classes*

Today we review what pontryagin classes are, and prove some results about their reductions mod 2 and mod 4. These results are helpful to have when dealing with topological terms generated by integrating out massive fermions.

Solution:

First some preliminaries on the Pontryagin classes.

In what follows we will frequently need to complexify real bundles, and realify complex ones. The sequence to keep in mind for complexifying and realifying is

$$U(n) \to SO(2n) \to U(2n). \tag{206}$$

The first map is used to turn a complex $n \times n$ matrix into a real $2n \times 2n$ one, while the second map is used to complexify a real $2n \times 2n$ matrix. The second map in the sequence comes from the inclusion $\mathbb{R} \to \mathbb{C}$, while the first map comes from

$$U(n) \ni A + iB \mapsto \mathbf{1} \otimes A + J \otimes B \in SO(2n), \qquad J = -iY, \tag{207}$$

with A, B real. Here J is how we represent i in $SO(2n)$. Why is the image of $A + iB$ in $SO(2n)$? For $A + iB$ to be unitary, we need

$$(A^T - iB^T)(A + iB) = \mathbf{1} \implies A^T A + B^T B = \mathbf{1}, \quad A^T B - B^T A = 0. \tag{208}$$

Now consider $\mathbf{1} \otimes A + J \otimes B$. Then since $J^T = -J$,

$$(\mathbf{1} \otimes A^T - J \otimes B^T)(\mathbf{1} \otimes A + J \otimes B) = \mathbf{1} \otimes (A^T A + B^T B) + J \otimes (A^T B - B^T A) = \mathbf{1} \otimes \mathbf{1}, \tag{209}$$

and so $A + JB$ is indeed orthogonal.

Anyway, on to Pontryagin classes. For a vector bundle E (usually a real vector bundle), they are defined by

$$p_j(E) = (-1)^j c_{2j}(E \otimes \mathbb{C}), \tag{210}$$

where $E \otimes \mathbb{C}$ is the complexification of E. Pontryagin classes almost obey the same sum formula as the Chern classes. Indeed (writing $\otimes$ for $\otimes_{\mathbb{R}}$),

$$p_j(E \oplus F) = (-1)^j c_{2j}(E \otimes \mathbb{C} \oplus F \otimes \mathbb{C}) = (-1)^j [c(E \otimes \mathbb{C}) \wedge c(F \otimes \mathbb{C})]_{2j} = [p(E) \wedge p(F)]_j + \dots, \tag{211}$$

where $\dots$ are terms that involve odd Chern classes. For example,

$$p_1(E \oplus F) = p_1(E) + p_1(F) - c_1(E \otimes \mathbb{C}) \wedge c_1(F \otimes \mathbb{C}). \tag{212}$$

Now the odd Chern classes of the complexification of a real bundle are 2-torsion[16], so that the Whitney sum formula holds for Pontryagin classes only up to 2-torsion elements. Another way to say this is to realize that if L is a real line bundle, then $L \otimes L$ is trivial, since $L^* \cong L$ by the reality of L means $L \otimes L \cong L \otimes L^* \cong \mathrm{Hom}(L, L)$, which always has a global section given by the identity map. Therefore any cohomology elements that classify real line bundles must be 2-torsion, and so the appropriate cohomology for describing real line bundles is $H^1(M; \mathbb{Z}_2)$. This means that when we map $H^1(M; \mathbb{Z}_2)$ into $H^2(M; \mathbb{Z})$, which classifies complex line bundles, we should get something that's 2-torsion. As we saw above, a similar statement holds for higher degrees.

The Pontryagin classes for a complex vector bundle E, defined by the Chern classes of $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ with $E_{\mathbb{R}}$ the realification of E, can easily be computed in terms of the Chern classes of E. If E is a complex vector bundle, then the isomorphism

$$E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E} \tag{215}$$

tells us that[17]

$$c(E \otimes_{\mathbb{R}} \mathbb{C}) = c(E \oplus \bar{E}) = (1 + c_1(E) + c_2(E) + \dots)(1 - c_1(E) + c_2(E) - \dots). \tag{221}$$

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[16]Proof: Let $\mathcal{L}$ be a complex line bundle. Then we claim that $c_1(\mathcal{L}) = -c_1(\bar{\mathcal{L}})$. Indeed, the first Chern class of a complex line bundle is the same as the Euler class of the underlying real bundle, which locally is expressible in terms of the logarithms of the transition functions. Since $\bar{\mathcal{L}}$ has transition functions which are conjugate to those of $\mathcal{L}$, the Euler class associated to $\mathcal{L}$ is the negative of the one associated to $\bar{\mathcal{L}}$.

Now we use the splitting principle: assume E splits as a direct sum of line bundles, so that

$$E = \bigoplus_j \mathcal{L}_j \implies c(E) = \prod_j (1 + c_1(\mathcal{L}_j)) \implies c(\bar{E}) = \prod_j (1 - c_1(\mathcal{L}_j)). \tag{213}$$

Thus

$$c_k(E) = (-1)^k c_k(\bar{E}). \tag{214}$$

Now suppose that $E = F \otimes \mathbb{C}$, for F a real line bundle. Then $E = F \otimes \mathbb{C} = F \oplus iF$ is isomorphic to $\bar{E} = F \oplus (-iF)$ (this isomorphism does *not* hold if E is a generic complex vector bundle). Therefore by the splitting principle we can conclude that $c_k(F \otimes \mathbb{C}) = -c_k(F \otimes \mathbb{C})$ for k odd, meaning that the odd Chern classes of the complexification of a real bundle are all 2-torsion.

[17]Why is this true? We just have to decipher the proof in Milnor. For any $z = x + iy \in E$, the corresponding element in $E_{\mathbb{R}}$ is obtained just by erasing the i and writing z as a tuple $(x, y) \in E_{\mathbb{R}}$. We create $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ (the fact that it's a $\otimes$ over $\mathbb{R}$ is important! We can't $\otimes$ a real bundle with something unless the tensor unit is $\mathbb{R}$) by forming the sum $E_{\mathbb{R}} \oplus E_{\mathbb{R}}$, and adding in the complex structure with the map $J : (z, w) \mapsto (w, -z)$, where

This relation shows that $c_{2k+1}(E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C})=0$ since all the odd-degree parts cancel in pairs, and so the Pontryagin classes which are not of degree a multiple of 4 vanish for the realification of a complex vector bundle.

This whole song and dance of defining the Pointryagin classes in terms of the Chern classes of a complexified bundle is mainly just so that we can show that the p_k are only nonzero for $k\in 4\mathbb{Z}$, and that we can show the whitney sum formula for the p_i's. A simpler way to define them would be to use Chern-Weil and just write down the p's explicitly, but then we'd have to do invariant polynomials and stuff to see which ones could be non-zero. This is often the better way to go in terms of computing things, since the complexification is pretty trivial: we just take our real curvature form F_A, and allow ourselves to e.g. diagonalize it using complex numbers. But this approach has the disadvantage that we'd miss torsion phenomena: for example, using the expansion of $\det(\mathbf{1}+F/2\pi)$ it's easy to see that the p_i's obey a Whitney sum formula modulo torsion, but to see the torsion effects we need to work with the complexification.

One such expression is as follows. The general claim is that for an (oriented?) vector bundle, we have [**?**]

$$P(w_{2k}(E))=p_i(E)+2\sum_{j=0}^{k-1}w_{2j}(E)\cup w_{4k-2j}(E)\mod 4. \tag{222}$$

Here, the pontryagin square is a map into $H^*(E;\mathbb{Z}_4)$; hence the mod 4 on the RHS. In particular,

$$P(w_2(E))=p_1(E)+2w_4(E)\mod 4. \tag{223}$$

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z,w are both regarded as tuples of their real and imaginary parts. Now define the following two maps:

$$\mathcal{I},\mathcal{I}^*:E\to E_\mathbb{R}\oplus E_\mathbb{R},\quad \mathcal{I}(z)=(z,-iz),\quad \mathcal{I}^*(z)=(z,iz). \tag{216}$$

Just to be clear, if $z=x+iy$, we have $\mathcal{I}(z)=((x,y),(y,-x))\in E_\mathbb{R}\oplus E_\mathbb{R}$. Anyway, note that

$$J(\mathcal{I}(z))=J(z,-iz)=(iz,-z)=\mathcal{I}(iz), \tag{217}$$

so that $\mathcal{I}$ is complex linear with respect to the complex structure on $E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}$ (again, just to be clear, $(iz,-z)=((-y,x),(-x,-y))$ as an element of $E_\mathbb{R}\oplus E_\mathbb{R}$). Similarly,

$$J(\mathcal{I}^*(z))=J(z,iz)=(-iz,z)=-(iz,-z)=-\mathcal{I}^*(iz), \tag{218}$$

so that $\mathcal{I}^*$ is complex anti-linear with respect to the complex structure on $E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}$. Note however that any $(z,w)\in E_\mathbb{R}\oplus E_\mathbb{R}\cong E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}$ can be written uniquely as

$$(z,w)=\frac{1}{2}\left[\mathcal{I}(z+iw)+\mathcal{I}^*(z-iw)\right]. \tag{219}$$

This means that we can take any element in $E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}$ and identify it with one element in the pre-image of $\mathcal{I}$, and one element in the pre-image of $\mathcal{I}^*$. The former is just E, while the latter is $\bar{E}$, since the complex structure on the pre-image of $\mathcal{I}^*$ is opposite to that of the complex structure on $E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}$. So finally we can conclude that

$$E_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}\cong E\oplus\bar{E} \tag{220}$$

as claimed.

Additionally, from the above general formula, we can conclude that

$$p_k(E) = P(w_{2k}) \mod 2. \tag{224}$$

Thus the mod 2 reduction of the Pointryagin class p_k is *not* given by w_{4k}, but rather by the square of w_{2k}.

This is easy to prove if we are dealing with the realification of a complex vector bundle. In that case, we can follow our earlier manipulations and write

$$p_k(E_{\mathbb{R}}) = c_{2k}(E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = c_{2k}(E \oplus \bar{E}) = [c(E) \wedge c(\bar{E})]_{2k}, \tag{225}$$

where the brackets instruct us to take the degree $2k$ part. Expanding out the RHS,

$$p_k(E_{\mathbb{R}}) = 2\sum_{j=1}^{k-1} c_{2k-2j}(E) \wedge c_{2j}(E) + c_k(E) \wedge c_k(E), \tag{226}$$

where all the terms involving odd Chern classes have canceled. Working mod 4, and using that the mod 2 reduction of the Chern classes for a complex vector bundle E are

$$c_k(E) = w_{2k}(E_{\mathbb{R}}) \mod 2, \tag{227}$$

we obtain (222) (we also need to use the Pontryagin square as the appropriate cohomology operation on the mod 2 reduction of $c_k(E)$).

This proof relied on using the Whitney product formula for the Chern classes of a direct sum of *complex* bundles. Thus we generically won't have $E \otimes \mathbb{C} \cong F \oplus \bar{F}$ for some complex F, unless E happens to be the realification of a complex bundle (viz. $E = F_{\mathbb{R}}$). Since in this case we can't apply the Whitney product theorem, the proof is trickier[18]. The mod 2 version of (222), however, *is* easy to prove when E is a real vector bundle. Indeed, letting E be real, we have

$$p_k(E) = c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) = c_{2k}(E \oplus iE). \tag{228}$$

As we mentioned earlier, we can't apply the Whitney product formula on this, since E and iE are *real* vector bundles. Anyway, writing $\rho_n(\cdot)$ for the reduction of $\cdot$ mod n, we have

$$\rho_2[p_k(E)] = \rho_2[c_{2k}(E \oplus iE)] = w_{4k}((E \oplus iE)_{\mathbb{R}}) = w_{4k}(E \oplus E) = P(w_{2k}(E)) \mod 2, \tag{229}$$

where in the last step we have used the Whitney product formula mod 2 on $w_{4k}(E \oplus E)$. If we change to working mod 4 it really seems like the natural thing that appears should be the rest of the terms in (222), but as of now I don't have a full proof.

For example, consider $\mathbb{CP}^n$, which has nontrivial cohomology in even degrees[19] The Chern

[18] Actually from the Chern-Weil point of view, such a product formula kind of makes sense, even though $c(E)$ is only defined for E complex. However since we are interested in torsion phenomena, thinking about expressions like $\mathrm{Tr}[F \wedge F]$ isn't really the road we want to take.

[19] This comes from the cell decomposition of $\mathbb{CP}^n$, which we can motivate in the following way. First of all, $\mathbb{CP}^n$ consists of nonzero $n+1$ tuples $(z_0, \ldots, z_n)$ modulo scaling by elements in $\mathbb{C}$. When $z_0 \neq 0$, we can normalize by it and get $(1, \widetilde{z}_1, \ldots, \widetilde{z}_n)$, with $\widetilde{z}_i = z_i/z_0$. This space is $\mathbb{C}^n$, and it covers all of $\mathbb{CP}^n$ except at "infinity", where $z_0 = 0$. Thus to cover $\mathbb{CP}^n$, we need to attached the space of all $(0, z_1, \ldots, z_n)$ to the $\mathbb{C}^n$ space of nonzero z_0. But we still have a re-scaling freedom on the $(0, z_1, \ldots, z_n)$ that we place at infinity, so $\mathbb{CP}^n$ is realized by taking $\mathbb{C}^n$ and gluing it up with a copy of $\mathbb{CP}^{n-1}$ at infinity. Iterating this process, which stops at $\mathbb{CP}^0 = \mathbb{C}^0$, we see that

$$\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^0, \tag{230}$$

which gives us the cell decomposition we're familiar with.

classes of the tangent bundle are determined by the Whitney sum formula by taking the product of $n+1$ $\mathbb{C}$ line bundles:

$$c(T\mathbb{CP}^n) = (1+z)^{\wedge(n+1)}, \tag{231}$$

where z is the generator for $H^2(\mathbb{CP}^n;\mathbb{Z})$ and we have to remember to set $c_i = 0$ if $i > n$ (i.e. if $i = n+1$). The SW classes are then obtained by taking the mod-2 reduction of this (the only nonzero SW classes are even, since the odd SW classes of the realification of a complex bundle vanish). For example, take $n = 2$. Then we see that

$$w_2(T\mathbb{CP}^2) = z, \qquad w_4(T\mathbb{CP}^2) = z \cup z, \tag{232}$$

where we are implicitly working mod 2. For $n = 3$ all of the coefficients in the binomial expansion $(1,4,6,4,1)$ are even except the first and the last, and so the total SW class is (there is no contribution from the last 1 since z^4 is an 8-form, which is too big to live on $\mathbb{CP}^3$)

$$w(\mathbb{CP}^3) = 1. \tag{233}$$

Finally for $n = 4$, the binomial expansion is $(1,5,10,10,5,1)$ and so

$$w(T\mathbb{CP}^4) = 1 + z + z^{\cup 4}. \tag{234}$$

Now we can use our characteristic classes formula to find out what the pontryagin square of w_2 is in each of these cases, using our knowledge of p_1. The Pointryagin classes are determined from the Chern classes (remember that we have to complexify the bundle first!)

$$c(T\mathbb{CP}^n \otimes_{\mathbb{R}} \mathbb{C}) = c(T\mathbb{CP}^n \oplus (T\mathbb{CP}^n)^*) = c(T\mathbb{CP}^n) \wedge c((T\mathbb{CP}^n)^*). \tag{235}$$

Since $c((T\mathbb{CP}^n)^*)$ is the same as $c(T\mathbb{CP}^n)$ but with the signs of all the odd degree terms flipped, $c((T\mathbb{CP}^n)^*) = (1-z)^{n+1}$. So then since $p_k = (-1)^k c_{2k}$, we have

$$1 - p_1 + p_2 - \cdots = (1+z)^{n+1}(1-z)^{n+1} = (1-z^2)^{n+1}, \tag{236}$$

where we remember to set $p_k = 0$ if $k \geq \lceil (n+1)/2 \rceil$ by dimensionality reasons. We see that the minus signs in the p_k with k odd on the LHS will always match a minus sign on the RHS for z^{2k}. Thus the non-zero p_k are

$$p_k(T\mathbb{CP}^n_{\mathbb{R}}) = \binom{n+1}{k} z^2, \qquad k < \lceil (n+1)/2 \rceil. \tag{237}$$

So for example,

$$p(T\mathbb{CP}^2_{\mathbb{R}}) = 1 + 3z^2, \quad p(T\mathbb{CP}^3_{\mathbb{R}}) = 1 + 4z^2, \quad p(T\mathbb{CP}^4_{\mathbb{R}}) = 1 + 5z^2 + 10z^4. \tag{238}$$

Now we can finally check our formula for p_1 mod 4 and the Pontryagin square. Our formula $P(w_2) = p_1 - 2w_4 \mod 4$ tells us that

$$P(w_2(T\mathbb{CP}^2_{\mathbb{R}})) = 3z^2 - 2z^2 \mod 4, \quad P(w_2(T\mathbb{CP}^3_{\mathbb{R}})) = 4z^2 - 2\cdot 0 \mod 4, \tag{239}$$

and

$$P(w_2(T\mathbb{CP}^4_{\mathbb{R}})) = 5z^2 - 2\cdot 0 \mod 4. \tag{240}$$

Thus for $\mathbb{CP}^2$ and $\mathbb{CP}^4$, since $w_2 = z$, we get $P(w_2) = w_2^2$, the usual cup square. For $\mathbb{CP}^3$ we get $P(w_2) = 0$, which we needed to get since $\mathbb{CP}^3$ is spin and has $w_2 = 0$. So, everything checks out!

18 *February 6 — Chirality of instanton-induced zero modes in four dimensions*

Consider some massless fermions coupled a background gauge field. The index theorem tells us the net chirality $\mathrm{ind}(\not{D}_A) = \nu_+ - \nu_-$ of the zero modes of the Dirac operator is determined by the instanton number (we are ignoring the gravitational contribution). However, it only tells us the difference in the left- and right-chirality zero modes; it does not tell us how many zero modes there are. Argue however that in an instanton field such that $\nu_+ - \nu_- = n$, we actually have $\nu_+ = n, \nu_- = 0$.

Solution:

Consider a zero mode of the Dirac operator with chirality $\pm$:

$$\not{D}_A(1 \pm \bar{\gamma})\psi_\pm = 0. \tag{241}$$

Now hit this with $\not{D}_A$, and use (the $\circ$ notation here is meant to emphasize that the derivatives in the left $\not{D}$ act on the A in the right $\not{D}$)

$$\begin{aligned}
\not{D}_A \circ \not{D}_A &= \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - A^a_\mu A^b_\nu \gamma^\mu\gamma^\nu T^a T^b - iA_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\} - i(\partial_\mu A_\nu)\gamma^\mu\gamma^\nu \\
&= \partial_\mu\partial^\mu - A_\mu A^\mu - i\partial_\mu A^\mu - \frac{1}{2}A^a_\mu A^b_\nu[\gamma^\mu,\gamma^\nu]T^aT^b - 2iA_\mu\partial^\mu - \frac{i}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\gamma^\mu\gamma^\nu \\
&= \partial_\mu\partial^\mu - A_\mu A^\mu - i\partial_\mu A^\mu - \frac{1}{2}if^{abc}A^b_\mu A^c_\nu T^a - \frac{i}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\gamma^\mu\gamma^\nu \\
&= (\partial_\mu - iA_\mu)^2 - \frac{i}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu,
\end{aligned} \tag{242}$$

where $(\partial_\mu - iA_\mu)^2$ means that the ∂_μ acts on the A_μ as well.

Using this, we have

$$0 = \not{D}_A \circ \not{D}_A(1 \pm \bar{\gamma})\psi_\pm = \left[(\partial_\mu - iA_\mu)^2 - \frac{i}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\right](1 \pm \bar{\gamma})\psi_\pm. \tag{243}$$

Now in Euclidean signature, $\bar{\gamma} = \prod_\mu \gamma^\mu$ (all of the γs, including $\bar{\gamma}$, are Hermitian). Thus we have

$$\gamma^\mu\gamma^\nu\bar{\gamma} = -\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\gamma^\lambda\gamma^\sigma. \tag{244}$$

We now multiply the field strength term in (711) by $(1 \pm \bar{\gamma})/2$, which is allowable since it's a projector. Thus the putative zero mode satisfies

$$\left[(\partial_\mu - iA_\mu)^2 - \frac{i}{2}F_{\mu\nu}\Sigma^{\mu\nu}_\pm\right](1 \pm \bar{\gamma})\psi_\pm = 0, \tag{245}$$

where we have defined

$$\Sigma^{\mu\nu}_\pm = \gamma^\mu\gamma^\nu \mp \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\gamma^\lambda\gamma^\sigma. \tag{246}$$

Note that Σ_+ is anti-self-dual while Σ_- is self-dual (note to self: missed a sign?):

$$(\star\Sigma_\pm)^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\Sigma_\pm^{\lambda\sigma} = \mp\Sigma_\pm^{\mu\nu}. \tag{247}$$

Let us write $F = \mathcal{F}_+ + \mathcal{F}_-$, where $\mathcal{F}_+ = (F + \star F)/2$ is self-dual and $\mathcal{F}_- = (F - \star F)/2$ is anti-self-dual (we are in Euclidean signature, with $\star^2 = (-1)^{p(4-p)}$ on p-forms). Now, the contraction of a SD with an ASD form vanishes, since $A \wedge \star B = B \wedge \star A$ means that $A \wedge \star B = -A \wedge \star B$ if only one of A, B is ASD. Thus we can write

$$\left[(\partial_\mu - iA_\mu)^2 - \frac{i}{2}\mathcal{F}_\mp^{\mu\nu}\Sigma_\pm^{\mu\nu}\right](1 \pm \bar{\gamma})\psi_\pm = 0. \tag{248}$$

Now we consider a field for which

$$\frac{1}{8\pi^2}\int \mathrm{Tr}[F \wedge F] = n. \tag{249}$$

Then

$$n = \frac{1}{8\pi^2}\int \left(\mathrm{Tr}[\mathcal{F}_+ \wedge \mathcal{F}_+] - |\mathrm{Tr}[\mathcal{F}_- \wedge \mathcal{F}_-]|\right). \tag{250}$$

Here we have used $\int \mathcal{F}_+ \wedge \mathcal{F}_- = \int \star\mathcal{F}_+ \wedge \star\mathcal{F}_- = -\int \mathcal{F}_+ \wedge \mathcal{F}_- = 0$, and the fact that $0 < \int \mathcal{F}_- \wedge \star\mathcal{F}_- = -\int \mathcal{F}_- \wedge \mathcal{F}_-$. Thus we see that the self-dual part of the field strength contributes positively to the instanton number, while the anti-self-dual part contributes negatively. Both SD and ASD parts contribute positively to the $\int \mathrm{Tr}[F \wedge \star F]$ YM action. This means that if we want to look for a minimal-action configuration with a given instanton number, we can restrict ourselves to purely SD or purely ASD fields[20].

Let us suppose $n > 0$, so that the minimal action configuration has $\mathcal{F}_+ \neq 0, \mathcal{F}_- = 0$. Then we see that a putative ψ_+ zero-mode obeys

$$(\partial_\mu - iA_\mu)^2\psi_+ = 0, \tag{252}$$

since there is no anti-self-dual field strength contribution. Now $(\partial_\mu - iA_\mu)$ is anti-Hermitian, so $(\partial_\mu - iA_\mu)^2$ is Hermitian with $\mathbb{R}$ eigenvalues. Furthermore, it is negative-definite, since the eigenvalue of an eigenspinor of $(\partial_\mu - iA_\mu)$ is purely imaginary (by anti-Hermitian-ness). Thus since all the eigenvalues of $(\partial_\mu - iA_\mu)$ have the same sign and only the non-normalizable choice $\psi_+ = 0$ has a zero eigenvalue, there are no normalizable solutions to the above equation, and we conclude that there are no + zero modes. Similarly, if we were to choose $n < 0$ so that

[20]This is being a bit glib, since there may be instantons with $n \neq 0$ and field strengths which are not purely SD or ASD, but which are still solutions to the equations of motion (just not minimal action ones). For example, consider $SU(N \geq 4)$ gauge theory. Then we can consider the field configuration

$$A^{SU(N)} = 0_{N-4\times N-4} \oplus A_{SD,k}^{SU(2)} \oplus A_{ASD,l}^{SU(2)}, \tag{251}$$

where $A_{SD,k}^{SU(2)}$ is a configuration with self-dual field strength and $SU(2)$ instanton number k, and similarly for $A_{ASD,l}^{SU(2)}$. This configuration has instanton number $k - l$ and is a solution to the equations of motion, but is not purely SD or ASD.

the minimal action configuration for the gauge fields resulted in a purely ASD field strength, we would find $(\partial_\mu - iA_\mu)^2\psi_- = 0$, meaning that there are no $-$ zero modes.

So, at least for minimal-action purely SD / ASD field configurations, not only does the instanton number determine the net difference in $+$ and $-$ chirality zero modes, but it also tells us that $\nu_- = 0$ if the instanton number is positive, while $\nu_+ = 0$ if the instanton number is positive, and so the chiral difference in zero modes is actually equal to the (signed) total number of zero modes. Now we can imagine slowly deforming the background fields away from the minimal action purely SD / ASD configuration, while keeping the instanton number fixed. Since the number of $\pm$ chirality zero modes cannot change continuously, we expect that all configurations with a given instanton number, not just the purely SD / ASD ones, have a total number of zero modes equal to the chiral difference in zero modes.

Finally, a comment on reflections. We know that reflections take left-handed fermions to right-handed ones—therefore in order for the zero mode situation to be invariant under reflections, we must have that a reflection takes a SD 2-form to an ASD 2-form. Indeed, this is true: one can see this by realizing that if we consider a reflection of the α coordinate, then

$$F_{\mu\nu} \mapsto [I_\alpha]_\mu^\lambda [I_\alpha]_\nu^\rho F_{\lambda\rho} \qquad [I_\alpha]^\lambda_\mu \equiv \delta^\lambda_\mu - 2\delta^\alpha_\mu \delta^\lambda_\alpha. \tag{253}$$

The result then follows after a little bit of algebra.

19 *February 7 — GSD for K matrix CS theory from phase space*

Today is a quickie: we show a cool way that I hadn't seen in the literature (I'm sure it exists somewhere though) for how to get the $|\det K|^g$ GSD on a Riemann surface of genus G for a CS theory with K-matrix K.

Solution:

The strategy we will take will be to compute the volume of phase space. First we need the symplectic form. We get this by taking a variation of $K(a, da) = a_i \wedge da_j K^{ij}$, integrating by parts, and looking at the boundary term. Choosing a Cauchy slice Σ_g on which to quantize, the sympletic potential is

$$\omega = \frac{1}{4\pi}\int_{\Sigma_g} K(a, \delta a). \tag{254}$$

This gives us the symplectic potential as

$$\Omega = \frac{1}{4\pi}\int_{\Sigma_g} K(\delta a, \delta a) = \frac{1}{4\pi}\int_{\Sigma_g} K_{ij}\delta a^i \wedge \delta a^j. \tag{255}$$

Here the wedge product takes place in both actual space and in variational space. Thus e.g.

$$\delta a^i \wedge \delta a^j = \delta_1 a^i_x \delta_2 a^j_y - \delta_2 a^i_x \delta_1 a^j_y - \delta_1 a^i_y \delta_2 a^j_x + \dots \tag{256}$$

where δ_1, δ_2 are two (orthogonal) variations in variational space.

The space of solutions to the equations of motion is the space of flat connections on Σ_g. We can thus write

$$\delta a^i = \sum_{C_\mu \in H_1(\Sigma_g;\mathbb{Z})} \delta_\alpha \theta_\mu \widehat{C}_\mu, \tag{257}$$

where the Poincare dual is taken in Σ_g, so that $\widehat{C}_\mu$ is a flat 1-form. Here the coefficients $\theta_\mu \in [0, 2\pi)$, since when $\theta_\mu \in \overline{\mathbb{Z}}$, $\theta_\mu \widehat{C}_\mu$ (no sum) is a large gauge transformation with $\overline{\mathbb{Z}}$ holonomy around the cycle C_μ.

Before plugging this in to the symplectic form, we note that for Σ_g of genus g, $H_1(\Sigma_g; \mathbb{Z}) = \mathbb{Z}^{2g}$, with generators $C_{0,\rho}, C_{1,\rho}$ for $\rho \in \mathbb{Z}_g$, such that

$$C_{\alpha,\rho} \cap C_{\beta,\sigma} = \delta_{\rho,\sigma}\delta_{\alpha,\beta+1}(-1)^\alpha, \tag{258}$$

with $\alpha, \beta \in \mathbb{Z}_2$ and consequently where $\beta + 1$ is taken mod 2. The minus sign here is indeed because the $\widehat{C}_\mu$'s anticommute when wedged together.

Anyway, the point is that the homology of Σ_g is just g powers of the homology of the torus (since Σ_g is a connected sum). Thus, doing the integral, we can write Ω as

$$\Omega = \frac{1}{4\pi} K_{ij} \sum_{\rho=1,\ldots,g} \sum_{\alpha=0,1} (-1)^\alpha \delta\theta^i_{\rho,\alpha} \wedge \delta\theta^j_{\rho,\alpha+1}, \tag{259}$$

where now $\wedge$ only takes place in variational space. Since the K matrix is symmetric (it has to be so that the off-diagonal parts add pairwise to give mutual CS terms that are properly quantized as $\sum_{i<j} a^i da^j/2\pi$), the antisymmetry of the sum on α cancels the antisymmetry of the wedge product in variational space, and we can write

$$\Omega = \frac{1}{2\pi} K_{ij} \sum_{\rho=1,\ldots,g} \delta\theta^i_{\rho,0} \wedge \delta\theta^j_{\rho,1}. \tag{260}$$

Thus, for each torus ρ in the connected sum and for each flavor index i, the holonomy around the longitudinal cycle of the ρth torus, namely $\theta^i_{\rho,0}$, will constitute a position variable in the phase space. Its corresponding canonically conjugate momentum variable is then a linear combination (in flavor space) of the holonomies around the other cycle on the ρth torus, namely $\sum_j K_{ij}\theta^j_{\rho,1}$.

To find the GSD, we need to look at the symplectic volume of the ground state subspace. From the sum over ρ, we see that this factors into a product over each torus in the connected sum, each of which have the same phase space volume. Thus the GSD will be $GSD_{\Sigma_g} = (GSD_{T^2})^g$, where T^2 is the torus.

So, we just have to compute GSD_{T^2}. This is evidently

$$GSD_{T^2} = \int \bigwedge_{i=1,\ldots,\dim K} \frac{K_{ij}}{4\pi^2} \delta\theta^i_0 \wedge \delta\theta^j_1, \tag{261}$$

where the integral is in variational space. Here we have remembered to divide by a further factor of 2π since the phase space volume form for a single degree of freedom is $dq \wedge dp/h$, and in our units $h = 2\pi$.

To see how $\det K$ arises from this, we just have to use the antisymmetry of the variational wedge product. Since $\delta\theta^i_\alpha \wedge \delta\theta^i_\alpha = 0$, the only terms which survive the product are those which contain the full volume form

$$V = \bigwedge_{i\in 1,\dots,\dim K} \delta\theta^i_0 \wedge \bigwedge_{j\in 1,\dots,\dim K} \delta\theta^j_1. \tag{262}$$

So, bringing the $\delta\theta$'s in our expression for GSD_{T^2} into this form,

$$GSD_{T^2} = \frac{1}{(4\pi^2)^{\dim K}} \int \bigwedge_{i=1,\dots,\dim K} \delta\theta^i_0 \wedge \sum_{\{j_\lambda\}\in\mathbb{Z}^{\dim K}_{\dim K}} K_{1j_1}\delta\theta^{j_1}_1 \wedge K_{2j_2}\delta\theta^{j_2}_1 \wedge\cdots\wedge K_{\dim K j_{\dim K}}\delta\theta^{j_{\dim K}}_1. \tag{263}$$

Moving all of the $\delta\theta_1$'s into order, which we do at the cost of an ϵ symbol, we have

$$GSD_{T^2} = \frac{1}{(4\pi^2)^{\dim K}} \int V \sum_{\{j_\lambda\}\in\mathbb{Z}^{\dim K}_{\dim K}} \epsilon^{j_1,\dots,j_{\dim K}} K_{1j_1}K_{2j_2}\cdots K_{\dim K j_{\dim K}} = \frac{\det K}{(4\pi^2)^{\dim K}} \int V. \tag{264}$$

Now since each of δ^i_0, δ^i_1 can be varied from 0 to 2π, the integral over V exactly cancels the factor in the denominator. Thus we get $GSD_{T^2} = |\det K|$, and hence $GSD_{\Sigma_g} = |\det K|^g$, as required.

20 *February 8 — Flavor symmetries of fermions*

First, some notation. In the following, we will let

$$J \equiv (-iY)\otimes \mathbf{1}_N \tag{265}$$

be the symplectic form preserved by elements in $Sp(2n;\mathbb{K})$, where $\mathbb{K}$ is some field. The compact subgroup of $Sp(2n;\mathbb{C})$ will be denoted

$$Sp(n) \equiv U(2n)\cap Sp(2n;\mathbb{C}). \tag{266}$$

Anyway, the question that motivated this entry was the following. Consider

$$\mathcal{L} = \sum_{i=1}^N \bar\psi_i \not{D}_A \psi_i, \tag{267}$$

where A is the connection on some gauge group (which may be trivial). What is the global internal symmetry group of the above theory? On one hand, it's clearly $U(N)$. On the other hand, write

$$\psi_i = \chi_i + i\eta_i, \tag{268}$$

where χ_i and η_i are majoranas. Then since the action of $O(2N)$ preserves the commutation relations of the Majorannas and leaves $\mathcal{L}$ invariant, the global symmetry is clearly $O(2N)$. But $U(N)\subset O(2N)$, so—what's up? In the following, we will work in the basis

$$\Psi^T \equiv (\chi_1,\chi_2,\dots,\chi_N,\eta_1,\dots,\eta_N)^T. \tag{269}$$

What sorts of constraints can break the $O(2N)$ down to the naive $U(N)$? As far as mass terms go, the Dirac mass is $\bar{\psi}\psi = \bar{\chi}\chi + \bar{\eta}\eta$, since $\bar{\eta}\chi = \bar{\chi}\eta$. Thus the Dirac mass is invariant under $O(2N)$ and hence also under $U(N)$. The fermion number operator however is $\psi^\dagger\psi = 2 + 2i\chi^T\eta$, which up to a constant is $\Psi^T J\Psi$, and therefore is *not* preserved by the full $O(2N)$. This term is of course preserved by the diagonal $U(1)$, since the action of $U(1)$ is by $\Psi \mapsto U\Psi$, with $U = S \otimes \mathbf{1}_{N\times N}$ and $S \in SO(2)$. Since $-iY \in SO(2)$ and $SO(2)$ is Abelian, we have

$$U^T JU = (S^T \otimes \mathbf{1})(-iY \otimes \mathbf{1})(S \otimes \mathbf{1}) = (\mathbf{1} \otimes \mathbf{1})(-iY \otimes \mathbf{1})(S^T S \otimes \mathbf{1}) = J. \tag{270}$$

Now while $\psi^\dagger\psi$ is not preserved by $O(2N)$, it is preserved by the full $U(N)$ (as should be obvious from how it acts on the complex fermions). Moreover, $U(N) \subset O(2N)$ is the maximal subgroup that preserves $\psi^\dagger\psi$. Indeed, preserving $\psi^\dagger\psi$ means preserving $\Psi^T J\Psi$, which means that if $R \in O(2N)$ is to preserve $\Psi^T J\Psi$, we need $R^T JR = J \implies R \in Sp(2N;\mathbb{R})$. Thus the group of transformations that preserve complex fermion number is

$$O(2N) \cap Sp(2N;\mathbb{R}) \cong U(N). \tag{271}$$

Here the last equality is a manifestation of the 2-in-3 property, namely that the intersection

$$O(2N) \cap GL(N;\mathbb{C}) \cap Sp(2N;\mathbb{R}) = U(N), \tag{272}$$

and that actually $U(N)$ is equal to the intersection of any two of the three groups on the LHS. Why is this? Let $V \in Sp(2n;\mathbb{R})$. Then $V^T JV = J$. Alternatively, let $V \in GL(N;\mathbb{C})$. Then when viewed as a $2N \times 2N$ real matrix, in order to have a legit complex structure, we need V to commute with some matrix i, such that $i^2 = -\mathbf{1}$ and $Vi = iV$. If the complex structure and symplectic structure being considered are compatible,[21] is described by a splitting of then we need to take $i = J$. Finally, if $V \in O(2N)$, then $V^T V = \mathbf{1}$. Thus if $V \in O(2N) \cap GL(N;\mathbb{C}) \cap Sp(2N;\mathbb{R})$, then $V^T = V^{-1}, V^T JV = J$, and $V^{-1}JV = J$. Then we see that any two of these properties implies the third; hence the 2-in-3 property. We can realize the matrices in $U(N)$ in this way by using our knowledge of the Lie algebra of $Sp(N)$, and taking only the real part. So we claim that all the elements in $U(N)$ can be written as

$$O(2N) \cap Sp(2N;\mathbb{R}) = U(N) = \{\exp\left(\mathbf{1} \otimes A + iY \otimes S\right)\}, \tag{273}$$

where A is anti-symmetric and S is symmetric. It's easy to check that the above matrices are orthogonal and preserve J. Have we missed any? No, let's count dimensions: there are $(N^2 - N)/2$ choices for A and $(N^2 + N)/2$ choices for S, and all of these choices give distinct elements in $O(2N) \cap Sp(2N;\mathbb{R})$. This adds up to N^2 total elements, which is the same as

[21] A complex structure on a vector space $\mathcal{W}$ means we can realize it as a direct sum $\mathcal{W} \cong V \oplus V$, where V is real (so e.g. the tangent bundle TM has an [almost] complex structure). A symplectic structure on $\mathcal{W}$ means that we can decompose it as $\mathcal{W} \cong W \oplus W^*$ for W real (this means choosing coordinates and momentum; e.g. T^*M has a symplectic structure with the dimensions coming from M being the coordinates and those coming from the fiber being the momenta), with the symplectic form being given by $\omega(v \oplus f, u \oplus g) = f(u) - g(v)$. The compatibility of the complex and symplectic structures means that we can choose V (known as the real subspace) and W (known as the Lagrangian subspace) to be equal, with the symplectic form corresponding to multiplication by i.

the number of generators for $U(N)$. So indeed, all the elements in $U(N)$ can be written as real matrices in this way.

Anyway, enough with that digression. Returning to the fermion problem, we see that $O(2N)$ contains elements which do not preserve $\psi^\dagger\psi$. Thus if we restrict to transformations that preserve the fermion number, we get that the symmetry groups is the naive $U(N)$. As example, consider the matrix $Z \otimes \mathbf{1} \in O(2N)$. In the Ψ basis this sends all the χ's to themselves, and it multiplies the η's by minus signs. Thus $Z \otimes \mathbf{1} : \psi_i \mapsto \psi_i^\dagger$, and so $Z \otimes \mathbf{1}$ is charge conjugation. This of course should preserve $\psi^\dagger\psi$, which it doesn't: while $Z \otimes \mathbf{1} \in O(2N)$, $Z \otimes \mathbf{1} \notin Sp(2N;\mathbb{R})$ and so $Z \otimes \mathbf{1} \notin U(N)$.

Another way to understand how the $O(2N) \to U(N)$ restriction of the symmetry group can come about is to remember that complex numbers are not simply two copies of $\mathbb{R}$: there is a complex structure that relates the two copies. Consider multiplication by i, $\psi_i \to i\psi_i$. We see that in the Ψ basis, this acts as J. Thus $i = J$ when acting on the Majorana fermions. Now if our flavor symmetry transformation R does not involve complex conjugation, then $Ri\psi = iR\psi$. But when written in terms of Majoranas, this means that $RJ = JR$, and so from the orthogonality of R, we have $R \in Sp(2N;\mathbb{R})$, and thus from the 2-in-3 property we know that $R \in U(N)$ (another way to say this is that $RJ = JR$ is the requirement of the existence of a complex structure, and tells us that $R \in GL(N;\mathbb{C})$. But from the 2-in-3 property, the orthogonality of R then implies that R is in $Sp(2N;\mathbb{R})$ as well. Since R is then both orthogonal and symplectic / complex structure preserving, must have $R \in U(N)$). So if we want R to preserve the complex structure, i.e. for R to not be anti-unitary, then R must be in $U(N)$ (which sounds kind of tautological).

$SU(2)$ gauge theory with N Dirac fermions

We now try to understand the global symmetry of N Dirac fermions, all coupled to an $SU(2)$ gauge field in the fundamental. This comes from wanting to understand the construction in [1]. The naive guess for what the internal part of the global symmetry should be, namely $U(N)/\mathbb{Z}_2$ (or maybe $[U(N)/\mathbb{Z}_2] \rtimes \mathbb{Z}_2$ for charge conjugation or something) is not correct. In fact the internal part of the global symmetry is actually $PSp(N)$!

Let's see how this comes about. Let $\psi_i = (\psi_{i\uparrow}, \psi_{i\downarrow})^T$ be one of the Dirac fermions in the fundamental of $SU(2)$. A single fermion in $SU(2)_f$ can be built from four majoranas. We can build it as a matrix field as follows:

$$\mathcal{X}_i = \frac{1}{\sqrt{2}}(\chi_i^1 \mathbf{1} + i\chi_i^a \sigma^a), \tag{274}$$

with $a \in \{x, y, z\}$. We build the constituent complex fermions from the Majoranas so that (dropping the flavor index for simplicity)

$$\mathcal{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^1 + i\chi^z & i\chi^x + \chi^y \\ i\chi^x - i\chi^y & \chi^1 - i\chi^z \end{pmatrix} = \begin{pmatrix} \psi_\uparrow & \psi_\downarrow \\ -\psi_\downarrow^\dagger & \psi_\uparrow^\dagger \end{pmatrix}. \tag{275}$$

With this one can check that $\mathrm{Tr}[\bar{\mathcal{X}} \not{D}_A \mathcal{X}]$ gives the correct Dirac Lagrangian. The mass term $\mathrm{Tr}[\bar{\mathcal{X}}\mathcal{X}]$ is $\sum_\alpha \bar{\chi}^\alpha \chi^\alpha$, as expected.

Consider the right action on $\mathcal{X}$ by $SU(2)$. Right multiplication by e.g. $e^{i\alpha Z}$ does

$$\mathcal{X} \mapsto \mathcal{X} e^{i\alpha Z} = \begin{pmatrix} e^{i\alpha}\psi_\uparrow & e^{-i\alpha}\psi_\downarrow \\ -e^{i\alpha}\psi_\downarrow^\dagger & e^{-i\alpha}\psi_\uparrow^\dagger \end{pmatrix}, \tag{276}$$

which is just what a gauge rotation about the z axis in $SU(2)$ should do. So, we see that the $SU(2)$ we want to gauge is the right action on $\mathcal{X}$ by $SU(2)$.

The left action then parametrizes the system's global flavor symmetry. In order for $\bar{\mathcal{X}} \not{D}_A \mathcal{X}$ to be left invariant, the U in $\mathcal{X} \mapsto U\mathcal{X}$ must be unitary, and since there are N flavors of Dirac fermions, $U \in U(2N)$. However, there is an additional restriction. Indeed, consider the fact that

$$\mathcal{X}^\dagger = (Y \otimes \mathbf{1})\mathcal{X}^T(Y \otimes \mathbf{1}). \tag{277}$$

Now take $\mathcal{X} \mapsto U\mathcal{X}$. Then we need

$$\mathcal{X}^\dagger U^\dagger = (Y\otimes\mathbf{1})\mathcal{X}^T U^T (Y\otimes\mathbf{1}) \implies \mathcal{X}^\dagger = (Y\otimes\mathbf{1})\mathcal{X}^T(Y\otimes\mathbf{1})^2 U^T(Y\otimes\mathbf{1})U = \mathcal{X}^\dagger(Y\otimes\mathbf{1})U^T(Y\otimes\mathbf{1})U. \tag{278}$$

In particular, this means that

$$U^T J U = J \implies U \in Sp(N) = U(2N) \cap Sp(2N;\mathbb{C}). \tag{279}$$

So, the global symmetry on the left action is actually $Sp(N)$.

Actually this is not completely true, since it may happen that elements of the global symmetry acting from the left act in the same way as elements of the gauge group acting from the left. Clearly this is true for $-\mathbf{1}$, which acts the same both as a $Sp(N)$ element from the left and an $SU(2)$ element from the right. But this is the only common element shared by the two actions. Indeed, consider a given element of $Sp(N)$ acting from the left, and ask if it is equivalent to an element of $SU(2)$ acting from the right. Since the $SU(2)$ acts in the same way on each Dirac fermion, we just need to look for elements of $Sp(N)$ that are diagonal on the flavor index, and so we can restrict ourselves to a single flavor wolog, and take the left action to be that of $Sp(1) = SU(2)$. Then consider the $U(1)$ rotation $e^{i\alpha Z}$ acting from the left. This multiplies both $\psi_\uparrow$ and $\psi_\downarrow$ by the same phase. This can never be done by an element of $SU(2)$ acting on the left: the only element which just multiplies $\psi_\uparrow$ and $\psi_\downarrow$ by phases does so in a gauge-invariant way, namely by multiplying $\psi_\uparrow$ by $e^{i\alpha}$ and $\psi_\downarrow$ by $e^{-i\alpha}$. So the left action by $e^{i\alpha Z}$ is only equivalent by the right action of something in $SU(2)$ if $e^{i\alpha Z} = -\mathbf{1}$. Since every element in the left $SU(2)$ can be written as $e^{i\alpha Z}$ in the right choice of basis, every element of the left $SU(2)$ action (except $-\mathbf{1}$) must also not be expressible as the action of some $SU(2)$ element from the right. Thus only the $-\mathbf{1}$ gets modded out, and the global symmetry acting on the left is in fact $PSp(N) = Sp(N)/\mathbb{Z}_2$.

This is kind of surprising! For example, take $N = 1$: the internal part of the global symmetry is then $PSp(1) = SU(2)/\mathbb{Z}_2 = SO(3)$. If we just looked at the Lagrangian $\bar\psi \not{D}_A \psi$, we might have thought that the global internal symmetry was $U(1)$, or maybe $O(2) = U(1) \rtimes \mathbb{Z}_2$ after including charge conjugation. But in fact the real global symmetry is bigger! This is because the conclusion that the symmetry is $U(1)$ came from requiring the global symmetry to act identically on both of the components of the $SU(2)$ doublet. This is a natural thing to do, since the global symmetry has to commute with the action of the gauge group. But we see from this example that we can actually have the global symmetry act nontrivially on the different components in the $SU(2)$ doublet! For example, consider the left action by $e^{i\alpha Z}$. This is the diagonal $U(1)$ that we would have guessed to be the naive global symmetry. But what about the left action by $e^{i\alpha X}$? One checks that this sends e.g $\psi_\uparrow \mapsto i\psi_\downarrow^\dagger, \psi_\downarrow \mapsto i\psi_\uparrow^\dagger$: so it mixes the two components of the doublet, but also charge-conjugates them; this allows it to be gauge invariant.

Thus the action of charge conjugation is built in to the $PSp(N)$ symmetry. Or more precisely, it's mixed up between the $PSp(N)$ and $SU(2)$ actions.

The fact that we get a $PSp(N)$ global symmetry can be understood through the decomposition

$$SO(4N) \supset \frac{SU(2) \times Sp(N)}{\mathbb{Z}_2}. \tag{280}$$

The relevance of this is that N Dirac fermions in the fundamental of $SU(2)$ can be written as $4N$ Majorannas, which are acted on by $SO(4N)$. The $Sp(N)$ factor in the above decomposition is the largest subgroup which commutes with the $SU(2)$, and so after gauging the $SU(2)$ we are left with $Sp(N)/\mathbb{Z}_2$'s worth of global symmetry.

Note that this inclusion is not an equality in general, as we check by computing dimensions: as a Lie algebra, $\dim \mathfrak{so}(4N) = (16N^2 - 4N)/2 = 8N^2 - 2N$, while

$$\dim[\mathfrak{su}(2) \times \mathfrak{sp}(N)] = 3 + (N^2 - N)/2 + 3(N^2 + N)/2 = 2N^2 + N + 3 \leq \dim \mathfrak{so}(4N). \tag{281}$$

In fact the equality does hold when $N = 1$ for which both Lie algebras are 6-dimensional, which is just a manifestation of

$$SO(4) = \frac{SU(2) \times Sp(1)}{\mathbb{Z}_2}, \tag{282}$$

since $Sp(1)$ has alias $SU(2)$[22]

Let's see how this decomposition works explicitly, at the level of Lie algebra generators. $Sp(N)$ is complex, and so in order to embed it in $SO(4N)$, we will need to send $i \mapsto J_2$, where now we are using the notation $J_2 \equiv -iY$. Recalling a diary entry from last year wherein

[22]Here's a way to see $SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2$: let $r^2 = x^2 + y^2 + z^2 + t^2 = 1$ define S^4, and consider the matrix $U = t\mathbf{1} + i(xX + yY + zZ)$. We will think of the coordinates $v = t + iz, w = y + ix$ as equivalent coordinates for the sphere in $\mathbb{C}^2$. With this,

$$U = \begin{pmatrix} v & w \\ -\bar{w} & \bar{v} \end{pmatrix} \tag{283}$$

Consider $[SU(2)_L \times SU(2)_R]/\mathbb{Z}_2$, with the first factor acting on U on the left and the second factor acting on the right. A generic element of $SU(2)$ is conjugate to $e^{i\theta Z}$. Acting on the left, these actions send

$$e^{i\theta Z_L} : v \mapsto e^{i\theta v}, \quad w \mapsto e^{i\theta w}, \qquad e^{i\theta Z_R} : v \mapsto e^{i\theta v}, \quad w \mapsto e^{-i\theta w}. \tag{284}$$

This means that the left $SU(2)$ action, after being conjugated so that it only has a Z part, rotates the tz plane and the xy plane by equal angles, while the right $SU(2)$ action rotates them by opposite angles. Since both actions preserve $\det U = 1$, they are symmetries of S^4. Furthermore since they only rotate planes (pairs of basis vectors) instead of individual basis vectors, they are orientation preserving (okay, also since they have $\det = 1$). We can generate all of $SO(4)$ by rotating arbitrary planes, and so after modding out by the common $-\mathbf{1}$ to both actions, we get the desired isomorphism.

the generators for $\mathfrak{sp}(N)$ were written down, we have that when embedded in $\mathfrak{so}(4N)$, the $\mathfrak{sp}(N)$ generators are

$$\mathfrak{sp}(N) \ni (A_N \otimes \mathbf{1}_2 + S_N^y \otimes iY) \otimes \mathbf{1}_2 + (S_N^x \otimes X + S_N^z \otimes Z) \otimes J_2, \tag{285}$$

where A_N is an antisymmetric $N \times N$ matrix, the S^a are symmetric $N \times N$ matrices, and $\mathbf{1}_2$ is the 2×2 unit matrix.

What about the $SU(2)$ factor? To embed this in $SO(4N)$, we need to turn the symmetric generators X, Z into antisymmetric matrices. We do this by tensoring with J_2. Since the exponential map is $\exp(\mathfrak{so}(4N)) = SO(4N)$ (no factor of i), we also want the $SU(2)$ generators to be i times the normal physicist-convention $SU(2)$ generators. One sees that the following three generators

$$\sigma^1 = \mathbf{1}_N \otimes J_2 \otimes X, \quad \sigma^2 = \mathbf{1}_N \otimes \mathbf{1}_2 \otimes iY, \quad \sigma^3 = \mathbf{1}_N \otimes J_2 \otimes Z \tag{286}$$

obey $[\sigma^a, \sigma^b] = \epsilon^{abc}\sigma^c$, and hence generate an $SU(2) \subset SO(4)$ subgroup. Furthermore, we see that all of these generators commute with the $\mathfrak{sp}(N)$ generators, telling us that indeed, $[SU(2) \times Sp(N)]/\mathbb{Z}_2 \subset SO(4N)$, where the quotient is because $-\mathbf{1}_{2N}$ is in both factors. Furthermore we see that $Sp(N)$ is the largest subgroup that commutes with $SU(2)$, so that when e.g. $SU(2)$ is gauged, $PSp(N)$ is the global symmetry that remains.

21 *February 11 — Properties of Clifford algebras, their representations, and their actions on fermions (i.e. how spacetime reflections act on fermions)*

Today we're going to talk about how spacetime symmetries, in particular spacetime reflections, act on fermions. We will try to be as general as possible, covering both even and odd spacetime dimensions, real and imaginary time, and different choices of signature. This was motivated by (finally!) feeling like I was understanding some of the material in [2] and wanting to put things into words / elaborate on pedantic details that aren't in the paper.

Solution:

A word on notation: in this diary entry we will let $J = -iY$ and let $\mathcal{K}$ denote complex conjugation. $d = s + t$ will denote the dimension of spacetime, with s the number of spacelike-signature indices (positive signs in $\eta_{\mu\nu}$) and t the number of timelike-signature indices (negative signs in $\eta_{\mu\nu}$). I guess this mnemonic is showing our prejudice for mostly positive signature, but oh well. We will assume throughout that there is only one time coordinate, so that the signature is either $(+,-,-,\dots)$, $(-,+,+,\dots)$, $(+,+,+,\dots)$, or $(-,-,-,\dots)$.

Therefore, statements like "the eigenvalue of a spinor under parity / time reversal" are totally meaningless: to talk about orientation-reversing symmetries, one must talk about their action on pinors, not spinors.

Representation theory prelude

Suppose the representation $R : G \to \mathrm{Aut}(V)$ that the fermions transform in is isomorphic to its complex conjugate $\bar{R}$ (here we will assume that V is a complex vector space). Then there exists some map $\mathscr{J}$.

Thus $\mathscr{J}$ must be an anti-linear map. Since $\mathscr{J}^2$ is a linear map taking R to itself, by Schur's lemma we must have $\mathscr{J}^2 \propto \mathbf{1}$; wolog we can re-scale $\mathscr{J}$ so that the constant of proportionality is ± 1 (but the sign cannot be eliminated with a rescaling by i due to the anti-linear nature of $\mathscr{J}$).

Suppose that $\mathscr{J}^2 = +\mathbf{1}$. In this case, we say that R is a *real* representation. This is because... Although we have been representing the group action as automorphisms on a complex vector space V, if the representation is real, we can also restrict the representation to a map $G \to \mathrm{Aut}(V_{\mathbb{R}})$, with $V_{\mathbb{R}}$ a real vector space.

If χ is a fermion field transforming in a real representation, .

Now suppose that $\mathscr{J}^2 = -\mathbf{1}$. In this case, we say that R is a *pseudoreal* or *quaternionic* representation; the reason for the former moniker is because the relations $\mathscr{J}^2 = (i\mathbf{1})^2 = -\mathbf{1}$ and $\mathscr{J}(i\mathbf{1}) = -(i\mathbf{1})\mathscr{J}$ give us a quaternionic structure formed from $i = i\mathbf{1}, j = \mathscr{J}, k = i\mathscr{J}$.

Another way to say this is that since $R \otimes \bar{R} = \mathbf{1} \oplus \dots$,

In the quaternionic case, we cannot restrict the representation to an action on $\mathrm{Aut}(V_{\mathbb{R}})$.

Note that if R is a real representation and $\widetilde{R}$ is a pseudoreal representation, $R \otimes R$ and $\widetilde{R} \otimes \widetilde{R}$ are both real, while $R \otimes \widetilde{R}$ is pseudoreal (just by looking at they [anti]symmetry of the $\otimes$ of bilinear forms).

Finally, if $R \not\cong \bar{R}$, we say the representation is complex. In this case there is no invariant bilinear form that pairs two fields transforming in R, and so to construct an action we need to use two fields: one transforming in $\bar{R}$ and one transforming in R.

One annoying thing which makes the fact that Euclidean time stuff works seem very miraculous is that the character of the representations of $\mathrm{Spin}(D, 0)$ and $\mathrm{Spin}(D-1, 1)$ are not the same (although $\mathrm{Spin}(D-1, 1)$ and $\mathrm{Spin}(1, D-1)$ have isomorphic representations). Indeed, fixing the spacetime dimension as D, the character of the spin representations as represented over $\mathbb{C}$ are as follows:

D	Euclidean Time	Real Time
1	$\mathbb{R}$	$\mathbb{R}$
2	$\mathbb{C}$	$\mathbb{R}$
3	$\mathbb{H}$	$\mathbb{R}$
4	$\mathbb{H}$	$\mathbb{C}$
5	$\mathbb{H}$	$\mathbb{H}$

(287)

These are easy enough to check: in real time, the first three entries follow from using the matrix $J = -iY$ for the negative-signature coordinate and X, Z for the others, while e.g. the complexity of the $D = 4$ entry can be checked in any QFT book and the $D = 5$ entry is from $\mathrm{Spin}(4, 1) = Sp(1, 1)$. The Euclidean time entries follow from $\mathrm{Spin}(1) = \mathbb{Z}_2, \mathrm{Spin}(2) = U(1), \mathrm{Spin}(3) = SU(2), \mathrm{Spin}(4) = SU(2) \times SU(2)$, and $\mathrm{Spin}(5) = Sp(2)$. Note how $\mathrm{Spin}(D, 0)$ has the same type of spinor representation as $\mathrm{Spin}([D+2]-1, 1)$. The pattern can be continued up to higher D by using Bott periodicity (so that only D mod 8 is relevant).

Now from the above, we see that the representation theory of the spin group strongly depends on our choice of signature. Thus if G is the full symmetry group, the type of representation that G acts on the fermions with will depend on the choice of signature. This does not mean that the symmetries in the Lorentzian and Euclidean theories are different, it just means that the way in which the symmetries act is dependent on the choice of signature: this is true for the action of the spin group and for the action of the pin group (for example, a $\mathbb{R}$ time theory with T symmetry will continue to an $i\mathbb{R}$ time theory with T symmetry, but T will act differently in the $i\mathbb{R}$ time theory).

Generalities on representations of Clifford algebras

Let $\mathcal{C}(s,t)$ denote the Clifford algebra generated by the γ matrices γ_μ, $\mu \in \mathbb{Z}_{s+t}$, with

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{288}$$

where η has s positive diagonal entries and t negative ones. The γ^μ are always built out of tensor products of Pauli matrices (with possible factors of i), and so will always be unitary.

We will let $\mathcal{C}_\pm(d)$ represent the elements in $\mathcal{C}(s,t)$ that are even / odd under $\gamma_\mu \mapsto -\gamma_\mu \,\forall\, \mu$, respectively. $\mathcal{C}(s,t)$ splits as

$$\mathcal{C}(s,t) \cong \mathcal{C}_+(s,t) \oplus \mathcal{C}_-(s,t). \tag{289}$$

$\mathcal{C}_+(s,t)$ forms a subalgebra of $\mathcal{C}(s,t)$, which we identify as the algebra of $\text{Spin}(s,t)$. The representation theory of $\mathcal{C}(s,t)$ depends on the parity of d, and is sorted out by using the matrix

$$\bar{\gamma} \equiv i^{(d^2-d)/2+t} \prod_\mu \gamma_\mu. \tag{290}$$

The dumb factor out front is to ensure that $\bar{\gamma}^2 = \mathbf{1}$ in every dimension and signature, so that (in even dimensions) it can be employed in the projectors $(\mathbf{1} \pm \bar{\gamma})/2$. Note that in other diary entries $\bar{\gamma}$ doesn't have the prefactor, so be careful.

If $d \in 2\mathbb{Z}+1$, $\bar{\gamma}$ commutes with all the γ_μ, and so it will be $\pm\mathbf{1}$. The map $\gamma_\mu \mapsto -\gamma_\mu \,\forall\, \mu$ changes the sign of $\bar{\gamma}$ and preserves the $\mathcal{C}(s,t)$ anticommutaion relations: thus we get two distinct representations, differing my the signs of the γ matrices. When $d \in 2\mathbb{Z}$ we only have one irreducible representation, since the map which changes the sign of all the γ_μ can be obtained by conjugating with $\bar{\gamma}$: $\bar{\gamma}^\dagger \gamma_\mu \bar{\gamma} = -\gamma_\mu$.

$\text{Spin}(s,t)$ is defined as the elements in $\mathcal{C}_+(s,t)$ of unit norm. Since $\text{Spin}(s,t) \subset \mathcal{C}_+(s,t)$, the two distinct representations of $\mathcal{C}(s,t)$ when $d \in 2\mathbb{Z}+1$ are indistinguishable in $\text{Spin}(s,t)$, and in fact the spinor representation of $\text{Spin}(s,t)$ is irreducible. When $d \in 2\mathbb{Z}$, we can form chiral projectors with $\bar{\gamma} \in \mathcal{C}_+(d)$, which commutes with everything in $\text{Spin}(s,t)$. This means we can decompose the representation matrices of $\text{Spin}(s,t)$ in a form which is block-diagonal in the ± 1 eigenspaces of $\bar{\gamma}$, meaning that the spinor representation of $\text{Spin}(s,t)$ is reducible, with the spinor bundle splitting as $S_+ \oplus S_-$. Including spacetime reflections (elements in $\mathcal{C}_-(s,t)$) mixes sections of S_+ with those of S_- since the reflections all anticommute with $\bar{\gamma}$, and leaves us with only one irreducible representation.

The group $\mathrm{Pin}(s,t)$ is defined as the elements of $\mathcal{C}(s,t)$ of unit norm. We will be interested in a representation (the pinor representation) of $\mathrm{Pin}(s,t)$ on $\mathcal{C}(s,t)$, since this representation is what will allow us to determine how spacetime symmetries act on the fields (pinors) in the Lagrangian. This action is determined by the homomorphism

$$\Omega : \mathrm{Pin}(s,t) \to O(s,t) \tag{291}$$

defined for every $\Lambda \in \mathrm{Pin}(s,t)$ by

$$\Lambda^{-1}\gamma_\mu\Lambda = R_{\mu\nu}\gamma^\nu, \tag{292}$$

where $R_{\mu\nu} \in O(s,t)$. This transformation law is what allows $\bar{\psi}\partial\!\!\!/\psi$, with ψ a pinor, to be invariant under the action of Lorentz transformations[23], since it means that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector. The fact that $R_{\mu\nu} \in O(s,t)$ is required can be seen from requiring the anticommutation relations of the Clifford generators to be invariant under the action of $\mathrm{Pin}(s,t)$. From applying the action of $\mathrm{Pin}(s,t)$ on $\eta^{\mu\nu}$, we find

$$2\eta_{\mu\nu} = \{\gamma_\mu,\gamma_\nu\} \mapsto \Lambda^{-1}\{\gamma_\mu,\gamma_\nu\}\Lambda = R_{\mu\lambda}R_{\nu\sigma}\{\gamma^\lambda,\gamma^\sigma\} = 2R_{\mu\lambda}\eta^{\lambda\sigma}[R^T]_{\sigma\nu} \implies [R^TR]_{\mu\nu} = \eta_{\mu\nu}. \tag{293}$$

The homomorphism Ω is obviously not injective, since both Λ and $-\Lambda$ are associated with the same matrix R (this is why the Pin groups are $\mathbb{Z}_2$ extensions of the orthogonal groups). Whether or not Ω is surjective actually depends on whether d is even or odd. Indeed, consider the action on $\bar{\gamma}$. Then

$$\Lambda^{-1}\bar{\gamma}\Lambda = i^{(d^2-d)/2+t}R_{1\mu_1}R_{2\mu_2}\cdots R_{d\mu_d}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_d} = R_{1\mu_1}\cdots R_{d\mu_d}\epsilon^{\mu_1\dots\mu_d}\bar{\gamma} = (\det R)\bar{\gamma}. \tag{294}$$

Here we have used that if any $\mu_i = \mu_j$ but $i \neq j$, then we get a product $R_{i\mu_j}R_{k\mu_j}$ for $i \neq k$, which vanishes by the orthogonality of R. The transformation rule $\Lambda^{-1}\bar{\gamma}\Lambda = (\det R)\bar{\gamma}$ is why $\bar{\gamma}$ is a pseudoscalar.

Now in odd dimensions, $\bar{\gamma}$ commutes with all of $\mathcal{C}(s,t)$, and so in odd dimensions we have $\bar{\gamma} = \det R\bar{\gamma} \implies \det R = 1$. Thus in odd dimensions we can only generate an action of $SO(s,t)$. In even dimensions $\bar{\gamma}$ anticommutes with $\mathcal{C}_-(s,t)$, and so if we take Λ to be generated by something in $\mathcal{C}_-(s,t)$, we can pick up matrices with $\det R = -1$, and we get the full $O(s,t)$ algebra. This is basically coming from the fact that unlike in even d, in odd d the matrix $-\mathbf{1}$ is the generator of the $\det R = -1$ part of $O(d)$, which is central and so $O(d) = SO(d) \times \mathbb{Z}_2$: the action of $\mathrm{Pin}(s,t)$ is unable to generate the decoupled $\mathbb{Z}_2$ factor.[24]

Now specialize to $d \in 2\mathbb{Z}$ and look at how the various reflections in $O(s,t)$ are realized. First consider $P = 1 \oplus (-\mathbf{1}_{d-1})$, which acts as parity. We see that this is generated by $\Lambda_P = \gamma_0$. Now for $R_0 = -1 \oplus \mathbf{1}_{d-1}$, which reverses time: this is accomplished with $\Lambda_T = \prod_j \gamma_j$, where the product is over spatial indices. In general, $\Lambda = \gamma_\mu$ reflects all the axes of spacetime

[23]here $\bar{\psi}$ is a copinor so that $\psi \mapsto \Lambda\psi \implies \bar{\psi} \mapsto \bar{\psi}\Lambda^{-1}$. More on this later.

[24]We can "fix" this and make Ω surjective by defining it instead through

$$(-1)^{s(\Lambda)}\Lambda^{-1}\gamma_\mu\Lambda = R_{\mu\nu}\gamma^\nu, \tag{295}$$

where $s(\Lambda) = 0$ if $\Lambda \in \mathcal{C}_+(s,t)$ and $s(\Lambda) = 1$ if $\Lambda \in \mathcal{C}_-(s,t)$.

except for μ, and so a reflection about the axis μ is performed with $\Lambda_{R_\mu} = \prod_{\nu\neq\mu}\gamma_\nu$. We will find it slightly more convenient to choose the convention

$$\Lambda_{R_\mu} = \gamma_\mu\bar{\gamma}, \tag{296}$$

which differs from the previous equation by a c-number (if at all), and is easier to remember.

The different choices of signature affect what reflections square to. Clearly

$$\Lambda_{R_\mu}^2 = \gamma_\mu\bar{\gamma}\gamma_\mu\bar{\gamma} = -\gamma_\mu^2 = -\eta_{\mu\mu} \tag{297}$$

holds for all reflections about a single axis. When we talk about parity — reflection of all the spatial indices — we have to make a choice between simplicity and consistency of notation. We will opt for the former and define

$$\Lambda_P \equiv \gamma_0 \implies \Lambda_P^2 = \eta_{00}. \tag{298}$$

This is particularly nice since it means that spatial reflections and parity square to the same thing in real time, and lets the reversal of the time coordinate always square to the negative of Λ_P^2.[25]

Now some general comments before getting into specific examples. While $\mathrm{Spin}(1,d-1) \cong \mathrm{Spin}(d-1,1)$, and $O(1,d-1) \cong O(d-1,1)$, as we can see by the above calculations, $\mathrm{Pin}(1,d-1) \ncong \mathrm{Pin}(1,d-1)$, and $\mathrm{Pin}(0,d) \ncong \mathrm{Pin}(d,0)$. The fact that pin groups in different signatures aren't isomorphic holds in even the simplest case of 0+1 dimensions, where $\mathrm{Pin}(1,0) \cong \mathbb{Z}_2^2$, while $\mathrm{Pin}(0,1) \cong \mathbb{Z}_4$ (in this case both Spin groups are trivial, and both orthogonal groups are $\mathbb{Z}_2$).

Action of Pin(s,t) on fermions

We now elaborate on the general procedure for determining how C, P, and T act on fermion fields. One technical comment first: none of C, P, or T are connected to the identity in the Lorentz group. Thus their actions are only really defined up to arbitrary phases (which will always cancel out in Lorentz-invariant quantities). There are certain canonical choices to make and in what follows we will make them, but it is important to keep this ambiguity in mind.

[25] The reason why this is a less consistent choice is that ideally, we would have defined

$$\Lambda_P = \prod_j \Lambda_{R_j}, \tag{299}$$

which after some combinatorial shenanigans is

$$\Lambda_P = i^{d^2/2-2d-d+1+d/2-t}\eta_{00}\gamma_0, \tag{300}$$

which is disgusting. It squares to, when d is even,

$$\Lambda_P^2 = (-1)^{1-t+d/2}\eta_{00}, \tag{301}$$

which is really not very pretty. Hence we have used the simpler $\Lambda_P = \gamma_0$ in the main text.

In this section, we will be somewhat abstract and field-theory-centric, and will make use of the technology introduced in the previous section. The free fermion action, in real time, is

$$S = \int d^{d-1}x\, dt\, \bar{\psi} i \not{D}_A \psi. \tag{302}$$

In this expression, ψ is a pinor, and $\bar{\psi}$ is a copinor. This means that the action of $\mathrm{Pin}(s,t)$ is represented on the pinors via

$$\mathrm{Pin}(s,t) \ni g : \psi \mapsto \Lambda_g \psi, \quad \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_g. \tag{303}$$

We will *almost* be setting the barred representation to be the inverse of Λ_g representation, so that $\bar{\Lambda}_g$ is almost Λ_g^{-1}. There are some subtleties involved with reflections in odd dimensions, though, which means this relation won't always hold.

Now in general, Λ_g and $\bar{\Lambda}_g$ will be distinct representations of $\mathrm{Pin}(s,t)$. This means that $\psi, \bar{\psi}$ are generally *distinct* fields in the path integral. If the representation of $\mathrm{Pin}(s,t)$ the pinor ψ transforms under is isomorphic to its dual through some isomorphism R, then we can use R to relate ψ and $\bar{\psi}$: this is the case where ψ is a Majorana, and because there is only one variable being integrated over in the path integral, the resulting partition function is a Pfaffian, not a determinant. In what follows though, we will simply define $\bar{\psi}$ as a copinor which transforms in the way defined above. We will largely avoid writing $\bar{\psi} = \psi^\dagger \gamma^0$, since this will not be true if the fermions are Majorana and just adds more clutter. If the fermion is not Majorana then ψ and $\psi^\dagger$ are independent anyway, but $\bar{\psi}, \psi$ is a conceptually nicer set of independent fields to work with than $\psi^\dagger, \psi$.

First for reflections / parity. First consider a reflection R_μ about the coordinate x^μ (if $\mu = 0$ we are reversing the flow of time, but not doing anything antilinear). When $d \in 2\mathbb{Z}$, the appropriate action is to take $\Lambda_{R_\mu} = \gamma_\mu \bar{\gamma}$. Indeed, the invariance of the kinetic term is demonstrated by

$$\bar{\gamma}^\dagger \gamma_\mu^\dagger (-\partial_\mu \gamma^\mu + \sum_{\nu \neq \mu} \partial_\nu \gamma^\nu) \bar{\gamma} \gamma_\mu = \not{\partial}. \tag{304}$$

This works because when $d \in 2\mathbb{Z}$, $\bar{\gamma}$ anticommutes with all of the γ_μ.

When $d \in 2\mathbb{Z}+1$, we need something different: as we saw above, the homomorphism $\Omega : \mathrm{Pin}(s,t) \to O(s,t)$ is not surjective, and we cannot generate things with odd determinant. The solution to this is to twist the action of $\mathrm{Pin}(s,t)$ on $\bar{\psi}$ by a minus sign. So, we should do something like $\psi \mapsto \Lambda_{R_\mu}\psi$ and $\bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_{R_\mu} = -\bar{\psi}\Lambda_{R_\mu}^{-1}$. Said differently, recall that when d is odd, there are two distinct $\mathrm{Pin}(s,t)$ representations, which differ by the map $\gamma_\mu \mapsto -\gamma_\mu$. That is, if one representation is represented by the matrices $\Lambda_g, g \in \mathrm{Pin}(s,t)$, then the other is represented by the matrices $(\det \Lambda_g)\Lambda_g$, where $\det \Lambda_g$ is the function which splits apart $\mathcal{C}_+(s,t)$ and $\mathcal{C}_-(s,t)$.

If the pinor ψ transforms in one representation, the copinor $\bar{\psi}$ is taken to transform in the other representation, so that it picks up an extra minus sign when acted on by orientation-reversing elements on $\mathrm{Pin}(s,t)$. When d is odd we will thus take $\Lambda_{R_\mu} = \gamma_\mu$ and $\bar{\Lambda}_{R_\mu} = -\gamma_\mu$.[26]

[26]The reason for the absence of the $\bar{\gamma}$ here is just because we want the $\bar{\Lambda}_{R_\mu}$ matrix to be obtained from the $\Lambda_{R_\mu}^\dagger$ matrix through the substitution $\gamma_\mu \mapsto -\gamma_\mu$, which interchanges the representations, and since $\bar{\gamma} \mapsto -\bar{\gamma}$ under this map, we would not get the right minus sign if we included the $\bar{\gamma}$.

From a slightly different point of view, the difference between odd and even dimensions can be understood in the following way (this paragraph will be in Euclidean signature, for simplicity). When reflections are included, we can consider pinors that are ± 1 eigenpinors under reflection (we can choose ± 1 wolog since $\gamma_\mu^2 = \mathbf{1}$ for all μ). These pinors will be sections of two pinor bundles, that we will denote as $\mathcal{P}$ (+1 eigenvalue) and $\mathcal{P}'$ (-1 eigenvalue). These two pinor bundles are related via $\mathcal{P} = \mathcal{P}' \otimes \varepsilon$, where ε is the orientation bundle; basically this is because sections of pinor bundles are glued with reflections along orientation-reversing transition functions, and so changing the signs of these transition functions by tensoring with ε is equivalent to sending $\gamma_\mu \mapsto -\gamma_\mu$. Anyway, in even dimensions, the action of a reflection, $\gamma_\mu \bar{\gamma}, x^\mu \mapsto -x^\mu$, commutes with the Dirac operator $i\partial\!\!\!/$. This means that $i\partial\!\!\!/ : \mathcal{P} \to \mathcal{P}$, $\mathcal{P}' \to \mathcal{P}'$, and the Dirac operator is self-adjoint, giving us a determinant.[27] In odd dimensions though, reflections *anticommute* with $i\partial\!\!\!/$, and there is no way to ameliorate this with a factor of $\bar{\gamma}$. Thus in odd dimensions $i\partial\!\!\!/ : \mathcal{P} \to \mathcal{P}'$, $\mathcal{P}' \to \mathcal{P}$, and sections of $\mathcal{P}$ must get paired with sections of $\mathcal{P}'$ in the action: the Dirac operator is not self-adjoint, and the parition function is a Pfaffian rather than a determinant. This is just another way of saying that in odd dimensions, the invariant pairing is constructed between pinors transforming in the two different representations of $\mathrm{Pin}(s,t)$.

Summarizing, a reflection about the x^μ axis acts as

$$\begin{aligned} R_\mu : \psi \mapsto \Lambda_{R_\mu}\psi = \gamma_\mu \bar{\gamma}\psi, \qquad & \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_{R_\mu} = \bar{\psi}(\gamma_\mu\bar{\gamma})^\dagger \qquad d \in 2\mathbb{Z}, \\ R_\mu : \psi \mapsto \Lambda_{R_\mu}\psi = \gamma_\mu \psi, \qquad & \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_{R_\mu} = -\bar{\psi}\gamma_\mu^\dagger \qquad d \in 2\mathbb{Z}+1. \end{aligned} \tag{305}$$

Note that if we take the copinor to be $\bar{\psi} = \psi^\dagger \gamma^0$ and the reflection to be about a spatial axis, the $(-1)^d$ factor for the $\bar{\Lambda}_{R_j}$ transformation is picked up when moving the $\gamma_j \bar{\gamma}$ through the γ^0.

Reflection P of all the spatial axes is easy: (we are calling it "parity" even though when d is odd it has determinant 1) in the conventions defined in the previous section,

$$P : \psi \mapsto \Lambda_P \psi = \gamma_0 \psi, \qquad \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_P = \bar{\psi}\gamma_0^\dagger. \tag{306}$$

There are no complications in the odd d case since if d is odd P is an element of $\mathrm{Spin}(s,t)$ and is represented identically by both pinor representations of $\mathrm{Pin}(s,t)$.

Now one thing to note here is that we can always work with different reflection / parity operators defined as $\widetilde{\Lambda}_{R_\mu} = \Lambda_{R_\mu}\bar{\gamma}$. If d is odd then $\bar{\gamma}$ is central and this modification obviously does nothing, provided we modify the transformation of the copinor in the same way. Thus, if d is odd, this doesn't give us anything new. If d is even though, we can then take the copinor to transform under the matrix $-(\widetilde{\Lambda}_{R_\mu})^{-1}$; this minus sign is naturally generated if the copinor is defined though $\psi^\dagger \gamma_0$. Thus this gives us an alternative way to represent reflections in even dimensions:

$$R_\mu : \psi \mapsto \widetilde{\Lambda}_{R_\mu}\psi = \gamma_\mu \psi, \qquad \bar{\psi} \mapsto -\bar{\psi}\gamma_\mu^\dagger \qquad d \in 2\mathbb{Z}. \tag{307}$$

[27]In Witten's paper he does something different: the $\bar{\gamma}$ is not included in his reflection action, but he modifies his Dirac operator by a factor of $\bar{\gamma}$ in a compensating way.

The difference between this representation and the tilde one is simply that in this representation, the reflection squares to something different:

$$(\widetilde{\Lambda}_{R_\mu})^2 = \eta_{\mu\mu} = -\Lambda_{R_\mu}^2. \tag{308}$$

In keeping with this change, we can also define a primed parity $\widetilde{\Lambda}_P = \gamma_0\bar{\gamma}$, so that

$$(\widetilde{\Lambda}_P)^2 = -\eta_{00}. \tag{309}$$

Thus in the primed representation, parity and reflection square to the same thing in real time.

The full antiunitary reversal is much messier: its antiunitary nature means that while it involves the reversal of the time coordinate (which is a transformation in $\mathrm{Pin}(s,t)$), it itself does not act on the (co)pinors through a linear representation of $\mathrm{Pin}(s,t)$. First consider the element R_0 of $\mathrm{Pin}(s,t)$ which reverses time, but does not act with complex conjugation. By the same argument we used for spatial reflections, this acts as

$$R_0 : \psi \mapsto \Lambda_{R_0}\psi = \gamma_0\bar{\gamma}\psi, \quad \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_{R_0} = (-1)^d\bar{\psi}(\gamma_0\bar{\gamma})^\dagger. \tag{310}$$

Unlike for spatial reflections, this action is not compatible with $\bar{\psi} = \psi^\dagger\gamma^0$ in odd dimensions. Similarly to the above, we could also define the transformation through $\widetilde{\Lambda}_{R_0} = \bar{\gamma}\Lambda_{R_0}$, provided that we also took the copinor to transform as $-(-1)^d[\Lambda_{R_0}]^{-1}$.

Now suppose we let time reversal act on pinors by $T : \psi \mapsto \mathcal{K}\Lambda_{R_0}\mathcal{C}\psi$, where $\mathcal{C}$ is a unitary to be determined. In the following, we will assume that the Dirac adjoint is used, so that $\bar{\psi} = \psi^\dagger\gamma^0$. The Majorana adjoint will be treated later. Anyway, requiring that $\bar{\psi}i\partial\!\!\!/\psi$ be invariant means that

$$\mathcal{C}^\dagger\Lambda_{R_0}^\dagger\gamma_0^*(-\partial^0\gamma_0^* + \partial^j\gamma_j^*)\Lambda_{R_0}\mathcal{C} = -\partial\!\!\!/. \tag{311}$$

Recalling that $\Lambda_{R_0} = \gamma_0\bar{\gamma}$, we have

$$\mathcal{C}^\dagger\gamma_0^*\partial\!\!\!/^*\mathcal{C} = \partial\!\!\!/ \implies \mathcal{C}^\dagger\gamma_\mu^*\mathcal{C} = \pm\gamma_\mu. \tag{312}$$

Fortunately the matrices γ_μ^* obey the same algebra as the $\pm\gamma_\mu$, and so such a unitary $\mathcal{C}$ will always exist (it is nearly the same unitary employed in charge conjugation, but depending on the signature it is not quite the same).

Euclidean time

We now briefly comment on what happens in Euclidean time. In Euclidean time the action is *not* Hermitian, and so we will avoid writing the copinor field as something dependent on $\psi^\dagger$, since this is confusing. We write the Lagrangian as[28]

$$\mathcal{L}_E = \chi_E^\dagger(\partial\!\!\!/ + m)\psi_E, \tag{314}$$

[28] Another choice would be to write

$$\mathcal{L}_E = \chi_E^\dagger\bar{\gamma}(\partial\!\!\!/ + m)\psi_E, \tag{313}$$

but then the reflections act as $\Lambda_{R_j} = \gamma_j$ (without the $\bar{\gamma}$), which is different from the $\mathbb{R}$ time case, so we avoid this.

where $\chi_E^\dagger$ is an independent field (not related to ψ_E by Hermitian conjugation). In even dimensions, the $\mathrm{Pin}(d)$[29] action $\psi \mapsto \Lambda_g \psi$ acts on χ as $\chi^\dagger \mapsto \chi^\dagger \Lambda_g^\dagger$. With this one checks that $\mathcal{L}_E$ is invariant under $\mathrm{Pin}(d)$, since the γ_E^μ matrices are all Hermitian. If d is odd, we have two different $\mathrm{Pin}(d)$ representations, differing by $\gamma_\mu \mapsto -\gamma_\mu$. In this case, we take χ_E and ψ_E to transform in opposite $\mathrm{Pin}(d)$ representations, so that $\chi_E^\dagger \mapsto \chi_E^\dagger \Lambda_g^\dagger \det \Lambda_g$, allowing the Lagrangian to be reflection-invariant.

Action of charge conjugation

Charge conjugation, in this context, is a unitary operator that relates ψ and $\bar{\psi}$. We take it to act as

$$\psi \mapsto \bar{\psi} C^\dagger, \qquad \bar{\psi} \mapsto -C\psi. \tag{315}$$

The reason for the minus sign is to ensure that the Dirac mass is always C-even:

$$\bar{\psi}\psi \mapsto -C_{\alpha\beta}\psi_\beta \bar{\psi}_\lambda [C^\dagger]_{\lambda\alpha} = +\bar{\psi} C^\dagger C \psi = \bar{\psi}\psi. \tag{316}$$

A general bilinear $\bar{\psi} M \psi$ (not involving derivatives) then transforms as

$$\bar{\psi} M \psi \mapsto \bar{\psi}(C^\dagger M^T C)\psi. \tag{317}$$

In order for the free term to be invariant, we need

$$\not{\partial} = (C^T i\not{\partial}[C^\dagger]^T)^T = C^\dagger [i\not{\partial}]^T C, \implies C^\dagger \gamma_\mu^T C = -\gamma_\mu, \tag{318}$$

since $\partial_\mu^T = -\partial_\mu$. Since $-\gamma_\mu^T$ and γ_μ obey the same algebra, such a matrix C always exists. Exact expressions for C depend on the choice of signature and dimension.

As we saw, the Diarc mass is always C invariant. When $d \in 2\mathbb{Z}$ we also have a chiral mass, which transforms as

$$\bar{\psi}\bar{\gamma}\psi \mapsto \bar{\psi}(C^\dagger \bar{\gamma}^T C)\psi. \tag{319}$$

Now

$$C^\dagger \bar{\gamma}^T C = \prod_{\mu=d,\dots,1} C^\dagger \gamma_\mu^T C = \prod_{\mu=d,\dots,1} \gamma_\mu = (-1)^{((d-1)^2+(d-1))/2}\bar{\gamma}. \tag{320}$$

So, we see that for $d \in 4\mathbb{Z}$ charge conjugation respects chirality and the chiral mass is C-even, while for $d \in 4\mathbb{Z}+2$ charge conjugation exchanges chiral components and the chiral mass is C-odd.

[29] Here by $\mathrm{Pin}(d)$ we really mean $\mathrm{Pin}(d,0)$. We take reflections to act as $\Lambda_{R_j} = \gamma_j \bar{\gamma}$ and $\bar{\Lambda}_{R_j} = (\gamma_j\bar{\gamma})^{-1}$ if d is even and as $\Lambda_{R_j} = \gamma_j$, $\bar{\Lambda}_{R_j} = -\gamma_j$ if d is odd. Just as in the indefinite signature case, were the squares of the reflections were fixed but could be changed by taking $\mathrm{Pin}(s,t) \to \mathrm{Pin}(t,s)$, here we can get different squares for reflections by taking $\mathrm{Pin}(d,0) \to \mathrm{Pin}(0,d)$. This amounts to taking $\gamma_\mu \to i\gamma_\mu$, and so switches the signs in the squares of the reflection matrices, both for d even and d odd.

Majorana fermions

To define Majorana fermions, we need to find a matrix that intertwines the spinor representation of $\mathrm{Spin}(s,t)$ with its dual. That is, if Λ_g is a representation matrix of $\mathrm{Spin}(s,t)$ (so that $\Lambda_g^{-1}\gamma^\mu\Lambda_g = R^{\mu\nu}\gamma_\nu$ for $R \in SO(s,t)$), we need to find a matrix $\mathscr{C}$ such that

$$\Lambda_g^T\mathscr{C} = \mathscr{C}\Lambda_g^{-1}. \tag{321}$$

If such a matrix exists, then given a spinor χ we can define a cospinor $\bar{\chi}$ which is linearly dependent on χ, with the two related via $\bar{\chi} \propto \chi^T\mathscr{C}$. The properties of $\mathscr{C}$ then guarantee that $\chi^T\mathscr{C}\chi$ is invariant under $\mathrm{Spin}(s,t)$ transformations.

Recall that the representation of any $g \in \mathrm{Spin}(s,t)$ can be written as

$$\Lambda_g = \exp\left(\frac{1}{8}[\gamma_\mu,\gamma_\nu]\theta_g^{\mu\nu}\right) \implies \Lambda_g^T = \exp\left(-\frac{1}{8}[\gamma_\mu^T,\gamma_\nu^T]\theta_g^{\mu\nu}\right), \tag{322}$$

by the antisymmetry of $\theta_g^{\mu\nu}$. We see that $\mathscr{C}$ will do the job provided that

$$\gamma_\mu^T\mathscr{C} = -\mathscr{C}\gamma_\mu. \tag{323}$$

But this is exactly what charge conjugation does, since C is unitary and satisfies $C^\dagger\gamma_\mu^T C = -\gamma_\mu$. Thus we can take $\mathscr{C} = C$.

Now in order for the theory to be nontrivial, we need $\bar{\chi}i\partial\!\!\!/\chi \neq 0$, i.e. we need $C\partial\!\!\!/$ to be an antisymmetric matrix. Now its transpose is

$$(C\partial\!\!\!/)^T = -\gamma_\mu^T\partial^\mu C^T. \tag{324}$$

Transposing $\gamma_\mu^T C = -C\gamma_\mu$ tells us that $\gamma_\mu^T C^T = -C^T\gamma_\mu$, so

$$(C\partial\!\!\!/)^T = +C^T\partial\!\!\!/, \tag{325}$$

and so if the action is to be nontrivial we need a charge conjugation which is antisymmetric: $C^T = -C$ (the antisymmetry also implies the existence of a nontrivial mass term for the Majoranas). Fortunately, this property is satisfied if $C^2 = \mathbf{1}$ when acting on fermions, since

$$C^2 : \psi_\alpha \mapsto [C(\bar{\psi}_\gamma[C^\dagger]_{\gamma\lambda})]_\alpha = -\psi_\rho C_{\gamma\rho}[C^\dagger]_{\gamma\lambda} = -C^T C^\dagger\psi, \tag{326}$$

and so $-C^T C^\dagger = \mathbf{1} \implies C = -C^T$.

Finally, in $\mathbb{R}$ time (and in $\mathbb{R}$ time only!), we need the action to be Hermitian. So far we have only used $\bar{\chi} \propto \chi^T C$. Now fix the proportionality constant as

$$\bar{\chi} = \lambda_\chi\chi^T C. \tag{327}$$

Now since $C\partial\!\!\!/$ is antisymmetric, in order to have a Hermitian action, we need $\lambda_\chi i\partial\!\!\!/$ to be purely imaginary. Evidently this is only possible if the γ matrices are either all real (in which case we take $\lambda_\chi = 1$) or all imaginary (in which case we take $\lambda_\chi = i$). This property holds in all the dimensions we're usually interested in (2, 3, and 4). This requirement is why we can impose a reality condition on Majorana spinors in $\mathbb{R}$ time (when we Wick-rotate to $i\mathbb{R}$ time, there is no Hermiticity requirement, and no reality condition for Majoranas).

The above shows how to define Majorana spinors. What about Majorana pinors? This is trickier. For example, consider the reflection of all spatial indices. This is usually represented in $\mathrm{Pin}(s,t)$ by $\chi \mapsto \gamma_0\chi$. But we see that $\chi^T C\chi$ is only invariant under this action if $\gamma_0^2 = -\mathbf{1}$, since

$$\gamma_0^T C\gamma_0 = C \implies C^\dagger\gamma_0^T C = \gamma_0^{-1} \implies -\gamma_0 = \gamma_0^{-1}. \tag{328}$$

So, whether or not we can define a Majorana pinor depends on the signature (or better, on the Pin structure) that we choose! Evidently we can only have Majorana pinors with this definition if they are "Kramers doublets" under parity.

Now consider the action of time reversal. The element in $\mathrm{Pin}(s,t)$ which reverses time is, as we saw, $\gamma_0\bar{\gamma}$. Thus we need to look at

$$(\gamma_0\bar{\gamma})^T C\gamma_0\bar{\gamma} = (-1)^{d-1}\prod_{j=1}^{d}\gamma_j^2 \tag{329}$$

Examples

We now look at examples in low dimensions. We first look at $i\mathbb{R}$ time, and then at $\mathbb{R}$ time. For the later, we find the action of time reversal and other symmetries.[30] Note that the choice of operator for time reversal, i.e. the way we represent it in the pinor representation of $\mathrm{Pin}(s,t)$, doesn't depend on the choice (mostly + or mostly −) of signature.[31] However, the action of T^2 will depend on the signature convention, since e.g. γ_μ^2 has different signs for different signatures. The same holds for parity.

We will also consider mass terms: in odd dimensions the only Lorentz scalar is $\bar{\psi}\psi$ ($\bar{\gamma} \propto \mathbf{1}$), and so there is only one type of mass term. In even dimensions $\bar{\gamma}$ is nontrivial, and we also have the chiral mass $\bar{\psi}\bar{\gamma}\psi$.

Two dimensions

In Euclidean signature, we can take our γ matrices to be X and Z. This means that $\mathrm{Spin}(2)$ consists of unit-norm linear combinations of $\mathbf{1}$ and $XZ = J$, so that $\mathrm{Spin}(2) \cong U(1)$. The splitting $S_+ \oplus S_-$ in this case corresponds to spinors that transform as $e^{i\theta}$ or $e^{-i\theta}$. Another basis we commonly use is to take $\gamma^0 = Y, \gamma^1 = X$. Then $\mathrm{Spin}(2)$ has generators $\mathbf{1}, iZ$, the diagonality of which make it clear that the representation is reducible, with the splitting $S_+ \oplus S_-$ being a splitting into left- and right-moving spinors.

In real time with signature $(-,+)$, we may take

$$\gamma^0 = J, \qquad \gamma^1 = X. \tag{331}$$

[30]Reminder: the adjoint of an anti-unitary like T is defined via

$$\langle v|Tw\rangle = \langle T^\dagger v|w\rangle^*. \tag{330}$$

[31]Regardless of signature, the free term in the action is $\bar{\psi}i\not{D}\psi$, and so we need $T^\dagger\gamma^0\gamma^j T = -\gamma^0\gamma^j$ for all spatial indices j. Changing between the two signature conventions can be done by multiplying all the γ matrices by i, under which the above condition is invariant.

Again, Spin(2) is generated by diagonal elements $\mathbf{1}$ and Z, which makes the $S_+ \oplus S_-$ decomposition manifest.

The charge conjugation matrix needs to satisfy $C^\dagger J C = J$ and $C^\dagger X C = -X$; hence we take $C = Y$, which squares to $\mathbf{1}$ and is properly antisymmetric.

Writing time reversal as $T = U_T \mathcal{K}$ with U_T unitary, we need $U_T^\dagger (JX) U_T = -JX$. Thus we may take $U_T = X$, so that $T^2 = \mathbf{1}$. However, we may also take $U_T = J$, so that $T^2 = (-1)^F$! Either option is consistent.

Likewise, for parity, we need $P^\dagger Z P = -Z$, so that any of $P = X, P = Y, P^2 = \mathbf{1}$ or $P = J, P^2 = -\mathbf{1}$ is consistent. Finally, for charge conjugation, we need (remembering the minus sign from integrating by parts!) $[C^\dagger Z C]^T = Z$. We'd also like C to preserve the Dirac mass, so that $-[C^\dagger J C]^T = J$, and we'd like $C^2 = \mathbf{1}$. Thus we will fix $C : \psi \mapsto \psi^\dagger$,

As a sanity check, consider the action of CPT: invariance of $\bar{\psi} i \not{D}_A \psi$ tells us that (remember the minus sign from the integration by parts!)

$$-[U_T^\dagger \mathcal{K} P^\dagger C^\dagger Z C P \mathcal{K} U_T]^T = Z. \tag{332}$$

Now all of the choices give C, P, U_T which anticommute with Z. Thus

$$[Z U_T^\dagger P^\dagger C^\dagger C P U_T]^T = Z \tag{333}$$

Thus CPT holds since all the matrices involved are unitary.

The two Hermitian mass terms we can add are the Dirac mass $im\bar{\psi}\psi$ or the chiral mass $m\bar{\psi}Z\psi$. If we let $T = J\mathcal{K}$ so that $T^2 = (-1)^F$, then both mass terms are T-odd. On the other hand, if we let $T = X\mathcal{K}$ so that $T^2 = \mathbf{1}$, then both mass terms are T-even. If we take $P = X$ then the Dirac mass is P-odd while the chiral mass is even, or (more sensibly) if $P = J$ then the opposite is true. Both masses are even under C, as they should be.

The γ matrices are real, so we expect to be able to define Majorana fermions.

If we took the signature $(+, -)$, then we could use the same γ matrices, just with their indices exchanged, and so the full structure is the same, just with the unitary matrices associated with T and P switched: in both cases, we are looking at the representation theory of $\mathrm{Pin}(1, 1)$.

Three dimensions

In Euclidean signature, we can take our γ matrices to just be the Pauli matrices. Consider then the antilinear map $\mathscr{J} = J\mathcal{K}$: this anticommutes with every γ matrix, and so it commutes with all products of an even number of γ matrices—thus, the spinor representation of Spin(3) is quaternionic (duh, since Spin(3) $= SU(2)$, but still).

If we pass to $\mathrm{Pin}(3, 0)$, $\mathscr{J}$ is no longer an invariant form, since it anticommutes with each individual γ matrix. Thus the pinor representations of $\mathrm{Pin}^+(3)$ are complex. We say representations because there are two, since we are in an odd dimension. Recall that the representation matrices Λ_g of the two representations differ by a factor of $\det \Lambda_g$: $\Lambda'_g = (\det \Lambda_g) \Lambda_g$. From our general discussion, we know that these two representations should give an invariant pairing. Indeed they do: if ψ is a pinor transforming under Λ_g and ψ' is a pinor transforming under Λ', then the antisymmetry of the fermions means that

$$\mathrm{Pin}(3, 0) \ni g : \psi'_\alpha J^{\alpha\beta} \psi_\beta \mapsto [\Lambda'_g]_{\beta\alpha} \psi'_\alpha J^{\beta\gamma} [\Lambda_g]_{\gamma\lambda} \psi_\lambda = (\det \Lambda_g) J_{\alpha\beta} \Lambda_g^{\alpha\lambda} \Lambda_g^{\beta\rho} \psi'_\lambda \psi_\rho = (\det g)^2 J^{\alpha\beta} \psi'_\alpha \psi_\beta, \tag{334}$$

which is invariant since $\det(\Lambda_g)^2 = 1$. Now since $\mathrm{Pin}(3,0)$ is compact, this invariant pairing gives us an isomorphism between the Λ' representation and the complex conjugate of the Λ representation. To find this isomorphism, we look for a unitary $\mathcal{U}$ such that

$$\mathcal{U}^\dagger \Lambda_g^* \mathcal{U} = (\det \Lambda_g)\Lambda_g = \Lambda'_g. \tag{335}$$

Indeed, we see that such an isomorphism is provided by taking $\mathcal{U} = Y$.

In $\mathbb{R}$ time with signature $(-,+,+)$, we will take the γ matrices to be (J, X, Z). They are all real, and so evidently the pinor representation of $\mathrm{Pin}(2,1)$ is real. We need time reversal to act as $T = U_T\mathcal{K}$, where $U_T^\dagger(JX)U_T = -JX, U_T^\dagger(JZ)U_T = -JZ$. Thus we may choose $T = J\mathcal{K}$ or $T = Y\mathcal{K}$. Either way, $T^2 = (-1)^F$. Note that unlike in two dimensions, we *cannot* choose something like $U_T = X$ that gives $T^2 = \mathbf{1}$.

Now since d is odd, $\mathrm{Pin}(2,1)$ has two irreducible representations. Just as in the $\mathrm{Pin}(3,0)$ case, a fermion ψ transforming in one and a fermion ψ' transforming in the other have an invariant antisymmetric pairing provided by J. However, since $\mathrm{Pin}(2,1)$ is *not compact*, this invariant pairing doesn't imply that the Λ_g representation is related to the complex conjugate of the Λ'_g representation through a unitary transformation. Indeed, now that J is among the γ matrices there is no matrix $\mathcal{U}$ such that $\mathcal{U}^\dagger\gamma_\mu^*\mathcal{U} = -\gamma_\mu$, so we can not relate the two representations by (335).

Charge conjugation is unchanged, with $\psi \mapsto \bar\psi C^\dagger$, where $C = Y$. In terms of the chiral components, this is

$$C : \psi_{L/R} \mapsto i\psi^\dagger_{L/R}. \tag{336}$$

Particle-hole symmetry CT therefore acts as

$$CT : \psi_L \mapsto -\psi_R^\dagger, \qquad \psi_R \mapsto +\psi_R^\dagger, \qquad i \mapsto -i. \tag{337}$$

In particular, $(CT)^2 = (-1)^F$, and so in three dimensions, both T and CT have to square to $(-1)^F$. From a representation theory point of view, this is because the spinor representation of $\mathrm{Spin}(3) = SU(2)$ is pseudoreal, and so the isomorphism between the spinor representation and its conjugate is an antilinear map which squares to -1.

Four dimensions

In Euclidean signature, we can take

$$\gamma^0 = X \otimes \mathbf{1}, \quad \gamma^1 = Y \otimes \mathbf{1}, \quad \gamma^2 = Z \otimes X, \quad \gamma^3 = Z \otimes Z. \tag{338}$$

Consider the antilinear map

$$\mathscr{J} = (X \otimes J)\mathcal{K}. \tag{339}$$

We see that $\mathscr{J}^2 = -\mathbf{1}$ and $\mathscr{J}$ commutes with all products of an even number of γ matrices, and so the spinor representation of $\mathrm{Spin}(4)$ is quaternionic, just like for $\mathrm{Spin}(3)$ (this is to be expected since $\mathrm{Spin}(4) = SU(2)^2$).

But in four dimensions, we get something better: $\mathscr{J}$ actually commutes with every individual γ matrix! This means that the pinor representation of $\mathrm{Pin}^+(4)$ is also pseudoreal.

In $\mathbb{R}$ time with $(-,+,+,+)$ signature, we can use the matrices

$$\gamma^0 = J \otimes X, \quad \gamma^1 = J \otimes J, \quad \gamma^2 = Z \otimes \mathbf{1}, \quad \gamma^3 = X \otimes \mathbf{1}, \tag{340}$$

which are all real! Thus we can have Majorana fermions, since all the elements in Spin$(1,3)$ are real (the equation $(i\not{D} - im)\psi = 0$ implies $(i\not{D} - im)\psi^* = 0$ if all the γ matrices are real). It will be useful to know that $\bar{\gamma} = iJ \otimes Z$.

Charge conjugation needs to satisfy

$$C^\dagger(J\otimes X)C = J\otimes X, \quad C^\dagger(J\otimes J)C = -J\otimes J, \quad C^\dagger(Z\otimes\mathbf{1})C = -Z\otimes\mathbf{1}, \quad C^\dagger(X\otimes\mathbf{1})C = -X\otimes\mathbf{1}, \tag{341}$$

and so we take

$$C = Y \otimes X, \tag{342}$$

which satisfies $C^T = -C$ and $C^2 = \mathbf{1}$ as required.

For time reversal, U_T needs to anticommute with all of $\mathbf{1} \otimes Z, X \otimes X$, and $Z \otimes X$. Thus we can take $U_T = \mathbf{1} \otimes J$ or $U_T = J \otimes X$. Either way, $T^2 = (-1)^F$.

The two mass terms are the Dirac mass $im\bar{\psi}\psi$ and the chiral mass $im\bar{\psi}\bar{\gamma}\psi$.[32] Annoyingly, if we choose $U_T = \mathbf{1} \otimes J$ then the Dirac mass is T-even while the chiral mass is T-odd, while the situation is reversed if we choose $U_T = J \otimes X = \gamma^0$.

In $\mathbb{R}$ time with $(+,-,-,-)$ signature, we can choose

$$\gamma^0 = X \otimes \mathbf{1}, \quad \gamma^1 = J \otimes X, \quad \gamma^2 = J \otimes Y, \quad \gamma^3 = J \otimes Z. \tag{343}$$

Note that the product of any two of these is block diagonal—this shows us explicitly that Spin$(1,3)$ acts reducibly on $S_+ \oplus S_-$. Alternatively, we may choose i times the matrices given for $(-,+,+,+)$ signature: this gives us a purely imaginary representation. This means that all products of two γ matrices are real, and so Spin$(1,3)$ acts with real matrices; again, we can have Majorana fermions (the equation $(i\not{D} - m)\psi = 0$ implies $(i\not{D} - m)\psi^* = 0$ if all the γ matrices are imaginary).

Actions

Consider first complex fermions, where a Lorentz invariant bilinear is formed (in $\mathbb{R}$ time) by adding on γ^0 to $\psi^\dagger$ in $\psi^\dagger\psi$ to get $\bar{\psi}\psi$. Now in $\mathbb{R}$ time the operator $\gamma^0 i\not{D}_A$ is Hermitian, regardless of signature, and so the action is Hermitian as required (we treat $\psi^\dagger$ and ψ as the (independent) integration variables). Now when we go to Euclidean time, we might be tempted to just naively do (independent of signature)

$$\gamma^0 i\not{D}_A \xrightarrow{t\mapsto it} \gamma^0 \not{D}_A, \tag{344}$$

and write the action as $\int d\tau\, d^{D-1}x\, \bar{\psi}\not{D}_A\psi$. However, in Euclidean time, while $\not{D}_A\psi$ transforms as a spinor under Spin(D), $\bar{\psi}\psi$ is not invariant under Spin(D), because now the γ^0 in $\bar{\psi}$ screws stuff up. So, we should really write the Euclidean action as

$$S_E = \int d\tau\, d^{D-1}x\, \chi^\dagger \not{D}_A \psi, \tag{345}$$

[32] the factor of i on the chiral mass is because $\bar{\gamma}$ is anti-Hermitian in this representation.

where if ψ transforms in a representation R of $\mathrm{Spin}(D)$, then $\chi^\dagger$ transforms in the representation $\bar{R}$. This does not actually lead to a doubling of the number of fields, since while $\chi^\dagger$ and ψ are independent, so too were $\psi^\dagger$ and ψ in the real time picture (notice that there is no $\psi^\dagger$ or χ field appearing in S_E). One thing to note however is that in Euclidean time, the action is no longer Hermitian (this is okay; recall e.g. theta angles).

22 *February 20 — Fermions, bundles, and Spin$_G$ structures*

The goal today is just to write down the right words explaining what bundles are relevant when dealing with fermions coupled to a gauge field. This is because I (like most physicists, probably) knew intuitively what equations to write down for fermions and spinc structures and all of that, but I didn't really know in a precise sense what mathematical structures were being used.

Solution:

In today's diary entry, we will work in Euclidean signature. In Euclidean signature there are no reality conditions on fermions, and so our spinors will always live in complex vector spaces.

Preliminaries

First we review what kind of bundles regular fermions (not coupled to any particular gauge field) are associated with. Let X be spacetime, $n = \dim X$, and let TX denote the tangent bundle of X. The fibers of the tangent space are acted on by $GL(n;\mathbb{R})$, which we can (with the metric) reduce to an action of $SO(n)$. Let LX, the frame bundle, be the principal $SO(n)$ bundle (we are assuming X is oriented) associated[33] to the tangent bundle (i.e. the principal

[33] The word "associated" used to describe a bundle can mean a few different things. First, suppose we are given a principal G bundle $\pi : P_G \to X$, and a vector space V which carries an action of G via some representation $R : G \to \mathrm{Aut}(X)$. Then the vector bundle E associated to P_G is denoted

$$E = P_G \times_R V, \tag{346}$$

and consists of pairs $(u, v) \in P_G \times V$ modulo the equivalence relation

$$(u, v) \sim (gu, R_g^{-1} v), \tag{347}$$

where gu is the action of g on the fiber. This is basically like a $\otimes$ of P_G and V. Here the inverse of the representation is chosen so that $(gu, v) \sim (u, R_g v)$. Anyway, we can use this to see that as we go from patch to patch, the vector space V gets acted on by $R(t_{\alpha\beta})$, where $t_{\alpha\beta}$ are the transition functions. Thus the transition functions for E are given by $R(t_{\alpha\beta})$.

Another notion of an associated bundle is the principal bundle associated to a vector bundle. If the fibers of the vector bundle $\pi : F \to X$ are $\mathbb{K}^n$ then we have transition functions valued in $G \subset GL(n;\mathbb{K})$ (the subgroup G depends on how much extra structure our manifold has, like orientability or a metric, etc), and these transition functions can be used to construct a principal G bundle over X, associated to the vector bundle F.

bundle obtained from TX's transition functions).

Spinors are constructed with the help of the "square root" of the frame bundle, namely a principal Spin(n) bundle that we will write as SLX. Forming SLX is done, if possible, by lifting the transition functions on the frame bundle from $SO(n)$-valued functions to Spin(n)-valued functions; this can be done provided the cocycle conditions for the transition functions in LX hold in a particular way (more on this in a sec).

Suppose that SLX is well-defined. Then we can then form the spinor bundle S, which is the associated vector bundle made from the principal bundle SLX, the vector space $\mathbb{C}^{2^{\lfloor n/2\rfloor}}$, and the spinor action of Spin(n) on $\mathbb{C}^{2^{\lfloor n/2\rfloor}}$.[34][35] To save on writing, we will define the vector space that spinor fields live in as

$$\Delta_n \equiv \mathbb{C}^{2^{\lfloor n/2\rfloor}}. \tag{349}$$

In our notation,

$$S = SLX \times_{1/2} \Delta_n, \tag{350}$$

where $1/2$ is the spinor representation. Fermion fields are sections of S.[36]

To address when the spin frame bundle SLX exists, we need to ask the following question: when is it possible to take the square root of a principal $SO(n)$-bundle? Roughly speaking, the square root of $G/\mathbb{Z}_2$ is G, since $\mathbf{1} \in G/\mathbb{Z}_2$ is the image of $\sqrt{\mathbf{1}} = \{\mathbf{1}, -\mathbf{1}\} \in G$. This means that we can form the bundle "$SLX = \sqrt{LX}$" when the structure group in LX can be lifted to Spin(n) without causing the cocycle condition to fail by $\pm\mathbf{1} \in$ Spin(n) at the triple overlaps. The second SW class makes an appearance here through

$$(\delta g^{\mathrm{Spin}(n)})_{\alpha\beta\gamma} = (-1)^{w_2(LX)_{\alpha\beta\gamma}}\mathbf{1}, \tag{352}$$

where the $g^{\mathrm{Spin}(n)}_{\alpha\beta}$ are spin lifts of the transition functions in LX. Here the notation $w_2(LX)$ is a bit sloppy, since the SW classes are defined only for vector bundles. What we really mean is $w_2(V)$, where V is the vector bundle associated to LX via $V = LX \times_1 \mathbb{R}^n$, with

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[34] Here $\lfloor n/2 \rfloor$ is the dimension of the spinor representation, which is what it is for the following reason. The spin group is generated by all elements in the Clifford algebra

$$\mathrm{Spin}(n) \ni \gamma_0^{j_0}\cdots\gamma_n^{j_n}, \qquad \sum_i j_i \in 2\mathbb{Z}. \tag{348}$$

There are 2^{n-1} possible products, and so the dimension of the spin group is 2^{n-1}. This means that it must be represented by matrices of dimension $d \geq 2^{(n-1)/2}$. When $n \in 2\mathbb{Z}$ we cannot have an equality. However when $n \in 2\mathbb{Z}$, the spinor representation is reducible, since we can project onto ± 1 eigenstates of $\bar{\gamma}$. In this case, the reducibility of the representation into two $d/2 \times d/2$ blocks means that $2(d/2)^2 = 2^{n-1}$, so that when d is even, we have $d^2 = 2^n$, and so $d = n/2$. Combining the odd and even case with the floor function, we get the stated result.

[35] if the representation of Spin(n) is real, then we can take the vector space to be real instead; we will ignore this possibility for simplicity

[36] In even dimensions, when the spinor representation is reducible, we can also form the bundles

$$S_\pm = SLX \times_{(1/2)_\pm} \Delta_n^\pm, \qquad \Delta_n^\pm = \mathbb{C}^{2^{\lfloor n/2\rfloor - 1}}, \tag{351}$$

where $(1/2)_\pm$ is the representation of Spin(n) on the positive / negative chirality reducible component. Sections of these bundles are Weyl fermions.

1 the fundamental (vector) representation. V is isomorphic to the tangent bundle TX,[37] and so $w_2(LX)$ is just another way of writing $w_2(TX)$. From the Cech 3-cochain one can construct a class in $H^2(X;\mathbb{Z}_2)$; more on this in the next section.

So roughly speaking, $SLX = \sqrt{LX}$. What about the associated vector bundles? We can use our knowledge of the representations of the spin group to conclude, sgain roughly speaking, that S is the "square root" of the associated bundle $V = LX \times_1 \mathbb{C}^n$ (with the 1 the vector representation; note to self: since our spinors are living in $\mathbb{C}$ I wrote $\mathbb{C}^n$, but should this be $\mathbb{R}^n$?), in the sense that

$$S \otimes S \supset V, \tag{354}$$

with the tensor product of associated bundles being performed by tensoring both the vector spaces and the representations. In the special case of $n = 2$ we have $\mathrm{Spin}(2) = U(1)$ and this is easy to understand, since we are just tensoring two line bundles together.[38]

The Chern class and w_2

The above discussion showed how $w_2(TX)$ appeared from a Cechian point of view. Later on we will need to consider $w_2(E)$ for other types of vector bundles E. In the case where E is a complex vector bundle (which will be relevant to us when discussing fermions coupled to gauge fields), $w_2(E)$ has a very simple relation with $c_1(E)$, which we now describe.

In the following, fix G to be some compact Lie group, containing $\mathbb{Z}_2$ in its center. Further let E be the complex vector bundle associated to a principal $G/\mathbb{Z}_2$ bundle over X, with a given representation of $G/\mathbb{Z}_2$, and with transition functions

$$g_{\alpha\beta} = e^{i2\pi\Lambda_{\alpha\beta}} \in GL(n;\mathbb{C}). \tag{355}$$

Since E is a legit $G/\mathbb{Z}_2$ associated bundle, we have

$$\delta\Lambda \in H^2(X;\mathbb{Z}), \tag{356}$$

where we have identified the Cech 3-cocycle $\delta\Lambda$ with a simplicial 2-cohomology class (more on this in a sec; $\delta\Lambda$ is not exact in $H^2(X;\mathbb{Z})$ since Λ isn't a $\mathbb{Z}$-valued cochain).

Taking the square root $E^{1/2}$ of E by passing to an associated G bundle means taking the square root of all of E's transition functions. Suppose first that $\delta\Lambda \in 2H^2(X;\mathbb{Z})$. If this is true, then the transition functions of $E^{1/2}$ will fail the cocycle condition by

$$(\delta g^{E^{1/2}})_{\alpha\beta\gamma} \in \mathbf{1}e^{2\pi i\mathbb{Z}}, \tag{357}$$

which is still acceptable. If however the $\delta\Lambda$ is just a class in $H^2(X;\mathbb{Z})$, then the cocycle condition can fail in $E^{1/2}$ by $\pm\mathbf{1}$ on each patch; this is not acceptable, and the bundle $E^{1/2}$ does not exist.

[37] The isomorphism is given by

$$V = LX \times_1 \mathbb{R}^n \ni (u,v) = ((p,g),v) \mapsto (p, [R_1(g)](v)) \in TX, \tag{353}$$

where $u \in LX$, p is a basepoint in X, $g \in SO(n)$, and $v \in \mathbb{R}^n$. Note that as required, both (u,v) and $(hu, [R_1^{-1}(h)]v)$ get mapped to the same point in TX.

[38] In this case the spinor bundle S can be thought of as containing an action in the "charge 1/2 representation" of $U(1)$ on $\mathbb{R}^2$ (or $\mathbb{C}^2$?).

This means that the square-root-ibility of E is determined by how the transition functions fail the cocycle condition. But we know that if $(\delta\Lambda)_{\alpha\beta\gamma} = n_{\alpha\beta\gamma}$, then the first Chern class of E is just the class in $H^2(X;\mathbb{Z})$ which counts the $n_{\alpha\beta\gamma}$. Now from our above discussion, we see that in this case, $w_2(E)_{\alpha\beta\gamma}$ is precisely the class which counts $n_{\alpha\beta\gamma}$ mod 2. Thus $w_2(E)$ is the mod-2 reduction of c_1,[39] and so the nontriviality of $w_2(E) = c_1(E) \mod 2$ means an obstruction to consistently defining a bundle with square root transition functions.

Relating this to trivializing the 2-skeleton

How does this cocycle-centric definition of w_2 relate to other definitions of it as a SW class? For example, we know that $w_2(TX) \neq 0$ means that there is an obstruction to a extending a trivialization of a $SO(n)$ principal bundle from the 1-skeleton of X to the 2-skeleton[40]: how is this connected to the failure of the square roots of the transition functions to be closed in Cech cohomology? Now one way (I'm not sure if this is a really general statement) to go between a skeleton and a Cechian patch-covering of a manifold is to associate patches to each node of the 0-skeleton, such that the patches are chosen to be the convex regions that extend slightly beyond the halfway point of each of the 1-cells emanating from the 0-cell they are centered on. Choosing the patches this way means that we can associate to each 1-cell a 2-fold overlap of patches; this is nice because we can thinking about moving between 0-cells on the 0-skeleton as moving between patches, applying a transition function when we move across each 1-cell. Anyway, this also means that each 2-cell is associated to a triple overlap of patches, and so on: in this construction, each k-cell in the k-skeleton is associated to a $(k+1)$-fold overlap of patches.

Most importantly for us, each 2-cell corresponds to a triple overlap of patches. Now, what prevents us from extending a trivialization of a principal $SO(n)$ bundle from the 1-skeleton into the 2-skeleton? Such an extension will not be possible at a given 2-cell if the framing winds by the nontrivial element in $\pi_1(SO(d))$ around the boundary of that 2-cell. But this exactly translates to the condition that the square root of the transition functions on the triple patch overlap at the center of that 2-cell fail the cocycle condition by $-\mathbf{1}$: the framing winds by an odd multiple of 2π around the 2-cell, which corresponds to $-\mathbf{1}$ in $\text{Spin}(n)$. Thus the Cechian way of thinking about $w_2(TX)$ and the skeleton way of thinking about $w_2(TX)$ are the same: the nontriviality of either indicates that we won't be able to take the square root of our principal $SO(n)$ bundle. The same applies to other groups G and $G/\mathbb{Z}_2$, where the nontriviality of $\pi_1(G/\mathbb{Z}_2)$ prevents the trivialization of a G bundle to be extended into the 2-skeleton.

Fermions with a gauge field

Now let the fermions be coupled to a gauge field A, with gauge group G. In general, if we have a field coupled to a background field for $G \times H$, then we will have a principal $G \times H$ bundle, and the vector space used to construct the associated bundle of which the field is a section will transform in a representation of $G \times H$. What representation we choose is up to us, but it will always be expressible as a direct sum of $\otimes$'s between an irrep of G and one of

[39] this is only true for complex vector bundles!

[40] Here by k-skeleton, we mean the k-th dimensional part of a cell complex.

H, since the irreps of $G \times H$ are constructed as the $\otimes$ of irreps of G with irreps of H.[41] An example where we have a principal $G \times H$ bundle but use a reducible representation for the action on the vector space is when we are considering fermions in e.g. four dimensions, and making use of the decomposition $\mathrm{Spin}(4) = SU(2) \times SU(2)$. Here we do not want to take the $\otimes$ of two spin 1/2 irreps; rather, we want to take the representation

$$(1/2)_L \otimes \mathbf{1}_R \oplus \mathbf{1}_L \otimes (1/2)_R, \tag{360}$$

which is reducible. In this case, the associated bundle we get is a direct sum of two associated $SU(2)$ bundles, rather than a tensor product:

$$S = (SLX \times_{(1/2)_+} \mathbb{C}^{2^2}) \oplus (SLX \times_{(1/2)_-} \mathbb{C}^{2^2}). \tag{361}$$

Anyway, back to fermions coupled to a gauge field for an internal symmetry. We start out with a principal $G \times SO(n)$ bundle. If $w_2(TX) = 0$ then we can lift the $SO(n)$ factor to $\mathrm{Spin}(n)$, and we get the spinor bundle which we will write as

$$S_G = P_{G\times \mathrm{Spin}(n)} \times_{R_G \otimes 1/2} \Delta_n^G, \qquad \Delta_n^G = \mathbb{C}^{(\dim R_G)2^{\lfloor n/2 \rfloor -1}}. \tag{362}$$

The spinors are sections of this bundle. We will also write this as

$$S_G = (P_G \times_{R_G} \Delta^G) \otimes (P_{\mathrm{Spin}(n)} \times_{1/2} \Delta_n), \qquad \Delta^G = \mathbb{C}^{\dim R_G}. \tag{363}$$

Spin$_G$ structures

There is another option which we use to construct spinors, that can be employed even when $w_2(TX) \neq 0$ in cohomology. This option is available to us when $G = \widetilde{G}/\mathbb{Z}_2$ for some Lie group $\widetilde{G}$, such that $\mathrm{Spin}(n)$ and $\widetilde{G}$ share a common central $\mathbb{Z}_2$ factor in the way they act on the fermions: this is possible if they act on the fermions in a tensor product representation of the spinor representation and a representation of $\widetilde{G}$ which includes this $\mathbb{Z}_2$ factor. For example if $G = SO(3)$, we might take the fermions to transform in the $(1/2)_{\mathrm{Spin}(n)} \otimes (1/2)_{SU(2)}$ representation, since the fundamental of $SU(2)$ has a $-\mathbf{1}$ factor. In this case, the group that couples to the fermions is really $(\widetilde{G} \times \mathrm{Spin}(n))/\mathbb{Z}_2$, and the transition functions in the full bundle are blind to the quotiented $\mathbb{Z}_2$ factor.

[41] That is, every rep of $G \times H$ can be written as

$$(G \times H \to \mathrm{Aut}(V)) \ni R = \bigoplus_i \rho_i^g \otimes \rho_i^h, \tag{358}$$

where ρ_i^g, ρ_i^h are irreps of G and H, respectively. Furthermore, each factor in the direct sum is irreducible (as an irrep of $G \times H$). To prove this, we use Peter-Weyl: irreps of a compact group are the same as L^2 functions on that group (which exist because of the assumed compactness condition). A basis for these L^2 functions are precisely the characters. Now $L^2(G \times H) = L^2(G) \otimes L^2(H)$, which since the bases for the L^2 functions are provided by the characters, is the same thing as saying

$$\chi_{\rho^g}(g)\chi_{\rho^h}(h) = \chi_{\rho^g \otimes \rho^h}(g \times h), \tag{359}$$

which is true because $\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\mathrm{Tr}[B]$. Since the L^2 functions of the product group is the $\otimes$ of the individual L^2's, using Peter-Weyl proves the claim.

Anyway, while we might not be able to construct a bundle $SLX \times_{1/2} \Delta_n$ because the cocycle condition fails as $\delta g^{\mathrm{Spin}(n)} = -\mathbf{1}$ at some points, we still may be able to form the bundle

$$E = P_{(\widetilde{G}\times \mathrm{Spin}(n))/\mathbb{Z}_2} \times_{R_{\widetilde{G}}\otimes 1/2} \Delta_n^{\widetilde{G}}. \tag{364}$$

Here the bundle does *not* split as a tensor product of vector bundles:

$$E \neq (P_{\widetilde{G}} \times_{R_{\widetilde{G}}} \Delta^{\widetilde{G}}) \otimes (SLX \times_{1/2} \Delta_n), \tag{365}$$

since the latter factor does not exist. If E is to be well-defined, since the latter factor in (365) is not well-defined, the principal bundle $P_{\widetilde{G}}$ must also not be well-defined, in a compensating way: the transition functions in G must fail to lift to $\widetilde{G}$-valued transition functions in a way that cancels the ill-defined-ness of the latter tensor factor.

Now the transition functions of E are given by the matrices

$$g^E_{\alpha\beta} = R_{\widetilde{G}}(g^{\widetilde{G}}_{\alpha\beta}) \otimes R_{1/2}(g^{\mathrm{Spin}(n)}_{\alpha\beta}). \tag{366}$$

This means that if we choose our bundles such that cocycle conditions in each of the factors in (365) fail in the same way, the transition functions above will satisfy the cocycle condition. From what we saw earlier, the condition for the transition functions in the SLX and $P_{\widetilde{G}}$ factors to not be closed in Cech cohomology in the same way is given by

$$w_2(TX) = w_2(E_G), \tag{367}$$

where E_G is the vector bundle associated to the principal bundle P_G in the representation R_G.[42] Such an E is called (idk if this is standard?) a Spin_G bundle. Remember that for this to work, we also need to impose the condition that the representation $R_G \otimes 1/2$ that the fermions transform under is such that the representation R_G includes the $\mathbb{Z}_2$ factor corresponding to $-\mathbf{1}$ in G (i.e., if $G = SU(2)$, we must choose a half-odd-integer spin representation).

[42]Since E_G is a complex vector bundle, let's be pedantic and elaborate on exactly what we mean by this. Since we have been working with complex spinors, E_G is formed by

$$E_G = P_G \times_{R_G} \mathbb{C}^{\dim R_G}. \tag{368}$$

Then we really mean

$$w_2(E_G) = w_2([E_G]_{\mathbb{R}}) = w_2([P_G \times_{R_G} \mathbb{C}^{\dim R_G}]_{\mathbb{R}}), \tag{369}$$

where the subscript denotes the realification, which is accomplished by taking $\mathbb{C}^{\dim R_G} \to \mathbb{R}^{2\dim R_G}$ and realifying the G action by

$$R_G \to [R_G]_{\mathbb{R}} : G \to GL(2\dim(R_G); \mathbb{R}) \tag{370}$$

via the inclusion $GL(n;\mathbb{C}) \to GL(2n;\mathbb{R})$.

Now as we have seen, the SW classes of the realification of a complex vector bundle E are the mod-2 reduction of that vector bundle's Chern classes:

$$c_j(E) \xrightarrow{\textit{reduction mod 2}} w_{2j}(E_{\mathbb{R}}), \tag{371}$$

and so we can equivalently write

$$w_2(E_G) = c_1(E_G) \mod 2. \tag{372}$$

The simplest case is when $G = U(1)$. Suppose that $w_2(TX) \neq 0$, so that SLX does not exist. Suppose also that the line bundle $L = P_{U(1)}$ is such that

$$w_2(L) = [c_2(L)]_2 = w_2(TX). \tag{373}$$

Then a fermion field of charge q will be a section of the spinc bundle

$$E = P_{[U(1)\times \mathrm{Spin}(n)]/\mathbb{Z}_2} \times_{q/2\otimes 1/2} (U(1)\otimes \Delta_n) \text{ `` = '' } (\sqrt{L}\times_{q/2} U(1)) \otimes (SLX \times_{1/2} \Delta_n), \tag{374}$$

where strictly speaking the second way of writing things is schematic, since neither tensor factor makes sense. Here in order for the representation q of $U(1)$ to include the central $\mathbb{Z}_2$, we also need to take $q \in (2\mathbb{Z}+1)$, so that the element -1 in $U(1)$ is represented nontrivially in the transition functions.

When $G = U(1)$, the classification of such Spin_G structures works just in the same way as for Spin structures, but up a dimension: $\mathrm{Spin}_{U(1)}$ structures, alias $\mathrm{Spin}_{\mathbb{C}}$ structures, are obstructed by $w_3(TX)$, and they are in (non-canonical) bijection with elements in $H^2(X;\mathbb{Z})$, which correspond to different large instantons that can be inserted into the $U(1)$ factor of $\mathrm{Spin}(n)\times U(1)$.

Digression on $\mathrm{Spin}_{\mathbb{C}}$ and CS

The biggest use of $\mathrm{Spin}_{\mathbb{C}}$ connections comes when defining $U(1)$ CS theories on non-spin 3-manifolds. Such connections always exist because

$$w_3(TX) = 0, \qquad \dim X = 3,\ X \,\text{compact} \tag{375}$$

We can prove this just by recalling that $w_3(TX) = [e(TX)]_2$, where $e(TX)$ is the Euler class, which is true since $\dim X = 3$. But the Euler characteristic for all *compact* 3-manifolds (orientable or not) vanishes:[43]

$$[\chi(X)]_2 = \sum_{i=0}^{3} \dim(-1)^i H^i(X;\mathbb{Z}_2) = 0, \tag{376}$$

where the sum vanishes by Poincare duality.

To see why this is useful, let a be a $U(1)$ gauge field, and $k = 2l+1$, $l \in \mathbb{Z}$. Then while the CS term at level k is not well-defined unless X is spin, for arbitrary X we may write

$$\mathcal{L} = \frac{k}{4\pi} a\wedge da + \frac{1}{2\pi} a \wedge dB, \tag{377}$$

where B is a $\mathrm{Spin}_{\mathbb{C}}$ connection, so that

$$\frac{1}{2}\int_M w_2(TX) = \int_M \frac{F_B}{2\pi} \mod 2, \tag{378}$$

[43] $\chi(\mathbb{R}^3) = 1$ shows that we need to stipulate compactness.

for any closed $M \subset X$. To see that $\mathcal{L}$ makes sense, we consider the following integral over a closed 4-manifold X_4:

$$I = \int_{X_4} \left(\frac{2l+1}{4\pi} F_a \wedge F_a + \frac{1}{2\pi} F_a \wedge F_B \right). \tag{379}$$

Working modulo things in $2\pi\mathbb{Z}$ and letting $\bar{F} = F/2\pi$,

$$\begin{aligned} I &= 2\pi \int_{X_4} \left(\frac{1}{2} \int \bar{F}_a \wedge \bar{F}_a + \bar{F}_a \wedge \bar{F}_B \right) \mod 2\pi \\ &= 2\pi \int_{X_4} \left(\frac{1}{2} \bar{F}_a \wedge w_2 + \frac{1}{2} \bar{F}_a \wedge (2F_B) \right) \mod 2\pi \\ &= 2\pi \int_{X_4} \bar{F}_a \wedge w_2 = 0 \mod 2\pi. \end{aligned} \tag{380}$$

Thus $\mathcal{L}$ is indeed well-defined.

23 *February 23 — Specific heat for (massive) Dirac fermions*

Calculate the specific heat of a Dirac fermion in 1+1D, for $m = 0$, $m \gg T$, and $m \ll T$.

Solution:

First for $m = 0$. The L and R components of a massless Dirac fermion are decoupled, and so we can just calculate C for a single component and then multiply the result by 2. The energy of excitations with respect to the Dirac sea where all negative energy states are filled is $v_F|k|$, and so we have, for a single component,

$$\begin{aligned} \langle E \rangle &= -\partial_\beta \ln Z = -\partial_\beta \ln \left[\sum_{n_k=0,1} e^{-\beta v_F \sum_k |k| n_k} \right] = -\partial_\beta \int_{\mathbb{R}} \frac{dk}{2\pi} \ln \left[1 + e^{-\beta v_F |k|} \right] = \frac{v_F}{\pi} \int_0^\infty \frac{k}{e^{\beta v_F k} + 1} \\ &= \frac{T^2}{\pi v_F} \int_0^\infty dl \frac{l}{e^l + 1} = \frac{\pi T^2}{12 v_F}. \end{aligned} \tag{381}$$

Thus the heat capacity for a single component is $\pi T/(6v_F)$, and so we have (letting $k_B = 1$)

$$C(T)_{m=0} = \frac{\pi T}{3v_F}. \tag{382}$$

Alternatively, we can do the calculation as follows: expand the modes of a chiral (say, holomorphic) Majorana fermion on a cylinder (treating the imaginary compact direction as space, with circumference L, and setting $v_F = 1$) as

$$\psi(z) = \frac{1}{\sqrt{L}} \sum_{k \in \mathbb{Z}+1/2} \psi_k z^k, \tag{383}$$

where z is the complex coordinate on the cylinder. Here the $1/\sqrt{L}$ is needed so that $\psi(z)$ has energy dimension 1/2, while ψ_k remains dimensionless (since the Hamiltonian is a sum, not integral, over k values). The modding in $\mathbb{Z}+1/2$ is because the fermions are taken to have APBC on the cylinder. Anyway, now we can calculate

$$\frac{1}{2}\int \langle\psi\partial\psi\rangle = -\frac{1}{2L}\sum_k \langle\psi_k(2\pi k/L)\psi_{-k}\rangle = \frac{\pi}{L^2}\sum_{k>0} k, \tag{384}$$

since ψ_k and ψ_{-k} are conjugate and $\psi_{k>0}$ annihilates the vacuum. Now we ζ-regularize and use

$$\sum_{k>0} k = \sum_{n\in\mathbb{Z}^{\geq 0}} \frac{1}{(n+1/2)^{-1}} = \zeta(-1,1/2) = \frac{1}{24} \implies \frac{1}{2}\int\langle\psi\partial\psi\rangle = \frac{\pi}{24L^2}. \tag{385}$$

This gives the expectation value of the energy on the cylinder for a single chiral Majorana. Summing over the two decoupled chiralities and undoing the breaking up of one Dirac fermion into two Majorana ones, we multiply by 4 and see that the vev of the cylinder energy for a Dirac fermion is $\pi/6L^2$. Since the circumference of the cylinder maps on to the inverse temperature, we get $\langle E\rangle = \pi T^2/6 \implies C = \pi T/3$, which agrees with our earlier result (after restoring v_F).

What happens when we add a mass? Now that we have another energy scale in the problem, we can form the dimensionless ratio m/T. When $m\to\infty$ the fermions disappear, so we should expect some dependence of C on m of the form $e^{-m/T}$. Indeed, taking into account both the particle-like and hole-like excitations (and setting $v_F = 1$ for simplicity),

$$\langle E\rangle = -\frac{2}{\pi}\partial_\beta \int_{k>0} \ln\left[1+e^{-\sqrt{k^2+m^2}}\right] = \frac{2}{\pi}\int_{k>0}\frac{\sqrt{k^2+m^2}}{e^{\beta\sqrt{k^2+m^2}}+1}, \tag{386}$$

which has a T dependence of $e^{-m/T}$ for $m\to\infty$, so that in this limit the specific heat has a T-dependence of something like $C(T)\sim T^{-2}e^{-m/T}$. The exact formula for the above integral isn't very edifying. However we can expand to pick up the non-relativistic $k^2/2m$ parts (which will be a bad approximation when $k\to\infty$ but those regions make small contributions to $\langle E\rangle$ anyway), and write

$$\begin{aligned}\langle E\rangle_{m\to\infty} &\approx \frac{2}{\pi}\int_{k>0}\frac{m+k^2/2m}{e^{\beta(m+k^2/2m)}+1} \approx -\sqrt{\frac{m}{2\pi\beta}}\left(2m\,\mathrm{PolyLog}(1/2,-e^{-\beta m}) + \beta^{-1}\,\mathrm{Polylog}(3/2,-e^{-\beta m})\right)\\ &\approx \sqrt{\frac{m}{2\pi\beta}}(2m+\beta^{-1})e^{-\beta m}.\end{aligned} \tag{387}$$

Thus the specific heat behaves as

$$C(T)_{m\to\infty} \approx \sqrt{\frac{m}{2\pi}}e^{-m/T}\left(\sqrt{T}+\frac{2m+T}{2\sqrt{T}}+\frac{m(2m+T)}{T^{3/2}}\right). \tag{388}$$

Alternatively we can look at happens when m is small (compared to T), and we're near the phase transition point. Here the integrals are messier so we will have to be a bit imprecise.

We will approximate E as

$$\langle E\rangle_{m\to 0} \approx \frac{2}{\pi}\left(\int_0^m \frac{m+k^2/2m}{e^{\beta m}+1} + \int_m^\infty \frac{k}{e^{\beta k}+1}\right), \tag{389}$$

so that we can actually do the integrals. Doing the them gives

$$\langle E\rangle_{m\to 0} \approx \frac{7m^2}{3\pi(1+e^{\beta m})} + \frac{\pi}{6\beta^2} - \frac{m^2}{2\pi} + \frac{\beta m^3}{6\pi}. \tag{390}$$

This means that the specific heat behaves near the phase transition point as

$$C(T)_{m\to 0} \approx \frac{\pi T}{3} + \frac{7m^3}{3\pi}\frac{T^{-2}}{(1+e^{\beta m})^2} - \frac{m^3}{6\pi T^2}, \tag{391}$$

where the numerical factors are not to be taken seriously.

24 *February 25 — The SSH model and polarization*

This is a problem from Senthil's 2019 class on correlated electronic systems. The Hamiltonian for the SSH model is

$$H = -\sum_i (t_1 c^\dagger_{iA} c_{iB} + t_2 c^\dagger_{iB} c_{i+1A} + h.c.). \tag{392}$$

Do several things. a) show that

$$\widetilde{C} : c_{iA} \mapsto c^\dagger_{iA}, \quad c_{iB} \mapsto -c^\dagger_{iB} \tag{393}$$

is a symmetry of H, and identify it in terms of C, P, T. b) Show that $\widetilde{C}$ implies that the spectrum of H comes in $\pm\lambda$ pairs. c) Write the Hamiltonian at a given momentum as $H(k) = h_i(k)\sigma^i$. Consider varying k around the S^1 of the BZ. How are the $t_1 = 0$ and $t_2 = 0$ insulators different? d) Calculate the polarization (alias Berry phase) for each band in the two insulating phases. e) Argue that in the presence of $\widetilde{C}$ symmetry, the polarization is quantized.

Solution:

a) Since $c^\dagger_{i\alpha}$ and $c_{j\beta}$ anticommute for all $\alpha \neq \beta$ (here $\alpha, \beta \in \{A, B\}$), we see that the map

$$\widetilde{C} : c_{iA} \mapsto c^\dagger_{iA}, \quad c_{iB} \mapsto -c^\dagger_{iB}, \qquad i \mapsto -i, \tag{394}$$

is a symmetry of H: the minus sign on the B sublattice is compensated for by the fermion anticommutation sign. Here the fact that $\widetilde{C}$ is antiunitary follows from us wanting to send $c^\dagger_{kA} \mapsto c_{kA}, c^\dagger_{kB} \mapsto -c^\dagger_{kB}$ in momentum space.

Pedantic discussion of the $\widetilde{C}$ symmetry: Some cmt people call this charge conjugation, but this is not what we'd call charge conjugation, since we usually think of charge conjugation as being a *unitary* symmetry. To identify what $\widetilde{C}$ actually does, it helps to look at the continuum theory so that we can compare to field theory conventions. In what follows we will need the continuum Hamiltonian near $t_1 \sim t_2 \sim t$, linearized around the gapless point $k = \pi$. Writing $t_1 = t + \delta/2, t_2 = t - \delta/2$ and $k = \pi + q$, we have to linear order in q,

$$\mathcal{H}(q) \approx \delta X + vqY, \qquad v = 2t. \tag{395}$$

From now on we will set $v = 1$. This gives us the Dirac Hamiltonian

$$H = \int dx\, \psi^\dagger(x)(-iY\partial_x + \delta X)\psi(x). \tag{396}$$

What γ matrices are we using? Since the Dirac mass is $m\bar{\psi}\psi$ (which is Hermitian in $(+,-)$ signature), we can take $\gamma^0 = X$. We will work in $\mathbb{R}$ time so that thinking about T is easier, so that means we should have γ^1 such that $-i\gamma^0\gamma^1 = -iY$, since the Lagrangian is $\bar{\psi}(i\partial\!\!\!/ + m)\psi$, meaning that H contains $-i\bar{\psi}\gamma^1\partial_x\psi$. Anyway, we see that if we take $\gamma^1 = iZ$, then we get $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and indeed, $-i\gamma^0\gamma^1 = XZ = -iY$.

Now we should identify how C, P, T act. Here C is charge conjugation, which we should avoid confusing with the anti-unitary particle-hole symmetry $\widetilde{C}$. Unfortunately the representation theory of Pin groups is a huge mess, and depends on the signature we're working in and all sorts of details. So, we will just find the action of these symmetries by explicit computations. P should satisfy $P^\dagger\gamma^0 P = \gamma^0$ and $P^\dagger\gamma^0\gamma^1 P = -\gamma^0\gamma^1$, and so we can take $P : \psi(t,x) \mapsto \gamma^0\psi(t,-x)$. This is indeed the generator of reflections that send $x \mapsto -x$ in the Clifford algebra generated by γ^0 and γ^1. Since $\gamma^0 = X$, this also agrees with how we know reflections act on the microscopic chain: by exchanging A and B sublattices.

We want C to be unitary, and so requiring that it preserve the kinetic term means that for $(C)\psi_\alpha = [C]_{\alpha\beta}\psi^*_\beta$, we need

$$-\bar{\psi}\gamma^\mu\psi = \psi^T C^\dagger\gamma^0\gamma^\mu C\psi^* = -\psi^\dagger[C^\dagger\gamma^0\gamma^\mu C]^T\psi \implies [C^\dagger\gamma^0\gamma^\mu C]^T = \gamma^0\gamma^\mu \tag{397}$$

for all μ (here the first minus sign on the left is since there is an extra minus sign in the transformation of the kinetic term after integrating by parts, which in my experience is incredibly easy to forget about). Setting $\mu = x$ means that $C^T ZXC^* = -ZX$, so that $C = Z$ or $C = X$ (since we want $C^2 = \mathbf{1}$). Now if C is to preserve $\bar{\psi}\psi$, then we need $[C^\dagger\gamma^0 C]^T = -\gamma^0$, and so we see that we can choose $C : \psi \mapsto Z\psi^*$. With this choice, C does *not* preserve the chiral mass, since $[C^\dagger YC]^T \neq -Y$, which is annoying.[44]

Now for T. To preserve the kinetic term $\bar{\psi}i\partial\!\!\!/\psi$, we need, for $T : \psi(t,x) \mapsto T_U K\psi(-t,x)$ with K complex conjugation and T_U unitary,

$$-iT_U^\dagger\gamma^0(-\partial_0\gamma^0 - \partial_x\gamma^1)T_U = +i\gamma^0\partial\!\!\!/. \tag{398}$$

Thus T_U needs to commute with $\gamma^0\gamma^1 = Y$, and so we can take either $T_U = J, Y$ or $T_U = \mathbf{1}$[45] Let T_+ be the symmetry that acts as K, and T_- the one that acts as JK. One can check

[44] If we chose $C = X$ instead, then both the Dirac mass and the chiral mass would be odd under C.

[45] This later option is a bit weird: if we were just thinking about the representation of the Ciifford group, we might have guessed that T multiplied the spinors by γ^1, since γ^1 reflects the γ matrices about the t axis, and is present in the action of T in Euclidean signature. This just goes to show how much of a mess the representations of the Pin groups are! When we choose a wonky representation for the γ matrices, the symmetries can act in wonky ways.

that the combination $CPT_\pm$ is also a symmetry of the theory, as it must be: the relevant thing to check for the kinetic term is that

$$-[(CPT_U)^\dagger Y(CPT_U)]^T = Y. \tag{399}$$

Now for T_+, $CPT_U = J$, while for T_-, $CPT_U = \mathbf{1}$: for both cases, the above equation holds (one can also check that it holds if we instead take $C = X$, and / or if we take $P = Z$).

Anyway, if these symmetry identifications are correct, then the operation $CT_+ : \psi \to Z\psi^* K$ is an anti-linear symmetry which functions in the same way as particle-hole symmetry $\widetilde{C}$.

To check the sensibility of this, let's look at how various fermion bilinears transform. The Dirac mass $c_A^\dagger c_B + c_B^\dagger c_A$ is even under CT_+, and odd under CT_-. The current is

$$j^\mu = \bar\psi\gamma^\mu\psi = (c_A^\dagger c_A + c_B^\dagger c_B,\, -i[c_A^\dagger c_B - c_B^\dagger c_A]) \implies \widetilde{C} : j^0 \mapsto -j^0,\, j^1 \mapsto j^1. \tag{400}$$

This action on the current makes sense for PH symmetry, and is also what we expect for CT_+ symmetry in field theory (it transforms in the same way under CT_-). Similarly, using that $\gamma^5 = iXZ = Y$ is Hermitian, the axial mass maps as

$$\bar\psi m_5\psi = i\bar\psi\gamma^5\psi = -\psi^\dagger Z\psi = -c_A^\dagger c_A + c_B^\dagger c_B \implies \widetilde{C} : m_5 \mapsto -m_5, \tag{401}$$

which is again what we expect from CT_+ symmetry in field theory (by contrast, the axial mass is even under CT_-). Note that here, the physical interpretation of the regular Dirac mass is an anisotropy between the strengths of t_1 and t_2, while the chiral mass becomes a anisotropic perturbation that encourages the densities on the A and B sublattices to be different.

Finally we can check that the axial current, being the hodge dual of the vector current, maps as (in our signature, there's no i out front in the axial current)

$$j_5^\mu = \bar\psi\gamma^\mu\gamma^5\psi \implies : \widetilde{C} : j_5^0 \mapsto j_5^0, j_5^1 \mapsto -j_5^1, \tag{402}$$

which yes, is just what we expect from CT_+ symmetry (it is acted on by CT_- in the same way).

To break $\widetilde{C}$ symmetry then, we would want to add a Lorentz-invariant term that breaks CT symmetry. From the options surveyed so far, we see that we can only do this with the chiral mass term $\bar\psi m_5\psi$. Since this term is $\psi^\dagger Z\psi$, the only way to break the symmetry in a Lorentz-invariant way is to make $h^z(k)$ in $\mathcal{H}(k)$ nonzero, thereby leading to different preferred occupation numbers on the two sublattices.

b) Diagonalizing by going to Fourier space (with sign convention $c_{r\alpha} \to \int_k e^{+ikr} c_{k\alpha}$), we see that (setting the lattice spacing $a = 1$ and setting $V = (2\pi)^d$ to make the Fourier transforms simpler)

$$H = \int_k c_k^\dagger \mathcal{H}(k) c_k, \qquad c_k = (c_{kA},\, c_{kB})^T, \tag{403}$$

where

$$\mathcal{H}(k) = \begin{pmatrix} 0 & t_1 + t_2 e^{-ik} \\ t_1 + t_2 e^{ik} & 0 \end{pmatrix} = t_1 X + t_2(X\cos k + Y\sin k) \tag{404}$$

We see that under charge conjugation $\widetilde{C}$,

$$\widetilde{C} : H \mapsto \int_k c_k^T Z\mathcal{H}^*(k)Z[c_k^\dagger]^T = -\int_k c_k^\dagger [Z\mathcal{H}^*(k)Z]^T c_k = -\int_k c_k^\dagger Z\mathcal{H}(k)Zc_k, \tag{405}$$

since $\mathcal{H}^*(k) = \mathcal{H}^T(k)$ on account of $\mathcal{H}(k)$'s Hermiticity and off-diagonal-ness. Now since $ZXZ = -X, ZYZ = -Y$, we have $Z\mathcal{H}(k)Z = -\mathcal{H}(k)$, which as before cancels the fermion anticommutation sign. Anyway, the point of this is that all eigenstates at a fixed k (with non-zero energy) come in pairs, since if $|E\rangle$ is an eigenstate with energy E, then $Z|E\rangle$ is an eigenstate with energy $-E$, since $\mathcal{H}Z|E\rangle = ZZ\mathcal{H}Z|E\rangle = -Z\mathcal{H}|E\rangle = (-E)Z|E\rangle$.

c) Now as we saw, $\mathcal{H}(k) = h^i(k)\sigma^i$, for

$$h^x(k) = t + 1 + t_2 \cos k, \quad h^y(k) = t_1 + t_2 \sin k, \quad h^z(k) = 0. \tag{406}$$

Now since $\det(\lambda\mathbf{1} - h^i(k)\sigma^i) = \lambda^2 - h^i(k)h_i(k)$, we will have a band gap provided that $|h(k)| > 0$, since the spectrum of $\mathcal{H}(k)$ is $\lambda_\pm = \pm|h(k)|$. For the SSH model then, if we want to have a gap closing, we need $h^z(k) = 0$. Note that this is protected by $\widetilde{C}$ symmetry: if $h^z(k) \neq 0$ then we no longer have $Z\mathcal{H}(k)Z = -\mathcal{H}(k)$, which means that $\widetilde{C}$ is no longer a symmetry of H.

First consider the $t_2 = 0$ insulator. In the x-y equatorial plane of the ball in $h^i(k)$ space, $\mathcal{H}(k)$ is just a single point on the x axis at $x = t_1$. On the other hand, the $t_1 = 0$ insulator is a circle (the image of the BZ) in the plane of radius t_2. Since we are only interested in classifying the insulating phases, we can work on the punctured plane $\mathbb{R}^2 \setminus \{0\}$, since if part of the BZ is mapped to the point $h^x = h^y = 0$, the system is gapless. The difference between these two insulators is then the homotopy class of the map of the BZ into $\mathbb{R}^2 \setminus \{0\}$: trivial in the case of the $t_2 = 0$ insulator, and nontrivial for the $t_1 = 0$ insulator. We see that the transition between these two cases is at the point $t_1 = t_2$, where the image of the $k = \pi$ point of the BZ touches the origin, where $\mathcal{H}(k) = 0$.

d) We now want to calculate the Berry phase along the S^1 of the BZ for the two insulators. First consider the case $t_2 = 0$. Then the eigenvectors of $\mathcal{H}(k)$ are just $(\pm 1,\, 1)^T$, and are momentum-independent: thus the Berry connection vanishes, and $\Phi_\pm \equiv \oint_{S^1} a_\pm = 0$, where

$$a_\pm = i\langle\chi_\pm|d|\chi_\pm\rangle \tag{407}$$

with $\chi_\pm(k)$ the eigenvectors of $\mathcal{H}(k)$ (we can label them by $\pm$ because of $\widetilde{C}$ symmetry). More precisely, $\Phi_\pm \in 2\pi\mathbb{Z}$: translation symmetry means that $\chi_\pm(k) \cong e^{i\theta(k)}\chi_\pm$, with $\theta(2\pi) = \theta(0) + 2\pi n$, $n \in \mathbb{Z}$. We have the freedom to perform this shift since the Hamiltonian, being translation-symmetric, contains no ∂_k factors.

What about when $t_1 = 0$? Then the eigenvectors of $\mathcal{H}(k) = t_2(X \cos k + Y \sin k)$ are $\chi_\pm(k) = (1,\, \pm e^{ik})^T/\sqrt{2}$. Then $\Phi_+ = \Phi_-$ is[46]

$$\Phi_+ = \frac{i}{2}\oint_{S^1} dk\,(1, \pm e^{-ik}) \cdot (0, i \pm e^{ik})^T = -\pi \sim \pi, \tag{408}$$

[46]It doesn't matter whether we calculate Φ_+ or Φ_-: as we saw, $\Phi_\pm$ is related to the product of reflection eigenvalues at the reflection-symmetric points $k = 0, \pi$. But the reflection eigenvalue on the $-$ band is the negative of the reflection eigenvalue on the $+$ band, since reflection (X) and particle-hole (ZK) anti-commute. Since there are two reflection symmetric points, the product of the two eigenvalues differs between the two bands by a factor of $(-1)^2 = 1$.

where the last step is because $\Phi_+ \sim \Phi_+ + 2\pi$.

We can then conjecture that for general t_1, t_2, the Berry phase will be equal to half the winding number in the h^x, h^y plane. To prove this, it's helpful to get a more general expression for the $\chi_\pm$. Now thinking in terms of $SU(2)$ generators, $\mathcal{H}(k)$ acts to perform a rotation about the vector $\hat{h}^i$. Thus its eigenstates will be proportional to h^i: taking them to be normalized properly, we can just take them to be $\pm\hat{h}^i$. To write this as a vector transforming in the fundamental of $SU(2)$, we need the coordinates θ, ϕ on the Bloch sphere such that $\cos\theta = h^z/|h|$ and $e^{i\phi} = (h^x + ih^y)/\sqrt{h_x^2 + h_y^2}$. Then

$$\chi_+ = (e^{i\phi}\cos(\theta/2),\, \sin(\theta/2))^T, \tag{409}$$

which can be checked to reduce to the correct eigenstates when only one of the h^i is non-zero. Another common choice of coordinates has a factor of $e^{i\phi/2}$ in the first slot and $e^{-i\phi/2}$ in the second slot, but this is not periodic in $\phi \mapsto \phi + 2\pi$ and as such is a bad coordinate choice for us. Our coordinates are singular when $\theta = 0$ at the north pole, but this is okay since in the SSH we will always be on the equatorial plane.[47] The other eigenstate is found by sending $\phi \mapsto \phi + \pi$ and $\theta \mapsto \pi - \theta$:

$$\chi_- = (e^{i\phi}\sin(\theta/2),\, -\cos(\theta/2))^T, \tag{410}$$

where we have done away with an un-important factor of i.

For the application to the SSH model, the $\widetilde{C}$ symmetry tells us that $h^z = 0$, and hence $\theta = \pi/2$. Therefore our eigenstates are

$$\chi_\pm = \frac{1}{\sqrt{2}}(e^{i\phi},\, \pm 1)^T. \tag{411}$$

The Berry phase is then

$$\Phi_\pm = i\oint dk\, a_k = \frac{i}{2}\oint dk\,(i\partial_k\phi). \tag{412}$$

Already, it's clear that this is (the negative of) the winding number, and independent of the choice of band $\pm$. To make the former more explicit,

$$\begin{aligned}\Phi_\pm &= -\frac{1}{2}\oint dk\,\partial_k(-i\ln[(h^x + ih^y)/\sqrt{h_x^2 + h_y^2}]) = -\frac{1}{2}\oint dk\,\partial_k \arctan(h_y/h_x) \\ &= -\frac{1}{2}\oint dk\,\frac{h_x\partial_k h^y - h_y\partial_k h_x}{h_x^2 + h_y^2} = -\frac{1}{2}\oint dk\,\left[\frac{h_x}{|h|}\partial_k\frac{h_y}{|h|} - \frac{h_y}{|h|}\partial_k\frac{h_x}{|h|}\right] = -\frac{1}{2}\oint dk\,(\hat{h}\times\partial_k\hat{h})_z,.\end{aligned} \tag{413}$$

which is the usual formula for (half) the winding number.

Since the winding number is quantized, and is unchanged for changes in t_1, t_2 which don't lead to a closing of the gap, we conclude that the Berry phase is also quantized. This means that the Berry curvature $F_{t_1 k} = \partial_{[t_1}a_{k]}$ vanishes everywhere except for at the degenerate

[47] We'll never be able to choose a single coordinate patch that will give a well-defined Berry connection on the whole Bloch sphere, since the connection is that of a bundle $SU(2) = S^3 \to S^2$ (the base space is the parameter space of the Bloch sphere, while the total space is S^3 since this is the space the spinors [eigenstates of $H(k)$] live in) with the $U(1)$ fibered over the S^2 with nonzero Chern number.

point where the bands touch, which evidently must be a 2π monopole of Berry curvature (likewise for $F_{t_2 k}$), since in the presence of a 2π monopole the Berry phase along an arc is $1/2$ of the enclosed solid angle, and our arc around the equator encloses a solid angle of 2π. We can check this explicitly using (it's the same for each band $\pm$)

$$F_{t_j k} = \partial_{[t_j} a_{k]} = \frac{1}{2}\partial_{[t_j}\partial_{k]}\phi. \tag{414}$$

The antisymmetrized derivative vanishes everywhere that ϕ is non-singular. Since $\phi = \arctan(h_y/h_x)$, which is singular only as $h_x^2 + h_y^2 \to 0$, we see that the Berry curvature indeed is a δ function of strength 2π supported at the origin of the equatorial plane.

We can also see the quantization of the Berry phase to π from reflection symmetry, which acts by conjugating $\mathcal{H}(k)$ by X. This is equivalent to sending $\mathcal{H}(k) \mapsto \mathcal{H}(-k)$, under which the Berry connection is odd (and under which $\phi \mapsto -\phi$, since $h^y \mapsto -h^y$): thus reflection symmetry tells us that $\Phi_\pm = -\Phi_\pm \mod 2\pi \implies \Phi_\pm \in \{0, \pi\}$.

e) As we saw earlier, the presence of $\widetilde{C}$ symmetry tells us that $h^z = 0$. In this case, as we have shown above, the Berry phase is determined by half the winding number in the equatorial plane, and hence is quantized. If we were to break $\widetilde{C}$ by adding an $h^z \neq 0$ term, we would be working on a parameter space $\mathbb{R}^3 \setminus \{0\}$ rather than $\mathbb{R}^2 \setminus \{0\}$: the former has trivial π_1, and hence the polarization (alias Berry phase) would no longer be quantized.

25 *February 27 — Wannier states and the Berry connection*

This is a problem from Senthil's 2019 class on correlated electronic systems. Find an expression for the Wannier centers in terms of the Berry connection (expressed in terms of the Bloch wavefunctions. Here the Wannier center for a band n is defined as

$$r_n = \langle nR|(r - R)|nR\rangle, \tag{415}$$

where $|nR\rangle$ are the Wannier states for the band n.

Solution:

We let

$$|nR\rangle = \frac{1}{\sqrt{N}}\sum_k e^{-ikR}|\psi_{nk}\rangle \tag{416}$$

denote the Wannier states, with

$$|\psi_{nk}\rangle = e^{ikr}|u_{nk}\rangle, \quad \langle r + a|u_{nk}\rangle = \langle r|u_{nk}\rangle \tag{417}$$

the Bloch states. Note how we are being sloppy and not writing dot products and not using vector notation—sorry. Letting $\langle\psi_{nk}|\psi_{n'k'}\rangle = \delta_{n,n'}\delta_{k.k'}$ this normalization gives $\langle nR|nR\rangle = N^{-1}\sum_k = 1$, since if the system has $N = (La)^d$ then there are N choices for k as $k \in \frac{2\pi}{La}\mathbb{Z}^d$ but $k \sim k + 2\pi/a$, so really $k \in \frac{2\pi}{La}\mathbb{Z}_L^d$, which is a lattice with N different k points.

First we assume $V = (La)^d$ is large. Then the sum goes over to an integral as $\sum_k \to V/(2\pi)^d \int k$. The reason for the factors is as follows: the points in the lattice over which we are summing k lie at the vertices of a cubic lattice. The smallest cube in this lattice has volume $(2\pi/La)^d$, since $2\pi/La$ is the smallest magnitude of k value in the lattice. If the volume is large, we can instead pretend we have a continuum of k values (within the BZ), and sum over the average of the k values in each unit cube of the lattice. Thus we should integrate over k in the BZ, and divide by $(2\pi/La)^d$, and $\sum_k \to (La/2\pi)^d \int_k = V/(2\pi)^d \int k$. Then we can write

$$(r-R)\langle r|Rn\rangle = \frac{V}{(2\pi)^d}\int_k (r-R)e^{ik(r-R)}\langle r|u_{nk}\rangle = \frac{V}{(2\pi)^d}\int_k \left(-i\frac{d}{dk}e^{ik(r-R)}\right)\langle r|u_{nk}\rangle. \quad (418)$$

Now we integrate by parts:

$$(r-R)\langle r|Rn\rangle = \frac{V}{(2\pi)^d}\int_k e^{ik(r-R)}\langle r|i\frac{d}{dk}|u_{nk}\rangle, \quad (419)$$

and take the inner product by multiplying by $\langle Rn|r\rangle$ and integrating over r: defining $r_n \equiv \langle Rn|(r-R)|Rn\rangle$, we have

$$r_n = \frac{V^2}{(2\pi)^{2d}}\int_{k,k',r} e^{ir(k-k')}f(r;k,k'), \quad (420)$$

where

$$f(r;k,k') \equiv e^{iR(k'-k)}\langle u_{nk'}|r\rangle i\frac{d}{dk}\langle r|u_{nk}\rangle. \quad (421)$$

Note that $f(r+a;k.k') = f(r;k,k')$ by the periodicity of the $|u_{nk}\rangle$'s.

Let $x \in [0,a]^d$, and write $r = Na + x$, with $N \in \mathbb{Z}^d$. Then because of the periodicity of f,

$$r_n = \frac{V^2}{(2\pi)^{2d}}\int_{k,k',x}\left(\sum_{N\in\mathbb{Z}^d} e^{i(k-k')Na}\right)e^{i(k-k')x}f(x;k,k'). \quad (422)$$

The sum over N tells us that $k - k' \in \frac{2\pi}{a}\mathbb{Z}^d$. But since we are only summing over k, k' in the BZ, this is equivalent to setting $k = k'$. We can then eliminate the k' integration, along with one of the factors of $V/(2\pi)^d$, which gets killed by a factor of the BZ volume. Thus using the resolution of the identity on the $|r\rangle\langle r|$ inside the f, we see that

$$r_n = \frac{V}{(2\pi)^d}\int_k \langle u_{nk}|i\frac{d}{dk}|u_{nk}\rangle. \quad (423)$$

26 *February 28 — Domain walls in the SSH model, index theorems, and CT symmetry*

This is yet another problem from Senthil's 2019 class on correlated electronic systems. a) Show that in the continuum formulation of the SSH model (near the phase transition), a domain wall in the sign of the hopping anisotropy (alias Dirac mass) localizes a fermion zero mode. b) What is the charge of this zero mode? Discuss ad nauseum.

Solution:

a) The continuum Hamiltonian is (setting $v = 1$ wolog; it's a non-universal parameter)

$$H = -iY\partial_x + m(x)X. \tag{424}$$

Zero modes $\psi = (\psi_L, \psi_R)^T$ satisfy

$$(-\partial_x + m)\psi_R = 0, \qquad (p_x + m)\psi_L = 0. \tag{425}$$

This is solved by

$$\psi_L(x) \propto e^{-\int_0^x dx'\, m(x')}, \qquad \psi_R(x) \propto e^{\int_0^x dx'\, m(x')}. \tag{426}$$

Let $m(\pm\infty) = \pm m_0$. Then we see that if $m_0 > 0$, $\psi_R(x)$ is not normalizable, while if $m_0 < 0$ then $\psi_L(x)$ is not normalizable. Therefore we can construct a normalizable (but not normalized; the normalization coefficient depends on the profile of $m(x)$) zero-mode solution by

$$\psi_0(x) = \begin{pmatrix} (1+\ \mathrm{sgn}(m_0))e^{-\int_0^x dx'\, m(x')} \\ (1-\ \mathrm{sgn}(m_0))e^{\int_0^x dx'\, m(x')} \end{pmatrix}. \tag{427}$$

b) What is the charge carried by the domain wall? The domain wall has two states, $|0\rangle, |1\rangle$, according to whether or not the zero-mode is filled. Particle-hole symmetry should exchange these, $\widetilde{C} : |0\rangle \propto |1\rangle$. For $k \in \mathbb{Z}_2$, since $\widetilde{C}^2 = \mathbf{1}$[48], we have

$$\widetilde{C} : |k\rangle \mapsto e^{i\phi}|-k+1\rangle, \tag{428}$$

with the antilinearity of $\widetilde{C}$ ensuring $\widetilde{C}^2 = \mathbf{1}$. Now consider the charge operator e^{iQ}, which when acting on $|k\rangle$ we define to have the eigenvalue e^{iq_k}. Since $\widetilde{C}$ anticommutes with Q, it commutes with e^{iQ}.[49] Thus we have

$$\widetilde{C}e^{iQ}|k\rangle = \widetilde{C}e^{iq_k}|k\rangle = e^{-iq_k+i\phi}|-k+1\rangle. \tag{429}$$

On the other hand,

$$\widetilde{C}e^{iQ}|k\rangle = e^{iQ}\widetilde{C}|k\rangle = e^{iq_{-k+1}+i\phi}|-k+1\rangle. \tag{430}$$

Now since $|0\rangle$ and $|1\rangle$ differ by a single electron which has charge 1, we know that $q_1 = q_0 + 1$. But on the other hand, the above equations tell us that $q_0 = -q_1$. These two equations can only be satisfied if $q_0 = -1/2$, $q_1 = 1/2$, and so the domain wall indeed carries fractional charge.

[48] We can see this from $\widetilde{C} = ZK$ for K complex conjugation, or by using $\widetilde{C} \sim P$ by CPT, which squares to $\mathbf{1}$ (rather than e.g. $(-1)^F$).

[49] Recall that $\widetilde{C}$ is really CT. Now $Q = \int \star j$. j is odd under both T as a differential form, so that $\star j$ is even. $\star j$ is odd under C, so Q is odd under CT, meaning that, because of the antilinearity of CT, $\widetilde{C}$ and e^{iQ} commute.

This analysis was focused on a single domain wall, but we can also consider a situation where the mass changes sign only for a finite region. So, consider a situation where the flipped-mass domain has endpoints x_L and x_R. Each endpoint will localize a zero mode, but since the wavefunctions $\psi_{0,L/R}(x)$ extend away from the domain wall endpoints, there will be a hybridization between the two of order

$$t_{LR} = \int dx\, \psi^*_{0,L}(x)\psi_{0,R}(x) \sim \int dx\, e^{-(x-x_L)/\xi} e^{-(x_R-x)/\xi} \sim e^{-L/\xi}, \quad \xi = 1/m \tag{431}$$

where $L = x_R - x_L$ is the length of the flipped-mass region. Going back to the lattice model, the Hamiltonian will include the term

$$H \ni H_{DW} = -t_{LR}(c^\dagger_{x_L A} c_{x_R B} + h.c.), \tag{432}$$

where wolog we have taken the lattice point x_L to be on the A sublattice and x_R to be on the B sublattice. The eigenstates of H_{DW} are, just focusing on the x_L and x_R sites,

$$E = \pm t_{DW}: \quad \frac{c^\dagger_{x_L A} \pm c^\dagger_{x_R B}}{\sqrt{2}}|0 \otimes 0\rangle, \qquad E = 0: \quad (c^\dagger_{x_L A} c^\dagger_{x_R B} \pm 1)|0 \otimes 0\rangle, \tag{433}$$

which are all eigenstates of the particle-hole symmetry, since we haven't broken it. Anyway, we see that in the ground state, there is a single delocalized electron, smeared out between the two domain walls. Since an electron has charge 1, each of the domain walls can be thought of as having charge 1/2. Thus even after we account for hybridization, we can say that the charge of each domain wall is 1/2.

Via the index theorem: More craftily, we can argue as follows: work in Euclidean signature and couple ψ to a $U(1)$ gauge field A. We will be a bit long-winded and pedantic here because it's fun. Quite generally, regularizing by the partition function by a heavy PV field of mass M, the partition function when ψ has mass m is[50]

$$Z[A;m] = \frac{\det(\not{D}_A + m)}{\det(\not{D}_A + M)} = \prod_\lambda \frac{i\lambda + m}{i\lambda + M}, \qquad |M| \to \infty \tag{434}$$

where $\lambda \in \mathbb{R}$ by the Hermiticity of $i\not{D}_A$.

Consider first $m \to \pm\infty$. Then

$$Z[A;m] = \prod_\lambda (-1)^{\Theta(m/M)-1} = \exp\left(i\pi \sum_\lambda \operatorname{sgn}(\lambda)(\Theta(m/M)-1)\right) = \exp\left(i\pi\eta(\Theta(m/M)-1)\right), \tag{435}$$

[50]A comment on the absence of is in the following equation, since it's important: in Euclidean signature, with all the γ matrices Hermitian, full Hermitian conjugation sends $\psi \to \psi^\dagger$ *and* $x^0 \mapsto -x^0$; it both conjugates the fields and reverses the time coordinate. Thus $\gamma^0\not{D}_A$ is Hermitian in this definition, while $i\gamma^0\not{D}_A$ is anti-Hermitian in this sense. Now we often write the Lagrangian as $\bar\psi(\not{D}_A + m)\psi$. This is slightly misleading, since $\bar\psi \neq \psi^\dagger\gamma^0$: indeed, if this were true, $\bar\psi\psi$ would not be invariant under the Loretnz group in Euclidean signature. Instead, $\bar\psi$ should be viewed as an independent field transforming in the representation conjugate to the representation which ψ transforms under. Since $\bar\psi$ and ψ are the integration variables in the path integral, the path integral produces $\det(\not{D}_A + m)$ for $Z[A;m]$. Now $\not{D}_A$ is anti-Hermitian as a matrix (with the usual definition of Hermitian conjugation that doesn't reverse time, since this is what's relevant for computing the determinant), which means the eigenvalues appearing in the determinant will be of the form $i\lambda + m$, with $\lambda \in \mathbb{R}$.

where η is the eta-invariant for the Dirac operator $i\not{D}_A$.[51]

First consider $\dim Y \in 2\mathbb{Z}+1$, where Y is our Euclidean spacetime. We will use the APS index theorem in the context of realizing Y as the boundary of some higher-dimensional manifold X over which the relevant structures (spin, gauge, etc) extend (this is always possible if the spacetime dimension is small enough). The index theorem says that

$$\frac{\eta}{2} + I = \operatorname{ind}(i\not{D}_A), \tag{436}$$

where I is the appropriate degree part of $\widehat{A} \wedge \operatorname{ch}(F_A)$ integrated over X, with $\widehat{A}$ the Dirac genus (which we will ignore). Since the index of $i\not{D}_A$ on X is integral, we can write

$$Z[A;m] = \exp\left(2\pi i(\Theta(m/M) - 1)I\right). \tag{437}$$

For example, suppose we are in three dimensions, so that X is four-dimensional. Then we obtain the effective action $S = \frac{k}{4\pi}\int F_A \wedge F_A$, where $k = 0$ if m and M have the same sign, and $k = -1$ else. This gives us a CS term on Y of level either 0 or -1. Similarly, suppose Y is 1-dimensional. Then we get the 1-dimensional version of the CS term on Y, namely $S = k\int_Y A$, where k is the same as before. Note that $Z[A;m]$ is independent of the choice of X, and so our use of X was just a calculational trick.[52]

Now suppose $\dim Y \in 2\mathbb{Z}$. Then we have a doubling of the spectrum for all non-zero modes of $\not{D}_A$, since $\bar{\gamma}$, the chirality operator, anticommutes with $\not{D}_A$. This means that only the zero modes of $\not{D}_A$ contribute to the partition function. Now if $\Theta(m/M) = 1$, $Z[A;m] = 1$. Suppose then that m and M have opposite signs. Then we get

$$\begin{aligned} Z[A;m] &= (-1)^{\dim\ker(i\not{D}_A)} = (-1)^{\dim\ker(i\not{D}_A[1+\bar{\gamma}]/2)+\dim\ker(i\not{D}_A[1-\bar{\gamma}]/2)} \\ &= (-1)^{\dim\ker(i\not{D}_A[1+\bar{\gamma}]/2)-\dim\ker(i\not{D}_A[1-\bar{\gamma}]/2)} = e^{i\pi\operatorname{ind}(i\not{D}_A)} \\ &= \exp\left(\frac{i\pi}{(\dim Y/2)!}\int (F_A/2\pi)^{\wedge(\dim Y/2)}\right), \end{aligned} \tag{438}$$

where in the last line we used the index theorem. This is a theta term at $\theta = \pi$.

Finally, consider $m = 0$. In this case, if $i\not{D}_A$ has zero modes, we get $Z[A;0] = 0$. So suppose $\det(i\not{D}_A) \neq 0$. Then we get

$$Z[A;0] = \prod_\lambda \frac{\lambda}{\lambda - iM} = \prod_\lambda \frac{1}{|1-iM/\lambda|}\frac{1+iM/\lambda}{|1-iM/\lambda|} = |Z[A;0]|\exp\left(\sum_\lambda \ln(z/|z|)\right), \tag{439}$$

where $z = 1 + iM/\lambda$. Taking $|M| \to \infty$, we get

$$Z[A;0] = |Z[A;0]|\exp\left(i\operatorname{sgn}(M)\sum_\lambda \operatorname{sgn}(\lambda)\tan^{-1}(|M|/|\lambda|)\right) = |Z[A;0]|\exp\left(i\frac{\pi}{2}\operatorname{sgn}(M)\eta\right). \tag{440}$$

[51] We really should be doing the regularization more carefully, e.g. with a $e^{-\lambda^2\epsilon}$ factor, but as long as we don't look too closely this won't affect the action we derive. If we were to look too closely, we would notice that our final expression for η is related to a CS term integrated over spacetime, which is $\mathbb{R}$-valued. Clearly this is incompatible with our naive definition of η; regulating the sum properly fixes this.

[52] We have been ignoring the gravitational contribution, and so we have to assume Y is spin. If we want to have non-spin Y then we can take A to be spinc, but then we need to put the gravitational contribution back in.

So, the phase of the partition function is determined just as for the massive case with $\mathrm{sgn}(m/M) = -1$, except there is a (crucial!) factor of $1/2$ in front of the η invariant. Now if we think of Y as being the boundary of some X, we can use the APS index theorem to express $Z[A;0]$ as a local functional of A on X, together with a sign determined by the index $i\not{D}_A$ on X. But because of the $1/2$, this will not be expressible as a local functional of A on Y, and our answer for $Z[A;0]$ will depend on the choice of X. If our fields only live on Y, then we are stuck with writing η, which will generally be a *singular* function of A.

Anyway, now we can apply this to the problem at hand. We first integrate out ψ away from the domain wall, where it is massive. Wolog let $m(-\infty) = -m$, $m(\infty) = +m$, $m > 0$. For $m \to \infty$, the effective action $S[A]$ contains the $\theta = \pi$ term

$$S[A] \ni \frac{1}{2} \int_{H^+} F, \tag{441}$$

where H^+ is the upper half-plane where $m > 0$, and we have used a regulator with large negative mass. Since θ terms in two dimensions act like electric fields with strength $\theta/2\pi$,[53] we see that the bulk fermions produce an effective contribution of $1/2$ to the electric flux in the region to the right of the domain wall.

Now we do the integral over the zero mode localized on the domain wall. This gives us the η invariant, whose contribution to $S[A]$ we can write as

$$S[A] \ni -\frac{\pi}{2}\eta = -\frac{\pi}{2}\left(\frac{1}{\pi}\int_{DW} A\right)_{[-1,1)}, \tag{442}$$

where DW is the domain wall (we will usually take it to be an S^1 in the Euclidean time direction). Here the subscript $[-1,1)$ means that we always take the thing in parenthesis to be in the range $[-1,1)$ by adding / subtracting 2 as need be, which ensures that η is invariant under large gauge transformations, as it must be. Note that if we had π instead of $\pi/2$ out front then we could take the integral of A to be defined mod 2 anyway, and could drop the parenthesis, obtaining the well-defined term $\int_{DW} A$. As it stands, we cannot do this. Also note that $\eta(A)$ is singular at $\int_{DW} A = \pi$, but that $Z[A;0]$ vanishes there anyway because of the zero mode: the DW theory will have $\det(\partial_t - iA) = 0$ when $\psi(t+\beta) = e^{i\int_{DW} A}\psi(t)$, where β is the radius of the compactified Euclidean time direction. With the usual NS spin structure we have $\psi(t+\beta) = -\psi(t)$, and so $Z[A;0]$ indeed vanishes when $\int_{DW} A = \pi$.[54] Anyway, since twice this term gives $\int_{DW} A$, we conclude that the DW effectively describes a particle of charge $-1/2$ living on the domain wall, since such a particle would have a response theory of $-\frac{1}{2}\int_{DW} A$, were that to be well-defined.

[53] This is because the canonical momentum of A is $\pi_A = \frac{\delta S}{\delta dA} = \star F/e^2 + \theta/2\pi$, so that the electric field at a given point in space is $\star F/e^2 + \theta/2\pi$.

[54] With the R spin structure, I believe the η invariant is modified to

$$\eta_R(A) = \left(\frac{1}{\pi}\int_{DW} A\right)_{[0,2)}, \tag{443}$$

which now has a singularity at $\int_{DW} A = 0$, in keeping with $Z[0;0] = 0$ for the R spin structure.

Putting these two pieces together gives us the effective action

$$S[A] = -\frac{\pi}{2}\left(\frac{1}{\pi}\int_{DW} A\right)_{[-1,1)} + \frac{1}{2}\int_{H^+} F. \tag{444}$$

If we integrate the second term and assume A dies off at ∞, we can combine the two terms and get $S[A] = -\int_{DW} A$, which is well-defined and gauge-invariant. The physical interpretation of this scenario is that we have a mode of fractional charge sitting on the domain wall, and an induced strength $e/2$ electric field to the right of the domain wall, coming from the difference in polarization between the two domains. Since we ultimately have an integer number of electrons in the system the total electric field must be integral, and it is: the domain wall contribution and the polarization contribution add up to give an integral result.[55]

Finally let's comment on the role of $\widetilde{C}$ symmetry. Now as we saw earlier,

$$\widetilde{C} = CT : j^0 \mapsto -j^0, \quad j^1 \mapsto j^1, \tag{445}$$

so that $\widetilde{C}$ is an antilinear symmetry which leaves the current as a differential form invariant; $\widetilde{C} : j \mapsto j$. The transformation rule for the gauge field A is then found by requiring that $\widetilde{C}$ leave $j_\mu A^\mu$ invariant: therefore $\widetilde{C} : A \mapsto A$ as a differential form, i.e. $A_0 \mapsto -A_0$, $A_1 \mapsto A_1$. This means that $\widetilde{C} : F \mapsto F$, and so the theta term $\exp(i\theta\int F/2\pi)$ has $\theta \mapsto -\theta$ under $\widetilde{C}$. But the domain wall part is also not invariant, and is also conjugated under $\widetilde{C}$. Thus our net action $\sim \int_{DW} A$ changes sign. This isn't a problem though, since we are always free to add the well-defined counterterm $k\int_{DW} A$ to $S[A]$ for any $k \in \mathbb{Z}$. In particular, choosing $k = +1$ gives us something that is $\widetilde{C}$ invariant. The statement is just that our theory must be invariant under $\widetilde{C}$ for some choice, but not all choices, of counterterms in the background field. Note that if we were missing either the bulk or the DW contribution, such a counterterm would not be able to render $S[A]$ invariant under $\widetilde{C}$.

27 *March 1 — How spin CS theory sees the spin structure*

Consider a CS theory which is spin. How does the partition function $Z[\sigma]$ depend on the spin structure σ? That is, under $\sigma \mapsto \sigma + \beta$ with $\beta \in H^1(X;\mathbb{Z}_2)$, how does $Z[\sigma]$ change?

Now suppose the fundamental representation f of the gauge group is real. Then since the fundamental of Spin(3) is pseudoreal, the bundle $S \otimes E$ is an associated bundle defined with a pseudoreal action on $\mathbb{C}^{\dim S + \dim E}$.[56] This means that there is an antilinear automorphism $J\mathcal{K} : R_{1/2\otimes f} \to R_{1/2\otimes f}$ that commutes with the Dirac operator. Since $(J\mathcal{K})^2 = -\mathbf{1}$, this leads to a doubling of the spectrum, and hence $\mathrm{ind}(i\not{D}_A) \in 2\mathbb{Z}$.

$$\frac{Z[\sigma+\beta]}{Z[\sigma]} = \exp\left(i\pi\int w_2(E)\wedge\beta\right). \tag{446}$$

[55] In one space dimension the potential from a point charge diverges linearly as $V(r) \sim r$, so the electric field is constant.

[56] Just because the $\otimes$ of a ps$\mathbb{R}$ rep and a $\mathbb{R}$ rep is ps$\mathbb{R}$.

Real line bundles are classified by $w_1(F) \in H^1(X;\mathbb{Z}_2)$. The coefficient is $\mathbb{Z}_2$ since $F \otimes F$ is trivial for any real line bundle: since $F \cong F^*$, $F \otimes F \cong F \otimes F^* \cong \mathrm{Hom}(L, L)$ admits the section identity $\mathbf{1} : L \to L$, which is nonvanishing everywhere.

28 *March 8 — Random stuff about stereographic projection and the Hopf fibration; the incredible 3-sphere*

First, show that stereographic projection is a conformal map, i.e. it preserves angles between lines. For the case of S^2, show that after stereographic projection the metric can be written in complex coordinates as $\partial\bar{\partial}V$, and hence show that S^2 is Kahler.

Show that geodesics on the sphere are projected onto either lines through the origin or circles, and that either way, the projected geodesics intersect the equator of the sphere at two antipodal points.

Now consider the Hopf fibration $S^3 \to S^2$. Let the S^3 be parametrized by $X^2 + Y^2 + Z^2 + T^2 = 1$, and let the projected coordinates be x, z, y. Describe how the generators $L_{XY} = X\partial_Y - Y\partial_X$ and $L_{TZ} = T\partial_Z - Z\partial_T$ act on the $\mathbb{R}^3$ of x, y, z, and describe the fixed points under these actions. Construct a Killing field on S^3 that has no fixed points.

Solution:

Stereographic projection stuff

In what follows we will use uppercase letters to denote coordinates on S^n, and lowercase letters to denote coordinates of a stereographic projection. We will let X^0 be the "vertical" coordinate in the stereographic projection (the one which controls radial distance in the projected coordinate system), so that the coordinates are related as[57]

$$x^i = \frac{X^i}{1 + X_0}. \tag{447}$$

Letting $r^2 = x_i x^i$, the fact that $X_\mu X^\mu = 1$ tells us that, after a bit of algebra,

$$X_0 = \frac{1 - r^2}{1 + r^2} \implies X^i = \frac{2x^i}{1 + r^2}. \tag{448}$$

We claim that in the projected coordinates x^i, the metric is

$$ds^2 = \frac{4}{(1 + r^2)^2} dx^2, \tag{449}$$

[57] This is a projection from the south pole, and is not well-defined there (where $X_0 = -1$). To switch to a projection from the north pole, just replace $X_0 \to -X_0$ in the following formulae.

which is conformally flat. To show this, note that

$$r^2 = \frac{X_i X^i}{(1+X_0)^2} \implies \frac{4}{(1+r^2)^2} = (1+X_0)^2. \tag{450}$$

Thus the above metric is

$$\begin{aligned} ds^2 &= (1+X_0)^2 \left(\frac{dX^i}{1+X^0} - \frac{X_i}{(1+X_0)^2} dX_0 \right)^2 \\ &= dX^i dX_i + \frac{1}{(1+X_0)^2} X^i X_i dX_0^2 - \frac{2}{1+X_0} X^i dX_i dX_0 \\ &= dX_i dX^i + dX_0 dX^0 \left(\frac{1-X_0^2}{(1+X_0)^2} + \frac{2X_0}{1+X_0} \right) \\ &= dX_\mu dX^\mu, \end{aligned} \tag{451}$$

which is the metric on the sphere.

Now specialize to the case of S^2. Then if x, y are the stereographic coordinates in $\mathbb{R}^2$, we can define $z = x + iy$, so that $r^2 = |z|^2$, and $dz d\bar{z} = dx^2 + dy^2$, $idz \wedge d\bar{z} = 2dx \wedge dy$. With these coordinates, one can show that S^2 is a complex manifold.[58] The metric is then

$$ds^2 = \frac{4}{(1+|z|^2)^2} dz d\bar{z}, \tag{454}$$

which is Hermitian, meaning that we can construct the symplectic form

$$\Omega = \frac{2i}{(1+|z|^2)^2} dz \wedge d\bar{z}. \tag{455}$$

Since $d\Omega = 0$, S^2 is Kahler. Since Ω is closed under both complex differentials, it can be locally written as $\Omega = \partial \bar{\partial} V(z, \bar{z})$. Indeed,

$$\Omega = 2i \partial \bar{\partial} \ln(1+|z|^2). \tag{456}$$

[58]A complex manifold has holomorphic transition functions. The assignment of coordinates above via stereographic projection from the south pole is not good enough to cover the whole sphere, since it misses the south pole. So, define the coordinate patch

$$\widetilde{x} = \frac{X^1}{1-X_0}, \quad \widetilde{y} = -\frac{X^2}{1-X_0}, \tag{452}$$

which covers the south pole, but not the north pole (the reason for the $-$ sign in the $\widetilde{y}$ definition will be come clear shortly). Away from the poles where both coordinate systems are defined, we see that

$$\widetilde{z} = \frac{X^1 - iX^2}{1-X_0} = 2\frac{x-iy}{1-(1-r^2)/(1+r^2)} = \frac{x-iy}{r^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{z}. \tag{453}$$

So indeed, the tilde'd and un-tilde'd coordinate systems are related holomorphically. If we had not used the minus sign in the definition of $\widetilde{y}$ we'd have gotten $\widetilde{z} = 1/z^*$, which would have made the transition function non-holomorphic.

Using $dz \wedge d\bar{z} = -2idx \wedge dy$ we have $\Omega = \sin\theta d\theta \wedge d\phi$ back in spherical coordinates, and so with this normalization the symplectic form is exactly equal to the volume form.

Now we'll go back to a general sphere S^n. Geodesics on the sphere are circles, and as such can be written as $\xi^\mu(\tau) = N^\mu \cos\tau + M^\mu \sin\tau$ for N and M two orthogonal unit vectors. The equator is where $X_0 = 0$, and so the points where ξ^μ meets the equator will be at time τ such that $N^0 \cos\tau + M^0 \sin\tau = 0$. Now if τ solves this equation then so does $\tau + \pi$, so each geodesic intersects the equator at two points. Since $\xi^\mu(\tau + \pi) = -\xi^\mu(\tau)$, this means that the two points where the geodesic intersects the equator are antipodal. Note that this will be true even after stereographic projection: the projected geodesics will meet the unit sphere in the projected plane (the image of the equator) at antipodal points.

The stereographic projection of *any* circle on S^n is either a straight line (if the circle in question passes through the south pole of S^n, from where we are projecting), or a circle. This is kind of crazy, since intuitively one might guess that circles that are not either the equator or geodesics passing through the poles would get projected to ellipses or something. For example, consider the stereographic projection of S^2. Circles are formed by intersecting a plane in $\mathbb{R}^3$ with the S^2.[59] The plane can be parametrized by $aX + bY + cZ + d = 0$. Going to stereographic coordinates, this reads

$$\frac{2}{1+r^2}(ax+by) + c\frac{1-r^2}{1+r^2} + d = 0 \implies r^2(d-c) = -2(ax+by) - c - d. \tag{457}$$

If $d = c$ then the point $(0, 0, -1)$ is on the circle, and the above tells us that we get a straight line in the xy plane. Therefore circles containing the south pole become lines (if the circle also contains the north pole then we need $c = -d$ and so $c = d = 0$, and the projected line passes through the origin). Else, after dividing by $d - c$, we get an equation $x^2 + y^2 = \dots$, where $\dots$ is linear in x, y. Therefore we can complete the square and get $(x - x_0)^2 + (y - y_0)^2 = r_0$, where x_0, y_0, r_0 are constants; this gives us a circle in the xy plane.

Hopf stuff

Now we specialize to the case $S^3 \to S^2$, letting the stereographic projection occur from the point $(X, Y, Z, T) = (0, 0, 0, 1)$, and letting it project onto the $T = 0$ subspace, whose coordinates we write as x, y, z.

Consider the rotation generators L_{XY} and L_{TZ}, which rotate the XY and TZ planes, respectively. Now

$$L_{XY} = X\partial_Y - Y\partial_X = \frac{2x}{1+r^2}\frac{1}{1+T}\partial_y - \frac{2y}{1+r^2}\frac{1}{1+T}\partial_x = x\partial_y - y\partial_x. \tag{458}$$

Thus $L_{XY} = L_{xy}$, which just acts to rotate the xy plane. The z axis, which is a geodesic on S^3, is fixed under this action. The only other geodesic that is mapped to itself under L_{xy} is the circle $x^2 + y^2 = 1$ (the equator of the equator). The flow lines of this killing field are circles wrapping the z axis.

[59] In general, we get circles in S^n by intersecting a codimension $n-1$ hyperplane with S^n. This is because S^n is codimension 1, so the intersection carves out a surface of codimension $1 + (n-1) = n$, which since S^n is embedded in $\mathbb{R}^{n+1}$ is one-dimensional (recall that codimensions are additive under intersection).

Now we can take a look at L_{TZ}. We write

$$L_{TZ} = T\partial_Z - Z\partial_T = \frac{1-r^2}{2}\partial_z - \frac{2z}{1+r^2}\frac{-X^i}{(1+T)^2}\partial_i = \frac{1-r^2}{2}\partial_z + zx^i\partial_i. \tag{459}$$

The first term causes vectors inside the unit sphere in $\mathbb{R}^3$ (the equator of the S^3) to flow up along the z axis, and those outside equator to flow down along the z axis. The second term causes vectors to flow radially outward / inward with a strength proportional to their z coordinate. The submanifold left invariant under L_{TZ} is evidently the one where $r^2 = 1$ and $z = 0$, i.e. the circle $x^2 + y^2 = 1$ (the equator of the equator). The flow lines under L_{TZ} are circles that wrap the $x^2 + y^2 = 1$ circle.

Now consider the Killing field $\Xi = L_{XY} + L_{ZT}$. Evidently, there are no fixed points under Ξ! This is why S^3 is parallelizable. The flow lines under Ξ are helix-shaped; twisted around both the z axis and the $x^2 + y^2 = 1$ circle. Each flow line defines a torus which it wraps around once, and the family of all such tori defined in this way fill the whole space (see Figure 2).

The cool thing about Ξ is that *all* of its flow lines are geodesics! Indeed, let us switch to complex notation where

$$z_1 = X + iY, \quad z_2 = Z + iT. \tag{460}$$

With this notation, we can project the S^3 onto the S^2 via the map $(z_1, z_2) \mapsto z_1/z_2$, which makes it clear that the fiber is $U(1)$ (the image of this map is S^2 realized as a copy of $\mathbb{R}^2$ compactified at infinity). With this notation, it is also clear that $S^2 = \mathbb{CP}^1$.

Now we can write any geodesic on the S^3 as

$$\xi^\mu(\tau) = A^\mu \cos\tau + B^\mu \sin\tau, \qquad A_\mu A^\mu = B_\mu B^\mu = 1, \quad A_\mu B^\mu = 0. \tag{461}$$

In complex notation, this is

$$\xi^\alpha(\tau) = A^\alpha \cos\tau + B^\alpha \sin\tau, \qquad A_\alpha \bar{A}^\alpha = B_\alpha \bar{B}^\alpha = 1, \quad A_\alpha \bar{B}^\alpha + B_\alpha \bar{A}^\alpha = 0, \tag{462}$$

where now $\alpha = 1, 2$.

Consider the family of geodesics

$$\xi^\alpha(\tau) = e^{i\tau} A^\alpha \tag{463}$$

for some choice of A^α. The tangent lines to this family are (using conventions where e.g. $\partial = \frac{1}{2}(\partial_x - i\partial_y)$)

$$\mathcal{T}_\xi = \partial_\tau \xi^\alpha \partial_\alpha + \partial_\tau \bar{\xi}^\alpha \bar{\partial}_\alpha = i(\xi^\alpha \partial_\alpha - \bar{\xi}^\alpha \bar{\partial}_\alpha). \tag{464}$$

Writing this out,

$$\mathcal{T}_\xi = \xi^X \partial_Y - \xi^Y \partial_X + \xi^Z \partial_T - \xi^T \partial_Z \tag{465}$$

Thus if we choose the unit vector $\xi^\mu = (X, Y, Z, T)$, we see that

$$\mathcal{T}_\xi = L_{XY} + L_{ZT} = \Xi. \tag{466}$$

Thus this family of geodesics are precisely the flow lines of the Killing field Ξ!

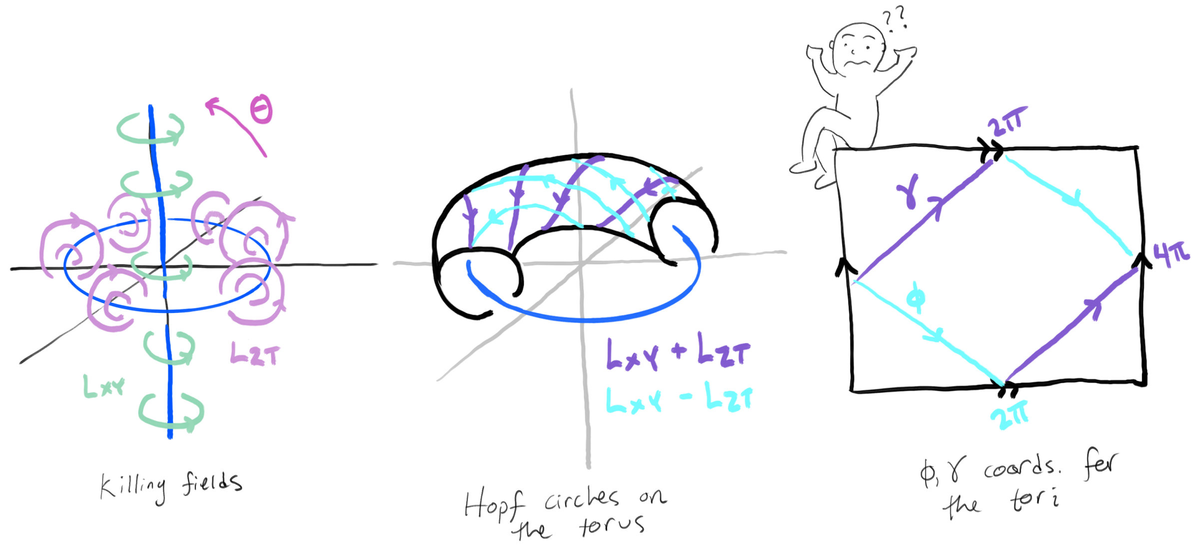


Figure 2: Left: the flow lines of the Killing fields L_{XY} and L_{ZT} in stereographic projection. Center: one torus at a fixed value of θ, with geodesic lines (flows of $L_{XY} \pm L_{ZT}$) drawn on top. Right: the same torus, split open. When we parametrize the torus with γ, ϕ, we only need ϕ to be 2π periodic.

Because these geodesics are also the flows of a Killing field, they never intersect. Pedantically, consider two geodesics $\xi^\alpha(\tau) = e^{i\tau} A^\alpha$, $\eta^\alpha(\tau) = e^{i\tau + i\tau_0} B^\alpha$. Here we parametrize the geodesics so that we move around both of them at the same speed (they have the same τ dependence), but we allow for a constant offset τ_0 that allows us to start the parametrization at different offsets between the two geodesics. Anyway, the dot product

$$\bar{\xi} \cdot \eta + \xi \cdot \bar{\eta} = e^{i\tau_0} \bar{A}_\alpha B^\alpha + e^{-i\tau_0} A_\alpha \bar{B}^\alpha \tag{467}$$

is independent of τ, and so the geodesics are a constant distance apart. In particular, any two distinct such geodesics never intersect. This is kind of remarkable, if we think about trying to do the same on the 2-sphere: there all geodesics have to intersect, since the positive curvature of S^2 means that two initially parallel geodesics must intersect eventually. S^3 is also positively curved, but it turns out that we can get non-intersecting geodesics by twisting them in such a way that the twist exactly compensates the tendency for the positive curvature to "pull" them together. The circular orbits under the above Killing field are known as Hopf circles.

Now we talk about why this is a fibration over S^2. First, we ask what the space of Hopf circles is. We can label them with two coordinates, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Here θ is a polar coordinate measured *up from the xy plane to the z axis*, with the value of θ defining the radii of the concentric tori that foliate the space. ϕ is an angle that measures the distance along each torus normal to the Hopf circles, and each pair θ, ϕ has an S^1 growing out of it (a Hopf circle), which wraps once around both cycles around the torus at the angle θ. Note that when $\theta = 0$ (the associated torus degenerates into the unit circle in the xy plane) or when $\theta = \pi$ (the associated torus degenerates into a circle formed by the z axis and the point at ∞), all of the ϕ values become degenerate, and so the θ, ϕ coordinates give us an S^2: we have an S^2's worth of Hopf circles. We will let the coordinate along the circles be denoted by $\gamma \in [0, 4\pi)$. Thus we have an S^1 bundle over S^2, and the coordinate along the fibers is γ.

This bundle is nontrivial, as there is no global choice of coordinates such that $\gamma = \gamma(\phi, \theta)$ is a well-defined function. Indeed, suppose we tried to smoothly pick out a point on each fiber by marking the points where each Hopf circle pierces the xy plane. This works fine for every Hopf circle except for the one at $\theta = 0$, which lies inside the xy plane. Similarly we could try to mark the points where the Hopf circles pierce the zx plane, but this fails because the Hopf circle at $\theta = \pi$ (viz. the z axis) lies inside this plane. No matter what choice we make, we always miss one (or more) fibers.

Before moving on, we comment on a concise way of understanding the actions of the various Killing fields. First, we package the coordinates z_i as a matrix in the following way:

$$S^3 \ni \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \tag{468}$$

Consider the left action of $e^{i\theta\sigma^z}$. This sends $z_i \mapsto e^{i\theta} z_i$, and hence rotates the XY and ZT planes. That is, we can identify

$$i\sigma^z = L_{XY} + L_{ZT}. \tag{469}$$

So, the action of $i\sigma^z$ is precisely that of the flow of the Killing field we identified earlier, which flows in a helix wrapped around the $x^2 + y^2 = 1$ circle. Of course, the other generators in $SU(2)$ perform similar flows: acting on the left with $i\sigma^x$ mixes up z_1 and iz_2, and so it mixes X with T and Y with Z. Likewise, $i\sigma^y$ mixes z_1 and z_2, and hence X with Z and Y with T. This means that (I'm not keeping very careful track of signs, sorry!)

$$i\sigma^x = L_{TX} + L_{YZ}, \qquad i\sigma^y = L_{XZ} + L_{YT}. \tag{470}$$

The right action of e.g. $e^{i\theta\sigma^z}$ is similar to the left action, except z_1 and z_2 get opposite phases, so that the XY and ZT planes are rotated in opposite directions. Thus to identify the Killing fields corresponding to the right action, we just change the relative signs between the two terms: $i\sigma^z$ acting on the right gives the field $L_{XY} - L_{ZT}$, and so on. The left and right actions commute, and together provide six linearly independent Killing fields. This identifications are also familiar from the fact that $SU(3) = \text{Spin}(3)$.

Now we return to looking at the structure of the $S^3 \to S^2$ bundle. For this it helps to make another coordinate redefinition, via

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i(\gamma+\phi)/2} \cos(\theta/2) \\ e^{i(\gamma-\phi)/2} \sin(\theta/2) \end{pmatrix} \tag{471}$$

Here $\theta \in [0, \pi]$, $\gamma \in [0, 4\pi)$, and $\phi \in [0, 2\pi)$. The γ coordinate parametrizes distance along the Hopf circles; i.e., distance along the fibers. As before, the θ coordinate measures the distance away from the $x^2 + y^2 = 1$ circle (the fixed point locus of the L_{ZT} action), and the subspace of constant θ picks out a torus foliated by Hopf circles. Finally, ϕ parametrizes distance along Hopf circles which wind in the *opposite* sense around each of the tori (see Figure 2). The γ, ϕ coordinates thus label points on the tori, while θ parametrizes a direction normal to the tori. As a sanity check on these identifications, consider $\theta = 0$, which is the unit circle in the xy plane. Then $z_2 = 0$, and $z_1 = e^{i\varphi}$, where φ is the polar angle in the xy plane. Now at $\theta = 0$ the two ways of winding around the torus are degenerate, and so $\gamma = \phi = \varphi$, meaning that $(z_1, z_2) = (e^{i\varphi}, 0) = (e^{i(\gamma+\phi)/2}, 0)$, which agrees with (471). A similar sanity check works for $\theta = \pi$: here we are in the ZT plane so that $z_1 = 0$ and $z_2 = e^{i\varphi'}$, with φ' the angle in the ZT plane. We also have $\phi = -\gamma$ (the torus at $\theta \to \pi$ degenerates in the opposite sense as the torus at $\theta \to 0$), and so $(z_1, z_2) = (0, e^{i(\gamma-\phi)/2})$, matching (471).

Why is γ 4π periodic, while ϕ is only defined in the range $[0, 2\pi)$? This is simply because in addition to the γ coordinate, only the first half of each opposite-winding Hopf circle is needed to parametrize the full torus. This is sketched in the right panel of Figure 2: the $\hat{\gamma}$ basis vector points diagonally up; the $\hat{\phi}$ basis vector points diagonally down. Working in units where the torus has sides of length $4\pi/\sqrt{2}$, every point on the torus can written as a linear combination $\alpha\hat{\gamma} + \beta\hat{\phi}$, where $\alpha \in [0, 4\pi), \beta \in [0, 2\pi)$. This is the reason for the different periodicity of the two variables.

In these coordinates, the projection $(z_1, z_2) \mapsto z_1/z_2$ is particularly transparent. Indeed, we have

$$(z_1, z_2) \mapsto e^{i\phi} \cot(\theta/2). \tag{472}$$

The γ coordinate no longer appears when after dividing out (since γ is the coordinate along the fibers), and we see that this mapping gives us a presentation of the S^2 where the north pole is located at ∞ and the south pole at the origin.

To get from these coordinates to a vector $n \in S^2$, we use

$$n^i = \vec{z}^\dagger \sigma^i \vec{z}. \tag{473}$$

Indeed, one checks that e.g. $n^z = \cos^2\theta/2 - \sin^2\theta/2 = \cos\theta$, and similarly for n^x and n^y.

The line element in complex coordinates is of course $ds^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$. In terms of the angular coordinates,

$$|dz_1|^2 = \frac{(d\gamma + d\phi)^2}{4}\cos^2\theta/2 + \frac{d\theta^2}{4}\sin^2\theta/2, \tag{474}$$

and

$$|dz_2|^2 = \frac{(d\gamma - d\phi)^2}{4}\sin^2\theta/2 + \frac{d\theta^2}{4}\cos^2\theta/2, \tag{475}$$

so that $(\cos^2\theta/2 - \sin^2\theta/2 = \cos\theta)$

$$ds^2 = \frac{d\gamma^2 + d\theta^2 + d\phi^2}{4} + \frac{1}{2}\cos\theta\, d\gamma\, d\phi. \tag{476}$$

The tori that foliate the S^3 are the fixed θ subspaces. We can confirm our earlier statement about γ being the coordinate along the Hopf circles and ϕ being the coordinate along the opposite-winding Hopf circles by looking at the metric restricted to a given torus at $\theta = \theta_0$. The metric on this torus $T^2_{\theta_0}$ is

$$ds^2|_{T^2_{\theta_0}} = \frac{1}{4}(d\gamma^2 + d\phi^2) + \frac{1}{2}\cos\theta_0\, d\gamma\, d\phi. \tag{477}$$

Now define the variables σ, ρ through $\gamma = (\sigma + \rho)/\sqrt{2}$, $\phi = (\sigma - \rho)/\sqrt{2}$. If our identification of γ, ϕ was correct, then σ should be (proportional to) the coordinate on the torus that travels along the flow of L_{XY}, while ρ should be (proportional to) the coordinate that travels along the flow of L_{ZT} (see Figure 2). Indeed, the metric in these coordinates is

$$ds^2|_{T^2_{\theta_0}} = \frac{1}{4}\left((1+\cos\theta_0)d\sigma^2 + (1-\cos\theta_0)d\rho^2\right). \tag{478}$$

We see that the σ coordinate disappears when $\theta_0 = \pi$, while the ρ coordinate disappears at $\theta_0 = 0$. Since the $\theta_0 = 0$ torus is just the unit circle in the xy plane, the σ coordinate evidently labels the angular direction in the xy plane, which is just as we said it would be based on our identification of γ, ϕ. Likewise, ρ labels the direction on the torus that runs along the flow lines of L_{ZT}, since the torus degenerates to a circle along one of these flow lines at $\theta_0 = \pi$.

We can determine the connection by requiring that it partition the fiber bundle into vertical and horizontal subspaces which are orthogonal with respect to the above metric, via a splitting $TM = TM_h \oplus TM_v$. We will write the connection as the 1-form $\omega = \omega_\mu dx^\mu$. Its goal in life is to give us a definition of the vertical subspace at each point in the bundle, and is chosen such that the contraction of ω with any vector field in the horizontal subspace vanishes: $\omega(X_\tau) = \omega_\mu \frac{dx^\mu}{d\tau} = 0$, for any horizontal vector field $X_\tau = \partial_\tau$. This equation yields a DE relating movement in the vertical subspace to movement in the horizontal subspace, which when solved gives us the holonomies, curvature, etc.

In our application, the vertical subspace is defined by the vector field $X_\gamma = \partial_\gamma$. Consider a curve $(\theta, \phi)(\tau)$ in the base S^2, parametrized by τ. We want to lift this curve into the full bundle in such a way that it is lifted into the horizontal subspace. Now the tangent vector to the lifted curve is of course

$$X_\tau = \dot{\gamma}\partial_\gamma + \dot{\theta}\partial_\theta + \dot{\phi}\partial_\phi, \tag{479}$$

with the overdot denoting differentiation with respect to τ. If this is to be lifted to the horizontal subspace, then we need

$$(X_\tau)_\mu (X_\gamma)^\mu = (X_\tau)_\mu (X_\gamma)_\nu g^{\mu\nu} = 0. \tag{480}$$

Using the form for the metric and $(X_\gamma)_\nu = (1, 0, 0)$ in the basis (γ, θ, ϕ), we have

$$g_{\gamma\gamma}\dot{\gamma} + g_{\gamma\theta}\dot{\theta} + g_{\gamma\phi}\dot{\phi} = \dot{\gamma} + \cos\theta\dot{\phi} = 0. \tag{481}$$

On the other hand, the above is equal to $\omega(X_\tau) = \omega_\mu dx^\mu/d\tau$. So evidently the connection is

$$\omega = d\gamma + \cos\theta d\phi. \tag{482}$$

Integrating the differential equation $\omega(X_\tau) = 0$ around a closed loop C in the base S^2 tells us that the holonomy is $\Delta\gamma = -\oint_C \cos\theta d\phi$.

The typical physicist usually uses "connection" to mean a 1-form on the base manifold. We can get a 1-form on the base S^2 by choosing a section, i.e. choosing a paremetrization $\gamma(\theta, \phi)$. Of course, since the bundle is nontrivial, we won't be able to choose a globally well-defined such function $\gamma(\theta, \phi)$. Indeed, one choice we might make is $\gamma = \phi$, in which case the 1-form A on S^2 would be

$$A_S = (1 + \cos\theta)d\phi. \tag{483}$$

As the subscript indicates, this is well-defined everywhere except the north pole of the S^2, i.e. the $x^2 + y^2 = 1$ circle in the stereographic $\mathbb{R}^3$. Likewise we might choose $\gamma = -\phi$, which would give $A_N = (-1+\cos\theta)d\phi$, which works as a section everywhere except the south pole.

Note that regardless of how we project ω onto the S^2, the curvature is

$$dA = \sin\theta \, d\phi \wedge d\theta, \tag{484}$$

which is proportional to the volume form on the S^2. This is why the Hopf fibration is the $U(1)$ bundle on an S^2 enclosing a monopole of unit strength.

Real and quaterionic Hopf fibrations

What we have talked about above is the complex Hopf fibration, since the base space was $S^2 = \mathbb{CP}^1$, and we had the sequence $S^1 \to S^3 \to S^2$, or

$$U(1) \to S^3 \to \mathbb{CP}^1, \tag{485}$$

which is the unit complex numbers, the unit sphere in $\mathbb{C}^2$, and the first complex projective space.

We can also get Hopf fibrations for $\mathbb{RP}^1$ and $\mathbb{HP}^1$. First for the real Hopf fibration, which is rather trivial. The relevant sequence here is $S^0 \to S^1 \to S^1$, or

$$U(1;\mathbb{R}) \to S^1 \to \mathbb{RP}^1. \tag{486}$$

In complete analogy with the complex Hopf fibration, we have the unit real numbers (of course $U(1;\mathbb{R})$ is just a suggestive way of writing $\mathbb{Z}_2$), the unit sphere in $\mathbb{R}^2$, and the first real projective space. Now $\mathbb{RP}^1 = S^1$ (do antipodal identification on S^1 to produce an S^1 of half the circumference—nothing unorientable about this, unlike the case of $\mathbb{RP}^n$ for even n), so we can just as well write $\mathbb{Z}_2 \to \mathrm{Spin}(2) \to S^1$, where Spin(2) is the double cover of S^1 (the boundary of a mobius band). Just like the $\mathbb{C}$ Hopf fibration, the projection from $\mathrm{Spin}(2) \to S^1$ is realized (on the patches where it is defined) by $(x, y) \to x/y$, which clearly has kernel $S^0 = \mathbb{Z}_2$.

Now for the quaterionic case. By analogy with the previous two, we look for the fibration

$$U(1;\mathbb{H}) \to S^7 \to \mathbb{HP}^1, \tag{487}$$

since S^7 is the unit sphere in $\mathbb{H}^2$. Such a fibration does exist, and can be written $S^3 \to S^7 \to S^4$. Here $U(1;\mathbb{H}) = Sp(1) = SU(2) = S^3$, while $\mathbb{HP}^1 = S^4$. That $\mathbb{HP}^1 = S^7/S^3$ can be seen by realizing that the space of quaternionic lines containing the origin in $\mathbb{H}^2$ can be visualized in the same way as the set of four-dimensional hyperplanes (containing the origin) in $\mathbb{R}^8$. Each hyperplane can be defined by the way in which it intersects the unit sphere S^7 in $\mathbb{R}^8$. But each hyperplane intersects the S^7 in a four-dimensional subspace of that S^7, which cuts out an S^3 inside of the S^7, and so $\mathbb{HP}^1 = S^7/S^3$. Finally, that $\mathbb{HP}^1 = S^4$ can be seen by writing the projection from $\mathbb{H}^2$ with two patches as either as $(q,r) \to q/r$ for $r \neq 0$ or $(q,r) \to r/q$ for $q \neq 0$, with $q, r \in \mathbb{H}$. When $r \neq 0$ we can use the first patch to get a full $\mathbb{R}^4$'s worth of points. When $r = 0$ we use the other patch, and we see that the $r = 0$ subspace gets mapped onto a single point. Putting the two patches together, we see that the quotient space is $\mathbb{R}^4$, but with the points at infinity identified: so, the quotient is indeed S^4.

29 *March 11 — Coherent states, Berry curvature, symplectic stuff, and Chern insulators*

Solution:

First we'll work generally, with coordinates on our phase space labeled by ζ^i, e.g. $\zeta = (q_1, \ldots, q_n, p_1, \ldots, p_n)$. Let $|\psi(\zeta)\rangle$ denote the complete set of single-particle states that we use to form the resolutions of the identity that we insert into the trotterization of e^{-iHT}. The partition function is then (keeping ∂ conds implicit)

$$Z = \int \prod_i \mathcal{D}\zeta^i \exp\left(\int dt \left[\langle\psi(\zeta)|\frac{d}{dt}|\psi(\zeta)\rangle - iH(\zeta)\right]\right), \tag{488}$$

with $H(\zeta) = \langle\psi(\zeta)|H|\psi(\zeta)\rangle$. We write the time derivative as

$$\langle\psi(\zeta)|\frac{d}{dt}|\psi(\zeta)\rangle = \dot{\zeta}^j\langle\psi(\zeta)|\partial_{\zeta^j}|\psi(\zeta)\rangle + \langle\psi(\zeta)|\partial_t|\psi(\zeta)\rangle = i(\dot{\zeta}^j\mathcal{A}_j + \mathcal{A}_0), \tag{489}$$

where $\mathcal{A}_j = -i\langle\psi(\zeta)|\partial_{\zeta^j}|\psi(\zeta)\rangle$ is the Berry connection, and the temporal part is $\mathcal{A}_0 = -i\langle\psi(\zeta)|\partial_t|\psi(\zeta)\rangle$. The factor of $-i$ is to make it real, since $\partial_{\zeta^j}^\dagger = -\partial_{\zeta^j}, \partial_t^\dagger = -\partial_t$. We then can write the action concisely as

$$S = \int(\mathcal{A}_j d\zeta^j - [H(\zeta) - \mathcal{A}_0]dt). \tag{490}$$

We now take a variation with respect to the phase space variables ζ^j. This gives us a term $-\delta\zeta^j(d\mathcal{A}_j/dt)$, which is $-\delta\zeta^j\dot{\zeta}^k\partial_{\zeta_k}\mathcal{A}_j - \delta\zeta^j\partial_t\mathcal{A}_j$. Thus

$$\delta S = \int \left(\mathcal{F}_{ij}d\zeta^j - [\partial_{\zeta^i}H(\zeta) + \partial_t\mathcal{A}_i - \partial_{\zeta^i}\mathcal{A}_0]dt\right)\delta\zeta^i, \tag{491}$$

where we have defined the Berry curvature $\mathcal{F} = d\mathcal{A}$, with the exterior derivative being taken in the ζ^i coordintes. This shows us why the Berry curvature is intimately related to the symplectic structure of phase space: the Berry curvature is the symplectic form, since the symplectic form ω_{ij} appears in δS via[60]

$$\delta S \ni \int \delta\zeta^i \omega_{ij} d\zeta^j. \tag{493}$$

To elaborate on the indentification between $\mathcal{A}$ and the sympletic structure, we can write quite generally the action as

$$S = \int d\zeta^i \theta_i - \int dt\, H, \tag{494}$$

which describes motion of a trajectory in phase space (θ is the symplectic potential). The condition that H generate time evolution is $\partial_{\zeta_i} H = \omega_{ij} V_H^j$, where $V_H = \dot{\zeta}^j \partial_{\zeta_j}$ is the Hamiltonian vector field (therefore $\int d\zeta^i \theta_i = \int dt\, \theta(V_H)$). This equation is obtained as the equation of motion of the above action provided that $d\theta = \omega$, where the d is in $d\zeta^i$ space. Comparing this to the action derived above via the usual Trotterization procedure, we see that $\mathcal{A}_i = \theta_i$: thus the Berry connection is the essentially the symplectic potential, while the Berry curvature is essentially the symplectic form.

Anyway, the equations of motion are

$$\frac{d\zeta^j}{dt} = [\mathcal{F}^{-1}]^{ji}(\partial_{\zeta^i} H(\zeta) + \partial_t \mathcal{A}_i - \partial_{\zeta^i}\mathcal{A}_0). \tag{495}$$

Now in classical mechanics we know that the time evolution of ζ^j is generated by taking the Poisson bracket of ζ^j with $H(\zeta)$. Since $\{\zeta^j, H(\zeta)\} = \{\zeta^j, \zeta^i\}\partial_{\zeta^i} H(\zeta)$, we can classically identify $[\mathcal{F}^{-1}]^{ij} = \{\zeta^i, \zeta^j\}$. In the quantum theory then, since the Poisson bracket goes to $-i$ times the commutator, the commutation relations in the time-independent case when $\partial_t \mathcal{A}_i = \partial_{\zeta^i}\mathcal{A}_0 = 0$ are determined from the Berry curvature via

$$[\zeta^i, \zeta^j] = i[\mathcal{F}^{-1}]^{ij}. \tag{496}$$

When canonical coordinates $\zeta = (q_1, \dots, q_n, p_1, \dots, p_n)$ are chosen, $\mathcal{F}$ has the canonical structure of a symplectic form, namely

$$\mathcal{F}_{ij} = \begin{pmatrix} 0 & -\mathbf{1}_{n\times n} \\ \mathbf{1}_{n\times n} & 0 \end{pmatrix}_{ij}. \tag{497}$$

In geometric terms, this means that the $U(1)$ bundles over the q and p subspaces are trivial, while the bundle is twisted in the pq plane. The inverse is $\mathcal{F}^{-1} = -\mathcal{F}$, and we of course get $[q^i, p^j] = i\delta^{ij}$.

[60]Another way to see this is to write the boundary term obtained in the course of finding the eom as $\mathcal{A}_j \delta\zeta^j|_{t_i}^{t_f}$. Taking a second variation gives

$$\delta(\mathcal{A}_j \delta\zeta^j) = \delta\mathcal{A}_j \wedge \delta\zeta^j = \partial_{\zeta^i}\mathcal{A}_j \delta\zeta^i \wedge \delta\zeta^j = \frac{1}{2}\mathcal{F}_{ij}\delta\zeta^i \wedge \delta\zeta^j, \tag{492}$$

where the wedge product is in variational space. Now we know that the above is $\omega = \frac{1}{2}\omega_{ij}\delta\zeta^i \wedge \delta\zeta^j$ where ω is the symplectic form, and so $\omega = d\mathcal{A}$: the symplectic form and Berry curvature are one and the same.

To be concrete for a second, consider the simple example of a particle coupled to a background electromagnetic field. The action is

$$S = \int \big((p_i + A_i)dq^i - (H(p,q) - A_0)dt \big) . \tag{498}$$

Working in the basis $(q_1, \dots, q_d, p_1, \dots, p_d)$, we see that the Berry curvature is given by (here $F = dA$ is the coordinate part of the Berry curvature)

$$\mathcal{F}_{ij} = \begin{pmatrix} F & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}_{ij} . \tag{499}$$

Thus the coordinate part of the Berry curvature is equal to the magnetic field. This of course tells us that the curvature in the coordinate part of phase space is nontrivial: translations in different directions in coordinate space don't commute. This in turn means that the momenta p_i will not commute with one another, which we can see formally by computing $\mathcal{F}^{-1} = (X \otimes \mathbf{1}_{d\times d})\mathcal{F}(X \otimes \mathbf{1}_{d\times d})$, which tells us that $[p_i, p_j] = iF_{ij}$. Of course, the symplectic form can be recast in as the canonical $J \otimes \mathbf{1}_{d\times d}$ form by changing basis: a glance at the action tells us that the right way to do this is to define the canonical momenta $\pi_j = p_i + A_i$: in the basis $(q_1, \dots, q_d, \pi_1, \dots, \pi_d)$, we have $\mathcal{F}_{ij} = J \otimes \mathbf{1}_{d\times d}$. In the non-canonical basis, the equations of motion are the familiar

$$\dot{p}^i = -F^{ij}\partial_{p^j}H - \partial_{q^i}H - \partial_{q^i}A_0 + \partial_t A_i, \qquad \dot{q}^i = \partial_{p^i}H, \tag{500}$$

which, assuming $\partial_{p^i}H(\zeta) = \dot{q}^i$ with $H(\zeta) = p^2/2m + V(q)$, we can re-write as

$$\dot{p}^i = \epsilon^{ijk}\dot{q}^j B^k + E^i - \partial_{q^i}V(q), \qquad \dot{q}^i = p^i/m. \tag{501}$$

Now consider an analogous situation in which the magnetic field is zero, but the momentum part of the Berry curvature is non-vanishing, so that the action contains the term $\mathscr{A}_i dp^i$ (our notation is such that the total Berry connection / curvature are written with mathcal, with the coordinate part written in roman and the momentum part written in script). Proceeding in the same way, we get the equations of motion

$$\dot{q}^i = \mathscr{F}^{ij}\partial_{q^i}H(\zeta) + \partial_{p^i}H(\zeta) + \partial_t\mathscr{A}_{p^i} - \partial_{p^i}\mathscr{A}_0, \qquad \dot{p}^i = -\partial_{q^i}H(\zeta). \tag{502}$$

Taking $H(\zeta) = p^2/2m + V(q)$ again, we can write this as

$$\dot{q}^i = \epsilon^{ijk}\mathscr{B}^j\partial_{q^k}V(q) + \mathscr{E}^i + p^i/m, \qquad \dot{p}^i = -\partial_{q^i}V(q), \tag{503}$$

where $\mathscr{E}_i = \partial_{p^i}\mathcal{A}_0 - \partial_t\mathscr{A}_i$ and $\mathscr{B}$ are momentum-space electric and magnetic fields, which are responsible for an "anomalous velocity".

Finally, consider the general case where both F and $\mathscr{F}$ are nonzero. To analyze this we will work semiclassically, to first order in $\hbar$. Restoring $\hbar$ momentarily, the action is

$$S = \int \big((p_i + A_i)dx^i + \hbar\mathscr{A}_i dp^i - [H(\zeta) - \mathcal{A}_0]dt \big) . \tag{504}$$

The $\hbar$ in front of $\mathscr{A}_i dp^i$ is due to the fact that $\mathscr{A}_i$ has dimensions of inverse momentum, so that $\mathscr{A}_i dp^i$ is dimensionless, and needs to be multiplied by $\hbar$ in order to have dimensions of action (on the other hand, A_i has dimensions of momentum so $A_i dx^i$ already has dimensions of action). This means that the semiclassical limit can be taken by working to first order in $\mathscr{A}$ and $\mathscr{F}$.

Now the full Berry curvature is (going back to $\hbar = 1$)

$$\mathcal{F}_{ij} = \begin{pmatrix} F & -\mathbf{1} \\ \mathbf{1} & \mathscr{F} \end{pmatrix}_{ij} \implies [\mathcal{F}^{-1}]^{ij} = \frac{1}{1+F\mathscr{F}} \begin{pmatrix} \mathscr{F} & \mathbf{1} \\ -\mathbf{1} & F \end{pmatrix}^{ij}, \tag{505}$$

where e.g. $F\mathscr{F}$ denotes matrix multiplication. In the semiclassical approximation then,

$$\mathcal{F}^{-1} = \begin{pmatrix} \mathscr{F} & \mathbf{1} - F\mathscr{F} \\ -\mathbf{1} + F\mathscr{F} & F - F\mathscr{F}F \end{pmatrix}. \tag{506}$$

Now we can get the equations of motion. We will consider the case when $\partial_{p^i}\mathcal{A}_0 = \partial_t \mathscr{A}_i = 0$ for simplicity. We then have

$$\begin{aligned} \dot{q}^j &= \mathscr{F}^{ji}(\partial_{q^i}H - E_i) + \partial_{p^j}H - F^{jk}\mathscr{F}^{ki}\partial_{p^i}H, \\ \dot{p}^j &= (F^{ji} - F^{jk}\mathscr{F}^{kl}F^{li})\partial_{p^i}H + (-\delta^{ji} + F^{jk}\mathscr{F}^{ki})(\partial_{q^i}H - E_i). \end{aligned} \tag{507}$$

We can rewrite these in a simpler form as

$$\begin{aligned} \dot{q}^j &= \partial_{p^j}H + \epsilon^{jik}(\partial_{q^i}H - E_i)\mathscr{B}^k - \epsilon^{jkl}\epsilon^{kim}\partial_{p^i}HB^l\mathscr{B}^m \\ \dot{p}^j &= -\partial_{q^j}H + E^j + \epsilon^{jik}\dot{q}^i B^k. \end{aligned} \tag{508}$$

For example, suppose $\partial_{q^i}H = 0$ and $B = 0, \mathscr{B} \neq 0$, with q^i denoting the position of an electron in some Bloch band (this is the context of a Chern insulator). Then, supposing that the $\mathscr{B}$ and $\mathscr{E}$ fields live in three dimensions, we have

$$\dot{\mathbf{q}} = \vec{\nabla}H(p) + \mathscr{B} \times \mathbf{E}, \qquad \dot{\mathbf{p}} = \mathbf{E}. \tag{509}$$

The consequences of the $\mathscr{B} \times \mathbf{E}$ term are seen by computing the current density (restoring the electric charge e momentarily):

$$\mathbf{j} = -e\int_{BZ} \frac{d^dp}{(2\pi)^d} f(H(p))(\vec{\nabla}H(p) + \mathscr{B} \times e\mathbf{E}), \tag{510}$$

where $f(H(p))$ is the Fermi function. Suppose we are in a filled band, so that $f(H(p)) = 1$ for all $p \in BZ$. The first term then vanishes upon integration, which tells us that in the absense of momentum-space Berry curvature, filled bands don't contribute to the current (duh).

Now consider the case where the electrons live in a $d = 2$ plane, with the $\mathscr{B}$ and $\mathbf{E}$ fields living in three dimensions. Further suppose that $\mathbf{E}$ is uniform and in the plane, and $\mathscr{B}$ is orthogonal to the plane.[61] Then we get

$$\mathbf{j}^i = \epsilon^{ij}E^j\frac{e^2}{2\pi}\int_{BZ}\frac{\mathscr{B}}{2\pi} = \epsilon^{ij}E^j\frac{e^2}{2\pi}C, \tag{511}$$

[61] if the problem is really two-dimensional then $\mathscr{B}$ is a scalar and we should just be writing $\mathscr{B} \times \mathbf{E} \to \mathscr{B}(-E_y, E_x)$. But we haven't done this since eh, cross products are nice-looking.

where C is the Chern number. Thus we can get a (quantized!) Hall conductance without a real space magnetic field—a momentum space one will do the job as well.

For example, suppose our single-particle states are the wavepackets[62]

$$|\psi(q,p)\rangle = \frac{1}{(\sigma^2\pi)^{1/4}} \int dx\, e^{ipx} e^{-(x-q)^2/2\sigma^2} |x\rangle. \tag{513}$$

Then $\mathcal{A}_q = 0$, while $\mathcal{A}_p = q$, so that $\mathcal{F}_{qp} = 1$. This of course gives the equations of motion $\dot{q} = \partial_q H$ and $\dot{p} = -\partial_q H$. The fact that $\mathcal{F}_{qp} = 1$ is just telling us that the symplectic volume of the pq plane is non-zero: translations in the q and p directions (implemented by p and q, respectively) do not commute, and the failure of their commutativity is measured by $\mathcal{F}_{qp}$.

Now consider a two-dimensional coordinate space x, y, and suppose that $\mathcal{F}_{xy} \neq 0$. This means that translations in the x and y directions don't commute, so that the commutator $[p_x, p_y] \neq 0$. This of course is what happens when there's a magnetic field in coordinate space. If x, y are coordinates on a compact manifold X and $\int_X \mathcal{F}_{xy} dx \wedge dy \neq 0$, we have a nontrivial $U(1)$ bundle over the coordinate part of phase space.

As the simplest example, take 2d electrons in a magnetic field, with vector potential $A = (0, By)$. Then the eigenstates of $H = -\partial_x^2 - (\partial_y - Bx)^2$ are $e^{ik_y y} e^{-(x-k_y/B)^2}$. The Berry curvature is then

$$\mathcal{A}_y = Bx \implies \mathcal{F}_{xy} = B. \tag{514}$$

A consequence of having a nonzero Chern number for the Berry connection in coordinate space is that the single-particle wavefunctions $|\psi(\zeta)\rangle$ cannot be chosen to be localized in momentum space. This is because a function localized in momentum space is smooth in real space, which $|\psi(\zeta)\rangle$ cannot be due to the fact that the $U(1)$ bundle over X admits no global section.

Now we specialize to the example that motivated this diary entry, namely that of a Chern insulator. A Chern insulator is characterized by a Berry curvature when restricted to the momentum subspace of phase space is nontrivial in second cohomology. Suppose for simplicity that the BZ is the torus T^d.

The states $|\psi(\zeta)\rangle$ that we use in the construction of the path integral are the Wannier states, which diagonalize H.

Now let's find the equations of motion.

30 *March 17 — Haldane's model of a Chern insulator*

Solution:

[62]These of course are not orthogonal; setting $\sigma^2 = 1/2$ for simplicity gives an overlap like

$$\langle\psi(p,q)|\psi(p',q')\rangle \propto \exp\left(-(p-p')^2 - (q-q')^2 + i(q+q')(p'-p)\right). \tag{512}$$

31 *March 18 — θ angles and deconfinement in two dimensions from the strong coupling expansion on the lattice*

Today we're going to elaborate on some of the details implicitly contained in [**?**]. The goal will be to consider Euclidean lattice $U(1)$ gauge theory with a θ term in two dimensions, and to find the free energy and Wilson loop vevs in the strong coupling limit. Results for the (weak-coupling) continuum limit can then be argued for on faith.

Solution: First we need to write down an appropriate Euclidean lattice action. We write it as

$$S = \beta S_{\text{matter}} + \frac{i\theta}{2\pi}\sum_{\square}\Phi_{\square}, \tag{515}$$

where the flux $\Phi_\square$ is, for a plaquette $\square$ with bottom-left corner at the lattice site i,

$$\Phi_\square = -i\ln e^{i\oint_{\partial\square}A} = [A_x(i_x,i_y)+A_y(i_x+1,i_y)-A_x(i_x+1,i_y+1)-A_y(i_x,i_y+1)]_{[-\pi,\pi)}, \tag{516}$$

where the subscript on the brackets indicates that we choose a branch of the logarithm such that $\Phi_\square$ is always valued in $[-\pi,\pi)$. In keeping with this branch, we also choose our lattice gauge fields $A_\mu(i)$ to be valued in $[-\pi,\pi)$.[63] Of course, with this convention, $\sum_\square \Phi_\square \in 2\pi\mathbb{Z}$ when summed over the whole lattice. This means $\theta \sim \theta + 2\pi$, and in what follows we will always take $|\theta| \le \pi$.

Let's first calculate the free energy $\mathcal{F}[\theta]$ at $\beta = 0$. This won't depend on the boundary conditions for the lattice in the thermodyamic limit, and so to be consist with the Wilson line calculations we'll do later, we take the lattice to be a cylinder of length L_x in the x direction, and circumference L_y in the y direction (the compact direction). We can then fix a gauge such that $A_x = 0$ (we can't choose $A_y = 0$ since $H^1(\text{Cyl};\mathbb{R}) \neq 0$; the holonomy $e^{i\oint dy\,A_y}$ is gauge invariant). The partition function is

$$Z[\theta] = \int \prod_{x,y=0}^{L_x,L_y} d\gamma_x^y \, \exp\left(\frac{i\theta}{2\pi}\sum_\square \Phi_\square\right), \tag{517}$$

where we have written $\gamma_x^y = A_y(x,y)$ to save on the notation.

Note that the γ_x^y for different y are completely decoupled in this gauge: thus we can write

$$Z[\theta] = (Z_1[\theta])^L_y = \left(\int \prod_{x=0}^{L_x} d\gamma_x \, e^{i\bar\theta\sum_\square \Phi_\square}\right)^{L_y}, \qquad \bar\theta \equiv \theta/2\pi. \tag{518}$$

We start with the integral over γ_0, since γ_0 only appears in one place. Thus

$$Z_1[\theta] = \int \prod_{x=0}^{L_x} d\gamma_x \, \exp\left(i\bar\theta[\gamma_1-\gamma_0]_{[-\pi,\pi)}\right)(\dots), \tag{519}$$

[63] The reason that we choose the $[-\pi,\pi)$ branch instead of $[0,2\pi)$ is because it will make it easier to work with the θ angle, on which the free energy will depend on in a way that's symmetric about $\theta = 0$, not $\theta = \pi$.

where ... involves things that don't contain γ_0.

The integral is easy to get confused about, so we will be pedantic. Suppose first that $\gamma_1 > 0$. Then the []s will come into affect when $\gamma_1 - \gamma_0 = \pi$, i.e. when $\gamma_0 = \gamma_1 - \pi$. Thus

$$\begin{aligned}\int_{-\pi}^{\pi} d\gamma_0\, e^{i\bar\theta[\gamma_1-\gamma_0]_{[-\pi,\pi)}} &= \int_{-\pi}^{\gamma_1-\pi} d\gamma_0\, e^{i\bar\theta(-2\pi+\gamma_1-\gamma_0)} + \int_{\gamma_1-\pi}^{\pi} d\gamma_0\, e^{i\bar\theta(\gamma_1-\gamma_0)} \\ &= \frac{i}{\bar\theta}\left(e^{i(-\theta+\gamma_1\bar\theta)}(e^{i\bar\theta(\pi-\gamma_1)} - e^{i\theta/2}) + e^{i\bar\theta\gamma_1}(e^{-i\theta/2} - e^{i\bar\theta(\pi-\gamma_1)})\right) \\ &= \frac{i}{\bar\theta}\left(e^{-i\theta/2} - e^{-i\theta/2+i\gamma_1\bar\theta} + e^{i\bar\theta\gamma_1-i\theta/2} - e^{i\theta/2}\right) \\ &= \frac{2}{\bar\theta}\sin(\theta/2).\end{aligned} \tag{520}$$

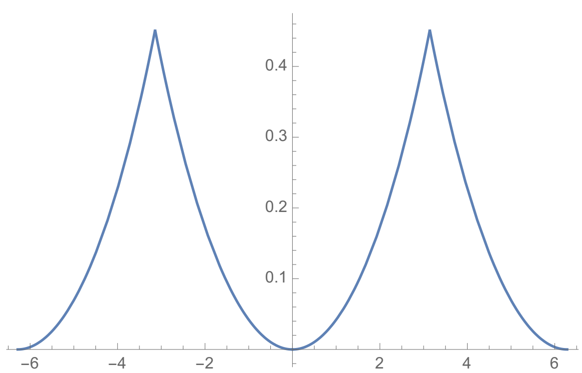
Since this is independent of γ_1, we of course get the same result if we take $\gamma_1 < 0$. The important thing here is that after γ_0 is integrated out, the resulting partition function looks *exactly* like the one for the partition function of a system with $L_x \mapsto L_x - 1$, multiplied by a factor of $2\sin(\theta/2)/\bar\theta$. Thus we can simply write[64]

$$Z[\theta] = \left(\frac{\sin(\theta/2)}{\theta/2}\right)^{L_xL_y}, \tag{521}$$

so that the free energy per unit area is

$$\mathcal{F}[\theta] = -\ln\left(\frac{\sin(\theta/2)}{\theta/2}\right). \tag{522}$$

Note that $\mathcal{F}[\theta] = \mathcal{F}[-\theta]$ as required. However, we also know that $\mathcal{F}[\theta] = \mathcal{F}[\theta \pm 2\pi]$: imposing this condition leads to a non-analyticity of $\mathcal{F}[\theta]$ at $\theta = \pm\pi$, which comes from the twofold GSD at the points where $\theta \in \pi(2\mathbb{Z}+1)$. $\mathcal{F}[\theta]$ looks like



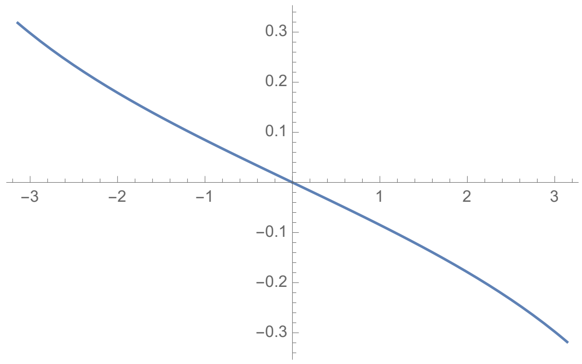
(523)

Now $\langle\Phi\rangle$ is odd under P and C (but unlike in 3+1D it is even under T if $A \mapsto -A$ as a differential form under T). The vev of the topological charge density is

$$\langle Q_{top}\rangle = -i\partial_\theta\mathcal{F}[\theta] = i\frac{\theta}{\sin(\theta/2)}\left(\frac{\cos(\theta/2)}{2\theta} - \frac{\sin(\theta/2)}{\theta^2}\right) = i\left(\frac{\cot(\theta/2)}{2} - \frac{1}{\theta}\right), \tag{524}$$

[64] we are replacing $\bar\theta \to \theta$ here, since in retrospect it is nicer to have a normalization factor of $1/2\pi$ multiplying each $d\gamma^y_x$ in the partition function.

which is non-vanishing (equal to $\mp i/\pi$) at the P symmetric points $\theta = \pm\pi$ (if we approach them from within $|\theta| \le \pi$): thus, we have SSB for P (as well as for C). $\langle Q_{top}\rangle \neq 0$ just indicates the presence of a nonzero background electric field. As expected, the points $\pm\pi$ are the points of largest $|\langle Q_{top}\rangle|$: the plot of $\langle Q_{top}\rangle$ looks like



(525)

Now let's diagnose confinement by looking at $\langle W_q(C_x)W_{-q}(C_{x+L})\rangle$, where $q \in \mathbb{Z}$ is a charge, C_x is a curve wrapping the y direction at x-coordinate x, and L is the lattice distance between the two Wilson lines. Because the partition function in the absence of Wilson lines factorizes as a product of partition functions on each x coordinate, when we calculate the expectation value we can, without loss of generality, take $x = 0$ and $L_x = L$. Therefore

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\int \prod_{x,y=0}^{L,L_y} \frac{d\gamma_x^y}{2\pi} \exp\left(iq\sum_{y=0}^{L_y}\gamma_0^y - iq\sum_{y=0}^{L_y}\gamma_L^y + i\bar\theta\sum_{x,y=0}^{L-1,L_y}[\gamma_{x+1}^y - \gamma_x^y]_{[-\pi,\pi)}\right). \tag{526}$$

As before, the γ_x^y variables for different y are completely independent—the only variables that are linked together are the ones in the brackets. So then

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\left[\int \prod_{x=0}^{L}\frac{d\gamma_x}{2\pi}\exp\left(iq(\gamma_0 - \gamma_L) + i\sum_{x=0}^{L-1}\bar\theta[\gamma_{x+1}^y - \gamma_x^y]_{[-\pi,\pi)}\right)\right]^{L_y}. \tag{527}$$

Let's look at just the part involving γ_0. Assume for simplicity that $\gamma_1 > 0$. Then similarly to before, the relevant integral is

$$\begin{aligned}\int_{-\pi}^{\gamma_1-\pi} d\gamma_0\, e^{iq\gamma_0 + i\bar\theta(-2\pi+\gamma_1-\gamma_0)} + \int_{\gamma_1-\pi}^{\pi} d\gamma_0\, e^{iq\gamma_0+i\bar\theta(\gamma_1-\gamma_0)} &= \frac{e^{i(-\theta+\bar\theta\gamma_1)}}{i(q-\bar\theta)}\left(e^{i(q-\bar\theta)(\gamma_1-\pi)} - e^{-i\pi(q-\bar\theta)}\right)\\ &\quad + \frac{e^{i\bar\theta\gamma_1}}{i(q-\bar\theta)}\left(e^{i\pi(q-\bar\theta)} - e^{i(q-\bar\theta)(\gamma_1-\pi)}\right)\\ &= -(-1)^q\frac{2e^{iq\gamma_1}\sin(\theta/2)}{q-\bar\theta}.\end{aligned} \tag{528}$$

Therefore after integrating out γ_0, we get (assuming for simplicity that $L_y \in 2\mathbb{Z}$ to get rid of the $(-1)^q$ [which is kinda weird])

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\left[\frac{2\sin(\theta/2)}{q-\bar{\theta}}\int\prod_{x=1}^{L}\frac{d\gamma_x}{2\pi}\exp\left(iq(\gamma_1-\gamma_L)+i\sum_{x=0}^{L-1}\bar{\theta}[\gamma^y_{x+1}-\gamma^y_x]_{[-\pi,\pi)}\right)\right]^{L_y}. \tag{529}$$

Note that this looks, up to the multiplicative factor, exactly the same as what we had before, just with the left Wilson line moved one lattice spacing closer to the right one. Therefore we can easily iterate and integrate out the rest of the γ_xs:

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\left[\frac{2\sin(\theta/2)}{2\pi q-\theta}\right]^{LL_y}. \tag{530}$$

This means that

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = e^{-L_yT}, \tag{531}$$

where the line tension is

$$T = L\ln\left|\frac{2\pi q-\theta}{\theta}\right|, \tag{532}$$

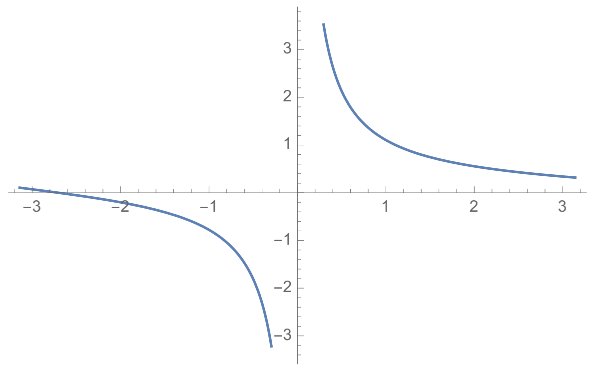
where as before $|\theta|\leq\pi$. Thus if $\theta\neq\pm\pi$, we will for sure have area-law confinement. However, suppose $\theta=\pi$: we get a completely tensionless and deconfined Wilson line, provided that $q=1$. If we choose $q=-1$, we get a strong line tension, and an area law. The opposite is true if we take $\theta=-\pi$, with $q=-1$ giving a tensionless Wilson line: these two choices are related by C, which sends both $\theta\to-\theta$ and $q\to-q$. What's going on here is the following: the θ term sets up a background electric field of strength $\bar{\theta}$, pointing along the x axis. This electric field causes charges to be linearly confined, except when $\theta=\pm\pi$: in this case, the ±1 strength electric flux created by the Wilson line insertions can screen the background electric field, leading to an electric field which is of uniform magnitude everywhere, but which has a reversal of sign in the domain enclosed by the two Wilson lines (the Wilson lines are domain walls for the spontaneously broken C symmetry, which is broken by the orientation of the electric field). The reason why one of the $q\pm1$ Wilson line charges is confined while one is tensionless at $\theta=\pm\pi$ is just because only one choice of q allows for an electric field that is everywhere uniform in strength.

To back up this conclusion, we can calculate the vev of the topological charge density as a function of x. For x to the left of C_x or to the right of C_{x+L}, $\langle Q_{top}(x)\rangle = i(\cot(\theta/2)/2-1/\theta)$ as before. But for x in between the two loops, we find

$$\langle Q_{top}(x)\rangle = i\partial_\theta\ln\left[\frac{2\sin(\theta/2)}{2\pi q-\theta}\right] = i\frac{2+(2\pi q-\theta)\cot(\theta/2)}{4\pi q-2\theta}. \tag{533}$$

The energetically favorable choices with P symmetry are $q=\pm1,\theta=\pm\pi$. For these choices we get $\langle Q_{top}(x)\rangle\to\pm i/\pi$. Note that this is *opposite in sign* to $\langle Q_{top}(x)\rangle$ for x not between the two Wilson lines: this is why the two Wilson lines are domain walls, across which the sign of $\langle Q_{top}(x)\rangle$ flips. At general θ, $\langle Q_{top}(x)\rangle$ for x in between the two Wilson lines (with

$q = 1$) looks like, as a function of θ,



(534)

The divergence at $\theta = 0$ comes from the fact that when $\theta = 0$, $\langle W_1(C_x)W_{-1}(C_{x+L})\rangle = 0$.

32 *March 20 — Zero modes of $i\not{D}_A$ on the sphere*

In today's diary entry we're going to a calculation that I've heard mentioned in a few papers, but have never actually seen worked out anywhere. We're going to explicitly construct the zero modes of the Dirac operator on a sphere, in the presence of some amount of magnetic flux.

Solution:

Spherical coordinates

In what follows, we will be using veilbeins, since that's the only method we have for dealing with fermions on curved spaces.[65] Recall that the veilbeins are found by taking the square root of the metric:

$$g_{\mu\nu} = e^a_\mu e^a_\nu, \qquad e^a_\mu e^b_\nu g^{\mu\nu} = \delta^{ab}. \tag{535}$$

Since we don't want to constantly be phantoming when writing stuff out, our convention will be that, when viewing the veilbeins as a matrix, the greek (spacetime) letter will always denote the row index of the matrix, and the roman (internal space) letter will always denote the column index. When we break apart the metric like this, we pick up a gauge redundancy, since the transformation $e^a_\mu \mapsto [O]^a_b e^b_\mu$ for $O \in O(s,t)$ leaves the splitting $g_{\mu\nu} = e^a_\mu e^a_\nu$ invariant (in what follows we will only be concerned with 2+0 dimensions, so that the relevant "gauge group" is $O(2)$).

[65]This is because fermion actions need γ^a matrices to be defined, which represent Clifford algebras. We want to represent a Clifford algebra with the relation $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ (or maybe η^{ab}), and definitely don't want to have the anticommutator be equal to $g^{ab}(x)$; this would be a mess. Thus we need veilbeins to switch between spacetime and a frame in which the Clifford generators can be defined.

The Dirac operator is (roman indices can be raised / lowered with impunity)

$$\not{D}_A = \gamma_a e^{\mu a}(\partial_\mu + i(\omega_\mu + A_\mu)), \tag{536}$$

where ω, A are the spin and gauge connections, with

$$\omega_\mu{}^a{}_b = e^a_\nu \partial_\mu e^\nu_b + e^a_\nu \Gamma^\nu_{\mu\lambda} e^\lambda_b. \tag{537}$$

We've tried to take a sign convention that is maximally simple; ours differs from the conventions in many other places though so be careful. The spin connection is needed to ensure that $\not{D}_A\psi$ transforms covariantly under local $O(2)$ gauge rotations of the coordinate frames.[66]

Since the generators of Spin(d) are $-i[\gamma_a, \gamma_b]/4$, the spin connection is, quite generally,

$$\omega_\mu = \frac{1}{2}\omega^{ab}_\mu \Sigma_{ab}, \qquad \Sigma_{ab} = \frac{-i}{4}[\gamma_a, \gamma_b]. \tag{542}$$

The veilbeins for spherical coordinates on the unit S^2 are easy to write down:

$$e^a_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \sin\theta \end{pmatrix}^a_\mu, \qquad e^{\mu a} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-1}\theta \end{pmatrix}^{\mu a}. \tag{543}$$

[66]It is also needed to ensure that the (real-time) action is Hermitian! The added spin-connection term is actually not Hermitian, and this compensates for the non-Hermiticity of $\bar\psi(i\not{\partial})\psi$ when working on a curved manifold. Indeed, under Hermitian conjugation,

$$\dagger : i\bar\psi e^{a\mu}\gamma_a\partial_\mu\psi \mapsto i\bar\psi e^{a\mu}\gamma_a\partial_\mu\psi + i\bar\psi(\partial_\mu e^{a\mu})\gamma_a\psi. \tag{538}$$

Now let's look at the spin connection part. For simplicity, we will work in Riemann normal coordinates around a certain point p, where the Christoffel symbols (but not their derivatives) can be chosen to vanish. The spin connection part of the Lagrangian density at this point is then

$$\begin{aligned} \mathcal{L} \ni -\frac{1}{2}\sum_{a,b\neq c}\psi^\dagger\gamma^0\gamma_a e^{a\mu}\omega^{bc}_\mu\Sigma_{bc}\psi &= \frac{i}{2}\sum_{a,b\neq c}\psi^\dagger\gamma^0\gamma_a e^{a\mu}(e^b_\nu\partial_\mu e^{c\nu})\gamma_b\gamma_c\psi \\ &= \sum_{a\neq b\neq c}\frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_a\gamma_b\gamma_c\psi + \sum_{a\neq c}\frac{i}{4}\bar\psi e^{a\mu}(e^a_\nu\partial_\mu e^{\nu c} - e^c_\nu\partial_\mu e^{\nu a})\gamma_c\psi \\ &= \sum_{a\neq b\neq c}\frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_a\gamma_b\gamma_c\psi + \frac{i}{2}\sum_c\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi \end{aligned} \tag{539}$$

The first term here is Hermitian: using $\gamma_a^\dagger\gamma_0^\dagger = \gamma_0\gamma_a$ (we are in $\mathbb{R}$ time, remember), we have

$$\dagger : \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\psi \mapsto -\frac{i}{4}\psi^\dagger e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}(\gamma_c^\dagger\gamma_b^\dagger\gamma_a^\dagger)\gamma_0^\dagger = \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_c\gamma_b\gamma_a = \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\psi, \tag{540}$$

since here $a \neq b \neq c$.

However, the second term at the end of (539) is actually anti-Hermitian:

$$\dagger : \frac{i}{2}\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi \mapsto -\frac{i}{2}\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi. \tag{541}$$

However, when we add in the $+i\bar\psi(\partial_\mu e^{a\mu})\gamma_a\psi$ from (538), we see that it combines with the RHS of the above equation to yield the LHS, giving an action that is Hermitian. Thus the second term at the end of (539) is a counterterm that ensures that the full action is Hermitian.

Here the fact that the tetrads are the "square root of the metric" is made manifest. Of course, there are infinitely many other choices, related by local $O(2)$ transformations.

To get the spin connection, we will need to know that the nonzero Christoffel symbols on the sphere are

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{544}$$

Then we calculate

$$\omega^{ab}_\theta = 0, \qquad \omega^{ab}_\phi = \cos\theta J^{ab}. \tag{545}$$

Then, if we adopt the gamma matrices $\gamma^1 = X, \gamma^2 = Y$ (this is the best choice since it makes the splitting $S = S_+ \oplus S_-$ manifest), we get

$$\omega_\theta = 0, \qquad \omega_\phi = \frac{1}{2}\cos\theta J^{ab}\Sigma_{ab} = -\frac{\cos\theta}{2}Z. \tag{546}$$

Finally, we need an expression for A_μ. We will make the usual choice for a monopole on S^2, namely

$$A^{N/S} = n\frac{\pm 1 - \cos\theta}{2}d\phi, \tag{547}$$

which gives $\int_{S^2} F = 2\pi n$. Note how similar the forms of the gauge and spin connections are. This means the covariant derivatives are

$$\nabla_\theta = \partial_\theta, \qquad \nabla_\phi = \partial_\phi - \frac{iZ}{2}\cos\theta + in\frac{\pm 1 - \cos\theta}{2}. \tag{548}$$

We can now finally write down the expression for $i\not{D}_A\psi = 0$, which is

$$\not{D}_A\psi^{(N/S)} = \left[X\left(\partial_\theta + \frac{\cot\theta}{2}\right) + Y\csc\theta\left(\partial_\phi + in\left(\frac{\pm 1 - \cos\theta}{2}\right)\right)\right]\psi^{(N/S)} = 0 \tag{549}$$

or written out, (note to self: oops, I think I goofed and wrote n instead of $-n$ in the following equation—if true, switch $R \leftrightarrow L$ in what follows)

$$\begin{aligned} &\left(\partial_\theta + \frac{\cot\theta}{2} - i\csc\theta\partial_\phi - n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)\right)\psi_R^{(N/S)} = 0 \\ &\left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi + n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)\right)\psi_L^{(N/S)} = 0 \end{aligned} \tag{550}$$

These equations are actually very easy to solve: the ϕ dependence is $e^{il\phi}$ by symmetry, while the θ dependence is figured out by the common factor of $1/\sin\theta$ in the equations.

For $n = 0$, a solution with $l = 0$ is $\psi_{R/L} = 1/\sqrt{\sin\theta}$. However, while normalizable, this is not differentible, and therefore is not an allowed solution.[67] So there are no zero modes

[67] We are looking only for solutions in the domain of $i\not{D}_A$, which by definition are C^∞ sections of the bundle $S \otimes E$, where S is the spinor bundle and E is the gauge bundle. Indeed, the Laplacian and the Dirac operator are only defined on infinitely smooth functions, and stuff can go wrong if our functions are not infinitely differentible. For example, consider $\psi_R(\theta,\phi) = 1/\sqrt{\sin(\theta)}$. This looks like a totally fine zero mode solution for the zero-flux case $n = 0$, since $i\not{D}_0\psi = 0$ and also $\int \psi_R^\dagger\psi_R\sin\theta = 2\pi^2$ is finite, so that ψ_R is L_2 on the S^2. However, weird pathologies come up due to the non-differentibility at $0, \pi$. For example, while

when $n = 0$. This follows from the fact that there are no zero modes on spaces where the curvature scalar is nowhere negative; more on this tomorrow.

Now consider $n = 1$. From the index theorem we know there should be one more L zero mode than R zero mode. And indeed, there is one L zero mode, and no R zero modes. The L zero mode is just

$$\psi_L^N(1, 1/2) = e^{i\phi/2}, \qquad \psi_L^S(1, 1/2) = e^{-i\phi/2}, \tag{552}$$

where we have adopted the notation $\psi_{L/R}^{N/S}(n, l)$. Note that the gluing transition function on the equator correctly recovers the $n = 1$ flux. Conversely, if $n = -1$ then we have an R zero mode and no L zero mode: the R zero mode is

$$\psi_R^N(-1, -1/2) = e^{-i\phi/2}, \qquad \psi_R^S(-1, -1/2) = e^{i\phi/2}. \tag{553}$$

Now take $n = \pm 2$. For $n = 2$ we expect two more L zero modes than R zero modes, and indeed, the two L zero modes are (there are no R zero modes)

$$\psi_L^N(2, 3/2) = \sin(\theta/2)e^{i3\phi/2}, \qquad \psi_L^S(2, -1/2) = \sin(\theta/2)e^{-i\phi/2} \tag{554}$$

and

$$\psi_L^N(2, 1/2) = \cos(\theta/2)e^{i\phi/2}, \qquad \psi_L^S(2, -3/2) = \cos(\theta/2)e^{-3i\phi/2}. \tag{555}$$

The situation is reversed for $n = -2$: the two zero modes are

$$\psi_R^N(-2, -3/2) = \sin(\theta/2)e^{-i3\phi/2}, \qquad \psi_R^S(-2, 3/2) = \sin(\theta/2)e^{i3\phi/2} \tag{556}$$

and

$$\psi_R^N(-2, -1/2) = \cos(\theta/2)e^{-i\phi/2}, \qquad \psi_R^S(-2, 3/2) = \cos(\theta/2)e^{3i\phi/2}. \tag{557}$$

In general, for flux n, there are n L zero modes and no R zero modes if $n > 0$, while there are n R zero modes and no L zero modes if $n < 0$. The n modes come in a series $\psi_{L/R}^N(n, l)$ with $l = n - 1/2, n - 3/2, \dots, \pm 1/2$, with the $+$ for L and the $-$ for R. The corresponding functions on the southern patch are related by $l \mapsto l - n$, which ensures that the transition function on the equator is a large gauge transformation with winding $2\pi n$.

One important point is that despite the fact that $l \in (2\mathbb{Z} + 1)/2$, the zero modes do *not* have half-odd-integer spin! To find out what their spins are, we need to calculate the L^2 angular momentum operator, which is modified by the spin and gauge connections; we will do this in a little bit.

We've been working with a uniform field strength, but of course (by the index theorem), we know that the zero modes must persist if we take an arbitrary field configuration with

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$\not{D}_0\psi = 0$,

$$\not{D}_0^2\psi = \left(-\csc\theta\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{4} + \frac{1}{4\sin^2\theta}\right)\psi = \sin^{-5/2}\theta(1 - \cos^2(\theta)/4) - \sin^{-1/2}(\theta)/4 \neq 0. \tag{551}$$

The problem here is that $\not{D}(\not{D}\psi) \neq (\not{D}^2)\psi$ and so the positive-definiteness arguments we made don't apply to ψ (this is why we don't want to let functions like ψ be part of the domain of definition of the Dirac operator).

Anyway, I'm still not really sure about the physicality of such a restriction. Should I really be bothered if ψ is non-differentible, provided that $\sqrt{|g|}|\psi(x)|^2$ is finite everywhere? Not sure.

the same value of $\int F$. For example, we might add the vector potential $\widetilde{A} = g(\theta)d\phi$ to the existing monopole potential, where $\widetilde{A}$ is globally defined on the sphere. Then we just have to modify our zero mode solution by $\psi_{L/R} \mapsto f_{L/R}(\theta)\psi_{L/R}$, where $f_{L/R}(\theta)$ satisfies

$$(\partial_\theta \pm \csc(\theta) g(\theta)) f(\theta) = 0 \implies \psi_{L/R} = \exp\left(\mp \int_0^\theta d\theta' \csc(\theta') g(\theta')\right) \psi^0_{L/R}, \tag{558}$$

where $\psi^0_{L/R}$ is the solution for the homogeneous field. From this expression we see that we get a well-defined answer only if $g(\theta) \to \theta$ as $\theta \to 0, \pi$; this is the condition that $\widetilde{A}$ go to zero at the poles, so that $\widetilde{A}$ is topologically trivial. For example, consider the L zero mode, and let $g(\theta) = \sin 2\theta$. Then we see that the zero-mode solution is modified by a factor of $f_L(\theta) = e^{-2\sin\theta}$, so that the weight of the wavefunction becomes concentrated near the poles, where the normalized field strength $F/\sin\theta$ becomes largest; this is because as we have seen, the L zero modes "like" positive field strength. More generally, for an arbitrary asymmetric $\widetilde{A}$, we just have to multiply our symmetric zero mode solution by a factor $f(\theta, \phi)$, where

Anyway, now some more comments on the symmetric (uniform field strength) case. we've seen that the zero mode states fall into half-odd-integer representations of $SU(2)$. This is no surprise given the symmetry of the problem (in fact, the whole spectrum of $\not{D}$ falls into $SU(2)$ representations, not just the zero modes).

To find the generators of the $SU(2)$, it is not simply enough to covariantize by making the replacement $\mathbf{L} = -i\mathbf{n} \times \partial \mapsto -i\mathbf{n} \times \nabla$, with $\mathbf{n}$ the unit vector on the sphere. Indeed, doing this leads to generators that fail to satisfy the correct $SU(2)$ commutation relations, since in the presence of background field strengths the covariant momenta $-i\nabla$ fail to commute (their commutator measures the field strength). The commutation relations are ruined not just by the gauge background field, but also by the field strength of the spin connection (the geometric curvature of the sphere). Now viewing the S^2 as living in three dimensions, the Hodge duals of the field strengths of both the gauge field and the spin connection are oriented along $\mathbf{n}$ (the magnetic field for both spin and gauge connections is radial), and so we can write $(d[\omega + A])_{\mu\nu} = \epsilon_{\mu\nu\lambda} n^\lambda B$, where in the present case $B = \mathbf{1}n/2 + Z/2$. One can check that in this case,

$$[(-i\mathbf{n} \times \nabla)^\mu, (-i\mathbf{n} \times \nabla)^\nu] = i\epsilon^{\mu\nu\lambda}((-i\mathbf{n} \times \nabla)_\lambda - Bn_\lambda). \tag{559}$$

This prompts us to take the ansatz

$$L_\mu = \epsilon_{\mu\nu\lambda} n^\nu \nabla^\lambda + Bn^\mu \tag{560}$$

for the angular momentum generators. Indeed, one can check (see the next diary for the calculation) that with this choice, the L_μ satisfy the usual $SU(2)$ commutation relations.

We can now write down the angular momentum generators explicitly. We have

$$\begin{aligned} L_z &= -i\nabla_\phi + \cos\theta B \\ L_\pm &= e^{\pm i\phi}\left(\pm\nabla_\theta + i\cot\theta\nabla_\phi + B\sin\theta\right). \end{aligned} \tag{561}$$

When simplifying this, we will use the covariant derivative $\nabla_\phi = \partial_\phi - i(Z\cos\theta + n\cos\theta)/2$, which differs from the one written above by the term $\pm in/2$, which doesn't affect the field strength and hence can be dropped without affecting the angular momentum commutation relations. However, one must use caution with this convention, since changing the convention for the gauge field *does* change the expressions for the eigenstates of $\not{D}_A$. This means that the zero mode eigenstates obtained above will *not* be appropriate eigenfunctions of the L_z and L^2 operators obtained below! Maybe someday I'll come back and redo this so that the conventions are the same. Anyway, plugging in and simplifying, we find

$$\begin{aligned} L_z &= -i\partial_\phi \\ L_\pm &= e^{\pm i\phi}\left(\pm\partial_\theta + i\cot\theta\partial_\phi + \frac{1}{2\sin\theta}(Z+n)\right). \end{aligned} \tag{562}$$

I've done a check in mathematica of the commutation relations, and they work! Yay! Note that if we had chosen to keep the factor of $\pm d\phi/2$ in the gauge connection, the eigenvalues of L_z would be shifted by $\pm 1/2$: therefore the eigenvalues of L_z are not a gauge-invariant thing to calculate, and they do not tell you about the spin of the zero mode.

As a reminder, one should not confuse the total angular momentum operator $L^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+)$ with the (negative of, depending on conventions) covariant Laplacian $-\nabla_\mu\nabla^\mu$. Indeed, the Laplacian is (we are working in conventions where the Laplacian is positive-definite)

$$\begin{aligned} -\Delta = \nabla_\mu\nabla^\mu &= \nabla_\theta^2 + \cot\theta\,\nabla_\theta + \csc^2\theta\,\nabla_\phi^2 \\ &= \partial_\theta^2 + \cot\theta\,\partial_\theta + \csc^2\theta\left(\partial_\phi^2 - i(Z+n)\cos\theta\partial_\phi - \frac{\cos^2\theta}{4}(Z+n)^2\right). \end{aligned} \tag{563}$$

In contrast, the angular momentum operator is

$$L^2 = \Delta + \frac{1}{4}(Z+n)^2. \tag{564}$$

We would like to use this result to figure out the angular momentum of the zero modes obtained previously. In our current conventions, the zero mode equation is, on the N hemisphere

$$\begin{aligned} \left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi + \frac{n}{2}\cot\theta\right)\psi_R = 0 \\ \left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi - \frac{n}{2}\cot\theta\right)\psi_L = 0 \end{aligned} \tag{565}$$

This choice for the gauge field makes finding the zero modes slightly easier. For example, if $n = 1$ we see that we get a solution where $\psi_R = 1/\sqrt{4\pi}, \psi_L = 0$. The fact that the zero mode is a constant suggests that it has spin zero: to check, we act on it with L^2:

$$L^2(n=1)\psi_R = \Delta(n=1)\psi_R = 0, \tag{566}$$

as expected.

The last thing we square a lot is the Dirac operator: more on this and its relation to Δ in a future diary entry.

Stereographic projection

Now we will go through the problem again using stereographic coordinates. These coordinates are nicer since they are less singular than spherical coordinates. We usually cover the sphere in two hemispherical patches, but doing it this way means that both patches will contain a coordinate singularity for ϕ, which is no good (although for us it's okay since the zero mode solutions vanish at the poles so we can still use two patches to construct a single-valued zero mode solution). Getting a good covering of S^2 requires 4 patches, and in order to escape coordinate singularities, the patches have to have different definitions of ϕ, θ such that the $\theta = 0, \pi$ points in a given patch's coordinate system don't occur within that patch itself. Gluing together zero mode solutions like this really is a hopeless mess.

By contrast, stereographic coordinates are great! We only need two patches to cover the S^2 (not goodly, but that's okay), and within each patch we can use a metric which is perfectly singularity-free, and in fact is conformally equivalent to flat Euclidean space. Indeed, recall from a few diary entries ago that in stereographic projection of S^2 onto the plane, the metric assumes the conformally flat form (assuming the sphere has radius 1 for simplicity)

$$ds^2 = \frac{4}{(1+r^2)^2}(dx^2 + dy^2). \tag{567}$$

Tetrad is simple is this coordinate system:

$$e^a_\mu = \frac{2}{1+r^2}\mathbf{1}^a_\mu, \qquad e^{\mu a} = \frac{1+r^2}{2}\mathbf{1}^{\mu a}. \tag{568}$$

The Christoffel symbols are easily calculated to be (I won't write out the algebra)

$$\Gamma^\mu_{\nu\lambda} = \frac{2}{1+r^2}(x^\mu \delta_{\nu\lambda} - x_\nu \delta^\mu_\lambda - x_\lambda \delta^\mu_\nu). \tag{569}$$

We can then calculate

$$\omega^{ab}_\mu = \frac{2x_\mu}{1+r^2}\delta^{ab} + \delta^a_\nu \delta^{b\lambda}\Gamma^\nu_{\mu\lambda}. \tag{570}$$

Only the off-diagonal part is non-zero:

$$\omega^{12}_\mu = -\omega^{21}_\mu = \frac{2}{1+r^2}(x\delta_{\mu y} - y\delta_{x\mu}), \tag{571}$$

and so the full spin connection is

$$\omega_\mu = \frac{1}{2}\omega^{ab}_\mu \Sigma_{ab} = -\frac{i}{2}\sum_{a<b}\omega^{ab}_\mu \gamma_a\gamma_b = \frac{Z}{1+r^2}(x\delta_{\mu y} - y\delta_{\mu x}). \tag{572}$$

Therefore the equation $\not{D}_A\psi = 0$ reads

$$\left[X\left(\partial_x + iA_x - iZ\frac{y}{1+r^2}\right) + Y\left(\partial_y + iA_y + iZ\frac{x}{1+r^2}\right)\right]\psi = 0 \tag{573}$$

Decomposing this with $\psi = (\psi_L, \psi_R)^T$,

$$\begin{aligned} \left(2\partial + 2iA - \frac{\bar{z}}{1+|z|^2}\right)\psi_R = 0 \\ \left(2\bar{\partial} + 2i\bar{A} - \frac{z}{1+|z|^2}\right)\psi_L = 0, \end{aligned} \tag{574}$$

where $z = x + iy$, $A = (A_x - iA_y)/2$, and $\partial = (\partial_x - i\partial_y)/2$.

Now we need to get an expression for A. We want the field strength to be proportional to the volume form, which is $4/(1+r^2)^2$. In complex coordinates,

$$\frac{4}{(1+r^2)^2}(dx^2 + dy^2) = \frac{4}{(1+|z|^2)^2} dz d\bar{z}. \tag{575}$$

Thus if we take

$$A = in\frac{\bar{z}}{2(1+|z|^2)}, \quad \bar{A} = -in\frac{z}{2(1+|z|^2)}, \tag{576}$$

then we have

$$F_{z\bar{z}} = \partial\bar{A} - \bar{\partial}A = i\frac{n}{1+|z|^2}\left(1 - \frac{|z|^2}{1+|z|^2}\right) = i\frac{n}{(1+|z|^2)^2}, \tag{577}$$

which gives

$$\int_{\mathbb{R}^2} F_{z\bar{z}}\, dz \wedge d\bar{z} = \int_{\mathbb{R}^2} \frac{n}{(1+r^2)^2} i(-2idx \wedge dy) = 2\pi n \int_0^\infty dr\, \frac{2r}{(1+r^2)^2} = 2\pi n \tag{578}$$

as desired. Thus the equations for the zero modes are (I think I missed a factor of 2 somehwere, which I am re-instating below in a very ad hoc manner: I think it is needed to get the right zero mode solutions)

$$\begin{aligned} \left(2\partial - (1/2 + n)\frac{\bar{z}}{1+|z|^2}\right)\psi_R = 0 \\ \left(2\bar{\partial} - (1/2 - n)\frac{z}{1+|z|^2}\right)\psi_L = 0. \end{aligned} \tag{579}$$

The solutions are then

$$\begin{aligned} \psi_R(z,\bar{z}) &= f_R(\bar{z})(1+|z|^2)^{(1/2+n)/2} \\ \psi_L(z,\bar{z}) &= f_L(z)(1+|z|^2)^{(1/2-n)/2}, \end{aligned} \tag{580}$$

where the Laurent series for $f_R(\bar{z})$ and $f_L(z)$ only involve terms of non-negative degree since we require $\psi_{R/L}$ to be finite at the origin. Now, requiring that the zero mode solutions be finite at ∞ tells us that no L zero modes exist if $n \leq 0$, while no R zero modes exist if $n \geq 0$, in agreement with what we found before. Wolog, take $n > 0$ and look at the $\psi_L(z,\bar{z})$ solutions. We can take them to be eigenstates of L_z, which in complex coordinates is

$$L_z = -i(X\partial_Y - Y\partial_X) = -i(x\partial_y - y\partial_x) = \frac{z+\bar{z}}{2}(\partial - \bar{\partial}) + \frac{z-\bar{z}}{2}(\partial + \bar{\partial}) = z\partial - \bar{z}\bar{\partial}, \tag{581}$$

where X, Y are coordinates in 3-space on the S^2 (note that the contribution from the connection to the covariant derivative has canceled with the modification of the angular momentum generators required in a magnetic field; see a subsequent diary entry for details). Note that functions only of $|z|$ have no angular momentum, as required. This means that z^α is an eigenfunction of L_z with eigenvalue $+\alpha$, while $\bar{z}^\alpha$ is an eigenfunction with eigenvalue $-\alpha$. Therefore in a basis in which L_z is diagonalized with eigenvalue $l \in \frac{1}{2}\mathbb{Z}$, the $f_L(z), f_R(\bar{z})$ will be proportional to z^l and $\bar{z}^l$, respectively. So, the zero mode eigenfunctions of L_z for $n > 0$ are of the form $z^l(1+|z|^2)^{1/4-n/2}$. At $r \to \infty$ this goes as $r^{l-n+1/2}$, so we require that $l \leq n - 1/2$ (having a constant is okay since constants have finite integrals due to the conformal factor in the metric). This recovers the situation where we have n L zero modes of angular momentum $l = n-1/2, n-3/2, \dots, 1/2$ if $n > 0$, and n R zero modes with $l = n+1/2, \dots, -1/2$ if $n < 0$.

Another way to derive the parity anomaly

Consider the case of a 2+1D theory on $S^2 \times \mathbb{R}$, with a unit of flux through the spatial S^2. Then the results above tell us that the Hamiltonian (which is just the spatial part of the Lagrangian) has a single zero mode; when we quantize we thus get two states $|0\rangle$ and $|1\rangle = \chi^\dagger|0\rangle$.

Now the charges of $|1\rangle$ and $|0\rangle$ must satisfy $q_0 = q_1 - 1$: this is just a consequence of $Q\chi^\dagger = \chi^\dagger(Q+1)$. Now, using CT symmetry,[68] we have (taking $CT|0\rangle = |1\rangle$; a possible phase factor here doesn't contribute to the discussion)

$$CTe^{iQ}|0\rangle = CTe^{iq_0}|0\rangle = e^{-iq_0}|1\rangle \tag{582}$$

but also

$$CTe^{iQ}|0\rangle = e^{iQ}CT|0\rangle = e^{q_1}|1\rangle, \tag{583}$$

so that $q_1 = -q_0$. Thus if CT really is a symmetry, we have $q_0 = -1/2, q_1 = +1/2$. This however means that both $|0\rangle$ and $|1\rangle$ are not gauge-invariant; a contradiction. Hence CT must actually be broken. Of course, this is the parity anomaly, and the way that CT gets broken is by T getting broken.

33 *March 25 — Zero modes for Dirac fermions on the torus*

Today's diary entry is simple, but hasn't appeared in the diary yet: finding the spectrum of Dirac fermions on a torus in the presence of nonzero net magnetic flux.

Solution:

We will work on a square torus with both side lengths set to 1 for simplicity. Suppose that over the torus, the integral of the field strength is $B \equiv \int F = 2\pi n$. We will split the torus

[68] The presence of $\int dxdy\, F_{xy} = 2\pi$ means that both C and T are broken, while CT is preserved. Thus it makes sense to ask about how CT acts within the subspace of the zero modes, but not C or T individually.

up into two cylindrical coordinate patches. The first will be $U_1 = \{0 \le x \le 1/2, 0 \le y \le 1\}$, and the second will be $U_2 = \{1/2 \le x \le 1, 1/2 \le y \le 1\}$, with $1 \sim 0$ in both limits. The two patches overlap along the line $x = 1/2$ and along the line $x = 0 \sim 1$. We will take the transition function to be trivial on the former overlap region, and to be the exponential of a function that winds by $2\pi n$ around the y direction, viz. $g_{12} = e^{iyB}$. The natural choice for the gauge field is then $A = (0, Bx)$. This gives the correct field strength and the way it is glued between the two patches is determined correctly as

$$A^2_\mu(1, y) = g_{12}^{-1}(A^1_\mu(0, y) - i\partial_\mu)g_{12}. \tag{584}$$

The Hamiltonian $H = -i(X[\partial_x - iA_x] + Y[\partial_y - iA_y])$ is then, in this gauge,

$$H = -i \begin{pmatrix} 0 & \partial_x - i\partial_y - Bx \\ \partial_x + i\partial_y + Bx & 0 \end{pmatrix}. \tag{585}$$

Define the operator

$$\gamma \equiv \frac{1}{\sqrt{2|B|}}(\partial_x + i\partial_y + Bx), \qquad H = -i\sqrt{2|B|}\begin{pmatrix} 0 & -\gamma^\dagger \\ \gamma & 0 \end{pmatrix}. \tag{586}$$

Then

$$[\gamma, \gamma^\dagger] = \operatorname{sgn}(B). \tag{587}$$

In what follows we will assume $B > 0$, and so $\gamma, \gamma^\dagger$ obey the usual harmonic oscillator algebra.

Now if we square $H\psi = E\psi$, we get

$$2B \begin{pmatrix} \gamma^\dagger\gamma & 0 \\ 0 & \gamma^\dagger\gamma + 1 \end{pmatrix} \psi = E^2\psi. \tag{588}$$

Therefore the energy levels are

$$E_n = \pm\sqrt{2Bm}, \qquad m \in \mathbb{N}. \tag{589}$$

Now since $\gamma^\dagger\gamma$ has only non-negative eigenvalues, we see that for $\psi = (\psi_L, \psi_R)^T$, we can have ψ_L zero modes, but cannot have ψ_R zero modes. If we were to change the sign of the flux $B > 0$ by $B \mapsto -B$, then in order to maintain the right commutation relations, we would need to interchange γ and $\gamma^\dagger$. This would then give $H = 2|B|(\gamma^\dagger\gamma + 1) \oplus \gamma^\dagger\gamma$, which is the same as for $B > 0$, but with left and right components switched. Therefore for $B > 0$ we can have only L zero modes, while for $B < 0$ we can only have R zero modes. This is in agreement with the index theorem.

Anyway, to solve for the eigenspectrum, we need to find $|0\rangle$ such that $\gamma|0\rangle = 0$. Since y doesn't appear in γ we can give the $|0\rangle$ wavefunction a y dependence of e^{iyk_y}. We then have $(\partial_x - k_y + Bx)\psi_0(x, y) = 0$, where $\langle x, y|0\rangle = \psi_0(x, y)$. This gives

$$\psi_0(x, y) = e^{iyk_y} e^{-(x-k_y/B)^2 B/2}. \tag{590}$$

If we set PBC in the y direction, then we need $k_y \in 2\pi\mathbb{Z}$. For the x direction, setting $x \sim x + 1$ means that $k_y \sim k_y + B$, Therefore $B = 2\pi n$ means that we have n different options for k_y, and so the degeneracy of the zero energy states is n. The zero modes are then $\psi = (\psi_0, 0)^T$. Note that the zero modes survive the introduction of a non-uniform perturbing flux, so long as that flux integrates to zero over the torus. For example, adding on $\widetilde{A} = \widetilde{A}_y(x)dy$ to A modifies the zero mode solution by a factor of $\exp(-\int_0^x dx'\, \widetilde{A}_y(x))$, which preserves the boundary conditions on ψ_0 provided that $\int d\widetilde{A} = 0$ (for example, we could take $\widetilde{A}_y(x) = \sin(2\pi x)$).

Excited states are constructed by acting with $(\gamma^\dagger)^n$ on $|0\rangle$ in the usual way. For example, $\gamma^\dagger|0\rangle = |1\rangle = \sqrt{2B}(x - k_y/B)\psi_0$. The state with energy $\pm\sqrt{2Bm}$, $m > 0$ is formed by taking $|m\rangle$ for the left component and $\mp i|m-1\rangle$ for the right component:

$$H\psi_{\pm m} = H\begin{pmatrix} |m\rangle \\ \mp i|m-1\rangle \end{pmatrix} = \sqrt{2B}\begin{pmatrix} 0 & i\gamma^\dagger \\ -i\gamma & 0 \end{pmatrix}\begin{pmatrix} |m\rangle \\ \mp i|m-1\rangle \end{pmatrix} = \pm\sqrt{2Bn}\begin{pmatrix} |m\rangle \\ \mp i|m-1\rangle \end{pmatrix}. \tag{591}$$

Suppose we require that the system be CT symmetric. Now in this basis, charge conjugation $C : \psi \mapsto C\psi^\dagger$ needs to satisfy $[C^\dagger(X, Y)C]^T = -(X, Y)$, and so we can take $C = Y$. If T acts as $T : \psi \mapsto Y\psi$, $i \mapsto -i$, then CT acts just as $\psi \mapsto \psi^\dagger, i \mapsto -i$. Note that $CT : \psi_{\pm m} \mapsto \psi_{\mp m}$, as expected for particle-hole symmetry.

Let $|+\rangle$ be the many-body state with the zero mode filled, and $|-\rangle$ be the many-body state with the zero mode unfilled. Since the charge operator $e^{iQ} = e^{i\int dx\, dy\, \psi^\dagger\psi}$ commutes with CT, by acting on $|\pm\rangle$ with CTe^{iQ} in two different ways (directly, or by moving the e^{iQ} to the left first), we get $e^{iq_+} = e^{-iq_-}$. Since $q_+ = q_- + 1$ (because $C : |\pm \leftrightarrow |\mp\rangle$ and because $e^{iQ}C = Ce^{-iQ}$), we then conclude that CT symmetry implies $q_\pm = \pm 1/2$. If we break C symmetry with e.g. $\delta H \propto \gamma^\dagger\gamma\mathbf{1}$, then since the charge is quantized, the charge assignment of the zero modes will remain unchanged. The fact that the preservation of T symmetry implies a charge of $\pm 1/2$ for the monopole can also be inuited from fact that T symmetry can be preserved in this model by adding "$\frac{1}{8\pi}\int AdA$" to the Lagrangian.

34 *March 24 — Details of the* $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$ *isomorphism*

Today we will work out the explicit mapping between the $\mathfrak{spin}(n)$ and $\mathfrak{so}(n)$ Lie algebras. Understanding exactly how this works is a prerequisite for figuring out the correct factors of 1/2 to write in the spin connection, which we will need for the next diary entry.

Let γ_j be Clifford generators, and fix the signature to be Euclidean so that $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. We can write a generic element in the odd part of the real Clifford algebra $Cl_n = Cl_n^+ \oplus Cl_n^-$ as $v^j\gamma_j$, where $v \in \mathbb{R}^n$. The group $\text{Pin}(n)$ consists of all elements of the form $\prod_{a=1}^k v_a$, where each v_a is a unit vector: $v_{a,i}v_a^i = 1$, and where $k \in \mathbb{N}$ is any integer. The $\text{Spin}(n)$ subgroup is the even subgroup of $\text{Pin}(n)$, i.e. those elements with $k \in 2\mathbb{N}$.

As discussed in the diary entry on Clifford algebras, the map $P : \text{Spin}(n) \to SO(n)$ is realized in the following way: for any $X = x^i\gamma_i \in Cl_n^-$ and any $Y \in \text{Spin}(n)$, we have

$$P : x_i \mapsto R_{ij}x_j, \tag{592}$$

where R is the orthogonal matrix defined by the adjoint action of Y:

$$P : X \mapsto Y^{-1}XY = R_{ij}x^j\gamma^i. \tag{593}$$

That R is orthogonal can be seen by applying the transformation to both $X = \gamma_j, X' = \gamma_k$ and considering the transformation of $\{X, X'\}$.[69]

The map P induces a map $P_* : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ via the differential. To find P_*, we first need to write elements in $\mathrm{Spin}(n)$ as points on parametrized curves passing through the identity at $\theta = 0$. To this end, consider the curves

$$\begin{aligned}\mathrm{Spin}(n) \ni Y_{ab}(\theta) &= (-\gamma_b\cos(\theta/2) + \gamma_a\sin(\theta/2))(\gamma_a\sin(\theta/2) + \gamma_b\cos(\theta/2)) \\ &= [\gamma_a, \gamma_b]\cos(\theta/2)\sin(\theta/2) + \mathbf{1}(\sin^2(\theta/2) - \cos^2(\theta/2)).\end{aligned} \tag{597}$$

Using $\cos(\theta/2)\sin(\theta/2) = \sin(\theta)/2$, this is

$$Y_{ab}(\theta) = \gamma_a\gamma_b\sin\theta + \mathbf{1}\cos\theta. \tag{598}$$

The important thing here is that $Y_{ab}(0) = \mathbf{1}$ and $Y'_{ab}(0) = \gamma_a\gamma_b$. Thus $\mathfrak{spin}(n)$ includes all the matrices of the form $\gamma_a\gamma_b$, with $a < b$ (taking products of $Y_{ab}(\theta)$'s for different choices of a, b allows us to pick up all such matrices). Furthermore, the $\gamma_a\gamma_b$s generate all of $\mathfrak{spin}(n)$, since there are $(n^2 - n)/2 = \dim\mathfrak{spin}(n)$ such matrices.

To find P_*, we just need to find how P acts on the $Y_{ab}(\theta)$, and then take the differential at $\theta = 0$. So for any $X = x^i\gamma_i$,

$$P : X \mapsto Y_{ab}^{-1}(\theta)XY_{ab}(\theta) \implies P_* : X \to \left(\frac{d}{d\theta}[Y_{ab}^{-1}(\theta)XY_{ab}(\theta)]\right)|_{\theta=0}. \tag{599}$$

The RHS is

$$\begin{aligned}\left(\frac{d}{d\theta}[Y_{ab}^{-1}(\theta)XY_{ab}(\theta)]\right)|_{\theta=0} &= -Y_{ab}(0)^{-1}Y'_{ab}(0)Y_{ab}(0)XY_{ab}(0) + Y_{ab}^{-1}(0)XY'_{ab}(0) \\ &= [X, \gamma_a\gamma_b]\end{aligned} \tag{600}$$

[69] We can find R_{ij} explicitly by doing

$$R_{ij} = \frac{1}{N}\mathrm{Tr}[Y^{-1}\gamma_i Y\gamma_j], \tag{594}$$

where N is a normalization factor for the trace. One might also like to write this as $R_{ij} = \langle\gamma_i|Y\gamma_j Y^{-1}\rangle$, where the γ_is are viewed as basis vectors and the inner product is $\langle X|Y\rangle = \mathrm{Tr}[X^\dagger Y]$; this is then saying that R_{ij} can be found by taking the ith component of the image of the jth basis vector.

As an example of how this works, consider $n = 3$. Using the Pauli matrices for the γ matrices, the claim is that the map $P : \mathrm{Spin}(3) = SU(2) \to SO(3)$ is realized through

$$P : SU(2) \ni U \to R \in SO(3), \qquad R_{ij} = \frac{1}{2}\mathrm{Tr}[U^\dagger\sigma_i U\sigma_j]. \tag{595}$$

As a check, consider the image of $U = e^{i\theta Z/2}$. Then $U = \cos(\theta/2) + iZ\sin(\theta/2)$, and so

$$[P(U)]_{ij} = R_{ij} = \frac{1}{2}\mathrm{Tr}\left[(\cos(\theta/2) - iZ\sin(\theta/2))\sigma_i(\cos(\theta/2) + iZ\sin(\theta/2))\sigma_j\right]. \tag{596}$$

If $i = z$ then the RHS is $\mathrm{Tr}[Z\sigma_j]/2 = \delta_{zj}$. If $i = j \neq z$, then we get $\mathrm{Tr}[e^{i\theta Z}]/2 = \cos\theta$, while if $i \neq j \neq z$ then we get $\mathrm{Tr}[e^{i\theta Z}i\epsilon_{jiz}Z]/2 = \epsilon_{ijz}\sin\theta$. Putting these all together gives the 3×3 rotation matrix that we expect.

The commutator is

$$[X,\gamma_a\gamma_b]=\sum_i x^i[\gamma_i,\gamma_a\gamma_b] \quad =\sum_i x^i[(-\gamma_a\gamma_i+2\delta_{ai})\gamma_b-\gamma_a\gamma_b\gamma_i]=2\sum_i x^i(\delta_{ai}\gamma_b-\delta_{bi}\gamma_a)$$
$$=\sum_{ij}(2A^{ab}_{ij}x^j)\gamma^i, \tag{601}$$

where A^{ab} is the antisymmetric matrix with a 1 in the bth row and ath column and a -1 in the ath row and bth column: $A^{ab}_{ij}=\delta_{bi}\delta_{aj}-\delta_{bj}\delta_{ai}$ (these matrices form a basis form $\mathfrak{so}(n)$). This means that on the generators of $\mathfrak{spin}(n)$, P_* acts as

$$P_*:\gamma_a\gamma_b\mapsto 2A^{ab}\in\mathfrak{so}(n). \tag{602}$$

This means that the mapping back from $\mathfrak{so}(n)$ to $\mathfrak{spin}(n)$ is, for a generic element $O=\sum_{a<b}\alpha_{ab}A^{ab}\in\mathfrak{so}(n)$ with $\alpha_{ab}=-\alpha_{ba}$,

$$P_*^{-1}:O\mapsto\frac{1}{2}\sum_{a<b}\alpha_{ab}\gamma_a\gamma_b=\frac{1}{4}\sum_{a,b}\alpha_{ab}\gamma_a\gamma_b=\frac{1}{8}\sum_{a,b}\alpha_{ab}[\gamma_a,\gamma_b]. \tag{603}$$

This equation tells us where the factors of 1/8 in the spin connection and the expression for elements in Spin(n) as exponentials of gamma matrix commutators come from.

35 *March 26 — The square of the Dirac operator and zero mode solutions; some stuff about veilbeins*

Today's diary entry is simple: we want to compute $(i\not{D}_A)^2$ for an arbitrary gauge connection A, on an arbitrary geometry. This will tell us cool stuff about when zero modes of $i\not{D}_A$ can exist.

Solution:

Preliminary veilbein things

Since we are dealing with fermions, we will be using tetrad methods, with $e^a=e^a_\mu dx^\mu$ denoting a 1-form (or using the metric, a vector field) component of the orthonormal basis at each point in spacetime. The roman indices can be raised and lowered with δ_{ab}, while the Greek indices must be raised and lowered using $g_{\mu\nu}$. When acting on e.g a $(1,1)$ tensor V^a_b in the orthonormal basis, the covariant derivative along a vector field X acts as using the spin connection ω_μ as

$$[\nabla_X V]^a_b=X^\mu(\delta^a_c\delta^d_b\partial_\mu+\omega_\mu{}^a{}_c\delta^d_b-\omega_\mu{}^d{}_b\delta^a_c)V^c_d. \tag{604}$$

As usual, covariant indices get minus signs: this is required in order to have contractions like V_aV^a get differentiated appropriately as $\nabla_\mu(V_aV^a) = \partial_\mu(V_aV^a)$. A consequence of this is that the spin connection $\omega_\mu{}^a$ is antisymmetric in a, b, which is a requirement of $\nabla_\mu(\delta^{ab}) = 0$.

One result we will need in the following is that $[\nabla_X e_b]^\mu = 0$: the veilbeins are covariantly constant along the flow of any vector field X. Now the covariant derivative of a veilbein is

$$[\nabla_X e_b]^\nu = X^\mu(\partial_\mu e^\nu_b - \omega_\mu{}^a{}_b e^\nu_a + \Gamma^\nu_{\mu\lambda}e^\lambda_b). \tag{605}$$

This turns out to identically vanish; to see this we need an expression for the spin connection in terms of the veilbeins and the Christoffel symbols. We get this by expressing $\nabla_X V$ in both the spacetime and orthonormal basis as $\nabla_X V = X^\mu\nabla_\mu V^\nu\partial_\nu = X^\mu[\nabla_\mu V]^a e_a$. In the spacetime basis, we of course have $[\nabla_X V]^\nu = X^\mu(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda)$. In the orthonormal frame basis, we have $[\nabla_X V]^a = X^\mu(\partial_\mu V^a + \omega_\mu{}^a{}_b V^b)$. So then setting $[\nabla_X V]^a e^\nu_a = [\nabla_X V]^\nu$, we have

$$\begin{aligned}\omega_\mu{}^a{}_b V^b e^\nu_a &= \partial_\mu(V^c e^\nu_c) - e^\nu_a\partial_\mu V^a + \Gamma^\nu_{\mu\lambda}V^b e^\lambda_b \\ &= (\partial_\mu e^\nu_b + \Gamma^\nu_{\mu\lambda}e^\lambda_b)V^b.\end{aligned} \tag{606}$$

Contracting both sides with e^c_ν and then renaming some indices,

$$\omega_\mu{}^a{}_b = e^a_\nu\partial_\mu e^\nu_b + e^a_\nu\Gamma^\nu_{\mu\lambda}e^\lambda_b. \tag{607}$$

so that the spin connection can be thought of as the 1-form

$$\omega^a{}_b = [\nabla^G_{e^a} e_b]_\mu dx^\mu \tag{608}$$

where ∇^G is the "Greek covariant derviative", i.e. the convariant derivative that acts only on Greek (spacetime) indices. Anyway, from (607), we see that the full covariant derivative of the vector field e_b vanishes (and hence so too does the covariant derivative of e^b).

In the following, we will be using the expression for the Riemann curvature tensor in terms of a commutator of two covariant derivatives along the two vector fields e^a, e^b. In terms of the spin connection, we have, for any vector field V^e in the orthonormal frame basis,

$$\begin{aligned}[\nabla_{e^a}, \nabla_{e^b}]V^d &= e^\mu_a e^\nu_b\left((\delta^d_c\partial_\mu + \omega_\mu{}^d{}_c)(\delta^c_e\partial_\nu + \omega_\nu{}^c{}_e) - (\mu \leftrightarrow \nu)\right)V^e \\ &= e^\mu_a e^\nu_b(\delta^d_c\partial_{[\mu}\omega_{\nu]}{}^c{}_e + \omega_{[\mu|}{}^d{}_c\omega_{\nu]}{}^c{}_e)V^e.\end{aligned} \tag{609}$$

Thus we can write the components of the Riemann curvature tensor in the orthonormal frame basis as

$$[R_{ab}]_{de} = [\nabla_{e^a}, \nabla_{e^b}]_{de} = e^\mu_a e^\nu_b[d\omega^d{}_e + \omega^d{}_c \wedge \omega^c{}_e]_{\mu\nu}. \tag{610}$$

Thinking of the curvature tensor as a matrix-valued 2-form on spacetime, this is written more succinctly as $R_{ab} = [d\omega + \omega \wedge \omega]_{ab}$.

Square of the Dirac operator

First we need to set the notation and establish a preliminary result. We will be interested in the Dirac operator $i\not{D}_A : \Gamma(S \otimes E) \to \Gamma(S \otimes E)$, where $\Gamma(S \otimes E)$ denotes C^∞ sections of the bundle $S \otimes E$ with connection A. Here $S = SXL \times_\rho \mathbb{C}^{2^{\lfloor n/2 \rfloor}}$ is the spinor bundle, with ρ the spinor representation and SLX the spin lift of the frame bundle. E is some gauge bundle, and we take the fermion fields (sections of $S \otimes E$) to transform under a $\otimes$ representation of the spinor representation and whatever representation they carry under the gauge group. The covariant derivative on $S \otimes E$ will be written as ∇, and we will write $\nabla^\dagger$ for the adjoint of ∇.

The Dirac operator is $i\not{D}_A = \gamma^a \nabla_{e^a}$. Its square is

$$(i\not{D}_A)^2 = -\gamma^a \nabla_{e^a} \gamma^b \nabla_{e^b} = -\gamma^a \gamma^b e_a^\mu (\nabla_\mu e_b^\nu) \nabla_\nu - \gamma^a \gamma^b e_a^\mu e_b^\nu \nabla_\mu \nabla_\nu = -\gamma^a \gamma^b \nabla_{e^a} \nabla_{e^b}, \tag{611}$$

since $\nabla e^a = 0$. So then

$$\begin{aligned} (i\not{D}_A)^2 &= -\sum_a \gamma^a \gamma^a (\nabla_{e^a})^2 - \frac{1}{2} \sum_{a \neq b} \gamma^a \gamma^b [\nabla_{e^a}, \nabla_{e^b}] \\ &= \nabla^\dagger \nabla - \frac{1}{2} \sum_{ab} \gamma^a \gamma^b R_{ab}, \end{aligned} \tag{612}$$

where R_{ab} is the curvature tensor (for each a, b, R_{ab} is an $n \times n$ antisymmetric matrix) for the full connection and $\nabla^\dagger = -\nabla$ is the adjoint of the covariant derivative.[70] We can split the curvature tensor into a gauge part and a spin connection part (since the full connection is $A = \omega \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{A}$, where $\mathcal{A}$ is the gauge connection, there are no cross terms) as $R_{ab} = R^S_{ab} + R^E_{ab}$, where R^S is the spin part and R^E is the gauge part.

Let's start with the spin part R^S_{ab}: for each a, b, we need to represent this as an $n \times n$ matrix in the spinor representation. Now R^S_{ab} is an antisymmetric matrix[71], and so we can write it as $R^S_{ab} = \sum_{c<d} R_{abcd} A^{cd} = \frac{1}{2} \sum_{cd} R_{abcd} A^{cd}$, where $[A^{cd}]_{ij} = \delta_{cj}\delta_{di} - \delta_{ci}\delta_{dj}$ and R_{abcd} is the Riemann curvature tensor for the spin part of the connection. To find the representation of R^S_{ab} on fermion fields, we need to represent the $\mathfrak{so}(n)$ matrix A^{cd} as a matrix in $\mathfrak{spin}(n)$. We can do this using the map $P_*^{-1} : \mathfrak{so}(n) \to \mathfrak{spin}(n)$ derived in the last diary entry: this gives

$$\begin{aligned} -\frac{1}{2} \sum_{ab} \gamma^a \gamma^b R^S_{ab} &= -\frac{1}{4} \sum_{abcd} R_{abcd} P_*^{-1}(A^{cd}) = -\frac{1}{8} \sum_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d R_{abcd} \\ &= -\frac{1}{8} \sum_a \gamma^a \left(\sum_{b \neq c \neq d} \gamma^b \gamma^c \gamma^d R_{abcd} + \sum_{bd} \left[\gamma^b \gamma^b \gamma^d R_{abbd} + \gamma^b \gamma^d \gamma^b R_{abdb} \right] \right), \end{aligned} \tag{614}$$

[70] $\nabla^\dagger$ is defined so that $\nabla^\dagger \nabla = -\nabla_\mu \nabla^\mu$ is a positive-definite operator. Indeed, for any compactly-supported ψ,

$$0 \leq \int \langle \nabla \psi, \nabla \psi \rangle = -\int \langle (\nabla)^\mu \nabla_\mu \psi, \psi \rangle = \int \langle \nabla^\dagger \nabla \psi, \psi \rangle. \tag{613}$$

[71] Remember that the matrix R_{ab} tells us how transform a vector into the vector that it becomes after being transported around a rectangle defined by the vector fields $e^a = e^{a\mu}\partial_\mu$ and $e^b = e^{b\mu}\partial_\mu$. A vector and its parallel-transported-around-a-rectangle image differ by a rotation, which is encoded by the matrix R_{ab}. Thus R_{ab} lives in $\mathfrak{so}(n)$, and can be built from a linear combination of the matrices A^{ab} that featured in the last diary entry.

where we used the antisymmetry of R_{abcd} in both the first and last pair of indices (so that we can sum over $b \neq c \neq d$, and then get the remaining terms by taking $c = b$ and $d = b$). Now the first term dies by the antisymmetry of the gamma matrices in front of it and the Bianchi identity, which says that $R_{a[bcd]} = 0$. Thus

$$-\frac{1}{2}\sum_{ab}\gamma^a\gamma^b R^S_{ab} = -\frac{1}{8}\sum_{abd}\gamma^a\gamma^d(R_{abbd} - R_{abdb}). \tag{615}$$

Now $R_{abdb} = R_{dbab}$ and $R_{abbd} = R_{dbba}$, so the only terms that survive are those where $a = d$. So, using $R_{abab} = -R_{abba}$,

$$-\frac{1}{2}\sum_{ab}\gamma^a\gamma^b R^S_{ab} = -\frac{1}{4}\sum_{ab} R_{abba} = \frac{1}{4}R. \tag{616}$$

In addition to the spin part, we also have the gauge part. Since the gauge part doesn't act on the spinor indices, there is really no more work to do: we just write it as

$$\sum_{ab}\gamma^a\gamma^b R^E_{ab} = -i\sum_{ab}\gamma^a\gamma^b e^\mu_a e^\nu_b \mathcal{F}_{\mu\nu}, \tag{617}$$

with $\mathcal{F} = d\mathcal{A}$ the curvature for the gauge part.[72] Putting everything together, we have

$$(i\not{D}_A)^2 = \nabla^\dagger\nabla + \frac{1}{4}R + \frac{i}{4}\sum_{ab}[\gamma^a, \gamma^b]e^\mu_a e^\nu_b \mathcal{F}_{\mu\nu}. \tag{618}$$

Now as we established above, $\nabla^\dagger\nabla$ is a positive-definite operator. This means that if the total (gauge + geometric) curvature is everywhere nonzero, there are no solutions to $\not{D}_A\psi = 0$, and hence no zero modes.

As an example, consider fermions on S^2 with $U(1)$ flux such that $\int_{S^2} F = 2\pi m$, as in the last diary entry. Letting $\gamma^1 = X, \gamma^2 = Y$ and $e^{x\theta} = 1, e^{y\phi} = \sin^{-1}\theta$, with $\mathcal{A} = \frac{1-\cos\theta}{2}d\phi$, we find

$$(i\not{D}_A)^2 = -\Delta_A + \frac{1}{2} - \frac{m}{2}Z, \tag{619}$$

where Δ_A is the gauged Laplacian on the sphere: $\Delta_A = (\partial_\mu + i(\omega_\mu + \mathcal{A}_\mu))(\partial^\mu + i(\omega^\mu + \mathcal{A}^\mu))$. This formula shows us that left-handed fermions see positive magnetic flux as negative curvature, while the opposite is true for right-handed fermions (our spinors are $(\psi_L, \psi_R)^T$). Since $-\Delta_A$ is positive definite, zero modes are only possible for non-zero magnetic flux, with the chirality of the zero modes depending on the sign of the flux.

Another fun application of these results is to apply them to the $d, d^\dagger$ differential complex, by taking the dirac operator to be the Laplacian $\Delta = d + d^\dagger$. This tells us that if $R \geq 0$ everywhere, Δ has no zero modes: there are no harmonic forms on a manifold which is everywhere positively curved.

[72] The factor of $-i$ is there because we take the covariant derivative to involve $i\mathcal{A}_\mu$, and there is a further sign coming from the i^2 in $(i\not{D}_A)^2$ that we need to include to get the signs straight.

36 March 27 — Riemann curvature tensor and parallel transport

Today's diary entry is pretty simple—just doing a simple calculation I hadn't done before. It's a problem in Carroll's GR book.

One way to define the curvature tensor R is via

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{620}$$

where $\nabla_{[X,Y]} Z = \nabla_{\nabla_X Y} Z - \nabla_{\nabla_Y X} Z$ vanishes if X and Y are coordinate vector fields.

Another way to define the curvature tensor $R(X,Y)$ is as the matrix which relates a vector Z to its image under parallel transport through a rectangle defined by the flow lines of the vector fields X and Y: if the difference between Z and its parallel-transported image is δZ, then

$$\delta Z^\rho = R^\rho_{\mu\nu\lambda} X^\mu Y^\nu Z^\lambda. \tag{621}$$

Why are these two definitions the same? (This isn't totally obvious to me; one might have thought that parallel transport along a rectangle would involve terms quartic in covariant derivatives)

Solution:

The proof proceeds in essentially the same way as the proof of the analogous statement for conventional gauge theories. Parallel transporting a vector Z along a curve $x(t)$ means that $Z(t)$ is covariantly constant:

$$[\nabla_{x(t)} Z]^\rho = \dot{x}^\mu (\partial_\mu Z^\rho + \Gamma^\rho_{\mu\lambda} Z^\lambda) = 0, \tag{622}$$

so that along the curve, $\partial_\mu Z^\rho = -\Gamma^\rho_{\mu\lambda} Z^\lambda$. Integrating this along the curve and then iterating, the usual procedure of integrating over simplices leads to

$$\begin{aligned} Z^\rho(t) &= Z^\rho(t_0) - \int_{t_0}^{t} dt'\, \dot{x}^\lambda \Gamma^\rho_{\mu\lambda} Z^\lambda(t') \\ &= \left[P \exp\left(- \int_{t_0}^{t} dt'\, \dot{x}^\lambda \Gamma_\lambda \right) \right]^\rho_\mu Z^\mu(t_0), \end{aligned} \tag{623}$$

where we are viewing Γ_λ as a matrix with one upper and one lower index.

To prove the equivalence between the two definitions of the curvature tensor, we need to consider the path t to be a small rectangular closed path such that $x(t) = x(t_0)$. We will let the sides of the rectangle have lengths δa and δb, and will wolog take the sides to be along the x^1 and x^2 axes, respectively. This makes doing the integrals easy, since we can parametrize the paths by x^1 and x^2, letting $t \to x^1, x^2$, depending on which side of the rectangle we are on.

With this setup, we then just need to expand the above equation to second order in $\delta a, \delta b$.[73] Let the four corners of the path be $1, 2, 3, 4$. For each path segment, we need to

[73] This will suffice for the proof if the claim is true, since the first formula for the curvature tensor given above is quadratic in derivatives.

expand the path-ordered exponential to second order. The second order term is, to our order, the matrix $\delta a^2 \Gamma^\rho_{1\mu}\Gamma^\mu_{1\lambda}/2$ or $\delta b^2 \Gamma^\rho_{2\mu}\Gamma^\mu_{2\lambda}/2$, depending on which part of the path we're on. The first-order terms for the 23 and 34 parts of the path can be expanded in terms of the first-order terms for the 12 and 41 parts of the path by taking a first-order derivative expansion of the Γ matrices. This gives

$$Z^\rho(t) = \left(\delta^\rho_\alpha - \int_1^2 \Gamma^\rho_{1\alpha} + \frac{\delta a^2}{2}\Gamma^\rho_{1\lambda}\Gamma^\lambda_{1\alpha}\right)\left(\delta^\alpha_\sigma - \int_2^3 \Gamma^\alpha_{2\sigma} + \frac{\delta b^2}{2}\Gamma^\alpha_{2\lambda}\Gamma^\lambda_{2\sigma}\right)$$
$$\times\left(\delta^\sigma_\beta + \int_1^2 (\Gamma^\sigma_{1\beta} - \delta b\partial_2\Gamma^\sigma_{1\beta}) + \frac{\delta a^2}{2}\Gamma^\sigma_{1\lambda}\Gamma^\lambda_{1\beta}\right)\left(\delta^\beta_\omega + \int_2^3 (\Gamma^\beta_{2\omega} + \delta a\partial_1\Gamma^\beta_{2\omega}) + \frac{\delta b^2}{2}\Gamma^\beta_{2\lambda}\Gamma^\beta_{2\omega}\right) Z^\omega(t_0). \tag{624}$$

The terms that involve only one integral of a single Γ matrix cancel pairwise, while the terms that have the explicit factors of $\delta a^2/2$ and $\delta b^2/2$ are canceled by terms that are quadtratic products of the integrals involving the single Γ matrices (for example, the terms with the explicit $\delta a^2/2$ dependence are canceled by the the product of the two $\int_1^2 \Gamma^\rho_{1\omega}$ terms).

The only terms that survive are then the ones involving derivatives, and ones that contain products of $\int_1^2$ integrals with $\int_2^3$ integrals. Some of these cancel, but two terms remain and we can then do the integrals to lowest order and get

$$Z^\rho(t) - Z^\rho(t_0) = \delta a\delta b\left(\partial_1\Gamma^\rho_{2\omega} - \partial_2\Gamma^\rho_{1\omega} + \Gamma^\rho_{1\lambda}\Gamma^\lambda_{2\omega} - \Gamma^\rho_{2\lambda}\Gamma^\lambda_{1\omega}\right) Z^\omega(t_0). \tag{625}$$

This exactly the transformation we expect from (621), with the vector fields X and Y chosen to be x^1 and x^2, respectively. Indeed, a quick check of the commutator of ∇_X with ∇_Y (no $\nabla_{[X,Y]}$ needed since in this case X, Y are coordinate vector fields) shows that we get the above expression of Christoffel symbols.

37 *March 29 — Haldane's model of a Chern insulator*

Today's diary entry is a homework problem from Senthil's cmt class. The problem has several parts: a) consider a hopping model on the honeycomb lattice. Find the dispersion and show where the two Dirac points are located. b) Now add second-nearest-neighbor hopping in a T-breaking manner, with opposite hopping phases for each sublattice. Also add a chemical potential that alternates signs between the two sublattices. What happens? c) Find the Dirac mass near the K, K' points as a function of the various parameters in the Hamiltonian. d) What are the Chern numbers of the different bands in the different regions of parameter space?

Solution:

a) The hopping Hamiltonian is (setting the hopping strength to 1 wolog)

$$H_0 = -\sum_{\langle rr'\rangle}(c^\dagger_r c_{r'} + h.c.). \tag{626}$$

To analyze this we need to fix conventions for the honeycomb lattice. The unit cell consists of two neighboring cites; the lattice of such unit cells is a triangular lattice. We will set the lattice spacing of the honeycomb lattice to be 1, meaning that nearest neighbors on a single sublattice are a distance $a = 2\cos(\pi/6) = \sqrt{3}$ apart. The basis vectors for a given sublattice are then

$$v_1 = (\sqrt{3}, 0), \qquad v_2 = (\sqrt{3}/2, 3/2). \tag{627}$$

The reciprocal lattice vectors are then, from $v_i \cdot u_j = 2\pi\delta_{ij}$,

$$u_1 = 2\pi(1/\sqrt{3}, -1/3), \qquad u_2 = 2\pi(0, 2/3). \tag{628}$$

The BZ is determined by taking these vectors, plus the vector orthogonal to $a_2 - a_1$, dividing their lengths by 2, and then taking the interior of the shape formed by the lines drawn perpendicular to the endpoints of each of these vectors, as well as their negatives. This gives us a BZ which is hexagonal, with a flat side parallel to the k_x axis in momentum space; this flat side hits the k_y axis at $(0, 2\pi/3)$. The BZ hits the $k_x > 0$ part of the k_x axis at[74]

$$K = (4\pi/(3\sqrt{3}), 0), \tag{629}$$

which is a corner of the BZ hexagon. There is only one other corner of the hexagon distinct from the K point, located at[75]

$$K' = \frac{2\pi}{3}(1/\sqrt{3}, 1). \tag{630}$$

Note that because of the identification of the BZ corners, $K \cong -K'$.

Anyway, let $c_r = (c_{rA}, c_{rB})^T$, where r labels a lattice site on the A sublattice. Then by looking at the geometry we see that the hopping Hamiltonian is

$$H_0 = -\sum_k c_k^\dagger \mathcal{H}_0(k) c_k \tag{631}$$

where

$$\begin{aligned} H_0(k) &= \begin{pmatrix} 0 & e^{-ik_y} + e^{-i(\cos(\pi/6)k_x - \sin(\pi/6)k_y)} + e^{i(\cos(\pi/6)k_x + \sin(\pi/6)k_y)} \\ h.c. & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{-ik_y} + 2e^{ik_y/2}\cos(\sqrt{3}k_x/2) \\ e^{ik_y} + 2e^{-ik_y/2}\cos(\sqrt{3}k_x/2) & 0 \end{pmatrix} \\ &= h_x(k)X + h_y(k)Y, \end{aligned} \tag{632}$$

where

$$h_x(k) = \cos k_y + 2\cos(k_y/2)\cos(\sqrt{3}k_x/2), \qquad h_y(k) = \sin k_y - 2\sin(k_y/2)\cos(\sqrt{3}k_x/2). \tag{633}$$

Note how this is symmetric under $k_x \to -k_x$, which is a consequence of C_2 symmetry.

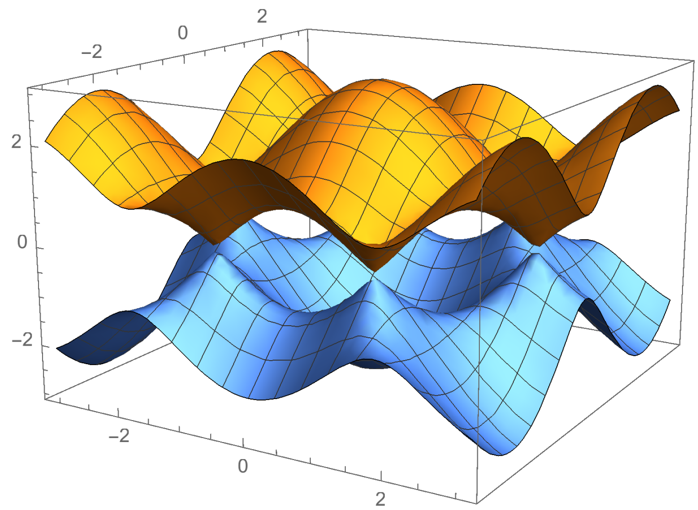
This gives us the spectrum

$$E_0(k) = \pm\sqrt{1 + 4\cos(\sqrt{3}k_x/2)(\cos(\sqrt{3}k_x/2) + \cos(3k_y/2))}. \tag{634}$$

[74] We find this by taking the line perpendicular to the midpoint of the u_1 vector, namely $\{(\pi/\sqrt{3} + t/\sqrt{3}, -\pi/3 + t)\}$, and setting its y-coordinate to 0.

[75] To find this, take the intersection of the lines $(t, -2\pi/3)$ and $(s + 4\pi/(3\sqrt{3}), \sqrt{3}s)$.

One can check that $E_0(K) = E_0(K') = 0$, so that the gap closes on the corners of the BZ. Graphically, the dispersion is



(635)

We can linearize the spectrum about the K, K' points: to first order this gives

$$
\begin{aligned}
H(K+q) &\approx \frac{3}{2}(q_y Y - q_x X), \\
H(K'+q) &\approx -\frac{3}{2}(-q_y Y - q_x X),
\end{aligned}
\tag{636}
$$

which are free Dirac Hamiltonians with the same speed of light (here we have chosen a representative momentum for the K' point so that $K' = -K$). The minus sign between the two of them follows from the fact that T (which is a symmetry of H_0) reverses momenta and the sign of Y, and exchanges K with K'.

Let's also take a look at the symmetries of the problem. We have "parity" (reflection of both axes; a π rotation), which sends $c_{A/B,k} \to c_{B/A,-k}$ and exchanges the A and B sublattices. This is a symmetry since

$$X^T H(k) X = H(-k). \tag{637}$$

We also have T symmetry, which just acts as complex conjugation, $T = \mathcal{K}$,[76] in addition to reversing momentum. This is a symmetry since $H^*(-k) = H(k)$. We also have charge conjugation: if $c \mapsto \mathcal{C}c^\dagger$, then we need

$$[\mathcal{C}^\dagger H(k)\mathcal{C}]^T = -H(k). \tag{638}$$

Thus we can choose $\mathcal{C} = Y$, so that $\mathcal{C}^2 = \mathbf{1}$, with $\mathcal{C}$ acting to exchange the A and B sublattices. Finally we have particle hole symmetry, which we write as $\mathcal{P} = ZT = Z\mathcal{K}$. This tells us that every state $|\psi\rangle$ of energy E comes with a partner of energy $-E$, since $H\mathcal{P}|\psi\rangle = \mathcal{P}ZH^*(-k)Z|\psi\rangle = -H(k)|\psi\rangle = -E|\psi\rangle$.

[76]No reason to have $T^2 = (-1)^F$ here; there is no spin in the problem.

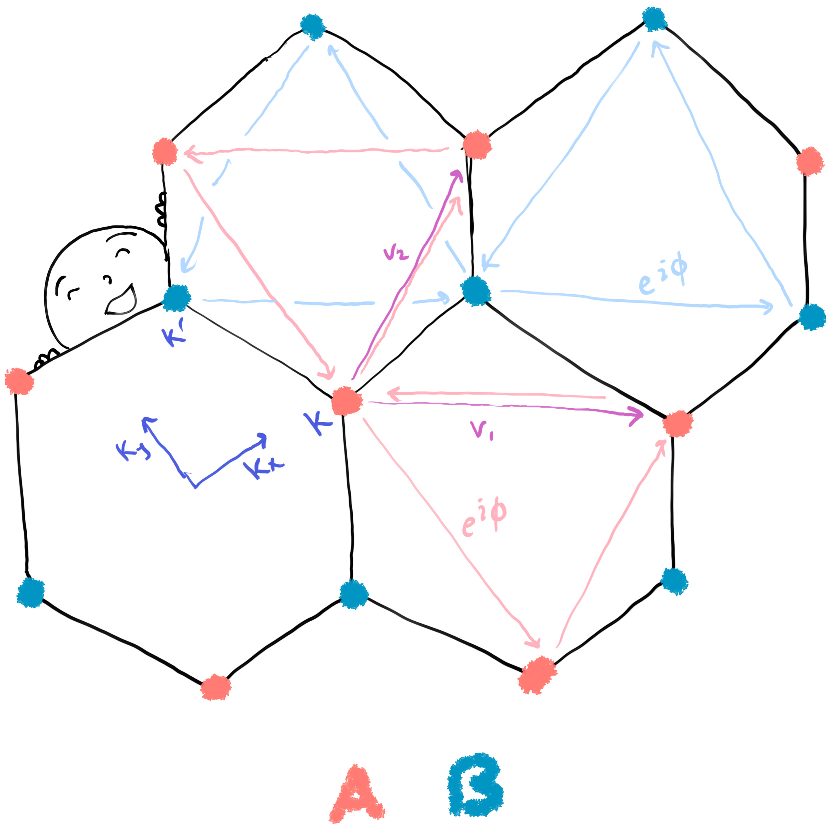


Figure 3: Our conventions for the graphene lattice. The light arrows show the conventions for nearest neighbor hopping, and the BZ (also a hexagon) is shown in the bottom left.

b) Now we add

$$H_1 = \Delta \sum_k c_k^\dagger Z c_k, \qquad H_2 = t_2 \sum_{\langle\langle rr'' \rangle\rangle} e^{i\phi} c_r^\dagger c_{r''} + h.c, \tag{639}$$

where $e^{i\phi}$ is a hopping phase. Note that H_1 breaks parity P since it treats the A and B sublattices differently. It also breaks $\mathcal{P}$, since $[Z\mathcal{K}, H_1(k)] = 0$. However, since it's real, it preserves time reversal.

Now for H_2. This term preserves P, breaks $\mathcal{C}$ and $\mathcal{P}$ unless $\phi = \pm\pi/2$ (because of the chemical potential term proportional to $\mathbf{1}\cos\phi$), and breaks T unless $\phi = 0, \pi$. The signs for the hopping phase are shown in figure 3: for each hop along an arrow, we get a phase $e^{i\phi}$; for each hop against an arrow we get $e^{-i\phi}$ (actually, oops—I think this is backward to what we did in class. Oh well). Note how if the A sublattice has a hopping along a vector v of phase $e^{i\phi}$, the B sublattice has a hopping along the same direction, but with phase $e^{-i\phi}$. Since the phase of the hopping around a closed loop is $e^{i\oint A}$, the total magnetic flux vanishes, since over a unit cell we have $e^{i\oint_{\Delta_A} A + i\oint_{\Delta_B} A} = 1$, where $\Delta_{A/B}$ are the triangles enclosed by similar hopping trajectories on the two sublattices. Anyway, writing out H_2,

$$H_2 = 2t_2 \sum_k \Big(c_{kA}^\dagger [\cos(\phi + k\cdot v_1) + \cos(\phi - k\cdot v_2) + \cos(\phi + k\cdot(v_2 - v_1))] c_{kA} + c_{kB}^\dagger [\phi \leftrightarrow -\phi] c_{kB} \Big). \tag{640}$$

We can re-write this as

$$\begin{aligned} H_2 = 2t_2 \sum_k c_k^\dagger \Big(&\mathbf{1}\cos(\phi)[\cos(k\cdot v_1) + \cos(k\cdot v_2) + \cos(k\cdot(v_2 - v_1))] \\ &+ Z\sin(\phi)[\sin(k\cdot v_1) - \sin(k\cdot v_2) + \sin(k\cdot(v_2 - v_1))] \Big) c_k. \end{aligned} \tag{641}$$

When $|\Delta| \gg |t_2|$ (only the relative sign of Δ and t_2 will be important, so in what follows we will fix $t_2 > 0$ wolog), we see that we get a gap at both K, K' points that goes as Δ. Since the $t_2 \to 0$ Hamiltonian preserves T we can't have edge modes, and since the presence / absence of edge modes is something that's quantized, we know that the large Δ phase is gapped in a trivial way.

c) Now we want to know what happens near the K, K' points. Linearizing H_2 about these points gives (to first order, H_2 is independent of q in $k = K + q$ and $k = K' + q$)

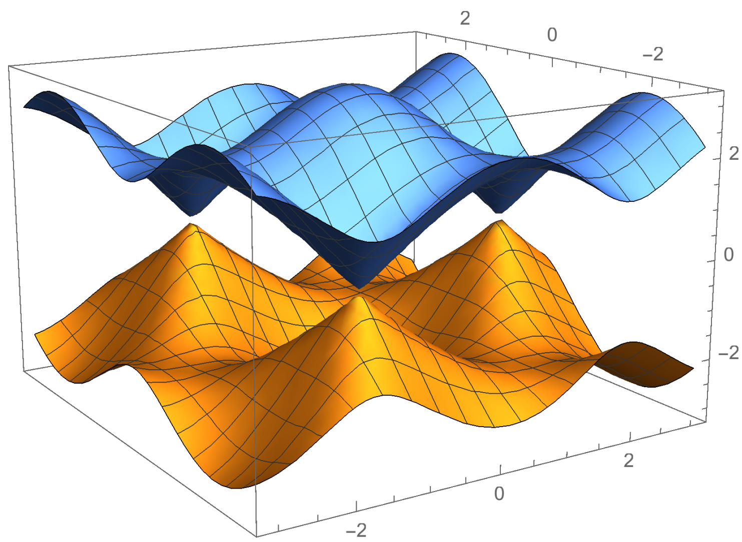
$$\begin{aligned} H_2(K) &\approx -3t_2\cos\phi\mathbf{1} - 3\sqrt{3}t_2\sin\phi Z, \\ H_2(K') &\approx -3t_2\cos\phi\mathbf{1} + 3\sqrt{3}t_2\sin\phi Z. \end{aligned} \tag{642}$$

Notice that the t_2 terms are not symmetric between the K and K' points if $\sin\phi \neq 0$; this is because unless $\phi = 0, \pi$ they break T, which relates K and K'.

Recapping, the full Hamiltonian around the two points of interest is

$$\begin{aligned} H(K+q) &\approx -3t_2\cos\phi\mathbf{1} + (\Delta - 3\sqrt{3}t_2\sin\phi)Z + \frac{3}{2}(q_y Y - q_x X) \\ H(K'+q) &\approx -3t_2\cos\phi\mathbf{1} + (\Delta + 3\sqrt{3}t_2\sin\phi)Z - \frac{3}{2}(q_y Y - q_x X) \end{aligned} \tag{643}$$

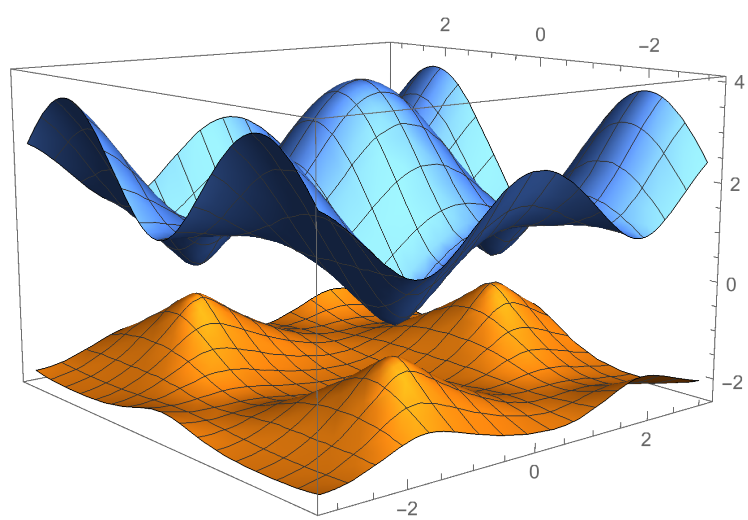
The Dirac mass is thus determined at each (former) Dirac cone by $m = \Delta \mp 3\sqrt{3}t_2 \sin\phi$. We see that if $\Delta \neq 0$, we never have a situation in which both Dirac fermions are massless. Indeed, the K point fermion is massless when $\Delta = 3\sqrt{3}t_2 \sin\phi$, while the K' point fermion is massless when $\Delta = -3\sqrt{3}t_2 \sin\phi$. A plot of the dispersion for the former situation at $\phi = \pi/2$ (where the K point is gapless and the K' point is maximally gapped) is as follows:



(644)

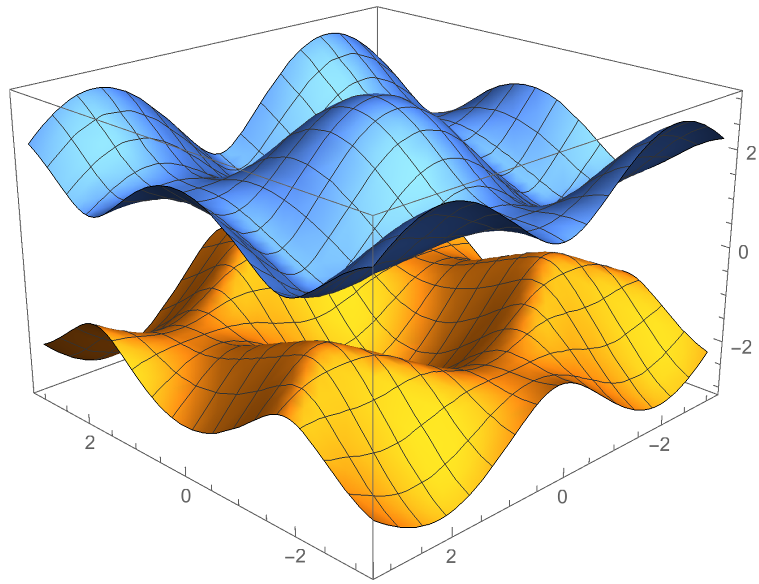
Since we know that $\Delta \to \pm\infty$ gives a trivially gapped state, which can only go to something with a nonzero Chern number if the gap closes, the only region in parameter space where the Chern number can be nonzero is the region in the $(\Delta/t_2, \phi)$ plane enclosed by the curves $\Delta/t_2 = \pm 3\sqrt{3}\sin\phi$. This is consistent with $C = 0$ if $\phi = 0, \pi$, since at these values of ϕ the system is T symmetric, and must have $C = 0$ for each band.

Anyway just to throw some more pictures in, here's what we get if we take $t_2 = 3^{-3/2}, \Delta = 1, \phi = \pi/4$, which is in the trivially gapped state:



(645)

If we keep $t_2 = 3^{-3/2}$ but now take $\phi = \pi/2, \Delta = 1/2$, which as we will see is in the topological phase, we have



$$(646)$$

d) We can map out the Chern numbers of the two bands for the various regions in $(\Delta/t_2, \phi)$ parameter space without actually calculating the Berry curvature. Consider first $0 < \phi < \pi$. As we bring $\delta \equiv \Delta/t_2$ down from $+\infty$, where the valence band must have $C = 0$, we cross a point at $\delta = 3^{3/2}\sin\theta$ where the band closes at the K point, with the Dirac mass changing from positive to negative. The Chern number must change by ± 1 at this point. The choice of ± 1 is determined by the chirality of the edge modes at domain walls where the mass changes sign; I think with the current choice for the hopping phases we obtain a change of $\Delta C = +1$ (ΔC can be reversed by sending $\phi \to -\phi$ in the Hamiltonian). The reason for the change is the following: we can compute the Berry curvature with the familiar formula $\mathcal{F}_{xy} \sim i\epsilon_{ijk}h^i\partial_{[k_x}h^j\partial_{k_y]}h^k$, where for the Dirac Hamiltonain $h = (k_x, k_y, m)$. Now for m very slightly greater than 0, the vector h essentially winds in a plane, and so integrating the Berry connection around a loop enclosing the Dirac cone in momentum space tells us that the Berry curvature is a slightly smeared delta function of strength $1/2$ concentrated at the Dirac cone ($1/2$ since the trajectory of h around the cone encloses a solid angle of $\approx 2\pi$). Changing the sign of m flips the sign of one of the components of h, under which the Berry curvature is odd (as we can see from the expression we wrote for it above). Thus a Berry curvature strength $1/2$ zone around the Dirac point is mapped to a strength $1/2$ zone when the mass changes sign, and so the net Chern number changes by 1.

So, we know that the region $-3^{3/2}\sin\phi < \delta < 3^{3/2}\sin\phi$ has a Chern number of $C = +1$. When δ is lowered past $-3^{3/2}\sin\phi$, the Dirac mass at the K' point also changes from positive to negative, but this process is accompanied by a shift of $\Delta C = -1$ change in the Chern number, instead of $+1$: this is because the K' point is the time-reversed image of the K point, and the Berry curvature is odd under T (recall that it has a factor of i out front!), so that the change in Chern number under taking the Dirac mass from positive to negative is opposite in sign between the K and K' points. Thus we revert to a phase with $C = 0$ when $\delta < -3^{3/2}\sin\phi$, as we knew we had to.

When $\phi = 0, \pi$ the system has T symmetry and the net Chern number of both bands is always zero. We can also realize this by noting that when $\phi = 0, \pi$ the gaps at the K, K' points close simultaneously: as we just saw the closing of the gaps leads to opposite changes in C, and so for these values of ϕ there is value of δ for which $C \neq 0$.

Finally we can look at $\pi < \phi < 2\pi$. In this case, when we decrease δ from $+\infty$, it is the K' point that becomes gapless first, at $\delta = 3^{3/2}|\sin\phi|$. From our above discussion, we know this produces a region with $C = -1$. After we decrease δ to $-3^{3/2}|\sin\phi|$ the gap closes, and brings us back into the trivial $C = 0$ phase.

Note that this whole story plays out in reverse for the other band: the assignment of $C = \pm 1$ regions of parameter space is simply swapped. This is because we started from a local $\mathbb{R}$-space hopping Hamiltonian, which can't give us anything with nonzero net (summed over all bands) Chern number. Alternatively, we can argue this by realizing that the vector h used in the computation of the Berry curvature for the Dirac Hamiltonian is odd under particle hole, and hence so too is the Berry curvature.

38 *March 30 — Making spinful particles from bosons and magnetic flux*

Today we have another problem from Senthil's class.

Solution:

a) Define

$$\Lambda_\mu = [\mathbf{n} \times \pi]_\mu, \qquad \pi = -i\nabla + \mathbf{A}. \tag{647}$$

The commutator $[\Lambda_\mu, \Lambda_\nu]$ has two contributions: one is the usual term coming from the standard angular momentum commutation relations, and the other is an additional piece picked up because the momenta π_μ do not commute with one another, due to the magnetic field. Writing it out, we have

$$[\Lambda_\mu, \Lambda_\nu] = \epsilon^{\mu\alpha\beta}\epsilon^{\nu\gamma\delta} n^\alpha n^\gamma (-i\partial_{[\beta} A_{\delta]}) + \dots, \tag{648}$$

where $\dots$ are terms that don't involve derivatives of A, and therefore can be computed as usual, by pretending that the π_μ commute.

We are considering the potential for a monopole placed at the center of the sphere. Thus the field strength is

$$F = B(\star n_\mu dx^\mu) \implies \int F = 4\pi B. \tag{649}$$

In component notation, and writing S instead of B to be more suggestive later on,

$$\partial_{[\mu} A_{\nu]} = S\epsilon_{\mu\nu\lambda} n^\lambda. \tag{650}$$

This means that we can simplify the first term in the commutator to

$$\begin{aligned}\epsilon^{\mu\alpha\beta}\epsilon^{\nu\gamma\delta}n^\alpha n^\gamma(-i\partial_{[\beta}A_{\delta]}) &= -i\epsilon^{\mu\alpha\beta}\epsilon^{\nu\gamma\delta}n^\alpha n^\gamma n^\lambda\epsilon_{\beta\delta\lambda}S \\ &= -iS\epsilon^{\nu\gamma\delta}(\delta_{\mu\delta}\delta_{\alpha\lambda}-\delta_{\mu\lambda}\delta_{\alpha\delta})n^\alpha n^\gamma n^\lambda \\ &= -iS\epsilon^{\mu\nu\gamma}n^\gamma.\end{aligned} \tag{651}$$

Now we need the regular angular momentum commutator. Pretending that the different components of π now commute with themselves, we can write the ... term as

$$\begin{aligned}[(\mathbf{n}\times\pi)^\mu,(\mathbf{n}\times\pi)^\nu] &= \epsilon^{\mu\beta\gamma}\epsilon^{\nu\rho\sigma}(n_\beta[\pi_\gamma,n_\rho]\pi_\sigma - n_\rho[\pi_\sigma,n_\beta]\pi_\gamma) \\ &= -i\epsilon^{\mu\beta\rho}\epsilon^{\nu\rho\sigma}n_\beta\pi_\sigma + i\epsilon^{\mu\sigma\gamma}\epsilon^{\nu\rho\sigma}n_\rho\pi_\gamma \\ &= -i(\delta_{\mu\sigma}\delta_{\beta\nu}-\delta_{\mu\beta}\delta_{\sigma\nu})n_\beta\pi_\sigma \\ &= i\epsilon^{\mu\nu\lambda}\epsilon^{\lambda\beta\sigma}n_\beta\pi_\sigma,\end{aligned} \tag{652}$$

where we used $\epsilon^{abc}\epsilon^{dec}=\delta^{ad}\delta^{be}-\delta^{ae}\delta^{bd}$ a bunch.

Putting this together with the part involving the field strength, we conclude that

$$[\Lambda_\mu,\Lambda_\nu] = i\epsilon^{\mu\nu\lambda}(\Lambda_\lambda - Sn_\lambda). \tag{653}$$

Thus the commutation relations for the naive angular momentum operators ($\mathbf{n}$ crossed with the gauge covariant derivative) do not satisfy an $SU(2)$ algebra.

b) Now define the operators

$$L_\mu = \Lambda_\mu + Sn_\mu. \tag{654}$$

Then

$$[L_\mu,L_\nu] = i\epsilon_{\mu\nu\lambda}L^\lambda - 2i\epsilon_{\mu\nu\lambda}Sn^\lambda + S([\Lambda_\mu,n_\nu]-[\Lambda_\nu,n_\mu]). \tag{655}$$

The commutators on the RHS are easy to evaluate:

$$\begin{aligned}[\Lambda_\mu,n_\nu] &= \epsilon_{\mu\alpha\beta}n^\alpha[\pi^\beta,n_\nu] = -i\epsilon_{\mu\alpha\beta}n^\alpha\delta^\beta_\nu \\ &= i\epsilon_{\mu\nu\alpha}n^\alpha.\end{aligned} \tag{656}$$

The other commutator evaluates to the negative of this by antisymmetry, and so these extra commutators kill the $-2i\epsilon_{\mu\nu\lambda}Sn^\lambda$ term. This means that the L_μ operators satisfy the $SU(2)$ algebra

$$[L_\mu,L_\nu] = i\epsilon_{\mu\nu\lambda}L^\lambda. \tag{657}$$

c) Since the L_μ satisfy the $SU(2)$ commutation relations, we know that the eigenvalues of L^2 are of the form $l(l+1)$ for $l\in\frac{1}{2}\mathbb{Z}^{>0}$. Now

$$L^2 = \Lambda^2 + S^2 + S(\Lambda_\mu n^\mu + n_\mu\Lambda^\mu). \tag{658}$$

The last term dies since $\mathbf{n}\times\pi$ is $\perp$ to $\mathbf{n}$, while the second to last simplifies to

$$\Lambda_\mu n^\mu = -i\epsilon_{\mu\nu\lambda}n_\mu\partial_\nu n_\lambda = -i\mathbf{n}\cdot(\partial\times\mathbf{n}) = 0, \tag{659}$$

since $\mathbf{n}$ is purely radial. This means that $L^2 = \Lambda^2 + S^2$. Since Λ^2 has only positive eigenvalues, this means that the only allowed eigenvalues of L are those such that

$$l(l+1) \geq S^2. \tag{660}$$

Thus S sets the minimum possible value for the angular momentum.

d) Now we can rewrite the Hamiltonian as

$$H = \frac{1}{2m}(L^2 - S^2). \tag{661}$$

The minimum value of l is S, and so the ground state energy for a given field strength S is $E_0 = \frac{1}{2m}(S(S+1) - S^2) = \frac{S}{2m}$. Evidently the ground state has angular momentum S, with degeneracy $2S+1$ coming from the $2S+1$ possible choices for the eigenvalue $m = -l, \dots, l$ of L_z. The next excited state has $l = S+1$, which gives $E_1 = \frac{3S+2}{2m}$, which has an energy gap of $(S+1)/m$ with respect to the ground state.

e) Now take $S = 1/2$, which is the minimal value allowed by Dirac quantization. There are two degenerate ground states, with L_z eigenvalues of $\pm 1/2$ and L^2 eigenvalues of $3/4$, and hence with Λ^2 eigenvalues of $1/2$. Now L_z is just[77]

$$L_z = -i\nabla_\phi + \frac{1}{2}\cos\theta = -i\left(\partial_\phi - i\frac{\cos\theta}{2}\right) + \frac{1}{2}\cos\theta = -i\partial_\phi, \tag{664}$$

which is the same as in the un-gauged case.

Let us write $\psi_\pm$ for the $\pm 1/2$ eigenfunctions of L_z. The easiest way to do this is to use the raising and lowering operators in $SU(2)$. They are

$$\begin{aligned} L_\pm &= e^{\pm i\phi}\left(\pm\nabla_\theta + i\cot\theta\,\nabla_\phi + (\sin\theta)/2\right) \\ &= e^{\pm i\phi}\left(\pm\partial_\theta + i\cot\theta\,\partial_\phi + \frac{1}{2\sin\theta}\right). \end{aligned} \tag{665}$$

Now ψ_+ should be annihilated by L_+, so

$$L_+\psi_+ = 0 \implies \left(\partial_\theta + \frac{1-\cos\theta}{2\sin\theta}\right)\psi_+ = 0 \implies \psi_+ = e^{i\phi/2}\cos(\theta/2). \tag{666}$$

[77]We have two options for what we might take the gauge field to be. The one we'll work with is

$$A = -\frac{\cos\theta}{2}d\phi, \tag{662}$$

which gives the correct field strength. This is singular at both poles, and can be fixed up by doing

$$A_{N/S} = \frac{\pm 1 - \cos\theta}{2}d\phi \tag{663}$$

on the two hemispheres. The $\pm d\phi/2$ part doesn't affect the field strength though, and its absence leads to more symmetric (though more singular) wavefunctions. For example, with the gauge we'll use, the $L_z = \pm 1/2$ eigenfunctions have a ϕ dependence of $e^{\pm i\phi/2}$, which is more symmetric but more singular than the $e^{i\phi}, 1$ (or $1, e^{-i\phi}$) dependence of the eigenfunctions with the $A_{N/S}$ potential.

Likewise, ψ_- is killed by L_-, so that

$$L_-\psi_- = 0 \implies \left(-\partial_\theta + \frac{1+\cos\theta}{2\sin\theta}\right)\psi_- = 0 \implies \psi_- = e^{-i\phi/2}\sin(\theta/2). \tag{667}$$

As a sanity check, we can look at how $\psi_\pm$ are acted on by Λ^2, the covariant Laplacian. We have

$$\begin{aligned} -\Lambda^2 &= \nabla_\theta^2 + \cot\theta\,\nabla_\theta + \csc^2\theta\,\nabla_\phi^2 \\ &= \partial_\theta^2 + \cot\theta\,\partial_\theta + \csc^2\theta(\partial\phi - i(\cos\theta)/2)^2 \\ &= \partial_\theta^2 + \cot\theta\,\partial_\theta + \csc^2\theta\,\partial_\phi^2 - i\csc\theta\,\cot\theta\,\partial_\phi + \frac{1-\csc^2\theta}{4}. \end{aligned} \tag{668}$$

So, acting on $\psi_\pm$,

$$\Lambda^2\psi_\pm = -(\partial_\theta^2 + \cot\theta\,\partial_\theta \pm (\csc\theta\,\cot\theta)/2 + 1/4 - (\csc^2\theta)/2)\psi_\pm. \tag{669}$$

Plugging in the form for $\psi_\pm$ above, one verifies that

$$\Lambda^2\psi_\pm = \frac{1}{2}\psi_\pm, \tag{670}$$

which from $\Lambda^2 = L^2 - S^2$ tells us that $L^2\psi_\pm = \frac{1}{2}(1+1/2)\psi_\pm$, as required. Summarizing, a general state in the ground state subspace is labeled by θ, ϕ and can be written as

$$\psi = (\psi_+, \psi_-)^T = (e^{i\phi/2}\cos\theta/2, e^{-i\phi/2}\sin\theta/2)^T. \tag{671}$$

This same strategy works for getting the eigenstates when $S > 1/2$. The ground states always have $l = S$, and are $(2l+1)$-fold degenerate. The highest-weight state is always $\psi_+ = e^{iS\phi}\cos^{2S}(\theta/2)$, and the lowest-weight state is always $e^{-iS\phi}\sin^{2S}(\theta/2)$. For example, for $S = 1$ we have $\psi_+ = e^{i\phi}\cos^2(\theta/2), \psi_0 = \sqrt{2}\sin(\theta/2)\cos(\theta/2), \psi_- = e^{-i\phi}\sin^2(\theta/2)$.

Anyway, going back to the case of $S = 1/2$, we want to find out what the operator $\mathbf{n}$ projects to in the ground state subspace. A quick way to argue is the following: since the ground state subspace is two-dimensional and $\mathbf{n}$ is Hermitian, we can write $\mathbf{n} = a\mathbf{1} + b_j\sigma^j$. Now by rotational symmetry all the b_js are equal, while again by rotational symmetry $\mathrm{Tr}[\mathbf{n}] = 0$, where the trace is in the ground state subspace. Thus $n^i \propto \sigma^i/2$.

A slightly less quick way to argue is to use the explicit expressions for the ground state wavefunctions. Let $S_j = \sigma_j/2$. We have

$$\langle\psi|S_z|\psi\rangle = \frac{1}{2}(\cos^2(\theta/2) - \sin^2(\theta/2)) = \frac{\cos\theta}{2}. \tag{672}$$

Similarly,

$$\langle\psi|S_x|\psi\rangle = \frac{1}{2}(e^{i\phi} + e^{-i\phi})\cos(\theta/2)\sin(\theta/2) = \frac{\cos\phi\,\sin\theta}{2} \tag{673}$$

and

$$\langle\psi|S_y|\psi\rangle = \frac{1}{2i}(e^{i\phi} - e^{-i\phi})\cos(\theta/2)\sin(\theta/2) = \frac{\sin\phi\,\sin\theta}{2}. \tag{674}$$

Thus $\langle\psi|\sigma^j|\psi\rangle = n^j$, and so on the ground state subspace, the components of $\mathbf{n}$ act as the Pauli matrices.

f) Consider an antiunitary map T which sends $\mathbf{n} \mapsto -\mathbf{n}$. In terms of the angles, T does

$$T : \theta \mapsto \pi - \theta,\ \phi \mapsto \phi - \pi. \tag{675}$$

Thus when acting on the ground state subspace,

$$T : \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix} \mapsto \begin{pmatrix} i\cos(\theta/2 - \pi/2)e^{-i\phi/2} \\ i\sin(\theta/2 - \pi/2)e^{i\phi/2} \end{pmatrix} = Y\mathcal{K} \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix}, \tag{676}$$

where $\mathcal{K}$ is complex conjugation. Therefore in the ground state subspace, T is represented by $T = Y\mathcal{K}$. Since $(Y\mathcal{K})^2 = -\mathbf{1}$, the ground state is a Kramers doublet.

If we were to go to $S = 1$ (or any $S \in \mathbb{Z}$), we would see instead that $T^2 = \mathbf{1}$ on the ground state subspace, while if $S \in \frac{1}{2}(2\mathbb{Z}+1)$, then we would still have $T^2 = -\mathbf{1}$. In general, the action of T sends $m \mapsto -m$ and so it inverts the ladder of states in the ground state subspace, but when $S \in \mathbb{Z}$ it does so in a way that squares to $\mathbf{1}$. Indeed, one only needs to check the action of T on the highest weight state, which in general is

$$\cos^{2S}(\theta/2)e^{iS\phi} \xrightarrow{T} e^{-i\pi S}\cos^{2S}(\theta/2 - \pi/2)e^{-i\phi S} = e^{-i\pi S - iS\phi}\sin^{2S}(\theta/2) \xrightarrow{T} e^{2\pi i S}\cos^{2S}(\theta/2)e^{iS\phi} \tag{677}$$

which tells us that $T^2 = e^{2\pi i S}$.

This also follows from representation theory. The spin 1/2 representation of $SU(2)$ is pseudoreal, and so it is isomorphic to its complex conjugate via an antilinear map which squares to $-\mathbf{1}$: this isomorphism is T. In contrast, the spin 1 representation of $SU(2)$ is real (since it's a representation of $SO(3)$), and so is isomorphic to its complex conjugate via an antilinear map T which squares to $+\mathbf{1}$. More generally, any half-odd-integer representation of $SU(2)$ is pseudoreal,[78] while any integer spin rep is real, since it descends to an $SO(3)$ rep. Thus in the spin S subspace, we indeed have $T^2 = (-1)^{2S}$.

g) We will take the Hamiltonian to be $H = \pi^2/2m + \lambda(q^2 - 1)$, with $\pi_\mu = (p_\mu + A_\mu)$ ($p_\mu \to -i\partial_\mu$ is the canonical momentum) and λ a Lagrange multiplier constraining the particle to lie on the sphere. To build the path integral we insert $\int dq(t)|q(t)\rangle\langle q(t)|$ and $\int dp(t)|p(t)\rangle\langle p(t)|$ into the Trotterization of e^{-iHt}:

$$Z = \int \mathcal{D}q\,\mathcal{D}p\,\mathcal{D}\lambda\, \exp\left(i\int dt(\,p_i\dot{q}^i - (p+A)^2/2m - \lambda(q^2-1))\right). \tag{678}$$

We can do the p integral exactly; after absorbing a constant into the integration measure we get

$$Z = \int \mathcal{D}q\,\mathcal{D}\lambda\, \exp\left(i\int dt(-A_i\dot{q}^i + m\dot{q}^2/2 - \lambda(q^2-1))\right). \tag{679}$$

Now doing the integral over λ,

$$Z = \int \mathcal{D}\theta\,\mathcal{D}\phi\, \exp\left(i\int dt(-A_\theta\dot{\theta} - A_\phi\dot{\phi} + m\dot{q}^2)\right). \tag{680}$$

[78] since it appears in an odd $\otimes$-power of the spin 1/2 irrep, and since the $\otimes$ of a ps$\mathbb{R}$ and ps$\mathbb{R}$ rep is $\mathbb{R}$, while the $\otimes$ of a ps$\mathbb{R}$ rep with a $\mathbb{R}$ rep is ps$\mathbb{R}$.

h) Now we can take the $m \to 0$ limit easily. We saw above that the energy gap between the ground states and the first excited states is S/m, and so taking $m \to 0$ is equivalent to restricting our attention to the ground state subspace (note that this restriction is easier to do after the momenta have already been integrated out, since it just amounts to dropping the kinetic term in the Lagrangian).

Now for a given magnetic field strength S, the vector potential can be chosen as above to be $A = S(1 - \cos)d\phi$ (on the northern hemisphere). The $Sd\phi$ piece contributes a total derivative to the action, and so we can drop it. Thus after $m \to 0$ we are left with

$$Z = \int \mathcal{D}\theta\, \mathcal{D}\phi\, \exp\left(iS \int dt\, \cos\theta\, \dot{\phi}\right). \tag{681}$$

This is the path integral for a particle of spin S in 0+1 dimensions: there is no Hamiltonian, and the entire action is the symplectic form piece. In particular, we see that the canonical variables on phase space are $\cos\theta, \phi$ (which gives the appropriate symplectic volume of $\Omega = \sin\theta d\theta \wedge d\phi$ and assigns uniform phase space density to the whole S^2), and tells us that when we quantize, we will have the commutator $[\cos\theta, \phi] = i$. One can then go back and check that these commutation relations imply that the components of the vector $\mathbf{n}$ in the ground state subspace obey the spin commutation relations. For example, using $[x, e^{ip}] = -e^{ip}$ for $[x, p] = i$, we have

$$[n^z, n^x] = [\cos\theta, \sin\theta\cos\phi] = \sin\theta[\cos\theta, e^{i\phi} + e^{-i\phi}]/2 = i\sin\theta(e^{i\phi} - e^{-i\phi})/2i = in^y. \tag{682}$$

Continuing in this way one sees that the components of n^μ obey the commutation relations of the Pauli algebra.

To arrive at this result, we first did the path integral over momentum, then implemented the $q^2 = 1$ constraint, and then took the $m \to 0$ limit. We could also do things in a different order by first implementing the constraint and projecting onto the ground state subspace, and then doing the path integral directly in this constrained subspace.

To do this approach, we use coherent states. First let's focus on the case of $S = 1/2$, since we've already found the eigenstates $|\psi\rangle$ of H, which were rather simple. Inserting resolutions of $\int_{S^2} |\psi\rangle\langle\psi|$, the path integral is

$$Z = \int \mathcal{D}\theta\, \mathcal{D}\phi\, \exp\left(\int dt\, \langle\psi|d_t|\psi\rangle\right), \tag{683}$$

where $-i\psi\ |d_t|\psi\rangle = \dot{\theta}A_\theta + \dot{\phi}A_\phi$, with A the Berry connection,[79] which we can evaluate explicitly as $A_\theta = -i\langle\psi|\partial_\theta|\psi\rangle = 0$, and

$$A_\phi = -i\langle\psi|\partial_\phi|\psi\rangle = \frac{1}{2}(\cos^2(\theta/2) - \sin^2(\theta/2)) = S(1 - 2(1 - \cos\theta)/2) = \frac{1}{2}\cos\theta, \tag{684}$$

which is of course exactly the right gauge potential for a magnetic field with $\int F = 2\pi$. This gives us back the path integral (681) for the case of $S = 1/2$.

[79] The factors of i are because $\langle\psi\ |d_t|\psi\rangle \in i\mathbb{R}$ due to the anti-Hermiticity of d_t.

We can also use coherent states to prove that (681) is the path integral for a spin in the spin S representation of $SU(2)$ (before, we just made a heuristic argument). Let $|g\rangle$ denote a coherent state in $SU(2)$ taken in the spin S representation, and let $\mathcal{D}g$ denote the Haar measure on $SU(2)$. Then the coherent state path integral is

$$Z = \int \mathcal{D}g \, \exp\left(\int dt \, \langle g|d_t|g\rangle\right). \tag{685}$$

To get this into the form of (681), we write

$$|g\rangle = e^{i\phi S_z} e^{-i\theta S_y} e^{i\gamma S_z}|+\rangle, \tag{686}$$

where $|+\rangle$ is the eigenstate of S_z in the S representation with the largest eigenvalue (namely S).[80] The three coordinates (ϕ, θ, γ) are the coordinates on $SU(2)$ adapted to the isomorphism $SU(2) \cong S^3$: θ, ϕ are the S^2 coordinates of the base space of the Hopf fibration, while γ is the coordinate along the fibers (the "gauge" direction). Since it is acting on the highest eigenstate of S_z, $e^{i\gamma S_z}$ becomes $e^{i\gamma S}$. The γ dependence of the path integral is then the term $iS \int dt \, (\partial_t \gamma)\langle g|g\rangle = iS \int dt \, \partial_t \gamma$, which is a total derivative and can be dropped. Thus as expected, the phase space only consists of the base S^2 of the Hopf fibration: the direction along the fibers, since it parametrizes gauge degrees of freedom, does not enter into the phase space.

Anyway, we can now go ahead and calculate the overlap $\langle g|d_t|g\rangle$:

$$\begin{aligned} \langle g|d_t|g\rangle &= i\langle +|e^{i\theta S_y} e^{-i\phi S_z}(\dot{\phi} S_z e^{i\phi S_z} - e^{i\phi S_z}\dot{\theta} S_y) E^{-i\theta S_y}|+\rangle \\ &= i\dot{\phi}\langle +|e^{i\theta S_y} S_z e^{-i\theta S_y}|+\rangle - i\dot{\theta}\langle +|S_y|+\rangle. \end{aligned} \tag{688}$$

The last term in zero, while the first term can be handled by using

$$e^{i\theta S_\mu} S_\nu e^{-i\theta S_\mu} = S_\nu \cos\theta + \epsilon_{\mu\nu\lambda} S_\lambda \sin\theta. \tag{689}$$

This gives us

$$\langle g|d_t|g\rangle = i\langle +|(S_z \cos\theta + S_x \sin\phi)|+\rangle = iS\cos\theta. \tag{690}$$

Putting this into the path integral gives us precisely (681), so (681) indeed describes the quantum mechanics of a spin S particle.

39 *March 31 — Better derivation of dyon charge and angular momentum quantization*

Today we're re-doing something which appeared scattered throughout several places in last year's diary entry but which wasn't done super concisely: showing that for $U(1)$ gauge

[80]For example, one checks that for $S = 1/2$,

$$|g\rangle = e^{i\gamma/2} e^{i\phi Z/2}(\cos(\theta/2)\mathbf{1} - i\sin(\theta/2)Y)|+\rangle = e^{i\gamma/2}\begin{pmatrix} e^{i\phi/2}\cos(\theta/2) \\ e^{-i\phi/2}\sin(\theta/2)\end{pmatrix}, \tag{687}$$

which is, up to an unimportant phase factor, exactly the coherent state we used above.

theory, two dyons of charge e, g and e', g' must be quantized so that $eg' - e'g \in \mathbb{Z}$, provided that conventions are such that the minimal magnetic charge is $g_{min} = 1/e$ (here magnetic charge is defined through $\int F = 2\pi g$), as well as finding the angular momentum for such a pair of dyons.

Solution:

We are going to work in units where the electric charge is dimensionless. We will also find it helpful to work in units where the quantization condition is such that $\int_\Sigma F \in 2\pi\mathbb{Z}$ for all closed Σ, since this is the most common choice in theoretical physics. With this normalization, the quantization on charges is $eg \in 2\pi\mathbb{Z}$. Now $E_i = F_{0i}$ and $B_i = \epsilon_{ijk}F^{jk}/2 = (\star F)_i$, where the $\star$ only occurs in space. We want these fields to be normalized so that for electric / magnetic charges of strength $e, g \in \mathbb{Z}$, we have

$$\int \star F = e, \qquad \int F = 2\pi g, \tag{691}$$

for appropriate surfaces surrounding the charges. This means that the fields for electric / magnetic charges of unit strength placed at the origin are

$$F_{emon} = \frac{\hat{r}}{4\pi r^2}, \qquad F_{mmon} = \star\frac{\hat{r}}{2r^2} \implies F_{mmon} = 2\pi \star F_{emon}. \tag{692}$$

Thus to obtain the magnetic monopole field strength, we just take the Hodge dual of the electric monopole field strength, and multiply by 2π. These conventions match with the usual ones used when talking about electromagnetic duality, where the dual field strength is $\widetilde{F} = 2\pi \star F$.

Now we will get an expression for the angular momentum of a charge + monopole system with this normalization. We have $\mathbf{L} = \int \mathbf{r} \times \mathbf{p}$, where $p_i = T_{0i}$. Using $T_{\mu\nu} = \delta_{g_{\mu\nu}} S$ we get $T_{0i} = F_{0j}F_i^j = E_j\epsilon^{ijk}\epsilon^{klm}F_{lm}/2 = (\mathbf{E}\times\mathbf{B})_i$, where we are in $(-+++)$ signature. So evaluated about a point $\mathbf{a}$, the angular momentum is

$$\mathbf{L_a} = \int_{\mathbf{x}} (\mathbf{x} - \mathbf{a}) \times (\mathbf{E} \times \mathbf{B}). \tag{693}$$

Now consider a situation in which we have an electric charge at the origin and a magnetic charge at position $\mathbf{R}$. Then using the triple product $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, we get

$$\begin{aligned} \mathbf{L_a} &= \frac{eg}{8\pi}\int_{\mathbf{x}} \frac{1}{x^3|\mathbf{x}-\mathbf{R}|^3}\left(\mathbf{x}((\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{R})) - (\mathbf{x}-\mathbf{R})((\mathbf{x}-\mathbf{a})\cdot\mathbf{x})\right) \\ &= \frac{eg}{8\pi}\int_{\mathbf{x}} \frac{1}{x^3|\mathbf{x}-\mathbf{R}|^3}\left(-\mathbf{x}(\mathbf{R}\cdot(\mathbf{x}-\mathbf{a})) + \mathbf{R}(\mathbf{x}\cdot(\mathbf{x}-\mathbf{a}))\right) \\ &= \frac{eg}{8\pi}\int_{\mathbf{x}} \left[\frac{1}{|\mathbf{x}-\mathbf{R}|^3}\left(-\frac{\mathbf{x}(\mathbf{R}\cdot\mathbf{x}) - \mathbf{R}x^2}{x^3}\right) + \frac{\mathbf{x}(\mathbf{R}\cdot\mathbf{a}) - \mathbf{R}(\mathbf{x}\cdot\mathbf{a})}{|\mathbf{x}-\mathbf{R}/2|^3|\mathbf{x}+\mathbf{R}/2|^3}\right], \end{aligned} \tag{694}$$

where in the last line we've broken the integral up and shifted the part containing $\mathbf{a}$ by $\mathbf{x} \mapsto \mathbf{x} + \mathbf{R}/2$. Now the second term is totally antisymmetric under $\mathbf{x} \mapsto -\mathbf{x}$, and so it dies.

Now we realize that

$$\partial_i \widehat{r}^j = \frac{\delta_{ij}}{|r|} - \frac{r_j r_i}{|r|^3}. \tag{695}$$

This means that the term in round parenthesis in the last integral is

$$-\frac{x^i(\mathbf{R}\cdot\mathbf{x}) - R^i x^2}{x^3} = \frac{-x^j + R^j}{|x|^3}(x^2\delta_{ij} - x_j x_i) = (R^j - x^j)\partial_j \widehat{x}^i. \tag{696}$$

So then (dropping the $\mathbf{a}$ subscript since $\mathbf{a}$ has disappeared from the integral) after integrating by parts,

$$\mathbf{L} = \frac{eg}{8\pi}\int_{\mathbf{x}} \left[\nabla \cdot \frac{\mathbf{x}-\mathbf{R}}{|\mathbf{x}-\mathbf{R}|^3}\right]\widehat{\mathbf{x}} = \frac{eg}{2}\widehat{\mathbf{R}}. \tag{697}$$

Since in these conventions both e, g are quantized in $\mathbb{Z}$, we see from the above that $\mathbf{L}\cdot\hat{\mathbf{R}} \in \mathbb{Z}/2$. Remember that here $\mathbf{R}$ is the vector pointing from the electric charge to the magnetic one.

Now, consider two dyons, and let $\mathbf{R}$ point from dyon 1 to dyon 2. Then we get a contribution to $\mathbf{L}$ of $e_1 g_2 \widehat{\mathbf{R}}/2$ from the above calculation. The rest of the angular momentum comes from that created by the e_2 and g_1 charges, which is calculated in the same way as that for the other set of charges, except for the replacement $\mathbf{R} \mapsto -\mathbf{R}$. Thus adding these two contributions, we get the correct quantization condition:

$$\mathbf{L} = \frac{1}{2}(e_1 g_2 - e_2 g_1)\widehat{\mathbf{R}} \implies e_1 g_2 - e_2 g_1 \in \mathbb{Z}. \tag{698}$$

40 *April 4 — $SU(2)$ gauge theory from an $O(5)$ σ model*

Solution:

First, a new notation appears! We will be using $k_{\mathbf{1}}$ to denote the elements of a field k that have unit norm, e.g. $\mathbb{C}_{\mathbf{1}} = U(1)$.

Preliminary quaternion things

We can write a quanternion as a pair of complex numbers $z_i = x_i + iy_i$, via the map $(z_1, z_2) \to z_1 + z_2 j$, or in terms of Pauli matrices, as

$$(z_1, z_2) \to \mathcal{O} = x_1\mathbf{1} + y_1 iZ - x_2 iY + y_2 iX = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}. \tag{699}$$

Since the quaternions are the unit S^3, they are acted on by $O(4) = [SU(2)_L \times SU(2)_R]/\mathbb{Z}_2$, via

$$\mathcal{O} \mapsto U_L \mathcal{O} U_R^\dagger. \tag{700}$$

The vector $z = (z_1, z_2)^T$ is in the fundamental representation of $SU(2)_L$: for example, if $U_L = e^{i\theta Z}$, then $z_1 \mapsto e^{i\theta} z_1, z_2 \mapsto e^{-i\theta} z_2$. z is not in the fundamental of $SU(2)_R$, since e.g. if $U_R = e^{i\theta Z}$, then $z_i \mapsto e^{-i\theta} z_i$ for both choices of i. Instead, the vector that is a fundamental under $SU(2)_R$ is $\widetilde{z} = (z_1^*, -z_2)$. For example, again taking $U_R = e^{i\theta Z}$, we have $\widetilde{z}_1 \mapsto e^{i\theta}\widetilde{z}_1, \widetilde{z}_2 \mapsto e^{-i\theta}\widetilde{z}_2$.[81] One can similarly check that the other generators of $SU(2)_R$ act on $\widetilde{z}$ in the fundamental.

Anyway, using this representation, we see that multiplication of quaternions proceeds via

$$(z_1, z_2) \cdot (w_1, w_2) = (z_1 w_1 - z_2 w_2^*, z_1 w_2 + z_2 w_1^*). \tag{701}$$

The same thing works for the octonions, except we replace pairs of complex numbers with pairs of quaternions. So an octonion can be written as a tuple $(v_1, v_2) \in \mathbb{H}^2$, with multiplication

$$(v_1, v_2) \cdot (u_1, u_2) = (v_1 u_1 - v_2 u_2^*, v_1 u_2 + v_2 u_1^*). \tag{702}$$

This multiplication law is non-associative since quaternions don't commute; hence it has no matrix representation.

$O(3) \leftrightarrow \mathbb{CP}^1$

Let's recall how we got a $\mathbb{CP}^1$ model from an $O(3)$ model. This mapping depended on the identification

$$n^i = z^\dagger \sigma^i z, \tag{703}$$

where $z \in \mathbb{C}^2_\times$. The reason for this identification was that the Hopf map does

$$H_{\mathbb{C}} : \mathbb{C}^2_{\mathbf{1}} \ni (z_1, z_2) \to (\Re[2\bar{z}_1 z_2], \Im[2\bar{z}_1 z_2], |z_1|^2 - |z_2|^2) \in S^2. \tag{704}$$

The RHS is in S^2 since $H_{\mathbb{C}}((z_1, z_2))^2 = (|z_1|^2 + |z_2|^2) = 1$. The identification (703) then works since e.g. for n^1, we have $\Re[2\bar{z}_1 z_2] = \bar{z}_1 z_2 + \bar{z}_2 z_1 = z^\dagger X z$. The preimage of the north pole $(0, 0, 1)$ is the circle $(e^{i\theta}, 0)$ (in the XY plane, in the notation of our earlier diary entry on the Hopf map), while the preimage of $(1, 0, 0)$ is the helix $(e^{i\theta}, e^{i\theta})$ (this is a helix about the unit circle in the XY plane).

A more mathy way of understanding this is the following. The z vectors in (703) are acted on the left by $SU(2) = \text{Spin}(3)$. The Pauli matrices are the generators for $Cl(3)$, and so this $SU(2)$ action acts adjointly on $Cl(3)$. Recalling facts from the diary entry on Clifford algebras and spin groups, we recall that the adjoint action ofSpin(n) on $Cl(n)$ is equivalent to an action of $SO(3)$ on $Cl(n)$: this is how we define the map $P : \text{Spin}(n) \to SO(n)$. So, for any $U \in SU(2)$, we can write $U^\dagger \sigma^i U = R^{ij}\sigma^j$, with $R \in SO(3)$; this means that $z^\dagger \sigma^i z$ transforms as a vector under this $SO(3)$, as it should. Furthermore, it is normalized correctly: taking e.g. $z = (e^{i\theta}, 0)$ one verifies that $n^i n_i = 1$; since the length of $\mathbf{n}$ is preserved under the $SU(2)$ action on z, which lets us rotate $(e^{i\theta}, 0)$ to any vector in $\mathbb{C}^2_{\mathbf{1}}$, we can conclude that $\mathbf{n}^2 = 1$ for any choice of z.

Of course, this presentation in terms of $z^\dagger, z$ suffers a gauge redundancy, which we can think of as a "right" action by $U(1)$. This

[81] This is just because if $\mathcal{O} \mapsto \mathcal{O}U_R^\dagger$ then $\mathcal{O}^\dagger \mapsto U_R \mathcal{O}^\dagger$, so $\mathcal{O}^\dagger$ transforms in the fundamental of $SU(2)_R$. Taking $\mathcal{O} \to \mathcal{O}^\dagger$ is equivalent to replacing $z \mapsto \widetilde{z}$.

$O(5) \leftrightarrow \mathbb{HP}^1$

Now we turn to the quaternionic version of the above construction. From the Hopf fibration $S^3 \to S^7 \to S^4$, we should be able to write a vector in S^4 in terms of one in S^7 modulo a gauge redundancy in $SU(2)$. For the complex case, we wrote things in terms of a vector $z \in \mathbb{C}^2_{\mathbf{1}}$. For the quaterionic case then, we will write things in terms of a vector $Q \in \mathbb{H}^2_{\mathbf{1}}$. We can think of Q as a 4×2 matrix with entries in $\mathbb{C}$, or as a 2×1 column vector with entries in $\mathbb{H}$:

$$Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \mathbf{1} + \sum_{j=1}^3 i\phi_j \sigma^j \\ \phi_5 \mathbf{1} + \sum_{k=1}^3 i\phi_{k+5}\sigma^k \end{pmatrix} = \begin{pmatrix} \phi_1 + \phi_2 \mathbf{i} + \phi_3 \mathbf{j} + \phi_4 \mathbf{k} \\ \phi_5 + \phi_6 \mathbf{i} + \phi_7 \mathbf{j} + \phi_8 \mathbf{k} \end{pmatrix}, \qquad \vec{\phi} \in S^8. \tag{705}$$

The condition that $\boldsymbol{\phi}^2 = 1$ is equivalent to

$$\frac{1}{2}\text{Tr}[Q^\dagger \cdot Q] = |q_1|^2 + |q_2|^2 = 1. \tag{706}$$

Now we need to figure out how to form a vector in S^4 out of Q. One way to do this is by doing the quaternionic analogue of the complex Hopf map:

$$H_{\mathbb{H}} : \mathbb{H}^2_{\mathbf{1}} \ni (q_1, q_2) \mapsto \mathbf{n} = ([2\bar{q}_1 q_2]_{\mathbf{i}}, [2\bar{q}_1 q_2]_{\mathbf{j}}, [2\bar{q}_1 q_2]_{\mathbf{k}}, [2\bar{q}_1 q_2]_{\mathbf{1}}, |q_1|^2 - |q_2|^2) \in S^4, \tag{707}$$

where $[x]_{\mathbf{1}}$ is the real part of x, $[x]_{\mathbf{i}}$ is the $\mathbf{i}$-part, and so on. The image of (q_1, q_2) is in S^4 just because as before, $\mathbf{n}^2 = (|q_1|^2 + |q_2|^2)^2$.

Now the last two components of $\mathbf{n}$ can be written as

$$n^4 = \bar{q}_1 q_2 + \bar{q}_2 q_1 = \frac{1}{2}\text{Tr}[Q^\dagger (X \otimes \mathbf{1}) Q], \qquad n^5 = \frac{1}{2}\text{Tr}[Q^\dagger (Z \otimes \mathbf{1}) Q]. \tag{708}$$

The others can be written as e.g.

$$n^1 = \mathbf{i}\bar{q}_1 q_2 - \bar{q}_2 q_1 \mathbf{i}, \tag{709}$$

and likewise for n^2, n^3. Unfortunately, I don't think these have a simple expression in terms of a trace, since the $\mathbf{i}$ does not appear between the two qs. So instead of using the above formula for $\mathbf{n}$, we will modify it slightly. We will still take n^4 and n^5 to be as above, but we will instead take

$$n^1 = -\bar{q}_1 \mathbf{i} q_2 + \bar{q}_2 \mathbf{i} q_1, \qquad n^2 = -\bar{q}_1 \mathbf{j} q_2 + \bar{q}_2 \mathbf{j} q_1, \qquad n^3 = -\bar{q}_1 \mathbf{k} q_2 + \bar{q}_2 \mathbf{k} q_1. \tag{710}$$

This allows us to write the first three components of $\mathbf{n}$ as traces:

$$n^i = \frac{1}{2}\text{Tr}[Q^\dagger \gamma^i Q], \qquad i = 1, 2, 3, \tag{711}$$

where

$$\gamma_1 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & -\mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & -\mathbf{k} \\ \mathbf{k} & 0 \end{pmatrix}. \tag{712}$$

Note that this does *not* give the same formula for $n^i, i = 1, 2, 3$ as (707). For example, (707) gives $n^1 = 2(\phi_1\phi_6 - \phi_2\phi_5 - \phi_3\phi_8 + \phi_4\phi_7)$, while (711) gives $n^1 = 2(-\phi_1\phi_6 + \phi_2\phi_5 - \phi_3\phi_8 + \phi_4\phi_7)$.

However, the $\mathbf{n}$ obtained with (711) does satisfy $\mathbf{n}^2 = 1$, which as far as I'm concerned is a miracle of algebra; I don't really know why this should be the case a priori.

Summarizing, we will use the Hopf map

$$n^i = \mathrm{Tr}[Q^\dagger \gamma^i Q], \tag{713}$$

where

$$\gamma_1 = Y \otimes X, \quad \gamma_2 = Y \otimes Y, \quad \gamma_3 = Y \otimes Z, \quad \gamma_4 = X \otimes \mathbf{1}, \quad \gamma_5 = Z \otimes \mathbf{1}. \tag{714}$$

Note that this presentation has a gauge redundancy under a right action of $SU(2)$ on Q, via $SU(2)_g : Q \mapsto QU$, or $q_i \mapsto q_i v$, where $v \in \mathbb{H}_{\mathbf{1}}$.

Of course, the notation γ^i is no accident; these are exactly the γ^i matrices used to generate $Cl(5)$! This representation of them is convenient since it's adapted to the splitting into the subgroup $SO(3) \times SO(2) \subset SO(5)$.

Why are the matrices that are used to construct Spin(5) showing up? They are showing up because

$$SO(8) = (\mathrm{Spin}(5) \times SU(2))/\mathbb{Z}_2. \tag{715}$$

The $SO(8)$ acts on the vector $\boldsymbol{\phi}$, with $SU(2)$ showing up in this representation as the gauge degree of freedom, and with the remaining $SO(5)$ part surviving to act (on the left) as the physical symmetries on the vector $\mathbf{n} \in S^4$. The way this works is just as we have described above: Q transforms in the spinor representation of Spin(5) (acting on the left), which gives an adjoint action on the γ^is appearing in n^i. This adjoint action is equivalent to having γ^i transform in the fundamental of $SO(5)$, as explained earlier. Thus the physical symmetry group, viz. $SO(5)$, is just what we need for a unit vector living in S^4.

This is a good opportunity to motivate the isomorphism $\mathrm{Spin}(5) \cong Sp(2)$. This is plausible since $Sp(2) = U(2; \mathbb{H})$ has a natural action on Q from the left, in the same way that Spin(5) does. To verify this isomorphism, consider the matrix $\mathcal{J} = -\gamma_1\gamma_3 = \mathbf{1} \otimes J$. Then $\mathcal{J}^T = -\mathcal{J}$ and $\mathcal{J}^2 = -\mathbf{1}$. Furthermore, since only γ_1 and γ_3 are antisymmetric, $\mathcal{J}$ satisfies

$$\gamma_i^T \mathcal{J} \gamma_i = \mathcal{J}\gamma_i^2 = \mathcal{J} \tag{716}$$

for all γ_i. In particular, $(\gamma_i\gamma_j)^T \mathcal{J} \gamma_i \gamma_j = \mathcal{J}$ for all pairs i, j, and so $S^T \mathcal{J} S = \mathcal{J}$ for any $S \in \mathrm{Spin}(5)$. Thus $\mathcal{J}$ is a 4-dimensional symplectic form preserved by Spin(5), establishing the isomorphism $\mathrm{Spin}(5) \cong Sp(2)$.

We've shown how to write a vector in S^4 in terms of a vector in S^7 plus an $SU(2)$'s worth of gauge redundancy. Now we need to show how to rewrite the nlσm action in the S^7 variables.

Since the $SU(2)_g$ transformations act on the right, the covariant derivative is

$$D_\mu Q = \partial_\mu Q - iQA_\mu, \qquad A_\mu = -iQ^\dagger \partial_\mu Q. \tag{717}$$

This expression for A gives the correct transformation rule, viz.

$$Q \mapsto QU \implies A \mapsto U^\dagger A U - iU^\dagger \partial_\mu U \, dx^\mu. \tag{718}$$

$$S = \frac{1}{2g^2} \int \mathrm{Tr}[(D_\mu Q)^\dagger D^\mu Q] + \frac{\theta}{2\pi} \int \mathrm{Tr}[F_A \wedge F_A]. \tag{719}$$

41 April 6 — $SU(2) \times SU(2)$ chiral symmetry breaking

Today we're going through the details of the setup of the chiral lagrangian for the breaking of $SU(2) \times SU(2)$ in QCD (with the first generation of quarks only). This is standard stuff; I just wanted to have the details that were skipped over in the book I was reading (Zinn-Justin) spelled out somewhere.

Solution: We will be adding things to the free action

$$S_\psi = \int \bar{\psi} i \partial\!\!\!/ \psi, \qquad \psi = (u,\, d)^T, \tag{720}$$

where u, d are the up and down quarks. This possesses a $U(2)^2$ flavor symmetry, where the two factors act separately on the left- and right-handed parts of the two quarks.

To discuss spontaneous chiral symmetry breaking, we will start by adding to the action the term

$$S_{\psi M} = -g \int \bar{\psi}(\Pi_+ M + \Pi_- M^\dagger)\psi, \qquad \Pi_\pm = \frac{1}{2}(\mathbf{1} \pm \bar{\gamma}), \tag{721}$$

where M is some 2×2 matrix field. We need to know how the discrete symmetries P and C are implemented on M (we are in Euclidean time, so we only care about C and P). We will take P to act as reflection of a single spacetime coordinate for simplicity—following the discussion in the diary entry on fermions, we recall that P is implemented by $P : \psi \to \bar{\gamma}\gamma_i\psi, \bar{\psi} \mapsto \bar{\psi}\gamma_i$ for some i ($\bar{\psi}$ is a field independent from ψ, transforming in the inverse way under $\mathrm{Spin}(d)$ as ψ). Requiring that $S_{\psi M}$ is P-invariant means that

$$\Pi_+ M + \Pi_- M^\dagger = \gamma_i(\Pi_+ P(M) + \Pi_- P(M^\dagger))\gamma_i = \Pi_- P(M) + \Pi_+ P(M^\dagger) \tag{722}$$

and so evidently P acts as

$$P(M) = M^\dagger. \tag{723}$$

Charge conjugation symmetry gives us the constraint

$$[C\Pi_+ C^\dagger \otimes C(M) + C\Pi_- C^\dagger \otimes C(M^\dagger)]^T = \Pi_+ M + \Pi_- M^\dagger. \tag{724}$$

Hopefully the notation here isn't too confusing: $C(M)$ is the charge-conjugated image of M, while the $\otimes$s are used since M and $\Pi_\pm$ act on different factors of the Hilbert space.[82] Now $C\gamma^\mu C^\dagger = -\gamma_\mu^T$, and thus $[C\bar{\gamma}C^\dagger]^T = \bar{\gamma}$ in $d = 4$ dimensions, while $[C\bar{\gamma}C^\dagger]^T = -\bar{\gamma}$ in $d = 2$. Thus $C\Pi_\pm C^\dagger = \Pi_\pm$ if $d = 4$, and $C\Pi_\pm C^\dagger = \Pi_\mp$ in $d = 2$. Therefore C symmetry tells us that

$$C(M) = M^T \quad (d = 4), \qquad\qquad C(M) = M^* \quad (d = 2). \tag{725}$$

With these transformations, $S_{\psi M}$ is symmetric.

[82]We will of course usually omit the $\otimes$s, but here we've written them since e.g. $C\Pi_\pm C^\dagger C(M)$ is likely to cause confusion.

The flavor symmetry of the free term is $U(2)_+ \times U(2)_-$, which acts as

$$U(2)_+ \times U(2)_- : \psi \mapsto U_+^{\Pi_+} U_-^{\Pi_-}, \qquad \bar{\psi} \mapsto \bar{\psi} P[(U_+^{\Pi_+} U_-^{\Pi_-})^\dagger] = \bar{\psi}(U_+^{\Pi_-} U_-^{\Pi_+})^\dagger, \qquad U_\pm = e^{i\theta_a^\pm t^a}, \tag{726}$$

where t^a are the (Hermitian) generators for the $\mathfrak{u}(2)$ Lie algebra. Since $U_\pm \Pi_\mp = \mathbf{1}$, we can also write this as

$$U(2)_+ \times U(2)_- : \psi \mapsto (\Pi_+ U_+ + \Pi_- U_-)\psi, \qquad \bar{\psi} \mapsto \bar{\psi}(\Pi_+ U_-^\dagger + \Pi_- U_+^\dagger). \tag{727}$$

Since $e^{i\Pi_\pm} \not{\partial} = \not{\partial} e^{-i\Pi_\pm}$, the free part of the action is invariant.

Under $U(2)_+ \times U(2)_-$, $S_{\psi M}$ transforms as

$$U(2)_+ \times U(2)_- : S_{\psi M} \mapsto -g \int \bar{\psi} \left(\Pi_+ U_-^\dagger M' U_+ + \Pi_- U_+^\dagger (M')^\dagger U_- \right) \psi, \tag{728}$$

where M' is the image of M under $U(2)_+ \times U(2)_-$. Therefore this interaction will be symmetric provided that M transforms as

$$U(2)_+ \times U(2)_- : M \mapsto U_- M U_+^\dagger. \tag{729}$$

The next terms we need to add to the action are a kinetic term for M, and a potential to give M a vev:

$$S_M = \frac{1}{2\alpha} \int \left[\mathrm{Tr}[\partial_\mu M \partial^\mu M^\dagger] - V(MM^\dagger) \right], \tag{730}$$

which preserves C and P.

We will also include a term that explicitly breaks the $U(2)_+ \times U(2)_-$ symmetry while preserving C and P. This is done with

$$S_B = -\frac{1}{\sqrt{2}} \int \mathrm{Tr}[B(M + M^\dagger)], \tag{731}$$

where B is some matrix. S_B preserves P for any choice of B since $M + M^\dagger$ is a scalar, while it preserves C if $B^T = B$. Note that $M - M^\dagger$ is a pseudoscalar though, so such a term would break P explicitly if added.

Let's now look at how all the terms we've introduced affect the axial and vector currents. The vector current is found by taking $U_+ = U_-$ and performing the variation $\psi \mapsto U\psi$, $M \mapsto UMU^\dagger$. Taking $U = e^{i\theta_a t^a}$ for θ_a small, we use $\partial_\mu M \mapsto i\partial_\mu \theta^a [t^a, M]$ to get (I'm not being super careful about signs and is)

$$\delta_V S = \int \left[\partial_\mu \theta^a \left(\bar{\psi}\gamma^\mu t^a \psi + i\mathrm{Tr}[t^a([M, \partial_\mu M^\dagger] + [M^\dagger, \partial_\mu M])] \right) + \frac{i}{\sqrt{2}} \theta_a \mathrm{Tr}[[t^a, B](M + M^\dagger)] \right], \tag{732}$$

where we used the cyclicity of the trace and that $M^\dagger \mapsto UM^\dagger U^\dagger$. The part contracted with $\partial_\mu \theta^a$ is the vector current, and so the Ward identity tells us

$$\partial_\mu j_V^{\mu a} = \partial_\mu \left(\bar{\psi}\gamma^\mu t^a \psi + i\mathrm{Tr}[t^a([M, \partial_\mu M^\dagger] + [M^\dagger, \partial_\mu M])] \right) = -\frac{i}{\sqrt{2}} \mathrm{Tr}[[t^a, B](M + M^\dagger)]. \tag{733}$$

The axial current is found by taking $U_- = U_+^\dagger$, so that $\psi \mapsto (\Pi_+ U + \Pi_- U^\dagger)\psi = e^{-i(\Pi_+ - \Pi_-)\theta_a t^a}\psi$ and $M \mapsto U^\dagger M U^\dagger$. This leads to a similar situation as the vector current, except with commutators replaced by anti-commutators:

$$\delta_A S = \int \left[\partial_\mu \theta^a \left(\bar{\psi}\gamma^\mu \bar{\gamma} t^a \psi + \mathrm{Tr}[t^a(\{M, \partial_\mu M^\dagger\} + \{M^\dagger, \partial_\mu M\})] \right) - \frac{i}{\sqrt{2}} \theta_a \mathrm{Tr}[\{t^a, B\}(M - M^\dagger)] \right], \tag{734}$$

so that

$$\partial_\mu j_A^{\mu a} = \frac{i}{\sqrt{2}} \mathrm{Tr}[\{t^a, B\}(M - M^\dagger)]. \tag{735}$$

Therefore the vector current is not conserved unless $B \propto \mathbf{1}$, while the axial current is not conserved for all $B \neq 0$.[83] Also note that as required, the divergence of j_V is a scalar, while the divergence of j_A is a pseudoscalar.

To discuss SSB, we need to pick an explicit form for B and B. For simplicity we will specialize to the case where $B = b\mathbf{1}$, which conserves j_V but explicitly breaks j_A. After M gets a vev from, this choice of B will give (equal) masses to the u and d quarks. For the potential, we take the usual ($m^2 < 0$)

$$V(x) = \frac{1}{2} m^2 x + \frac{1}{24} \lambda x^2. \tag{736}$$

Therefore we will parametrize M as

$$M = \frac{1}{\sqrt{2}} (\sigma \mathbf{1} + i \pi_a \sigma^a). \tag{737}$$

We can now solve for $p = \langle \pi \rangle$ and $s = \langle \sigma \rangle$. If $b \neq 0$ then $p = 0$, which then after some straightforward algebra gives

$$s(m^2 + \lambda s^2/6) = b. \tag{738}$$

Solving this to first order in b, we get

$$s = \sqrt{-6m^2/\lambda} + c\sqrt{-3/(2\lambda m^2)}. \tag{739}$$

We can then plug this back into the potential by taking $\sigma \mapsto \sigma - s$ to find the masses of the σ and the π—we get, to first order in b,

$$m_\pi^2 = m^2 + s^2 \lambda/6 = \frac{b}{4|m^2|}, \qquad m_\sigma^2 = m^2 + s^2\lambda/2 = 2|m^2| + \frac{3b}{4|m^2|}. \tag{740}$$

As required, the π mass is zero when $b = 0$, since when $b = 0$ the π is a Goldstone boson. This vev for M also gives rise to a mass term for the fermions via the Yukawa coupling in $S_{\psi M}$; in this simple case where $B \propto \mathbf{1}$, at $m_\sigma \to \infty$ both quarks have the same mass, $m_u = m_d \propto gs$ (we don't really think of $\langle \pi \rangle$ as contributing to the mass since it only really does so when b is large, but this is outside of our approximation scheme).

[83]This is because in order for $\partial_\mu j_A^{\mu a} = 0$ for all a with $B \neq 0$, we need B to anti-commute with all the t^a. This is impossible since this implies $B[t^a, t_b] = [t^a, t^b]B$, which cannot be true since $[t^a, t^b] = i f^{abc} t^c$ is linear in the t^as, and hence B must anti-commute with it—a contradiction.

A last comment is that this approach lets us easily deal with a possible θ term. If the quarks are coupled to an $SU(3)$ gauge field with $\theta \int c_2[A_{SU(3)}]$ in the action, then we can eliminate the θ term with a chiral rotation. This is the equivalent to performing the shift $M \mapsto e^{-2i\theta} M$, which then shows up equivalently as a shift $B \mapsto e^{i\theta Z} B$. This has the effect of doing $b \mapsto b\cos\theta$ for the purposes of computing $\langle \sigma \rangle, \langle \pi \rangle$.

42 *April 8 — More on the massless Schwinger model and its spectrum*

Applications of bosonization in 1+1D is becoming a bit of a tired topic in this diary, but today we have something slightly new: an alternate way of deriving confinement and the existence of a chiral condensate in the (massless) Schwinger model. We will be somewhat fast and loose with factors of i and signs relating to squares of $\star$ (which squares to -1 on 1-forms in 1+1D in Euclidean time).

Solution:

In 1+1D, we can fix $d^\dagger A = 0$ gauge and write $A = \star d\phi$ for ϕ a scalar. The kinetic term for the gauge field is then (here $\Box = -\partial_\mu \partial^\mu$ is positive-definite)

$$\int F_A \wedge \star F_A = \int d \star d\phi \wedge d^\dagger d\phi = \int d\phi \wedge \star \Box d\phi = \int \phi \Box^2 \phi. \tag{741}$$

The fermions then couple to the gauge field via the term

$$S \ni \frac{1}{2\pi} \int i\bar{\psi}\gamma^\mu \epsilon^{\mu\nu} \partial_\nu \phi \psi. \tag{742}$$

Now consider performing a chiral rotation by $\psi \mapsto e^{-i\bar{\gamma}\phi}$, where $\bar{\gamma} = -i\gamma_1\gamma_2 = -iXY = Z$. Then the kinetic term changes by

$$\delta(\bar{\psi}\not{\partial}\psi) = -i\bar{\psi}\gamma^\mu \bar{\gamma}\partial_\mu \phi \psi = i\bar{\psi}(\epsilon^{\mu\nu}\partial_\nu \phi)\gamma^\mu \psi, \tag{743}$$

which is just what is needed to kill the coupling of ψ to ϕ. Of course when we do this the chiral anomaly comes into play, and gives us a term $\frac{1}{2\pi}\int(\phi d \star d\phi) = -\frac{1}{2\pi}\int \phi \Box \phi$. Lastly, we bosonize the fermions with a compact field θ, sticking to Witten's conventions in QFTs and strings. Putting all the pieces together, the action is

$$S = \frac{1}{2} \int \left(\frac{1}{4\pi} |d\theta|^2 - \frac{1}{\pi} |d\phi|^2 + \frac{1}{e^2} \phi \Box^2 \phi \right). \tag{744}$$

The ϕ propagator is evidently

$$G^\phi(p) = \frac{1}{p^2/\pi + p^4/e^2} = \frac{1}{p^2} + \frac{1}{p^2 + e^2/\pi}. \tag{745}$$

Therefore we have two modes: one massive with mass $m^2 \equiv e^2/\pi$, and one massless. The gauge field propagator is then

$$G^A_{\mu\nu}(p) = \epsilon^{\mu\alpha}\epsilon^{\nu\beta}p_\alpha p_\beta\langle\phi(p)\phi(-p)\rangle = m^2(\delta^{\mu\nu} - p^\mu p^\nu/p^2)\frac{1}{p^2+m^2}, \tag{746}$$

and so the gauge field is rendered massive by its coupling to the fermions.

It turns out that the massless θ particle and the massless mode of ϕ cancel each other out, leaving behind only a massive mode. To see this, we first take $\theta \mapsto \theta + 2\phi$, which eliminates the $|d\phi|^2$ term and gives us a $\theta\Box\phi/\pi$ term. Then we take $\phi \mapsto \phi - m^2\Box^{-1}\theta$, which kills the mutual ϕ, θ coupling. This gives an effective action for θ which is

$$S = \frac{1}{8\pi}\int \theta(\Box + m^2)\theta. \tag{747}$$

This tells us that the theory consists of a single massive pseudo-scalar boson (it's "pseudo-scalar" since $\psi^\dagger_\pm\psi_\mp \to e^{\pm i\theta}$ means $P : \theta \mapsto -\theta$, at least in the simple case with a Pin$^+$ structure, where P acts as $X = \gamma^0$).

One important thing to realize is that the expectation value of the chiral fermion bilinear is nonzero:

$$\langle\psi^\dagger_L\psi_R\rangle = \frac{1}{a}e^{-\langle\theta^2\rangle/2} = e^{-G^\theta(x=0;m)/2} \approx \frac{1}{a}e^{\ln(ma)} = m, \tag{748}$$

where a is a short-distance cutoff.[84] Many people say that the fact that this is non-zero indicates that we have chiral symmetry breaking. This would not contradict the CMW theorem, since in this case we have long-ranged interactions, provided by the gauge field (the absence of a Goldstone is also okay—the massless part of the gauge field was eaten by the term that we had to add to the action in accordance with the chiral anomaly). But this is not really correct, since the anomaly means that we never actually had chiral symmetry in the first place, so there is nothing to break.

Now let us look at correlators of the bilinears $\sigma \equiv \bar\psi\psi$.[85] First, as a check of asymptotic freedom, we can compute the 2-point functions $\langle\sigma_s(x)\sigma_{s'}(0)\rangle$ at $x \to 0$, where $\sigma_\pm \equiv \psi^\dagger_\pm\psi_\mp$ and where s, s' are signs. If $s = s'$, then we find

$$\langle\sigma_\pm(x)\sigma_\pm(0)\rangle_{x\to 0} = \langle\sigma^2_\pm(0)\rangle e^{G^\theta(x\to 0)} \approx m^2 e^{-\ln(x^2/m^2)} = \frac{1}{x^2}. \tag{750}$$

On the other hand if $s = -s'$ then the sign in the exponent switches, and we get

$$\langle\sigma_\pm(x)\sigma_\mp(0)\rangle \approx x^2m^4 \to 0. \tag{751}$$

[84] With the current conventions,

$$G^\theta(0;m) = 4\pi\int_0^{a^{-1}}\frac{dp}{2\pi}\frac{p}{p^2+m^2} = \ln(a^{-2}/m^2+1) \approx -2\ln(am). \tag{749}$$

[85] Chirally-invariant correlators of two fermions, i.e. $\langle\psi^\dagger_\pm(x)\psi_\pm(0)\rangle$ are hard since the mass term for θ screws up a holomorphic / anti-holomorphic decomposition for θ, and means that the only correlators that are easy to compute are those of $\sigma_\pm = \psi^\dagger_\pm\psi_\mp$ ($\sigma = \sigma_+ + \sigma_-$), since $\sigma_\pm$ bosonizes to $e^{\pm i\theta}$.

Note that these results are exactly in accordance with what we'd get from free field theory; hence the model is asymptotically free. More complicated correlators are those involving the scalar σ. Using the usual manipulations for expectation values of exponentials, we get

$$\langle\sigma(x)\sigma(0)\rangle = \langle\sigma^2(0)\rangle 4\cosh(G^\theta(x;m)). \tag{752}$$

Now besides the scalar σ, we also have the pseudo-scalar $\widetilde{\sigma}_\pm \equiv \bar{\psi}\bar{\gamma}\psi = i(\psi_+^\dagger\psi_- - \psi_-^\dagger\psi_+)$. The minus sign turns the cosh into a sinh:

$$\langle\widetilde{\sigma}(x)\widetilde{\sigma}(0)\rangle = \langle\widetilde{\sigma}^2(0)\rangle 4\sinh(G^\theta(x;m)). \tag{753}$$

To evaluate these expressions, we expand the hyperbolic functions in powers of G^θ and then go to momentum space. Therefore we need to evaluate integrals like

$$\int\prod_i^k\left(\frac{d^2p_i}{4\pi^2}\frac{1}{p_i^2+m^2}\right)\frac{1}{\left(q-\sum_i p_i\right)^2+m^2} \tag{754}$$

for some fixed q^2. Terms with k even will appear in the expansion of the σ correlator, and terms with k odd will appear in the expansion of the $\widetilde{\sigma}$ correlator.

First, note that only the expansion of the $\sinh(G^\theta)$ will give an isolated pole in momentum space (only for $k=0$ in the above equation, which is the first term in the expansion of sinh, will we get a simple pole at $q^2=-m^2$; all other singularities are part of branch cuts, and are not isolated). This confirms the result that the boson in the spectrum is a pseudo-scalar, since there is a simple pole only in the correlation function of the pseudo-scalar $\widetilde{\sigma}$ field.

In general, I think it is true that the successive terms in the $\langle\sigma(x)\sigma(0)\rangle$ correlation function contribute branch cut singularities at $q^2=-(2n)^2m^2, n\in\mathbb{Z}$, while the successive terms in the $\langle\widetilde{\sigma}(x)\sigma(0)\rangle$ correlation function contribute branch cut singularities at $q^2 = -(2n+1)m^2, n\in\mathbb{Z}$. The first singular contribution to $\langle\sigma(x)\sigma(0)\rangle$ is determined by the integral (ignoring 2πs)

$$\begin{aligned} I &= \int_p \frac{1}{(p^2+m^2)((p-q)^2+m^2)} \\ &= \int_x\int_p \frac{1}{[x((p-q)^2+m^2)+(1-x)(p^2+m^2)]^2} \\ &= \int_x\int_p \frac{1}{(p^2+m^2+q^2(x-x^2))^2} \\ &= \frac{1}{2}\int_x \frac{1}{m^2+q^2(x-x^2)} \\ &= \frac{1}{q\beta}\ln\left(1+\frac{q}{2m^2}(q+\beta)\right), \qquad \beta\equiv\sqrt{4m^2+q^2}. \end{aligned} \tag{755}$$

This is singular precisely when $q=2im$, indicating the contribution of particle production involving a particle with mass m to the Greens function. In fact we can already see the singularity before we do the x integral: taking $q^2=-\lambda m^2$, the denominator vanishes when

$$x = \frac{1}{2}\pm\sqrt{\lambda^2-4\lambda}, \tag{756}$$

which tells us that we have singularities as soon as $\lambda > 4$: this gives us a branch point starting at $q = 2im$, as found above.

Now we turn to the leading term in the expansion for the $\widetilde{\sigma}$ correlator. With two integrals, things are much more heinous:

$$\begin{aligned} I &= \int_{p,k} \frac{1}{(p^2+m^2)(k^2+m^2)((q-p-k)^2+m^2)} \\ &= 2\int_{x,y}\int_{p,k} \frac{1}{(x(p^2+m^2)+y(k^2+m^2)+z((q-p-k)^2+m^2))^3} \qquad z \equiv 1-x-y \\ &= 2\int_{x,y}\int_{p,k} \frac{1}{(m^2+p^2+k^2+z[-2q\cdot(p+k)+q^2+2p\cdot k])^3}. \end{aligned} \tag{757}$$

Now we need to eliminate the dot product between q and $p+k$. Consider shifting $\delta p = \delta k = \alpha q$. Then the terms involving q in the denominator become

$$2q\cdot(p+k)[\alpha - z + \alpha z] + q^2(z + 2\alpha^2 z + 2\alpha^2) \implies \alpha = \frac{z}{1+z}. \tag{758}$$

Then we shift $\delta p = -zk$, ending up with

$$\begin{aligned} I &= 2\int_{x,y}\int_{p,k} \frac{1}{(m^2+p^2+k^2(1-z^2)+\gamma q^2)^3} \qquad \gamma \equiv (1+3z^2)/(1+z) \\ &= \int_{x,y} \frac{1}{2(1-z^2)(m^2+\gamma q^2)} \end{aligned} \tag{759}$$

The integral can't be done analytically, but if we look at when the denominator vanishes, we can check that it does so at $q^2 = -9m^2$, which is exactly what we'd expect for a contribution coming from a 3-particle intermediate state. (Note to self: come back and work this out more carefully sometime)

43 *April 9 — Gluon screening of Wilson lines in non-Abelian gauge theory; some representation theory computations*

Today we're going over something that took me forever to finally understand: how exactly gluons can screen Wilson lines in certain representations to turn area law behavior into perimeter law behavior.

Solution:

Since we are looking at pure gauge theory and are discussing screening, only non-Abelian gauge groups will be relevant, since only for non-Abelian groups are the gauge bosons charged. We will work on the lattice, since it will make the screening phenomenon most clear. We will show that for certain choices of representation R, the Wilson line $W_R(C)$

will have perimeter-law behavior in the strong-coupling expansion. Now in the weak coupling limit[86] the Wilson line always has perimeter law: we expand it to order g^2 and get $\langle W_R(C)\rangle \approx 1 + Ng^2 \oint\oint \langle A^a_\mu(x) A^a_\nu(y) dx^\mu \wedge dy^\nu\rangle$, where $N = \dim(R)C_2(R)/2$ and a is an arbitrary group index (no sum). We can replace the correlator to $O(g^2)$ with the Fourier transform of Π_T/p^2, which gives a perimeter law (in 3+1 D). This means that there are representations for which $W_R(C)$ has a perimeter law both at strong coupling and at weak coupling, and so the Wilson lines are tensionless throughout the phase diagram.

Anyway, for a given representation R, we want to compute

$$\langle W_R(C)\rangle = \left\langle \mathrm{Tr}_R \left[\prod_{l\in C} U_R(g_l)\right]\right\rangle = \int \prod_l \mathcal{D}g_l \,\mathrm{Tr}_R \left[\prod_{l\in C} U_l\right] \exp\left(-\beta \sum_\Box \mathrm{Tr}_f \left[\prod_{l\in\partial\Box} U_l\right]\right), \tag{761}$$

where f is the fundamental representation of the gauge group, and the sum over plaquettes includes a sum over plaquette orientations.[87] Here the notation is such that U_l is the representation of g_l in the fundamental or antifundamental, depending on the orientation of the link. A more verbose notation is $U_f(g_l)$, which we use for the representation matrices appearing in the Wilson loop. If a representation subscript on a $U(g)$ is omitted, it is implied that the representation is taken to be the fundamental. As usual, each plaquette term has two fundamental matrices and two antifundamental matrices: if the group elements on the links of a plaquette are $g_1,\dots,g_4$ labeled counterclockwise from the "bottom", then $U_\Box \equiv \prod_{l\in\partial\Box} U_l = U(g_1)U(g_2)U^*(g_3)U^*(g_4)$. Flipping the orientation of a plaquette is therefore equivalent to taking the trace in the conjugate representation f^*.

Since the weak-coupling limit gives a perimeter law, to address confinement we just need to look at strong coupling. For strong coupling, we expand the exponential in powers of β. Now $\int \mathcal{D}g_l\, U_l = 0$ by the properties of the Harr measure. If we were doing Abelian gauge theory, we would then derive an area law by expanding the exponential to Ath order (here A is the area enclosed by C), which would give us enough U_ls from the exponential to cancel those from the Wilson loop.

However, with non-Abelian gauge theories, we can do something different. Physically, this is because gluons are charged, so that in non-Abelian gauge theories, the gauge field

[86] The action is $\beta\sum_\Box(\dots)$. For large β we can fix a gauge in which the product of Us around a plaquette goes to $e^{-a^2F_{\mu\nu}+\cdots}$, where $\mu\nu$ label the plane the plaquette is in and a is the lattice spacing (and $U_l = e^{i\int_l A} \approx \mathbf{1}$). This gives $S \approx \beta a^4 \sum_\Box \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$, and so (in four dimensions)

$$\frac{1}{g^2} \leftrightarrow \beta a^4. \tag{760}$$

The continuum limit is thus the weak coupling ($\beta\to\infty$) limit. Note that we can't get access to the strong-coupling side of the gauge theory; β has to remain large for the continuum formulation to work.

[87] This is needed so that the action is real: if $\sum'_\Box$ is a sum over all plaquettes without counting the orientation separately, then

$$\sum_\Box \mathrm{Tr}_f \prod_{l\in\partial\Box} U_l = \sum_\Box{}' \mathrm{Tr}_f\left(\prod_{l\in\partial\Box} U_l + \prod_{l\in\partial\Box} U_l^*\right), \tag{762}$$

which is real (if the gauge group is real, like $SO(n)$, then a term like $\mathrm{Tr}_f(U_1U_2U_3U_4)$ has a partner $\mathrm{Tr}_f(U_4^{-1}U_3^{-1}U_2^{-1}U_1^{-1}) = \mathrm{Tr}_f[(U_1U_2U_3U_4)^T]$, and so the sum over orientations is redundant.

itself can screen sources. This allows us to form tubes of glue around the Wilson loop, which can potentially screen it, depending on its charge.

For concreteness, consider a 3D (Euclidean) theory, with a Wilson loop inserted as above in a representation R. Now form a tube out of cubes, with the Wilson line located along one of the sharp edges of the tube. A section of this tube is shown in Figure 4. This tube will appear at order β^{4P} in perturbation theory, where P is the perimeter of the Wilson line (each corner contributes an extra β^4).

This tube will screen the Wilson line if

$$R^* \in f \otimes f^*. \tag{763}$$

If the Wilson line weren't there, the tube would give a non-zero contribution to the partition function since each U_l on an edge appears with a corresponding U_l^* from a neighboring plaquette, allowing the integral over g_l to be nonzero. This is because $1 \in f \otimes f^*$. If $R^* \in f \otimes f^*$ then $1 \in R \otimes f^* \otimes f$, and so with a Wilson line with such an R can be added in the position shown without making the result vanish under integration.

To argue this precisely, we need to recall the orthogonality relation[88]

$$\int \mathcal{D}g\, [U_R(g)]^i_j [U_{R'}(g)]^k_l = \frac{1}{\dim R} \delta_{R^*,R'} \delta^i_l \delta^k_j. \tag{767}$$

The LHS is basically a group average over all similarity transforms of the matrix $[E^k_l]^i_j = \delta^i_l \delta^k_j$, and the RHS tells us that this average is zero if E^k_l has the 1 off of the diagonal, while E^k_l averages out to ($1/\dim R$ times) the identity if the 1 is on the diagonal. In particular, $\int \mathcal{D}g\, [U_R(g)]^i_j = 0$ unless $R = 1$.

Now consider expanding the expectation value to the order of β at which the tube geometry appears. We have

$$\langle W_R(C) \rangle = \int \prod_l \mathcal{D}g_l \, \mathrm{Tr}_R \left[\prod_{l \in C} U_R(g_l) \right] \prod_{\square \in T} \mathrm{Tr}[U_\square], \tag{768}$$

where T is the tube. Combining these into one trace,

$$\langle W_R(C) \rangle = \int \prod_l \mathcal{D}g_l \, \mathrm{Tr} \left[\prod_{l \in C} U_R(g_l) \otimes \bigotimes_{\square \in T} U_\square \right]. \tag{769}$$

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[88] The proof goes as follows: using the invariance of the Harr measure under shifts, one shows that

$$U_R(h) \int \mathcal{D}g\, U_R(g) E^k_l U_{(R')^*}(g^{-1}) = \left(\int \mathcal{D}G\, U_R(g) E^k_l U_{(R')^*}(g^{-1}) \right) U_{(R')^*}(h), \tag{764}$$

where $[E^k_l]^i_j = \delta^i_l \delta^k_j$. Therefore using Schur's lemma, since both R, R' were assumed to be irreducible,

$$\int \mathcal{D}g\, [U_R(g) E^k_l U_{R'}(g)]^i_j = \delta_{R^*,R'} \delta^i_j C(E^k_l), \tag{765}$$

where $C(E^k_l)$ is a constant. If we set $R^* = R'$, take the trace of both sides, and use the cyclicity of the trace, we get

$$\mathrm{Tr}[E^k_l] = \dim(R) C(E^k_l) \implies C(E^k_l) = \frac{1}{\dim R} \delta^k_l. \tag{766}$$

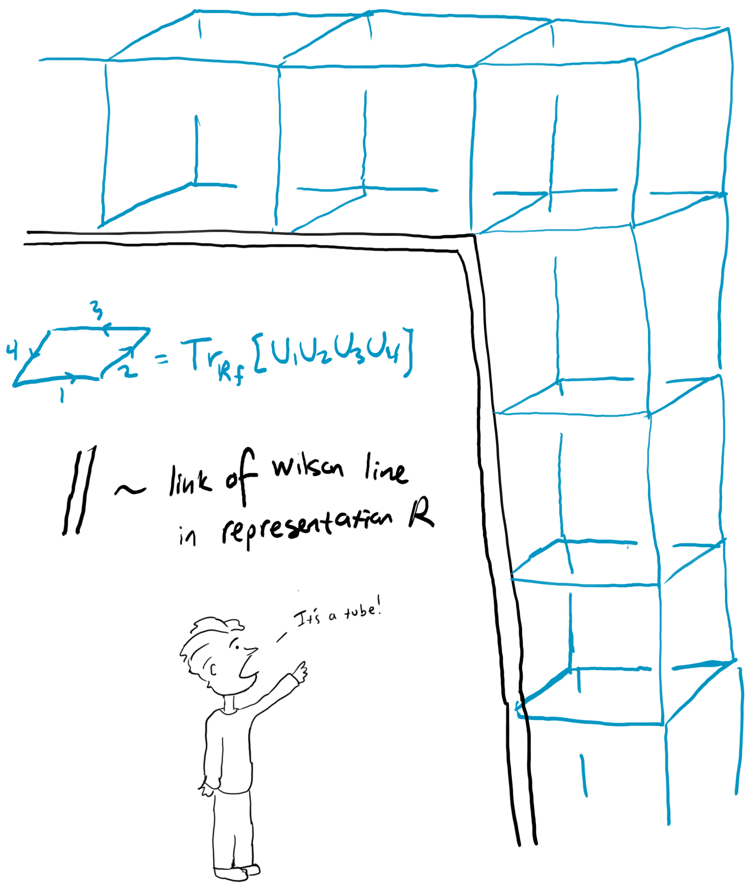

Figure 4: Geometry for how a tube of glue can screen a Wilson line.

Let us isolate just the terms that depend on g_1, where g_1 is the first link of the Wilson line:

$$\langle W_R(C)\rangle = \int \prod_l \mathcal{D}g_l \operatorname{Tr}\Bigg[(U_R(g_1)\otimes U(g_1)\otimes U^*(g_1)\otimes \mathbf{1})\cdot\Bigg(\prod_{l>1} U_R(g_l)\otimes \prod_{l\in\square_1\setminus l_1} U_l \\ \otimes \prod_{l\in\square_1'\setminus l_1} U_l \otimes \bigotimes_{T\ni\square\neq\square_1,\square_1'} U_\square\Bigg)\Bigg]. \tag{770}$$

Oh god, the notation. Sorry. Here, $\square_1, \square_1'$ are the two plaquettes in T that have the link l_1 as a side, and $\mathbf{1}$ is the identity acting on the tensor factors of every plaquette but these two.

Now let S be the matrix such that

$$S(U_R(g_1)\otimes U(g_1)\otimes U^*(g_1))S^{-1} = \bigoplus_{R_i\in R\otimes f\otimes f^*} U_{R_i}(g_1). \tag{771}$$

For example, for $1/2\otimes 1/2$ in $SU(2)$, we have (see this footnote for a reminder about how to decompose $\otimes$s of irreps[89])

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}, \tag{774}$$

which takes the basis $\chi^i\otimes\psi^j$ to the basis $(\chi^0\psi^0,\ [\chi^0\psi^1+\chi^1\psi^0]/\sqrt{2},\ \chi^1\psi^1,\ [\chi^0\psi^1-\chi^1\psi^0]/\sqrt{2})^T$. Actually, since we will want to calculate decompositions involving $f\otimes f^*$ and not $f\otimes f$ (in $SU(2)$ they happen to be isomorphic), we will want to work e.g. in the basis $\chi^i\otimes\psi_j$ instead. The matrix which takes the $\chi^i\otimes\psi_j$ basis to the $(\chi^0\psi_1,\ [\chi^0\psi_0-\chi^1\psi_1]/\sqrt{2},\ -\chi^1\psi_0,\ [\chi^0\psi_0+\chi^1\psi_1]/\sqrt{2})^T$ basis is

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix} \tag{775}$$

[89] The general strategy is to take a tensor product of tensors transforming in two given representations, and then look for invariant subspaces among the collection of tensors in the tensor product, with each invariant space constituting a term in the direct sum decomposition. For example, we can work out $3\otimes 8$ in $SU(3)$, with 3 the fundamental and 8 the adjoint. Consider then $\chi^i A^j_k$. First, we can contract the i with the k, obtaining a single fundamental index and implying a 3 in the $\oplus$ decomposition. Next, after taking out the contracted piece, form $S^{ij}_k = \chi^{(i}A^{j)}_k$ and $A^{ij}_k = \chi^{[i}A^{j]}_k$. We can now contract both of these with ϵ_{lmn}. The former dies, and hence gives us an irrep. With two symmetrized upper indices and one lower index, we naively have an 18-dimensional irrep. But we have to remember that we have taken out the triplet that came from the contraction, and so $S^{kj}_k = 0\,\forall j$. This means that the S tensors define a $18-3=15$ dimensional irrep. When A^{ij}_k is contracted with the epsilon symbol, we get the tensor $\widetilde{A}_{kn} = \epsilon_{nij}A^{ij}_k$. Contracting this again with ϵ^{nlm}, we get

$$\epsilon^{nlm}\epsilon_{nij}A^{ij}_k = (\delta^l_i\delta^j_m - \delta^m_i\delta^l_j)A^{ij}_k = 0, \tag{772}$$

since the trace part of A^{ij}_k has already been subtracted out. Thus $\widetilde{A}_{kn}$ is symmetric in its indices, and hence transforms as the 6^* irrep. Summarizing,

$$3\otimes 8 = 3\oplus 6^*\oplus 15. \tag{773}$$

Note that the trivial representation appears symmetrically in this basis instead of representation, since it is just the trace.

Now consider the integration over g_1. Since all matrix elements of any representation other than the trivial one have zero average over the group, we will only get something nonzero if the trivial representation appears in the above $\oplus$. We can then (not worrying about constant factors) do the g_1 integral and write

$$\langle W_R(C)\rangle = \int \prod_{l>1} \mathcal{D}g_l \mathrm{Tr}\left[\left(1 \oplus \bigoplus_{R_i \in R\otimes f\otimes f^*} 0_{d_{R_i}\times d_{R_i}}\right) S(\cdots)S^{-1}\right], \tag{776}$$

where $\cdots$ are the terms not containing g_1. We then write

$$S^{-1}\left(1 \oplus \bigoplus_{R_i \in R\otimes f\otimes f^*} 0_{d_{R_i}\times d_{R_i}}\right) S = \Pi_{\mathbf{1}}^{R\otimes f\otimes f^*}, \tag{777}$$

where $\Pi_{\mathbf{1}}^{R\otimes f\otimes f^*}$ is the projector onto the trivial representation in $R\otimes f\otimes f^*$, expressed in the $\otimes$ basis. For example, in the $SU(2)$ example,

$$\Pi_{\mathbf{1}}^{1/2\otimes 1/2^*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{778}$$

Therefore we have

$$\langle W_R(C)\rangle = \int \prod_{l>1} \mathcal{D}g_l \mathrm{Tr}\left[(\Pi_{\mathbf{1}} \otimes \mathbf{1})\cdot(\cdots)\right], \tag{779}$$

and we have successfully done the g_1 integral, getting a nonzero result.

Now we repeat this for all $g_l, l \in C$. Each integration gives us a factor of $\Pi_{\mathbf{1}}^{R\otimes f\otimes f^*}$ in the trace. We then do the integrals over the g_l for l a link in the tube not containing the Wilson line. These integrations make $\Pi_{\mathbf{1}}^{f\otimes f^*}$ matrices appear in the trace. When all is said and done, we get a trace over a bunch of products of things like $\mathbf{1}\otimes\ldots\otimes\mathbf{1}\otimes\Pi_{\mathbf{1}}^{R\otimes f\otimes f^*}\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}$ and $\mathbf{1}\otimes\ldots\otimes\mathbf{1}\otimes\Pi_{\mathbf{1}}^{f\otimes f^*}\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}$. These are giant matrices, and because of the projectors they are mostly made out of zeros. However, the trace does not vanish. Indeed, each matrix in the product always has a 1 in the upper-right hand corner: in the basis of the tensor product, the first basis vector in the Hilbert space that the matrix in the trace acts on is $T^{0\ldots0}_{0\ldots0}$, where T is the tensor product of the basis vectors for every single $\otimes$ factor. Since this entry always appears in the trace (the trivial representation), every matrix in the product appearing in the trace must not annihilate this vector; hence their product does not either. This gives us a positive, nonzero contribution to $\langle W_R(C)\rangle$. Since in this basis the matrix elements of the $\Pi_{\mathbf{1}}$ projectors are only either 0 or 1, this contribution cannot be canceled by any other parts of the trace, and so

$$\langle W_R(C)\rangle \sim \mathcal{N}\beta^{4P}, \tag{780}$$

where $\mathcal{N}$ is some number depending on R, the gauge group, etc. Therefore the Wilson line has a perimeter law, and is screened.

So, the Wilson line will get screened if $R^* \in f \otimes f^*$, or equivalently if $R \in f \otimes f^*$, since $f \otimes f^*$ is invariant under conjugation. In particular, adjoint Wilson lines are always screened, since $f \otimes f^* \otimes A \ni 1$, which can be proved either by noting that $A \in f \otimes f^*$ and recalling that A is self-dual[90], or by noting that the generator matrices $[T^a]^i_j$ constitute invariant symbols with one adjoint index, one fundamental index, and one anti-fundamental index.[91] Physically this is intuitive since sources that come from tensor products of the adjoints should be able to be screened by a sufficient number of gluons, which are in the adjoint.

However, something more general is true: if R is any representation appearing in $(f \otimes f^*)^{\otimes N}$, for any N, then R is screened. Mathematically, this follows from simply repeating the above procedure, but going to β^{4NP} in perturbation theory, where an N-sheeted tube along the Wilson line appears. This N-sheeted tube will then be enough to screen the Wilson line.

The irreps that appear in the $\oplus$ decomposition of $(f \otimes f^*)^N$ are precisely those which transform trivially under the center of the gauge group. To put it another way, let G be the gauge group (and f its fundamental representation), $\widetilde{G}$ be its universal cover, and let $G' = \widetilde{G}/Z(\widetilde{G})$. Then the claim is that irreps appearing in the $\oplus$ decomposition of

$$\langle f \otimes f^* \rangle = \{(f \otimes f^*)^{\otimes N} \,|\, N \in \mathbb{N}\} \tag{783}$$

are precisely the irreps of G'.

To show this pedantically, note that any irrep in $\langle f \otimes f^* \rangle$ transforms trivially under $Z(\widetilde{G})$, and hence is an irrep of G', and so $\langle f \otimes f^* \rangle \subset \mathrm{Rep}(G')$. Conversely, any irrep of G' will appear in $\langle f \otimes f^* \rangle$. To show this, first note that the irreps of G' are a subset of the irreps of G, the later of which is generated by taking tensor powers of f. Thus all the irreps of G' will appear in the decomposition of some $f^{\otimes N}$ for some N. Let $m = |Z(G)|$. Then since irreps of G' are invariant under $Z(G)$, every irrep of G' appears in the decomposition of $f^{\otimes mk}$, for some $k \in \mathbb{N}$. Furthermore, $1 \in f^{\otimes m}$, and so likewise $1 \in (f^*)^{\otimes m}$. Thus $(f \otimes f^*)^{mk} \supset f^{\otimes mk} \otimes \mathbf{1}^{\otimes k}$, and so $\mathrm{Rep}(G') \subset \langle f \otimes f^* \rangle$. Therefore we actually have that $\mathrm{Rep}(G') = \langle f \otimes f^* \rangle$.

Summing up, we can say that a Wilson line in a representation R will be screened iff $R \in \mathrm{Rep}(G')$. Again, physically this follows from the fact that the gluons are in the adjoint, and so the tube of glue can only screen things that transform trivially under $Z(G)$.

For example, if $G = SU(2)$, then integer-spin Wilson lines are screened, while half-odd-integer spin lines are not. In general, for $G = SU(N)$, $N \otimes N^* = \mathbf{1} \oplus A$, and so any irrep

[90] The adjoint representation is always self-dual. Indeed, from two tensors $\chi^i_j \eta^l_m$, we can always contract both pairs of indices to form a singlet, and so $A \otimes A \ni 1$. Alternatively, one could note that the Killing form on the Lie algebra provides an isomorphism between A and its dual.

[91] Actually something more general is true, namely $R \otimes R^* \otimes A \ni 1$ for any irrep R. Indeed, consider an infinitesimal transformation by $U = \mathbf{1} + i\theta^b T^b_R$. Then thinking of a in T^a as an adjoint index, to $O(\theta)$ we have

$$\begin{aligned}[T^a]^i_j \mapsto &(\delta^i_k + i\theta^b [T^b_R]^i_k)(\delta^{ad} + i\theta^b [T^b_A]^{ad})[T^d_R]^k_l(\delta^l_j - i\theta^b [T^b_R]^l_j) \\ &= [T^a]^i_j + i\theta^b \left([T^b_R, T^a_R]^i_j + (-if^{bad})[T^d_R]^i_j\right) \\ &= [T^a]^i_j,\end{aligned} \tag{781}$$

since $[T^a_A]^{bc} = -if^{abc}$. In particular, taking $R = A$ and using the reality of A tells us that

$$A \otimes A = \mathbf{1} \oplus A \oplus \dots . \tag{782}$$

appearing in $\langle A \rangle$ will be screened. These are precisely the irreps coming from tensors with an equal number of upper and lower indices.

For example, in $SU(3)$, the $3, 3^*, 6, 6^*, 15, 15^*, \dots$ representations are un-screened, while e.g. the $A = 8, 10, 10^*, 27, \dots$ representations are screened. The adjoint representation $A = 8$ can be screened by a single tube of glue, but e.g. if $R = 10$ then a 2-sheeted tube will do the job: the two sheets provide[92]

$$(3 \otimes 3^*)^{\otimes 2} = (1 \oplus 8)^{\otimes 2} = 1^{\oplus 2} \oplus 8^{\oplus 4} \oplus 10 \oplus 10^* \oplus 27, \tag{785}$$

and the 10^* on the RHS gives us the representation needed to screen the Wilson line. Of course, for $G = PSU(N)$, all Wilson lines are screened.

So, we see that confinement is very much dependent on the topological properties of the gauge group! This is perhaps not too surprising, since some sort of monopole condensation is usually the mechanism whereby confinement occurs, and the different types of monopoles which can exist depend on what the topology of the gauge group is.

(note to self: is the tube geometry really needed? Here's an alternative geometry: first take any "wall" of plaquettes that have the Wilson line on one edge, and the overlay the wall with its dual, forming a double-layer wall with a factor of $\mathrm{Tr}_{f\otimes f^*}[U_\square \otimes U^*_\square]$ for each plaquette. If $R^* \in (f \otimes f^*)^N$, then we take N copies of this wall, which provides enough glue to screen the Wilson line. This lets us do screening with slightly fewer plaquettes, and also gives us an approach which works in two dimensions)

44 *April 13 — Robust characterization of 2+1D topological insulators*

This is a problem from Senthil's class. Consider a quantum spin Hall state on a spatial disk. We will look at what happens when a π flux is threaded through some plaquette in the disk.

[92]To derive this, we just need to calculate $8 \otimes 8$. From two adjoint tensors $A^i_j B^k_l$ we can take traces in two ways to create adjoints, and in one way to create a singlet, giving us $1 \oplus 8 \oplus 8$. Then we can take off the trace pieces and symmetrize / antisymmetrize the top two indices. Let $\widetilde{A}^{ij}_{kl}$ be antisymmetric in the upper two indices. Then form $\epsilon_{ijm}\widetilde{A}^{ij}_{kl}$, and contract with ϵ^{kln}; this gives zero since $\epsilon_{ijm}\epsilon^{kln}$ can be expanded as δ functions between the first and second triplet of indices, each term of which then vanishes by the tracelessness of $\widetilde{A}$. The same vanishing act happens if we instead contract with ϵ^{mkn} or ϵ^{mln}, and so $\epsilon_{ijm}\widetilde{A}^{ij}_{kl}$ is totally symmetric in mkl, and hence $\widetilde{A}^{ij}_{kl}$ gives us an irrep. A totally symmetric mkl gives us 10 independent tensors, and so this irrep is 10 dimensional. Antisymmetrizing the bottom two indices gives us another irrep, with three symmetric upper indices; this is also 10-dimensional. These two irreps are 10^* and 10, respectively. Note that 10 and 10^* are distinct, despite being invariant under the $\mathbb{Z}_3$ center of the gauge group (this is weird?); this is because the contraction $\widetilde{A}^{ij}_{kl}\widetilde{A}^{kl}_{ij} = 0$ due to the mixed symmetry / antisymmetry of the upper and lower sets of indices. Finally, we can symmetrize both the upper and lower indices: this gives 36 index choices, but there are 3×3 constraints on them coming from tracelessness, hence this irrep is 27-dimensional (and self-dual). Putting all of these together, we get

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10^* \oplus 27. \tag{784}$$

We will a) show that as a π flux is adiabatically turned on, it nucleates no net charge, but a net spin of $\pm 1/2$. Since the bulk of the system obeys the spin-charge relation, this half spin but zero charge quantum number assignment is fractional. We will then b) use the edge theory to show that the π flux is a Kramer's doublet under time reversal.

Solution:

a) Consider a background electromagnetic field A, such that

$$dA = \frac{R(t)\pi}{T}\delta(\mathbf{x})dx \wedge dy, \tag{786}$$

where $R(t)$ is the ramp function

$$R(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < T \\ T & T < t \end{cases}. \tag{787}$$

A quantum spin Hall state is basically two copies of a Chern insulator with opposite Hall responses and opposite spin polarizations. Assume the $\uparrow$ Chern insulator has $C = 1$ and the $\downarrow$ has $C = -1$, wolog. The expectation value of the spin σ current is

$$j_{\mu,\sigma}(x) = \pm \frac{\delta}{\delta A^\mu(x)} S_{CS}, \tag{788}$$

where S_{CS} is the level-1 Chern-Simons action and the $+$ sign is for $\sigma =\uparrow$. Working in units where $\hbar = e = c = 1$, we have

$$j_{\mu,\sigma} = \pm \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda = \pm \frac{1}{2\pi}(\star F)_\mu. \tag{789}$$

The change in charge $Q_\sigma(R)$ for a spin component σ in a region R enclosing the origin during the time over which the flux is turned on (assuming the flux insertion is done adiabatically so that the gap is never closed and the CS electromagnetic response is valid for all t) is

$$\Delta Q_\sigma(R) = \int_{-\infty}^{\infty} dt \int_R d^2\mathbf{x}\, \partial_0 j^0 = \pm \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_R d^2\mathbf{x}\, \frac{(\Theta(t) - \Theta(t-T))\pi}{2T}\delta(\mathbf{x}) = \pm \frac{1}{2}. \tag{790}$$

Therefore the flux insertion leads to the flux gaining half-charge from the $\uparrow$ layer, and loosing half-charge from the $\downarrow$ layer. This results in the net acquisition of zero electric charge, but a net acquisition of $+1/2$ spin. If we were to instead send $R(t) \mapsto -R(t)$ and insert the flux in the time-reversed fashion, the flux would acquire $-1/2$ spin instead, consistent with $T : \sigma \mapsto -\sigma$.

b) The gapless modes on the edge are described by

$$H_e = \int dx\, (\psi_L^\dagger(i\partial_x + A_x)\psi_L - \psi_R^\dagger(i\partial_x + A_x)\psi_R). \tag{791}$$

We will take the fermions to have periodic boundary conditions in the absence of any flux.[93] There are two equivalent ways to proceed: one is to have the fermion momentum always be quantized in $\mathbb{Z}$ (taking the radius of the circle to be 1 wolog), and to have the gauge connection show up in the Hamiltonian. With this way of doing things, the boundary fermions are sections of a bundle with trivial transition functions, i.e., they are well-defined by using a single patch for the whole boundary circle. The other is to take the fermions to be defined on a bundle with two patches, related on one of their overlaps by a transition function $e^{i\oint A_x}$. In this case, we can schematically think of the momenta as being quantized in $\mathbb{Z} + \oint A_x/2\pi$, with the gauge field not appearing in H_e. Either way, writing $A_x = \Phi/2\pi$ where Φ is the flux, we see that the spectrum contains right-moving modes with dispersion $E_R = n - \Phi/2\pi$, $n \in \mathbb{Z}$, and left-moving modes with $E_L = n + \Phi/2\pi$, $n \in \mathbb{Z}$. So when $\Phi \in 2\pi\mathbb{Z}$ we have two zero modes, and when $\Phi \in \pi(2\mathbb{Z}+1)$, when the momenta are quantized in half-odd-integers, we have no zero modes.

When $\Phi = 0$, there are two zero modes; one L and one R. This gives four degenerate states, since each mode can either be filled or unfilled. C symmetry here reflects the spectrum, and so we can take it to act as $C : \psi \mapsto Y\psi^\dagger$, where $\psi = (\psi_L, \psi_R)^T$. This means that $C|L,R\rangle = Cf_L^\dagger f_R^\dagger|0,0\rangle = f_R f_L(C|0,0\rangle)$, and so $C|0,0\rangle = |L,R\rangle$ (a possible constant phase here cancels out in the end) and likewise $C|L,R\rangle = |0,0\rangle$. Using this and $Ce^{iQ} = e^{-iQ}C$ with $Q = Q_L + Q_R$ tells us that the charge of $|L,R\rangle$ is the negative of the charge of $|0,0\rangle$, and that the two states differ in charge by 2. Therefore $Q_{|L,R\rangle} = +1, Q_{|0,0\rangle} = -1$.

The two singly occupied states $f_{L/R}^\dagger|0,0\rangle$ are related by C, and have charge 0. For T acting as $T = Y\mathcal{K}$ on fermions, the two states form a Kramers doublet pair, since repeated application of T yields $T : f_L^\dagger \mapsto if_R^\dagger \mapsto -i(-if_L^\dagger) = -f_L^\dagger$, implying $T^2 = -\mathbf{1}$ on $f_{L/R}^\dagger|0,0\rangle$.

Now when we increase Φ to π, there is a unique ground state. Of the four degenerate modes at $\Phi = 0$, two (one L and one R) move to negative energy at $E = -1/2$, while the other two move to positive energy at $E = +1/2$. The ground state has the former two modes filled, and is not degenerate; hence while it is T symmetric as the filled states are $L \leftrightarrow R$ symmetric, it cannot be a Kramers doublet. Therefore changing the flux toggles the edge states between being a Kramers doublet and a Kramers singlet. Correspondingly, the π flux in the bulk is a Kramers doublet.

45 *April 14 —Two-site Hubbard model at half-filling*

This is another homework problem from Senthil's class. The problem is simple: solve the two-site Hubbard model at half-filling. Use conventions where the potential term is $H_U = U(n_1^2 + n_2^2)$.

Solution:

[93] This is slightly weird since the natural boundary conditions (i.e. the ones that can be smoothly extended into the disk) are anti-periodic, since only the spin structure with anti-periodic boundary conditions is bounding.

First, a note on notation: $\mathbf{1}_n$ means the $n \times n$ unit matrix; if there is no subscipt, the default is $\mathbf{1}_2$.

There are six states in the half-filled Hilbert space $\mathcal{H}_{1/2}$, which we label as

$$\mathcal{H}_{1/2} = \langle e_1, \dots e_6 \rangle, \quad e_1 = |\uparrow, \uparrow\rangle, e_2 = |\downarrow, \downarrow\rangle, e_3 = |\uparrow\downarrow, 0\rangle, e_4 = |0, \uparrow\downarrow\rangle, e_5 = |\uparrow, \downarrow\rangle, e_6 = |\downarrow, \uparrow\rangle. \tag{792}$$

Note that $|\uparrow\downarrow, 0\rangle$ and $|\downarrow\uparrow, 0\rangle$ are both perfectly good basis vectors; we will therefore find it helpful to fix conventions whereby at a single site, up spins always appear to the left of down spins in basis vectors. We will also fix conventions where states are created by a series of creation operators are always ordered in the left-to-right order obtained from the bra-ket notation of the states. For example,

$$|\uparrow\downarrow, 0\rangle = c^\dagger_{\uparrow 1} c^\dagger_{\downarrow 1} |0, 0\rangle, \qquad |\uparrow, \downarrow\rangle = c^\dagger_{\uparrow 1} c^\dagger_{\downarrow 2} |0, 0\rangle. \tag{793}$$

In this basis, the Hamiltonian is $H = H_t + H_U$, with

$$H_t = -t\left(0_{2\times 2} \oplus P\right), \qquad H_U = 2U[\mathbf{1}_2 \oplus (2\mathbf{1}_2) \oplus \mathbf{1}_2], \qquad P = \begin{pmatrix} & & 1 & -1 \\ & & 1 & -1 \\ 1 & 1 & & \\ -1 & -1 & & \end{pmatrix}. \tag{794}$$

The minus signs on H_t are super important, and come from keeping careful track about the ordering of creation operators in the second-quantized expressions of the various basis vectors.

Since H is diagonal in spin indices, it is $SU(2)$-symmetric and commutes with both S^2_{tot} and S^z_{tot}. In our basis, we find

$$S^z_{tot} = 2Z \oplus 0_{4\times 4}, \qquad S^2_{tot} = 2\mathbf{1}_2 \oplus 0_{2\times 2} \oplus (\mathbf{1}_2 + X). \tag{795}$$

The former is easy to see, while the latter can be found by noting that, restricted to the subspace $V = (e_1, e_5, e_6, e_1)$, S^2_{tot} is

$$S^2_{tot}|_V = (S^i \otimes \mathbf{1} + \mathbf{1} \otimes S^i)^2 = \frac{3}{2}\mathbf{1}_4 + \frac{1}{2}(X \otimes X + Y \otimes Y + Z \otimes Z) = 2 \oplus (\mathbf{1} + X) \oplus 2. \tag{796}$$

Using this and using $S^2_{tot} e_3 = S^2_{tot} e_4 = 0$,[94] we arrive at the above expression for S^2_{tot}.

S^z_{tot} is already diagonal, but we'd like to go into a basis in which S^2_{tot} is diagonal as well. This is easy: defining $e'_5 = (e_5 + e_6)/\sqrt{2}$ and $e'_6 = (e_5 - e_6)/\sqrt{2}$, we get, in the new basis,

$$S^2_{tot} = 2\mathbf{1} \oplus 0_{2\times 2} \oplus (\mathbf{1} + Z). \tag{797}$$

Therefore take $H \mapsto SHS^{-1}$, where

$$S = \mathbf{1}_4 \oplus \frac{1}{\sqrt{2}}(Z + X). \tag{798}$$

[94]Since $S^2_{tot} e_3 = |\uparrow\downarrow, 0\rangle + |\downarrow\uparrow, 0\rangle = 0$, and likewise for $S^2_{tot} e_4$.

Therefore

$$H_t \mapsto -t(\mathbf{1}_4 \oplus \frac{Z+X}{\sqrt{2}})(0_{2\times 2} \oplus P)(\mathbf{1}_4 \oplus \frac{Z+X}{\sqrt{2}}) = -t(0_{2\times 2} \oplus M), \tag{799}$$

where

$$M = \sqrt{2}\begin{pmatrix} & & & 1 \\ & & & 1 \\ & & & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{800}$$

H_U is invariant, since the Coulomb repulsion doesn't care about spin. To make things slightly nicer, define $e_0 = e_5'$, and change basis so that the basis vectors are ordered in sequence—this just moves the triplet which is annihilated by H_t up to the first basis vector. Then in this basis,

$$S^2_{tot} = 2\cdot\mathbf{1}_3 \oplus 0_{3\times 3}, \quad S^z_{tot} = 0_{1\times 1} \oplus Z \oplus 0_{3\times 3}, \quad H_t = -t(0_{3\times 3} \oplus M'), \quad H_U = 2U(\mathbf{1}_3 \oplus 2\cdot\mathbf{1}_2 \oplus 1), \tag{801}$$

where M' is M with the 3rd row and 3rd column removed. Letting $t' = \sqrt{2}t$, we have

$$H = 2U\mathbf{1}_3 \oplus \begin{pmatrix} 4U & 0 & -t' \\ 0 & 4U & -t' \\ -t' & -t' & 2U \end{pmatrix}. \tag{802}$$

Let $\mathcal{E} \equiv \sqrt{4t^2 + U^2}$. Then diagonalizing H gives

$$H = \mathcal{S}H_D\mathcal{S}^{-1}, \qquad H_D = 2U \cdot \mathbf{1}_3 \oplus \begin{pmatrix} 3U+\mathcal{E} & & \\ & 4U & \\ & & 3U-\mathcal{E} \end{pmatrix}, \tag{803}$$

where the transformation is accomplished with the matrix

$$\mathcal{S} = \mathbf{1}_3 \oplus \begin{pmatrix} t/(U-\mathcal{E}) & -1 & t/(U+\mathcal{E}) \\ t/(U-\mathcal{E}) & 1 & t/(U+\mathcal{E}) \\ 1 & 0 & 1 \end{pmatrix}. \tag{804}$$

The state with the lowest energy is e_5', a singlet, which has energy $3U-\mathcal{E}$. This is followed by the symmetric triplet e_0 and the two "jammed" triplet states e_1, e_2, the three of which are triply degenerate in energy, with energy $2U$. From our expression for S^2_{tot}, we verify that all three of these states are triplets. The next two states up have energy $4U$ and $3U+\mathcal{E}$, and are both singlets.

The energy difference between the ground state singlet and the excited triplet is, for the limit of $t/U \ll 1$,

$$\Delta E = \mathcal{E} - U \approx 2t^2/U. \tag{805}$$

From the matrix $\mathcal{S}$, we see that the ground state wavefunction is (after normalizing)

$$\psi_G = \frac{U+\mathcal{E}}{\sqrt{t^2 + (U+\mathcal{E})^2}}\left[\frac{t}{U+\mathcal{E}}(c^\dagger_{\uparrow 1}c^\dagger_{\downarrow 1} + c^\dagger_{\uparrow 2}c^\dagger_{\downarrow 2}) + \frac{1}{\sqrt{2}}(c^\dagger_{\uparrow 1}c^\dagger_{\downarrow 2} - c^\dagger_{\downarrow 1}c^\dagger_{\uparrow 2})\right]|0,0\rangle. \tag{806}$$

When $U \to \infty$ the doubly-occupied terms vanish as expected, and the singlet superposition becomes degenerate with the triplet superposition, since as $U \to \infty$ the AF exchange interaction that is responsible for the singlet having lower energy goes away.

46 *April 16 — Why is the periodic spin structure on S^1 non-bounding?*

Today we have a super short diary entry — just explaining the question in the title, in a down-to-earth way.

Solution:

Since $\mathrm{Spin}(1) = \mathbb{Z}_2$, and since there are two $\mathbb{Z}_2$ bundles over the circle, the circle has two spin stuctures. One spin structure corresponds to the trivial product bundle, and periodic boundary conditions, while the other corresponds to the "edge of a mobius strip" bundle, and has anti-periodic boundary conditions. It turns out that only the latter defines a spin structure which can be extended into a disk bounded by the circle.

To see this, note that a framing on S^1 is a choice of tangent vector $\pm\hat{\phi}$ at every point on S^1, when the S^1 is thought of as living in the plane. The P spin structure has a definite choice of tangent vector and trivial transition functions, while the AP ßis defined with two patches, with a transition on one of the patch overlaps that flips the framing by -1.

To see whether these extend into the disk, we draw the disk as two patches 1 and 2, with their overlap constituting a thin strip running along a diameter of the disk. The spin framing is restricted to be tangent to the disk boundary at the disk boundary, and the transition function g_{12} between the two patches is required to be $\mathbf{1}$ on the left edge of the overlap, and either $\mathbf{1}$ (P spin structure) or $-\mathbf{1}$ (AP spin structure) on the right edge of the overlap.

Now $\mathrm{Spin}(2) = U(1)$, so we can draw the spin framing in the interior of the disk as a vector field in the plane. Requiring that the framing be nonzero everywhere on each patch, given the requirement that the framing become tangential on the disk boundary, essentially fixes the framing on each patch to be as shown in Figure 5. Given that the transition function is $\mathbf{1}$ on the left side of the patch overlap, we see that the transition function must become a π rotation at the center of the disk, and then grow to a 2π rotation on the right edge of the disk. Now a 2π rotation in $\mathrm{Spin}(2)$ is identified with $-\mathbf{1}$, and so the transition function must be $-\mathbf{1}$ on the right edge of the disk (if we walk around the disk boundary, following the framing in the spin bundle, we must walk around twice before we get back to where we started).

This assignment for the transition function g_{12} at each point on the overlap is only compatible if the circle has the AP spin structure, and so we conclude that only the AP spin structurebounds. If we tried to get the P spin structureto bound, we'd run into a problem of defining the spin framing at the center of the disk.

47 *April 17 — Anomalous scale invariance in quantum mechanics*

Solution:

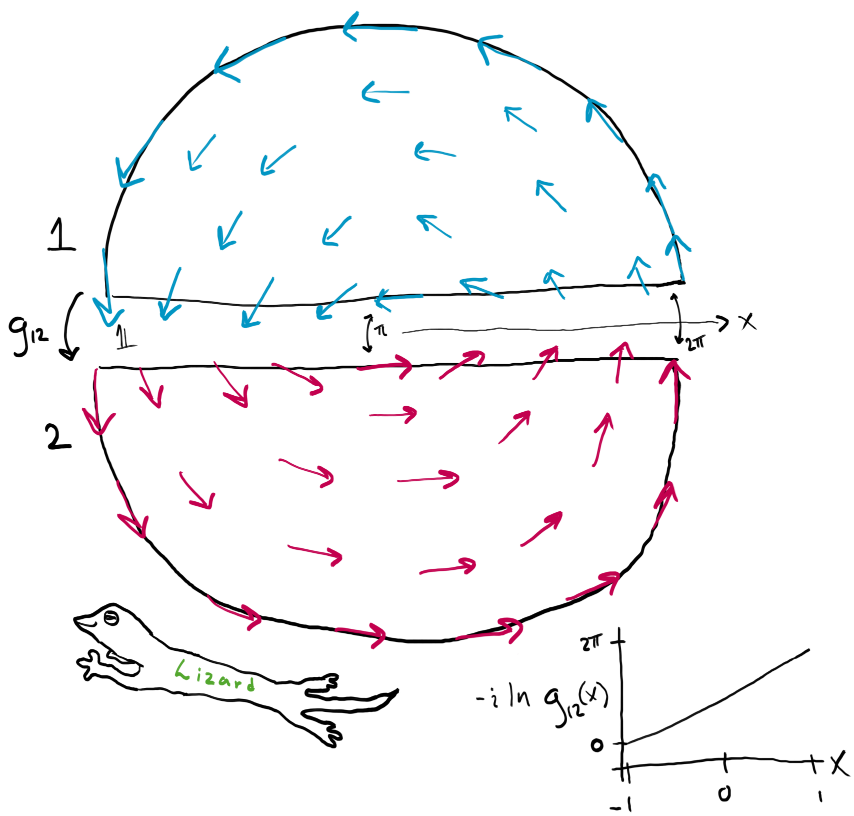

Figure 5: Spin framings on the two halves of the disk, the gluing function between them, and a lizard.

Consider a Schrodinger equation of the form

$$\left(-\nabla^2 - \frac{\alpha}{r^2}\right)\psi = i\partial_t \psi. \tag{807}$$

This equation is left invariant under the re-scaling $r \mapsto \gamma r$, $t \mapsto \gamma^2 t$. Note that this scale invariance depends on the potential scaling as $1/r^2$, so that α is dimensionless. The scaling is generated by the operator $\mathcal{O} = \mathbf{r}\cdot\nabla + 2t\partial_t$, which does an anisotropic dilitaion in spacetime. The symmetry also gives rise to a conserved quantity, $\rho \equiv -\frac{1}{2}\mathbf{r}\cdot\nabla + t\partial_t$, since

$$[\rho, Ht] = 0, \tag{808}$$

meaning that ρ commutes with the time evolution operator e^{-iHt} and hence is conserved.

Scale invariance here manifests itself in the fact that the S-matrix is independent of E. Indeed, this can already be seen in the WKB approximation, where the phase shift in a wavepacket passing through the potential relative to the shift in the absence of the potential is

$$\delta = 2\int_0^\infty dr\,\left(\sqrt{E+\alpha/r^2} - \sqrt{E}\right) = 2\int_0^\infty dr\,\left(\sqrt{1+\alpha/r^2} - 1\right), \tag{809}$$

which (although divergent) is independent of E.

In two dimensions we have another option for a scale-invariant problem: the δ function well:

$$\left(-\nabla^2 - \alpha\delta(\mathbf{r})\right)\psi = i\partial_t\psi. \tag{810}$$

Here the binding energy of the bound state depends on a regulator in the same way that e.g. the BCS gap or the mass in $1+1D$ large N σ models does: we have

$$k^2\psi_k - \alpha\psi(0) = E\psi_k \implies \frac{1}{\alpha} = \int d^2k \frac{1}{k^2 - E} = \pi\ln(\Lambda/E) \implies E = \Lambda e^{-1/\pi\alpha}, \tag{811}$$

where Λ is our UV cutoff. Thus even in quantum mechanics, divergences can manifest themselves in ways similar to the ones in QFT.

48 *April 20 — The effective potential and thermodynamics*

I've always found the manipulations regarding the definition of the effective potential to be confusing, since (as of writing) the way the Legendre transform is usually presented doesn't make a lot of sense to me. Today, in an attempt to clarify things, we'll therefore look at analogies with thermodynamics, and how the whole construction is really just one of Lagrange multipliers.

Solution:

First let's remind ourselves of how it works in thermodynamics. Our goal is to maximize the entropy, subjected in e.g. the microcanonical picture to the constraints that $\sum E_i p_i = E$ and $\sum_i p_i = 1$, where p_i is the weight of the ith state. Then we add a Lagrange multiplier we'll call $-\beta$ for the energy constraint and one we'll call $-\ln Z+1$ for the normalization, and try to extremize

$$-\sum_i p_i \ln p_i - \beta\left(\sum_i p_i E_i - E\right) + (-\ln Z+1)\left(\sum_i p_i - 1\right). \tag{812}$$

Varying wrt p_i tells us that

$$-\ln p_i - \beta E_i - \ln Z = 0 \implies p_i = \frac{e^{-\beta E_i}}{Z}. \tag{813}$$

Okay, duh. No surprises here.

Therefore, we see that the current plays the role of a Lagrange multiplier enforcing the expectation value of ϕ be φ. Just as the temperature is an implicit function of the average energy, the current is an implicit function of the vev of the field to which it couples.

Summarizing, the correspondence is

$$\begin{aligned} J &\leftrightarrow \beta \\ \Gamma[\varphi] &\leftrightarrow E \end{aligned} \tag{814}$$

49 *April 21 — Higgs effective potential*

This is a problem from Schwartz's QFT book, chapter 34. Consider a theory where a scalar field ϕ is coupled to some other fields. We want to calculate, to 1-loop, the contribution of these other fields to the effective potential for ϕ. As an example, consider the example where the other field is a fermion ψ, with the action

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi - V(\phi) + i\bar{\psi}\partial\!\!\!/\psi - Y\phi\bar{\psi}\psi. \tag{815}$$

What is the effective potential for the ϕ field, including the contributions from the fermions and the self-coupling of the ϕ field? How does this generalize when we couple ϕ to arbitrarily many different fields?

Solution:

First let's recall how the effective action approach works, working to 1-loop order. We will first include a current term $\int J\phi$ in the action, and then expand the action about the vev of ϕ, writing $\phi \mapsto \eta + \varphi$, with $\varphi = \langle\phi\rangle_J$. The generating functional of connected correlation functions is then

$$e^{-iW[J]} = \int \mathcal{D}\eta \exp\left(i\int\left[\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi+\eta] + (J_1+\delta J)(\varphi+\eta) + \frac{\delta\mathcal{L}}{\delta\phi}\Big|_\varphi \eta + \frac{1}{2}\int \eta\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi \eta\right]\right) \tag{816}$$

Here $\delta\mathcal{L}$ contains counterterms, and we've stopped the expansion of the Lagrangian at quadratic order since we will only be interested in 1-loop effects when integrating out η.

As in P&S, J_1 is defined to be the current such that $\delta_\phi \mathcal{L}|_\varphi = -J_1$, so that J_1 is the current which in the classical limit gives an expectation value of φ. We then adjust the "counterterm" δJ order-by-order in perturbation theory to ensure that the expectation value of ϕ in the presence of the current really is φ, i.e. that the expectation value is not changed by a nonzero $\langle\eta\rangle$. This means that δJ is chosen to precisely cancel the tadpole diagrams involving η, and so we can write

$$e^{-iW[J]} = \int \mathcal{D}\eta \exp\left(i\int\left[\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi+\eta] + (J_1+\delta J)\varphi + \frac{1}{2}\int \eta \frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi \eta\right]\right) \tag{817}$$

Doing the integral over η gives

$$-W[J] = \int\left(\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi] + J\varphi + \frac{i}{2}\mathrm{Tr}\left[\ln -\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi\right]\right), \tag{818}$$

Using the definition of the effective action as $\Gamma[\varphi] + W[J] = -\int J\varphi$, we get

$$\Gamma[\varphi] = \int\left(\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi] + \frac{i}{2}\mathrm{Tr}\left[\ln -\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_\varphi\right]\right), \tag{819}$$

which is independent of J as required (and which again is only correct to 1-loop).

Let's now consider what happens when we have a scalar coupled via a Yukawa coupling to a massless fermion, with the Lagrangian written in the problem statement. Taking $\phi \mapsto \varphi+\eta$ and then dropping the terms linear in η (since they get canceled in the effective action by the condition on the current), we see that after doing the integral over the fermions,

$$\Gamma[\varphi] = \int\left(\frac{1}{2}\varphi\Box\varphi - V(\varphi) + \delta\mathcal{L}[\varphi] + \frac{i}{2}\mathrm{Tr}\left[\ln\left(-\Box + \frac{\delta^2 V}{\delta\phi\delta\phi}\Big|_\varphi\right)\right] - i\mathrm{Tr}\left[\ln(-i\partial\!\!\!/ + Y\varphi)\right]\right). \tag{820}$$

Taking φ to be a constant lets us evaluate the traces. Let

$$m_\varphi^2 \equiv V''(\varphi) \tag{821}$$

be the effectivce mass-squared of the φ field (it may or may not be positive). Then the bosonic trace is (here V is the spacetime volume)

$$\mathrm{Tr}[\ln(-\Box + m_\phi^2)] = VI, \qquad I = \int_p \ln(-p^2+m_\phi^2). \tag{822}$$

The integral becomes convergent if we go to Euclidean time and differentiate it three times wrt m_ϕ^2:

$$\partial^3_{m_\phi^2} I \to i\frac{2\cdot 2\pi^2}{16\pi^4}\int_0^\infty dp\,\frac{p^3}{(p^2+m_\phi^2)^3} = \frac{i}{8\pi^2}\int_0^\infty du\frac{u}{(u+m_\phi^2)^3} = \frac{1}{16\pi^2 m^2}, \tag{823}$$

where the $\to$ means that we rotated to $i\mathbb{R}$ time to do the integral. Integrating three times then gives

$$I = \frac{i}{16\pi^2}(Am_\varphi^4 + Bm_\varphi^2 + C + [m_\varphi^4/2]\ln(m_\varphi^2/\phi_R^2)), \tag{824}$$

where A, B, C are some (infinite) constants—they will be dealt with using the counterterms. ϕ_R is a dimensionful scale introduced during the renormalization process, and is fixed as one of our renormalization conditions. Since the trace we just computed appeared in the effective action with a 1/2 coefficient, the effective potential is ($V_{eff}[\varphi] = \Gamma[\varphi]/V$)

$$V_{eff}[\varphi] = C + V_R[\varphi] + \frac{1}{64\pi^2} m_\varphi^4 \ln(m_\varphi^2/\phi_R^2) + \dots, \tag{825}$$

where C is a cosmological constant, $V_R[\varphi]$ is the renormalized potential (into which the terms $Am_\varphi^4 + Bm_\varphi^2$ have been absorbed), and the $\dots$ signify the contribution from the fermions (note that C and the renormalized parameters in $V_R[\varphi]$ will need to be adjusted further after we calculate the fermion contribution, which will have its own divergences as well).

Now we need to deal with the fermion contribution. We write (dropping an infinite constant)

$$\mathrm{Tr}[\ln(-i\partial\!\!\!/ + Y\varphi)] = V \int_p \mathrm{Tr}[\ln(1 - p\!\!\!/ Y\varphi/p^2)] = V \int_p \sum_{n=0}^{\infty} \frac{1}{n} \mathrm{Tr}[(p\!\!\!//p^2)^n](Y\varphi)^n. \tag{826}$$

Now the trace can be evaluated as[95]

$$\mathrm{Tr}[(p\!\!\!//p^2)^n] = \begin{cases} 4p^{-n}, & n \in 2\mathbb{Z} \\ 0 & n \in 2\mathbb{Z}+1 \end{cases}. \tag{830}$$

Therefore

$$\mathrm{Tr}[\ln(-i\partial\!\!\!/ + Y\varphi)] = 4V \int_p \sum_{n \in 2\mathbb{Z}^{\geq 0}} \frac{1}{n} (Y\varphi/p)^n = 2V \int_p \ln(1 - Y^2\varphi^2/p^2) = 2V \int_p \ln(-p^2 + Y^2\varphi^2), \tag{831}$$

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[95] "Proof": if n is odd then we are taking a trace of an odd number of γ matrices, which vanishes since we are in four dimensions. If n is even, we have to calculate the sum

$$\sum_{\mu_1 \cdots \mu_n} \mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] p_{\mu_1} \cdots p_{\mu_n}. \tag{827}$$

Now this sum will only be nonzero if all of the μ_i group off in pairs, so that $\mu_i = \mu_j$ for some pair i, j. The trace will then produce $\pm\mathrm{Tr}[\mathbf{1}] = \pm 4$, depending on the way in which the indices get paired up. The number of ways N_n to pair up indices for a given n can be calculated inductively. For $n = 2$, there is only one way ($\mu_1 = \mu_2$), means $N_2 = 1$ and $\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}] = 4\eta^{\mu_1\mu_2}$. For $n = k+2$, the μ_1 index can be paired with $k+1$ different other indices. Once this decision is made, the remaining k indices can pair with each other in N_k ways, and so using the base case, we see that

$$N_{k+2} = (k+1)N_k \implies N_n = \prod_{i=0}^{n/2-1} (2i+1). \tag{828}$$

Now most of these N_n different pairings give different signs for the trace, and cancel out. However, since N_n is always odd, this cancellation is never complete, and always leaves behind one term. So then

$$\sum_{\mu_1 \cdots \mu_n} \mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] p_{\mu_1} \cdots p_{\mu_n} = 4\eta^{\mu_1\mu_2} \cdots \eta^{\mu_{n-1}\mu_n} p_{\mu_1} \cdots p_{\mu_n} = 4p^n. \tag{829}$$

where we "un-dropped" the infinite constant from $\mathrm{Tr}\ln i\partial\!\!\!/$ in the last step (this whole rigamarole of expanding the log and then un-expanding it was just to deal with the spin trace). Thus we are back to calculating the same integral as we did when we integrated out the bosonic field: we get

$$\mathrm{Tr}[\ln(i\partial\!\!\!/ - Y\varphi)] = \frac{1}{16\pi^2}(Y\varphi)^4 \ln\frac{(Y\varphi)^2}{\Lambda^2} + \dots, \tag{832}$$

where the ... are polynomials in $Y\varphi$ with infinite coefficients that will go into determining the correct counterterms to be used for determining the renormalized coupling constants. Keeping track of the sign that the functional determinant entered into $\Gamma[\varphi]$ with, we then have

$$V_{eff}[\varphi] = C' + V_R[\varphi] + \frac{1}{64\pi^2}m_\varphi^4\ln(m_\varphi^2/\phi_R^2) - \frac{1}{16\pi^2}(Y\varphi)^4\ln(Y^2\varphi^2/\phi_R^2). \tag{833}$$

Note how the contribution from the fermion has a numerical coefficient that is a factor of 4 greater than the boson one, and is negative. The factor of 4 ultimately comes from the fact that a Dirac fermion has 4 components; the minus sign is from the properties of fermionic functional integration. From these remarks, it is clear how to generalize the above potential to include arbitrarily many fields that couple to ϕ: let Ξ_i be a field that couples to ϕ through some interaction $\mathcal{L}_{\Xi_i\phi}$. Then after we expand about φ and drop tadpoles, to one-loop the integral over Ξ_i will produce a functional determinant, with the analogue of m_φ or $Y\varphi$ being defined as $m_{\Xi_i} = (\delta^2\mathcal{L}_{\Xi_i\phi}/\delta\phi\delta\phi)|_\varphi$ if Ξ_i is a fermion (with Yukawa coupling to ϕ) or $m_{\Xi_i}^2 = (\delta^2\mathcal{L}_{\Xi_i\phi}/\delta\phi\delta\phi)|_\varphi$ if Ξ_i is a boson, with a coupling to ϕ like $\mathcal{L}_{\Xi_i\phi} \sim \phi^2\Xi_i^2 + \dots$. Then the effective potential becomes

$$V_{eff}[\varphi] = C + V_R[\varphi] + \sum_i (-1)^{2\sigma_i}\frac{n_i}{64\pi^2}m_{\Xi_i}^4\ln\frac{m_{\Xi_i}^2}{\phi_R^2}, \tag{834}$$

where σ_i is the spin of Ξ_i and n_i is the number of real dof that Ξ_i carries (e.g. 1 for a boson, 8 for an $SU(2)$ fundamental Dirac fermion, etc.).

One more helpful remark to make about the effective potential. Recall that the physical meaning of $V_{eff}[\varphi]$ is the minimum energy density state of the theory, given that the expectation value of ϕ is fixed at φ. Depending on the choice of potential $V_R[\phi]$, there is no reason that m_φ^2 (going back to the case of a single scalar field) should always be positive—indeed, for the usual Mexican hat potential, it is negative near $\varphi = 0$. Then from the above, we see that the argument of the logarithm is negative, and we get an imaginary effective potential. What does this mean? Actually, the meaning is quite physical: an imaginary part means that the time evolution factor $e^{-iV_{eff}[\varphi]T}$ is exponentially damped[96], which means the state with $\langle\phi\rangle = \varphi$ is unstable. This is totally reasonable, since e.g. for the Mexican hat potential, this tells us that regions where $V''[\varphi] < 0$ are unstable: we can't have a theory where the field has zero expectation value, since such a theory is unstable. What's happening here is the ϕ^4 analogue of particle production by strong electric fields with $\langle E^2\rangle > 2m_ec^2$: forcing the vev of the electric field to be too high results in an unstable state, and screening will occur

[96] to check that the sign is such that it is indeed damped, we need to go back and keep track of $i\epsilon$ factors. However clearly the opposite possibility, viz. that it grows exponentially, is obviously not physical.

until the vev of E^2 is brought down. This is a quantum effect, so we needed to calculate the 1-loop contribution to $V_{eff}[\varphi]$ in order to see it.

As another super simple example of how this works, consider a quantum particle moving in a potential $V(x)$. Following the procedure above, we get an effective potential of (here x, like φ, is just a number, not a coordinate to be integrated over in the path integral)

$$V_{eff}(x) = V(x) + \frac{1}{2}\int dk\, \ln(k^2 + V''(x)). \tag{835}$$

The integral is divergent but can be made convergent by differentiating once with respect to V''. Then integrating with respect to V'', we get (may or may not have gotten numerical factors right)

$$V_{eff} = V_R(x) + \frac{1}{16\sqrt{\pi}}\sqrt{V''(x)}, \tag{836}$$

where we absorbed a (divergent; unimportant) constant into $V(x)$ and wrote the result as $V_R(x)$. The point is that if $V''(x) < 0$, the effective potential is imaginary (keeping track of the $i\epsilon$ would inform us that the imaginary part is negative), and tells us that the particle decays with a decay rate that goes as $\sqrt{V''(x)}$, since the time evolution of the particle at this position is damped by $\sim e^{-\sqrt{V''(x)}}$ if $V''(x) < 0$. This e.g. happens for a quantum particle in the Mexican hat potential when we consider small x. Classically the particle can balance on the maximum at $x = 0$; quantum mechanically it cannot. Of course this is obvious, but here we have actually computed the precise degree to which it cannot!

50 *April 22 — Freedom of the Schwinger model without bosonization*

Today we're reading the original Schwinger model paper [?] and seeing how he was able to derive the spectrum without using bosonization. The original paper is a bit abstruse in some aspects, so we will just elaborate on some things and provide details of calculations that aren't in the paper (we're also using different notation, so watch out).

Solution:

The argument goes in two steps. First, we compute an exact expression for the current, and then we use this and the ward identity to derive the spectrum.

By considering the action coupled to sources as $S \ni \int(\bar{J}\psi + \bar{\psi}J)$, shifting ψ by $i\not{D}_A^{-1}J$ gives

$$Z[J] = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i\int_x \left[\bar{\psi}i\not{D}_A\psi + \frac{1}{2e^2}|F_A|^2\right] - \int_{x,y}\bar{J}(x)\mathcal{G}(x-y)J(y)\right), \tag{837}$$

where the exact Green's function for ψ is $\mathcal{G}$, with

$$\not{D}_A(x)\mathcal{G}(x-y) = \delta(x-y). \tag{838}$$

The thing that's special about 1+1D is that we can actually get a tractable expression for $\mathcal{G}$, at least at small $|x-y|$. Indeed, take the ansatz

$$\mathcal{G}(x-y) = G_0(x-y)e^{i(\phi(x)-\phi(y))}, \tag{839}$$

where $\phi(x)$ is a "Wilson line function" such that $\partial\!\!\!/\phi = A\!\!\!/$ (ϕ is a matrix in spin space), and $G_0(x-y)$ is the free propagator, i.e.

$$\partial\!\!\!/_x G_0(x-y) = \delta(x-y) \implies G_0(x-y) = \int_p e^{ip\cdot(x-y)}\frac{-ip\!\!\!/}{p^2}. \tag{840}$$

This ansatz works, since

$$(\partial\!\!\!/-iA\!\!\!/)\left(G_0(x-y)e^{i(\phi(x)-\phi(y))}\right) = \left(\delta(x-y)+G_0(x-y)[i\partial\!\!\!/\phi(x)-iA\!\!\!/]\right)e^{i(\phi(x)-\phi(y))} = \delta(x-y). \tag{841}$$

Now we can get an exact expression for the current, using point splitting. We have

$$j^\mu(x) = -\lim_{\epsilon\to 0}\langle\psi_\alpha(x+\epsilon/2)\bar{\psi}_\beta(x-\epsilon/2)\rangle\gamma^\mu_{\beta\alpha}\exp\left[-i\int_{x-\epsilon/2}^{x+\epsilon/2} d\lambda^\mu\, A_\mu(\lambda)\right], \tag{842}$$

where the Wilson line has been inserted to maintain gauge invariance. So this is then

$$j^\mu(x) = \lim_{\epsilon\to 0}\left(\mathrm{Tr}[\mathcal{G}(\epsilon)\gamma^\mu](1+i\epsilon^\mu A_\mu+\dots)\right). \tag{843}$$

Using our expression for $\mathcal{G}$, and using that

$$G_0(x) = -\partial\!\!\!/_x\int_p\frac{e^{ip\cdot x}}{p^2} = -\frac{1}{2\pi}\partial\!\!\!/[\ln(|x|\Lambda)+\dots] = -\frac{1}{2\pi}\frac{x\!\!\!/}{|x|^2}, \tag{844}$$

we find, taking the trace,

$$j^\mu(x) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\frac{\epsilon_\nu}{\epsilon^2}(2g^{\mu\nu}[1-i\epsilon^\lambda A_\lambda(x)]+i\epsilon^\lambda\mathrm{Tr}[\gamma^\mu\gamma^\nu\partial_\lambda\phi(x)]\right). \tag{845}$$

Taking the limit symmetrically, (and removing a factor of i that I must have goofed on)

$$j^\mu(x) = -\frac{1}{\pi}A^\mu(x)+\frac{1}{2\pi}\mathrm{Tr}[\partial^\mu\phi(x)]. \tag{846}$$

We can deal with the ϕ term by hitting both sides of $\partial\!\!\!/\phi = A\!\!\!/$ with $\partial\!\!\!/$ and taking the trace. On one hand,[97]

$$\mathrm{Tr}[(\partial\!\!\!/)^2\phi] = \Box\mathrm{Tr}[\phi], \tag{848}$$

[97]Here $\Box = -\partial_\mu\partial^\mu$ is positive-definite. In terms of differential forms, the positive-definiteness means the sign is fixed as $\Box = +(d^\dagger d+dd^\dagger)$, which is positive definite since, letting $A = d\alpha+d^\dagger\beta+\omega$ with ω harmonic,

$$\int A\wedge\star\Box A = \int(d\alpha+d^\dagger\beta)\wedge\star(d^\dagger dd^\dagger\beta+dd^\dagger d\alpha) = ||dd^\dagger\beta||^2+||d^\dagger d\alpha||^2 \tag{847}$$

while on the other hand,

$$\mathrm{Tr}[\partial\!\!\!/ A\!\!\!/] = 2d^\dagger A. \tag{849}$$

Therefore

$$\mathrm{Tr}[d\phi] = 2d\Box^{-1}d^\dagger A, \tag{850}$$

and so, as differential forms,

$$j = -\frac{1}{\pi}(1-d\Box^{-1}d^\dagger)A = -\frac{1}{\pi}\Box^{-1}(\Box - dd^\dagger)A = -\frac{1}{\pi}\Box^{-1}d^\dagger dA \implies j^\mu = -\frac{1}{\pi}[\Pi_T]^{\mu\nu}A_\nu, \tag{851}$$

where Π_T is the transverse projector. Because of the presence of the projector, this result is manifestly conserved and gauge-invariant. Now in this expression, A^μ is a dynamical field, and this expression only makes sense inside of $\int \mathcal{D}A$. However, we can pull it out the path integral by coupling the gauge field to a source current J_μ and then taking $A \mapsto -ie\delta_J$.[98] Therefore we can write

$$j^\mu(x) = i\frac{e}{\pi}[\Pi_T]^{\mu\nu}\frac{\delta}{\delta J}Z[J], \tag{853}$$

where $j^\mu(x)$ is now the current in the presence of the source J, which may or may not be turned off after taking the functional derivative. Note that regardless, we will always need to have $d^\dagger J = 0$ so as to retain gauge invariance.

Anyway, while the form of the current looks simple, note that this is a *non-perturbative* result! It's also the result we'd have gotten if we computed the current using the usual 1-loop polarization bubble diagram—evidently the 1-loop result is exact, something which is made possible by a miraculous cancellation between all the other diagrams. Also, note that this result is compatible with the chiral anomaly: taking $j \mapsto \star j$ gives

$$j_A = -\frac{1}{\pi}(d^\dagger)^{-1} \star F, \tag{854}$$

giving the correct result for $d^\dagger j_A$. In fact, knowledge that the chiral anomaly is 1-loop exact, plus the above formula, would have also been a sufficient starting point to derive our expression for j^μ, since in two dimensions the vector and axial currents are just $\star$s of one another. This fact makes the cancellation of all the diagrams in the computation of j^μ less mysterious.

Now consider the consequences of the Ward identity for changing variables in the $\mathcal{D}A$ measure. The Ward identity reads (I have a sign difference from the original paper, but I think this is due to different metric signature choices)

$$\left(-i\frac{1}{e}d^\dagger d\frac{\delta}{\delta J^\mu} + (J_\mu + j_\mu)\right) Z[J] = 0, \tag{855}$$

[98]The e here appears since in our conventions the JA coupling is

$$S \ni ie^{-1}\int J \wedge \star A. \tag{852}$$

where here $j_\mu(x)$ is to be understood as a function of $-i\delta/\delta J^\mu(x)$. Now since both currents are conserved, we can freely insert the projector Π_T in front of the $J+j$ term. Plugging in our expression for the current, and writing the $d^\dagger d$ in the above equation as $\Pi_T\Box$, we have

$$[\Pi_T]^{\mu\nu}\left(-i\Box\frac{\delta}{\delta J^\nu}+i\frac{e^2}{\pi}\frac{\delta}{\delta J^\nu}+J_\nu\right)Z[J]=0, \tag{856}$$

and so, defining the massive Greens function

$$\mathcal{G}_m=\frac{1}{\Box-m^2}, \qquad m^2\equiv e^2/\pi, \tag{857}$$

we can multiply by $\mathcal{G}_m$ (since $\Box$ commutes with Π_T) and conclude that the generating functional obeys

$$i[\Pi_T]^{\mu\nu}\frac{\delta}{\delta J^\nu(x)}Z[J]=\int_y[\Pi_T]^{\mu\nu}\mathcal{G}_m(x-y)J_\nu(y)Z[J]. \tag{858}$$

Solving this, we conclude that the generating functional of connected correlation functions has the *exact* expression

$$W[J]=\frac{1}{2}\int_{x,y}J_\mu(x)[\Pi_T]^{\mu\nu}\mathcal{G}_m(x-y)J_\nu(y). \tag{859}$$

Therefore the only nonzero connected correlation function in the theory is the two-point function, given by $\mathcal{G}_m(x-y)$. Therefore the model is exactly equivalent to a free massive scalar of mass m. Since $\mathcal{G}_m$ is a massive propagator, the above equation tells us that two (probe) charges stuck at positions x,y see an exponentially screened potential—this is kind of what happens after the photon is made massive through Higgsing, but of course here there was no photon to begin with, since we are in 1+1D (and likewise, there was no Goldstone boson, since although $\bar\psi\psi$ gets a vev, there is no Goldstone since the symmetry doesn't exist due to the chiral anomaly).

A natural question to ask is to what extent these features are changed when we add a mass (which ruins the soluability of the model). When we have a mass we can also have a θ term, which complicates things, but basically what happens is that the model ceases to be in a "Higgs" regime—the potential between incommensurate charges (like $J=\frac{1}{2}\widehat{C}-\frac{1}{2}\widehat{C}'$ for some curves C,C') *is* long-ranged, and leads to confinement of these charges—but also that there is screening, so that the potential between sources with integral charges is finite-ranged. More on this in a later diary entry.

51 *April 24 — $SO(3)$ monopoles and zero modes*

Since $\pi_1(SO(N))=\mathbb{Z}_2$, $SO(N)$ gauge theories have $\mathbb{Z}_2$ monopoles. Consider e.g. an $SO(3)$ gauge theory coupled to a Dirac fermion (in the spin 1 representation) on a spatial S^2, T^2, etc. Does the Hamiltonian generically have zero modes?

Solution:

We can basically answer this question by using what we know about the index of $\not{D}_A$. In two dimensions (spatial dimensions, or 1+1D in Euclidean time), the Dirac operator $\not{D} : \Gamma(S^\pm \otimes E \to) \to \Gamma(S^\mp \otimes E)$ has a pseudoreal structure if the associated gauge bundle $E = P_G \times_\rho \mathbb{C}^{\dim \rho}$ is such that the representation ρ is either real or pseudoreal (here P_G is a principal G-bundle, with G the gauge group).

The reason for this is as follows: if ρ is real, then we can choose a connection such that $\not{A}^a T^a \mathcal{K} = -\mathcal{K} \not{A}^a T^a$, where A is the gauge connection (we are working in physicist conventions where the T^a are Hermitian). Then choosing the γ matrices to be X, Y, we see that the matrix $\mathscr{J} = \mathcal{K}(J \otimes \mathbf{1})$ (notation is spin $\otimes$ gauge) is such that

$$i\not{D}_A \mathscr{J} = \mathscr{J}(i\not{D}_A), \qquad \mathscr{J}^2 = -\mathbf{1}. \tag{860}$$

If ρ is pseudoreal, then we can find a unitary U_G such that

$$\not{A}^a T^a \mathcal{K} U_G = -\mathcal{K} U_G \not{A}^a T^a, \qquad (\mathcal{K} U_G)^2 = -\mathbf{1}. \tag{861}$$

Then we can choose the γ matrices to be the real matrices X, Z, which tells us that the operator $\mathscr{J}' = \mathcal{K}(\mathbf{1} \otimes U_G)$ satisfies

$$i\not{D}_A \mathscr{J}' = -\mathscr{J}'(i\not{D}_A), \qquad (\mathscr{J}')^2 = -\mathbf{1}. \tag{862}$$

Therefore if ρ is not complex, we can find an antilinear operator $\mathcal{J}$ that squares to $-\mathbf{1}$ and either commutes or anticommutes with $i\not{D}_A$. This means that if $\not{D}_A \psi = 0$, then $\mathscr{J}\psi$ is also a zero mode. Since $\mathscr{J}\psi$ has opposite chirality to ψ,[99] the index of $\not{D}_A$ must vanish. Therefore if the fermions transform in a representation of the gauge group that is not complex, $\mathrm{ind}\not{D}_A = 0$, and there is nothing that protects zero modes, if they do exist, from being lifted.[100] Therefore in a generic situation, we expect no zero modes. The remainder of the diary entry is just an attempt to confirm this and to make sure we aren't missing any other symmetry that might protect the zero modes from being lifted.

We should also point out that this conclusion is rather special to two dimensions. This is because the γ matrices in two dimensions admit both a real and a pseudoreal structure, which meant that we could get a pseudoreal structure for the full connection on $S \otimes E$ with either a real or pseudoreal gauge connection. We also used the fact that the pseudoreal structure $\mathscr{J}$ anticommutes with $\bar{\gamma}$; in four dimensions this is not true (more on this later).

On the plane / torus

We will now specialize to $SO(3)$ gauge theory with 1 Dirac fermion in the spin 1 representation. On the plane / torus, we will work in Landau gauge, where $A_x = 0$ and A_y is a

[99] In the case where ρ is real, this is clear since the J tensor factor in $\mathscr{J}$ is off-diagonal, and in this basis $\bar{\gamma} = Z$. For the pseudoreal case, the choice of γ matrices means $\bar{\gamma} = Y$, so that eigenspinors of $\pm$ chirality look like $(1, \pm i)^T$. The complex conjugation in $\mathscr{J}$ exchanges these, and hence $\mathscr{J}$ anticommutes with $\bar{\gamma}$.

[100] Of course, another way to derive this would just have been to say that since in two dimensions the only gauge-invariant 2-forms for the gauge curvature that are non-vanishing are those from $U(1)$ groups, so that gauge groups like $SO(N)$ can make no contribute to $\mathrm{ind}\not{D}_A$, by the index theorem. However the argument given in the main text doesn't depend on taking the index theorem for granted, which I think is nice.

function of x only. Then $\not{D}_A\psi = 0$ reads

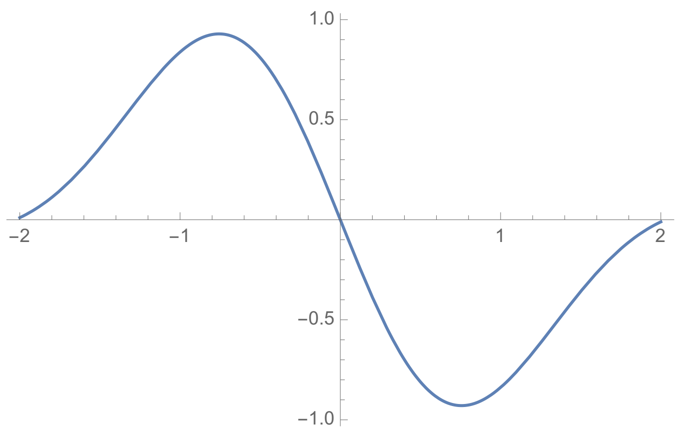
$$(\partial_x - k_y + A_y)\psi_L = 0, \qquad (\partial_x + k_y - A_y)\psi_R = 0, \tag{863}$$

where k_y is the y component of the momentum. For a uniform $U(1)$ flux, we would take $A_y = Bx$, where B is the flux density. For $SO(3)$, we let $A_y = BxT^3$.[101] After diagonalizing T^3 from $Y \oplus 0$ to $T^3 = Z \oplus 0$ and writing the spinors in flavor space as (f_1, f_2, f_3) with each f_i a two-component spinor, this gives a left-handed zero mode $(f_L, 0, 0)$ and a right-handed zero mode $(0, f_R, 0)$.

We would like to know whether these zero modes still exist after we perturb with some field strength that does not point uniformly in one direction in flavor space.[102] To do this, consider as an example adding the connection $\widetilde{A} = \widetilde{A}_y dy$, where

$$\widetilde{A}_y = \widetilde{A}_y^2 T^2 = \epsilon \cos(x) e^{-x^2/2} T^2. \tag{864}$$

This has a field strength which has zero integral over the plane[103], and so it is topologically trivial. The field strength as a function of x looks like



(865)

After diagonalizing T^3, the matrix T^2 in flavor space becomes

$$T^2 = \begin{pmatrix} & & i \\ & & \\ -i & & \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} & & i \\ & & i \\ -i & -i & \end{pmatrix}. \tag{866}$$

Now since the fermions are not in a complex representation, we have $\mathrm{ind}\not{D}_A = 0$ and we

[101]For other gauge groups G with $\pi_1(G) \neq 0$, we can just take $A_y = A_y^a T^a = BxT^a$ for some particular generator T^a.

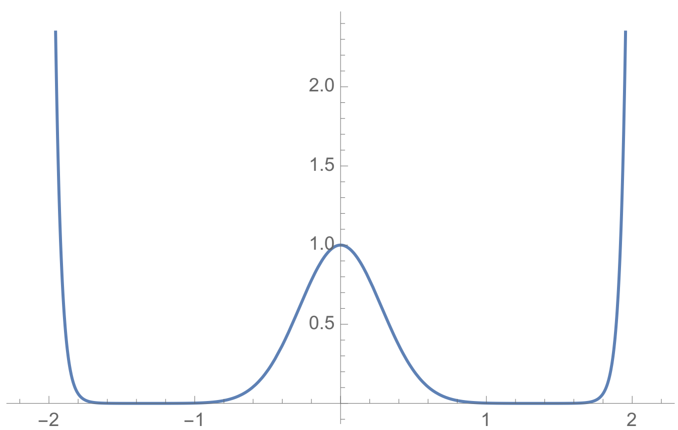
[102]This wording isn't very precise, since the fact that F transforms adjointly under gauge transformations means that we can perform a gauge transformation to take our uniform flux field to one in which $F_{\theta\phi}(\theta,\phi)$ has constant $\mathrm{Tr}[F \wedge \star F]$, but which has a direction in flavor space that is an arbitrary function of θ, ϕ. So we are really interested in making a perturbation that changes $\mathrm{Tr}[F \wedge \star F]$.

[103]Or torus. If we are on the torus, we take it to be big enough that the usual $e^{-(x-k_y/B)^2 B/2}$ zero mode wavefunctions have support only within width δx which is small compared to the size of the torus, so that the fact that the above wavefunctions are technically speaking not smooth over the torus doesn't really matter.

know that there will always be as many left zero modes as right zero modes.[104] Therefore to see whether adding the $\widetilde{A}_y$ term to the connection does anything to $\ker i\not{D}_A$, we can focus wolog on a certain chirality, which we will take to be L for definiteness. Therefore we are interested in whether we can find normalizable solutions to the following equations (setting $k_y = 0$ for simplicity)

$$\begin{aligned}(\partial_x + Bx)f_1 + i\frac{\widetilde{A}_y}{\sqrt{2}}f_3 &= 0\\ (\partial_x - Bx)f_2 + i\frac{\widetilde{A}_y}{\sqrt{2}}f_3 &= 0\\ \partial_x f_3 - i\frac{\widetilde{A}_y}{\sqrt{2}}(f_1+f_2) &= 0.\end{aligned} \tag{868}$$

If $\widetilde{A}_y = 0$ then we just take $f_2 = f_3 = 0$, and let f_1 be the usual harmonic oscillator solution. However, if $\widetilde{A}_y \neq 0$, this is not possible: the last equation means that either f_2 or f_3 must be nonzero if f_1 is nonzero, and the second equation then ensures that in fact $f_2 \neq 0$. Now f_2 is the mode that doesn't have a normalizable solution when $\widetilde{A}_y = 0$, and so we might expect that the $\widetilde{A}_y$ coupling ruins the normalizability of the solution. Indeed, this is what appears to happen: using the form of $\widetilde{A}_y$ above with $B = 2\pi$ and $\epsilon = .1\sqrt{2}$, a plot of the magnitude $\sum_i f_i^* f_i$ as a function of x shows a divergence:



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There of course may be something I've missed, or some tricky choice of initial conditions (the above plot was for $f_1(0) = 1$, $f_2(0) = f_3(0) = 0$; modifying the latter two to be nonzero makes the divergence worse) that allow this divergence to be avoided, but for now it seems to be a generic consequence of taking $\widetilde{A} \neq 0$.

[104] An operator that provides the (pseudo)real structure here is $\mathcal{K}(Y \otimes \mathbf{1})$, where the first tensor factor is for the spin indices and the second is for the gauge indices. Indeed, as in the previous section, using X and Y as the γ matrices, and working in a basis where the gauge generator matrices are purely imaginary and antisymmetric, we have

$$[i\not{D}_A, \mathcal{K}(Y \otimes \mathbf{1})] = 0, \tag{867}$$

and so $\mathcal{K}(Y \otimes \mathbf{1})$ provides a way to take a zero mode of a certain chirality and construct another zero mode with opposite chirality.

On the sphere

We first need to choose a gauge connection. For a $U(1)$ monopole of flux n, the standard choice is

$$A^{N/S} = n\frac{\pm 1 - \cos\theta}{2} d\phi, \tag{870}$$

which gives $\int_{S^2} F = 2\pi n$. For a gauge group with $\pi_1(G) \neq 0$, the simplest choice for a monopole field is the above but with a T^a tacked on, where T^a is a particular (Hermitian) generator of $\mathfrak{g}$. Using results from our earlier entry on zero modes, the covariant derivatives are

$$\nabla_\theta = \partial_\theta, \qquad \nabla_\phi = \partial_\phi - \frac{iZ \otimes \mathbf{1}}{2}\cos\theta + in\frac{\pm 1 - \cos\theta}{2}\mathbf{1} \otimes T^a, \tag{871}$$

where the first tensor factor is the spin indices and the second is the gauge indices (we won't bother to explicitly write the $\otimes$ in what follows). The expression $i\not{D}_A\psi = 0$, is then, for our uniform monopole field,

$$\not{D}_A\psi^{(N/S)} = \left[X\left(\partial_\theta + \frac{\cot\theta}{2}\right) + Y\csc\theta\left(\partial_\phi + in\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\right]\psi^{(N/S)} = 0 \tag{872}$$

or written out in chiral components,

$$\begin{aligned} \left(\partial_\theta + \frac{\cot\theta}{2} - i\csc\theta\partial_\phi + n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\psi_R^{(N/S)} = 0 \\ \left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi - n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\psi_L^{(N/S)} = 0 \end{aligned} \tag{873}$$

In what follows, we will take $n = 1$ for concreteness. Then in the $U(1)$ case, we see that we get a single R zero mode, $\psi_R = e^{-i\phi/2}$ (the reason we get an R zero mode and not an L one is because of our sign conventions for the covariant derivative). Also note that this zero mode actually has spin zero; to see this one needs to properly calculate the angular momentum generators, which I won't go into here.

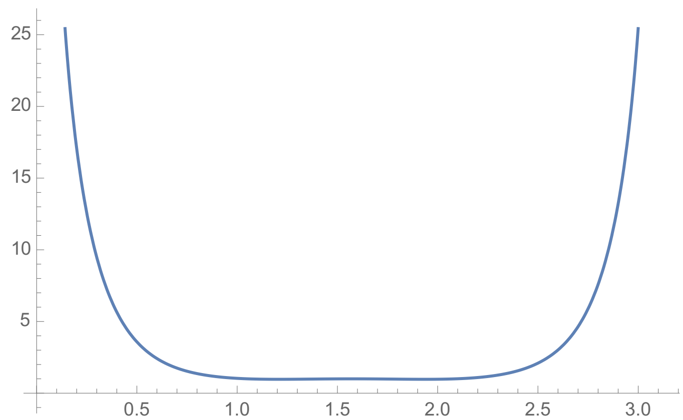
Now for $SO(3)$. If we let the field strength point in the T^3 direction, we see that we get a single L and a single R zero mode, as expected. Do these zero modes survive when a perturbation is added? Let us add the potential

$$\widetilde{A} = \widetilde{A}^2_\phi T^2\, d\phi = \epsilon\sin(2\theta)T^2\, d\phi. \tag{874}$$

This is well-defined on the sphere since $\widetilde{A}(\theta = 0, \pi) = 0$, and it is topologically trivial since $\int_{S^2} d\widetilde{A} = 0$. As mentioned before, since the zero modes for real gauge groups always come in left-right pairs, we can focus on a single handedness (we will look at R) wolog. We therefore want to find normalizable solutions to (working on the N coordinate patch)

$$\begin{aligned} \left(\partial_\theta - i\csc\theta\partial_\phi + \frac{1}{2}\csc\theta\right)f_1 + i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}f_3 = 0 \\ \left(\partial_\theta + \cot\theta - i\csc\theta\partial_\phi - \frac{1}{2}\csc\theta\right)f_2 + i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}f_3 = 0 \\ \left(\partial_\theta + \frac{1}{2}\cot\theta - i\csc\theta\partial_\phi\right)f_3 - i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}(f_1 + f_2) = 0 \end{aligned} \tag{875}$$

When $\epsilon = 0$ we just take $f_1 = e^{-i\phi/2}$, $f_2 = f_3 = 0$.[105] When $\epsilon \neq 0$ the coupling between the different modes kicks in, and as in the planar case we seem to run into normalizability problems caused by the troublesome modes f_2, f_3 being forced to be nonzero. For example, set $\epsilon = \sqrt{2}$. The natural choices for the ϕ dependence of the three modes is $f_1 \propto e^{-i\phi/2}$, $f_2 \propto e^{i\phi/2}$, and with f_3 having no ϕ dependence. With these assignments of ϕ dependence, the volume-element-normalized magnitude $\sum_i f_i^* f_i \sin\theta$ as a function of θ looks like



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So, it blows up at the poles, and we don't get a legit zero mode solution. This seems to be the generic behavior for any $\epsilon \neq 0$.

52 *April 25 — Non relativistic limit in ϕ^4 theory*

Today we have something super simple that's not usually talked about in QFT books. This is basically an exercise in Duncan's "The conceptual framework of QFT". First, we will show how to take the non-relativistic of ϕ^4 theory, with an attractive ϕ^4 potential (with a very weak ϕ^6 term added for stability reasons, which we won't keep track of). Then we will find the energy of the lowest bound state, which exists in two and three spacetime dimensions.

Solution:

The action is (mostly negative signature)

$$S = \int (\phi(\Box - m^2)\phi - \lambda\phi^4/6). \tag{879}$$

[105] Here the f_3 mode has no normalizable solution when $\epsilon = 0$, since taking the ϕ dependence to be trivial means

$$f_3 \propto \frac{1}{\sqrt{\sin\theta}}, \tag{876}$$

which is not acceptable (it integrates to something finite because of the $\sin\theta$ in the measure, but it is not differentible). Similarly the f_2 mode has to be zero, since otherwise we have

$$f_2 \propto e^{i\phi/2}\csc(\theta), \tag{877}$$

which is also no good.

Our philosophy here will be one of effective field theory. We will assume all the momentum modes of energy $k^0 > m$ have been integrated out, and that the coupling constants in the above Lagrangian are those generated during this integrating out procedure. Define the field

$$\psi(x) = \sqrt{2m}e^{imt}\int_k \theta(k^0)\phi_k e^{-ikx}, \tag{880}$$

The frequency components of ψ are all constrained to be less than m, and so when we write the action in terms of ψ, any times that do not contain an equal number of ψs and $\psi^\dagger$s will vanish, since no combination of frequencies from the Fourier modes of the ψs will be able to cancel the time dependence of the $e^{\pm imt}$ factor such a term will have, and the time integration will kill the term in question. so that

$$\phi(x) = \int_k \theta(k^0)(\phi_k e^{-ikx} + \phi^\dagger_{-k}e^{ikx}) = \frac{1}{\sqrt{2m}}(e^{-imt}\psi(x) + e^{imt}\psi^\dagger(x)). \tag{881}$$

Finally, we can drop $\psi^\dagger\partial_t^2\psi$, since it goes as k_0^2/m, which is negligible in comparison to the other terms. So then we get

$$S = \int \left(\psi^\dagger i\partial_t\psi + \frac{1}{2m}\psi^\dagger\nabla^2\psi + \lambda\psi^\dagger\psi^\dagger\psi\psi\right), \tag{882}$$

which is just what we expect for $p\dot{q} - H$.

Lets now use this action to find, QFT-style, the existence of bound states for attractive potentials, $\lambda < 0$.

53 *April 28 — Self energy and particle production in fields*

Solution:

We will be working in $\mathbb{R}$ time and with a mostly negative signature metric.

We write the exact propagator as

$$\mathcal{G}(p^2) = \frac{1}{p^2 - \lambda^2 + i\epsilon + (p^2 + i\epsilon)\Pi(p^2)}, \tag{883}$$

where

$$\Pi(p^2) = \int_0^\infty dm^2 \frac{s(m^2)}{p^2 - m^2 + i\epsilon}. \tag{884}$$

The goal now is to explain what $s(m^2)$ measures. To do this, note that the partition function in the presence of a current J is, for sufficiently weak J,

$$Z[J] \approx \exp\left(\frac{i}{2}\int dx\, dy\, J(x)\mathcal{G}(x-y)J(y)\right). \tag{885}$$

This reproduces the two point function correctly but $\ln Z[J]$ written in this way does not allow us to reproduce the higher-point functions; to do this we would need to go beyond quadratic order in J.

Now define

$$\Phi(x) \equiv \int dy\, \mathcal{G}^{-1}(x-y)J(y). \tag{886}$$

Then to our approximation,

54 *May 1 — Pseudoreality and the index of $\not{D}_A$*

Today is a short one. We're going to be looking at what we can say about the index of the Dirac operator when we have fermions which transform in a pseudoreal representation of the combined pin + gauge group. We'll show that the index vanishes in two dimensions, and that it is always even in four dimensions. The difference between these two cases will come down to the fact that charge conjugation operates differently in different dimensions.

Solution:

We will be working in Euclidean signature throughout, since we are mainly interested in either computing zero modes of a Dirac Hamiltonian in even-dimensional space, or calculating Euclidean partition functions involving $\not{D}_A$ in even-dimensional spacetime. We will just look at dimensions 2 and 4, but all statements hold if we take our dimensions to be modulo 8, by Bott periodicity.

First for two dimensions.

Talk about coupling to the spin connection for both dimensions! Distinguish between pin and spin as the thing that needs to be pseudoreal.

Now we go to four dimensions. The Clifford algebra on four generators is $M_2(\mathbb{H})$, and we can take our gamma matrices to be

$$\gamma^0 = X \otimes \mathbf{1}, \quad \gamma^1 = Y \otimes \mathbf{1}, \quad \gamma^2 = Z \otimes X, \quad \gamma^3 = Z \otimes Z. \tag{887}$$

With these choices, $\bar{\gamma}$ is the (unfortunately non-block-diagonal) matrix

$$\bar{\gamma} = Z \otimes Y. \tag{888}$$

Now since $\text{Spin}(4) = SU(2) \times SU(2)$, the spinor representation is pesudoreal and we expect our fermions to transform in a pseudoreal representation (unlike two dimensions, the Clifford algebra just comes with a pseudoreal structure; there is no real structure as well). Indeed, consider the antilinear map

$$\mathscr{J} = (J \otimes \mathbf{1})\mathcal{K}. \tag{889}$$

Then $\mathscr{J}^2 = -\mathbf{1}$ and $\gamma^\mu \mathscr{J} = -\mathscr{J}\gamma^\mu$, giving us the pseudoreal structure. Consequently, $i\not{\partial}\mathscr{J} = \mathscr{J} i\not{\partial}$. If we twist the spinor bundle by an associated gauge bundle in a real representation with purely imaginary generators T^a, then $\not{A}^a T^a \mathscr{J} = \mathscr{J} \not{A}^a T^a$, and so $[\not{D}_A, \mathscr{J}] = 0$.

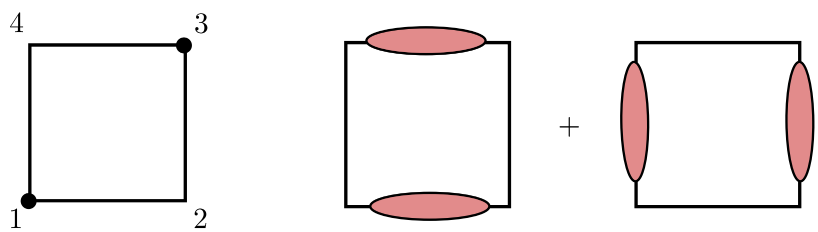


Figure 6: Left: our conventions. The roman numerals indicate how the tensor products in our Fock space are ordered. The black dots indicate how we determine the signs for the singlets: in each singlet, the term with a spin $\uparrow$ on the site with the black dot is given a positive sign. Right: the RVB ground state in the 4-electron subspace. Each oval is a singlet, with signs determined according to the conventions on the left.

Note that unlike in two dimensions, $\mathscr{J}$ preserves chirality:

$$[\mathscr{J}, \bar{\gamma}] = 0. \tag{890}$$

If $\not{D}_A\psi = 0$, then $\psi' = \mathscr{J}\psi$ is also killed by $\not{D}_A$. However, the above property means that both ψ and ψ' have the same $\bar{\gamma}$ eigenvalue. Therefore in this case, the pseudoreal structure means that $\mathrm{ind}\not{D}_A \in 2\mathbb{Z}$: all the zero modes are doubled, but they are not necessarily doubled in a left-right symmetric fashion.

55 *May 8 — RVB on a single plaquette and why one should expect $d_{x^2-y^2}$ superconductivity in the cuprates*

Today we will consider the t-J model on a single square plaquette. We will see that the "undoped" sector with four fermions has an RVB ground state, which will lead us to an understanding of why a d-wave SCing order parameter is generically expected in the cuprates. I read about this in a long review of high T_c [**?**], where the result was written down without derivation—the goal today is to work out the details of where the result comes from.

Solution:

First, consider the half-filled subspace, with four electrons, and restrict to the low-energy subspace where double-occupancy is forbidden. Therefore the Hamiltonian restricted to this subspace is just $H = J\sum_i S_i \cdot S_{i+1}$, with $J > 0$. We can determine on fairly general grounds what the ground state should be. First, it should be a singlet, since the only thing we need to optimize is the AF interaction. This rules out e.g. the Neel states, which are not eigenstates of the $S_i \cdot S_j$ operators.[106] Therefore we should look for a ground state built from (a superposition of) nearest-neighbor singlets.

[106]The $S_i \cdot S_j$ operators entangle the $|\uparrow\downarrow\rangle$ tensor product states on adjacent sites, and so do not map the Neel states (which are $\otimes$ states and have no entanglement) to Neel states.

Let $V_{12;34}$ be the horizontal valence bond state as shown in the left term of the right half of figure 6. We will write this state in hopefully transparent matrix notation as

$$2V_{12;34} = -\begin{pmatrix}\downarrow & \uparrow \\ \downarrow & \uparrow\end{pmatrix} - \begin{pmatrix}\uparrow & \downarrow \\ \uparrow & \downarrow\end{pmatrix} + \begin{pmatrix}\downarrow & \uparrow \\ \uparrow & \downarrow\end{pmatrix} + \begin{pmatrix}\uparrow & \downarrow \\ \downarrow & \uparrow\end{pmatrix}. \tag{891}$$

The signs here are determined using the left half of figure 6: in each singlet, the term where the spin on the lattice site with the $\bullet$ is $\uparrow$ is given a positive sign. The state with the vertical valence bonds is similarly (just rotate the above by $\pi/2$)

$$2V_{41;23} = -\begin{pmatrix}\downarrow & \downarrow \\ \uparrow & \uparrow\end{pmatrix} - \begin{pmatrix}\uparrow & \uparrow \\ \downarrow & \downarrow\end{pmatrix} + \begin{pmatrix}\downarrow & \uparrow \\ \uparrow & \downarrow\end{pmatrix} + \begin{pmatrix}\uparrow & \downarrow \\ \downarrow & \uparrow\end{pmatrix}. \tag{892}$$

Note how we have $\downarrow\leftrightarrow\uparrow$ symmetry; therefore we can only need to explicitly write half of the terms in the manipulations that follow.

The operator $S_1 \cdot S_2$, acting on the basis $\mathbb{C}^2_1 \otimes \mathbb{C}^2_2$, has the form

$$S_1 \cdot S_2 = \frac{1}{4}\left(1 \oplus \begin{pmatrix}-1 & 2 \\ 2 & -1\end{pmatrix} \oplus 1\right). \tag{893}$$

Therefore on a singlet, $S_1 \cdot S_2$ acts as $(-3/4)\mathbf{1}$. Therefore $V_{12;34}$ is an eigenstate of $S_1 \cdot S_2 + S_3 \cdot S_4$, with eigenvale $-3/2$. Likewise, $V_{41;23}$ is an eigenstate of $S_4 \cdot S_1 + S_2 \cdot S_3$, with the same eigenvalue. However, the other spin interactions break the valence bonds. Therefore if we are to build an eigenstate of H, we should try a linear combination of the two valence bond configurations. From the above, we see that in order to determine what happens to our putative ground state $V_{12;34} \pm V_{41;23}$,[107] we should calculate $(S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} \pm (S_1 \cdot S_2 + S_3 \cdot S_4)V_{41;23}$. This is straightforward enough: operating on the first and third terms in (891), we get

$$8(S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} = -2\begin{pmatrix}\downarrow & \uparrow \\ \downarrow & \uparrow\end{pmatrix} - 2\begin{pmatrix}\downarrow & \uparrow \\ \uparrow & \downarrow\end{pmatrix} + 4\begin{pmatrix}\uparrow & \uparrow \\ \downarrow & \downarrow\end{pmatrix} + \dots, \tag{894}$$

where the $\dots$ are the terms related by spin-flip symmetry. Therefore these terms have the effect of, among other things, flipping the valence bonds with a minus sign, so that $V_{12;34} \mapsto -\frac{1}{2}V_{41;23} + \dots$, where the $\dots$ are other terms (not related by spin-flips). Likewise, we have

$$8(S_1 \cdot S_2 + S_3 \cdot S_4)V_{41;23} = -2\begin{pmatrix}\downarrow & \downarrow \\ \uparrow & \uparrow\end{pmatrix} - 2\begin{pmatrix}\downarrow & \uparrow \\ \uparrow & \downarrow\end{pmatrix} + 4\begin{pmatrix}\uparrow & \downarrow \\ \uparrow & \downarrow\end{pmatrix} + \dots. \tag{895}$$

Adding these up,

$$\begin{aligned} 8[(S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} \pm (S_1 \cdot S_2 + S_3 \cdot S_4)V_{41;23}] = (-2 \pm 4)\begin{pmatrix}\downarrow & \uparrow \\ \downarrow & \uparrow\end{pmatrix} + (-2 \mp 2)\begin{pmatrix}\downarrow & \uparrow \\ \uparrow & \downarrow\end{pmatrix} \\ + (4 \mp 2)\begin{pmatrix}\uparrow & \uparrow \\ \downarrow & \downarrow\end{pmatrix} + \dots \end{aligned} \tag{896}$$

[107]We can restrict to a $\pm$ sign since our wavefunction needs to transform in an irrep of D_4, and since the two valence bond configurations are symmetric under a π rotation.

If we choose the + sign, then we get

$$
\begin{aligned}
(S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} + (S_1 \cdot S_2 + S_3 \cdot S_4)V_{41;23} &= -\frac{1}{4}\left[2\begin{pmatrix}\downarrow & \uparrow \\ \uparrow & \downarrow\end{pmatrix} - \begin{pmatrix}\downarrow & \uparrow \\ \downarrow & \uparrow\end{pmatrix} - \begin{pmatrix}\uparrow & \uparrow \\ \downarrow & \downarrow\end{pmatrix} + \dots\right] \\
&= -\frac{1}{2}(V_{12;34} + V_{41;23}).
\end{aligned}
\tag{897}
$$

So when applied to the linear combination of the valence bond states, the spin-spin interactions have the effect of exchanging the two valence bond configurations Therefore if we define $|\Phi_0\rangle$ as the symmetric superposition of the two valence bond states, viz.

$$
|\Phi_0\rangle \equiv \frac{1}{\sqrt{2}}(V_{12;34} + V_{41;23}), \tag{898}
$$

then

$$
J\sum_{\langle ij\rangle} S_i \cdot S_j|\Phi_0\rangle = -2J|\Phi_0\rangle. \tag{899}
$$

One can then check numerically that this is indeed the ground state.

What is the momentum carried by $|\Phi_0\rangle$? It looks to be zero, because of the + sign in the linear combination. However, we have to remember that the spins are coming from electrons, which give us minus signs. These minus signs mean that while the spin wavefunction is s-wave, the full wavefunction actually changes under rotations. To find out how it transforms, consider how the following term changes under a rotation $R_{\pi/2}$:

$$
\begin{aligned}
|\Phi_0\rangle \ni \frac{1}{\sqrt{2}}\begin{pmatrix}\downarrow & \uparrow \\ \uparrow & \downarrow\end{pmatrix} &= \frac{1}{\sqrt{2}}c^\dagger_{\uparrow 1}c^\dagger_{\downarrow 2}c^\dagger_{\uparrow 3}c^\dagger_{\downarrow 4}|0\rangle \xrightarrow{R_{\pi/2}} \frac{1}{\sqrt{2}}c^\dagger_{\uparrow 2}c^\dagger_{\downarrow 3}c^\dagger_{\uparrow 4}c^\dagger_{\downarrow 1}|0\rangle \\
&= -\frac{1}{\sqrt{2}}c^\dagger_{\downarrow 1}c^\dagger_{\uparrow 2}c^\dagger_{\downarrow 3}c^\dagger_{\uparrow 4}|0\rangle \in -|\Phi_0\rangle,
\end{aligned}
\tag{900}
$$

and similarly for the other terms. Therefore $R_{\pi/2}|\Phi_0\rangle \mapsto -|\Phi_0\rangle$, and the wavefunction changes sign under a $\pi/2$ rotation (by contrast, one can check that choosing the $-$ sign for the linear combination would have given an s-wave wavefunction).

A superconducting pairing term will connect the 4-electron subspace to the 2-electron subspace (and not to the 6-electron space, since that's bad for Hubbard U reasons). Therefore to establish the symmetry of the SCing gap, we need to look at the ground state wavefunction in the 2-electron sector.

This can be done numerically, but we can get the information we need from a simple heuristic analysis. First, when there is no Hubbard U (just t), the ground state is clearly (the minimum of the $-t\cos(k)$ dispersion favors a uniform real-space wavefunction)

$$
|\Psi_0(t=0)\rangle = \frac{1}{4}\sum_{i,j} c^\dagger_{\uparrow i}c^\dagger_{\downarrow j}|0\rangle. \tag{901}
$$

The action of D_4 just re-organizes the summation indices, and so this state is s-wave. Now when we turn on U and stay within the 2-electron subspace, the weights of the various terms will change, so that the electrons stay away from one another. However, the fact that the sign of the gs wavefunction is invariant under $R_{\pi/2}$ will not change: the irrep of D_4 that the gs transforms under is fixed. One way to motivate this is that after we project onto the singly-occupied subspace, the $-t$ hopping amplitude will always be larger than the induced AF $J \sim t^2/U$ coupling, and so it will always be favorable to have a low-momentum state (in the trivial irrep of D_4), rather than a high-momentum state that takes advantage of the AF coupling.

From this we can now conclude that any SCing pairing, should it exist, should be d-wave. Indeed, if Δ_k is a pairing term (built from a sum of $c_\uparrow c_\downarrow$ operators; it has to be singlet pairing since both 4-particle and 2-particle ground states are spin singlets), we want the overlap $|\langle\Psi_0|\Delta_k|\Phi_0\rangle|$ to be maximized. Since $|\Phi_0\rangle$ changes sign under $R_{\pi/2}$ and $|\Psi_0\rangle$ doesn't, in order for the overlap to be nonzero, Δ_k must also change sign under $R_{\pi/2}$, and so the SCing gap must indeed be d-wave.

Of course, a single plaquette is not the same as an infinite square lattice, but the basic physics at work here gives us strong reason to believe that superconductivity in the cuprates, arising as it does from a doped Mott insulator, should have a d-wave gap.

56 *May 9 — JW details*

This is a bit basic, but I had to write it up for a pset for Senthil's class, and so I'm including it in the diary since it hasn't appeared as a diary entry before.

Consider the XX chain, with Hamiltonian

$$H = J\sum_i (X_iX_{i+1} + Y_iY_{i+1}) = J\sum_i (P_iM_{i+1} + M_iP_{i+1}), \tag{902}$$

with $J > 0$ and

$$P \equiv (X + iY)/2, M \equiv (X - iY)/2. \tag{903}$$

Do several things. a) show that $J > 0$ and $J < 0$ are related by a unitary transformation on H. b) show that total S^z is conserved. c) show how the JW mapping to a fermion chain works. d) show that the ground state is gapless. e) what happens to the $U(1)$ S^z conservation symmetry on the fermion side? g) explain how the JW transformation is modified for a periodic chain of length L.

a) If we want to get a ferromagnetic chain, we can perform the unitary transformation $H \mapsto U^\dagger HU$, where

$$U = \exp\left(i\sum_{i\in 2\mathbb{Z}} \frac{\pi}{2} Z_i\right) \tag{904}$$

performs a rotation by π about the $\hat{z}$ axis on every even site. Then we have

$$\begin{aligned}
H \mapsto U^\dagger H U &= J\sum_j e^{-i\sum_{i\in 2\mathbb{Z}}\frac{\pi}{2}Z_i}\left(X_jX_{j+1}+Y_jY_{j+1}\right)e^{i\sum_{i\in 2\mathbb{Z}}\frac{\pi}{2}Z_i} \\
&= J\sum_j e^{-i\sum_{i\in 2\mathbb{Z}}\frac{\pi}{2}Z_i+i\sum_{k\in 2\mathbb{Z}}\frac{\pi}{2}(-1)^{\delta_{k,j}j}Z_k}(X_jX_{j+1}+Y_jY_{j+1}) \\
&= J\sum_j e^{-i\pi Z_j}(X_jX_{j+1}+Y_jY_{j+1}) = -H,
\end{aligned} \tag{905}$$

so that this rotation has the effect of sending $J\mapsto -J$. Therefore wolog we can take $J>0$.

b) Since the Hamiltonian involves only P_iM_{i+1} and its Hermitian conjugate, both of which flip a pair of spins and hence conserve total S^z, the Hamiltonian conserves $\sum_i Z_i$. Indeed, the commutator is

$$\begin{aligned}
\left[\sum_i Z_i, H\right] &= J\sum_{i,j}[Z_i, P_jM_{j+1}+M_jP_{j+1}] = J\sum_j [Z_j, P_jM_{j+1}+M_jP_{j+1}+P_{j-1}M_j+M_{j-1}P_j] \\
&= 2J\sum_j (P_jM_{j+1}-M_jP_{j+1}-P_{j-1}M_j+M_{j-1}P_j) = 0.
\end{aligned} \tag{906}$$

c) We do the mapping via[108]

$$Z_i = -2c_i^\dagger c_i + 1 = (-1)^F_i, \qquad P_i = \prod_{j<i}(-1)^F_j c_i. \tag{907}$$

The inverse mapping is then

$$c_i^\dagger \mapsto \prod_{j<i} Z_j M_i, \qquad c_i \mapsto \prod_{j<i} Z_j P_i. \tag{908}$$

Since $[(-1)^F_i, c_j] = 2\delta_{ij}c_i$, we have $[Z_i, P_j] = 2P_i\delta_{ij}$ as required. Likewise since $[(-1)^F_i, c_j^\dagger] = -2\delta_{ij}c_j^\dagger$, we have $[Z_i, M_j] = [Z_i, P_j^\dagger] = -2\delta_{ij}M_j$. Finally we have

$$\begin{aligned}
[M_i, P_j] &= \delta_{j>i}[c_i^\dagger, (-1)^F_i c_j] + \delta_{i>j}[(-1)^F_j c_i^\dagger, c_j] + \delta_{i,j}[c_i^\dagger, c_j] \\
&= \delta_{i,j}[c_i^\dagger, c_i] = \delta_{i,j}(2c_i^\dagger c_i - 1) = -2\delta_{ij}Z_i,
\end{aligned} \tag{909}$$

as required.

d) In terms of fermion variables, H becomes

$$H = J\sum_i (c_i^\dagger(-1)^F_i c_{i+1} + hc) \tag{910}$$

[108] n.b. we are not associating P_i with $c_i^\dagger$, as might seem more notationally logical. This is done to avoid stupid minus signs, and to allow the fermion basis to be (empty, filled)^T instead of the other way around.

and so we get a simple hopping model, viz.

$$H = J\sum_j (c_j^\dagger c_{j+1} + hc) = 2J\sum_k \cos(k) c_k^\dagger c_k, \tag{911}$$

which gives us a half-filled band. We actually already knew that the filling had to be half: this is the value of the filling for which we have particle-hole symmetry, which is performed by taking $J \mapsto -J$ using the rotation described previously. Adding a term like $\sum_i Z_i$ breaks the $J \mapsto -J$ mapping, and consequently shows up in the fermion variables as a chemical potential, breaking particle-hole symmetry. Anyway, since when such terms are absent the Fermi level cuts the band halfway up, the system is gapless and is half-filled.

e) In the fermion variables, the operator which generates the $U(1)$ symmetry transformation becomes

$$\sum_j Z_j \to \sum_j -(-1)^F_j, \tag{912}$$

which measures the total particle number (fermions minus holes) and which clearly commutes with the fermion Hamiltonian. The ground state has a definite value of total particle number, and so the $U(1)$ symmetry is preserved in the ground state (as it must be; we are in one dimension).

f) Now let's consider a closed chain of length L. The mapping is the same as before, expect this time we fix the tails in the JW mapping to start at the site $i = 1$. The only subtlety occurs when we examine the mapping of operators which wrap around the chain. There are a few ways to fix conventions; we will use

$$c_{i+L}^\dagger \mapsto \prod_{\text{all}\, j} Z_j \prod_{k<i} Z_k M_i \mapsto (-1)^F c_i^\dagger \implies c_{i+L} = c_i(-1)^F. \tag{913}$$

This isn't surprising; hopping a fermion around a chain should give a minus sign for every fermion it passes. Note that the proper anticommutation relations are preserved, e.g.

$$\{c_{j+L}, c_{k+L}^\dagger\} = -\{(-1)^F c_j, (-1)^F c_k^\dagger\} = (-1)^{2F}\{c_j, c_k^\dagger\} = \delta_{jk}. \tag{914}$$

Now as long as we are sticking to things which preserve fermion number mod 2 (so e.g. $P_i P_{i+1}$ is okay but P_i is not; luckily our chosen H preserves fermion number), these sign issues don't come up. However, to write the Hamiltonian we'd still like to know how to decompose c_i into Fourier modes—since the relation between c_{i+L} and c_i involves an operator, how are we supposed to do this? We do it by realizing that $(-1)^F$ commutes with the Hamiltonian, and hence we can decompose the Hilbert space $\mathcal{H}$ into $(-1)^F$ superselection sectors. In each sector, the fermions have well-defined boundary conditions, and a Fourier decomposition is well-defined.

57 *May 10 — $\mathbb{Z}_2$ gauge theory on the Kagome lattice and the TFIM*

This is another problem from Senthil's class. Consider the Hamiltonian $H_\triangle$ on the Kagome lattice:

$$H_\triangle = -J\sum_i Z_i - h\sum_{ijk\in\partial\triangle} X_iX_jX_k, \tag{915}$$

where the second sum runs over the vertices at the points of each triangle on the Kagome lattice.

Discuss the various phases of this model, and show that there is a single phase transition, by mapping the theory to a (gauged) TFIM. Show that in the topological phase there are four nearly degenerate ground states on the torus, and find the splitting between them. Then introduce dynamical magnetic fluxes by adding a term like $-\Gamma\sum_i X_i$ to $H_\triangle$, with Γ/J small. The four nearly degenerate ground states are now no longer orthogonal; find the splitting between them.

Solution:

This is a $\mathbb{Z}_2$ gauge theory, with Z_i the Wilson link variables (think $e^{i\int_l A}$, where l is a link of the associated $\hexagon$ lattice), and X_i the electric field variables. Therefore we will think of the XXX term as measuring the electric flux coming out of a particular triangle in the Kagome, with Z the operator giving the charges dynamics by hopping electric charges between triangles.

We note that in accordance with $H_\triangle$ being a gauge theory without magnetic matter, $H_\triangle$ possesses a 1-form symmetry generated by the operators

$$W_C = \prod_{l\in C} Z_l, \tag{916}$$

for any closed loop C in the $\hexagon$ lattice associated to the Kagome lattice, which all commute with $H_\triangle$. The distinct symmetry generators can be labeled by classes $C\in H_1(X_{\hexagon};\mathbb{Z}_2)$—we will call this a "magnetic" 1-form symmetry, since its charge operator measures the magnetic flux.

The W_C operators for C contractible give us an extensive number of operators which commute with the Hamiltonian, and measure the magnetic flux at each $\hexagon$ plaquette. This extensive number of commuting operators give us a gauge redundancy, and in order to eliminate the gauge redundancy, we will fix our Hilbert space by fixing a particular value for W_C on every $\hexagon$ plaquette.[109] If the value of the magnetic flux on a plaquette $\hexagon$ is measured by $\widetilde{\tau}^x_{\hexagon}$, then the operator $W_{\hexagon}\widetilde{\tau}^x_{\hexagon}$ performs a magnetic gauge transformation (magnetic since it involves Wilson line variables, not electric field variables), and acts trivially on the physical Hilbert space (in order to also talk about electric gauge transformations, we'd need to use

[109]again, we are treating the W_C operators for noncontractible C rather differently since they generate global symmetries and not gauge transformations; more on this in a sec.

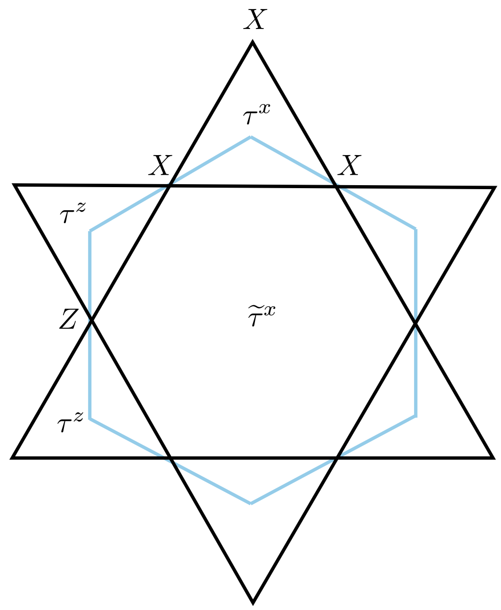


Figure 7: The Kagome lattice and its associated ⬡ lattice, with the locations of the support of various operators in various Hamiltonians shown. Matter fields are indicated with τs and gauge field variables with captial letters.

a formulation $H'_\triangle$ with explicit electric matter fields, which we will in a sec). Therefore the full Hilbert space dimension, given a particular magnetic flux configuration, is

$$\dim \mathcal{H}_\triangle = N_L - N_F = N_V, \tag{917}$$

where N_L, N_F, and N_V are the number of links, faces, and vertices of the ⬡ lattice, respectively, and we've used that on a flat surface (torus or plane, so that we can tile the ⬡s appropriately), the Euler characteristic is zero. If we want to include a magnetic flux at a certain plaquette, we must then *modify the Hilbert space* to modify the Gauss's law constraint at that plaquette. This is because the fluxes are not dynamical; there are no operators in $\mathcal{H}_\triangle$ that create or hop fluxes, and so the flux configuration has to be put in by hand by constraints on $\mathcal{H}_\triangle$. If we wanted to make the fluxes dynamical, we could write

$$H'_\triangle = H_\triangle - \Gamma \sum_{\langle \hexagon_m \hexagon_n \rangle} \widetilde{\tau}^z_m X_{mn} \widetilde{\tau}^z_n, \tag{918}$$

where X in the added term creates magnetic fluxes on two neighboring plaquettes, and the $\widetilde{\tau}^z$s create the corresponding magnetic charges. As mentioned above, the magnetic Gauss's law is then that when acting on physical states, we have

$$\prod_{i \in \partial \hexagon} Z_i = \widetilde{\tau}^x_{\hexagon}, \tag{919}$$

since now the operator which performs the (magnetic) gauge transformation is the product of the RHS and the LHS. Adding either the RHS or the LHS to the Hamiltonian would be the magnetic analogue of the $XXX = \tau^x$ term, which would explicitly gap the magnetic fluxes. Of course, making the fluxes dynamical breaks the 1-form symmetry explicitly, since now the W_C do not commute with the Hamiltonian.

Anyway, let's return to the model $H_\triangle$ without matter fields, with the magnetic flux configuration is fixed as a Hilbert space constraint. When $h = 0$, the ground state is the non-degenerate product state $\bigotimes_i |\uparrow\rangle_i$. When $J = 0$, a ground state is the "loop liquid" formed by closed paths on the associated honeycomb lattice of $|-\rangle$ states:

$$|\Phi_0\rangle = \sum_{C\in B_1(X_{\hexagon};\mathbb{Z}_2)} \prod_{l\in C} Z_l|\emptyset\rangle, \qquad |\emptyset\rangle = \bigotimes_l |+\rangle_l, \tag{920}$$

where $B_1(X_{\hexagon};\mathbb{Z}_2)$ denotes the closed unoriented *contractible* loops on the associated $\hexagon$ lattice $X_{\hexagon}$, with the vertices of the Kagome becoming links l of the $\hexagon$ lattice. We say "a" ground state because depending on the topology of space, there may be multiple degenerate ground states. The different ground states differ by the eigenvalues of W_C along different cycles, and specializing to the torus, we can write them as states transforming in a representation of the 1-form symmetry group $\mathbb{Z}_2^2$:

$$|\pm_x \pm_y\rangle_M \equiv (\mathbf{1} \pm_x W_{C_x})(\mathbf{1} \pm_y W_{C_y})|\Phi_0\rangle, \tag{921}$$

for $C \in H_1(X_{\hexagon};\mathbb{Z}_2)$, with C_x, C_y denoting representatives for the nontrivial classes in $H_1(T^2;\mathbb{Z}_2)$.

Note that for $J = 0$, there is a second 1-form symmetry, generated by the operators

$$T_C = \prod_{l\in C} X_l, \tag{922}$$

which also commute with the Hamiltonian when $J = 0$. This is an electric symmetry, coming from the conservation of electric flux, which is broken when the term containing electric matter is added. The eigenstates of this symmetry are

$$|\pm_x \pm_y\rangle_E \equiv W_{C_x}^{(1\mp_x 1)/2} W_{C_y}^{(1\mp_y 1)/2}|\Phi_0\rangle, \tag{923}$$

and satisfy $T_{C_j}|\pm_x\pm_y\rangle_E = \pm_j|\pm_x\pm_y\rangle_E$. Note that the $|\pm_x\pm_y\rangle_E$ are just linear combinations of the $|\pm_x\pm_y\rangle_M$, and so in either basis we just have four ground states. In the E basis states the expectation value of the W_C operators vanish since the W_C permute different E basis states, while in the M basis the expectation value of the T_C operators vanish, since they permute different M basis states. Therefore the M basis describe SSB for the electric symmetry, while the E basis states describe SSB for the magnetic symmetry. Note that we can't choose basis states that are in charge eigenstates for both symmetries, since the operator which measures the charge for one symmetry changes the charge of the other. When we turn on a nonzero J the electric symmetry is explicitly broken since the T_C operators no longer commute with the Hamiltonian, and so we will write the eigenstates in the M basis, since this basis is the eigenbasis of the remaining magnetic symmetry.

Anyway, the point is that since the phases in the two limits are both gapped and have different GSDs on T^2, there must be a phase transition between them at some intermediate value of h/J.

To think about the phase transition, it's helpful to re-write stuff in a more suggestive way by making a mapping to an Ising theory. We do this simply by explicitly including the electric matter that is implicitly present in the $H_\triangle$ Hamiltonian, using matter variables τ_i. The mapping is

$$Z_l \mapsto \tau_i^z Z_l \tau_j^z, \qquad \prod_{l \in \partial i} X_l \mapsto \tau_i^x, \tag{924}$$

where now i, j indicate sites of the $\hexagon$ lattice and l indicates links of the $\hexagon$ lattice, i.e. sites of the Kagome lattice. Here the presence of Z_l on the RHS of the mapping is needed to account for the $\mathbb{Z}_2$ gauge redundancy which appears when we try to map $Z_l \mapsto \tau_i^z \tau_j^z$.

This identification means that we map the Hamiltonian as

$$H_\triangle \mapsto H_{\hexagon} = -J \sum_{l=\langle ij \rangle} \tau_i^z Z_l \tau_j^z - h \sum_i \tau_i^x. \tag{925}$$

This Hamiltonain, just like $H_\triangle$, represents electric charges hopping on the $\hexagon$ lattice. In this formulation, the electric charges are just made explicit though the τ variables, whereas before their presence was implicit via violations of the XXX term. In the first term in $H_{\hexagon}$, the two τ^z operators create electric charges, and the Z operator creates electric flux between them. In this formulation, Guass's law means that when acting on physical states,

$$\prod_{l \in \partial i} X_l = \tau_i^x, \tag{926}$$

since the operator $\tau_i^x \prod_{l \in \partial i} X_l$ performs electric gauge transformations.

As a check that we have properly accounted for the degrees of freedom, we can count Hilbert space dimensions. For the Kagome lattice model, we saw that $\ln_2[\dim \mathcal{H}_\triangle] = N_L - N_F = N_V$. For the theory with the matter fields, we have "extra" variables on the sites of the $\hexagon$ lattice, but also a Gauss's law constraint at each site, so that $\dim \mathcal{H}_{\hexagon} = \dim \mathcal{H}_\triangle$. Note that since $\dim \mathcal{H}_{\hexagon} = N_V$ (modulo topological issues), the full Hilbert space is really just that of the matter fields; the gauge fields are only there to enforce constraints / carry topological information.

As another sanity check, we can look at the GSD: when $h = 0$ we are in the Higgs phase where $Z_{\langle ij \rangle} = \tau_i^z \tau_j^z$ on all links; the ground state is non-degenerate. When $J = 0$ the Hamiltonian looks like it's in the product state $\bigotimes_i |+\rangle$, but we have to remember the gauge constraint, which means that not all the τ_i^x are independent. Writing the h term as $\sum_i \prod_{l \in \partial i} X_l$, we see that we get $\dim H_1(X_{\hexagon}; \mathbb{Z}_2)$ ground states as before, distinguished by the eigenvalues of the W_{C_j} operators.

The fact that we've mapped this model to the gauged TFIM means that, since we know how the TFIM behaves, our model only has a single phase transition as a function of h/J, which is the usual 2d Ising transition. One might worry that the gauge field complicates things, but actually since there is no dynamical magnetic flux, the gauge field only keeps track of topological information, and can be ignored for the purposes of thinking about phase transitions. This is because if we ignore topological information, our Hilbert space constraint on the values of the W_C for contractible C completely determines the value of the Z_l operators (as we saw, the Hilbert space dimension is determined solely from the number of matter field sites). For example, if we work on the plane and work in the sector with no magnetic flux, we can just set all of the Z_l variables to $\mathbf{1}$, thereby recovering the regular TFIM model. Working in a different Hilbert space with nonzero magnetic flux just amounts to changing the sign of the Js along paths in the $\triangle$ular lattice dual to the $\hexagon$ lattice, but the statements we can make about the phase transition are unchanged.

As mentioned above, we get four eigenstates of the different W_C operators. Eigenstates in which $\langle W_{C_j}\rangle = \pm$ indicate that the electric charges experience periodic / antiperiodic boundary conditions around C_j, respectively.[110] Now in the $J/h \gg 1$ phase, these eigenstates will have vastly different energies, with the state $|+_x +_y\rangle_M$ being the true ground state. This is because we can think of $J/h \gg 1$ as the "Higgs" phase, where the matter field likes to follow in lockstep with the gauge field. If W_{C_j} has a negative expectation value then the magnitude of the Higgs field cannot be uniform around C_j, and in particular must pass through 0 at some point, since there are no global sections of the nontrivial $\mathbb{Z}_2$ bundle over S^1. This means that e.g. a value of $\langle W_{C_x}\rangle = -1$ implies an energetically costly line along the y direction where the Higgs field changes sign, giving an energy cost that scales as $L_y J$.

However, in the $h/J \gg 1$ limit, the four states are nearly degenerate. This is because the matter has a gap $\sim h$, and so since the boundary conditions will only be felt by processes which tunnel a charge around a cycle, we expect that twisted boundary conditions around the cycle C_j will result in an energy splitting that goes as (we are working in units where L_j is dimensionless, with the lattice spacing set to 1)

$$\Delta E \sim J e^{-L_j/\xi}, \qquad \xi = \frac{1}{\ln(h/J)}. \tag{927}$$

To prove this, we first take $H\psi = E\psi$ and write $\psi = \psi_G + \psi'$, where the Hilbert space splits as $\mathcal{H} = \mathcal{H}_G \oplus \mathcal{H}'$ into a ground-state subspace and the collection of excited states. We write $H = H_0 + H'$ with H_0 the h term that acts within $\mathcal{H}_G$ and H' the J term, which takes states in $\mathcal{H}_G$ to states in $\mathcal{H}'$. Then acting with $(\mathbf{1} - \mathcal{P}) = \mathcal{P}'$ on the Schrodinger equation, where $\mathcal{P}$ is the projector onto $\mathcal{H}_G$ and $\mathcal{P}'$ the projector onto $\mathcal{H}'$, we have

$$\mathcal{P}' H' \psi = (E - H_0)\psi'. \tag{928}$$

Solving for ψ' and adding it to ψ_G gives

$$\psi = \psi_G + \frac{1}{E - H_0}\mathcal{P}' H' \psi \tag{929}$$

so that

$$\psi = \sum_{n=0}^{\infty} \left[\frac{1}{E - H_0}\mathcal{P}' H'\right]^n \psi_G. \tag{930}$$

Acting on this with $\mathcal{P}H$ and then subtracting off the GS energy $H_0\psi_G$ gives us the effective Hamiltonian, which acts only within the $\mathcal{H}_G$ subspace:

$$H_{eff}\psi = \mathcal{P}H' \sum_{n=1}^{\infty} \left[\frac{1}{E - H_0}\mathcal{P}' H'\right]^n \mathcal{P}\psi. \tag{931}$$

[110] Just because the difference between a configuration with two charges and the configuration resulting from moving one of the charges around the cycle C_j differ by an application of W_{C_j}.

Now we apply this the the problem at hand: the terms on the RHS that survive the left-most $\mathcal{P}$ will be those where the products of H' terms form closed loops. At each intermediate process when the loops are being formed, we can write $E - H_0 = -2h + O(J)$, and so the effective Hamiltonian in $\mathcal{H}_G$ is, keeping the leading order J term for each loop (i.e. taking $E - H_0 \approx -2h$ for each term individually, just so that we can actually compute things)

$$H_{eff} = -J \sum_{C \in Z_1(X_{\hexagon};\mathbb{Z}_2)} \epsilon^{L(C)} W_C, \qquad \epsilon \equiv J/2h, \tag{932}$$

with $L(C)$ the length of the loop C.

Now the expectation value of W_C for contractible C is the same in all four eigenstates of the $\mathbb{Z}_2$ 1-form symmetry, since we have fixed the values of the fluxes on all the plaquettes by our Hilbert space constraint. Therefore the splitting between the different 1-form symmetry eigenstates will only come from terms where C wraps a cycle. Therefore we see that the $|+_x +_y\rangle_M$ state has the lowest energy, but that the splitting to the other states is only of order (taking $L_x = L_y$)

$$\Delta E \sim J\epsilon^{L_x}, \tag{933}$$

which vanishes exponentially quickly in the thermodynamic limit.

Now we add the term

$$\Delta H = -\Gamma \sum_l X_l, \tag{934}$$

which has the effect of creating a pair of magnetic fluxes, since it anticommutes with Wilson lines that run through the links l. We could also keep track of the fluxes as actual matter fields themselves; doing this would mean writing $\widetilde{\tau}^z_a X_{ab} \widetilde{\tau}^z_b$ instead of X_l, with $l = ab$ and a, b denoting plaquettes on the $\hexagon$ lattice.

Since magnetic flux lines are no longer conserved, we no longer have a $\mathbb{Z}_2$ 1-form symmetry, and can no longer index the states by eigenvalues of the W_C operators; consequently we can no longer restrict our Hilbert space to a specific flux configuration. However, the different $|\pm_x \pm_y\rangle_M$ states can only be connected by the T_C operators, which require the application of L_x different δHs. Now the magnetic charges are gapped, since as we have seen perturbation theory generates the term $-\sum_{\hexagon} W_{\hexagon} = -\sum_{\hexagon} \widetilde{\tau}^x_{\hexagon}$. Therefore the hybridization between the different states will be exponentially small in L for the same reason as in the last part of the problem. If we write (still in the phase where $h \gg J$ so that we have (nearly) degenerate groundstates) the Hamiltonian as

$$H \approx -J\epsilon^5 \sum_{\hexagon} W_{\hexagon} - h \sum_i \prod_{l \in \partial i} X_l - \Gamma \sum_l X_l \tag{935}$$

and take $J/\Gamma \gg 1$, then we can repeat the procedure in the last part of the problem with $H' = \delta H$. This gives

$$H_{eff} \approx -J\epsilon^5 \sum_{\hexagon} W_{\hexagon} - \Gamma \sum_{C \in Z_1(X^*_{\triangle};\mathbb{Z}_2)} \eta^{L(C)} T_C, \qquad \eta \equiv \Gamma/2J, \tag{936}$$

where $X^*_{\triangle}$ is the triangular lattice dual to the Honeycomb lattice. All of the terms in the second sum have the same expectation value in each of the W_C eigenstates, except for the ones where C is non-contractible, which lead to hybridization between the different W_C eigenstates. Therefore the energy splitting coming from the δH term goes as

$$\Delta E \sim \Gamma \eta^{L_x}, \tag{937}$$

which again vanishes in the thermodynamic limit.

58 *May 11 — Anomalies in $SO(3) \times T$ symmetric $\mathbb{Z}_2$ spin liquids*

This is another problem from Senthil's class. Consider a gapped $\mathbb{Z}_2$ spin liquid in 2+1D, enriched with $SO(3)$ symmetry and time-reversal. Show that as long as T is preserved, assigning both e and m spin 1/2 is anomalous. Find out how to cancel the anomaly with a 3+1D bulk.

Solution:

As a field theory, the action for the case where both e and m particles carry integer spin can be written (the notation here is $\bar{S} \equiv S/2\pi$)

$$\bar{S} = \frac{1}{2} \int a \cup \delta b. \tag{938}$$

Here the Poincare dual of δa is the worldline of an e particle, while that of δb is the worldline of an m particle; we will use a and e quasi-interchangably, and likewise for b and m.

Now suppose that e.g. e has spin 1/2. This means that an a worldline $e^{i \oint a}$ is not gauge invariant under certain gauge transformations in $SO(3)$ (i.e., certain re-arrangements of the transition functions in the $SO(3)$ bundle). In this context, consider a gauge transformation which changes the $SO(3)$ bundle in a manner such that the product of transition functions along a closed loop changes by the nontrivial element of $\pi_1(SO(3)) = \mathbb{Z}_2$. Then since the spin of a is 1/2, a Wilson line for a, when wrapped around this loop, changes by a minus sign (see figure 8).

Now consider the theory in the background of an $SO(3)$ monopole, and let ω_2 be the second SW class of the $SO(3)$ bundle. Physically, the Poincare dual of ω_2 is the Dirac string coming out of the monopole; under the gauge transformations described above, this string changes shape. Therefore, we see that such a gauge transformation sends

$$a \mapsto a + \lambda, \omega_2 \mapsto \omega_2 + \delta\lambda, \tag{939}$$

with λ some $\mathbb{Z}_2$ 1-cochain.

We can make the action invariant under this transformation if we couple b to ω_2:

$$\bar{S} = \frac{1}{2} \int_X b \cup (\delta a - \omega_2). \tag{940}$$

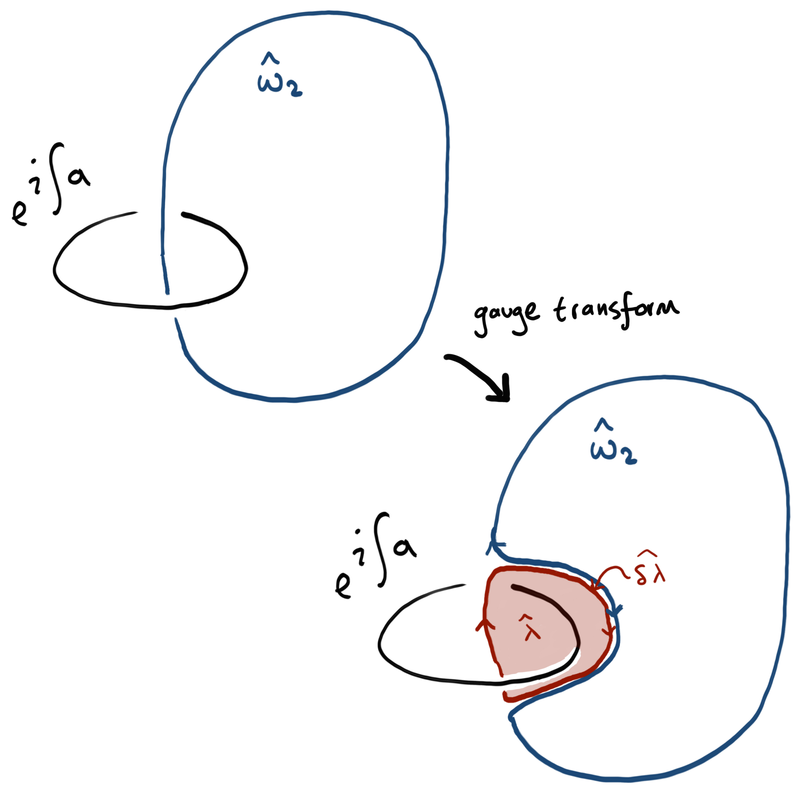


Figure 8: Schematic illustration of of what happens to an a Wilson line when we do a gauge transformation by fiddling with the transition functions in the $SO(3)$ bundle. When we shift the $SO(3)$ background field by $\delta\lambda$, the Poincare dual $\widehat{\omega}_2$ of ω_2 changes by the dual of $\delta\lambda$. Since λ is a 1-chain it is dual to an open surface, with $\delta\lambda$ dual to that surface's boundary. After the gauge transformation, $\widehat{\omega}_2$ changes accordingly—in the figure, the gauge-transformed $\widehat{\omega}_2$ no longer links with the Wilson line. Since $\widehat{\omega}_2$ changes by the boundary of a surface, its topological class cannot change. The difference between the phase of $e^{i\int_C a}$ before and after the gauge transformation is measured by the intersection number between $\widehat{\lambda}$ and C, and so can be obtained by shifting $a \mapsto a + \lambda$.

Note that the modification of the action is only non-trivial when ω_2 is nontrivial in $H^2(X;\mathbb{Z}_2)$, i.e. when the $SO(3)$ gauge configuration has a monopole.

A physical perspective on why this is the right thing to do is the following: in order for the gauge field to be a legit $SO(3)$ gauge field, the Dirac string emanating from the monopole needs to be invisible. When a has spin $1/2$ the string is not invisible, since an a line will detect it. However, consider adding the coupling

$$\frac{1}{2}\int b \cup \omega_2 = \frac{1}{2}\int_{\widehat{\omega}_2} b, \tag{941}$$

where $\widehat{\omega}_2$ is the Poincare dual of ω_2, alias the Dirac string. We see that the Dirac string $\widehat{\omega}_2$ now carries b charge, which because of the braiding phase between e and m, renders the $\widehat{\omega}_2$ worldline properly transparent.

This is all well and good, and the same construction works if b has spin $1/2$ while a has integer spin, provided that we instead add the coupling $\frac{1}{2}\int a \cup \omega_2$ to the action.

Problems occur when we take both e and m to have spin $1/2$. On one hand, we can check explicitly that the putative action

$$\bar{S} = \frac{1}{2}\int (a \cup \delta b - (a+b) \cup \omega_2) \tag{942}$$

is not gauge-invariant: it changes as

$$\delta \bar{S} = -\frac{1}{2}\int \lambda \cup \delta\lambda. \tag{943}$$

This means that the symmetry is anomalous: this is a contradiction, since we assumed we were working with a spin system where the $SO(3)$ symmetry acted in an on-site fashion, and such an on-site symmetry action can always be coupled to a background gauge field in a gauge-invariant way. Physically, the anomaly happens for the following reason: in order to make the Dirac string transparent to the spin $1/2$ charges, we need to make the $SO(3)$ monopole carry charge under both a and b gauge fields. But then the mutual statistics of the e and m particles means that the monopole is a fermion, and hence the Dirac string can detect itself, which breaks $SO(3)$ gauge invariance.

This gauge-noninvariance problem can be solved by realizing our theory at the boundary of some 4-manifold B. This is possible without extending the a or b fields into B, since luckily there are no a or b fields appearing in the expression for $\delta\bar{S}$. We consider the bulk action

$$\bar{S}_B = \frac{1}{2}\int_B \mathcal{P}(\omega_2), \tag{944}$$

where $\mathcal{P}$ is the Pointryagin square, which maps $\mathbb{Z}_2$ classes to $\mathbb{Z}_4$ classes. The variation of this bulk term is precisely what is needed to cancel $\delta\bar{S}$, and it has the effect of giving an extra minus sign to processes during which ω_2 worldlines are braided on the boundary, rendering the $SO(3)$ monopoles properly bosonic.

Now since $\mathcal{P}(\omega_2)$ is an even class when evaluated on a spin manifold, the action S_B is actually independent of the choice of B when B is spin. However, we are dealing with a bosonic system—all the local operators carry integer spin—and so our construction needs to be completely independent of the existence and choice of a spin structure. Therefore $SO(3)$ gauge invariance cannot be restored through a term that does not depend on a choice of bulk 4-manifold, since we cannot make the assumption that B is spin.

So far we have not used time reversal at all. The reason why the preservation of T is important here is the following. First, note that in terms of the curvature of the $SO(3)$ bundle, the bulk action can be written

$$S_B = 4\pi l = \frac{4\pi}{4 \cdot 8\pi^2} \int_B \mathrm{Tr}[F \wedge F], \tag{945}$$

where l is the instanton number, which would be valued in $\frac{1}{4}\mathbb{Z}$ if B where closed, and would be such that $l \in \mathbb{Z}$ if B were closed and the $SO(3)$ bundle in question could be lifted to an $SU(2)$ bundle. Anyway, the point is that S_B represents a θ term at $\theta = 4\pi$ (θ is 8π periodic on a closed 4-manifold). Since θ can be tuned continuously, in the absence of time reversal symmetry, it can be smoothly tuned to zero. Hence the spin liquid is not anomalous in the absence of time reversal, as the bulk action can then be eliminated with a symmetric counterterm in the background field.

One potentially confusing aspect of this is that S_B, taken by itself, breaks time-reversal (it would be T invariant if B were closed, but it's not). Indeed,

$$T : S_B \mapsto S_B - \frac{1}{4\pi} \int_B \mathrm{Tr}[F \wedge F] = S_B - \int_{\partial B} \mathcal{L}_{SO(3)_2}, \tag{946}$$

so that the bulk action changes by a level-2 $SO(3)$ Chern-Simons term.[111] Therefore if the theory really is T invariant, the boundary partition function must also change by an $SO(3)_2$ CS term under T, in a way rather similar to the change in the partition function of a massless Dirac fermion coupled to $U(1)$ that we know from the context of the parity anomaly.

This change in the boundary partition function is a bit surprising if one just looks at the action, since it's unclear how a change in the fields can reproduce the $SO(3)_2$ term. But it's also reasonable that the boundary theory should break T by itself, since the fact that the monopoles are fermions and the boundary is three dimensional means that the monopoles are Kramers doublets, and so the relation $T^2 = \mathbf{1}$ when acting on states of the full boundary theory is broken in the presence of a monopole.

One rather crappy way of finding the variation of the boundary partition function is as follows. Consider an $SO(3)$ monopole with a gauge potential of the form $A = \mathcal{A}_\mu T^3 dx^\mu$, where $\mathcal{A}$ is a $U(1)$ monopole configuration and $T^3 = \mathrm{diag}(1, 0, -1)$ is a diagonalized $\mathfrak{so}(3)$ generator. This form of the $SO(3)$ gauge field can be taken wolog, since such configurations can realize the minimal-strength $SO(3)$ instanton, where we can have the minimal

[111] The $SO(N)_k$ CS term is normalized as

$$\mathcal{L}_{SO(N)_k} = \frac{k}{8\pi} \int_M \mathrm{Tr}\left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right]. \tag{947}$$

The coefficient in front is chosen so that when $k = 1$, we get a well-defined action provided that M is spin. Note that e.g. for $SO(3)$ this reduces to $U(1)_k$ when we restrict the field strength to lie in a particular direction in flavor space.

$\frac{1}{8\pi^2}\int \mathrm{Tr}[F\wedge F] = \int (F_{\mathcal{A}}/2\pi \wedge F_{\mathcal{A}}/2\pi) = 1$ on a closed manifold. Anyway, with this choice, the integral class $\widetilde{F}_{\mathcal{A}} \equiv F_{\mathcal{A}}/2\pi$ reduces mod 2 to ω_2. Therefore we can write the full action as

$$S = \frac{2\pi}{2}\int_{\partial B}(a\cup\delta b - (a+b)\cup \widetilde{F}_{\mathcal{A}}) + \pi\int_B \widetilde{F}_{\mathcal{A}}\wedge\widetilde{F}_{\mathcal{A}}. \tag{948}$$

Let's now look at the variation of the first term under T. Integrating out a tells us that $\delta b = \widetilde{F}_{\mathcal{A}}$, and so the boundary action is, switching to a continuum notation,

$$S_{\partial B} = -\frac{1}{4\pi}\int_{\partial B}\mathcal{A}\wedge F_{\mathcal{A}}. \tag{949}$$

This level-1 $U(1)$ CS term[112] is actually quite reasonable, since it confirms that the $SO(3)$ monopoles are fermions.

When we act with T, we see that $S_{\partial B}\mapsto S_{\partial B} + \int_{\partial B}\mathcal{L}_{U(1)_2}$, or, generalizing back to the case with an arbitrary $SO(3)$ field configuration,

$$T : S_{\partial B}\mapsto S_{\partial B} + \int_{\partial B}\mathcal{L}_{SO(3)_2}. \tag{950}$$

This extra term precisely cancels the extra term picked up from the T-variation of the bulk term, and the full theory is therefore T-invariant.

59 *May 12 — Fermions and magnetic moments and stuff*

Today we're doing problem 10.1 in Schwartz, which is something basic that I'd never worked through before. The goal is to take the non-relativistic limit of the Dirac equation and derive how the electron couples to EM fields in this limit. Unfortunately the method which Schwartz suggests in the problem statement doesn't work (at least, I don't think it works) when the electric field is nonzero (due to an incorrect sleight of hand involving differential operators and square roots). We will follow the outline of the problem but will use more careful (and longer) methods to get the electric field dependence right.

Solution:

Our sign conventions for the Dirac equation will be $(i\hbar c\not{\partial} - ec\not{A} - mc^2)\psi = 0$, so that $i\partial_t\psi = H_D\psi$ means[113]

$$H_D = -i\gamma^0\gamma^j D_j + eA^0 + \gamma^0 mc^2, \qquad D_j = \hbar c(\partial_j + i\hbar^{-1}eA_i). \tag{951}$$

[112] Note that this action doesn't actually mean that the theory is spin; the pre-integrating-out-a theory is clearly not spin *waves hands*.

[113] The only potentially hard-to-remember assignment of cs and $\hbar$s is the one for the vector potential. A telsa is a kg /(s C), so that eA_j is valued in kg m/s, meaning that ecA_j has units of energy.

Schwartz tells you to subtract off eA_0 and square, but doing this makes you liable to forget about terms like $\partial_j A_0$ which need to be retained to get e.g. the SOC term and the $\nabla^2 A_0$ term. Choosing the non-Weyl basis (so that the rest energy $\gamma^0 mc^2$ is diagonal)

$$\gamma^0 = Z \otimes \mathbf{1}, \qquad \gamma^j = J \otimes \sigma^j, \tag{952}$$

we have

$$H_D = (Z \otimes \mathbf{1})mc^2 + \phi + i(X \otimes \sigma^j)D_j, \qquad \phi \equiv eA_0. \tag{953}$$

From now on, all $\otimes$s and $\mathbf{1}$s will be omitted: X, Y, Z will be understood to live in the first $\otimes$ factor, and σ^j will be understood to live in the second.

The strategy is now to subject H to a SW transformation and systematically eliminate all the terms which are off-diagonal in the first tensor factor, so that the transformed Hamiltonian contains only things like $\mathbf{1} \otimes (\dots)$ and $Z \otimes (\dots)$. The offending off-diagonal terms will usually come in the form of $X\sigma^j \partial_j$s and $J\sigma^j\partial_j$s. Eliminating these terms will give us a Hamiltonian that we can easily find the spectrum of. We will work in the non-relativistic limit where $p^2/2m \ll mc^2$—this facilitates the diagonalization process described above because the off-diagonal terms will generally go as ratios of momenta to powers of mc^2.

To this end, rewrite the Schrodinger equation as

$$e^\Lambda i\hbar\dot{\psi} = e^\Lambda H e^{-\Lambda}(e^\Lambda \psi) \implies i\hbar\partial_t \widetilde{\psi} = \left(e^\Lambda H e^{-\Lambda} + i\hbar(\partial_t e^\Lambda)e^{-\Lambda}\right)\widetilde{\psi} = \widetilde{H}\widetilde{\psi}, \qquad \widetilde{\psi} \equiv e^\Lambda \psi. \tag{954}$$

Here Λ is some anti-Hermitian matrix that we will need to solve for. In tomorrow's diary entry we will prove the tools needed to show that when expanded in Λ, the Schrodinger equation for $\widetilde{\psi}$ is

$$i\hbar\partial_t\widetilde{\psi} = \sum_{k=0}^{\infty}\left(\frac{\mathcal{N}_k}{k!} + i\hbar\frac{\mathcal{C}_k}{(k+1)!}\right)\widetilde{\psi}, \tag{955}$$

where we have defined the nested commutators

$$\mathcal{N}_k \equiv [\Lambda, [\cdots, [\Lambda, H]\cdots]], \qquad \mathcal{C}_k \equiv [\Lambda, [\cdots, [\Lambda, \dot{\Lambda}]\cdots]], \tag{956}$$

where both terms contain k powers of Λ.

We will now work out an expression for Λ, order-by-order in the non-relativistic limit). Because I'm feeling slightly masochistic tonight, we will go to order $(pc)^4/(mc^2)^3$. This will entail performing three SW transformations—one to push all the p terms to order p^2, one to get to p^3, and then a final one to get to p^4. Now keeping all the terms that can possibly contribute, we have

$$\begin{aligned}\widetilde{H} = H + ([\Lambda, H] + i\hbar\dot{\Lambda}) + \frac{1}{2}([\Lambda,[\Lambda,H]] + i\hbar[\Lambda,\dot{\Lambda}]) + \frac{1}{6}([\Lambda,[\Lambda,[\Lambda,H]]] + i\hbar[\Lambda,[\Lambda,\dot{\Lambda}]])\\ + \frac{1}{24}([\Lambda,[\Lambda,[\Lambda,[\Lambda,H]]]] + i\hbar[\Lambda,[\Lambda,[\Lambda,\dot{\Lambda}]]]).\end{aligned} \tag{957}$$

Now split up the Hamiltonian as

$$H_D = H_0 + H', \qquad H_0 = \phi + Zmc^2, \qquad H' = iX\not{D}_A. \tag{958}$$

To lowest order, we should look for a Λ such that $[\Lambda, H_0] = -H' + \ldots$, where $\ldots$ does not contain $\not{D}_A$. Such a Λ is

$$\Lambda_1 = \frac{i}{2mc^2} J \not{D}_A = -\frac{1}{2mc^2} Z H' \implies [\Lambda_1, H_0] = \frac{1}{2}[Z, iJ\not{D}_A] + \Omega = -iX\not{D}_A + \Omega, \tag{959}$$

where we've defined

$$\Omega \equiv [\Lambda_1, \phi] = \frac{i}{2mc^2} J\hbar c \boldsymbol{\sigma} \cdot \nabla\phi. \tag{960}$$

Note that Λ_1 is properly anti-Hermitian, so that e^{Λ} implements a unitary transformation on H. We will also need

$$[\Lambda_1, H'] = \frac{1}{mc^2} Z(\not{D}_A)^2 = 4mc^2 Z\Lambda_1^2 \tag{961}$$

and

$$[\Lambda_1, Z\Lambda_1^2] = \frac{iX}{4(mc^2)^3}\not{D}_A^3 = -2Z\left(iJ\frac{\not{D}_A}{2mc^2}\right)^3 = -2Z\Lambda^3. \tag{962}$$

Similarly,

$$[\Lambda_1, Z\Lambda_1^3] = -2Z\Lambda_1^4 \tag{963}$$

To save space, define the rest energy as $\alpha \equiv mc^2$. Then we have

$$\begin{aligned}
[\Lambda_1, [\Lambda_1, H]] &= -4\alpha Z\Lambda_1^2 - 8\alpha Z\Lambda_1^3 + [\Lambda_1, \Omega], \\
[\Lambda_1, [\Lambda_1, [\Lambda_1, H]]] &= 8\alpha Z\Lambda_1^3 + 16\alpha Z\Lambda_1^4 + [\Lambda_1, [\Lambda_1, \Omega]] \\
[\Lambda_1, [\Lambda_1, [\Lambda_1, [\Lambda_1, H]]]] &= -16\alpha Z\Lambda_1^4 + [\Lambda_1, [\Lambda_1[\Lambda_1, \Omega]]],
\end{aligned} \tag{964}$$

where in the last line we only kept terms of order Λ_1^4. To make the expansion less messy, we will work in the limit where the momenta of both the $\widetilde{\psi}$ field and the EM field are small compared to the rest energy α, so that we're basically doing an expansion in $1/\alpha$, and dropping everything that goes as $1/\alpha^{n\geq 4}$. Therefore in our order of approximation, we plug everything into $\widetilde{H}$ and get

$$\begin{aligned}
\widetilde{H} &= H_0 + 2\alpha Z\left(\Lambda_1^2 - \frac{4}{3}\Lambda_1^3 + \Lambda_1^4\right) + \Omega + \frac{1}{2}[\Lambda_1, \Omega] + \frac{1}{6}[\Lambda_1, [\Lambda_1, \Omega]] \\
&\quad + i\hbar\left(\dot{\Lambda}_1 + \frac{1}{2}[\Lambda_1, \dot{\Lambda}_1] + \frac{1}{6}[\Lambda_1, [\Lambda_1, \dot{\Lambda}_1]]\right) \\
&= \widetilde{H}_0 + \widetilde{H}',
\end{aligned} \tag{965}$$

where $\widetilde{H}_0$ is diagonal in the first $\otimes$ factor and $\widetilde{H}'$ is off-diagonal.

We aren't done, because there are still terms which are off-diagonal in the first $\otimes$ factor. Therefore we must perform another SW transformation to get rid of these terms. For our first transformation, note that we chose $\Lambda_1 = -\frac{1}{2\alpha}ZH'$, where H' was the term that was off-diagonal and needed to be killed. This prompts us to try a second transformation with

$$\Lambda_2 = -\frac{1}{2\alpha}Z\widetilde{H}' \approx -\frac{1}{2\alpha}Z\left(-\frac{8}{3}\alpha Z\Lambda_1^3 + \Omega + i\hbar\dot{\Lambda}_1\right), \tag{966}$$

where we dropped higher-order terms that are killed by the $1/\alpha$ in front.

To check that this works, one just needs to check that $[\Lambda_2, \widetilde{H}_0] = -\widetilde{H}' + \dots$, where $\dots$ don't involve bare $\not{D}_A$s. For example, let's check that the Λ_1^3 term in $\widetilde{H}'$ gets killed:

$$[\Lambda_2, \widetilde{H}_0] \ni [\Lambda_2, H_0] \ni \frac{4}{3}[\Lambda_1^3, Z\alpha] = \frac{8Z\alpha}{3}\Lambda_1^3, \tag{967}$$

which is exactly the right term needed to kill the $-\frac{8\alpha Z}{3}\Lambda_1^3$ appearing in $\widetilde{H}'$. The full commutator is then checked to be (still to order $1/\alpha^3$)

$$[\Lambda_2, \widetilde{H}_0] \approx -\widetilde{H}' - \frac{4}{3}[\phi, \Lambda_1^3]. \tag{968}$$

Since all the terms in Λ_2 go as at least $1/\alpha^2$, we have

$$e^{\Lambda_2}\widetilde{H}e^{-\Lambda_2} = \widetilde{H}_0 + \frac{1}{2}[\Lambda_2, \widetilde{H}'] - \frac{4}{3}[\phi, \Lambda_1^3]. \tag{969}$$

The commutator is worked out to be, to this order[114]

$$\frac{1}{2}[\Lambda_2, \widetilde{H}'] = -\frac{1}{2\alpha}[Z(\Omega + i\hbar\dot{\Lambda}_1), \Omega + i\hbar\dot{\Lambda}_1] = \frac{Z}{\alpha^3}(\hbar ce)^2(\mathbf{E}\cdot\boldsymbol{\sigma})^2 = \frac{Z}{\alpha^3}(\hbar ce)^2|\mathbf{E}|^2. \tag{971}$$

A final SW transform removes this, at the expense of generating further higher-order terms which are then thrown away. So in conclusion, to this order, the transformed Hamiltonian is

$$H_{\text{eff}} = \phi + Z\alpha + 2\alpha Z(\Lambda_1^2 + \Lambda_1^4) + \frac{1}{2}[\Lambda_1, \Omega] + \frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] + \frac{Z}{\alpha^3}(\hbar c)^2(\mathbf{E}\cdot\boldsymbol{\sigma})^2. \tag{972}$$

Now we just need to calculate the remaining commutators. The first is

$$\frac{1}{2}[\Lambda_1, \Omega] = \frac{1}{2(2\alpha)^2}[\not{D}_A, \hbar c\boldsymbol{\sigma}\cdot\nabla\phi] = \frac{e(\hbar c)^2}{2(2\alpha)^2}\sigma^i\sigma^j\partial_i\partial_j A_0 + \frac{\hbar ce}{2(2\alpha)^2}[\sigma^i, \sigma^j]\nabla_i A_0 D_j. \tag{973}$$

The second is

$$\frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] = \frac{ec\hbar}{2(2\alpha)^2}[\not{D}_A, \boldsymbol{\sigma}\cdot\dot{\mathbf{A}}] = \frac{(\hbar c)^2 e\sigma^j\sigma^i\partial_j\dot{A}_i}{2(2\alpha)^2} + \frac{\hbar ce}{2(2\alpha)^2}[\sigma^i, \sigma^j]A_i D_j. \tag{974}$$

Adding these up and simplifying the σ commutators, we get

$$\begin{aligned}\frac{1}{2}[\Lambda_1, \Omega] + \frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] &= \frac{e}{2}\left(\frac{\hbar c}{2\alpha}\right)^2\sigma^i\sigma^j\partial_j(\partial_i A_0 - \dot{A}_i) - i\frac{ec\hbar}{4\alpha^2}\sigma^k\epsilon_{ijk}(\partial_i A_0 - \dot{A}_i)D_j \\ &= \frac{e}{2}\left(\frac{\hbar c}{2\alpha}\right)^2(\nabla\cdot\mathbf{E} + i\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{E})) - i\frac{ec\hbar}{(2\alpha)^2}\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{D}).\end{aligned} \tag{975}$$

These terms, which we've worked so hard for, are the ones that we miss out on if we take the approach in e.g. Schwartz.

[114] To get this, we used

$$\dot{\Lambda}_1 = -\frac{ec}{2\alpha}\boldsymbol{\sigma}\cdot\dot{\mathbf{A}}. \tag{970}$$

The last thing we need to do then is to calculate Λ_1^2. It is

$$2\alpha\Lambda_1^2 = \frac{1}{2\alpha}\left[(\hbar c\partial_i + iecA_i)^2 + \frac{i[\sigma^i,\sigma^j]}{4}\hbar c^2 eF_{ij}\right] = \frac{1}{2\alpha}\left[(\hbar c\partial_i + iecA_i)^2 - 2ec^2\mathbf{S}\cdot\mathbf{B}\right], \quad (976)$$

with $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$. Therefore

$$2\alpha Z(\Lambda_1^2 + \Lambda_1^4) = -Z\left[\frac{|\boldsymbol{\pi}|^2}{2m} + 2\mu_B\mathbf{S}\cdot\mathbf{B}\right] + Z\frac{1}{8\alpha^3}(c^4|\boldsymbol{\pi}|^4 + 8\hbar^2e^2c^4|\mathbf{B}|^2), \quad (977)$$

where $\boldsymbol{\pi} = \mathbf{p} + ec\mathbf{A}$ is the canonical momentum and $\mu_B = e/(2m)$ is the Bohr magneton. Note that we have dropped the $\boldsymbol{\pi}$-$\mathbf{B}$ cross terms in the expansion above, but retained the $\mathbf{B}^2$ term: this is just because the later will fit nicely in with the $\mathbf{E}^2$ term we derived above (notice that its coefficient is $\hbar^2e^2c^2/(m^3c^4)$, which is c^2 times the coefficient of the $\mathbf{E}^2$ part). In principle we should keep the mixed term; we're just dropping it cause it's relatively high-order, and ugly.

We have finally calculated everything we need to calculate. For aesthetic purposes we will rename $Z \mapsto -Z$, just so that the $|\pi|^2$ term has a coefficient $+Z$ instead of $-Z$. Adding everything together, we get our final Hamiltonian:[115]

$$\begin{aligned} H_{\text{eff}} = Zmc^2 + eA_0 + Z\left(\frac{|\boldsymbol{\pi}|^2}{2m} - \frac{|\boldsymbol{\pi}|^4}{8m^3c^2}\right) - \frac{e\hbar}{4m^2c^2}\left(\hbar\nabla\cdot\mathbf{E} + i\hbar\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{E}) + \sigma\cdot(\mathbf{E}\times\boldsymbol{\pi})\right) \\ - 2Z\mu_B\mathbf{S}\cdot\mathbf{B} + Z\frac{\hbar^2e^2}{m^3c^2}(c^2|\mathbf{B}|^2 + |\mathbf{E}|^2). \end{aligned} \quad (978)$$

From this Hamiltonian, we can read off a lot. For example, we can find the gyromagnetic ratio g for the electron by choosing a rotationaly-symmetric background field and looking at the relative sizes of the spin $\mathbf{S}\cdot\mathbf{B}$ and orbital $\mathbf{L}\cdot\mathbf{B}$ interactions. The former appears just as $2\mu_B$. For the later, fix a gauge in which we have a uniform field along the z direction, viz. $\mathbf{A} = \frac{B^z}{2}(-y, x, 0)$. The lowest order term which involves $\mathbf{L}\cdot\mathbf{B}$ comes from the expansion of $|\pi|^2$, and we see that the coefficient in front of B^zL^z is $e/2m = \mu_B$. Therefore the ratio of the two couplings tells us that $g = 2$.

Another thing one can do is to find the spin angular momentum of the electron, by checking that $[\mathbf{L} + a\mathbf{S}, H_{\text{eff}}] = 0$ for $a = 1$, provided that the EM potential is spherically symmetric. We can also read off the SOC interaction for a Coulomb potential; choosing $\mathbf{E} = -\partial_r A_0\widehat{\mathbf{r}}$ the $\mathbf{E}\times\boldsymbol{\pi}$ term becomes

$$+\frac{e\hbar}{4m^2c^2}\sigma^i\left(r^{-1}\partial_r A_0[\mathbf{r}\times\boldsymbol{\pi}]_i\right) = \frac{e\hbar}{4m^2c^2}r^{-1}\partial_r A_0\boldsymbol{\sigma}\cdot\mathbf{L}. \quad (979)$$

The remaining terms in H_{eff} include the relativistic $|\boldsymbol{\pi}|^4$ correction to the kinetic energy, the magnetic interaction between the electron and the magnetic field produced by $\nabla\times\mathbf{E}$, a quantum-mechanical correction to the potential energy caused by the zero-point motion of the charge density (the $\nabla\cdot\mathbf{E}$ term), and an induced kinetic term for the EM fields (the last term—I haven't seen it in any textbooks before so it may be suspect).

[115] I tried, but I think slightly failed, to get all the minus signs consistent in the above. I might come back and fix them later, but for now I'll just fix the signs in the answer by what we know they should be on physical grounds (of course, we can take $Z \mapsto -Z$ without affecting anything).

60 *May 12 — SW transformation and derivatives of matrix exponentials*

Today we're just working out some math facts that we need to do SW transformations on Hamiltonians. Nothing fancy, but I thought it'd be nice to have around as a reference.

Solution:

First, we want an expression for $\partial_t e^\Lambda$ which doesn't assume that $[\Lambda, \dot{\Lambda}] = 0$. First, expand the exponential: there are n ways to take the time derivative in the nth term, and so

$$\partial_t e^\Lambda = \sum_{n\in\mathbb{N}} \sum_{k=0}^{n} \frac{1}{(n+1)!} \Lambda^k \dot{\Lambda} \Lambda^{n-k}. \tag{980}$$

To deal with this, we systematically commute the time derivative to the front of each term:

$$\begin{aligned}
\Lambda^k \dot{\Lambda} \Lambda^{n-k} &= \Lambda^{k-1}(\dot{\Lambda}\Lambda + [\Lambda, \dot{\Lambda}])\Lambda^{n-k} \\
&= \Lambda^{k-2}(\dot{\Lambda}\Lambda + [\Lambda, \dot{\Lambda}])\Lambda^{n-k} + \Lambda^{k-2}([\Lambda, \dot{\Lambda}]\Lambda + [\Lambda, [\Lambda, \dot{\Lambda}]])\Lambda^{n-k},
\end{aligned} \tag{981}$$

and so on. Let us define the nested commutator

$$\mathcal{C}_n \equiv [\Lambda, [\cdots, [\Lambda, \dot{\Lambda}] \cdots]], \tag{982}$$

where $\mathcal{C}_n$ contains n nested commutators; $\mathcal{C}_0 = \dot{\Lambda}, \mathcal{C}_1 = [\Lambda, \dot{\Lambda}]$, etc. Playing with this a bit, we see that if the derivative occurs in a spot with k powers of Λ before it, then there are k commutation steps needed, and k choose l ways to produce the term $\mathcal{C}_l$. Therefore

$$\partial_t e^\Lambda = \sum_{n\in\mathbb{N}} \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{k}{l} \mathcal{C}_l \Lambda^{n-l}. \tag{983}$$

Now we recall the identity

$$\sum_{k=l}^{n} \binom{k}{l} = \binom{n+1}{l+1}. \tag{984}$$

For a fixed n, consider a fixed l. It will appear in all sums with $k \geq l$. Therefore we can re-arrange the sum by l and write

$$\partial_t e^\Lambda = \sum_{n\in\mathbb{N}} \sum_{l=0}^{n} \binom{n+1}{l+1} \mathcal{C}_l \Lambda^{n-l} = \sum_{n\in\mathbb{N}} \sum_{l=0}^{n} \frac{\mathcal{C}_l}{(l+1)!} \frac{\Lambda^{n-l}}{(n-l)!}. \tag{985}$$

Again consider a fixed l. It appears in every summand with $n \geq l$, and these summands contain $\Lambda^m/m!$, where $m = n - l$ is summed over all $\mathbb{N}$. Therefore we can re-write the sum over n as an exponential, yielding

$$\partial_t e^\Lambda = \sum_{l\in\mathbb{N}} \frac{\mathcal{C}_l}{(l+1)!} e^\Lambda. \tag{986}$$

Another related expression that we'll need is one for $e^\Lambda H e^{-\Lambda}$, which comes up when doing SW transformations. This one is comparatively easy:

$$e^\Lambda H = \sum_{n\in\mathbb{N}} \sum_{k=0}^{n} \frac{1}{n!} \mathcal{N}_k \binom{n}{k} \Lambda^{n-k}, \tag{987}$$

where we have defined

$$\mathcal{N}_k = [\Lambda, [\cdots, [\Lambda, H] \cdots]], \tag{988}$$

where there are k total Λs appearing. To get the expression for $e^\Lambda H$, we used the same strategy: after expanding the exponential we move the H through the Λs until it gets to the left. For the nth term in the expansion, the kth order nested commutator has $\binom{n}{k}$ ways of appearing; hence the RHS of (987). Then using the same strategy as above,

$$e^\Lambda H = \sum_{n\in\mathbb{N}} \sum_{k=0}^{n} \frac{\mathcal{N}_k}{k!} \frac{1}{(n-k)!} \Lambda^{n-k} = \sum_{k\in\mathbb{N}} \frac{\mathcal{N}_k}{k!} e^\Lambda, \tag{989}$$

and so

$$e^\Lambda H e^{-\Lambda} = \sum_{k\in\mathbb{N}} \frac{\mathcal{N}_k}{k!}. \tag{990}$$

61 *March 14 — Nonrenormalizable theories and Fourier transforms*

This is basically a problem in Schwartz.

Solution:

Therefore we have to do the Fourier transform

Now this looks like a disaster, but it really isn't: after being transformed back into real space, these terms all become $\delta(x)$s and derivatives of $\delta(x)$s! Therefore these divergences only affect correlation functions at coincident points, which we don't ascribe physical significance to anyway.

Now let's look at some examples. First consider a logarithmic singularity. The Fourier transform is

$$\begin{aligned} \mathcal{F}[\ln(p^2/\Lambda)] &= \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot x} \ln(p^2/\Lambda^2) \\ &= -\frac{1}{\pi^2} \int dp\, p^2 \ln(p^2/\Lambda) \frac{\sin(px)}{px} \\ &= \frac{1}{\pi^2} \int dp\, \frac{\sin(px)}{px^2}, \end{aligned} \tag{991}$$

where we integrated by parts and used the first Bessel function

$$\int dp\, p \sin p = -\frac{p\cos(px)}{x} + \frac{\sin(px)}{x^2}, \tag{992}$$

and dropped the cosine on the grounds that it "integrates to zero". If we remember to put back in the convergence factor $i\epsilon$, the integral is

$$\begin{aligned} \mathcal{F}[\ln(p^2/\Lambda)] &= \frac{1}{2\pi^2 x^3}\int_{\mathbb{R}} dt\, \frac{\sin t}{t+i\epsilon} = -i\frac{1}{4\pi^2 x^3}\oint_L dt\, \frac{e^{-it}}{t+i\epsilon} \\ &= -\frac{1}{2\pi x^3}, \end{aligned} \tag{993}$$

where we closed the integral in the lower half plane (the minus sign is from the clockwise path of the integration contour). Therefore a logarithmic singularity produces a power law in real space, which is certainly important for the purposes of looking at correlation functions.

Another example which is closely related is the square of a logarithm. Since products in Fourier space are convolutions in real space,

$$\mathcal{F}[\ln^2(p^2/\Lambda)] = \frac{1}{\pi}\int d^3y\, \frac{1}{|y|^3|x-y|^3}. \tag{994}$$

Sadly I don't know how to do this integral!

62 *May 18 — Fixed points for QCD β function*

Today we're taking a very simple look at the BZ fixed point. This was suggested as an exercise at one of the bootstrap schools; I found the problem statement written online (by Komargodski).

Solution:

First, some comments on why $\beta(\alpha_*) = 0$ means that α_* is a fixed point. For $\beta = \beta(\alpha, t)$ a general differential equation in terms of α and the RG time t, points where $\beta = 0$ are not in general fixed points. For example, suppose that $\beta = 2(t-1)$, so that $\alpha(t) = (t-1)^2$. Then $\beta = 0$ when $\alpha = 0$, but this is not a fixed point, since $\alpha''(t) \neq 0$ at $t = 1$. The reason why zeros of the β function are fixed points in practice is that the β function is always an autonomous DE—it never depends explicitly on the RG time (the RG flow only depends on the choice of coupling constants, not on what path was taken to get to them). Then since $\beta = \beta(\alpha)$, the root α_* with $\beta(\alpha_*) = 0$ has $\alpha(t) = \alpha_*$ as a constant solution; if β also depended on t then α_* would depend on t, and wouldn't be a fixed point.

The beta function for non-Abelian gauge theory coupled to n fermions in a representation R is, to 2-loop, (in cond-mat conventions where $\beta > 0$ indicates that a coupling is relevant)

$$\beta(\alpha) = \frac{d\alpha}{dt} = 2(\beta_0 + \beta_1/\alpha^2), \tag{995}$$

where $t = -\ln \Lambda/\Lambda_0$ is the RG time and $\alpha = 16\pi^2/g^2$. The first term is

$$\beta_0 = 4n\frac{T_2(R)}{3} - \frac{11}{3}C_2(G). \tag{996}$$

The first order correction term is

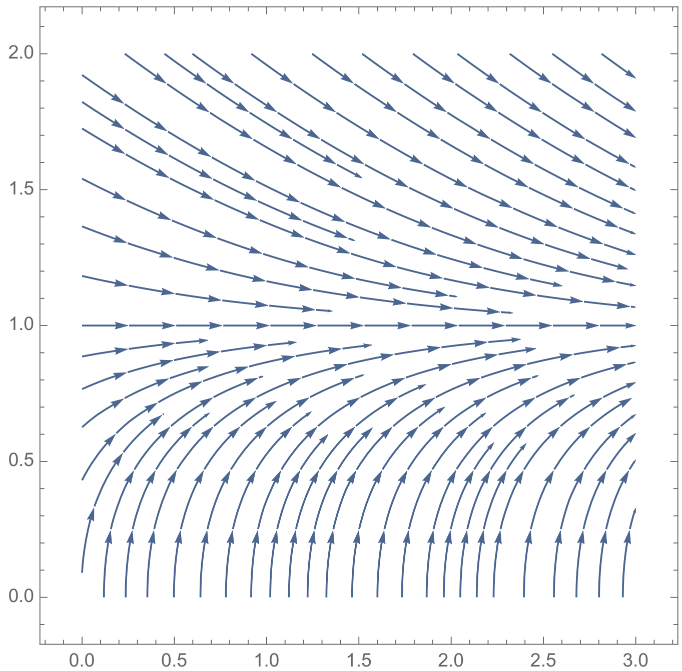
$$\beta_1 = \left(\frac{20}{3}C_2(G) + 4C_2(R)\right) nT_2(R) - \frac{34}{3}(C_2(G))^2. \tag{997}$$

We will mainly be interested in having R be either then fundamental / anti-fundamental, or the adjoint. Note that $T_2(Ad) = C_2(G) > T_2(F)$, so that adjoint fermions push the theory closer to being IR-free (just because more fermion flavors mean more screening and hence less strongly coupled IR physics, and replacing fundamental fermions with adjoint fermions is roughly the same as increasing the number of flavors).

First take $\beta_0 < 0$. This is where the IR fixed point is stable. Mathematically, this is seen by nothing that for the fixed point to exist, $\beta_0 < 0 \implies \beta_1 > 0$. Then we see that

$$\frac{\partial \beta(\alpha)}{\partial \alpha} = -2\beta_1/\alpha^2 < 0. \tag{998}$$

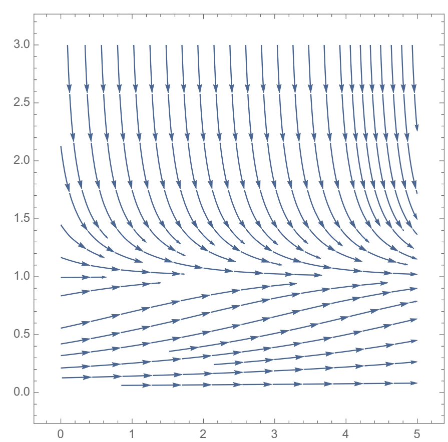
The fact that the derivative of the beta function is negative at the fixed point guarantees IR stability. Anyway, we can solve for $\alpha(t)$ explicitly in terms of a productlog function, but just plotting the β function is more illuminating: for e.g. $\beta_0 = -1/2, \beta_1 = 1/2$, (this is different from a usual RG flow diagram involving two different couplings—here the y axis is α and the x axis is RG time. This is a bit redundant since the flow of α doesn't depend on RG time, but the plot looks cool so what the hell)[116]



. (1000)

Now suppose $\beta_0 > 0$—this is where the fixed point is IR unstable, since then we have

[116]If you prefer to have a plot of g^2 instead of α, here it is, in mini-form:



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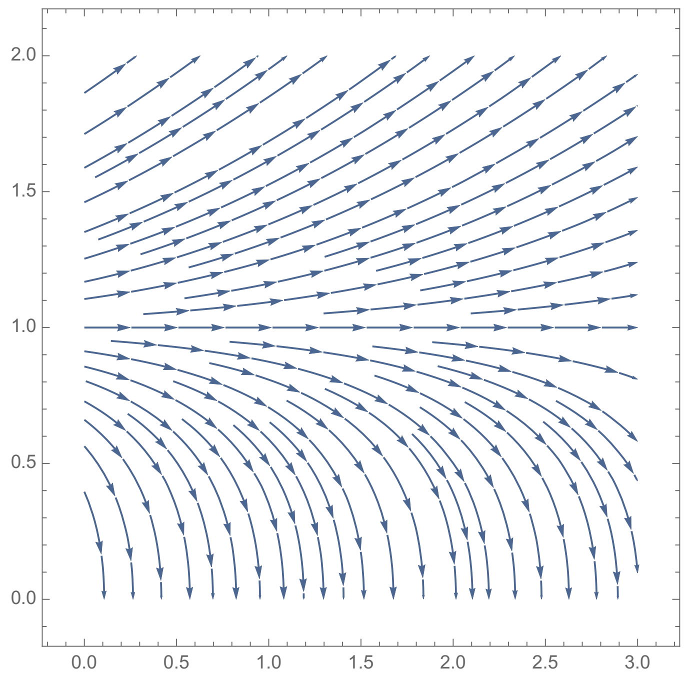
$\partial_\alpha \beta(\alpha) > 0$ at the fixed point. Indeed, the RG flow is (for $\beta_0 = 1/2, \beta_1 = -1/2$)



(1001)

Now we will focus on $G = SU(N)$, for which $C_2(G) = C_2(Ad) = N$. The beta function for $R = F$ the fundamental representation is

$$\beta_0^F = \frac{1}{3}\left(2n - 11N\right), \qquad \beta_1^F = -\frac{34}{3}N^2 + N\frac{13n}{3} - \frac{n}{N}, \tag{1002}$$

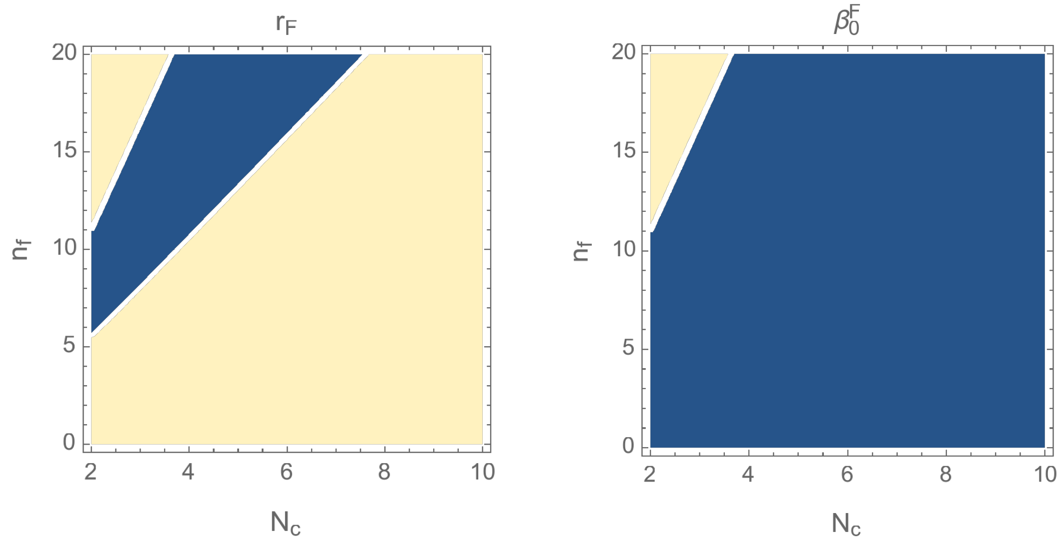
while for the adjoint,

$$\beta_0^{Ad} = \frac{1}{3}(4Nn - 11N), \qquad \beta_1^{Ad} = N^2\left(-\frac{34}{3} + 26n\right) - 2n. \tag{1003}$$

To have a fixed point, we need $\beta_1/\beta_0 < 0$. Therefore we define the two ratios

$$r_F \equiv (\beta_1/\beta_0)_F = \frac{-34N^2 + 13nN - 3n/N}{2n - 11N}, \qquad r_{Ad} \equiv (\beta_1/\beta_0)_{Ad} = \frac{(26n - 34)N - 6n/N}{4n - 11}. \tag{1004}$$

We can plot the sign of the above ratios to figure out when a fixed point will occur. The rs will change sign when either β_0 or β_1 passes through zero. For the fundamental representation, we have (here the plots are just the signs of the relevant quantities; negative is blue and positive is cream)



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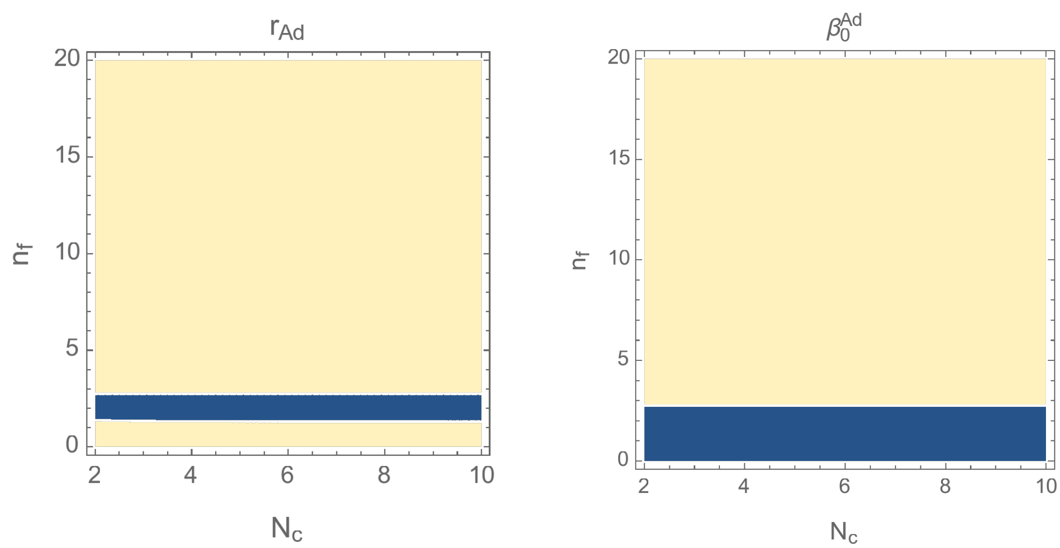
For the adjoint,



(1006)

Let N be fixed, let n^R_{*0} be the number of flavors at which $\beta^R_0 = 0$ (this is when the theory passes from conformal to deconfined), and let n^R_{*1} be the number of flavors at which $\beta^R_1 = 0$ (this is when the theory passes from being confined to being conformal). Then some algebra gives

$$n^F_{*0} = \frac{11N}{2}, \qquad n^F_{*1} = \frac{34N}{13 - 3/N^2} \tag{1007}$$

for the fundamental and

$$n^{Ad}_{0*} = \frac{11}{4}, \qquad n^{Ad}_{*1} = \frac{34}{78 - 6/N^2} \tag{1008}$$

for the adjoint. A sanity check is that $n^R_{0*} > n^R_{1*}$, since we know that when $n^R = 0$ we should have confinement.

Finally some comments on these two critical fermion numbers. The point n^R_{*0} is the exact location of the upper boundary of the conformal window, i.e. its location doesn't depend on our ignorance of the higher order terms in the expansion of β. To see this, consider n bigger than n^R_{*0}, so that $\beta_0 > 0$. Then if we start the flow at $1/\alpha$ infinitesimal, only β_0 contributes to the flow, and since here $\beta_0 > 0$, $1/\alpha$ is driven back to zero—the higher-order terms have no chance to contribute. On the other hand, suppose n is very slightly less than n^R_{*0}. If we start from $g^2 = 0$ then $\beta_0 < 0$ tells us that g^2 initially increases. This is true regardless of what the higher-order terms in β are. Now the fixed point we identified is at $(\alpha^*)^{-1} = -\beta_0/\beta_1$, which can be made arbitrarily small by tuning n arbitrarily close to n^R_{*0}. Therefore we can reach the fixed point after an arbitrarily short RG flow, which means that the fixed point can be reached, at least If we are infinitesimally close to n^R_{*0}, before any of the other terms in the β function expansion have a chance to contribute. Therefore n^R_{*0} indeed marks a sharp boundary between a region with a trivial free IR fixed point and a nontrivial (weakly) interacting one. This is similar to the WF fixed point in the ϵ expansion—we don't need to know the full expression for the β function in order to guarantee the existence of a nontrivial IR fixed point, since by tuning ϵ we can make the WF fixed point arbitrarily close to the Gaussian one.

In contrast, n^R_{*1} is not a sharp lower boundary for the conformal window. This is where $\beta_1 \to 0$, and so our expression for the fixed point is $(\alpha^*)^{-1} \to \infty$. If we start from $\alpha^{-1} = 0$ then the flow takes a long time to get to α_*, and by the time we reach it, the higher-order terms in the β function will contribute, shifting the location of the fixed point. Hence, the lower bound of the conformal window cannot be calculated in perturbation theory.

63 *May 19 — Some examples of simple RG flows*

Today's problem is again inspired by a problem in Schwartz (23.8). We're going to be looking at the effect of

Solution:

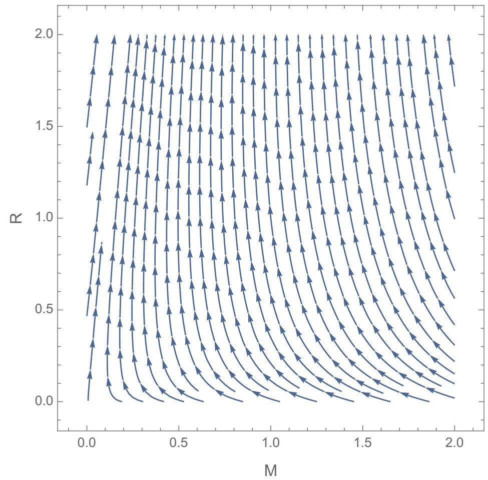
Marginal and relevant couplings

Let us consider one dimension 2 coupling $\Lambda^2\mathcal{R}$ (with $\mathcal{R}$ dimensionless) and one marginal coupling $\mathcal{M}$. In order to be able to make general analytic statements, we will consider the linearized β functions, where we keep only the leading terms in $\mathcal{R}, \mathcal{M}$ for each β function. The RG equations are then, in terms of the RG time $t = -\ln \Lambda/\Lambda_0$,

$$\begin{aligned} \frac{d\mathcal{R}}{dt} &= (2+a)\mathcal{R} + b\mathcal{M} \\ \frac{d\mathcal{M}}{dt} &= c\mathcal{R} + d\mathcal{M} \end{aligned} \tag{1009}$$

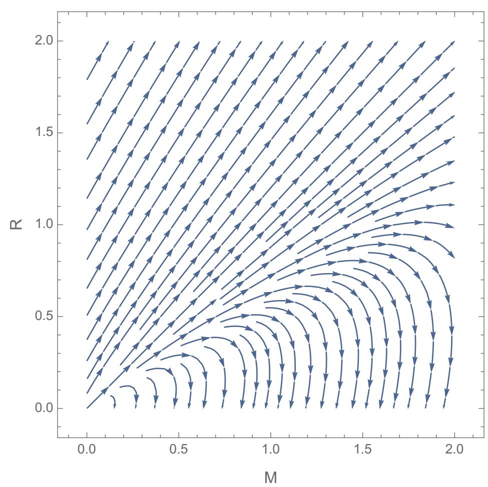
for some arbitrary numbers a, b, c, d.

Before we solve them, let's take a look at the RG flows. a and d are roughly speaking the anomalous dimensions of $\mathcal{R}$ and $\mathcal{M}$, and are usually going to be smallish. For positive couplings, the generic situation is that $\mathcal{R}$ rapidly gets large (whether or not it can make it to infinity depends on how it couples to $\mathcal{M}$), as expected from a relevant coupling.[117] For example, if $\mathcal{M}$ is marginally irrelevant, we might have something like



(1010)

However, if we make some of the couplings negative and e.g. take $b < 0, c > 0$, then the marginal coupling can take over the flow and drive the relevant one to zero, before $\mathcal{R}$ has a chance to get large: this typically looks like



(1011)

Therefore "relevant couplings always get big" is oversimplified—marginal couplings can dictate where they flow (even marginally irrelevant ones). We could go on and on with different plots, but won't.

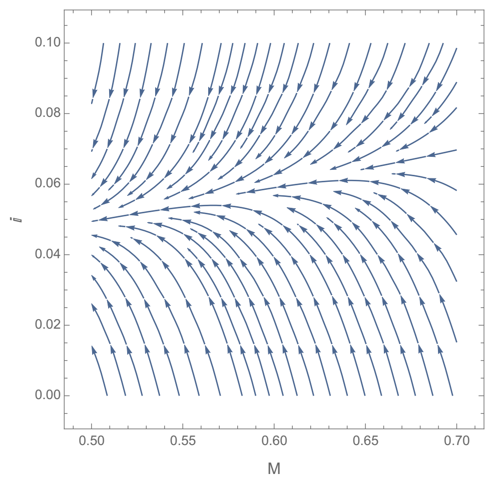
[117]Technically we should only be talking about relevance / irrelevance for eigenvectors of the beta function, but you know what I mean—relevance / irrelevance of $\mathcal{O}$ is determined by asking whether $\mathcal{O}$ would be relevant / irrelevant if the couplings to all other operators were turned off.

Marginal and irrelevant couplings

$$\begin{aligned}\frac{d\mathcal{I}}{dt} &= (-2+a)\mathcal{I} + b\mathcal{M} \\ \frac{d\mathcal{M}}{dt} &= c\mathcal{I} + d\mathcal{M}.\end{aligned} \tag{1012}$$

We won't go through the whole rigamarole of solving these equations, since it was done in the previous section. The moral of the story is that the value of $\mathcal{I}_f = \mathcal{I}(\Lambda_f)$ is completely insensitive to $\mathcal{I}_0 = \mathcal{I}(\Lambda_0)$ for appropriately large Λ_0/Λ_f. Instead, $\mathcal{I}_f$ is determined solely through the way in which it couples to $\mathcal{M}$ and the initial condition $\mathcal{M}_0$: since $\mathcal{I}$ is irrelevant, the point it flows to under RG is not determined by where it starts, but only by how it interacts with the non-irrelevant couplings.

Anyway, we will just content ourselves by looking at some RG flows. We will take the couplings $c, b > 0$ so that there is positive feedback between the two couplings. If $\mathcal{M}$ is marginally irrelevant, so that $d < 0$, then we get something that looks like



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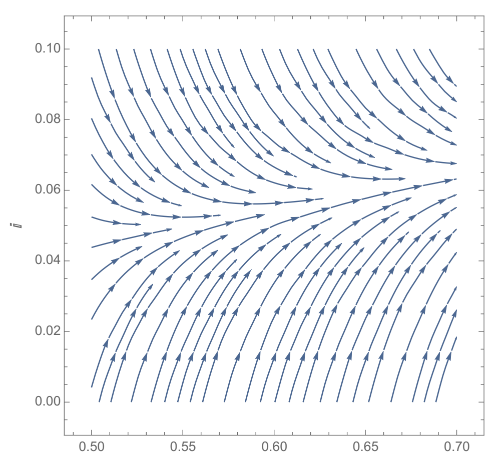
On the other hand if $\mathcal{M}$ is marginally relevant, then the flow gets reversed, and the trajectories have $\mathcal{M}$ diverging to ∞ (and taking $\mathcal{I}$ along with it):



(1014)

With sufficiently strong coupling between the two parameters, the limiting flow for the marginally relevant and marginally irrelevant cases can be reversed.

64 *May 20 — Current-current correlators for N scalar fields*

Today we're doing a simple problem suggested by Pufu for the attendees of one of the bootstrap schools. Nothing complicated, but I thought it would be good to have around as a reference.

Consider a theory of N free scalars. Find an expression for the 2-point function of the currents, $\langle J^\mu_{ij}(x) J^\nu_{kl}(0)\rangle$. What aspects of this 2-point function are unchanged if $O(N)$ symmetry-preserving interactions are added?

Solution:

First let us recall what the currents are. Taking $\phi_i \mapsto \phi_i + \epsilon_a(x) A^a_{ij}\phi_j$ where $A^a_{ij} = |e_i\rangle\langle e_j| - |e_j\rangle\langle e_i| \in \mathfrak{so}(N)$ tells us that

$$\delta S = \int d^dx\, \partial_\mu \epsilon_a (A^a_{ij}\phi_j \partial^\mu \phi_i + Aij(\partial^\mu\phi_i)\phi_j) = \int d^dx\, (\partial_\mu \epsilon_{ij}) J^\mu_{ij}, \qquad J^\mu_{ij} = \phi_i\partial^\mu\phi_j - \phi_j \partial^\mu \phi_i. \tag{1015}$$

We want to compute the current-current correlators. Since we care about the coefficients, we need to remember exactly what the propagator is. We invert ∂^2 by requiring that $G_{ij}(r) = \langle \phi_i(r)\phi_j(0)\rangle$ go as $\alpha\delta_{ij}/|r|^\gamma$, where

$$\partial_\mu \frac{\alpha}{r^\gamma} = \frac{\widehat{r}^\mu}{A(S^{d-1})r^{d-1}}, \tag{1016}$$

so that $\partial^2 G_{ij}(r) = \delta_{ij}\delta(r)$. Therefore

$$G_{ij}(r) = \frac{1}{(d-2)A(S^{d-1})}\frac{1}{|r|^{d-2}} \equiv E r^{2-d}. \tag{1017}$$

To get the current 2pt function, we need to compute things like

$$\langle : \phi^i(x)\partial_\mu\phi^j(x) :: \phi_k(y)\partial_\nu\phi_l(y) :\rangle = (\partial^x_\mu G_{jk}(x-y))(\partial^y_\nu G_{il}(x-y)) + G_{ik}(x-y)\partial^x_\mu\partial^y_\nu G_{jl}(x-y), \tag{1018}$$

where the superscripts on the derivatives just indicate which variable they are being taken with respect to. Taking the derivatives,

$$\langle : \phi^i(x)\partial_\mu\phi^j(x) :: \phi_k(y)\partial_\nu\phi_l(y) :\rangle = \delta_{ik}\delta_{jl}\frac{E^2(d-2)}{r^{2d-2}}\left(\delta_{\mu\nu} - \frac{dr_\mu r_\nu}{r^2}\right) - \delta_{il}\delta_{jk}\frac{E^2(d-2)^2}{r^{2d}} r_\mu r_\nu. \tag{1019}$$

The full current-current correlator comes from taking the above and adding $-(i\leftrightarrow j) - (k \leftrightarrow l) + (i,j \leftrightarrow k,l)$. Therefore

$$\langle J^\mu_{ij}(r) J^\nu_{kl}(0)\rangle = \frac{2}{(d-2)A^2(S^{d-1})r^{2d-2}}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})\left(\delta^{\mu\nu} - 2\frac{r^\mu r^\nu}{r^2}\right). \tag{1020}$$

The dimensionality is just what is needed to ensure that the current has no anomalous dimension, and the factor of 2 in the last factor is just what is needed to ensure that the RHS is divergenceless.

We can also get the dimension of the current operator with a Ward identity. This is a slightly better way of finding A since it doesn't make any assumptions about the Lagrangian. The fact that this method recovers the same result as above means that conserved currents always have an anomalous dimension of zero.

Let $\mathcal{O}$ be any operator such that is charged under the symmetry generated by J. Let us consider varying the fields as

$$\phi \mapsto \phi + \eta(x)\delta_S\phi, \tag{1021}$$

where $\delta_S\phi$ is a symmetry and $\eta(x)$ is an indicator function equal to ϵ for $x \in R$ and 0 else. Letting $\mathcal{O}$ be supported at a point $y \in R$, we have (taking ϵ infinitesimal)

$$\langle \delta\mathcal{O}(y)\rangle = \int_X d^dx\,(\partial_\mu\eta(x))\langle J^\mu(x)\mathcal{O}(y)\rangle = \int_{\partial R} d^{d-1}x_\perp^\mu\,\langle J_\mu(x)\mathcal{O}(y)\rangle. \tag{1022}$$

Since this equation must hold regardless of what ∂R is, on dimensional grounds we must have $[J] = d-1$, so that J must have zero anomalous dimension.

When the current is associated with a non-Abelian symmetry as in the above context, the current itself provides us with such an $\mathcal{O}$. If the structure constant $f^{abc} \neq 0$, then $\delta_a\langle J_\mu^b(x)J_\nu^c(0)\rangle \neq 0$ (here δ_a is the variation which sends $\delta_a J_\mu^b = \eta i f_a{}^b{}_c J_\mu^c$, and the currents are such that $\langle J^a(x)J^b(0)\rangle \propto \delta^{ab}$), and so taking $\mathcal{O}(y) = J_\mu^b(y)J_\nu^c(0)$ gives (not keeping track of numerical factors)

$$f^a{}_{bc}\langle J_\mu^b(y)J^c(0)\rangle \sim \int_{\partial R} d^{d-1}x_\perp^\lambda\langle J_\lambda^a(x)J_\mu^b(y)J_\nu^c(0)\rangle. \tag{1023}$$

Requiring this to hold for arbitrary R again tells us the dimension of J and that $\langle J(x)J(0)\rangle \sim 1/|x|^{2(d-1)}$ exactly, as long as J is conserved.

65 *May 21 — ϵ expansion and β functions in the anisotropic $O(N)$ model*

Today we're doing an extended version of a problem suggested by Pufu for the attendees of one of the bootstrap schools. We will be considering the $O(N)$ model deformed with an anisotropy v which reduces the symmetry to the "cubic" symmetry $\phi_i \mapsto -\phi_i, \phi_i \mapsto \phi_{\rho(i)}, \rho \in S_N$:

$$S = \int d^dx\left(\frac{1}{2}\nabla_\mu\boldsymbol{\phi}\cdot\nabla^\mu\boldsymbol{\phi} + \frac{1}{2}t_0\boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{1}{4!}\sum_{i,j=1}^N(u_0 + v_0\delta_{ij})\phi_i^2\phi_j^2\right). \tag{1024}$$

The numbers in the action aren't the ones given in Pufu's assignment, but I think these factors lead to the simplest manipulations.

Anyway, our goal is to study the RG properties of this model in $d = 4 - \epsilon$. We will perturb in ϵ, and let N be arbitrary. Pufu provides students with the 1-loop beta functions and asks them to analyze the RG flow. We will do this but will instead also calculate the 1-loop beta functions ourself: it's fun and this way we know that the factors we get in the beta functions are correct (the result Pufu writes down is probably correct, but one has to make some variable-re-definitions to get to his result that he doesn't mention, and which I couldn't quite figure out).

Solution:

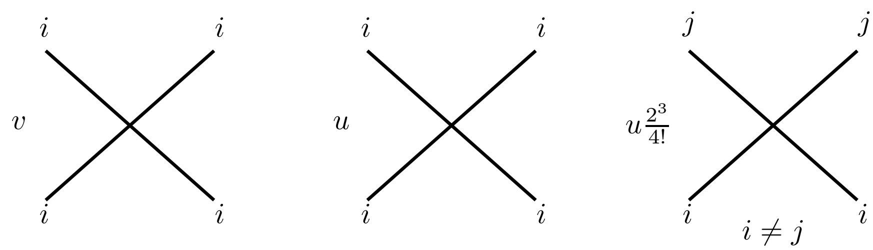
First let's define the dimensionless couplings that we will be doing RG with. The dimension of ϕ_i is $(d-2)/2 = 1-\epsilon/2$. This means that we can define a dimensionless renormalized mass through $t_0 = a^{-2}Z_t t$, where a^{-1} is the UV cutoff. Since $[\phi_i]^4 = 4-2\epsilon$, the bare couplings for the quartic terms are determined through dimensionless couplings u, v via

$$u_0 = uZ_u a^{-\epsilon}, \qquad v_0 = vZ_v a^{-\epsilon} \tag{1025}$$

with $Z_u = 1 + \delta_u$ and likewise for Z_v (the dimensions work out since $(4-2\epsilon)+\epsilon = d$).

We will determine the beta functions by using the fact that the bare couplings are independent of the UV cutoff a^{-1}. Our mission is to find the appropriate counterterms δ_g, since they will carry cutoff-dependence. We will just go to 1-loop, and will work to quadratic order in all the couplings. N will be arbitrary. Note that since we are going to 1-loop, an easier way to determine the β functions would be to use conformal perturbation theory—this just requires knowing the OPEs, which are easy to derive since we would be determining them at the Gaussian fixed point. However, in the name of practicing a method that can be easily generalized to higher loops, we'll stick with the Feynman diagrams + counterterms approach.

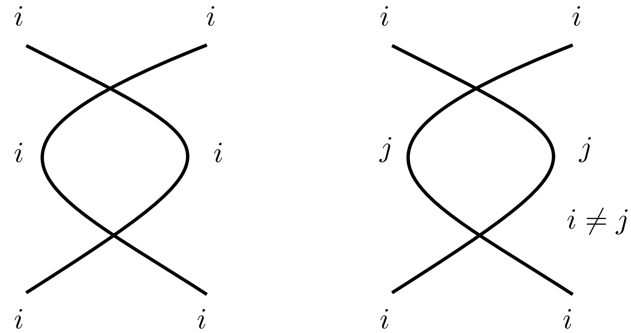
The basic 4-point vertices are



(1026)

The factor of $2^3/4! = 1/3$ comes from the symmetry factor of the diagram $(2!2!)$ and the fact that $\phi_i^2\phi_j^2, i \neq j$ appears in the Lagrangian with coefficient $2u/4!$.

With all the outgoing legs labeled by the same index, the t-channel graphs are



$$\tfrac{1}{2}(u^2+v^2+2uv) \qquad \tfrac{N-1}{2}\left(u\tfrac{2^3}{4!}\right)^2 \tag{1027}$$

The s and u channel processes are the same. So all the 1-loop graphs with identical external legs give us the 4-point function

$$G^{(4)}_{iiii} = -i\left((1+\delta_v)v + (1+\delta_u)u + \frac{1}{2}\left(u^2+v^2+2uv+\frac{N-1}{9}u^2\right)(V(s)+V(t)+V(u))\right)\prod_k \frac{i}{p_k^2}, \tag{1028}$$

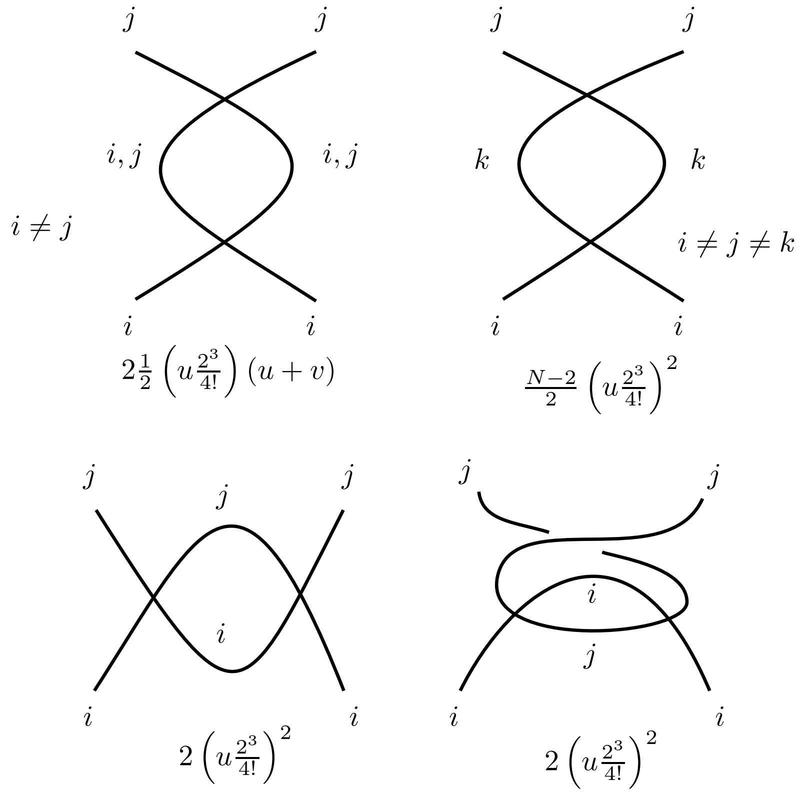
where p_k are the external momenta and

$$V(p^2) = \int_k \frac{i^2}{k^2(k+p)^2}. \tag{1029}$$

Note that $V(p^2)$ does not include a $1/2$ symmetry factor. Also note that there is no t appearing in the above propagator—we are going to be treating t as a vertex, i.e. as a (small) coupling in its own right. This is done wolog since summing up the geometric series for a straight propagator line with all possible t insertions recovers $i/(p^2-t)$.

Anyway, the above formula for $G^{(4)}_{iiii}$ does not determine the counterterms δ_u, δ_v uniquely. To find them, we need to also renormalize the vertex with two i legs and two j legs, for

$i \neq j$. The contributing diagrams are



(1030)

Adding these all up,

$$G^{(4)}_{iijj} = -i\left(\frac{1}{3}(1+\delta_u)u + \left(\frac{2}{2\cdot 3}(u^2+vu) + \frac{N-2}{2\cdot 3^2}\right)V(t) + \frac{1}{3^2}u^2(V(s)+V(t))\right)\prod_k \frac{i}{p_k^2}. \tag{1031}$$

Note the absence of the 1/2 symmetry factor in the last term, since the s and t channel processes don't have internal legs with the same index. As another example of how the counting works, look at e.g. the (u^2+vu) term: the 2 in the numerator comes from taking the internal loop to be either i or j, the 2 in the denomenator is a symmetry factor, and the 3 is $2^3/4!$.

Ensuring that the divergences in $G^{(4)}_{iijj}$ are canceled lets us determine the counterterm δ_u. Setting our renormalization conditions for momenta with $s=t=u=-M^2$, some algebra tells us that the divergences in $G^{(4)}_{iijj}$ are canceled, and that the 4-point function reduces to just $-iu/(\prod p^2)$ at the scale $-M^2$, provided that

$$\delta_u = -V(-M^2)\left(\frac{N+8}{6}u + v\right). \tag{1032}$$

We can now substitute this counterterm into the expression for the $G^{(4)}_{iiii}$ Greens function to solve for δ_v. Some algebra gives

$$\delta_v = -V(-M^2)\left(2u + \frac{3}{2}v\right), \tag{1033}$$

which ensures that $G^{(4)}_{iiii}$ reduces to $-i(u+v)/(\prod p^2)$ at the scale $-M^2$.

Calculating $V(-M^2)$ is standard. For simplicity we will set the RG conditions to be at zero momentum. Then the integral for $V(0)$ is logarithmically divergent: when we cut it off at the UV cutoff a^{-1}, the a-dependence is

$$V(0) \supset \frac{1}{16\pi^2}\ln(a^{-2}). \tag{1034}$$

We only care about the a dependence, since this is what gives the counterterms their a dependence, which is the thing that's needed to compute the beta functions.

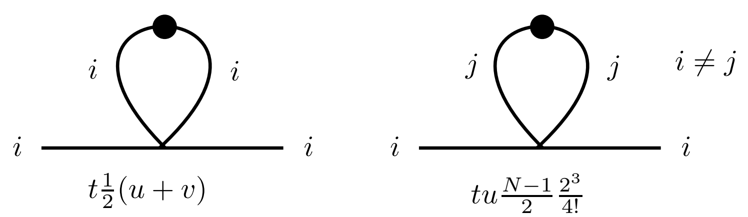
Now in pure $\lambda\phi^4/4!$ theory (i.e. if $N=1, v=0$), a standard calculation shows that the beta function is $\beta_\lambda = \epsilon\lambda - 3\lambda^2/(16\pi^2)$. Therefore it is helpful to re-define λ by absorbing the $3/16\pi^2$ factor. We will do the same thing, by defining the new variables

$$\mathsf{u} \equiv \frac{3}{16\pi^2}u, \qquad \mathsf{v} \equiv \frac{3}{16\pi^2}v. \tag{1035}$$

With these conventions, the a-dependent parts of the counterterms are, after some algebra, (note how no a^ϵs have been entering the expressions for the counterterms—both the δs and the couplings constants are dimensionless)

$$\delta_{\mathsf{v}} = \left(\frac{4}{3}\mathsf{u} + \mathsf{v}\right)\ln a, \qquad \delta_{\mathsf{u}} = \left(\frac{N+8}{9}\mathsf{u} + \frac{2}{3}\mathsf{v}\right)\ln a. \tag{1036}$$

Finally, we need the mass counterterm, working to quadratic order in all the couplings. Now the corrections to the propagator that go as u, v are zero in dimensional regularization, since they are $\sim \int_k k^{-2}$ and diverge as a power law. However, since we are treating the mass as an interaction vertex, we do get a logarithmic divergence from diagrams which go as tu, tv. There are only two such 1PI diagrams where the mass appears: they are[118]



, (1037)

where the dot is the mass insertion. The two-point function at momentum p then contains the terms

$$G^{(2)}_{ii} = -\frac{i^2}{p^2}t(1+\delta_t)\left(\frac{v}{2} + \frac{N+2}{6}u\right)\int_k \frac{i^2}{k^4}, \tag{1038}$$

which is exactly the integral that we did above for the 1-loop calculations. Therefore the counterterm is determined as (going back to the u, v variables)

$$\delta_t = \left(\frac{1}{3}\mathsf{v} + \frac{N+2}{9}\mathsf{u}\right)\ln a. \tag{1039}$$

[118]The δ_v and δ_u counterterms don't contribute to this order since their contributions would be third order in the couplings.

Now we can compute the beta functions, for example for u, as follows:

$$0 = \frac{d}{d\ln a}(a^{-\epsilon} Z_{\mathsf{u}} \mathsf{u}) \implies \beta_{\mathsf{u}} = \epsilon \mathsf{u} - \frac{\mathsf{u}}{Z_{\mathsf{u}}} \frac{d\delta_{\mathsf{u}}}{d\ln a}, \tag{1040}$$

and likewise for v. For t, the only change is in the first term:

$$\beta_t = 2t - \frac{t}{Z_t} \frac{d\delta_t}{d\ln a}. \tag{1041}$$

To quadratic order in the coupling constants we can take $1/Z_g \to 1$ in the above expressions for each coupling g, and therefore we derive

$$\begin{aligned} \beta_{\mathsf{u}} &= \epsilon \mathsf{u} - \frac{N+8}{9}\mathsf{u}^2 - \frac{2}{3}\mathsf{uv}, \\ \beta_{\mathsf{v}} &= \epsilon \mathsf{v} - \mathsf{v}^2 - \frac{4}{3}\mathsf{uv}, \\ \beta_t &= 2t - \frac{1}{3}t\mathsf{v} - \frac{N+2}{9}t\mathsf{u}. \end{aligned} \tag{1042}$$

I found a reference for the above β functions in [**?**] (although the details of how they arrived at them aren't given), and amazingly, after converting to our normalization conventions, they agree with the above! What are the odds of that?! I can't tell you how amazed I was when I discovered this. Evidently we actually kept track of all the symmetry factors correctly—a small miracle.

Some preliminary things to notice about these beta functions: first, t only appears in the beta function for itself. This is general, and is simply because while n-valent vertices for $n > 2$ can combine to give larger-valence vertices or contract to give smaller valence vertices, 2-valent vertices can only combine to make more 2-valent vertices. This means that the mass flow won't affect where the interaction vertices flow.[119]

Second, the beta function for v is independent of N, roughly because the v interaction is diagonal and identical to the one in $N = 1$ ϕ^4 theory. However, the beta function for u does depend on N, which tells us that the qualitative behavior of the RG will likely depend on how big N is. Also note that the coefficients of the uv terms in the beta functions for u and v aren't the same.

[119]This is part of a more general statement: a collection of dimensionless coupling constants g_β, with dimensionful coupling constants $g_{0\beta} = \Lambda^{[g_{0\beta}]} g_\beta$, will only appear in the beta function for a coupling g_α if

$$[g_{0\alpha}] \geq \sum_\beta [g_{0\beta}]. \tag{1043}$$

In particular, this means that marginal and irrelevant operators, for which $[g_{0\alpha}] < 0$, are never renormalized by relevant operators (for which $[g_{0\beta}] > 0$); hence the absence of t in the beta functions for u, v. To prove this, note that since the counterterm δ_{g_α} is dimensionless, the counterterm will appear with in diagrams in the form $\delta_{g_\alpha} \Lambda^{[g_{0\alpha}]}$. On the other hand, a correction to the g_α interaction coming from operators with couplings g_β will have a Λ dependence of $\Lambda^{\sum_\beta [g_{0\beta}] + l}$, where $l > 0$ is a cutoff dependence coming from doing loop integrals. In order for the counterterm to cancel divergences coming from these operators, we then need $[g_{0\alpha}] = \sum_\beta [g_{0\beta}] + l$, proving the claim.

Now let's analyze the beta functions. First, we will determine the fixed points. This is easy enough to do, and we will do it with slightly more generality than we have to. Let us write the generalized beta functions as

$$\beta_{g_i} = d_i g_i - \sum_j \Gamma_{ij} g_i g_j, \tag{1044}$$

where the $\Gamma_{g_i g_j}$ are some numbers.[120] In the case of three couplings u, v, t where t doesn't appear in the beta functions for u, v (as above), we find four fixed points: all of them have $t_* = 0$, with the values of u and v being determined as

$$\begin{aligned}
\mathcal{G}: &\quad u_* = v_* = 0, \\
\mathcal{U}: &\quad u_* = \frac{d_u}{\Gamma_{uu}}, \quad v_* = 0, \\
\mathcal{V}: &\quad u_* = 0, \quad v_* = \frac{d_v}{\Gamma_{vv}}, \\
\mathcal{M}: &\quad u_* = \frac{d_v \Gamma_{uv} - d_u \Gamma_{vv}}{\Gamma_{uv}\Gamma_{vu} - \Gamma_{uu}\Gamma_{vv}}, \quad v_* = \frac{d_u \Gamma_{vu} - d_v \Gamma_{uu}}{\Gamma_{uv}\Gamma_{vu} - \Gamma_{uu}\Gamma_{vv}}.
\end{aligned} \tag{1045}$$

$\mathcal{G}$ is the Gaussian fixed point. In our model, $\mathcal{U}$ is the nontrivial $O(N)$-symmetric fixed point, $\mathcal{V}$ is an $O(N)$-breaking fixed point where the theory splits into a sum of N decoupled ϕ^4 theories, while at the mixed fixed point $\mathcal{M}$, $O(N)$ is broken but the theory is not diagonal.

To analyze each fixed point, we need the linearized beta functions, viz.

$$\bar{\beta}_{g_i} = d_i \bar{g}_i - \sum_j \Gamma_{ij}(g_{j*}\bar{g}_i + \bar{g}_j g_{i*}), \tag{1046}$$

where $\bar{g}_i \equiv g_i - g_{i*}$. Diagonalizing these equations determines the scaling variables at the appropriate fixed point.

At the Gaussian fixed point $\mathcal{G}$, we see that the (ir)relevance of the couplings are of course entirely determined by the d_g—for our $O(N)$ model this means that all of the couplings are relevant for $\epsilon > 0$, and so $\mathcal{G}$ is very unstable. At the $\mathcal{U}$ fixed point, we have

$$\begin{aligned}
\bar{\beta}^{\mathcal{U}}_t &= \left(2 - \frac{\Gamma_{ut}}{\Gamma_{uu}} d_u\right) \bar{t}, \\
\bar{\beta}^{\mathcal{U}}_u &= -d_u \bar{u} - d_u \frac{\Gamma_{uv}}{\Gamma_{uu}} \bar{v}, \\
\bar{\beta}^{\mathcal{U}}_v &= \left(d_v - \frac{\Gamma_{vu}}{\Gamma_{uu}} d_u\right) \bar{v},
\end{aligned} \tag{1047}$$

and similarly for the $\mathcal{V}$ fixed point. At the $\mathcal{M}$ fixed point, well... I won't write it out, since we are going to be specializing back to the $O(N)$ model now.

[120] This is done wolog since Γ_{kj} with $j, k \neq i$ will never appear in β_{g_i} at quadratic order—this is just because all terms in the expression for β_{g_i} must contain a g_i, as we saw in the derivation above.

For the $O(N)$ model, the fixed points are

$$\begin{aligned}
\mathcal{G}: &\quad \mathsf{u}_* = \mathsf{v}_* = 0, \\
\mathcal{U}: &\quad \mathsf{u}_* = \frac{9\epsilon}{8+N}, \quad \mathsf{v}_* = 0, \\
\mathcal{V}: &\quad \mathsf{u}_* = 0, \quad \mathsf{v}_* = \epsilon, \\
\mathcal{M}: &\quad \mathsf{u}_* = \frac{3\epsilon}{N}, \quad \mathsf{v}_* = \frac{\epsilon(N-4)}{N}.
\end{aligned} \tag{1048}$$

We are working with arbitrary N, but consider for a moment taking $N \to \infty$. We see then that the $\mathcal{U}$ fixed point merges with $\mathcal{G}$, while $\mathcal{V}$ merges with $\mathcal{M}$. Therefore the theory in the large N limit behaves as the decoupled sum of N copies of the ϕ^4 theory, up to $1/N$ corrections: this is exactly what we expect from the usual large N story, where in the $N \to \infty$ limit the different vector components decouple from one another.

Now we need to examine the stability of the different fixed points for different N. Notice that when $N < 4$, $\mathsf{v}_* < 0$—therefore we should look at the stability of the model to make sure that this is okay. While either v, u can be negative, the potential must still be bounded from below. To find the region of stability for the potential, we look at its derivative wrt ϕ_k, where $\phi_k^2 \geq \phi_i^2 \, \forall \, i$. Requiring that this be positive for positive ϕ_k means that

$$\mathsf{v}\phi_k^2 + \mathsf{u}\sum_j \phi_j^2 > 0 \tag{1049}$$

in the limit $\phi_k \to \infty$. If $\mathsf{v} < 0$, then we simply need $\mathsf{u} > -\mathsf{v}$. If $\mathsf{u} < 0$, then having $\mathsf{v} > -\mathsf{u}$ isn't enough: the strongest constraint comes from when all the ϕ_i^2 are equal, and tells us that in fact $\mathsf{v} > -N\mathsf{u}$. These conditions define the region of stability for our model.

Anyway, we see that $\mathsf{v}_* = \epsilon(N-4)/N$ is allowed when $\mathsf{u}_* = 3\epsilon/N$ for all $N > 1$, so that this fixed point is indeed always within the range of stability. Since the sign of v_* changes at $N = 4$, we expect that $N = 4$ might be a critical value across which the nature of the RG flow changes. Indeed, this suspicion is confirmed when we notice that at $N = 4$, the $\mathcal{U}$ and $\mathcal{M}$ fixed points become degenerate. To see exactly what happens, we linearize the beta functions, obtaining for $\mathcal{U}$ (omitting those for t since it's always relevant and hence not so interesting)

$$\bar{\beta}^{\mathcal{U}}_{\mathsf{u}} = -\epsilon\left(\bar{\mathsf{u}} + \frac{6}{N+8}\bar{\mathsf{v}}\right), \qquad \bar{\beta}^{\mathcal{U}}_{\mathsf{v}} = \epsilon\frac{N-4}{N+8}\bar{\mathsf{v}}, \tag{1050}$$

for $\mathcal{V}$

$$\bar{\beta}^{\mathcal{V}}_{\mathsf{u}} = \frac{\epsilon}{3}\bar{\mathsf{u}}, \qquad \bar{\beta}^{\mathcal{V}}_{\mathsf{v}} = -\epsilon\left(\bar{\mathsf{v}} + \frac{4}{3}\bar{\mathsf{u}}\right), \tag{1051}$$

and for $\mathcal{M}$,

$$\bar{\beta}^{\mathcal{M}}_{\mathsf{u}} = -\frac{\epsilon}{3}\bar{\mathsf{u}}(1+8/N), \qquad \bar{\beta}^{\mathcal{M}}_{\mathsf{v}} = \epsilon\left(\frac{4}{N} - 1\right)\left(\bar{\mathsf{v}} - \frac{4}{3}\bar{\mathsf{u}}\right). \tag{1052}$$

We could now proceed by diagonalizing these equations to get the scaling variables at each fixed point, but instead we'll just turn to pictures to better visualize things. However, we can at least see from the equations that the $\mathcal{V}$ fixed point is always unstable wrt adding u, and that the stability of the other two fixed points depends on whether N is bigger or less

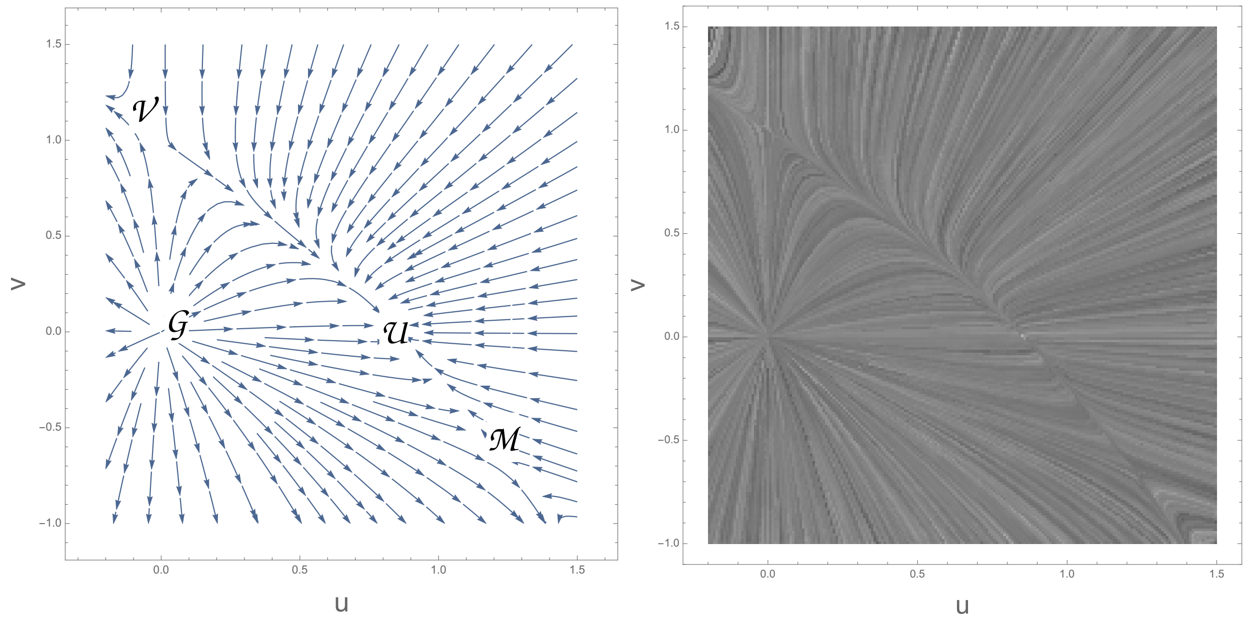


Figure 9: Left: RG flow for $N = 2, \epsilon = 1$. Right: same thing, but shown with a pretty convolution plot instead.

than 4, which in the 1-loop approximation is the critical N for which the behavior changes qualitatively.

For $N < 4$, the $\mathcal{U}$ point is the stable IR fixed point. The $\mathcal{M}$ point is located at negative v, and positive-v deviations away from it flow into $\mathcal{U}$. In figure 9, we show an example of the flow for $N = 2, \epsilon = 1$.

On the other hand, when $N > 4$, the $\mathcal{M}$ fixed point moves to positive v, u, and usurps the $\mathcal{U}$ point as the IR fixed point, with small positive v perturbations away from $\mathcal{U}$ now leading to $\mathcal{M}$. As an example, the flow with $N = 7$ is shown in figure 10.

So, we see that for small $N < 4$, the IR fixed point is $O(N)$ symmetric, despite the fact that the microscopic UV Lagrangian contains an interaction which breaks the $O(N)$ symmetry explicitly. This is therefore an example of an emergent symmetry. In line with our intuition from large N, this no longer occurs when N is made sufficiently large.

Another thing to note is that if $N > 4$ and the flow starts with either $\mathsf{u}, \mathsf{v} < 0$, or if $N < 4$ and the flow starts with either $\mathsf{u} < 0$ or with v below the line connecting $\mathcal{G}$ to $\mathcal{M}$, then the flow takes us to negative values of u, v that lie outside of the region of stability for the model. When we leave the stability region this means that a previously neglected $\boldsymbol{\phi}^6$ term needs to be kept, and that a first-order transition occurs: this then provides us with an example of a "fluctuation-induced" first order transition.

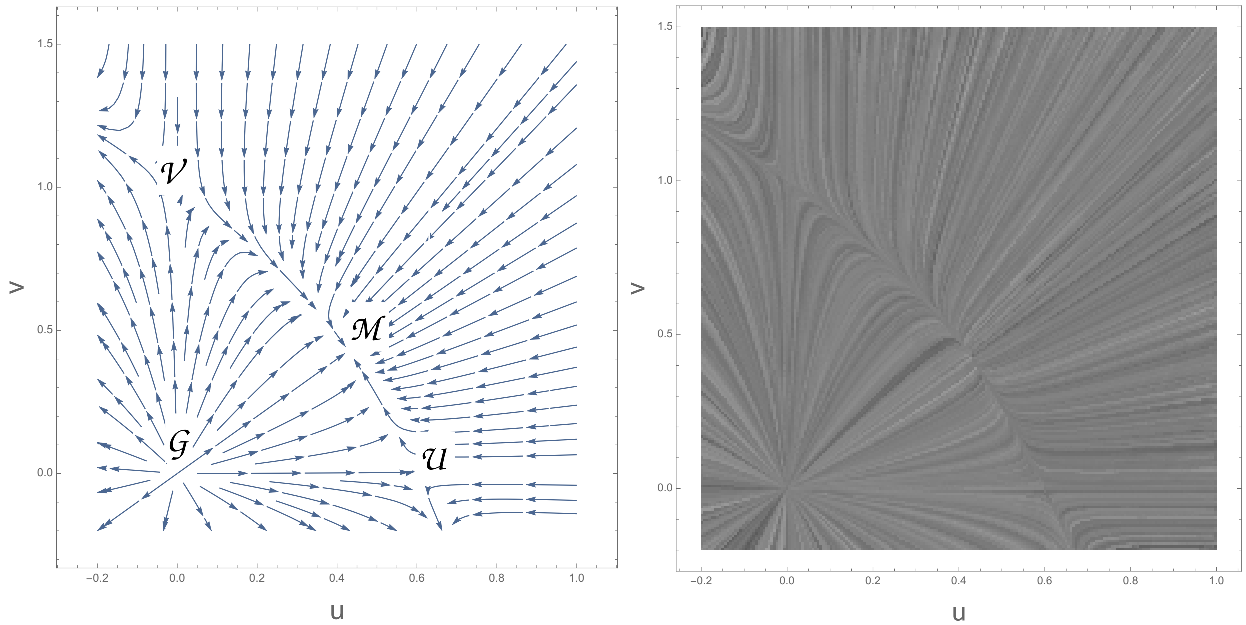


Figure 10: Same thing as the last figure, but now with $N = 7, \epsilon = 1$.

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