# Current diary projects

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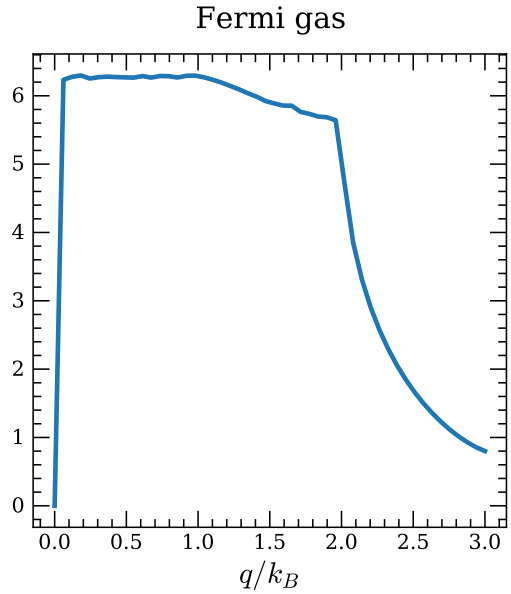
## 1 Fermi liquid susceptibilities in 2d

Today's diary entry is rather trivial — we will be numerically calculating the zero-frequency susceptibility $\Pi(q)$ for a 2d Fermi gas. This came up as a sanity check when doing research and is included just for posterity's sake.

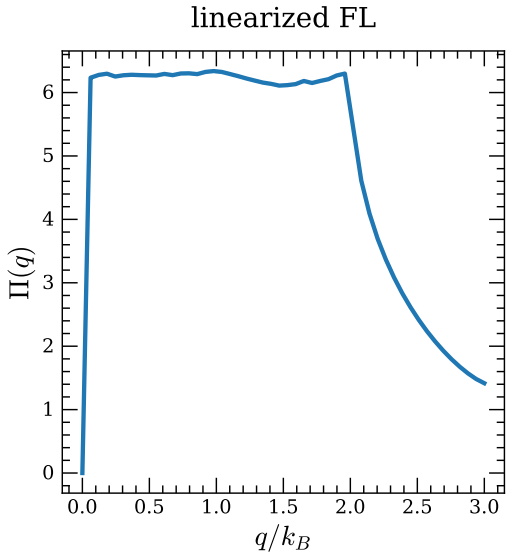
In the following we will set $m = 1$ and also $N(0) = 1$. In a Fermi gas with a pure quadratic dispersion, the susceptibility is




(1)

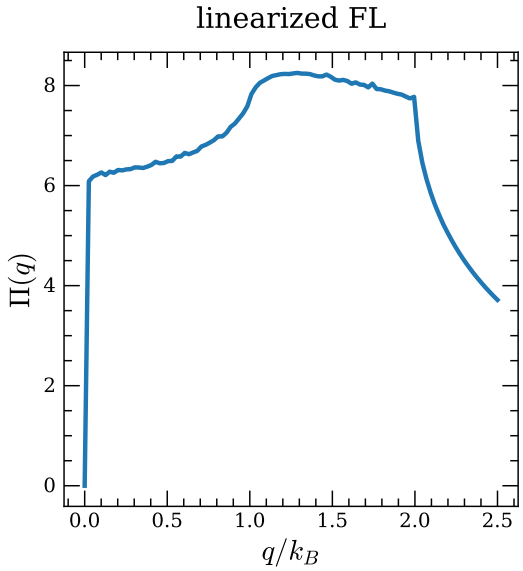
where the small jagged bits are due to finite resolution and after $q = 2k_B$ we have a nice $-\sqrt{q - 2k_F}$ dependence. If we use a different UV completion where the dispersion

is purely linear for all momenta, we get




(2)

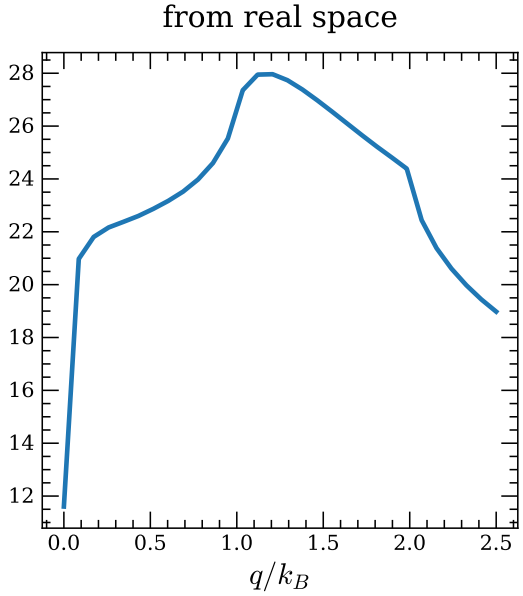
One approximation we often make is to replace the $k$ in $k\,dk$ with $k_F$. Doing this, and sticking with the linearized dispersion, we get




(3)

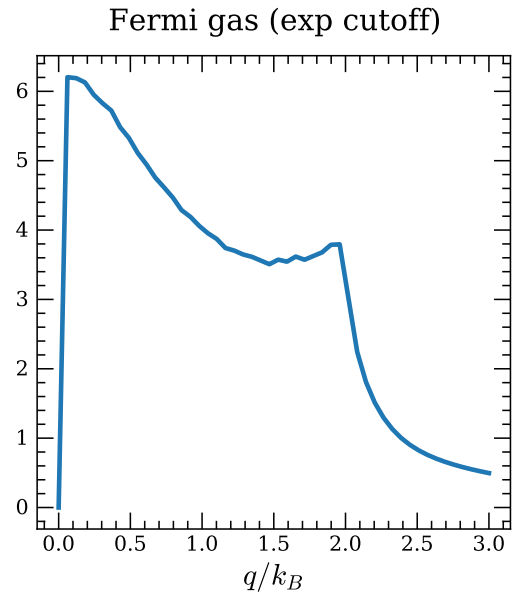
We can also bosonize the Fermi liquid by treating it as a bunch of compact bosons at radius $R^2 = 2$. Finding the real-space correlation functions in this model and then

Fourier transforming, we get




(4)

which is essentially the same as the above (ignore the scale), since we ignore the $k$ in the measure when bosonizing (on grounds of irrelevance). If we take a Fermi gas and apply a soft cutoff using a term $e^{-\alpha(k-k_F)^2}$, we produce things like




(5)

As expected, changing the details of the UV completion don't affect the universal aspects of the susceptibility, e.g. the location of the kink at $2k_F$ and the $1/2$ power law at $q > 2k_F$ — it only affects contributions to $\Pi(q)$ analytic in $q$, overall normalization, etc.

## 2 FL through bosonization

Today we will look at how some features of FLs can be understood through bosonization, focusing on 2+1D for concreteness. Of course the idea to treat FLs with bosoniza-

tion is not new, but the papers that I've read treat this issue rather mathematically / from an operator algebra perspective. At the time of writing my opinion is that the main point can be conveyed using EFT principles and a few heuristic arguments, and so in what follows we will sketch the outline of how this works.

As we will discuss shortly, in the FL the appearance of the $\Lambda_\perp$ in the FS interactions is argued to be why FL behavior is so generic—in that case, one is starting from a FL state and adding small interactions: there the FS interactions can significantly modify the scaling behavior only if $v_F \to 0$ or if $g_\gamma$ is (unnaturally) large. In contrast, bosonization needs interactions to work, and since the FS interactions are relevant until we get to the crossover into the relativistic scaling (at which point we aren't scaling $\Lambda_\perp$ anyway, so $\Lambda_\perp$ is not getting smaller as we flow to the IR), they get larger by a factor of $\sim k_B/\Lambda_\perp$ by the time we get to the IR theory. ethan: *again: need to explain more clearly, especially the $\Lambda_\perp$ vs $\Lambda_\parallel$ issue*

In this section we briefly discuss how to bosonize a 2+1D Fermi liquid. The purpose here is to compare the result to the theories studied previously in this section. In particular, our goal is to show that systems with BSs are not just doubled (viz. non-chial) versions of FLs.

When bosonizing Fermi liquids one often takes an algebraic approach (e.g. a la Haldane), where one stays within the Hamiltonian formalism, identitifes the appropriate algebra for the shape deformation modes of the Fermi surface, and so on. As we mentioned earlier, this approach should be contrasted to the point of view we've taken here, which is one of EFT. The reason for this is that the algebraic approach is predicated on having a limit in which knows the correlation functions of the fields one wishes to bosonize. All the operator relations for the currents in the bosonized description of the FL are obtained by having a known starting state, viz. the FL, evaluate expectation values in. As mentioned in a footnote above, this is not workable for the present case, since any stable phase in the IR must be (perhaps strongly) interacting when written in terms of the UV bosonic variables — without the interactions we just get a condensate.

Let's now see if we can correctly reproduce the known beta functions of the FS and BCS couplings from bosonization. We will continue to assume a circular FS, and will be using the patch approach, which is the method of choice for bosonizing FLs.[1]

The leading pieces in IR bosonization of the fermion operators gives a bosonization

[1]Luther's original approach used 1+1D fields. Some people say that this approach isn't general since it doesn't allow couplings between generic points on the FS. It does, but they just don't appear in terms that preserve $(-1)^F$ for each angle, which makes the operator-centric bosonization approach more difficult.

map[2]

$$\psi_\gamma \to \sqrt{\Lambda_\perp k_F} e^{i\varphi_\gamma}, \tag{7}$$

where again the $\Lambda_\perp$ is for dimensional reasons (the combination $\Lambda_\perp k_F$ is the density assigned to a single species of fermion, since each species contributes a fraction $\Lambda_\perp/k_F$ of the total density $\propto k_F^2$). The free fermions by themselves then bosonize to

$$S_0 \to \frac{\Lambda_\perp}{2\pi} \oint d\gamma \int dt\, d^2x \, \left( (\partial_t \varphi_\gamma \nabla_\gamma \varphi_\gamma - v_{F\gamma} (\nabla_\gamma \varphi_\gamma)^2 \right), \tag{8}$$

where the factors of $k_F$ (inverse of the UV cutoff) get absorbed when making the derivatives. Note that here there are no parameters we can tune to make the radius of the boson be a function of $\gamma$, unlike in the BL case.

Let's first look at the forward-scattering interactions. These appear in the action as

$$S_{FS} \sim \Lambda_\perp^2 \oint d\gamma\, d\gamma' \int dt\, d^2x\, V_{FS}(\gamma - \gamma') \nabla_\gamma \varphi_\gamma \nabla_{\gamma'} \varphi_{\gamma'} \tag{9}$$

Do these terms do anything to the scaling behavior? While strictly speaking the beta function for the FS terms is nonzero if we use the scaling from only $S_0$ (this is just telling us that we should be diagonalizing the quadratic parts to get the correct scaling), the $V_{FS}$ terms won't significantly modify the scaling unless $V_{FS}$ is very large. More precisely, we see that the FS term only makes a significant contribution to the Hamiltonian when $V_{FS}(\gamma - \gamma')\Lambda_\perp / \sqrt{v_{F\gamma} v_{F\gamma'}}$ is of order 1. Since we are taking $\Lambda_\perp$ to be small compared to $k_B$ in the IR limit, this requires either that $v_{F\gamma} \to 0$ for some $\gamma$ or that the interaction strength is large in $k_B/\Lambda_\perp$ (which generically occurs in the bosonic case); both scenarios take us outside the realm of a weakly interacting Fermi liquid.

Now for the BCS terms. These get bosonized as

$$S_{BCS} \sim {\Lambda_\perp}^2 k_F^2 \int_0^\pi d\gamma\, d\gamma' \int dt\, dx\, V_{BCS}(\gamma - \gamma') \cos(\varphi_\gamma + \varphi_{\gamma+\pi} - \varphi_{\gamma'} - \varphi_{\gamma'+\pi}). \tag{10}$$

The dimension of this operator is

$$\Delta_{BCS} = 2\left( \frac{1}{R^2} + \frac{R^2}{4} \right), \tag{11}$$

where $R = \sqrt{2}$ at the free fixed point. Therefore at the free fixed point we have $\Delta_{BCS} = 2$. To determine the first non-zero term in the $\beta$ function, we just do conformal perturbation theory: the leading term in the effective action which contributes to the

[2] We could also do things in terms of the non-chiral fields $\Phi$ and $\Theta$, via

$$\psi_\gamma \to \sqrt{\Lambda_\perp k_F} e^{i(\Phi_\gamma + \Theta_\gamma)}, \qquad \psi_{\gamma+\pi} \to \sqrt{\Lambda_\perp k_F} e^{i(\Phi_\gamma - \Theta_\gamma)}, \qquad \gamma \in [0, \pi). \tag{6}$$

Since these fields are non-chiral, all the symmetry-preserving cosine interactions we can consider adding to the action can be written in terms of them. This however only really works with circular FSs, and leads to more cumbersome notation. Therefore in what follows we will stick to the "chiral" fields $\varphi_\gamma$.

renormalization of $V_{BCS}$ is (Euclidean signature)

$$dS \supset -\frac{1}{2}\int_{\gamma\gamma'\gamma''} V(\gamma-\gamma'')V_{BCS}(\gamma''-\gamma')\int_{\Lambda^{-1}<|x-y|<\Lambda^{-1}(1+dt)} d^2x d^2y \times \cos(\varphi_\gamma(x)+\varphi_{\gamma+\pi}(x)-\varphi_{\gamma'}(y)-\varphi_{\gamma'+\pi}(y))\frac{1}{|x-y|^2}, \tag{12}$$

where $dt$ is the infinitesimal RG time. From this we get

$$\beta_{V_{BCS}(\gamma-\gamma')} \propto -\int_{\gamma''} V_{BCS}(\gamma-\gamma'')V_{BCS}(\gamma''-\gamma'), \tag{13}$$

as expected from the usual fermion calculation.

Do we have to consider cosines of linear combinations of $\Theta_\gamma \equiv \varphi_\gamma - \varphi_{\gamma+\pi}$ to assess the stability of the fixed point? First, translation acts on $\Theta_\gamma$ by $U(1)_T : \Theta_\gamma \mapsto \Theta_\gamma + k_F\boldsymbol{\gamma}\cdot\boldsymbol{\lambda}$ for $\mathbf{x}\mapsto\mathbf{x}+\boldsymbol{\lambda}$. The $U(1)_T$-invariant cosine with the lowest momentum transfer we can make is $\cos(\Theta_\gamma - \Theta_{\gamma+\Lambda_\perp/k_F} - \mathbf{x}\cdot\boldsymbol{\gamma}_\perp\Lambda_\perp)$, where we used $\boldsymbol{\gamma} - R_{\Lambda_\perp/k_F}\boldsymbol{\gamma} \approx (\Lambda_\perp/k_B)\boldsymbol{\gamma}_\perp$, where $R_\theta$ is a rotation matrix. This cosine can only be integrated to something nonzero if the $e^{i\Theta_\gamma}$s have momentum transfer of $\Lambda_\perp$, but this is only possible on a domain of $\sim$ measure zero, and hence we needn't worry about any cosines in $\Theta_\gamma$ so long as $U(1)_T$ is preserved (this is just another way of stating that only BCS and FS interactions are operative).

All of the usual FL phenomenology can be derived straightforwardly in the bosonized description. For example, let us look at the compressibility, for which we need to compute $\langle|\rho(\mathbf{k},\omega)|^2\rangle$. A nice geometrically-summed expression for this can only really be worked out in the case where $V_{FS}$ is a constant, which we will write as $V_{FS}(\gamma) = V$. Since the low-momentum part of the density maps to $\Lambda_\perp\sum_\gamma \nabla_\gamma\varphi_\gamma$, we have (in $i\mathbb{R}$ time with $v_F = 1$ for all $\gamma$, and with $R^2$ a stand-in for the number 2)

$${\Lambda_\perp}^{-2}\langle|\rho(\mathbf{k},\omega)|^2\rangle \sim \sum_\gamma \frac{2\pi}{R^2\Lambda_\perp}\frac{k_\gamma^2}{k_\gamma(i\omega-k_\gamma)} + V{\Lambda_\perp}^2\sum_{\gamma\gamma'}\frac{(2\pi)^2}{R^4{\Lambda_\perp}^2}\frac{k_\gamma^2 k_{\gamma'}^2}{k_\gamma(i\omega-k_\gamma)k_{\gamma'}(i\omega-k_{\gamma'})} + \cdots, \tag{14}$$

which nicely becomes a geometric series, and is simpler than the analogous calculation in the fermionic case since there are no loop integrals other than the angles. We then use the sum

$$2\pi\sum_\gamma \frac{k\cos\gamma}{i\omega - k\cos\gamma} = N\left(1-\frac{\omega}{\sqrt{\omega^2+k^2}}\right). \tag{15}$$

Since $N = 2\pi k_F/\Lambda_\perp$, the sum goes as ${\Lambda_\perp}^{-1}$. One can then check that each term in $\langle|\rho(\mathbf{k},\omega)|^2\rangle$ is independent of $\Lambda_\perp$, as required in this context on physical grounds. Therefore

$$\chi_{\rho\rho}^{-1}(k,\omega) = \frac{R^2}{2\pi k_F}\left(1-\frac{\omega}{\sqrt{\omega^2+k^2}}\right)^{-1} - V \tag{16}$$

Taking $\omega = 0, k\to 0$ we get a constant compressibility proportional to $k_F$, with the form for $\chi_{\rho\rho}^{-1}$ in exact agreement with the usual FL theory result.

# 3 Atom-field Hamiltonian, Markovian master equations, and lack of relaxation

Today we're doing a slight elaboration on an awesome problem in Altland and Simons, from the nonequilibrium (quantum) chapter. see (74)

Define the partial trace projector $\mathcal{P}$ as the superoperator

$$\mathcal{P}\mathcal{O} = \bar{\rho}_b \mathrm{Tr}_b[\mathcal{O}] = (\bar{\rho}_b \otimes \mathbf{1}_s)(\mathbf{1}_b \otimes \mathrm{Tr}_b[\mathcal{O}]), \tag{17}$$

where the last version is the more verbose notation that we won't be using.

Let's first prove some useful identities with this projector. First,

$$\mathcal{P}L_b = L_b\mathcal{P} = 0, \tag{18}$$

which holds since

$$\mathcal{P}L_b\mathcal{O} = \bar{\rho}_b \mathrm{Tr}_b[H_b, \mathcal{O}] = 0, \quad [H_b, \bar{\rho}_b \mathrm{Tr}_b[\mathcal{O}]] = 0, \tag{19}$$

where the first equality comes from the cyclicity of the partial trace[3] and the second from $[H_b, \bar{\rho}_b] = 0$.

Second, we have

$$[L_s, \mathcal{P}] = 0. \tag{22}$$

This makes sense because time evolution by $H_s$ won't affect the results of the trace over the bath, since $H_s$ doesn't touch the bath degrees of freedom—or more formally, because

$$[L_s, \mathcal{P}]\mathcal{O} = [H_s, \bar{\rho}_b \mathrm{Tr}_b \mathcal{O}] - \bar{\rho}_b \mathrm{Tr}_b\left[[H_s, \mathcal{O}]\right] = \bar{\rho}_b[H_s, \mathrm{Tr}_b\mathcal{O}] - \bar{\rho}_b[H_s, \mathrm{Tr}_b\mathcal{O}] = 0, \tag{23}$$

where we pulled $\bar{\rho}_b$ out of the commutator since it commutes with $H_s$ and pulled $H_s$ out of the trace. In keeping with the theme, we are abusing notation by writing $\mathrm{Tr}_b\mathcal{O}$ when we really mean $\mathbf{1}_b \otimes \mathrm{Tr}_b\mathcal{O}$.

Note that $\mathcal{P}\rho = \bar{\rho}_b \otimes \rho_s$ is a perfectly good density matrix: since both factors have unit trace, it has unit trace; since both factors are positive, it is positive. However,

[3] If $X, Y$ are two linear operators of $\mathcal{H}_a \otimes \mathcal{H}_b$ and either one is of the form $\mathbf{1}_a \otimes Z_b$ for some operator $Z_b$ (e.g. $H_b$), then it's easy to check that we have

$$\mathrm{Tr}_b[XY] = \mathrm{Tr}_b[YX]. \tag{20}$$

Indeed, if $X = \mathbf{1} \otimes \mathcal{X}$ and we let $|a\rangle$ be a basis for $\mathcal{H}_a$ and $|i\rangle$ a basis for $\mathcal{H}_b$ (so that e.g. $Y = \sum_{abij} Y_{abij}|ai\rangle\beta bj\rangle$), then one can check that both sides of the above evaluate to

$$\mathrm{Tr}_b[XY] = Y_{acmj}X_{jm}|a\rangle\beta c. \tag{21}$$

We of course also have cyclicity if both operators tensor factorize: $X = X_a \otimes X_b, Y = Y_a \otimes Y_b$. However, if the operators are not of this form, cyclicity does *not* generically hold.

$\rho^\perp \equiv \mathcal{Q}\rho$ is *not* a density matrix: indeed, its trace vanishes, and furthermore $\rho^\perp(0) = 0$, since at $t = 0$ the fact that $\rho = \bar{\rho}_b \otimes \rho_s(0)$ means $\mathcal{P}\rho(0) = \rho(0)$.

blah blah

$$\begin{aligned} \partial_t \rho_s &= L_s \rho_s + \mathcal{P} L_i \rho^\perp \\ \partial_t \rho^\perp &= \mathcal{P}^\perp L \rho^\perp + \mathcal{P}^\perp L_i \rho_s. \end{aligned} \tag{24}$$

discuss why neither of these obey louiville equations!

Finally, we have

$$\mathcal{P} L_i \mathcal{P} = 0, \tag{25}$$

since

$$\mathcal{P} L_i \mathcal{P} \mathcal{O} = \bar{\rho}_b \mathrm{Tr}_b \left( [H_i, \bar{\rho}_b \mathrm{Tr}_b \mathcal{O}] \right) = \bar{\rho}_b \left( \mathrm{Tr}_b[H_i \bar{\rho}_b] \mathrm{Tr}_b[\mathcal{O}] - \mathrm{Tr}_b[\mathcal{O}] \mathrm{Tr}_b[\bar{\rho}_b H_i] \right) = 0, \tag{26}$$

since $\mathrm{Tr}_b[H_i \bar{\rho}_b] = 0$ by assumption (note that $\mathrm{Tr}_b[\mathcal{O}]$ does not necessarily commute with the other trace; nor does $\bar{\rho}_b$ commute with $H_i$).

The interaction picture is compatible with using the partial trace to get reduced density matrices:

$$\begin{aligned} \mathrm{Tr}_b[\rho_I(t)] &= \mathrm{Tr}_B[\mathcal{U}_0(t)^\dagger \rho_S(t) \mathcal{U}_0(t)] \\ &= \mathcal{U}_{0,s}^\dagger(t) \mathrm{Tr}_B[\mathcal{U}_{0,b}^\dagger(t) \rho_S(t) \mathcal{U}_{0,b}(t)] \mathcal{U}_{0,s}(t) \\ &= \mathcal{U}_{0,s}^\dagger(t) \rho_{s,S}(t) \mathcal{U}_{0,s}(t) \\ &= \rho_{s,I}(t), \end{aligned} \tag{27}$$

since the free time evolution $\otimes$ factors over the system and bath as

$$\mathcal{U}_0(t) = \mathcal{U}_{0,b}(t) \otimes \mathcal{U}_{0,s}(t). \tag{28}$$

, since the interaction picture is "Schrodinger for the interactions, Heisenberg for the free term."[4]

after writing integral eqn: The integral here is telling us that in order to do the full time evolution exactly, we need to keep track of what the reservoir is doing throughout the evolution.

# 4 *Entanglement negativity basics*

The point of the negativity is that it's a measure of entanglement that works for *mixed* states. The regular vN entropy is a great measure of entanglement for pure states, but it does not distinguish classical from quantum correlations, since it does not vanish when evaluated on classical thermal states. On the other hand, the negativity does vanish for classical (diagonal) density matrices.

[4]The reason why we do it like this and not the other way around is because we usually want to have a simple picture for what operators $\mathcal{O}(t)$ do — creation / annihilation operators etc....

### Gauging $SU(N_c > 2) \subset O(2N_cN_f)$

Now we do a relatively simple example which determines the global symmetry group of e.g. $N_f$ Dirac fermions coupled to an $SU(N_c > 2)$ gauge field. On one hand the answer is kind of obvious, but on the other hand we know that when $N_c = 2$ we get a larger symmetry group than naively expected, and so it's best to work through the logic carefully. We can write $SU(N_c) \subset O(2N_cN_f)$ as

$$SU(N_c) \ni (V_{\mathbb{R}} \otimes \mathbf{1}_2 + V_{\mathbb{I}} \otimes J) \otimes \mathbf{1}_{N_f}. \tag{29}$$

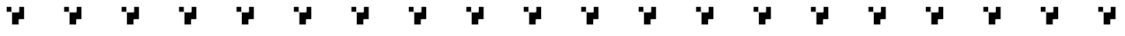
.[5] Clearly $U(N_f)$ is in the normalizer if we write it as

$$U(N_f) \ni \mathbf{1}_{N_c} \otimes (\mathbf{1}_2 \otimes U_{\mathbb{R}} + J \otimes U_{\mathbb{I}}), \tag{31}$$

since it commutes with the $SU(N_c)$ matrices.

ethan: *need to show that there's nothing else in the normalizer but charge conj*

## Walking RG flows



The prototypical example of walking is the RG flow

$$d_t g = \varepsilon + g^2, \qquad \varepsilon > 0. \tag{32}$$

This occurs e.g. in the context of the XY model, where $\varepsilon = g^2 - y^2$ parametrizes the distance away from the separatrix along which one flows to the endpoint of the critical LL line.[6] The pseudocritical nature of this RG flow is examined by computing the dimensionless correlation length $\xi(t_f) = e^{-t_f}\xi(0)$, where $t_f$ is the time along the

[5] In this case the $\rtimes$ was nontrivial and the automorphism was outer, but this is of course not always the case. As an unnecessarily complicated example, consider the automorphism $\rho$ used to define the unitary group as $U(N) \cong SU(N) \rtimes_\rho U(1)$. The existence of this nontrivial $\rtimes$ doesn't mean that $\rho$ is nontrivial in $\mathrm{Out}(SU(N))$. Indeed, the group elements added to do the extension can be taken to be of the form $M_\theta = e^{i\theta} \oplus \mathbf{1}_{N-1}$, with the $U(1)$ in the $\rtimes$ acting on matrices $\mathcal{U} \in SU(N)$ as

$$\rho_\theta[\mathcal{U}] = M_\theta^{-1}\mathcal{U}M_\theta. \tag{30}$$

While $M_\theta$ isn't in $SU(N)$, this doesn't mean $\rho$ is an outer automorphism. For example, consider poo

[6] $x$ is the distance of the spin stiffness away from the endpoint of the critical line, and $y$ is the fugacity for the vortices. If $\varepsilon < 0$, then the flow would end up on the fixed LL line where $y = 0$; for $\varepsilon > 0$ the flow runs away to an ordered phase.

RG flow.[7] We are interested in computing $\xi(0)$—if this is large, then the theory will appear critical at a large range of distance scales (pseudocritical).

The amount of time $t_f$ it takes to flow from some negative initial UV value of $-g_0 \ll -\sqrt{\varepsilon}$ to large $g_0 \gg \sqrt{\varepsilon}$ is

$$t_f = \frac{1}{\varepsilon}\int_{-g_0}^{g_0} \frac{dg}{1+(g/\sqrt{\varepsilon})^2} = \frac{2}{\varepsilon}\tan^{-1}(g_0/\sqrt{\varepsilon}) = \frac{\pi}{\varepsilon} + a, \tag{33}$$

where $a \sim \mathcal{O}(g_0/\varepsilon^2) \ll 1$ goes to zero as $g_0/\varepsilon \to \infty$. Since the RG time is the log of the length scale along the flow via $\xi(t)e^t = \xi(0)$, we have

$$\xi(0) = \xi(t_f)e^{\pi/\varepsilon}. \tag{34}$$

Now $\xi(t_f)$ is the correlation length in some non-critical renormalized theory and hence is presumably of order 1 (the correlation length $\approx$ the lattice spacing far away from the critical point). Hence

$$\xi(0) \approx e^{\pi/\varepsilon} \gg 1, \tag{35}$$

and so the correlation length in the initial theory is exponentially large in $\varepsilon^{-1}$, so that pseudocritical behavior is exponentially strong in $\varepsilon^{-1}$.

# 5 Philisophical comments meaning of correlations / fluctuations and the continuum limit of lattice models

Today we have a few short philosophical comments on the meaning of the correlation length.

*Corrections to mean field theory come only from fluctuations at scales larger than $\xi$:* We run RG on our model while keeping the correlation length in lab units fixed. We flow all the way until the cutoff is at $\Lambda^{-1} = \xi$—at this point the dimensionless correlation length is zero, and hence different "lattice sites" are non-interacting, and MFT applies; All that fluctuations on scales $< \xi$ do are to renormalize the parameters of the theory—they do not change anything qualitatively.

fractals?

Systems near criticality with finite $\xi$ are often treated as being composed of mutually non-correlated blocks of size $\xi^d$.

[7] Remember that the dimensionless correlation length *decreases* along the flow. Picture a massive free theory to remind yourself of this—as we do the rescaling by eliminating lattice sites the correlation length stays fixed (the theory is free), and eventually becomes smaller than the lattice spacing; hence $\xi \to 0$ as required of a massive theory. Another way to remember this is that the physical correlation length (measured by some scientist with a ruler) is always fixed (this is the whole point of RG—low energy observables are held fixed under the flow), and so the dimensionless correlation lengths obey $\xi(t)e^t = \xi(0)$.

## The continuum limit

To say what we mean by the continuum limit, we have to introduce a notion of a dimensionful length scale (so far, everything has been measured with respect to the lattice spacing, which we've set to 1). The continuum limit means working in a limit where all physically relevant length scales in the theory are much larger than the lattice spacing. We often say that "the lattice spacing $a$ is being sent to zero", but this is not a dimensionally correct thing to say: better is that "the ratio $a/l$, where $l$ is any physical length scale, is being sent to zero".

We then need to define what we mean by "any physical length scale".

Therefore in order to have a continuum limit, we need the RG towards the UV in the lattice model to converge to a set of couplings such that all dimensionless physical length scales are infinite—that is, the theory must flow to some UV fixed point with infinite correlation length.

If our theory has a continuum limit, then the UV correlation length must be infinite—we are doing the RG with boundary conditions in the IR (or at intermediate energies) that fix the physical correlation length.

The fact that a collection of theories with different choices of $\xi$ all come from a continuum limit with $\xi_{UV} = \infty$ shouldn't be surprising; this is what happens when defining RG flows by fixing IR boundary conditions on irrelevant parameters (like a length scale). For example, the electric charge in 3+1D QED is irrelevant, meaning that the "bare" electric charge in the formal continuum limit is also infinite.

As a simple example, consider the Ising model (note: probably balls and springs is simpler). Upon fermionizing, we know the UV lattice model looks like

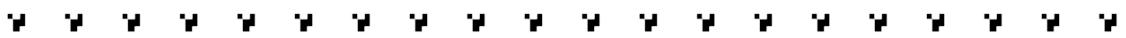
$$H \sim J \sum_i (c_i^\dagger c_{i+1} + c_i^\dagger c_i^\dagger + h.c.) - 2h \sum_i c_i^\dagger c_i. \tag{36}$$

Let's assume that a continuum limit can be taken, and write $c_i \to \sqrt{a}\psi_i$, with $\psi_i$ slowly varying on the lattice scale and $a$ a lattice spacing. Then

$$H \to 2Ja \sum_i a\psi_i^\dagger \partial \psi_i - 2(h-J) \sum_i a\psi^\dagger\psi \to \int \left(2Ja\psi^\dagger\partial\psi + 2(J-h)\psi^\dagger\psi\right), \tag{37}$$

where we need $a \to 0$. Therefore if we want an IR Hamiltonian with finite parameters, we need $J \to \infty$ and hence $h \to J$! This just means that we get a continuum limit in the IR only when the UV system goes to its critical point (with infinite coupling)...

# 6 2/5/2020 — Solutions to the wave equation and the parity of the spacetime dimension



# 7 Some chemistry

One general heuristic for why the transition metals + rare earths are good places to look for strongly correlated physics is that the wavefunctions for their valence electrons are typically relatively localized. This is essentially because their high angular momentum + principal quantum number means that their wavefunctions have many nodes, and therefore require less integration over the radial coordinate $r$ to be orthogonal with the lower $L$ / $n$ wavefunctions. Because of their localized wavefunctions the hopping terms in any putative Hamiltonian for a material involving these elements will be relatively small compared to the electron repulsion $U$, with the small value of $t/U$ then promoting correlated physics.

## Transition metals

- Examples: Cu2+, Ti3+, . . . , Herbertsmithite)
- Valence shells are 3d; electrons are usually fairly well localized
- Very small SOC; $SU(2)$ symmetry usually a good approximation
- Typical exchange constants are very large,

$$J_{TM} \sim 100 - 1000\,K \tag{38}$$

  This means that applying magnetic fields to polarize the system is often extremely hard / impossible to do

## Rare earths

- Examples: Ytterbium 3+,
- Valence shells are 4f.
- Very large SOC with strengths around 1eV; always want to use $J = L + S$ to label multiplets
- Typically have crystal field environments that will reduce things to sets of degenerate levels (often Kramers pairs) separated by 30 meV ish

- Electron orbitals are extremely localized. As a result, typical exchange constants are very small;
$$J_{RE} \sim 1\,K \tag{39}$$
This has the advantage of easily being able to apply magnetic fields that completely polarize the system.

- Since orbitals are so compact, spin Hamiltonians for these elements are usually strictly local to a very good approximation, i.e. nnn and higher terms can be omitted.

- Usually have large $g$ factors (the effective moments arise from rather complicated atomic physics) which means that the magnetizations they produce are relatively large — therefore e.g. neutron scattering experiments with rare earth compounds require smaller samples than those with transition metals

Going beyond the rare earths to the 4d / 5d transition metals, one gets things with strong SOC that can give interesting physics – this is home to e.g. the Ruthenates (Kitaev materials).

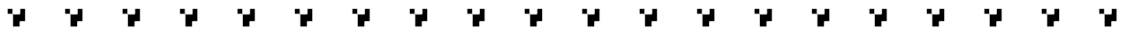
# 8 Random hopping model in 0+1D and heuristics about quasiparticles

Today's diary entry is motivated by wanting to go through in a bit more detail some topics discussed by Subir Sachdev during his lectures at the 2019 advanced winter school in Jerusalem. We will be looking at the $N \to \infty$ limit of the free Hamiltonian

$$H = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu Q, \tag{40}$$

where the $t_{ij}$s are the entries of a random Hermitian matrix with entries distributed according to Gaussians of standard deviation $t$, viz. $\langle t_{ij}^* t_{kl} \rangle \sim t^2 \delta_{ik} \delta_{jl}$, and where $Q$ is the number operator ($Q$ and not $N$ since $N$ is already in use).

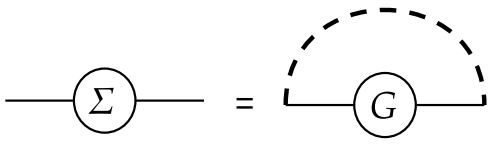
We will first compute the spectrum of this model. This has already been done in the diary entry on matrix models and geometry, but here our route to the Wigner semi-circle will be much more physics-flavored. We will then discuss in what sense this model has quasiparticles, and compute some simple thermodynamic quantities.



ethan: *quasiparticles are $not$ defined by the low energy level spacing. The low energy level spacing determines the thermodynamics since it determines the entropy, but there are plenty of nfls (for example?) which have e.g. $S \propto T$ etc. In this specific example of SYK models, it is true that the low energy level spacing makes a difference. But in general looking at the low energy spacing is really never a useful thing to do.*

### *Getting to the Wigner semicircle*

We will find the spectrum of $t_{ij}$ by using $\rho(\omega) \propto \mathrm{Im}[G(\omega)]$. The propagator is computed as follows. Graphically, $\langle c_i^\dagger c_j \rangle$ is given by a sum of $1/\omega$ propagators with with all possible numbers of $t_{ij}$s inserted. In the large $N$ limit the $t_{ij}$ will effectively provide us with a sample of a Gaussian probability distribution with standard deviation $t$, and so when computing the correlator we can average over the $t_{ij}$, treating them as free bosonic fields. For the usual diagrammatic reasons the leading terms in the $N \to \infty$ limit are those which are formed by $t_{ij}$ insertions connected by $t_{ij}$ propagators which, when all arranged on one side of the base line of $c$ propagators, do not cross. The self energy is computed by selecting the 1PI diagrams, which satisfy the recursion relation




where the dashed line is the $t_{ij}$ propagator. The $t_{ij}$ propagator just becomes a factor of $t^2$, and so the above diagram gives the relation

$$\Sigma(\omega)/t^2 = G(\omega). \tag{41}$$

Since $G(\omega) = \frac{1}{\omega - \Sigma(\omega)}$, we then get a quadratic equation for $\Sigma(\omega)$, which tells us that

$$\Sigma(\omega) = \frac{1}{2}\left(\omega + \sqrt{\omega^2 - t^2}\right). \tag{42}$$

Therefore

$$G(\omega) = \frac{2}{\omega - \sqrt{t^2 - \omega^2}}. \tag{43}$$

Evidently then

$$\mathrm{Im}[G(\omega)] = 2\Theta(t^2 - \omega^2)\sqrt{1 - \omega^2/t^2}. \tag{44}$$

Since the imaginary part of the correlator gives the spectral density, the distribution of eigenvalues of the Hamtiltonian, i.e. of $t_{ij}$, is indeed semicircular as claimed. That's it!

This is very easy to see in the case where we have large symmetries in the problem, like translation invariance. For example,

*Why this model has quasiparticles*

On one hand, the cmt texbook approach tells us that we certainly cannot have a qp description in the present case — the spectral function has a branch cut, and there are no poles in sight.

Note that this model being free does not necessarily imply that it has a quasi particle description, at least for our present definition of "quasiparticle description." It may be possible that the many-body level spacing near the chemical potential is exponentially small, even without interactions.

However, having a qp description available does not require there to be a (sub)extensive number of conserved quantities.

This Hamiltonian definitely does not have an extensive number of symmetries, having only the single global $U(1)$ charge conservation symmetry. This Hamiltonian The eigenvalues of the matrix $t_{ij}$ are distributed according to the Wigner semicircle at large $N$, and since $t_{ij}$ is an $N \times N$ matrix with eigenvalues distributed in the range $[-t, t]$, the eigenvalue spacing goes as $1/N$ in the interior of the semicircle. Because of the large dimension of the Hilbert space, he energy levels of the full Hamiltonian acting on Fock space are generically spaced as $\Delta\varepsilon \sim t \sim 2^{-N}$ (ignoring prefactors polynomial in $N$ which arise since e.g. the maximum energy depends on $N$). However near the bottom of the spectrum, the spacing instead scales as $1/N$. This is because the way in which we combine single-particle energies to form low-energy states is much more constrained than the way we in which we combine them to form high energy states (there are many more ways to express the number $10^6$ as a sum of distinct positive integers than there are ways to express the number 10). As such the spacing goes as $1/N$, since this is the spacing of the eigenvalues of the matrix appearing in the Hamiltonian.

Now the statistics of the energy levels might be an interesting diagnostic for theorists, but it is not useful if one would like to do an experiment to determine whether or not a given material has quasiparticles or not. Instead, one should look at e.g. thermodynamic quantities and look to see what kinds of features are expected from theories with quasiparticle descriptions. We know that the free energy (per particle) will look at low temperatures like $F/N = E_0 - \mu + \cdots$, where the $\cdots$ are $T$-dependent (we are at fixed chemical potential rather than fixed density, so $\mu$ is $T$-independent for now). Therefore we can calculate $F$ at $\mu = 0$, and then restore $\mu$ at the end by adding it on to the answer. This means that (after a change of variables) we have to calculate the following expression:

$$F/N = -2T \int_{-1}^{1} dx \, \sqrt{1-x^2} \ln(1 + e^{-\beta x t}). \tag{45}$$

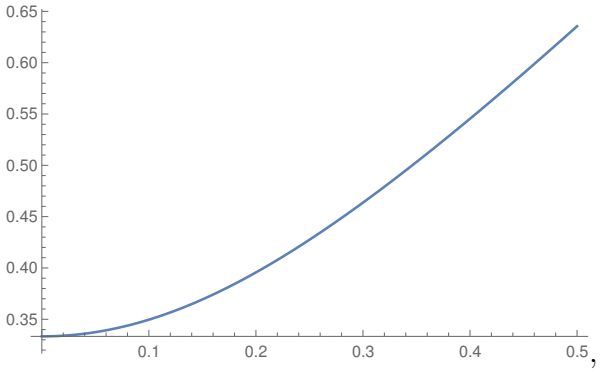
In the $T/t \to 0$ limit we may proceed by breaking up the log with the approximation

$$\begin{aligned} F/N &\approx T \int_0^1 dx \, \sqrt{1-x^2} \left( e^{-\beta x t} + \beta x t \right) \\ &\approx T \left( \int_0^\infty dx \, (1 - x^2/2 + \dots) e^{-\beta x t} + \int_0^1 dx \, \beta x t \right) \\ &\approx E_0 + \frac{\gamma}{2} T^2 + \cdots , \end{aligned} \tag{46}$$

where $E_0$ is $T$-dependent, $\gamma \propto 1/t$, and where the $\cdots$ go as $T^{n>2}$. Therefore the leading $T$-dependence of the entropy at low $T$ is

$$S(T \to 0) = \gamma T + \cdots , \tag{47}$$

which is the same as in a Fermi liquid. To check that the approximations above were valid, we can evaluate the free energy numerically: as a function of $T$, it looks like




which is indeed quadratic at low $T$.

## 9 Quantum criticality heuristics

- Another way to think of this is in terms of the gap of $\Delta \sim 1/\xi \sim t^{\nu}$ that the (former) critical fluctuations have away from the critical point. When we flow to the IR at $T = 0$, we end up at energies below $\Delta$, and can't "see" the critical fluctuations. Once we increase $T$ such that $T > \Delta$ however, the critical fluctuations at $\Delta$ are thermally activated, and we have a chance to see their effects on low-energy physics. This reasoning basically assumes that the effect of non-zero $t$ is solely to displace the spectrum at criticality upwards by opening a gap, while preserving the features of the critical modes.

- order parameter getting a vev iff thermo quantities are singular, provided that the derivative of $\langle\mathcal{O}\rangle$ wrt $\delta g$ is nonzero at the phase transition (in fact, maybe it has to be infinite?). Is it possible for $\langle\mathcal{O}\rangle$ to go smoothly to zero at the critical point?

- Sachdev motivates things by comparing the scale of $k_BT$ to $\Delta$, where $\Delta$ is the "typical energy scale". In a gapped phase this is clear; what about when we have continuous SSB? Should $\Delta$ be the mass of the Higgs mode? Why? Is $\Delta$ the part where the level spacing switches over from being sparse $L^{-d}$ (goldstones) to dense $2^{-L}$?

- how to make more precise this relaxation time Sachdev mentions, and how to show that it is $\gg \hbar/kT$ in the "classical" regime? Is the critical fan simply the area where $k_BT\tau \sim \hbar$ approaches the uncertainty limit?

- ETH says that pure states contain a temperature within them. Is this temperature a function of the energy density of these states? Can it be computed in any reasonable way? Is this too non-universal a question?

- basically: going to high $T$ takes you into the UV, in the sense that . For a fixed distance in coupling space away from the quantum critical point, the theory will look critical, i.e. will have power law correlations and such, at scales below some $\xi$. If $T$ takes you far enough into the UV that $\xi$ can be taken to $\infty$ as far as transport questions go, then the transport of the system will be governed by the quantum critical point. But, you need to ask questions that are concerned with relaxation times and stuff and transport. Maybe it's like: $R(T)$ and transporty things are controlled by physics below $\min(1/T, \xi)$. So if $1/T > \xi$ transport is that of a non-critical state, while in the other case it is that of a critical state cutoff by $1/T$?

- The argument about $\xi > \lambda_T$ tells us why the fan is critical — but why is it quantum? Basically because in this regime, the spacing between particles (which is $\sim \xi^{-1}$ for $\xi$ the dimensionless correlation length [think about e.g. the particles being domain walls, with the correlation length being schematically(!) given by the distance between domain walls]) is $\leq$ the inverse de brogile wavelength for the particles. The latter is determined by $\lambda_B^{-2} \sim \langle\varepsilon\rangle 2\Delta$, where $\Delta$ is the gap to the excitations / domain walls. $\langle\varepsilon\rangle$ is determined by equipartition to be $T$, and so $\lambda_B \sim 1/\sqrt{\Delta T}$.

- talk about $\tau_\phi$ the phase coherence time. This is identified with the relaxation time. In the quantum regime, $\theta_R = i\chi(0,0)\partial_\omega\chi^{-1}(0,\omega)|_{\omega=0}$, where we assume a form for the susceptibility like

$$\chi(k,\omega) = \frac{\chi(0,0)}{1 - i\omega\tau_\varphi + k^2\xi^2 - (\omega\tau')^2 + \ldots}, \tag{48}$$

  logarithmic singularity free energy with the linear imaginary part controlling the relaxation. (the linear part has to be imaginary because of the reality of $\chi(x,t)$). In the classical regimes, $\theta_\phi$ (and $\xi$) are determined by classical reasoning.

First, if we buy the argument that $\xi \sim \rho^{-1}$ with $\rho$ the density of domain walls, (this is clear because the correlation function in space will be $e^{-\rho|x|}$, due to the fact that higher densities of domain walls lead to more sign cancellations in the correlator) then using Boltzmann statistics tells us that $\rho \sim \int dk\, e^{-(\Delta+k^2/2\Delta)/T} \sim \sqrt{T\Delta}e^{-\Delta/T}$. For example, in the 1+1D Ising model, they are determined by computing correlators by thinking about the domain walls as particles whose worldlines can intercept the line linking the two spins in the correlator. The chance for a given particle with velocity $v_k$ to intersect a line in the time direction is $|v_k t|/L$, and so Boltzmann averaging over all such particles we get $\theta_\phi^{-1} \sim \int_0^\infty dk\, v_k\, e^{-\varepsilon_k/T} \sim Te^{-|\Delta|/T}$. ethan: *See sachdev pg 154*

## General expectations for the phase diagram

In this subsection we will be interested in QCPs controlled by two relevant parameters: the temperature $T$ and a relevant (dimensionless) coupling $g$, with the QCP being at $g, T = 0$. We will use scaling theory to explain the typical shapes of the phase boundaries in $T$-$g$ phase diagrams with a QCP on the $T = 0$ axis.

Finite size scaling (which is what we'll be doing in finite-$T$ quantum problems) works essentially by matching scaling forms for thermodynamic quantities (viz. the free energy $f$) in the limits where the length of the compact direction (alias $T^{-1}$—we will be sloppy and let $T$ correspond to the *dimensionless* scaling variable of the inverse size of the thermal circle) goes to either 0 or $\infty$. First, consider the $T = 0$ QCP, where $T, g$ have RG eigenvalues $z, y$. We know from general arguments that after $n$ RG steps of size $s$, the singular part of the free energy satisfies[8]

$$f_s(g, T) = s^{-n(d+z)} f_s(s^{ny} g, s^{nz} T). \tag{49}$$

Choosing $n$ such that $s^{-n} \sim g^{1/y} = g^\nu$, we get the scaling form

$$f_s(g, T) \sim g^{(d+z)\nu} \Phi_g(Tg^{-z\nu}). \tag{50}$$

We can also choose the scale at which $s^{-n} \sim T^{1/z}$:

$$f_s(g, T) \sim T^{d/z+1} \Phi_T(gT^{-1/z\nu}). \tag{51}$$

Note that these two scaling forms are useful in different limits: the first when we are sitting at $T = 0$ and tuning $g$ towards the critical point, and the second when we are sitting at $g = 0$ and tuning $T$. There is no nice limit where both scaling functions are useful, since if $Tg^{-z\nu}$ is small then $gT^{-1/z\nu}$ is large, and vice versa. Evidently the order of limits for $T, g \to 0$ do not commute, and so extending these scaling functions to find $f_s(g, T)$ as a full function in the plane is nontrivial. Nonetheless, these scaling functions can be used to make quite general predictions about scaling of thermodynamic quantities near the critical point in various limits. For example, at the critical coupling $g = 0$ (for which $\Phi_T(0)$ is a constant) we have $f_s \sim T^{d/z+1}$. The

[8] Note that this is exact: it is always possible to define a set of variables which agree with $g, T$ right near the critical point, and which scale *exactly* under RG via the $s^{ny}$ factors below (proved in another diary entry). In the following we will assume that this has been done.

singular part of the specific heat is computed with $T\partial_T^2 f_s$,[9] so that we have on very general grounds that

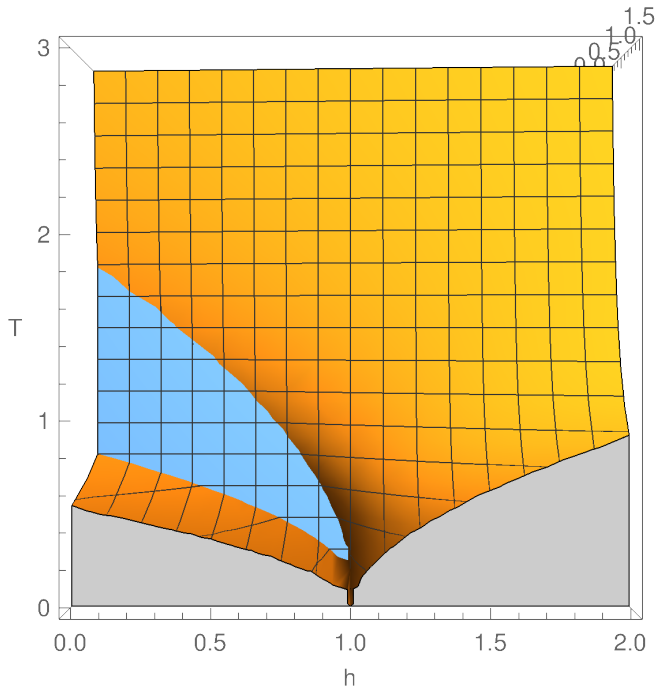
$$C_s(T, g = 0) \sim T^{d/z}. \tag{52}$$

This characterization for $f_s$ uses "UV information" from the scaling form near the $T = 0$ critical point. However, being a relevant variable, the flow will be towards (we assume for the sake of concreteness) a critical point at finite $T$ described by classical model in $d$ dimensions. Since for small $g$ most of the RG time will be spent by this critical point, we can alternatively constrain the scaling behavior of $f_s$ by using the known scaling rules at this fixed point. ethan: *talk about why the fixed point will be at finite $T < \infty$, even though we expect the exponents to be those of the (formally $T = \infty$) $d - 1$ dimensional model.*

The region where a classical description is valid is then determined by[10]

$$|g - g_c(T)|^{z\nu} \lesssim T. \tag{53}$$

Using that $g_c(T) \sim T^{1/z\nu}$, we find the expected result of a classical region which shrinks to zero width at the QCP and grows as the finite-$T$ point is approached:




(54)

where the region controlled by classical scaling is in blue (of course the exact boundaries are not sharp; it is just a crossover). The width of the classical region is just determined by dimensional analysis, with the width growing as $\Delta T \sim |g - g_c|^{\nu z}$.

[9] Since $T\partial_T^2 f = \partial_{\ln T} S = C$.

[10] The $\lesssim$ here means that the constant of proportionality is some number strictly less than unity; usually one chooses $1/2$ or similar. Replacing the $\lesssim$ with an $=$ is no good since we would then conclude the classical region extends to all $t$ when $g = 0$.

### How to deal with systems with Fermi surfaces

The general scaling arguments we presented above don't carry over directly to systems with Fermi surfaces, since in such systems the usual way of doing the rescaling during RG does not match the one assumed above. For example, consider the specific heat near the FL fixed point: we argued above that it must scale as $T^{d/z}$. In the FL context, we know that we must have $z = 1$. However then we are certainly not allowed to use the physical space dimension for $d$, since this would give us the wrong result in all but $d = 1$. Instead, we must use $d = 1$ regardless of the physical dimension, due to the fact that the scaling when doing RG near the FL FP happens along along one (radial) direction in momentum space. In fact the whole scaling procedure in this case doesn't really translate well into $\mathbb{R}$ space, casting the above formalism in this case into doubt, since it relied on having a $\mathbb{R}$ space scaling procedure.

## 10 *Basic black hole thermodynamics ✓*

This is another homework problem, this time from Hong's AdS/CFT class. We will need to make use of the Kerr-Newman metric, viz.

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma}dt^2 + \frac{\Sigma}{\rho^2}\sin^2\theta(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \tag{55}$$

where $J$ is the angular momentum, $Q$ the charge, $M$ the mass, and constants are

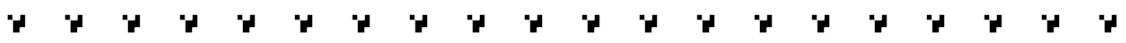
$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr, \quad a = J/M, \tag{56}$$

and

$$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \quad \omega = \frac{a}{\Sigma}(r^2 + a^2 - \Delta). \tag{57}$$

The problem statement is as follows:

a) verify the first law of black hole thermodynamics for the Kerr-Newmann metric. b) Take $Q = 0$. Suppose the black hole looses all of its angular momentum through some classical process. How much of its initial mass can be radiated away? c) Take $J = 0$ and $M > Q$. Suppose we want to reach extremality by throwing charged particles into the black hole. Show that if the particles being thrown in are not themselves black holes, an infinite number of particles need to be thrown in before the black hole becomes extremal.



a) The first law of BH thermodynamics is

$$dM = TdS + \Phi dQ + \Omega dJ, \tag{58}$$

where $S = A/4$. Now for us, the electric potential at the horizon is $\Phi = Er_+ = 4\pi Qr_+/A$. Putting this into the first law, we get

$$dM = \frac{r_+ - M}{2A}dA + \frac{4\pi J}{MA}dJ + \frac{4\pi Qr_+}{A}dQ. \tag{59}$$

Re-arranging,

$$dA = \frac{8\pi}{r_+ - M}\left((r_+^2 + J^2/M^2)dM - \frac{J}{M}dJ - Qr_+dQ\right). \tag{60}$$

This had better agree with our formula for the area. Using the area formula, we have

$$dA = 8\pi(r_+dr_+ + ada) = 8\pi\left[r_+\left(dM + \frac{MdM - ada - QdQ}{r_+ - M}\right) + ada\right]. \tag{61}$$

Now

$$ada = \frac{J}{M}\left(\frac{dJ}{M} - \frac{JdM}{M^2}\right), \tag{62}$$

so that

$$dA = 8\pi\Big(dM\left(r_+ + \frac{Mr_+}{r_+ - M} - \frac{J^2}{M^3} + \frac{J^2}{M^3}\frac{r_+}{r_+ - M}\right) + dJ\left(-\frac{r_+}{r_+ - M}\frac{J}{M^2} + \frac{J}{M^2}\right) - QdQ\frac{r_+}{r_+ - M}\Big). \tag{63}$$

After pulling out the common factor of $r_+ - M$, this agrees with the above expression for $dA$ derived from the first law. So, the first law checks out.

b) The second law of BH thermodynamics says that classically, $A$ is monotonically increasing: $A_f \geq A_i$ for some process $i \to f$. If we want to radiate away as much mass as possible, we should take our initial black hole to be spinning as fast as possible. For $Q = 0$, the extremal solution thus has $a = M$, or $J = M^2$. The initial area is then

$$A_i = 4\pi(M_i^2 + J^2/M_i^2) = 8\pi M_i^2. \tag{64}$$

On the other hand, when the black hole radiates to the point where $J = 0$, we have the Schwarzchild relation

$$A_f = 4\pi r_+^2 = 16\pi M_f^2. \tag{65}$$

Setting $A_f = A_i$ to get the biggest possible amount of radiated mass, we get

$$M_f^2 = M_i^2/2, \tag{66}$$

or $M_f/M_i = 1/\sqrt{2} \approx 0.71$. Thus at most 29 percent of the black hole's initial mass can be radiated away.

c) If the particles with charge, mass $q, m$ we are throwing into the black hole are pushing the black hole closer to extremality, we need $q/m > 1$. But in order for them to fall in we need $qQ < Mm$, so that $1 < q/m < M/Q$. Define

$$\chi \equiv Q/M. \tag{67}$$

Now

$$\dot{\chi} = \chi(\partial_t \ln Q - \partial_t \ln M) = \frac{q}{M} - \frac{mQ}{M^2}, \tag{68}$$

where $\dot{Q} = q$ and $\dot{M} = m$ (each timestep is signified by throwing one particle $q, m$ into the BH). Now we re-write this as

$$\dot{\chi} = \chi \frac{m}{M}\left(\frac{q}{m}\chi^{-1} - 1\right) < \chi\left(\frac{q}{m}\chi^{-1} - 1\right). \tag{69}$$

Here we used $m/M < 1$: this is true since if $m \geq M$, the fact that $q > m$ would mean that the "particle" being added to the black hole was a super-extremal black hole itself. Now since $q/m < M/Q$,

$$\dot{\chi} < \chi\left(\chi^{-2} - 1\right) = \chi^{-1} - \chi. \tag{70}$$

This means that the time it takes to bring the black hole up to $Q = M$ is bounded below by

$$\Delta t \geq \int_{\chi_0}^{1} d\chi \frac{1}{\chi^{-1} - \chi} = -\frac{1}{2}\int_{\chi_0}^{1} \frac{-2\chi d\chi}{1 - \chi^2} = -\frac{1}{2}\ln(1 - \chi^2)|_{\chi_0}^{1}, \tag{71}$$

which is divergent for all $\chi_0 < 1$. Thus it takes an infinite amount of time to build up to the point $Q/M = 1$.

# 11 Exactly WTF is up with fine structure of atoms? How do you compute the degeneracies of various shells?

blah blah . We will be ignoring hyperfine splitting and all stuff coming from nuclear magnetic moments throughout.

First, notation. We will use $L$ to denote the total angular momentum quantum number, i.e. $|\boldsymbol{\lambda}|^2 = L(L+1)\mathbf{1}$. We will always use subscripts to denote a particular component of the angular momentum, e.g. $\mathbf{L}^z$ (indices will be raised/lowered at convenience), which when diagonalized acts as $L^z\mathbf{1}$. When adding multiple angular momenta, we will write $L^z$ for the eigenvalue of $\sum \mathbf{L}_i^z$, where the $i$s run over different tensor factors and $\otimes\mathbf{1}$s are suppressed. We will never use the horrible convention where $L = \sum_i L_i^z$; this is disgusting. All of the above notational comments are covariant under replacing $L$ with $J$ or $S$ or any other matrices generating a representation of $SU(2)$.

To this order, the energy levels are labeled only by $n$. Since the number of states at each $n$ is

$$2\sum_{L=0}^{n-1}\sum_{L_z=-L}^{L} 1 = 2\sum_{L=0}^{n-1}(2L+1) = 2(n + (n-1)^2 + (n-1)) = 2n^2 \tag{72}$$

This is unfortunately totally in contradiction with observed distribution of the Hydrogen level spacings. Furthermore, if we neglected the influence of electron interactions on the level distributions and pictured the higher $Z$ elements as being constructed by filling up Hydrogen levels with multiple electrons then we would get something totally in conflict with the periodic table — the elements with atomic numbers $Z = \sum n = 1^N 2n^2$ for $N \in \mathbb{N}$ do not display any kind of similar behaviors (these elements would be where we'd predict the noble gasses to be).

However, once we take into account relativistic corrections, this $2n^2$ degeneracy will get (partially lifted) and things will look a lot more realistic. Instead of labeling states by $L, L_z, S, S_z$, it will be better to label them by an equivalent maximally commuting set of operators (this is a maximal torus?), viz. $J, J_z, L, S$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ (checking that this is a maximally commuting set is easy). The advantage of doing this is that $[\mathbf{J}, \mathbf{L} \cdot \mathbf{S}] = 0$ (actually do it! Note that $\mathbf{L} - \mathbf{S}$ actually commutes with $\mathbf{L} \cdot \mathbf{S}$!). When we take into account relativistic corrections a SOC term $\mathbf{L} \cdot \mathbf{S}$ will appear in $H$, and so this is a better basis to work in.

Now since we are tensoring the $L$ angular momentum with the spin, the possible values for $J$ are $L \pm 1/2$. After a bit of algebra, the SOC + kinetic energy $p^4$ corrections + Darwin term[11] combine to give an energy splitting of

$$\Delta E_{n,J=L\pm 1/2,J_z,L} = \frac{1}{2} mc^2 \left(\frac{\alpha Z}{n}\right)^4 \left(\frac{3}{4} - \frac{n}{J + 1/2}\right), \tag{73}$$

which is independent of $J_z$. Note how it also does not depend on $L$, but rather only on how we take the tensor product between $L$ and $S$,. Finally, note how for a single electron the biggest $J$ can be is $L + 1/2$, so that the biggest the denominator in the second term can be is $n$ — hence $\Delta E$ is always negative, with $|\Delta E|$ being smaller for larger $J$.

Some easy examples of this splitting are the following. For $n = 1$ we have $J = 1/2$, $J_z = \pm 1/2$, and hence two degenerate states — so the relativistic corrections just shift the doublet down in energy. For $n = 2$ we non-relativistically have 8 degenerate states. The $L = 0$ states will have $J = 1/2$ and hence will still give a degenerate doublet. The $L = 1$ states will have $J = 1/2$ or $J = 3/2$. The former gives another doublet at the same energy as the $L = 0$ doublet,[12] while the latter tells us that the remaining $2(3/2) + 1 = 4$ states are in a quadruplet at a slightly higher energy. For $n = 3$, we again get two degenerate $J = 1/2$ doublets. We also have two ways to get to $J = 3/2$, which give us $2 \cdot (2(3/2) + 1) = 8$ degenerate states, and then finally one $J = 5/2$ multiplet with 6 states. All together this gives us $4 + 8 + 6 = 18 = 2 \cdot 3^2$ states, as required. The general pattern is similar: all $J$ values except $J = n - 1/2$ can be obtained in two ways (while there is only a single $J = n - 1/2$ multiplet), and each multiplet gives $2J + 1$ states.

Therefore we see that the reason why energy levels with different $L$ values have different energies is due to SOC.

[11] The one that comes from the electron spending time at the origin; only relevant for zero angular momentum states.

[12] Neglecting QFT self-energy corrections of course — the fact that these two doublets have different energies is the famous Lamb shift!

This has just been for Hydrogen. How do we figure out how energy levels get filled up in more complicated atoms? We will proceed as usual by assuming that the electrons don't interact with each other, so that we can treat the problem as filling up a Hydrogen-like spectrum of levels with multiple electrons.

Since the quadratic Casimirs used to define $L$ and $S$ are non-linear,

The claim then is that the degeneracy of the partially-filled shell is $2J_z + 1$. Why does $J_z$ appear here and not $J$?

For a shell with angular momentum $L$ with only one electron or with only one electron missing, this tells us that the degeneracy is

We haven't been focused on numbers here, but here's the rough guide. Orbital and kinetic energies ($n$ quantum number) determine the gross level spacings; this is on the $O(1)$ eV scale. Accounting for the relativistic corrections gives shifts of $10^{-4}$eV ish, or about 1K. Things like hyperfine structure, which generically completely remove degeneracies, are a further factor of $10^3$ish suppressed. This is all also in vacuum; the effects of SOC can be enhanced in crystal environments, and of course there are other things like crystal field splitting that will break apart degenerate multiplets.

## 12 Various large-$N$ limits of the $O(3)$ model



The large-$N$ limit we will consider here is obtained by extending the $\mathbf{n}$ field of (**??**) to a $(2N)^2 - 1$-component field, whose components we then fractionalize as

$$n^a = 2z^\dagger T^a z, \tag{74}$$

with the $z$ fields transforming in the fundamental of $SU(2N)$, and with $T^a, a = 1, \ldots, (2N)^2 - 1$ the generators of the defining representation of $\mathfrak{su}(2N)$, which ensure that $\mathbf{n}$ transforms as the $A$ in $\bar{N} \otimes N = 1 \oplus A$.[13] The $z$ fields in the above decomposition satisfy the normalization condition $|z|^2 = 1$ (from which one can check that $|n|^2 =$), and have a $U(1)$ gauge redundancy by virtue of (74); as such they live in the space $\mathbb{CP}^{2N-1}$, with the case of physical interest is obtained by setting $N = 1$. We

[13] Indeed, consider transforming the $z$ fields by $z \mapsto e^{\theta^a T^a} z$. For infinitesimal $\theta^a$, we then have

$$n^a \mapsto 2z^\dagger (T^a + \theta^b [T^a, T^b]) z = 2z^\dagger (T^a + i\theta^b f^b{}_{ac} T^c) z = (\delta_{ac} + i\theta^b f^b{}_{ac}) n^c, \tag{75}$$

which shows that $\mathbf{n}$ indeed transforms in the adjoint of $SU(2N)$.

will take the magnetic field to point along the direction of the matrix $Z^{\oplus N}$, so that we get back the spin-1 results when $N \to 1$.[14]

Using this decomposition for $\mathbf{n}$, one obtains the $\mathbb{CP}^{2N-1}$ model with action

$$S = \int d^d x\, d\tau\, z^\dagger \left( m_0 \partial_\tau - \rho D_{\mathbf{a}}^2 + \lambda - m_0 H Z^{\oplus N} \right) z - \int d^d x\, d\tau\, \lambda, \tag{76}$$

where $\lambda$ is a Lagrange multiplier enforcing $|z|^2 = 1$ and $\mathbf{a}$ is a gauge field introduced when decoupling the interactions of the $z$ fields. To see how this mapping works, consider the kinetic term $\rho|\nabla \mathbf{n}|^2/2$. In terms of the $z$ fields, this becomes

$$\frac{\rho}{2}|\nabla \mathbf{n}|^2 = \rho \left( \delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl} \right) z^*_{(i}\nabla z_{j)} z^*_{(k}\nabla z_{l)} = 2\rho \left( (z^\dagger \nabla z)^2 + |\nabla z|^2 \right), \tag{77}$$

where we have used $T^a_{ij}T^a_{kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}/2N)$ and $z^*_{(i}\nabla z_{i)} = \nabla |z|^2 = 0$. We then decouple the quartic term by integrating in a vector field $\mathbf{a}$, which has the effect of replacing $|\nabla z|^2$ with $|D_{\mathbf{a}} z|^2 = |\nabla z - i\mathbf{a}z|^2$. The equation of motion for $\mathbf{a}$ reads $\mathbf{a} = -iz^\dagger \nabla z$, which indeed transforms as a $U(1)$ gauge field under $z \mapsto e^{i\theta(x)} z$.

In terms of the $\mathbf{n}$ field, [15]

The factor of 2 here is because we will be working with the standard normalization $\mathrm{Tr}(T^a T^b) = \delta_{ab}/2$ The awkward factor of $C_N$ here ensures that $|\mathbf{n}|^2 = 1$, with $C_1 = 2$ giving the expected $\mathbf{n} = z^\dagger \boldsymbol{\sigma} z$ in the physical $SU(2)$ case. Indeed, we check that

$$\begin{aligned} n^a n^a &= C_N^2 T^a_{ij} T^a_{kl} z_i^\dagger z_k^\dagger z_j z_l = \frac{C_N^2}{2} \left( \delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl} \right) z_i^\dagger z_k^\dagger z_j z_l \\ &= \frac{C_N^2}{2}|z|^4(1 - 1/2N) = C_N^2(N/2 - 1/4) = 1 \ \checkmark \end{aligned} \tag{80}$$

[14] The matrix $Z^{\oplus N}$ can always be chosen as a generator of $\mathfrak{su}(2N)$ since the generators of $\mathfrak{su}(2N)$ form a basis for the set of $2N \times 2N$ traceless Hermitian matrices.

[15] The WZW is well-defined here because $\pi_2(\mathbb{CP}^n) = \mathbb{Z}$ for all $n$. Indeed, since $\mathbb{CP}^n = S^{2n+1}/U(1)$ we have the long exact sequence

$$\cdots \to \pi_2(S^{2n+1}) \to \pi_2(\mathbb{CP}^n) \to \pi_1(S^1) \to \pi_1(S^{2n+1}) \to \cdots \tag{78}$$

which reads

$$\cdots \to 0 \to \pi_2(\mathbb{CP}^n) \to \mathbb{Z} \to 0 \to \cdots, \tag{79}$$

implying $\pi_2(\mathbb{CP}^n) = \mathbb{Z}$. Note that this homotopy group is nontrivial only because of the $U(1)$ gauge redundancy, which means that the generators of $\pi_2(\mathbb{CP}^n)$ can be identified with monopole configurations of the gauge field $\mathbf{a}$. Therefore the WZW term can be written as $2m_0 i \int d^d x \int_{B^2} d\mathbf{a}$, where $\partial B^2 = S^1_\tau$ with $S^1_\tau$ the thermal circle, and where $d$ is the exterior derivative on $B^2$ (this is independent of $B^2$ provided $m_0 \in \frac{1}{2}\mathbb{Z}$). This then gives $2m_0 i \int d^d x \int d\tau\, \mathbf{a}_\tau = 2m_0 \int d^d x\, d\tau\, z^\dagger \partial_\tau z$ as required.

fact: t hooft lines in coulomb phase are P law; this follows from noticing that magnetic fluxes are not confined in the real world. This is a bit puzzling from the perspective of the lattice $\mathbb{Z}_N$ theory, and from the lattice $U(1)$ theory as well — perturbation theory in $\lambda$ would naively indicate an area law. But this is because of some fundamental asymmetry between electric and magnetic stuff that's not really there...

lineon condensate: can form $m^a$ sheets using links in the $a$ direction. Only weird thing about this is that the line tension for the magnetic sheets is anisotropic — on one boundary there are a bunch of fractons; on the other there aren't.

Problem is how to think of the $\mathfrak{e}$ lines in the magnetic dipole condensate in terms of surfaces — should only have $m$s be able to intersect them from one direction, but this doesn't seem to be true. Where do we draw the line between deconfined $\mathbb{Z}_N$ gauge theory and something trivial? When the mass of the gauge particles is large? What if the particles aren't that massive but the magnetic sheets have infinite surface tension?

*pure gauge theory*

| | 2+1D | | 3+1D | | |
|---|---|---|---|---|---|
| Operator | deconfined | confined | deconfined | confined | Coulomb |
| closed Wilson | P | A | P | A | P |
| open t Hooft | A | P | A | P | P |

*with matter*

| | 2+1D | | 3+1D | | |
|---|---|---|---|---|---|
| Operator | deconfined | confined | deconfined | confined | Coulomb |
| closed Wilson | P | A | P | A | P |
| open Wilson | | | | | |
| closed t Hooft | | | | | |
| open t Hooft | | | | | |

# 13 Skyrmion supercondutors

Today we're going to be following [1] and seeing why skyrmions in a quantum spin Hall system have electric charge 2.

The microscopic realization is fermions on the honeycomb lattice at half-filling. We will assume that the interactions are such that within mean field theory the electrons

couple via a mass term to a vector field $\mathbf{N}$, which represents the expectation value of the bilinear $\bar{\psi}\boldsymbol{\sigma}\psi\rangle$. Here $\boldsymbol{\sigma}$ operates in physical $SU(2)$ spin space, while the Dirac matrices for $\psi$ are constructed in the sublattice space. The Lagrangian is

$$\mathcal{L} = i\bar{\psi}(\not{\partial} + m\mathbf{N}\cdot\boldsymbol{\sigma})\psi. \tag{81}$$

We will assume that the interactions are such that a ferromagnetic state for $\mathbf{N}$ is favored, with $\langle\mathbf{N}\rangle = (0,0,1)$. This order then gives opposite Dirac masses to the two sublattice components of $\psi$. In the ordered state, and coupling $\psi$ to background $U(1)$ fields for charge conservation ($A^c$) and rotation about $\sigma^z$ ($A^s$), we have

$$\mathcal{L} = \bar{\psi}\gamma_\mu(i\partial_\mu + A^c_\mu + \frac{1}{2}\sigma^z A^s_\mu)\psi + im\bar{\psi}\sigma^z\psi. \tag{82}$$

Now consider the response theory we obtain upon integrating out the fermions. Since the two spin components have opposite masses, there will be no $A^c \wedge dA^c$ or $A^s \wedge dA^s$ responses. However, since the two components also have opposite charges under $U(1)_s$, the mixed terms $A^c \wedge dA^s$ will have the same sign. This means

$$\mathcal{L}_{\text{eff}} = \frac{1}{2\pi}A^c \wedge dA^s, \tag{83}$$

which of course is the statement of the quantum spin Hall effect.

Now consider a Skyrmion texture in $\mathbf{N}$. We want to show that it has charge 2. This can be done by integrating out the fermions in the large $m$ expansion a la Abanov; see an earlier diary entry for the details. The way that [1] argues it is as follows. First, consider a skyrmion texture that goes from $\mathbf{N} = -\hat{\mathbf{z}}$ in the core to $\mathbf{n} = \mathbf{z}$ at infinity. We can write

$$\mathbf{N} = (\sin(\pi f(r))\cos(\phi), \sin(\pi f(r))\sin(\phi), \cos(\pi f(r))), \tag{84}$$

where $f(0) = 0$, $f(r \to \infty) \to 1$, and where $\phi$ is the angle in the spatial plane.

We then perform a local rotation to make the mass term have the simple $\bar{\psi}\sigma^z\psi$ form (for which we know the response theory), at the expense of adding in an $A^s$ gauge field which corrects for this rotation. This is done by finding $U$ such that

$$U^\dagger \mathbf{N} U = \hat{\mathbf{z}}. \tag{85}$$

For the skyrmion we are considering, $U$ is a rotation by $\pi f(r)$ about $\hat{\boldsymbol{\phi}}$, viz.

$$U = \cos(\pi f(r)/2)\mathbf{1} + i\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma}\sin(\pi f(r)/2). \tag{86}$$

Rotating by $U$ introduces the $A^s$ gauge field $-iU^\dagger dU$. At long distances, this is

$$dU(r \to \infty) \approx i\frac{1}{r}(\partial_\phi\hat{\boldsymbol{\phi}})\cdot\boldsymbol{\sigma}, \tag{87}$$

with the corrections (proportional to $\partial_r f(r)$) falling off faster than $1/r$. Therefore

$$\begin{aligned} -iU^\dagger dU &= -i\frac{1}{r}(\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\sigma})(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}) \\ &= \frac{1}{r}(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{r}})\cdot\boldsymbol{\sigma} \\ &= \frac{1}{r}\sigma^z. \end{aligned} \tag{88}$$

The skyrmion background therefore has the effect of generating (at long distances) a $4\pi$ flux tube vector potential for the up spins and a $-4\pi$ potential for the down spins ($4\pi$ since the vector potential couples to the fermions as $\sigma^z A^s/2$), so that the response theory to $A^c$ at distances far from the skyrmion core is

$$\mathcal{L}_{\text{eff}} = \frac{2}{2\pi} A^c \delta(r). \tag{89}$$

From the factor of two out front, we conclude that the Skyrmions have charge 2 — condensing them can then produce a superconductor.

So first, if we think of going from $C = 1$ to $C = 2$ as simply doubling $\Theta$, we do get a level 2 CS term $\frac{2}{4\pi} ada$ in commutative theory, which through a shift of $a$ by $-A$ gives the correct Hall conductance.

The CS term comes from the constant term in Lagrangian multiplier $ia_0\underline{\rho}$, which after SW map becomes $i\underline{\rho}\Theta\frac{1}{2}ada$. So it does scale with $\Theta$.

However I also had doubts on legitimacy of this hand waving treatment, which may be related to your question.

So, previously for $C = 1$ case, we constructed the composite boson by binding a microscopic fermion $c$ to an auxillary fermionic vortex $f$ in an inverted LLL. Basically, $b_{mn} = c_m f_n$, where physcial spin is supressed and $m, n$ are indices for orbits in LLL.

But in $C = 2$ case, the magnetic field is reduced by one half while density is fixed. Therefore one has to include 2 LLLs for the $f$ fermions to fill.

In the meantime the same needs to be done to the "left"/physical degrees of freedom, so that the composite boson $b_{im,in} = c_{im} f_{in}, \quad i = 1, 2$ sees no magnetic field, so that we can transform to momentum space.

Now the problem basically becomes a spinful composite bosons talking to two gauge fields $aandA$. Here the "spin" is just the index oop of LLL, $i = 1, 2$.

First look at the intergnal gauge filuctuation. The gauge constraint is now $f^\dagger_{i,n} f_{i',n'} = \delta_{ii'}\delta_{nn'}$, where $i = 1, 2$ is the lable of auxillary LLLs. So there are now 4 constraints.

Now let's call Landau level degeneracy $N$, then the gauge constraint should enforce the "right" indices $(i, n)$ to be SU(2N) singlet. Then I guess the gauge structure in long wavelength limit should really be $U(1) \times SU(2)$ instead of $U(1)$ or $U(1) \times U(1)$.

Meanwhile, the external U(1) gauge field $A$ of course still couples to left coordinates of composite bosons same as before.

Also, if I naively look at the CS term, assuming this term is still only contributed by the constant term in Lagrangian multiplier for the right density, it seems to give two level-1 CS terms, which leads to the correct $\sigma_{xy}$. So we have $ia_0^{ij}\underline{\rho_{ij}}$, where summation is assumed, and I have written the 4 components of internal gauge field as $a^{ij}, \quad i, j = 1, 2$.

There are two nonzero terms $\underline{\rho_{11}} = \underline{\rho_{22}} = \underline{\rho}/2$. I am keeping the convention of setting total density to be one, "/2" because the right density in one LLL is halved, the same as the magnetic field.

But $\Theta$ is doubled. In the end we get after SW map: $i\underline{\rho}/2\Theta\frac{1}{2}(a^{11}da^{11}+a^{22}da^{22})$. When shifting the field, $a^{11}$ and $a^{22}$ both shift by $-A$. Then we get the correct Hall conductance.

# 14 Reminder about magnetic translations

$$\Phi = 2\pi\frac{p}{q}, \qquad (p,q)=1. \tag{90}$$

In what follows we will be working exclusively on a square lattice with PBCs. The minimal-sized lattice we may consider is one of size $L=q$. The magnetic translation generators satisfy

$$T_x^q = T_y^q = \mathbf{1}, \qquad T_xT_y = \zeta_q T_yT_x. \tag{91}$$

One representation can be taken to be $T_x = Z, T_y = X$, with $Z, X$ the $\mathbb{Z}_q$ clock and shift matrices. In fact this is the unique irreducible representation: it is $q$-dimensional and the order of the group is $q^2$ (hence there are no other nontrivial representations)

These act on functions as

$$\begin{aligned} T_xf(x,y) &= f(x-1,y)\\ T_yf(x,y) &= e^{i\Phi x}f(x,y-1). \end{aligned} \tag{92}$$

The basis functions are of course

$$\psi^n_{k_y} = \phi_n(x-k_y/\Phi)e^{ik_yy} \tag{93}$$

where $\phi_n$ is the $n$th oscillator mode.

On the basis functions,

$$\begin{aligned} T_x\psi^n_{k_y} &= \psi^n_{k_y+\Phi}\\ T_y\psi^n_{k_y} &= e^{i(\Phi x-k_y)}\psi^n_{k_y}. \end{aligned} \tag{94}$$

Invariance under translations around the whole lattice in the $y$ direction tell us that $x$ needs to be the position of a lattice site (and not a continuous variable), and that $k_y\in(2\pi/L)\mathbb{Z}$ as usual. $T_y^L$ then acts on the $\psi$s as $\mathbf{1}$, but the same is not true for $T_x^L$, since as we are in the continuum we are not identifying $k_y$ with $k_y+2\pi$. To work with basis functions on which $T_x^L$ is represented trivially, we take linear combinations of the $\psi^n_{k_y}$ as

$$\Psi^n_{k_y} \equiv \sum_{m\in\mathbb{Z}} e^{imq} \tag{95}$$

# 15 Inverting a tensor product of two Dirac cones

Today's entry is recording a math fact that was useful when doing manipulations with multi-flavor Dirac fermions. We will be writing down the inverse of the matrix

$$M = a\mathbf{1}_4 + \mathbf{b}\cdot(\boldsymbol{\sigma}\otimes\mathbf{1}) + \mathbf{c}\cdot(\mathbf{1}\otimes\boldsymbol{\sigma}). \tag{96}$$

When $\mathbf{b}$ or $\mathbf{c}$ vanish, we know what the inverse is, e.g.

$$M(\mathbf{c}=0)^{-1} = \frac{-a + \mathbf{b}\cdot\boldsymbol{\sigma}}{-a^2+b^2}, \tag{97}$$

which follows from the anticommutativity of the different $\sigma^a$ matrices. Note that this matrix is singular when $|a| = |b|$.

For the general case where $\mathbf{b}, \mathbf{c} \neq 0$, we know we will need to get something that reduces to the above expression when $\mathbf{b}$ or $\mathbf{c}$ vanishes. It is also easy to see that we will also need a term proportional to $\sigma^a \otimes \sigma^b$. Since we only have the vectors $\mathbf{b}, \mathbf{c}$ to work with, each $\sigma$ in this term must be contracted with a $\mathbf{b}$ or a $\mathbf{c}$. These considerations fix

$$M^{-1} = \frac{1}{N}\left(A\mathbf{1} + B\mathbf{b}\cdot(\boldsymbol{\sigma}\otimes\mathbf{1}) + C\mathbf{c}\cdot(\mathbf{1}\otimes\boldsymbol{\sigma}) + D\mathbf{b}\cdot\boldsymbol{\sigma}\otimes\mathbf{c}\cdot\boldsymbol{\sigma}\right), \tag{98}$$

where the constants $A, B, C, D, N$ are to be determined. They can be fixed by multiplying things out and using the known answers in the $\mathbf{b} = 0$ or $\mathbf{c} = 0$ limits. The algebra is un-illuminating and leads to the result

$$M^{-1} = \frac{a(a^2-b^2-c^2) + (b^2-a^2-c^2)\mathbf{b}\cdot(\boldsymbol{\sigma}\otimes\mathbf{1}) + (c^2-a^2-b^2)\mathbf{c}\cdot(\mathbf{1}\otimes\boldsymbol{\sigma}) + 2a\mathbf{b}\cdot\boldsymbol{\sigma}\otimes\mathbf{c}\cdot\boldsymbol{\sigma}}{a^2(a^2-b^2-c^2) + b^2(b^2-a^2-c^2) + c^2(c^2-a^2-b^2)}. \tag{99}$$

That's all!

## 16 Longitudinal and transverse reponse in metals and SCs

Today's entry is something simple that I was never able to fully understand from textbooks.

## 17 Field theory perspective on the Fu-Kane proposal for TI surfaces

## 18 Time reversal, conductivity, and Onsager reciprocity

Today's diary entry is a subtlety that didn't seem to have been explained to me before. If we think like a field theory person, we might get confused about the behavior of the equation $\mathbf{j} = \sigma\mathbf{E}$ under (the usual definition of) $\mathcal{T}$, since $\mathbf{j}$ is odd while $\mathbf{E}$ is even, even though $\sigma$ can be nonzero in a $\mathcal{T}$-invariant system. What's up?

The zeroth order answer to the question comes from using $\mathbf{j} = \sigma\mathbf{E}$ to say that

$$\sigma(\mathbf{A}) = -\sigma^*(-\mathbf{A}). \tag{100}$$

For a $\mathcal{T}$-invariant system, this tells us that $\sigma_{\mathbb{R}} = 0$. This is reasonable as we know that it is $\sigma_{\mathbb{R}}$ (not $\sigma_{\mathbb{I}}$) which relates to dissipation, which is a $\mathcal{T}$-breaking phenomenon (e.g. in the Drude approach, $\sigma_{\mathbb{R}} \neq 0$ arises from the $\mathcal{T}$-breaking $\mathbf{k}/\tau$ drag term in the kinetic equation). This explanation is however not fully satisfactory since at a more microscopic level, the mechanisms that give rise to a nonzero $\sigma_{\mathbb{R}}$ (e.g. disorder scattering) can be perfectly $\mathcal{T}$-invariant. It is furthermore not obvious why $\sigma_{\mathbb{I}}$, which controls the dissipationless part of the response, should be $\mathcal{T}$-invariant, as conceptually the generation of a $\mathcal{T}$-odd $\mathbf{j}$ in response to a $\mathcal{T}$-even $\mathbf{E}$ seems to require $\mathcal{T}$-breaking regardless of whether or not the flow of current is dissipationless.

ethan: *note that it's not as simple as real part ↔ dissipation, is it? By KK we always need to have a nonzero real part to match the integral of the imaginary part... but maybe dissipationless stuff requires the real part to be a delta function in $\omega$?*

ethan: *when taking the limit $\mathbf{q} = 0, \omega \to 0$, one is not working in thermal equilibrium, since in this limit excitations cannot propagate over a distance equal to the wavelength of the background field over a time of order $1/\omega$. However we could still ask the same question about time reversal in the limit where $\omega$ is taken to zero first.*

The resolution to this "puzzle" comes from actually writing out the Kubo formula for the conductivity. Since we are interested in how things transform under $\mathcal{T}$, it is in practice best to stay in $\mathbb{R}$ time throughout.

## 18.1 General constraints

While $\sigma$ is not an actual causal response function as it relates $\langle\mathbf{j}\rangle$ to $i\omega\mathbf{A}$ rather than $\mathbf{A}$, the fact that it only differs from a proper response function by a factor of $-1/i\omega$ means that like legit response functions it is constrained to satisfy several symmetry requirements. First, since $\sigma^{ab}(x,t)$ is real, we trivially have

$$\sigma^{ab}_{\mathbf{A}}(\omega,\mathbf{q})^* = \sigma^{ab}_{\mathbf{A}}(-\omega,-\mathbf{q}). \tag{101}$$

To prove this as a check that our factors of $i$ are correct, we write

$$\begin{aligned}\sigma^{ab}_{\mathbf{A}}(\omega,\mathbf{q}) &= \frac{1}{\omega}\int_{t,t'}\Theta(t-t')e^{i\omega(t-t')}\langle[j^a_{\mathbf{q}}(t), j^b_{-\mathbf{q}}(t')]\rangle_{\mathbf{A}} \\ &= \frac{i}{\omega}\int_{\omega'}\frac{1}{\omega'-\omega+i\eta}\langle[j^a_{\omega',\mathbf{q}}, j^b_{-\omega',-\mathbf{q}}]\rangle_{\mathbf{A}},\end{aligned} \tag{102}$$

and then use the Hermiticity of $j^a(t,x)$.

Less trivial constraints follow from gauge invariance. In what follows we will assume rotational invariance, for simplicity. First of all, we have

$$\sigma^{ab}(\omega,\mathbf{q}) = \frac{1}{\omega}\left(\frac{\delta^2}{\delta\widetilde{A}^a_{\omega,\mathbf{q}}\delta\widetilde{A}^b_{-\omega,-\mathbf{q}}}\ln Z[\mathbf{A},\widetilde{\mathbf{A}}]\right)_{\widetilde{\mathbf{A}}=0}, \tag{103}$$

where $\widetilde{\mathbf{A}}$ is a probe field. By gauge invariance, the functional derivative must be constructed from $\Pi_T, \star d$, and $\Box$. Thus we may write

$$\sigma^{ab}(\omega, \mathbf{q}) = \sigma^{xx}(\omega, \mathbf{q}) \left( \delta^{ab} - \frac{q^a q^b}{\omega^2 - \mathbf{q}^2} \right) + \varepsilon^{0ab} \sigma^{xy}(\omega, \mathbf{q}) \tag{104}$$

### *18.1.1 Onsager reciprocity*

To think about the differences between the longitudinal and Hall conductivities, we would like constraints relating $\sigma^{ab}$ with $\sigma^{ba}$. Just using the antisymmetry of the commutator is not enough, as we only get

$$\sigma^{ab}_{\mathbf{A}}(\omega, \mathbf{q}) = -\frac{1}{\omega} \int_{t,t'} \Theta(t - t') e^{i\omega(t-t')} \langle [j^b_{-\mathbf{q}}(-t), j^a_{\mathbf{q}}(-t')] \rangle_{\mathbf{A}}, \tag{105}$$

with the $\Theta$ function factor and $+i\eta$ coming from causality preventing us from changing variables and relating this to $\sigma^{ba}_{\mathbf{A}}(-\omega, -\mathbf{q})$. To proceed we need to make use of $\mathcal{T}$, which allows us to flip the sign of $t, t'$ in the commutator. Provided that $\mathcal{T}$ is only broken by the background field, we have[16]

$$\begin{aligned} \sigma^{ab}_{\mathbf{A}}(\omega, \mathbf{q}) &= -\frac{1}{\omega} \int_{t,t'} \Theta(t - t') e^{i\omega(t-t')} \langle [j^b_{\mathbf{q}}(t), j^a_{-\mathbf{q}}(t')] \rangle^*_{-\mathbf{A}} \\ &= \frac{1}{\omega} \int_{t,t'} \Theta(t - t') e^{i\omega(t-t')} \langle [j^b_{-\mathbf{q}}(t), j^a_{\mathbf{q}}(t')] \rangle^*_{-\mathbf{A}} \\ &= \sigma^{ba}_{-\mathbf{A}}(\omega, -\mathbf{q}). \end{aligned} \tag{107}$$

Taking $\mathbf{A} \to 0$ then gives the usual formulation of Onsager reciprocity applied to the conductivity.

### *18.1.2 Dissipation*

Figuring out the parts of $\sigma^{ab}$ which relate to dissipation is done by finding the singularities of $\sigma$ across the real axis. From the second line of (102), we see that

$$\begin{aligned} \sigma^{ab}_{\mathbf{A}}(\omega - i\varepsilon)^* &= -\frac{i}{\omega + i\varepsilon} \int_{\omega'} \frac{1}{\omega' - \omega - i\varepsilon} \langle [j^b_{\omega', \mathbf{q}}, j^a_{-\omega', -\mathbf{q}}] \rangle_{\mathbf{A}} \\ &= -\sigma^{ba}(\omega + i\varepsilon), \end{aligned} \tag{108}$$

where $\varepsilon$ is a positive infinitesimal.

We thus see that $\sigma^{aa}_{\mathbb{I}}$ is smooth across the real axis, while $\sigma^{aa}_{\mathbb{R}}$ is not. This means that it is the *real* part of $\sigma^{aa}$ which is dissipative, while the imaginary part is non-dissipative (unlike a regular response function). On the other hand, which part of $\sigma^{xy}$ is dissipative depends on the relation between $\sigma^{xy}$ and $\sigma^{yx}$. Using Onsager reciprocity,

$$\sigma^{xy}_{\mathbf{A}}(\omega - i\varepsilon)^* = -\sigma^{xy}_{-\mathbf{A}}(\omega + i\varepsilon). \tag{109}$$

[16] This works by using, for any operator $\mathcal{O}$,

$$\langle n|\mathcal{O}|m\rangle = \langle n|\mathcal{T}^\dagger \mathcal{T}(\mathcal{O})|\mathcal{T}m\rangle = \langle \mathcal{T}n|\mathcal{T}(\mathcal{O})|\mathcal{T}m\rangle^*, \tag{106}$$

since $\mathcal{T}$ is antiunitary.

For $\sigma^{xy} \propto \mathrm{sgn}(\nabla \times \mathbf{A})$, we then have that unlike the longitudinal part, it is the *imaginary* part of $\sigma^{xy}$ which is dissipative (as in an ordinary causal response function). As a sanity check, note that $\sigma^{xy}$ for a quantum Hall system with a response at level $k = |k|\mathrm{sgn}(\nabla \times \mathbf{A})$ is

$$\sigma^{ab}_{\mathbf{A}}(\omega) = k\sigma_0 \varepsilon^{ab}, \tag{110}$$

where $\sigma_0 = e^2/h$. The point here is that this is real, so that $\sigma^{xy}$ is *non-dissipative* (as required in a gapped system).

ethan: *should do SC example — there you sort of clearly can get a real part at finite $\omega$, maybe via a term like $i\sqrt{\Delta^2 - \omega^2}$.*

## 19 1-form propagators in 2+1d

Today's entry is mega-short: we will write down a general propagator for a vector field in 2+1d, to serve as a useful reference.

Consider a vector field $X$ with Lagrangian

$$\mathcal{L} = \frac{1}{2} X \wedge \star(\alpha \Pi_T + \beta \Pi_L + \gamma \star d) X, \tag{111}$$

with $\Pi_T = d^\dagger d/\Box, \Pi_L = dd^\dagger/\Box$. We invert this by writing the propagator as

$$D_X = A\Pi_T + B\Pi_L + C \star d. \tag{112}$$

We then use the orthogonality of $\Pi_{T/L}$ as well as (as usual, $\Box = d^\dagger d + dd^\dagger$ is negative-definite)

$$(\star d)^2 = -\Box \Pi_T, \qquad \star d \Pi_T = \Pi_T \star d = \star d, \tag{113}$$

where $\star d$ is viewed as a matrix with vector indices. The sign on this first equation is important, and follows from the fact that when acting on $p$-forms in $D$-dimensional Euclidean space, the adjoint of $d$ is

$$d^\dagger = (-1)^{Dp+D+1} \star d \star . \tag{114}$$

For us $D = 3$ and $p = 1$, so that $d^\dagger = - \star d \star$ (alternatively one can just write out $\star d \star d$ explicitly).

This gives the conditions

$$\begin{aligned} \alpha A - \gamma C \Box &= 1 \\ \beta B &= 1 \\ \gamma A + \alpha C &= 0 \end{aligned} \tag{115}$$

so that

$$D_X = \frac{1}{\Box + \alpha^2/\gamma^2} \left( \frac{\alpha}{\gamma^2} \Pi_T - \frac{1}{\gamma} \star d \right) + \frac{1}{\beta} \Pi_L, \tag{116}$$

or in momentum space,

$$D_X^{\mu\nu} = \frac{1}{q^2 - \alpha^2/\gamma^2}\left[-\frac{\alpha}{\gamma^2}\left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{i}{\gamma}\varepsilon^{\mu\nu\lambda}q_\lambda\right] + \frac{1}{\beta}\frac{q^\mu q^\nu}{q^2}. \tag{117}$$

This is such that $D_X$ is the inverse of the kernel in (111). If we just want to e.g. invert the kernel on coexact forms (viz. those with $\Pi_L X = 0$), we simply need drop the last $1/\beta$ term in the above expression.

As a check, note that this gives the correct topologically massive propagator when we take $\alpha = -\Box/e^2, \gamma = ik/2\pi, \beta = 0$.

# 20 Resolving IHQE edge theory confusion: boundary modes and factors of 2

In today's diary we will be discussing how to write down the bulk + edge theory for the IQHE, working in the hydrodynamic (bosonized) description for the edge. In particular, we will be interested in how to couple this theory to a background gauge field. This seems like it should be something given in textbooks, but it isn't (the discussions given in e.g. Fradkin + Tong's notes are very dubious).

The following is basically the writing up of a conclusion reached in an email chain with Chen Jing-Yuan. Thanks also to Michael Demarco and Hart Goldman for discussions.

## 20.1 Approach from bosonization

In the bosonization approach, we start form the knowledge that the IHQE can be described by a bulk $A \wedge dA$ term and a chiral fermion $\chi$ on the edge. We thus write

$$S = \frac{k}{2}\int A \wedge dA + \int_\partial \chi^\dagger i(sD_t + vD_x)\chi, \tag{118}$$

where $\int_\partial$ denotes integration over the boundary (with the boundary always assumed to be parametrized by the coordinates $x, t$), the $D_\mu$ are covariant derivatives, and have abused notation by defining $S$ to be $2\pi$ times the action. The $\chi$ fermions are assumed to be defined using a regulator which renders the combined theory gauge-invariant without any additional counterterms for $A$. Finally, $s = \pm 1$ in the above is a sign which fixes the chirality of $\chi$, which will be fixed after bosonizing by requiring anomaly cancellation. We will restrict to $k = \pm 1$ in the following.

Let us now write down the bosonized form of this action, using the usual bosonization rules derived from matching correlation functions. Note that since we are coupling to a background field, the bosonization map does not need to be modified by e.g. inserting Wilson lines when point splitting, or anything like that. We therefore simply send

$$\chi^\dagger\chi \mapsto \partial_x\phi, \qquad \chi^\dagger i\partial_\mu\chi \mapsto -\frac{1}{2}\partial_x\phi\partial_\mu\phi, \tag{119}$$

as can be worked out using standard techniques. This tells us that we should write

$$S = \frac{k}{2}\int A\wedge dA + \frac{1}{2}\int_\partial \left(-s\partial_t\phi\partial_x\phi - v(\partial_x\phi)^2 + 2\partial_x\phi(sA_0 + vA_x)\right) + c.t., \tag{120}$$

where $c.t.$ is a local counterterm on the edge involving just the background field (the bosonization mapping is derived by matching correlation functions at non-coincident points; hence an additional local counterterm may be needed to properly reproduce the full free energy). The counterterm can be determined by requiring the partition function be gauge-invariant. The variation of the above action without the counterterm is

$$\delta(S - c.t.) = \int_\partial \left(A_x\left(-k\partial_0\lambda/2 + v\partial_x\lambda\right) + A_0\partial_x\lambda(s + k/2)\right). \tag{121}$$

From how the gauge field appears in this expression, we may write

$$c.t. = \int_\partial A_x(\alpha A_x + \beta A_0). \tag{122}$$

Matching the variation above gives $\alpha = -v/2$ and $\beta = k/2$; this furthermore matches the variation only if $s = -k$. This restriction on $s$ is good news; after all we know that the chirality of the edge fermion should be fixed by the sign of the Hall conductance $k$. Therefore we may write the full action as

$$S = \frac{k}{2}\int A\wedge dA + \frac{1}{2}\int_\partial \left(k\partial_t\phi\partial_x\phi - v(\partial_x\phi)^2 - 2\partial_x\phi(kA_0 - vA_x) + A_x(kA_0 - vA_x)\right). \tag{123}$$

This can be alternately written in an obviously gauge-invariant fashion as

$$S = \frac{k}{2}\int (A - d\phi)\wedge dA + \frac{1}{2}\int_\partial \left(kD_t\phi D_x\phi - v(D_x\phi)^2\right), \tag{124}$$

with $D_\mu = \partial_\mu - A_\mu$. Note that the structure of the counterterm, in particular the $kA_xA_0/2$ part, means that $A_0$ appears in the action as $k\int A_0\nabla\times\mathbf{A} - k\int_\partial A_0\partial_x\phi$ (following from an integration by parts)

Note that the gauge variation of the boundary theory (viz. the gauge variation of the fermion partition function) is given by

$$\delta_\lambda \ln Z[A] = -i\frac{k}{2}\int_\partial \lambda dA. \tag{125}$$

From thinking about the chiral anomaly, the factor of 1/2 might seem a bit dubious. This is because for a normal 1+1D Dirac fermion, we are used to a chiral rotation by $\lambda$ producing a phase of $2\int\lambda dA$ (as chiral charge is conserved only mod 2 in the presence

of nonzero $dA$). Since a Dirac fermion is built from one left- and one right-moving chiral fermion, we might have expected the variation of the chiral fermion partition function to be given by half of the Dirac fermion result, viz. $k\int \lambda dA$.

This is however just an issue of choosing the wrong regularization for the problem in question, viz. regulating with a charge-conserving UV completion, like a Fermi sea. In the present context the correct regularization

## 20.2 Approach from integrating out the dynamical gauge field

An alternate starting point is in terms of the dynamical gauge field $a$. We write down the action

$$S = k\int \left(-\frac{1}{2}a\wedge da + a\wedge dA\right). \tag{126}$$

Since $A$ is a fixed background field, its value on the boundary can be arbitrary. $a$ however is dynamical, and we must therefore choose boundary conditions for it. We will choose the boundary condition $a_0|_\partial = 0$. Note that this boundary condition is left invariant under gauge transformations which are time-independent on the boundary, viz. those which send $a\mapsto a+d\lambda$ with $(\partial_t\lambda)|_\partial = 0$. The variation of (126) under such gauge transformations is

$$\delta S = k\int_\partial \partial_x\lambda A_0. \tag{127}$$

It is impossible to cancel this variation with a local boundary term while preserving invariance under gauge transformations of $A$.

We now integrate out $a_0$. To do this we perform an integration by parts, with the boundary term canceling the $\int_\partial \mathbf{a}a_0$. This gives

$$\begin{aligned} S &= \frac{k}{2}\int (-a_0\nabla\times\mathbf{a} + a_0\partial_y a_x + a_x\partial_0 a_y - a_y\partial_x a_0 + a_y\partial_0 a_x + 2a\wedge dA) \\ &= k\int \left(\frac{1}{2}a_0[-\nabla\times\mathbf{a}+\nabla\times\mathbf{A}] + \frac{1}{2}\mathbf{a}\times\partial_0\mathbf{a} - \mathbf{a}\times\mathbf{E}\right) \end{aligned} \tag{128}$$

so that $a_0$ can be easily integrated out, giving the constraint

$$\mathbf{a} = -\nabla\phi + \mathbf{A} \tag{129}$$

for some scalar $\phi$. Note that since $a$ is neutral under the global $U(1)$ symmetry, $e^{i\phi}$ must have charge 1 (this is the reason for the minus sign above). Putting this constraint in,

$$S = k\int \left(\frac{1}{2}(\nabla\phi-\mathbf{A})\times\partial_0(\nabla\phi-\mathbf{A}) - (\nabla\phi-\mathbf{A})\times\mathbf{E}\right). \tag{130}$$

When simplifying this expression, one finds that the terms only involving $A$ are not in fact just $\frac{1}{2}AdA$, but contain an extra boundary piece. Some algebra gives

$$\begin{aligned} S &= k\int \left(\frac{1}{2}\nabla\phi\times\partial_t\nabla\phi - \nabla\phi\times\nabla A_0 + \frac{1}{2}A\wedge dA\right) + \frac{k}{2}\int_\partial A_0 A_x \\ &= \frac{k}{2}\int A\wedge dA + \frac{k}{2}\int_\partial (\partial_t\phi\partial_x\phi - 2\partial_x\phi A_0 + A_0 A_x). \end{aligned} \tag{131}$$

This gives an edge theory with a boson with zero velocity. However in general, instead of choosing the boundary condition $a_0 = 0$, we may choose $a_0 - kva_x = 0$ for some (nonuniversal) velocity $v$. To work with this boundary condition, we simply make a coordinate transform where $\partial_t \mapsto \partial_t - kv\partial_x$ (here $k$ appears in the coordinate transformation by the requirement that the Hamiltonian in the absence of $A$ be positive). This gives

$$S = \frac{k}{2}\int A \wedge dA + \frac{1}{2}\int_\partial \left(k\partial_t\phi\partial_x\phi - v(\partial_x\phi)^2 - 2\partial_x\phi(kA_0 - vA_x) + A_x(kA_0 - vA_x)\right), \tag{132}$$

which is precisely what we had concluded from the bosonization logic above.

## 21 4th order perturbation theory

Today we will work out the expression for the 4th order correction to the ground state energy of a Hamiltonian perturbed by a Hermitian operator $\mathcal{O}$, where $\mathcal{O}$ does not act in the ground state, $\langle 0|\mathcal{O}|0\rangle = 0$. The expression we will derive can actually be found on Wikipedia, but I was not able to find a derivation. Therefore to be safe, we will work things out ourselves.

The way I like to organize perturbation theory is via a path integral flavored approach. Consider the imaginary time evolution of the perturbed system for an imaginary time $T \to \infty$. We may then write

$$e^{-T\sum_{i=1} E_i} = \langle e^{-\int d\tau \mathcal{O}(\tau)}\rangle, \tag{133}$$

where $E_i$ is the $i$th order correction to the ground state energy, working perturbatively in powers of $\mathcal{O}$. By expanding both exponentials, we find to fourth order

$$\begin{aligned}
\chi_1 &= E_1 T \\
\frac{1}{2}\chi_2 &= \frac{1}{2}E_1^2T^2 - E_2T \\
\frac{1}{6}\chi_3 &= \frac{1}{6}E_1^3T^3 - E_1E_2T^2 + E_3T \\
\frac{1}{24}\chi_4 &= \frac{1}{24}E_1^4T^4 - \frac{1}{2}E_1^2E_2T^3 + E_1E_3T^2 + \frac{1}{2}E_2^2T^2 - E_4T,
\end{aligned} \tag{134}$$

where we have defined

$$\chi_n \equiv \int_{\mathbb{R}} \prod_{i=1}^n d\tau_n \langle \mathcal{T}[\prod_{j=1}^n \mathcal{O}(\tau_j)]\rangle, \tag{135}$$

with $\mathcal{T}$ denoting time ordering.

All that remains is the calculate the $\chi_n$. In most situations we are interested in, $\chi_1 = 0$. $\chi_2$ is the familiar

$$\begin{aligned}\chi_2 &= T\sum_{l\neq 0}\int_{\mathbb{R}} d\tau \left(\Theta(\tau)e^{\tau E_{0l}} + \Theta(-\tau)e^{\tau E_{l0}}\right)|\mathcal{O}_{0l}|^2 \\ &= 2T\sum_{l\neq 0}\frac{|\mathcal{O}_{0l}|^2}{E_{0l}},\end{aligned} \tag{136}$$

where $T$ appears as our way of regulating the integral over all times (viz. $T = \int_{\mathbb{R}} d\tau$).

To derive $\chi_3$ and $\chi_4$, it is helpful to define a more compact notation by writing

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\langle p_1\cdots p_n\rangle \equiv \int_{a_1}^{b_1} d\tau_1\cdots\int_{a_n}^{b_n} d\tau_n\langle 0|\mathcal{O}(\tau_{p_1})\cdots\mathcal{O}(\tau_{p_n})|0\rangle \tag{137}$$

We may then use this notation to write (after using time translation invariance to fix one of the times at 0)

$$\begin{aligned}\chi_3/T &= \int_{-\infty}^{0}\left(\int_{-\infty}^{1}\langle 012\rangle + \int_{1}^{0}\langle 021\rangle + \int_{0}^{\infty}\langle 201\rangle\right) + \int_{0}^{\infty}\left(\int_{-\infty}^{0}\langle 102\rangle + \int_{0}^{1}\langle 120\rangle + \int_{1}^{\infty}\langle 210\rangle\right) \\ &\equiv \sum_{k,l\neq 0}\mathcal{O}_{0k}\mathcal{O}_{kl}\mathcal{O}_{l0}\sum_{i=1}^{6}K_i\end{aligned} \tag{138}$$

where 1 in the integration limits means $\tau_1$, the integration variable of the outermost integral. In this case in fact all six terms are exactly the same:

$$K_i = \frac{1}{E_{0k}E_{0l}}, \tag{139}$$

so that

$$\chi_3 = 6T\sum_{k,l\neq 0}\frac{\mathcal{O}_{0k}\mathcal{O}_{kl}\mathcal{O}_{l0}}{E_{0k}E_{0l}}. \tag{140}$$

The calculation of $\chi_4$ is a bit more involved. There are a few easy symmetries of the integrations we can take advantage of to write

$$\begin{aligned}\chi_4/T = 4\int_0^\infty \Big(&\int_{-\infty}^{0}\int_{-\infty}^{2}\langle 1023\rangle + \int_{0}^{1}\int_{-\infty}^{0}\langle 1203\rangle + \int_{0}^{1}\int_{0}^{2}\langle 1230\rangle \\ &- \int_{1}^{\infty}\int_{-\infty}^{0}\langle 2103\rangle + \int_{1}^{\infty}\int_{0}^{1}\langle 2130\rangle + \int_{1}^{\infty}\int_{1}^{2}\langle 2310\rangle\Big) \\ \equiv 4\sum_{k,l,m\neq 0}&\mathcal{O}_{0k}\mathcal{O}_{km}\mathcal{O}_{ml}\mathcal{O}_{l0}\sum_{i=1}^{6}L_i + 4\sum_{k,l\neq 0}|\mathcal{O}_{0k}|^2|\mathcal{O}_{0l}|^2\sum_{i=1}^{6}M_i.\end{aligned} \tag{141}$$

The $L_i$ are in fact all the same; a straightforward calculation checks that

$$L_i = -\frac{1}{E_{0k}E_{0l}E_{0m}}. \tag{142}$$

For the $M_i$, we find (the subscripts defined in order of how the terms appear in the expression for $\chi_4$ above)

$$\begin{aligned} M_1 = M_4 &= \frac{T}{2E_{0k}E_{0l}} \\ M_2 = M_5 &= \frac{1}{E_{0k}E_{0l}}\left(\frac{T}{2}+\frac{1}{E_{0k}}\right) \\ M_3 = M_6 &= \frac{T}{2E_{0l}E_{0k}}+\frac{2}{E_{0l}E_{0k}^2}, \end{aligned} \tag{143}$$

where we have used symmetry in $k \leftrightarrow l$. Adding these up, we find

$$\chi_4 = 24T\sum_{k,l,m\neq 0}\frac{\mathcal{O}_{0k}\mathcal{O}_{km}\mathcal{O}_{ml}\mathcal{O}_{l0}}{E_{k0}E_{l0}E_{m0}} + 24T\sum_{k,l\neq 0}\frac{|\mathcal{O}_{0k}|^2|\mathcal{O}_{0l}|^2}{E_{0k}E_{0l}}\left(\frac{1}{E_{0k}}+\frac{T}{2}\right). \tag{144}$$

Some simple algebra then leads to an expression for the energy shifts. As we are assuming $E_1 = 0$, we find

$$\begin{aligned} E_2 &= \sum_{l\neq 0}\frac{|\mathcal{O}_{0l}|^2}{E_{0l}} \\ E_3 &= \sum_{l,k\neq 0}\frac{\mathcal{O}_{0k}\mathcal{O}_{kl}\mathcal{O}_{l0}}{E_{0l}E_{0k}} \\ E_4 &= \sum_{k,l,m\neq 0}\frac{\mathcal{O}_{0k}\mathcal{O}_{km}\mathcal{O}_{ml}\mathcal{O}_{l0}}{E_{0k}E_{0m}E_{0l}} - E_2\sum_{l\neq 0}\frac{|\mathcal{O}_{0l}|^2}{E_{0l}^2}, \end{aligned} \tag{145}$$

which reassuringly matches the expressions on Wikipedia.

# 22 Duality and entropy in the 2d Ising model

# 23 How Bogoliubov transformations act on states

In today's entry we will discuss an aspect of Bogoliubov transformations (BTs; aka canonical transformations) that somehow is not covered in any of the books I own:[17][18] how they act on states.

Suppose we are interested in a problem defined on a Hilbert space $\mathcal{H}$, which is acted on by the creation / annihilation operators $a_\lambda^\dagger, a_\lambda$, with $\lambda$ a flavor index. Define

[17] Except in the discussions on BTs in the context of superconductivity; here we will be interested in more general (e.g. bosonic) examples.

[18] Update: this actually seems to be fairly well-known in quant-ph land. This diary entry should therefore just be viewed as a verbose discussion explaining where the formulae in e.g. [2] come from.

the operator $\boldsymbol{a} \equiv \bigoplus_\lambda (a_\lambda,\, a_\lambda^\dagger)^T$, which is a vector of operators of length $|\{\lambda\}|$. One part of a BT is a canonical transformation on $\boldsymbol{a}$, viz. one that sends

$$BT \,:\, \boldsymbol{a} \mapsto V\boldsymbol{a}, \qquad V^\dagger(\mathbf{1}\otimes C)V = \mathbf{1}\otimes C, \tag{146}$$

where $C = Z$ if the particles destroyed by $a_\lambda$ are bosons, and $C = \mathbf{1}_2$ if they are fermions (the condition on $V$ ensures that the C(A)CR are preserved).

Now if $V$ is of the form $V_f \otimes \mathbf{1}_2$, the BT acts on operators by unitary conjugation. However if it exchanges creation and annihilation operators then the action on operators *cannot* be that of conjugation by a unitary matrix. This then raises the question of how BTs act on states, since we know that the action on states should be that of a unitary matrix.

To understand how to derive the action on states, let us first look at the case of a single flavor of bosons, so that the allowed BTs are defined by a 2×2 determinant-1 matrix $V$ satisfying $V^\dagger Z V = Z$, viz. by the group $SU(1,1)$. As with $SU(2)$, any matrix in $SU(1,1)$ can be parametrized in terms of two complex numbers $z, w$, with

$$V = \begin{pmatrix} z & w \\ w^* & z^* \end{pmatrix}, \qquad |z|^2 - |w|^2 = 1. \tag{147}$$

Since the overall phase of $V$ is redundant, we may thus write $V$ in terms of a real number $\theta$ and a $U(1)$ phase $\zeta$ as

$$V = \begin{pmatrix} \cosh\theta & \zeta \sinh\theta \\ \zeta^* \sinh\theta & \cosh\theta \end{pmatrix}. \tag{148}$$

## 23.1 Action on the vacuum

Let us first ask how the vacuum is mapped under the BT. The new vacuum $|\widetilde{0}\rangle$ should satisfy $\widetilde{a}|\widetilde{0}\rangle = 0$, where $\widetilde{a} \equiv c_\theta a + \zeta s_\theta a^\dagger$ is the transformed annihilation operator and $c_\theta \equiv \cosh\theta, s_\theta \equiv \sinh\theta$. Writing $|\widetilde{0}\rangle = \sum_n \alpha_n |n\rangle$, some algebra shows that

$$\frac{\alpha_{n+1}}{\alpha_{n-1}} = -\zeta t_\theta \sqrt{\frac{n}{n+1}}, \qquad \alpha_1 = 0, \tag{149}$$

where $t_\theta \equiv \tanh(\theta)$. This gives

$$|\widetilde{0}\rangle = \alpha_0 \sum_{n=0}^{\infty} (t_\theta \zeta)^n \sqrt{\frac{(2n-1)!!}{(2n)!!}} |2n\rangle. \tag{150}$$

Note in particular that the new vacuum is not built out of combinations of coherent states, and contains only states of *even* boson number (this can be anticipated by re-reading the diary entry on thermofield dynamics). Using $(2n)!! = 2^n n!$ and $|2n\rangle = (a^\dagger)^{2n}|0\rangle/\sqrt{(2n)!}$, some algebra then lets us write

$$|\widetilde{0}\rangle = \alpha_0 e^{\frac{\zeta t_\theta}{2}(a^\dagger)^2}|0\rangle, \tag{151}$$

with $\alpha_0$ determined by normalization. We have

$$\begin{aligned}|\langle\widetilde{0}|\widetilde{0}\rangle|^2 &= \alpha_0^2 \sum_{n=0}^{\infty} (\zeta t_\theta/2)^{2n} \frac{(2n)!}{(n!)^2} \\ &= \alpha_0^2 \sum_{n=0}^{\infty} \frac{(\zeta t_\theta)^{2n}}{2^n} \frac{(2n-1)!!}{n!} \\ &= \frac{\alpha_0^2}{\sqrt{1-(\zeta t_\theta)^2}},\end{aligned} \tag{152}$$

so that

$$|\widetilde{0}\rangle = c_\theta^{-1/2} e^{\frac{\zeta t_\theta}{2} a^\dagger a^\dagger} |0\rangle. \tag{153}$$

## 23.2 General action

To determine the action on general states, strictly speaking one can proceed as above, by solving $\widetilde{a}^m|\widetilde{m}\rangle = 0$. This however involves rather heinous normal-ordering manipulations of the expansion of $\widetilde{a}^m$ and the commutation of these terms through the exponential in the above expression for $|\widetilde{0}\rangle$.

A better strategy is to simply guess the form of the unitary based on the above expression for $|\widetilde{0}\rangle$. To do this, it helps to first recall some things about coherent states. The usual way of writing down a coherent state on which $a$ acts as $\alpha$ is to write

$$(a-\alpha)|\alpha\rangle = 0 \implies \left(\frac{\delta}{\delta a^\dagger} - \alpha\right)|\alpha\rangle = 0 \implies |\alpha\rangle = e^{\alpha a^\dagger - |\alpha|^2/2}|0\rangle, \tag{154}$$

where the $e^{-|\alpha|^2/2}$ comes from normalization. This is just how one obtains the coherent state $|\alpha\rangle$, but in the present context it is more helpful to think of this as a shift map which sends $a \mapsto a - \alpha$, $a^\dagger \mapsto a^\dagger - \alpha^*$ (which of course preserves the CCR).

How should this shift map be extended to an action on all states? Of course it cannot simply be via the operator $e^{\alpha a^\dagger - |\alpha|^2/2}$, which is not unitary. However, using the BCH formula we may write

$$e^{\alpha a^\dagger - |\alpha|^2/2}|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle \equiv D(\alpha), \tag{155}$$

where $D(\alpha)$ is apparently known as the "displacement operator" (according to Wikipedia). $D(\alpha)$ is unitary, and it is natural to guess that $D(\alpha)$ is the correct unitary action on all states. This can be checked by observing that conjugation by $D(\alpha)$ implements the desired shift on $a$:

$$D(\alpha) a D(\alpha)^\dagger = e^{\alpha a^\dagger} a e^{-\alpha a^\dagger} = a + e^{\alpha a^\dagger}[a, e^{-\alpha a^\dagger}] = a - \alpha. \tag{156}$$

The reason why the above comments about coherent states are useful is that the Bogoliubov ground state is essentially a coherent state of boson pairs (think about the BCS wavefunction), and we can use essentially the same strategy as above to guess the unitary that acts on the states. In particular, a bit of playing around shows that the natural guess is the unitary

$$\mathcal{U}_{\theta,\zeta} = e^{\frac{\theta}{2}(\zeta a^\dagger a^\dagger - \zeta^* a a)}, \tag{157}$$

which is basically a coherent state of boson pairs (a "squeezed state", presumably because the matrix $V$ implements a squeezing of phase space). The first sanity check on this is that it obeys the composition law

$$\mathcal{U}_{\theta,\zeta}\mathcal{U}_{\theta',\zeta} = \mathcal{U}_{\theta+\theta',\zeta} \tag{158}$$

which is a requirement based on the multiplication law of the matrices in (148).

As another sanity check, we can see whether or not the action of $\mathcal{U}_{\theta,\zeta}$ reduces to that of $c_\theta^{-1/2} e^{\theta_\theta \zeta a^\dagger a^\dagger}$ when acting on $|0\rangle$. To do this, we need to normal-order $\mathcal{U}_{\theta,\zeta}$. The fact that $\mathcal{U}^\lambda_{\theta,\zeta} = \mathcal{U}_{\lambda\theta,\zeta}$ for $\lambda \in \mathbb{R}$ is also imp

# 24 Gottesman-Knill

Today we are doing a problem (10.40) from Mike+Ike which is basically a demonstration of the Gottesman-Knill theorem, which states that classical computers can efficiently simulate any computation involving: state preparations in the computational basis, operations in the Clifford group (the normalizer of the Pauli group, involving $H, S, CNOT$, and Pauli gates), measurements of observables in the Pauli group, and classical control conditioned on such measurement outcomes.

Let $\mathcal{C}_n$ be the Clifford group on $n$ qubits and let $G_n$ be the appropriate Pauli group. We first construct $\mathcal{C}_1$, which we claim is generate by $H$ and $S$. This can just be demonstrated explicitly: any $U_1 \in \mathcal{C}_1$ is determined by $U_1^\dagger Z U_1$ and $U_1^\dagger X U_1$. Since both of these need to be in $G_1$, our only options are $\mathbf{1}$, $H$ ($X \leftrightarrow Z$), $S$ ($X \leftrightarrow Y$), or $HS$ ($Y \leftrightarrow Z$). Since the center of $G_1$ is trivial, there is no further ambiguity in determining $U_1$. Now we construct $\mathcal{C}_n$ inductively. Suppose $U_{n+1}$ is a $n+1$ qubit gate in $\mathcal{C}_{n+1}$ with $U_{n+1}^\dagger(Z_1 \otimes \mathbf{1}_n)U_{n+1} = X_1 \otimes g$ and $U_{n+1}^\dagger(X_1 \otimes \mathbf{1}_n)U_{n+1} = Z_1 \otimes g'$ where $\mathbf{1}_n$ is the identity on $n$ qubits and $g, g' \in G_n$. Let $U_n$ be determined by the action of $U_{n+1}$ on states of the form $|0\rangle \otimes |\psi\rangle$.

We can construct

# 25 Fano's inequality: classical and quantum (P10.9, P10.10)

Today we are doing two problems from Preskill concerned with Fano's inequality and its quantum generalization.

The first is to demonstrate Fano's inequality:

**Proposition 1** (classical Fano inequality)**.** *Let $X$ be a random variable on an alphabet of $d$ letters, and $Y$ be another random variable correlated with $X$. Let $F(y)$ be a function used to estimate the outcome of $X$ based on an outcome $y$ of $Y$. Furthermore, let $E = \{e, P_e\}$ with $e \in \{0, 1\}$ be a random variable such that $e = 0$ if $x = F(y)$ and $e = 1$ else. Then*

$$H(X|Y) \leq H_2(P_e) + P_e \log(d-1), \tag{159}$$

*where $H_2(x)$ is the binary Shannon entropy.*

*Proof.* We first show an inequality between various entropies of interest. We claim

$$H(X|Y) = H(X|YE) + H(E|Y) - H(E|XY). \tag{160}$$

This just follows from liberal use of $H(X,Y) = H(X) + H(Y|X)$. Indeed, writing $H(E|XY) = H(E,X,Y) - H(X,Y)$ and $H(X|YE) = H(E,X,Y) - H(Y,E)$, the above claim translates to

$$\begin{aligned} H(X|Y) &= -H(Y,E) + H(E|Y) + H(X|Y) \\ &= H(X,Y) - H(Y), \end{aligned} \tag{161}$$

which is indeed true (in the second step we have expanded $H(Y,E) = H(Y) + H(E|Y)$).

Since $E$ is a function of $X, Y$, $H(E|XY) = 0$. Furthermore since $H(E|Y) \leq H(E)$, we have

$$H(X|Y) \leq H(X|YE) + H(E). \tag{162}$$

Now

$$H(X|YE) = (1-P_e)H(X|Y, e=0) + P_e H(X|Y, e=1) = P_e H(X|Y, e=1), \tag{163}$$

since $H(X|Y, e=0) = 0$ on account of $X$ being fully determined by $Y$ when $e = 0$. Now given that $e = 1$, the outcome of $X$ given an outcome $y$ of $Y$ could be any of the $d-1$ states other than $F(y)$. Therefore the above entropy has the upper bound

$$H(X|YE) \leq P_2 \log(d-1). \tag{164}$$

Finally, using the above and (162),

$$H(X|Y) \leq H_2(P_e) + P_e \log(d-1). \tag{165}$$

□

Fano's equality finds application in proving the bound on classical noisy channel capacity. In this context $X$ is taken to be a codeword sent through the channel using a duplication code which repeats $n$ times, $Y$ the recieved version of the codeword, and $P_e$ the error probability. Taking $n \to \infty$, this is usually written as

$$\frac{1}{n} H(X|Y) \leq \frac{1}{n} H_2(P_e) + P_e R, \tag{166}$$

where $R = k/n$ is the code rate, with $k = \log(d)$ the number of logical bits.

**Proposition 2** (quantum Fano inequality)**.** *In a d-dimensional system, suppose $\rho$ approximates a given pure state $|\psi\rangle\langle\psi|$ with fidelity*

$$\langle\psi|\rho|\psi\rangle = 1 - \varepsilon. \tag{167}$$

*Then*

$$H(\rho) \leq H_2(\varepsilon) + \varepsilon \log(d-1). \tag{168}$$

*Proof.* The strategy is to make use of the fact that if a complete orthogonal measurement performed on $\rho$ has a distribution of outcomes $X$, then

$$H(\rho) \leq H(X). \tag{169}$$

Proving this is an easy exercise in Preskill: let $E_a$ be the projectors associated with the measurement outcomes $a$, which occur with probability $p_a = \mathrm{Tr}[E_a\rho]$. Let $U$ diagonalize $\rho$, with $[U\rho U^\dagger]_{ij} = \lambda_i\delta_{ij}$. Then

$$p_a = \Gamma_{ai}\lambda_i, \qquad \Gamma_{ai} = \sum_j U^\dagger_{ji}E^a. \tag{170}$$

It is easy to check that $\Gamma$ is doubly stochastic using $\sum_a E^a = \mathbf{1}$ and $\mathrm{Tr}[E^a] = 1\,\forall\, a$ on account of $E_a$ being a projector onto a 1d space. Therefore

$$\{p_a\} \prec \{\lambda_j\} \tag{171}$$

and the claim follows.

Consider then the case where $E^1 = |\psi\rangle\langle\psi|$, and $E^{2,\ldots,d}$ are projectors onto other states orthogonal to $|\psi\rangle$. Then

$$\begin{aligned} H(X) &= H_2(\varepsilon) + \varepsilon\log\varepsilon - \sum_{a\neq 1} p_a \log(p_a) \\ &= H_2(\varepsilon) + \varepsilon\log\varepsilon - \varepsilon\sum_{a\neq 1}\frac{p_a}{\varepsilon}\left(\log\left(\frac{p_a}{\varepsilon}\right) + \log\varepsilon\right) \\ &= H_2(\varepsilon) - \varepsilon\sum_{a\neq 1}\widetilde{p}_a\log(\widetilde{p}_a), \end{aligned} \tag{172}$$

where the $\widetilde{p}_a \equiv p_a/\varepsilon$ themselves a probability distribution as long as $\varepsilon \neq 0$. Since there are $d-1$ possible outcomes for $\widetilde{p}_a$, we have

$$H(X) \leq H_2(\varepsilon) + \varepsilon\log(d-1), \tag{173}$$

which combined with (169) gives the desired result. □

Note that the above was rather different than the proof of the classical Fano equality (that we did following the hints in Preskill) — maybe there's a better way of doing the classical case?

## 26 Quantum bit committment (P2.3)

Today we are doing a very short but educational problem from Preskill about quantum bit committment.

## 27 Building stuff from CNOTs

Today we are doing a simple problem from Xiaodi Wu's QI class.

## 28 Reductions for several NP-complete graph problems

Today we are solving a few exercises in Arora + Borak's book which together show the following reductions:

$$\texttt{3Sat} \to \texttt{indSet} \to \texttt{vertexCover} \to \texttt{maxCut}, \quad \texttt{indSet} \to \texttt{clique}, \quad \texttt{3Sat} \to \texttt{3Color} : \tag{174}$$

$\texttt{3Sat} \to \texttt{indSet}$

This is farily simple and is basically described in the book, so we will not go into much detail. For each clause in a given 3-sat instance, we draw 7 nodes, labeled by the 7 ways that $x_1 \wedge x_2 \wedge x_3$ can be true. Each 7 nodes in each clause cluster are then fully connected with edges, so that any independent set of the resulting graph $G$ can contain at most one vertex from each clause cluster. We then draw edges between all pairs of vertices which are impossible to simultaneously satisfy (i.e. connecting up $x$s with $\neg x$s). Then the given 3-sat instance is satisfiable iff $G$ has an independent set of size equal to the number of clauses (which by construction is the largest independent set possible).

`indSet` $\rightarrow$ `vertexCover`

This is the simplest reduction in the list. We accomplish this simply by noting that an independent set is basically dual to a vertex cover. We claim that finding a size $l \geq k$ independent set on $G$ is the same as finding a size $l' \leq |G| - k$ vertex cover (the inequalities flip since small independent sets are easy to find, while the easy vertex covers are large) . Let $\mathcal{V}_k$ be a size $k$ independent set. Then $G \setminus \mathcal{V}_k$ defines a vertex cover. Indeed, if $G \setminus \mathcal{V}_k$ were not a vertex cover, then there would necessarily exist an edge with both ends in $\mathcal{V}_k$ — a contradiction.

`indSet` $\rightarrow$ `clique`

Consider a graph $G$ and let $\mathcal{V}_k \subset G$, $|\mathcal{V}_k| = k$. Define a new graph $G'$ as follows. If an edge $E_{ij} \in G$, then $E_{ij} \notin G'$. Conversely, if $E_{ij} \notin G$, then $E_{ij} \in G'$. $G'$ can be obtained from $G$ in at most $O(|G|^2)$ operations, coming from looping over $\binom{|G|}{2}$ vertex pairs.

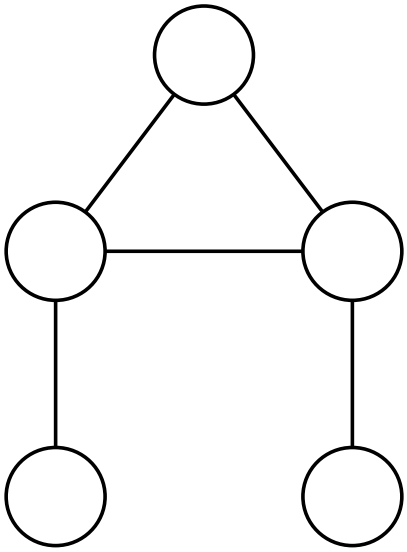
We claim that $\mathcal{V}_k$ is an independent set in $G$ iff $G'$ has a $k$-cliquie. First suppose $\mathcal{V}_k$ is an independent set in $G$. Then by construction $\mathcal{V}_k$ defines a $k$-clique in $G'$, since all of the vertices in $\mathcal{V}_k$ are disconnected in $G$, and are consequently fully connected in $G'$. Next, suppose $G'$ has a $k$-clique $\mathcal{K}_k$. Again by construction, all of the vertices in $\mathcal{K}_k$ must be disconnected in $G$, proving the claim.

`3Sat` $\rightarrow$ `3Color`

When reducing an instance of `3Sat` to one of `3Color`, the variables of `3Sat` will of course get mapped to the vertices of some graph $G$ that may or may not be 3-colorable, depending on the solvability of the instance in question. Because the variables in 3Sat are all either 1 or 0 (true or false), we will need to find a way of restricting the colors on these vertices to take only one of two values. We will take the 'colors' to be $0, 1, +$, and construct $G$ as follows. First, we create a vertex of $G$ for each of the variables in each of the clauses of the `3Sat` instance. The colors of these vertices are restricted to be $0, 1$ by connecting all such vertices to a single vertex of an additional triangle that we add to $G$, with this vertex being assigned the color $+$.

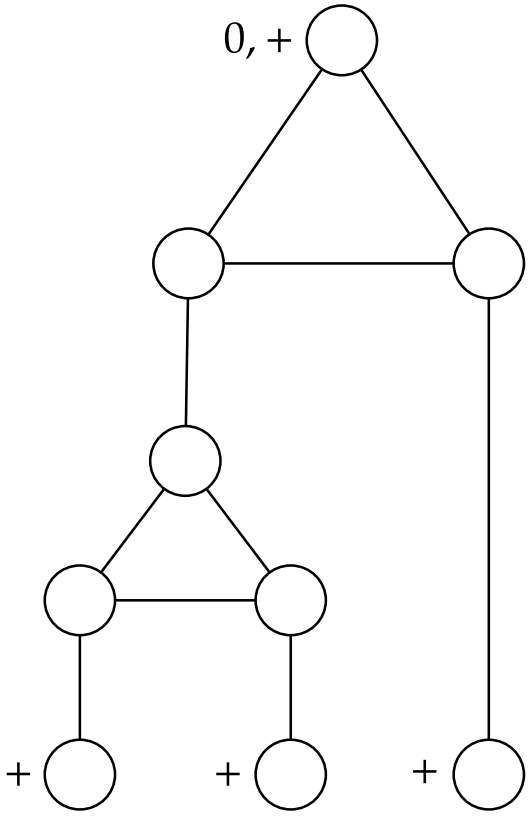
To deal with the logic of the $\vee$s in each clause, we will use a hint provided in a similar exercise in Sipser's book, which is to make use of the following subgraph called

an 'OR gadget':



(175)

The utility of this gadget in the present setting is that if the two bottom vertices of the gadget have the same color, a 3-coloring is possible only if the top vertex shares that color. For each clause, we then stack two OR gadgets together to produce the subgraph




(176)

Here the symbols next to the bottom and top vertices indicate which of the colors in the added triangle the vertices are connected to. Connecting the bottom vertices to the + vertex means that they must all be colored with $0, 1$, while also connecting

the top vertex (the truth value of the clause in question) to 0 ensures that a legit 3-coloring is possible only if the top 'output' vertices of each clause subgraph are colored in as 1. By staring at this subgraph for a while, one checks that a coloring of 1 for the top vertex is indeed impossible iff all of the input vertices are colored 0 — and hence this subgraph dutifuly computes the OR of the bottom three vertices. The only thing remaining in the construction is to draw edges between mutually contradictory variables (viz. pairs of vertices assigned to $x$ and $\neg x$). Once this is done, it is easy to see that a 3-coloring is possible iff all clauses are satisfiable.

`vertexCover` $\rightarrow$ `maxCut`

To show this reduction we will make use of the same OR gadget used in the previous problem.

Consider the Hamiltonian

$$H = \sum_i \left( -t(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) - \mu n_i + \frac{U}{2} n_i(n_i - 1) + V i n_i \right) \equiv H_t + H_U + H_V, \quad (177)$$

where $H_U$ includes the chemical potential. Our goal is to perform a rotation into a basis in which the Hamiltonian commutes with the linear potential term $V \sum_i i n_i$ up to some fixed order in $t/V, U/V$. The usual way of rotating the Hamiltonian is through an SW transformation, wherein $H$ becomes

$$e^\Lambda H e^{-\Lambda} = \sum_{k \geq 0} \frac{1}{k!} \mathrm{Ad}_\Lambda^k(H), \quad (178)$$

where $\mathrm{Ad}_\Lambda(\cdot) = [\Lambda, \cdot]$ and $\Lambda$ is anti-Hermitian, with $\Lambda$ chosen so as to eliminate the offending single-particle hopping term $-t(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i)$ at lowest order.

Note that it is already clear that interactions are required for producing a nonzero dipolar hopping term. Indeed, without the interaction term, $H$ is built solely of 2-body terms — we can thus choose $\Lambda$ to be a 2-body operator, and $\mathrm{Ad}_\Lambda^k(H)$ will consequently always itself be built from 2-body operators, which can only either be purely onsite or dipole non-conserving. In fact if we just take

$$\Lambda = \Lambda_t \equiv \frac{t}{V} \sum_i (b_i^\dagger b_{i+1} - b_{i+1}^\dagger b_i), \quad (179)$$

it is easy to check that when $U = 0$,

$$[\Lambda_t, H] = [\Lambda_t, H_V] = t \sum_i (b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) = -H_t. \quad (180)$$

Since this is just the negative of the hopping term, the first order part $\mathrm{Ad}_\Lambda(H_V)$ dutifully kills $H_t$. Moreover, since $[\Lambda_t, [\Lambda_t, H_t + H_V]] = 0$, the effective Hamiltonian stops at linear order, and we simply obtain $H_{eff} = e^{\Lambda_t}(H_t + H_V)e^{-\Lambda_t} = H_V$, which is purely onsite. This means that when $U = 0$, no effective dipole hopping terms are generated — there is perfect destructive interference between all putative hopping processes, and no such processes are generated to all orders in perturbation theory.

Let us then bring back a nonzero $U$. We take

$$\Lambda = \sum_{n=1}^{\infty} \Lambda_n, \tag{181}$$

where $\Lambda_n$ is order $n$ in $t/V, U/V$, and we set $\Lambda_1 = \Lambda_t$. We fix the second order term $\Lambda_2$ by requiring that it cancel the terms generated when commuting $\Lambda_1 = \Lambda_t$ against $H_U$. Specifically, we require

$$[\Lambda_2, H_V] = -[\Lambda_t, H_t + H_U]. \tag{182}$$

Keeping terms to third order in this expansion, we find that $H_{eff}$ becomes

$$H_{eff} = H_V + H_U + [\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2 - \Lambda_t, H_t] + [\Lambda_3, H_V]. \tag{183}$$

We then need the commutators

$$\begin{aligned} [\Lambda_t, H_t] &= \frac{t^2}{V} \sum_i \left( b_{i+2}^\dagger b_i + 2n_{i+1} + b_i^\dagger b_{i+2} - (i \to i+1) \right) = 0, \\ [\Lambda_t, H_U] &= \frac{tU}{V} \sum_i \{ b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i, n_{i+1} - n_i \} \end{aligned} \tag{184}$$

which together determines $\Lambda_2$ as (it is easy to guess this using the facts that the terms in $\Lambda_2$ must have support on two adjacent sites, consist only of 4-body operators, and must not commute with $H_V$)

$$\Lambda_2 = -\frac{tU}{V^2} \sum_i \{ b_i^\dagger b_{i+1} - b_{i+1}^\dagger b_i, n_{i+1} - n_i \}. \tag{185}$$

The effective Hamiltonian to cubic order is then

$$H_{eff} = H_V + H_U + [\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2, H_t] + [\Lambda_3, H_V]. \tag{186}$$

The commutators $[\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2, H_t]$ will include terms which do not commute with $H_V$, with $\Lambda_3$ chosen to cancel these terms. Now since $H_U$ is purely on-site while $\Lambda_2$ is purely off-site and only supported on nearest neighbors, $[\Lambda_2, H_U]$ must consist only of dipole non-conserving terms. However, $[\Lambda_2, H_t]$ may contain terms that commute with $H_V$, as both $\Lambda_2$ and $H_t$ are supported on nearest neighbor sites (allowing us to get e.g. $b_i^\dagger b_{i+1}^2 b_{i+2}^\dagger$). $\Lambda_3$ cannot be chosen to cancel these terms, as if $[A, H_V] \neq 0$ then $[H_V, [A, H_V]] \neq 0$. [19] The part of $\frac{1}{2}[\Lambda_2, H_t]$ which commutes with $H_V$ thus unambiguously provides the leading dipole hopping terms. An unilluminating calculation shows

[19] One can see this by writing a general $A$ as a linear combination of terms of the form $C = \sum_i C_{\mathbf{n}} \prod_{k \in \mathbb{N}} b_{i+k}^{n_k}$, where each $n_k \in \mathbb{Z}$ (we have ignored number operators $n_i$ since they commute with $H_V$). Since the action of $\mathrm{Ad}_{H_V}$ does not change the powers of $b_i$ operators it acts on, the only way for $[H_V, [A, H_V]]$ to vanish is for $[H_V, [C, H_V]] = 0$ for all such operators $C$ appearing in $A$. But $[H_V, [C, H_V]]$ takes the same form as $C$, except with $C_{\mathbf{n}}$ replaced with $C_{\mathbf{n}} \mathrm{dip}(\mathbf{n})^2$ with $\mathrm{dip}(\mathbf{n})$ the dipole moment of $\prod_k b_k^{n_k}$; thus this vanishes only if $\mathrm{dip}(\mathbf{n}) = 0 \implies [C, H_V] = 0$.

that these terms are

$$\begin{aligned}\frac{1}{2}[\Lambda_2, H_t] &\supset -\frac{t^2U}{V^2}\sum_{i,j}\{[b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i, n_{j+1} - n_j], b_j^\dagger b_{j+1} - b_{j+1}^\dagger b_j\} \\ &\supset -\frac{t^2U}{V^2}\sum_i b_i^\dagger b_{i+1}^2 b_{i+} + h.c,\end{aligned} \tag{187}$$

where $\supset$ denotes those terms which commute with $H_V$. Thus the effective Hamiltonian to cubic order in $t/V, U/V$ is

$$H_{eff} = \sum_i \left(-\frac{t^2U}{V^2} b_i^\dagger b_{i+1}^2 b_{i+2} + h.c. - \mu n_i + \frac{U}{2} n_i(n_i - 1) + V i n_i\right) \tag{188}$$

## 29 Sanov's theorem and applications

This entry deals with a basic result in large deviation theory (viz. Sanov's theorem), which gives us a way of determining how typical an iid sequence $\mathbf{s} = (s_1, \ldots, s_N), s_i \in \mathcal{X}$ drawn from a distribution $p \in \mathcal{P}_\mathcal{X}$ is (here $\mathcal{P}$ is the set of distributions on the alphabet $\mathcal{X}$).

We first need a definition.

**Definition 1.** Given an iid sequence of variables $\mathbf{s}$, the *type* $q_\mathbf{s} \in \mathcal{P}_\mathcal{X}$ of $\mathbf{s}$ (aka the empirical distribution of $\mathbf{s}$) is defined as

$$q_\mathbf{s}(x) = \frac{1}{N}\sum_{i=1}^N \delta_{x,s_i}. \tag{189}$$

First we give a proof of Sanov's theorem, which says that

**Proposition 3.** *Let $K \in \mathcal{P}_\mathcal{X}$ be a set of probability distributions on $\mathcal{X}$. Then the probability that the empirical distribution $q$ of a length-$N$ sequence $\mathbf{s}$ lies in $K$ is given to leading exponential order in $N$ as*

$$\mathbb{P}[q \in K] = e^{-ND(q_*||p)}, \tag{190}$$

*where the information projection of $q$ is*

$$q_* = \arg\min_{q\in K} D(q||p) \tag{191}$$

*and $D(q||p) = \mathbb{E}[\ln(q/p)]$ is the KL divergence.*

The reason for having the minimization here is just that the probability of an empirical distribution being in $K$ is dominated at large $N$ by the single most likely distribution $q_*$ in $K$.

*Proof.* □

## 30 Entropies and negativities for stabilizer states

In today's entry we will review some stabilizer-based methods used to compute entropies and negativities in stabilizer states, which are useful when one wants to do numerics with Clifford circuits. The first part of this diary entry works out some background material, and in the second part we prove some relations quoted in an appendix of [3] (I have not tried to track down original sources where these relations were first proved).

## 31 Impurity bound states in superconductors

In today's entry we will be studying bound states that form around different types of impurities in different types of superconductors. The motivation for doing these calculations came from wanting to derive and then extend (from conventional to unconventional SCs) a classic result by Shiba from the 60s.

Consider a SC with Bogoliubov Hamiltonian

$$\mathcal{H}(\mathbf{k}) = \xi(\mathbf{k})\lambda^z + \lambda^+\Delta(\mathbf{k}) + \lambda^-\Delta(\mathbf{k})^\dagger, \tag{192}$$

where $\Delta(\mathbf{k}) = d_\mu(\mathbf{k})s^\mu s^y$, with $d_\mu$ a complex 4-vector, $s^\mu$ Pauli matrices in spin space, and $\lambda^\mu$ Pauli matrices in Nambu space.

*Prerequisite: Greens functions for triplet SCs*

A quasi-nontrivial problem is to obtain the free Greens function $G(\mathbf{k},\omega) = (\omega - \mathcal{H}(\mathbf{k}))^{-1}$ for a general (viz. non-unitary) triplet SC ($d_0 = 0$). We begin with the observation

$$\Delta\Delta^\dagger = \mathbf{d}\cdot\mathbf{s}\,\mathbf{d}^*\cdot\mathbf{s} = |d|^2 + \mathbf{M}\cdot\mathbf{s}, \tag{193}$$

where $|d|^2 = \sum_{a=1}^3 d_a^* d^a$ and we have defined the pair magnetization

$$\mathbf{M} \equiv i\mathbf{d}\times\mathbf{d}^*, \tag{194}$$

which vanishes for unitary pairing (where $\mathbf{d} \parallel \mathbf{d}^*$) and is maximal for fully spin-polarized pairing (where $\mathbf{d} \perp \mathbf{d}^*$).

For unitary pairing, $\Delta\Delta^\dagger \propto s^0$. This makes finding $G$ easy, as it takes on essentially the same form as in a singlet SC. Non-unitary pairing requires a bit more work. We find the normal $G_n$ and anomalous $G_a$ (often written as $F$) parts of the free Greens function using the consistency conditions

$$\begin{aligned}(\omega - \xi)G_n - \Delta G_a^\dagger &= 1 \\ -\Delta^\dagger G_n + (\omega + \xi)G_a^\dagger &= 0.\end{aligned} \tag{195}$$

Now $G_n$ has zero charge and zero spin; thus we take the ansatz

$$G_n = A + B\mathbf{M}\cdot\mathbf{s}. \tag{196}$$

Likewise $G_a$ has charge two and spin given by that of $\Delta$; thus we write

$$G_a^\dagger = C\Delta^\dagger + Ds^y(\mathbf{M}\times\mathbf{d}^*)\cdot\mathbf{s}. \tag{197}$$

We thus need to solve

$$\begin{aligned}-\Delta^\dagger(A + B\mathbf{M}\cdot\mathbf{s}) + (\omega+\xi)(C\Delta^\dagger + D(\mathbf{M}\times\mathbf{d}^*)\cdot s^y\,\mathbf{s}) &= 0 \\ (\omega-\xi)(A + B\mathbf{M}\cdot\mathbf{s}) - \Delta(C\Delta^\dagger + D(\mathbf{M}\times\mathbf{d}^*)\cdot s^y\,\mathbf{s}) &= 1.\end{aligned} \tag{198}$$

for $A, B, C, D$.

Consider the first line of (198). Looking at the part that is nonzero when $\mathbf{M} = 0$, we obtain

$$A = (\omega + \xi)C. \tag{199}$$

Then looking at the part containing $\mathbf{M}$, we use

$$\Delta^\dagger\mathbf{M}\cdot\mathbf{s} = is^y(\mathbf{d}^*\times\mathbf{M})\cdot\mathbf{s} \tag{200}$$

to obtain

$$B = iD(\omega + \xi). \tag{201}$$

Now consider the second line of (198). We simplify this by way of using $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ to write $\mathbf{M}\times\mathbf{d}^* = i(|d|^2\mathbf{d}^* - (d^*)^2\mathbf{d})$, giving

$$\begin{aligned}\Delta s^y(\mathbf{M}\times\mathbf{d}^*)\cdot\mathbf{s} &= \mathbf{d}\cdot(\mathbf{M}\times\mathbf{d}^*) + i[\mathbf{d}\times(\mathbf{M}\times\mathbf{d}^*)]\cdot\mathbf{s} \\ &= i(M^2 + |d|^2\mathbf{M}\cdot\mathbf{s}),\end{aligned} \tag{202}$$

where we used $M^2 = |d|^4 - (d^*)^2d^2$. This then gives

$$1 = A(\omega-\xi) - C|d|^2 - iDM^2 + \mathbf{M}\cdot\mathbf{s}((\omega-\xi)B - C - iD|d|^2). \tag{203}$$

Plugging in for $B, A$, we have

$$C = -iD(-\omega^2 + \xi^2 + |d|^2) \tag{204}$$

and

$$C(\omega - \xi^2 - |d|^2) - iDM^2 = 1, \tag{205}$$

so that with a little bit of algebra one obtains

$$D = \frac{-i}{(\omega^2 - E_+^2)(\omega^2 - E_-^2)}, \qquad E_\pm \equiv \sqrt{\xi^2 + |d|^2 \pm M}. \tag{206}$$

Note that if $|M| = |d|^2$ — so that the pairs have maximal magnetization and only one spin component is involved in the pairing — we correctly get that one of the $E_\pm = \xi$.

At this point we can then also find $A, B, C$; we omit the algebra. The result is that the Greens functions are

$$\begin{aligned} G_n &= \frac{-\omega^2 + \xi^2 + |d|^2 + \mathbf{M}\cdot\mathbf{s}}{(\omega^2 - E_+^2)(\omega^2 - E_-^2)}(\omega + \xi) \\ G_a &= \frac{(-\omega^2 + \xi^2 + |d|^2)\mathbf{d} + i(\mathbf{M}\times\mathbf{d})}{(\omega^2 - E_+^2)(\omega^2 - E_-^2)} \cdot \mathbf{s}\, s^y. \end{aligned} \tag{207}$$

*Impurities: generalities*

We will study what happens when an impurity is placed in the SC, which we capture with the Hamiltonian

$$I = U\lambda^z + \frac{1+\lambda^z}{2}\mathbf{J}\cdot\mathbf{s} + \frac{1-\lambda^z}{2}\mathbf{J}\cdot s^y \mathbf{s} s^y. \tag{208}$$

where the scattering matrix element is assumed to be completely momentum non-conserving.

To find the impurity bound states (if they exist), we write down the full Greens function (omitting frequency dependence when possible) as

$$\mathcal{G}(\mathbf{k}, \mathbf{k}') = G(\mathbf{k})\delta_{\mathbf{k},\mathbf{k}'} + G(\mathbf{k})TG(\mathbf{k}'), \tag{209}$$

where $G$ is the free Greens function, and the T matrix is easily seen to be

$$T(\omega) = \frac{1}{\mathbf{1} - IG_{loc}(\omega)} I = \frac{1}{\mathbf{1} - (IG_{loc}(\omega))^2}(I + IG_{loc}(\omega)I), \tag{210}$$

where the local Greens function is

$$G_{loc}(\omega) = \int_{\mathbf{k}} G(\mathbf{k}, \omega). \tag{211}$$

The bound states we are looking for correspond to the poles of $T(\omega)$; thus we are looking for frequencies such that (it is often easier to compute the spectrum of $(IG_0)^2$ rather than $IG_0$; this is why we have written $T(\omega)$ in this form)

$$\det[\mathbf{1} - (IG_{loc}(\omega))^2] = 0. \tag{212}$$

*$s$-wave*

First consider the $s$-wave case, taking $d_0$ to be real wolog. We will make all the usual assumptions having a constant density of states $\nu$ and approximate particle-hole symmetry near the Fermi level. Then the local Greens function is

$$G_{loc}(\omega) = \frac{\pi\nu}{\sqrt{d_0^2 - \omega^2}}(\omega + \lambda^x s^y d_0). \tag{213}$$

We know abstractly that simple potential disorder should not be pair-breaking, and thus should not induce a bound state. Indeed, set $\mathbf{J} = 0$. Then a bound state would require

$$\det[\mathbf{1}/U^2 - (\lambda^z G_{loc}(\omega)\lambda^z G_{loc}(\omega))] = 0. \tag{214}$$

But

$$\lambda^z G_{loc}(\omega)\lambda^z G_{loc}(\omega) = -(\nu\pi)^2\mathbf{1} \tag{215}$$

obviously has only negative eigenvalues. Hence $T(\omega)$ has no poles, and no bound states form.

Contrast this with a magnetic impurity, which we know should be pair-breaking. Indeed, it is easy to check that now $[G_{loc}(\omega), I] = 0$ and $I^2 = J^2\mathbf{1}$, so that we just need to solve

$$\det[\mathbf{1}/J^2 - G_{loc}(\omega)^2] = 0. \tag{216}$$

The eigenvalues of $G_{loc}(\omega)$ are $\pi\nu(\omega \pm d_0)/\sqrt{d_0^2 - \omega^2}$, and so a little bit of algebra yields poles at

$$\omega_* = \pm d_0 \frac{1 - (\pi\nu J)^2}{1 + (\pi\nu J)^2}. \tag{217}$$

Note that $\omega_*$ is always inside the gap, and that the bound state lies at *zero* frequency when $J = 1/(\pi\nu)$. This is the result from Shiba that we wanted to derive.

### *nonzero angular momentum*

Now consider pairing with nonzero angular momentum. Our first task is to determine $G_{loc}(\omega)$, which we will do under the assumptions of approximate particle-hole symmetry near the FS and the existence of only a single angular harmonic $l$ in the pairing function. We will furthermore assume that $\xi(\mathbf{k}) = \xi(R_{\pi/l}\mathbf{k})$. Since $\mathbf{d}(\mathbf{k}) = -\mathbf{d}(R_{\pi/l}\mathbf{k})$ and $\mathbf{M}(\mathbf{k}) = +\mathbf{M}(R_{\pi/l}\mathbf{k})$, the anomalous part of the local Greens function vanishes,

$$G_{a,loc} = 0. \tag{218}$$

The normal part is rather more complicated:

$$G_{n,loc}(\omega) = K(\omega)s^0 + \mathbf{L}(\omega)\cdot\mathbf{s}, \tag{219}$$

where

$$\begin{aligned} K(\omega) &= \omega\nu\pi \int_{FS} \left( \frac{1}{\sqrt{|d|^2 - \omega^2 + M}} + \frac{1}{\sqrt{|d|^2 - \omega^2 - M}} \right) \left( 1 + \frac{|d|^2 - \omega^2}{\sqrt{(|d|^2 - \omega^2)^2 - M^2}} \right) \\ \mathbf{L}(\omega) &= \omega\nu\pi \int_{FS} \left( \frac{1}{\sqrt{|d|^2 - \omega^2 + M}} + \frac{1}{\sqrt{|d|^2 - \omega^2 - M}} \right) \frac{\mathbf{M}}{\sqrt{(|d|^2 - \omega^2)^2 - M^2}}, \end{aligned} \tag{220}$$

where $\int_{FS}$ denotes an angular average over the FS, with the $\mathbf{d}$ appearing on the RHS implicitly evaluated at the appropriate FS point.

Finding the bound states is easy because $G_{loc}$ is diagonal in Nambu space. Therefore we only need to solve

$$\det[\mathbf{1} - IG_{n,loc}(\omega)] = 0. \tag{221}$$

Since the eigenvalues of $\mathbf{L}\cdot\mathbf{s}$ are $\pm L$,[20] in the case of potential disorder we need only solve

$$1/U = K(\omega) \pm L(\omega), \tag{222}$$

while for magnetic disorder we need to solve

$$1 = \mathbf{J}\cdot\mathbf{L}(\omega) \pm \sqrt{(K(\omega)\mathbf{J} + i\mathbf{J}\times\mathbf{L}(\omega))^2} \tag{223}$$

where we write the square root like this to emphasize that we are not taking a complex conjugate when computing the norm.

The unitary case is of course the simplest. Here

$$\mathbf{L}(\omega) = 0, \qquad K(\omega) = 4\omega\nu\pi \int_{FS} \frac{1}{\sqrt{|d|^2 - \omega^2}}. \tag{224}$$

Then for potential disorder, we find a bound state by solving

$$4\pi\nu \int_{FS} \frac{\omega_*}{\sqrt{|d|^2 - \omega_*^2}} = \frac{1}{U} \tag{225}$$

for $\omega_*$ (for a magnetic impurity we simply replace $U$ by $J$).

We can see right away that $\omega$ will necessarily have an imaginary part — due to the gapless qp excitations present near the nodes — and that $\omega_* = 0$ is never a solution (unlike with the singlet case). An analytic solution is possible when one expands the integral above near the nodes, and one can confirm that the imaginary part is perturbatively small (in $1/\log(\Lambda\alpha)$, where $\alpha$ is proportional to $U$ and the LDOS near the nodes) compared to the real part. Out of laziness however we will not write this out, and be content with the more general expression above.

## 32 Bound states in graphene superconductors

The full matrix Greens function is

$$\mathcal{G} = \begin{pmatrix} G_{k,\omega} & F_{k,\omega} \\ F^\dagger_{k,\omega} & G^T_{-k,-\omega} \end{pmatrix}. \tag{226}$$

The T matrix is

$$T(\omega) = \frac{1}{\mathbf{1} - I\int_{\mathbf{k}} \mathcal{G}_{loc}(\omega)} I, \tag{227}$$

where the local Greens function is defined as

$$\mathcal{G}_{loc}(\omega) \equiv \int_{\mathbf{k}} \mathrm{Tr}_\tau \left[(\mathbf{1} + \alpha X)\mathcal{G}_{k,\omega}\right]. \tag{228}$$

[20] Note that $L \equiv \sqrt{\mathbf{L}\cdot\mathbf{L}}$ and not $\sqrt{\mathbf{L}^*\cdot\mathbf{L}}$.

## 32.1 IVC

Suppose that we polarize the valley according to the projector $|v\rangle\langle v|$. Writing $v$ in the Bloch sphere representation $v = \cos(\theta/2)e^{i\phi/2}|0\rangle + \sin(\theta/2)e^{-i\phi/2}|1\rangle$,

$$|v\rangle\langle v| = \begin{pmatrix} c_{\theta/2}^2 & \frac{1}{2}s_\theta e^{-i\phi} \\ \frac{1}{2}s_\theta e^{i\phi} & s_{\theta/2}^2 \end{pmatrix}, \qquad |v^*\rangle\langle v| = \begin{pmatrix} c_{\theta/2}^2 e^{i\phi} & \frac{1}{2}s_\theta \\ \frac{1}{2}s_\theta & s_{\theta/2}^2 e^{-i\phi} \end{pmatrix}. \tag{229}$$

These matrices satisfy

$$\mathrm{Tr}[(\mathbf{1} + \alpha X)|v\rangle\langle v|] = 1 + \alpha s_\theta c_\phi, \qquad \mathrm{Tr}[(\mathbf{1} + \alpha X)|v^*\rangle\langle v|] = c_\phi + \alpha s_\theta + i c_\theta s_\phi. \tag{230}$$

### *32.1.1 singlet*

Here the order parameter is simply $\Delta = d_0 i s^y$. The Greens functions are

$$F = |v^*\rangle\langle v|\frac{\Delta}{\omega^2 + E^2}, \qquad G = |v\rangle\langle v|\frac{i\omega + \xi}{\omega^2 + E^2}. \tag{231}$$

Integrating over momentum and tracing against $\tau^0 + \alpha\tau^x$ to get the local Greens functions, we find

$$\begin{aligned} F_{loc} &= i s^y (c_\theta + \alpha s_\theta + i c_\theta s_\phi)\nu \oint_\gamma \frac{d_0(\gamma)}{\sqrt{\omega^2 + |d_0(\gamma)|^2}} \\ G_{loc} &= (1 + \alpha s_\theta c_\phi)\nu \oint_\gamma \frac{i\omega}{\sqrt{\omega^2 + |d_0(\gamma)|^2}}. \end{aligned} \tag{232}$$

Consider first potential scattering, where $I = U\lambda^z$.

## 32.2 SVL

# 33 measuring the optical conductivity

I recently found myself having to teach some students about measurements of the optical conductivity, the KK relations, and sum rules; in the process of preparing I realized I had forgotten many of the relevant details. This diary entry is a summary of the final logical flow I ended up with. All calculations will be done assuming a homogenous + isotropic medium.

First, some terminology: we are measure the "optical" conductivity when we look at an object (i.e. probe it with electromagnetic radiation) with frequency $\omega > 0$. At $\omega > 0$, the charges in the object will just slosh back and forth (in a direction determined by the electric field), and thus the material behaves as a dielectric — all of the charges / currents are essentially "bound" in the language of electromagnetism.

The easiest thing to measure when we look at an object with frequency $\omega$ is the reflectance

$$r = \frac{1-n}{1+n}, \tag{233}$$

where $n$ is the (complex and $\omega$-dependent) index of refraction of the material. This follows from Fresnel's law, stating the reflection across a barrier between an incident medium with index $n_i$ and a transmission medium with index $n_t$ is $r = (n_i - n_t)/(n_i + n_t)$ (see e.g. wikipedia), which reduces to the above after taking the incident medium to be the vacuum (recall $n = c_{\text{vacuum}}/c_{\text{medium}}$). Note that (233) holds only when the incoming light is at normal incidence, i.e. whose momentum is normal to the interface (so that we are probing the $\mathbf{q} = 0$ response in the plane of the interface); for more general situations various cosines of the angles involved appear.

Now (233) inverts to give $n = (1-r)/(1+r)$, so that knowing $r$ tells us $n$ and the dielectric "constant" (sometimes aka relative permittivity) $\varepsilon$ from $n = \sqrt{\varepsilon}$ (assuming the material has $\mu = \mu_0$). We can then use $\varepsilon$ to get $\sigma$ as follows. Recall that the definition of the polarization density $\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}$, which microscopically is to be thought of as the density of bound dipole moments present in the material ($\varepsilon$ is big when you have a lot of screening, so that a big $\varepsilon$ and big $\mathbf{P}$ give a small $\mathbf{E}$). Also recall that $\partial_t \mathbf{P} = \mathbf{J}_b$, where $\mathbf{J}_b$ is the bound current. The correctness of this relation can be seen just by thinking of a picture of a dipole that expands / contracts in time. Finally, we have $\mathbf{J}_b = \sigma\mathbf{E}$. While one might be tempted to instead write $\mathbf{J}_f = \sigma\mathbf{E}$, recall that we are working at $\omega \neq 0$: hence all the charge is being moved around by carriers which are effectively bound (thinking about the DC "electric conductivity" is slighly different, although note even there we take $\mathbf{q} = 0$ before $\omega \to 0$, so that this bound current picture still makes sense).

The relations in the previous paragraph tell us that

$$\sigma\mathbf{E} = \partial_t \mathbf{P} \implies \frac{\sigma}{-i\omega} = \varepsilon - \varepsilon_0. \tag{234}$$

Thus knowledge of $r(\omega)$ gives us $\sigma(\omega)$. However, in many cases we are only able to measure the real number $|r(\omega)|^2$ (getting the full $r(\omega)$ in fact requires using polarized light and shining it at different angles using "ellipsometry", which we will not delve into). While $|r(\omega)|^2$ is a function of both the real and imaginary parts of $\varepsilon(\omega)$, it still does not directly give us access to both, but rather only a single (unfortunately rather non-linear) combination of them.

Nevertheless, $\sigma(\omega)$ can be obtained from $|r(\omega)|^2$ by way of the KK relations and the optical sum rule, plus a fair amount of (seemingly rather complicated) nonlinear data processing. These relations were detailed in a previous diary entry, so here we will be rather laconic. The KK relations follow from causality, viz. $\sigma(t) = \sigma(t)\Theta(t)$, so that

$$\sigma(z) = \int_{\mathbb{R}} dt\, e^{izt}\sigma(t) \tag{235}$$

is manifestly analytic in the UHP. This implies

$$\sigma(\omega + i\delta) = \oint dz\, \frac{\sigma(z)}{z - (\omega + i\delta)}. \tag{236}$$

$$\sigma(\omega - i\delta) = -\sigma(\omega + i\delta)^* \tag{237}$$

The usual way of writing the Drude conductivity is as

$$\sigma_D(\omega) = \frac{\omega_p^2}{\nu - i(\omega + i\delta)} \tag{238}$$

where $\omega_p = \sqrt{ne^2/m}$ is the plasma frequency. As wrriten however this cannot be correct, since it does *not* satisfy (237). Thus it seems reasonable that we should instead write

$$\sigma_D(\omega) = \frac{\omega_p^2}{\nu \mathrm{sgn}(\delta) - i(\omega + i\delta)} \tag{239}$$

which I think is indeed what is borne out in a more detailed Greens function approach a la Altland + Simons.

We should also check that $\sigma_D(\omega)$ obeys the KK relations. We will just check that $\sigma_R(\omega)$ is given by $P\int \frac{dz}{\pi}\frac{\sigma_I(z)}{z-\omega}$. Now in the present setting

$$\sigma_R(\omega) = \nu\frac{\omega_p^2}{\nu^2+\omega^2}, \qquad \sigma_I(\omega) = \omega\frac{\omega_p^2}{\nu^2+\omega^2}. \tag{240}$$

The integral we need the principal part of is thus

$$I_a^b \equiv \int_a^b \frac{dz}{\pi}\frac{z}{\nu^2+z^2}\frac{1}{z-\omega} = \frac{2\nu\arctan(z/\nu) + 2\omega\ln(z-\omega) - \omega\ln(\nu^2+z^2)}{2\pi(\nu^2+\omega^2)}\Big|_a^b. \tag{241}$$

We then compute $I_{\omega+\varepsilon}^{\infty} + I_{-\infty}^{\omega-\varepsilon}$. We find

$$\begin{aligned} I_{\omega+\varepsilon}^{\infty} + I_{-\infty}^{\omega-\varepsilon} &= \frac{\nu}{\nu^2+\omega^2} - \frac{i\omega}{\nu^2+\omega^2} + \frac{\omega(\ln(-\varepsilon) - \ln(\varepsilon))}{\pi(\nu^2+\omega^2)} \\ &= \frac{\nu}{\nu^2+\omega^2}. \end{aligned} \tag{242}$$

Therefore

$$P\int \frac{dz}{\pi}\frac{\sigma_I(z)}{z-\omega} = \omega_p^2(I_{\omega+\varepsilon}^{\infty} + I_{-\infty}^{\omega-\varepsilon}) = \sigma_R(\omega) \qquad \checkmark. \tag{243}$$

There is one important subtlety to keep in mind when employing the sum rule in a many-body system (thanks to Leonid Levitov for pointing this out). What values should we use for $n$ and $m$? From the way the derivation went (arguing that $\sigma(t=\delta)$ probes the response of electrons as if they were in vacuum), we should take $n$ to be the density of *all* electrons (including even those trapped deep in inner energy levels), and $m$ to be the bare electron mass. If we have to do this the sum rule is clearly useless; we instead want an expression that only involves the number density of the 'active' electrons (and perhaps with $m$ taken to be an appropriate band mass).

Strictly speaking, this is true. However also strictly speaking, the sum rule includes an integral up to infinite frequencies.

# 34 One-axis twisting

Today I attended a very nice talk by Norm Yao on spin squeezing. In his talk he briefly introduced the "one axis twisting" model as a simple model for understanding spin squeezing. In this diary entry we will try to derive the formulae he quoted on his slides for the squeezing rate. I have made no attempt to dig through the literature and hence won't provide citations to the original sources. Any mistakes in the following discussion are due to me not understanding the talk fully.

Simplifying a bit, the goal of quantum metrology is to use (an appropriate type of) quantum entanglement to perform a measurement of an external field which is more precise than what could be achieved with classical methods. The most common setting is where one aims to measure the strength of a magnetic field; if this field is represented by the Hamiltonian $\sum_i Z_i$ then the GHZ state provides an optimal state for performing the sensing, basically due to the fact that the GHZ state involves a coherent superposition of states with macroscopically different values of the quantum number we want to sense (viz. $S^z$).

One challenge with this is that the GHZ state is hard to prepare. To this end we might look for states which are good for sensing (i.e. involve superpositions of states with relatively large differences in magnetic moments), but which are still easy to prepare. For us, a state of $N$ qubits will count as being easy to prepare if it is obtained by quenching a product state by a simple Hamiltonian and evolving for time poly($N$) with some reasonably small polynomial.

The OG example of such a state is apparently the one constructed by quenching a state with maximal spin along the $\mathbf{a}$ direction by the Hamiltonian $\frac{1}{N}(\mathbf{S}_T \cdot \mathbf{b})^2$, where $\mathbf{b} \cdot \mathbf{a} = 0$ ($\mathbf{S}_T$ is the total spin operator of the $N$ qubits). For concreteness, we will consider performing a quench on the state $|+\rangle^{\otimes N}$ with the Hamiltonian

$$H = \frac{1}{N} \sum_{i,j} Z_i Z_j = \frac{1}{N} Z_T^2, \tag{244}$$

where $Z_T = \sum_i Z_i$ is the "total $S^z$ spin" (in quotes because we are not including factors of 1/2).

Thinking back to when we learned about squeezed states in optics, it is perhaps reasonable that $|\psi(t)\rangle = e^{-itH}|+\rangle^{\otimes N}$ is a squeezed state of some form (recall that photonic squeezed states are created using squeezing operators like $e^{\zeta a^2 - \zeta^* a^{\dagger 2}}$ or by including interactions like this into a Hamiltonian; the resemblance between the non-linearities here and the $Z_T^2$ Hamiltonian should be apparent). That $|\psi(t)\rangle$ is indeed squeezed can be 'proven' with a simple physical argument. At $t = 0$, the total spin $\mathbf{S}$ is prepared in a 'cloud' spread out near the point $\hat{\mathbf{x}}$ on the Bloch sphere (with radius $\sim L/2$). Evolution with $H$ makes the cloud of spin precess around the $\hat{\mathbf{z}}$ direction *with a rate that depends on the z-component of the spin.* Thus the part of the cloud lying at positive $z$ will precess one way around $\hat{\mathbf{z}}$, while the part at negative $z$ will precess

the opposite way. The precession is faster for larger $|z|$, and the result is a dynamics which 'shears' the could, distending it by wrapping it around the sphere. Pictorially, this shearing process then leads to a squeezing of the cloud, and getting a squeezed spin state for long enough times is then quite reasonable (this was the logic given in Norm's talk).

Now we make this intuition more precise. We will aim to compute a figure of merit introduced in the talk, which adapted to the present notation is

$$\xi^2(t) \equiv N \frac{\min_\phi \mathrm{var}(Z_T(t)\cos\phi + Y_T(t)\sin\phi)}{\langle X_T(t)\rangle^2}. \tag{245}$$

Here the minimum over $\phi$ acts to select out the "most squeezed" direction of the spin in the plane normal to the initial magnetization vector $\parallel \hat{\mathbf{x}}$. On one of Norm's slides he claimed that the minimum value $\xi_s^2$ of $\xi^2(t)$ is $\xi_s^2 \sim N^{-2/3}$, and that this minimum is reached at a time $t \sim N^{1/3}$. These results are what we will aim to reproduce below.

To calculate the variance appearing in $\xi^2$, we start off with the useful relation

$$S_T^s(t) = e^{itZ_T^2/N} S_T^s e^{-itZ_T^2/N} = S_T^s e^{\frac{it}{N}[(Z_T+2s)^2 - Z_T^2]} = S_T^s e^{s\frac{4it}{N}Z_T + \frac{4it}{N}} = e^{s\frac{4it}{N}Z_T - \frac{4it}{N}} S_T^s \tag{246}$$

together with $Z_T(t) = Z_T$. Thus defining $\lambda \equiv 4t/N$, we have

$$\begin{aligned}
\langle X_T(t)\rangle &= \sum_s \langle e^{is\lambda Z_T - i\lambda} S_T^s\rangle \\
&= N \sum_s \langle +|S^s|+\rangle (\langle +|\cos(\lambda Z)|+\rangle)^{N-1} \\
&= N\cos(\lambda)^{N-1}
\end{aligned} \tag{247}$$

where the second line follows from an explicit expansion of $S_T^s$. On the other hand we obviously have

$$\langle Y_T(t)\rangle = \langle Z_T(t)\rangle = 0. \tag{248}$$

Now for the two point functions. We start with

$$\begin{aligned}
\langle S_T^s(t) S_T^{-s}(t)\rangle &= \frac{1}{4}\sum_{j,k} \langle +|(X+sJ)_j (X-sJ)_k|+\rangle \\
&= \frac{1}{4}\sum_{j,k} \langle +|2\delta_{j,k} + (1-\delta_{j,k}) X_j X_k|+\rangle \\
&= \frac{N(N+1)}{4}.
\end{aligned} \tag{249}$$

The other combination is

$$\begin{aligned}
\langle S_T^s(t)^2\rangle &= \langle S_T^s e^{2i\lambda Z_T} S_T^s\rangle \\
&= \cos(2\lambda)^{N-2} N(N-1) \langle +|S^s e^{2i\lambda Z}|+\rangle \langle +|e^{2i\lambda Z} S^s|+\rangle \\
&= \frac{N(N-1)}{4}\cos(2\lambda)^{N-2}.
\end{aligned} \tag{250}$$

The $ZZ$ correlators are of course $\langle Z_T(t)^2\rangle = N$ since the initial state has $\sqrt{N}$ fluctuations in both $Z_T$ and $Y_T$, and $Z_T$ commutes with the Hamiltonian. The last correlator we will need is

$$\begin{aligned}\langle Z_T(t)S_T^s(t)\rangle &= \langle Z_T e^{i\lambda s Z_T - i\lambda} S_T^s\rangle \\ &= \sum_{j,k}\left(\delta_{j,k}\cos(\lambda)^{N-1}\frac{s}{2} + (1-\delta_{j,k})\frac{\cos(\lambda)^{N-2}}{2}\langle +|Ze^{i\lambda sZ}|+\rangle\right) \\ &= \frac{s}{2}\left(N\cos(\lambda)^{N-1} + iN(N-1)\cos(\lambda)^{N-2}\sin(\lambda)\right).\end{aligned} \tag{251}$$

Define the operator $W_\phi \equiv \cos(\phi)Z_T + \sin(\phi)Y_T$. Since $\langle W_\phi(t)\rangle = 0$ for all $t$, $\mathrm{var}(W_\phi(t))^2 = \langle W_\phi(t)^2\rangle$, which we compute via

$$\begin{aligned}\langle W_\phi(t)^2\rangle &= \cos(\phi)^2\langle Z_T(t)^2\rangle + \sin(\phi)^2\langle Y_T(t)^2\rangle + \cos(\phi)\sin(\phi)\langle\{Z_T(t), Y_T(t)\}\rangle \\ &= \cos(\phi)^2 N + \sin(\phi)^2\sum_s \langle S_T^s(t)S_T^{-s}(t) - S_T^s(t)S_T^s(t)\rangle + 2\mathrm{Im}\langle Z_T(t)(S_T^+(t) - S_T^-(t))\rangle \\ &= \cos(\phi)^2 N + \sin(\phi)^2\left(N^2\frac{1-\cos(2\lambda)^{N-2}}{2} + N\frac{1+\cos(2\lambda)^{N-2}}{2}\right) \\ &\quad + 2\sin(\phi)\cos(\phi)N(N-1)\cos(\lambda)^{N-2}\sin(\lambda).\end{aligned} \tag{252}$$

Note that the RHS is equal to $N$ at $t = 0$, as required by the isotropic (in the $yz$ plane) $\sqrt{N}$ fluctuations present in the initial state.

To simplify the above expression, we bring all of the trig functions involving $\phi$ into a common form. One of the intermediate steps along the way involves writing $\cos(2\phi)\alpha + \sin(2\phi)\beta = \sqrt{\alpha^2+\beta^2}\cos(2\phi - 2\theta)$, where $\theta = \arctan(\beta/\alpha)$. Some unilluminating algebra along these lines leads to

$$\langle W_\phi(t)^2\rangle = \frac{1}{4}\left(N^2(1-\cos(2\lambda)^{N-2}) + N(3+\cos(2\lambda)^{N-2})\right) + \sqrt{\Xi^2+\Upsilon^2}\cos(2\phi - 2\theta), \tag{253}$$

where we have defined

$$\begin{aligned}\theta &\equiv \arctan(\Xi/\Upsilon) \\ \Xi &\equiv N(N-1)\cos(\lambda)^{N-2}\sin(\lambda) \\ \Upsilon &\equiv N(N-1)\frac{\cos(2\lambda)^{N-2}-1}{4}.\end{aligned} \tag{254}$$

The most squeezed direction is clearly the direction defined by $\phi = \theta - \pi/2$ (while $\phi = \theta$ is accordingly the least squeezed). We now need to evaluate the squeezing factor $\xi^2$ in the case where $\phi = \theta - \pi/2$, and find the time $t$ at which $\xi^2$ is maximized.

By either plotting $\xi^2(\lambda)$ or by recognizing that we have a $\langle X_T(t)\rangle^2 = N^2\cos(\lambda)^{2N-2}$ in the denominator, one sees that the maximum of $\xi^2$ is attained at small $\lambda \ll 1$. Simply performing a series expansion of $\xi^2$ in small $\lambda$ doesn't work however, as doing so yields a squeezing time $t_s \equiv \mathrm{argmin}_t(\xi^2)$ which scales as $t_s = \Theta(N^0)$ and a gives a squeezing parameter $\xi_s^2 \equiv \xi^2(t = t_s)$ of $\xi_s^2 = \Theta(N^{-1/2})$ — which is has the same scaling with $N$ as $\xi^2(0)$!

Playing around with the plots of $\xi^2(t)$ more carefully (by rescaling $t$ by $N$) shows that $t_s$ is not small (and in fact grows with $N$), but that $t_s/N$ always is. Thus we can

find the minimum by expanding in $\lambda \ll 1$ while at the same time taking $\lambda N \gg 1$. Doing so yields

$$\langle W_\phi(t)^2 \rangle \sim N \left( \frac{A}{(N\lambda)^2} + B\lambda^2 (N\lambda)^2 \right), \tag{255}$$

where (don't quote me on this) $A = 1, B = 384$ (really?). Thus

$$\xi_s^2 \sim \frac{A}{(N\lambda)^2} + B\lambda^2 (N\lambda)^2, \tag{256}$$

which is minimized when $\lambda = (2B/A)^{-1/6} N^{-2/3}$; writing this in terms of $t$ gives

$$t_s = \gamma N^{1/3}, \qquad \gamma \equiv \frac{A^{1/6}}{4(2B)^{1/6}}, \tag{257}$$

which correctly gives the $N^{1/3}$ scaling quoted on Norm's slide. By plugging this back in, we see that

$$\xi_s = \gamma' N^{-2/3}, \qquad \gamma' \equiv \frac{A}{\gamma^2} + B\gamma^4, \tag{258}$$

which also correctly gives the claimed $N^{-2/3}$ scaling (and hence does better than the $N^{-1/2}$ that we get in the unsqueezed $t = 0$ state).

Consider a situation in which the PDA at the midpoint is in the 'reverse' state and has $k$ characters in the stack $z$. The left halves of the words which lead to the PDA being in this configuration have $k+k'$ nontrivial characters, followed by an $\alpha$, followed by $k'$ nontrivial characters. The number of such strings is

$$N_L(z) = \sum_{p=k+1}^{l} \sum_{k'=0}^{\min(l-p,p-1-k)} c^{k'} \binom{p-1}{k+k'} \binom{l-p}{k'}. \tag{259}$$

Given this, the number of matching right strings is the simpler

$$N_R(z) = \binom{l}{k}. \tag{260}$$

We now calculate an asymptotic expression for $p_z$ valid in the large-$l$ limit. First we derive an expression for the number of palindromes $N_P$. This is

$$N_P = 2 \sum_{p=1}^{l} \sum_{k=0}^{p-1} c^k \binom{p-1}{k} \binom{2l-p}{k} \tag{261}$$

The binomial coefficients are maximized when $p \approx l$, $k \approx l/2$. The weighting by $c^k$ shifts the maximum to higher values of $k$ if $c > 1$, but for $c = \Theta(1)$ it still keeps the

most important values of $k, p$ to be such that $|k-p+1|, |k-2l+p| = \Theta(l)$. We can then use the approximation

$$\binom{n}{k} \approx \frac{2^n}{\sqrt{\pi n/2}} e^{-(n-2k)^2/2n}. \tag{262}$$

Using a saddle point analysis for the sums over both $k$ and $p$ and defining

$$\gamma \equiv 4 + \ln c, \tag{263}$$

we have

$$\begin{aligned} N_P &\approx \sum_{p=1}^{l} \sum_{k=0}^{p-1} \frac{2^{2l+1}}{\pi\sqrt{(p-1)(2l-p)}} \exp\left(k \ln c - \frac{(p-1-2k)^2}{2(p-1)} - \frac{(2l-p-2k)^2}{2(2l-p)}\right) \\ &\approx \sum_{p=1}^{l} \frac{2^{2l}}{\sqrt{2\pi(2l-1)}} \exp\left(-\frac{(p-1)(p-2l)\gamma^2}{8(2l-1)} - \frac{2l-1}{2}\right) \\ &\approx \frac{2^{2l+1}}{\gamma} \exp\left(\frac{2l-1}{32}(\gamma^2 - 16)\right). \end{aligned} \tag{264}$$

At the saddle point, the number of non-empty characters in each half of the palindrome is

$$k_* = \gamma \frac{(p_*-1)(2l-p_*)}{4(2l-1)} = \frac{\gamma}{8}(l - 1/2), \tag{265}$$

where $p_* = l + 1/2$ is the saddle point value for $p$ (which lies in the middle of the chain by symmetry). The fact that $k_* = \Theta(l)$ is important because it means the expected size of the stack for a bipartite cut is proportional to $l$ (rather than $\sqrt{l}$); this is ultimately what is responsible for the volume-law scaling of the entanglement entropy. Note however that for very large $c > \lceil e^4 \rceil$ we have $k_* > l$, and almost all palindromes have no empty spaces.

We can use this estimate of $k_*$ to get a bound on the entanglement entropy as

$$\begin{aligned} S &= -2 \sum_z c^k \frac{N_L(z) N_R(z)}{N_P} \log\left(\frac{N_L(z) N_R(z)}{N_P}\right) \\ &= -\sum_z p_k \ln\left(\frac{p_k}{2c^k}\right) \\ &= H(p(k)) + \ln 2 + \log c \sum_k k p(k) \\ &\geq \langle k \rangle \log c, \end{aligned} \tag{266}$$

where we have defined the probability distribution

$$p(k) \equiv 2c^k \frac{N_L(z) N_R(z)}{N_P}. \tag{267}$$

From the above estimate, we know that $\langle k \rangle \approx k_* = \Theta(l)$. Thus

$$S \geq \frac{\ln(c)(4 + \ln(c))}{8} l, \tag{268}$$

implying that the entanglement entropy is volume law for all $c > 1$.

In arriving at (266) we dropped the term $H(p(k)) + \ln 2$, which we expect to contribute a term logarithmic in $l$. This can be checked by estimating $N_L(z)$ in a similar way to how we estimated $N_P$. We have

$$\begin{aligned} N_L(z) &\approx \sum_{p=k+1}^{l} \sum_{k'=0}^{\min(l-p,p-1-k)} \frac{c^{k'} 2^l}{\pi\sqrt{(p-1)(l-p)}} \exp\left(-\frac{(p-1-2(k+k'))^2}{2(p-1)} - \frac{(l-p-2k')^2}{2(l-p)}\right) \\ &\approx \sum_{p=k+1}^{l} \frac{2^l}{\sqrt{2\pi(l-1)}} \exp\left(-\frac{p^2\gamma^2 - p\gamma(8k+\gamma(l+1)) + 16k^2 + 8k(2+l(\gamma-2)) + l(\gamma^2-8) + 4}{8(l-1)}\right. \\ &\approx \frac{2^l}{\gamma} c^{\frac{\gamma(l-1)+4(l-1-4k)}{32}} \end{aligned} \tag{269}$$

Combining this with $N_R(z) \approx 2^l e^{-(l-2k)^2/2l}/\sqrt{\pi l/2}$ and our previous expression for $N_P$, we find

$$p(k) \approx \frac{1}{\sqrt{\pi l/2}} \exp\left(-\frac{2k^2}{l} + \frac{\gamma k}{2} - \frac{l\gamma^2}{32}\right) \tag{270}$$

which produces the Shannon entropy

$$H(p(k)) \approx \frac{1 + \log(l\pi/2)}{2}, \tag{271}$$

which scales with $\log(l)$ as expected. Thus the full entanglement entropy at large $l$ is approximately

$$S \approx \frac{\ln(c)(4+\ln(c))}{8} l + \frac{1 + \log(2\pi) + \log(l)}{2}. \tag{272}$$

# 35 Langevin dynamics from the Caldeira-Leggett model

This diary entry was written when preparing to teach a class on Brownian motion and the FDT. In the course of preparing I came accross some subtleties relating to sum rules and the Langevin equation, and thus decided to write up the lecture notes for posterity's sake.

## Spectral functions

Standard cond-mat textbooks are rife with confusion surrounding the objects that come up when studying linear response. The notation is often abstruse and misleading (especially when it involves things like $\chi'$ and $\chi''$), and sometimes simplifications are made which obfuscate important results. A common example of this arises in discussions of the conductivity, whose *real* part controls dissipation, despite us often being told that spectral functions are defined by the imaginary parts of susceptibilities.

In this section we will try to set some of this confusion straight. We define the spectral function

$$A_{XY}(t-t') \equiv \frac{1}{2}\langle [X(t), Y(t')]\rangle, \tag{273}$$

with the factor of $1/2$ chosen so that

$$A_{XY}(\omega) = \pi \sum_{n,m} (\pi_n - \pi_m) X_{nm} Y_{mn} \delta(E_{mn} - \omega), \tag{274}$$

where $\pi_n$ is the equilibrium occupation of the eigenstate $|n\rangle$. $A_{XY}(t)$ looks almost like the susceptibility, but in real time it differs by a $\Theta(t)$ and a factor of $2i$:

$$\chi_{XY}(t) = 2i\Theta(t) A_{XY}(t). \tag{275}$$

In frequency space, it is *not* always true that $A_{XY}(\omega)$ is the imaginary part of $\chi_{XY}(\omega)$. The correct relation (as follows from the spectral representation) is

$$A_{XY}(\omega) = \frac{1}{2i}(\chi_{XY}(\omega) - \chi_{Y^\dagger X^\dagger}(\omega)^*), \tag{276}$$

which gives the imaginary part only when $\chi_{XY}(\omega) = \chi_{Y^\dagger X^\dagger}(\omega)$.

The operators we are interested in are usually either even or odd under time reversal. Let $t_X$ denote the $\mathcal{T}$-parity of $X$. Then it is easy to check that

$$\chi_{XY}(\omega) = t_X t_Y \chi_{YX}(\omega). \tag{277}$$

Thus, letting $X, Y$ be Hermitian for simplicity, we have

$$A_{XY}(\omega) = \frac{1}{2i}(\chi_{XY}(\omega) - t_X t_Y \chi_{XY}(\omega)^*) = \begin{cases} \mathrm{Im}\chi_{XY}(\omega) & t_X = t_Y \\ -i\,\mathrm{Re}\,\chi_{XY}(\omega) & t_X = -t_Y \end{cases}. \tag{278}$$

The second line on the RHS is relevant when we are computing the conductivity, for which $X = J$ is the current and $Y = D$ is the dipole operator (since $D$ is what the electric field couples to). This explains why the real part of $\sigma$ is what controls dissipation, since $t_J = -1$ while $t_D = +1$.

## Sum rules

The simplest sum rule comes from realizing that the equal-time spectral function $A_{XY}(0)$ is just

$$\int_\omega A_{XY}(\omega) = \frac{1}{2}\langle [X(0), Y(0)]\rangle. \tag{279}$$

In particular, the RHS is *independent* of the Hamiltonian, requiring that the integrated spectral weight be a function only of the operators under consideration, and not on the nature of the dynamics (it also vanishes when $X = Y$ which is why the sum rule is often not discussed in this way). For the conductivity, it is easy to see that $[J, D]$ gives the expected $ne^2/m$.[21] For a fixed choice of $X, Y$ we can generate other sum

[21] One could argue that this actually does depend on the Hamiltonian since $J$ is determined by the commutator of $H$ with the charge density — but you get the point.

rules by differentiating (273) with respect to both $t$ and $t'$ before setting $t = t' = 0$, giving constraints on all moments of $A_{XY}(\omega)$:

$$\begin{aligned}\int_\omega \omega^{n+m} A_{XY}(\omega) &= \frac{i^{m-n}}{2}\langle[\frac{d^m X}{dt^m}, \frac{d^n Y}{dt'^n}]|_{t=t'=0}\rangle \\ &= \frac{(-1)^m}{2}\langle[\mathrm{Ad}_H^m(X(0)), \mathrm{Ad}_H^n(Y(0))]\rangle.\end{aligned} \tag{280}$$

The RHS usually easy (in principle) to compute for any $m, n$, and is usually constant. We will now explain why the $m, n > 0$ versions of this equation will force us to critically re-examine the Langevin equation.

## Langevin dynamics and the problem with Lorentzians

The canonical Langevin equation is[22]

$$\ddot{x} + \Omega x + \gamma \dot{x} = \frac{1}{m}\xi, \tag{281}$$

where $\xi$ has zero mean and a variance fixed by $\gamma, T, m$ via the FDT. Consider for illustration the simple case of $\Omega = 0$. The above equation then only involves the velocity $v \equiv \dot{x}$, and so

$$v = \frac{1}{m(-i\omega + \gamma)}\xi. \tag{282}$$

Since the field $\xi$ acts as a random force which couples to the position $x$, we have

$$\chi_{vx}(\omega) = \frac{1}{m(-i\omega + \gamma)}. \tag{283}$$

Since $v$ and $x$ have opposite transformations under time reversal (recall our discussion of the conductivity), the spectral function is a Lorentzian:

$$A_{vx}(\omega) = -i\,\mathrm{Re}\,\chi_{vx}(\omega) = -i\frac{\gamma}{m(\omega^2 + \gamma^2)}. \tag{284}$$

The dangerous thing about Lorentzians is that all of their moments (beyond the zeroth) are infinite! Thus while $A_{vx}(\omega)$ can (and does) fulfill the sum rule $\int A_{vx}(\omega) = -i/(2m)$, it violates all of the higher sum rules where at least one of $m, n$ is nonzero. We will eventually see that the problem lies in the assumption that the drag force $-v\gamma$ is local in time — this is a good enough assumption when one only cares about long times, but is problematic at short times and leads to bad behavior at large $\omega$ which produces the diverging higher moments.[23]

[22] We are somewhat abusing notation here by letting $x$ denote $\langle\hat{x}\rangle$ in the quantum case.

[23] This can be seen by noting that the Greens function for the velocity is $\mathcal{G}_{vv}(t) = e^{-\gamma t}\Theta(t)$. The instantaneous response at $t = 0^+$ is the source of the problem.

## The Caldeira-Leggett model

We can understand how to fix the above issue with the naive Langevin equation (the one with constant $\gamma$) by considering a model in which the emergence of Langevin behavior from the underlying microscopic dynamics can be explicitly calculated. This is what Calderia and Leggett did in their paper (they were mostly interested in how decoherence — viz. the coupling of a quantum system to a bath — modifies the probability of objects to tunnel through barriers; we will not have anything to say about this problem here).

The model is the simplest system one could imagine using to study thermalzation and Brownian motion from a microscopic perspective. It consists of $N \gg 1$ "bath oscillators" (with phase space coordinates $(q_i, p_i)$, fundamental frequencies $\omega_i$, and masses $m_i$) coupled to one "big" oscillator (with coordinates $(x, p)$, mass $M$, and fundamental frequency $\Omega$). The coupling between the "system" (just a single oscillator!) and the bath is linear and parametrized by constants $C_i$, with total Hamiltonian

$$H = \frac{1}{2}M\Omega^2 x^2 + \frac{p^2}{2M} + \sum_i \frac{p_i^2}{2m_i} + \frac{1}{2}\sum_i m_i\omega_i^2\left(q_i - x\frac{C_i}{m_i\omega_i^2}\right)^2. \tag{285}$$

Our strategy in what follows will be to integrate out the bath oscillators and arrive at a Langevin-esque equation for the system oscillator. We start by writing down Newton's law for the system oscillator, which is

$$\ddot{x} = \frac{-i}{M}[p, H] = -x\left(\Omega^2 + \frac{1}{M}\sum_i \frac{C_i^2}{m_i\omega_i^2}\right) + \frac{1}{M}\sum_i C_i q_i. \tag{286}$$

Defining an effective frequency

$$\widetilde{\Omega} \equiv \sqrt{\Omega^2 + M^{-1}\sum_i C_i^2/(m_i\omega_i^2)} \tag{287}$$

and an effective "noise field"

$$\xi_q(t) \equiv \sum_i C_i q_i(t), \tag{288}$$

we can write this in the more suggestive form

$$\ddot{x} + x\widetilde{\Omega}^2 = \xi_q(t). \tag{289}$$

So far it is not entirely obvious how the noise field $\xi_q(t)$ will end up producing a viscous term like $-\gamma\dot{x}$. We can investigate this by simply solving for the full evolution of each of the $q_i(t)$. Their EOM is simply

$$\partial_t^2 q_i + \omega_i^2 q_i = \frac{C_i}{m_i}x. \tag{290}$$

This EOM is solved by $q_i(t) = q_i^h(t) + q_i^p(t)$, where $q_i^{h,p}$ are homogenous and particular solutions, respectively. The homogenous solution is

$$q_i^h(t) = q_i(0)\cos(\omega_i t) + \dot{q}_i(0)\frac{\sin(\omega_i t)}{\omega_i}. \tag{291}$$

The particular solution is

$$q_i^p(t) = \frac{C_i}{m_i}\int_0^\infty dt'\,\mathcal{G}_i(t-t')x(t') \tag{292}$$

where the $i$th oscillator Greens function satisfies $(\partial_t^2+\omega_i^2)\mathcal{G}_i(t-t') = \delta(t-t')$ and is consequently

$$\mathcal{G}_i(t) = \Theta(t)\frac{\sin(\omega_i t)}{\omega_i}. \tag{293}$$

Since we want to write $\xi_q(t)$ in terms of $\dot{x}$, we will choose to integrate by parts and write

$$\begin{aligned} q_i^p(t) &= \frac{C_i}{m_i}\int_0^\infty dt'\,\Theta(t-t')\frac{\sin(\omega_i(t-t'))}{\omega_i}x(t) \\ &= \frac{C_i}{m_i\omega_i^2}\left(-\int_0^t dt'\,\cos(\omega_i(t-t'))\partial_t x(t') + (x(t)-x(0)\cos(\omega_i t))\right). \end{aligned} \tag{294}$$

We now substitute this expression for $q_i(t)$ into the $\xi_q(t)$ appearing in the EOM for $x$. Some algebra yields

$$\ddot{x} + \Omega^2 x + \int_0^t dt'\,\gamma(t-t')\dot{x}(t') = \frac{1}{M}\xi(t), \tag{295}$$

where we have defined the relaxation kernel

$$\gamma(t) \equiv \frac{1}{M}\sum_i \frac{C_i^2}{m_i\omega_i^2}\cos(\omega_i t) \tag{296}$$

and the effective noise field

$$\xi(t) \equiv \sum_i C_i\left(\cos(\omega_i t)\delta q_i(0) + \frac{p_i(0)}{m_i\omega_i}\sin(\omega_i t)\right) \tag{297}$$

with $\delta q_i(0) \equiv q_i(0) - \frac{C_i}{m_i\omega_i^2}x(0)$. The retarted nature of the relaxation kernel cures the sharp short-time jump in Langevin Greens function $\Theta(t)e^{-\gamma t}$, and renders higher moments of the spectral function finite.

Note that the effective noise field is completely deterministic, depending as it does solely on the initial conditions $q_i(0), p_i(0), x(0)$. Of course in reality $\xi$ effectively becomes a random variable in the $N\to\infty$ limit, which we deal with by averaging over the initial coordinates of the bath oscillators. Writing this average with angle brackets, we have $\langle\xi\rangle = 0$ if the initial conditions are random across the different oscilators (just having $N$ be large isn't enough — if e.g. for some reason all of the initial momenta where colinear, $\xi$ could potentially not behave as a random variable, and the motion of the system oscillator would not be of the Brownian form we expect).

Now we check the noise correlation function. Is it in accordance with the FDT? To compute the noise power spectrum we need

$$\begin{aligned} \langle\delta q_i(0)\delta q_j(0)\rangle &= \frac{1}{2m_i\omega_i}\delta_{i,j}(2n_B(\omega_i)+1) \\ \langle p_i(0)\delta q_j(0)\rangle &= -\frac{i}{2}\delta_{i,j}. \end{aligned} \tag{298}$$

Therefore

$$\begin{aligned}\langle \xi(t)\xi(0)\rangle &= \sum_i \frac{C_i^2}{2m_i\omega_i}\left(\cos(\omega_i t)(2n_B(\omega_i)+1) - i\sin(\omega_i t)\right) \\ &= \sum_i \frac{C_i^2}{2m_i\omega_i}\left(n_B(\omega_i)e^{i\omega_i t} + (n_B(\omega_i)+1)e^{-i\omega_i t}\right).\end{aligned} \tag{299}$$

Fourier transforming,

$$\begin{aligned}\langle |\xi(\omega)|^2\rangle &= \sum_i \frac{C_i^2}{2m_i\omega_i}\left(n_B(\omega_i)\delta(\omega+\omega_i) + (n_B(\omega_i)+1)\delta(\omega-\omega_i)\right) \\ &\equiv \int_0^\infty d\omega'\, A^B(\omega')\left(n_B(\omega')\delta(\omega+\omega') + (n_B(\omega')+1)\delta(\omega-\omega')\right),\end{aligned} \tag{300}$$

where we defined the bath spectral density

$$A^B(\omega) \equiv \sum_i \frac{C_i^2}{2m_i\omega_i}\delta(\omega-\omega_i), \tag{301}$$

with the integral over $\omega'$ going from 0 to $\infty$ just because all the $\omega_i \geq 0$.

Note that the bath spectral density also determines the relaxation kernel, since we may write

$$\gamma(t) = \frac{2}{M}\int_0^\infty d\omega'\,\frac{A^B(\omega')}{\omega'}\cos(\omega' t). \tag{302}$$

The naive Langevin equation corresponds to taking $\gamma(t) = \gamma\delta(t-t')$, which can be engineered via the choice

$$A^B_{naive}(\omega) = \frac{\omega M\gamma}{2\pi}. \tag{303}$$

This choice gives a large number of oscillators with large frequencies, leading to the unphysical behavior of the $vx$ spectral function at short times, but of course correctly captures long-time physics. Note that regardless, for Langevin-esque behavior to emerge, it is essential for $A^B(\omega)$ to be a rather *broad* function of frequency. If we happened to find ourselves in a situation where e.g. $A^B(\omega) = \delta(\omega-\omega_0)$, with all of the bath oscillators at the same frequency, we would most certainly not thermalize, as the "damping" term would be purely oscillatory and long-ranged in time. Thus for the oscillators to act as an effective thermal bath, it is essential that they contain a relatively diverse range of fundamental frequencies.

We also expect that the choice $A^B_{naive}(\omega)$ should lead to an Einstein relation between $\gamma$ and $T, M$, at least in the limit where we expect such a relation to hold. This is the high-temperature limit, where we set $\omega_i/T \ll 1\,\forall i$, so that $n_B(\omega) \approx T/\omega_i$ (of course since $A^B_{naive}(\omega)$ is not damped at large $\omega$ this is not true for the whole integration domain, but we will not go through the hassle of introducing a soft cutoff). Indeed, it is easy to see that in this limit the noise power spectrum is simply

$$\langle |\xi(\omega)|^2\rangle \to 2M\gamma T, \tag{304}$$

in agreement with FDT.

## 35.1 Periodic chains and symmetry bundles

In the discussion so far we have restricted our attention to infinite or open chains. With periodic boundary conditions, the nature of dipole symmetry becomes more subtle. On a periodic chain of length $L$, the dipole operator $D$ is consistent with the periodic boundary condition $X_{j+L} = X_j$ only when $L$ is divisible by $N$, so that $X_j^{j+L} = X_j^j$. When $L$ is not a multiple of $N$, $D$ itself is no longer a symmetry; the reamining symmetry is instead generated by $D^q$, where $q$ is the smallest integer such that $(X^q)^L = 1$, i.e. $q = N/\gcd(L, N)$. As a result, the global symmetries are not $\mathbb{Z}_N \times \mathbb{Z}_N$, but rather $\mathbb{Z}_N \times \mathbb{Z}_{\gcd(L,N)}$. In the most extreme case of $L, N$ co-prime, i.e. $\gcd(L, N) = 1$, the dipole symmetry is absent altogether.

This raises the following question: does the SPT phase change — or perhaps disappear entirely — when we work with periodic boundary conditions? On one hand, the global symmetry group protecting the SPT phase is certainly affected by the choice of $L$. On the other hand, the ground state $|\Psi_D\rangle$ and its attendant string order are well-defined for all $L$, even for a periodic chain. Furthermore, it is intuitively clear that features of reduced density matrices, such as the entanglement spectrum degeneracy, should look identical with both open and periodic boundary conditions, since after cutting open a periodic chain of any length $L$ we unambiguously obtain a ground state with SPT order. To develop vocabulary allowing us to talk about this type of situation in more a precise way, we now turn to defining a mathematical object that we dub a *symmetry bundle*.

Consider a local Hamiltonian $H$ which acts on a spatial lattice $M$ with local Hilbert space dimension $d$. Let $\{A_\alpha\}$ denote a collection of contractible subregions which together provide an open cover for $M$ (so that all of the intersections $A_\alpha \cap A_\beta$ are themselves contractible). ethan: *this is abusing terminology in several ways; if I was to be legalistic I'd say that $M$ is a collection of points, the locality of $H$ endows $M$ with the structure of a cell complex, that the $A_\alpha$ are defined wrt that cell complex, blah blah blah — but this seemed too pedantic* ethan: *symmetry bundles are also $G$ bundles, but I didn't think it necessary to discuss the group action on the fibers* A *symmetry bundle* $E_G$ over $M$ is defined by a projection $\pi : E_G \to M$ and a certain collection of operators defined on the $A_\alpha$ which we will define momentarily. On each patch $A_\alpha$, these operators are defined using local inverses of $\pi$, which identify the preimage of $A_\alpha$ under $\pi$ with the product space $U(d) \times A_\alpha$, with $U(d)$ the $d$-dimensional unitary group. We will write these local inverses as $\pi_g^{-1}$, where for now $g$ is a formal symbol indexing the different inverses. For each $\pi_g^{-1}$, we define the *symmetry section* $\varphi_g$ as the following map from patches to operators:

$$\varphi_g(A_\alpha) = \bigotimes_{i \in A_\alpha} \pi_g^{-1}(i)_1, \tag{305}$$

where $\pi_g^{-1}(i)_1$ denotes the restriction of $\pi_g^{-1}(i)$ to the (first) $U(d)$ factor of $U(d) \times A_\alpha$, and where for all $A_\alpha$ we require that $\varphi_g(A_\alpha)$ behave like an internal symmetry of $H$ on the interior of $A_\alpha$, meaning that

$$\mathrm{supp}([H, \varphi_g(A_\alpha)]) = \mathsf{LocNeigh}(\partial A_\alpha), \tag{306}$$

where $\mathsf{LocNeigh}(\partial A_\alpha)$ is a local neighborhood of $\partial A_\alpha$. ethan: *again, not sure how legalistic we want to be here; we could also make the definition such that for increasing $|A_\alpha|$ we have*

$||[H, \varphi_g(A_\alpha)]||/|A_\alpha| \to 0$ *or something else along these lines* Note that the set of operators satisfying (306) are closed under matrix multiplication and taking inverses. The $\varphi_g$ thus form a group, explaining our choice to index them by the symbol $g$ (and allowing us to write e.g. $\varphi_g\varphi_h \equiv \varphi_{gh}$ and $\varphi_{g^{-1}} \equiv \varphi_g^\dagger$). Finally, for each pair of patches $A_\alpha, A_\beta$ with $A_\alpha \cap A_\beta \neq \emptyset$, we define the transition operators

$$t_{g\alpha,h\beta}(i \in A_\alpha \cap A_\beta) \equiv \varphi_g^\dagger(i)\varphi_h(i). \tag{307}$$

Note that $t_{g\alpha,h\beta}$ is itself a local symmetry section on $A_\alpha \cap A_\beta$.

In the language of symmetry bundles, a global symmetry is thus a global section of $E_G$, viz. a symmetry section $\varphi_g$ which admits an extension of its range from the collection of patches $A_\alpha$ to the entire spatial manifold $M$, with $[\varphi_g(M), H] = 0$. More precisely, a global symmetry is a symmetry section characterized by the property that $t_{g\alpha,g\beta} = \mathbf{1}$ for all patches $A_\alpha, A_\beta$ with nonzero intersection. For sections $\varphi_g$ which do not admit an extension to $M$, the associated "symmetry" can only be defined on a patchwise basis. If such an extension is obstructed for any $\varphi_g$, we will call the symmetry bundle "nontrivial".

1d systems with charge and dipole conservation provide simple examples of nontrivial symmetry bundles. In this context we are given a group $G$ (with $G = \mathbb{Z}_N$ in the example discussed previously), ethan: *in progress...*

More examples can be constructed with the models obeying conservation laws defined from quadratic- and exponentially-varying functions of space to be considered later, or with the more general class of systems with "spatially modulated symmetries" considered in [**?**]. In the future it could be interesting to explore the topology of symmetry bundles in more detail, e.g. constructing examples in higher dimensions with magnetic monopole-like topologies.

ethan: *working on this offline; will update...*

For a periodic chain of length $L$, the stabilizer operators at $j = 1, L$ are $a_1 = Z_L Z_1^\dagger X_1 Z_1^\dagger Z_2$ and $a_L = Z_{L-1} Z_L^\dagger X_L Z_L^\dagger Z_1$ since the coordinate $j$ is defined mod $N$. The charge operator $Q$ commutes with $H_D$, but the commutation of $D = \prod_{j=1}^L (X_j)^j$ yields

$$Da_1D^{-1} = \omega^{-L}a_1, \quad Da_LD^{-1} = \omega^L a_L. \tag{308}$$

Indeed, $D$ commutes with $a_1, a_L$ only when $L$ is an integer multiple of $N$, or when $D^q$ with $q = N/\gcd(L, N)$ is used instead of $D$.

ethan: *should have a look at this* Instead of $D$ used above, one may well have chosen

$$\begin{aligned} D(j_0) &= \prod_{j=1}^{L} (X_{j+j_0})^j \\ &= (X_1)^{L-j_0+1} \cdots (X_{j_0})^L (X_{j_0+1}) \cdots (X_L)^{L-j_0} \end{aligned} \tag{309}$$

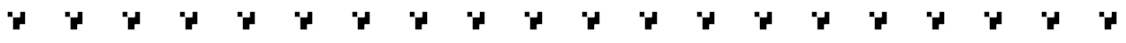
for arbitrary integer $j_0$ as the new dipole symmetry operator since the coordinate choice is after all arbitrary on a periodic chain. Instead of $a_1, a_L$, it is now $a_{j_0}, a_{j_0+1}$ that fail to commute with $D(j_0)$. In other words, one can shift the position of the 'singularity' (but never remove it) by a re-definition of the dipole symmetry operator. Among the different dipole symmetry operators $D(j_0), D(j_0')$ there is a relation

$$D(j_0') = Q^{j_0 - j_0'} D(j_0). \tag{310}$$

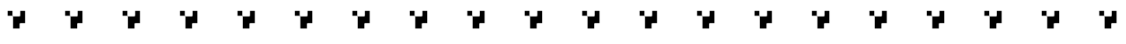
relating one dipole operator to another through the 'transition operator' $Q$. Rather than being a single global symmetry operator, $D$ should be viewed as a collection of operators $D(j_0)$ $(1 \leq j_0 \leq N)$ related to each other through (310). We are reminded of the gauge fields defined on different sections of the manifold, with transition functions connecting the two gauge field choices over the overlapping regions. In analogy to the nomenclature in differential geometry and gauge theory, we propose that $D$ is a *symmetry bundle* rather than an ordinary global symmetry operator.

The utility of the symmetry bundle concept is most easily seen in cases of a periodic chain of length $L$ such that $q = N$ and the global dipolar symmetry operator $D^q = D^N = 1$ seemingly disappears. Rather than saying that one has lost the symmetry and consequently the SPT order due to a poor choice of the system size, one can say that $D$ in the sense of the symmetry bundle continues to protect the SPT for any length of the chain.

# 36 Wannier-Stark and Landau-Zener



# 37 Notes on pinned wigner crystals



- we are most interested in the case where the disorder correlation length is very long! like super long! it is much more accurate to consider the situation of a very dilute number of very strong impurities. each impurity is strongly localized on a scale $\lesssim$ the lattice spacing.
- how essential is the disorder average? can we just consider the situation with a single pinning impurity / a lattice of pinning impurities?
-

We now want to construct a variational free energy that works as the best quadratic approximation to the true free energy. To this end we consider the variational Hamiltonian

$$H_\Theta = \frac{1}{2}\int u^T \Theta^{-1} u, \tag{311}$$

where $\Theta$ carries no spatial indices and has a replica structure to be determined; explicitly, $\langle u_a^i(x) u_b^j(0)\rangle = \delta^{i,j}\Theta_{ab}(x)$. To ensure that $H_\Theta$ yields a probability distribution

$p_\Theta$ which is as close as possible to the true distribution $p_e$ (in the sense of $p_\Theta$ and $p_e$ having minimal KL divergence), we chose $\Theta$ to minimize

$$F_{var} = F_e + D(p_\Theta||p_e) = F_\Theta + \langle H_e - H_\Theta\rangle_\Theta. \tag{312}$$

Since $H_\Theta$ is quadratic, we have (dropping $\Theta$-independent constants)

$$F_\Theta = -T\ln(\det\sqrt{\Theta})^m = -\frac{mT}{2}\mathrm{Tr}\ln\Theta \implies \delta F_\Theta = -\frac{mT}{2}\delta\Theta_{ab}[\Theta^{-1}]_{ba}. \tag{313}$$

Now $\langle H_\Theta\rangle_\Theta$ is $\Theta$-independent. Thus we only need to know

$$\langle H_e\rangle_\Theta = \sum_{a,b}\left(\frac{cq^2mT}{2T}\Theta_{ab} - \sum_{K\neq 0}\frac{\rho_0^2\Delta_K}{2T}e^{-\frac{K^2}{2}B_{ab}(0)}\right), \tag{314}$$

where

$$B_{ab}(x) = \frac{1}{m}\langle(u_a(x)-u_b(0))^2\rangle = T\int_q(\Theta_{aa}(q)+\Theta_{bb}(q)-2\cos(qx)\Theta_{ab}(q)). \tag{315}$$

Thus

$$\delta\langle H_e\rangle_\Theta = \delta\Theta_{ab}\left(\frac{cq^2m}{2T} - \sum_{K\neq 0}\frac{\rho_0^2\Delta_K K^2}{2}(\delta_{a,b}-1)\right), \tag{316}$$

and the condition on $\Theta_{ab}$ is

$$[\Theta^{-1}]_{ab} = cq^2 - \sum_K\frac{\rho_0^2\Delta_K K^2}{mT}(\delta_{a,b}-1)e^{-\frac{K^2}{2}B_{ab}(0)}. \tag{317}$$

Consider a replica-symmetric solution, where $B_{ab} = (1-\delta_{a,b})B$. Since $B(0)$ is then divergent in 2d, the $K>0$ modes of the disorder do not contribute. This is not what we want in a situation where the WC is pinned.

Note that here $\Delta$ has units of energy$^2$ $\times$ length$^2$. The shear modulus is

$$c = \alpha\frac{\rho_c^2 a}{\varepsilon} = \frac{4\alpha}{\pi a^2}E_C, \qquad E_C \equiv \frac{e^2}{4\pi a}, \tag{318}$$

where $\alpha \approx 0.03$ is a dimensionless constant, the value of which is obtained from classical WC calculations (Bonsall + Maradudin 75).[24]

$$R_a = \frac{ca^2}{n\sqrt{\Delta}} = \frac{4\alpha E_C}{\pi n\sqrt{\Delta}}. \tag{319}$$

$$\Sigma = c\lambda R_a^{-2}(a/\xi_0)^6, \tag{320}$$

where $\lambda$ is an $O(1)$ constant (looks like $\lambda = 1/(2\pi^2)^{1/6}$).

$$\omega_p = \frac{\Sigma}{\rho_m\omega_c} = \frac{\lambda}{\pi e}\frac{\Delta}{\xi_0^6 cB} = \frac{\lambda}{\pi}\frac{\Delta\ell_B^2}{\xi_0^6 c\hbar}. \tag{321}$$

[24]The expression obtained there is $c \approx 0.25 n^{3/2}e^2/4\pi\varepsilon$.

## Krylov methods

Let $\{|i\rangle\}$ be a set of Krylov vectors associated with a Hamiltonian $H$ and a particular initial state $|0\rangle$. To construct the Krylov vectors associated with a generic matrix, one must use a GS orthogonalization process, with

$$|i\rangle \propto \left(\mathbf{1} - \sum_{j<i} \Pi_j\right) H|j\rangle. \tag{322}$$

Since $H$ is Hermitian, the orthogonalization simplifies and one need only orthogonalize against the previous two states:

$$|i\rangle \propto (\mathbf{1} - \Pi_{i-1} - \Pi_{i-2})H|i-1\rangle. \tag{323}$$

We prove this by induction. Let $\{|j\rangle,\, j = 0, \ldots, i-1\}$ be a set of orthogonal vectors constructed as above. We claim that if $|i\rangle$ is chosen as in the above equation, then it is automatically orthogonal to all of the $|j\rangle$. Clearly $|i\rangle$ is orthogonal to $|i-1\rangle, |i-2\rangle$. So consider $|j\rangle$ with $j < i-2$. Then

$$\begin{aligned} \langle j|i\rangle &= \langle j|(\mathbf{1} - \Pi_{i-1} - \Pi_{i-2})H|i-1\rangle \\ &= \langle j|H|i-1\rangle \\ &= \langle j|H^\dagger(1 - \Pi_j - \Pi_{j-1})|i-1\rangle \\ &= \langle j+1|i-1\rangle \\ &= 0, \end{aligned} \tag{324}$$

where the last line follows from the fact that all of the $\{|j<i\rangle\}$ were assumed orthogonal. □

## Bounds on mixing and relaxation times

In this diary entry we prove two basic facts about the mixing and relaxation times of Markov chains.

Consider an irreducible aperiodic Markov chain with transition matrix $M$ and unique steady state $|\pi\rangle$. We quantify the distance between $M^t$ and $M^\infty = |\pi\rangle\langle\pi|$ in terms of the maximum total variational distance of the probability distribution $(M^t - M^\infty)|w\rangle$, with the maximum taken over initial states $|w\rangle$:

$$d(M^t, M^\infty) \equiv \max_{|w\rangle} ||M^t|w\rangle - |\pi\rangle||_{TV} = \frac{1}{2} \max_{|w\rangle} \sum_{|w\rangle'} |\langle w'|M^t|w\rangle - \langle w'|\pi\rangle| \tag{325}$$

where we used that $||p-q||_{TV} = \frac{1}{2}||p-q||_1$ for probability distributions $p, q$. The *mixing time* of $M$ is defined as the first time at which $d(M^t, M^\infty)$ drops below some fixed threshold $\varepsilon$:

$$t_{\text{mix}}(\varepsilon) \equiv \min\{t \,:\, d(M^t, M^\infty) < \varepsilon\}. \tag{326}$$

Since the exact value of $\varepsilon$ isn't so important, we will fix $t_{\text{mix}} \equiv t_{\text{mix}}(1/3)$. Roughly speaking, the mixing time is reached when the slowest-diffusing initial state $|w_s\rangle$ diffuses out to cover a $\sim 1-\varepsilon$ fraction of the state space. Suppose for example that after $t$ steps, $M^t|w_s\rangle$ is a bump function supported on a fraction $f$ of the state space. Then one easily checks that the mixing time is indeed reached when $f > 1-\varepsilon$.

We begin with the rather obvious result that $M$ cannot mix in a time shorter than the time it takes to traverse state space:

**Proposition 4.** *Let $L$ be the diameter of the state space in question. Then*

$$t_{\text{mix}} \geq L/2. \tag{327}$$

*Proof.* We will find it helpful to use an alternate characterization of $t_{\text{mix}}$ using the distance measure

$$d'(M^t, M^\infty) \equiv \max_{|w\rangle,|w'\rangle} ||M^t|w\rangle - M^t|w'\rangle||_{TV}, \tag{328}$$

which makes no explicit reference to the equilibrium distribution. We claim that

$$t_{\text{mix}}(\varepsilon) \geq \min\{t \,:\, d'(M^t, M^\infty) \leq 2\varepsilon\}. \tag{329}$$

This follows from the fact that

$$d(M^t, M^\infty) \geq \frac{1}{2} d'(M^t, M^\infty), \tag{330}$$

which is true by virtue of the triangle inequality for the TV:

$$\begin{aligned} \max_{|w\rangle,|w'\rangle} ||M^t|w\rangle - M^t|w'\rangle||_{TV} &\leq \max_{|w\rangle,|w'\rangle} \left(||M^t|w\rangle\pi||_{TV} + ||M^t|w'\rangle - |\pi\rangle||_{TV}\right) \\ &= 2d(M^t, M^\infty), \end{aligned} \tag{331}$$

so that if $d(M^t, M^\infty) \leq \varepsilon$ then for sure $d'(M^t, M^\infty) \leq 2\varepsilon$.

Now let $|w\rangle, |w'\rangle$ be two states a distance $L$ apart. Then $M^t|w\rangle, M^t|w'\rangle$ have disjoint support for all $t \leq L/2$, and hence $||M^t|w\rangle - M^t|w'\rangle||_{TV} = 1$ for all $t \leq L/2$. □

We now characterize the relaxation time of $M$ in terms of the mixing time. The former is defined by the inverse gap of $M$, viz.

$$t_{\text{rel}} \equiv \frac{1}{1-\lambda_2}, \tag{332}$$

where $\lambda_2 < 1$ is the second largest eigenvalue of $M$. The operational meaning of this is that it puts a limit on how large fluctuations can be.

The following proposition lets us bound $t_{\text{rel}}$ in terms of $t_{\text{mix}}$, and hence also in terms of $L$:

**Proposition 5.** *Assume that the equilibrium distribution of $M$ is uniform, $|\pi\rangle = |1\rangle$ with $|1\rangle \equiv \frac{1}{D}\sum_w |w\rangle$ and $D$ the dimension of state space. Then*[25]

$$t_{\rm rel} \geq \frac{t_{\rm mix}(\varepsilon)}{\ln[D/\varepsilon]}. \tag{333}$$

*Proof.* Let $M = \sum_j \lambda_j |j\rangle\langle j|$ be $M$'s eigendecomposition, with $\lambda_1 = 1$ and $|1\rangle = |\pi\rangle$. Then

$$\begin{aligned} d(M^t, M^\infty) &= \frac{1}{2}\max_{|w\rangle}\sum_{|w'\rangle}\left(\sum_{j\geq 1}\lambda_j^t \langle w|j\rangle\langle j|w'\rangle - \frac{1}{D}\right) \\ &\leq \frac{1}{2}\max_{|w\rangle}\sum_{|w'\rangle}\sum_{j>1}\lambda_j^t \langle w|j\rangle\langle j|w'\rangle \\ &\leq \frac{1}{2}\max_{|w\rangle}\sum_{|w'\rangle}\lambda_2^t\sqrt{\sum_{j>1}\langle w|j\rangle^2\sum_{k>1}\langle w'|k\rangle^2} \\ &\leq \frac{\lambda_2^t D}{2} \end{aligned} \tag{334}$$

since $\sum_{j>1}\langle w|j\rangle^2 = \langle w|(\mathbf{1} - |\pi\rangle\langle\pi|)|w\rangle \leq 1$ for all $|w\rangle$. The claim then follows upon taking $t = t_{\rm mix}(\varepsilon)$, setting the LHS above equal to $\varepsilon/2$, and using

$$\lambda_2^t = (1 - 1/t_{\rm rel})^t \leq e^{-t/t_{\rm rel}}. \tag{335}$$

□

## Models for the Mpemba effect

A system described by

$$\begin{aligned} \partial_t E_A &= -(sE_B + \lambda_A)E_A \\ \partial_t E_B &= -E_B\lambda_B \end{aligned} \tag{336}$$

displays the ME for $s = 1$ and the IME for $s = -1$ iff $\lambda_A < \lambda_B$, assuming for simplicity that $E_A = E_B$ in equilibrium (and taking $E_{A/B} = 0$ to be infinite temperature in the IME case). Unfortunately, adding a $-sE_BE_A$ term to $\partial_t E_B$ seems to kill the ME, regardless of $\lambda_{A,B}$ (might have thought one would be okay if $\lambda_B \gg 1$, but this appears not to be the case). We need to find some physical system where this kind of an equation is realistic.

An idea for this is to take $B$ to be the bulk of a FM (AFM) and $A$ the boundary with a AFM (FM) interaction. Then $B$ just does its own thing since it's the bulk, while $A$ evolves in the manner shown above; the relaxation rate cares about $B$ since it is slower when $E_B$ is smaller: the magnetic order in the bulk creates a field that frustrates the relaxation of domain walls on the boundary. Question is whether there is some balance of coupling constants whereby such a situation can be engineered in 1d as well (need to get one of the chains oblivious to the other).

An example of this can be found when $J_e = -2, J_o = 2, J' = -0.4, h_e = 2.5, h_o = 0$ and performing a quench from $T_h = 50$ and $T_c = 5$.

[25] When $\pi$ is not uniform I believe $D$ in the log should be replaced by $1/\min_{|w\rangle}\langle w|\pi\rangle$.

In antiferromagnets: a longitudinal field *does* appear to slow down the relaxation.

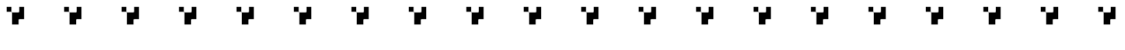
$$H = \frac{1}{2}\left(\sum_i J_e s_{2i}(s_{2i-2}+s_{2i+2}) + J_o s_{2i+1}(s_{2i-1}+s_{2i+3}) + J s_i(s_{i+1}+s_{i-1}) + 2h_e s_{2i} + 2h_o s_{2i+1}\right) \tag{337}$$

# 38 Relaxation times for Glauber dynamics on $\mathbb{Z}^d$, $\mathbb{T}_k$, and $\mathbb{G}_k(N,\alpha)$

The present diary entry is an elaboration on a collection of exercises in Mezard and Montanari's book relating to Glauber dynamics for Ising models on different types of graphs. Let $\mathbb{T}_k$ denote the degree-$k$ tree, following my—apparently ideosycratic—notation where each vertex has degree $k$ (and not $k+1$). Let also $\mathbb{G}_k(N,\alpha)$ denote the ensemble of Erdos-Renyi random factor graphs: each graph contains $N$ spins, with each of the $\binom{N}{k}$ distinct $k$-body interactions included in the factor graph with probability $\alpha$. Our aim will be to argue that the relaxation time $\tau$ of Glauber dynamics in the ordered phase of these models satisfies[26]

$$\tau \sim \begin{cases} \Theta(N^0) & \mathbb{Z}^1 = \mathbb{T}_2 \\ \Theta(e^{\beta c N^{d-1}}) & \mathbb{Z}^{d>1} \\ \Theta(e^{\beta N}) & \mathbb{G}_k(N,\alpha N) \\ \Theta(N^{c\beta}) & \mathbb{T}_{k>2} \end{cases} \tag{338}$$

We will not actually show the $\Theta$s on the RHS, we will rather only show $\Omega$s.



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The arguments will all proceed using (the easy side of) Cheeger's inequality, which says that for any subset $A$ of configuration space which has probability weight $\pi(A)$ in equilibrium,

$$\tau \geq 2\frac{\pi(A)(1-\pi(A))}{P(A \to A^c)}, \tag{339}$$

where

$$P(A \to A^c) = \sum_{s\in A,\, s'\in A^c} \pi(s) P(s \to s'), \tag{340}$$

where $P(s \to s')$ is the probability that Glauber dynamics proceeds from $s$ to $s'$, given that it starts in $s$. The name of the game is to then find an appropriate choice of $A$ that makes the RHS of the above bound as large as possible. In all of what follows, we will choose $A$ to be the set of all spin configurations with positive magnetization, and will choose $N$ to be odd so that $A^c$ is the set of configurations with negative magnetization. Then a crude bound for $P(A \to A^c)$ is

$$P(A \to A^c) \leq \frac{N}{2}\pi(M_{+1}), \tag{341}$$

[26] The number of spins in the system is always $N$; here by $\mathbb{Z}^d$ we really mean a $d$-dimensional torus with linear size $N^{1/d}$.

where $\pi(M_{+1})$ is the total probability mass assigned to configurations with magnetization exactly equal to $+1$ (this bound follows from Glauber dynamics flipping only one spin at each step; there are thus at most $N/2$ ways to transition from $x \in A$ to $y \in A^c$, and each of these transitions happens with probability at most 1 [conditioned on being in $x$]). So then

$$\tau \geq \frac{1}{N\pi(M_{+1})}, \tag{342}$$

where we used $\pi(A) = 1/2$ by symmetry. We may write this as

$$\tau \geq \frac{1}{N} e^{\beta(F_{+1}-F)}, \tag{343}$$

where $F$ is the free energy and

$$F_{+1} = -\beta^{-1} \ln \left( \sum_{s\,:\sum_i s_i = 1} e^{-\beta E(s)} \right) \tag{344}$$

is the free energy restricted to configurations with magnetization $+1$. Thus the bound on $\tau$ comes simply from the free energy cost of imposing a near-zero magnetization. In most ordered phases imposing zero magnetization is done by inserting a domain wall bisecting the system, so that $\tau$ is essentially lower-bounded by the inverse probability of nucleating such a domain wall.

### *Lattices*

On $\mathbb{Z}$ there is no ordered phase, and entropy always wins out over energy. The free energy cost of maintaining zero magnetization is thus the entropy cost of fixing the domain wall positions in such a way that the magnetization cancels; since domain walls have $\log(N)$ entropy the free energy cost is also $\log(N)$. Since this is a purely entropic effect its contribution to $\beta(F_{+1} - F)$ is $\beta$-independent, and we get[27]

$$\tau_{\mathbb{Z}} = \Omega(N^0). \tag{345}$$

Of course we know that it should really be $\Theta(N^0)$, but out of laziness we will not give a matching upper bound (the same remark applies to all of the following examples).

On $\mathbb{Z}^{d>1}$ energy is minimized at large enough $\beta$; here $F_{+1} - F$ corresponds to the energy cost of inserting a domain wall that bisects the system. Thus

$$\tau_{\mathbb{Z}^{d>1}} \geq \frac{1}{N} e^{c\beta N^{1-1/d}} \tag{346}$$

for some $O(1)$ constant $c$.

[27] This looks like the correct lower bound, but that is because we are using the math definition of relaxation time; with the physicist's definition (taking advantage of the fact that Glauber updates can be applied in parallel) we would get $\Omega(1/N)$. This would in turn be fixed by realizing that our bound of $P(A \to A^c)$ was too crude: while there are $N/2$ places to flip a spin, the probabilities for each of these to occur will be $\Theta(1/N)$; including this gives a physicist-definition bound of $\Omega(1)$. We will not bother with keeping track of this below since it is the $N$-dependence of $F_{+1} - F$ that we really care about.

*Trees*

The situation on the (finite) tree is rather subtle: on one hand $\mathbb{T}_k$ resembles $\mathbb{Z}$ in some respects (since it can be bisected by cutting a single link; this implies certain domain wall energies will be only $O(1)$, like on $\mathbb{Z}$), while in other respects it resembles a random graph (because of the locally constant expansion, and because of soon-to-be-mentioned entropic considerations).

*Random graphs*

The case of $\mathbb{G}_k(N,\alpha)$ is most interesting since although these random graphs look like trees locally (and nearly globally), the relaxation times scale exponentially with $N$, rather than polynomially (like the tree) or quasi-exponentially (like $\mathbb{Z}^{d>1}$). This ultimately comes down to the fact that while finite random graphs and finite trees are locally very similar, finite trees are maximally bad expanders (entirely because of their boundaries), while random graphs—at least those from the ensembles we are considering—are good ones.

In what follows we will bound the typical relaxation time, viz. we will give a bound that is satisfied by $\tau$ with probability 1 over the choice of $G \in \mathbb{G}_k(N,\alpha)$ in the $N \to \infty$ limit.

# 39 Entangled pair representation for $\mathbb{Z}_N^2$ SPTs

In today's entry we will do a short calculation of the unitary which transforms the standard fixed-point wavefunction for the $\mathbb{Z}_N \times \mathbb{Z}_N$ SPT labeled by the cohomology class $\omega \in \mathbb{Z}_N$ from the "standard" representation into the entangled pairs representation. Here the standard representation is the one in which the symmetry operator at each pair of sites is

$$R_g = Z^{g_1} \otimes Z^{g_2}. \tag{347}$$

To make our lives easier we will assume $\gcd(\omega, N) = 1$.

In the entangled pairs representation, the representation of the symmetry is $(V_g^{(\omega)})^* \otimes V_g^{(\omega)}$, where for $\mathbb{Z}_N$ the basis we will choose for the projective representation is

$$V_g^{(\omega)} = X^{g_1} Z^{\omega g_2}. \tag{348}$$

We want to then find the unitary $U_\omega$ which satisfies

$$U_\omega R_g U_\omega^\dagger = X^{g_1} Z^{-\omega g_2} \otimes X^{g_1} Z^{\omega g_2}. \tag{349}$$

We will find it useful to define the matrix

$$F_\alpha \equiv \sum_{a,b \in \mathbb{Z}_N} \zeta_N^{-ab\alpha} |a\rangle\langle b|, \tag{350}$$

which is unitary if $\alpha \in \mathbb{Z}_N^\times$. This matrix satisfies

$$F_\alpha Z = X^{1/\alpha} F_\alpha, \qquad F_\alpha X = Z^{-\alpha} F_\alpha. \tag{351}$$

We will write $U_\omega$ in terms of the $F_\alpha$ and the $\mathbb{Z}_N$ version of the CZ gate, defined in the obvious way as

$$CZ_N = \sum_{h_1,h_2} \zeta^{-h_1 h_2} |h_1, h_2\rangle\langle h_1, h_2|. \tag{352}$$

**Proposition 6.** *The unitary in question is*

$$U_\omega = (\mathbf{1} \otimes F_\omega) \circ (CZ_N)^\omega \circ (F_1 \otimes F_1). \tag{353}$$

*Proof.* This is of course just simple algebra. After the first step,

$$F_1^{\otimes 2} R_g = (X^{g_1} \otimes X^{g_2}) F_1^{\otimes 2}. \tag{354}$$

Now one readily checks that

$$(CZ_N)^\omega (X^{g_1} \otimes X^{g_2}) = \zeta^{\omega g_1 g_2} (Z^{-\omega g_2} X^{g_1} \otimes Z^{-\omega g_1} X^{g_2}) (CZ_N)^\omega = (X^{g_1} Z^{-\omega g_2} \otimes Z^{-\omega g_1} X^{g_2}) (CZ_N)^\omega. \tag{355}$$

Finally, hitting this with $\mathbf{1} \otimes F_{-\omega}$, we have

$$F_{-\omega} Z^{-\omega g_1} X^{g_2} = X^{g_1} Z^{\omega g_2}. \tag{356}$$

Putting the three steps together gives the desired action of $U_\omega$. □

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