

CMT diary

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Preface:

This is a diary containing worked-out problems in condensed matter theory. These problems are either elaborations on calculations in papers which I wanted to work out in detail, explanations of well-known facts that I wanted to remember, problems which arose when doing research, or problems assigned in grad classes. There are doubtless many typos, and I have not been very diligent about adding citations. Moreover, some entries were written near the beginning of grad school, when I didn't understand much about condensed matter physics. I take no responsibility for any misguided beliefs that my younger self decided to write down.

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Partons in spin systems and a parton analysis of a single spin

To examine exact what sort of approximations are made when doing partons and mean field theory, we will take a look at what happens when we use this technology to compute the free energy of a single $SU(2)$ spin in 0+1D. By comparing with the exact answer (0+1D is the only dimension in which this is really possible), we will be able to see which aspects of the thermodynamics are missed by the partons at the mean field level.



Basics about the parton approach applied to spin systems

In today's diary entry we will mostly be interested in studying $SU(2)$ spins in the representation s . If we want to fractionalize the spin operator in terms of Schwinger bosons, we write¹

$$S_\alpha^\beta = b_\alpha^\dagger b_\alpha^\beta - s\delta_\alpha^\beta, \quad (1)$$

where we are constrained to work in the subspace in which

$$b_\alpha^\dagger b_\alpha^\alpha = 2s, \quad (2)$$

which ensures that the S are properly traceless and cuts down the Hilbert space to the correct dimension of $2s + 1$. Here the spin components are

$$S^i = \frac{1}{2} \text{Tr}[S\sigma^i], \quad (3)$$

and with this one checks that $\mathbf{S} \cdot \mathbf{S} = S(S+1)\mathbf{1}$ when acting on the constrained Hilbert space: indeed, using $\sigma_{\alpha\beta}^a \sigma_{\gamma\delta}^a = 2(\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{2}\delta_{\alpha\beta}\delta_{\gamma\delta})$,

$$\mathbf{S} \cdot \mathbf{S} = \frac{1}{2} S_\alpha^\beta S_\gamma^\delta \left(\delta_\alpha^\delta \delta_\beta^\gamma - \frac{1}{2} \delta_\alpha^\beta \delta_\gamma^\delta \right) = \frac{1}{2} \text{Tr}[S^2] = \frac{1}{2} \left(-4s^2 + 2s^2 + b_\alpha^\dagger (1 + b_\beta^\dagger b_\beta^\beta) b_\alpha^\alpha \right) = s(s+1) \checkmark \quad (4)$$

where we have been careful to only use $b_\alpha^\dagger b_\alpha^\alpha = 2s$ when acting on \mathcal{H}_{phys} : this is *not* an operator identity, and so in particular we cannot replace the $b_\beta^\dagger b_\beta^\beta$ in the inner parenthesis in the fourth term above with $2s$.

Note also that we could also fractionalize the spin into fermions. In this case, instead of the normalization condition (2), the s -dependence enters by introducing a flavor index $a = 1, \dots, 2s$, with

$$S_\alpha^\beta = c_{\alpha a}^\dagger c_{\alpha a}^\beta - s\delta_\alpha^\beta, \quad (5)$$

where we are required to work in the $c_{\alpha a}^\dagger c_{\alpha a}^{\alpha b} = \delta_a^b$ subspace. Checking the commutation relations and $\mathbf{S} \cdot \mathbf{S} = s(s+1)\mathbf{1}$ on \mathcal{H}_{phys} is done similarly to the bosonic case.²

¹In this diary entry all lower indicies are row indicies.

²The constrained Hilbert space for the fermions seems a bit wonkier though, and when realizing the constraints through Lagrange multipliers it seems to be easier to use the bosonic representation.

The difference in representation here actually is actually just the $N = 2$ case of a general relation between representations of $SU(N)$ spins: $SU(N)$ operators acting on a Hilbert space transforming in a representation whose Young tableaux has n columns and m rows can be represented either by m flavors³ of bosons satisfying $b_{\alpha a}^\dagger b^{\alpha c} = n\delta_a^c$, or by n flavors of fermions satisfying $c_{\alpha a}^\dagger c^{\alpha c} = m\delta_a^c$ —see Ref. [13] for a good discussion of this. The equivalence between these two representations essentially boils down to the fact that columns in Young tableaux represent antisymmetrization while rows represent symmetrization; hence exchanging bosons with fermions swaps rows and columns.

In a given situation, how do we know whether we should use bosonic or fermionic partons? Consider for example the $SU(2)$ Heisenberg Hamiltonian on a bipartite lattice. Then we have

$$H \sim -J \sum_{\langle ij \rangle} a_{i\alpha}^\dagger a_{i\beta} a_{j\beta}^\dagger a_{j\alpha} \sim -J \sum_{\langle ij \rangle} (-1)^\eta a_{i\alpha}^\dagger a_{j\alpha} a_{j\beta}^\dagger a_{i\beta} \equiv -J \sum_{\langle ij \rangle} B_{ij}^\dagger B_{ij} \quad (6)$$

where \sim denotes equality up to constants and $(-1)^\eta$ is -1 for fermions and $+1$ for bosons.

If we have a ferromagnet and the a s are bosons, then H is proportional to a sum of negative-definite terms. This is good, because it allows us to decouple the interaction by integrating in a bosonic HS field (we need to integrate in something like $\int \mathcal{D}A_{ij} e^{-A_{ij}^2}$; for $A_{ij} \in \mathbb{R}$ the decoupling only works if the interaction in H has a minus sign), and it also tells us physically what the system likes to do: it likes to have links in the $+1$ eigenstate of the “triplet operator” $B_{ij} = a_{i\uparrow}^\dagger a_{j\uparrow} + a_{i\downarrow}^\dagger a_{j\downarrow}$. The same is true if we have an antiferromagnet and the a s are fermions.

If we e.g. have an antiferromagnet and want to use bosons, this construction does not work. Instead, we perform a PH transformation on the B sublattice by sending $\mathbf{S}_B \mapsto J^\dagger \mathbf{S}_B J$ (with $J = -iY$ as usual). This has the effect of sending $\sigma \mapsto -\sigma^*$ on the B sublattice, so that

$$H \sim +J \sum_{\langle ij \rangle} a_{i\alpha}^\dagger a_{i\beta} a_{j\alpha}^\dagger a_{j\beta} \sim J \sum_{\langle ij \rangle} a_{i\alpha}^\dagger a_{j\alpha}^\dagger a_{j\beta} a_{i\beta} \equiv J \sum_{\langle ij \rangle} A_{ij}^\dagger A_{ij}, \quad (7)$$

where we have used that the matrix elements of \otimes s of generators for the fundamental and its conjugate are given as

$$\sigma_{\alpha\beta}^a (\sigma^a)_{\gamma\delta}^* = 2 \left(\delta_{\alpha\gamma} \delta_{\beta\delta} - \frac{1}{2} \delta_{\alpha\beta} \delta_{\gamma\delta} \right). \quad (8)$$

The interaction can then be decoupled with a \mathbb{R} HS field if $J < 0$. Note that in the original unrotated basis the operator A_{ij} is $A_{ij} = a_{i\uparrow} a_{j\downarrow} - a_{i\downarrow} a_{j\uparrow}$, which gives us a concrete way of understanding why antiferromagnets like spins to be locked up in singlets.

Parton analysis of a single spin

Bosonic partons

Now we will look at the single-spin path integral, first starting in the Schwinger boson representation. We will choose the magnetic field to point along the Z direction, so that the

³Meaning that the flavor index runs from 1 to m ; adding in the physical $SU(N)$ index gives a total of Nm bosons.

field term is (in our normalization of H) $H\text{Tr}[SZ]/2 = Hb^\dagger Zb/2$. Writing the gauge field as $\lambda - ia$ with λ the saddle point, the partition function is

$$\begin{aligned} Z &= \int \mathcal{D}b(\tau) \mathcal{D}a(\tau) \exp \left(- \int d\tau [b^\dagger (\partial_\tau + \lambda - ia - HZ/2)b - 2s(\lambda - ia)] \right) \\ &= \int \mathcal{D}a(\tau) e^{-S_{eff}[a]}, \end{aligned} \quad (9)$$

with

$$S_{eff}[a(\tau)] = \sum_{\sigma=\pm 1} \int d\tau \ln(\partial_\tau + \lambda - ia - \sigma H/2) - 2\bar{\lambda}s + 2is \int d\tau a, \quad (10)$$

where $\bar{\lambda} \equiv \lambda/T$. Note that the time derivative appears here as ∂_τ and *not* $2s\partial_\tau$, which we might have naively thought would be the case if we thought about the WZW term arising when quantizing a particle on a sphere. The mean-field equation has $\bar{a} = 0$ with λ determined by the constraint on the total density:

$$2s = \sum_{\sigma=\pm 1} n_B(\lambda + \sigma h/2), \quad (11)$$

where in this section $h \equiv H/T$. Defining $q \equiv e^{\bar{\lambda}}$, we rewrite the above as a quadratic equation for q in terms of s, h :

$$(l-1)q^2 - 2lq \cosh(h/2) + l+1 = 0, \quad l \equiv 2s+1. \quad (12)$$

The solution is

$$q = \frac{1}{1-1/l} \left(\cosh(h/2) + \sqrt{l^{-2} + \sinh^2(h/2)} \right). \quad (13)$$

At large h this is $e^{h/2}/(1-1/l)$, while at $h=0$ it is $(l+1)/(l-1)$. When evaluated on the mean-field solution, the free energy is

$$f_{MF}/T = \ln[(1 - e^{-h/2}/q)(1 - e^{h/2}/q)] - 2s\bar{\lambda}. \quad (14)$$

Therefore we can write the mean-field partition function as

$$Z_{MF} = \sum_{n,m=0}^{\infty} \exp(-\bar{\lambda}(n+m-2s) + h(n-m)/2). \quad (15)$$

Since this is a product of up-spin and down-spin partition functions, it receives contributions from an infinite number of Boltzmann weights, with terms proportional to $e^{kh/2}$ for all $k \in \mathbb{Z}$.

The mean-field approximation becomes exact in the large- N limit, where we add a flavor index $a = 1, \dots, N$ to the Schwinger bosons. To generalize what we have done so far to $N > 1$, we need only replace S_{eff} with NS_{eff} . In the following we will study the theory at different values of N , which allows us to interpolate between the exact $N = 1$ solution and the MF approximation.

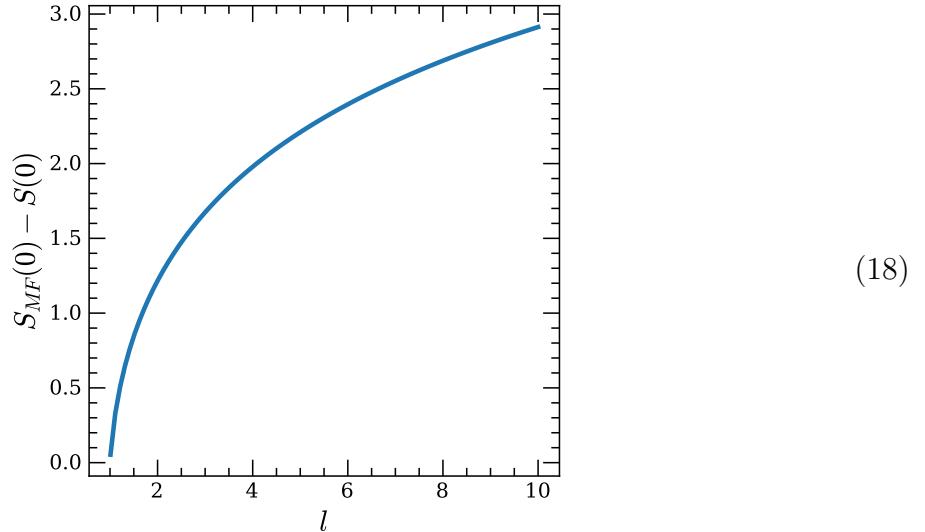
The mean-field and exact ($N = 1$) entropies are

$$\begin{aligned} S_{MF,s} &= -\ln \left(1 + \frac{1}{q^2} - \frac{2}{q} \cosh(h/2) \right) + \frac{2q \ln(q) \cosh(h/2) - qh \sinh(h/2) - 2 \ln(q)}{1 + q^2 - 2q \cosh(h/2)}, \\ S_{1,s} &= \ln \left(\sum_{n=0}^{2s} e^{h(m-s)} \right) - \frac{\sum_{m=0}^{2s} (m-s) h e^{h(m-s)}}{\sum_{n=0}^{2s} e^{h(n-s)}} \end{aligned} \quad (16)$$

with $S_{MF,s} \equiv \lim_{N \rightarrow \infty} S_{N,s}/N$. To see the “problems” with S_{MF} , consider first the limit $h \rightarrow 0$. In this limit, $q \rightarrow (l+1)/(l-1)$, and after some algebra we find

$$S_{MF,s}(h=0) = \ln \left(\frac{(1+l)^2}{4} \right) + (l-1) \ln \left(\frac{l+1}{l-1} \right). \quad (17)$$

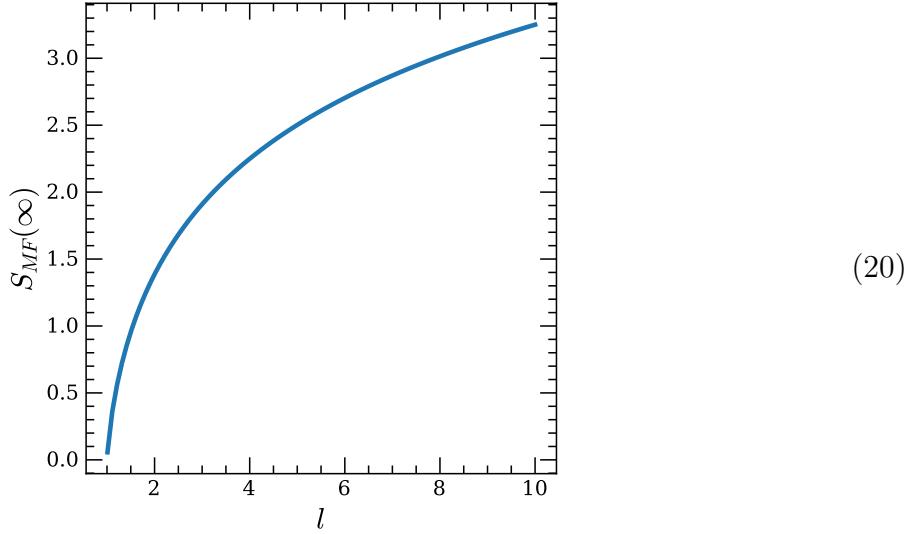
Of course, the exact answer is $S(h=0) = \ln(l)$. The difference between these two functions looks like



$S_{MF,s}$ also gives unphysical results at large fields. Here we have

$$S_{MF,s}(h \gg 1) \approx \ln(l) + (l-1) \ln \left(\frac{l}{l-1} \right) + \mathcal{O}(e^{-h}), \quad (19)$$

which looks almost the same as $S_{MF,s}(0) - S_{1,s}(0)$:



The fact that we get a constant as $h \rightarrow \infty$ is unphysical. Here by ‘unphysical’ we mean ‘not a feature of the $N = 1$ model’. In fact this $h \rightarrow \infty$ entropy is a completely physical feature of the $N > 1$ models, in that it is present even after exactly integrating out the gauge field. This is just due to the fact that taking $N > 1$ gives $N > 1$ degenerate flavors of bosons in the large h limit, which has nonzero entropy (which cannot be taken care of within the $N > 1$ models without breaking the large- N structure of the effective action). To better understand these issues, we now turn to calculating the exact partition function at general N .

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The exact partition function is obtained by integrating over the ‘fluctuations’ of the gauge field. The most convenient choice of gauge is one in which $\partial_\tau a = 0$, so that $a(\tau) = a$ is just a constant. Since in this gauge a is a constant, to obtain the full partition function we need only replace $\bar{\lambda} \mapsto \bar{\lambda} - i\bar{a}$ in Z_{MF} , and then integrate over a .⁴ With $N = 1$, we expand the terms in the log appearing in f_{MF} in a power series and get

$$Z_{1,s} = \int d\bar{a} \sum_{n,m=0}^{\infty} \exp \left(-(\bar{\lambda} - i\bar{a})(n + m - 2s) + h(n - m)/2 \right). \quad (21)$$

Note that the integrand is invariant under $\bar{a} \mapsto \bar{a} + 2\pi$, as required. The integral over the zero mode a then cuts down the infinite number of summands appearing in Z_{MF} to just those allowed by spin quantization and doing the integral gives the required result, viz.

$$Z_{1,s} = \sum_{n=0}^{2s} e^{h(n-s)}. \quad (22)$$

⁴Technically we also need to add the $\det(-\partial_\tau^2)$ from the gauge-fixing; however since this is h -independent it will be ignored.

For larger N , the combinatorics is slightly more complicated. For general N , we have

$$Z_{N,s} = \int d\bar{a} \left[\sum_{n,m=0}^{\infty} \exp(i\bar{a}(n+m) + h(n-m)/2) \right]^N e^{-i2sN\bar{a}}. \quad (23)$$

We then need to select out the terms in the brackets whose total power of $e^{i\bar{a}}$ is $2sN$ and determine their h dependence. The combinatorics works out as follows. We can think about selecting terms from the expansion by looking at the number of ways to have a total occupancy of $2sN$ in a system with $2N$ flavors of bosons. Here the bosons are divided into two groups A and B , with A the first N flavors and B the second N flavors. Each A boson comes with a factor of $r \equiv e^{h/2}$, and each B boson comes with a factor of r^{-1} . Each allowed configuration of $2sN$ bosons then comes with a factor of $r^{i-(2sN-i)}$, with $i \in \mathbb{Z}_{2sN}$ representing the number of A bosons. The number of ways to give the N bosons in A total filling i is $\binom{N+i-1}{N-1}$, with the corresponding combinatorial factor for the B bosons being $\binom{N+2sN-i-1}{N-1}$. Therefore the partition function is

$$Z_{N,s} = \sum_{i=0}^{2sN} r^{2i-2sN} \binom{N+i-1}{N-1} \binom{N+2sN-i-1}{N-1}. \quad (24)$$

For $s = 1/2$ and small N , some more explicit expressions are

$$\begin{aligned} Z_{2,1/2} &= 3(r^2 + r^{-2}) + 4 \\ Z_{3,1/2} &= 10(r^3 + r^{-3}) + 18(r + r^{-1}) \\ Z_{4,1/2} &= 35(r^4 + r^{-4}) + 80(r^2 + r^{-2}) + 100. \end{aligned} \quad (25)$$

Entropies are then obtained from the above expressions by using $S = -f/T - \frac{hr}{2}\partial_r(f/T)$. As an example, one finds

$$S_{2,1/2} = \ln(3(r^2 + r^{-2}) + 4) - 3h \left(\frac{r^2 + r^{-2}}{3(r^2 + r^{-2}) + 4} \right). \quad (26)$$

In particular, the *exact* $h \rightarrow \infty$ entropy is non-zero for all $N > 1$. Indeed, as $h \rightarrow \infty$ all the bosons which create spins not parallel to the field disappear from the thermodynamics, but this still leaves N degenerate species of bosons which do not get projected out. The $h \rightarrow \infty$ entropy is then just the log of the number of ways to have a total occupation number of $2sN$ in a system of N bosons, which is $\binom{lN-1}{N-1}$ with $l = 2s + 1$ as before. Therefore

$$S_{N,s}(h \rightarrow \infty) = \ln[(lN-1)!] - \ln[((l-1)N)!] - \ln[(N-1)!], \quad (27)$$

and at large N ,

$$S_{N \rightarrow \infty, s}(h \rightarrow \infty) \approx lN \ln \left(\frac{lN-1}{(l-1)N} \right) + N \ln \left(\frac{(l-1)N}{N-1} \right) + \ln \left(\frac{N-1}{lN-1} \right) \quad (28)$$

so that

$$S_{MF,s}(h \rightarrow \infty) = l \ln l - (l-1) \ln(l-1), \quad (29)$$

which agrees with the result above.

The entropy in the other limit of $h = 0$ is obtained by counting the number of ways for $2N$ flavors of bosons to have an occupation number of $2sN$, so that

$$S_{N,s}(h = 0) = \ln[(lN + N - 1)!] - \ln[((l - 1)N)!] - \ln[(2N - 1)!], \quad (30)$$

which at large N gives

$$S_{N \rightarrow \infty, s}(h = 0)/N \approx Nl \ln \left(\frac{(l + 1)N - 1}{(l - 1)N} \right) + N \ln \left(\frac{((l + 1)N - 1)(l - 1)N}{(2N - 1)^2} \right), \quad (31)$$

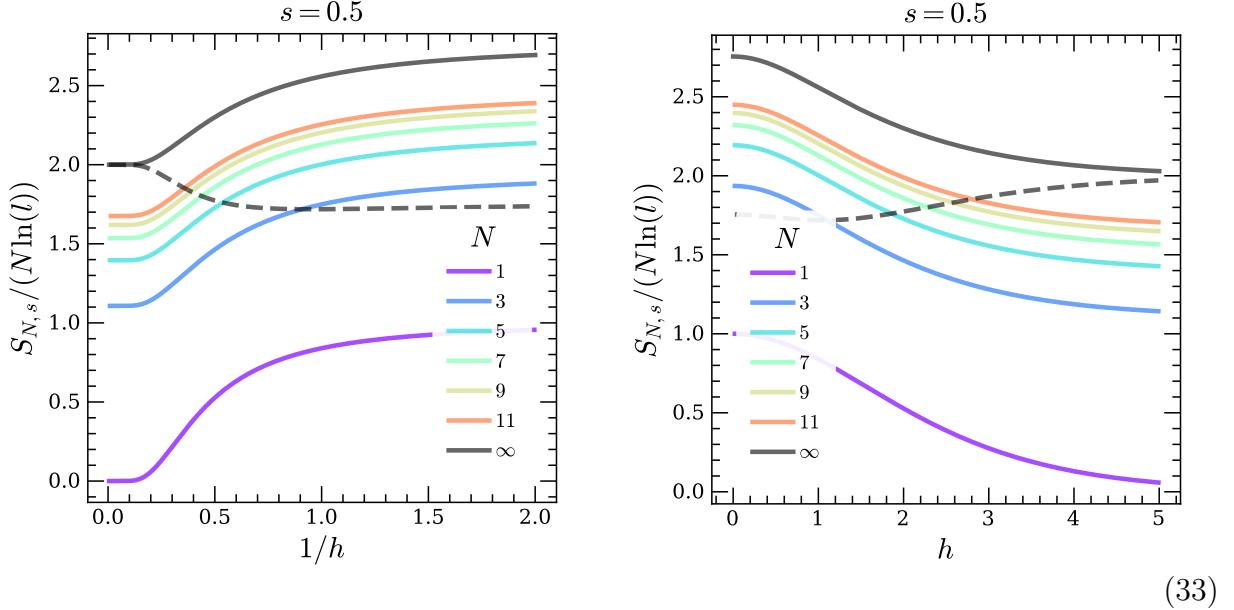
which agrees with the MF result derived above.

These results can also be derived in a potentially clearer way without ever bringing in the notion of N . It is easy to show that a single species of boson in 0+1D with $\langle b^\dagger b \rangle = n$ has entropy

$$S(n) = \ln(1 + n) + n \ln(1 + 1/n). \quad (32)$$

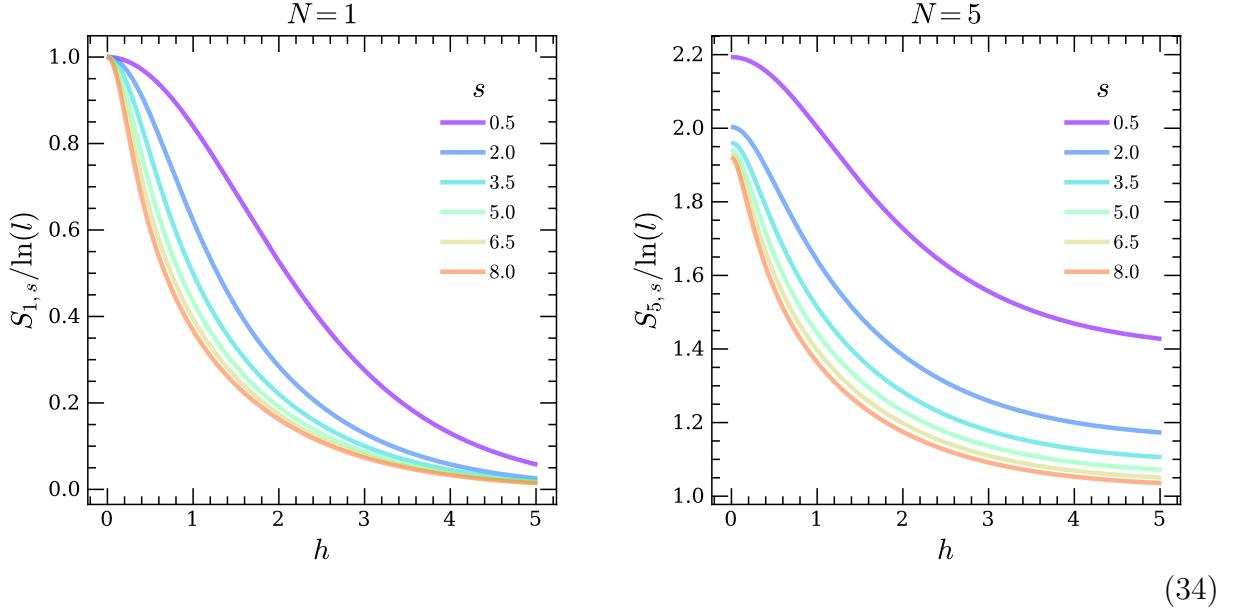
At $h \rightarrow \infty$ mean field (in the $N = 1$ model of physical interest) we have a single species of boson with $n = 2s$, while at $h = 0$ we have two species of bosons each with $n = s$. Plugging these values into the above equation then produces the $N \rightarrow \infty$ results above.

In pictures, the entropy as a function of N for $s = 1/2$ is



where the dashed line is the difference between the mean field and $N = 1$ results. For

$N = 1, 5$ as a function of s , we get



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Let us also briefly discuss the magnetization. Unlike the entropy, the magnetization at $N > 1$ provides a good estimate of the $N = 1$ magnetization, and its large- and small- h limits are physical, and the unphysical N -fold degeneracy of each σ^z eigenvalue can be completely accounted for by dividing the magnetization by N .

The mean-field and $N = 1$ magnetizations are calculated as

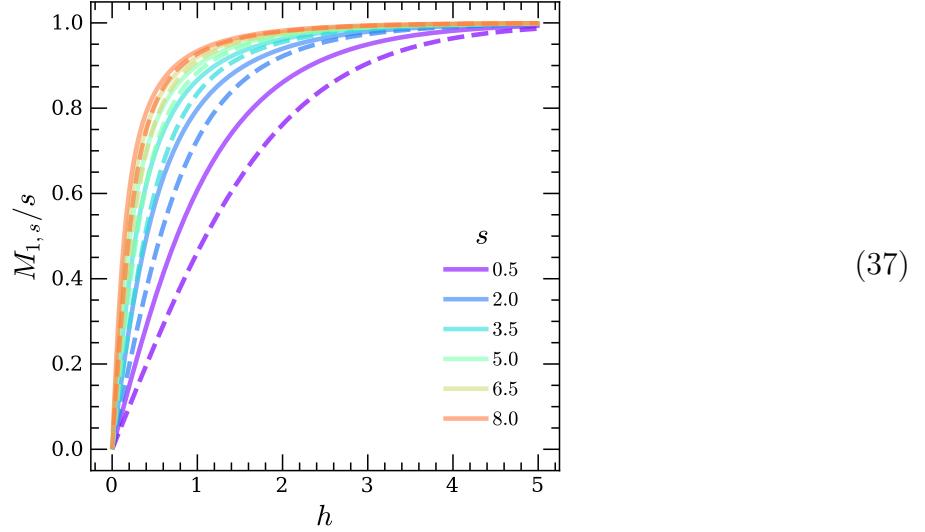
$$M_{MF,s} = \frac{1}{2} \left(\frac{1}{qe^{-h/2} - 1} - \frac{1}{qe^{h/2} - 1} \right),$$

$$M_{1,s} = \frac{\sum_{m=0}^{2s} (m-s) e^{h(m-s)}}{\sum_{n=0}^{2s} e^{h(n-s)}}.$$
(35)

For $h \gg 1$, the mean field solution gives

$$M_{MF}(h \gg 1) \approx \frac{1}{2} \frac{1}{\frac{1}{1-e^{-h}} - 1} + \mathcal{O}(e^{-h}) = s + \mathcal{O}(e^{-h}),$$
(36)

which is the expected large-field limit. Plotting M_{MF} as solid lines and M as dashed lines,



so that M_{MF} indeed does better at large s , as expected. Also note that $M_{MF} > M$ for all h at fixed s ; this is in agreement with the fact that including fluctuations about the saddle point will reduce the magnetization.

Fermionic partons

Suppose now that we break apart the spin into fermions. The structure of the Fock space for fermions means that the mean-field solution will not have any entropy in the $h \rightarrow \infty$ limit, unlike the case with bosons. The disadvantage is that the mean-field magnetization and the field-dependence of the mean-field entropy at intermediate fields are worse than the bosonic results.

To work with $s > 1/2$ this requires introducing a flavor index (even at $N = 1$), and the constraints become rather awkward. Therefore we will specialize to $s = 1/2$ for now.

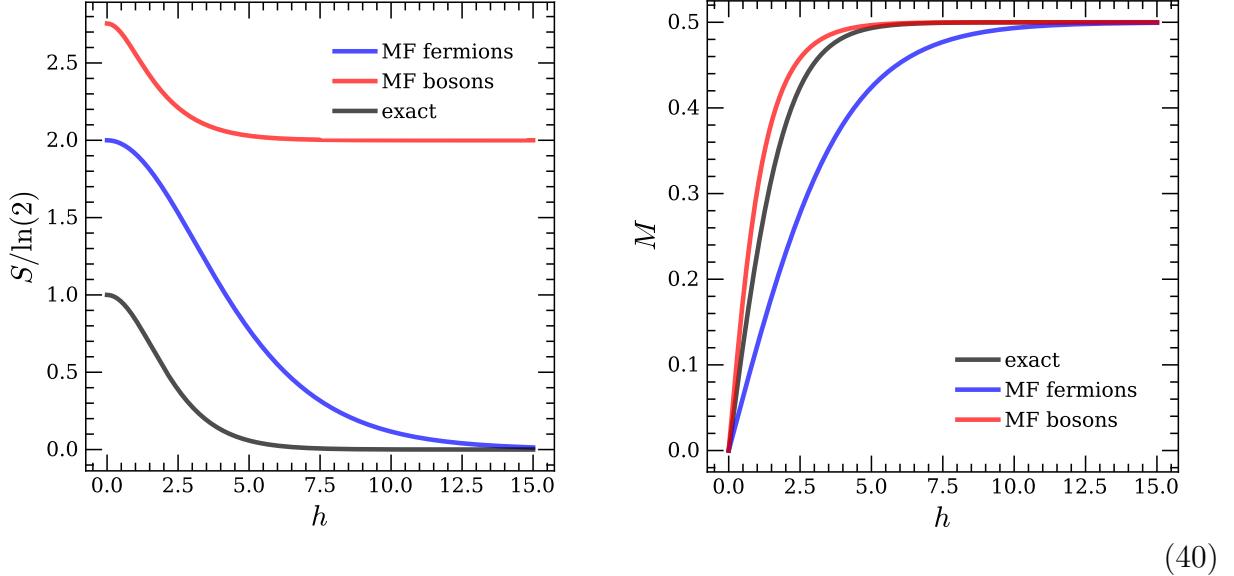
The mean-field equation, viz. $1 = n_F(\lambda + h/2) + n_F(\lambda - h/2)$, is very simple; since $n_F(x) + n_F(-x) = 1$ we just have $\bar{\lambda} = 0$. The mean field free energy, entropy, and magnetization are

$$\begin{aligned} f_{MF} &= -T \ln(2) - T \ln(1 + \cosh(h/2)) \\ S_{MF} &= \ln(2) + \ln(1 + \cosh(h/2)) - \frac{h}{2} \frac{\sinh(h/2)}{1 + \cosh(h/2)} \\ M_{MF} &= \frac{\sinh(h/2)}{2(1 + \cosh(h/2))}. \end{aligned} \quad (38)$$

The exact partition function is recovered by integrating over a as in the previous section:

$$Z = \int da e^{-ia} (1 + e^{-h/2+ia}) (1 + e^{h/2+ia}) = 2 \cosh(h/2) \quad \checkmark \quad (39)$$

These functions look like



While the fermions don't suffer from the same $h \rightarrow \infty$ entropy that the bosons do, if this part is subtracted off, the bosons do a better job at capturing the functional form of $S(h)$.



Anomalies in $SO(3) \times T$ symmetric \mathbb{Z}_2 spin liquids

Today we will consider a gapped \mathbb{Z}_2 spin liquid in 2+1D, enriched with $SO(3)$ symmetry and time-reversal T . We will show that as long as T is preserved, assigning both e and m spin 1/2 under $SO(3)$ leads to a t' Hooft anomaly. We will then find out how to cancel the anomaly with a 3+1D bulk. This is a rather long-winded elaboration on a problem assigned in Senthil's 2019 class on correlated electrons.



As a field theory, the action for the case where both e and m particles carry integer spin can be written (the notation for the action here is $\bar{S} \equiv S/2\pi$)

$$\bar{S} = \frac{1}{2} \int a \cup \delta b. \quad (41)$$

Here the Poincare dual of δa is the worldline of an e particle, while that of δb is the worldline of an m particle; we will use a and e quasi-interchangably, and likewise for b and m .

Now suppose that e.g. e has spin 1/2. This means that an a worldline $e^{i\oint a}$ is not gauge invariant under certain gauge transformations in $SO(3)$ (i.e., certain re-arrangements of the transition functions in the $SO(3)$ bundle). In this context, consider a gauge transformation which changes the $SO(3)$ bundle in a manner such that the product of transition functions along a closed loop changes by the nontrivial element of $\pi_1(SO(3)) = \mathbb{Z}_2$. Then since the spin of a is 1/2, a Wilson line for a , when wrapped around this loop, changes by a minus sign (remember that the Wilson lines include the transition functions between patches); see figure 1.

Now consider the theory in the background of an $SO(3)$ monopole, and let ω_2 be the second SW class of the $SO(3)$ bundle. The Poincare dual of ω_2 is the Dirac string coming out of the monopole; under the gauge transformations described above, this string changes shape. Therefore, we see that such a gauge transformation sends

$$a \mapsto a + \lambda, \quad \omega_2 \mapsto \omega_2 + \delta\lambda, \tag{42}$$

with λ some \mathbb{Z}_2 1-cochain.

We can make the action invariant under this transformation if we couple b to ω_2 :

$$\bar{S} = \frac{1}{2} \int_X b \cup (\delta a - \omega_2). \tag{43}$$

Note that the modification of the action is only non-trivial when ω_2 is nontrivial in $H^2(X; \mathbb{Z}_2)$, i.e. when the $SO(3)$ gauge configuration has a monopole. This is the usual story for t' Hooft anomalies, which are usually only activated by topologically non-trivial background field configurations.

A physical perspective on why this is the right thing to do is the following: in order for the gauge field to be a legit $SO(3)$ gauge field, the Dirac string emanating from the monopole needs to be invisible. When a has spin 1/2 the string is not invisible, since an a line will detect it. However, consider adding the coupling

$$\frac{1}{2} \int b \cup \omega_2 = \frac{1}{2} \int_{\widehat{\omega}_2} b, \tag{44}$$

where $\widehat{\omega}_2$ is the Poincare dual of ω_2 , alias the Dirac string. We see that the Dirac string $\widehat{\omega}_2$ now carries b charge, which because of the braiding phase between e and m , renders the $\widehat{\omega}_2$ worldline properly transparent.

This is all well and good, and the same construction works if b has spin 1/2 while a has integer spin, provided that we instead add the coupling $\frac{1}{2} \int a \cup \omega_2$ to the action.

Problems occur when we take both e and m to have spin 1/2. On one hand, we can check explicitly that the putative action

$$\bar{S} = \frac{1}{2} \int (a \cup \delta b - (a + b) \cup \omega_2) \tag{45}$$

is not gauge-invariant: it changes as

$$\delta \bar{S} = -\frac{1}{2} \int \lambda \cup \delta \lambda. \tag{46}$$

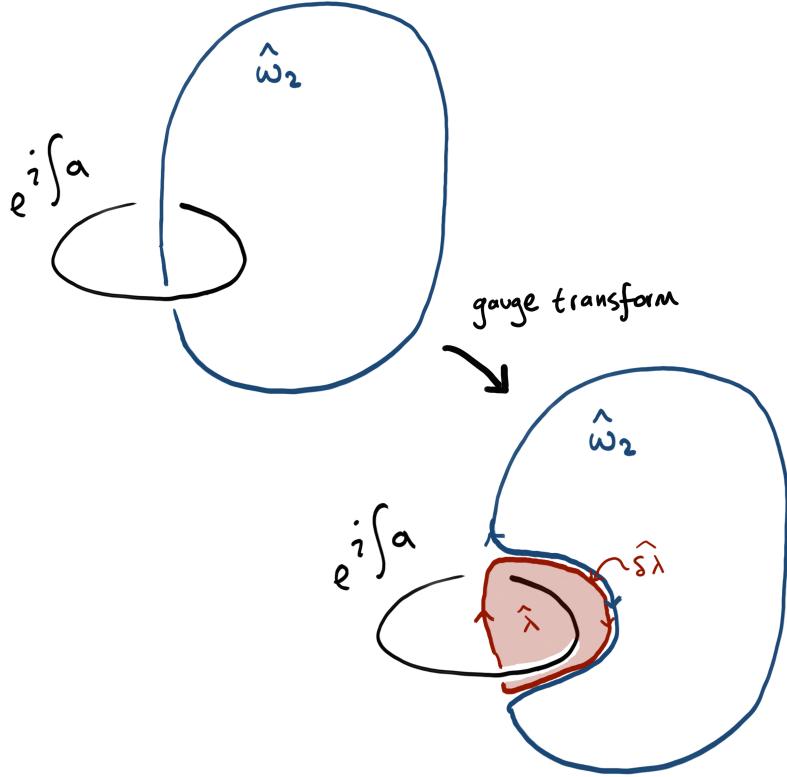


Figure 1: Schematic illustration of what happens to an a Wilson line when we do a gauge transformation by fiddling with the transition functions in the $SO(3)$ bundle. When we shift the $SO(3)$ background field by $\delta\lambda$, the Poincaré dual $\hat{\omega}_2$ of ω_2 changes by the dual of $\delta\lambda$. Since λ is a 1-chain it is dual to an open surface, with $\delta\lambda$ dual to that surface's boundary. After the gauge transformation, $\hat{\omega}_2$ changes accordingly—in the figure, the gauge-transformed $\hat{\omega}_2$ no longer links with the Wilson line. Since $\hat{\omega}_2$ changes by the boundary of a surface, its topological class cannot change. The difference between the phase of $e^{i\int_C a}$ before and after the gauge transformation is measured by the intersection number between $\hat{\lambda}$ and C , and so can be obtained by shifting $a \mapsto a + \lambda$.

This indicates that the symmetry is anomalous: this is a contradiction, since we assumed we were working with a spin system where the $SO(3)$ symmetry acted in an on-site fashion, and such an on-site symmetry action can always be coupled to a background gauge field in a gauge-invariant way. Physically, the anomaly happens for the following reason: in order to make the Dirac string transparent to the spin 1/2 charges, we need to make the $SO(3)$ monopole carry charge under both a and b gauge fields. But then the mutual statistics of the e and m particles means that the monopole is a fermion, and hence the Dirac string can detect itself, which breaks $SO(3)$ gauge invariance.

This gauge-noninvariance problem can be solved by realizing our theory at the boundary of some 4-manifold B . This is possible without extending the a or b fields into B , since luckily there are no a or b fields appearing in the expression for $\delta\bar{S}$. We consider the bulk action

$$\bar{S}_B = \frac{1}{2} \int_B \mathcal{P}(\omega_2), \quad (47)$$

where \mathcal{P} is the Pointryagin square, which maps \mathbb{Z}_2 classes to \mathbb{Z}_4 classes. The variation of this bulk term is precisely what is needed to cancel $\delta\bar{S}$, and it has the effect of giving an extra minus sign to processes during which ω_2 worldlines are braided on the boundary, rendering the $SO(3)$ monopoles properly bosonic.

Now since $\mathcal{P}(\omega_2)$ is an even class when evaluated on a spin manifold, the action S_B is actually independent of the choice of B when B is spin. However, we are dealing with a bosonic system—all the local operators carry integer spin—and so our construction needs to be completely independent of the existence and choice of a spin structure. Therefore $SO(3)$ gauge invariance cannot be restored through a term that does not depend on a choice of bulk 4-manifold, since we cannot make the assumption that B is spin.

So far we have not used time reversal at all. The reason why the preservation of T is important here is the following. First, note that in terms of the curvature of the $SO(3)$ bundle, the bulk action can be written

$$S_B = 4\pi l = \frac{4\pi}{4 \cdot 8\pi^2} \int_B \text{Tr}[F \wedge F], \quad (48)$$

where l is the instanton number, which would be valued in $\frac{1}{4}\mathbb{Z}$ if B were closed, and would be such that $l \in \mathbb{Z}$ if B were closed and the $SO(3)$ bundle in question could be lifted to an $SU(2)$ bundle. Anyway, the point is that S_B represents a θ term at $\theta = 4\pi$ (θ is 8π periodic on a closed 4-manifold). Since θ can be tuned continuously, in the absence of time reversal symmetry, it can be smoothly tuned to zero. Hence the spin liquid is not anomalous in the absence of time reversal, as the bulk action can then be eliminated with a symmetric counterterm in the background field.

One potentially confusing aspect of this is that S_B , taken by itself, breaks time-reversal (it would be T invariant if B were closed, but it's not). Indeed,

$$T : S_B \mapsto S_B - \frac{1}{4\pi} \int_B \text{Tr}[F \wedge F] = S_B - \int_{\partial B} \mathcal{L}_{SO(3)_2}, \quad (49)$$

so that the bulk action changes by a level-2 $SO(3)$ Chern-Simons term.⁵ Therefore if the theory really is T invariant, the boundary partition function must also change by an $SO(3)_2$ CS term under T , in a way rather similar to the change in the partition function of a massless Dirac fermion coupled to $U(1)$ that we know from the context of the parity anomaly.

This change in the boundary partition function is a bit surprising if one just looks at the action, since it's unclear how a change in the fields can reproduce the $SO(3)_2$ term. But it's also reasonable that the boundary theory should break T by itself, since the fact that the monopoles are fermions and the boundary is three dimensional means that the monopoles are Kramers doublets, and so the relation $T^2 = \mathbf{1}$ when acting on states of the full boundary theory is broken in the presence of a monopole.

One rather unsatisfying way of finding the variation of the boundary partition function is as follows. Consider an $SO(3)$ monopole with a gauge potential of the form $A = \mathcal{A}_\mu T^3 dx^\mu$, where \mathcal{A} is a $U(1)$ monopole configuration and $T^3 = \text{diag}(1, 0, -1)$ is a diagonalized $\mathfrak{so}(3)$ generator. This form of the $SO(3)$ gauge field can be taken wolog, since such configurations can realize the minimal-strength $SO(3)$ instanton, where we can have the minimal $\frac{1}{8\pi^2} \int \text{Tr}[F \wedge F] = \int (F_A/2\pi \wedge F_A/2\pi) = 1$ on a closed manifold. With this choice, the integral class $\tilde{F}_A \equiv F_A/2\pi$ reduces mod 2 to ω_2 . Therefore we can write the full action as

$$S = \frac{2\pi}{2} \int_{\partial B} (a \cup \delta b - (a + b) \cup \tilde{F}_A) + \pi \int_B \tilde{F}_A \wedge \tilde{F}_A. \quad (51)$$

Let's now look at the variation of the first term under T . Integrating out a tells us that $\delta b = \tilde{F}_A$, and so the boundary action is, switching to a continuum notation,

$$S_{\partial B} = -\frac{1}{4\pi} \int_{\partial B} \mathcal{A} \wedge F_A. \quad (52)$$

This level-1 $U(1)$ CS term⁶ is actually quite reasonable, since it confirms that the $SO(3)$ monopoles are fermions.

When we act with T , we see that $S_{\partial B} \mapsto S_{\partial B} + \int_{\partial B} \mathcal{L}_{U(1)_2}$, or, generalizing back to the case with an arbitrary $SO(3)$ field configuration,

$$T : S_{\partial B} \mapsto S_{\partial B} + \int_{\partial B} \mathcal{L}_{SO(3)_2}. \quad (53)$$

This extra term precisely cancels the extra term picked up from the T -variation of the bulk term, and the full theory is therefore T -invariant.

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⁵The $SO(N)_k$ CS term is normalized as

$$\mathcal{L}_{SO(N)_k} = \frac{k}{8\pi} \int_M \text{Tr} \left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right]. \quad (50)$$

The coefficient in front is chosen so that when $k = 1$, we get a well-defined action provided that M is spin. Note that e.g. for $SO(3)$ this reduces to $U(1)_k$ when we restrict the field strength to lie in a particular direction in flavor space.

⁶Note that this action doesn't actually mean that the theory is spin; the pre-integrating-out- a theory is clearly not spin *waves hands*.

Before ending, we note that all of the above field theory stuff can be said in words in the following way.⁷ If e (m) has spin 1/2 and m (e) has spin zero, we can add an m (e) to the monopole to cancel the braiding phase, and make the monopole operator local as required. If both e, m have spin 1/2, we would have to add an ε to the monopole to render it local. The monopole would then be a fermion. This then means that \mathcal{T} symmetry is broken, since it acts projectively



Classical $O(n)$ model in one dimension

Consider the $O(n)$ model in one dimension. The Hamiltonian is

$$H = -J \sum_i S_i \cdot S_{i+1}. \quad (54)$$

We will be finding the correlation length to first order in $(\beta J)^{-1}$, which we will do by calculating the first two eigenvalues of the transfer matrix.



From the definition of the correlation length, we can get the correlation length by computing

$$\xi = - \lim_{l \rightarrow \infty} \frac{|l|}{\ln |\langle S_0 S_l \rangle_c|}. \quad (55)$$

Because of the $l \rightarrow \infty$ limit, it will help to do a spectral decomposition on the transfer matrices implicitly appearing in the expectation value. When we compute the log of the connected correlator in this limit, we get λ_0/λ_1 , where the λ_i are the eigenvalues of the transfer matrix \mathbb{T} , listed in descending order. This basically comes from taking the expression $\lambda_0^l + \lambda_1^l + \dots$, dividing by λ_0^l for normalization, taking off the first term to get the connected piece, and then taking the log (details are in another diary entry). We then have

$$\xi^{-1} = \ln(\lambda_0/\lambda_1). \quad (56)$$

The transfer matrix \mathbb{T} acts on a function $\psi(S)$ of the vector S as

$$\mathbb{T}\psi(S) = \int dS' \delta(S'^2 - 1) e^{\beta JS \cdot S'} \psi(S'). \quad (57)$$

⁷Thanks to Dan Mao for the help.

Note that \mathbb{T} has $O(n)$ symmetry, since it only depends on the dot product. Thus its eigenvectors will transform in various irreps of $O(n)$. The lowest energy irrep is the trivial irrep, the second lowest is the fundamental S irrep, and so on. The largest eigenvalue of \mathbb{T} will come from an eigenvector $\psi_0(S)$ in the trivial representation—that is, a constant. So to get λ_0 , we just need to compute

$$\lambda_0 \psi_0(S) = \int dS' \delta(S'^2 - 1) e^{\beta JS \cdot S'} \psi_0(S'), \quad (58)$$

where $\psi_0(S') = 1$. We do this by choosing $S = (1, 0, 0, \dots, 0)$. A convenient parametrization of S' is then

$$S' = (\cos \phi, \mathbf{n} \sin \phi), \quad \mathbf{n} \in S^{n-2}. \quad (59)$$

We then write

$$dS' \delta(S'^2 - 1) = d\phi d\mathbf{n} \sin^{n-2}(\phi) \delta(\mathbf{n}^2 - 1), \quad (60)$$

where n powers of $\sin \phi$ came from the Jacobian in the change of variables and -2 powers came from the δ function.

Doing the integral over \mathbf{n} gives the surface area of the sphere, which we will just write as Γ . So

$$\lambda_0 = \Gamma \int d\phi e^{\beta J \cos \phi} \sin^{n-2}(\phi). \quad (61)$$

We are taking $\beta J \gg 1$, so we can do a saddle point approximation. Since $\beta J \gg 1$, the integrand only cares about small ϕ . Thus we can expand in ϕ and take the integral from 0 to ∞ rather than 0 to π . We write

$$\begin{aligned} e^{\beta J \cos \phi} \sin^{n-2}(\phi) &\approx e^{\beta J} e^{-\beta J \phi^2/2} (1 + \beta J \phi^4/4!) (\phi^{n-2} - (n-2)\phi^n/3!) \\ &\approx e^{\beta J} e^{-\beta J \phi^2/2} \phi^{n-2} (1 - (n-2)\phi^2/3! + \beta J \phi^4/4!). \end{aligned} \quad (62)$$

The integral can then be done since the integrand is Gaussian and we're integrating up to ∞ . We get

$$\lambda_0 = \frac{C}{(\beta J)^{(n-1)/2}} e^{\beta J} \left(1 - \frac{1}{8\beta J} (n^2 - 4n + 3) + O([\beta J]^{-2}) \right), \quad (63)$$

where C is a βJ -independent but n -dependent constant with 2s and π s and stuff (it won't matter since we're calculating λ_0/λ_1 and as we will see the same such constant appears in λ_1).

Now we get λ_1 . We expect that the next-to-lowest energy state will be in the fundamental representation of $O(n)$. Thus the appropriate eigenfunction of \mathbb{T} will be $\psi(S) = S$ (remember that S is a vector in S^{n-1}). The eigenvalue equation reads

$$\lambda_1 S = \int dS' \delta(S'^2 - 1) e^{\beta JS \cdot S'} S'. \quad (64)$$

We can again without loss of generality set $S = (1, 0, \dots, 0)$. Working again in the parametrization $S' = (\cos \phi, \mathbf{n} \sin \phi)$, we get

$$\lambda_1(1, 0, \dots, 0) = \int d\mathbf{n} d\phi \delta(\mathbf{n}^2 - 1) \sin^{n-2}(\phi) e^{\beta J \cos \phi} (\cos \phi, \mathbf{n} \sin \phi). \quad (65)$$

The integration over the S^{n-2} kills all but the first component on the RHS, demonstrating that indeed, S is an eigenvector of \mathbb{T} . So then we have

$$\lambda_1 = \Gamma \int d\phi \sin^{n-2}(\phi) e^{\beta J \cos \phi} \cos \phi = \partial_{\beta J} \lambda_0. \quad (66)$$

So then

$$\lambda_1 = \frac{1-n}{2\beta J} \lambda_0 + \lambda_0 + O([\beta J]^{-2}), \quad (67)$$

telling us that

$$\xi^{-1} = -\ln \left(1 - \frac{n-1}{2\beta J} \right) \implies \xi \approx \frac{2\beta J}{n-1}. \quad (68)$$



Global properties of KW duality

Consider the usual KW duality for a one-dimensional transverse field Ising model, with Hamiltonian chosen as

$$H = - \sum_i (Z_i Z_{i+1} + h X_i). \quad (69)$$

As usually formulated the duality only makes sense locally, since it maps something degenerate to something non-degenerate and vice versa. We will show how to minimally modify the statement of the duality so that the mapping is exact. We will do so for both an open chain and a circle, consider the Majorana fermion representation, and figure out where the ground states map. (note: this was written a long time ago; I think there are a few new papers that treat this issue better)



The usual statement of the operator mapping for the duality is (dual operators are in mathcal)

$$X_i = \mathcal{Z}_i \mathcal{Z}_{i+1}, \quad \mathcal{X}_i = Z_i Z_{i+1}, \quad (70)$$

where the indices on the dual operators are thought of as being displaced by half a lattice spacing relative to the original operators. Let's first consider a closed chain. These theories can't be actually dual, since the global symmetry generated by $\prod_i X_i$ maps to $\prod_i (\mathcal{Z}_i)^2 = \mathbf{1}$ on the other side. Similarly, the putative global symmetry of the dual theory actually doesn't exist, since it would be generated by $\prod_i \mathcal{X}_i$, but this is also equal to $\mathbf{1}$. So, the dual theory has no information about the global symmetry of the X, Z theory. This is to be expected

since \mathcal{X} measures the presence of a domain wall, and since there must always be an even number of domain walls, $\prod_i \mathcal{X}_i = \mathbf{1}$.

Related problems come when we try to split the operator mapping and find expressions for individual Z and \mathcal{Z} operators. We would usually write

$$\mathcal{Z}_i = \mathcal{Z}_0 \prod_{j < i} X_j, \quad (71)$$

but we run into a problem when we run around the full length of a chain since the periodic ∂ conditions imply $\prod_i X_i = \mathbf{1}$, which isn't true. (we could also take the chain of operators to run off in the other direction—these two choices are related by the global \mathbb{Z}_2 symmetry.)

To get the full \mathbb{Z}_2 symmetry in the dual theory and fix these issues, we introduce a \mathbb{Z}_2 gauge field A defined on the links of the dual lattice. The operator mapping is upgraded to (writing out the $1/2$ s in the indices for once)

$$X_i \mapsto \mathcal{Z}_{i-1/2} e^{iA_i} \mathcal{Z}_{i+1/2}, \quad (72)$$

with A_i taking values in $\pi\mathbb{Z}_2$. Products of X operators then become two charges connected with a Wilson line:

$$X_i X_{i+1} \dots X_{i+n} \mapsto \mathcal{Z}_{i-1/2} W_{i,i+n} \mathcal{Z}_{i+n+1/2}, \quad (73)$$

where $W_{i,i+n}$ is the product of the gauge field variables along the path. In particular, the global symmetry in the original model is mapped to a Wilson line spanning the whole system in the dual theory:

$$\prod_i X_i \mapsto \exp(i \text{hol}(A)). \quad (74)$$

Hopefully the notation $\text{hol}(A)$ is clear. If our ground states are $\otimes |\uparrow\rangle \pm \otimes |\downarrow\rangle$ then in the dual theory we are in a basis which diagonalizes the Wilson loop, and so the single \mathbb{Z}_2 degree of freedom we added by gauging (viz. $\text{hol}(A)$) makes up for the lost \mathbb{Z}_2 symmetry. Likewise if our ground states are $\otimes |\uparrow\rangle$ and $\otimes |\downarrow\rangle$ then on the dual side the Wilson loop takes us between ground states, and so the ground states are labeled by the \mathbb{Z}_2 electric flux. If one wants to gauge-fix, we can choose A so that $A_i = 0$ for all sites on the original lattice except for one, where \mathbb{Z}_2 degree of freedom gets trapped. Finally, gauge invariance in the dual model means that the operator \mathcal{X}_i is equivalent to $E_l E_{l+1}$, where the variables E_l (defined on the links of the dual lattice) are conjugate to the A_i . So, we can write the dual Hamiltonian as

$$\mathcal{H} = - \sum_i (h \mathcal{Z}_i e^{iA_i} \mathcal{Z}_{i+1} + E_l E_{l+1}). \quad (75)$$

Hopefully the notation here is clear: the *is* and *ls* are dual lattice indices — I just didn't want to write the $\pm 1/2$ s everywhere.

Let's now see in more detail how the gauge field fixes the issues about ground state degeneracy. When $h \gg 1$ the original theory has the unique ground state $\otimes |+\rangle$. The dual coupling is thus very small, and so the gauge field degree of freedom gets frozen out (if we gauge-fix so that A only appears in one term $\mathcal{Z}_0 e^{iA} \mathcal{Z}_1$, then the ground state has $A = 0$: if A were non-trivial we would need $\mathcal{Z}_0 = -\mathcal{Z}_1$ [recall $h \gg 1$ means diagonalizing \mathcal{Z}], but then this would mean that some $\mathcal{Z}_j = -\mathcal{Z}_{j+1}$ for $j \neq 0$, which because of the term $-h \mathcal{Z}_j \mathcal{Z}_{j+1} \subset H$

is very energetically costly). Note that the dual theory no longer has a \mathbb{Z}_2 symmetry being broken in this phase, since this symmetry has been gauged, and so the $g \gg 1$ phase has a non-degenerate ground state on both sides of the duality (the operator which performs the symmetry action is $\prod_i X_i \mapsto \prod_i (E_l E_{l+1}) = 1$, which indeed acts trivially). Note that if we were on a manifold with boundary or on an infinite line, boundary conditions would come into play and things would be slightly more involved.

Now consider $h \ll 1$. In the original spin model, we have a \mathbb{Z}_2 ground state degeneracy. In the dual model, we see that we also have a \mathbb{Z}_2 degeneracy, with the electric flux taking on values $E_l = \pm 1$. The \mathbb{Z}_2 symmetry in the original model is generated by $\prod_i X_i$, which maps to $\text{hol}(A)$ as we have said. Does this generate the \mathbb{Z}_2 symmetry that takes us between different electric flux sectors? Of course; inserting a Wilson line changes the electric flux between $+1$ and -1 .

Thus a consistent picture emerges: by adding a \mathbb{Z}_2 gauge field to the dual theory, we eliminate one \mathbb{Z}_2 symmetry, which is spontaneously broken when $h \gg 1$, creating an unphysical degeneracy. Upon gauging, this symmetry is replaced with another \mathbb{Z}_2 symmetry generated by the Wilson lines, which is spontaneously broken in the $h \ll 1$ phase, giving a \mathbb{Z}_2 ground state degeneracy which matches that of the original spin model. The fact that the Wilson lines are the generators of the dual symmetry is quite a general fact, and comes up a lot in the diary entries on quantum field theory.

We can also look at open chains. There are several ways of dealing with things. We can map a chain with L sites to one with $L - 1$ sites provided we fix boundary conditions on the first chain, or we can do the reverse, or we can do things slightly weirdly by mapping an L site chain to another L site chain, but then the Hamiltonians have single Z operators and dangling spins and stuff gets annoying. Perhaps most aesthetically, we can map an L -site chain to an $L + 1$ site chain, and fix the dual spins on both dual boundaries to be $|\uparrow\rangle$ (this is okay since $\mathcal{X}_0, \mathcal{X}_L \notin \tilde{H}$). This cuts down the Hilbert space to $\mathbb{C}^{2(L-1)}$, and we get the extra degree of freedom back by adding a \mathbb{Z}_2 gauge field so that $\prod_i e^{iA_i}$ becomes a physical variable. Since the boundary conditions are fixed, gauge transformations cannot change the sign of the $\mathcal{Z}_0 e^{iA_1/2} \mathcal{Z}_1$ or $\mathcal{Z}_L e^{iA_{L+1}/2} \mathcal{Z}_{L+1}$ terms, and the gauge field carries one degree of freedom even though the topology of the chain is trivial.

Finally we make a few statements about fermions. We define Majorana operators as (carefully putting the tails to the left)

$$\eta_i = \prod_{j < i} X_j Y_i, \quad \gamma_i = \prod_{j < i} X_j Z_i. \quad (76)$$

The Hamiltonian is then

$$H = -i \sum_i (\eta_i \gamma_{i+1} + h \gamma_i \eta_i). \quad (77)$$

Since these are fermions, they can hop around on the sheets of $SU(2)$ as they go around the circle and we can let them have either periodic or anti-periodic boundary conditions (non-bounding or bounding spin structures). Acting on the ground state wavefunction,

$$\eta_L |\psi\rangle = \prod_i X_i \eta_0 |\psi\rangle = -\eta_0 \prod_i X_i |\psi\rangle = \mp \eta_0 |\psi\rangle. \quad (78)$$

So, we have

$$|\psi\rangle = \otimes |\uparrow\rangle - \otimes |\downarrow\rangle \implies \eta_L = \eta_0, NB, \quad |\psi\rangle = \otimes |\uparrow\rangle + \otimes |\downarrow\rangle \implies \eta_L = -\eta_0, B. \quad (79)$$

Thus the antisymmetric ground state, which gets lifted to higher energy after splitting in a finite chain, is the one with non-bounding spin structure (and the Majorana zero mode). The fermion parity of these states is determined by $(-1)^{f_i} = 2c_i^\dagger c_i - 1 = i\eta_i \gamma_i$, so that for the whole system, the total fermion parity is the same as the global \mathbb{Z}_2 symmetry: $(-1)^F = \prod_i X_i$. This gives the bounding spin structure even parity and the non-bounding one odd parity (from the MZM).

So, the two different ground states of the TFIM, which are in the same Hilbert space and related by a global \mathbb{Z}_2 , map to two different Majorana fermion theories in two different Hilbert spaces, distinguished by their spin structures (and as a consequence, by their fermion parities as well).

Comment on how the gauge field is needed even when we have open BCs — still have a gauge inv. wilson line going along the whole chain. In any case the gauge field is needed, since we need to keep track of the degenerate ground states in the SSB phase.



Staggered Hubbard model at half-filling and criticality on Mott lobes

This is a problem from a pset in a class on quantum matter taught by Sachdev in spring 2018.

Consider the boson Hubbard model defined on a bipartite (hypercubic) lattice, with differing chemical potentials on the two sublattices:

$$H = \sum_i \left((-\mu + \epsilon(-1)^{|i|}) b_i^\dagger b_i + \frac{U}{2} b_i^\dagger b_i (b_i^\dagger b_i - 1) \right) - v \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i), \quad (80)$$

where $|i| = 1$ on the sublattice A and $|i| = 2$ on the sublattice B . Take $U > 2\epsilon$, and work in the regime where lattice is at half-filling (one boson per unit cell). The goal is to find a field theory description of this model that reduces to the WF CFT at tip of the Mott “lobe”.

Do several things: first, find the imaginary-time correlator $\chi_\alpha(\tau) = \langle Tb(\tau)b^\dagger(0) \rangle$ at $v = 0$ (and zero temperature), where α is the sublattice index and the b 's are taken to act on a single site. Next, decouple the hopping term in H with a HS transformation, and integrate out the bosons, stopping at one-loop order. Determine the mass variable in the field theory expanded about the critical point in terms of the χ_α 's and v (and express the critical value of v in terms of the χ_α 's). This is most easily accomplished by going over to frequency + momentum space.



Since $U > 2\epsilon$, at half-filling (where the A sublattice is filled), new bosons which are added will go into filling up the B sublattice, since $-\mu + \epsilon < -\mu - \epsilon + U$. Also since we have assumed half-filling, we know that $\epsilon > \mu$.

When $v = 0$, we get the boson correlator (at zero temperature) by the usual spectral decomposition (just focusing on a single site wolog and dropping the α sublattice index on the b 's):

$$\begin{aligned}\chi_\alpha(\tau) &= \langle T(b(\tau)b^\dagger(0)) \rangle \\ &= \sum_n (\Theta(\tau)e^{-\tau E_{n0}} \langle 0|b|n\rangle \langle n|b^\dagger|0\rangle + \Theta(-\tau)e^{\tau E_{n0}} \langle 0|b^\dagger|n\rangle \langle n|b|0\rangle),\end{aligned}\quad (81)$$

where E_{n0} is the energy difference between $|n\rangle$ and $|0\rangle$ on the α sublattice. In frequency space,

$$\chi_\alpha(\omega) = \sum_n \left(-\frac{\langle 0|b|n\rangle \langle n|b^\dagger|0\rangle}{i\omega - E_{n0}} + \frac{\langle 0|b^\dagger|n\rangle \langle n|b|0\rangle}{i\omega + E_{n0}} \right) \quad (82)$$

At zero temperature, we get only the $n = \pm 1$ terms. For $\alpha = A$ we have $E_{10} = U - \epsilon - \mu$ and $E_{-10} = \mu + \epsilon$, while for $\alpha = B$ we have $E_{10} = \epsilon - \mu$ and $E_{-10} = 0$. So we write (using $\langle b^\dagger b \rangle = 1$ for the A sublattice and $\langle b^\dagger b \rangle = 0$ for the B sublattice)

$$\begin{aligned}\chi_A(\omega) &= \frac{2}{-i\omega + U - \mu - \epsilon} + \frac{1}{i\omega + \mu + \epsilon}, \\ \chi_B(\omega) &= \frac{1}{i\omega + \mu - \epsilon}.\end{aligned}\quad (83)$$

Writing

$$\chi_\alpha(\omega) = -r_\alpha + iK_{1\alpha}\omega - K_{2\alpha}\omega^2, \quad (84)$$

we expand in small ω and get

$$r_A = \frac{2}{\mu + \epsilon - U} - \frac{1}{\mu + \epsilon}, \quad K_{1A} = \frac{2}{(U - \mu - \epsilon)^2} - \frac{1}{(\mu + \epsilon)^2}, \quad K_{2A} = \frac{2}{(U - \mu - \epsilon)^3} + \frac{1}{(\mu + \epsilon)^3}, \quad (85)$$

as well as

$$r_B = \frac{1}{\epsilon - \mu}, \quad K_{1B} = \frac{1}{(\mu - \epsilon)^2}, \quad K_{2B} = \frac{1}{(\mu - \epsilon)^3}. \quad (86)$$

Thus we can verify that

$$\partial_\mu r_A = -K_{1A}, \quad \partial_\mu r_B = -K_{1B}. \quad (87)$$

When we turn this into a mean field theory, these relations will be essential for guaranteeing a $U(1)$ gauge invariance that will appear in our description.

We now do mean field theory, by introducing a HS field Ψ to decouple the inter-site hopping (not the interactions! Since we're doing MFT, we want the Hamiltonian to be as

on-site as possible (and we're working at large U anyway)). Doing this in the usual way, we get

$$L = \sum_i \left(b_i^\dagger \partial_\tau b_i - (\mu - \epsilon(-1)^{|i|}) b_i^\dagger b_i + \frac{U}{2} : n_i n_i : + b_i^\dagger \Psi_i + b_i \Psi_i^\dagger \right) + \sum_{ij} \frac{1}{v} \Psi_i^\dagger A_{ij}^{-1} \Psi_j. \quad (88)$$

Here A_{ij} is the adjacency matrix, with entries which are equal to one if i and j are nearest neighbors and zero otherwise. Note that in this parametrization we have the time-dependent (but not space-dependent) gauge transformation (for this to make sense we need to think of μ as the expectation value of a dynamical field)

$$b_i \mapsto e^{i\phi(\tau)}, \quad \Psi \mapsto e^{i\phi(\tau)} \Psi, \quad \mu \mapsto \mu + i\partial_\tau \phi(\tau). \quad (89)$$

We get the effective action for the mean fields Ψ by integrating out the bosons. For the Feynman rules, we see that we can convert Ψ s into b bosons, and we can draw four-point boson vertices. The two-point interactions for Ψ in its effective potential are then determined just by the boson propagators at $v = 0$, which we already know. The four-point boson vertex also gives us a Ψ^4 interaction, while there are no odd n -point interactions (one- or three-point) since they aren't invariant under the gauge transformation (similarly, the gauge invariance means that Ψ s and Ψ^\dagger s appear in equal numbers in each term). Thus the relevant terms are

$$L = \sum_{ij} \frac{1}{v} \Psi_i^\dagger A_{ij}^{-1} \Psi_j + \sum_{i \in A} \chi_A(\omega) |\Psi_i(\omega)|^2 + \sum_{i \in B} \chi_B(\omega) |\Psi_i(\omega)|^2 + \sum_i u |\Psi|^4. \quad (90)$$

Now we go to momentum space to simplify the first term. Since there are two sublattices, and each lattice gets identified into a point in the BZ, A_{ij}^{-1} in band space becomes a two-by-two off-diagonal matrix proportional to σ^x . For a square lattice the entries are proportional to $\cos(k_x) + \cos(k_y)$ since we can hop in the $\pm x$ and $\pm y$ directions, and for more general coordination numbers we get some function of various cosines that goes as $1/Z$, where Z is the coordination number. Thus the quadratic part of the Lagrangian is (sticking with the square lattice for simplicity)

$$L = \sum_k (\Psi_A^\dagger, \Psi_B^\dagger) \begin{pmatrix} \chi_A(\omega) & (\cos(k_x) + \cos(k_y))/4v \\ (\cos(k_x) + \cos(k_y))/4v & \chi_B(\omega) \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}. \quad (91)$$

The phase transition will happen when Ψ gets a vev, that is when the above matrix acquires a negative eigenvalue at $\omega, k = 0$. This after all is how the effective mass is defined, by the spacetime-independent eigenvectors of the second derivative of the effective potential with respect to Ψ . When such an eigenvector has a negative eigenvalue the mass squared is negative, the system condenses, and we enter the SF phase. The lower eigenvalue at $\omega = 0, k = 0$ is (sending $4 \rightarrow Z$ for a more general lattice)

$$\lambda_- = \frac{1}{2} \left(\chi_A(0) + \chi_B(0) - \sqrt{(\chi_A - \chi_B)^2 + 4/(Z^2 v^2)} \right). \quad (92)$$

Thus we get a condensate when

$$\frac{1}{Z^2 v^2} = \chi_A(0) \chi_B(0) = r_A r_B. \quad (93)$$

Near the phase transition, we can focus on just the lower eigenmode, and so the effective Lagrangian is approximately

$$L = K_1 \Psi_i^\dagger \partial_\tau \Psi_i + K_2 |\partial_\tau \Psi|^2 + b |\nabla \Psi|^2 + \tilde{r} |\Psi|^2 + u |\Psi|^4, \quad (94)$$

where I'm not writing out all the coefficients explicitly. The linear time derivative comes from the $K_{1\alpha}$ terms in $\chi_\alpha(\omega)$, the quadratic time derivative comes from the $K_{2\alpha}$ terms, and so on. The most important one is the mass \tilde{r} . We define the parameter η to measure where we are in relation to the critical point:

$$\eta \equiv \frac{1}{v^2 4^2} - r_A r_B. \quad (95)$$

Expanding the ω and k independent piece of the eigenvalue of the matrix written in (91), a bit of algebra gives something like

$$\tilde{r} \approx -\frac{\eta}{r_A + r_B}. \quad (96)$$

Note that this effective action needs to retain the invariance under the space-independent but time-dependent gauge transformations written down previously, which is only possible since relations like (87) are obeyed. In particular, we see that when \tilde{r} is independent of μ (on the tip of the Mott lobe), the gauge invariance tells us that we must have $K_1 = 0$. At the tips of the Mott lobes then, the theory gains a relativistic invariance and the critical point is governed by the Wilson Fischer CFT. This state of affairs is made possible by an emergent particle-hole symmetry that forbids the linear time derivative term.



Basic setup of elasticity theory

Consider some sort of quantum solid in three spacetime dimensions. Write down the Lagrangian in terms of displacement fields, assuming a rotationally-invariant Hamiltonian (I was thinking about dislocation-mediated melting when writing this, and wanted to ignore disclinations, the presence of which restore the rotational symmetry).

Assuming an isotropic situation, write down the propagators for the transverse and longitudinal modes in terms of the compression and shear elastic moduli κ and μ . Identify the stress tensor and then write down the theory in a dual form where the stress tensor is treated as the fundamental variable. Finally, solve the conservation equation on the stress energy tensor and write the Lagrangian in terms of “elastic gauge fields”. See [3] for inspiration if desired.



Let R_{ij} and R_{ij}^0 be the (vectorial) positions and equilibrium displacements between the atoms at sites i and j . Let u_i^a denote the a -direction displacement of the i th atom from its equilibrium position. We will expand the potential $V(\{R_{ij}\})$ about a state in which all separations are R_{ij}^0 (ignoring the binding energy of the solid):

$$\begin{aligned} V &= \frac{1}{2} \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} (R_{ij}^a - R_{ij}^{0a})(R_{kl}^b - R_{kl}^{0b}) + \dots \\ &= \frac{1}{2} \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} (u_i^a - u_j^a)(u_k^b - u_l^b), \end{aligned} \quad (97)$$

where there is no linear term since we are expanding about the equilibrium positions and where we used $R_{ij}^a = R_{ij}^{0a} + u_i^a - u_j^a$. Here all the indices are spatial, but it is very helpful for QFTy people like me to regard the a, b, \dots indices as “flavor” indices, with the u^a representing different goldstone fields.

We assume that V contains only interactions between neighboring sites, so that we can expand u_i^a about u_j^a , writing

$$u_i^a = u_j^a + R_{ij}^{0m} \partial_m u^a + \frac{1}{2} R_{ij}^{0m} R_{ij}^{0n} \partial_n \partial_m u^a + \dots \quad (98)$$

The potential energy thus goes to

$$V = \frac{1}{2} C_{abmn} \partial_m u^a \partial_n u^b + \frac{1}{2} C_{abmnrs} \partial_m \partial_r u^a \partial_n \partial_s u^b, \quad (99)$$

where

$$C_{abmn} = \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} R_{ij}^{0m} R_{kl}^{0n}, \quad C_{abmnrs} = \frac{1}{4} \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} R_{ij}^{0m} R_{ij}^{0n} R_{kl}^{0r} R_{kl}^{0s}. \quad (100)$$

We will ignore the four-derivative part in V for now. Also note that we’ve dropped the cross terms which are cubic in derivatives: this is because they are antisymmetric under $R_{ij} \rightarrow -R_{ij}^0 = R_{ji}^0$.

It’ll be good to have a more specific form for C_{abmn} in terms of various more specific elastic moduli. We can a few different types of things that can happen to the lattice which lead to an increase in energy: compressions, shears, and rotational things (I don’t know the right words to use). Compressions are scalars, in that compressions are radially-symmetric things that transform trivially under rotations (i.e. are in the trivial rep of $O(2) \times O(2)$ —one $O(2)$ for the ab indices and one for the mn indices). We are going to be assuming that the energy V is invariant under rotations, and so we won’t have to worry about rotational things (spin 1 representations) appearing in the decomposition of C_{abmn} . However, we do have to worry about shear forces, where one direction is compressed and the other direction is elongated. Because shear configurations are symmetric under π rotations, they come in the spin 2 representation (think of gravity waves and gravitons). We thus decompose C_{abmn} in terms of minimal projectors as

$$C_{abmn} = 2 (\kappa \Pi_{abmn}^0 + \mu \Pi_{abmn}^2), \quad (101)$$

where κ and μ are the compression and shear elastic moduli, and the projectors are

$$\Pi_{abmn}^0 = \frac{1}{2}\delta_{am}\delta_{bn}, \quad \Pi_{abmn}^2 = \frac{1}{2}(\delta_{mn}\delta_{ab} - \delta_{ma}\delta_{nb} + \delta_{mb}\delta_{na}), \quad (102)$$

which project onto the trivial representation and the 2 representation, respectively.

Now we can write down the Lagrangian (in Euclidean signature). Here it is:

$$\mathcal{L} = \frac{\rho}{2}(\partial_t u)^2 + \frac{1}{2}\partial_m u^a (\kappa\delta_{ma}\delta_{nb} + \mu[\delta_{mn}\delta_{ab} - \delta_{ma}\delta_{nb} + \delta_{mb}\delta_{na}]) \partial_n u^b. \quad (103)$$

Now the δ structure of the various terms becomes clearer: for example, the compression term with κ looks like $(\nabla \cdot u)^2$, which makes sense since compressions should be captured by a divergence in the displacements.

We can write \mathcal{L} more concisely by extending C_{abmn} to $C_{ab\mu\nu}$, which incorporates the time index, through

$$C_{ab\mu\nu} = C_{abmn} + \rho\delta_{ab}\delta_{\mu 0}\delta_{\nu 0}. \quad (104)$$

Hopefully the notation is clear: the first part is the purely spatial part and the second is relevant if both μ and ν are temporal. Thus

$$\mathcal{L} = \frac{1}{2}C_{\mu\nu}^{ab}\partial^\mu u^a \partial^\nu u^a. \quad (105)$$

The stress tensor is thus

$$T_\mu^a = C_{\mu\nu}^{ab}\partial^\nu u^b. \quad (106)$$

Since u^t doesn't make sense, it is good to segregate the indices on T in this way. This form for the stress tensor makes sense because $\partial^\nu u_b$ is, well, a stress. When we source the displacement field by adding a $\int J_a u^a$ term to the action, we get the expected relation

$$\partial^\mu T_\mu^a = J^a, \quad (107)$$

saying that external forces lead to divergences in the stress tensor. Also note that T_μ can be thought of as a 1-form field itself, with a acting as a flavor index.

Let's find the propagators by going to momentum space:

$$\mathcal{L} = \frac{\rho}{2}\omega^2 u^a p^2 u^a + \frac{\kappa}{2}u^a p^a p^b u^a u^b + \frac{\mu}{2}u^a p^2 u^a. \quad (108)$$

Now we invert the kernel with transverse and longitudinal projectors in the usual way. We write

$$\mathcal{L} = \frac{1}{2} [\Pi_{ab}^T(\rho\omega^2 + p^2\mu) + \Pi_{ab}^L(\rho\omega^2 + (\mu + \kappa)p^2)], \quad (109)$$

where as usual $\Pi_{ab}^T = \delta_{ab} - p^a p^b / p^2$ and $\Pi^L = p^a p^b / p^2$. So, the full propagator is

$$G_{ab} = \rho^{-1} \left(\frac{\Pi_{ab}^L}{\omega^2 + \frac{\mu+\kappa}{\rho}p^2} + \frac{\Pi_{ab}^T}{\omega^2 + \frac{\mu}{\rho}p^2} \right). \quad (110)$$

Strictly speaking, this expression for $\langle u^a u^b \rangle$ will only make sense when u^a is single-valued, i.e. when there are no dislocations. If there are dislocations around, we will need instead to compute $\langle \partial_\mu u^a \partial_\nu u^b \rangle$, which is well-defined.

Anyway, from the expression for G_{ab} we identify the speed of the modes as

$$c_T = \sqrt{\mu/\rho}, \quad c_L = \sqrt{(\mu + \kappa)/\rho}. \quad (111)$$

Thus the longitudinal compression mode is always faster than the transverse one. We also see that κ only appears in the longitudinal mode: this is because the longitudinal mode is a compressional wave, while the transverse modes are not. The transverse mode is better to focus on because it is more discriminatory when it comes to dealing with different phases: both liquids and solids have compressional longitudinal modes, but the liquid doesn't have transverse modes, since it is dissipative.

Now we will write the theory in a dual presentation in terms of the stress tensor, rather than in terms of the displacement variables (the strains). This can be done easily since T_μ^a is essentially the momentum conjugate to u^a . To invert $T_\mu^a = C_{\mu\nu}^{ab}\partial^\nu u^b$, we need to invert C . This is strictly speaking not possible, since the coefficient of the Π^1 projector onto the spin 1 representation in the decomposition of C is zero, and so C will annihilate spin 1 vectors. This is okay for us though, since the spin 1 representation is antisymmetric (think $r_n\partial_m - r_m\partial_n$), while in the absence of external torques the stress tensor will be symmetric, and so at least when acting on T it is possible to invert C . So we can write

$$[C^{-1}]_{ab\mu\nu} = \frac{1}{\rho}\delta_{ab}\delta_{\mu 0}\delta_{\nu 0} + \frac{1}{2}(\kappa^{-1}\Pi_{abmn}^0 + \mu^{-1}\Pi_{abmn}^2). \quad (112)$$

The Lagrangian in terms of the stresses is then

$$\mathcal{L} = \frac{1}{2}T_\mu^a[C^{-1}]_{\mu\nu}^{ab}T_\nu^b + iT_\mu^a\partial_\mu u^a, \quad (113)$$

which comes from either doing the canonical procedure for fields and their momenta or from doing a HS transformation on the $\partial u C \partial u$ term.

Let us perform a Hodge decomposition on the 1-form $\partial_\mu u^a$ by writing $u = u_s + u_d$, where u_s is smooth and u_d is a discontinuous part due to the presence of dislocations. This means that $\partial_\mu u^a = du_s^a + du_d^a$, where du_d^a is not generically either closed or exact. Note that the path integral over u_s enforces conservation of energy-momentum. This means that

$$d^\dagger T^a = 0, \quad (114)$$

which we can enforce by writing $T^a = d^\dagger \alpha^a$ for some 2-form α^a . To do this, we have made use of the decomposition

$$\ker(d_p^\dagger) = \text{im}(d_{p+1}^\dagger) \oplus H^p(X, \partial X), \quad (115)$$

where the cohomology group contains all cohomology classes which vanish on the boundary of X . We assume that this cohomology group is trivial, meaning that $d^\dagger T = 0$ implies that T is coexact.

Actually it will be more convenient to write things in terms of the 1-form $A^a \equiv \star \alpha^a$, since this will give us something more gauge-theory-like. Thus the integration of u_s sets

$$T^a = \star dA^a, \quad (116)$$

where A is a gauge field since $A \mapsto A + d\lambda$ doesn't affect the physical variable T^a . If the above cohomology group were nontrivial, then we would have $T^a = \star(dA + c)^a$, where $c \in H^2(X; \mathbb{Z})$ is a 2-form parametrizing the nontriviality of the A gauge bundle.

Thus we can write the Lagrangian as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\star dA^a)_\mu [C^{-1}]_{\mu\nu}^{ab} (\star dA^b)_\nu + i(\star dA^a)_\mu \partial_\mu u_d^a \\ &= \frac{1}{2}(\star F^a)_\mu [C^{-1}]_{\mu\nu}^{ab} (\star F^b)_\nu + iA^a \wedge d^2 u^a \\ &= \frac{1}{2}(\star F^a)_\mu [C^{-1}]_{\mu\nu}^{ab} (\star F^b)_\nu + iA^a \wedge \star J_d^a,\end{aligned}\tag{117}$$

where we have defined the dislocation current as

$$J_d^a \equiv \star d^2 u^a.\tag{118}$$

This is the main reason why working in the gauge field formulation is preferred: the dislocation current couples directly to the dual gauge field, rather than to the field-strength variable T_μ^a . We see that dislocations act as electric charges, and interact amongst themselves by exchanging photons. This is very similar to the “spin waves + vortices” picture in the XY model, except with an extra flavor index a and additional elastic moduli added in to the kinetic term in the form of C^{-1} .



Landau parameters and mass renormalization with magnetization

This is one of the problems in Piers Coleman's book (there are a few typos in the chapter and one in the problem though, so watch out). The problem statement is as follows:

Derive Landau's mass renormalization formulae in the case where the Fermi liquid has a nonzero magnetization. The starting point is the Hamiltonian in the presence of an external field for both the current and the z component of the spin current:

$$H[\mathbf{v}, \mathbf{w}] = \int \frac{1}{2m} \Psi^\dagger [-i\nabla - \mathbf{v} - Z\mathbf{w}]^2 \Psi + V,\tag{119}$$

where $\Psi = (\psi_\uparrow, \psi_\downarrow)^T$, Z is the third Pauli matrix, and V denotes interactions (which we assume conserve the spin current). \mathbf{v} is an external velocity that couples to the particle number current, and \mathbf{w} is an external field which couples to the spin current.

Find the renormalized mass parameters associated with the background fields in terms of the Landau parameters F_l^a, F_l^s , where F_l^s is the l th multipole moment of the density-density spin-independent part of the interaction, and F_l^a is the l th multipole part of the

spin-dependent part (that depends on the dot product of the two spins involved). Assume that there is no net magnetization in the system. Next, find the renormalized masses of the two spin species in the case of a net magnetization, when the density of states N_σ is not independent of σ . See Coleman's book for background on these parameters.

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We start by expanding the energy of the Fermi liquid in terms of the densities in the usual way, viz.

$$E \approx E_0 + \sum_{p\sigma} \epsilon_{p\sigma}^{(0)} n_{p\sigma} + \frac{1}{2} \sum_{p\sigma, p'\sigma'} f_{p\sigma, p'\sigma'} n_{p\sigma} n_{p'\sigma'}, \quad (120)$$

where $\epsilon_{p\sigma}^{(0)}$ is measured relative to the Fermi surface and is the energy of a single isolated quasiparticle. We define the first order variation of E as $\epsilon_{p\sigma}$:

$$\frac{\delta E}{\delta n_{p\sigma}} = \epsilon_{p\sigma} = \epsilon_{p\sigma}^{(0)} + \sum_{p'\sigma'} f_{p\sigma, p'\sigma'} n_{p'\sigma'}, \quad (121)$$

which captures the energy of a single quasiparticle when interactions with others are included, at least to quadratic order in the n_s .

Varying this equation gives us an expression for $\delta\epsilon_{p\sigma}$ in terms of $\delta n_{p\sigma}$. We will be calculating the renormalized parameters by expressing $\delta\epsilon_{p\sigma}$ in two ways, one in terms of the bare parameters and one in terms of the renormalized ones. From the previous expression, we see that to get $\delta\epsilon_{p\sigma}$ in terms of the bare parameters, we need to know the $\delta n_{p\sigma}$ that occurs for small departures of ϵ from the equilibrium value $\epsilon^{(0)}$. We find this by writing

$$n_{p\sigma} = f(\epsilon_{p\sigma}^{(0)}) + \frac{\partial f}{\partial \epsilon_{p\sigma}}(\epsilon_{p\sigma}^{(0)}) \delta\epsilon_{p\sigma}. \quad (122)$$

f here is the Fermi distribution, so at zero temperature we get

$$n_{p\sigma} = n_{p\sigma}^{(0)} - \delta_D(\epsilon_{p\sigma}^{(0)}) \delta\epsilon_{p\sigma}, \quad (123)$$

where δ_D is the Dirac delta. So putting this expression for $\delta n_{p\sigma}^{(0)}$ into the formula for $\delta\epsilon_{p\sigma}$, we get

$$\delta\epsilon_{p\sigma} = \delta\epsilon_{p\sigma}^{(0)} - \sum_{p'\sigma'} f_{p\sigma, p'\sigma'} \delta_D(\epsilon_{p\sigma}^{(0)}) \delta\epsilon_{p'\sigma'}. \quad (124)$$

The variation we will consider is that imposed by external fields which couple to the particle number current and spin current, which we will assume are conserved. For small applied fields, we see from the problem statement that

$$\delta\epsilon_{p\sigma}^{(0)} = -\frac{1}{m} \mathbf{p} \cdot (\mathbf{v} + \sigma \mathbf{w}). \quad (125)$$

Wolog, we can fix \mathbf{v} and \mathbf{w} to point in e.g. the \hat{z} direction, so that $\delta\epsilon_{p\sigma}^{(0)} \propto \cos\theta_{\mathbf{p}}$.

Now we decompose the interaction into spherical harmonics / Legendre polynomials:

$$f_{p\sigma,p'\sigma'} = \sum_{\alpha=s,a} \sum_{l=0}^{\infty} \frac{1}{2N_{\sigma}(0)} (2l+1) F_l^{\alpha} P_l(\cos \theta), \quad (126)$$

where θ is the angle between \mathbf{p} and \mathbf{p}' . Here the sum over α is over the two representations that the interactions can be in as far as the spin degrees of freedom are concerned. $\alpha = s$ is the trivial representation, which is independent of spin, while $\alpha = a$ (a for adjoint) gives a matrix in spin space which is dependent on the dot product of the two spins involved (this is the only other possibility by rotational invariance, or from the fact that $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$), and so we can rotate to a basis where $F_l^a \propto \sigma \sigma'$. We have also divided by the density of states so that the F_l^{α} are dimensionless. The choice of $2N_{\sigma}(0)$ will be explained later.

Now we insert this into the expression for $\delta\epsilon_{p\sigma}$. The Dirac delta makes us integrate over the Fermi surface, and we do so with

$$\sum_p \delta_D(\epsilon_{p\sigma}^{(0)}) = N_{\sigma}(0) \int \frac{d\Omega_p}{4\pi}, \quad (127)$$

where $N_{\sigma}(0)$ is the appropriate density of states. Now since we are integrating over the sphere against $\delta\epsilon_{p\sigma}$ which is proportional to $\cos \theta$, we select out the $l = 1$ component of the decomposition of the interaction (again by spin rotational invariance and actual rotational invariance, we can choose these two θ 's, viz. the angle between the two spins and the angle on the sphere, to be the same). That the $l = 1$ component gets selected is no surprise, since a uniform velocity field will displace the FS linearly in some direction, which is a dipolar perturbation. The normalization factors ensure that all the 4π 's and stuff cancel. So then

$$\delta\epsilon_{p\sigma} = \delta\epsilon_{p\sigma}^{(0)} - \sum_{\sigma'} \sum_{\alpha} \frac{1}{2N_{\sigma}(0)} N_{\sigma'}(0) F_1^{\alpha} \delta\epsilon_{p\sigma'}. \quad (128)$$

First let us assume that there is no net magnetization, so that $N_{\uparrow}(0) = N_{\downarrow}(0)$. We plug in the form for $\delta\epsilon_{p\sigma}^{(0)}$, and define the renormalized masses by

$$\delta\epsilon_{p\sigma} \equiv -\mathbf{p} \left(\frac{\mathbf{v}}{m^*} + \sigma \frac{\mathbf{w}}{m_s^*} \right), \quad (129)$$

where we have a separate renormalization for the spin part. Putting this in and equating coefficients of the background fields on both sides of the equation, one sees that the coefficient of \mathbf{v} contains only F_1^s (since the F_1^a part dies by antisymmetry under the spin sum on σ'), while the coefficient of \mathbf{w} contains only F_1^a (for the same reason). After a little algebra, we get

$$m^* = (1 + F_1^s)m, \quad m_s^* = (1 + F_1^a)m. \quad (130)$$

Thus, the phrase “mass renormalization” makes sense here, since the F_1^{α} 's appear just like counterterms that enter when renormalizing a QFT.

Now we consider the case where there is a net magnetization, with the density of states $N_{\uparrow} \neq N_{\downarrow}$. We want to find the renormalized masses m_{σ}^* , so we just turn on an external velocity field \mathbf{v} . The renormalized masses for each spin are of course defined by

$$\delta\epsilon_{p\sigma} = -\mathbf{p} \cdot \frac{\mathbf{v}}{m_{\sigma}^*}. \quad (131)$$

We again use (128), and set the coefficient of \mathbf{v} to vanish. Because the density of states are different, the antisymmetric F_1^a piece survives and makes a finite contribution. We thus end up having to solve

$$0 = -\frac{1}{m} + \frac{1}{m_{\downarrow}^*} + \frac{1}{2N_{\downarrow}} \left(F_1^s \left[\frac{N_{\uparrow}}{m_{\uparrow}^*} + \frac{N_{\downarrow}}{m_{\downarrow}^*} \right] + F_1^a \left[-\frac{N_{\uparrow}}{m_{\uparrow}^*} + \frac{N_{\downarrow}}{m_{\downarrow}^*} \right] \right), \quad (132)$$

together with the companion equation where the second term is $(m_{\downarrow}^*)^{-1}$. Putting these equations in mathematica and doing a bit of simplification,

$$\frac{1}{m_{\sigma}^*} = \frac{1}{2mn_{\sigma}} \left[\frac{1}{1 + F_1^s} + \sigma \frac{M}{1 + F_1^a} \right], \quad (133)$$

where $M = n_{\downarrow} - n_{\uparrow}$ and $n_{\sigma} = N_{\sigma}/(N_{\uparrow} + N_{\downarrow})$. One sees that when $M = 0$ we recover the expected answer, since then $n_{\sigma} \rightarrow 1/2$.

Note that in the book, he tells you to derive something different. The answer he tells you to get is probably (?) not right since it doesn't have the factor of σ in the second term on the LHS, and it also doesn't have the factor of $1/n_{\sigma}$ in front. I think this may be related to the somewhat ad-hoc choice of making F_1^{α} dimensionless by dividing by $2N_{\sigma}$, which reproduces the correct result in the $M = 0$ case but might not be the most universal choice. After playing around in Mathematica for far too long I couldn't come up with any way to make the interaction dimensionless that gave the answer listed in the book, so I'll just go with this. Plus, this factor kind of makes sense, since we expect a larger density of states to give a larger effective mass (there's more stuff to "get in the way" of quasiparticles moving around the FS, and in general [depending on how exactly we define the effective mass] we expect $m^* \propto 1/v_F \propto n$).



Photon-atom interactions and stimulated emission

Another simple problem today—this is from Coleman's many body theory book. Consider an atom that can be in two different electronic states, with energies $E_+ > E_-$. The Hamiltonian is

$$H = H_{ph} + H_a + H_I, \quad (134)$$

with

$$\begin{aligned} H_{ph} &= \sum_{\mathbf{q}} \omega_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \\ H_a &= E_+ c_+^\dagger c_+ + E_- c_-^\dagger c_- \\ H_I &= \sum_{\mathbf{q}} g_{\mathbf{q}} (c_+^\dagger c_- + c_-^\dagger c_+) (a_{-\mathbf{q}} + a_{\mathbf{q}}^\dagger), \end{aligned} \quad (135)$$

where the photon dispersion is $\omega_{\mathbf{q}} = c|\mathbf{q}|$.

By computing the self energies to second order in $g_{\mathbf{q}}$ and taking their imaginary parts, find the lifetimes of the + and - states. You can assume a non-degenerate scenario and can take $f(E_{\pm}) \approx 0$ (where f is the Fermi distribution). Your answer should be in terms of the Bose distribution at frequency $\omega_c = E_+ - E_-$.



We need to compute the self energies of the two states, which will come from the usual polarization bubble type of diagram, whereby the atom emits/absorbs a photon and hops to the other state, and then absorbs/emits the photon and returns to the original state. Since the photon coupling in the relevant vertex is $a_{-\mathbf{q}} + a_{\mathbf{q}}^\dagger$, it helps to get the propagator for the combination $b_{\mathbf{q}} = a_{-\mathbf{q}} + a_{\mathbf{q}}^\dagger$ instead of the propagator for $a_{\mathbf{q}}$. We have

$$\begin{aligned} D(\mathbf{q}, t) &= -i\langle 0 | T\{b_{\mathbf{q}}(t)b_{\mathbf{q}}^\dagger(0)\} | 0 \rangle \\ &= -i\langle 0 | T\{a_{-\mathbf{q}}(t)a_{-\mathbf{q}}^\dagger(0) + a_{\mathbf{q}}^\dagger(t)a_{\mathbf{q}}(0)\} | 0 \rangle \\ &= -i\Theta(t)e^{-i\omega_{\mathbf{q}}t} - i\Theta(-t)e^{i\omega_{\mathbf{q}}t}. \end{aligned} \quad (136)$$

Fourier transforming and not writing the convergence factor,

$$D(\mathbf{q}, \omega) = \frac{1}{\omega - \omega_{\mathbf{q}}} - \frac{1}{\omega + \omega_{\mathbf{q}}} = \frac{2\omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2}. \quad (137)$$

The self energy for the - state is then, now at imaginary frequencies,

$$\Sigma_-(\nu) = -T \sum_{\omega} \int_{\mathbf{q}} g_{\mathbf{q}}^2 \frac{2\omega_{\mathbf{q}}}{(i\omega)^2 - \omega_{\mathbf{q}}^2} \frac{1}{i(\nu - \omega) - E_+}. \quad (138)$$

Here ν is a fermionic Matsubara frequency, while ω is a bosonic one.

Rewriting the product of the propagators in a more useful form, and then separating the integrand into positive and negative frequency parts,

$$\begin{aligned} \Sigma_- &= -T \sum_{\omega} \int_{\mathbf{q}} g_{\mathbf{q}}^2 \left(\frac{1}{i\omega - \omega_{\mathbf{q}}} - \frac{1}{i\omega + \omega_{\mathbf{q}}} \right) \frac{1}{i(\nu - \omega) - E_+} \\ &= T \sum_{\omega} \int_{\mathbf{q}} \left(g_{\mathbf{q}}^2 \frac{1}{i\nu - (\omega_{\mathbf{q}} + E_+)} \left[\frac{1}{i\omega - \omega_{\mathbf{q}}} - \frac{1}{i\omega - (i\nu - E_+)} \right] - \dots \right), \end{aligned} \quad (139)$$

where ... indicates the same term but with $\omega_{\mathbf{q}} \rightarrow -\omega_{\mathbf{q}}$. To do the sum, we do the usual contour integral trick, by integrating against the Bose function $n(z)$. We do a counterclockwise integral around the imaginary axis, which since the sum of all the residues is zero, becomes the negative of the counterclockwise integral around all the other poles, namely the poles of the function in the integrand above. Let's just look at the positive $\omega_{\mathbf{q}}$ part. The poles in $z = i\omega$ are at $\omega_{\mathbf{q}}$ for the first term, and at $i\nu - E_+$ for the second term. So the positive $\omega_{\mathbf{q}}$ part gives

$$- \int_{\mathbf{q}} g_{\mathbf{q}}^2 \frac{n(\omega_{\mathbf{q}}) - n(i\nu - E_+)}{i\nu - \omega_{\mathbf{q}} - E_+}, \quad (140)$$

where there is no T since T is included in the poles of the integrand. Since ν is a fermionic Matsubara frequency,

$$n(i\nu - E_+) = \frac{1}{-e^{-\beta E_+} - 1} = f(E_+) - 1. \quad (141)$$

So the above term is

$$-\int_{\mathbf{q}} g_{\mathbf{q}}^2 \frac{1 + n(\omega_{\mathbf{q}}) - f(E_+)}{i\nu - \omega_{\mathbf{q}} - E_+}. \quad (142)$$

For the negative $\omega_{\mathbf{q}}$ part, the only thing that changes is that the pole shifts from being at $\omega_{\mathbf{q}}$ to being at $-\omega_{\mathbf{q}}$. Then we use

$$n(-\omega_{\mathbf{q}}) = \frac{e^{\beta \omega_{\mathbf{q}}}}{1 - e^{\beta \omega_{\mathbf{q}}}} = -n(\omega_{\mathbf{q}}) - 1. \quad (143)$$

Putting this together, the full self energy is

$$\Sigma_-(\nu) = -\int_{\mathbf{q}} g_{\mathbf{q}}^2 \left(\frac{1 + n(\omega_{\mathbf{q}}) - f(E_+)}{i\nu - \omega_{\mathbf{q}} - E_+} + \frac{n(\omega_{\mathbf{q}}) + f(E_+)}{i\nu + \omega_{\mathbf{q}} - E_+} \right). \quad (144)$$

Now let's find the lifetime. We can get the lifetime with

$$\tau_{\sigma}^{-1} = 2\text{Im} [\Sigma_{\sigma}(E_{\sigma} - i\eta)]. \quad (145)$$

Putting this in, and taking $f(E_+) \approx 0$,

$$\begin{aligned} \tau_{-}^{-1} &= 2\pi \int \frac{d^3 q}{(2\pi)^3} g_{\mathbf{q}}^2 ((1 + n(\omega_{\mathbf{q}}))\delta(-\omega_c - \omega_{\mathbf{q}}) + n(\omega_{\mathbf{q}})\delta(\omega_c - \omega_{\mathbf{q}})) \\ &= \frac{1}{\pi c^3} g(\omega_c)^2 n(\omega_c) \omega_c^2. \end{aligned} \quad (146)$$

To get the lifetime for the other state, we just have to swap plus signs and minus signs in the above. This leads to the same expression except with ω_c swapped with $-\omega_c$ in the first line of the above equation. Thus

$$\tau_{+}^{-1} = \frac{1}{\pi c^3} g(\omega_c)^2 [1 + n(\omega_c)] \omega_c^2. \quad (147)$$

As a sanity check, note that the lifetime in the lower level is longer by a factor

$$\frac{\tau_{-}}{\tau_{+}} = \frac{1 + n(\omega_c)}{n(\omega_c)} = e^{\beta(E_+ - E_-)}, \quad (148)$$

which makes sense as it's equal to the ratio of the Boltzmann weights for the two states.



AFM spin waves and itinerant magnetism

This is yet again from Coleman's many body theory book, chapter 13. Consider a three-dimensional tight-binding model on a cubic lattice, and develop the mean field theory for the AFM commensurate order parameter

$$M_i^\mu = M^\mu e^{iQ \cdot r_i}, \quad (149)$$

where μ is a vector index and i is a site on the lattice, and Q is the AFM vector $Q = (\pi, \pi, \pi)$.

Find the mean-field free energy and the gap equation for the magnitude M . Argue that at half-filling, we will always have a transition to a spin-density wave state even for arbitrarily small interaction strength (hint: nesting).



First we need to get the MF Hamiltonian. We start from (being sloppy about the index notation)

$$H = -t \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + \mu c_j^\dagger c_j) + U \sum_j n_{\uparrow j} n_{\downarrow j}, \quad (150)$$

and then fiddle with the interaction in the usual way so as to facilitate the mean-field decoupling, writing it more democratically in spin space as (using $n_{\sigma j}^2 = n_{\sigma j}$)

$$U n_{\uparrow j} n_{\downarrow j} = -\frac{U}{6} (c_{\alpha j}^\dagger \sigma_{\alpha\beta}^\mu c_{\beta j})^2 + \frac{U}{2} (n_{\uparrow j} + n_{\downarrow j}) \rightarrow -\frac{I}{2} (c_{\alpha j}^\dagger \sigma_{\alpha\beta}^\mu c_{\beta j})^2, \quad (151)$$

where $I = U/3$ and we've absorbed the $n_\uparrow + n_\downarrow$ term into the chemical potential. We then decouple the interaction with the mean field M . Because M^μ is at momentum Q , in momentum space we end up with a coupling like

$$M^\mu (c_{\alpha k+Q}^\dagger \sigma_{\alpha\beta}^\mu c_{\beta k}). \quad (152)$$

It thus helps to split up the electron operators in the BZ according to \mathbf{Q} . Consider the mini-BZ that stretches from 0 to $\pm\pi/2$ for all coordinates. Any point in the BZ can be reached by adding a \mathbb{Z}_2 -multiple of Q to a point in the mini-BZ, and furthermore almost all (in the mathematical sense) of the points outside the mini-BZ can be associated with a unique point in the mini-BZ after taking off \mathbf{Q} . So then we can just sum over the mini-BZ instead, and keep track of the operators located outside the mini-BZ by stacking on the \mathbf{Q} vector. So then define the four-component spinor

$$\Psi_{\mathbf{k}} = (c_{\uparrow \mathbf{k}}, c_{\downarrow \mathbf{k}}, c_{\uparrow \mathbf{k}+\mathbf{Q}}, c_{\downarrow \mathbf{k}+\mathbf{Q}})^T. \quad (153)$$

The Hamiltonian has the free c^\dagger part, the part coupling the order parameter to the fermions (which connects stuff in the mini-BZ to stuff outside of it, so it will be off-diagonal when

acting on $\Psi_{\mathbf{k}}$), and the $M^2/2I$ quadratic part. Neglecting to write the “volume” of the lattice, we have

$$H = \sum_{\mathbf{k} \in \text{mini-BZ}} \Psi_{\mathbf{k}}^\dagger ((\epsilon_{\mathbf{k}} - \mu) \mathbf{1} \oplus (\epsilon_{\mathbf{k}+\mathbf{Q}} - \mu) \mathbf{1} + X \otimes M_\mu \sigma^\mu) \Psi_{\mathbf{k}} + \frac{1}{2} M^2. \quad (154)$$

The dispersion is, just read off from the tight-binding model,

$$\epsilon_{\mathbf{k}} = -2t \sum_{\mu} \cos k_{\mu}, \quad (155)$$

and so we see that we have the important nesting property, namely that $\epsilon_{\mathbf{k}+\mathbf{Q}} = -\epsilon_{\mathbf{k}}$. Note also that the eigenvalues of the $\Psi^\dagger \Psi$ term are

$$E_{\mathbf{k}s} = s \sqrt{\epsilon_{\mathbf{k}}^2 + M^2} - \mu, \quad s = \pm 1. \quad (156)$$

Now let's get the free energy. We get the usual log det, and then have to evaluate

$$\sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \sum_{\omega} \ln(i\omega + E_{\mathbf{k}s}). \quad (157)$$

We do the Matsubara sum by integrating against the Fermi distribution. Besides the Matsubara frequencies, we have a branch cut for the log, starting at $E_{\mathbf{k}s}$ and e.g. going out to ∞ along the \mathbb{R} axis. The change in the branches across the \mathbb{R} axis is $2\pi i$, and so we get

$$- \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \int_{E_{\mathbf{k}s}}^{\infty} d\alpha f(\alpha). \quad (158)$$

Now

$$- \int_a^{\infty} dx \frac{1}{e^{\beta x} + 1} = a - T \ln(1 + e^{\beta a}), \quad (159)$$

which you can get by differentiating with respect to a . Using this and putting in $E_{\mathbf{k}s}$ for a , we see that the first term above gives us -4μ , since we have four eigenvalues for each k . The sum over s kills the $s\sqrt{\epsilon^2 + M^2}$ part, so we can borrow two of the μ 's in the -4μ to write the term coming from a on the RHS of the last equation as

$$\sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \left(-2\mu + \frac{\beta E_{\mathbf{k}s}}{2} \right). \quad (160)$$

Why did we do this dumb manipulation, adding in terms that vanish after summing over s anyway? Because it simplifies the $\ln(1 + e^{\beta a})$ part of the integral. We take the $\frac{\beta E_{\mathbf{k}s}}{2}$ term above and write

$$\sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \left[\frac{\beta E_{\mathbf{k}s}}{2} + \ln(1 + e^{-\beta E_{\mathbf{k}s}}) \right] = \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \ln[2 \cosh(\beta E_{\mathbf{k}s}/2)]. \quad (161)$$

So, combining this with the remaining -2μ part and the M^2 term, we finally get

$$F = -\frac{1}{\beta} \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \ln \left[2 \cosh \left(\frac{E_{\mathbf{k}s}\beta}{2} \right) \right] + \frac{M^2}{2I} - 2\mu. \quad (162)$$

Now we can differentiate with respect to M to find the gap equation. This is straightforward, and we get

$$\frac{1}{I} = \frac{1}{2} \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \tanh\left(\frac{\beta E_{\mathbf{k}s}}{2}\right) \frac{1}{\sqrt{\epsilon_{\mathbf{k}}^2 + M^2}}. \quad (163)$$

Now we see why the nesting property is important. Consider some small value of the interaction strength I . Since $\epsilon_{\mathbf{k}+Q} = -\epsilon_{\mathbf{k}}$, $\epsilon_{\mathbf{k}}$ must vanish on a codimension-1 surface in the (full) BZ. Thus at $M = 0$ the RHS diverges, since we are integrating something which has a $1/r$ -type singularity on a codimension-1 submanifold of the integration domain (note to self: this is true for this example and other simple ones — is it true in general? Should be simple to figure out). This means that by making M small, we can make the RHS of the above equation arbitrarily large (provided we also choose T to be small enough so that the tanh piece doesn't cancel the divergence). In particular, at least at $\mu = 0$, we can always find a non-zero solution for M of the above equation, no matter how small I is (think of superconductivity), and so we will always have a transition to a spin-density wave phase even at arbitrarily small I .



Consider the Anderson impurity Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \left(\epsilon_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + V_{\mathbf{k}} f_\sigma^\dagger c_{\mathbf{k}\sigma} + V_{\mathbf{k}}^* c_{\mathbf{k}\sigma}^\dagger f_\sigma \right) + \sum_{\sigma} \epsilon_f n_{f\sigma} + U n_{f\uparrow} n_{f\downarrow}. \quad (164)$$

Here the f (f for the f -orbitals that are operative in the heavy-fermion context) operators add / remove electrons from the impurities, and ϵ_f is assumed to lie below the Fermi energy while $2\epsilon_f + U$ is assumed to lie above. Both ϵ_f and U are to be taken to be large parameters.

By splitting the total wavefunction into $\psi_0 + \psi_1 + \psi_2$ where the subscripts indicate the occupancy of the impurity, we will use perturbation theory in V to find the effective Hamiltonian for the subspace in which the impurity is singly occupied. The point of this is just to establish a simple intermediary result that we'll need in the next diary entry.



Decomposing the wavefunction as suggested, the orthogonality of the ψ_i s and the Schrodinger equation tells us that

$$\begin{aligned} (H_{00} + H_{01})(\psi_0 + \psi_1) &= E\psi_0, \\ (H_{11} + H_{12} + H_{10})(\psi_0 + \psi_1 + \psi_2) &= E\psi_1, \\ (H_{22} + H_{21})(\psi_1 + \psi_2) &= E\psi_2, \end{aligned} \quad (165)$$

where H_{ij} connects the j occupancy subspace to the i occupancy subspace. To get the effective Hamiltonian acting on just the singly-occupied subspace, we then use the first equation to write $\psi_0 = (H_{00} - E)^{-1} H_{01} \psi_1$ and the last equation to write $\psi_2 = (E - H_{22})^{-1} H_{12} \psi_1$, which when together substituted into the middle equation yields the expected Schrodinger equation on the singly-occupied subspace:

$$\left(H_{12} \frac{1}{E - H_{22}} H_{21} + H_{11} + H_{10} \frac{1}{E - H_{00}} H_{01} \right) \psi_1 = E \psi_1. \quad (166)$$

From the Hamiltonian given above, we know that each term is

$$H_{00} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma}, \quad H_{11} = H_{00} + \epsilon_f, \quad H_{22} = H_{00} + 2\epsilon_f + U \quad (167)$$

for the diagonal terms and

$$H_{01} = \sum_{\mathbf{k}\sigma} V_{\mathbf{k}}^* c_{\mathbf{k}\sigma}^\dagger f_\sigma, \quad H_{21} = \sum_{\mathbf{k}\sigma} V_{\mathbf{k}} f_\sigma^\dagger c_{\mathbf{k}\sigma}, \quad (168)$$

for the off-diagonal ones, where $-\sigma$ is the opposite spin to σ . We also have $H_{10} = H_{01}^\dagger$ and $H_{12} = H_{21}^\dagger$ so that e.g.

$$H_{12} = \sum_{\mathbf{k}\sigma} V_{\mathbf{k}}^* c_{\mathbf{k}\sigma}^\dagger f_\sigma. \quad (169)$$

Now we need to simplify the effective Hamiltonian. Consider first the leftmost term. We write

$$\frac{1}{E - H_{22}} = \frac{1}{E - U - 2\epsilon_f - H_{00}} = \left[(-U - \epsilon_f) \left(1 - \frac{E - \epsilon_f - H_{00}}{U + \epsilon_f} \right) \right]^{-1}. \quad (170)$$

Since the Fermi level was chosen to be above ϵ_f but below $2\epsilon_f + U$, the $E - \epsilon_f - H_{00}$ term will be small when acting on ψ_1 and dividing by $U + \epsilon_f$ will make it smaller. Thus we can write

$$\frac{1}{E - H_{22}} \approx -\frac{1}{U + \epsilon_f} \left(1 + \frac{E - \epsilon_f - H_{00}}{U + \epsilon_f} \right). \quad (171)$$

We then put this in the Hamiltonian and move the fraction through H_{21} so that it acts on ψ_1 . When we move H_{00} through H_{21} we pick up the extra term

$$[H_{00}, H_{21}] = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} V_{\mathbf{k}} c_{\mathbf{k}\sigma} f_\sigma^\dagger, \quad (172)$$

and so the first term in the effective Hamiltonian is

$$- H_{12} \frac{1}{U + \epsilon_f} \left(H_{21} + H_{21} \frac{E - \epsilon_f - H_{00}}{U + \epsilon_f} - \frac{1}{U + \epsilon_f} \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} V_{\mathbf{k}} c_{\mathbf{k}\sigma} f_\sigma^\dagger \right). \quad (173)$$

The second term in the above equation will be droppable after acting on ψ_1 . We can then also take the $\epsilon_{\mathbf{k}}$ part and put it back in the denominator. So then we can write

$$- H_{12} \frac{1}{U + \epsilon_f - \epsilon_{\mathbf{k}}} H_{21}, \quad (174)$$

and we get the expected energy transfer in the denominator (the \mathbf{k} in the denominator makes sense after putting in the sums for H_{12} and H_{21} —it is the momentum of the latter).

We do the exact same thing for the other term in the effective Hamiltonian (which takes us through the unoccupied intermediate state). The algebra is the same as before, and we get

$$-H_{10} \frac{1}{\epsilon_{\mathbf{k}} - \epsilon_f} H_{01}. \quad (175)$$

Putting this together, the effective Hamiltonian is (ignoring constants)

$$H_{\text{eff}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_{\mathbf{kk}'\sigma\sigma'} V_{\mathbf{k}} V_{\mathbf{k}'}^* \left(\frac{f_\sigma^\dagger f_{\sigma'} c_{\mathbf{k}\sigma} c_{\mathbf{k}'\sigma'}}{\epsilon_{\mathbf{k}'} - \epsilon_f} + \frac{f_{\sigma'}^\dagger f_\sigma c_{\mathbf{k}'\sigma'}^\dagger c_{\mathbf{k}\sigma}}{U + \epsilon_f - \epsilon_{\mathbf{k}}} \right). \quad (176)$$

To write this in a more transparent form, we define the spin operators through the usual

$$S_f^\mu = \frac{1}{2} \sum_{ab} f_a^\dagger [\sigma^\mu]_{ab} f_b, \quad s_{\mathbf{kk}'}^\mu = \frac{1}{2} \sum_{ab} c_{\mathbf{k}a}^\dagger [\sigma^\mu]_{ab} c_{\mathbf{k}'b}. \quad (177)$$

Using

$$\sigma_{ab}^\mu \sigma_{cd}^\mu = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd} \quad (178)$$

to split the interaction up into singlet and triplet channels, we can write

$$\sum_{\sigma\sigma'} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}'\sigma'} f_{\sigma'}^\dagger f_\sigma = 2s_{\mathbf{kk}'}^\mu S_f^\mu + \frac{1}{2} \sum_{\sigma\sigma'} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}'\sigma} f_{\sigma'}^\dagger f_\sigma, \quad (179)$$

We now use this identity to get the terms in H_{eff} into a form like the form of the LHS above. This just involves commuting a few operators past each other: when we commute the c 's in the middle term of H_{eff} we get a number operator in f 's, which when acting on the low energy space is equal to the identity, and so we just get an extra constant. When commuting the f 's in the third term, we get an extra term with the operators $c_{\mathbf{k}'\sigma}^\dagger c_{\mathbf{k}\sigma}/(U + \epsilon_f - \epsilon_{\mathbf{k}})$. This comes in with a minus sign relative to the terms of the form $c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}'\sigma'} f_{\sigma'}^\dagger f_\sigma$, and so we get

$$H_{\text{eff}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{kk}'} \left(2J_{\mathbf{kk}'} S_f^\mu s_{\mathbf{kk}'}^\mu + T_{\mathbf{kk}'} \sum_{\sigma} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}'\sigma} \right), \quad (180)$$

where the coefficients are

$$J_{\mathbf{kk}'} = V_{\mathbf{k}} V_{\mathbf{k}'}^* \left(\frac{1}{U + \epsilon_f - \epsilon_{\mathbf{k}}} + \frac{1}{\epsilon_{\mathbf{k}'} - \epsilon_f} \right) \quad (181)$$

and

$$T_{\mathbf{kk}'} = -\frac{1}{2} V_{\mathbf{k}} V_{\mathbf{k}'}^* \left(\frac{1}{U + \epsilon_f - \epsilon_{\mathbf{k}}} - \frac{1}{\epsilon_{\mathbf{k}'} - \epsilon_f} \right), \quad (182)$$

with the relative minus sign in the last expression coming from the commuting we did to the c 's. The form of J is of course expected, since it's just the strength of the tunneling between the impurity and the conduction band times the sum of the energy denominators for the two relevant virtual processes. Since the interactions between the impurity and the

conduction band are happening in the singlet channel, we get an *antiferromagnetic* exchange coupling between the impurity and the spin density of the conduction electrons.



More about the Kondo problem and doing RG with the SW transformation

Today's diary entry uses the tools of yesterday's diary entry to implement an RG analysis on the Anderson impurity Hamiltonian, generalized to allow for anisotropic exchange couplings. We start with

$$H = H_0 + \sum_{\mathbf{kq}} \left(J_z S_f^z [c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{q}\uparrow} - c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{q}\downarrow}] + J_+ S_f^+ c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{q}\uparrow} + J_- S_f^- c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{q}\downarrow} \right), \quad (183)$$

where

$$H_0 = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \quad (184)$$

and where the S_f^a operators are the spin operators for the (spin 1/2) impurity. The problem statement is as follows:

Let the conduction electron band extend from $-\Lambda$ to Λ . The high energy degrees of freedom that need to be integrated out during the RG step are the electrons at the top of the band and holes at the bottom of the band. The RG will proceed in the Wilsonian perspective, by shaving off small regions around the band edges at $\pm\Lambda$.

Do a SW transformation on the Hamiltonian by decomposing $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, where \mathcal{H}_0 contains state vectors with at least one hole within some small energy window $\delta\Lambda$ of the lower band edge, \mathcal{H}_2 contains vectors with at least one electron within a small window $\delta\Lambda$ of the upper band edge, and \mathcal{H}_1 is the low-energy submanifold. Proceeding in the usual way, find the effective Hamiltonian and obtain the beta functions for J_z and J_\pm . Show that in the antiferromagnetic case of the Kondo problem, the theory is asymptotically free and find the Kondo temperature (the dimensionally transmuted scale in the problem—think Λ_{QCD}).



As we saw in the previous diary entry, the Schrodinger equation reads

$$\left(H_{12} \frac{1}{E - H_{22}} H_{21} + H_{11} + H_{10} \frac{1}{E - H_{00}} H_{01} \right) \psi_1 = E \psi_1. \quad (185)$$

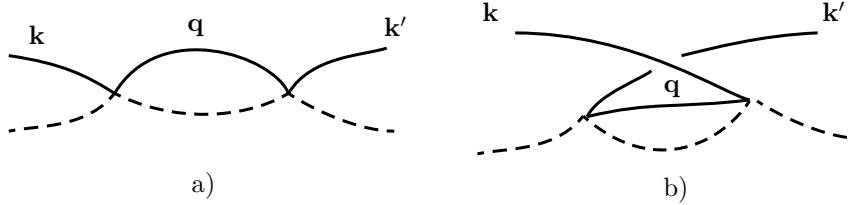


Figure 2: The two kinds of processes we need to worry about. The dashed lines are the impurity spins and the solid lines are conduction band electrons.

From the definition of the 0, 1, 2 decomposition, the term with $1/(E - H_{00})$ is essentially the particle-hole conjugate of the term with $1/(E - H_{22})$. Let's look at the latter first. We have a bunch of terms, that go as $J_z^2, J_+J_-, J_+J_z, J_-J_z$, and flip-flops thereof.

Let's just start by looking at the $S_f^- S_f^+$ term. It is

$$J_- J_+ \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} S_f^- c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{q}\downarrow} \frac{1}{E - H_{22}} S_f^+ c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{k}\uparrow}, \quad (186)$$

where \mathbf{k}, \mathbf{k}' are slow momenta and \mathbf{q} is a fast momentum. We were able to write the momentum structure in the way we did above since H_{22} is zero when acting on states with no fast electrons and since H_{22} conserves momentum.

Now we do the integral over \mathbf{q} : letting $\rho(\Lambda)$ be the density of states at the band edge, we get (using the fact that $E - H_{22}$ becomes $E - \Lambda + \epsilon_{\mathbf{k}} - H_0$ when acting on a state with an electron taken up to the band edge from a state with energy $\epsilon_{\mathbf{k}}$)

$$J_- J_+ \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda S_f^- S_f^+ c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{k}\uparrow} \frac{1}{E - \Lambda + \epsilon_{\mathbf{k}} - H_0}, \quad (187)$$

where $\delta\Lambda$ is the small energy window we've integrated out, over which the density of states is assumed constant. When we take the energy relative to the ground state and keep in mind that the cutoff Λ is much larger than any other scales we're interested in, we can write (using $S_f^+ S_f^- = 0 \oplus 1 = 1/2 - S_f^z$)

$$J_- J_+ \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} - S_f^z \right) c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{k}\uparrow} \frac{1}{E - \Lambda}. \quad (188)$$

We see that the S_f^z piece of this term will go towards renormalizing J_z , since it's diagonal in the spin index of the conduction band electrons. If we were to look at the $S_f^+ S_f^-$ term, we would get (following the same argument)

$$J_+ J_- \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} + S_f^z \right) c_{\mathbf{k}'\downarrow}^\dagger c_{\mathbf{k}\downarrow} \frac{1}{E - \Lambda}. \quad (189)$$

Now we look at the particle-hole conjugated versions of these two contributions. To find them, we do the same thing as in the above but with $E - H_{00}$ in the denominator. The process mediated by this term involves a creation of a hole at the lower band edge

together with a slow-momentum electron, and then a subsequent annihilation of the hole with a different slow-momentum electron. This is shown in part b of Figure 2 (time is read left-to-right), with the diagram leading to the previous contributions shown in part a. As we see from the figure, the particle-hole conjugate version has an extra overall minus sign (the crossed fermion lines) and involves the spin operators changing places, since the slow electron which escapes now participates in the interaction at an earlier time. This means that the ph conjugates contribute

$$- J_- J_+ \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} + S_f^z \right) c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{k}\uparrow} \frac{1}{E - \Lambda} \quad (190)$$

and

$$- J_+ J_- \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} - S_f^z \right) c_{\mathbf{k}'\downarrow}^\dagger c_{\mathbf{k}\downarrow} \frac{1}{E - \Lambda}. \quad (191)$$

The actual denominators are of these terms are slightly different than their ph conjugate partners, but since we're sending $E - \Lambda \pm \epsilon_{\mathbf{k}} \rightarrow E - \Lambda$ for slow \mathbf{k} , the differences don't show up.

Adding up all of the terms collected so far, we see that the $1/2$ terms cancel while the $\pm S_f^z$ terms add, so that the effective J_z coupling becomes

$$J_z \mapsto J_z - 2J_+ J_- \rho(\Lambda) \delta\Lambda \frac{1}{E - \Lambda}. \quad (192)$$

Now we need to find the renormalization of the J_{\pm} couplings, which will come from spin-flipping terms that go as either $J_z J_+$ or $J_z J_-$. Again we first look at the terms involving $1/(E - H_{22})$. Choosing e.g. the $J_- J_z$ term, we get a contribution

$$- J_- J_z \sum_{\mathbf{k}\mathbf{k}'} \frac{S_f^-}{2} \rho(\Lambda) \delta\Lambda c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}'\downarrow} \frac{1}{E - \Lambda}, \quad (193)$$

where we used $S_f^z S_f^- = -S_f^-/2$. The ph-conjugate has switched spin operators and an extra minus sign which cancel, so it adds with the above term. The $J_+ J_z$ term is obtained in the same way. Thus the J_{\pm} couplings change as

$$J_{\pm} \mapsto J_{\pm} - 2J_z J_{\pm} \rho(\Lambda) \delta\Lambda \frac{1}{E - \Lambda}. \quad (194)$$

Now we can get the β functions, since we know the effective couplings: they are⁸

$$\beta_{J_s} = \frac{dJ_s}{d\ln\Lambda} = -2J_s J_z \rho \quad s = \pm 1, \quad \beta_{J_z} = \frac{dJ_z}{d\ln\Lambda} = -2J_+ J_- \rho, \quad (195)$$

where we replaced $E - \Lambda$ with $-\Lambda$ since if the Wilsonian RG we're doing is going to work we need the “average energy scale” associated with the slow momentum modes (the eigenvalue

⁸We've ignored the terms quadratic in J_z since they are independent of the impurity spins and just lead to a renormalization of the kinetic energy term H_0 .

of ψ_1 under H) to be super small compared to Λ . Taking the ratio of the β functions, we see that

$$\frac{dJ_z}{dJ_{\pm}} = \frac{J_{\mp}}{J_z} \implies J_z^2 - J_+ J_- = \text{const.} \quad (196)$$

Consider the case where J_+ , J_- , and J_z all have the same sign. Note that then J_z is monotonically increasing along the RG trajectory (flowing from small to large distances). If we start with a ferromagnetic coupling ($J_z < 0$) and start with $J_z^2 > J_+ J_-$, then the flow is towards trivial couplings. If $J_z^2 < J_+ J_-$ then it seems like J_z is allowed to cross through zero with the J_{\pm} afterwards becoming increasingly negative, which is weird.

However, if we have an antiferromagnetic coupling (which as we saw above, we do) then the β function for J_z is asymptotically free and J_z increases without bound (meaning it increases until our perturbative calculation breaks down). This is of course the cause of the resistivity minimum.

Finally, let us consider for simplicity the case where $J = J_{\pm} = J_z$ are all antiferromagnetic. Introducing the dimensionless variable (at a particular energy scale we suppress)

$$g \equiv J\rho, \quad (197)$$

we see from the β function for g that at a given energy scale T , (here Λ is an arbitrary reference point for the RG)

$$g(T) = \frac{g(\Lambda)}{1 + 2g(\Lambda) \ln(T/\Lambda)}. \quad (198)$$

The dimensionally transmuted scale at which we lose perturbation theory (the Kondo temperature) is the root of the numerator, and so T_K is the familiar expression

$$T_K = \Lambda \exp\left(-\frac{1}{2g(\Lambda)}\right). \quad (199)$$



Beta function in the Kondo problem to 2-loop order

One last problem on the Kondo problem. Today we'll look at the beta function in more detail. The result is standard, but I thought it'd be helpful to work things out in detail and keep the calculations around for future reference.

Recall that the Hamiltonian we're interested in is

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} JS_f^a \left(\frac{1}{2} c_{\mathbf{k}\alpha}^\dagger [\sigma^a]_{\alpha\beta} c_{\mathbf{k}'\beta} \right) \quad (200)$$

Defining the dimensionless constant

$$g = J\rho, \quad (201)$$

where ρ is the density of states at the band edge (for a given cutoff), we will show that the beta function for g is

$$\beta_g = -2g^2 + 2g^3 + O(g^4). \quad (202)$$



To handle the impurity spins in Feynman diagrams, we will fractionalize the impurity spin as $S_f^a = \frac{1}{2}f_\alpha^\dagger\sigma_{\alpha\beta}^af_\beta$. We need to then project onto the singly-occupied f states, which we do by adding the chemical potential term $\lambda_f(f_\alpha^\dagger f_\alpha - 1) = [i\pi/(2\beta)](f_\alpha^\dagger f_\alpha - 1)$ to the Hamiltonian. This works since the normal Hamiltonian is zero for unoccupied or doubly-occupied f fermion states, and it ensures that the unphysical occupations cancel out in the partition function (only “unphysical” since we are restricting ourselves to the singly-occupied space—physically, the impurity spin could be unoccupied). However since we will be taking $T \rightarrow 0$ we won’t usually have to worry about adding this chemical potential term into the impurity spin propagator.

First let’s get the self energy for the impurity spin. One of the diagrams is taken care of by normal ordering, and so the diagram we need to compute looks like



where the dashed lines represent the impurity spin and the solid lines are the electrons.

First we will find the polarization bubble $\Pi_2(i\omega)$, which has one impurity line and one electron line. We will assume that all the lines in the bubble are pointing to the right, so that the diagram is (note: we will not keep careful track of the spin operators σ and S_f for a bit—they only produce some factors of 2 when summed over which can fairly easily be put back in later)

$$\Pi_2(\omega) = \frac{1}{\beta} \sum_{\omega} \int_{\epsilon} \rho(\epsilon) \frac{1}{i\omega - \epsilon} \frac{1}{i\nu - i\omega - \lambda_f}. \quad (204)$$

We’ve written it in this way since even though we’re taking $\beta \rightarrow \infty$, it actually turns out to be slightly easier to do the Matsubara sum compared to the integrals. The Matsubara sum is standard, and after doing it we do integral over ϵ from the lower band edge $-\Lambda$ up to Λ , and then set $\lambda_f = 0$. This gives (here $\omega > 0$)

$$\Pi_2(\omega) = \rho \ln \left(\frac{\Lambda}{|\omega|} \right), \quad (205)$$

where we took a flat density of states.

Now we can calculate the self energy using the melon we drew above. The top two lines of the melon are just Π_2 , while the bottom line just contributes an extra electron

propagator. There are three different melons (three places to put the dashed line) and each has a symmetry factor of 2, so the self energy is (here we take the top two lines in the melon to point right and the bottom line to point left)

$$\Sigma(\omega) = -\frac{3J^2}{2} \sum_{\mathbf{k}} \int \frac{d\nu}{2\pi} \Pi_2(\omega + \nu) \frac{1}{i\nu - \epsilon_{\mathbf{k}}}. \quad (206)$$

When we do the integral over energy we get a contribution from the half-residue of $-i\pi \text{sgn}(\nu)\rho$, so

$$\Sigma(\omega) = \frac{3i}{4} g^2 \int_{-\Lambda}^{\Lambda} d\nu \ln \left(\frac{\Lambda}{|\omega + \nu|} \right) \text{sgn}(\nu), \quad (207)$$

which gives us (for $\omega > 0$)

$$\Sigma(\omega) = -\frac{3i}{2} g^2 \omega \ln(\Lambda/\omega). \quad (208)$$

The self energy determines the wavefunction renormalization for the conduction electrons. We get it from the residue of $[i\omega - \Sigma(\omega)]^{-1}$, and so at an energy scale μ where we're doing the renormalization, to $O(g^2)$, we have

$$Z(\mu) = 1 - \frac{3}{2} g^2 \ln(\Lambda/\mu). \quad (209)$$

The last two-loop diagram we need to know is the 2-loop vertex correction. This contribution is illustrated by the worst looking figure I've ever made:



To do RG, we need to evaluate this diagram at a single energy scale. The RG condition we will choose is to have all the legs at the same energy μ (all the legs except for the top one in the electron bubble are assumed to be pointing to the right). With this choice, the two internal f impurity spin legs have the same frequency. Thus we need to compute something like

$$\frac{1}{2} J^2 (-JS_f^a \sigma^a) \int_{\mathbf{k}, \mathbf{k}'} \int_{\nu, \gamma} \frac{1}{[i(\mu + \gamma - \nu)]^2} \frac{1}{i\gamma - \epsilon_{\mathbf{k}}} \frac{1}{i\nu - \epsilon_{\mathbf{k}'}}, \quad (211)$$

where due to our choice of arrows in the diagrams there are no minus signs from fermion bubbles, the $1/2$ is a symmetry factor, and the σ^a is the spin operator for the conduction electrons (there are also spin operators for the other vertices which we are omitting). Note that except for the $-JS_f\sigma$ interaction piece, the above integral is just the derivative with respect to $i\mu$ of the diagram we computed for $\Sigma(i\omega)$, expect without the factor of 3. So then the diagram of (210), evaluated at the scale μ , is

$$-\frac{1}{2} (JS_f^a \sigma^a) g^2 \ln(\Lambda/\mu). \quad (212)$$

Now only the easy part is left: getting the 1-loop corrections to the vertex. We actually essentially already did this in the last problem. They can just be computed using our knowledge of the polarization bubble. The contributions from the particle (s-channel) and hole (t-channel) processes have opposite signs, but the ordering of the σ^a operators is also different. So the 1-loop diagrams give the counterterms

$$- J^2 \rho \ln(\Lambda/\mu) (\sigma^a \sigma^b - \sigma^b \sigma^a) S_f^a S_f^b. \quad (213)$$

We can antisymmetrize the $S_f^a S_f^b$ by adding a factor of $1/2$. The product of the two commutators is $-\epsilon^{abc} \epsilon^{abd} \sigma^c S_f^d = -2\sigma^a S_f^a$, and so the end result of the 1-loop terms is to produce the counterterm

$$2J^2 \rho \ln(\Lambda/\mu) \sigma^a S_f^a. \quad (214)$$

Collecting all of the counterterms and noting that $1/2 + 3/2 = 2$, the counterterms for the $J\sigma^a S_f^a$ interaction are

$$\text{counterterms} = 2 \ln\left(\frac{\Lambda}{\mu}\right) (J^2 \rho - J^3 \rho^2). \quad (215)$$

Multiplying everything by ρ gives us the β function for g by differentiating the RHS of the above equation with respect to $\ln \mu$:

$$\beta_g = -2(g^2 - g^3) + O(g^4). \quad (216)$$

Alternately, from a more Wilsonian point of view, we could say that the effective coupling constant for the $\sigma^a S_f^a$ term with the RG conditions imposed at the energy scale μ is

$$g(\mu) = g - 2g^2 \ln(\Lambda/\mu) + 2g^3 \ln(\Lambda/\mu), \quad (217)$$

and so taking $\Lambda \mapsto \Lambda + d\Lambda$, $g(\mu) \mapsto g(\mu) + dg$, and $\ln(1 + d\Lambda/\Lambda) = d \ln \Lambda$, we have

$$dg(\mu) = d \ln \Lambda (-2g^2 + 2g^3), \quad (218)$$

which of course gives us the same β function.



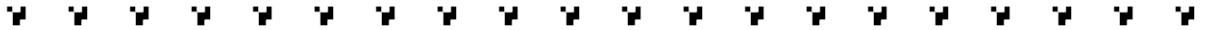
First, consider a bunch of 1+1D free fermion wires ($2N$ of them) indexed by j and coupled together in the following fashion (our notation for the hopping departs slightly from McGreevy and the original paper by Senthil + Vishwanath):

$$S = \int dx d\tau \sum_{j=0}^{2N-2} \left(\psi_j^\dagger (\partial_\tau + (-1)^j v \partial_x) \psi_j - (-1)^j w_j (\psi_{j+1}^\dagger \psi_j + \psi_j^\dagger \psi_{j+1}) \right), \quad (219)$$

where v is some velocity. Suppose w_j takes values w_e or w_o depending on $j \bmod 2$. Describe the different phases we can have and find the continuum limit when $t_e = t_o$.

Describe a similar thing that occurs upon coupling chains of $SU(2)_1$ theories with alternating signs for their WZW terms. You should find a theory with a theta term at $\theta = \pi$.

Finally, do the same thing for coupled critical majorana chains. You should produce a $p \pm ip$ superconductor.



First for the fermion lasagna (note that the noodles of the lasagna are in spacetime). We are basically doing the Kitaev wire construction, just in one higher dimension and with complex fermions. Since we are starting at $j = 0$, the leftmost edge of the lasagna is left-moving, while since we are ending the sum at an even value of j , the rightmost edge of the lasagna is right-moving. Thus if we consider the case when $w_e \gg w_o$, the left-moving edge is strongly coupled to the $j = 1$ chain and the right-moving edge is strongly coupled to the $j = 2N - 1$ chain, and similarly for the interior chains. This hybridizes all of the fermions and leaves us with something that is gapped and boring (the w_e terms become a mass term for a Dirac fermion built out of the even and odd wire fermions). However, in the opposite case where $w_o \gg w_e$, while the interior chains hybridize, the $j = 0$ and the $j = 2N - 1$ boundary chains are uncoupled, and give us a lasagna with $\sigma_{xy} = e^2/h$ coming from the single Dirac fermion on the edge, which is not connected to any mass term.

When $w_e = w_o = w$, we obtain a massless theory. Indeed, consider going to the continuum limit by taking $N \rightarrow \infty$ with Na held fixed. We can write things in terms of a Dirac fermion by defining, for j even,

$$\Psi_j = (\psi_{L,j}, \psi_{R,j})^T = (\psi_j, \psi_{j+1})^T. \quad (220)$$

We then collect the terms in the sum in the last term of S into a sum over only e.g. even j , which then prompts us to expand $\psi_{j+2} = \psi_j + 2a\partial_y \psi_j$, where y is the direction normal to the chains. Putting this in and integrating by parts, we get

$$S = \int d^3x \left(\psi_L^\dagger (\partial_\tau + v \partial_x) \psi_L + \psi_R^\dagger (\partial_\tau - v \partial_x) \psi_R - w \psi_L^\dagger \partial_y \psi_R + w \psi_R^\dagger \partial_y \psi_L \right). \quad (221)$$

Note that to end up with something that is massless, the $(-1)^j$ factor in the action is crucial—without it, we would get a mass that goes as $2w$. Letting the gamma matrices be X, Y, Z , we can rewrite this in the usual way, viz.

$$S = \int d^3x \bar{\Psi} (\gamma^0 \partial_x + v \gamma^1 \partial_x + w \gamma^2 \partial_y) \Psi. \quad (222)$$

In particular, at $w = v$ we get a massless Dirac cone.

Now we do a different variant related to bosonic IQH transitions, where the layers of the lasagna are $SU(2)_1$ WZW models. Each WZW layer (indexed by j) is

$$S_j = \frac{1}{\lambda} \int_L \text{Tr}[dg^\dagger \wedge \star dg] - \frac{i}{6 \cdot 2\pi} \int_{B_3} \text{Tr}[\omega \wedge \omega \wedge \omega], \quad \omega = g^\dagger dg, \quad (223)$$

where B_3 is a 3-ball bounding the lasagna layer L and g is the $SU(2)$ variable (or really, $U(2)$ variable) being integrated over. The check on the numerical factors is to note that

$$\frac{1}{12 \cdot \Omega_3} \int_{S^3} \text{Tr}[\omega^3] \in \mathbb{Z}, \quad \Omega_3 = 2\pi^2. \quad (224)$$

Here $1/12$ is really $1/6 \cdot 1/2$, where (I believe) the $1/2$ comes from cancelling $\text{Tr}[\mathbf{1}] = 2$ and the $1/6 = 1/3!$ is from the antisymmetrization on the derivatives.

Each layer is the edge theory of an SPT, and we will reconstruct that SPT by making a lasagna out of the edge theories. This will illustrate the general strategy this problem is trying to make clear (see McGreevy's notes):

- Start with a bunch of SPT- or SPT-like states, and take them to criticality. That is, tune them so that they are almost the trivial phase. For example, for the Kitaev wire, tune to the critical point where all the couplings in the chain are equal.
- These critical states will form the edge theory of a new SPT. Stack them together in some “alternating” way and couple them together by adding some new term to the Hamiltonian.
- The result will be a higher-dimensional SPT. One can then repeat the process by taking this SPT to the critical point separating it from the trivial phase, and then perform the stacking again (note to self: what does this have to do with the dimensional-reduction approach to classifying anomalies with splittable symmetries?)

Anyway, back to the current example: the setup is to consider a lasagna of $SU(2)_1$'s with alternating chiralities, like with the fermions. Here the chirality is set by the sign of the WZW term (they're not really chiral in the true sense, but they become chiral in the quantum hall context when only one of the bosons that constitute g is coupled to the $U(1)$ gauge theory). We also need to add a coupling between the chains, which we do with a “ferromagnetic” coupling $t \text{Tr}[g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j]$. If we write g as (see the paper by Senthil and Vishwanath)

$$g = \begin{pmatrix} a_1 & -a_2^\dagger \\ a_2 & a_1^\dagger \end{pmatrix} \quad (225)$$

where the a_i are boson operators, then after multiplying the matrices we see that the coupling term is a boson hopping term like $t(a_{I,j}^\dagger a_{I,j+1} + h.c.)$, for $I = 1, 2$. Anyway, the lasagna is (allowing for different even/odd hoppings as before)

$$S = \sum_{j=0}^{2N-2} \left(\frac{1}{\lambda} \int_{L_j} \text{Tr}[dg^\dagger \wedge \star dg] + (-1)^j \frac{i}{12\pi} \int_{B_{3,j}} \text{Tr}[\omega_j^3] - t_j \int_{L_j} \text{Tr}[g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j] \right). \quad (226)$$

Note that unlike with the fermions, we don't need the $(-1)^j$ factor in front of the interchain hopping term.

As before, if the even t_j 's are all strong compared to the odd ones, then we "hybridize" the whole system and are left with nothing interesting, while in the opposite limit we are left with surface chains of free $SU(2)_1$ theories at the edges (a bosonic IQH state). The most interesting case is when we have uniform coupling. Let's first look at the coupling term. The ferromagnetic coupling means that we want $g_{j+1} \sim g_j$ for all j , and so we can do

$$g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j \rightarrow 2 \cdot \mathbf{1} + g_j^\dagger \partial_z g_j + g_j \partial_z g_j^\dagger + \frac{1}{2}(g_j^\dagger \partial_z^2 g_j + g_j \partial_z^2 g_j^\dagger) \rightarrow -\partial_z g_j^\dagger \partial_z g_j, \quad (227)$$

where z is the direction normal to the layers and in the last step we dropped a constant, integrated by parts, and used $g_j^\dagger \partial_z g_j + g_j \partial_z g_j^\dagger = \partial_z(g_j^\dagger g_j) = 0$.

As for the WZW terms, we've already done a similar computation in a different diary entry with an alternating sum of WZW models in the AFM spin chain, where we saw that they combined to give a θ term (the extra dimension used to define the WZW actions became the dimension along the chain). Here we have the same thing, but one dimension up. Now S_{WZW} is the volume in S^3 enclosed (or "exclosed", our choice of coefficient means that it doesn't matter) by the image of the S^3 vector field n used to define g . We can write a given WZW term as

$$S_{WZW}[g] = \frac{2\pi i}{\Omega^3} \int_{B_3} d^2 x du \epsilon^{abcd} n^a \partial_t n^b \partial_x n^c \partial_u n^d, \quad (228)$$

where u is the extra coordinate of the 3-ball. The factor of $2\pi i/\Omega_3$ ensures that if the image of n for a given Lasagna layer bounds all of S^3 (which is the same as bounding none of it), we get something in $2\pi i \mathbb{Z}$.

Therefore, defining an n -field n_j at each site, to first order in $|n_{2j} - n_{2j+1}|$, we can compute the difference between two successive WZW terms as

$$S_{WZW}[g_{2j}] - S_{WZW}[g_{2j-1}] \approx \frac{2\pi i}{\Omega_3} \int_{S^3} d^3 x \epsilon^{abcd} (n_{2j}^a - n_{2j-1}^a) \partial_t n_{2j}^b \partial_x n_{2j}^c n_{2j}^d. \quad (229)$$

We arrive at this expression the following reasoning: the difference in the volume bounded by the two spheres defined by the trajectories of n_{2j} and n_{2j-1} (i.e. the volumes bounded by the images of each two-dimensional lasagna layer $L_j \approx S^2$) is equal to $n_{2j}^a - n_{2j-1}^a$ (the distance between the two spheres embedded in S^3) wedged with the pullback of the volume element in S^2 by n (the volume element of the image of a patch of two-dimensional spacetime mapped into S^3 by n). Drawing a picture may help—it's just the analogue of the drawable S^1 into S^2 example relevant for the AFM chain.

Replacing the difference in n vectors by a derivative and summing over all j , we get

$$\sum_j (-1)^j S_{WZW}[n_j] = \frac{2\pi i}{2 \cdot 6 \Omega_3} \int_M \epsilon^{abcd} n^a \wedge dn^b \wedge dn^c \wedge dn^d, \quad (230)$$

where M is the full spacetime manifold. The factor of $1/6$ is the price we pay for antisymmetrizing when taking the wedge product, while the all-important prefactor of $1/2$ comes when we write

$$n_{2j}^a \approx n_{2j-1}^a + \frac{\delta z}{2} \partial_z n_{2j-1}^a, \quad (231)$$

where z is the direction normal to the chains and $\delta z/2$ is the spacing between chains. The actual lattice constant is δz , namely twice the spacing between chains, since the fact that the chains come in with alternating signs means that we only have translation symmetry by a distance of twice the chain separation (you can also think that since we are summing up terms of the form $S_{WZW}[n_{2j}] - S_{WZW}[n_{2j-1}]$, we are only summing over half of the chains, namely the even ones). Without this factor of $1/2$, the topological term we get would disappear, so it is very important.

Written in terms of the g variables then, we can write⁹

$$\sum_j (-1)^j \frac{i}{12\pi} \int_{B_{3,j}} \text{Tr}[\omega_j^3] = \frac{\pi i}{2 \cdot 6\Omega^3} \int_M \text{Tr}[\omega^3] = \frac{1}{2} \frac{i}{12\pi} \int_M \text{Tr}[\omega^3]. \quad (232)$$

Since our boundary conditions on g will be such that $M \approx S^3$, the integral is a θ term at $\theta = \pi$ and as such is valued in ± 1 (since if we were to multiply the RHS by 2, we would get a WZW term integrated over a closed manifold, which we know has to give something in $2\pi i\mathbb{Z}$). This is clearest from the expression of the integral in terms of the n field: after the $1/6$ is removed by the antisymmetrization in the wedge product, we see that we get the term

$$S_{\theta=\pi} = i\pi \int_M n^*(\text{vol}_{S^3}) \in \pi i\mathbb{Z}. \quad (233)$$

Now for majorana lasagna. The simplest lasagna to make is the Kitaev wire. We start with a collection of majoranas in 0+1 dimensions, and couple them together in the same way as before, expect with factors of i to make the whole thing Hermitian, and without the $(-1)^j$ notation we used for the Dirac fermions:

$$H = i \sum_j (t_e \chi_j^L \chi_j^R + t_o \chi_j^R \chi_{j+1}^L). \quad (234)$$

Here L, R denote the left and right spots in a given site j , which contains places for two majoranas (each j is a site for a single \mathbb{C} fermion). When $t_e \gg t_o$ all of the majoranas are part of a $i\chi^L \chi_R$ mass term (which becomes $\sim c_j^\dagger c_j$), and so we get a boring gapped phase. When $t_o \gg t_e$ then we define $d_j = (\chi_j^R + i\chi_{j+1}^L)/2$ and get a Hamiltonian which is gapped in the bulk ($H \sim \sum d_j^\dagger d_j$), but which contains dangling edge modes. When $t_e = t_o$ we have a critical chain. In the continuum limit, we define $t_e/2 = t_o/2 = v$ and get

$$H = iv \int dx (\chi_L \partial_x \chi_R + \chi_R \partial_x \chi_L). \quad (235)$$

The next step up from this is to use critical majorana chains as the layers of a 2+1-dimensional lasagna, with the chains alternating in the sign of v . The coupled Hamiltonian is

$$H = i \sum_j \int dx (v \chi_j^L \partial_x \chi_j^L - v \chi_j^R \partial_x \chi_j^R + w_e \chi_j^L \chi_j^R + w_o \chi_j^R \chi_{j+1}^L). \quad (236)$$

⁹Note that this presentation of the result in terms of the g variables has an “extra” factor of $1/2$ relative to the prefactor for the presentation in terms of the n fields, due to the fact that $\text{Tr}[1] = 2$.

Here as before, each j label contains two majorana chains, one that is left-moving (χ_j^L) and one that is right-moving (χ_j^R). As usual, when $w_e \gg w_o$ then all of the χ_j 's are involved in a mass term, and we get something which is gapped. When $w_o \gg w_e$, the bulk is gapped but we have surviving Majorana modes on the edge: this is a $p \pm ip$ topological superconductor (with the sign in $p \pm ip$ set by the sign of v).

As with the C fermions, the critical point is where $w_e = w_o = w$, which describes the phase transition. We can take the continuum limit of the coupling terms in the same way that we did for constructing the critical Kitaev chain, and we get

$$H = i \int dx dy [v(\chi_L \partial_x \chi_L - \chi_R \partial_x \chi_R) + w(\chi_L \partial_y \chi_R + \chi_R \partial_y \chi_L)], \quad (237)$$

with y the direction normal to the chains.



The quantum n -state Potts chain

Today's diary entry is a homework problem from Ashvin's condensed matter class.

Consider the n -state Potts model. The Hamiltonian is

$$H_P = -J \left[\sum_j \sum_{l=0}^{n-1} Z_j^{-l} Z_{j+1}^l + g \sum_j \sum_{l=0}^{n-1} X_j \right]. \quad (238)$$

Here Z and X are the \mathbb{Z}_n generalizations of the Pauli matrices, and e.g. commute as

$$ZX = \omega^{-1} XZ. \quad (239)$$

Note that the first term is a Kronecker delta on \mathbb{Z}_n when acting on the Z eigenbasis.

Today we will answer the following questions:

- (a) What is the global symmetry? What are the eigenvalues / eigenstates of X ?
- (b) What are the ground states in the limit $g \rightarrow \infty$?
- (c) in the limit $g \rightarrow 0$?
- (d) What is the critical value g_c for the phase transition?
- (e) Do a mean-field analysis and plot the mean-field energy for a few different g s.
- (f) Minimize the mean field energy as a function of g and locate the phase transition. What order is it?



a) X measures discrete electric flux. We can thus find states with definite electric flux by Fourier transforming those of Z : they are

$$|Q\rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} e^{iQl} |l\rangle, \quad (240)$$

where $|l\rangle$ are the eigenstates of Z . Acting with X does $X : |l\rangle \mapsto |l-1\rangle$, so that

$$X|Q\rangle = e^{iQ}|Q\rangle. \quad (241)$$

The model has an internal \mathbb{Z}_n symmetry rotating the “spins” on every site by $2\pi/n$, which is performed by the operator $\prod_j X_k$.

b) When $g \rightarrow \infty$, we must work in a state with zero electric flux. Thus the system is in a unique ground state

$$|GS\rangle_{g \rightarrow \infty} = |Q = 0\rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} |l\rangle. \quad (242)$$

c) When $g \rightarrow 0$, we need to satisfy the interaction term. Since there is no X in the Hamiltonian we can work in the Z eigenbasis. If site j is in the eigenstate $|l_j\rangle$ and site $j+1$ is in $|l_{j+1}\rangle$, then the interaction term between them is

$$\sum_m \omega^{m(l_{j+1}-l_j)} = \delta_{l_j, l_{j+1}}. \quad (243)$$

Since this term appears with negative coefficient in the Hamiltonian, we want the δ constraint to be satisfied. Thus this forces neighboring sites to be in identical Z eigenstates, and we have n degenerate ground states

$$|GS_l\rangle_{g \rightarrow 0} = \bigotimes_j |l_j\rangle \quad (244)$$

(here $l_j \in \mathbb{Z}_n$ is the same for all j).

d) Let $j+$ denote the site $j + 1/2$ on the dual lattice and let $j-$ denote the dual lattice site $j - 1/2$. We define dual operators by

$$\tilde{X}_{j+} = Z_j^{-1} Z_{j+1}, \quad \tilde{Z}_{j-}^{-1} \tilde{Z}_{j+} = X_j. \quad (245)$$

The latter equation is satisfied if we write $\tilde{Z}_{j\pm}$ as the string

$$\tilde{Z}_{j\pm} = \prod_{j < j_\pm} X_j. \quad (246)$$

Now we want to compute the commutator between \tilde{X}_{j+} and \tilde{Z}_{k+} . Suppose first that the two dual sites are equal, $j+ = k+$. To move \tilde{X}_{j+} to the right of \tilde{Z}_{j+} , we need to pass the two

Z 's in the definition of \tilde{X} through the chain of X operators created by \tilde{Z} . Since we put the string in \tilde{Z} “to the left”, the Z_{j+1} operator goes through for free, while the Z_j^{-1} operator picks up an ω . Thus we have

$$\tilde{X}_{j+} \tilde{Z}_{k+} = \omega \tilde{Z}_{j+} \tilde{X}_{j+}. \quad (247)$$

Now suppose that the two dual operators are not at the same site. If the \tilde{X} is at a site to the left of the \tilde{Z} , it commutes though for free since the operators involved act on different \otimes factors. If it is to the left of the \tilde{Z} , then the Z it contains picks up a factor of ω^{-1} , while the Z^{-1} picks up a factor of ω which cancels the ω^{-1} , and so the two operators commute. Thus the commutation relation is

$$\tilde{X}_{j+} \tilde{Z}_{k+} = \omega^{\delta_{j,k}} \tilde{Z}_{k+} \tilde{X}_{j+}, \quad (248)$$

which is the same relation as the one the X, Z operators satisfy.

We know that there's a phase transition at some g_c because the ground state degeneracy must change. Note that in terms of the dual variables, the Hamiltonian is

$$H = -J \sum_j \left[g \sum_{l=0}^{n-1} \tilde{Z}_j^{-l} \tilde{Z}_{j+1}^l + \sum_{l=0}^{n-1} \tilde{X}_j \right], \quad (249)$$

which is the same as the original Hamiltonian when $g = 1$. Therefore we identify $g_c = 1$ as the self-dual point where the phase transition happens (as usual, we are assuming that there is only one critical point).

e) We take the mean field ansatz

$$|\Psi(x_1, \dots, x_n)\rangle = \bigotimes_j |\hat{n}(\{x_i\})\rangle_j, \quad |\hat{n}(\{x_i\})\rangle = \sum_{i=1}^n x_i |0\rangle, \quad \vec{x} \in S^{n-1}. \quad (250)$$

Here the fact that $\vec{x} \in S^{n-1}$ ensures that $|\Psi\rangle$ is normalized correctly.

What is the variational energy? Let's first look at the X term. Because of the sum over l , this is a projector onto the eigenstate of X with eigenvalue 1. In the basis where Z is diagonal, this is a projector onto the uniform sum $\sum_j |j\rangle$. It is represented by a matrix with 1 in every entry. Thus

$$\langle \Psi(\vec{x}) | X | \Psi(\vec{x}) \rangle = \left(\sum_j x_j \right)^2. \quad (251)$$

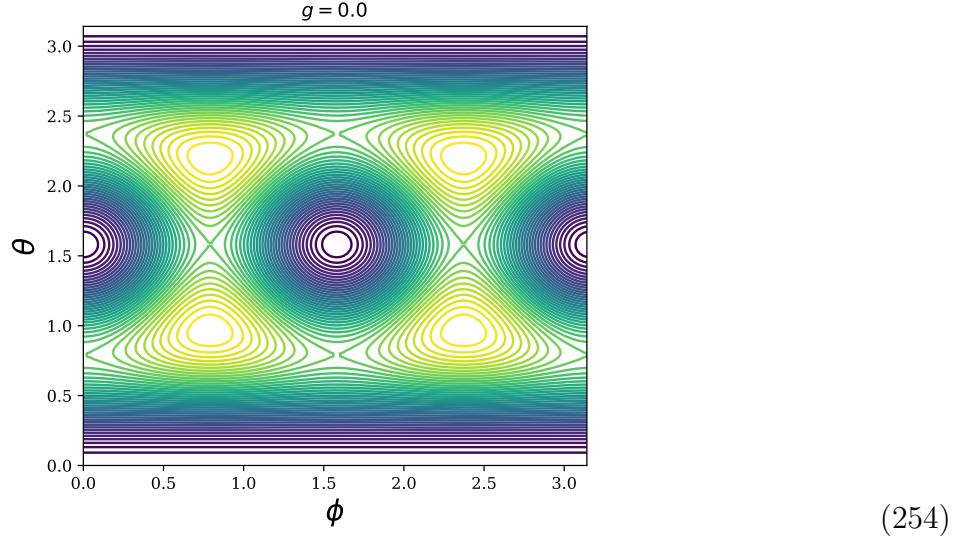
Now for the nearest neighbor term. Indexing the entries of $Z_j^{-l} \otimes Z_{j+1}^l$ as (a, b) , we see from $\sum_j \omega^{jk} \propto \delta k, 0$ that the matrix $Z_j^{-l} \otimes Z_{j+1}^l$ is diagonal with entries $(a, b) = \omega^{l(a-b)}$. Summing over l gives a non-zero result only when $a = b$, and so $\sum_l Z_j^{-l} \otimes Z_{j+1}^l$ has entries $(a, b) = \delta_{a,b} n$. Thus

$$\langle \Psi(\vec{x}) |_j \otimes \langle \Psi(\vec{x}) |_{j+1} \sum_l Z_j^{-l} \otimes Z_{j+1}^l | \Psi(\vec{x}) \rangle_j \otimes | \Psi(\vec{x}) \rangle_{j+1} = n \sum_i x_i^4. \quad (252)$$

Putting this together, on a length L chain we just get the contribution above for every j , and so the variational energy is

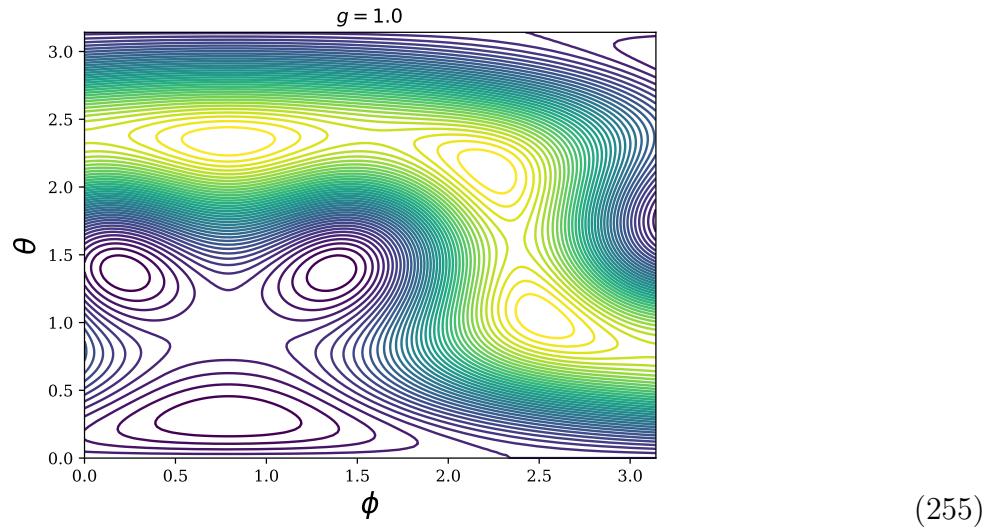
$$E(\vec{x}) = -JN \left[n \sum_i x_i^4 + g \left(\sum_j x_j \right)^2 \right]. \quad (253)$$

Now we specialize to the case of the 3-state Potts model, parametrizing the coordinates on S^2 by θ, ϕ . Only half of the sphere gives a physically distinct wavefunction, since $|\Psi\rangle$ and $-|\Psi\rangle$ are equivalent minima. We will choose the hemisphere $(\theta, \phi) \in [0, \pi]^2$. Setting $JN = 1$, when $g = 0$ $E(\theta, \phi)$ looks like

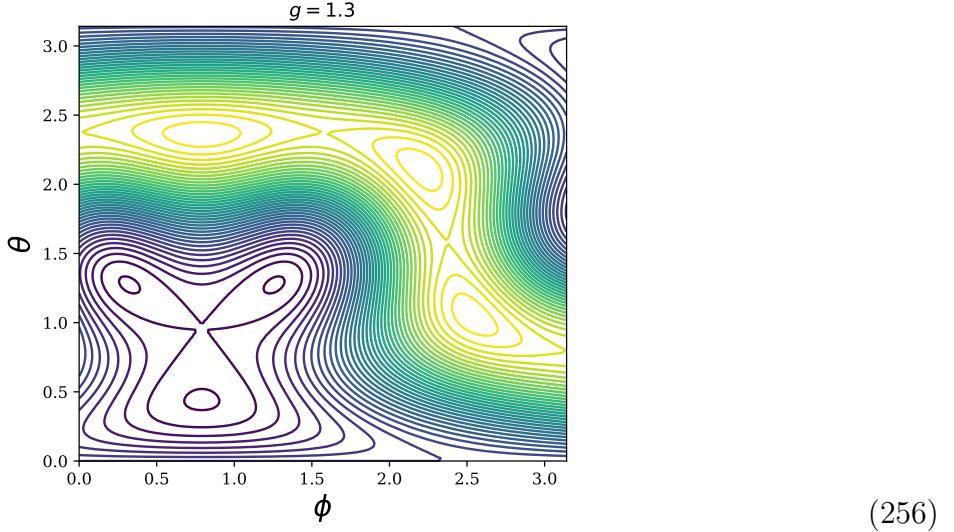


We see that there are three distinct minima, in agreement with what we predicted earlier (there is a single minimum at the top of the figure (the point $(0, 0, 1)$), which is identified with the minimum at the bottom of the figure. Likewise, the two minima at $\phi = 0, \pi$ are identified). They are the eigenstates of the Z operators, and consequently appear at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

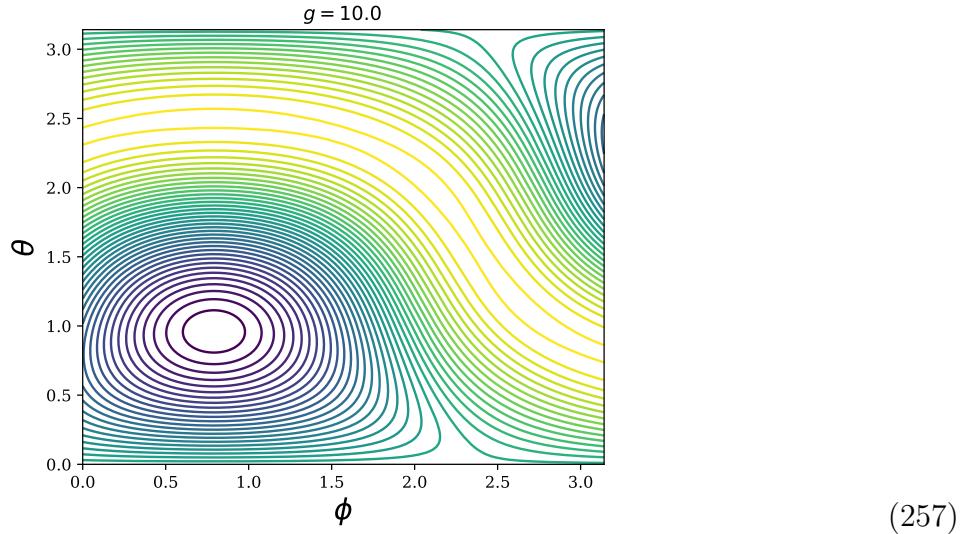
When we set $g = 1$ the plot looks like



The three minima start to merge as g is increased:

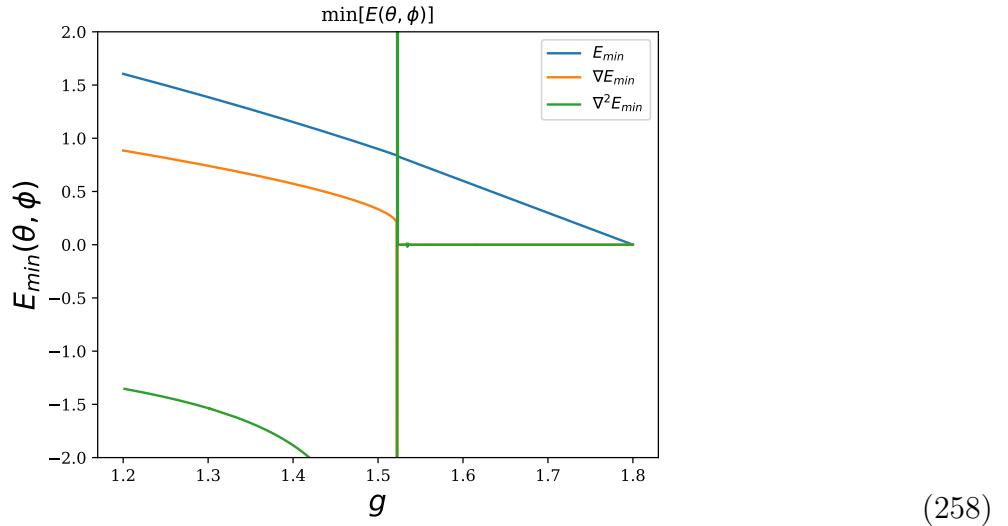


When g is large we need to get a \otimes state from satisfying the $\sum_l X^l$ term. Since this state will be one in the eigenvalue 1 eigenstate of X , it will have coordinates on the sphere $x = y = z$. Thus we expect a single minimum at $\theta = \pi/4, \phi = \pi/4$. Indeed, this is what happens at large g :



f) We now plot $\min[E(\theta, \phi)]$, again with $JN = 1$. The interesting behavior happens

around $g = 1.5$:



Here we have subtracted constants from E_{\min} and ∇E_{\min} so that they both hit 0 at $g = 1.8$. The behavior for $g < 1.2$ and $g > 1.8$ is what you would expect from the figure. We see that we have a second order phase transition at $g \approx 1.5225$, since $\nabla^2 E_{\min}$ is singular there. We know the self-dual point is at $g = 1$, so we see that mean-field theory over-estimates the critical value of g . This is what we expect from a mean-field treatment: the mean-field ansatz neglects fluctuations and assumes a \otimes state, so that it is biased towards ordered states (for us, small g).



Symmetries of Hubbard model and applications of non-Abelian bosonization

Consider the Hubbard model in arbitrary dimensions on a bipartite lattice. What are its global symmetries? Pay special attention to half filling.



We will work in one dimension for simplicity, although the generalization is straightforward as long as the lattice is bipartite. The Hubbard model Hamiltonian is

$$H = -t \sum_{i\sigma} (c_{i\sigma}^\dagger c_{i+1,\sigma} + h.c.) + U \sum_i n_{i\uparrow} n_{i\downarrow} - \sum_{i\sigma} \mu n_{i\sigma}. \quad (259)$$

Up to a constant, we can also write it as

$$H = -t \sum_{i\sigma} (c_{i\sigma}^\dagger c_{i+1,\sigma} + h.c) + U \sum_i (n_{i\uparrow} - u)(n_{i\downarrow} - u), \quad u \equiv \mu/U. \quad (260)$$

Now define Majoranas by

$$c_{i\sigma} = \frac{\chi_{i\sigma} + i\eta_{i\sigma}}{2}. \quad (261)$$

Here the $1/2$ means that the Majoranas satisfy the Clifford algebra relation with the 2 on the RHS, in order for $\{c, c^\dagger\}$ to work out properly. Then the number operator is

$$n_{i\sigma} = \frac{1}{2}(1 + i\chi_{i\sigma}\eta_{i\sigma}) = \frac{1}{4}(2 + i\lambda_{i\sigma}^T iY \lambda_{i\sigma}), \quad (262)$$

where $\lambda = (\chi, \eta)^T$. This means the potential term is

$$U \sum_i (n_{i\uparrow} - u)(n_{i\downarrow} - u) = U \sum_i \left(\alpha + \frac{i}{4} \lambda_{i\uparrow}^T iY \lambda_{i\uparrow} \right) \left(\alpha + \frac{i}{4} \lambda_{i\downarrow}^T iY \lambda_{i\downarrow} \right), \quad (263)$$

where we have defined

$$\alpha \equiv \frac{1}{2} - \frac{\mu}{U}. \quad (264)$$

We also need the Majorana form for the kinetic term, which is

$$-t \sum_{i\sigma} (c_{i\sigma}^\dagger c_{i+1,\sigma} + h.c) = -\frac{ti}{2} \sum_i \Lambda_i^T (iY \oplus iY) \Lambda_{i+1}, \quad \Lambda_i \equiv (\lambda_{i\uparrow}, \lambda_{i\downarrow})^T. \quad (265)$$

Now, what are the global symmetries of H ? Regardless of what μ is, H conserves spin and particle number, so we have at least a $G = (SU(2) \times U(1))/\mathbb{Z}_2$ symmetry (this is a global symmetry and not a local one because of the kinetic term, and the $1/\mathbb{Z}_2$ is needed to avoid over-counting the common \mathbb{Z}_2 subgroup of $SU(2)$ and $U(1)$). These are (as far as I can tell) the only symmetries of the model for generic μ .

Consider now the special case when $\mu = U/2$ so that $\alpha = 0$. Now

$$(\lambda_{i\sigma}^T iY \lambda_{i\sigma})^2 = 4(\chi_{i\sigma}\eta_{i\sigma})^2 = -4. \quad (266)$$

This means that when $\alpha = 0$, the potential term is, up to a constant,

$$-\frac{U}{16} \sum_i (\lambda_{i\uparrow}^T iY \lambda_{i\uparrow} + \lambda_{i\downarrow}^T iY \lambda_{i\downarrow})^2 = -\frac{U}{16} \sum_i (\Lambda_i^T (iY \oplus iY) \Lambda_i)^2. \quad (267)$$

Define $A \equiv iY \oplus iY$, so that at $\alpha = 0$ the Hamiltonian is

$$H = -\frac{it}{2} \sum_i \Lambda_i^T A \Lambda_{i+1} - \frac{U}{16} \sum_i (\Lambda_i^T A \Lambda_i)^2. \quad (268)$$

Consider $\Lambda_i \mapsto \mathcal{O}_i \Lambda_i$. In order to preserve the commutation relations for the Majoranas, we need $\mathcal{O} \in O(4)$. Consider first the case when $\mathcal{O}_i = \mathcal{O}$ is the same on every site. Then we also need $\mathcal{O}^T A \mathcal{O} = A$, i.e. we need

$$\mathcal{O} \in O(4) \cap Sp(4; \mathbb{R}) \cong U(2) \cong (SU(2) \times U(1))/\mathbb{Z}_2. \quad (269)$$

This is the spin rotation + charge conservation symmetry we found before.

However, there is another sneaky symmetry. Because the potential term has the square of the bilinear form, we can also let $\mathcal{O}_i^T A \mathcal{O}_i = -A$, provided that we choose \mathcal{O}_{i+1} such that $\mathcal{O}_i^T A \mathcal{O}_{i+1} = A$. This can be done by staggering the operators by taking $\mathcal{O}_i = -\mathcal{O}_{i+1}$. Now, let $\tilde{\mathcal{O}} \in O(4) \cap Sp(4; \mathbb{R})$, and define

$$\mathcal{O}_i = (-1)^i \mathcal{Z} \tilde{\mathcal{O}}, \quad \mathcal{Z} \equiv Z \oplus Z. \quad (270)$$

Multiplying $\tilde{\mathcal{O}}$ by \mathcal{Z} implements a particle-hole conjugation in addition to the rotation performed by $\tilde{\mathcal{O}}$. Since a particle-hole conjugation doesn't leave the kinetic term invariant (it changes it by a minus sign), we add the compensating factor of $(-1)^i$, which ensures that $\mathcal{O}_i^T A \mathcal{O}_{i+1} = A$. Since the definition of \mathcal{O}_i can be done for each $\tilde{\mathcal{O}} \in U(2)$, this gives us another $U(2)$ "pseudospin" symmetry. Note that this only works when $\mu = U/2$ (this choice favors $\langle n_{i\sigma} \rangle = 1/2$ for both σ , which corresponds to an average occupation of one particle per site: this is half-filling). Thus at half-filling we have two $U(2)$ s (quotiented by a discrete subgroup), while away from half-filling we just have the $SU(2) \times U(1)$. The basic picture here is that when we're at half-filling we have a charge conjugation symmetry which gets absorbed into a new $SU(2)$ symmetry—this is exactly the same reason behind the $Sp(N_f)$ symmetry of N_f Dirac fermions coupled to an $SU(2)$ gauge field as discussed in e.g. my paper with Zhen and Senthil.¹⁰

We can also look at this symmetry from a bosonization point of view. Recall how we would bosonize the $U = 0$ theory: working at long distances we can write $c_j = e^{ik_F j} \psi_{R,j} + e^{-ik_F j} \psi_{L,j}$ and keep the single-derivative terms, giving us an action of two Dirac fermions: $S \propto \int \sum_\sigma \bar{\Psi}_\sigma \not{\partial} \Psi_\sigma$, where $\Psi = (\psi_L, \psi_R)^T$. This gets bosonized to the $k = 1$ WZW model (which keeps track of the $SU(2)$ spin sector), plus a decoupled free boson that represents the $U(1)$ charge sector. A check that $k = 1$ is the right level is that the central charge of the $SU(2)_k$ WZW model is $c = 3k/(k+2)$, which gives the correct $c = 1$ to combine with the free $U(1)$ sector to produce a total of $c = 2$ when $k = 1$, matching the two Dirac fermions. From the way the WZW model is constructed, we have a manifest $(SU(2) \times SU(2))/\mathbb{Z}_2$ symmetry by $g \mapsto V^\dagger g U$, plus a $U(1)$ symmetry for the compact boson.

What happens when we turn on U , for an arbitrary filling (chemical potential)? We need to know what the U term maps to under bosonization. We can get the form of the answer by symmetry: the U term preserves $U(1)$ particle number conservation, as well as the diagonal $SU(2)$ symmetry which rotates the spins (diagonal in that it acts identically on ψ_L and ψ_R). First, define the currents

$$j_s = \psi_{s,\alpha}^\dagger \psi_{s,\alpha}, \quad J_s^a = \frac{1}{2} \psi_{s,\alpha}^\dagger \sigma_{\alpha\beta}^a \psi_{s,\beta}, \quad s \in \{L, R\}. \quad (271)$$

Recall from some diary entry way back that the $U(1)$ currents bosonize to the charge sector boson ϕ as

$$j_L \mapsto -i\bar{\partial}\phi, \quad j_R \mapsto i\partial\phi, \quad (272)$$

¹⁰The spin index in this context is the flavor index in that paper, while the L/R index for the low energy fermion fields in this context is that paper's spin index.

while the $SU(2)$ currents go to (the numerical factors are almost for sure wrong, but this won't matter now)

$$J_L^a \mapsto -i\text{Tr}[(\bar{\partial}g)g^\dagger\sigma^a], \quad J_R^a \mapsto i\text{Tr}[g\partial g^\dagger\sigma^a]. \quad (273)$$

So, what kind of interactions can appear while preserving the diagonal $U(1)$ and $SU(2)$? We can have $j_s j_s$ terms, as well as $J_s^a J_s^a$ terms. Since the stress tensor is $T \sim v : \partial\phi\partial\phi :$ (and likewise for \bar{T} , and likewise for the WZW sector), adding these terms just changes the velocity v and doesn't affect the WZW theory. Up to four fermion terms, we can add the current interactions

$$S_{current} = \int (\gamma_1 j_L j_R + \gamma_2 J_L^a J_R^a). \quad (274)$$

The γ_1 term gets bosonized to the free action for ϕ and so just contributes to a change in the radius of the $U(1)$ charge sector boson. The γ_2 term (which is invariant under the diagonal $SU(2)$ since the $SU(2)$ rotation preserves the dot product $J_L^a J_R^a$) is more serious and we will examine it separately. In addition to these, we can also get something invariant under the $SU(2)$ by using the intertwiner (invariant symbol) coming from $2 \otimes 2 \ni \mathbf{1}$. The invariant symbol is $\epsilon^{\alpha\beta}$ since

$$\epsilon^{\alpha\beta} \mapsto U_{\alpha\gamma} U_{\beta\delta} \epsilon^{\gamma\delta} = \epsilon^{\alpha\beta} \det U = \epsilon^{\alpha\beta}, \quad (275)$$

where the second-to-last step just uses the definition of the determinant. Thus $\psi_{s,\alpha} \epsilon^{\alpha\beta} \psi_{s,\beta}$ is invariant under the $SU(2)$ spin symmetry (one of the ψ 's can't be daggered since otherwise the bilinear wouldn't be invariant). To make this invariant under the diagonal $U(1)$ we need to add a corresponding term with $\psi_{s',\alpha}^\dagger$ s. If we choose $s' = s$ then we again get a redefinition of the velocity v , and so the most general interesting term along these lines we can add is

$$S_\epsilon = \int \gamma_3 \left(\epsilon^{\alpha\beta} \psi_{L,\alpha}^\dagger \psi_{L,\beta}^\dagger \cdot \epsilon^{\gamma\lambda} \psi_{R,\gamma} \psi_{R,\lambda} + h.c. \right). \quad (276)$$

Under bosonization, we use

$$\psi_{L,\alpha}^\dagger \psi_{R,\beta} \mapsto g_{\alpha\beta} e^{i\phi}, \quad (277)$$

where the $g_{\alpha\beta}$ keeps track of the spins and the $e^{i\phi}$ is because the ϕ parts are already normal-ordered since ψ_L^\dagger, ψ_R contract to give zero. Thus the γ_3 term maps as

$$\gamma_3 \left(\epsilon^{\alpha\beta} \psi_{L,\alpha}^\dagger \psi_{L,\beta}^\dagger \cdot \epsilon^{\gamma\lambda} \psi_{R,\gamma} \psi_{R,\lambda} + h.c. \right) \mapsto \gamma_3 g_{\alpha\gamma} g_{\beta\lambda} \epsilon^{\alpha\beta} \epsilon^{\gamma\lambda} \cos\phi = \gamma_3 \cos\phi, \quad (278)$$

since $\det g = 1$. Thus the γ_3 term is actually g -independent and it induces a mass for ϕ (it is relevant, which can be seen by re-scaling the kinetic term for ϕ , which was changed from the canonical normalization by the γ_1 term), which causes the $U(1)$ CDW part to be Mott-insulated away and disappear from the IR physics.

A final possible term that could appear which is allowed by the diagonal $U(1) \times SU(2)$ is the relevant $\text{Tr}[g]$, but this is actually not allowed as long as we preserve translation symmetry, which since it maps $\psi_L \mapsto i\psi_L$ and $\psi_R \mapsto -i\psi_R$ sends $g \mapsto -g$, and hence forbids $\text{Tr}[g]$. Since we have a WZW model at $k = 1$ only degree 1 polynomials in g are independent, and so we don't have to worry about a $\text{Tr}[g^2]$ term (which dies by Fermi statistics).

So we have accounted for all the possible things that the U term can add except for the γ_2 term, which is now the only term remaining that can reduce the spin part of the symmetry

down from $SU(2)_L \times SU(2)_R$. Since the γ_2 term breaks the $SU(2)_L \times SU(2)_R$ symmetry of the $U = 0$ free theory down to the diagonal $SU(2)$, the symmetry of the theory at long distances will depend on what γ_2 flows to in the IR. Now since the $J_{L/R}^a$ s are currents, they have dimension 1 and hence the γ_2 term is marginal. We can compute the leading term in the β function for gamma since we know the current-current OPE. Recall from an earlier diary entry that for a marginal perturbation $\sum_i \int \lambda_i \mathcal{O}_i$, we have, in an appropriate normalization,

$$\beta_{\lambda_i} = \frac{d\lambda_i}{d \ln \Lambda} = -C_{ijk}\lambda_j\lambda_k + O(\lambda^3). \quad (279)$$

Here C_{ijk} is the coefficient of \mathcal{O}_i appearing in the OPE of \mathcal{O}_j with \mathcal{O}_k . Since the $J_{L/R}^a$ s are currents, we can get their OPEs immediately from their commutation relations (write the commutator of the charge operator created by integrating the current with a local operator, deform the commutator to a contour integral enclosing the local operator, and use the OPE). Indeed, suppose under J_s^a we have $[Q_s^a, \mathcal{O}_i] = R_s^a \mathcal{O}_i$, where R_s^a is some representation of the symmetry defined by the current and $Q_s^a = \oint dz J_s^a(z)$. Then following the instructions in the above parenthetical, we get the OPE

$$J_s^a(z) \mathcal{O}_i(w) = \frac{1}{2\pi i} \frac{R_s^a \mathcal{O}_i}{z - w} + \dots \quad (280)$$

In particular, when we take $\mathcal{O}_i(w)$ to be one of the $SU(2)$ currents, we recover the correct OPE (again, we are not really paying attention to numerical factors)

$$J_s^a(z) J_s^b(w) \supset \epsilon^{abc} \frac{J_s^c(z)}{z - w}. \quad (281)$$

This means that the β function for γ_2 is

$$\beta_{\gamma_2} \propto \gamma_2^2. \quad (282)$$

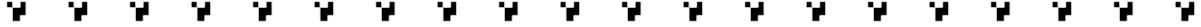
Since we are using the “high-energy” definition of the β function, this means that provided γ_2 is positive, γ_2 decreases as we flow to the IR, and so adding the γ_2 term actually does nothing in the IR: we just flow back to the $\gamma_2 = 0$ WZW fixed point, provided that there are no other intermediate fixed points along the way. This means that the $SU(2)_L \times SU(2)_R$ symmetry is actually likely to survive in the IR for the Hubbard model, even at finite U !

The last thing remaining to do is to check that γ_2 is indeed positive for the Hubbard model. This can be checked to indeed be the case (actually, I think that we have the simple $\gamma_2 = U$) by expanding out the definition of J_s^a and using the identity $\sigma_{\alpha\beta}^a \sigma_{\gamma\lambda}^a = 2\delta_{\alpha\gamma}\delta_{\beta\lambda} - \delta_{\alpha\beta}\delta_{\gamma\lambda}$.



Weakly interacting bosons

Consider bosons interacting through a potential V that is weak enough so that the bosons are mostly condensed. Let the boson annihilation operator be written as $\psi(r) = \psi + \phi(r)$, where the $\phi(r)$ are fluctuations about the condensate. Working quadratically in ϕ , we will diagonalize the Hamiltonian and find the dispersion of the low-lying excitations.



Let $\psi(r) = \psi + \phi(r)$, where ψ is a constant parametrizing the condensate and $\phi(r) \equiv \delta\psi(r)$ represents fluctuations about the condensate. Expanding the Hamiltonian to quadratic order in ϕ , we have (after normal-ordering and assuming $V(r) = V(|r|)$)

$$H = \int_r \left[\frac{|\nabla\phi|^2}{2m} - \mu(\psi^*\phi + \psi\phi^\dagger) - \mu|\psi|^2 - \mu|\phi|^2 \right] + \int_{r,r'} V(r-r') \left(\frac{1}{2}|\psi|^4 + |\psi|^2(\psi\phi(r')^\dagger + \psi^*\phi(r')) \right. \\ \left. + |\psi|^2|\phi(r')|^2 + \frac{1}{2}(\psi^2\phi(r)^\dagger\phi(r')^\dagger + (\psi^*)^2\phi(r)\phi(r')) + |\psi|^2\phi(r)^\dagger\phi(r') \right) + \mathcal{O}(\phi^3) \quad (283)$$

We want to choose ψ so that $\langle\phi(r)\rangle = 0$. Linear terms in ϕ are currents which source an expectation value for ϕ , and so in order to ensure that this expectation value vanishes we need the Hamiltonian to be independent of terms linear in ϕ, ϕ^\dagger . Taking a look at e.g. the ϕ^\dagger piece, we see that for the terms in $\delta H/\delta\phi^\dagger(r)$ which are c numbers to vanish for all r , we need

$$\mu\psi = \int_{r'} V(r')|\psi|^2\psi \implies \psi = \sqrt{\frac{\mu}{V_0}}e^{i\alpha}, \quad \alpha \in [0, 2\pi), \quad V_0 = \int_{r'} e^{-i0\cdot r'}V(r'). \quad (284)$$

We have a $U(1)$'s worth of choices for the phase of ψ , and wolog we will take it to be real by setting $\alpha = 0$. Thus the “condensate strength” is determined by the chemical potential and the $q = 0$ component of the potential.

Putting in this value for ψ , the Hamiltonian is (dropping constants)

$$H = \int_r \frac{|\nabla\phi|^2}{2m} + \frac{\mu}{V_0} \int_{r,r'} V(r-r') \left[\phi(r)^\dagger\phi(r') + \frac{1}{2}(\phi(r)^\dagger\phi(r')^\dagger + \phi(r)\phi(r')) \right], \quad (285)$$

where the $-\mu|\phi|^2$ part has canceled with one of the potential terms (the fluctuations have no chemical potential). We write this in momentum space by taking $\phi(r) = \int_q e^{iqr}\phi_q$, so that

$$H = \int_q \frac{q^2}{2m} \phi_q^\dagger\phi_q + \frac{\mu}{V_0} \int_{q,q',r,r'} V(r-r') \left(\phi_q^\dagger\phi_{q'} e^{i(-qr+q'r')} + \phi_q\phi_{q'} e^{i(qr+q'r')} + \phi_q^\dagger\phi_{q'}^\dagger e^{-i(qr+q'r')} \right) \\ = \int_q \frac{q^2}{2m} \phi_q^\dagger\phi_q + \frac{\mu}{V_0} \int_q V(q) \left(\phi_q^\dagger\phi_q + \frac{1}{2}[\phi_q\phi_{-q} + \phi_q^\dagger\phi_{-q}^\dagger] \right) \\ = \frac{1}{2} \int_q : \Phi_q^\dagger \mathcal{H}_q \Phi_q :, \quad (286)$$

where we have defined

$$\Phi_q \equiv (\phi_q, \phi_{-q}^\dagger)^T, \quad \mathcal{H}_q = \begin{pmatrix} \frac{q^2}{2m} + \frac{\mu}{V_0} V(q) & \frac{\mu}{V_0} V(q) \\ \frac{\mu}{V_0} V(q) & \frac{q^2}{2m} + \frac{\mu}{V_0} V(q) \end{pmatrix}. \quad (287)$$

To diagonalize this, we want a family of matrices M_q such that $M_q^\dagger \mathcal{H}_q M_q$ is diagonal. The diagonalized boson operators are then defined through $\Gamma_q = M \Phi_q$, with $\Gamma_q = (\gamma_q, \gamma_{-q}^\dagger)^T$. Their commutation relations are

$$[\gamma_q, \gamma_{q'}^\dagger] = i\delta_{q+q'} \det M_q, \quad (288)$$

and so we should use a boost

$$M_q = \begin{pmatrix} \cosh(\theta_q) & \sinh(\theta_q) \\ \sinh(\theta_q) & \cosh(\theta_q) \end{pmatrix}. \quad (289)$$

Since we are doing a boost and not a rotation (i.e. since M_q is not unitary), the eigenvalues of the diagonalized Hamiltonian $M_q \mathcal{H}_q M_q$ will not be the same as those of \mathcal{H}_q . This means we actually have to find θ_q in order to get the spectrum.

Writing $\mathcal{H}_q = a\mathbf{1} + bX$, we have

$$M_q \mathcal{H}_q M_q = [a \cosh(2\theta_q) + b \sinh(2\theta_q)]\mathbf{1} + [b \cosh(2\theta_q) + a \sinh(2\theta_q)]X. \quad (290)$$

Requiring that the coefficient of X vanish means that

$$\theta_q = \frac{1}{2} \operatorname{atanh}(-b/a). \quad (291)$$

Plugging this in gives, after some algebra

$$M_q \mathcal{H}_q M_q = \sqrt{a^2 - b^2} \mathbf{1}. \quad (292)$$

The diagonalized Hamiltonian is then

$$H = \int_q \gamma_q^\dagger \epsilon(q) \gamma_q, \quad \epsilon(q) = \sqrt{\frac{q^2}{2m} \left(\frac{q^2}{2m} + \frac{2\mu V(q)}{V_0} \right)}. \quad (293)$$

If we simplify by assuming a contact interaction so that $V(q) \approx V_0$, the dispersion simplifies. In particular, for small q we have

$$\epsilon(q \rightarrow 0) \approx |q| \sqrt{\mu/m} = c|q|, \quad (294)$$

as expected for phonons (the sound mode in the superfluid). On the other hand, when q is very large, we have the regular free particle $\epsilon(q) \approx q^2/2m$.



Tower of states

This was on a problem set for Ashvin's condensed matter class. We will explain what the Anderson tower of states is. We will go over the mega-simple example of a spin 1/2 AFM on a bipartite lattice by reducing the Hamiltonian to an effective Hamiltonian for the two "spins" made by combining the spins on all the sites of each of the sublattices.



Consider a (spin 1/2, for simplicity) AFM spin system on a bipartite lattice. The Hamiltonian is

$$H = J \sum_{\langle ij \rangle} S_i \cdot S_j = \frac{J}{N} \sum_{\langle ij \rangle} \sum_{p,q} S_p \cdot S_q e^{i(qi+pj)} = 2J \sum_{\mu} \sum_q S_q \cdot S_{-q} \cos(q_{\mu}), \quad (295)$$

where N is the number of lattice sites and $J > 0$.

The two most relevant wave vectors are $\vec{0}$ and (π, π, \dots, π) . The BZ is such that the latter wave vector is $S_{\pi} \propto \sum_j (-1)^{|j|} S_j$, where the minus sign takes on alternating values on the two sublattices. Thus the former q measures the total spin, while the latter measures the extent of the Neel ordering. If we only take these two wave vectors, we get the Hamiltonian

$$H = 2J(S_0 \cdot S_0 - S_{\pi} \cdot S_{\pi}) = \frac{2J}{N}(S_{tot} \cdot S_{tot} - S_{Neel} \cdot S_{Neel}) = \frac{4J}{N}S_A \cdot S_B, \quad (296)$$

where A, B are the two sublattices and e.g. $S_A = \sum_{j \in A} S_j$. We can write this more suggestively as

$$H = \frac{2J}{N} [(S_A + S_B)^2 - S_A^2 - S_B^2]. \quad (297)$$

As usual, we are writing $S_A + S_B$ for $S_A \otimes \mathbf{1} + \mathbf{1} \otimes S_A$ ¹¹. So, the ground state is evidently one annihilated by $S_A + S_B$ and where the spins on each sublattice add together to maximize

¹¹A little details on adding spins, since I remember always getting confused from the lack of explicit tensor products in physics books. The generators for a tensor product of representations R_1, R_2 for a Lie algebra are

$$T_{R_1 \otimes R_2} = \mathbf{1}_1 \otimes T_2 + T_1 \otimes \mathbf{1}_2. \quad (298)$$

Thus e.g. for two spin 1/2s, A and B , we have e.g. $S_{A \otimes B}^x = \frac{1}{2}(\mathbf{1} \otimes X + X \otimes \mathbf{1})$ and

$$S_{A \otimes B}^2 = \frac{1}{4} \sum_a (\mathbf{1} \otimes \sigma^a + \sigma^a \otimes \mathbf{1})^2 = \frac{3}{2}\mathbf{1} \otimes \mathbf{1} + \sum_a \frac{1}{2}\sigma^a \otimes \sigma^a. \quad (299)$$

Evaluating this on $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ and $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ shows that the quantum numbers are assigned in the way we know they are. Generalizing to N spin 1/2s tensored together, we get

$$S_{1/2^{\otimes N}} = \frac{1}{4} \sum_a \sum_{\{i_j\} | \sum i_j = 2} \bigotimes_j (\sigma^a)^{i_j}. \quad (300)$$

The raising / lowering operators for the N spin 1/2s are accordingly $\sigma^{\pm} \otimes \mathbf{1} \otimes \dots + \mathbf{1} \otimes \sigma^{\pm} \otimes \mathbf{1} \otimes \dots$, with highest / lowest weight states the expected $\bigotimes |\uparrow\rangle$ and $\bigotimes |\downarrow\rangle$, respectively. Evaluating the above expression

S_A, S_B . Here we are assuming that the number of spins on A and B are the same (namely $N/2$), so that the total number of spin 1/2s is even, allowing us to form a singlet. Also since $S_A + S_B$ commutes with the quadratic terms, we can both maximize S_A^2, S_B^2 and simultaneously ensure that $S_A + S_B$ acts as 0. Anyway, this means that the ground state is a spin singlet, and preserves the spin symmetry: there is no spontaneous symmetry breaking in the ground state, consistent with us working in finite volume.

This means in particular that the naive Neel state

$$|Neel\rangle \propto \bigotimes_{i \in A} |\uparrow\rangle_i \otimes \bigotimes_{j \in B} |\downarrow\rangle_j \quad (302)$$

is *not* the ground state of H . In fact, it is not even an eigenstate, since it transforms nontrivially under S^2 . Rather, it is a superposition of eigenstates

$$|Neel\rangle = \sum_S C_{(N/4, N/4), (N/4, -N/4)}^{(S, 0)} |S, 0\rangle, \quad (303)$$

where the C 's are the CG coefficients in an appropriate normalization (e.g. the relevant C 's for the $1/2 \otimes 1/2$ case are both $1/\sqrt{2}$). All the components in the sum have $S^z = 0$ and so $|Neel\rangle$ correctly has $S^z = 0$, but it is not an eigenstate of H since it is built out of a linear combination of all possible values for $S \in 0, \dots, N/2$.

What are the low-energy excitations? First we can consider what happens when we violate the $(S_A + S_B)^2$ term in H but satisfy the $S_A^2 + S_B^2$ term. Now $(S_A + S_B)^2$ can take values in $0, \dots, N(N+1)$ (renaming $N/2$ as N), and so one of these excitations costs

$$\Delta E_{S_A + S_B} = \frac{J}{N}, \quad (304)$$

and in particular the energy gap goes to zero as $1/L^d$ in d dimensions, since this is how the total spin scales. These are the excitations in the tower of states; in this case the tower is the same as the ladder formed with the raising / lowering operators of $SU(2)$.

The other excitations are ones in which the spins on a given sublattice do not combine to form the maximum possible spin of $N/2 \cdot (N/2 + 1)$ (recall the maximum total spin on each sublattice is $N/2$ in our current notation). The difference in S_A^2 between the highest weight and next-to-highest weight states is N , and so one of these excitations costs energy

$$\Delta E_{S_A^2} = 2J, \quad (305)$$

which is independent of N and thus independent of system size (normally we expect the energy of the ‘‘Goldstone mode’’ to vanish as $1/L$, which we can’t see in this model since our neglect of all wave vectors except 0, π has left us with pathological non-local couplings between the spins).

for S^2 on $\otimes |\uparrow\rangle$ shows that the X and Y terms cancel, leaving

$$S_{1/2^{\otimes N}}^2 |\uparrow \cdots \uparrow\rangle = \left(\frac{3N}{4} + 2 \binom{N}{2} \right) \mathbf{1}^{\otimes N} |\uparrow \cdots \uparrow\rangle = \frac{N}{2} \left(\frac{N}{2} + 1 \right) |\uparrow \cdots \uparrow\rangle, \quad (301)$$

as expected.

Basically the same thing happens in a model where we replace the $S_A^2 + S_B^2$ term with an external field, now thinking of S_A, S_B as fundamental spin $N/2$'s (and not composite):

$$H = \frac{J}{N} S_A \cdot S_B - h(S_A^z + S_B^z). \quad (306)$$

Since S_A^2, S_B^2 are now fixed as quadratic casimirs (unlike before where they specified the representation we selected out from $(1/2)^{\otimes N}$), we can also write H up to constants as

$$H = \frac{J}{2N} (S_A + S_B)^2 - h(S_A^z + S_B^z). \quad (307)$$

Here again the ground state is a singlet with both spins pointing in the Z direction. The lowest excitation in the “tower of states” is at $\Delta E = J/2N$, while the excitation created by changing the S_A^z eigenvalue by 1 has a constant energy gap of h .

Let's compute the susceptibility $\delta\langle S^z \rangle / \delta h$, now generalizing slightly to the case where each sublattice has N sites of spin S each. To do this we need the partition function. Since S_A and S_B are spins of the same magnitude N , we have the decomposition $N \otimes N = 0 \oplus 1 \oplus \dots \oplus 2NS$, where each representation appears with unit multiplicity. Choosing a value for $(S_A + S_B)^2$ selects out a given irrep in the direct sum decomposition, while choosing a value for $(S_A^z + S_B^z)$ selects out a given basis state within that irrep. So, specifying the pair S_{tot}^2, S_{tot}^z uniquely fixes a state in the direct sum decomposition. Thus the partition function is

$$Z = \sum_{j=0}^{2NS} e^{-\frac{\beta J}{2N} j(j+1)} \sum_{l=-j}^j e^{\beta hl}. \quad (308)$$

The sum over l can be done easily but the remaining sum is hard, even if we take $2NS \rightarrow \infty$. At zero temperature though, we can say something. The ground state will have S_{tot}^z maximized for a given value of S_{tot} , so that $S_{tot}^z = S_{tot}$. Thus the energy in the ground state is $JS_{tot}^2/N - hS_{tot}$, which means that

$$\langle S_{tot}^z \rangle_0 = \frac{hN}{J} \implies \chi = \frac{N}{J}. \quad (309)$$

Sanity check: when $N/J \rightarrow \infty$ there is no antiferromagnetic term, so that any external field causes the spins to align, giving $\chi \rightarrow \infty$. On the other hand, when $N/J \rightarrow 0$ the system is locked into antiferromagnetic ordering and can't respond to external fields, so that $\chi \rightarrow 0$.





The example we'll be focusing on is the bosonization of the XY chain, with Hamiltonian

$$H = -\frac{J}{2} \sum_i (X_i X_{i+1} + Y_i Y_{i+1}) = -J \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+). \quad (310)$$

Identifying the fermions as $c_i = S_i^- \prod_{j>i} Z_j = S_i^- \prod_{j< i} (-1)_j^F$, we have

$$H = -J \sum_i (c_i^\dagger (-1)_i^F c_{i+1} + c_i (-1)_i^F c_{i+1}^\dagger) = -J \sum_i (c_i^\dagger c_{i+1} + h.c.), \quad (311)$$

which is free fermions at half-filling. We go to the continuum via $c_j = e^{-ik_F j} \psi_L + e^{ik_F j} \psi_R$, and get the usual Dirac Lagrangian. Translation acts on the fields as $\psi_L \mapsto -i\psi_L, \psi_R \mapsto i\psi_R$; since we assume the ψ_σ fields vary slowly on the scale of a lattice spacing we drop the $a\partial_x \psi_\sigma + \dots$ part that comes after doing translation (so that translation acts as an internal symmetry).

Unlike in previous diary entries, we will actually be careful about Klein factors. There are a jillion different normalization conventions for the various factors that appear everywhere; we will bosonize via

$$\mathcal{B}[\psi_{L/R}] = \gamma_{L/R} e^{-i\phi_{L/R}}. \quad (312)$$

This is probably closest to the conventions in QFT and strings part II, although here the bosonizations of both ψ_L and ψ_R have the same sign in the exponent. This ensures that $e^{i\phi}$ changes the total fermion number, while $e^{i\theta}$ changes the eigenvalue of the total γ^5 , so that θ measures the total number of fermions while ϕ measures $N_L - N_R$. Here our conventions are

$$\phi = \phi_L + \phi_R, \quad \theta = \phi_L - \phi_R. \quad (313)$$

Here also the γ_σ 's are Klein factors that are representations of $Cl(2)$:

$$\{\gamma_\sigma, \gamma_\rho\} = 2\delta_{\rho,\sigma}. \quad (314)$$

In our conventions we have the commutators

$$[\phi_\sigma(x), \phi_\rho(y)] = -i\pi\delta_{\rho,\sigma}(-1)^\sigma \text{sgn}(x-y), \quad (315)$$

where we have defined $(-1)^R = -1, (-1)^L = +1$ ($(-1)^\sigma$ has alias γ_5). Note that with this commutation relation, ϕ_L and ϕ_R commute! Thus we need the Klein factors in order to have normal anticommutation relations between $\mathcal{B}[\psi_L]$ and $\mathcal{B}[\psi_R]$. One also checks that the commutation relation above ensures that the ψ_σ are self-fermionic. In this notation the canonical momentum is thus

$$\pi_\sigma = (-1)^\sigma \frac{\partial_x \phi_\sigma}{2\pi}. \quad (316)$$

This means that the canonical momentum for the ϕ field has an unaesthetic factor of $1/4\pi$:

$$\pi_\phi = \frac{\partial_x \theta}{4\pi}. \quad (317)$$

There are two symmetries in the problem that will be of interest to us. The first is translation, which as we have seen leaves $\psi_\sigma^\dagger \psi_\sigma$ invariant but acts as -1 on $\psi_L^\dagger \psi_R, \psi_R^\dagger \psi_L$. In particular, we see that $\phi \mapsto \phi, \theta \mapsto \theta + \pi$ under translation. We also have a type of particle-hole symmetry since we are at half-filling. The particle-hole transformation is

$$C : c_j \mapsto c_j^\dagger \implies \psi_L \mapsto \psi_R^\dagger, \psi_R \mapsto \psi_L^\dagger. \quad (318)$$

The fact that it exchanges left and right movers is familiar from the usual form of charge conjugation as $\psi \mapsto \gamma_2 \psi^*$. This of course is not a symmetry of the Hamiltonian, since it sends H to $-H$ (we are reflecting the $\cos k$ band structure about the zero-energy line). In order to make this a symmetry, we need to translate the flipped band structure by π . This is done by staggering the PH transformation on each site. We write this symmetry as \tilde{C} :

$$\tilde{C} : c_j \mapsto (-1)^j c_j^\dagger \implies c_j \mapsto e^{3ik_F j} \psi_R^\dagger + e^{ik_F j} \psi_L^\dagger \implies \psi_L \mapsto \psi_L^\dagger, \psi_R \mapsto \psi_R^\dagger, \quad \tilde{C} H \tilde{C} = H. \quad (319)$$

The fact that L and R are not exchanged is simply due to the fact that \tilde{C} involves a π momentum shift, which takes L fermions to R fermions and vice versa. In the spin language, this symmetry acts as a π rotation about X , so that

$$\tilde{C} : X \mapsto X, Y \mapsto -Y, Z \mapsto -Z. \quad (320)$$

Since the π rotation sends S^\pm to S^\mp , one might think that it does $c_j \mapsto c_j^\dagger$ for all j . But it also acts nontrivially on the $\prod_{j < i} Z_j$ strings attached to the S_i^\pm operators. It acts trivially on strings with an even number of Z_j operators, and as -1 on those with an odd number; hence the $(-1)^j$ in the action of \tilde{C} .

In bosonized language, translation acts trivially on ϕ and sends $\theta \mapsto \theta + \pi$, while \tilde{C} does

$$\tilde{C} : \phi \mapsto -\phi, \quad \theta \mapsto -\theta. \quad (321)$$

The combination of both \tilde{C} and translation does

$$\tilde{C}T : \phi \mapsto -\phi, \quad \theta \mapsto -\theta + \pi. \quad (322)$$

The \tilde{C} and T symmetries prevent mass terms from being added. Indeed, both, the $\bar{\psi}\psi = \psi_L^\dagger \psi_R + h.c.$ and the $i\bar{\psi}\gamma^5\psi$ mass terms get mapped to minus themselves under translation. On the other hand, we see that the $\bar{\psi}\psi$ mass term is mapped to minus itself under \tilde{C} , while the chiral mass is invariant. Conversely, the $\bar{\psi}\psi$ mass term is invariant under $\tilde{C}T$, while the chiral mass term gets multiplied by -1 . This suggests that the two mass terms are

$$\mathcal{B}[\bar{\psi}\psi] \sim \sin \theta, \quad \mathcal{B}[i\bar{\psi}\gamma^5\psi] \sim \cos \theta, \quad (323)$$

where we have only written \sim because of Klein factor issues. Indeed, doing this carefully we have

$$\mathcal{B}[\bar{\psi}\psi] = \mathcal{B}[\psi_L^\dagger \psi_R + h.c.] = \gamma_L \gamma_R e^{i\phi_L - i\phi_R} + \gamma_R \gamma_L e^{i\phi_R - i\phi_L} = 2i\gamma_L \gamma_R \sin \theta. \quad (324)$$

Note the crucial Klein factors, without which we would have concluded that the bosonized mass term was a cosine, as is often erroneously done in the literature. Similarly, we have

$$\mathcal{B}[i\bar{\psi}\gamma^5\psi] = i\gamma_L \gamma_R e^{i\phi_L - i\phi_R} - i\gamma_R \gamma_L e^{i\phi_R - i\phi_L} = 2i\gamma_L \gamma_R \cos \theta. \quad (325)$$

Again, the Klein factors are crucial. Note that this matches with what we guessed from the symmetry analysis above.

We will be interested in what happens when a $\Delta Z_i Z_{i+1}$ term is added to H . Since Z_i goes to a term involving the fermion density, this term will generate interactions for the fermions. In particular, such a term will contain a term

$$\sum_j \Delta Z_j Z_{j+1} \ni \int dx \lambda \cos(2\theta). \quad (326)$$

This is invariant under both translation symmetry and \tilde{C} , neither of which are broken by the added ZZ term. It is generated by the Umklapp term $(\psi_R^\dagger \psi_L)^2 + h.c.$ which is translation invariant since it carries momentum 2π and which is non-zero since the terms being squared are secretly point-split. Doing the point-splitting procedure leads to some derivative terms and also the $\cos(2\theta)$ term above. The other terms in the fermionization of the ZZ term will a) renormalize the velocity of light and b) change the radius of the boson (larger Δ will make it larger). The degree to which the radius of the boson is changed determines the relevance of the $\cos(2\theta)$ term, and so whether or not the $\cos(2\theta)$ term is relevant depends on how large Δ is (since as the radius gets larger, $\cos(2\theta)$ gets more relevant, as vortices become less energetically costly at large boson radii: small radii project onto zero winding, just as only the zero momentum mode of a field on a circle becomes relevant when the radius of the circle goes to zero, while a continuum of momenta open up when the radius becomes large).

Let us now assume that Δ is large enough so that the added $\cos(2\theta)$ is relevant. The relevance of the cosine drives the theory into a Mott insulating phase. The point though is that there are two types of Mott insulators, since we have two scenarios depending on the sign of λ :

$$\lambda > 0 \implies \langle \theta \rangle \in \{\pm\pi/2\}, \quad \lambda < 0 \implies \langle \theta \rangle \in \{0, \pi\}. \quad (327)$$

Both scenarios spontaneously break translation symmetry in a \mathbb{Z}_2 fashion, since the degenerate vacua are in both cases mapped into one another under $\theta \mapsto \theta + \pi$. In addition, we see that the $\lambda > 0$ scenario spontaneously breaks \tilde{C} particle-hole symmetry ($\theta \mapsto -\theta$), while the ground states remain symmetric under \tilde{CT} . On the other hand, the $\lambda < 0$ scenario has SSB for the \tilde{CT} symmetry, and preserves \tilde{C} .

Since the $\cos(2\theta)$ term leads to SSB for the translation symmetry, it effectively generates mass terms for the fermions that were previously forbidden by translation invariance. Indeed, let us sneakily write 2θ as $\langle \theta \rangle + \theta$. Then we have

$$\lambda \cos(2\theta) \rightarrow \lambda (\cos\langle \theta \rangle \cos \theta - \sin\langle \theta \rangle \sin \theta). \quad (328)$$

Thus depending on the sign of λ , which controls $\langle \theta \rangle$, we get different mass terms: we can write the generated mass term as

$$\bar{\psi} e^{i\gamma^5 \Omega} \psi = \cos \Omega \bar{\psi} \psi + i \sin \Omega \bar{\psi} \gamma^5 \psi \mapsto \cos \Omega (i\gamma_L \gamma_R \sin \theta) + \sin \Omega (i\gamma_L \gamma_R \cos \theta), \quad \Omega = \langle \theta \rangle + \pi/2. \quad (329)$$

This means that the mass term is

$$\lambda > 0 \implies H \ni \pm M \int dx i\gamma_L \gamma_R \sin \theta, \quad \lambda < 0 \implies H \ni \pm M \int dx i\gamma_L \gamma_R \cos \theta, \quad (330)$$

where the \pm signs are determined by which ground state is chosen by SSB. Note that in what we're doing, we can always fix a “gauge” in which the states we're acting on have eigenvalue 1 under $i\gamma_L\gamma_R$, so that the Klein factors become unimportant at this stage.

Let's now do a sanity check and identify these two Mott insulators. the $\lambda > 0$ gives our fermions a non-chiral $\bar{\psi}\psi$ mass, which as we have seen breaks translation symmetry and \tilde{C} symmetry. Now the $\bar{\psi}\psi$ mass is easily checked to correspond in the c_j language to a modulated chemical potential term, so that

$$\lambda > 0 \implies H \ni \pm M \sum_j (-1)^j c_j^\dagger c_j. \quad (331)$$

As a sanity check, we see that this term is odd under T (translation) and under \tilde{C} (remember to normal-order the $c_j^\dagger c_j$ to check the latter), while it preserves $T\tilde{C}$. The physical picture of the two ground states are chains where the electrons are localized to either all even sites, or all odd sites. Acting either with T or with \tilde{C} exchanges these two ground states. This is exactly what is expected for a strong anti-ferromagnetic coupling between the spins (since electron occupation \leftrightarrow spin up in Z basis), which makes sense because $\lambda \sim \Delta$, so that $\lambda > 0$ corresponds to an antiferromagnetic coupling.

Similarly, having $\lambda < 0$ gives our fermions a chiral mass, which breaks T and $T\tilde{C}$ but is invariant under charge conjugation \tilde{C} . Indeed, mapping this mass term back to the c_j language gives

$$\lambda < 0 \implies H \ni \pm M \sum_j ((-1)^j c_j^\dagger c_{j+1} + h.c.), \quad (332)$$

which is a modulated hopping. This term breaks T but is invariant under \tilde{C} , so that it also breaks $T\tilde{C}$, as required. The physical picture here is of a Mott insulator coming from a dimerized phase, with the two ground states being related by translation since they differ in whether the dimers form on the even or odd links.

Summarizing, we have two types of Mott insulators. The first type (type I from here on) is one where the fermion occupation number is staggered, while the second (type II) is a dimerized phase. The type I insulator has SSB for \tilde{C} , but preserves $T\tilde{C}$, while the type II insulator has SSB for $T\tilde{C}$ but preserves \tilde{C} .

For each insulator of a given type (there are two such insulators of each type, related by translation), we can view one insulator as a trivial vacuum and the other as an SPT protected by the relevant unbroken symmetry.

Since SPTs are diagnosed by their edges, in order to elucidate this claim we should study domain walls where M changes sign, corresponding to boundaries between the vacuum and the putative type I / type II SPTs (as usual, the choice of whether M or $-M$ corresponds to the vacuum is arbitrary; only the relative sign is important [since the relative sign is chosen by the way in which T is spontaneously broken]).

Consider a first a domain wall for the type I Mott insulator. The domain wall corresponds to a pair of adjacent lattice sites that are either both occupied or both unoccupied. Evidently this domain wall carries charge $1/2$, since it can be created by inserting a single occupied / unoccupied site into a chain with uniform M , which is at half-filling so that each site carries charge $1/2$. That the domain wall (edge of the SPT) carries fractional charge under \tilde{C} is a

consequence of the fact that the symmetry protecting the SPT is $T\tilde{C}$. More precisely, we see that a domain wall with M changing sign at y is created by the operator

$$U_\pi(y) = \exp \left(i\pi \int_{-\infty}^y dx \frac{\partial_x \phi}{4\pi} \right). \quad (333)$$

Indeed, using the commutation relations we see that acting on a state with a definite spatial profile for θ , we have (using $\hat{\theta}$ to denote an operator just for now)

$$U_\pi(y)e^{i\hat{\theta}(x)}|\theta\rangle = e^{i\theta(x)}|U_\pi(y)\theta\rangle, \quad U_\pi(y)e^{i\hat{\theta}(x)} = e^{i\pi\Theta(y-x)}e^{i\hat{\theta}(x)}|U_\pi(y)\theta\rangle. \quad (334)$$

Evidently the θ value of $|U_\pi(y)\theta\rangle$ is the same as that of $|\theta\rangle$ at all $x > y$, while it is increased by θ for all $x < y$. Thus $U_\pi(y)$ does indeed create a domain wall across which θ jumps by π . Also, we see that this domain wall carries charge $1/2$. One way to make this precise is to check that the domain wall has particle number $1/2$: since $-\partial_x \theta / 2\pi$ is the total fermion number density¹², we see that, letting $N(y-a, y+a)$ denote the total fermion number in the interval $[y-a, y+a]$,

$$e^{2\pi i \hat{N}(y-a, y+a)}U_\pi(y)|\theta\rangle = e^{-2\pi i \int_{y-a}^{y+a} \partial \theta / 2\pi}U_\pi(y)|\theta\rangle = e^{-i\pi}U_\pi(y)e^{2\pi i \hat{N}(y-a, y+a)}|\theta\rangle. \quad (341)$$

¹²We might naively have guessed that the total density was instead $\partial_x \phi_L + \partial_x \phi_R = \partial_x \phi$. To identify the density correctly, we can think about determining the phase of the fermion operators ψ_σ through the WKB approximation, where the phase is determined through $e^{i \int_0^x dx' k(x')}$. The right-moving part of the electron creation operator is

$$\psi_R^\dagger(x)e^{ik_F x} \sim e^{i \int_0^x dx' k(x')} = e^{ik_F x + i \int^x dx' \delta k_F(x')}, \quad (335)$$

where $\delta k_F(x')$ is the variation of the fermi level caused by a variation in electron density (I know mixing \mathbb{R} space and k space like this is weird, but we're working in the WKB approximation where we're assuming that the density varies sufficiently slowly that we can define a local Fermi level in this way). The density of right-moving electrons is $(k_F + \delta k_F)/2\pi$, so that $\delta k_F(x) = 2\pi \delta n_R$. Thus

$$\psi_R^\dagger(x)e^{ik_F x} \sim e^{ik_F x} e^{i2\pi \int^x dx' \delta n_R(x)}. \quad (336)$$

Since we defined ϕ_R through $\mathcal{B}[\psi_R^\dagger] = e^{i\phi_R}$, we see that

$$\phi_R(x) = 2\pi \int^x dx' \delta n_R(x') \implies \frac{\partial_x \phi_R}{2\pi} = \delta n_R(x). \quad (337)$$

The left moving electrons are treated in a similar way: since the ψ_L fields live near the point of the FS at $-k_F + \delta k_F$, we have

$$\psi_L^\dagger(x)e^{-ik_F x} \sim e^{-ik_F x + i \int^x dx' \delta k_F(x')}. \quad (338)$$

Now a positive δk_F corresponds to a smaller density of L fermions in our conventions. Thus $\delta k_F(x') = -2\pi \delta n_L(x')$, and so

$$\psi_L^\dagger(x) \sim e^{-i2\pi \int^x dx' \delta n_L(x')} \implies \frac{\partial_x \phi_L(x)}{2\pi} = -\delta n_L(x), \quad (339)$$

since we have $\mathcal{B}[\psi_L^\dagger] = e^{i\phi_L}$. Thus the total deviation of the total density away from $2k_F/2\pi$ is

$$\delta n_R + \delta n_L = \frac{1}{2\pi}(-\partial_x \phi_L + \partial_x \phi_R) = -\frac{\partial_x \theta}{2\pi}. \quad (340)$$

This means that the particle number in the interval $[y - a, y + a]$ in the state $|U_\pi(y)\theta\rangle$ is different by $1/2$ from the particle number in the state $|\theta\rangle$: thus the domain wall carries $U(1)$ charge $1/2$.

The type II insulator is entirely analogous. Here the domain wall is a location where two dimers share an end on a single site, or where two dimers are missing from consecutive links. The operator creating the domain wall is the same as before (since as for the type I case the domain wall is a π shift of θ), and as above it carries fractional $U(1)$ charge.



Skyrmion energy in the 2+1D $O(3)$ sigma model

I couldn't find the derivation of the skyrmion energy in the 2+1D $O(3)$ nlsm online anywhere (although I didn't try that hard), so today we will calculate it ourselves.



By symmetry, the minimal-charge skyrmion solution $\mathbf{n}(\mathbf{r})$ will only be a function of r . Since we are in two dimensions the skyrmion action will satisfy $S[\mathbf{n}(r/\xi)] = S[\mathbf{n}(r)]$ for any ξ ; hence we must have a one-parameter family of solutions related by scale transformations. Since the generator of $\pi_2(S^2)$ is the unit map, the skyrmion solution will therefore take the form of a conformal map from S^2 onto the plane.

As is the case with any sigma model, the action can be written as

$$S = \int_{\mathbb{R}^2} \text{Tr}[\mathbf{n}^*(g)\eta], \quad (342)$$

with η the (in this case flat) metric on \mathbb{R}^2 and g the metric on the target space (here S^2), with $\mathbf{n}^*(g)$ denoting the pullback by $\mathbf{n} : \mathbb{R}^2 \rightarrow S^2$. Since \mathbf{n} must be a conformal map, the pullback metric $\mathbf{n}^*(g)$ must have the conformally flat form

$$[\mathbf{n}^*(g)]_{ij} = \frac{4\xi^2}{(\xi^2 + r^2)^2} \delta_{ij}, \quad (343)$$

for some scale parameter ξ . We then have

$$S[\mathbf{n}] = \frac{\rho}{2} \int_0^\infty dr 2\pi r \text{Tr}[\mathbf{n}^*(g)] = \frac{\rho}{2} \int_0^\infty dr 2\pi r \frac{8}{(1+r^2)^2} = 4\pi\rho. \quad (344)$$

Note that we never had to know the explicit coordinate representation for \mathbf{n} !

We can check this result by writing out the explicit form of the conformal map. The conformal maps from $S^2 \rightarrow \mathbb{R}^2$ correspond to doing stereographic projections with different

scale factors ξ . Therefore the explicit coordinate expressions for the components of \mathbf{n} can be obtained from

$$x^i/\xi = \frac{n^i}{1+n^z}, \quad (345)$$

where $i = x, y$. Letting $r^2 = x^2 + y^2$, the fact that $|\mathbf{n}| = 1$ tells us that, after a bit of algebra, the skyrmion solution has components

$$n^z = \frac{\xi^2 - r^2}{\xi^2 + r^2} \implies (n^x, n^y) = \frac{2}{1+r^2/\xi^2}(x/\xi, y/\xi). \quad (346)$$

Now we only need to plug this into the action, and see what we get. Since we know all choices for ξ will have the same action, we will now set $\xi = 1$. The Laplacian of the above solution is

$$-\nabla^2 \mathbf{n} = \frac{8}{(1+r^2)^3} (2r \cos(\phi), 2r \sin(\phi), 1-r^2). \quad (347)$$

This then gives

$$-\mathbf{n} \cdot \nabla^2 \mathbf{n} = \frac{8}{(1+r^2)^2}, \quad (348)$$

which then is integrated with the same integral as above. ✓

For posterity's sake, we list the derivatives of n :

$$\begin{aligned} \partial_i n^z &= -\frac{4x^i \xi^2}{(\xi^2 + r^2)^2} \\ \partial_x n^x &= 2 \frac{\xi^3 + \xi(y^2 - x^2)}{(\xi^2 + r^2)^2} \\ \partial_y n^x &= -\frac{4xy\xi}{(\xi^2 + r^2)^2} \end{aligned} \quad (349)$$



Today's a fast one: working out something that I heard Senthil mention at lunch, which apparently came from a physics today article by Witten. The idea is to solve the Schrodinger equation for some given atomic potential as an expansion in $1/d$, where d is the space dimension. This kind of approximation is useful since while the spectrum for the Hydrogen atom is of course solvable the spectra of (even slightly) more complicated atoms is not, and so finding a simple large- N flavored approach is useful.



The reason why large d is useful is because it simplifies the Laplacian. We will define the Laplacian as the negative-definite operator $\Delta = \nabla_i \nabla^i$, and consider the Schrodinger equation

$$(-\alpha \Delta + V(r))\psi = E\psi, \quad (350)$$

where we have assumed the potential to be rotationally symmetric. We claim that

$$\Delta = \partial_r^2 + \frac{d-1}{r} \partial_r + \Delta_{S^{d-1}}, \quad (351)$$

where $\Delta_{S^{d-1}}$ is the Laplacian on the unit S^{d-1} sphere. Accepting that this is true, we see why $d \rightarrow \infty$ simplifies things, since the radial part of the Laplacian becomes a simple linear derivative term.

Let's now prove this presentation of the Laplacian. To do this, we need to remember how to take divergences on curved spaces. The claim is that for any vector field X , we have in components

$$\nabla \cdot X = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} X^i) = \partial_i X^i + (\partial_i \ln \sqrt{\det g}) X^i, \quad (352)$$

which is equal to the flat-space $\partial_i X^i$ plus a correction term. This correction term is needed to ensure that the divergence theorem works, namely that $\int_V \text{vol} \nabla_i X^i = \int_{\partial V} d^{d-1}x^i \sqrt{g|_{\partial V}} X^i$ (here vol is the volume form on V). This extra term can equivalently be derived by recognizing that the above divergence theorem formula holds provided that we define the divergence of a vector field by

$$\text{vol} \nabla_i X^i = \mathcal{L}_X(\text{vol}), \quad (353)$$

where \mathcal{L}_X is the Lie derivative along X (draw a picture of flow lines along X to see why this is the right definition for the divergence!). We then use

$$\begin{aligned} \mathcal{L}_X(\text{vol}) &= \frac{1}{2} \text{Tr}[g^{-1} \mathcal{L}_X g] \sqrt{g} = \frac{1}{2} (X^k \text{Tr}[g^{-1} \partial_k g] + 2g^{ij} \partial_i X^k g_{kj}) \sqrt{\det g} \\ &= \sqrt{\det g} (\partial^j X_j + X^j \partial_j \ln \sqrt{\det g}), \end{aligned} \quad (354)$$

which gives us the claimed formula for $\nabla \cdot X$.

Next we need to know the appropriate way to take the gradient of a scalar. But this is easy, since there's no parallel-transport stuff to worry about: we simply use $\nabla^i \psi = g^{ij} \partial_j \psi$. Therefore putting this together with the above formula for the gradient, we have shown that for any scalar ψ ,

$$\Delta \psi = \frac{1}{\sqrt{\det g}} \partial_i (g^{ij} \partial_j \psi \sqrt{\det g}). \quad (355)$$

Now we apply this to the problem of \mathbb{R}^d space with the metric given in spherical coordinates, viz.

$$g = 1 \oplus r^2 g_{S^{d-1}}, \quad (356)$$

where $[g_{S^{d-1}}]_{ij}$ is the metric on the unit S^{d-1} sphere. From this, we see that $\sqrt{\det g} \propto r^{2(d-1)}$. Therefore we have

$$\Delta\psi = r^{1-d}\partial_r(r^{d-1}\partial_r\psi) + \Delta_{S^{d-1}}\psi = \partial_r^2\psi + \frac{d-1}{r}\partial_r\psi + \Delta_{S^{d-1}}\psi, \quad (357)$$

proving the earlier claim.

Solving the Schrodinger equation as $d \rightarrow \infty$ is then easy (we will ignore the S^{d-1} part of Δ on the grounds of it being subleading in $1/d$, but it can easily be included using spherical harmonics). The Schrodinger equation goes over approximately to

$$\left(-\alpha\frac{d}{r}\partial_r + V(r)\right)\psi = E\psi, \quad (358)$$

telling us that

$$\psi(r) = \psi(0) \exp\left(-\frac{1}{d\alpha} \int_0^r dr' r'(E - V(r'))\right). \quad (359)$$

Apparently this kind of approach gets spectra correct within $\sim 10\%$ at $d = 3$ for things like e.g. Helium.



Yet more on the $SU(2)$ point of the $c = 1$ CFT

This is a fast one, and comes from a problem in a pset assigned in Ashvin Vishwanath's fall 2018 class on quantum matter. The problem statement is as follows:

Consider the fermionized description of the anisotropic XXZ chain:

$$H = \sum_i \left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + g S_i^z S_{i+1}^z \right]. \quad (360)$$

Write down an expression for $S^z(r)$ in terms of the low energy fermion fields, including components both at zero momentum and at momentum $q = \pi$.

What value of g corresponds to the self-dual $SU(2)$ point? Find out by requiring that the $q = \pi$ component of the S^z spin density have the same power law exponent as the 2-point function of the S^+ operator. We will work in conventions where the bosonized Lagrangian is written as $\mathcal{L} = (\partial_\mu\theta/2\pi)^2/(2K)$.



Since $S_r^\pm = (-1)^r(-1)^{\sum_{r' < r} n_{r'}}$ (the $(-1)^r$ is needed to cancel a minus sign from moving a string operator past a creation operator in $S_i^+ S_{i+1}^-$ in order to produce a correct-sign hopping), we have $S_r^z = S_r^+ S_r^- - 1/2 = n_r - 1/2$. Thus in terms of the low energy fields,

$$\begin{aligned} S_r^z \rightarrow c_r^\dagger c_r - 1/2 &\sim (e^{-ik_F r} R^\dagger(r) + e^{ik_F r} L^\dagger(r))(e^{ik_F r} R(r) + e^{-ik_F r} L(r)) - 1/2 \\ &= n_R(r) + n_L(r) + (-1)^r (L^\dagger(r)R(r) + R^\dagger(r)L(r)) - 1/2, \end{aligned} \quad (361)$$

since $k_F = \pi/2$.

The part of the S^z 2-point function that goes as $(-1)^r$ is the part involving scattering from one of the Fermi points to the other:

$$\langle S_r^z S_0^z \rangle \supset (-1)^r \langle L^\dagger(r)R(r)R^\dagger(0)L(0) \rangle \rightarrow (-1)^r \langle e^{-i\theta(r)} e^{i\theta(0)} \rangle, \quad (362)$$

since in our conventions $L^\dagger(x) \rightarrow e^{i\phi_L(x)}$, $R(x) \rightarrow e^{-i\phi_R(x)}$ and $\theta = \phi_R - \phi_L$. From the action we read off that the propagator for the θ field is

$$G_\theta(r) = -2\pi K \ln |r|, \quad (363)$$

and so

$$\langle S_r^z S_0^z \rangle \rightarrow \frac{1}{|r|^{2\pi K}}. \quad (364)$$

Now we look at the S^\pm correlator. We can figure out the image of S_r^\pm under bosonization from its commutation relation with Z_r , which we know maps to $n_r - 1/2 = \partial_x \theta / (2\pi) - 1/2$. Since $[Z_r, S_r^\pm] = \pm S_r^\pm$, we guess that $S_r^\pm \rightarrow e^{\pm i\phi(r)}$. This identification is natural since we know $\phi(r)$ gets shifted by the $U(1)$ symmetry of rotations about the z axis, in the same way that S^+ does. Indeed, from the number-phase relation $[\phi(x), \partial_y \theta(y)/2\pi] = i\delta(x-y)$, we check that

$$[Z_r, S_r^\pm] \rightarrow [\partial_r \theta, e^{\pm i\phi(r)}] = \pm e^{i\pm\phi(r)} \rightarrow \pm S_r^\pm, \quad (365)$$

as required. Actually this doesn't completely fix the bosonization of S_r^\pm . We will actually include an explicit factor of $(-1)^r$ in its bosonization (as we wrote down above), so that

$$S_r^\pm \rightarrow (-1)^r e^{\pm i\phi(r)}. \quad (366)$$

Anyway, this means that

$$\langle S_r^+ S_0^- \rangle = (-1)^r \langle e^{i\phi(r)} e^{-i\phi(0)} \rangle = (-1)^r e^{G_\phi(r)}. \quad (367)$$

The propagator $G_\phi(r)$ for ϕ is determined by T duality: if the coefficient of the free θ action is $R^2/4\pi$, then the coefficient for the free ϕ action is $1/(4\pi R^2)$. For us $R^2 = 1/2\pi K$, and so the coefficient for the ϕ action will be $K/2$. Thus we have

$$G_\phi(r) = -\frac{1}{2\pi K} \ln |r| \implies \langle S_r^+ S_0^- \rangle \rightarrow (-1)^r \frac{1}{|r|^{1/2\pi K}}. \quad (368)$$

If we require that the power law exponents in the $(-1)^r$ parts of the S^\pm and S^z two point functions match, which is a necessary requirement if the theory is to have $SU(2)$ symmetry, then we require that $2\pi K = 1/(2\pi K)$, so that we predict the $SU(2)$ point to be located at

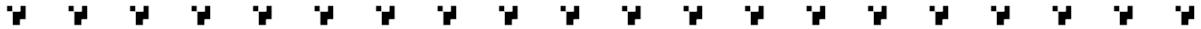
$K = 1/2\pi$. We know from above that the operator $\cos 2\theta$ becomes marginal when $K = 1/2\pi$, so that the $SU(2)$ point is characterized by the radius at which the $\cos 2\theta$ Umklapp term crosses over between relevance and irrelevance.



MFT for Mott insulators and SFs in two dimensions

Today's entry is pretty basic but is something that I hadn't worked out before. The problem statement is as follows:

Show that in the 1+1D Fermi-Hubbard model at half filling, mean field theory predicts a CDW state for arbitrarily small repulsive interactions, and an SCing state for arbitrarily small attractive interactions. Of course both of these conclusions are actually incorrect, with a gapless phase surviving for finite interactions—what goes wrong with the mean field prediction?



First, a note that I'm writing this super late and without paper at hand, so there will likely be numerical mistakes. The starting Hamiltonian is the usual

$$H = -\frac{1}{2} \sum_j (c_j^\dagger c_{j+1} + h.c.) + U \sum_j (n_j - 1/2)(n_{j+1} - 1/2). \quad (369)$$

First let's examine the CDW state, which we expect to occur for some large enough U . We take as our MFT ansatz

$$n_j = \frac{1}{2}(1 + \Delta(-1)^j) + : \delta n_j :, \quad (370)$$

where $: \delta n_j :$ has zero expectation value and is assumed to be small. Putting this into H and dropping the term quadratic in δn_j ,

$$H = -\frac{1}{2} \sum_j (c_j^\dagger c_{j+1} + h.c.) - U \sum_j \left(\frac{\Delta^2}{4} + \Delta(-1)^j : \delta n_j : \right). \quad (371)$$

Now going to momentum space,

$$H \rightarrow \sum_{0 \leq k < \pi} \Psi_k^\dagger \mathcal{H}_k \Psi_k - \Delta^2 UV/4, \quad \mathcal{H}_k = \begin{pmatrix} \cos k & -\Delta U \\ -\Delta U & -\cos(k) \end{pmatrix}, \quad \Psi_k = (c_k, c_{k+\pi})^T. \quad (372)$$

The eigenvalues of \mathcal{H}_k are $E_{\pm}(k) = \pm\sqrt{\cos^2 k + \Delta^2 U^2}$, and so getting the ground state by filling all of the $E_-(k)$ modes means the ground state energy is

$$\mathcal{E}_0 \rightarrow - \int dk \sqrt{\cos^2 k + \Delta^2 U^2} - VU\Delta^2/4. \quad (373)$$

Minimizing this (and normalizing by $V = 1$ for simplicity),

$$\frac{1}{4U} = \int \frac{dk}{\sqrt{\cos^2 k + U^2\Delta^2}}, \quad (374)$$

which has a solution for arbitrarily small positive U since $\cos^2(k)$ vanishes at $k = \pm\pi/2$. We can approximate (ignoring numerical constants so that the important part of the integral is $\int_k dk/\sqrt{k^2 + U^2\Delta^2} \rightarrow \ln(k^2 + U^2\Delta^2)|_0^\Lambda \implies \ln(\Lambda^2/U^2\Delta^2) \approx 1/U$)

$$\Delta \approx \frac{\Lambda}{U} e^{-1/U}. \quad (375)$$

Does this solution for Δ give a lower energy than the $\Delta = 0$ state? Indeed it does: the energy with Δ turned on is schematically

$$\mathcal{E}_\Delta = \alpha\Delta^2 - \beta \int_{k,\Delta} \frac{U^2\Delta}{\sqrt{\cos^2 k + U^2\Delta^2}} \approx \alpha\Delta^2 - \gamma \int_\Delta \Delta \ln(\Lambda^2/U^2\Delta^2) = \Delta^2(\lambda + \omega \ln \Delta), \quad (376)$$

where all the greek letters are *positive* numerical constants that we don't care about. Now when $\Delta = 0$ we get $\mathcal{E}_\Delta \rightarrow 0$, but because of the logarithm, we see that we can always choose Δ to be small enough so that $\lambda + \omega \ln \Delta < 0$, meaning that \mathcal{E}_Δ is most negative when $\Delta \neq 0$. Thus MFT predicts a CDW at any finite $U > 0$, a result that also goes under the name of the Pierles instability (since the mean field acts as a periodic $2k_F$ -wavevector potential).

We can also see the CDW instability by looking at what happens when we add a perturbation H_Δ to H_0 , where H_0 is the hopping term and H_Δ carries momentum π , e.g.

$$H = H_0 + H_\Delta, \quad H_\Delta = \Delta \sum_j (-1)^j c_j^\dagger c_j \rightarrow \Delta \int dx (R^\dagger L + L^\dagger R), \quad (377)$$

where $c_j = e^{-ik_F j} R + e^{ik_F j} L$ (we are setting the Fermi velocity to unity). We now go to second-order perturbation theory, with the second order correction to the energy being (the first-order correction vanishes, since in the ground state $|0\rangle$ where all states below $k_F = \pi/2$ are filled the matrix element $\langle 0|H_\Delta|0\rangle$ vanishes)

$$E_2 = \sum_{l \neq 0} \frac{|\langle l|H_\Delta|0\rangle|^2}{\mathcal{E}_0 - \mathcal{E}_l}, \quad (378)$$

where l run over the non-groundstate eigenstates of the free Hamiltonian and \mathcal{E}_l are the free Hamiltonian energies. Now since the energy denominator will be small only when $|l\rangle$ is a state with excitations near the Fermi surface, we can take $H_0 \approx \int dx (iR^\dagger \partial_x R - iL^\dagger \partial_x L)$ for our purposes, so that the energies of the $|l\rangle$ states are linear in momentum. Now the mass

term H_Δ has momentum transfer π , so that it moves a hole just below the left Fermi point to a particle just above the right Fermi point, and vice versa. Thus when the relevant matrix element $\langle l | H_\Delta | 0 \rangle$ is non-zero, $\mathcal{E}_l \sim l$, where l is a momentum. So rather schematically we see that the second order correction to the energy is

$$E_2 \sim \Delta \int_0^\Lambda dk \frac{1}{k} \rightarrow \infty, \quad (379)$$

so that E_2 diverges logarithmically. Here Λ is some unimportant upper cutoff (note that we have a divergence here from the *small* momentum modes right by the Fermi surface).

What happens to the CDW instability away from half-filling? Let us more generally write the interaction term as $\sum_j U(n_j - \nu)(n_{j+1} - \nu)$. This adds to the MF Hamiltonian (still assuming a CDW at momentum π since this is where the divergence in the susceptibility is) a term $\sum_j U\delta\nu n_j$, where $\delta\nu = \nu - 1/2$. The negative-energy branch of the spectrum is then $E_- = U\delta\nu - \sqrt{U^2(\delta\nu)^2 + \cos^2 k + \Delta^2 U^2}$, which in turn leads to the gap equation (assuming $\Delta \neq 0$)

$$\frac{1}{4U} = \int \frac{dk}{\sqrt{U^2(\delta\nu)^2 + \cos^2 k + \Delta^2 U^2}}. \quad (380)$$

Does this lead to an instability for arbitrarily small U ? Because of the $U^2(\delta\nu)^2$ term in the denominator, the integrand can no longer be made arbitrarily large by making Δ appropriately small. If we make the same approximation to the integral as before, and drop unimportant numerical factors, the self-consistent equation ends up reading (I think)

$$\Delta^2 \sim -(\delta\nu)^2 + (\Lambda/U)^2 e^{-1/U}. \quad (381)$$

Now for a fixed cutoff Λ , a small enough U will lead to $\Delta^2 < 0$, which is a contradiction. Thus away from half-filling the system is not immediately unstable to a CDW, as expected (one needs a CDW ansatz at wavevector $\sim 2k_F$ to get an instability).

The same analysis can be applied for the SCing instability, which also comes about from a diverging susceptibility. Consider adding the term

$$H_\Delta = \Delta \sum_j (c_j c_{j+1} + h.c.) \rightarrow 2\Delta \int dx (iLR - iR^\dagger L^\dagger) + \dots, \quad (382)$$

where \dots denotes less relevant operators / ones that oscillate rapidly. The matrix element $\langle 0 | H_\Delta | l \rangle$ will be non-zero if $|l\rangle$ contains a pair of particles or holes with opposite momenta $k, -k$. The energy of such a pair is linear in k , and so we again get a (logarithmically) divergent second-order correction to the energy in the same way as before.

Doing MFT for the SCing instability works in essentially the same way as for the CDW instability; we just replace $c_j c_{j+1} = \Delta + (c_j c_{j+1} - \Delta)$, and drop terms quadratic in the deviation about Δ . The symmetry $\cos(k) = \cos(-k)$ is now what allows us to derive the instability by defining the spinor $\Psi_k = (c_k, c_{-k}^\dagger)$, with the associated Hamiltonian

$$\mathcal{H}_k = \begin{pmatrix} \cos k - \mu & e^{ik} \Delta U \\ e^{-ik} \Delta U & -[\cos(k) - \mu] \end{pmatrix}, \quad (383)$$

where we took $\Delta \in \mathbb{R}$ for simplicity and where μ is a chemical potential determined by the filling. For a filling fraction of ν , we have $\mu = 2\nu U$. Going through the analysis yields the exact same gap equation as in the CDW case, except with $\cos(k) \mapsto \zeta(k) \equiv \cos(k) - \mu$ and $U \mapsto -U$. As long as the chemical potential is not so large that $\zeta(k)$ the gap equation is solved in the same way as in the CDW case, by linearizing $\zeta(k)$ about its vanishing point. Thus as the SCing instability only relies on the $k \mapsto -k$ TRS symmetry of the diagonal part of the Hamiltonian, it is insensitive to the precise value of the filling ν (at the technical level, the difference is that the chemical potential caused by $\nu \neq 1/2$ shows up in \mathcal{H}_k as proportional to $\mathbf{1}$ in the CDW case, but proportional to Z in the SC case).

Now we know in fact that both these mean field results are incorrect, with a gapless LL phase surviving up to finite interaction strengths. This is of course best seen by bosonization. The basic idea is that the two ordering tendencies compete with one another: forming a crystal at a large wavevector is exactly the opposite of what you'd want to do when forming a SF (which orders at zero wavevector), and vice versa. The cool thing is that the system can't decide whether to be a crystal or a SF over a finite range of parameters, leaving us with a gapless region in parameter space.



Topological robustness of Fermi surfaces

Today we will explain why the existence of a fermi surface is protected by a topological invariant relating to the winding of a fermion Greens function. Nothing about this is new, with the present diary entry simply being a recapitulation of chapter 8 of Volovik's book.



For a Fermi “surface” of codimension $p + 1$ in (ω, k) space (we will always write k for both $|\vec{k}|$ and \vec{k} , hopefully the distinction will be clear from context), a topological invariant is obtained by integrating a p -form around an S^p that links the Fermi surface. The p -form is constructed by taking the trace of wedge products of the Maurer-Cartan form

$$\omega_{\mathcal{G}} = \mathcal{G} d\mathcal{G}^{-1}, \quad \mathcal{G}(\omega, k) = \frac{1}{i\omega - \mathcal{H}}, \quad (384)$$

where \mathcal{H} is the Hamiltonian density. Note that here we are writing $\mathcal{G}(\omega, k)$ for the full interacting Greens function at *imaginary* frequency. The nontriviality of the topological invariant will come from the properties of the singularities of \mathcal{G} in ω, k space. Since we want to characterize the robustness of the Fermi surface, we want the only singularities in

\mathcal{G} to come at the Fermi surface at zero frequency, where \mathcal{H} and ω vanish. If we were at \mathbb{R} frequencies this would not be the case, since then we'd have singularities in \mathcal{G} whenever we had quasiparticles going on-shell. A general topological invariant is constructed for each p via

$$N_p = \frac{1}{\mathcal{N}} \oint_{S^p} \text{Tr}[\omega_{\mathcal{G}}^{\wedge p}], \quad (385)$$

with \mathcal{N} an (imaginary) normalization constant. N_p is invariant under smooth deformations of the Hamiltonian density for the usual reasons.

In fact, this only provides a nontrivial index when p is odd, since ω^p defines a nontrivial class in $H^\bullet(X)$ only for odd p : when $p \in 2\mathbb{Z}$, we in fact have $\text{Tr}(\omega^p) = -d\text{Tr}(\omega^{p-1})$ (there may be an omitted combinatorial prefactor). This suggests that Fermi surfaces of even codimension (in full ω, \vec{k} space) are stable (like regular Fermi surfaces in spatial dimension $d = 3$), while those of odd codimension are unstable (e.g. Fermi lines in $d = 3$, which are codimension 3).

The simplest example of this is a prototypical codimension-2 Fermi surface (i.e. an S^2 in three spatial dimensions) for a single free fermion. In this case we have (writing N for N_1 in what follows)

$$N = \frac{1}{2\pi i} \oint_C dz^\mu \frac{1}{i\omega - \mathcal{H}(k)} \partial_\mu(i\omega - \mathcal{H}(k)). \quad (386)$$

Here z^μ is a stand-in for the coordinate along the contour. We will choose the contour to be a small circle of radius R linking the Fermi sphere in the (ω, k_x) plane. Taking R to be small, we can expand $\mathcal{H}(k)$ about the FS as $\mathcal{H}(k) \approx v_F k$, where now k is measured relative to the FS. Wolog we can set $v_F = 1$, so that the invariant is

$$N = -\frac{1}{2\pi i} \oint dz^\mu \frac{i\omega + k}{\omega^2 + k^2} \partial_\mu(i\omega - k). \quad (387)$$

Parametrizing the contour in the (ω, k_x) plane by $(R \cos \theta, R \sin \theta)$, we have

$$\begin{aligned} N &= -\frac{1}{2\pi i} \oint d\theta (-\sin \theta, \cos \theta)_j (i, -1)^j [i \cos \theta + \sin \theta] \\ &= -\frac{1}{2\pi i} \oint d\theta (-i \sin^2 \theta - i \cos^2 \theta) = 1. \end{aligned} \quad (388)$$

Thus the winding around the Fermi surface is nontrivial, and the FS is topologically protected (N is independent of the contour since $\omega_{\mathcal{G}}$ is closed: $d\mathcal{G} \wedge d\mathcal{G}^{-1} = 0$).

How do we get $N = -1$? We just have to turn left-movers to right-movers by changing the sign of v_F to $v_F = -1$. Indeed, if we do this then we have

$$N = -\frac{1}{2\pi i} \oint dz^\mu \frac{i\omega - k}{\omega^2 + k^2} \partial_\mu(i\omega + k) = -\frac{1}{2\pi i} \oint d\theta (-\sin \theta, \cos \theta)_j (i, 1)^j [i \cos \theta - \sin \theta] = -1. \quad (389)$$

So, changing the sign of $\mathcal{H}(k)$ flips the index N .

This quantization of course does not change when e.g. weak interactions are switched on. For example, as long as the singular part of the propagator near the FS is of the form $Z/(i\omega - v_F k)$, the quantization of $N = \pm 1$ is quite insensitive to the exact values of Z and

$|v_F|$: in FL theory changing the strength of interactions only changes the residue Z and renormalizes the Fermi velocity, neither of which change N .

We can then use N to essentially *define* what we mean by a Fermi surface: it is the codimension-1 manifold such that curves C which have linking number one with it in (ω, k) space have $N = +1$. The sign of the linking number (or equivalently, the orientation of the Fermi surface) let us distinguish between particle-like and hole-like Fermi surfaces, since the two signs are related by either reversing the orientation of the contour or reversing the sign of the Hamiltonian density.

The quantization of the index and this definition of the FS continues to hold even in some non-FL scenarios, viz. the Luttinger liquid in 1+1D. Indeed, let $z = k + i\omega$; a general LL then has

$$\mathcal{G}(\omega, k) = \frac{\omega_0^{\alpha+\beta-1}}{(i\omega - k)^\alpha (i\omega + k)^\beta} \sim \frac{1}{z^\alpha \bar{z}^\beta}, \quad (390)$$

where ω_0 is a constant needed to get the dimensions right. The index is then just

$$N = \frac{1}{2\pi i} \oint (\alpha z^{-1} dz + \beta \bar{z}^{-1} d\bar{z}) = \alpha - \beta. \quad (391)$$

From this, we confirm that only chiral parts of correlators contribute to the index N .

In these examples, this result about the quantization of the index is pretty trivial if we think about it for a second, and in fact will hold whenever the ψ fields continue to represent the IR fields with a dispersion linearized about the FS. Indeed, the difference between powers of $x - t$ and $x + t$ in the propagator is fixed simply by fermion statistics and the fact that when the theory is massless and we work in the deep IR, the field ψ is a scaling field. We know that in the IR, the ψ_i fields will have the (Euclidean-space) scaling (here i labels points on the FS)

$$\langle \psi_i(z, \bar{z}) \psi_i^\dagger(z, \bar{z}) \rangle \sim \frac{1}{z^{2h} \bar{z}^{2\bar{h}}} = \frac{1}{(z\bar{z})^{h+\bar{h}} (z/\bar{z})^{h-\bar{h}}} = \frac{1}{|z|^{2\Delta} (z/\bar{z})^s}, \quad (392)$$

with $\Delta = h + \bar{h}$ and $s = h - \bar{h}$. The statistics of ψ is determined by s ; for spin 1/2 fermions this must always be $s = \pm 1/2$. and therefore when interactions render the ψ_i fields non-chiral by making the ψ_i correlator depend on both z and \bar{z} , they can only do so in a way such that $\langle \psi_i(z, \bar{z}) \psi_i^\dagger(z, \bar{z}) \rangle \sim z^{-\alpha} \bar{z}^{-\beta}$, with $\alpha - \beta = \pm 1$. Therefore regardless of interactions (and regardless of whether or not the low energy excitations have any overlap with the microscopic electrons), a FS can always be defined by looking for the surface around which the N index is ± 1 .

Instabilities of the FS and the topological invariant

We know of many examples where Fermi surfaces can be destroyed by various types of instabilities, like the CDW or SCing instabilities. Evidently for this to happen, we need to have some way of changing the Greens function holonomy we calculated above so that it becomes trivial. Indeed, if we take a free theory with $N \neq 0$ and then adiabatically turn on interactions we will still get something with $N \neq 0$, i.e. we will still get something with a

FS. Therefore $N = 0$ is required if we want to get an instability of the FS which leads to a gap (since in a gapped phase \mathcal{G} has no singularities and N accordingly vanishes).

Since we have seen that taking $\mathcal{H}(k) \mapsto -\mathcal{H}(k)$ changes the sign of N , we see that one potential way to create an instability is to find some way of pairing up Fermi surfaces of particles ($N = 1$) and Fermi surfaces of holes ($N = -1$) to create a Fermi surface with net zero winding number, which then has no obstruction to being destroyed by an instability that creates a gap.

As simple examples, both the CDW and SCing instabilities in 1+1D are created in this way. In each case, we write the Hamiltonian (as we would do in e.g. a MFT analysis) as a bilinear in terms of the spinors

$$\Psi_{CDW} = (c_k, c_{k+\pi})^T, \quad \Psi_{SC} = (c_k, c_{-k}^\dagger)^T. \quad (393)$$

The crucial property of both of these spinors is that the Hamiltonian matrices they are associated with have a diagonal component proportional to Z (the Pauli matrix), e.g. $\cos(k)Z$ (because e.g. in the CDW case we have $\cos(k + \pi) = -\cos(k)$; in the SCing case the minus sign is due to the mixed creation / annihilation nature of Ψ_{SC} and the symmetry $\cos(-k) = \cos(k)$). More generally, suppose we can write the Hamiltonian as a bilinear in spinors such that after diagonalizing $\mathcal{H}(k)$ ¹³, we have $\mathcal{H}(k) = E(k)Z \approx v_F k Z$. Again setting $v_F = 1$, the index is then

$$\begin{aligned} N &= -\frac{1}{2\pi i} \text{Tr} \left[\oint dz^\mu \begin{pmatrix} \frac{1}{i\omega-k} & 0 \\ 0 & \frac{1}{i\omega+k} \end{pmatrix} \partial_\mu \begin{pmatrix} i\omega-k & 0 \\ 0 & i\omega+k \end{pmatrix} \right] \\ &= -\frac{1}{2\pi i} \oint dz^\mu \frac{1}{\omega^2 + k^2} ((i\omega - k)\partial_\mu(i\omega + k) + (i\omega + k)\partial_\mu(i\omega - k)) \\ &= 0. \end{aligned} \quad (394)$$

So in this case, the index when interactions are switched off is actually zero (and hence not topologically protected), which means that introducing interactions adiabatically does have the possibility of creating a gap and leading to an instability of the FS.

Studying the N_p invariants for $p > 1$ is a nice way to learn about Chern insulators and other concepts in topological band theory; more on this in a future diary entry.



Luttinger liquid correlators

Today we will calculate the form of the correlators in a Luttinger liquid. This is rather straightforward, but I thought it would be useful to have a set of conventions laid out somewhere.

¹³Sending $\mathcal{H} \mapsto U^\dagger \mathcal{H} U$ for unitary U leaves $\text{Tr}(\omega_{\mathcal{G}})$ invariant, which is a relatively quick thing to check.



In the free case, we can use bosonization to write the Hamiltonian in the Sugawara form

$$H_0 = \pi v \int dx (J_+ J_+ + J_- J_-), \quad (395)$$

where $J_{\pm} = \frac{1}{2\pi} \partial_x \phi_{\pm}$, with $\psi_{\pm} \sim :e^{\pm i\phi_{\pm}}:$ (using quantum fields and strings bosonization conventions). Four-fermion interaction terms are current bilinears, and so a generic¹⁴ quartically interacting theory will have the Hamiltonian

$$H = \int dx (\pi \tilde{v} (J_+ J_+ + J_- J_-) + \lambda J_+ J_-), \quad (396)$$

with $\tilde{v} = v + \delta v$.

This Hamiltonian mixes left and right currents, but we can get it back into the Sugawara form by doing a rotation¹⁵

$$j_{\pm} = J_{\pm} \cosh \theta + J_{\mp} \sinh \theta, \implies J_{\pm} = j_{\pm} \cosh \theta - j_{\mp} \sinh \theta. \quad (397)$$

Substituting this in, we require that

$$H = \pi v \int dx (j_+ j_+ + j_- j_-). \quad (398)$$

This form will mean that the fields that make up the j currents will have vertex operators $e^{\pm i\varphi_{\pm}}$ that have $\Delta_{\pm} = 1/2$. This gives us the constraints (using $\cosh^2 \theta + \sinh^2 \theta = \cosh(2\theta)$)

$$(\cosh^2 \theta + \sinh^2 \theta)v = \tilde{v} \implies 2 \cosh^2 \theta = \tilde{v}/v + 1 \implies \cosh(2\theta) = \tilde{v}/v, \quad (399)$$

and

$$4\pi v \sinh \theta \cosh \theta = \lambda \implies \frac{4\pi \tilde{v} \sinh \theta \cosh \theta}{\cosh(2\theta)} = \lambda \implies \tanh(2\theta) = \frac{\lambda}{2\pi \tilde{v}}, \quad (400)$$

where in the last line we used $\sinh \theta \cosh \theta = \sinh(2\theta)/2$. Since $|\tanh(2\theta)| < 1$, the model only makes sense when $|\lambda| < 2\pi \tilde{v}$,¹⁶ in particular changing the velocity isn't just something trivial than can be undone by working in relativistic notation—if \tilde{v} is too small compared to the mixing λ , the theory stops making sense.

Fermions are still bosonized according to $\psi_{\pm} \sim :e^{\pm i\phi_{\pm}}:$, but now the ϕ are not related to the currents j which appear in the free Hamiltonian. We then need to rewrite the ϕ vertex operators in terms of the vertex operators of the field associated to the j currents, since this field is the one with easy-to-compute correlators. Letting $j_{\pm} = \frac{1}{2\pi} \partial_x \varphi_{\pm}$, we have¹⁷

$$\psi_{\pm} \rightarrow :e^{\pm i(\cosh \theta \varphi_{\pm} - \sinh \theta \varphi_{\mp})}: \quad (401)$$

¹⁴Not quite generic; we will be taking the left and right movers to have equal velocities.

¹⁵This form of the rotation preserves the Kac-Moody commutators for the $J_{\pm,k}$ s appearing in the Laurent expansion of the J_{\pm} s.

¹⁶which isn't obvious on the fermion side.

¹⁷This is in conventions where the correlator of φ_{\pm} is $\sim \ln(x \pm t)$: this means that φ_+ is left-moving, while φ_- is right-moving, in keeping with us usually assigning left-movers positive chirality.

This gives us the correlators

$$\langle \psi_{\pm}(t, x) \psi_{\pm}^{\dagger}(0, 0) \rangle \sim \frac{1}{(x \pm t)^{\cosh^2 \theta} (x \mp t)^{\sinh^2 \theta}} = \frac{1}{(x \pm t)(x^2 - t^2)^{\sinh^2 \theta}}, \quad (402)$$

which shows that the effect of interactions is to add on a left-right mixed part to the correlation function, while keeping the conformal spin of the correlator (viz. the number of powers of $x + t$ minus the number of powers of $x - t$) unchanged, as is required if the fermionic character of the low energy excitations is to be preserved. From the above expression, we see that the scaling dimensions of the fermions are

$$\Delta_{\pm} = \frac{\cosh^2 \theta + \sinh^2 \theta}{2} = \frac{\tilde{v}}{2v}, \quad s_{\pm} = h - \bar{h} = \pm \frac{1}{2}. \quad (403)$$

Now we would like to Fourier transform this expression for the correlator. To do this, define

$$w \equiv x + t, \quad \bar{w} \equiv x - t. \quad (404)$$

Then in general, the Fourier transform of interest is

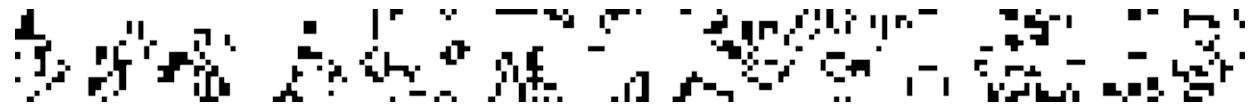
$$\int dw \wedge d\bar{w} \frac{e^{-i\omega(w-\bar{w})/2+ik(w+\bar{w})/2}}{w^{\alpha} \bar{w}^{\beta}} = \left(\int dw \frac{e^{iw(k-\omega)/2}}{w^{\alpha}} \right) \left(\int d\bar{w} \frac{e^{i\bar{w}(k+\omega)/2}}{\bar{w}^{\beta}} \right), \quad (405)$$

for some α, β . By dimensional analysis, this gives

$$\int dw \wedge d\bar{w} \frac{e^{-i\omega(w-\bar{w})/2+ik(w+\bar{w})/2}}{w^{\alpha} \bar{w}^{\beta}} \sim \frac{1}{(k - \omega)^{1-\alpha} (k + \omega)^{1-\beta}}. \quad (406)$$

Applying this to the problem at hand, we get, after some algebra

$$\langle \psi_{\pm}(\omega, k) \psi_{\pm}^{\dagger}(\omega, k) \rangle \sim \frac{(k^2 - \omega^2)^{\sinh^2 \theta}}{k \pm \omega}. \quad (407)$$



The SSH model and polarization

This is a problem from Senthil's 2019 class on correlated electronic systems. The Hamiltonian for the SSH model is

$$H = - \sum_i (t_1 c_{iA}^{\dagger} c_{iB} + t_2 c_{iB}^{\dagger} c_{i+1A} + h.c.), \quad (408)$$

where A/B labels the two sites in the unit cell.

Do several things:

(a) show that

$$\tilde{C} : c_{iA} \mapsto c_{iA}^\dagger, \quad c_{iB} \mapsto -c_{iB}^\dagger \quad (409)$$

is a symmetry of H , and identify it in terms of C, P, T .

- (b) Show that \tilde{C} implies that the spectrum of H comes in $\pm\lambda$ pairs.
- (c) Write the Hamiltonian density at a given momentum as $H(k) = h_i(k)\sigma^i$. Consider varying k around the S^1 of the BZ. How are the $t_1 = 0$ and $t_2 = 0$ insulators different?
- (d) Calculate the polarization (alias Berry phase) for each band in the two insulating phases.
- (e) Argue that in the presence of \tilde{C} symmetry, the polarization is quantized.



a) Since $c_{i\alpha}^\dagger$ and $c_{j\beta}$ anticommute for all $\alpha \neq \beta$ (here $\alpha, \beta \in \{A, B\}$), we see that the map

$$\tilde{C} : c_{iA} \mapsto c_{iA}^\dagger, \quad c_{iB} \mapsto -c_{iB}^\dagger, \quad i \mapsto -i, \quad (410)$$

is a symmetry of H : the minus sign on the B sublattice is compensated for by the fermion anticommutation sign. Here the fact that \tilde{C} is antiunitary follows from us wanting to send $c_{kA}^\dagger \mapsto c_{kA}, c_{kB}^\dagger \mapsto -c_{kB}^\dagger$ in momentum space.

Pedantic discussion of the \tilde{C} symmetry: Some cmt people call this charge conjugation, but this is not in keeping with the definition of charge conjugation in hep-th or in the rest of this diary, where it is a *unitary* symmetry. To identify what \tilde{C} actually does, it helps to look at the continuum theory so that we can compare to field theory conventions. In what follows we will need the continuum Hamiltonian near $t_1 \sim t_2 \sim t$, linearized around the gapless point $k = \pi$. Writing $t_1 = t + \delta/2, t_2 = t - \delta/2$ and $k = \pi + q$, we have to linear order in q ,

$$\mathcal{H}(q) \approx \delta X + vqY, \quad v = 2t. \quad (411)$$

From now on we will set $v = 1$. This gives us the Dirac Hamiltonian

$$H = \int dx \psi^\dagger(x)(-iY\partial_x + \delta X)\psi(x). \quad (412)$$

What γ matrices are we using? Since the Dirac mass is $m\bar{\psi}\psi$ (which is Hermitian in $(+, -)$ signature), we can take $\gamma^0 = X$. We will work in \mathbb{R} time so that thinking about T is easier, so that means we should have γ^1 such that $-i\gamma^0\gamma^1 = -iY$, since the Lagrangian is $\bar{\psi}(i\cancel{\partial} + m)\psi$, meaning that H contains $-i\bar{\psi}\gamma^1\partial_x\psi$. Anyway, we see that if we take $\gamma^1 = iZ$, then we get $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and indeed, $-i\gamma^0\gamma^1 = XZ = -iY$. This choice of basis is nice because it makes the decomposition into different chiralities manifest.

Now we should identify how C, P, T act. Here C is charge conjugation, which we should avoid confusing with the anti-unitary particle-hole symmetry \tilde{C} . Unfortunately the representation theory of Pin groups is a huge mess, and depends on the signature we're working

in and all sorts of details. So, we will just find the action of these symmetries by explicit computations. P should satisfy $P^\dagger \gamma^0 P = \gamma^0$ and $P^\dagger \gamma^0 \gamma^1 P = -\gamma^0 \gamma^1$, and so we can take $P : \psi(t, x) \mapsto \gamma^0 \psi(t, -x)$. This is indeed the generator of reflections that send $x \mapsto -x$ in the Clifford algebra generated by γ^0 and γ^1 . Since $\gamma^0 = X$, this also agrees with how we know reflections act on the microscopic chain: by exchanging A and B sublattices.

We want C to be unitary, and so requiring that it preserve the kinetic term means that for $(C)\psi_\alpha = [C]_{\alpha\beta}\psi_\beta^*$, we need

$$-\bar{\psi} \gamma^\mu \psi = \psi^T C^\dagger \gamma^0 \gamma^\mu C \psi^* = -\psi^T [C^\dagger \gamma^0 \gamma^\mu C]^T \psi \implies [C^\dagger \gamma^0 \gamma^\mu C]^T = \gamma^0 \gamma^\mu \quad (413)$$

for all μ (here the first minus sign on the left is since there is an extra minus sign in the transformation of the kinetic term after integrating by parts, which in my experience is incredibly easy to forget about). Setting $\mu = x$ means that $C^T Z X C^* = -ZX$, so that $C = Z$ or $C = X$ (since we want $C^2 = \mathbf{1}$). Now if C is to preserve $\bar{\psi}\psi$, then we need $[C^\dagger \gamma^0 C]^T = -\gamma^0$, and so we see that we can choose $C : \psi \mapsto Z\psi^*$. With this choice, C does *not* preserve the chiral mass, since $[C^\dagger Y C]^T \neq -Y$.¹⁸

Now for T . To preserve the kinetic term $\bar{\psi} i \not{\partial} \psi$, we need, for $T : \psi(t, x) \mapsto T_U K \psi(-t, x)$ with K complex conjugation and T_U unitary,

$$-iT_U^\dagger \gamma^0 (-\partial_0 \gamma^0 - \partial_x \gamma^1) T_U = +i\gamma^0 \not{\partial}. \quad (414)$$

Thus T_U needs to commute with $\gamma^0 \gamma^1 = Y$, and so we can take either $T_U = J, Y$ or $T_U = \mathbf{1}$ ¹⁹. Let T_+ be the symmetry that acts as K , and T_- the one that acts as JK . One can check that the combination CPT_\pm is also a symmetry of the theory, as it must be: the relevant thing to check for the kinetic term is that

$$-[(CPT_U)^\dagger Y (CPT_U)]^T = Y. \quad (415)$$

Now for T_+ , $CPT_U = J$, while for T_- , $CPT_U = \mathbf{1}$: for both cases, the above equation holds (one can also check that it holds if we instead take $C = X$, and / or if we take $P = Z$).

Anyway, if these symmetry identifications are correct, then the operation $CT_+ : \psi \rightarrow Z\psi^* K$ is an anti-linear symmetry which functions in the same way as particle-hole symmetry \tilde{C} .

To check the sensibility of this, let's look at how various fermion bilinears transform. The Dirac mass $c_A^\dagger c_B + c_B^\dagger c_A$ is even under CT_+ , and odd under CT_- . The current is

$$j^\mu = \bar{\psi} \gamma^\mu \psi = (c_A^\dagger c_A + c_B^\dagger c_B, -i[c_A^\dagger c_B - c_B^\dagger c_A]) \implies \tilde{C} : j^0 \mapsto -j^0, j^1 \mapsto j^1. \quad (416)$$

This action on the current makes sense for PH symmetry, and is also what we expect for CT_+ symmetry in field theory (it transforms in the same way under CT_-). Similarly, using that $\gamma^5 = iXZ = Y$ is Hermitian, the axial mass maps as

$$\bar{\psi} m_5 \psi = i\bar{\psi} \gamma^5 \psi = -\psi^\dagger Z \psi = -c_A^\dagger c_A + c_B^\dagger c_B \implies \tilde{C} : m_5 \mapsto -m_5, \quad (417)$$

¹⁸If we chose $C = X$ instead, then both the Dirac mass and the chiral mass would be odd under C .

¹⁹This later option is a bit weird: if we were just thinking about the representation of the Clifford group, we might have guessed that T multiplied the spinors by γ^1 , since γ^1 reflects the γ matrices about the t axis, and is present in the action of T in Euclidean signature. This just goes to show how much of a mess the representations of the Pin groups are! When we choose a wonky representation for the γ matrices, the symmetries can act in wonky ways.

which is again what we expect from CT_+ symmetry in field theory (by contrast, the axial mass is even under CT_-). Note that here, the physical interpretation of the regular Dirac mass is an anisotropy between the strengths of t_1 and t_2 , while the chiral mass becomes a anisotropic perturbation that encourages the densities on the A and B sublattices to be different.

Finally we can check that the axial current, being the hodge dual of the vector current, maps as (in our signature, there's no i out front in the axial current)

$$j_5^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi \implies : \tilde{C} : j_5^0 \mapsto j_5^0, j_5^1 \mapsto -j_5^1, \quad (418)$$

which yes, is just what we expect from CT_+ symmetry (it is acted on by CT_- in the same way).

To break \tilde{C} symmetry then, we would want to add a Lorentz-invariant term that breaks CT symmetry. From the options surveyed so far, we see that we can only do this with the chiral mass term $\bar{\psi} m_5 \psi$. Since this term is $\psi^\dagger Z \psi$, the only way to break the symmetry in a Lorentz-invariant way is to make $h^z(k)$ in $\mathcal{H}(k)$ nonzero, thereby leading to different preferred occupation numbers on the two sublattices.

b) Diagonalizing by going to Fourier space (with sign convention $c_{r\alpha} \rightarrow \int_k e^{+ikr} c_{k\alpha}$), we see that (setting the lattice spacing $a = 1$ and setting $V = (2\pi)^d$ to make the Fourier transforms simpler)

$$H = \int_k c_k^\dagger \mathcal{H}(k) c_k, \quad c_k = (c_{kA}, c_{kB})^T, \quad (419)$$

where

$$\mathcal{H}(k) = \begin{pmatrix} 0 & t_1 + t_2 e^{-ik} \\ t_1 + t_2 e^{ik} & 0 \end{pmatrix} = t_1 X + t_2 (X \cos k + Y \sin k) \quad (420)$$

We see that under charge conjugation \tilde{C} ,

$$\tilde{C} : H \mapsto \int_k c_k^T Z \mathcal{H}^*(k) Z [c_k^\dagger]^T = - \int_k c_k^\dagger [Z \mathcal{H}^*(k) Z]^T c_k = - \int_k c_k^\dagger Z \mathcal{H}(k) Z c_k, \quad (421)$$

since $\mathcal{H}^*(k) = \mathcal{H}^T(k)$ on account of $\mathcal{H}(k)$'s Hermiticity and off-diagonal-ness. Now since $Z X Z = -X$, $Z Y Z = -Y$, we have $Z \mathcal{H}(k) Z = -\mathcal{H}(k)$, which as before cancels the fermion anticommutation sign. Anyway, the point of this is that all eigenstates at a fixed k (with non-zero energy) come in pairs, since if $|E\rangle$ is an eigenstate with energy E , then $Z|E\rangle$ is an eigenstate with energy $-E$, since $\mathcal{H}Z|E\rangle = ZZ\mathcal{H}Z|E\rangle = -Z\mathcal{H}|E\rangle = (-E)Z|E\rangle$.

c) Now as we saw, $\mathcal{H}(k) = h^i(k)\sigma^i$, for

$$h^x(k) = t + 1 + t_2 \cos k, \quad h^y(k) = t_1 + t_2 \sin k, \quad h^z(k) = 0. \quad (422)$$

Now since $\det(\lambda \mathbf{1} - h^i(k)\sigma^i) = \lambda^2 - h^i(k)h_i(k)$, we will have a band gap provided that $|h(k)| > 0$, since the spectrum of $\mathcal{H}(k)$ is $\lambda_\pm = \pm|h(k)|$. For the *SSH* model then, if we want to have a gap closing, we need $h^z(k) = 0$. Note that this is protected by \tilde{C} symmetry: if $h^z(k) \neq 0$ then we no longer have $Z \mathcal{H}(k) Z = -\mathcal{H}(k)$, which means that \tilde{C} is no longer a symmetry of H .

First consider the $t_2 = 0$ insulator. In the x - y equatorial plane of the ball in $h^i(k)$ space, $\mathcal{H}(k)$ is just a single point on the x axis at $x = t_1$. On the other hand, the $t_1 = 0$ insulator is a circle (the image of the BZ) in the plane of radius t_2 . Since we are only interested in classifying the insulating phases, we can work on the punctured plane $\mathbb{R}^2 \setminus \{0\}$, since if part of the BZ is mapped to the point $h^x = h^y = 0$, the system is gapless. The difference between these two insulators is then the homotopy class of the map of the BZ into $\mathbb{R}^2 \setminus \{0\}$: trivial in the case of the $t_2 = 0$ insulator, and nontrivial for the $t_1 = 0$ insulator. We see that the transition between these two cases is at the point $t_1 = t_2$, where the image of the $k = \pi$ point of the BZ touches the origin, where $\mathcal{H}(k) = 0$.

d) We now want to calculate the Berry phase along the S^1 of the BZ for the two insulators. First consider the case $t_2 = 0$. Then the eigenvectors of $\mathcal{H}(k)$ are just $(\pm 1, 1)^T$, and are momentum-independent: thus the Berry connection vanishes, and $\Phi_{\pm} \equiv \oint_{S^1} a_{\pm} = 0$, where

$$a_{\pm} = i \langle \chi_{\pm} | d | \chi_{\pm} \rangle \quad (423)$$

with $\chi_{\pm}(k)$ the eigenvectors of $\mathcal{H}(k)$ (we can label them by \pm because of \tilde{C} symmetry). More precisely, $\Phi_{\pm} \in 2\pi\mathbb{Z}$: translation symmetry means that $\chi_{\pm}(k) \cong e^{i\theta(k)} \chi_{\pm}$, with $\theta(2\pi) = \theta(0) + 2\pi n$, $n \in \mathbb{Z}$. We have the freedom to perform this shift since the Hamiltonian, being translation-symmetric, contains no ∂_k factors.

What about when $t_1 = 0$? Then the eigenvectors of $\mathcal{H}(k) = t_2(X \cos k + Y \sin k)$ are $\chi_{\pm}(k) = (1, \pm e^{ik})^T / \sqrt{2}$. Then $\Phi_+ = \Phi_-$ is²⁰

$$\Phi_+ = \frac{i}{2} \oint_{S^1} dk (1, \pm e^{-ik}) \cdot (0, i \pm e^{ik})^T = -\pi \sim \pi, \quad (424)$$

where the last step is because $\Phi_+ \sim \Phi_+ + 2\pi$.

We can then conjecture that for general t_1, t_2 , the Berry phase will be equal to half the winding number in the h^x, h^y plane. To prove this, it's helpful to get a more general expression for the χ_{\pm} . Now thinking in terms of $SU(2)$ generators, $\mathcal{H}(k)$ acts to perform a rotation about the vector \hat{h}^i . Thus its eigenstates will be proportional to h^i : taking them to be normalized properly, we can just take them to be $\pm \hat{h}^i$. To write this as a vector transforming in the fundamental of $SU(2)$, we need the coordinates θ, ϕ on the Bloch sphere such that $\cos \theta = h^z / |h|$ and $e^{i\phi} = (h^x + ih^y) / \sqrt{h_x^2 + h_y^2}$. Then

$$\chi_+ = (e^{i\phi} \cos(\theta/2), \sin(\theta/2))^T, \quad (425)$$

which can be checked to reduce to the correct eigenstates when only one of the h^i is non-zero. Another common choice of coordinates has a factor of $e^{i\phi/2}$ in the first slot and $e^{-i\phi/2}$ in the second slot, but this is not periodic in $\phi \mapsto \phi + 2\pi$ and as such is a bad coordinate choice for us. Our coordinates are singular when $\theta = 0$ at the north pole, but this is okay since in the

²⁰It doesn't matter whether we calculate Φ_+ or Φ_- : as we saw, Φ_{\pm} is related to the product of reflection eigenvalues at the reflection-symmetric points $k = 0, \pi$. But the reflection eigenvalue on the $-$ band is the negative of the reflection eigenvalue on the $+$ band, since reflection (X) and particle-hole (ZK) anti-commute. Since there are two reflection symmetric points, the product of the two eigenvalues differs between the two bands by a factor of $(-1)^2 = 1$.

SSH we will always be on the equatorial plane.²¹ The other eigenstate is found by sending $\phi \mapsto \phi + \pi$ and $\theta \mapsto \pi - \theta$:

$$\chi_- = (e^{i\phi} \sin(\theta/2), -\cos(\theta/2))^T, \quad (426)$$

where we have done away with an un-important factor of i .

For the application to the SSH model, the \tilde{C} symmetry tells us that $h^z = 0$, and hence $\theta = \pi/2$. Therefore our eigenstates are

$$\chi_{\pm} = \frac{1}{\sqrt{2}}(e^{i\phi}, \pm 1)^T. \quad (427)$$

The Berry phase is then

$$\Phi_{\pm} = i \oint dk a_k = \frac{i}{2} \oint dk (i\partial_k \phi). \quad (428)$$

Already, it's clear that this is (the negative of) the winding number, and independent of the choice of band \pm . To make the former more explicit,

$$\begin{aligned} \Phi_{\pm} &= -\frac{1}{2} \oint dk \partial_k (-i \ln[(h^x + ih^y)/\sqrt{h_x^2 + h_y^2}]) = -\frac{1}{2} \oint dk \partial_k \arctan(h_y/h_x) \\ &= -\frac{1}{2} \oint dk \frac{h_x \partial_k h^y - h_y \partial_k h_x}{h_x^2 + h_y^2} = -\frac{1}{2} \oint dk \left[\frac{h_x}{|h|} \partial_k \frac{h_y}{|h|} - \frac{h_y}{|h|} \partial_k \frac{h_x}{|h|} \right] = -\frac{1}{2} \oint dk (\hat{h} \times \partial_k \hat{h})_z, . \end{aligned} \quad (429)$$

which is the usual formula for (half) the winding number.

Since the winding number is quantized, and is unchanged for changes in t_1, t_2 which don't lead to a closing of the gap, we conclude that the Berry phase is also quantized. This means that the Berry curvature $F_{t_1 k} = \partial_{[t_1} a_{k]}$ vanishes everywhere except for at the degenerate point where the bands touch, which evidently must be a 2π monopole of Berry curvature (likewise for $F_{t_2 k}$), since in the presence of a 2π monopole the Berry phase along an arc is $1/2$ of the enclosed solid angle, and our arc around the equator encloses a solid angle of 2π . We can check this explicitly using (it's the same for each band \pm)

$$F_{t_j k} = \partial_{[t_j} a_{k]} = \frac{1}{2} \partial_{[t_j} \partial_{k]} \phi. \quad (430)$$

The antisymmetrized derivative vanishes everywhere that ϕ is non-singular. Since $\phi = \arctan(h_y/h_x)$, which is singular only as $h_x^2 + h_y^2 \rightarrow 0$, we see that the Berry curvature indeed is a δ function of strength 2π supported at the origin of the equatorial plane.

We can also see the quantization of the Berry phase to π from reflection symmetry, which acts by conjugating $\mathcal{H}(k)$ by X . This is equivalent to sending $\mathcal{H}(k) \mapsto \mathcal{H}(-k)$, under which the Berry connection is odd (and under which $\phi \mapsto -\phi$, since $h^y \mapsto -h^y$): thus reflection symmetry tells us that $\Phi_{\pm} = -\Phi_{\pm} \bmod 2\pi \implies \Phi_{\pm} \in \{0, \pi\}$.

²¹We'll never be able to choose a single coordinate patch that will give a well-defined Berry connection on the whole Bloch sphere, since the connection is that of a bundle $SU(2) = S^3 \rightarrow S^2$ (the base space is the parameter space of the Bloch sphere, while the total space is S^3 since this is the space the spinors [eigenstates of $H(k)$] live in) with the $U(1)$ fibered over the S^2 with nonzero Chern number.

e) As we saw earlier, the presence of \tilde{C} symmetry tells us that $h^z = 0$. In this case, as we have shown above, the Berry phase is determined by half the winding number in the equatorial plane, and hence is quantized. If we were to break \tilde{C} by adding an $h^z \neq 0$ term, we would be working on a parameter space $\mathbb{R}^3 \setminus \{0\}$ rather than $\mathbb{R}^2 \setminus \{0\}$: the former has trivial π_1 , and hence the polarization (alias Berry phase) would no longer be quantized.



Wannier states and the Berry connection

Today's entry is short and simple: we will be deriving an expression for the Wannier centers in terms of the Berry connection, expressed in terms of the Bloch wavefunctions.



Recall that the Wannier center for a band n is defined as

$$r_n = \langle nR | (r - R) | nR \rangle, \quad (431)$$

where $|nR\rangle$ are the Wannier states for the band n .

We let

$$|nR\rangle = \frac{1}{\sqrt{N}} \sum_k e^{-ikR} |\psi_{nk}\rangle \quad (432)$$

denote the Wannier states (with $N = L^d$ the number of sites), with

$$|\psi_{nk}\rangle = e^{ikr} |u_{nk}\rangle, \quad \langle r + a | u_{nk} \rangle = \langle r | u_{nk} \rangle \quad (433)$$

the Bloch states. Note how we are being sloppy and not writing dot products and not using vector notation—sorry. Letting $\langle \psi_{nk} | \psi_{n'k'} \rangle = \delta_{n,n'} \delta_{k,k'}$ this normalization gives $\langle nR | nR \rangle = N^{-1} \sum_k = 1$.

First we assume $V = (La)^d$ is large. Then the sum goes over to an integral as $\sum_k \rightarrow V/(2\pi)^d \int_k$. Then we can write

$$(r - R) \langle r | Rn \rangle = \frac{V}{(2\pi)^d} \int_k (r - R) e^{ik(r-R)} \langle r | u_{nk} \rangle = \frac{V}{(2\pi)^d} \int_k \left(-i \frac{d}{dk} e^{ik(r-R)} \right) \langle r | u_{nk} \rangle. \quad (434)$$

Now we integrate by parts:

$$(r - R) \langle r | Rn \rangle = \frac{V}{(2\pi)^d} \int_k e^{ik(r-R)} \langle r | i \frac{d}{dk} | u_{nk} \rangle, \quad (435)$$

and take the inner product by multiplying by $\langle Rn|r\rangle$ and integrating over r : with $r_n \equiv \langle Rn|(r - R)|Rn\rangle$ as above, we have

$$r_n = \frac{V^2}{(2\pi)^{2d}} \int_{k,k',r} e^{ir(k-k')} f(r; k, k'), \quad (436)$$

where

$$f(r; k, k') \equiv e^{iR(k'-k)} \langle u_{nk'}|r\rangle i \frac{d}{dk} \langle r|u_{nk}\rangle. \quad (437)$$

Note that $f(r + a; k, k') = f(r; k, k')$ by the periodicity of the $|u_{nk}\rangle$ s.

Let $x \in [0, a]^d$, and write $r = ma + x$, with $m \in \mathbb{Z}^d$. Then because of the periodicity of f ,

$$r_n = \frac{V^2}{(2\pi)^{2d}} \int_{k,k',x} \left(\sum_{m \in \mathbb{Z}^d} e^{i(k-k')ma} \right) e^{i(k-k')x} f(x; k, k'). \quad (438)$$

The sum over m tells us that $k - k' \in \frac{2\pi}{a} \mathbb{Z}^d$. But since we are only summing over k, k' in the BZ, this is equivalent to setting $k = k'$. We can then eliminate the k' integration, along with one of the factors of $V/(2\pi)^d$, which gets killed by a factor of the BZ volume. Thus using the resolution of the identity on the $|r\rangle\langle r|$ inside the f , we see that

$$r_n = \frac{V}{(2\pi)^d} \int_k \langle u_{nk}|i \frac{d}{dk}|u_{nk}\rangle. \quad (439)$$



Domain walls in the SSH model, index theorems, and CT symmetry

This is another problem from Senthil's 2019 class on correlated electronic systems. We will answer the following two questions:

- Show that in the continuum formulation of the SSH model (near the phase transition), a domain wall in the sign of the hopping anisotropy (alias Dirac mass) localizes a fermion zero mode.
- What is the charge of this zero mode? Discuss ad nauseum.



Existence of the zero mode

The continuum Hamiltonian is (setting $v = 1$ wolog; it's a non-universal parameter)

$$H = -iY\partial_x + m(x)X. \quad (440)$$

Zero modes $\psi = (\psi_L, \psi_R)^T$ satisfy

$$(-\partial_x + m)\psi_R = 0, \quad (p_x + m)\psi_L = 0. \quad (441)$$

This is solved by

$$\psi_L(x) \propto e^{-\int_0^x dx' m(x')}, \quad \psi_R(x) \propto e^{\int_0^x dx' m(x')}. \quad (442)$$

Let $m(\pm\infty) = \pm m_0$. Then we see that if $m_0 > 0$, $\psi_R(x)$ is not normalizable, while if $m_0 < 0$ then $\psi_L(x)$ is not normalizable. Therefore we can construct a normalizable (but not normalized; the normalization coefficient depends on the profile of $m(x)$) zero-mode solution by

$$\psi_0(x) = \begin{pmatrix} (1 + \text{sgn}(m_0))e^{-\int_0^x dx' m(x')} \\ (1 - \text{sgn}(m_0))e^{\int_0^x dx' m(x')} \end{pmatrix}. \quad (443)$$

Charge of the zero mode

What is the charge carried by the domain wall? The domain wall has two states, $|0\rangle, |1\rangle$, according to whether or not the zero-mode is filled. Particle-hole symmetry should exchange these, $\tilde{C} : |0\rangle \propto |1\rangle$. For $k \in \mathbb{Z}_2$, since $\tilde{C}^2 = \mathbf{1}$ ²², we have

$$\tilde{C} : |k\rangle \mapsto e^{i\phi}| - k + 1\rangle, \quad (444)$$

with the antilinearity of \tilde{C} ensuring $\tilde{C}^2 = \mathbf{1}$. Now consider the charge operator e^{iQ} , which when acting on $|k\rangle$ we define to have the eigenvalue e^{iq_k} . Since \tilde{C} anticommutes with Q , it commutes with e^{iQ} .²³ Thus we have

$$\tilde{C}e^{iQ}|k\rangle = \tilde{C}e^{iq_k}|k\rangle = e^{-iq_k+i\phi}| - k + 1\rangle. \quad (445)$$

On the other hand,

$$\tilde{C}e^{iQ}|k\rangle = e^{iQ}\tilde{C}|k\rangle = e^{iq_{-k+1}+i\phi}| - k + 1\rangle. \quad (446)$$

Now since $|0\rangle$ and $|1\rangle$ differ by a single electron which has charge 1, we know that $q_1 = q_0 + 1$. But on the other hand, the above equations tell us that $q_0 = -q_1$. These two equations can only be satisfied if $q_0 = -1/2$, $q_1 = 1/2$, and so the domain wall indeed carries fractional charge.

This analysis was focused on a single domain wall, but we can also consider a situation where the mass changes sign only for a finite region. So, consider a situation where the

²²We can see this from $\tilde{C} = ZK$ for K complex conjugation, or by using $\tilde{C} \sim P$ by CPT , which squares to $\mathbf{1}$ (rather than e.g. $(-1)^F$).

²³Recall that \tilde{C} is really CT . Now $Q = \int \star j$. j is odd under both T as a differential form, so that $\star j$ is even. $\star j$ is odd under C , so Q is odd under CT , meaning that, because of the antilinearity of CT , \tilde{C} and e^{iQ} commute.

flipped-mass domain has endpoints x_L and x_R . Each endpoint will localize a zero mode, but since the wavefunctions $\psi_{0,L/R}(x)$ extend away from the domain wall endpoints, there will be a hybridization between the two of order

$$t_{LR} = \int dx \psi_{0,L}^*(x) \psi_{0,R}(x) \sim \int dx e^{-(x-x_L)/\xi} e^{-(x_R-x)/\xi} \sim e^{-L/\xi}, \quad \xi = 1/m \quad (447)$$

where $L = x_R - x_L$ is the length of the flipped-mass region. Going back to the lattice model, the Hamiltonian will include the term

$$H \ni H_{DW} = -t_{LR}(c_{x_L A}^\dagger c_{x_R B} + h.c.), \quad (448)$$

where wolog we have taken the lattice point x_L to be on the A sublattice and x_R to be on the B sublattice. The eigenstates of H_{DW} are, just focusing on the x_L and x_R sites,

$$E = \pm t_{DW} : \frac{c_{x_L A}^\dagger \pm c_{x_R B}^\dagger}{\sqrt{2}} |0 \otimes 0\rangle, \quad E = 0 : (c_{x_L A}^\dagger c_{x_R B}^\dagger \pm 1) |0 \otimes 0\rangle, \quad (449)$$

which are all eigenstates of the particle-hole symmetry, since we haven't broken it. Anyway, we see that in the ground state, there is a single delocalized electron, smeared out between the two domain walls. Since an electron has charge 1, each of the domain walls can be thought of as having charge 1/2. Thus even after we account for hybridization, we can say that the charge of each domain wall is 1/2.

Via the index theorem: More craftily, we can argue as follows: work in Euclidean signature and couple ψ to a $U(1)$ gauge field A . We will be a bit long-winded and pedantic here because it's fun. Quite generally, regularizing by the partition function by a heavy PV field of mass M , the partition function when ψ has mass m is²⁴

$$Z[A; m] = \frac{\det(i\mathcal{D}_A + m)}{\det(i\mathcal{D}_A + M)} = \prod_\lambda \frac{i\lambda + m}{i\lambda + M}, \quad |M| \rightarrow \infty \quad (450)$$

where $\lambda \in \mathbb{R}$ by the Hermiticity of $i\mathcal{D}_A$.

Consider first $m \rightarrow \pm\infty$. Then

$$Z[A; m] = \prod_\lambda (-1)^{\Theta(m/M)-1} = \exp \left(i\pi \sum_\lambda \text{sgn}(\lambda)(\Theta(m/M) - 1) \right) = \exp(i\pi\eta(\Theta(m/M) - 1)), \quad (451)$$

²⁴A comment on the absence of *is* in the following equation, since it's important: in Euclidean signature, with all the γ matrices Hermitian, full Hermitian conjugation sends $\psi \rightarrow \psi^\dagger$ and $x^0 \mapsto -x^0$; it both conjugates the fields and reverses the time coordinate. Thus $\gamma^0 \mathcal{D}_A$ is Hermitian in this definition, while $i\gamma^0 \mathcal{D}_A$ is anti-Hermitian in this sense. Now we often write the Lagrangian as $\bar{\psi}(\mathcal{D}_A + m)\psi$. This is slightly misleading, since $\bar{\psi} \neq \psi^\dagger \gamma^0$: indeed, if this were true, $\bar{\psi}\psi$ would not be invariant under the Lorentz group in Euclidean signature. Instead, $\bar{\psi}$ should be viewed as an independent field transforming in the representation conjugate to the representation which ψ transforms under. Since $\bar{\psi}$ and ψ are the integration variables in the path integral, the path integral produces $\det(\mathcal{D}_A + m)$ for $Z[A; m]$. Now \mathcal{D}_A is anti-Hermitian as a matrix (with the usual definition of Hermitian conjugation that doesn't reverse time, since this is what's relevant for computing the determinant), which means the eigenvalues appearing in the determinant will be of the form $i\lambda + m$, with $\lambda \in \mathbb{R}$.

where η is the eta-invariant for the Dirac operator $i\mathcal{D}_A$.²⁵

First consider $\dim Y \in 2\mathbb{Z} + 1$, where Y is our Euclidean spacetime. We will use the APS index theorem in the context of realizing Y as the boundary of some higher-dimensional manifold X over which the relevant structures (spin, gauge, etc) extend (this is always possible if the spacetime dimension is small enough). The index theorem says that

$$\frac{\eta}{2} + I = \text{ind } (i\mathcal{D}_A), \quad (452)$$

where I is the appropriate degree part of $\widehat{A} \wedge \text{ch}(F_A)$ integrated over X , with \widehat{A} the Dirac genus (which we will ignore). Since the index of $i\mathcal{D}_A$ on X is integral, we can write

$$Z[A; m] = \exp(2\pi i(\Theta(m/M) - 1)I). \quad (453)$$

For example, suppose we are in three dimensions, so that X is four-dimensional. Then we obtain the effective action $S = \frac{k}{4\pi} \int F_A \wedge F_A$, where $k = 0$ if m and M have the same sign, and $k = -1$ else. This gives us a CS term on Y of level either 0 or -1 . Similarly, suppose Y is 1-dimensional. Then we get the 1-dimensional version of the CS term on Y , namely $S = k \int_Y A$, where k is the same as before. Note that $Z[A; m]$ is independent of the choice of X , and so our use of X was just a calculational trick.²⁶

Now suppose $\dim Y \in 2\mathbb{Z}$. Then we have a doubling of the spectrum for all non-zero modes of $i\mathcal{D}_A$, since $\bar{\gamma}$, the chirality operator, anticommutes with $i\mathcal{D}_A$. This means that only the zero modes of $i\mathcal{D}_A$ contribute to the partition function. Now if $\Theta(m/M) = 1$, $Z[A; m] = 1$. Suppose then that m and M have opposite signs. Then we get

$$\begin{aligned} Z[A; m] &= (-1)^{\dim \ker(i\mathcal{D}_A)} = (-1)^{\dim \ker(i\mathcal{D}_A[1+\bar{\gamma}]/2) + \dim \ker(i\mathcal{D}_A[1-\bar{\gamma}]/2)} \\ &= (-1)^{\dim \ker(i\mathcal{D}_A[1+\bar{\gamma}]/2) - \dim \ker(i\mathcal{D}_A[1-\bar{\gamma}]/2)} = e^{i\pi \text{ind } (i\mathcal{D}_A)} \\ &= \exp\left(\frac{i\pi}{(\dim Y/2)!} \int (F_A/2\pi)^{\wedge(\dim Y/2)}\right), \end{aligned} \quad (454)$$

where in the last line we used the index theorem. This is a theta term at $\theta = \pi$.

Finally, consider $m = 0$. In this case, if $i\mathcal{D}_A$ has zero modes, we get $Z[A; 0] = 0$. So suppose $\det(i\mathcal{D}_A) \neq 0$. Then we get

$$Z[A; 0] = \prod_{\lambda} \frac{\lambda}{\lambda - iM} = \prod_{\lambda} \frac{1}{|1 - iM/\lambda|} \frac{1 + iM/\lambda}{|1 - iM/\lambda|} = |Z[A; 0]| \exp\left(\sum_{\lambda} \ln(z/|z|)\right), \quad (455)$$

where $z = 1 + iM/\lambda$. Taking $|M| \rightarrow \infty$, we get

$$Z[A; 0] = |Z[A; 0]| \exp\left(isgn(M) \sum_{\lambda} \text{sgn}(\lambda) \tan^{-1}(|M|/|\lambda|)\right) = |Z[A; 0]| \exp\left(i\frac{\pi}{2} \text{sgn}(M)\eta\right). \quad (456)$$

²⁵We really should be doing the regularization more carefully, e.g. with a $e^{-\lambda^2\epsilon}$ factor, but as long as we don't look too closely this won't affect the action we derive. If we were to look too closely, we would notice that our final expression for η is related to a CS term integrated over spacetime, which is \mathbb{R} -valued. Clearly this is incompatible with our naive definition of η ; regulating the sum properly fixes this.

²⁶We have been ignoring the gravitational contribution, and so we have to assume Y is spin. If we want to have non-spin Y then we can take A to be spinc, but then we need to put the gravitational contribution back in.

So, the phase of the partition function is determined just as for the massive case with $\text{sgn}(m/M) = -1$, except there is a (crucial!) factor of $1/2$ in front of the η invariant. Now if we think of Y as being the boundary of some X , we can use the APS index theorem to express $Z[A; 0]$ as a local functional of A on X , together with a sign determined by the index $i\hat{\mathcal{D}}_A$ on X . But because of the $1/2$, this will not be expressible as a local functional of A on Y , and our answer for $Z[A; 0]$ will depend on the choice of X . If our fields only live on Y , then we are stuck with writing η , which will generally be a *singular* function of A .

Anyway, now we can apply this to the problem at hand. We first integrate out ψ away from the domain wall, where it is massive. Wolog let $m(-\infty) = -m$, $m(\infty) = +m$, $m > 0$. For $m \rightarrow \infty$, the effective action $S[A]$ contains the $\theta = \pi$ term

$$S[A] \ni \frac{1}{2} \int_{H^+} F, \quad (457)$$

where H^+ is the upper half-plane where $m > 0$, and we have used a regulator with large negative mass. Since θ terms in two dimensions act like electric fields with strength $\theta/2\pi$ ²⁷, we see that the bulk fermions produce an effective contribution of $1/2$ to the electric flux in the region to the right of the domain wall.

Now we do the integral over the zero mode localized on the domain wall. This gives us the η invariant, whose contribution to $S[A]$ we can write as

$$S[A] \ni -\frac{\pi}{2}\eta = -\frac{\pi}{2} \left(\frac{1}{\pi} \int_{DW} A \right)_{[-1,1]}, \quad (458)$$

where DW is the domain wall (we will usually take it to be an S^1 in the Euclidean time direction). Here the subscript $[-1, 1]$ means that we always take the thing in parenthesis to be in the range $[-1, 1]$ by adding / subtracting 2 as need be, which ensures that η is invariant under large gauge transformations, as it must be. Note that if we had π instead of $\pi/2$ out front then we could take the integral of A to be defined mod 2 anyway, and could drop the parenthesis, obtaining the well-defined term $\int_{DW} A$. As it stands, we cannot do this. Also note that $\eta(A)$ is singular at $\int_{DW} A = \pi$, but that $Z[A; 0]$ vanishes there anyway because of the zero mode: the DW theory will have $\det(\partial_t - iA) = 0$ when $\psi(t + \beta) = e^{i\int_{DW} A} \psi(t)$, where β is the radius of the compactified Euclidean time direction. With the usual NS spin structure we have $\psi(t + \beta) = -\psi(t)$, and so $Z[A; 0]$ indeed vanishes when $\int_{DW} A = \pi$.²⁸ Anyway, since twice this term gives $\int_{DW} A$, we conclude that the DW effectively describes a particle of charge $-1/2$ living on the domain wall, since such a particle would have a response theory of $-\frac{1}{2} \int_{DW} A$, were that to be well-defined.

²⁷This is because the canonical momentum of A is $\pi_A = \frac{\delta S}{\delta dA} = \star F/e^2 + \theta/2\pi$, so that the electric field at a given point in space is $\star F/e^2 + \theta/2\pi$.

²⁸With the R spin structure, I believe the η invariant is modified to

$$\eta_R(A) = \left(\frac{1}{\pi} \int_{DW} A \right)_{[0,2]}, \quad (459)$$

which now has a singularity at $\int_{DW} A = 0$, in keeping with $Z[0; 0] = 0$ for the R spin structure.

Putting these two pieces together gives us the effective action

$$S[A] = -\frac{\pi}{2} \left(\frac{1}{\pi} \int_{DW} A \right)_{[-1,1]} + \frac{1}{2} \int_{H^+} F. \quad (460)$$

If we integrate the second term and assume A dies off at ∞ , we can combine the two terms and get $S[A] = -\int_{DW} A$, which is well-defined and gauge-invariant. The physical interpretation of this scenario is that we have a mode of fractional charge sitting on the domain wall, and an induced strength $e/2$ electric field to the right of the domain wall, coming from the difference in polarization between the two domains. Since we ultimately have an integer number of electrons in the system the total electric field must be integral, and it is: the domain wall contribution and the polarization contribution add up to give an integral result.²⁹

Finally let's comment on the role of \tilde{C} symmetry. Now as we saw earlier,

$$\tilde{C} = CT : j^0 \mapsto -j^0, \quad j^1 \mapsto j^1, \quad (461)$$

so that \tilde{C} is an antilinear symmetry which leaves the current as a differential form invariant; $\tilde{C} : j \mapsto j$. The transformation rule for the gauge field A is then found by requiring that \tilde{C} leave $j_\mu A^\mu$ invariant: therefore $\tilde{C} : A \mapsto A$ as a differential form, i.e. $A_0 \mapsto -A_0$, $A_1 \mapsto A_1$. This means that $\tilde{C} : F \mapsto F$, and so the theta term $\exp(i\theta \int F/2\pi)$ has $\theta \mapsto -\theta$ under \tilde{C} . But the domain wall part is also not invariant, and is also conjugated under \tilde{C} . Thus our net action $\sim \int_{DW} A$ changes sign. This isn't a problem though, since we are always free to add the well-defined counterterm $k \int_{DW} A$ to $S[A]$ for any $k \in \mathbb{Z}$. In particular, choosing $k = +1$ gives us something that is \tilde{C} invariant. The statement is just that our theory must be invariant under \tilde{C} for some choice, but not all choices, of counterterms in the background field. Note that if we were missing either the bulk or the DW contribution, such a counterterm would not be able to render $S[A]$ invariant under \tilde{C} .



Coherent states, Berry curvature, and the semiclassical EOM

Today we are going to review the definition of the Berry connection and its effects on the semiclassical equations of motion from a rather symplectic point of view. [5] is a good reference.



²⁹In one space dimension the potential from a point charge diverges linearly as $V(r) \sim r$, so the electric field is constant.

We will label coordinates on our phase space by ζ^i , e.g. $\zeta = (q_1, \dots, q_n, p_1, \dots, p_n)$. Let $|\psi(\zeta)\rangle$ denote the complete set of single-particle states that we use to form the resolutions of the identity that we insert into the trotterization of e^{-iHT} . The partition function is then (keeping ∂ cons implicit)

$$Z = \int \prod_i \mathcal{D}\zeta^i \exp \left(\int dt \left[\langle \psi(\zeta) | \frac{d}{dt} |\psi(\zeta)\rangle - iH(\zeta) \right] \right), \quad (462)$$

with $H(\zeta) = \langle \psi(\zeta) | H | \psi(\zeta) \rangle$. We write the time derivative as

$$\langle \psi(\zeta) | \frac{d}{dt} |\psi(\zeta)\rangle = \dot{\zeta}^j \langle \psi(\zeta) | \partial_{\zeta^j} |\psi(\zeta)\rangle + \langle \psi(\zeta) | \partial_t |\psi(\zeta)\rangle = i(\dot{\zeta}^j \mathcal{A}_j + \mathcal{A}_0), \quad (463)$$

where $\mathcal{A}_j = -i\langle \psi(\zeta) | \partial_{\zeta^j} |\psi(\zeta)\rangle$ is the Berry connection, and the temporal part is $\mathcal{A}_0 = -i\langle \psi(\zeta) | \partial_t |\psi(\zeta)\rangle$. The factor of $-i$ is to make it real, since $\partial_{\zeta^j}^\dagger = -\partial_{\zeta^j}, \partial_t^\dagger = -\partial_t$. We then can write the action concisely as

$$S = \int (\mathcal{A}_j d\zeta^j - [H(\zeta) - \mathcal{A}_0] dt). \quad (464)$$

We now take a variation with respect to the phase space variables ζ^j . This gives us a term $-\delta\zeta^j(d\mathcal{A}_j/dt)$, which is $-\delta\zeta^j\dot{\zeta}^k\partial_{\zeta^k}\mathcal{A}_j - \delta\zeta^j\partial_t\mathcal{A}_j$. Thus

$$\delta S = \int (\mathcal{F}_{ij} d\zeta^j - [\partial_{\zeta^i} H(\zeta) + \partial_t \mathcal{A}_i - \partial_{\zeta^i} \mathcal{A}_0] dt) \delta\zeta^i, \quad (465)$$

where we have defined the Berry curvature $\mathcal{F} = d\mathcal{A}$, with the exterior derivative being taken in the ζ^i coordinates. This shows us why the Berry curvature is intimately related to the symplectic structure of phase space: the Berry curvature is the symplectic form, since the symplectic form ω_{ij} appears in δS via³⁰

$$\delta S \ni \int \delta\zeta^i \omega_{ij} d\zeta^j. \quad (467)$$

To elaborate on the identification between \mathcal{A} and the symplectic structure, we can write the action as

$$S = \int d\zeta^i \theta_i - \int dt H, \quad (468)$$

which describes motion of a trajectory in phase space (θ is the symplectic potential). The condition that H generate time evolution is $\partial_{\zeta^i} H = \omega_{ij} V_H^j$, where $V_H = \dot{\zeta}^j \partial_{\zeta^j}$ is the Hamiltonian vector field (therefore $\int d\zeta^i \theta_i = \int dt \theta(V_H)$). This equation is obtained as the equation

³⁰Another way to see this is to write the boundary term obtained in the course of finding the eom as $\mathcal{A}_j \delta\zeta^j|_{t_i}^{t_f}$. Taking a second variation gives

$$\delta(\mathcal{A}_j \delta\zeta^j) = \delta\mathcal{A}_j \wedge \delta\zeta^j = \partial_{\zeta^i} \mathcal{A}_j \delta\zeta^i \wedge \delta\zeta^j = \frac{1}{2} \mathcal{F}_{ij} \delta\zeta^i \wedge \delta\zeta^j, \quad (466)$$

where the wedge product is in variational space. Now we know that the above is $\omega = \frac{1}{2} \omega_{ij} \delta\zeta^i \wedge \delta\zeta^j$ where ω is the symplectic form, and so $\omega = d\mathcal{A}$: the symplectic form and Berry curvature are one and the same.

of motion of the above action provided that $d\theta = \omega$, where the d is in $d\zeta^i$ space. Comparing this to the action derived above via the usual Trotterization procedure, we see that $\mathcal{A}_i = \theta_i$: thus the Berry connection is the symplectic potential, while the Berry curvature is the symplectic form.

Anyway, the equations of motion are

$$\frac{d\zeta^j}{dt} = [\mathcal{F}^{-1}]^{ji}(\partial_{\zeta^i} H(\zeta) + \partial_t \mathcal{A}_i - \partial_{\zeta^i} \mathcal{A}_0). \quad (469)$$

Now in classical mechanics we know that the time evolution of ζ^j is generated by taking the Poisson bracket of ζ^j with $H(\zeta)$. Since $\{\zeta^j, H(\zeta)\} = \{\zeta^j, \zeta^i\}\partial_{\zeta^i} H(\zeta)$, we can classically identify $[\mathcal{F}^{-1}]^{ij} = \{\zeta^i, \zeta^j\}$. In the quantum theory then, since the Poisson bracket goes to $-i$ times the commutator, the commutation relations in the time-independent case when $\partial_t \mathcal{A}_i = \partial_{\zeta^i} \mathcal{A}_0 = 0$ are determined from the Berry curvature via

$$[\zeta^i, \zeta^j] = i[\mathcal{F}^{-1}]^{ij}. \quad (470)$$

When canonical coordinates $\zeta = (q_1, \dots, q_n, p_1, \dots, p_n)$ are chosen, \mathcal{F} has the canonical structure of a symplectic form, namely

$$\mathcal{F}_{ij} = \begin{pmatrix} 0 & -\mathbf{1}_{n \times n} \\ \mathbf{1}_{n \times n} & 0 \end{pmatrix}_{ij}. \quad (471)$$

In geometric terms, this means that the $U(1)$ bundles over the q and p subspaces are trivial, while the bundle is twisted in the pq plane. The inverse is $\mathcal{F}^{-1} = -\mathcal{F}$, and we of course get $[q^i, p^j] = i\delta^{ij}$.

In two dimensions, nonzero Chern numbers for the real space / momentum space components of the Berry curvature have consequences for the forms of the wavefunctions one deals with. If the coordinate space Chern number is nonzero then the single-particle wavefunctions $|\psi(\zeta)\rangle$ cannot be chosen to be localized in momentum space: this is because a function localized in momentum space is smooth in real space, which $|\psi(\zeta)\rangle$ cannot be due to the fact that the $U(1)$ bundle in question admits no global section. This fact is familiar from the usual solution of the Landau level wavefunctions, which in the usual gauge are completely delocalized along one direction in momentum space. Likewise, a nonzero momentum-space Chern number (usually just called “the” Chern number) means that one will not be able to construct localized Wannier functions in \mathbb{R} space.

Now we will look at a few examples. The simplest case is to suppose our single-particle states are the wavepackets³¹

$$|\psi(q, p)\rangle = \frac{1}{(\sigma^2 \pi)^{1/4}} \int dx e^{ipx} e^{-(x-q)^2/2\sigma^2} |x\rangle. \quad (473)$$

³¹These of course are not orthogonal; setting $\sigma^2 = 1/2$ for simplicity gives an overlap like

$$\langle \psi(p, q) | \psi(p', q') \rangle \propto \exp(-(p-p')^2 - (q-q')^2 + i(q+q')(p'-p)). \quad (472)$$

Then $\mathcal{A}_q = 0$, while $\mathcal{A}_p = q$, so that $\mathcal{F}_{qp} = 1$. This of course gives the equations of motion $\dot{q} = \partial_q H$ and $\dot{p} = -\partial_q H$. The fact that $\mathcal{F}_{qp} = 1$ is just telling us that the symplectic volume of the pq plane is non-zero: translations in the q and p directions (implemented by p and q , respectively) do not commute, and the failure of their commutativity is measured by \mathcal{F}_{qp} .

Consider now the simple example of a particle coupled to a background electromagnetic field. The action is

$$S = \int ((p_i + A_i)dq^i - (H(p, q) - A_0)dt). \quad (474)$$

Working in the basis $(q_1, \dots, q_d, p_1, \dots, p_d)$, we see that the Berry curvature is given by (here $F = dA$ is the coordinate part of the Berry curvature)

$$\mathcal{F}_{ij} = \begin{pmatrix} F & -1 \\ 1 & 0 \end{pmatrix}_{ij}. \quad (475)$$

Thus the coordinate part of the Berry curvature is equal to the magnetic field. This of course tells us that the curvature in the coordinate part of phase space is nontrivial: translations in different directions in coordinate space don't commute. This in turn means that the momenta p_i will not commute with one another, which we can see formally by computing $\mathcal{F}^{-1} = (X \otimes \mathbf{1}_{d \times d})\mathcal{F}(X \otimes \mathbf{1}_{d \times d})$, which tells us that $[p_i, p_j] = iF_{ij}$. Of course, the symplectic form can be recast in as the canonical $J \otimes \mathbf{1}_{d \times d}$ form by changing basis: a glance at the action tells us that the right way to do this is to define the canonical momenta $\pi_j = p_i + A_i$: in the basis $(q_1, \dots, q_d, \pi_1, \dots, \pi_d)$, we have $\mathcal{F}_{ij} = J \otimes \mathbf{1}_{d \times d}$. In the non-canonical basis, the equations of motion are the familiar

$$\dot{p}^i = -F^{ij}\partial_{p^j}H - \partial_{q^i}H - \partial_{q^i}A_0 + \partial_t A_i, \quad \dot{q}^i = \partial_{p^i}H, \quad (476)$$

which, assuming $\partial_{p^i}H(\zeta) = \dot{q}^i$ with $H(\zeta) = p^2/2m + V(q)$, we can re-write as

$$\dot{p}^i = \epsilon^{ijk}\dot{q}^jB^k + E^i - \partial_{q^i}V(q), \quad \dot{q}^i = p^i/m. \quad (477)$$

Now consider an analogous situation in which the magnetic field is zero, but the momentum part of the Berry curvature is non-vanishing, so that the action contains the term $\mathcal{A}_i dp^i$ (our notation is such that the total Berry connection / curvature are written with mathcal, with the coordinate part written in roman and the momentum part written in script; retrospectively this seems like an appalling choice). Proceeding in the same way, we get the equations of motion

$$\dot{q}^i = \mathcal{F}^{ij}\partial_{q^i}H(\zeta) + \partial_{p^i}H(\zeta) + \partial_t \mathcal{A}_{p^i} - \partial_{p^i}\mathcal{A}_0, \quad \dot{p}^i = -\partial_{q^i}H(\zeta). \quad (478)$$

Taking $H(\zeta) = p^2/2m + V(q)$ again, we can write this as

$$\dot{q}^i = \epsilon^{ijk}\mathcal{B}^j\partial_{q^k}V(q) + \mathcal{E}^i + p^i/m, \quad \dot{p}^i = -\partial_{q^i}V(q), \quad (479)$$

where $\mathcal{E}_i = \partial_{p^i}\mathcal{A}_0 - \partial_t \mathcal{A}_i$ and \mathcal{B} are momentum-space electric and magnetic fields, which are responsible for the anomalous velocity.

Finally, consider the general case where both F and \mathcal{F} are nonzero. To analyze this we will work semiclassically, to first order in \hbar . Restoring \hbar momentarily, the action is

$$S = \int ((p_i + A_i)dx^i + \hbar\mathcal{A}_i dp^i - [H(\zeta) - \mathcal{A}_0]dt). \quad (480)$$

The \hbar in front of $\mathcal{A}_i dp^i$ is due to the fact that \mathcal{A}_i has dimensions of inverse momentum, so that $\mathcal{A}_i dp^i$ is dimensionless, and needs to be multiplied by \hbar in order to have dimensions of action (on the other hand, A_i has dimensions of momentum so $A_i dx^i$ already has dimensions of action). This means that the semiclassical limit can be taken by working to first order in \mathcal{A} and \mathcal{F} .

Now the full Berry curvature is (going back to $\hbar = 1$)

$$\mathcal{F}_{ij} = \begin{pmatrix} F & -1 \\ 1 & \mathcal{F} \end{pmatrix}_{ij} \implies [\mathcal{F}^{-1}]^{ij} = \frac{1}{1 + F\mathcal{F}} \begin{pmatrix} \mathcal{F} & 1 \\ -1 & F \end{pmatrix}^{ij}, \quad (481)$$

where e.g. $F\mathcal{F}$ denotes matrix multiplication. In the semiclassical approximation then,

$$\mathcal{F}^{-1} = \begin{pmatrix} \mathcal{F} & 1 - F\mathcal{F} \\ -1 + F\mathcal{F} & F - F\mathcal{F}F \end{pmatrix}. \quad (482)$$

Now we can get the equations of motion. We will consider the case when $\partial_{p^i}\mathcal{A}_0 = \partial_t\mathcal{A}_i = 0$ for simplicity. We then have

$$\begin{aligned} \dot{q}^j &= \mathcal{F}^{ji}(\partial_{q^i}H - E_i) + \partial_{p^j}H - F^{jk}\mathcal{F}^{ki}\partial_{p^i}H, \\ \dot{p}^j &= (F^{ji} - F^{jk}\mathcal{F}^{kl}F^{li})\partial_{p^i}H + (-\delta^{ji} + F^{jk}\mathcal{F}^{ki})(\partial_{q^i}H - E_i). \end{aligned} \quad (483)$$

We can rewrite these in a simpler form as

$$\begin{aligned} \dot{q}^j &= \partial_{p^j}H + \epsilon^{jik}(\partial_{q^i}H - E_i)\mathcal{B}^k - \epsilon^{jkl}\epsilon^{kim}\partial_{p^i}HB^l\mathcal{B}^m \\ \dot{p}^j &= -\partial_{q^j}H + E^j + \epsilon^{jik}\dot{q}^iB^k. \end{aligned} \quad (484)$$

For example, suppose $\partial_{q^i}H = 0$ and $B = 0, \mathcal{B} \neq 0$, with q^i denoting the position of an electron in some Bloch band (this is the context of a Chern insulator). Then, supposing that the \mathcal{B} and \mathcal{E} fields live in three dimensions, we have

$$\dot{\mathbf{q}} = \vec{\nabla}H(p) + \mathcal{B} \times \mathbf{E}, \quad \dot{\mathbf{p}} = \mathbf{E}. \quad (485)$$

The consequences of the $\mathcal{B} \times \mathbf{E}$ term are seen by computing the current density (restoring the electric charge e momentarily):

$$\mathbf{j} = -e \int_{BZ} \frac{d^dp}{(2\pi)^d} f(H(p))(\vec{\nabla}H(p) + \mathcal{B} \times e\mathbf{E}), \quad (486)$$

where $f(H(p))$ is the Fermi function. Suppose we are in a filled band, so that $f(H(p)) = 1$ for all $p \in BZ$. The first term then vanishes upon integration, which tells us that in the absense of momentum-space Berry curvature, filled bands don't contribute to the current (duh).

Now consider the QAH case where the electrons live in a $d = 2$ plane, with the \mathcal{B} and \mathbf{E} fields living in three dimensions. Further suppose that \mathbf{E} is uniform and in the plane, and \mathcal{B} is orthogonal to the plane.³² Then as expected, we get

$$\mathbf{j}^i = \epsilon^{ij} E^j \frac{e^2}{2\pi} \int_{BZ} \frac{\mathcal{B}}{2\pi} = \epsilon^{ij} E^j \frac{e^2}{2\pi} C, \quad (487)$$

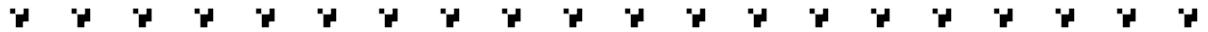
where C is the Chern number.



Haldane's model of a Chern insulator

Today's diary entry is a homework problem from Senthil's cmt class. The problem has several parts:

- (a) Consider a TB model on the honeycomb lattice. Find the dispersion and show where the two Dirac points are located.
- (b) Now add second-nearest-neighbor hopping in a T -breaking manner, with opposite hopping phases for each triangular sublattice. Also add a chemical potential that alternates signs between the two sublattices. What happens?
- (c) Find the Dirac mass near the K, K' points as a function of the various parameters in the Hamiltonian.
- (d) What are the Chern numbers of the different bands in the different regions of parameter space?



- a) The TB Hamiltonian is (setting the hopping strength to 1 wolog)

$$H_0 = - \sum_{\langle rr' \rangle} (c_r^\dagger c_{r'} + h.c.). \quad (488)$$

To analyze this we need to fix conventions for the honeycomb lattice. The unit cell consists of two neighboring cites, and the lattice of such unit cells is a triangular lattice. We will

³²if the problem is really two-dimensional then \mathcal{B} is a scalar and we should just be writing $\mathcal{B} \times \mathbf{E} \rightarrow \mathcal{B}(-E_y, E_x)$. But we haven't done this since eh, cross products are nice-looking.

set the lattice spacing of the honeycomb lattice to be 1, meaning that nearest neighbors on a single sublattice are a distance $a = 2 \cos(\pi/6) = \sqrt{3}$ apart. The basis vectors for a given sublattice are then

$$v_1 = (\sqrt{3}, 0), \quad v_2 = (\sqrt{3}/2, 3/2). \quad (489)$$

The reciprocal lattice vectors are then, from $v_i \cdot u_j = 2\pi\delta_{ij}$,

$$u_1 = 2\pi(1/\sqrt{3}, -1/3), \quad u_2 = 2\pi(0, 2/3). \quad (490)$$

The BZ is determined by taking these vectors, plus the vector orthogonal to $u_2 - u_1$, dividing their lengths by 2, and then taking the interior of the shape formed by the lines drawn perpendicular to the endpoints of each of these vectors, as well as their negatives. This gives us a BZ which is hexagonal, with a flat side parallel to the k_x axis in momentum space; this flat side hits the k_y axis at $(0, 2\pi/3)$. The BZ hits the $k_x > 0$ part of the k_x axis at³³

$$K = (4\pi/(3\sqrt{3}), 0), \quad (491)$$

which is a corner of the BZ hexagon. There is only one other corner of the hexagon distinct from the K point, located at³⁴

$$K' = \frac{2\pi}{3}(1/\sqrt{3}, 1). \quad (492)$$

Note that because of the identification of the BZ corners, $K \cong -K'$.

Let $c_r = (c_{rA}, c_{rB})^T$, where r labels a lattice site on the A sublattice. Then by looking at the geometry we see that the hopping Hamiltonian is

$$H_0 = - \sum_k c_k^\dagger \mathcal{H}_0(k) c_k \quad (493)$$

where

$$\begin{aligned} H_0(k) &= \begin{pmatrix} 0 & e^{-ik_y} + e^{-i(\cos(\pi/6)k_x - \sin(\pi/6)k_y)} + e^{i(\cos(\pi/6)k_x + \sin(\pi/6)k_y)} \\ h.c. & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{-ik_y} + 2e^{ik_y/2} \cos(\sqrt{3}k_x/2) \\ e^{ik_y} + 2e^{-ik_y/2} \cos(\sqrt{3}k_x/2) & 0 \end{pmatrix} \\ &= h_x(k)X + h_y(k)Y, \end{aligned} \quad (494)$$

where

$$h_x(k) = \cos k_y + 2 \cos(k_y/2) \cos(\sqrt{3}k_x/2), \quad h_y(k) = \sin k_y - 2 \sin(k_y/2) \cos(\sqrt{3}k_x/2). \quad (495)$$

Note how this is symmetric under $k_x \rightarrow -k_x$, which is a consequence of C_2 symmetry.

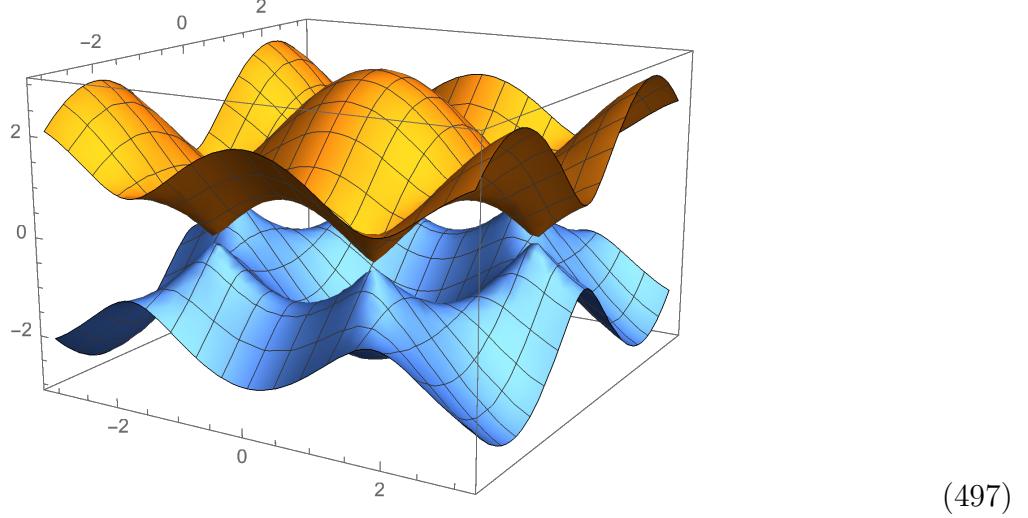
This gives us the spectrum

$$E_0(k) = \pm \sqrt{1 + 4 \cos(\sqrt{3}k_x/2)(\cos(\sqrt{3}k_x/2) + \cos(3k_y/2))}. \quad (496)$$

³³We find this by taking the line perpendicular to the midpoint of the u_1 vector, namely $\{(\pi/\sqrt{3} + t/\sqrt{3}, -\pi/3 + t)\}$, and setting its y -coordinate to 0.

³⁴To find this, take the intersection of the lines $(t, -2\pi/3)$ and $(s + 4\pi/(3\sqrt{3}), \sqrt{3}s)$.

One can check that $E_0(K) = E_0(K') = 0$, so that the gap closes on the corners of the BZ. Graphically, the dispersion is



We can linearize the spectrum about the K, K' points: to first order this gives

$$\begin{aligned} H(K + q) &\approx \frac{3}{2}(q_y Y - q_x X), \\ H(K' + q) &\approx -\frac{3}{2}(-q_y Y - q_x X), \end{aligned} \quad (498)$$

which are free Dirac Hamiltonians with the same speed of light (here we have chosen a representative momentum for the K' point so that $K' = -K$). The minus sign between the two of them follows from the fact that T (which is a symmetry of H_0) reverses momenta and the sign of Y , and exchanges K with K' .

Let's also take a look at the symmetries of the problem. The spatial part of the symmetry group includes π rotations about the hexagon centers. For a given choice of rotation center, this symmetry sends $c_{A/B,k} \rightarrow c_{B/A,-k}$ and exchange the A and B sublattices. This is a symmetry since

$$X^T H(k) X = H(-k). \quad (499)$$

We also have T symmetry, which just acts as complex conjugation, $T = \mathcal{K}$, in addition to reversing momentum. This is a symmetry since $H^*(-k) = H(k)$. We also have charge conjugation: if $c \mapsto \mathcal{C}c^\dagger$, then we need

$$[\mathcal{C}^\dagger H(k) \mathcal{C}]^T = -H(k). \quad (500)$$

Thus we can choose $\mathcal{C} = Y$, so that $\mathcal{C}^2 = \mathbf{1}$, with \mathcal{C} acting to exchange the A and B sublattices. Finally we have particle hole symmetry, which we write as $\mathcal{P}\mathcal{H} = ZT = Z\mathcal{K}$. This tells us that every state $|\psi\rangle$ of energy E comes with a partner of energy $-E$, since $\mathcal{H}\mathcal{P}\mathcal{H}|\psi\rangle = \mathcal{P}\mathcal{H}ZH^*(-k)Z|\psi\rangle = -H(k)|\psi\rangle = -E|\psi\rangle$.

b) Now we add

$$H_1 = \Delta \sum_k c_k^\dagger Z c_k, \quad H_2 = t_2 \sum_{\langle\langle rr'' \rangle\rangle} e^{i\phi} c_r^\dagger c_{r''} + h.c., \quad (501)$$

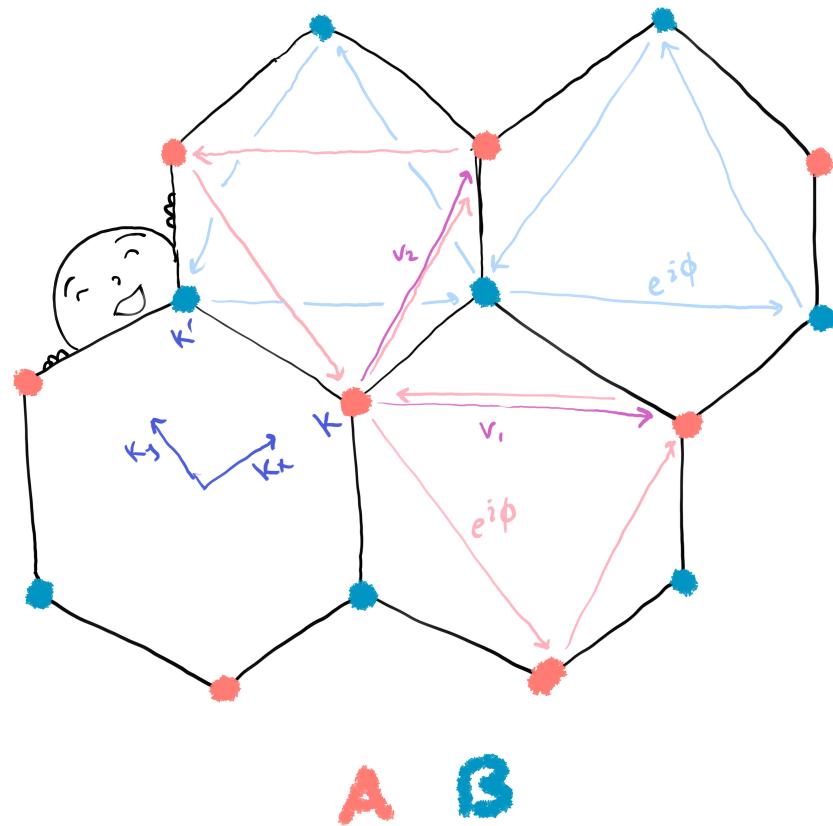


Figure 3: Our conventions for the graphene lattice. The light arrows show the conventions for nearest neighbor hopping, and the BZ (also a hexagon) is shown in the bottom left.

where $e^{i\phi}$ is a hopping phase. Note that H_1 breaks the C_2 rotation symmetry since it treats the A and B sublattices differently. It also breaks \mathcal{PH} , since $[Z\mathcal{K}, H_1(k)] = 0$. However, since it's real, it preserves time reversal.

Now for H_2 . This term preserves C_2 , breaks \mathcal{C} and \mathcal{PH} unless $\phi = \pm\pi/2$ (because of the chemical potential term proportional to $\mathbf{1} \cos \phi$), and breaks T unless $\phi = 0, \pi$. The signs for the hopping phase are shown in figure 3: for each hop along an arrow, we get a phase $e^{i\phi}$; for each hop against an arrow we get $e^{-i\phi}$. Note how if the A sublattice has a hopping along a vector v of phase $e^{i\phi}$, the B sublattice has a hopping along the same direction, but with phase $e^{-i\phi}$. Since the phase of the hopping around a closed loop is $e^{i\oint_A A}$, the total magnetic flux vanishes, since over a unit cell we have $e^{i\oint_{\Delta_A} A + i\oint_{\Delta_B} A} = 1$, where $\Delta_{A/B}$ are the triangles enclosed by similar hopping trajectories on the two sublattices. Thus a generic value of ϕ gives us a way of breaking T without introducing a net magnetic field.

Writing out H_2 ,

$$H_2 = 2t_2 \sum_k \left(c_{kA}^\dagger [\cos(\phi + k \cdot v_1) + \cos(\phi - k \cdot v_2) + \cos(\phi + k \cdot (v_2 - v_1))] c_{kA} + c_{kB}^\dagger [\phi \leftrightarrow -\phi] c_{kB} \right). \quad (502)$$

We can re-write this as

$$\begin{aligned} H_2 = 2t_2 \sum_k c_k^\dagger & \left(\mathbf{1} \cos(\phi) [\cos(k \cdot v_1) + \cos(k \cdot v_2) + \cos(k \cdot (v_2 - v_1))] \right. \\ & \left. + Z \sin(\phi) [\sin(k \cdot v_1) - \sin(k \cdot v_2) + \sin(k \cdot (v_2 - v_1))] \right) c_k. \end{aligned} \quad (503)$$

When $|\Delta| \gg |t_2|$ (only the relative sign of Δ and t_2 will be important, so in what follows we will fix $t_2 > 0$ wolog), we see that we get a gap at both K, K' points that goes as Δ . Since the $t_2 \rightarrow 0$ Hamiltonian preserves T we can't have edge modes, and since the presence / absence of edge modes is something that's quantized, we know that the large Δ phase is gapped in a trivial way.

c) Now we want to know what happens near the K, K' points. Linearizing H_2 about these points gives (to first order, H_2 is independent of q in $k = K + q$ and $k = K' + q$)

$$\begin{aligned} H_2(K) & \approx -3t_2 \cos \phi \mathbf{1} - 3\sqrt{3}t_2 \sin \phi Z, \\ H_2(K') & \approx -3t_2 \cos \phi \mathbf{1} + 3\sqrt{3}t_2 \sin \phi Z. \end{aligned} \quad (504)$$

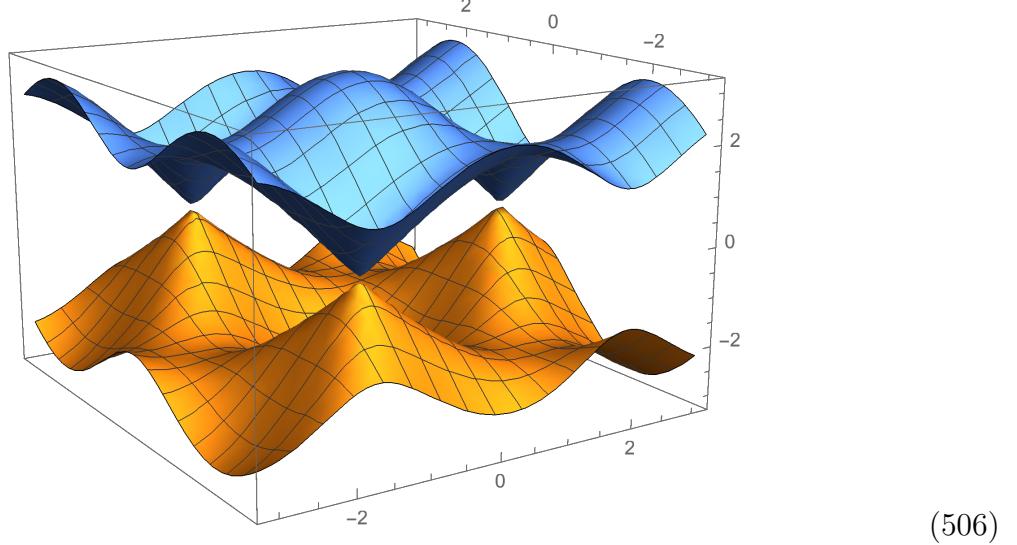
Notice that the t_2 terms are not symmetric between the K and K' points if $\sin \phi \neq 0$; this is because unless $\phi = 0, \pi$ they break T , which relates K and K' .

Recapping, the full Hamiltonian around the two points of interest is

$$\begin{aligned} H(K + q) & \approx -3t_2 \cos \phi \mathbf{1} + (\Delta - 3\sqrt{3}t_2 \sin \phi) Z + \frac{3}{2}(q_y Y - q_x X) \\ H(K' + q) & \approx -3t_2 \cos \phi \mathbf{1} + (\Delta + 3\sqrt{3}t_2 \sin \phi) Z - \frac{3}{2}(q_y Y - q_x X) \end{aligned} \quad (505)$$

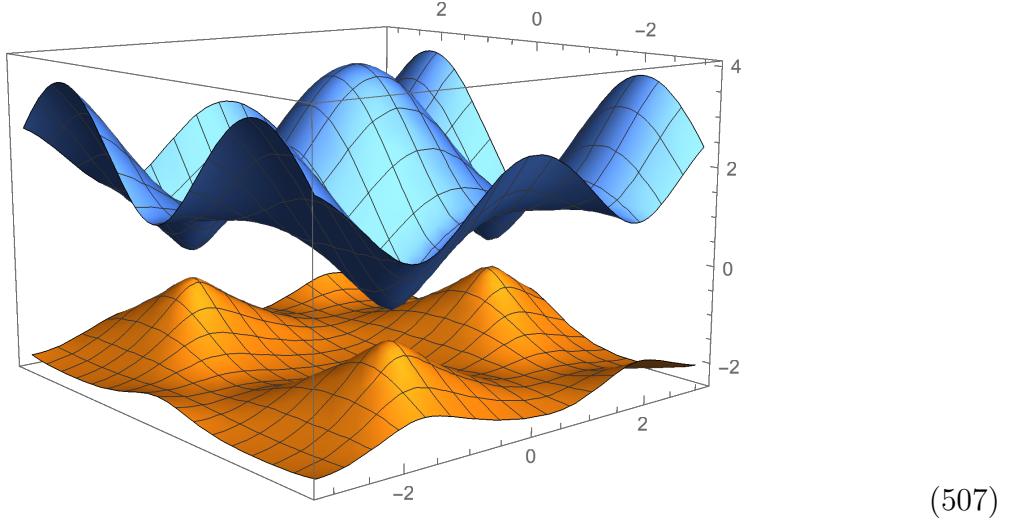
The Dirac mass is thus determined at each (former) Dirac cone by $m = \Delta \mp 3\sqrt{3}t_2 \sin \phi$. We see that if $\Delta \neq 0$, we never have a situation in which both Dirac fermions are massless.

Indeed, the K point fermion is massless when $\Delta = 3\sqrt{3}t_2 \sin \phi$, while the K' point fermion is massless when $\Delta = -3\sqrt{3}t_2 \sin \phi$. A plot of the dispersion for the former situation at $\phi = \pi/2$ (where the K point is gapless and the K' point is maximally gapped) is as follows:



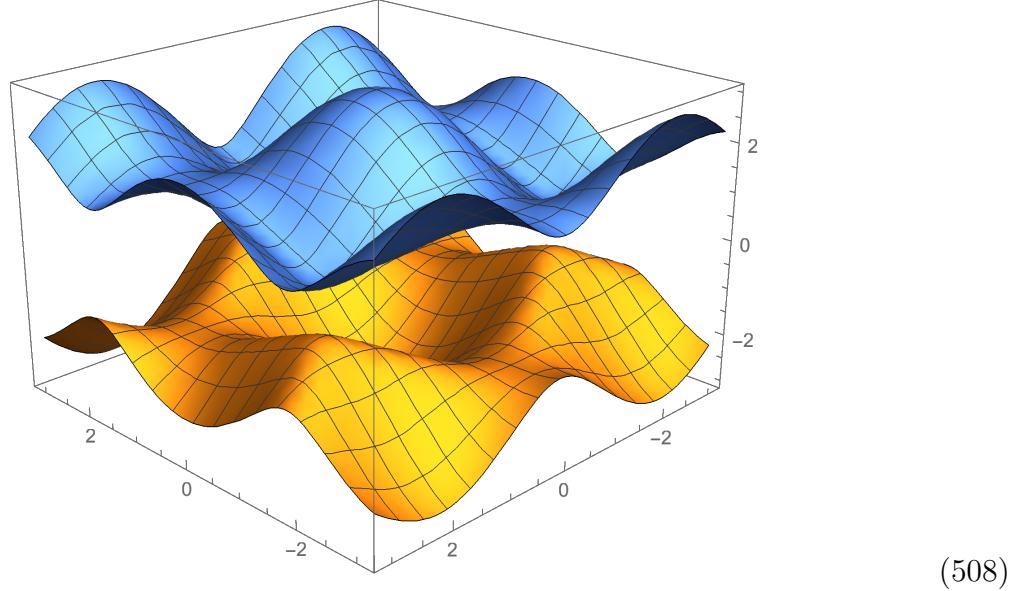
Since we know that $\Delta \rightarrow \pm\infty$ gives a trivially gapped state, which can only go to something with a nonzero Chern number $C \neq 0$ if the gap closes, the only region in parameter space where the Chern number can be nonzero is the region in the $(\Delta/t_2, \phi)$ plane enclosed by the curves $\Delta/t_2 = \pm 3\sqrt{3} \sin \phi$. This is consistent with $C = 0$ if $\phi = 0, \pi$, since at these values of ϕ the system is T symmetric, and must have $C = 0$ for each band.

Anyway just to throw some more pictures in, here's what we get if we take $t_2 = 3^{-3/2}$, $\Delta = 1$, $\phi = \pi/4$, which is in the trivially gapped state:



If we keep $t_2 = 3^{-3/2}$ but now take $\phi = \pi/2$, $\Delta = 1/2$, which as we will see is in the topological

phase, we have



d) We can map out the Chern numbers of the two bands for the various regions in $(\Delta/t_2, \phi)$ parameter space without actually calculating the Berry curvature. Consider first $0 < \phi < \pi$. As we bring $\delta \equiv \Delta/t_2$ down from $+\infty$, where the valence band must have $C = 0$, we cross a point at $\delta = 3^{3/2} \sin \theta$ where the band closes at the K point, with the Dirac mass changing from positive to negative. The Chern number must change by ± 1 at this point. The choice of ± 1 is determined by the chirality of the edge modes at domain walls where the mass changes sign; I think with the current choice for the hopping phases we obtain a change of $\Delta C = +1$ (ΔC can be reversed by sending $\phi \rightarrow -\phi$ in the Hamiltonian). The reason for the change is the following: we can compute the Berry curvature with the familiar formula $\mathcal{F}_{xy} \sim i\epsilon_{ijk}h^i\partial_{[k_x}h^j\partial_{k_y]}h^k$, where for the Dirac Hamiltonian $h = (k_x, k_y, m)$. Now for m very slightly greater than 0, the vector h essentially winds in a plane, and so integrating the Berry connection around a loop enclosing the Dirac cone in momentum space tells us that the Berry curvature is a slightly smeared delta function of strength $1/2$ concentrated at the Dirac cone ($1/2$ since the trajectory of h around the cone encloses a solid angle of $\approx 2\pi$). Changing the sign of m flips the sign of one of the components of h , under which the Berry curvature is odd (as we can see from the expression we wrote for it above). Thus a Berry curvature strength $1/2$ zone around the Dirac point is mapped to a strength $1/2$ zone when the mass changes sign, and so the net Chern number changes by 1.

So, we know that the region $-3^{3/2} \sin \phi < \delta < 3^{3/2} \sin \phi$ has a Chern number of $C = +1$. When δ is lowered past $-3^{3/2} \sin \phi$, the Dirac mass at the K' point also changes from positive to negative, but this process is accompanied by a shift of $\Delta C = -1$ change in the Chern number, instead of $+1$: this is because the K' point is the time-reversed image of the K point, and the Berry curvature is odd under T (recall that it has a factor of i out front!), so that the change in Chern number under taking the Dirac mass from positive to negative is opposite in sign between the K and K' points. Thus we revert to a phase with $C = 0$ when $\delta < -3^{3/2} \sin \phi$, as we knew we had to.

When $\phi = 0, \pi$ the system has T symmetry and the net Chern number of both bands is always zero. We can also realize this by noting that when $\phi = 0, \pi$ the gaps at the K, K' points close simultaneously: as we just saw the closing of the gaps leads to opposite changes in C , and so for these values of ϕ there is value of δ for which $C \neq 0$.

Finally we can look at $\pi < \phi < 2\pi$. In this case, when we decrease δ from $+\infty$, it is the K' point that becomes gapless first, at $\delta = 3^{3/2}|\sin \phi|$. From our above discussion, we know this produces a region with $C = -1$. After we decrease δ to $-3^{3/2}|\sin \phi|$ the gap closes, and brings us back into the trivial $C = 0$ phase.

Note that this whole story plays out in reverse for the other band: the assignment of $C = \pm 1$ regions of parameter space is simply swapped. This is because we started from a local \mathbb{R} -space hopping Hamiltonian, which can't give us anything with nonzero net (summed over all bands) Chern number. Alternatively, we can argue this by realizing that the vector h used in the computation of the Berry curvature for the Dirac Hamiltonian is odd under particle hole (which exchanges the bands), and hence so too is the Berry curvature.



Making spinful particles from bosons and magnetic flux

Today we have another problem from Senthil's class, which is basically a long-winded discussion about why a quantum particle moving on a sphere in the presence of a monopole flux background is equivalent to a spin 1/2 particle. The focus will be algebraic, and we will be doing several things:

- (a) Define the operators Λ_μ as

$$\Lambda_\mu = [\mathbf{n} \times \boldsymbol{\pi}]_\mu, \quad \boldsymbol{\pi} = -i\nabla + \mathbf{A} \quad (509)$$

where \mathbf{A} is the vector potential for a monopole living inside the sphere and \mathbf{n} is a unit vector in S^2 , with $\boldsymbol{\pi}$ the conjugate momentum. Compute the commutators of the Λ_μ s and show that they are *not* angular momentum generators.

- (b) Now define the operators

$$L_\mu = \Lambda_\mu + S n_\mu. \quad (510)$$

What are their commutation relations?

- (c) What are the eigenvalues of L^2 ?
- (d) Write the Hamiltonian $H = \pi^2/2m$ in terms of L^2 — what is the gap to the first excited state, and what is its angular momentum?

- (e) Consider the case of minimal flux where $\int F = 2\pi$. Find the ground states (there will be two). Show that in this ground state subspace, the position operator \mathbf{n} acts as $\boldsymbol{\sigma}$, the vector of Pauli matrices.
- (f) Define an antiunitary "time reversal" sending $T : \mathbf{n} \mapsto -\mathbf{n}$. Show that this acts on the ground state subspace in the way one would expect it on a spin 1/2 particle.
- (g) Write down this theory in a path integral representation.
- (h) What happens when one takes the limit $m \rightarrow 0$?

• •

a) The commutator $[\Lambda_\mu, \Lambda_\nu]$ has two contributions: one is the usual term coming from the standard angular momentum commutation relations, and the other is an additional piece picked up because the momenta π_μ do not commute with one another, due to the magnetic field. Writing it out, we have

$$[\Lambda_\mu, \Lambda_\nu] = \epsilon^{\mu\alpha\beta} \epsilon^{\nu\gamma\delta} n^\alpha n^\gamma (-i\partial_{[\beta} A_{\delta]}) + \dots, \quad (511)$$

where \dots are terms that don't involve derivatives of A , and therefore can be computed as usual, by pretending that the π_μ commute.

We are considering the potential for a monopole placed at the center of the sphere. Using suggestive notation, we take the field strength to be uniform and write

$$F = S(\star n_\mu dx^\mu) \implies \int F = 4\pi S \quad (512)$$

for some constant S , which by Dirac quantization needs to be in $\frac{1}{2}\mathbb{Z}$. Therefore one has

$$\partial_{[\mu} A_{\nu]} = S\epsilon_{\mu\nu\lambda} n^\lambda. \quad (513)$$

This means that we can simplify the first term in the commutator to

$$\begin{aligned} \epsilon^{\mu\alpha\beta} \epsilon^{\nu\gamma\delta} n^\alpha n^\gamma (-i\partial_{[\beta} A_{\delta]}) &= -i\epsilon^{\mu\alpha\beta} \epsilon^{\nu\gamma\delta} n^\alpha n^\gamma n^\lambda \epsilon_{\beta\delta\lambda} S \\ &= -iS\epsilon^{\nu\gamma\delta} (\delta_{\mu\delta}\delta_{\alpha\lambda} - \delta_{\mu\lambda}\delta_{\alpha\delta}) n^\alpha n^\gamma n^\lambda \\ &= -iS\epsilon^{\mu\nu\gamma} n^\lambda. \end{aligned} \quad (514)$$

Now if the gauge field vanished so that the different components of π commuted with themselves, we would of course have

$$\begin{aligned} [(\mathbf{n} \times \pi)^\mu, (\mathbf{n} \times \pi)^\nu] &= \epsilon^{\mu\beta\gamma} \epsilon^{\nu\rho\sigma} (n_\beta [\pi_\gamma, n_\rho] \pi_\sigma - n_\rho [\pi_\sigma, n_\beta] \pi_\gamma) \\ &= -i\epsilon^{\mu\beta\rho} \epsilon^{\nu\rho\sigma} n_\beta \pi_\sigma + i\epsilon^{\mu\sigma\gamma} \epsilon^{\nu\rho\sigma} n_\rho \pi_\gamma \\ &= -i(\delta_{\mu\sigma}\delta_{\beta\nu} - \delta_{\mu\beta}\delta_{\sigma\nu}) n_\beta \pi_\sigma \\ &= i\epsilon^{\mu\nu\lambda} \epsilon^{\lambda\beta\sigma} n_\beta \pi_\sigma, \end{aligned} \quad (515)$$

where we used $\epsilon^{abc}\epsilon^{dec} = \delta^{ad}\delta^{be} - \delta^{ae}\delta^{bd}$ a bunch. But this determines the \dots term in (511), and so putting this together with the part involving the field strength, we conclude that

$$[\Lambda_\mu, \Lambda_\nu] = i\epsilon^{\mu\nu\lambda}(\Lambda_\lambda - Sn_\lambda). \quad (516)$$

Thus the commutation relations for the naive angular momentum operators (\mathbf{n} crossed with the gauge covariant derivative) do not satisfy an $SU(2)$ algebra, and hence the Λ_μ s are not in fact angular momentum generators. We will see that the reason here is essentially that one needs to couple to the geometry as well as the $U(1)$ field.

b) We have

$$[L_\mu, L_\nu] = i\epsilon_{\mu\nu\lambda}L^\lambda - 2i\epsilon_{\mu\nu\lambda}Sn^\lambda + S([\Lambda_\mu, n_\nu] - [\Lambda_\nu, n_\mu]). \quad (517)$$

The commutators on the RHS are easy to evaluate:

$$\begin{aligned} [\Lambda_\mu, n_\nu] &= \epsilon_{\mu\alpha\beta}n^\alpha[\pi^\beta, n_\nu] = -i\epsilon_{\mu\alpha\beta}n^\alpha\delta_\nu^\beta \\ &= i\epsilon_{\mu\nu\alpha}n^\alpha. \end{aligned} \quad (518)$$

The other commutator evaluates to the negative of this by antisymmetry, and so these extra commutators kill the $-2i\epsilon_{\mu\nu\lambda}Sn^\lambda$ term. This means that the L_μ operators satisfy the $SU(2)$ algebra

$$[L_\mu, L_\nu] = i\epsilon_{\mu\nu\lambda}L^\lambda \quad (519)$$

and hence it is *these* operators which can be interpreted as angular momenta operators.

c) Since the L_μ satisfy the $SU(2)$ commutation relations, we know that the eigenvalues of L^2 are of the form $l(l+1)$ for $l \in \frac{1}{2}\mathbb{N}$. Now

$$L^2 = \Lambda^2 + S^2 + S(\Lambda_\mu n^\mu + n_\mu \Lambda^\mu). \quad (520)$$

The last term dies since $\mathbf{n} \times \pi$ is \perp to \mathbf{n} , while the second to last simplifies to

$$\Lambda_\mu n^\mu = -i\epsilon_{\mu\nu\lambda}n_\mu\partial_\nu n_\lambda = -i\mathbf{n} \cdot (\partial \times \mathbf{n}) = 0, \quad (521)$$

since \mathbf{n} is purely radial. This means that $L^2 = \Lambda^2 + S^2$. Since Λ^2 has only positive eigenvalues, this means that the only allowed eigenvalues of L are those $\sqrt{l(l+1)}$ such that

$$l(l+1) \geq S^2. \quad (522)$$

Thus S sets the minimum possible value for the angular momentum.

d) Now we can rewrite the Hamiltonian as

$$H = \frac{1}{2m}(L^2 - S^2). \quad (523)$$

The minimum value of l is S , and so the ground state energy for a given field strength S is $E_0 = \frac{1}{2m}(S(S+1) - S^2) = \frac{S}{2m}$. Evidently the ground state has angular momentum S , with degeneracy $2S+1$ coming from the $2S+1$ possible choices for the eigenvalue $m = -l, \dots, l$ of L_z . The next excited state has $l = S+1$, which gives $E_1 = \frac{3S+2}{2m}$, which has an energy

gap of $(S + 1)/m$ with respect to the ground state. For more intuition on why the ground state in the flux background has angular momentum proportional to the flux, see the diary entries on fermionic zero modes in the presence of magnetic fluxes.

e) Now take $S = 1/2$, which is the minimal value allowed by Dirac quantization. There are two degenerate ground states, with L_z eigenvalues of $\pm 1/2$ and L^2 eigenvalues of $3/4$, and hence with Λ^2 eigenvalues of $1/2$. Now L_z is just³⁵

$$L_z = -i\nabla_\phi + \frac{1}{2} \cos \theta = -i \left(\partial_\phi - i \frac{\cos \theta}{2} \right) + \frac{1}{2} \cos \theta = -i\partial_\phi, \quad (526)$$

which is the same as in the un-gauged case.

Let us write ψ_{\pm} for the $\pm 1/2$ eigenfunctions of L_z . The easiest way to do this is to use the raising and lowering operators in $SU(2)$. They are

$$\begin{aligned} L_{\pm} &= e^{\pm i\phi} (\pm \nabla_\theta + i \cot \theta \nabla_\phi + (\sin \theta)/2) \\ &= e^{\pm i\phi} \left(\pm \partial_\theta + i \cot \theta \partial_\phi + \frac{1}{2 \sin \theta} \right). \end{aligned} \quad (527)$$

Now ψ_+ should be annihilated by L_+ , so

$$L_+ \psi_+ = 0 \implies \left(\partial_\theta + \frac{1 - \cos \theta}{2 \sin \theta} \right) \psi_+ = 0 \implies \psi_+ = e^{i\phi/2} \cos(\theta/2). \quad (528)$$

Likewise, ψ_- is killed by L_- , so that

$$L_- \psi_- = 0 \implies \left(-\partial_\theta + \frac{1 + \cos \theta}{2 \sin \theta} \right) \psi_- = 0 \implies \psi_- = e^{-i\phi/2} \sin(\theta/2). \quad (529)$$

As a sanity check, we can look at how ψ_{\pm} are acted on by Λ^2 , the covariant Laplacian. We have

$$\begin{aligned} -\Lambda^2 &= \nabla_\theta^2 + \cot \theta \nabla_\theta + \csc^2 \theta \nabla_\phi^2 \\ &= \partial_\theta^2 + \cot \theta \partial_\theta + \csc^2 \theta (\partial_\phi - i(\cos \theta)/2)^2 \\ &= \partial_\theta^2 + \cot \theta \partial_\theta + \csc^2 \theta \partial_\phi^2 - i \csc \theta \cot \theta \partial_\phi + \frac{1 - \csc^2 \theta}{4}. \end{aligned} \quad (530)$$

³⁵We have two options for what we might take the gauge field to be. The one we'll work with is

$$A = -\frac{\cos \theta}{2} d\phi, \quad (524)$$

which gives the correct field strength. This is singular at both poles, and can be fixed up by doing

$$A_{N/S} = \frac{\pm 1 - \cos \theta}{2} d\phi \quad (525)$$

on the two hemispheres. The $\pm d\phi/2$ part doesn't affect the field strength though, and its absence leads to more symmetric (though more singular) wavefunctions. For example, with the gauge we'll use, the $L_z = \pm 1/2$ eigenfunctions have a ϕ dependence of $e^{\pm i\phi/2}$, which is more symmetric but more singular than the $e^{i\phi}$, 1 (or $1, e^{-i\phi}$) dependence of the eigenfunctions with the $A_{N/S}$ potential.

So, acting on ψ_{\pm} ,

$$\Lambda^2 \psi_{\pm} = -(\partial_{\theta}^2 + \cot \theta \partial_{\theta} \pm (\csc \theta \cot \theta)/2 + 1/4 - (\csc^2 \theta)/2) \psi_{\pm}. \quad (531)$$

Plugging in the form for ψ_{\pm} above, one verifies that

$$\Lambda^2 \psi_{\pm} = \frac{1}{2} \psi_{\pm}, \quad (532)$$

which from $\Lambda^2 = L^2 - S^2$ tells us that $L^2 \psi_{\pm} = \frac{1}{2}(1 + 1/2)\psi_{\pm}$, as required. Summarizing, a general state in the ground state subspace is labeled by θ, ϕ and can be written as

$$\psi = (\psi_+, \psi_-)^T = (e^{i\phi/2} \cos \theta/2, e^{-i\phi/2} \sin \theta/2)^T. \quad (533)$$

This same strategy works for getting the eigenstates when $S > 1/2$. The ground states always have $l = S$, and are $(2l + 1)$ -fold degenerate. The highest-weight state is always $\psi_+ = e^{iS\phi} \cos^{2S}(\theta/2)$, and the lowest-weight state is always $e^{-iS\phi} \sin^{2S}(\theta/2)$. For example, for $S = 1$ we have $\psi_+ = e^{i\phi} \cos^2(\theta/2)$, $\psi_0 = \sqrt{2} \sin(\theta/2) \cos(\theta/2)$, $\psi_- = e^{-i\phi} \sin^2(\theta/2)$.

Anyway, going back to the case of $S = 1/2$, we want to find out what the operator \mathbf{n} projects to in the ground state subspace. A quick way to argue is the following: since the ground state subspace is two-dimensional and \mathbf{n} is Hermitian, we can write $\mathbf{n} = a\mathbf{1} + b_j \sigma^j$. Now by rotational symmetry all the b_j s are equal, while again by rotational symmetry $\text{Tr}[\mathbf{n}] = 0$, where the trace is in the ground state subspace. Thus $n^i \propto \sigma^i/2$.

A slightly less quick way to argue is to use the explicit expressions for the ground state wavefunctions. Let $S_j = \sigma_j/2$. We have

$$\langle \psi | S_z | \psi \rangle = \frac{1}{2}(\cos^2(\theta/2) - \sin^2(\theta/2)) = \frac{\cos \theta}{2}. \quad (534)$$

Similarly,

$$\langle \psi | S_x | \psi \rangle = \frac{1}{2}(e^{i\phi} + e^{-i\phi}) \cos(\theta/2) \sin(\theta/2) = \frac{\cos \phi \sin \theta}{2} \quad (535)$$

and

$$\langle \psi | S_y | \psi \rangle = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}) \cos(\theta/2) \sin(\theta/2) = \frac{\sin \phi \sin \theta}{2}. \quad (536)$$

Thus $\langle \psi | \sigma^j | \psi \rangle = n^j$, and so on the ground state subspace, the components of \mathbf{n} act as the Pauli matrices.

f) Consider an antiunitary map T which sends $\mathbf{n} \mapsto -\mathbf{n}$. In terms of the angles, T does

$$T : \theta \mapsto \pi - \theta, \phi \mapsto \phi - \pi. \quad (537)$$

Thus when acting on the ground state subspace,

$$T : \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix} \mapsto \begin{pmatrix} i \cos(\theta/2 - \pi/2)e^{-i\phi/2} \\ i \sin(\theta/2 - \pi/2)e^{i\phi/2} \end{pmatrix} = Y\mathcal{K} \begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} \end{pmatrix}, \quad (538)$$

where \mathcal{K} is complex conjugation. Therefore in the ground state subspace, T is represented by $T = Y\mathcal{K}$. Since $(Y\mathcal{K})^2 = -\mathbf{1}$, the ground state is a Kramers doublet.

If we were to go to $S = 1$ (or any $S \in \mathbb{Z}$), we would see instead that $T^2 = \mathbf{1}$ on the ground state subspace, while if $S \in \frac{1}{2}(2\mathbb{Z} + 1)$, then we would still have $T^2 = -\mathbf{1}$. In general, the action of T sends $m \mapsto -m$ and so it inverts the ladder of states in the ground state subspace, but when $S \in \mathbb{Z}$ it does so in a way that squares to $\mathbf{1}$. Indeed, one only needs to check the action of T on the highest weight state, which in general is

$$\cos^{2S}(\theta/2)e^{iS\phi} \xrightarrow{T} e^{-i\pi S} \cos^{2S}(\theta/2 - \pi/2)e^{-i\phi S} = e^{-i\pi S - iS\phi} \sin^{2S}(\theta/2) \xrightarrow{T} e^{2\pi i S} \cos^{2S}(\theta/2)e^{iS\phi} \quad (539)$$

which tells us that $T^2 = e^{2\pi i S}$.

This also follows from representation theory. The spin 1/2 representation of $SU(2)$ is pseudoreal, and so it is isomorphic to its complex conjugate via an antilinear map which squares to $-\mathbf{1}$: this isomorphism is T . In contrast, the spin 1 representation of $SU(2)$ is real (since it's a representation of $SO(3)$), and so is isomorphic to its complex conjugate via an antilinear map T which squares to $\mathbf{1}$. More generally, any half-odd-integer representation of $SU(2)$ is pseudoreal,³⁶ while any integer spin rep is real, since it descends to an $SO(3)$ rep. Thus in the spin S subspace, we indeed have $T^2 = (-1)^{2S}$.

g) We will take the Hamiltonian to be $H = \pi^2/2m + \lambda(q^2 - 1)$, with $\pi_\mu = (p_\mu + A_\mu)$ ($p_\mu \rightarrow -i\partial_\mu$ is the canonical momentum) and λ a Lagrange multiplier constraining the particle to lie on the sphere ($\mathbf{q} \in S^2$). To build the path integral we insert $\int dq(t)|q(t)\rangle\langle q(t)|$ and $\int dp(t)|p(t)\rangle\langle p(t)|$ into the Trotterization of e^{-iHt} :

$$Z = \int \mathcal{D}q \mathcal{D}p \mathcal{D}\lambda \exp \left(i \int dt (p_i \dot{q}^i - (p + A)^2/2m - \lambda(q^2 - 1)) \right). \quad (540)$$

We can do the p integral exactly; after absorbing a constant into the integration measure we get

$$Z = \int \mathcal{D}q \mathcal{D}\lambda \exp \left(i \int dt (-A_i \dot{q}^i + m\dot{q}^2/2 - \lambda(q^2 - 1)) \right). \quad (541)$$

Now doing the integral over λ ,

$$Z = \int \mathcal{D}\theta \mathcal{D}\phi \exp \left(i \int dt (-A_\theta \dot{\theta} - A_\phi \dot{\phi} + m\dot{\phi}^2) \right). \quad (542)$$

h) Now we can take the $m \rightarrow 0$ limit easily. We saw above that the energy gap between the ground states and the first excited states is S/m , and so taking $m \rightarrow 0$ is equivalent to restricting our attention to the ground state subspace (note that this restriction is easier to do after the momenta have already been integrated out, since it just amounts to dropping the kinetic term in the Lagrangian).

Now for a given magnetic field strength S , the vector potential can be chosen as above to be $A = S(1 - \cos)\phi$ (on the northern hemisphere). The $Sd\phi$ piece contributes a total derivative to the action, and so we can drop it. Thus after $m \rightarrow 0$ we are left with

$$Z = \int \mathcal{D}\theta \mathcal{D}\phi \exp \left(iS \int dt \cos \theta \dot{\phi} \right). \quad (543)$$

³⁶since it appears in an odd \otimes -power of the spin 1/2 irrep, and since the \otimes of a ps \mathbb{R} and ps \mathbb{R} rep is \mathbb{R} , while the \otimes of a ps \mathbb{R} rep with a \mathbb{R} rep is ps \mathbb{R} .

This is the path integral for a particle of spin S in 0+1 dimensions: there is no Hamiltonian, and the entire action is the symplectic form piece. In particular, we see that the canonical variables on phase space are $\cos\theta, \phi$ (which gives the appropriate symplectic volume of $\Omega = \sin\theta d\theta \wedge d\phi$ and assigns uniform phase space density to the whole S^2), and tells us that when we quantize, we will have the commutator $[\cos\theta, \phi] = i$. One can then go back and check that these commutation relations imply that the components of the vector \mathbf{n} in the ground state subspace obey the spin commutation relations. For example, using $[x, e^{ip}] = -e^{ip}$ for $[x, p] = i$, we have

$$[n^z, n^x] = [\cos\theta, \sin\theta \cos\phi] = \sin\theta[\cos\theta, e^{i\phi} + e^{-i\phi}]/2 = i\sin\theta(e^{i\phi} - e^{-i\phi})/2i = in^y. \quad (544)$$

Continuing in this way one sees that the components of n^μ obey the commutation relations of the Pauli algebra.

To arrive at this result, we first did the path integral over momentum, then implemented the $q^2 = 1$ constraint, and then took the $m \rightarrow 0$ limit. We could also do things in a different order by first implementing the constraint and projecting onto the ground state subspace, and then doing the path integral directly in this constrained subspace.

To do this approach, we use coherent states. First let's focus on the case of $S = 1/2$, since we've already found the eigenstates $|\psi\rangle$ of H , which were rather simple. Inserting resolutions of $\int_{S^2} |\psi\rangle\langle\psi|$, the path integral is

$$Z = \int \mathcal{D}\theta \mathcal{D}\phi \exp \left(\int dt \langle \psi | d_t | \psi \rangle \right), \quad (545)$$

where $-i\psi |d_t|\psi\rangle = \dot{\theta}A_\theta + \dot{\phi}A_\phi$, with A the Berry connection,³⁷ which we can evaluate explicitly as $A_\theta = -i\langle\psi|\partial_\theta|\psi\rangle = 0$, and

$$A_\phi = -i\langle\psi|\partial_\phi|\psi\rangle = \frac{1}{2}(\cos^2(\theta/2) - \sin^2(\theta/2)) = S(1 - 2(1 - \cos\theta)/2) = \frac{1}{2}\cos\theta, \quad (546)$$

which is of course exactly the right gauge potential for a magnetic field with $\int F = 2\pi$. This gives us back the path integral (543) for the case of $S = 1/2$.

We can also use coherent states to prove that (543) is the path integral for a spin in the spin S representation of $SU(2)$ (before, we just made a heuristic argument). Let $|g\rangle$ denote a coherent state in $SU(2)$ taken in the spin S representation, and let $\mathcal{D}g$ denote the Haar measure on $SU(2)$. Then the coherent state path integral is

$$Z = \int \mathcal{D}g \exp \left(\int dt \langle g | d_t | g \rangle \right). \quad (547)$$

To get this into the form of (543), we write

$$|g\rangle = e^{i\phi S_z} e^{-i\theta S_y} e^{i\gamma S_z} |+\rangle, \quad (548)$$

³⁷The factors of i are because $\langle\psi | d_t |\psi\rangle \in i\mathbb{R}$ due to the anti-Hermiticity of d_t .

where $|+\rangle$ is the eigenstate of S_z in the S representation with the largest eigenvalue (namely S).³⁸ The three coordinates (ϕ, θ, γ) are the coordinates on $SU(2)$ adapted to the isomorphism $SU(2) \cong S^3$: θ, ϕ are the S^2 coordinates of the base space of the Hopf fibration, while γ is the coordinate along the fibers (the “gauge” direction). Since it is acting on the highest eigenstate of S_z , $e^{i\gamma S_z}$ becomes $e^{i\gamma S}$. The γ dependence of the path integral is then the term $iS \int dt (\partial_t \gamma) \langle g | g \rangle = iS \int dt \partial_t \gamma$, which is a total derivative and can be dropped. Thus as expected, the phase space only consists of the base S^2 of the Hopf fibration: the direction along the fibers, since it parametrizes gauge degrees of freedom, does not enter into the phase space.

Anyway, we can now go ahead and calculate the overlap $\langle g | d_t | g \rangle$:

$$\begin{aligned} \langle g | d_t | g \rangle &= i \langle + | e^{i\theta S_y} e^{-i\phi S_z} (\dot{\phi} S_z e^{i\phi S_z} - e^{i\phi S_z} \dot{\theta} S_y) E^{-i\theta S_y} | + \rangle \\ &= i \dot{\phi} \langle + | e^{i\theta S_y} S_z e^{-i\theta S_y} | + \rangle - i \dot{\theta} \langle + | S_y | + \rangle. \end{aligned} \quad (550)$$

The last term is zero, while the first term can be handled by using

$$e^{i\theta S_\mu} S_\nu e^{-i\theta S_\mu} = S_\nu \cos \theta + \epsilon_{\mu\nu\lambda} S_\lambda \sin \theta. \quad (551)$$

This gives us

$$\langle g | d_t | g \rangle = i \langle + | (S_z \cos \theta + S_x \sin \phi) | + \rangle = iS \cos \theta. \quad (552)$$

Putting this into the path integral gives us precisely (543), so (543) indeed describes the quantum mechanics of a spin S particle.



Better derivation of dyon charge and angular momentum quantization

Today we’re going to work through more carefully a basic result that has been derived schematically in a couple other diary entries. We will show that for $U(1)$ gauge theory, two dyons of charge e, g and e', g' must be quantized so that $eg' - e'g \in \mathbb{Z}$, where our conventions are such that magnetic charge is defined through $\int F = 2\pi g$ and the minimal electric charge is 1. This is of course totally trivial if we know what bundles are, but that’s not the point of

³⁸For example, one checks that for $S = 1/2$,

$$|g\rangle = e^{i\gamma/2} e^{i\phi Z/2} (\cos(\theta/2) \mathbf{1} - i \sin(\theta/2) Y) |+\rangle = e^{i\gamma/2} \begin{pmatrix} e^{i\phi/2} \cos(\theta/2) \\ e^{-i\phi/2} \sin(\theta/2) \end{pmatrix}, \quad (549)$$

which is, up to an unimportant phase factor, exactly the coherent state we used above.

today's diary entry, the goal of which is to phrase this result in the more physical language of angular momentum quantization.



We are going to work in units where the electric charge is dimensionless, and since we are working with a $U(1)$ gauge theory we will fix conventions such that the minimal electric charge is 1. Here the electric and magnetic charges are defined by

$$\int \star F = e, \quad \int F = 2\pi g, \quad (553)$$

for appropriate surfaces surrounding the charges. This means that the fields for electric / magnetic charges of unit strength placed at the origin are

$$F_{emon} = \frac{\hat{r}}{4\pi r^2}, \quad F_{mmon} = \star \frac{\hat{r}}{2r^2} \implies F_{mmon} = 2\pi \star F_{emon}. \quad (554)$$

Thus to obtain the magnetic monopole field strength, we just take the Hodge dual of the electric monopole field strength, and multiply by 2π . These conventions match with the usual ones used when talking about electromagnetic duality, where the dual field strength is $\tilde{F} = 2\pi \star F$.

Now we will get an expression for the angular momentum of a charge + monopole system with this normalization. We have $\mathbf{L} = \int \mathbf{r} \times \mathbf{p}$, where $p_i = T_{0i}$. Using $T_{\mu\nu} = \delta_{g_{\mu\nu}} S$ we get $T_{0i} = F_{0j} F_i^j = E_j \epsilon^{ijk} \epsilon^{klm} F_{lm} / 2 = (\mathbf{E} \times \mathbf{B})_i$, where we are in $(-+++)$ signature. So evaluated about a point \mathbf{a} , the angular momentum is

$$\mathbf{L}_a = \int_{\mathbf{x}} (\mathbf{x} - \mathbf{a}) \times (\mathbf{E} \times \mathbf{B}). \quad (555)$$

Now consider a situation in which we have an electric charge at the origin and a magnetic charge at position \mathbf{R} . Then using the triple product $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, we get

$$\begin{aligned} \mathbf{L}_a &= \frac{eg}{8\pi} \int_{\mathbf{x}} \frac{1}{x^3 |\mathbf{x} - \mathbf{R}|^3} (\mathbf{x}((\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{R})) - (\mathbf{x} - \mathbf{R})((\mathbf{x} - \mathbf{a}) \cdot \mathbf{x})) \\ &= \frac{eg}{8\pi} \int_{\mathbf{x}} \frac{1}{x^3 |\mathbf{x} - \mathbf{R}|^3} (-\mathbf{x}(\mathbf{R} \cdot (\mathbf{x} - \mathbf{a})) + \mathbf{R}(\mathbf{x} \cdot (\mathbf{x} - \mathbf{a}))) \\ &= \frac{eg}{8\pi} \int_{\mathbf{x}} \left[\frac{1}{|\mathbf{x} - \mathbf{R}|^3} \left(-\frac{\mathbf{x}(\mathbf{R} \cdot \mathbf{x}) - \mathbf{R}x^2}{x^3} \right) + \frac{\mathbf{x}(\mathbf{R} \cdot \mathbf{a}) - \mathbf{R}(\mathbf{x} \cdot \mathbf{a})}{|\mathbf{x} - \mathbf{R}/2|^3 |\mathbf{x} + \mathbf{R}/2|^3} \right], \end{aligned} \quad (556)$$

where in the last line we've broken the integral up and shifted the part containing \mathbf{a} by $\mathbf{x} \mapsto \mathbf{x} + \mathbf{R}/2$. Note that the second term is totally antisymmetric under $\mathbf{x} \mapsto -\mathbf{x}$; hence it dies.

Now we write

$$\partial_i \widehat{r}^j = \frac{\delta_{ij}}{|r|} - \frac{r_j r_i}{|r|^3}. \quad (557)$$

This means that the term in round parenthesis in the last integral is

$$-\frac{x^i(\mathbf{R} \cdot \mathbf{x}) - R^i x^2}{x^3} = \frac{-x^j + R^j}{|x|^3} (x^2 \delta_{ij} - x_j x_i) = (R^j - x^j) \partial_j \hat{x}^i. \quad (558)$$

So then (dropping the \mathbf{a} subscript since \mathbf{a} has disappeared from the integral) after integrating by parts,

$$\mathbf{L} = \frac{eg}{8\pi} \int_{\mathbf{x}} \left[\nabla \cdot \frac{\mathbf{x} - \mathbf{R}}{|\mathbf{x} - \mathbf{R}|^3} \right] \hat{\mathbf{x}} = \frac{eg}{2} \hat{\mathbf{R}}. \quad (559)$$

Since in our conventions e is quantized in \mathbb{Z} , we see from the above that the magnitude of the angular momentum $L = eg/2$ is quantized in $\frac{1}{2}\mathbb{Z}$ only if g is quantized in \mathbb{Z} .

Now, consider two dyons, and let \mathbf{R} point from dyon 1 to dyon 2. Then we get a contribution to \mathbf{L} of $e_1 g_2 \hat{\mathbf{R}}/2$ from the above calculation. The rest of the angular momentum comes from that created by the e_2 and g_1 charges, which is calculated in the same way as that for the other set of charges, except for the replacement $\mathbf{R} \mapsto -\mathbf{R}$. Thus adding these two contributions, we find the total angular momentum³⁹

$$\mathbf{L} = \frac{1}{2}(e_1 g_2 - e_2 g_1) \hat{\mathbf{R}}. \quad (560)$$

Requiring that the magnitude of this be in $\frac{1}{2}\mathbb{Z}$, we find the quantization condition claimed in the problem heading, viz.

$$e_1 g_2 - e_2 g_1 \in \mathbb{Z}. \quad (561)$$

Note that the minus sign is a bit nontrivial, and as we said arises from the fact that the angular momentum of the electric-magnetic charge pair points from the electric charge to the magnetic one. This minus sign would be a little tricky to derive based on thinking about braiding dyons around one another, since the handedness of the motion is the same for the braiding about both the electric and magnetic charges.



Robust characterization of 2+1D topological insulators

This is a problem from Senthil's 2019 cond-mat class. Consider a quantum spin Hall state on a spatial disk. We will look at what happens when a π flux is threaded through some plaquette in the disk. We will a) show that as a π flux is adiabatically turned on, it nucleates no net charge, but a net spin of $\pm 1/2$. Since the bulk of the system obeys the spin-charge relation, this half spin but zero charge quantum number assignment is fractional. We will then b) use the edge theory to show that the π flux is a Kramer's doublet under time reversal.

³⁹We are ignoring the spin angular momentum of the two dyons since \mathbf{L} is the orbital angular momentum.



a) Consider a background electromagnetic field A , such that

$$dA = \frac{R(t)\pi}{T} \delta(\mathbf{x}) dx \wedge dy, \quad (562)$$

where $R(t)$ is the ramp function

$$R(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < T \\ T & T < t \end{cases}. \quad (563)$$

A quantum spin Hall state is basically two copies of a Chern insulator with opposite Hall responses and opposite spin polarizations. Assume the \uparrow Chern insulator has $C = 1$ and the \downarrow has $C = -1$, wolog. The expectation value of the spin σ current is

$$j_{\mu,\sigma}(x) = \pm \frac{\delta}{\delta A^\mu(x)} S_{CS}, \quad (564)$$

where S_{CS} is the level-1 Chern-Simons action and the $+$ sign is for $\sigma = \uparrow$. Working in units where $\hbar = e = c = 1$, we have

$$j_{\mu,\sigma} = \pm \frac{1}{2\pi} (\star F)_\mu. \quad (565)$$

The change in charge $Q_\sigma(R)$ for a spin component σ in a region R enclosing the origin during the time over which the flux is turned on (assuming the flux insertion is done adiabatically so that the gap is never closed and the CS electromagnetic response is valid for all t) is

$$\Delta Q_\sigma(R) = \int_{-\infty}^{\infty} dt \int_R d^2\mathbf{x} \partial_0 j^0 = \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_R d^2\mathbf{x} \frac{(\Theta(t) - \Theta(t-T))\pi}{2T} \delta(\mathbf{x}) = \pm \frac{1}{2}. \quad (566)$$

Therefore the flux insertion leads to the flux gaining half-charge from the \uparrow layer, and loosing half-charge from the \downarrow layer. This results in the net acquisition of zero electric charge, but a net acquisition of $+1/2$ spin. If we were to instead send $R(t) \mapsto -R(t)$ and insert the flux in the time-reversed fashion, the flux would acquire $-1/2$ spin instead, consistent with $T : \sigma \mapsto -\sigma$.

b) The gapless modes on the edge are described by

$$H_e = \int dx (\psi_L^\dagger(i\partial_x + A_x)\psi_L - \psi_R^\dagger(i\partial_x + A_x)\psi_R). \quad (567)$$

We will take the fermions to have periodic boundary conditions in the absence of any flux.⁴⁰ There are two equivalent ways to proceed: one is to have the fermion momentum always

⁴⁰This is slightly weird since the natural boundary conditions (i.e. the ones that can be smoothly extended into the disk) are anti-periodic, since only the spin structure with anti-periodic boundary conditions is bounding.

be quantized in \mathbb{Z} (taking the radius of the circle to be 1 wolog), and to have the gauge connection show up in the Hamiltonian. With this way of doing things, the boundary fermions are sections of a bundle with trivial transition functions, i.e., they are well-defined by using a single patch for the whole boundary circle. The other is to take the fermions to be defined on a bundle with two patches, related on one of their overlaps by a transition function $e^{i\oint A_x}$. In this case, we can schematically think of the momenta as being quantized in $\mathbb{Z} + \oint A_x/2\pi$, with the gauge field not appearing in H_e . Either way, writing $A_x = \Phi/2\pi$ where Φ is the flux, we see that the spectrum contains right-moving modes with dispersion $E_R = n - \Phi/2\pi$, $n \in \mathbb{Z}$, and left-moving modes with $E_L = n + \Phi/2\pi$, $n \in \mathbb{Z}$. So when $\Phi \in 2\pi\mathbb{Z}$ we have two zero modes, and when $\Phi \in \pi(2\mathbb{Z} + 1)$ we have no zero modes.

When $\Phi = 0$, there are two zero modes; one L and one R . This gives four degenerate states, since each mode can either be filled or unfilled. C symmetry here reflects the spectrum, and so we can take it to act as $C : \psi \mapsto Y\psi^\dagger$, where $\psi = (\psi_L, \psi_R)^T$. This means that $C|L, R\rangle = Cf_L^\dagger f_R^\dagger |0, 0\rangle = f_R f_L(C|0, 0\rangle)$, and so $C|0, 0\rangle = |L, R\rangle$ (a possible constant phase here cancels out in the end) and likewise $C|L, R\rangle = |0, 0\rangle$. Using this and $Ce^{iQ} = e^{-iQ}C$ with $Q = Q_L + Q_R$ tells us that the charge of $|L, R\rangle$ is the negative of the charge of $|0, 0\rangle$, and that the two states differ in charge by 2. Therefore $Q_{|L,R\rangle} = +1$, $Q_{|0,0\rangle} = -1$.

The two singly occupied states $f_{L/R}^\dagger |0, 0\rangle$ are related by C , and have charge 0. For T acting as $T = Y\mathcal{K}$ on fermions, the two states form a Kramers doublet pair, since repeated application of T yields $T : f_L^\dagger \mapsto if_R^\dagger \mapsto -i(-if_L^\dagger) = -f_L^\dagger$, implying $T^2 = -\mathbf{1}$ on $f_{L/R}^\dagger |0, 0\rangle$.

Now when we increase Φ to π , there is a unique ground state. Of the four degenerate modes at $\Phi = 0$, two (one L and one R) move to negative energy at $E = -1/2$, while the other two move to positive energy at $E = +1/2$. The ground state has the former two modes filled, and is not degenerate; hence while it is T symmetric as the filled states are $L \leftrightarrow R$ symmetric, it cannot be a Kramers doublet. Therefore changing the flux toggles the edge states between being a Kramers doublet and a Kramers singlet. Correspondingly, the π flux in the bulk is a Kramers doublet.



Two-site Hubbard models at half-filling and their entanglement entropies

Today we will be solving the two-site Hubbard model at half-filling, both for spinful fermions and spinful bosons. After finding the ground states we will compute the entanglement entropy in the ground state as a function of U/t . We will use conventions where the potential term is $H_U = U(n_1^2 + n_2^2)$, so that the Hamiltonian is

$$H = -t \sum_\sigma (c_{\sigma 1}^\dagger c_{\sigma 2} + h.c.) + U \sum_{i\sigma} n_{i\sigma}^2. \quad (568)$$



Fermions

There are six states in the half-filled Hilbert space $\mathcal{H}_{1/2}$, which we label as

$$\mathcal{H}_{1/2} = \langle e_1, \dots, e_6 \rangle, \quad e_1 = |\uparrow, \uparrow\rangle, e_2 = |\downarrow, \downarrow\rangle, e_3 = |\uparrow\downarrow, 0\rangle, e_4 = |0, \uparrow\downarrow\rangle, e_5 = |\uparrow, \downarrow\rangle, e_6 = |\downarrow, \uparrow\rangle. \quad (569)$$

Note that $|\uparrow\downarrow, 0\rangle$ and $|\downarrow\uparrow, 0\rangle$ are both perfectly good basis vectors; we will therefore find it helpful to fix conventions whereby at a single site, up spins always appear to the left of down spins in basis vectors. We will also fix conventions where states are created by a series of creation operators are always ordered in the left-to-right order obtained from the bra-ket notation of the states. For example,

$$|\uparrow\downarrow, 0\rangle = c_{\uparrow 1}^\dagger c_{\downarrow 1}^\dagger |0, 0\rangle, \quad |\uparrow, \downarrow\rangle = c_{\uparrow 1}^\dagger c_{\downarrow 2}^\dagger |0, 0\rangle. \quad (570)$$

This pedantry is needed just to get fermion-ordering minus signs under control.

In this basis, the Hamiltonian is $H = H_t + H_U$, with

$$H_t = -t (0_{2 \times 2} \oplus P), \quad H_U = 2U[\mathbf{1}_2 \oplus (2\mathbf{1}_2) \oplus \mathbf{1}_2], \quad P = \begin{pmatrix} & & 1 & -1 \\ & & 1 & -1 \\ 1 & 1 & & \\ -1 & -1 & & \end{pmatrix}. \quad (571)$$

The minus signs on H_t are super important, and come from keeping careful track about the ordering of creation operators in the second-quantized expressions of the various basis vectors.

Since H is diagonal in spin indices, it is $SU(2)$ -symmetric and commutes with both S_{tot}^2 and S_{tot}^z . In our basis, we find

$$S_{tot}^z = 2Z \oplus 0_{4 \times 4}, \quad S_{tot}^2 = 2\mathbf{1}_2 \oplus 0_{2 \times 2} \oplus (\mathbf{1}_2 + X). \quad (572)$$

The former is easy to see, while the latter can be found by noting that, restricted to the subspace $V = (e_1, e_5, e_6, e_1)$, S_{tot}^2 is

$$S_{tot}^2|_V = (S^i \otimes \mathbf{1} + \mathbf{1} \otimes S^i)^2 = \frac{3}{2}\mathbf{1}_4 + \frac{1}{2}(X \otimes X + Y \otimes Y + Z \otimes Z) = 2 \oplus (\mathbf{1} + X) \oplus 2. \quad (573)$$

Using this and using $S_{tot}^2 e_3 = S_{tot}^2 e_4 = 0$,⁴¹ we arrive at the above expression for S_{tot}^2 .

S_{tot}^z is already diagonal, but we'd like to go into a basis in which S_{tot}^2 is diagonal as well. This is easy: defining $e'_5 = (e_5 + e_6)/\sqrt{2}$ and $e'_6 = (e_5 - e_6)/\sqrt{2}$, we get, in the new basis,

$$S_{tot}^2 = 2\mathbf{1} \oplus 0_{2 \times 2} \oplus (\mathbf{1} + Z). \quad (574)$$

⁴¹Since $S_{tot}^2 e_3 = |\uparrow\downarrow, 0\rangle + |\downarrow\uparrow, 0\rangle = 0$, and likewise for $S_{tot}^2 e_4$.

Therefore take $H \mapsto SHS^{-1}$, where

$$S = \mathbf{1}_4 \oplus \frac{1}{\sqrt{2}}(Z + X). \quad (575)$$

Therefore

$$H_t \mapsto -t(\mathbf{1}_4 \oplus \frac{Z+X}{\sqrt{2}})(0_{2 \times 2} \oplus P)(\mathbf{1}_4 \oplus \frac{Z+X}{\sqrt{2}}) = -t(0_{2 \times 2} \oplus M), \quad (576)$$

where

$$M = \sqrt{2} \begin{pmatrix} & & 1 \\ & & 1 \\ & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \quad (577)$$

H_U is invariant, since the Coulomb repulsion doesn't care about spin. To make things slightly nicer, define $e_0 = e'_5$, and change basis so that the basis vectors are ordered in sequence—this just moves the triplet which is annihilated by H_t up to the first basis vector. Then in this basis,

$$S_{tot}^2 = 2 \cdot \mathbf{1}_3 \oplus 0_{3 \times 3}, \quad S_{tot}^z = 0_{1 \times 1} \oplus Z \oplus 0_{3 \times 3}, \quad H_t = -t(0_{3 \times 3} \oplus M'), \quad H_U = 2U(\mathbf{1}_3 \oplus 2 \cdot \mathbf{1}_2 \oplus 1), \quad (578)$$

where M' is M with the 3rd row and 3rd column removed. Letting $t' = \sqrt{2}t$, we have

$$H = 2U\mathbf{1}_3 \oplus \begin{pmatrix} 4U & 0 & -t' \\ 0 & 4U & -t' \\ -t' & -t' & 2U \end{pmatrix}. \quad (579)$$

Let $\mathcal{E} \equiv \sqrt{4t'^2 + U^2}$. Then diagonalizing H gives

$$H = \mathcal{S}H_D\mathcal{S}^{-1}, \quad H_D = 2U \cdot \mathbf{1}_3 \oplus \begin{pmatrix} 3U + \mathcal{E} & & \\ & 4U & \\ & & 3U - \mathcal{E} \end{pmatrix}, \quad (580)$$

where the transformation is accomplished with the matrix

$$\mathcal{S} = \mathbf{1}_3 \oplus \begin{pmatrix} t/(U - \mathcal{E}) & -1 & t/(U + \mathcal{E}) \\ t/(U - \mathcal{E}) & 1 & t/(U + \mathcal{E}) \\ 1 & 0 & 1 \end{pmatrix}. \quad (581)$$

The state with the lowest energy is e'_5 , a singlet, which has energy

$$E_G = 3U - \mathcal{E}. \quad (582)$$

This is followed by the symmetric triplet e_0 and the two “jammed” triplet states e_1, e_2 , the three of which are triply degenerate in energy, with energy $2U$. From our expression for S_{tot}^2 , we verify that all three of these states are triplets. The next two states up have energy $4U$ and $3U + \mathcal{E}$, and are both singlets.

The energy difference between the ground state singlet and the excited triplet is, for the limit of $t/U \ll 1$,

$$\Delta E = \mathcal{E} - U \approx 2t^2/U. \quad (583)$$

From the matrix \mathcal{S} , we see that the ground state wavefunction is (after normalizing)

$$\psi = \frac{U + \mathcal{E}}{\sqrt{4t^2 + (U + \mathcal{E})^2}} \left[\frac{\sqrt{2}t}{U + \mathcal{E}} (c_{\uparrow 1}^\dagger c_{\downarrow 1}^\dagger + c_{\uparrow 2}^\dagger c_{\downarrow 2}^\dagger) + \frac{1}{\sqrt{2}} (c_{\uparrow 1}^\dagger c_{\downarrow 2}^\dagger - c_{\downarrow 1}^\dagger c_{\uparrow 2}^\dagger) \right] |0, 0\rangle. \quad (584)$$

When $U \rightarrow \infty$ the doubly-occupied terms vanish as expected, and the singlet superposition becomes degenerate with the triplet superposition, since as $U \rightarrow \infty$ the AF exchange interaction that is responsible for the singlet having lower energy goes away. Note that the $\langle S_{tot}^z \rangle \neq 0$ basis vectors do not appear in the ground state wavefunction.

Now let us compute the entanglement entropy. We first find the reduced density matrix for the first site. Since the Hilbert space we've been working with has a global constraint, it is not separable in the form $\mathcal{H}_{full} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We can however easily embed it into the (16-dimensional) space \mathcal{H}_{full} , simply by giving the wavefunction zero support on all the remaining 10 basis vectors. After doing the embedding, \mathcal{H}_2 can then be traced out to yield ρ_1 in the usual way. In the basis

$$\mathcal{H}_1 = \text{span}\{|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle\}, \quad (585)$$

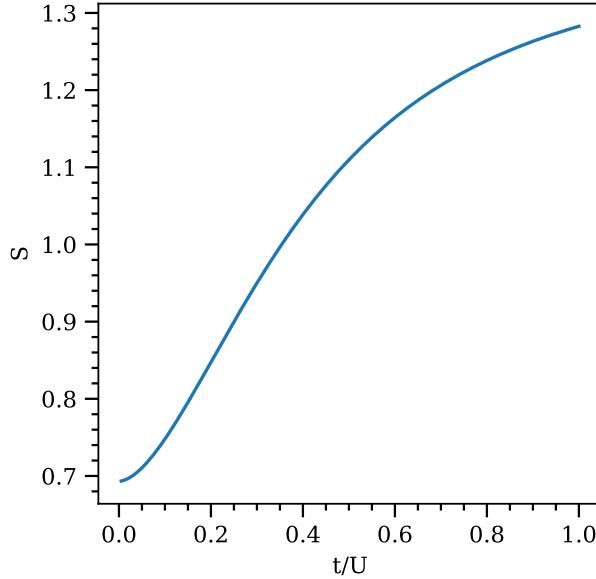
the reduced density matrix is

$$\begin{aligned} \rho_1 = \text{Tr}_{\mathcal{H}_2}[\rho] &= \begin{pmatrix} \langle 0, \uparrow\downarrow | \psi \rangle \langle \psi | 0, \uparrow\downarrow \rangle & & & \\ & \langle \uparrow, \downarrow | \psi \rangle \langle \psi | \uparrow, \downarrow \rangle & & \\ & & \langle \downarrow, \uparrow | \psi \rangle \langle \psi | \downarrow, \uparrow \rangle & \\ & & & \langle \uparrow\downarrow, 0 | \psi \rangle \langle \psi | \uparrow\downarrow, 0 \rangle \end{pmatrix} \\ &= \frac{(U + \mathcal{E})^2}{4t^2 + (U + \mathcal{E})^2} \begin{pmatrix} \frac{2t^2}{(U + \mathcal{E})^2} & & & \\ & 1/2 & & \\ & & 1/2 & \\ & & & \frac{2t^2}{(U + \mathcal{E})^2} \end{pmatrix} \end{aligned} \quad (586)$$

This is diagonal because the basis vectors appearing in the ground state wavefunction all have zero S_{tot}^z and an even number of particles: therefore the only terms that could contribute to the off-diagonal elements of ρ_1 would need to come from elements of ρ off-diagonal in the \mathcal{H}_2 factor, but these elements of course do not appear in the trace over \mathcal{H}_2 . The entanglement entropy is thus easy to compute:

$$S = 2 \frac{(U + \mathcal{E})^2}{4t^2 + (U + \mathcal{E})^2} \left(\frac{2t^2}{(U + \mathcal{E})^2} \ln \left[\frac{4t^2 + (U + \mathcal{E})^2}{2t^2} \right] + \frac{1}{2} \ln \left[\frac{8t^2 + 2(U + \mathcal{E})^2}{(U + \mathcal{E})^2} \right] \right). \quad (587)$$

Plotting this as a function of t/U , we find



This passes the sanity checks at the two extremes: when $t \rightarrow 0$ we know that we get a singlet from the superexchange interaction, and we see that the entropy accordingly goes to $\ln 2$ in the $t/U \rightarrow 0$ limit (we are using the natural log). On the other hand when $t/U \rightarrow \infty$ all four basis states above will be populated evenly, and indeed we see that $S(t/U \rightarrow \infty)$ asymptotes to $\ln 4$.

Bosons

Now we consider the case of spinful bosons. We will be more succinct, and won't go into great detail about how the diagonalization works. There are now ten states in the two-particle Hilbert space: we order the basis vectors in a way that makes the representation of the kinetic term easy:

$$\mathcal{H}_{2\text{-particle}} = \text{span}\{| \uparrow\uparrow, 0\rangle, | \uparrow, \uparrow\rangle, | 0, \uparrow\uparrow\rangle, | \uparrow, \downarrow\rangle, | \downarrow, \uparrow\rangle, | 0, \uparrow\downarrow\rangle, | \uparrow\downarrow, 0\rangle, | \downarrow\downarrow, 0\rangle, | \downarrow, \downarrow\rangle, | 0, \downarrow\downarrow\rangle\} \quad (588)$$

In this basis the two parts of the Hamiltonian are

$$H_t = -t \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} \sqrt{2} & 1 & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix} \quad (589)$$

and

$$H_U = 2U (2 \oplus 1 \oplus 2 \oplus \mathbf{1}_2 \oplus 2\mathbf{1}_2 \oplus 2 \oplus 1 \oplus 2), \quad (590)$$

which admittedly is rather hard to look at when written like this.

Given that the symmetries are the same as the fermion problem, with the only difference coming in the minus signs in the hopping and the extra four states, we expect that the

spectrum and ground state will look fairly similar to the fermion case. In particular, we expect the ground state to only be built out of basis vectors with $\langle S_{tot}^z \rangle = 0$, which gives us just the same set of four basis vectors as in the fermion problem. Without going into details, one finds that the ground state energy and ground state wavefunction are

$$E_G = 3U - \mathcal{E}, \quad (591)$$

which is the same as in the fermionic case, and

$$\psi = \frac{\mathcal{E} - U}{\sqrt{4t^2 + (\mathcal{E} - U)^2}} \left[\frac{1}{\sqrt{2}}(b_{\uparrow 1}^\dagger b_{\downarrow 1}^\dagger + b_{\uparrow 2}^\dagger b_{\downarrow 2}^\dagger) + \frac{\sqrt{2}t}{\mathcal{E} - U}(b_{\uparrow 1}^\dagger b_{\downarrow 2}^\dagger + b_{\downarrow 1}^\dagger b_{\uparrow 2}^\dagger) \right] |0, 0\rangle, \quad (592)$$

which is the ground state wavefunction in the fermionic case, except that we have replaced $\mathcal{E} \mapsto -\mathcal{E}$, the second term has been properly symmetrized, and the prefactors of the two groups of creation operators in parenthesis have been swapped. Let us run a quick sanity check on this wavefunction: first, in the $t/U \rightarrow \infty$ limit, we get the uniform $\psi = (1, 1, 1, 1)^T / 4$ as expected. More interesting is the $t/U \rightarrow 0$ limit: here we have

$$\psi \xrightarrow{t/U \ll 1} \frac{1}{\sqrt{2}}(b_{\uparrow 1}^\dagger b_{\downarrow 1}^\dagger + b_{\uparrow 2}^\dagger b_{\downarrow 2}^\dagger), \quad (593)$$

which is the symmetrized version of the fermionic wavefunction in the same limit. This is telling us that the sign of the superexchange interaction is determined by the statistics of the particles involved: antiferromagnetic if the particles are fermions (so that the ground state is the singlet), and ferromagnetic if the particles are bosons (so that the ground state is the symmetric member of the spin-1 triplet, selected out from the other two for kinetic energy reasons). That the superexchange is ferromagnetic can also be checked by doing the algebra for the usual second-order perturbation theory calculation; we won't write it out here.

The computation of the density matrix and entanglement entropy for the ground state are then done in essentially the exact same way as for the fermionic case: if we again work in the basis (585), the density matrix is

$$\rho_1 = \frac{(U - \mathcal{E})^2}{4t^2 + (U - \mathcal{E})^2} \begin{pmatrix} 1/2 & & & \\ & \frac{2t^2}{(U - \mathcal{E})^2} & & \\ & & \frac{2t^2}{(U - \mathcal{E})^2} & \\ & & & 1/2 \end{pmatrix} \quad (594)$$

while the entanglement entropy is the same formula for the fermionic case, just with $\mathcal{E} \mapsto -\mathcal{E}$:

$$S = 2 \frac{(U - \mathcal{E})^2}{4t^2 + (U - \mathcal{E})^2} \left(\frac{2t^2}{(U - \mathcal{E})^2} \ln \left[\frac{4t^2 + (U - \mathcal{E})^2}{2t^2} \right] + \frac{1}{2} \ln \left[\frac{8t^2 + 2(U - \mathcal{E})^2}{(U - \mathcal{E})^2} \right] \right). \quad (595)$$

Interestingly enough, it turns out that S is actually an *even* function of \mathcal{E} , so that the entanglement entropy in the ground state is the exact same as the fermionic case! This is not too surprising given the above: in the $t \rightarrow \infty$ limit we know we have to get the same $\ln 4$

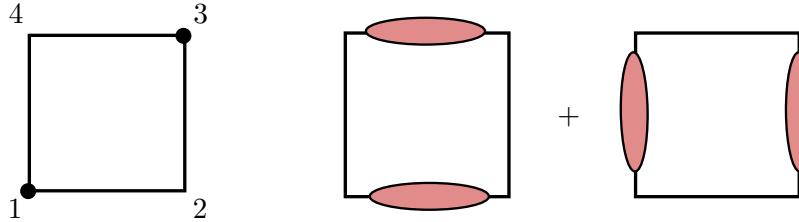


Figure 4: Left: our conventions. The roman numerals indicate how the tensor products in our Fock space are ordered. The black dots indicate how we determine the signs for the singlets: in each singlet, the term with a spin \uparrow on the site with the black dot is given a positive sign. Right: the RVB ground state in the 4-electron subspace. Each oval is a singlet, with signs determined according to the conventions on the left.

answer as we got for the fermions, while in the $t \rightarrow 0$ limit we know we also need to get $\ln 2$, since the ground state is again a single Bell pair (just with a different symmetry compared to the fermionic case).



RVB on a single plaquette and heuristics as to why one should expect $d_{x^2-y^2}$ superconductivity in the cuprates

Today we will consider the t - J model on a single square plaquette. We will see that the “undoped” sector with four fermions has an RVB ground state, which will lead us to an understanding of why a d -wave SCing order parameter is generically expected in the cuprates. I read about this in a long review of high T_c [4], where the result was written down without derivation—the goal today is to work out the details of where the result comes from.



First, consider the half-filled subspace, with four electrons, and restrict to the low-energy subspace where double-occupancy is forbidden. Therefore the Hamiltonian restricted to this subspace is just $H = J \sum_i S_i \cdot S_{i+1}$, with $J > 0$. We can determine on fairly general grounds what the ground state should be. First, it should be a singlet, since the only thing we need to optimize is the AF interaction. This rules out e.g. the Neel states, which are not eigenstates of the $S_i \cdot S_j$ operators,⁴² and are especially not that close to states which

⁴²The $S_i \cdot S_j$ operators entangle the $|\uparrow\downarrow\rangle$ tensor product states on adjacent sites, and so do not map the Neel states (which are \otimes states and have no entanglement) to Neel states.

would be energetically favorable for antiferromagnetism. While the Neel states look like the have spins which are "pointing in opposite directions", this is just the result of incorrectly thinking of spin as a classical vector: while the z components are indeed oppositely directed, the x and y components are totally scrambled, and not pointing in different directions at all. For a single pair of sites, the state which really has the spins oppositely oriented is the singlet state: the two spins in this state point oppositely in every basis (seen from the fact that the singlet is rotationally invariant). While we can only have two spins entangled in an antiparallel state like this, it would still seem reasonable to look for an AF ground state by using singlet building blocks—this is at least a better starting point than working with Z -basis \otimes states. We will therefore look for a ground state built from (a superposition of) nearest-neighbor singlets.

Let $V_{12;34}$ be the horizontal valence bond state as shown in the left term of the right half of figure 4. We will write this state in hopefully transparent matrix notation as The signs here are determined using the left half of figure 4: in each singlet, the term where the spin on the lattice site with the \bullet is \uparrow is given a positive sign. The state with the vertical valence bonds is similarly (just rotate the above by $\pi/2$) Note how we have $\downarrow \leftrightarrow \uparrow$ symmetry; therefore we can only need to explicitly write half of the terms in the manipulations that follow.

The operator $S_1 \cdot S_2$, acting on the basis $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2$, has the form

$$S_1 \cdot S_2 = \frac{1}{4} \left(1 \oplus \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \oplus 1 \right). \quad (596)$$

Therefore on a singlet, $S_1 \cdot S_2$ acts as $(-3/4)\mathbf{1}$ (this is just $(-1/2) \cdot (1/2)$ for each spin direction). Therefore $V_{12;34}$ is an eigenstate of $S_1 \cdot S_2 + S_3 \cdot S_4$, with eigenvalue $-3/2$. Likewise, $V_{41;23}$ is an eigenstate of $S_4 \cdot S_1 + S_2 \cdot S_3$, with the same eigenvalue. However, the other spin interactions break the valence bonds. Therefore if we are to build an eigenstate of H , we should try a linear combination of the two valence bond configurations, so as to minimize the kinetic energy. From the above, we see that in order to determine what happens to our putative ground state $V_{12;34} \pm V_{41;23}$,⁴³ we should calculate $(S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} \pm (S_1 \cdot S_2 + S_3 \cdot S_4)V_{41;23}$. This is straightforward enough: operating on the first and third terms in (??), we get

$$8(S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} = -2 \begin{pmatrix} \downarrow & \uparrow \\ \downarrow & \uparrow \end{pmatrix} - 2 \begin{pmatrix} \downarrow & \uparrow \\ \uparrow & \downarrow \end{pmatrix} + 4 \begin{pmatrix} \uparrow & \uparrow \\ \downarrow & \downarrow \end{pmatrix} + \dots, \quad (597)$$

where the \dots are the terms related by spin-flip symmetry. Therefore these terms have the effect of, among other things, flipping the valence bonds with a minus sign, so that $V_{12;34} \mapsto -\frac{1}{2}V_{41;23} + \dots$, where the \dots are other terms (not related by spin-flips). Likewise, we have Adding these up,

$$\begin{aligned} 8[(S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} \pm (S_1 \cdot S_2 + S_3 \cdot S_4)V_{41;23}] &= (-2 \pm 4) \begin{pmatrix} \downarrow & \uparrow \\ \downarrow & \uparrow \end{pmatrix} + (-2 \mp 2) \begin{pmatrix} \downarrow & \uparrow \\ \uparrow & \downarrow \end{pmatrix} \\ &\quad + (4 \mp 2) \begin{pmatrix} \uparrow & \uparrow \\ \downarrow & \downarrow \end{pmatrix} + \dots \end{aligned} \quad (598)$$

⁴³We can restrict to a \pm sign since our wavefunction needs to transform in an irrep of D_4 , and since the two valence bond configurations are symmetric under a π rotation.

If we choose the + sign, then we get

$$\begin{aligned} (S_4 \cdot S_1 + S_2 \cdot S_3)V_{12;34} + (S_1 \cdot S_2 + S_3 \cdot S_4)V_{41;23} &= -\frac{1}{4} \left[2 \begin{pmatrix} \downarrow & \uparrow \\ \uparrow & \downarrow \end{pmatrix} - \begin{pmatrix} \downarrow & \uparrow \\ \downarrow & \uparrow \end{pmatrix} - \begin{pmatrix} \uparrow & \uparrow \\ \downarrow & \downarrow \end{pmatrix} + \dots \right] \\ &= -\frac{1}{2}(V_{12;34} + V_{41;23}). \end{aligned} \quad (599)$$

So when applied to the linear combination of the valence bond states, the spin-spin interactions have the effect of exchanging the two valence bond configurations. Therefore if we define $|\Phi_0\rangle$ as the symmetric superposition of the two valence bond states, viz.

$$|\Phi_0\rangle \equiv \frac{1}{\sqrt{2}}(V_{12;34} + V_{41;23}), \quad (600)$$

then

$$J \sum_{\langle ij \rangle} S_i \cdot S_j |\Phi_0\rangle = -2J|\Phi_0\rangle. \quad (601)$$

One can then check numerically that this is indeed the ground state.

What is the momentum carried by $|\Phi_0\rangle$? It looks to be zero, because of the + sign in the linear combination. However, we have to remember that the spins are coming from electrons, which give us minus signs. These minus signs mean that while the spin wavefunction is s-wave, the full wavefunction actually changes under rotations. To find out how it transforms, consider how the following term changes under a rotation $R_{\pi/2}$:

$$\begin{aligned} |\Phi_0\rangle \ni \frac{1}{\sqrt{2}} \begin{pmatrix} \downarrow & \uparrow \\ \uparrow & \downarrow \end{pmatrix} &= \frac{1}{\sqrt{2}} c_{\uparrow 1}^\dagger c_{\downarrow 2}^\dagger c_{\uparrow 3}^\dagger c_{\downarrow 4}^\dagger |0\rangle \xrightarrow{R_{\pi/2}} \frac{1}{\sqrt{2}} c_{\uparrow 2}^\dagger c_{\downarrow 3}^\dagger c_{\uparrow 4}^\dagger c_{\downarrow 1}^\dagger |0\rangle \\ &= -\frac{1}{\sqrt{2}} c_{\downarrow 1}^\dagger c_{\uparrow 2}^\dagger c_{\downarrow 3}^\dagger c_{\uparrow 4}^\dagger |0\rangle \in -|\Phi_0\rangle, \end{aligned} \quad (602)$$

and similarly for the other terms. Therefore $R_{\pi/2}|\Phi_0\rangle \mapsto -|\Phi_0\rangle$, and the wavefunction changes sign under a $\pi/2$ rotation (by contrast, one can check that choosing the – sign for the linear combination would have given an s-wave wavefunction).

A superconducting pairing term will connect the 4-electron subspace to the 2-electron subspace (and not to the 6-electron space, since that's bad for Hubbard U reasons). Therefore to establish the symmetry of the SCing gap, we need to look at the ground state wavefunction in the 2-electron sector.

This can be done numerically, but we can get the information we need from a simple heuristic analysis. First, when there is no Hubbard U (just t), the ground state is clearly (the minimum of the $-t \cos(k)$ dispersion favors a uniform real-space wavefunction)

$$|\Psi_0(t=0)\rangle = \frac{1}{4} \sum_{i,j} c_{\uparrow i}^\dagger c_{\downarrow j}^\dagger |0\rangle. \quad (603)$$

The action of D_4 just re-organizes the summation indices, and so this state is s-wave. Now when we turn on U and stay within the 2-electron subspace, the weights of the various terms will change, so that the electrons stay away from one another. However, the fact that the

sign of the gs wavefunction is invariant under $R_{\pi/2}$ will not change: the irrep of D_4 that the gs transforms under is fixed. One way to motivate this is that after we project onto the singly-occupied subspace, the $-t$ hopping amplitude will always be larger than the induced AF $J \sim t^2/U$ coupling, and so it will always be favorable to have a low-momentum state (in the trivial irrep of D_4), rather than a high-momentum state that takes advantage of the AF coupling.

From this we can now conclude that any SCing pairing, should it exist, should be d-wave. Indeed, if Δ_k is a pairing term (built from a sum of $c_\uparrow c_\downarrow$ operators; it has to be singlet pairing since both 4-particle and 2-particle ground states are spin singlets), we want the overlap $|\langle \Psi_0 | \Delta_k | \Phi_0 \rangle|$ to be maximized. Since $|\Phi_0\rangle$ changes sign under $R_{\pi/2}$ and $|\Psi_0\rangle$ doesn't, in order for the overlap to be nonzero, Δ_k must also change sign under $R_{\pi/2}$, and so the SCing gap must indeed be d-wave.

Of course, a single plaquette is not the same as an infinite square lattice, but the basic physics at work here gives us strong reason to believe that superconductivity in the cuprates, arising as it does from a doped Mott insulator, should have a d-wave gap.



JW details

This is a bit basic, but I had to write it up for a pset for Senthil's class, and so I'm including it in the diary since it hasn't appeared as a diary entry before.

Consider the XX chain, with Hamiltonian

$$H = J \sum_i (X_i X_{i+1} + Y_i Y_{i+1}) = J \sum_i (P_i M_{i+1} + M_i P_{i+1}), \quad (604)$$

with $J > 0$ and where we have used the somewhat nonstandard notation

$$P \equiv (X + iY)/2, \quad M \equiv (X - iY)/2. \quad (605)$$

Do several things. a) show that $J > 0$ and $J < 0$ are related by a unitary transformation on H . b) show that total S^z is conserved. c) show how the JW mapping to a fermion chain works. d) show that the ground state is gapless. e) what happens to the $U(1)$ S^z conservation symmetry on the fermion side? g) explain how the JW transformation is modified for a periodic chain of length L .



a) If we want to get a ferromagnetic chain, we can perform the unitary transformation $H \mapsto U^\dagger H U$, where

$$U = \exp \left(i \sum_{i \in 2\mathbb{Z}} \frac{\pi}{2} Z_i \right) \quad (606)$$

performs a rotation by π about the \hat{z} axis on every even site. Then we have

$$\begin{aligned} H \mapsto U^\dagger H U &= J \sum_j e^{-i \sum_{i \in 2\mathbb{Z}} \frac{\pi}{2} Z_i} (X_j X_{j+1} + Y_j Y_{j+1}) e^{i \sum_{i \in 2\mathbb{Z}} \frac{\pi}{2} Z_i} \\ &= J \sum_j e^{-i \sum_{i \in 2\mathbb{Z}} \frac{\pi}{2} Z_i + i \sum_{k \in 2\mathbb{Z}} \frac{\pi}{2} (-1)^{\delta_{k,j}} Z_k} (X_j X_{j+1} + Y_j Y_{j+1}) \\ &= J \sum_j e^{-i\pi Z_j} (X_j X_{j+1} + Y_j Y_{j+1}) = -H, \end{aligned} \quad (607)$$

so that this rotation has the effect of sending $J \mapsto -J$. Therefore wolog we can take $J > 0$.

b) Since the Hamiltonian involves only $P_i M_{i+1}$ and its Hermitian conjugate, both of which flip a pair of spins and hence conserve total S^z , the Hamiltonian conserves $\sum_i Z_i$. Indeed, the commutator is

$$\begin{aligned} \left[\sum_i Z_i, H \right] &= J \sum_{i,j} [Z_i, P_j M_{j+1} + M_j P_{j+1}] = J \sum_j [Z_j, P_j M_{j+1} + M_j P_{j+1} + P_{j-1} M_j + M_{j-1} P_j] \\ &= 2J \sum_j (P_j M_{j+1} - M_j P_{j+1} - P_{j-1} M_j + M_{j-1} P_j) = 0. \end{aligned} \quad (608)$$

c) We do the mapping via⁴⁴

$$Z_i = -2c_i^\dagger c_i + 1 = (-1)_i^F, \quad P_i = \prod_{j < i} (-1)_j^F c_i. \quad (609)$$

The inverse mapping is then

$$c_i^\dagger \mapsto \prod_{j < i} Z_j M_i, \quad c_i \mapsto \prod_{j < i} Z_j P_i. \quad (610)$$

Since $[(-1)_i^F, c_j] = 2\delta_{ij}c_i$, we have $[Z_i, P_j] = 2P_i\delta_{ij}$ as required. Likewise since $[(-1)_i^F, c_j^\dagger] = -2\delta_{ij}c_j^\dagger$, we have $[Z_i, M_j] = [Z_i, P_j^\dagger] = -2\delta_{ij}M_j$. Finally we have

$$\begin{aligned} [M_i, P_j] &= \delta_{j>i}[c_i^\dagger, (-1)_i^F c_j] + \delta_{i>j}[(-1)_j^F c_i^\dagger, c_j] + \delta_{i,j}[c_i^\dagger, c_j] \\ &= \delta_{i,j}[c_i^\dagger, c_i] = \delta_{i,j}(2c_i^\dagger c_i - 1) = -2\delta_{ij}Z_i, \end{aligned} \quad (611)$$

as required.

⁴⁴n.b. we are not associating P_i with c_i^\dagger , as might seem more notationally logical. This is done to avoid stupid minus signs, and to allow the fermion basis to be $(\text{empty}, \text{filled})^T$ instead of the other way around.

d) In terms of fermion variables, H becomes

$$H = J \sum_i (c_i^\dagger (-1)_i^F c_{i+1} + hc) \quad (612)$$

and so we get a simple hopping model, viz.

$$H = J \sum_j (c_j^\dagger c_{j+1} + hc) = 2J \sum_k \cos(k) c_k^\dagger c_k, \quad (613)$$

which gives us a half-filled band. We actually already knew that the filling had to be half: this is the value of the filling for which we have particle-hole symmetry, which is performed by taking $J \mapsto -J$ using the rotation described previously. Adding a term like $\sum_i Z_i$ breaks the $J \mapsto -J$ mapping, and consequently shows up in the fermion variables as a chemical potential, breaking particle-hole symmetry. Anyway, since when such terms are absent the Fermi level cuts the band halfway up, the system is gapless and is half-filled.

e) In the fermion variables, the operator which generates the $U(1)$ symmetry transformation becomes

$$\sum_j Z_j \rightarrow \sum_j -(-1)_j^F, \quad (614)$$

which measures the total particle number (fermions minus holes) and which clearly commutes with the fermion Hamiltonian. The ground state has a definite value of total particle number, and so the $U(1)$ symmetry is preserved in the ground state (as it must be; we are in one dimension).

f) Now let's consider a closed chain of length L . The mapping is the same as before, except this time we fix the tails in the JW mapping to start at the site $i = 1$. The only subtlety occurs when we examine the mapping of operators which wrap around the chain. There are a few ways to fix conventions; we will use

$$c_{i+L}^\dagger \mapsto \prod_{\text{all } j} Z_j \prod_{k < i} Z_k M_i \mapsto (-1)^F c_i^\dagger \implies c_{i+L} = c_i (-1)^F. \quad (615)$$

This isn't surprising; hopping a fermion around a chain should give a minus sign for every fermion it passes, and the natural BCs for a single fermion are AP. Note that the proper anticommutation relations are preserved, e.g.

$$\{c_{j+L}, c_{k+L}^\dagger\} = -\{(-1)^F c_j, (-1)^F c_k^\dagger\} = (-1)^{2F} \{c_j, c_k^\dagger\} = \delta_{jk}. \quad (616)$$

Now as long as we are sticking to things which preserve fermion number mod 2 (so e.g. $P_i P_{i+1}$ is okay but P_i is not; luckily our chosen H preserves fermion number), these sign issues don't come up. However, to write the Hamiltonian we'd still like to know how to decompose c_i into Fourier modes—since the relation between c_{i+L} and c_i involves an operator, how are we supposed to do this? We do it by realizing that $(-1)^F$ commutes with the Hamiltonian, and hence we can decompose the Hilbert space \mathcal{H} into $(-1)^F$ superselection sectors. In each sector, the fermions have well-defined boundary conditions, and a Fourier decomposition is well-defined.



\mathbb{Z}_2 gauge theory on the Kagome lattice and the TFIM

Consider the Hamiltonian H_Δ on the Kagome lattice:

$$H_\Delta = -J \sum_i Z_i - h \sum_{ijk \in \partial\Delta} X_i X_j X_k, \quad (617)$$

where the second sum runs over the vertices at the points of each triangle on the Kagome lattice.

We will discuss the various phases of this model and show that there is a single phase transition, done by mapping the theory to a (gauged) TFIM. We will show that in the topological phase there are four nearly degenerate ground states on the torus, and will find the splitting between them. We will then introduce dynamical magnetic fluxes by adding a term $-\Gamma \sum_i X_i$ to H_Δ , with Γ/J small. The four nearly degenerate ground states are now no longer orthogonal, and we will find the splitting between them.

This is an elaboration on a problem assigned in Senthil's 2019 class on quantum matter, and the results are likely in the literature somewhere.



This is a \mathbb{Z}_2 gauge theory, with Z_i the Wilson link variables (think $e^{i \int_l A}$, where l is a link of the associated \diamond lattice), and X_i the electric field variables. Therefore we will think of the XXX term as measuring the electric flux coming out of a particular triangle in the Kagome, with Z the operator giving the charges dynamics by hopping electric charges between triangles.⁴⁵

We note that in accordance with H_Δ being a gauge theory without magnetic matter, H_Δ possesses a 1-form symmetry generated by the operators

$$W_C = \prod_{l \in C} Z_l, \quad (618)$$

for any closed loop C in the \diamond lattice associated to the Kagome lattice, which all commute with H_Δ . The distinct symmetry generators can be labeled by classes $C \in H_1(X_\diamond; \mathbb{Z}_2)$ —we will call this a “magnetic” 1-form symmetry, since its charge operator measures the magnetic flux.

The W_C operators for C contractible give us an extensive number of operators which commute with the Hamiltonian, and measure the magnetic flux at each \diamond plaquette. This

⁴⁵The language here is a little bit flipped compared to conventional taste, but of course both this choice and its EM dual are on equal footing.

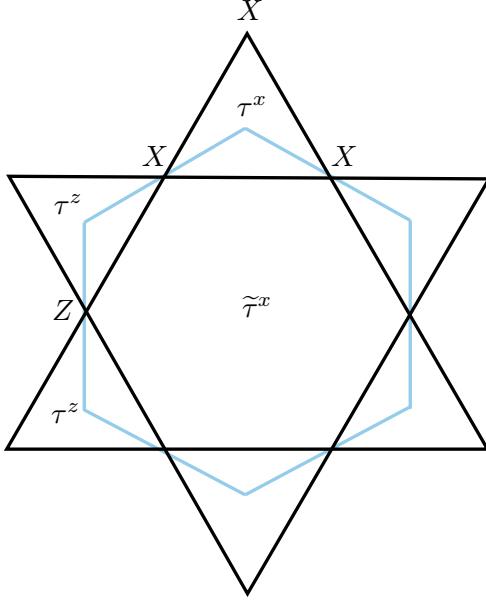


Figure 5: The Kagome lattice and its associated \diamond lattice, with the locations of the support of various operators in various Hamiltonians shown. Matter fields are indicated with τ s and gauge field variables with captial letters.

extensive number of commuting operators give us a gauge redundancy, and in order to eliminate the gauge redundancy, we will fix our Hilbert space by fixing a particular value for W_C on every \diamond plaquette.⁴⁶ If the value of the magnetic flux on a plaquette \diamond is measured by $\tilde{\tau}_{\diamond}^x$, then the operator $W_{\diamond}\tilde{\tau}_{\diamond}^x$ performs a magnetic gauge transformation (magnetic since it involves Wilson line variables, not electric field variables), and acts trivially on the physical Hilbert space (in order to also talk about electric gauge transformations, we'd need to use a formulation H'_Δ with explicit electric matter fields, which we will in a sec). Therefore the full Hilbert space dimension, given a particular magnetic flux configuration, is

$$\dim \mathcal{H}_\Delta = N_L - N_F = N_V, \quad (619)$$

where N_L , N_F , and N_V are the number of links, faces, and vertices of the \diamond lattice, respectively, and we've used that on a flat surface (torus or plane, so that we can tile the \diamond s appropriately), the Euler characteristic is zero. If we want to include a magnetic flux at a certain plaquette, we must then *modify the Hilbert space* to modify the Gauss's law constraint at that plaquette. This is because the fluxes are not dynamical; there are no operators in \mathcal{H}_Δ that create or hop fluxes, and so the flux configuration has to be put in by hand by constraints on \mathcal{H}_Δ . If we wanted to make the fluxes dynamical, we could write

$$H'_\Delta = H_\Delta - \Gamma \sum_{\langle \diamond_m \diamond_n \rangle} \tilde{\tau}_m^z X_{mn} \tilde{\tau}_n^z, \quad (620)$$

where X in the added term creates magnetic fluxes on two neighboring plaquettes, and the $\tilde{\tau}$'s create the corresponding magnetic charges. As mentioned above, the magnetic Gauss's

⁴⁶again, we are treating the W_C operators for noncontractible C rather differently since they generate global symmetries and not gauge transformations; more on this in a sec.

law is then that when acting on physical states, we have

$$\prod_{i \in \partial \diamond} Z_i = \tilde{\tau}_{\diamond}^x, \quad (621)$$

since now the operator which performs the (magnetic) gauge transformation is the product of the RHS and the LHS. Adding either the RHS or the LHS to the Hamiltonian would be the magnetic analogue of the $XXX = \tau^x$ term, which would explicitly gap the magnetic fluxes. Of course, making the fluxes dynamical breaks the 1-form symmetry explicitly, since now the W_C do not commute with the Hamiltonian.

Anyway, let's return to the model H_{Δ} without matter fields, with the magnetic flux configuration is fixed as a Hilbert space constraint. When $h = 0$, the ground state is the non-degenerate product state $\bigotimes_i |\uparrow\rangle_i$. When $J = 0$, a ground state is the loop liquid formed by closed paths on the associated honeycomb lattice of $|-\rangle$ states:

$$|\Phi_0\rangle = \sum_{C \in B_1(X_{\diamond}; \mathbb{Z}_2)} \prod_{l \in C} Z_l |\emptyset\rangle, \quad |\emptyset\rangle = \bigotimes_l |+\rangle_l, \quad (622)$$

where $B_1(X_{\diamond}; \mathbb{Z}_2)$ denotes the closed unoriented *contractible* loops on the associated \diamond lattice X_{\diamond} , with the vertices of the Kagome becoming links l of the \diamond lattice. We say “a” ground state because depending on the topology of space, there may be multiple degenerate ground states. The different ground states differ by the eigenvalues of W_C along different cycles, and specializing to the torus, we can write them as states transforming in a representation of the 1-form symmetry group \mathbb{Z}_2^2 :

$$|\pm_x \pm_y\rangle_M \equiv (\mathbf{1} \pm_x W_{C_x})(\mathbf{1} \pm_y W_{C_y})|\Phi_0\rangle, \quad (623)$$

for $C \in H_1(X_{\diamond}; \mathbb{Z}_2)$, with C_x, C_y denoting representatives for the nontrivial classes in $H_1(T^2; \mathbb{Z}_2)$.

Note that for $J = 0$, there is a second 1-form symmetry, generated by the operators

$$T_C = \prod_{l \in C} X_l, \quad (624)$$

which also commute with the Hamiltonian when $J = 0$. This is an electric symmetry, coming from the conservation of electric flux, which is broken when the term containing electric matter is added. The eigenstates of this symmetry are

$$|\pm_x \pm_y\rangle_E \equiv W_{C_x}^{(1 \mp_x 1)/2} W_{C_y}^{(1 \mp_y 1)/2} |\Phi_0\rangle, \quad (625)$$

and satisfy $T_{C_j} |\pm_x \pm_y\rangle_E = \pm_j |\pm_x \pm_y\rangle_E$. Note that the $|\pm_x \pm_y\rangle_E$ are just linear combinations of the $|\pm_x \pm_y\rangle_M$, and so in either basis we just have four ground states. In the E basis states the expectation value of the W_C operators vanish since the W_C permute different E basis states, while in the M basis the expectation value of the T_C operators vanish, since they permute different M basis states. Therefore the M basis describe SSB for the electric symmetry, while the E basis states describe SSB for the magnetic symmetry. Note that we can't choose basis states that are in charge eigenstates for both symmetries, since the operator which measures

the charge for one symmetry changes the charge of the other. When we turn on a nonzero J the electric symmetry is explicitly broken since the T_C operators no longer commute with the Hamiltonian, and so we will write the eigenstates in the M basis, since this basis is the eigenbasis of the remaining magnetic symmetry.

Anyway, the point is that since the phases in the two limits are both gapped and have different GSDs on T^2 , there must be a phase transition between them at some intermediate value of h/J .

To think about the phase transition, it's helpful to re-write stuff in a more suggestive way by making a mapping to an Ising theory. We do this simply by explicitly including the electric matter that is implicitly present in the H_Δ Hamiltonian, using matter variables τ_i . The mapping is

$$Z_l \mapsto \tau_i^z Z_l \tau_j^z, \quad \prod_{l \in \partial i} X_l \mapsto \tau_i^x, \quad (626)$$

where now i, j indicate sites of the \diamond lattice and l indicates links of the \diamond lattice, i.e. sites of the Kagome lattice. Here the presence of Z_l on the RHS of the mapping is needed to account for the \mathbb{Z}_2 gauge redundancy which appears when we try to map $Z_l \mapsto \tau_i^z \tau_j^z$.

This identification means that we map the Hamiltonian as

$$H_\Delta \mapsto H_\diamond = -J \sum_{l=\langle ij \rangle} \tau_i^z Z_l \tau_j^z - h \sum_i \tau_i^x. \quad (627)$$

This Hamiltonian, just like H_Δ , represents electric charges hopping on the \diamond lattice. In this formulation, the electric charges are just made explicit through the τ variables, whereas before their presence was implicit via violations of the XXX term. In the first term in H_\diamond , the two τ^z operators create electric charges, and the Z operator creates electric flux between them. In this formulation, Gauss's law means that when acting on physical states,

$$\prod_{l \in \partial i} X_l = \tau_i^x, \quad (628)$$

since the operator $\tau_i^x \prod_{l \in \partial i} X_l$ performs electric gauge transformations.

As a check that we have properly accounted for the degrees of freedom, we can count Hilbert space dimensions. For the Kagome lattice model, we saw that $\ln_2[\dim \mathcal{H}_\Delta] = N_L - N_F = N_V$. For the theory with the matter fields, we have "extra" variables on the sites of the \diamond lattice, but also a Gauss's law constraint at each site, so that $\dim \mathcal{H}_\diamond = \dim \mathcal{H}_\Delta$. Note that since $\dim \mathcal{H}_\diamond = N_V$ (modulo topological issues), the full Hilbert space is really just that of the matter fields; the gauge fields are only there to enforce constraints / carry topological information.

As another sanity check, we can look at the GSD: when $h = 0$ we are in the Higgs phase where $Z_{\langle ij \rangle} = \tau_i^z \tau_j^z$ on all links; the ground state is non-degenerate. When $J = 0$ the Hamiltonian looks like it's in the product state $\bigotimes_i |+\rangle$, but we have to remember the gauge constraint, which means that not all the τ_i^x are independent. Writing the h term as $\sum_i \prod_{l \in \partial i} X_l$, we see that we get $\dim H_1(X_\diamond; \mathbb{Z}_2)$ ground states as before, distinguished by the eigenvalues of the W_{C_j} operators.

The fact that we've mapped this model to the gauged TFIM means that, since we know how the TFIM behaves, our model only has a single phase transition as a function of h/J ,

which is the usual 2d Ising transition. One might worry that the gauge field complicates things, but actually since there is no dynamical magnetic flux, the gauge field only keeps track of topological information, and can be ignored for the purposes of thinking about phase transitions. This is because if we ignore topological information, our Hilbert space constraint on the values of the W_C for contractible C completely determines the value of the Z_l operators (as we saw, the Hilbert space dimension is determined solely from the number of matter field sites). For example, if we work on the plane and work in the sector with no magnetic flux, we can just set all of the Z_l variables to $\mathbf{1}$, thereby recovering the regular TFIM model. Working in a different Hilbert space with nonzero magnetic flux just amounts to changing the sign of the J_s along paths in the Δ lular lattice dual to the \diamond lattice, but the statements we can make about the phase transition are unchanged.

As mentioned above, we get four eigenstates of the different W_C operators. Eigenstates in which $\langle W_{C_j} \rangle = \pm$ indicate that the electric charges experience periodic / antiperiodic boundary conditions around C_j , respectively.⁴⁷ Now in the $J/h \gg 1$ phase, these eigenstates will have vastly different energies, with the state $|+_x +_y\rangle_M$ being the true ground state. This is because we can think of $J/h \gg 1$ as the “Higgs” phase, where the matter field likes to follow in lockstep with the gauge field. If W_{C_j} has a negative expectation value then the magnitude of the Higgs field cannot be uniform around C_j , and in particular must pass through 0 at some point, since there are no global sections of the nontrivial \mathbb{Z}_2 bundle over S^1 . This means that e.g. a value of $\langle W_{C_x} \rangle = -1$ implies an energetically costly line along the y direction where the Higgs field changes sign, giving an energy cost that scales as $L_y J$.

However, in the $h/J \gg 1$ limit, the four states are nearly degenerate. This is because the matter has a gap $\sim h$, and so since the boundary conditions will only be felt by processes which tunnel a charge around a cycle, we expect that twisted boundary conditions around the cycle C_j will result in an energy splitting that goes as (we are working in units where L_j is dimensionless, with the lattice spacing set to 1)

$$\Delta E \sim J e^{-L_j/\xi}, \quad \xi = \frac{1}{\ln(h/J)}. \quad (629)$$

To prove this, we first take $H\psi = E\psi$ and write $\psi = \psi_G + \psi'$, where the Hilbert space splits as $\mathcal{H} = \mathcal{H}_G \oplus \mathcal{H}'$ into a ground-state subspace and the collection of excited states. We write $H = H_0 + H'$ with H_0 the h term that acts within \mathcal{H}_G and H' the J term, which takes states in \mathcal{H}_G to states in \mathcal{H}' . Then acting with $(\mathbf{1} - \mathcal{P}) = \mathcal{P}'$ on the Schrodinger equation, where \mathcal{P} is the projector onto \mathcal{H}_G and \mathcal{P}' the projector onto \mathcal{H}' , we have

$$\mathcal{P}' H' \psi = (E - H_0) \psi'. \quad (630)$$

Solving for ψ' and adding it to ψ_G gives

$$\psi = \psi_G + \frac{1}{E - H_0} \mathcal{P}' H' \psi \quad (631)$$

so that

$$\psi = \sum_{n=0}^{\infty} \left[\frac{1}{E - H_0} \mathcal{P}' H' \right]^n \psi_G. \quad (632)$$

⁴⁷Just because the difference between a configuration with two charges and the configuration resulting from moving one of the charges around the cycle C_j differ by an application of W_{C_j} .

Acting on this with $\mathcal{P}H$ and then subtracting off the GS energy $H_0\psi_G$ gives us the effective Hamiltonian, which acts only within the \mathcal{H}_G subspace:

$$H_{eff}\psi = \mathcal{P}H' \sum_{n=1}^{\infty} \left[\frac{1}{E - H_0} \mathcal{P}'H' \right]^n \mathcal{P}\psi. \quad (633)$$

Now we apply this to the problem at hand: the terms on the RHS that survive the leftmost \mathcal{P} will be those where the products of H' terms form closed loops. At each intermediate process when the loops are being formed, we can write $E - H_0 = -2h + O(J)$, and so the effective Hamiltonian in \mathcal{H}_G is, keeping the leading order J term for each loop (i.e. taking $E - H_0 \approx -2h$ for each term individually, just so that we can actually compute things)

$$H_{eff} = -J \sum_{C \in Z_1(X_{\bigcirc}; \mathbb{Z}_2)} \epsilon^{L(C)} W_C, \quad \epsilon \equiv J/2h, \quad (634)$$

with $L(C)$ the length of the loop C .

Now the expectation value of W_C for contractible C is the same in all four eigenstates of the \mathbb{Z}_2 1-form symmetry, since we have fixed the values of the fluxes on all the plaquettes by our Hilbert space constraint. Therefore the splitting between the different 1-form symmetry eigenstates will only come from terms where C wraps a cycle. Therefore we see that the $|+_x+_y\rangle_M$ state has the lowest energy, but that the splitting to the other states is only of order (taking $L_x = L_y$)

$$\Delta E \sim J\epsilon^{L_x}, \quad (635)$$

which vanishes exponentially quickly in the thermodynamic limit.

Now we add the term

$$\Delta H = -\Gamma \sum_l X_l, \quad (636)$$

which has the effect of creating a pair of magnetic fluxes, since it anticommutes with Wilson lines that run through the links l . We could also keep track of the fluxes as actual matter fields themselves; doing this would mean writing $\tilde{\tau}_a^z X_{ab} \tilde{\tau}_b^z$ instead of X_l , with $l = ab$ and a, b denoting plaquettes on the \bigcirc lattice.

Since magnetic flux lines are no longer conserved, we no longer have a \mathbb{Z}_2 1-form symmetry, and can no longer index the states by eigenvalues of the W_C operators; consequently we can no longer restrict our Hilbert space to a specific flux configuration. However, the different $|\pm_x \pm_y\rangle_M$ states can only be connected by the T_C operators, which require the application of L_x different δH s. Now the magnetic charges are gapped, since as we have seen perturbation theory generates the term $-\sum_{\bigcirc} W_{\bigcirc} = -\sum_{\bigcirc} \tilde{\tau}_{\bigcirc}^x$. Therefore the hybridization between the different states will be exponentially small in L for the same reason as in the last part of the problem. If we write (still in the phase where $h \gg J$ so that we have (nearly) degenerate groundstates) the Hamiltonian as

$$H \approx -J\epsilon^5 \sum_{\bigcirc} W_{\bigcirc} - h \sum_i \prod_{l \in \partial i} X_l - \Gamma \sum_l X_l \quad (637)$$

and take $J/\Gamma \gg 1$, then we can repeat the procedure in the last part of the problem with $H' = \delta H$. This gives

$$H_{eff} \approx -J\epsilon^5 \sum_{\bigcirc} W_{\bigcirc} - \Gamma \sum_{C \in Z_1(X_{\Delta}^*; \mathbb{Z}_2)} \eta^{L(C)} T_C, \quad \eta \equiv \Gamma/2J, \quad (638)$$

where X_{Δ}^* is the triangular lattice dual to the Honeycomb lattice. All of the terms in the second sum have the same expectation value in each of the W_C eigenstates, except for the ones where C is non-contractible, which lead to hybridization between the different W_C eigenstates. Therefore the energy splitting coming from the δH term goes as

$$\Delta E \sim \Gamma \eta^{L_x}, \quad (639)$$

which (duh) vanishes in the thermodynamic limit.



Lightning review of Cuprate phenomenology

Today's diary is something a bit different: a lightning review of (hole-doped) cuprate phenomenology (written in spring 2019). The focus is on facts from experiments rather than on trying to explain things with toy models and numerics. Our strategy will be to provide brief vignettes on several different experimental results, rather than going into detail about any one particular technique. We will also be emphasizing older / more established experimental results—they already offer a wealth of information that theorists have yet to explain.



NMR

NMR has been used in several contexts for studying HTSCs. Since it can probe the magnetic environment that each distinct element in a given cuprate compound lives in, it can determine precisely where the holes that contribute to the spin interactions are located. For example, consider $\text{YBa}_2\text{Cu}_4\text{O}_8$, which is doped by changing the O concentration in the layers between the CuO planes. In principle the doped holes could be located in O orbitals that are unrelated to the holes in the Cu sites. However, it was found [17] that (for UD samples), the Cu and O Knight shifts have the same (unusual; more on this in a sec) T -dependence. This tells us that the spin susceptibility in fact comes from a single band involving strongly hybridized

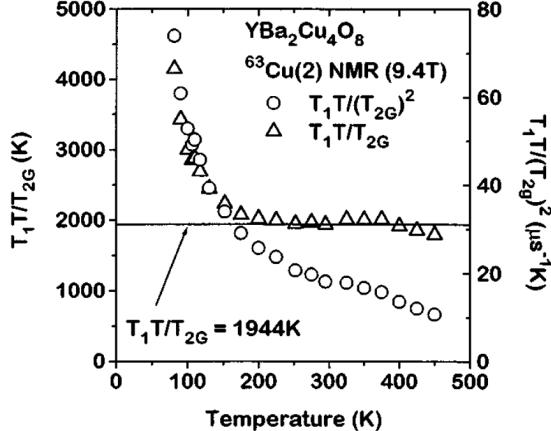


Figure 6: NMR relaxation rate data for the Cu line in UD YBCO, which has $T_C \sim 80$ K. Here the plot is the ratio of longitudinal relaxation time T_1 for the Cu nuclei (relaxation along the magnetizing field direction—depending on χ'' ; dominated by AF fluctuations) and the transverse relaxation time T_{2G} (relaxation in the plane normal to the magnetizing field—depending on χ') times the temperature T ($1/T_1$ depends linearly on T in typical metals). Taken from Ref. [6]

Cu-O orbitals, the knowledge of which is important for setting up theoretical models for the HTSCs.

More prominently, NMR was the first tool used to identify the PG present in the UD side of the phase diagram. Figure 6 shows Cu spin relaxation rates in UD YBCO (chosen for NMR since it is the most ordered of the cuprates, which ensures that all the Cu atoms live in essentially the same magnetic field). For this material, $T_C \approx 80$ K. Below T_C the electron spins are locked into singlets and so they cannot provide a channel for the nuclear spins to relax back to the magnetizing field after being perturbed by the NMR pulse; hence the diverging relaxation time below 80 K. However, the behavior at $T \lesssim T_C$ isn't what we expect from a BCS SC, for two reasons: 1) the nuclear relaxation rate falls off only as a power law below T_C , not as an exponential, implying that there are still some spin degrees of freedom active below T_C , so that the gapping is not complete, and 2) the absence of a sharp decrease in the relaxation rate just below T_C , which occurs in BCS theory because of the added spectral weight right above the gap.

The plot shows something else unusual: the loss in spin susceptibility with decreasing T happens well before T_C is reached (typical UD cuprates see a decrease in their spin susceptibilities by a factor of $\sim 3\text{-}5$ between room temperature and T_C), and indeed on the plot, nothing special happens at 80 K. Instead, the scaling changes at $T^* \sim 200$ K $\gg T_C$. Therefore some nonzero fraction of the spins (but not all of them) must be getting gapped and locked away into singlets well before SC occurs—hence the reason why this region is given the moniker “pseudogap”. In what follows, we will continue to use T^* to refer to the temperature marking the upper boundary of the PG—different experimental probes see different manifestations of the PG phenomenon, but for the most part all agree on a common temperature at which it starts to occur.

Photoemission

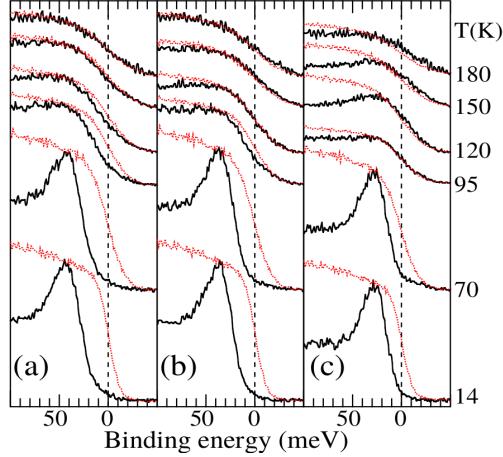


Figure 7: One of the cooler ARPES plots. The black curves are photoemission spectra from lightly UD BSCCO ($T_C = 85$ K), and the red curves are from a sample of platinum attached to the sample to get a reference for the zero on the energy axis. The leftmost panel is taken at an antinode $\sim (\pi, \pi/5)$, the rightmost is taken closer to the Fermi arc location at $\sim (2\pi/3, \pi/5)$, and the middle panel is taken between the two. Taken from Ref. [12]

ARPES is the method of choice for studying the band structure of the cuprates—by bombarding the sample with photons incident on the ab plane and measuring the kinetic energy of the ejected electrons at various ejection angles, one can build up a comprehensive map of the density of states below the Fermi level. In the SCing state, it is generically found that gapless points survive along the nodal lines $(\pm\pi/2, \pm\pi/2)$, with gapped antinodes spaced in between; this helps confirm the nature of the d-wave pairing symmetry. Above T_C a gap is found to remain at the antinodes for UD samples, with Fermi arcs at the node locations—this is the PG regime, which persists up to essentially the same temperature T^* as seen in NMR studies.

Photoemission spectra for UD $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$ (which is well-suited for ARPES studies since it cleaves easily; the energies employed in typical experiments mean that the photons don't penetrate far into the bulk, so a very clean surface is needed) is shown in Figure 7. The leftmost panel shows spectra (black) for various T , taken at k_F for an antinodal point in the BZ. This is where the PG is biggest—the midpoint of the sample's leading edge doesn't meet zero energy until the hottest curve, so that we can identify $T^* \sim 180$ K. By looking for the value of T at which the coherent peak disappears, we identify the critical temperature as $T_C = 85$ K. Note that the disappearance of the peak means that not only Cooper pairs loose their coherence above T_C , but that single electrons loose their coherence as well. The middle and right panels show spectra from closer to the node, where both the SCing gap and the PG are seen to be suppressed.

More generally, ARPES studies on UD samples show that the momentum dependence of the PG tracks that of the SCing gap and evolves smoothly into it as T is lowered. This can be taken as evidence that the PG is characterized by the existence of bound fermion pairs which lack phase coherence. This is true within a finite temperature window above T_C , but

not likely true all the way up to T^* , as we will see. Interestingly, the size of the SCing gap is found to increase with underdoping even though T_C decreases, indicating that the two gaps might be coming from competing ordering mechanisms. ARPES has also been used to map out the OD side of the phase diagram, where metallic states with large FSs are generically found.

Transport

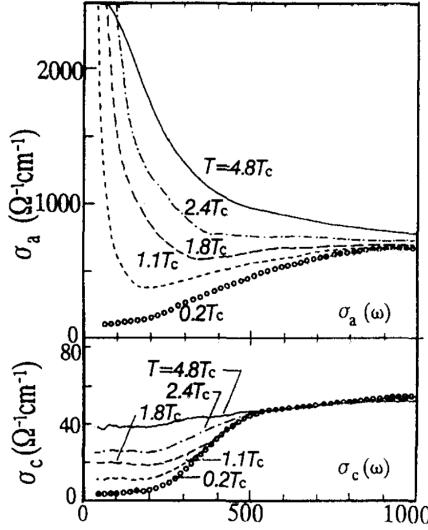


Figure 8: Conductivity in-plane (upper) and out-of-plane (lower) as a function of ω (in cm^{-1}) for UD YBCO ($T_C \sim 80$ K). Several optical phonon lines have been removed in the lower plot to display the electronic background more clearly. Taken from Ref. [18]

Optical conductivity stands out among the different transport-related experiments in the wide variety of information it offers. Figure 8 shows optical conductivity measurements for UD YBCO, chosen because it is the cleanest of the commonly studied cuprates (so that it has nice shiny surfaces). The upper panel shows the ab-plane conductivity: above T_C we see a standard Drude peak that starts at $\sim 500 \text{ cm}^{-1}$ and narrows for lower temperatures. At intermediate temperatures above T_C we see a depression around 400 cm^{-1} ; in OD samples this feature is absent, suggesting that it is related to the PG (which NMR tells us occurs at $T^* \sim 3T_C \sim 250$ K).

The lower panel shows the optical conductivity along the c-axis. Overall the conductance is low and flat, telling us that electron hopping in this direction is largely incoherent. There is not even remotely a sign of a Drude peak like the one seen in the in-plane data. Evidently there is more to the difference between the two conductivities than just the fact that the material is anisotropic.

The c-axis conductivity shows very clear signs of the PG: at low frequencies there is a depressed gap starting at T^* , with no qualitative change as T_C is crossed. Consistent with ARPES measurements, the gap is frequency-independent with changing T . The plots for near-optimally-doped samples in the SCing state turn out to look similar to the ones for UD samples in the normal state, telling us that the PG and the SCing gap are likely intimately

connected. Also note that taken together, the conductance data jives with the RVB idea: transport out of the plane is suppressed since it is only possible if singlets are broken (which necessitates paying a nonzero energy cost), while in-plane transport looks more typical, given that it is facilitated by the gapless holes.

One word of caution here is that different cuprates have very different transport properties vis-a-vis the magnitudes and ratios of the in-plane and out-of-plane conductivities. For example, the c-axis conductivities for YBCO and BSCCO differ[2] by a factor of $\sim 4 \times 10^4$! Even more distressingly, the general behavior of the T dependence of the normal-state c-axis transport seems to vary appreciably across the different cuprates—as far as I’m aware, the reason for this, and its possible significance to the problem at hand, is still not well-understood.

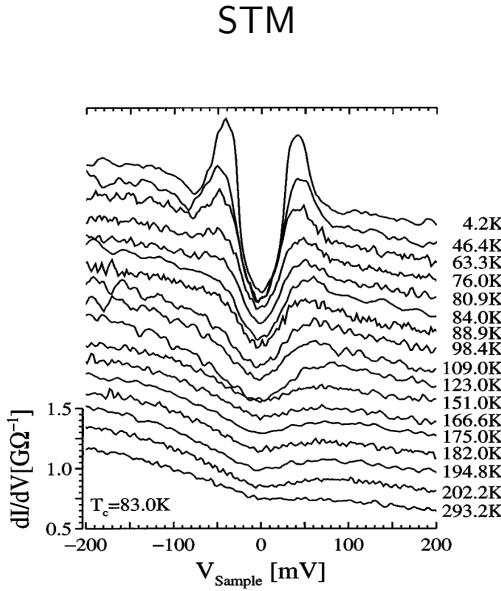


Figure 9: dI/dV (alias local DOS) as a function of sample bias in an STM measurement on UD BSCCO. Note that the width of the gap is unchanged above $T_C = 83$ K—the only special feature at T_C is the disappearance of the coherent peaks on either side of the gap. Taken from Ref. [14]

STM techniques, like ARPES studies, allow a glimpse into what the electrons near the Fermi level are doing. Figure 9 shows tunneling conductance data for UD BSCCO (ideal for STM because it cleaves easily—STM needs good surface quality since the electrons in HTSCs have short coherence times and hence STM only sees the first few layers of a material). This material has $T_C = 83$ K, and we see temperatures below this are accompanied by broad peaks separated by $2\Delta \approx 90$ meV. As far as we’re aware, the exact interpretation of the incoherent features beyond the peak, particularly the dips just below the peak (which are corroborated by ARPES studies) is still rather controversial.

The PG manifests itself as the gap in the DOS at zero bias, which occurs smoothly out of the SCing gap and persists to well above T_C . Going against BCS intuition, the gap is pretty much frequency independent across the entire temperature range, and when doping is increased, the gap decreases (even though T_C increases).

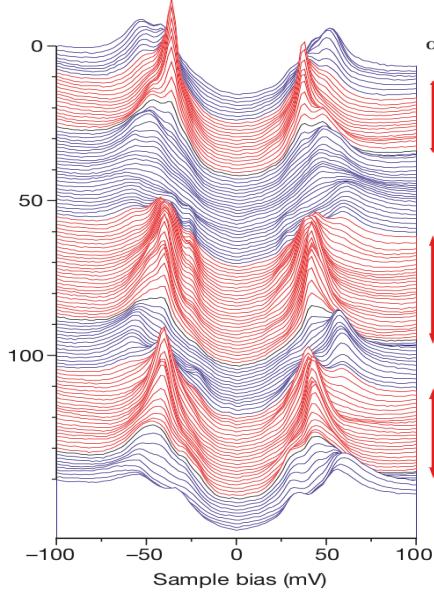


Figure 10: Very pretty dI/dV curves as a function of sample bias and spatial position (in Å) along a 140 Å path in a sample of UD BSCCO. The red areas are SCing; the blue areas have PG-y features. Taken from Ref. [9]

STM can do more than just confirm ARPES results, though—it has also been used to reveal the significantly heterogeneous nature of the SCing state in the UD region. Figure 10 shows tunneling conductance curves as a function of spatial position in a sample of UD BSCCO, scanning a length of 140 Å. Along the path of the STM scan, we see SCing regions loose their peaks and turn into PG-like regions (while maintaining the size of the gap). These results indicate the presence of inhomogeneities in the SCing state on the scale of $O(10)$ Å. Like the pseudogap itself, these inhomogeneities are seen to become larger with underdoping; for severely UD samples they persist to very low temperatures, indicating that these regions may be some zero-temperature manifestation of the PG (see e.g. Ref[8]). Also note that if one restricts to low (~ 10 meV) energies, the density of states looks homogeneous—this suggests that the low energy modes are spatially coherent, while the higher energy modes are not.

STM can also be used to look for things like CDW order and can be used to peer into the states inside of vortex cores; for space reasons we will sadly not get to discuss these applications.

Magnetic fields

The fate of superconducting fluctuations near T_C can be studied by utilizing the Nernst effect: in the presence of a thermal gradient, magnetic field, and nonzero pairing amplitude, the entropy-carrying magnetic vortices will flow along $-\nabla T$, producing a transverse electric response due to the “phase flux” carried by the vortices. Figure 11 shows the magnitude of the Nernst effect (the strength of the electric response) at different points in the phase diagram of LSCO. From the extent of the Nerst region, we can infer a residual nonzero pairing

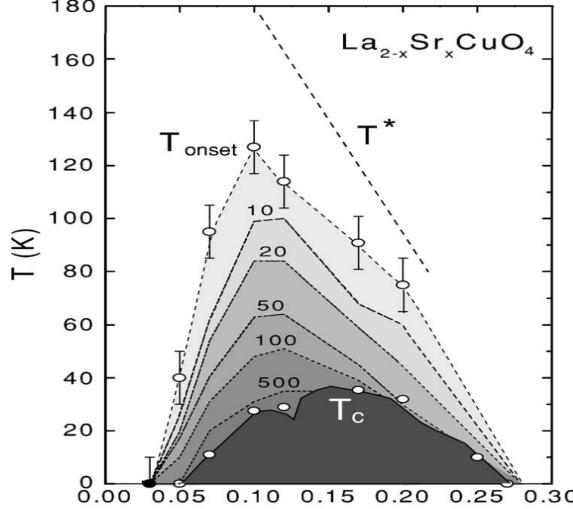


Figure 11: Nernst effect in the phase diagram (T vs x) of $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$; the contours indicate the value of the Nernst coefficient. The important thing is that the Nernst region does not extend up to T^* . Taken from Ref. [19]

amplitude for a large range of temperatures above T_C , implying that SC is killed at T_C by lack of phase coherence and the establishment of a vortex liquid, and not by the breaking-apart of pairs. Another important point is the fact that profile of the Nernst region has a shape similar to the T_C dome (in contrast to T^*), and that the Nernst region stops before T^* is reached—therefore residual SCing phase fluctuations cannot be the sole explanation for the PG.

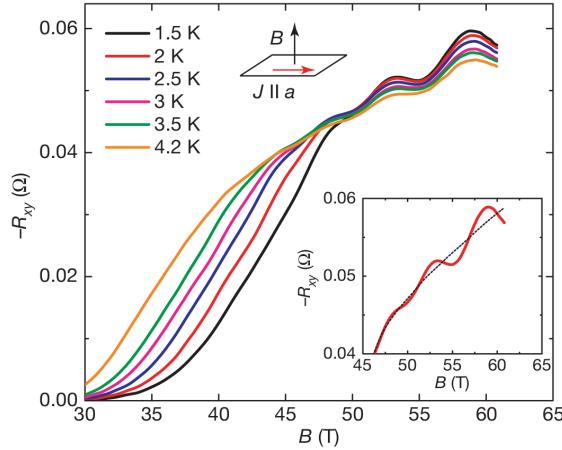


Figure 12: Hall resistance in UD YBCO at several different temperatures, showing quantum oscillations. After subtracting out the linear part, one finds a frequency for the $T = 2$ K curve of $f \approx 500$ (in Tesla). Taken from Ref. [7]

The measurement of quantum oscillations is a relatively contemporary experimental probe, but has offered a lot of interesting information about the potential phases “hiding” behind the SC dome. In Figure 12 we show quantum oscillations observed in the Hall resis-

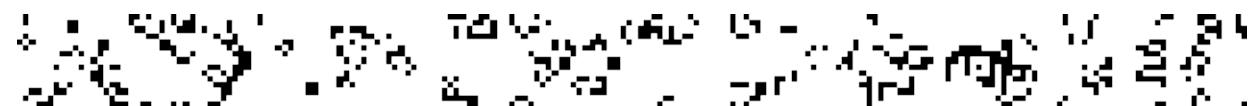
tivity for a sample of UD YBCO (chosen because YBCO more easily yields cleaner+more ordered crystals relative to the other cuprates; quantum oscillations are very sensitive to disorder). These measurements are taken well below the upper critical field H_{c2} , which is likely a factor of ~ 3 larger than the field at which the resistive transition out of a SCing state happens.

One things that immediately jumps out is the fact that the Hall coefficient is *negative*, indicating that the charge carriers have negative charge—to zeroth order this is unexpected, on account of the material being doped with holes, not electrons. Since the period of the oscillations tells us the in-plane FS area, one can conclude that in the UD sample shown, the FS occupies a measly $\sim 2\%$ of the BZ, implying the existence of small electron pockets.

Another somewhat surprising result is the fact that we see quantum oscillations at all: this implies the existence of a FS, but we know that the PG state doesn't have one (it has open Fermi arcs, which can't give closed orbits to contribute to the oscillations). Therefore the magnetic field which destroys the SCing state doesn't “reveal” the underlying $T = 0$ PG state. Evidently either the PG doesn't survive down to $T = 0$, or the magnetic field does more than just killing superconductivity.

In contrast to the UD sample shown above, OD samples show a much higher oscillation frequency with positive R_{xy} , implying the existence of the large (hole-like, in this case) FS that tools like ARPES tell us to expect from that part of the phase diagram.⁴⁸ Therefore the outstanding question is how the small electron pockets evolve into the large hole-like FS, and whether the FSs seen at high fields are actually germane to the zero field problem, and not just a consequence of the existence of the applied field.

While these measurements can only tell us the area of the FS (and not its shape or position within the BZ), experience tells us that one likely scenario is for the UD side to contain small electron pocket(s) arising from some sort of density wave order⁴⁹, since translational symmetry breaking lets us get around the constraints of Luttinger's theorem, which we need to do in order to reconcile the UD and OD FS measurements. The presence of topological order is another way to circumvent Luttinger's theorem, and is at least a possibility in not too UD YBCO, where no ordered state has been seen. [1]



⁴⁸High frequency oscillations means closely spaced energy levels, which means large orbits and hence a large FS.

⁴⁹For a typical example, consider SDW order at the AFM vector $Q = (\pi, \pi)$ occurring on top of a large circular hole-like FS centered on Q —this is the FS seen in the OD side of the phase diagram, as revealed by ARPES. Building up SDW order means identifying Q with 0 in the BZ, which has the effect of superimposing a copy of the FS centered at Q on top of the original FS. The places where the original FS and its double meet then pinch off, leaving behind a collection of electron pockets at $(0, \pi), (\pi, 0)$ and hole pockets at $(\pm\pi/2, \pm\pi/2)$.

Luttinger's theorem: old-school proof

Today we're going to review the OG proof of Luttinger's theorem. I found the proof in Abrikosov's book but I felt it warranted a bit of elaboration, and so here we will work out the details.



2PI generating functional

Before getting into the specifics, we give a few general comments on some technology we will need to make use of later on. Recall that the 1PI effective action is defined via

$$\Gamma[\phi] = W[J] - \int J\phi, \quad (640)$$

where ϕ is some arbitrary field, $W[J] = -i\hbar \ln Z[J]$, and where on the RHS, the current J is regarded as a functional of ϕ —it is the current that needs to be added to the action in order for the expectation value of the field in question to be ϕ . The vertices read off from $\Gamma[\phi]$ constitute all the exact 1PI diagrams. This is because we may write

$$W[J] = \lim_{\hbar \rightarrow 0} (-i\hbar) \ln \left[\int \mathcal{D}\phi e^{\frac{i}{\hbar}(\Gamma[\phi] + \int J\phi)} \right], \quad (641)$$

showing that the tree level contributions from $\Gamma[\phi]$ (tree-level is selected out by the $\hbar \rightarrow 0$ limit) produce the full generating functional of connected correlation functions—hence the tree level calculations with $\Gamma[\phi]$ produce the exact results calculated with the normal action, and so the vertices of $\Gamma[\phi]$ must indeed consist of all 1PI diagrams.

We will be interested in what is roughly the 2PI generalization of this. Define the functional (now dispensing with the \hbar s, going to imaginary time to hide from the i s, and using ψ s since we'll be focusing on fermions from now on)

$$W[\mathcal{J}] = -\ln \left[\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left(-S[\bar{\psi}, \psi] - \int \bar{\psi} \mathcal{J} \psi \right) \right], \quad (642)$$

where if there are multiple flavors of fermions the product is to be understood as $\sum_{ij} \bar{\psi}_i \mathcal{J}_{ij} \psi_j$.

Just as we defined the effective action to be a functional of the vev of the field ϕ in the presence of J , here we will define $\Gamma[G]$ to be a functional of the vev of $\bar{\psi}\psi$, namely a functional of G , the exact Greens function in the interacting theory in the presence of the source \mathcal{J} :

$$\Gamma[G] = W[\mathcal{J}] - \text{Tr}[\mathcal{J}G], \quad (643)$$

where the trace includes integration over spacetime along with the summation of any relevant internal indices. As usual, the fact that this is a Legendre transform means

$$\frac{\delta \Gamma[G]}{\delta G} = -\mathcal{J}, \quad \frac{\delta W[\mathcal{J}]}{\delta \mathcal{J}} = -G. \quad (644)$$

While $\Gamma[\phi]$ was the generating functional for 1PI diagrams (n th-order derivatives wrt ϕ yield 1PI graphs appearing in the irreducible parts of the n -point correlators), $\Gamma[G]$ is not quite the 2PI functional (the 2PI functional is the functional for which functional derivatives with respect to G produce all possible 2PI diagrams—such diagrams are called “non-anomalous” in Abrikosov’s book for some abstruse reason).

The reason why $\Gamma[G]$ as it stands isn’t the 2PI generating functional is because it contains two-particle reducible graphs that are needed to allow $\delta\Gamma/\delta G = \mathcal{J}$ to work for free theories. In order to subtract off the 2P-reducible graphs, it turns out that we can get away with just subtracting off what $\Gamma[G]$ evaluates to in the absence of interactions (i.e. by subtracting off $\Gamma_{\text{free}}[G]$, where Γ_{free} is defined by turning off all the interactions).

So, consider a free theory, where $G^{-1} = G_0^{-1} + \mathcal{J}$. We may write

$$\text{Tr}[\mathcal{J}G] = \text{Tr}[\mathbf{1} - G_0^{-1}G]. \quad (645)$$

On the other hand, we can evaluate $W[\mathcal{J}]$ explicitly, since $e^{-W[\mathcal{J}]} = Z[\mathcal{J}]$:

$$W[\mathcal{J}] = -\ln \det[G_0^{-1} + \mathcal{J}] = -\text{Tr} \ln[G]. \quad (646)$$

Therefore in a free theory, the functional $\Gamma[G]$ assumes the form

$$\Gamma[G] = -\text{Tr} \ln[G] - \text{Tr}[\mathbf{1} - G_0^{-1}G]. \quad (647)$$

Thus if we subtract of the result that one gets for a free theory from the full interacting functional, we get something which we will define as $\Phi[G]$:

$$\Phi[G] \equiv \Gamma[G] - \Gamma_{\text{free}}[G] = \Gamma[G] + \text{Tr} \ln[G] + \text{Tr}[1 - G_0^{-1}G]. \quad (648)$$

We claim that this is the generating functional for 2PI diagrams. Indeed, consider the fact that

$$\frac{\delta\Phi}{\delta G} = -\mathcal{J} + G^{-1} - G_0^{-1} = -\Sigma, \quad (649)$$

where we used Dyson’s formula, $G^{-1} = G_0^{-1} - \mathcal{J} - \Sigma$. Since Σ corresponds to 1PI graphs, and taking the functional derivative wrt G corresponds to cutting a propagator, it must be that Φ is 2PI, as claimed.

Note here that we are assuming there is a unique $J[G]$ such that $\Gamma[G] = W[J] - \text{Tr}[JG]$. In practice this may not be true (we are ultimately interested in $G[J=0]$, but there may be some $J \neq 0$ for which $G[0] = G[J]$) and I believe there are some examples of models where people have demonstrated non-uniqueness. We will ignore this subtlety in what follows.

Diagrammatic proof of Luttinger’s theorem

This proof of Luttinger’s theorem relies on general properties of the exact Greens function in any theory that flows to a Fermi liquid in the IR. The proof involves relating the electronic density (a microscopic UV quantity) to the area of the FS, appropriately defined (an IR quantity). The fact that Luttinger’s theorem relates a UV quantity to an IR one is another manifestation of the many UV / IR connections present in systems that flow to Fermi liquids in the IR.

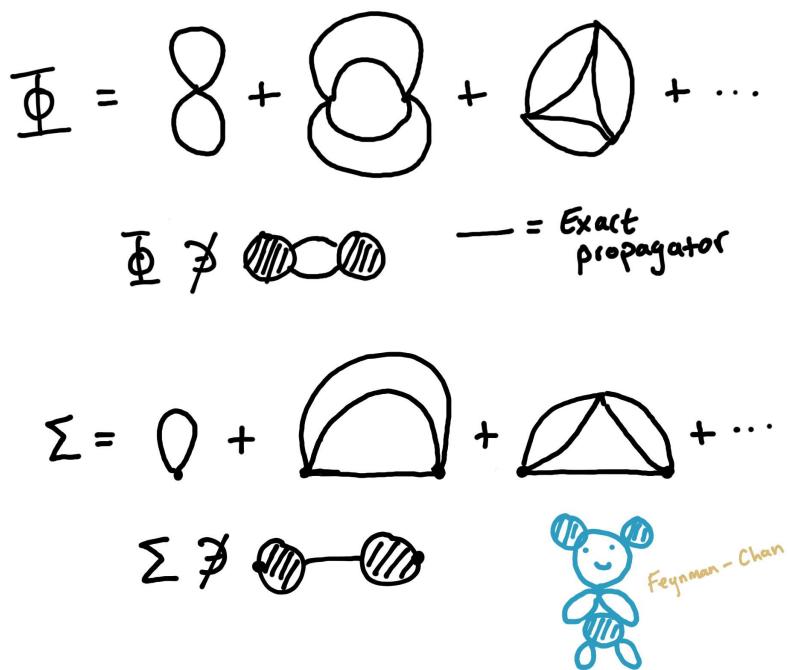


Figure 13: A few diagrams appearing in the Luttinger-Ward functional and the self-energy, drawn with two-body interactions only, for concreteness. The former is formed from closed 2PI diagrams; the latter is formed from 1PI diagrams with two insertion points where propagators can get stuck in. Shaded blobs represent arbitrary interactions.

First, the electronic density is computed as

$$\rho(\mathbf{x}) = \lim_{t \rightarrow 0} \langle \psi^\dagger(t, \mathbf{x}) \psi(0, \mathbf{x}) \rangle = G(\eta, 0) = \int_k G(\omega, \mathbf{k}) e^{i\omega\eta}, \quad (650)$$

where $\eta = 0^+$,⁵⁰ G is the exact interacting Greens function in the theory, and $\int_k = \int_\omega \int_{\mathbf{k}} \frac{d\omega d\mathbf{k}}{(2\pi)^{d+1}}$ means an integral over all d momenta and frequency. We will craftily rewrite this integral as

$$\langle \rho(\mathbf{x}) \rangle = - \int_k e^{i\omega\eta} (\partial_\omega \ln G - G \partial_\omega \Sigma), \quad (651)$$

which works since

$$\partial_\omega G = -\frac{1}{(\omega - \varepsilon_{\mathbf{k}} - \Sigma)^2} (1 - \partial_\omega \Sigma) = -G^2 (1 - \partial_\omega \Sigma), \quad (652)$$

so that

$$\partial_\omega \ln G - \partial_\omega \Sigma G = -G (1 - \partial_\omega \Sigma) - G \partial_\omega \Sigma = -G. \quad (653)$$

The claim is that the second term in the above expression for $\langle \rho(\mathbf{x}) \rangle$ vanishes. To prove this, we start by integrating by parts. We can drop the boundary term if $G\Sigma \rightarrow 0$ when $\omega \rightarrow \pm\infty$. To prove this rather intuitive fact, we will need a result about G which come from its spectral representation. First, recall that the $T = 0$ time-ordered Greens function $G(\omega, \mathbf{k})$ is determined from the spectral density as⁵¹

$$G(\omega, \mathbf{k}) = \sum_a |\langle 0 | c_{\mathbf{k}} | a \rangle|^2 \left(\frac{1}{\omega - E_a + i\eta} + \frac{1}{\omega + E_a - i\eta} \right). \quad (654)$$

This means that

$$G(\omega \rightarrow \pm\infty, \mathbf{k}) \rightarrow \pm 1/\omega. \quad (655)$$

Since $G \rightarrow 1/\omega$ for large frequencies, $\Sigma(\omega \rightarrow \pm\infty)$ must at least remain finite, and therefore the boundary term in the above IBP indeed vanishes. Hence we may write

$$\int_k \partial_\omega \Sigma G = - \int_k \Sigma \partial_\omega G = \delta_\omega \Phi, \quad (656)$$

where we used

$$\delta_\omega \Phi = \int_k \frac{\delta \Phi}{\delta G} \delta G = - \int_k \Sigma \delta G, \quad (657)$$

and choose the variation δ_ω to be such that $\delta G = \partial_\omega G$. The effect of this variation is to take the diagrammatic expression for Φ computed in terms of integrals of G propagators, and

⁵⁰The sign of η corresponds to whether we put the ψ^\dagger before or after the ψ in time, and is there because of our need to regulate the divergent frequency integral in the usual point-splitting fashion (the momentum integral doesn't need to be similarly regulated since we won't actually be doing it and since unlike the frequency integral, it doesn't extend over an unbounded domain). The choice of sign of η has consequences for the sign we get in Luttinger's theorem, and I'm pretty sure 0^+ is the correct one.

⁵¹ G is the time-ordered Greens function, since it appeared in the context of an expectation value in a path integral—hence it follows the Feynman prescription for the poles. The plus sign between the two terms is a consequence of fermionic time-ordering involving a minus sign.

replace every $G(\omega, \mathbf{k})$ line with a $G(\omega + \epsilon, \mathbf{k})$ line, with ϵ infinitesimal. However, doing such a shift does not shift the value of Φ : frequencies always appear in energy-conserving delta functions in positive-negative pairs (e.g. $\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$), which means the support of the integrals in Φ are left invariant under the variation. Therefore $\delta_\omega \Phi = 0$, and so in fact

$$\int_k G \partial_\omega \Sigma = 0, \quad (658)$$

as claimed.

Now all that remains in computing the density is to compute the integral with $\partial_\omega \ln G$. From the definitions of the various correlators, we know that the time-ordered and retarded Greens functions are related by $G(\omega, \mathbf{k}) = G_R(\omega, \mathbf{k})$ for $\omega > 0$, while the time-ordered correlator is related to the advanced one $G(\omega, \mathbf{k}) = G_A(\omega, \mathbf{k})$ for $\omega < 0$ (the poles of $G(\omega, \mathbf{k})$ are above the \mathbb{R} line for $\omega < 0$, and below the \mathbb{R} line for $\omega > 0$). For the conjugate Greens function $G^*(\omega, \mathbf{k})$ the assignment is opposite, since the half plane the poles are in is opposite. Therefore we may write

$$\begin{aligned} \int_{\mathbb{R}} d\omega \partial_\omega \ln G(\omega, \mathbf{k}) &= \int_{-\infty}^0 d\omega \partial_\omega \ln G_A(\omega, \mathbf{k}) + \int_0^\infty d\omega \partial_\omega \ln G_R(\omega, \mathbf{k}) \\ &= \int_{\mathbb{R}} d\omega \partial_\omega \ln G_R(\omega, \mathbf{k}) - \int_{-\infty}^0 d\omega \partial_\omega \ln [G_R(\omega, \mathbf{k})/G_A(\omega, \mathbf{k})] \\ &= \int_{\mathbb{R}} d\omega \partial_\omega \ln G_R(\omega, \mathbf{k}) + \int_{-\infty}^0 d\omega \partial_\omega \ln [G(\omega - i\eta, \mathbf{k})/G(\omega + i\eta, \mathbf{k})]. \end{aligned} \quad (659)$$

Now in our expression for $\rho(\mathbf{x})$, we have a convergence factor of $e^{i\omega\eta}$ (which means the integrand is actually not a total ∂). This means that when doing the ω integral, the contour will be closed in the upper half plane. The non-analyticities in G_R , as well as the branch cut from taking the log, are all in the LHP, and so this integral gives a vanishing contribution to the density. Therefore

$$\langle \rho(\mathbf{x}) \rangle = \int_{\mathbf{k}} \int_{-\infty}^0 \frac{d\omega}{2\pi} e^{i\omega\eta} \partial_\omega \ln [G(\omega - i\eta, \mathbf{k})/G(\omega + i\eta, \mathbf{k})]. \quad (660)$$

Doing a polar decomposition of G as

$$G(\omega + i\eta, \mathbf{k}) = r_{\mathbf{k}}(\omega) e^{i\phi_{\mathbf{k}}(\omega)}, \quad G(\omega - i\eta, \mathbf{k}) = r_{\mathbf{k}}(\omega) e^{-i\phi_{\mathbf{k}}(\omega)}, \quad (661)$$

since the spectral representation tells us that changing the sign of the $i\eta$ part of the frequency is equivalent to conjugating the Greens function. We then have⁵²

$$\langle \rho(\mathbf{x}) \rangle = \frac{1}{\pi} \int_{\mathbf{k}} (\phi_{\mathbf{k}}(-\infty) - \phi_{\mathbf{k}}(0)). \quad (662)$$

⁵²Yes, there's an i missing here. But there has been an i missing for a while because of the fact that our analysis of the Φ functional was done in imaginary time, and this i cancels that i , up to a minus sign. Should probably come back and fuss around with this, but we can fix the overall factors of -1 and i by general sanity checks.

Note that possible subtleties with the branch of the logarithm (i.e. possible factors of 2π in the above) are avoided since when $\omega \leq 0$, the sign of $\text{Im}[G]$ never changes—therefore we can always stay away from the branch of the logarithm by always working in one half plane of \mathbb{C} and avoid any 2π pieces. The reason that $\text{Im}[G]$ has constant sign is simply because of its relation to the spectral density, which is always positive.⁵³

Now we said above, $G(\omega \pm i\eta, \mathbf{k}) \rightarrow -1/\omega$ as $\omega \rightarrow -\infty$. Therefore $\phi(-\infty) = \pi$, and so

$$\langle \rho(\mathbf{x}) \rangle = \int_{\mathbf{k}} (1 - \phi_{\mathbf{k}}(0)/\pi). \quad (663)$$

Now exactly at $\omega = 0$, we have $\text{Im}[G(\pm i\eta, \mathbf{k})] = 0$ for all \mathbf{k} . This can be seen from the spectral representation for G : a nonzero imaginary part at $\omega = 0$ would require there to be eigenstates of H with exactly zero energy, which contradicts our (implicit) assumption of a unique ground state. Therefore $G(\omega = 0, \mathbf{k}) = -(\varepsilon_{\mathbf{k}} + \text{Re}[\Sigma(0, \mathbf{k})])^{-1}$. From the above, we then see that the integrand will only be nonzero when $\varepsilon_{\mathbf{k}}^* \equiv \varepsilon_{\mathbf{k}} + \text{Re}[\Sigma(0, \mathbf{k})] < 0$, since it is only in this region that $\phi_{\mathbf{k}}(0) = 0$. Therefore we may write

$$\langle \rho(\mathbf{x}) \rangle = \int_{\mathbf{k}} \theta(\mathbf{k} \in V_F) = \frac{V_F}{(2\pi)^d}, \quad (664)$$

where V_F is the volume enclosed by the FS, which for our purposes is defined as the locus in \mathbf{k} space across which $\phi_{\mathbf{k}}(0)$ changes between 0 and π (the interior of the FS is defined to have $\phi_{\mathbf{k}}(0) = 0$). This is Luttinger's theorem.

However, we should stress that calling the locus across which $\phi_{\mathbf{k}}$ changes a Fermi surface is strictly speaking a bit hasty. $\phi_{\mathbf{k}}$ can jump either when $G(0, \mathbf{k})$ passes through zero or when it becomes infinite. At a Fermi surface, $G(0, \mathbf{k}) = \infty$ for all $\mathbf{k} \in FS$. However, having $G(0, \mathbf{k}) = 0$ would also give us an appropriate surface to use the Luttinger relation with. This is what happens e.g. in some superconductors: the \mathbb{R} space Greens function can often have a nontrivial angular structure which when integrated during the Fourier transform can give a vanishing result for certain values of \mathbf{k} .



Different types of effective masses

Today we're doing part of a problem from Aschroft and Mermin: near a band maximum / minimum, the dispersion can be expanded as

$$E(\mathbf{k}) \approx C + \frac{1}{2}(\mathbf{k} - \mathbf{k}_0)^T M^{-1}(\mathbf{k} - \mathbf{k}_0), \quad (665)$$

⁵³Said another way, $\text{Im}[G]$ has constant sign for $\omega \leq 0$ since the $i\eta$ convergence factor has a constant sign and Σ_I has constant sign since the sign of Σ_I is fixed by the sign of the spectral density, which is always positive.

where M is a real positive-definite symmetric mass matrix, and C is a constant. We will find an expression for the cyclotron mass in terms of the matrix elements of M .



Recall that the cyclotron mass is calculated by finding $\partial_E A$, where A is the area swept out by the semiclassical electron orbits in the plane perpendicular to \hat{k}_z , at a fixed value of k_z . In particular,

$$m_*(E, k_z) = \frac{\hbar^2}{2\pi} \partial_E A(E, k_z). \quad (666)$$

Now for the purposes of getting $\partial_E A$ we may drop \mathbf{k}_0 . This quadratic expansion for the dispersion gives an $E(\mathbf{k})$ whose isoenergy surfaces define ellipsoids in \mathbf{k} space. The k_x - k_y plane at a certain chosen value of k_z (since $\partial_E A$ is invariant under translations $\mathbf{k} \mapsto \mathbf{k} - \mathbf{k}_0$ in momentum space, we can take the k_z value in question to be at $k_z = 0$) cuts through this ellipsoid, and the intersection of the plane and the ellipsoid defines the area A , which we want to compute.

Our strategy for finding this A is to note that any cross-section of an ellipsoid is an ellipse. Thus we just need to find the area of the ellipse bounded by the Fermi surface in the k_x - k_y plane. Now since we can translate in k_z without affecting the answer, we can assume wolog that there is no value of \mathbf{k} in the k_x - k_y plane such that $E(\mathbf{k}) = 0$. If we let \tilde{M}^{-1} denote the upper 2×2 block of M^{-1} (the k_x - k_y plane part of M^{-1}), then this requirement means that \tilde{M}^{-1} is itself invertible, and can be diagonalized. Let R be the matrix that accomplishes this partial diagonalization, viz.

$$(R^T \oplus 1) M^{-1} (R \oplus 1) = \begin{pmatrix} \lambda_1 & 0 & * \\ 0 & \lambda_2 & * \\ * & * & * \end{pmatrix}, \quad (667)$$

where the $*$ s are numbers that we don't care about. Now define \mathbf{P} by $(R \oplus 1)\mathbf{P} = \mathbf{p}$. When evaluated in the k_x - k_y plane, the energy takes on the form

$$E(\mathbf{P}) = \frac{1}{2}(P_x^2 \lambda_1 + P_y^2 \lambda_2), \quad (668)$$

which is indeed the formula for an ellipse. The semi-axes are

$$a = \sqrt{2E/\lambda_1}, \quad b = \sqrt{2E/\lambda_2}, \quad (669)$$

and so the area A is

$$A(E) = \frac{2\pi E}{\sqrt{\lambda_1 \lambda_2}}. \quad (670)$$

Therefore m_* is actually independent of both E and k_z , and given by

$$m_* = \hbar^2 (\lambda_1 \lambda_2)^{-1/2}. \quad (671)$$

The exact value then needs to be determined by partially diagonalizing M^{-1} to find the λ_i .

Note that this is *not* the expression give in Ashcroft and Mermin, who list $m_* = \sqrt{\det M/M_{zz}}$. If the dispersion happens to be such that the k_z axis is a principal axis of the ellipsoid, so that $M^{-1} = \widetilde{M}^{-1} \oplus M_{zz}^{-1}$, then the formula in Ashcroft and Mermin is correct, since in that case we have $\det M = M_{zz}/(\lambda_1\lambda_2)$, but in general it is wrong.



Lity catastrophe

Today we're doing a problem I found posed on Andy Lucas's old webpage at Stanford, which goes over the argument behind the Lity catastrophe. The goal will be to examine what happens to a Fermi liquid when the Hamiltonian is perturbed by a weak interaction, which is supposed to be thought of as the effect generated by the addition of an extra electron into the system.



Our approach to thinking about the initial FL ground state will be to think of the unperturbed system as a collection of single-particle energy levels $|i\rangle$ that get filled up by qps. We will construct the ground state by filling up the $|i\rangle$ just as we would do for a Fermi gas, with the resulting ground state $|0\rangle$ being a Slater determinant. Of course, qps are only well-defined right near the FS, and so this treatment is inaccurate for energy levels not close to the Fermi energy. However, the point of FL theory is that the UV completion we choose doesn't really matter—we can choose a UV completion of a fake Fermi gas of qps, and still get the right answers for the IR physics.

With this in mind, suppose the system size is such that the first N levels are occupied in the $T = 0$ ground state. In position space, the many-body gs wavefunction is then

$$\langle x_1, \dots, x_N | 0 \rangle = \det[A_{\mathbf{x}}] = \epsilon^{\mathbf{i}} \langle x_1, \dots, x_N | i_{i_1}, \dots, i_{i_N} \rangle, \quad (672)$$

where \mathbf{i} stands for the set i_1, \dots, i_N . The entries of the $N \times N$ matrix $A_{\mathbf{x}}$ are of course just the position-space projections of the single-particle levels $|i\rangle$:

$$[A_{\mathbf{x}}]_{x_i, j} = \langle x_i | j \rangle \quad j = 1, \dots, N. \quad (673)$$

Now we imagine perturbing the system by sending $H \mapsto H + V$; when we do this the single-particle levels are assumed to map bijectively to new slightly perturbed levels:

$|i\rangle \mapsto |i'\rangle$. The overlap between the new and old groundstates is (sorry for the profusion of i indices; you know what I mean though)

$$\langle 0|0' \rangle = \int \prod_{i=1}^N dx_i \varepsilon^i \varepsilon^j \langle i_{i_1}, \dots, i_{i_N} | x_1, \dots, x_N \rangle \langle x_1, \dots, x_N | i'_{j_1}, \dots, i'_{j_N} \rangle = \det[A'], \quad [A']_{ij} = \langle i|j' \rangle, \quad (674)$$

where we have ignored normalization.

<Digression: needed math fact>

To proceed, we will need the following math fact: if the columns (or rows) of a matrix A are unit vectors, so that $\sum_j A_{ij}^* A_{ij} = 1$ for all i , then $|\det A| \leq 1$. To show this, consider

$$|\det A|^2 = (\det A)(\det A^T)^* = \det[AA^\dagger]. \quad (675)$$

Now the matrix entries of AA^\dagger are $[AA^\dagger]_{ij} = R_i \cdot R_j^*$, where R_i is the i th row vector of A ; i.e. $[R_i]_j = A_{ij}$. Now by assumption, $R_i \cdot R_i^* = 1 \forall i$, and so the diagonals of AA^\dagger are all 1s.

We now use the bound

$$(\det[M])^{1/\dim M} \leq \frac{1}{\dim M} \text{Tr}[M], \quad (676)$$

for any matrix invertible M with positive eigenvalues. This can be proved as follows: first we start by diagonalizing M ; both the \det and Tr are invariant under the diagonalization, and so the above bound becomes $(\prod_{i=1}^{\dim M} \lambda_i)^{1/\dim M} \leq \frac{1}{\dim M} \sum_{i=1}^{\dim M} \lambda_i$, where λ_i are the eigenvalues. This is therefore just equivalent to the statement that the arithmetic mean is always greater than or equal to the geometric mean: for any collection of n positive numbers x_1, \dots, x_n , we have $\sum x_i/n \geq (\prod x_i)^{1/n}$, with equality only when all the x_i are equal. We can prove this with Lagrange multipliers: subjecting the function $f(\mathbf{x}) = (\prod x_i)^{1/n}$ to the constraint $g(\mathbf{x}) = \sum x_i/n - S = 0$ for constant S , we find that $f(\mathbf{x})/(nx_i) = \omega/n$ for all i and some constant ω —since the LHS depends on i but the RHS doesn't, we must take all the x_i to be the same. This extremum is a local minimum of $F(\mathbf{x}) = \sum x_i/n - (\prod x_i)^{1/n}$, since the second derivative evaluated at the extremum is $(n^{-1} - n^{-2})/x > 0$; QED.

Applying this bound (which is legit since AA^\dagger is Hermitian and hence all the eigenvalues are real, as required) and using that $\text{Tr}[AA^\dagger] = N$, we find $\det[AA^\dagger]^{1/N} \leq 1$, implying $\det[AA^\dagger] \leq 1$ and hence $|\det A| \leq 1$, as claimed. This whole song and dance is just a mathematical way of expressing something rather obvious: when we take a bunch of orthogonal unit vectors and map them to a bunch of other (not necessarily perpendicular) unit vectors, the volume spanned by the collection of unit vectors can only decrease.

</Digression

With this math fact at hand, we can now get a bound on the overlap $|\langle 0|0' \rangle|$. We turn the matrix A' into a matrix A'_n whose rows are unit vectors by forming the matrix

$$A'_n = \mathcal{N}A' \equiv \begin{pmatrix} \left(\sum_{k < N} |\langle 1|k' \rangle|^2 \right)^{-1/2} & & \\ & \left(\sum_{k < N} |\langle 2|k' \rangle|^2 \right)^{-1/2} & \\ & & \ddots \end{pmatrix} A'. \quad (677)$$

Note that \mathcal{N} is not the identity, since we are only summing over the gs-occupied subset of levels, not the whole collection of levels needed to use the resolution of **1**. Using the math fact then, we have

$$|\langle 0|0' \rangle| = |\det[A']| = |\det[\mathcal{N}^{-1}A'_n]| \leq \frac{1}{\det[\mathcal{N}]} = \sqrt{\prod_{i < N} \left(\sum_{k < N} |\langle i|k' \rangle|^2 \right)}. \quad (678)$$

Rewriting this by exponentiating, we have

$$|\langle 0|0' \rangle| = \exp \left[\frac{1}{2} \sum_{i < N} \ln \left(1 - \sum_{k > N} |\langle i|k' \rangle| \right) \right] \approx \exp \left[-\frac{1}{2} \sum_{i < N < k} |\langle i|k' \rangle| \right], \quad (679)$$

where we have used a resolution of the identity in the first step, and in the second step we have assumed that the weight of an unperturbed state below the Fermi level is mostly contained within perturbed states that are also below the fermi level.

We'll generically end up getting a divergence in the sum in the exponent coming from the extensive number of states that are occupied below the Fermi level. To get a rough idea of how this works, we can estimate the overlap in perturbation theory. Running perturbation theory on the Schrodinger equation to get the first order correction to the wavefunction gives, in the usual fashion,

$$\langle i|k' \rangle = \frac{|\langle i|V|k' \rangle|}{|E_i - E_k|}. \quad (680)$$

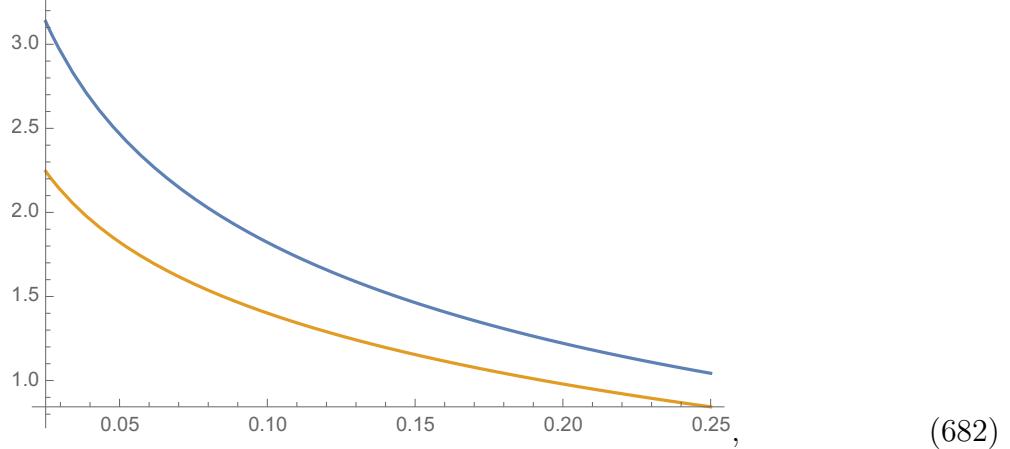
Therefore we have

$$\ln [|\langle 0|0' \rangle|^2] \approx - \int_{-\Lambda}^{-v_F/L} dE_1 \int_{v_F/L}^{\Lambda} dE_2 \rho(E_1) \rho(E_2) \frac{|V_{12}|}{|E_1 - E_2|}, \quad (681)$$

where $\rho(E)$ is the density of states, the Fermi level is at $E = 0$, and the band extends from $-\Lambda$ to Λ . Here the integral has been regulated by assuming that the system is in a finite-sized L^d box, so that the energy levels only ever get within $v_F \delta k \sim v_F/L$ of the Fermi surface.

To get the right philosophical take on this, it will suffice to take the density of states to be approximately constant (taking the Fermi energy to be much larger than the bandwidth, basically); in practice, any rather smooth choice for $\rho(E)$ is good enough for our conclusions to hold. Furthermore, we can take V_{12} to just be a constant—in reality a slightly better choice would be $V_{12} \sim e^{-\alpha|E_1 - E_2|}$, but this isn't really important: the thing we're really interested in here is the divergence in the system size, which is entirely determined by the $E_1 \rightarrow E_2 \rightarrow 0$ part of the integral, i.e. by the modes right near the FS. Choosing V_{12} to be an exponential or to be a constant doesn't really affect the behavior of this logarithmic divergence—for example, choosing an exponential or a constant produces curves for the integral like the two

in the following plot (the y-axis is the integral, the x-axis is L^{-1})



and so the characteristic behavior of the divergence as $L \rightarrow \infty$ doesn't really care about the exact choice of V_{12} .

Anyway, the point of this is that the integral over $dE_1 dE_2$ above has a logarithmic divergence, and so we get

$$|\langle 0|0' \rangle|^2 \sim (L\Lambda/v_F)^{-\alpha\Lambda|V_{12}|\rho^2}, \quad (683)$$

with $\alpha > 0$ some numerical constant. The point of this is that the overlap is power-law suppressed in the system size!⁵⁴ This is the orthogonality “catastrophe”, since it means that generically a weak perturbation will produce a ground state which is orthogonal to the original state in the thermodynamic limit.

Is this that surprising? After all, Hilbert space is enormous. To think about this, let us analyze the degree of orthogonality. Suppose the effect of the perturbation was to rotate the ground state to some randomly chosen vector in the Hilbert space \mathcal{H} . If we write $|0\rangle$ in a basis such that it is the vector $|1, 0, \dots\rangle$, then if $|0'\rangle$ were random, we'd have typically something like $|0'\rangle \sim \frac{1}{\sqrt{\dim \mathcal{H}}}|1, 1, \dots, 1\rangle$, giving $|\langle 0|0' \rangle|^2 \sim (\dim \mathcal{H})^{-1}$. Now $\dim \mathcal{H} \sim 2^{L^d} \gg L^{\alpha\Lambda|V_{12}|\rho^2}$, and so the overlap would be *much* smaller than the power-law-suppressed overlap we derived above. Therefore the L_{ity} catastrophe is telling us that the new state $|0'\rangle$ is nearly orthogonal, but not to the same extent that two randomly chosen vectors in \mathcal{H} are—the perturbation moves us around in \mathcal{H} space “further” than we might have expected, but as far as the full Hilbert space is concerned, it barely moves us around at all.

Additionally, one cool consequence of the L_{ity} catastrophe is that when an electron is added into a typical FL, it takes the system “a long time” to “absorb” the electron into the new many-body ground state. This is just by Fermi's golden rule: the matrix element between the state with the bare electron added ($|0'\rangle$) and the actual many-body ground state ($|0\rangle$) is very small, and hence the transition amplitude is similarly tiny. This produces “contact resistance” in metals, which is a resistance that comes from the “absorbability” of electrons into the FL, rather than from transport properties of the FL per se. Manifestations of the L_{ity} catastrophe also apparently can be seen in x-ray photoemission experiments: one bombards a metal with x-rays that are designed to eject electrons lying in filled shells below

⁵⁴The suppression when $\Lambda \rightarrow \infty$ isn't important—we're keeping Λ fixed (not sending it to infinity), and in any case it doesn't change as the thermodynamic limit is taken.

the conduction band. However, ejecting such an electron creates a local positive charge, which acts as a perturbation of the form just discussed to the surrounding conduction Fermi sea. The \perp ity catastrophe makes the matrix element between the “electron has just been ejected” state and the true ground state of the system very small, which suppresses the emission process, so that the intensity of the ejected electrons turns on more slowly than expected.



Real-space correlators for generically shaped Fermi surfaces at $T = 0$

Today we’re doing a calculation that I “needed” to do for research. Our goal is to calculate the real-space 2-point function for a Fermi liquid with a generically-shaped Fermi surface. We will work at $T = 0$ and stay in two dimensions for now (I might come back and do three dimensions later).



First, let’s derive the general expression for the correlator as an integral in \mathbf{k} space in a rather overly-verbose but instructive way.

We are interested in the time-ordered correlator; hence in \mathbf{k} space, (not really sure why I defined it with the c acting first—oh well)

$$G(\mathbf{k}, t) = i\langle T c_{\mathbf{k}}^\dagger(0) c_{\mathbf{k}}(t) \rangle = i [\theta(-t)\theta(\mathbf{k} \in FS) - \theta(t)\theta(\mathbf{k} \in FS^c)] e^{-iE_{\mathbf{k}}t}, \quad (684)$$

where FS^c is the compliment of the Fermi sea,⁵⁵ and we have used the correct sign convention $c_{\mathbf{k}}(t) = e^{iH_{\mathbf{k}}t} c_{\mathbf{k}}(0) e^{-iH_{\mathbf{k}}t}$. We can then take the Fourier transform: with the sign convention fixed from a few diary entries back, and $\eta = 0^+$,

$$\begin{aligned} G(\mathbf{k}, \omega) &= i \int_{\mathbb{R}} dt e^{i(\omega - E_{\mathbf{k}})t - |t|\eta} (\theta(-t)\theta(\mathbf{k} \in FS) - \theta(t)\theta(\mathbf{k} \in FS^c)) \\ &= i \frac{\theta(\mathbf{k} \in FS)}{i(\omega - E_{\mathbf{k}}) + \eta} + i \frac{\theta(\mathbf{k} \in FS^c)}{i(\omega - E_{\mathbf{k}}) - \eta} \\ &= \frac{1}{\omega - E_{\mathbf{k}} + i\eta_{\mathbf{k}}}, \quad \eta_{\mathbf{k}} = \theta(\mathbf{k} \in FS)0^- + \theta(\mathbf{k} \in FS^c)0^+. \end{aligned} \quad (685)$$

⁵⁵I just noticed a notational disaster that I’m too lazy to fix. When we write “FS” in upright font in the text, we mean “fermi surface”. However, when we write FS in equations / mathfont italicized, we mean “fermi sea”. Sorry!

The \mathbb{R} -space, \mathbb{R} -time correlator is then

$$G(t, \mathbf{x}) = \int_{\mathbb{R}} \frac{d\omega}{2\pi i} \int_{\mathbf{k}} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x} - |\omega|\delta} \frac{1}{\omega - E_{\mathbf{k}} + i\eta_{\mathbf{k}}}, \quad (686)$$

where $\delta = 0^+$ comes from the small amount of imaginary time evolution at infinity. If we take $t < 0$ (so that we're looking at hole propagation) then the contour must be closed in the UHP, and so we pick up only the poles where $\mathbf{k} \in FS$: if $\mathbf{k} \in FS^c$, the integral gives zero with this contour prescription. Therefore since the residue of the pole is just 1, we have

$$G(t, \mathbf{x}) = \int_{\mathbf{k}} e^{-iE_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}} \theta(\mathbf{k} \in FS). \quad (687)$$

We will be interested in what follows in the equal-time correlator, which is therefore just the Fourier transform of the Fermi-surface θ function:⁵⁶

$$G(0, \mathbf{x}) = \int_{\mathbf{k} \in FS} e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (692)$$

Note also that we would get the same answer (up to a sign) if we had integrated over the complement of the Fermi sea, since $\int_{\mathbf{k} \in BZ} e^{i\mathbf{k}\cdot\mathbf{x}}$ gives zero for $x \gg a$, with a the lattice spacing. This already tells us that the expression for $G(0, \mathbf{x})$ should be computable as an integral over the FSurface.

Our task is now to evaluate this for a generically shaped FS in two dimensions. We will work in the limiting regime where $|\mathbf{x}|k_F \gg 1$, where k_F is some average Fermi momentum.

⁵⁶We are interested in the equal-time correlator at various spatial separations because spatial correlations are dependent on the shape of the FS, while temporal correlation functions are not: $G(t, 0)$ only depends on the DOS near the FS. Indeed,

$$G(t, 0) = \int_{\omega, \mathbf{k}} e^{-i\omega t} \frac{1}{\omega - E_{\mathbf{k}} + i\eta_{\mathbf{k}}}. \quad (688)$$

The contour must be closed in the LHP (UHP) if $t > 0$ (if $t < 0$). If $t < 0$ then we get

$$G(t, 0) \sim \int_{\mathbf{k} \in FS} e^{-iE_{\mathbf{k}}t}. \quad (689)$$

Now at long times, the integral will fluctuate wildly for all values of \mathbf{k} that lie deep within the FSea, and so the dominant contributions will come from right near the FS. Then we may write

$$G(t, 0) \sim \rho(0) \int_{-\infty}^0 d\varepsilon e^{-i\varepsilon t}, \quad (690)$$

with $\rho(0)$ the DOS at the FS (or the angular average thereof, or similar). This means that (the integral is well-defined because of the η that we've been neglecting in the exponent)

$$G(t, 0) \sim \frac{\rho(0)}{t}, \quad (691)$$

and so the scaling of the correlator is $1/t$, regardless of the details of the FS geometry. If $t > 0$ so that the correlation function is particle-like, then we repeat the procedure above, but now integrate over the complement of the FSea. However, since the integral localizes to the FS at large t anyway, we have the same $\rho(0)/t$ behavior of both Greens functions.

Since Luttinger's theorem tells us V_{FS} is proportional to the electronic density, we know that $1/k_F$ is a measure of the average interparticle spacing—our limit is thus the limit in which the distance over which we are measuring the correlation is much larger than the microscopic average distance between electrons.

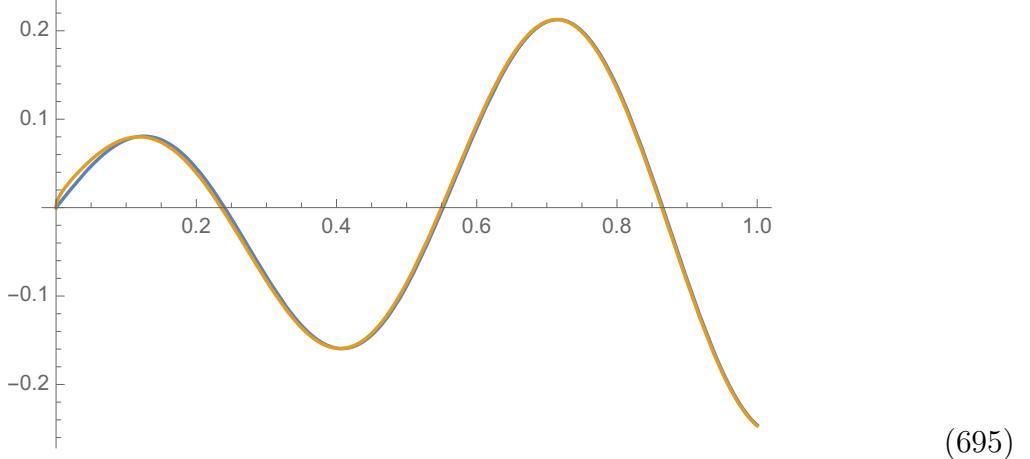
In our first calculation, we will restrict to a circular Fermi surface—this result will be useful in deriving the general case. For a circular FS, we have

$$G(0, \mathbf{x}) = \int_0^{K_f} dk k J_0[kx], \quad (693)$$

where $J_0[y]$ is a Bessel function. Since $kx \gg 1$ in most of the integration region, we may hope to approximate the Bessel function $J_0[y]$ by its large- y limit, viz.

$$kJ_0[kx] \approx k \sqrt{\frac{2}{\pi kx}} \cos(kx - \pi/4). \quad (694)$$

Indeed this is a pretty fantastic approximation; for e.g. $x = 10, k \in [0, 1]$, the two functions are



I won't tell you which is which. Therefore we have

$$G(0, x) \approx \int_0^{k_F} dk \sqrt{\frac{2k}{\pi x}} \cos(kx - \pi/4). \quad (696)$$

This integral can be done exactly in terms of Fresnel functions, plus an oscillating sin+cos term in a power-law envelope. It turns out that for $xk_F \gg 1$ the Fresnel function part is essentially zero (it goes as $1/x^2$ and has a coefficient that vanishes in the $xk_F \rightarrow \infty$ limit), with the remaining part being given by

$$G(0, x) \approx \sqrt{2k_F} \frac{\cos(k_F x - 3\pi/4)}{x^{3/2}}. \quad (697)$$

Sanity check: the dimension is $[G] = 2, \checkmark$. However, the square root of k_F means that the scaling dimension ethan: By this we just mean (half of) the power of x^{-1} when the oscillations from the $k_F x$ term are taken out; we are not trying to define any type of RG transformation at this point in time. is actually $3/4$, which is halfway between the case of $1/2$ that we would expect from 1+1D relativistic

fermions and that of 2+1D non-relativistic ones. For thinking about this, it might be helpful to compare to the case for 1+1D free fermions: if the Fermi points are at $\pm k_F$, then we get $G(0, x) \propto x^{-1} \cos(k_F x)$ (so that we indeed get scaling dimension 1/2).

Another sanity check: we get an oscillatory correlator at wavevector k_F , which is expected since singularities in \mathbf{k} space translate to oscillations of definite periods in \mathbb{R} space; the power-law envelope comes from the interference from singularities at different \mathbf{k} values.

Now we'll do the calculation for a generically-shaped FS. The key is to realize that the integral in the large xk_F limit localizes to only those areas of the FS whose normals are parallel or anti-parallel to \mathbf{x} . Indeed, let \mathbf{x}_{\parallel} and \mathbf{x}_{\perp} be the directions parallel and normal to \mathbf{x} , and consider doing the integral over the FS in thin rectangular slices of thickness δ , with each slice having a major axis which runs along the \mathbf{x}_{\perp} direction. The integrand is independent of the location along the slice in the \mathbf{x}_{\perp} direction, and so the contributions from two adjacent such slices will nearly cancel out in the integral, provided that the momentum-space scale on which the phase $e^{ik_{\parallel}x}$ oscillates, viz. $1/x$, is much less than the scale on which the size of the lengths of the rectangular slices changes. For a fixed δ , the momentum-space distance scale controlling how fast the length of the strips changes is set roughly by κ^{-1} , the inverse Gaussian curvature of the FS at the point at which the rectangle intersects the FS. Therefore, if we assume that at all points on the FS we have

$$1/x \ll 1/\kappa, \quad (698)$$

(which becomes $xk_F \gg 1$ for a circular Fermi surface), then the contributions from the majority of the rectangular strips cancels. The only strips which are un-canceled are those that are (nearly) tangent to the FS, and so the integral localizes to regions of the FS where the group velocity is \parallel to the chosen direction $\hat{\mathbf{x}}$ (which of course makes total sense from a physical point of view).

Note that this means that if there is no patch on the FS whose normal points along the $\hat{\mathbf{x}}$ direction, then there will be no such un-canceled rectangles, and we expect the correlator to decay exponentially (for this to happen we would require an open Fermi surface). Suppose however that this is not the case, and let \mathbf{k}_F denote a point on the FS whose normal is parallel to \mathbf{x} (so that \mathbf{k}_F is really a function of \mathbf{x} !). Now, for simplicity we will assume that there are two and only two points on the FS that meet this criterion, viz. $\pm \mathbf{k}_F$ (this is not the same as assuming the FS is circular!). If we assume the FS has a generic shape around these points (so that the curvature is not zero, but is still much smaller than $1/x$), then the integral will be dominated by contributions from the two regions around $\pm \mathbf{k}_F$, and we can assume that in these two regions the FS looks like part of a circle with curvature κ . Since the rest of the FS won't contribute to the integral in this limit, we can then instead equivalently do the calculation for a FS which is a circle of radius $1/\kappa$. All of this is illustrated in Figure 14.

Anyway, the point of this is that we can just use the circular FS calculation that we've already done. We need to replace k_F with $1/\kappa$ to account for the proper radius of the fake circle we're using, and we need to replace the $k_F x$ in the cosine with $\mathbf{k}_F \cdot \mathbf{x}$ (since \mathbf{k}_F as defined will generically not be parallel to \mathbf{x} ; an example is the right part of the figure); therefore the general result is

$$G(0, \mathbf{x}) \approx \sqrt{2\kappa^{-1}} \frac{\cos(\mathbf{k}_F \cdot \mathbf{x} - 3\pi/4)}{x^{3/2}}, \quad (699)$$

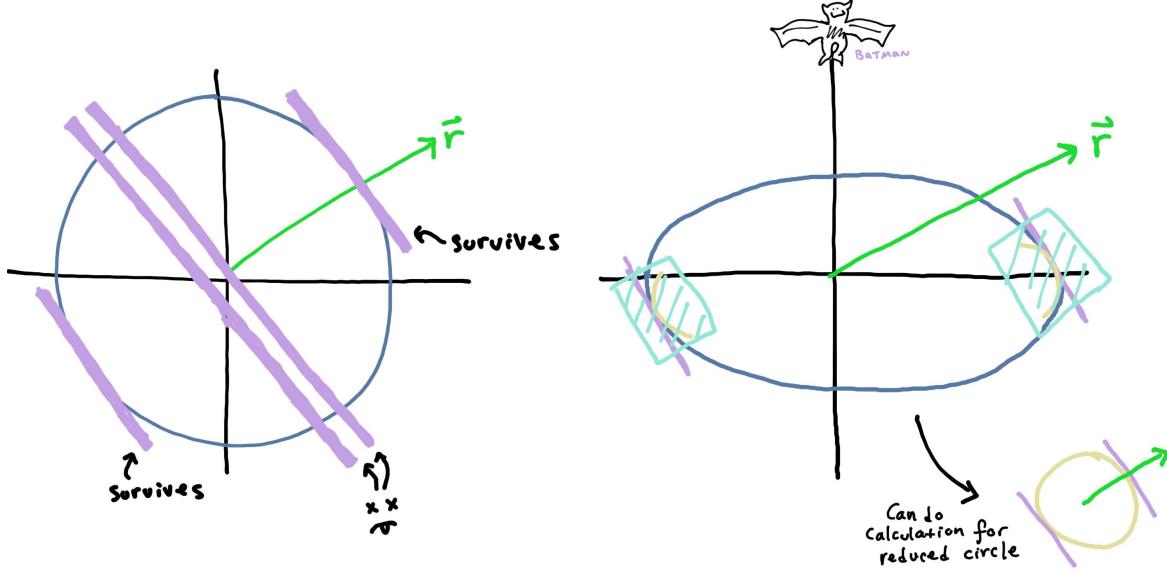


Figure 14: Left: doing the integral in rectangular bands (purple) for a circular FS. All but the bands that are tangent to the FS get canceled in the large xk_F limit. Right: what happens for a non-circular FS. Only the indicated rectangular regions contribute, and so for the purposes of doing the calculation we can pretend that the FS is circular, with a radius determined by the curvature κ at the two tangent points.

where again, both \mathbf{k}_F and κ are functions of \mathbf{x} .

Of course, the oscillatory behavior at the $2k_F$ wavevectors are Friedel oscillations, which evidently can occur just because of the Fermi degeneracy pressure—they don’t need interactions, even though they are often discussed in this context. In fact, one might take Friedel oscillations in correlators as the real definition of the FS, since the knowledge of these oscillations allows one to reconstruct the form of the FS.

Localization to the Fermi surface and the 1+1D nature of the FL

Since the calculation of the correlator localized to just narrow tangential strips around certain parts of the FS and not on anything that goes on inside the FSea, the result for the correlator is unchanged when we take $t > 0$, which corresponds to particle, rather than hole propagation, and consequently involves an integral over FS^c instead of FS . In fact this statement doesn’t depend on this splitting into rectangles argument, because of the argument we made at the beginning: the correlator at distances much larger than the lattice spacing must be computable as an integral over *only* the Fermi surface. Another way of saying this is that at long distances, the FL possesses particle-hole symmetry.

As a test of this, we now show that $G(0, \mathbf{x})$ can be reproduced by approximating the FL as a bunch of 1+1D Dirac fermions. In 1+1D this is obvious: we have

$$G(0, x) = \langle (e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x)) (\psi_R^\dagger(0) + \psi_L^\dagger(0)) \rangle \propto e^{ik_F x} \frac{1}{i2x} + e^{-ik_F x} \frac{1}{-2ix} = \frac{\sin(k_F x)}{x}, \quad (700)$$

which is exactly the right answer, even though we ignored all the details of the band structure except for k_F . In 2+1D we write the UV fermions as (still assuming a circular FS for simplicity)

$$c(x) = \int_0^{2\pi} d\gamma e^{ik_F \gamma \cdot \mathbf{x}} \psi_\gamma(\mathbf{x}), \quad (701)$$

where γ is a unit vector and all of the $\psi_\gamma(\mathbf{x})$ s are fermions with a Dirac dispersion along γ .⁵⁷ Then

$$G(0, \mathbf{x}) = G(0, x) = i \int_0^{2\pi} d\gamma e^{ik_F x \cos \gamma} \frac{1}{x \cos \gamma}. \quad (702)$$

The $\cos \gamma$ in the denominator is a bit troubling since the integral naively diverges—the phase is oscillating quickly in the exponent, but stops oscillating at the angle $\gamma = \pi$ where the integrand is singular. But in fact the oscillation of the phase is still sufficient to render the integral finite: not paying attention to factors of 2π and such,

$$G(0, x) = \int_{k_F}^{\infty} dk' \int_0^{2\pi} d\gamma e^{ik' \cos \gamma - k' 0^+} = \int_{k_F}^{\infty} dk' J_0[k' x] e^{-k' 0^+}. \quad (703)$$

Since $k' x \gg 1$ everywhere in the integration region, we can expand $J_0[k' x]$ about $k' x = \infty$, as above. The integral can then be done exactly in terms of gamma functions like $\Gamma(1/2, \pm ik_F x)$, which when expanded at large $k_F x$ give exactly the same $\cos(k_F x - 3\pi/4)/x^{3/2}$ answer as before.



Low-energy action for a Fermi patch and the curvature of the FS

Today we're doing a quasi-trivial math problem that came up in the context of figuring out the low-energy effective action for a certain patch of a two-dimensional Fermi surface. In several papers (I saw it first in Max and Sachdev's paper from 2010), people write the the low-energy Hamiltonian for the patch as

$$H \approx \int \psi^\dagger \left(s v_F (-i \partial_x) + \frac{v_F k}{2} (-i \partial_y)^2 \right) \psi, \quad (704)$$

where k is some constant related to the curvature of the Fermi surface at the patch, and where the coordinates are such that x is normal to the FS at the patch and y is tangential (and where s is the sign of the Fermi velocity). Today we will show that k is in fact exactly the (signed) FS curvature.

⁵⁷For the non-circular case, they disperse along the direction normal to the FS at $\mathbf{k}_F(\gamma)$.



Let's start with some elementary math facts about the (extrinsic) curvature of a parametric curve $\gamma = (x(s), y(s)) \subset \mathbb{R}^2$, where s is the arc length along the curve. First for the definition: let \mathbf{v} be the unit tangent vector to γ . Since it has unit length at every point on γ , the only way it can change from point to point on γ is by rotating in place; therefore its t derivative \mathbf{v}' must be normal to \mathbf{v} , and hence proportional to the unit normal to the curve, \mathbf{n} . The constant of proportionality is the (signed) curvature:

$$\mathbf{v}'(s) = \kappa(t)\mathbf{n}(s), \quad (705)$$

where $\kappa = \pm\kappa$ is positive or negative depending on how we orient the unit normal. Note here that it is important for the parametrization to be in terms of s , i.e. such that equal amount of arc length is swept out in equal times. Generic parametrizations don't have this property, so we have to be careful in applying the above formula (i.e. parametrizing an ellipse with $\mathbf{v} = (-a \sin s, b \cos s)$ doesn't work).

Unfortunately, most of the time using a parametrization in terms of the arc length is annoying. If we have a generic parametrization $\gamma = (x(t), y(t))$, we will instead need to use the formula

$$\mathbf{v}'(t) = \kappa|\gamma'(t)|\mathbf{n}(t). \quad (706)$$

Here \mathbf{v} is still a normalized unit tangent vector, viz. $\mathbf{v} = (x'(t), y'(t))/\sqrt{x'^2 + y'^2} = \gamma'/|\gamma'|$. The extra factor here just comes from converting the arc length derivative to a t derivative, and using $ds/dt = |\gamma'(t)|$.⁵⁸

First let's find the curvature at an arbitrary point. The unit tangent vector to the curve is

$$\mathbf{v} = \frac{(x', y')}{\sqrt{x'^2 + y'^2}}. \quad (707)$$

After some algebra, taking the derivative gives

$$\mathbf{v}' = \frac{(-y'(x''y' - y''x'), x'(x''y' - y''x'))}{(x'^2 + y'^2)^{3/2}} = \frac{x''y' - y''x'}{x'^2 + y'^2}\mathbf{n}. \quad (708)$$

This then tells us that the curvature is

$$\frac{|\mathbf{v}'|}{|\gamma'|} = \kappa = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}. \quad (709)$$

In a region of the curve where we can choose a parametrization of one coordinate by the other, e.g. if we take $x = x(y)$, then we just get

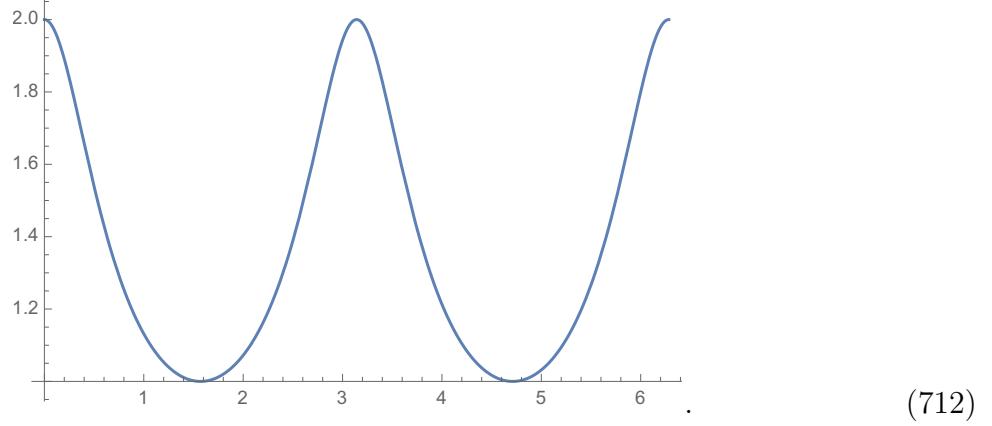
$$\kappa(x(y)) = \frac{|\partial_y^2 x|}{(1 + (\partial_y x)^2)^{3/2}}. \quad (710)$$

⁵⁸Sanity check: giving arc length dimensions of length means that κ has dimensions of length⁻¹ (as expected), and since the derivative in the above definition is with respect to a (thought of as dimensionless) parametrization time t instead of with respect to a length s , the extra factor of $|\gamma'|$ is needed to get the dimensions straight.

For example, let $x = a \cos t, y = b \sin t$ parametrize an ellipse. Then

$$\kappa(t) = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \quad (711)$$

is the curvature at an angle θ . Making the plot $\kappa(t)$ confirms this: for $a = 2, b = 1$,



Now we will go back to the problem about Fermi liquids. Consider the Fermi surface condition $\varepsilon(x, y) = \varepsilon_F$ (here x and y are momentum for simplicity; sorry). We will only be interested in a quadratic expansion of the dispersion near the Fermi level at a given point, and so near some given point (x_*, y_*) , at fixed energy ε_F we must have

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = \varepsilon_F, \quad (713)$$

where now x, y are coordinates relative to (x_*, y_*) . We will of course find it convenient to set up the coordinates so that $(x_*, y_*) = 0$; this means that the coordinate $(x, y) = (0, 0)$ must be on the FS, i.e. it must solve the above equation. This tells us that $F = \varepsilon_F$. Our freedom to rotate the coordinate system means that we can take the y axis to be a level curve of $\varepsilon(x, y)$ at the origin; this constraint gives $E = 0$. So, we are left with finding the curvature at the origin of the implicit curve

$$Ax^2 + By^2 + Cxy + Dx = 0. \quad (714)$$

This is easy enough to do using the above discussion. We obtain the rather simple⁵⁹

$$\kappa(0, 0) = |2B/D|. \quad (717)$$

Note that here, and in what follows, we will be assuming that the curvature does not vanish, so that the y axis is tangent to the FS at the origin, but that the y axis and the FS differ at quadratic order in the coordinates.

⁵⁹Just for posterity's sake, if we had instead

$$Ax^2 + By^2 + Cxy + Dx = E, \quad (715)$$

then we'd get

$$\kappa(0, 0) = \text{abs} \left(\frac{2(-2C^2E + B(D^2 + 4AE))}{(D^2 + 4AE)^{3/2}} \right). \quad (716)$$

The low energy Hamiltonian we're interested in is

$$H = \int \psi^\dagger (D(-i\partial_x) + B(-i\partial_y)^2) \psi, \quad (718)$$

where we are dropping the $\partial_x^2, \partial_x \partial_y$ terms (they *do* exist, but when doing RG we scale k_x and k_y differently so that such terms are irrelevant).

Now the Fermi velocity at the point of interest is $\mathbf{v}_F = \nabla \varepsilon|_0 = (D, 0) = (sv_F, 0)$ with $s = \text{sgn}(\mathbf{v}_F)$ tells us whether the part of the x -axis in \mathbf{k} space at small negative x is filled with electrons or with holes. Also, it turns out that $\text{sgn}(B) = \text{sgn}(\kappa)$, so that we may in fact write $B = \frac{1}{2}\kappa|D| = (\kappa v_F)/2$. Therefore we may write the above as

$$H = \int \psi^\dagger \left(sv_F(-i\partial_x) + \frac{v_F \kappa}{2} (-i\partial_y)^2 \right) \psi, \quad (719)$$

which is what we wanted to show. Note that unlike for systems of bosons, for fermions we are allowed to consider Hamiltonians with negative energies. Therefore both signs of κ are admissible.

The last thing we'll do is justify $\text{sgn}(B) = \text{sgn}(\kappa)$, which actually requires a non-zero amount of thinking about geometry. First, note that the interior of the FS can always be defined through requiring that the following hold: near a minimum of the surface $\varepsilon(x, y) \subset \mathbb{R}^3$, if we choose the convention that the unit normal to the surface ε points *out from* the FSea, then both principal curvatures of ε at the minimum are *positive*. Likewise, near a local maximum in ε (assuming the local maximum occurs at $\varepsilon_* < \varepsilon_F$), the principal curvatures are both negative.⁶⁰

Anyway, the FS is of course defined as the intersection of the ε surface with the xy plane (setting $\varepsilon_F = 0$ and always translating so that the FS point p we're interested lies at the origin, wolog), and using the above we can always determine an orientation for it. From the above discussion, we see that the sign of the FS curvature κ at p is determined as follows: if the vector pointing from p to the center of the circle defined by the curvature of the FS at p is such that it points outside the FS, then $\kappa < 0$, while if it points into the FS, then $\kappa > 0$.

What about the sign of B ? Now κ is the curvature of the curve $\gamma = \varepsilon \cap P_{xy}$, with P_{xy} the xy plane. Similarly, consider the curvature $\tilde{\kappa}$ of $\tilde{\gamma} = \varepsilon \cap P_{yz}$, where z is the energy (ε) direction. The sign of $\tilde{\kappa}$ is determined by whether or not the vector pointing to the center of the circle defined by the curvature of $\tilde{\gamma}$ points up along \hat{z} (in which case $\tilde{\kappa} > 0$) or down along $-\hat{z}$ (in which case $\tilde{\kappa} < 0$). Now the curve $\tilde{\gamma}$ is obtained by taking our above parametrization of the dispersion ε and setting $x = 0$: this gives the parabola $\tilde{\gamma} = (y, By^2)$

$$\tilde{\kappa} = 2B. \quad (720)$$

Finally, we claim that

$$\text{sgn}(\kappa) = \text{sgn}(\tilde{\kappa}) \implies \text{sgn}(\kappa) = \text{sgn}(B). \quad (721)$$

⁶⁰This is possible since ε is a well-defined function—if the FSea were somehow allowed to be diffeomorphic to S^2 , then both conditions on the principal curvatures couldn't hold simultaneously.

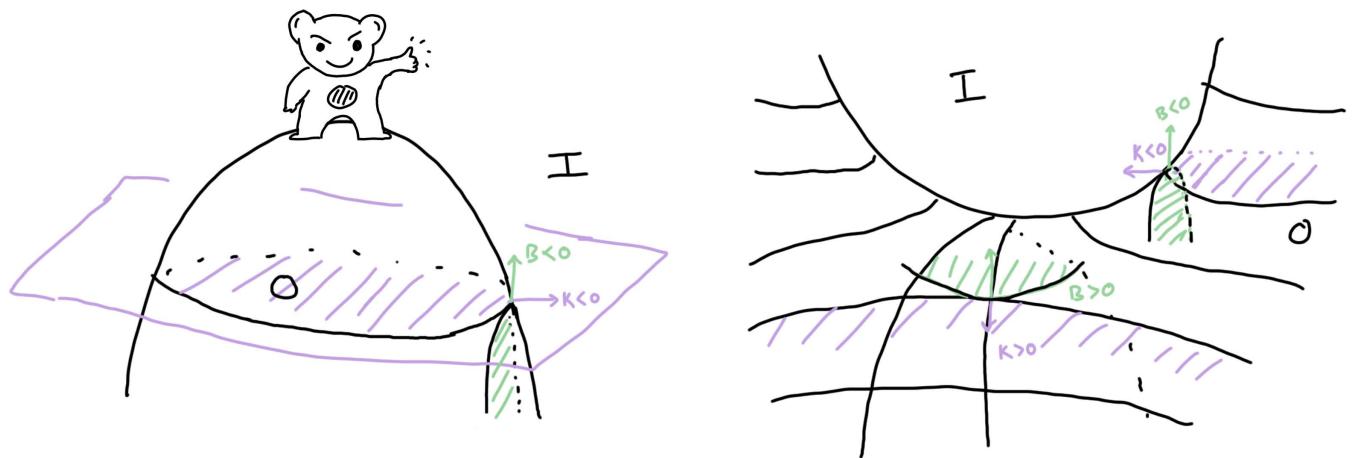


Figure 15: Some pictures to help with the visualization. The purple shaded regions indicate various xy planes that are intersected with the ε surface; I is placed in regions that can be filled up in a FSea, while O is placed in regions that are never part of a FS. The purple vectors are drawn to be pointing into the FS, and the signs of the curvatures are indicated. The green hatched regions are various yz planes that are intersected with the ε surface. From the picture it should be clear why B has the same sign as the curvature of the intersections of these planes with the ε surface, and why B and κ have the same sign.

To test out this claim, one can just draw pictures near the three different types of critical points and convince oneself that it is true; see Figure 15. More formally, consider rotating the P_{xy} plane into the P_{yz} plane by performing a rotation about the y axis. The rotation is a $+\pi/2$ rotation, with the handedness determined by requiring that the unit vector $-\mathbf{n}$ pointing *into* the FSea (viz. $-\mathbf{n} = -\mathbf{v}_F/|\mathbf{v}_F|$) rotates into $+\hat{z}$. Since $\varepsilon(x, y)$ is a well-defined function, this rotation can always be done without having the rotated image of $-\mathbf{n}$ be tangent to ε at any point (this is hard to describe in words; drawing pictures helps). Now, define the curvature vector \mathbf{k} to be such that $|\mathbf{k}| = \kappa$, and such that \mathbf{k} points towards the center of the circle defined by the curvature (the sign of $\mathbf{k} \cdot \mathbf{v}_F$ determines the sign of κ , and the sign of $\tilde{\mathbf{k}} \cdot \hat{z}$ determines the sign of $\tilde{\kappa}$). The claim is that the rotated image of \mathbf{k} (at each angle θ along the rotation, \mathbf{k} is defined as the curvature vector of $\varepsilon \cap R_\theta[P_{xy}]$) is parallel to $\tilde{\mathbf{k}}$. Since \mathbf{k} is either parallel or anti-parallel to \mathbf{n} , \mathbf{k} is never tangent to ε during the rotation (it stays “on the same side of ε during the whole rotation”), and so if this claim is true, then the signs of $\tilde{\kappa}$ and κ must agree. The claim that \mathbf{k} rotates to something \parallel to $\tilde{\mathbf{k}}$ is true for the following reason: if it weren’t true, then there would be some point along the rotation where \mathbf{k} vanished. This would mean that we could find a plane $R_\theta[P_{xy}]$ such that the curvature of $\varepsilon \cap R_\theta[P_{xy}]$ vanished; this in turn would mean that the y axis would have to meet ε degenerately, which would necessitate that the curvature of $\varepsilon \cap P_{xy}$ vanish, contrary to our assumption.

Phew! That was a lot of words. Hopefully looking at the figure helps; this is really a pretty visually intuitive statement, it’s just writing it down that takes a lot of work.



Dispersion of excitations and behavior of Wilson lines in \mathbb{Z}_2 gauge theory

Consider \mathbb{Z}_2 gauge theory on a square lattice in two dimensions:

$$H = -K \sum_{\square} \prod_{l \in \square} Z_l - g \sum_l X_l. \quad (722)$$

We will first diagnose the two phases (weak coupling and strong coupling, where “coupling” refers to g , viz. the coefficient of the kinetic term $\sum_l X_l$) by computing the scaling of the expectation value of the Wilson line $W_\gamma = \prod_{l \in \gamma} Z_l$ in the ground state. Next, we will find the dispersion of the lowest-energy excitations to order g in the deconfined phase ($K \gg g$) and to order K^2 in the confined phase ($g \gg K$).



Diagnosing the two phases

In this section, we will work along the line $K = 1/g$, which helps make contact with continuum intuition and suffices for diagnosing the two phases. We want to compute the behavior of the ground-state expectation value $\langle W_\gamma \rangle$ at $g \gg 1$ and $g \ll 1$.

The confining case, namely $g \gg 1$, is straightforward. Here we perturb about the ground state $|P\rangle = \bigotimes |+\rangle$. Of course $\langle W_\gamma \rangle_P = 0$, and so we need to compute corrections to the ground state $|P\rangle$ as a function of $1/g$. Following the usual high-temperature-expansion arguments (here $T = g$), we see that the term of lowest order in $1/g$ that contributes is the one where a product of $g^{-1}B_\square$ operators fills up the interior of the Wilson line, canceling it out. Therefore we have

$$\langle W_\gamma \rangle \sim (1/g)^A = e^{-A \ln g}, \quad (723)$$

so that the Wilson line obeys an area law, befitting us calling this the confining phase.

The weak coupling $g \ll 1$ case is slightly more tricky. Here the $g = \infty$ ground state is created by acting on $|\uparrow\rangle$ with strings of X operators along paths on the dual lattice:⁶¹

$$|\text{绳}\rangle = \sum_{\gamma \in Z_1(L^\vee)} \prod_{l \in \gamma} X_l |\uparrow\rangle \quad (725)$$

In this ground state, $\langle \text{绳} | W_\gamma | \text{绳} \rangle = 1$ for all γ ($|\text{绳}\rangle$ is just a superposition of all flat gauge field configs). Doing the appropriate high-temperature expansion (computing an expectation value of stuff with powers of $\sum_l X_l$ in the state $|\text{绳}\rangle$) doesn't get us anywhere, since the terms that keep us within the low-energy subspace are just string operators $\prod_{l \in \gamma} X_l$ with $\gamma \in Z_1(L^\vee)$, which don't affect the expectation value of the Wilson line since the Wilson line being topologically trivial means all such loops will have zero signed intersection number with it. Therefore to get corrections to the expectation value we need to look at perturbations to the ground state wavefunction.

The effects of the nonzero line tension when $g > 0$ on the wavefunction appear at first order in perturbation theory: the perturbed ground state is⁶²

$$|\text{绳}\rangle = |\text{绳}\rangle + \frac{g^2}{2} \sum_l X_l |\text{绳}\rangle + \dots, \quad (727)$$

where \dots are higher order in g^2 . The $O(g^2)$ term comes from all possible ways to insert two flux excitations on neighboring plaquettes. The terms higher order in g^2 come from fluxes

⁶¹This is just a pretentious way of writing the manifestly gauge-invariant state

$$|\text{绳}\rangle = \prod_v \mathcal{P}_v |\uparrow\rangle, \quad (724)$$

where $\mathcal{P}_v = \mathbf{1} + \prod_{l \in \partial_v} X_l$ is the projector onto the gauge-invariant subspace.

⁶²This just comes from the fact that the first-order correction to the wavefunction is

$$\delta |\text{绳}\rangle = \sum_{\{2f\}} \frac{\langle 2f | g \sum_{l'} X_{l'} | \text{绳} \rangle}{2g^{-1}} |2f\rangle, \quad (726)$$

where $\{2f\}$ stands for a sum over all configurations with two fluxes (eigenstates of the un-perturbed tensionless Hamiltonian), with $2g^{-1}$ being the energy of each two-flux configuration. Since all the two-flux terms that survive come from acting with X_l for some l on $|\text{绳}\rangle$, we can re-write this as in (727).

that live further apart from one another, as well as from states with more than two fluxes. The combinatorics for these contributions is hard, but we only care that the corrections are a series in powers $g^{n>2}$. To lowest order then, after canceling out the vacuum terms from the denominator, we just get a single term for each link in the path γ :

$$\frac{\langle \text{W} | W_\gamma | \text{W} \rangle}{\langle \text{W} | \text{W} \rangle} \sim 1 + g^4 \langle \text{绳} | \sum_{l \in \gamma} X_l W_\gamma X_l | \text{绳} \rangle + \dots \quad (728)$$

This means that the dominant scaling behavior is

$$\langle W_\gamma \rangle \sim e^{-(g^2)|\gamma|}, \quad (729)$$

where (g^2) is a polynomial vanishing smoothly when $g \rightarrow 0$. This gives us the perimeter law expected in the deconfined phase.

Dispersion of excitations

First consider the deconfined phase, where the minimal excitations are magnetic fluxes which violate a single Wilson-loop plaquette term. The electric field operator X_l creates a pair of magnetic fluxes on the two plaquettes whose boundaries include l , and so the electric operators are responsible for giving the fluxes dynamics. Let $|r\rangle$ represent a state with a π flux at plaquette r (of course, technically the fluxes can only come in pairs), and let $|k\rangle$ denote the state where the π flux is delocalized in a wave of momentum k . Then to first order, the energy relative to the ground state is

$$E_k = \langle k | -K \sum_{\square} \prod_{l \in \square} Z_l - g \sum_l X_l | k \rangle = \frac{1}{V} \sum_{r,s} \langle r | 2K \delta_{r,s} - e^{ik(r-s)} g \sum_l X_l | s \rangle. \quad (730)$$

At first order in g , the only hopping processes which survive are those that move the flux excitation to an adjacent plaquette, and so

$$E_k = 2K - 2g(\cos(k_x) + \cos(k_y)). \quad (731)$$

Now for the trickier confined case. It is easiest to get the dispersion by first mapping to the dual transverse field Ising model (technically this mapping only works locally), with the operator mapping

$$\prod_{l \in \square} Z_l \mapsto \mathcal{X}_i, \quad X_{\langle ij \rangle} \mapsto \mathcal{Z}_i \mathcal{Z}_j, \quad (732)$$

so that

$$H = -g \sum_{\langle ij \rangle} \mathcal{Z}_i \mathcal{Z}_j - K \sum_i \mathcal{X}_i. \quad (733)$$

Let $|i\rangle$ denote a state with a spin flip (in the \mathcal{Z} basis) at site i , which are the minimal excitations of the dual Ising model when $g \gg K$. In the gauge theory, a spin flip maps to a configuration where the electric field terms $-gX_l$ on all the links dual to the links emanating from i are violated — this corresponds to a small electric flux loop encircling the dual vertex i . Note that we couldn't have a smaller excitation that e.g. just violated a single $-gX_l$

term on one link, since that would violate gauge invariance (the electric flux lines, which are tensionfull and constitute the excitations in the system, must form closed loops).

Anyway, let's find the dispersion. Let $|r\rangle$ denote the state in the Ising model with a spin-flip at site r (choosing the spin flip to be measured relative to the global vacuum state $\otimes|\uparrow\rangle$), let $|r, s\rangle$ be the state with spin-flips at sites r and s , and so on. We find the dispersion using the effective Hamiltonian, which is designed to be an effective Hamiltonian to a given order in K/g for the eigenstates $|0\rangle$, $|r\rangle$, $|r, s\rangle$, etc. of the $K = 0$ Hamiltonian. To $O(K^2/g)$, the diagonal pieces are (after subtracting off the ground-state energy)

$$\begin{aligned}\langle r|H_{eff}|r\rangle &= \langle r|H|r\rangle + \langle r|H|0\rangle\langle 0|H|r\rangle \frac{1}{E_r - E_0} + \sum_s \langle r|H|r, s\rangle\langle r, s|H|r\rangle \frac{1}{E_r - E_{r,s}}, \\ &= -2Vg + 4g + \frac{K^2}{4g} + \frac{K^2(V-5)}{4g-8g} + \frac{4K^2}{4g-6g} \\ &= -(2g + K^2/4g)V + 4g - 2K^2/4g.\end{aligned}\tag{734}$$

where V is the number of sites and $2V$ is the number of links. On the other hand, if $r \neq s$ then

$$\langle r|H_{eff}|s\rangle = \frac{1}{2}\langle r|H|0\rangle\langle 0|H|s\rangle \left(\frac{1}{E_r - E_0} + \frac{1}{E_s - E_0} \right) + \frac{1}{2}\langle r|H|r, s\rangle\langle r, s|H|s\rangle \left(\frac{1}{E_r - E_{r,s}} + \frac{1}{E_s - E_{r,s}} \right).\tag{735}$$

If r and s are not nearest neighbors, then the first term in parenthesis is $1/2g^2$ while the second is $-1/2g$, and so we get zero. If r and s are nearest neighbors, then the second term is instead $-1/g$, and so we get

$$\langle r|H_{eff}|s\rangle = \left(\frac{K^2}{4g} - \frac{K^2}{2g} \right) \delta_{\langle rs \rangle} = -\frac{K^2}{4g} \delta_{\langle rs \rangle},\tag{736}$$

where $\delta_{\langle rs \rangle}$ is 1 if r, s are nearest neighbors and zero else.

Now we can get the dispersion by going to Fourier space. We see that, dropping the volume-dependent part and the ground state energy $-Vg$ of the $K = 0$ Hamiltonian, the energy of a single spin flip is

$$E_\downarrow(k) = 4g - \frac{K^2}{2g} (2 + \cos(k_x) + \cos(k_y)).\tag{737}$$

After un-doing the duality we see that this represents the dispersion of a small loop of electric flux, which is what we wanted to find.



Polarization bubbles and self-energies for patch fermions

Today we're basically just writing down the details for a calculation that I've seen mentioned in a few papers: we'll be computing the 1-loop self-energies for a theory of a (2+1)D FL coupled to a $U(1)$ gauge field, working within the patch approximation.



We'll be working in a patch approach, where we zoom in on just a single FS patch, and expand the dispersion so that in patch coordinates we have $\varepsilon(\mathbf{k}) = v_F k_x + \frac{\kappa v_F}{2} k_y^2 \equiv ak_x + bk_y^2$, where κ is the (signed!) FS curvature and where we are working to quadratic order in k_y , but linear order in k_x (the Fermi velocity is $\mathbf{v}_F = sv_F \hat{x}$, with $s = \pm 1$). Fermions are easiest to move around by giving them momentum in the k_y direction, and so a gauge mode with momentum \mathbf{q} will most readily couple to the patch of FS for which $k_y \parallel \mathbf{q}$; this is the justification for focusing on a single patch.

First we'll compute the 1-loop polarization bubble correction to the gauge propagator. In \mathbb{R} time, the fermion propagator is

$$G(\omega, \mathbf{k}) = \frac{1}{\omega - \varepsilon_{\mathbf{k}} + i\eta_{\mathbf{k}}}, \quad (738)$$

where $\eta_{\mathbf{k}} = \pm 0^+$, with the sign being positive if \mathbf{k} is outside the FS, and negative otherwise. This sign issue makes integrals over components of \mathbf{k} relatively annoying to deal with, and so we will instead be working in $i\mathbb{R}$ time, since the analytic structure is simpler. In $i\mathbb{R}$ time, the loop integral relevant for computing the polarization bubble is

$$I = \int_{\omega p} \frac{1}{(-i(\omega + \omega_q) + a(q_x + p_x) + b(q_y + p_y)^2)(-i\omega + ap_x + bp_y^2)}. \quad (739)$$

We will do the integral over p_x first. We can close the contour either way, and so we will only get a nonzero answer when both poles are in opposite half planes, i.e. when $\text{sgn}(\omega) = -\text{sgn}(\omega + \omega_q)$. If we have $\omega > 0, \omega + \omega_q < 0$, then we get the pole on the second factor, and the residue is $a^{-1}(-i\omega_q + aq_x + 2bq_y p_y + bq_y^2)^{-1}$. On the other hand, if we have $\omega < 0, \omega + \omega_q > 0$, then we get a pole on the first factor, with a residue that's the negative of the one just mentioned. Therefore the integral is in fact proportional to $-(\text{sgn}(\omega) - \text{sgn}(\omega + \omega_q))/2$ (this projects onto the frequencies with poles in both half planes, and has the right sign due to the opposite signs of the residues), and so

$$I = -\frac{i}{2a} \int_{\omega p_y} \frac{\text{sgn}(\omega + \omega_q) - \text{sgn}(\omega)}{-i\omega_q + aq_x + 2bq_y p_y + bq_y^2}. \quad (740)$$

ω only appears in the numerator, and so (note that I think the analogous formula in chapter 18 of Sachdev has a typo vis-a-vis the absolute value bars?)

$$I = -i \frac{\omega_q}{2a(2\pi)} \int_{p_y} \frac{1}{-i\omega + aq_x + 2bq_y p_y + bq_y^2}. \quad (741)$$

This integral is then

$$I = -i \frac{\omega_q}{4(2\pi)^2 abq_y} \ln \left(\frac{\infty + i\omega_q/(2bq_y)}{-\infty + i\omega_q/(2bq_y)} \right). \quad (742)$$

If we take the branch cut of the logarithm to pass along the negative half of the \mathbb{R} axis, then the imaginary part of the log of the denominator is $i\pi \text{sgn}(\omega_q/(2bq_y))$, while the real parts of the logs of the numerator and denominator cancel. In the problem we're interested in, $a = sv_F$ while $b = \kappa v_F/2$ with κ the signed FS curvature, and so (don't trust the numerical prefactor)

$$I = \frac{|\omega_q|}{8\pi v_F^2 \kappa |q_y|} \equiv \Gamma \frac{|\omega_q|}{|q_y|}. \quad (743)$$

This gives a correction to the gauge field propagator that is singular at low momentum. The usual thing people then do, which to me still seems like a rather big leap, is to take the RPA approximation for the gauge propagator, and then feed that back into the action, subsequently using this new kinetic term to derive an RG scaling scheme. The point of doing this as far as I understand is as follows: in 2+1D, if the scaling dimensions of the FS fermions and the gauge field are fixed by the free fermion / gauge kinetic terms, then the scaling

$$k_y \mapsto \lambda k_y, \quad k_x \mapsto \lambda^2 k_x, \quad \omega \mapsto \lambda^2 \omega \quad (744)$$

for $\lambda < 1$ which arises during the restoration of the original cutoff scale (which is anisotropic so that the curvature remains invariant; recall the FS is defined locally as $k_x + \kappa v_F k_y^2/2 = 0$) means that⁶³

$$\psi \mapsto \lambda^{-7/2} \psi, \quad a \mapsto \lambda^{-7/2} a, \quad (745)$$

and so the term $\int_{k,q} \psi_k^\dagger a_{q-k} \psi_q$ scales as $\lambda^{10-21/2} = \lambda^{-1/2}$, which is relevant. We don't really know to what IR theory this theory will flow to under the flow of the $\bar{\psi}a\psi$ coupling. However, we can make a guess and hope that the results obtained from RPA get us part of the way to the eventual fixed point: the idea is that this RPA approximation can get us to a point that is closer to the true IR fixed point than the free theory is, so that the amount of flow that we'll have to do will be less and will hence be more well-controlled, by virtue of our shorter distance to the true interacting fixed point.

Anyway, RPA tells us that the effective gauge field propagator is (the sign comes from the minus sign from the fermion bubbles, meaning that the 1- and 2-bubble terms differ in sign)

$$G_a^{RPA}(q) = \frac{1}{G_0^{-1}(q) + I(q)} = \frac{1}{\omega^2 + \chi(q_y^2 + q_x^2) + \Gamma|\omega|/|q_y|}, \quad (746)$$

where $\chi > 0$ is some phenomenological parameter. We then feed this result back into the effective action, by changing the free gauge term to $S_a \sim \int [G_a^{RPA}(q)]^{-1} |a|^2$.

However, not all of the terms in G_a^{RPA} are on equal footing. The logic is the following: when we do RG, we will choose to use $\Gamma|\omega|/|q_y|$ and χq_y^2 as the free terms that define how the RG scaling is performed; the motivation for this being that q_y is the most relevant momentum direction for the gauge fields in the full theory since it controls how a interacts

⁶³We determine the scaling of a through the k_y^2 and ω^2 terms; the k_x^2 term will be irrelevant.

with the fermions, and because if $\Gamma|\omega|/|q_y|$ is preserved under RG then the curvature of the FS does not renormalize.

Anyway, if we let $a \mapsto \lambda^\rho a$ and $\omega \mapsto \lambda^\alpha \omega$, then the invariance of the two aforementioned terms means that

$$5 + 2\rho + \alpha = 0, \quad 2\alpha + 2 + 2\rho = 0 \implies \rho = -4, \alpha = +3. \quad (747)$$

With this scaling, one can then check that the $k_x^2|a|^2$ scales as λ^2 , and the $\omega^2|a|^2$ term scales as λ^4 ; hence both are irrelevant. If we then omit these terms, the gauge field appears in the action as

$$S \supset \int_{\mathbf{q}\omega} \left(\Gamma \frac{|\omega|}{|q_y|^2} + \chi |q_y|^2 \right) |a|^2. \quad (748)$$

Again, the hope is that the scaling rules derived from this free term will be closer to the scaling rules of the true IR fixed point, which is prohibitively far away from the free theory—thus this RPA result is supposed to act as a stepping stone on the way to deriving the scaling laws of the true IR fixed point. A nice reference where this action is employed is e.g. [11] (although in my understanding the detailed calculations there haven't really held up?).

Now let's calculate the 1-loop diagram involving a single gauge field line that appears in the fermion self energy. We will use the singular form of the gauge propagator that takes into account the fermion bubble, and so the relevant integral is

$$J = \int_p \frac{1}{[-i(\omega_q - \omega_p) + a(q_x - p_x) + b(q_y - p_y)^2] [\Gamma|\omega_p|/|p_y| + \chi p_y^2]}. \quad (749)$$

Now p_x only appears in the fermion propagator, and its integral yields a $\log \ln[(\infty + i(\omega_q - \omega_p))/(-\infty + i(\omega_q - \omega_p))]$, which gives us $-i\pi \text{sgn}(\omega_q - \omega_p)$. So

$$J = -i \frac{1}{2a} \int_{p_y \omega_p} \frac{\text{sgn}(\omega_p - \omega_q)|p_y|}{\Gamma|\omega_p| + \chi|p_y|^3}. \quad (750)$$

If $\omega_q > 0$, then we get

$$\begin{aligned} & - \int_{p_y} \frac{i|p_y|}{2\pi a \Gamma} \left(- \int_{-\infty}^0 \frac{1}{-\omega + \chi|p_y|^3/\Gamma} - \int_0^{\omega_q} \frac{1}{\omega + \chi|p_y|^3/\Gamma} + \int_{\omega_q}^{\infty} \frac{1}{\omega + \chi|p_y|^3/\Gamma} \right) \\ &= - \int_{p_y} \frac{i|p_y|}{2\pi a \Gamma} \left(\ln \left(\frac{-\chi|p_y|^3/\Gamma}{-\infty} \right) - \ln \left(\frac{\omega_q + \chi|p_y|^3/\Gamma}{\chi|p_y|^3/\Gamma} \right) + \ln \left(\frac{\infty}{\omega_q + \chi|p_y|^3/\Gamma} \right) \right) \\ &= \int_{p_y} \frac{i|p_y|}{a\pi \Gamma} \ln \left(1 + \frac{\omega_q}{\chi|p_y|^3/\Gamma} \right) \end{aligned} \quad (751)$$

A similar computation shows that if $\omega_q < 0$, we get a minus sign out front and a minus sign in front of the ω_q in the logarithm. Therefore

$$J = \int_{p_y} \text{sgn}(\omega_q) \frac{i|p_y|}{a\pi \Gamma} \ln \left(1 + \frac{|\omega_q|}{\chi|p_y|^3/\Gamma} \right). \quad (752)$$

We then use the integral

$$\int_0^\infty dx x \ln(1 + \alpha/x^3) = \frac{\alpha^{2/3}\pi}{\sqrt{3}} \quad (753)$$

to conclude that

$$J = \frac{i}{2\sqrt{3}\pi v_F \Gamma^{1/3} \chi^{2/3}} \text{sgn}(\omega_q) |\omega_q|^{2/3}. \quad (754)$$

In the same spirit as the RPA approximation used for the gauge field propagator, this term has the effect of modifying the effective fermion propagator to

$$G_\psi(q) = \frac{1}{-i\omega(1 + |\omega|^{-1/3}\gamma) + sv_F q_x + \frac{\kappa v_F}{2} q_y^2}, \quad \gamma = \frac{1}{2\sqrt{3}\pi v_F \Gamma^{1/3} \chi^{2/3}}, \quad (755)$$

with $s = \text{sgn}(\mathbf{v}_F)$. One subtlety one might wonder about is whether this $|\omega|^{2/3}$ piece needs to be taken into account self-consistently when deriving the earlier formula for the gauge field self-energy bubble. In fact it doesn't: if we instead use the above non-analytic form of the propagator to compute the polarization bubble, we get the same result as before: the only affect of the $|\omega|^{2/3}$ piece is to slightly shift the support of the integrand when doing the ω integral, but its presence does not change the result of the integration.

Anyway, the non-analytic contribution becomes the dominant one at small frequencies, and is what is used to determine the RG procedure at low energies. In the same spirit as the discussion above, let's look at how we should be doing the rescaling in the RG. If we let $\psi \mapsto \lambda^\sigma \psi$, then we see that the $q_x + \kappa q_y^2/2$ term defining the curvature of the FS is invariant provided that $\sigma = -4$. With this choice the scaling of the $|\omega|^{2/3}$ term is $\lambda^{6+2-8} = 1$; hence this term is marginal. That means that the usual $\psi^\dagger \omega \psi$ term actually scales as λ , and hence is irrelevant. That said, even though it's irrelevant, we need to keep it around when doing manipulations in order to be able to do the contour integrals correctly.



Today is simple, just clearing up a small conceptual glitch. One often sees the analysis of the XY done by using the sine-Gordon QFT. On the other hand, one often sees the analysis done by using an action like $(\partial\phi)^2 + \cos\theta$, with θ dual to ϕ . We'll elaborate on why exactly this is possible.



Conceptually of course $\cos \phi$ and $\cos \theta$ are very different things to add to the action. The natural way of taking vortices into account in the XY model is via

$$\mathcal{L} = \frac{R^2}{4\pi} (\partial\phi)^2 + g \cos \theta. \quad (756)$$

Let's check that the coefficients are correct. The canonical momentum is $(R^2/2\pi)\partial_t\phi = (1/2\pi)\partial_x\theta$,⁶⁴ meaning that $[\theta, \phi] \sim 2\pi i \Theta_H(x - y)$. The 2π here means that $e^{i\theta}$ inserts a 2π branch cut for ϕ , and so the $\cos \theta$ term in \mathcal{L} naturally incorporates the minimally-charged vortices into the theory.

We could therefore expect to use this Lagrangian in place of $\frac{R^2}{4\pi}(\partial\Phi)^2$, where $\Phi = \phi_a + \phi_v$ with ϕ_v the singular field configurations that produce the vortices and ϕ_a the smooth (subscript is for analytic lol) spin wave part. This $(\partial\Phi)^2$ Lagrangian is equivalent to a sine-Gordon theory. Recall how this works: $d\phi_a$ is exact, while $d\phi_v = d^\dagger\sigma$ is actually co-exact, since we can do a Hodge decomposition, assume that we are on \mathbb{R}^2 so that there are no normalizable harmonic forms, and push any exact contribution to the Hodge decomposition into the ϕ_a field. Therefore $\int d\phi_a \wedge \star d\phi_v = 0$, and so after dropping the trivial free spin wave part we get simply

$$S = \frac{R^2}{4\pi} \int d\phi_v \wedge \star d\phi_v = \frac{R^2}{4\pi} \int d^\dagger\sigma \wedge \star d^\dagger\sigma = \frac{R^2}{4\pi} \int dd^\dagger\sigma \wedge \star \square^{-1} dd^\dagger\sigma, \quad (757)$$

where we used that \square commutes with d, d^\dagger and that \square acting on $d^\dagger\sigma$ is $d^\dagger d$. For $d^2\phi_v(x) = dd^\dagger\sigma = 2\pi \sum_i q_i \delta(x - r_i)$, we get

$$S = \frac{R^2}{2} \sum_{r_i \neq r_j} q_i q_j \ln |r_i - r_j| + \dots, \quad (758)$$

which is the expected (regularized) potential energy for the Coulomb gas (the dots are the core energies). Anyway, now we can go back and decouple the vortex-vortex term using a new field θ :

$$S = \frac{1}{4\pi R^2} \int d\theta \wedge \star d\theta + \frac{1}{2\pi} \int d\theta \wedge d^\dagger\sigma. \quad (759)$$

This gives us back the original action after shifting θ appropriately. We now want to show that the second term can actually be written as a cosine term for θ , after integrating out σ . This is easy enough to do:

$$Z = \int \mathcal{D}\sigma \left\langle \exp \left(\frac{i}{2\pi} \int \theta dd^\dagger\sigma \right) \right\rangle, \quad (760)$$

with the expectation value over the free θ action. Writing the σ path integral as a sum over vortex configurations,

$$Z = \sum_{n \in \mathbb{N}} \frac{1}{n!^2} \sum_{\{q_i\}} \int \prod_{j=1}^{2n} d^2 r_j \left\langle \prod_{k=1}^{2n} e^{i\theta(r_k)q_k} \right\rangle, \quad (761)$$

⁶⁴Duality relates $d\phi \leftrightarrow \frac{1}{R^2} \star d\theta$ in these conventions.

where n is the number of vortex / anti-vortex pairs, $q_i = \pm 1$, and r_j are the positions of the vortices. Note how we are only summing over even number of vortices; this is because all odd-numbered vortex configurations will die under the expectation value. For the same reason, we may sum over all possible choices for the q_i , without worrying about charge neutrality, which will be enforced automatically. Therefore we get $\langle \sum_n \frac{1}{n!^2} [\int \cos(\theta)]^{2n} \rangle$, which re-exponentiates to a cosine for θ . Therefore we get the action

$$S = \int \left(\frac{1}{4\pi R^2} d\theta \wedge \star d\theta + \cos \theta \right), \quad (762)$$

where the coefficient in front of the $\cos \theta$ term hasn't been kept track of. So, we see that the original model with the cosine for the dual field is equivalent to the sine-Gordon model, except now with a T -dual radius of $1/R$. Superficially these models are rather different because of the fact that in one of them the operator in the cosine doesn't commute with the field in the kinetic term, but from the T -duality perspective of course it's rather trivial. Also note that the $R \mapsto 1/R$ replacement is just right to ensure that the scaling dimension of the two cosines is the same in both theories (viz. $\Delta = R^2/2$).



Basics on bandstructure of Weyl semimetals

Today we're going to be doing some simple band structure stuff for Weyl semimetals, inspired by listening to Adolfo Grushin's 2019 Les Houches lectures on Fermi arcs and nonlinear optics. I'm sure all of this exists in detail in the literature, but I wanted to try going through stuff without looking at any of the original papers in detail.



A general discussion of Weyl semimetals in our context starts by considering the most general Lagrangian involving a 4-component Dirac fermion that one can write down in 3+1 dimensions by adding on quadratic terms to the Dirac Lagrangian (we start with degenerate Dirac cones, and then see what terms we can add that change where the cones are in \mathbf{k} space). The terms that could appear are characterized according to their transformation properties under T and P , and these symmetries dictate the phenomenology of the system.

We will adopt the choice $\gamma^0 = X \otimes \mathbf{1}$, $\gamma^j = J \otimes \sigma^j \implies \bar{\gamma} = Z \otimes \mathbf{1}$ (remember that $\bar{\gamma}$ has an i in addition to the product over the γ matrices!). Time reversal will be taken to act as the historically used T with $T^2 = (-1)^F$, viz.

$$T : \psi(t, \mathbf{x}) \mapsto (\mathbf{1} \otimes J)\psi(-t, \mathbf{x}), \quad i \mapsto -i. \quad (763)$$

We then check that T preserves chirality (since it reverses both momentum and spin), with $T^{-1}\bar{\gamma}T = \bar{\gamma}$. We also have parity P , alias inversion I . It just acts as $P = \gamma^0 = X \otimes \mathbf{1}$, which hence satisfies $P^2 = \mathbf{1}$. It switches chirality on account of $P^{-1}\bar{\gamma}P = -\bar{\gamma}$ (expected since it reverses momentum but not spin). Charge conjugation can be worked out to act as

$$C : \psi \mapsto -(J \otimes Y)\psi^*. \quad (764)$$

Therefore the mass is C -even, and since $d/2 \in 2\mathbb{Z}$, we have $C^\dagger \bar{\gamma}^T C = \bar{\gamma}$ as expected.

The number of independent 4×4 Hermitian matrices is 4^2 . We can enumerate them with the γ matrices (plus $\mathbf{1}$): we have $\mathbf{1}, \gamma^0, i\gamma^j, \gamma^0\gamma^j, i\gamma^j\gamma^k$ ($j < k$), $i\bar{\gamma}\gamma^0, \bar{\gamma}\gamma^j, \bar{\gamma}$. This gives us 16 matrices, and so evidently we have exhausted the list. The T properties are examined by taking each matrix M and looking at $(\gamma^1\gamma^3)M^*(\gamma^1\gamma^3)$: we find

$$\text{T-even: } \mathbf{1}, \gamma^0, i\gamma^j, \bar{\gamma} \quad \text{T-odd: } \gamma^0\gamma^j, i\gamma^j\gamma^k, i\bar{\gamma}\gamma^0, \bar{\gamma}\gamma^j. \quad (765)$$

The ones that commute with γ^0 are even under parity:

$$\text{P-even: } \mathbf{1}, \gamma^0, i\gamma^j\gamma^k, \bar{\gamma}\gamma^j, \quad \text{P-odd: } \bar{\gamma}, i\gamma^k, \gamma^0\gamma^j, i\bar{\gamma}\gamma^0. \quad (766)$$

An important thing to note here is that there are only two matrices which preserve both P and T separately.

The most general free Lagrangian we could consider is

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m - b_\alpha M^\alpha)\psi, \quad (767)$$

where α runs over all the Hermitian matrices listed above (except γ^0 , which we already have in the form of the mass term). This is a bit too much to analyze in a concrete way, and so we will specialize a bit. To start, we will select one term that has $(+, -)$ eigenvalues under T, P and one term which has $(-, +)$ eigenvalues. For the former, we choose $\bar{\gamma}$, while for the latter we choose the three terms $i\gamma^j\gamma^k$ (with $j < k$), which will appear in a Langrangian as the vector $\bar{\psi}\not{b}\bar{\gamma}\psi$. Therefore we take

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m - \not{b}\bar{\gamma})\psi. \quad (768)$$

The Hamiltonian is then

$$H = \int_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) \psi_{\mathbf{k}}, \quad \mathcal{H}(\mathbf{k}) = \begin{pmatrix} b_0 \mathbf{1} + (\mathbf{k} + \mathbf{b}) \cdot \boldsymbol{\sigma} & m \mathbf{1} \\ m \mathbf{1} & -b_0 \mathbf{1} - (\mathbf{k} - \mathbf{b}) \cdot \boldsymbol{\sigma} \end{pmatrix}, \quad (769)$$

or more concisely,

$$\mathcal{H}(\mathbf{k}) = (\mathbf{1} \otimes \boldsymbol{\sigma}) \cdot (\bar{\gamma}\mathbf{k} + \mathbf{b}) + \gamma^0 m + \bar{\gamma} b_0. \quad (770)$$

Since $\bar{\gamma}$ commutes with all the matrices in this expression except for m , the mass is the only term which mixes chiralities. As we said, the \not{b} term has both T -even and T -odd components:

$$T : b_0 \gamma^0 \bar{\gamma} \mapsto b_0 \gamma_0 \bar{\gamma}, \quad \mathbf{b} \cdot \boldsymbol{\gamma} \bar{\gamma} \mapsto -\mathbf{b} \cdot \boldsymbol{\gamma} \bar{\gamma}. \quad (771)$$

and

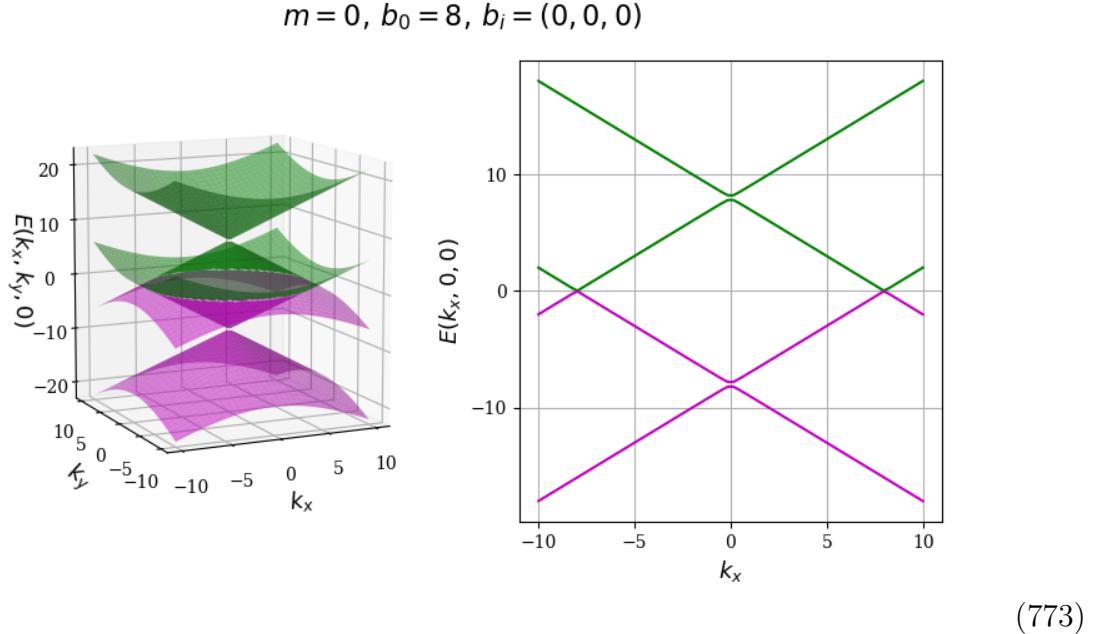
$$P : b_0 \gamma^0 \bar{\gamma} \mapsto -b_0 \gamma_0 \bar{\gamma}, \quad \mathbf{b} \cdot \boldsymbol{\gamma} \bar{\gamma} \mapsto \mathbf{b} \cdot \boldsymbol{\gamma} \bar{\gamma}, \quad (772)$$

which is the opposite eigenvalue assignment as T . Therefore PT symmetry will be broken for generic values of b . Also note that the $\not{b}\bar{\gamma}$ term is C -even (in keeping with the fact that the chiral current is C -even when $d \in 4\mathbb{Z}$).

The breaking of PT is essential to get a Weyl semimetal, because if PT is preserved then the bands are guaranteed to be doubly degenerate, coming in degenerate opposite-chirality pairs. This degeneracy ensures that we can never have Weyl nodes—the two Weyl nodes get trapped in pairs as Dirac points, and separating them requires breaking PT symmetry. The degeneracy here comes from the fact that PT preserves momentum, but is antiunitary and squares to $(-1)^F$ (on the Lorentz indices, it acts as $X \otimes J$); hence we get a Kramer's degeneracy of the bands at each \mathbf{k} value.

Note that a similar statement about degeneracies cannot be made in even spatial dimensions. For example, work in 2+1D. If we consider TI symmetry with I inversion, then we find that $(TI)^2 = R_{2\pi}T^2 = (-1)^{2F} = \mathbf{1}$, since $I = R_\pi$ is a spatial rotation by π . Therefore Kramer's theorem doesn't apply, and there is no doubling. Using a reflection R and considering TR wouldn't work either, since TR doesn't leave momenta invariant. Indeed, consider a free Dirac fermion in 2+1D: this certainly preserves TI symmetry, but it has two non-degenerate bands, so obviously TI symmetry doesn't enforce any kind of doubling in even dimensions.

Let's now illustrate the possible band structures with some pictures. Of course when $b = 0 = m$, we have a single boring Dirac cone. The m term will gap the this while preserving the spin degeneracy. The purpose of the b term is to separate the two Weyl cones that constitute the Dirac cone in momentum space. When we just have $b_0 \neq 0$, we get a line node semi-metal with band touchings on a circle, plus two Weyl points:

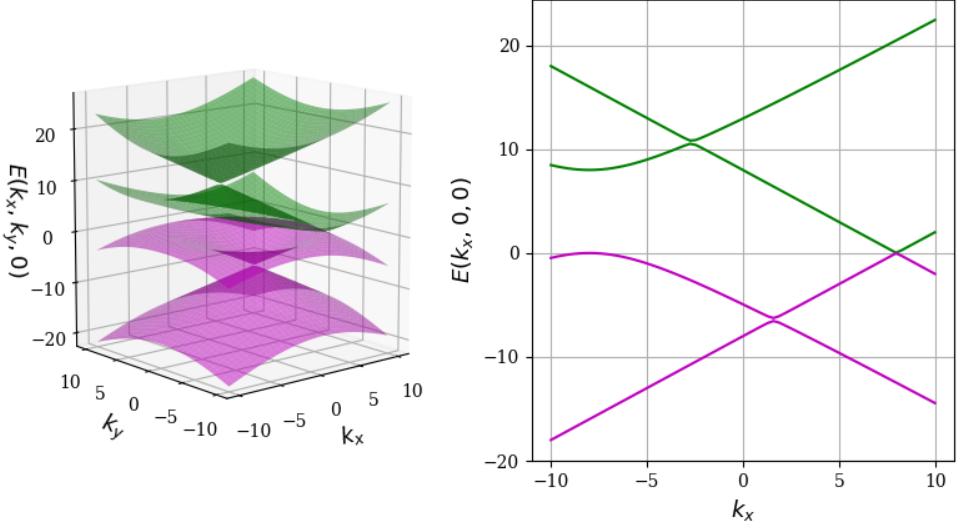


Note that since $\mathbf{b} = 0$, the TRS of the system is preserved. T sends a fermion at \mathbf{k} to one at $-\mathbf{k}$ with the same chirality, and we see that this doubling is reflected in the band structure.

When we turn on \mathbf{b} the degeneracy of the circle breaks into discrete points, with the \mathbf{b}

term separating the Weyl points in \mathbf{k} space:

$$m = 0, b_0 = 8, b_i = (2, 0, 0)$$

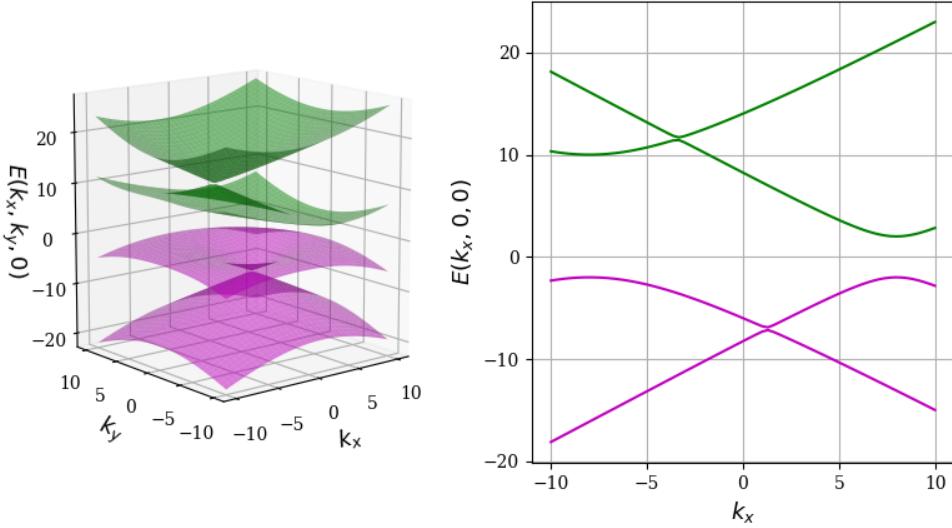


(774)

Now both P and T are broken; the former as evidenced by the fact that now the spectrum is asymmetric under $\mathbf{k} \mapsto -\mathbf{k}$ (this also allows the two Weyl points to be at different energies—in general, \mathbf{b} roughly controls the energy difference between these two Weyl points).

When we add a mass to this, we create a gap at the point where the purple band touches the green band (while the Weyl points are left alone since the mass only couples modes of opposite chirality)

$$m = 2, b_0 = 8, b_i = (2, 0, 0)$$



(775)

As another foray into the huge parameter space, we will consider using the Hermitian matrices $i\gamma^1\gamma^2$ and $\gamma^1\bar{\gamma}$, which is the choice made in the review by Armitage, Mele, and

Vishwanath adapted to our conventions (I don't really know where this choice was motivated from, but it lets us draw a few new pictures in any case). Therefore we consider

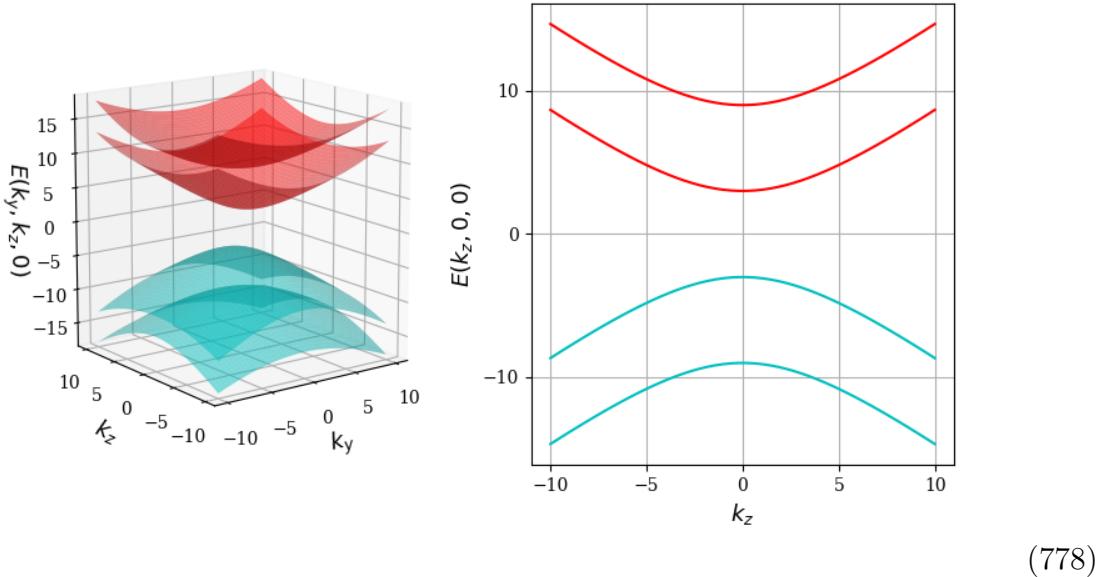
$$\mathcal{L} = \bar{\psi}(i\partial + ib\gamma^0\gamma^1\gamma^2 + b'\gamma^0\gamma^1\bar{\gamma})\psi, \quad (776)$$

which gives

$$\mathcal{H}(\mathbf{k}) = (\mathbf{1} \otimes \boldsymbol{\sigma}) \cdot \bar{\gamma}\mathbf{k} + b(\mathbf{1} \otimes Z) - b'(X \otimes X). \quad (777)$$

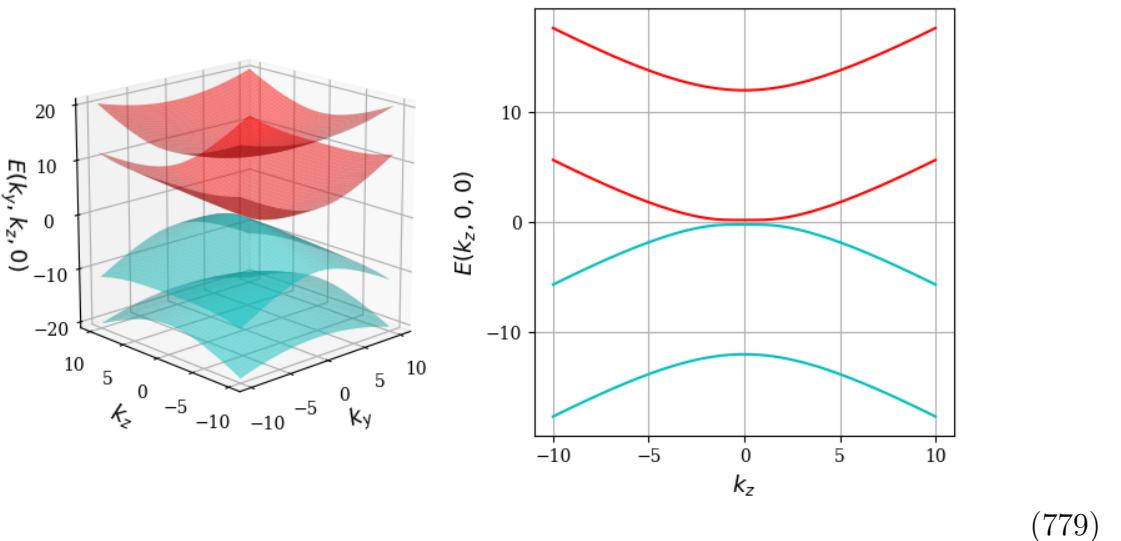
Note that both b, b' terms are T -odd and P -even; therefore all of the plots to follow will display the same symmetries (the P -evenness will make the plots invariant under $\mathbf{k} \mapsto -\mathbf{k}$). We know what happens when $b, b' = 0$. When we turn on $b < m$ we get a breaking of degeneracy of the former Kramers pairs, producing what we might call a magnetic semiconductor band structure:

$$m = 6, b = 3, b' = 0$$



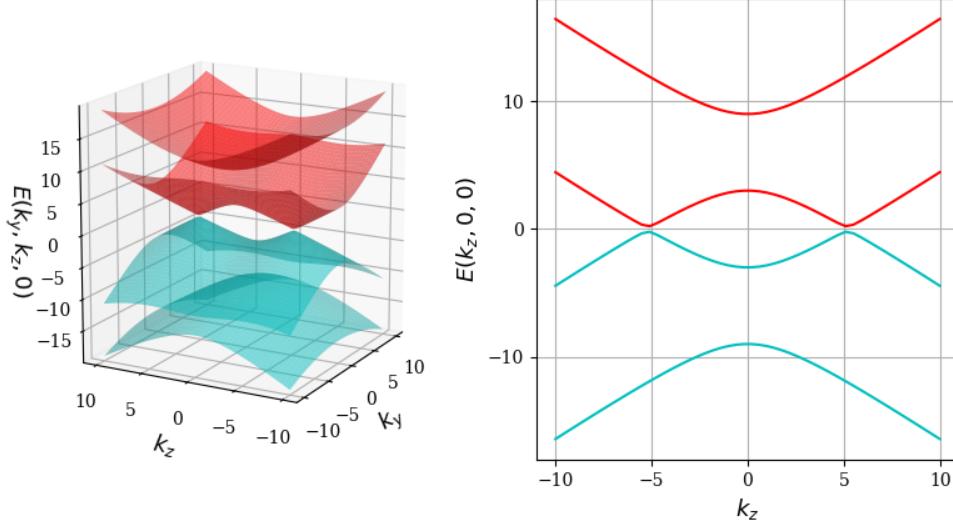
Increasing b to $b = m$ gives us a degenerate band touching:

$$m = 6, b = 6, b' = 0$$



If we then turn b past m so that $b > m$, we get a semimetal with two Weyl points, with positions and chiralities related by P :

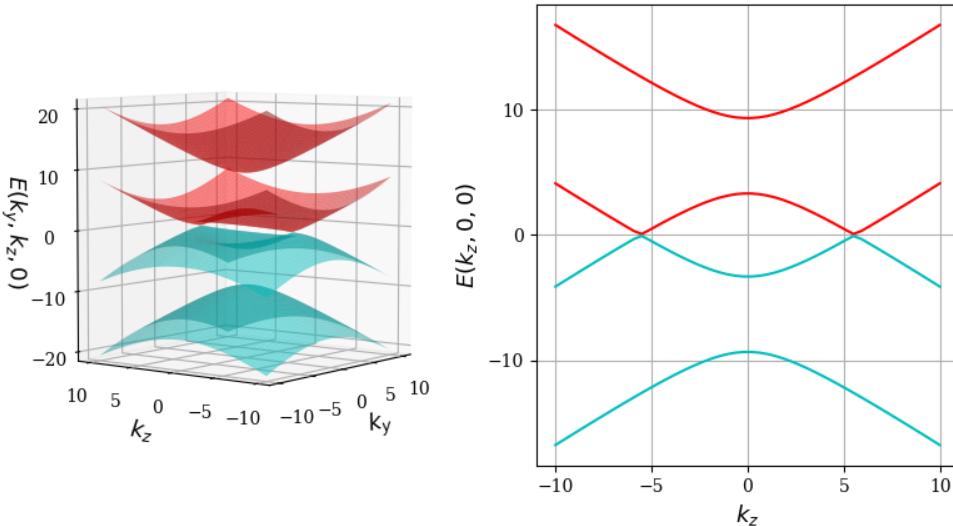
$$m = 3, b = 6, b' = 0$$



(780)

The larger we make b , the further apart the Weyl points get, making them harder and harder to gap out through the mass term which couples the two chiralities together. Setting $m = b = 0$ with $b' \neq 0$ gives the same circular nodal line dispersion as in the previous set of examples. The situation in the previous plot is actually kind of typical for the general case when m is not \gg than b and b' : choosing the parameters at random, we again get a semimetallish thing:

$$m = 3, b = 2, b' = 6$$



(781)

Now let's talk about Fermi arcs. The idea (a la Witten's lectures on topological phases) is to consider solving the Weyl Hamiltonian for a fermion mode with appropriate boundary

conditions on the half-space $\Sigma = \{x > 0\} \subset \mathbb{R}^3$. The Hamiltonian for a single Weyl node will be $H = iv_F \boldsymbol{\sigma} \cdot \nabla$. The appropriate boundary conditions for the Weyl field ψ can be figured out by requiring the Hamiltonian be Hermitian when evaluated on ψ :

$$\langle H\psi|\psi\rangle = \langle\psi|H\psi\rangle \implies \int_{\partial\Sigma} \psi^* X \psi = 0. \quad (782)$$

The simplest way to solve this is to require that $\psi|_{\partial\Sigma}$ be composed solely of eigenstates of Z and eigenstates of Y . By rotational invariance we can pick a direction at will, and so we will require that

$$Z\psi|_{\partial\Sigma} = \pm\psi|_{\partial\Sigma}. \quad (783)$$

In the full 4-component theory the action of Z on ψ is the same as the action of $\pm\gamma^0\gamma^3$ on ψ (with the sign depending on the chirality of the Weyl point), and hence the meaning of the boundary condition is that when restricted to the boundary, ψ must be a \pm eigenstate under a reflection of the x and y axes.

We can actually get solutions to the Schrodinger equation that respect this boundary condition throughout all of Σ . Suppose first that we choose a $+1$ eigenstate of Z . Then we need to solve

$$iv_F \partial_z \psi_+ = E\psi_+, \quad iv_F (\partial_x + i\partial_y) \psi_+ = 0. \quad (784)$$

Doing so is easy; we find the solutions

$$\psi_+ \propto (e^{-iEz/v_F + ik_y y} e^{k_y x}, 0)^T. \quad (785)$$

That we get a solution with well-defined momentum k_y in the y direction is no surprise, since it is the only non-special direction left (the boundary conditions made z special). Now ψ_+ is defined for $x > 0$, and so for it to not blow up we need $k_y < 0$, which consequently localizes all of these modes to the boundary.

The consequence of this is that we obtain separate zero-mode solutions for every $k_y < 0$, which are localized to the boundary. This means that when we project to the boundary BZ (which has only k_y, k_z momenta), we will get a line of gapless states extending from the Weyl node (at the origin) out to negative k_y values.

Now suppose instead we had boundary conditions such that $\psi|_{\partial\Sigma}$ was a -1 eigenstate of Z . The solution is then

$$\psi_- \propto (0, e^{iEz/v_F + ik_y y} e^{-k_y x})^T, \quad (786)$$

which now only works for $k_y > 0$. Therefore the projection of a Weyl point with these boundary conditions will also have an arc of gapless points extending out from it, but this time it will point in the $+k_y$ direction.

It should then be clear that by changing the boundary conditions so that $\psi|_{\partial\Sigma}$ is an eigenstate of $Y \cos\phi + Z \sin\phi$, we can get boundary Fermi arcs that extend out from the projection of the Weyl point at arbitrary angles. The boundary conditions are non-universal and will depend on the exact details of the surface termination, but the fact that the projection of each Weyl point must have a Fermi arc emanating from it in some direction is not. Thus only the existence of the Fermi arcs, and not the topology of their connectivity, is universal.



Building Dirac fermions on the lattice

Today we are going to be discussing a few different ways of providing lattice regularizations for Dirac fermions, in 1+1D up to 3+1D.



1+1 dimensions

A Dirac fermion in 1+1 dimensions has two components, so we will need to introduce hopping parameters such that there are two sites per unit cell. How to do this is an old story in this diary by now. We take the SSH model with the two hoppings equal in magnitude and opposite in sign: denoting the second-quantized operators on the two sublattices by c and d ,

$$H = \sum_j (td_j^\dagger c_j - tc_{j+1}^\dagger d_j + h.c.) = \int_k \psi_k^\dagger \begin{pmatrix} 0 & -te^{-ik} + t^* \\ t - t^* e^{ik} & 0 \end{pmatrix} \psi_k, \quad (787)$$

with $\psi_k = (c_k, d_k)^T$. We will let $t \in \mathbb{R}$. Therefore the spectrum is

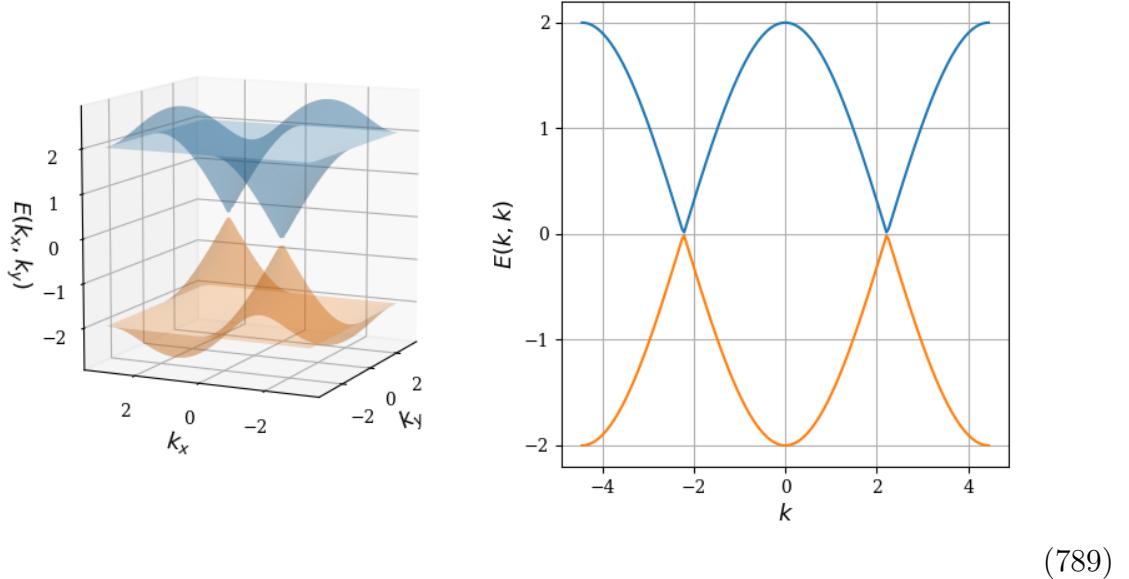
$$E(k) = \pm |2t \sin(k/2)|, \quad (788)$$

which has a single Dirac point at the origin. The Dirac point is gapped when a mass is turned on, either the chiral mass (an alternating chemical potential on the two sublattices) or the dirac mass (an alternating strength for the intra- and inter-site hopping). T symmetry (defined in the historically more often used way, i.e. T and not CT) will forbid both mass terms, but P or C will only forbid one (with the appropriate choice of P so that $P^2 = \mathbf{1}$; with this choice CPT preserves all the possible Lorentz-invariant terms).

2+1 dimensions

A Dirac fermion in 2+1 dimensions has two components, so we will need to introduce hopping parameters such that there are two sites per unit cell. Given our success with the SSH model, we can try building a 2+1D Dirac fermion by coupling together SSH chains. Stacking the chains so that the $-t$ bonds all line up vertically doesn't work; this leads to a hopping problem that is gauge-equivalent to something with all positive hoppings (of course there is no real gauge field; we just mean that a field redefinition by local $U(1)$ phase factors, which doesn't affect the spectrum of the theory since it is just a change of variables, can

be made which renders all the hoppings positive). However, we can stagger the SSH chains by one lattice site so that the $-t$ bonds form a checkerboard pattern—this gives the π flux per plaquette that we want for making the unit cell bigger (and which because of the field strength can't be deformed to something with only positive hoppings). The exact expression for the Hamiltonian isn't very illuminating, so we will just show the dispersion, which is

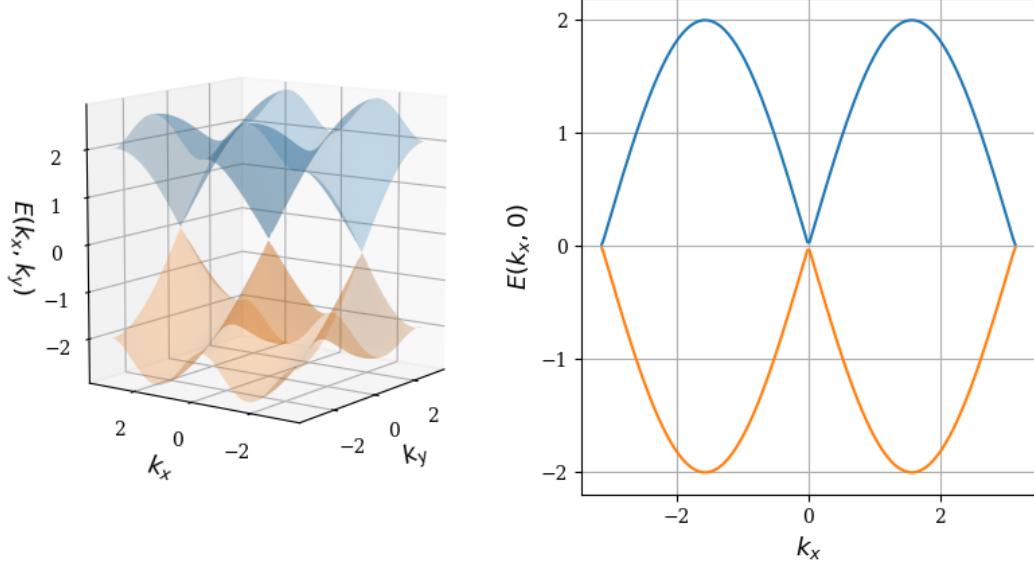


We can also use a different “gauge choice” to get π flux per unit cell, which is the classic choice where all the bonds in the y direction on lines with even x coordinate have negative hopping, with all the rest having positive hopping. This model produces

$$H(\mathbf{k}) = 2t \begin{pmatrix} -\sin(k_y) & \frac{e^{ik_x}-1}{2i} \\ \frac{1-e^{-ik_x}}{2i} & \sin(k_y) \end{pmatrix} \cong 2t(-Z \sin(k_y) + X \sin(k_x/2)), \quad (790)$$

where the \cong means that the two Hamiltonians are related by a unitary transformation (viz. $\mathcal{U} = \text{diag}(e^{ik_x/4}, e^{-ik_x/4})$) and hence have the same spectra (while $\sin(k_x/2)$ changes sign under $\delta k_x \in 2\pi\mathbb{Z}$, this sign cancels out when computing the energy since the sine appears squared, hence the spectrum is still well-defined). The spectrum is (there's no real need to

plot this since it's basically the same as the staggered SSH chains model)



(791)

A Dirac mass can be added by adding an extra intra-site hopping term to H , viz.

$$\delta H = \sum_j i t (c_j^\dagger d_j - d_j^\dagger c_j). \quad (792)$$

The mass is forbidden by T and R , just as in the usual story.

Could we ever cook up a lattice model with a single Dirac fermion? No: if we could, then we could gap it. We know that a single Dirac fermion in 2+1D produces a half-integer CS term when integrated out, which is not an allowed response. Therefore any putative Dirac theory coming from the IR limit of a lattice model needs to have at least two Dirac points in 2+1D.

3+1 dimensions

Now we turn to three dimensions. There are again multiple ways of getting Dirac fermions by modifying hoppings with π fluxes. In any case, we will need a four-site unit cell, which is a bit awkward since as in 2+1D it involves breaking rotation symmetry in the choice of hoppings. One possibility is

$$H_A/t = \sin(k_x)(Z \otimes X) \oplus (\mathbf{1} \otimes X) + \sin(k_y)(X \otimes \mathbf{1}) \oplus (X \otimes Z) + \sin(k_z)X \otimes (Z \oplus \mathbf{1}). \quad (793)$$

The drawback to this Hamiltonian is that the matrices multiplying the sines don't \otimes factorize into Pauli matrices.

To get something that \otimes factorizes into Pauli matrices, we need something that gives us a number of minus signs in each matrix that is in $4\mathbb{N}$. This is just because the tensor factors will multiply any minus sign present in a single Pauli matrix by four. Therefore the

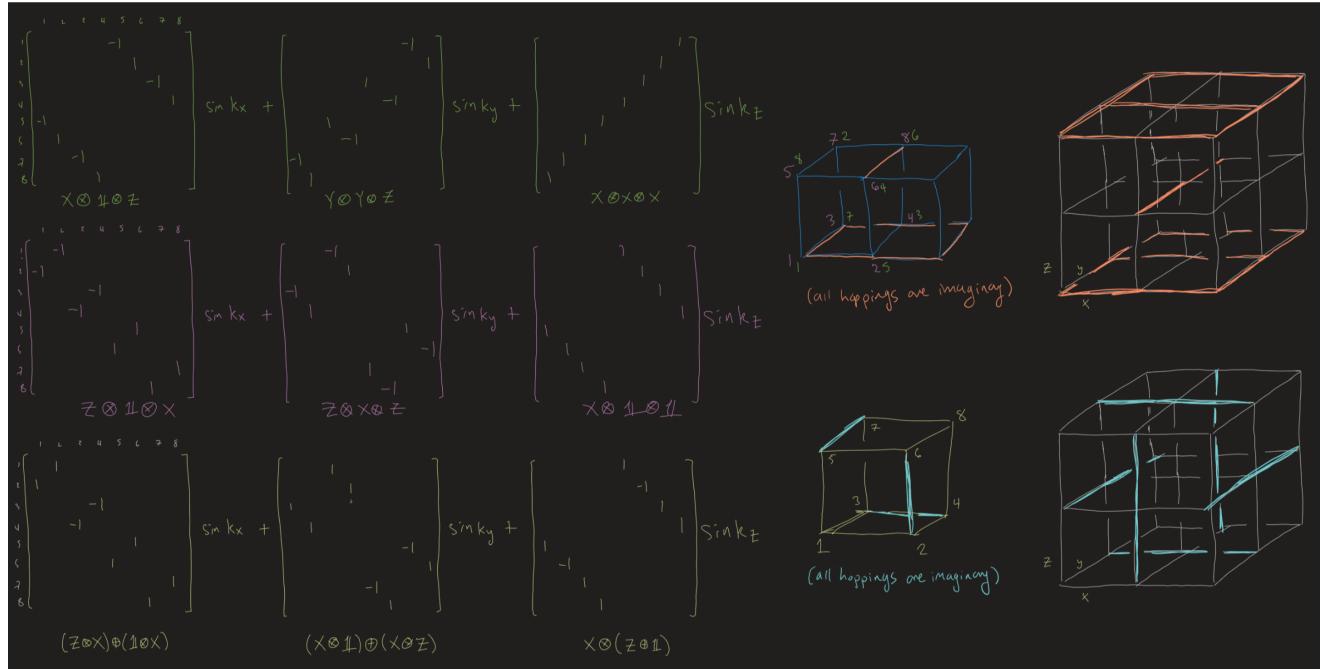


Figure 16: I do apologize for the truly reprehensible color scheme here—I made it with my tablet set to some OG-Kahn-academy-esque color scheme and this was the result. Anyway the first set of matrices gives the Hamiltonian in the green basis for the set of hopping amplitudes given in the upper right drawings—the orange links have negative hoppings. The second set gives the same Hamiltonian in the regular \otimes -according-to-Cartesian-coordinates basis, while the third set gives the Hamiltonian for the choice of hoppings as indicated in the bottom right drawings.

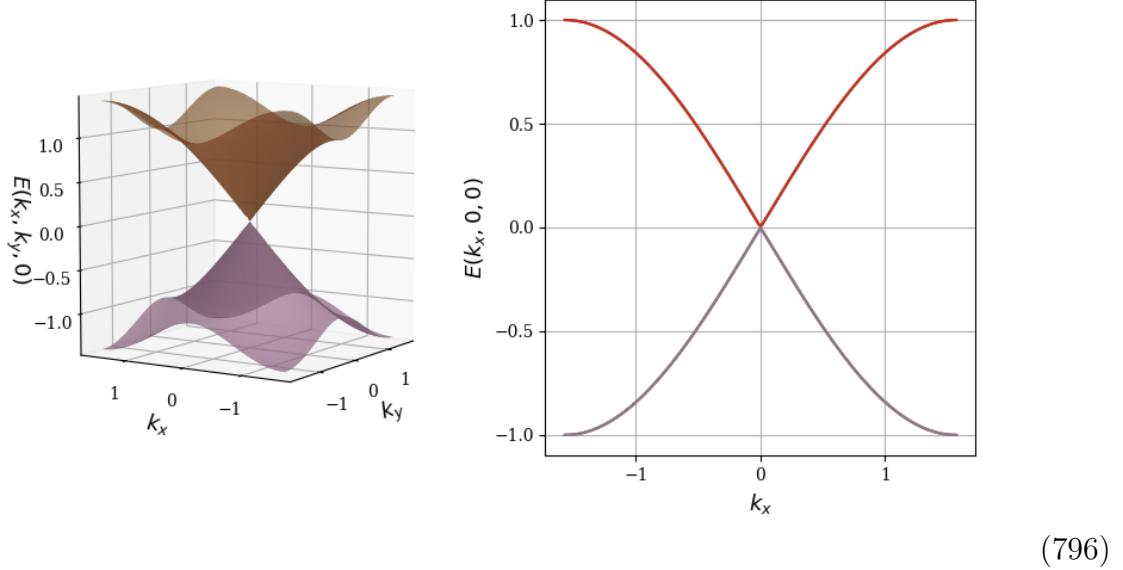
above choice of hopping won't work, since each k_i term in the Hamiltonian will have only two minus signs in it. To get something with $4N$ minus signs in each term, we need to add more minus signs in the hoppings. The first pattern I came up with is in the top right of the figure. Here we see that the x and y hoppings both have two minus signs (that will become four in the matrices appearing in H), while the z hopping is totally positive. If we use the basis indicated by the pink numbers (the usual \otimes basis) then we get

$$H_B/t = \sin(k_x)Z \otimes \mathbf{1} \otimes X + \sin(k_y)Z \otimes X \otimes Z + \sin(k_z)X \otimes \mathbf{1} \otimes \mathbf{1}. \quad (794)$$

If we choose another basis, which is green in the figure, then (this weird choice was made so that the resulting matrices would all be block off-diagonal)

$$H'_B/t = \sin(k_x)X \otimes \mathbf{1} \otimes X + \sin(k_y)Y \otimes Y \otimes Z + \sin(k_z)X \otimes X \otimes X. \quad (795)$$

Of course in either basis we get the same dispersion, which gives us two degenerate Dirac cones at the origin:



This means that at the origin we have a pair of degenerate Dirac cones. We also get the exact same dispersion if we use H_A , so even though H_A has a slightly less nice Hamiltonian, it's equivalent to the H_B ones.

This is fine, but it's given us two degenerate Dirac fermions—could we write down a model with only one? Unlike in one dimension down, there is no obvious CS argument stopping us from doing so. One way of doing this would be to try and gap out only one of the two Dirac fermions in the above example—but finding the appropriate mass term would be annoying since we'd have to do a change of basis in order to clearly separate out the two Dirac points.

The better point of view (thank you Senthil!) is to just write down the continuum theory exactly (still regulated by the lattice in that we're working inside the BZ). Here it is:

$$H = \int_{\mathbf{k}} \left(\sum_i \sin(k_i) \sigma^i \otimes Z + m(\mathbf{k}) \mathbf{1} \otimes X \right), \quad (797)$$

where

$$m(\mathbf{k}) = m_0 + m_1 \sum_i \cos(\mathbf{k}_i). \quad (798)$$

We know from our diary entry on Weyl semimetals that if we assume P and T separately then there are only two mass terms to write down, and so evidently the Hamiltonian above cannot have any extra terms added to it without breaking T or P . If we write this Hamiltonian down in terms of a lattice hopping model, it's slightly messy: we need to have a 4-site unit cell with some rather contrived looking hoppings between different sites—so starting straight from the field theory is the better approach.

For $m_0 > 3m_1 > 0$, this gives a spectrum which is gapped everywhere in the BZ ($\mathbf{k} \in [-\pi, \pi]^3$), with all the mass terms having the same sign. At the point $3m_1 = m_0$ the gap closes at the (π, π, π) point, and we get a single massless Dirac fermion. Decreasing m_0 slightly more gives a massive phase except now the mass of the Dirac point at (π, π, π) is flipped relative to the others, giving us a TI. Decreasing m_0 further lets us pass through various different band touchings, eventually ending in another trivially gapped phase where all the eight Dirac points have acquired a large negative mass.



Dipole moments and ferromagnetism

A friend of mine explained to me last night that she thought that dipolar interactions between electronic magnetic moments were responsible for magnetism. In what follows we'll provide an estimate that shows why this classically-intuitive explanation is unreasonable. A good reference is Auerbach's book.



First, recall how we get the field for a magnetic dipole with dipole moment $\boldsymbol{\mu}$: we imagine two magnetic charges separated by a small distance \mathbf{d} , with \mathbf{d} pointing from the negative charge (at the origin) to the positive charge (so that $\boldsymbol{\mu} \propto \mathbf{d}$). Then we determine the field with

$$-\frac{\mathbf{r}}{r^3} + \frac{\mathbf{r} - \mathbf{d}}{|\mathbf{r} - \mathbf{d}|^3} \approx -\frac{\mathbf{r}}{r^3} + \frac{\mathbf{r} - \mathbf{d}}{r^3} (1 + 3\mathbf{r} \cdot \mathbf{d}/r^2) = -\frac{1}{r^3} (\mathbf{d} - 3(\hat{\mathbf{r}} \cdot \mathbf{d})\hat{\mathbf{r}}). \quad (799)$$

This means that the Hamiltonian for two magnetic dipoles $\boldsymbol{\mu}_i$ is (SI units)

$$H = -\frac{\mu_0}{4\pi r^3} (3(\hat{\mathbf{r}} \cdot \boldsymbol{\mu}_1)(\hat{\mathbf{r}} \cdot \boldsymbol{\mu}_2) - \boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2), \quad (800)$$

with \mathbf{r} the vector pointing between the two dipoles. Considering two electrons with magnetic moments pointing in the directions $\hat{\boldsymbol{\ell}}_i$,

$$H = -\frac{\mu_0}{4\pi r^3} g_J^2 \left(\frac{e\hbar}{2m_e} \right)^2 (3\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\ell}}_1 \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\ell}}_2 - \hat{\boldsymbol{\ell}}_1 \cdot \hat{\boldsymbol{\ell}}_2). \quad (801)$$

Let's check units: in SI, μ_0 is a microhenry per meter, i.e. $A^{-2} \text{ kg m s}^{-2}$. So then the expression on the RHS has units of $C^{-2} \text{ s}^2 \text{ kg}^2 \text{ m s}^{-2} \text{ m}^{-3} C^2 \text{ J}^2 \text{ s}^2 \text{ kg}^{-2}$, which reduces to just Joules, as required. This of course favors antiferromagnetism; this can be checked by drawing the field lines. So regardless of their strengths, dipolar interactions definitely can't be responsible for ferromagnetism (and hence since ferromagnetism exists, the dipolar interactions can't be very strong, since ferromagnetic regions are maximally bad as far as dipolar energy goes).

We can now plug in numbers for all of the stuff above: for a separation of $r = 2 \text{ \AA}$, we get a dipolar interaction energy of 10^{-4} eV , ish. Now a Kelvin is about 10^{-4} eV , and so the dipolar interaction could be important below about 1K. Since the dipolar interaction favors antiferromagnetism, in order to determine its possible importance we should look at the ordering temperature (Neel temperature) for some antiferromagnets. According to Wikipedia typical T_N s are of the order of a few hundred K (i.e. $O(10)$ meV). So the dipolar energy is lower than the Neel temperature by a factor of about

$$T_{dip}/T_N \sim 10^{-2} \text{ or } 10^{-3}, \quad (802)$$

indicating that dipolar interactions can't be playing a role in setting the AF ordering (for materials with lower T_N , we can't make definitive statements). Note however that dipolar interactions can play a significant role when you have a bunch of aligned moments in a ferromagnet—the AF dipole-dipole interaction is the driving force behind the creation of magnetic domains at low temperatures (but still, this only happens at rather large scales).

◇ ◇ ◇

Anyway, evidently magnetism must be caused by electronic exchange. Consider first ferromagnetism. For simplicity, we will focus on the case where two electrons occupy two distinct orbitals. If the electron orbitals are such that they are orthogonal and their spatial overlap is large, then we get a ferromagnetic exchange interaction in order that the potential energy of the electrons to be minimized. This applies to multiple orbitals on the same site (in which case it is responsible for one of Hund's rules), or to large orbitals on different sites (in which case it produces ferromagnetism).

We can make the heuristic picture precise by considering an interacting Hamiltonian containing the term

$$H \supset \int_{x,y} \psi_\alpha^*(x) \psi_\beta^*(y) K(x-y) \psi_\beta(y) \psi_\alpha(x), \quad (803)$$

which conserves spin locally ($K(x-y)$ is some interaction kernel, e.g. $e^2/|x-y|$, and α is the spin index). Suppose the field operators are decomposed into two single-particle operators $\psi_\alpha(x) = \sum_{a=1,2} \lambda_a(x) c_{a\alpha}$, with $\int_x \lambda_a(x)^* \lambda_b(x) = \delta_{ab}$. We will assume that the two orbitals carry distinct quantum numbers under some symmetry preserved by the Hamiltonian (e.g. they could have different angular momenta), so that in the expansion for H in terms of the

constituents of the field operators, there will no terms of the form $c_{a\alpha}^\dagger c_{a\beta}^\dagger c_{a\alpha} c_{b\beta}$ with $a \neq b$. Then the part of the interaction which determines the spin exchange interaction is the Fock part

$$H \supset H_J = \sum_{\alpha\beta, a \neq b} \int_{x,y} \lambda_a^*(x) \lambda_a(y) \lambda_b^*(y) \lambda_b(x) K(x-y) c_{a\alpha}^\dagger c_{b\beta}^\dagger c_{a\beta} c_{b\alpha} \equiv J c_{a\alpha}^\dagger c_{b\beta}^\dagger c_{a\beta} c_{b\alpha}. \quad (804)$$

The reason that this determines the spin exchange is because

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{4} (c_1^\dagger \boldsymbol{\sigma} c_1) \cdot (c_2^\dagger \boldsymbol{\sigma} c_2) = \frac{1}{2} \sum_{\alpha\beta} c_{1\alpha}^\dagger c_{1\beta} c_{2\beta}^\dagger c_{2\alpha} - \frac{1}{4} n_1 n_2 \quad (805)$$

where the density operators are $n_a = n_{a\uparrow} + n_{a\downarrow}$ and where we have used the usual completeness relation:

$$\sigma_{\alpha\beta}^\mu \sigma_{\gamma\delta}^\mu = 2 \left(\delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{1}{2} \delta_{\alpha\beta} \delta_{\gamma\delta} \right). \quad (806)$$

Therefore we get a term in $H \supset -2J\mathbf{S}_1 \cdot \mathbf{S}_2$ (the minus sign from normal ordering). This is then ferromagnetic if $J > 0$, which we expect to usually be the case. For example, if the interaction is a completely screened delta function $K = \delta(x-y)$, then we have $J = \int_x |\lambda_1(x)\lambda_2(x)|^2$, which is manifestly positive and hence the interaction is ferromagnetic. Likewise, if we take the unscreened $K(r) \sim e^2/r$, then as an operator $(\hat{K}\psi)(x) = \int_{x,y} K(x-y)\psi(y)$, which is positive-definite since it is diagonalized in Fourier space as $4\pi e^2/q^2$, with all of the eigenvalues being positive. Hence J must again be positive, leading to a ferromagnetic exchange as expected. This is the pedantic reason why Hund's rules are a thing.

On the other hand, if the electron orbitals have a comparatively small overlap but are not orthogonal so that they hybridize, then we expect that an antiferromagnetic exchange interaction to be induced. The simplest argument for this is the usual argument for superexchange a la Anderson. Since this has been derived in detail elsewhere in the diary we won't go in to any details here.



Today we're doing some simple problems from chapter 7 of Coleman's many-body theory book. We consider a perturbative treatment of the electron gas in three dimensions. To first order, there are two interaction contributions to the ground state energy: one comes from a direct interaction (two bubbles connected by an interaction line), and the other comes from the exchange interaction (a bubble with an interaction line dividing it in two). The former contribution only selects out the $\mathbf{q} = 0$ component of the interaction, and hence is a

uniform charging energy that is canceled by the ionic background (in field theory language, all tadpoles are canceled by a simple mass counterterm). Therefore the ground state energy density is determined by the kinetic part and the exchange interaction: remembering to account for spin degeneracy,

$$E/V = 2 \int_{\mathbf{k}} \frac{k^2}{2m} f_{\mathbf{k}} - \int_{\mathbf{k}, \mathbf{k}'} \frac{e^2}{\varepsilon_0 |\mathbf{k} - \mathbf{k}'|^2} f_{\mathbf{k}} f_{\mathbf{k}'} \quad (807)$$

with the minus sign coming from the fermion bubble or from the antisymmetrization in the Slater determinants used as the ground state wavefunctions, whichever you prefer. In what follows we will calculate this and compare the relative contributions of the kinetic and potential terms.

* * * * *

The kinetic term contributes

$$2 \int_{\mathbf{k}} \frac{k^2}{2m} f_{\mathbf{k}} = \frac{3}{5} \left[\frac{k_F^5}{6\pi^2 m} \right] = \frac{3}{5} \left[\frac{k_F^3}{3\pi^2} \varepsilon_F \right] = \frac{3\rho\varepsilon_F}{5}, \quad (808)$$

since the density is

$$\rho = \frac{4}{3}\pi k_F^3 \frac{2}{(2\pi)^3}. \quad (809)$$

Writing the momenta as $\mathbf{k} = \mathbf{x}k_F$, the interaction term I is

$$I = \frac{8\pi^2 e^2 k_F^4}{\varepsilon_0 (2\pi)^6} \int dx dy d\cos\theta \frac{x^2 y^2}{x^2 + y^2 - 2xy \cos\theta}, \quad (810)$$

with the x, y integrals going from 0 to 1 and the cos integral going from -1 to 1 . A little algebra shows that the prefactor is $3e^2 k_F \rho / (8\pi^2 \varepsilon_0)$, while the integral is

$$\begin{aligned} \int dx dy d\cos\theta \frac{x^2 y^2}{x^2 + y^2 - 2xy \cos\theta} &= \int dx dy (-xy/2) \ln \left| \frac{(x^2 + y^2)/2xy - 1}{(x^2 + y^2)/2xy + 1} \right| \\ &= \frac{1}{2} \int dx dy xy \ln \left| \frac{x+y}{x-y} \right| = \frac{1}{4}, \end{aligned} \quad (811)$$

where the integral was done with Mathematica and gives $1/2$.

Adding up the two contributions, we see that the energy per particle is (factor of two compared to Coleman?!)

$$\frac{E}{\rho V} = \frac{3}{5} \varepsilon_F - \frac{3e^2 k_F}{32\pi^2 \varepsilon_0}. \quad (812)$$

We define the average separation of electrons in a Fermi gas by

$$R_e = \left[\frac{3}{4\pi\rho} \right]^{1/3}. \quad (813)$$

We further take $r_s = R_e/a_B$ with $a_B = 4\pi\varepsilon_0/(me^2)$ as a dimensionless measure of the electron density. Some easy algebra shows that $k_F^{-1} = \beta r_s a_B$, with $\beta = (4/9\pi)^{1/3}$. Recalling the Rydberg energy $R_Y = 1/(2ma_B^2) = 13.6\text{eV}$, we can then re-write the parameters in the expression for the energy density as

$$\frac{E}{\rho V} = 3R_Y \left[\frac{1}{5\beta^2 r_s^2} - \frac{1}{2\pi\beta r_s} \right], \quad (814)$$

which reveals that βr_s can be taken as a expansion parameter in the expression for the ground-state energy of the interacting electron gas (note that as long as r_s is sufficiently small, the above expression is positive). The point of this is that the interactions are *weaker when r_s is smaller*, meaning that the *high-density* limit is the weakly-interacting limit—perhaps contrary to what might naively expect, with higher density meaning that the electrons are closer together and hence have higher Coulomb energies. This calculation shows however that the increasing kinetic energy of the electrons mandated by the increasing of k_F with density more than compensates for the growing Coulomb interactions, with kinetic energy dominating at higher densities.



Today's diary entry was written while studying for the oral exam. We will derive the ω dependence of $\sigma(\omega)$ in the Drude model and in a BCS SC. For the SC we will also derive the weight of the delta function in limit of large SC gap and comment on its physical significance.



Metals

First let's recall how things work in metals under the purview of the Drude model. The conductivity just comes from the equation of motion of free electrons in an electric field, with an appropriate dissipative term $-\mathbf{k}/\tau$ to account for the scattering. If we want to be pedantic, we can use the Boltzmann equation with the collisional term $\partial_t g_{\mathbf{k}}|_{coll} = (g_{\mathbf{k}}^0 - g_{\mathbf{k}})/\tau$ with $g_{\mathbf{k}}^0$ the equilibrium distribution: the time derivative of the expectation value of the total momentum \mathbf{K} is

$$\partial_t \mathbf{K} = \int_{\mathbf{k}} \partial_t g_{\mathbf{k}} \mathbf{k} = \int_{\mathbf{k}} (e\mathbf{E}(\partial_{\mathbf{k}} g_{\mathbf{k}})\mathbf{k} + \tau^{-1}(g_{\mathbf{k}}^0 - g_{\mathbf{k}})\mathbf{k}) = -e\mathbf{E} - \frac{\mathbf{K}}{\tau}, \quad (815)$$

where we used that $\mathbf{K} = 0$ in the equilibrium distribution. The point of writing the equation of motion like this is that it stresses that \mathbf{K} isn't supposed to be thought of as the momentum of a particular electron, but rather a "hydrodynamic" momentum label attached to the whole distribution of electrons as a whole. This then gives (I guess we are using conventions where $e = |e|$)

$$(\partial_t + 1/\tau)\mathbf{J} = \frac{e^2 n}{m}\mathbf{E} \implies \sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \quad \sigma_0 \equiv ne^2\tau/m, \quad (816)$$

with n the number density. $\omega = 0$ correctly reproduces the DC result, while at large ω we get the τ -independent $\omega(w \rightarrow \infty) = ine^2/(m\omega)$. The large ω limit is generally the reliable one in the relaxion time approximation (where $\partial_t g_{\mathbf{k}}|_{coll}$ has the form given above, with the scattering matrix elements $W_{\mathbf{k} \rightarrow \mathbf{k}'}$ independent of the distribution function). This is because if the timescale ω^{-1} of the background fields is much shorter than the timescale τ it takes for electrons to scatter, then scattering events won't be super important in the response, and so our crude treatment of them won't be such a disaster. Also note that $\sigma_{\mathbb{R}}$ and $\sigma_{\mathbb{I}}$ each separately have ω integrals that are independent of τ_s , in accordance with the sum rules.

We can use this to find the dielectric properties of the metal. We start with the time derivative maxwells equations:

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad \mu_0^{-1} \nabla \times \mathbf{B} = \varepsilon_0 \partial_t \mathbf{E} + \mathbf{J}. \quad (817)$$

Taking $\nabla \times$ of the latter to get the wave equation and using the definition of the conductivity gives the dispersion

$$k^2 = \omega^2/c^2 + i\mu_0\omega\sigma(\omega). \quad (818)$$

We can rewrite this in terms of the dielectric function as

$$k^2 = \omega^2\mu_0\varepsilon(\omega), \quad \varepsilon(\omega) = \varepsilon_0 + \frac{i\sigma(\omega)}{\omega}. \quad (819)$$

In the large frequency regime, which is where we said the relaxion time approximation is usually legit, the point $\varepsilon = 0$ is reached when $\omega = \omega_p$ with the plasma frequency ⁶⁵

$$\omega_p \equiv \sqrt{\frac{e^2 n}{m\varepsilon_0}}. \quad (822)$$

ω_p is the roughly the point at which the metal switches from being shiny to being transparent (the criterion for transparency is that $\varepsilon_{\mathbb{R}} > 0$, while for shininess it is $\varepsilon_{\mathbb{R}} < 0$).

⁶⁵Recall that the plasma frequency can also be found in a more pedestrian way by considering the jiggling of a coherent block of electrons on top of a positively charged ionic background. To do this, use that the electric field a distance d away from an infinite charged slab of charge density ρ is $2\pi/\varepsilon$ (indep. of d).

Also recall that since the plasma frequency is an instability of the motions of the charges in the metal, it can also be found by examining the charge distribution more directly, by Fourier transforming the continuity equation:

$$d^\dagger j = 0 \implies i\omega\rho(\omega) = \nabla \cdot \mathbf{J}. \quad (820)$$

Taking the divergence of both sides,

$$i\omega\varepsilon_0 = \frac{\sigma_0}{1 - i\omega\tau}, \quad (821)$$

which at large $\omega\tau$ tells us that $\omega = \omega_p$.

Superconductors

Now we come to the case of a superconductor. A first approximate model is to treat the superconductor in a two-fluid scheme, with the lifetime τ_s of the superfluid component going to infinity. At low frequencies $\omega < 2\Delta$ (and at $T = 0$), the SCing and normal contributions to the conductivity are respectively

$$\sigma_s(\omega) = \frac{ne^2\tau_s/m}{1 + i\omega\tau_s}, \quad \sigma_n(\omega) \approx ne^2\tau_n/m, \quad (823)$$

where we assumed that $\omega\tau_n \ll 1$. As $\tau_s \rightarrow \infty$, and writing $\sigma_s = \sigma_s^{\mathbb{R}} + i\sigma_s^{\mathbb{I}}$, we find

$$\sigma_s^{\mathbb{R}}(\tau_s \rightarrow \infty) = \frac{\pi ne^2}{m} \delta(\omega), \quad (824)$$

which is checked by doing the integral and getting the arctan. This gives what we expect at low frequencies, namely a δ function concentrated on the DC response (plus a broader dissipative part from the normal fluid component). The strength of this pole is related to various quantities of interest. For example, consider the penetration depth. Recall that this is obtained from Maxwell's equations in the static limit and the London equation $\nabla \times \mathbf{J} = -(ne^2/m)\mathbf{B} \implies \nabla^2\mathbf{B} = (ne^2\mu_0/m)\mathbf{B} \implies \lambda_L = \sqrt{m/(ne^2\mu_0)}$, so that the weight of the $\delta(\omega)$ DC response pole is $\propto \lambda_L^{-2}$.

The imaginary part of the conductivity is

$$\sigma_s^{\mathbb{I}}(\tau_s \rightarrow \infty) = -\frac{ne^2}{m\omega}, \quad (825)$$

which is in accordance with the sum rule $\sigma^{\mathbb{I}} = -\omega/\pi P \int d\omega' \sigma^{\mathbb{R}}(\omega')/(\omega^2 - \omega'^2)$.

These expressions can of course also be derived though e.g. the Kubo formula by way of the LG action: the current correlator term vanishes at zero momentum since the linear realization of the $U(1)$ on the low energy fields means it goes as $q^2/(\omega^2 - q^2)$ —thus only the diamagnetic term remains, and when evaluated at a frequency $\omega + i\eta$ reproduces the results derived more heuristically above.

When $\omega \geq 2\Delta$ we expect the \mathbb{R} part of the conductivity to be nonzero, since $\sigma^{\mathbb{R}}$ is related to dissipation in the SC in response to external fields, which becomes possible when the electric fields are strong enough to excite quasiparticles across the gap. The rough behavior of $\sigma_s^{\mathbb{R}}$ in this domain can be estimated just by knowledge of the density of states $\rho(\omega)$, which goes as $N(0)\omega/\sqrt{\omega^2 - \Delta^2}$ when $\omega \geq 2\Delta$. To eliminate dependence on non-universal parameters one usually divides $\sigma_s^{\mathbb{R}}(\omega)$ by $\sigma_n^{\mathbb{R}}(\omega)$, with the later the conductivity one calculates from a normal metal.⁶⁶ This gives (letting $\Delta \in \mathbb{R}$ wolog and working in

⁶⁶Following the same steps, this is just

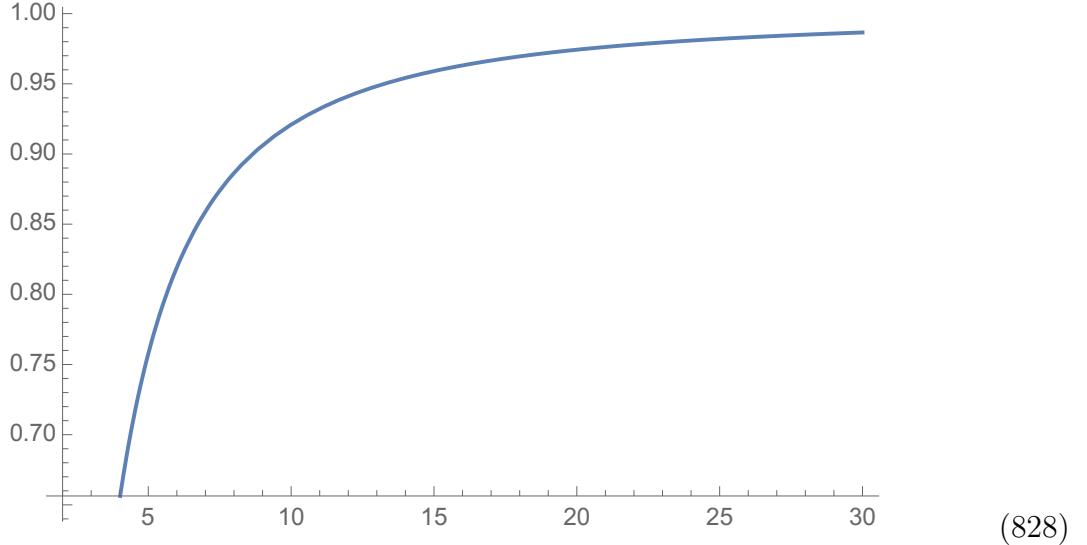
$$\sigma_n^{\mathbb{R}} \propto \int_{\mathbb{R}} dE (f(E) - f(E + \omega)) N(0)^2 = \omega \quad (826)$$

at $T = 0$.

“intrinsic semiconductor conventions” where the zero of energy is right in the middle of the gap)

$$\begin{aligned} \frac{\sigma_s^R(\omega > 2\Delta)}{\sigma_n^R(\omega > 2\Delta)} &= \frac{1}{\omega} \int_{\mathbb{R}} dE \frac{|E(E + \omega) + \Delta^2|}{\sqrt{E^2 - \Delta^2} \sqrt{(E + \omega)^2 - \Delta^2}} \theta(-E - \Delta) \theta(E + \omega - \Delta) \\ &= \frac{1}{\omega} \int_{\Delta-\omega}^{-\Delta} dE \frac{|E(E + \omega) + \Delta^2|}{\sqrt{E^2 - \Delta^2} \sqrt{(E + \omega)^2 - \Delta^2}}. \end{aligned} \quad (827)$$

Here the $+\Delta^2$ in the numerator is a little subtle, and comes from coherence factors stemming from the nature of the BCS wavefunction (basically, matrix elements connecting two qp states are added coherently with their particle-hole conjugates; see Tinkham’s book for the details). When we plot this ratio as a function of ω from $\omega = 2\Delta$ up, we see that $\sigma_s^R(2\Delta) = 0$ but that $\partial_\omega \sigma_s^R|_{2\Delta} > 0$, and that $\sigma_s^R \rightarrow \sigma_n^R$ as $\omega \rightarrow \infty$, as expected:



Miscellaneous comment: how would these results be affected by the presence of disorder? In a superconductor, non-magnetic impurities shouldn’t really change anything: the coherence length $\xi_0 \sim \hbar v_F / kT_c$ (the UV cutoff for GL theory) is typically big enough that the effects of the impurities average out on scales \gtrsim than ξ_0 , which doesn’t affect the SCing DOS or charge transport significantly. Magnetic impurities are a different story though, since they can break Cooper pairs, and so we expect some finite concentration of them to destroy SCvity all together.



We just want the functional form of ρ , and so in the following will be very fast and loose with numerical factors. We will get the DOS through the imaginary part of the retarded Greens function, which is tractable since we'll be working within the purview of MFT. We are interested in the DOS at a given frequency, and so we need to sum $\rho(\omega, q)$ over all momenta q (we want the \mathbb{R} -space local DOS at finite ω). Therefore we will start by computing the time-ordered Greens function

$$G_T(\omega, 0) = \int_q \text{Tr} \begin{pmatrix} \omega + \xi & \Delta \\ \Delta^* & \omega - \xi \end{pmatrix}^{-1}, \quad (829)$$

where we are in \mathbb{R} time and as usual ξ is the energy measured wrt the chemical potential. Note how the signs on the ξ s are swapped due to the choice of the Nambu spinors, but the signs of the ω s are the same since the action is $\int \Psi^\dagger (i\partial_t - H) \Psi$ when we do the path integral over fields which are eigenstates of H .

Inverting and taking the trace,

$$G_T(\omega, 0) = \int_q \frac{\omega}{-\omega^2 + \xi^2 + |\Delta|^2} \quad (830)$$

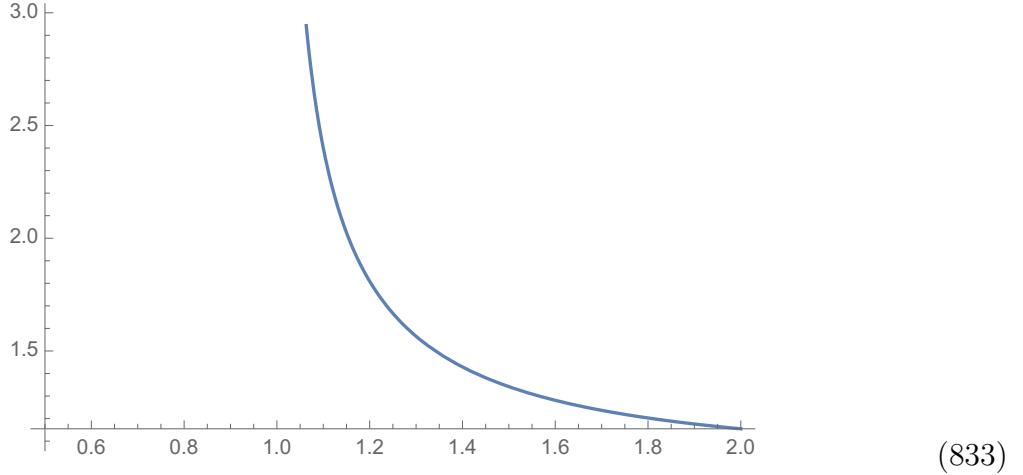
where as warned we are not keeping track of overall prefactors. Changing to an integral over ξ and extending the integral to one over \mathbb{R} as usual, we have

$$G_T(\omega, 0) \approx N(0) \int_{\xi \in \mathbb{R}} \frac{\omega}{-\omega^2 + \xi^2 + |\Delta|^2} \approx \frac{N(0)\omega}{\sqrt{-\omega^2 + \xi^2}}, \quad (831)$$

since the integral is an arctan. To get the causal response $G_+(\omega, 0)$, we need to evaluate this guy at $\omega + i\eta$ with $\eta \rightarrow 0^+$, and then we need to take the imaginary part of G_+ to get the DOS. The $i\eta$ in the numerator doesn't make a difference in the limit $\eta \rightarrow 0$ (easy to check), and neither does the $i\eta$ in the denominator. Therefore we just pick up the imaginary part of $(-\omega^2 + |\Delta|^2)^{-1/2}$, which gives

$$\rho(\omega) \propto \begin{cases} 0 & \omega < |\Delta| \\ \frac{N(0)\omega}{\sqrt{\omega^2 - |\Delta|^2}} & \omega > |\Delta| \end{cases} \quad (832)$$

For $\Delta = 1$ this looks like



As we could have anticipated, the spectral weight gets pushed up to the gap edge, and then falls back down to the free fermion value of $N(0)$ at higher frequencies.



Specific heat in various types of superconductors

Today is simple: calculating the specific heat in various (nodal and not) types of superconductors, both well below T_c and right below T_c . We will mostly just be interested in the T dependence, and so will largely neglect keeping track of numerical factors.



The usual way to proceed is via BCS theory. We start from

$$C = \partial_{\ln T} S = -\partial_{\ln \beta} S = \beta \partial_\beta \int_{\mathbf{k}} ((1 - f_{\mathbf{k}}) \ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}} \ln f_{\mathbf{k}}), \quad (834)$$

where $f_{\mathbf{k}}$ is the Fermi function, implicitly dependent on $\Delta(T)$. Remember that we have to include the holes here because they specify information distinct from the particles, e.g. just knowing that you have a particle a certain distance above the FS doesn't tell you how much energy the state you're looking at has; you need to know where in the FSea the hole is located as well. Taking the derivative,

$$C = -\beta \int_{\mathbf{k}} \partial_\beta f_{\mathbf{k}} \ln \frac{1 - f_{\mathbf{k}}}{f_{\mathbf{k}}} = -\beta^2 \int_{\mathbf{k}} (\partial_\beta f_{\mathbf{k}}) E_{\mathbf{k}} \quad (835)$$

In BCS theory, $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$, with $\Delta \in \mathbb{R}$ wolog. The unusual part about this energy is that it itself is temperature dependent. Therefore the specific heat gets two contributions: one from how the qps redistribute among the energy levels as T changes, and one from how the energy levels themselves change with T :

$$C = \beta^2 \int_{\mathbf{k}} \frac{1}{\cosh^2(\beta E_{\mathbf{k}}/2)} (E_{\mathbf{k}}^2 + \beta \partial_{\beta} \Delta) \approx \beta^2 \nu \int_{\xi} d\xi \frac{1}{\cosh^2[\beta \sqrt{\xi^2 + \Delta^2}/2]} (\xi^2 + \Delta^2 + \Delta \partial_{\ln \beta} \Delta), \quad (836)$$

with ν as usual the DOS at the FS.

First consider the $T > T_c$ regime, where $\Delta = 0$. Then we get

$$C(T > T_c) = \beta^2 \nu \int_{\xi} \frac{d\xi \xi^2}{\cosh^2[\beta \xi/2]} = \beta^{-1} \nu \int_x \frac{dx x^2}{\cosh^2[x/2]} \propto \nu T, \quad (837)$$

which is the usual Fermi liquid result. Now consider what happens when $T \ll T_c$. In this regime Δ is approximately independent of T (to be substantiated below), and so for an s-wave superconductor we have

$$C(T \ll T_c) \approx \beta^2 \nu \int_{\xi} d\xi \frac{\xi^2 + \Delta^2}{\cosh^2[\beta \sqrt{\xi^2 + \Delta^2}/2]}. \quad (838)$$

Since there is now no region of ξ in which the argument of the \cosh^2 is small, the integral will be exponentially small, and will go as

$$C(T \ll T_c) \propto P(T) e^{-\Delta/T}, \quad (839)$$

where $P(T)$ is some polynomial in T^{67} —the important part is the exponential decay, as expected of a gapped system. For a nodal superconductor we instead expect a power law, rather than an exponential. For example, consider the case where we are in two dimensions, with the gap going like $\Delta(\phi) \propto \Delta \sin \phi$ for some angle ϕ . Then at $\Delta/T \gg 1$ we approximately have

$$C(T \ll T_c) \propto \beta^2 \nu \int_{\xi} d\xi \int_{-\phi_*}^{\phi_*} d\phi \frac{\xi^2 + \Delta^2 \phi^2}{\cosh^2[\beta \sqrt{\xi^2 + \Delta^2 \sin^2 \phi}/2]}, \quad \phi_* = \xi/\Delta. \quad (842)$$

Doing the integral over ϕ simply by evaluating the integrand at ϕ_* and multiplying by ϕ_* (as we don't care about constants anyway) gives

$$C(T \ll T_c) \propto \beta^2 \nu \Delta^{-1} \int d\xi \xi^3 \cosh^{-2}[\beta \xi] \propto \frac{T^2}{\Delta} \nu, \quad (843)$$

⁶⁷More precisely, we write the \cosh at small ξ/Δ approximately as

$$\cosh^2[\beta \sqrt{\xi^2 + \Delta^2}/2] \approx \exp(-\beta(\Delta + \xi^2/2\Delta)). \quad (840)$$

The $e^{-\beta\Delta}$ gives us the exponential decay, while the $e^{-\beta\xi^2/2\Delta}$ gets integrated to give a factor of $\sqrt{\Delta T}$, which means that

$$C(T \ll T_c) \propto \frac{\Delta^2}{T^2} \sqrt{\Delta T} e^{-\Delta/T} \quad (841)$$

which gives us a T^2 power law. Similar considerations apply to different types of gaps. For example, a three-dimensional SC with point nodes would have a specific heat scaling as (taking e.g. $\Delta(\phi, \theta) = \Delta \sin \theta$)

$$C(T \ll T_c) \propto \beta^2 \nu \int_{-\theta_*}^{\theta_*} d\theta \sin \theta \frac{\xi^2 + \Delta^2 \sin^2 \theta}{\cosh^2[\beta \sqrt{\xi^2 + \Delta^2 \sin^2 \theta}/2]} \approx \frac{\beta^2 \nu}{\Delta^2} \int d\xi \frac{\xi^3}{\cosh^2[\beta \xi/2]} \propto \frac{T^3}{\Delta^2} \nu, \quad (844)$$

which gives us a T^3 power law.

Finally we examine the regime $T \lesssim T_c$ and look for the discontinuity in C across T_c . Here Δ^2 is small, but $\partial_{\ln \beta} \Delta$ is not. Indeed, we will show in a sec that

$$\Delta(T) \sim \sqrt{(T_c - T)T_c} = T_c \sqrt{-t} \quad (845)$$

for $T \lesssim T_c$, in accordance with MFT. Therefore $\partial_{\ln \beta} \Delta = -T \partial_T \Delta \implies (\Delta \partial_{\ln \beta} \Delta)|_{T_c} \sim T_c^2$. So right below T_c , we have

$$C(T \lesssim T_c) \approx \frac{\nu}{T_c^2} \int_\xi d\xi \frac{\xi^2 + T_c^2}{\cosh^2[\xi/2T_c]} \sim \nu T_c \int dx \frac{x^2 + 1}{\cosh^2[x/2]}. \quad (846)$$

Therefore $C_{SC}(T_c - \varepsilon)/C_{\text{normal metal}}(T_c)$ is some $O(1)$ number independent of T_c that can be computed by actually doing the integrals over x (which turns out to be 2.43-ish). So, the overall picture is that as we decrease T starting from above T_c , we have a linear decrease, then a discontinuous (but finite) jump at T_c , and then an exponential decay down to zero below T_c .

Our claim about the jump in C relied on knowing that $\Delta(T) \sim T_c \sqrt{-t}$ near T_c . This is obvious from GL theory, but let's establish it here by solving the gap equation

$$\frac{1}{g\nu} = \int_{-\Lambda}^{\Lambda} d\xi \sum_{\omega} \frac{1}{(i\omega - E(\xi))(-i\omega - E(\xi))}, \quad (847)$$

where Λ is the UV cutoff, g the interaction strength, and the propagators here come from the 1-loop bubble encountered when getting the quadratic part of the action for Δ after doing mean field theory (the opposite signs on the ω s is because it's a BCS-channel bubble). The sum is over $\omega \in 2\pi T \mathbb{Z}$, and we treat this in the usual way by introducing $z = i\omega$, multiplying by $f(z)$, and integrating z along a contour encircling the $i\mathbb{R}$ axis (with f the Fermi function). There are two poles not on the $i\mathbb{R}$ axis, viz at $z = \pm E(\xi)$, and so we get

$$\frac{1}{g\nu} = \int_{-\Lambda}^{\Lambda} d\xi \frac{f(E(\xi)) - f(-E(\xi))}{2E(\xi)} = \int_0^{\Lambda} d\xi \frac{\tanh(E(\xi)\beta/2)}{E(\xi)}. \quad (848)$$

The integral here is actually not that straightforward. Getting the critical temperature is easy, at least: setting $\Delta = 0$, we have

$$\frac{1}{g\nu} = \int_0^{\Lambda/2T_c} dx \frac{\tanh(x)}{x} \approx \ln(\Lambda/2T_c) \implies T_c \sim \Lambda e^{-1/g\nu}, \quad (849)$$

which is the usual dimensional transmutation result (here we used $\Lambda/2T_c \gg 1$).

Now we'd like to get $\Delta(T)$ near T_c . Unfortunately even though the integral above for T_c is dominated by large x , i.e. large ξ , we cannot just do an expansion in Δ/ξ in the gap equation integral—when we're trying to get the T dependence of $\Delta(T)$, the small ξ regions actually do contribute to the integral. So we will split up the integral into a small ξ regime and a large ξ regime. Taking the variation of the gap equation wrt Δ and setting $\tanh(x > 1) \approx 1$, we have

$$\frac{1}{g\nu} \approx \ln(\Lambda/2T) + \Delta\delta_\Delta \left(\int_0^1 dx \frac{\tanh \sqrt{x^2 + d^2}}{\sqrt{x^2 + d^2}} + \int_1^{\Lambda/2T} dx \frac{1}{\sqrt{x^2 + d^2}} \right), \quad (850)$$

where $x \equiv \xi/2T$, $d \equiv \Delta/2T$. We will crudely treat the last integral as $\ln(\Lambda/2T)$ which dies under the δ_Δ . The first integral is roughly $\int_0^1 dx(y - y^3/3)/y$ with y the argument of \tanh ; hence

$$\frac{1}{g\nu} \sim \ln(\Lambda/2T) + \Delta\delta_\Delta \int_0^1 dx(1 - (x^2 + d^2)/3) \implies \frac{1}{g\nu} \sim -\ln(2T_c(1+t)/\Lambda) - \Delta^2/T_c^2, \quad (851)$$

where we took t small and dropped $O(\Delta^2 t)$. Expanding the log then tells us that

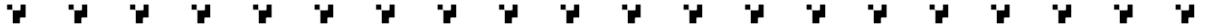
$$\Delta \sim T_c \sqrt{-t} \quad (852)$$

as claimed earlier.



Getting the Landau-Ginzburg coefficients microscopically

Today we're going to show how the coefficients in the LG action are obtained from a microscopic action, and will confirm that the ϕ^2 coefficient is indeed $\nu(T - T_c)/T_c$, with ν the FS DOS. Doing this calculation from the microscopic action is somewhat against the EFT philosophy of which I am a strong subscriber to, but I think it's still useful as a consistency check.



The way this works is by now tediously straightforward and barely worth a diary entry, so we'll be rather telegraphic. We start by decoupling the microscopic 4-fermion interaction $g\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow$ in the BCS channel with a bosonic field Δ . Then after integrating out the fermions, we have

$$S[\Delta] = \int \frac{1}{g} |\Delta|^2 + \text{Tr} \ln \begin{pmatrix} i\omega - \xi & \Delta \\ \Delta^* & i\omega + \xi \end{pmatrix}^{-1} \quad (853)$$

Separating out the part of the log that doesn't involve Δ and dropping it as usual, this is

$$S[\Delta] = \int_{\mathbf{k}} \left(\frac{1}{g} |\Delta_{\mathbf{k}}|^2 - T \sum_{\omega} \text{Tr} \ln \left[\mathbf{1} + \begin{pmatrix} \frac{1}{i\omega - \xi_{\mathbf{k}}} & \\ & \frac{1}{i\omega + \xi_{\mathbf{k}}} \end{pmatrix} \begin{pmatrix} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & \end{pmatrix} \right] \right) \quad (854)$$

Since we are interested in the parameters in LG theory, we can expand this in powers of Δ . Furthermore if we just want to get a flavor of how the coefficients are determined by computing the mass term in the LG action, we can content ourselves with a quadratic expansion in Δ , and furthermore may set the momenta of all Δ s to zero. This then gives (the structure of the particle-particle propagators in the bubble comes from the off-diagonal-ness of the Δ matrix in the log)

$$S[\Delta] \supset \nu \int_{\xi} |\Delta|^2 \left(\frac{1}{g} + T \sum_{\omega} \frac{1}{(i\omega - \xi)(-i\omega - \xi)} \right). \quad (855)$$

To do the sum, we multiply by $f(z)/T$ with $z = i\omega$ and f the Fermi distribution, and then deform the contour to pick up the poles at $z = \pm\xi$. This then gives

$$S[\Delta] \supset \nu \int_{\xi} |\Delta|^2 \left(\frac{1}{g} + \frac{f(\xi) - f(-\xi)}{2\xi} \right). \quad (856)$$

Now we write $f(\xi) - f(-\xi) = 2f(\xi) - 1$ and expand $f(\xi)$ about T_c . Since

$$\partial_T f(\xi) = -T^{-2} \partial_{\beta} f(\xi) = -T^{-2} (\partial_{\xi} f(\xi)) (\xi/\beta) = -T^{-1} \partial_{\xi} f(\xi), \quad (857)$$

we have after some algebra

$$S[\Delta] \supset 2\nu \int_0^{\Lambda} d\xi |\Delta|^2 \left(\frac{1}{g} - \frac{1 - 2f(\xi; T_c)}{2\xi} - t \partial_{\xi} f(\xi; T_c) \right) \quad (858)$$

with $t = (T - T_c)/T_c$ the reduced temperature.

The first two terms cancel by the definition of T_c , while the third just integrates to νt since $\int_0^{\Lambda} d\xi \partial_{\xi} f(\xi) = -1/2$. Therefore the quadratic term in the LG action is in fact

$$S_{LG} \supset \int t \nu |\Delta|^2. \quad (859)$$

Higher orders of $|\Delta|^2$ in the expansion come with coefficients of the form (from now on we will drop unimportant numerical factors)

$$S[\Delta] \supset \sum_n T \sum_{\omega} \int_{\xi} \nu \frac{|\Delta|^{2n}}{[(i\omega - \xi)(i\omega + \xi)]^n}, \quad (860)$$

where there is only a single frequency / momentum sum since we've taken Δ to be constant, and there are only even terms due to the off-diagonal-ness of the Δ matrix in the log. Also note that when $n \in 2\mathbb{Z}$, the $|\Delta|^{2n}$ term has a positive coefficient—therefore e.g. unlike the $|\Delta|^2$ term, the $|\Delta|^4$ term is guaranteed to be positive, meaning that regardless of the sign of the $|\Delta|^2$ term, we can generically stop at order $|\Delta|^4$ in the expansion and get a stable theory.

To find the T dependence of this term, we do something a bit slick. If we do the frequency sum first as usual, we get an integral that includes stuff like $\int d\xi \xi^{-2n} f(\xi)$, which has a bad IR divergence. So instead, we integrate over ξ first. Extending the upper limit to ∞ and dropping constants (in the form of Γ functions) we get

$$S[\Delta] \supset \sum_n \nu \oint dz \frac{f(z)|\Delta|^{2n}}{z^{2n-1}}, \quad (861)$$

where the contour on z encloses all the poles except for the order $2n - 1$ pole at $z = 0$.⁶⁸ The residue of the (non-Matsubara) pole at $z = 0$ comes from taking $2n - 2$ derivatives of the Fermi function in the numerator, and so we get

$$S[\Delta] \supset \sum_n \nu (\partial_\xi^{2n-2} f(\xi)|_{\xi=0}) |\Delta|^{2n}. \quad (862)$$

Each derivative of the Fermi function brings down one power of T^{-1} , and so

$$S[\Delta] \supset \sum_n \nu T^2 |\Delta/T|^{2n}. \quad (863)$$

Note that this power counting is consistent with our more careful evaluation of the $n = 1$ bubble, in terms of the powers of T appearing in the n th order expression. Also note that unlike the quadratic term, all of the higher order terms do not vanish near T_c —it is only the quadratic term that generically controls what phase the system is in. This is all expected on general EFT reasoning, but it's nice to have explicit confirmation in the form of the above calculations.

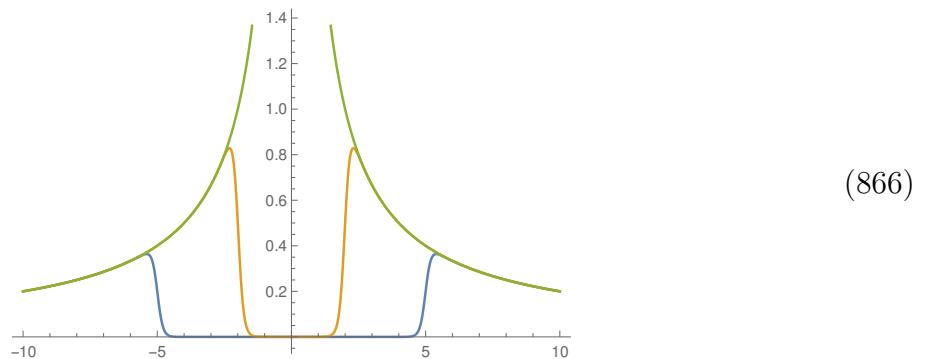
We can also look at how a magnetic field enters things. For singlet pairing, we now have

$$S[\Delta] \supset \nu \int_\xi |\Delta|^2 T \sum_\omega \frac{1}{(-i\omega + \xi - B)(-i\omega - \xi - B)}, \quad (864)$$

where we have set $g\mu_B/2 = 1$. Doing the frequency sum,

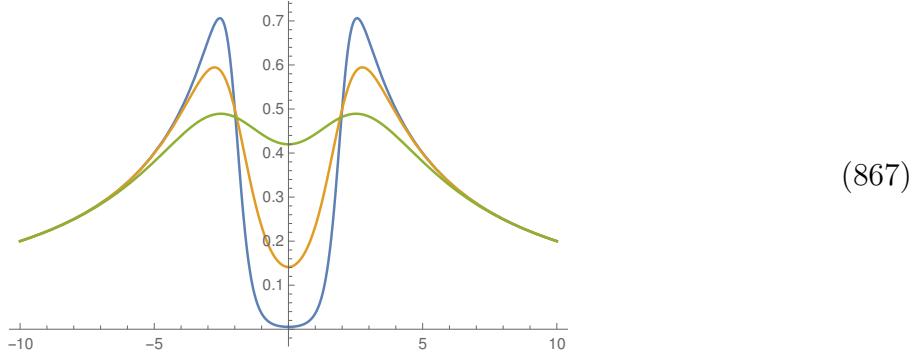
$$S[\Delta] \supset -\nu \int_\xi |\Delta|^2 \frac{\tanh[(\xi + B)/2T] + \tanh[(\xi - B)/2T]}{4\xi}. \quad (865)$$

The function in the integrand looks like, for varying B ,



⁶⁸This is a little bit subtle since this pole lies on top of one of the Matsubara frequencies—I think this is the reason for the IR divergence that comes up if you do the $\oint dz$ first.

Here larger B removes area from the integrand, with the integrand essentially vanishing at low T when $|\xi| < B$. This means that it is no longer possible to make the integral arbitrarily large by taking T to be arbitrarily small, which means that there will be a B_c at which $T_c = 0$. For varying T ,



which does what we expect.

To see how T_c should behave with B , we expand the above term to order B^2 . The linear term vanishes (can't depend on the sign of B), while the quadratic term is a rather complicated integral — the important part is it gives the term

$$S[\Delta] \supset +\nu C |\Delta|^2 B^2, \quad (868)$$

where $C > 0$. This decreases the condensate fraction by an amount B^2 . This is another way of deriving that $T_c \propto \sqrt{1 - B^2}$.



Today we are doing an elaboration on two problems in Simon's solid state book and one from A&M. We'll be deriving and commenting on the spectrum of spin waves in (anti)-ferromagnets.



Note: in this journal entry only, X, Y, Z will denote spin operators, i.e. $1/2$ times the appropriate Pauli matrix.

Ferromagnets

The Hamiltonian is

$$H = - \sum_{\langle \mathbf{r} \mathbf{r}' \rangle} J(\mathbf{r} - \mathbf{r}') S_{\mathbf{r}}^a S_{\mathbf{r}'}^a \quad (869)$$

with $J(\mathbf{r}) \geq 0$ (n.b. we will not assume short-ranged J).

First we will do the "quantum" approach, whereby we start from the state $\otimes |\uparrow\rangle$ (which is a *provably exact* ground state) and look at the excitations on top of it that carry minimal S^z . These excitations are obtained by flipping a single spin. In \mathbb{R} space, we then define the states

$$|\mathbf{r}\rangle = \frac{1}{\sqrt{2S}} S_{\mathbf{r}}^- |\otimes \uparrow\rangle, \quad (870)$$

where $|\otimes \uparrow\rangle$ is the all \uparrow ground state and the prefactor is for normalization. Since the $XX + YY$ term means that H isn't diagonal in this basis, we need to Fourier transform. Hence define

$$|\mathbf{k}\rangle = \frac{1}{\sqrt{N}} \sum_{\mathbf{r}} e^{i\mathbf{k} \cdot \mathbf{r}} |\mathbf{r}\rangle \quad (871)$$

where $N = L^d$ is the number of sites. In these states, we have $\langle Z_{\mathbf{r}} \rangle_{\mathbf{k}} = S - 1/N$, so that the $\delta Z = 1$ of the spin flip is spread out uniformly over the system.

We want to compute $E_{\mathbf{k}} - E_0$. The difference in the ZZ part of the interaction is, using translation invariance and orthogonality to do two of the sums,

$$E_{\mathbf{k}} - E_0 \supset - \sum_{\mathbf{r}} \sum_{\mathbf{R}} J(\mathbf{R}) (\langle \mathbf{r} | Z_0 Z_{\mathbf{R}} | \mathbf{r} \rangle - \langle \uparrow | Z_0 Z_{\mathbf{R}} | \uparrow \rangle). \quad (872)$$

The difference comes when \mathbf{r} is 0 or \mathbf{R} (we are assuming $J(0) = 0$ wolog since this would just add a trivial constant to the Hamiltonian), with an energy difference of $S(S-1) - S^2 = -S$. Hence

$$E_{\mathbf{k}} - E_0 \supset 2S \sum_{\mathbf{R}} J(\mathbf{R}). \quad (873)$$

The other part contributing to the energy difference is the hopping term, which has zero vev in the ground state. Hence

$$\begin{aligned} E_{\mathbf{k}} - E_0 &\supset -\frac{1}{N} \sum_{\mathbf{R}\mathbf{R}'\mathbf{r}\mathbf{r}'} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} J(\mathbf{R} - \mathbf{R}') \langle \mathbf{r}' | \frac{S_{\mathbf{R}}^+ S_{\mathbf{R}'}^- + S_{\mathbf{R}'}^+ S_{\mathbf{R}}^-}{2} | \mathbf{r} \rangle \\ &= -\frac{1}{2NS} \sum_{\mathbf{R}\mathbf{R}'\mathbf{r}\mathbf{r}'} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} J(\mathbf{R} - \mathbf{R}') \langle \uparrow | \frac{\delta_{\mathbf{R}\mathbf{r}} \delta_{\mathbf{R}'\mathbf{r}'} S_{\mathbf{R}}^+ S_{\mathbf{R}'}^- S_{\mathbf{R}'}^+ S_{\mathbf{R}}^- + (\mathbf{R} \leftrightarrow \mathbf{R}')}{2} | \uparrow \rangle \quad (874) \\ &= -2S \sum_{\mathbf{R}} J(\mathbf{R}) \cos(\mathbf{R} \cdot \mathbf{k}). \end{aligned}$$

Therefore if we add a constant to H so that the ground state is at zero energy, we have

$$E_{\mathbf{k}} = 2S \sum_{\mathbf{R}} J(\mathbf{R}) (1 - \cos(\mathbf{R} \cdot \mathbf{k})). \quad (875)$$

The spin waves are therefore gapless, but perhaps counter to our naive expectations based on symmetry breaking, they are *quadratically* dispersing near zero momentum.

All this was for the isotropic chain. What happens when we break the $SO(3)$ (and it is $SO(3)$ not $SU(2)$, since the operators in the Hamiltonian transform adjointly under $SU(2)!$) symmetry by generalizing to

$$H = - \sum_{\langle \mathbf{R} \mathbf{R}' \rangle} [(J(\mathbf{R} - \mathbf{R}') + \Delta(\mathbf{R} - \mathbf{R}')) Z_{\mathbf{R}} Z_{\mathbf{R}'} + J(\mathbf{R} - \mathbf{R}')(X_{\mathbf{R}} X_{\mathbf{R}'} + Y_{\mathbf{R}} Y_{\mathbf{R}'})] ? \quad (876)$$

The answer is fairly obvious if $\Delta > 0$, which we will assume, since then the symmetry is explicitly broken down to $U(1)$, with $\otimes | \uparrow \rangle$ remaining the ground state. Since the symmetry is broken explicitly down to $U(1)$ (while $U(1)$ is presumed to be unbroken in our trial spin wave vacuum state), there should be no Goldstones, with the spectrum totally gapped. This is checked by running the calculation above for $E_{\mathbf{k}}$ —the only change from the anisotropy is a $2S \sum_{\mathbf{R}} \Delta(\mathbf{R})$ from the ZZ term, and so

$$\delta E_{\mathbf{k}} = 2S \sum_{\mathbf{R}} \Delta(\mathbf{R}), \quad (877)$$

which gaps the spectrum. Note that while the spin wave spectrum just gets shifted up in energy by a constant, this is of course not a trivial shift in the Hamiltonian by a constant—it only looks like this when we look at the ground state or the single spin wave states.

The "classical" way of doing this is to take the eoms and expand them about a reference ordered state. For simplicity, we will assume only nearest-neighbor interactions—general long-ranged J s are harder to deal with in this approach. From the Heisenberg eoms, we find

$$i\hbar d_t S_i^a = [S_i^a, H] = -Ji \sum_j \epsilon_{abc} S_i^c S_j^b, \quad (878)$$

where the sum is over the neighbors of i . To get spin waves, we use the ansatz that

$$S^a = \begin{pmatrix} S + O(\varepsilon^2) \\ x \\ y \end{pmatrix}, \quad (879)$$

where x, y are of order ε , where $\varepsilon = 1/N$ with N the chain length. Putting this ansatz in and keeping only the terms linear in x, y , we have

$$\begin{aligned} d_t x_i &= -JS \sum_j (y_j - y_i) \\ d_t y_i &= -JS \sum_j (x_i - x_j) \end{aligned} \quad (880)$$

Now we can add the first to i times the second to get

$$d_t S_i^+ = iJS \sum_j (S_j^+ - S_i^+). \quad (881)$$

The fact that we get only one equation of motion for the spin waves despite them naively having two degrees of freedom (spontaneously breaking $SO(3)$ down to $U(1)$) should give us two goldstone modes since the coset space is locally S^2) is due to the fact that the two putative Goldstone modes are actually canonically conjugate variables, and hence are not independent. Said another way, the way in which the x and y components of the spin wave time evolve are not independent from one another: they differ by a $\pi/2$ phase shift,⁶⁹ so that the operators S^\pm remain well-defined under time evolution.

Anyway, from the S^+ eom we see that on a cubic lattice,

$$\omega = 2JS \left(d - \sum_j \cos k_j \right). \quad (883)$$

This gives us the same gapless quadratic dispersion (with a mass determined by $1/4JS$) as the "quantum" computation above.

How does the anisotropy result in a mass term here? If we add the term

$$H \supset -\Delta \sum_{\langle ij \rangle} Z_i Z_j \quad \Delta > 0 \quad (884)$$

then we see that the eom for S^+ is now

$$d_t S^+ = iJS \sum_j (S_j^+ - S_i^+ (1 - \Delta)). \quad (885)$$

Therefore the dispersion is now

$$\omega = 2JS \left(d - \sum_j \cos k_j + \Delta \right), \quad (886)$$

which is gapped by $2JS\Delta$, as expected.

Antiferromagnets

With AFMs the situation is trickier—in the FM case we started by introducing minimal excitations on top of a symmetry-broken state, which was an exact ground state of the system. For AFMs in general dimensions we don't even know what the AFM ground state is, and so we lack a suitable starting point.

The equations of motion approach is still useful though, if we make the assumption that we can describe the spin waves as perturbations away from a Neel-ordered reference state. Therefore we take the same equations of motion as above, but insert the ansatz

$$\mathbf{S}_i = \begin{pmatrix} x_i \\ y_i \\ (-1)^i S \end{pmatrix}, \quad (887)$$

⁶⁹The equations of motion in Fourier space look like

$$\begin{pmatrix} \omega & \lambda \\ -\lambda & \omega \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0. \quad (882)$$

One finds that $\lambda/\omega = \pm i$, which means that $X = \pm iY$; hence the phase shift.

where $(-1)^i$ takes on different signs on the two sublattices. Putting this into the eom and using that $(-1)^{i+j} = -1$ if i, j are nearest neighbors, we find

$$\begin{aligned} d_t x_i &= (-1)^i JS \sum_j (-y_i - y_j) \\ d_t y_i &= (-1)^i JS \sum_j (x_i + x_j). \end{aligned} \quad (888)$$

Adding i times the second one to the first to get an eom for the raising operator,

$$d_t S_i^+ = i(-1)^i JS \sum_j (S_i^+ + S_j^+). \quad (889)$$

There are two notational ways one can deal with solving problems like this, where the lattice has an enlarged unit cell in the ansatz under consideration. One is to solve it with momenta variables as if there was no enlarged unit cell, and then after computing the spectrum to realize that the periodicity of the momenta variables you're working with needs to be changed to account for the different unit cell. The other is to work in coordinates where each unit cell defines a single point on the lattice, with hoppings in between sites of a unit cell picking up no momentum phases. We will use the first approach. Therefore we write the above in Fourier space as

$$\omega \begin{pmatrix} S_A^+ \\ S_B^+ \end{pmatrix} = \begin{pmatrix} 2dJS & -2JS \sum_j \cos k_j \\ 2JS \sum_j \cos k_j & -2dJS \end{pmatrix} \begin{pmatrix} S_A^+ \\ S_B^+ \end{pmatrix} \quad (890)$$

where now A denotes a sublattice, not a lattice site. Diagonalizing, the dispersion is

$$\omega = \pm 2JS \sqrt{d^2 - \left(\sum_{i=1}^d \cos(k_i) \right)^2}. \quad (891)$$

In one dimension, this reduces to the simple

$$\omega = \pm 2JS |\sin(k)|. \quad (892)$$

In contrast to the ferromagnet, we have the expected number of Goldstones, namely 2, and as in the usual scenario, they are linearly dispersing near zero momenta. One can check in a similar way that explicitly breaking the symmetry by adding e.g. a ZZ term gaps the modes.



(Most of) the different critical fields for type I and type II superconductors

Today we will explain the criterion for distinguishing between type I and type II SCs, and will give derivations for the values of the various critical fields H_{c1} , H_c , and H_{c2} .



H_c —the thermodynamic critical field

When we think about the electrodynamics and thermodynamics of SCs, we are usually interested in situations where the SC is placed in an external field.

The field H , being a background field, is constant everywhere⁷⁰ regardless of what phase the system is in, and its purpose in life is to generate correlation functions of the magnetization M through functional differentiation of $\ln Z$.

Let's now define the thermodynamic critical field H_c . One makes the assumption that inside of the material, the free energy at fixed B contains the term

$$\mathcal{F}(B) \supset \frac{1}{2\mu_0} \int d^d x B^2, \quad (893)$$

and that $\mathcal{F}(B)$ has no other B dependence. Now in the context of thinking about magnetism in SCs, we would rather work at fixed H than at fixed B . Therefore we define the fixed- H free energy by doing a Legendre transform as

$$F(H) = \mathcal{F}(B) - \int BH, \quad (894)$$

where we have $\partial_H F = -B$, $\partial_B \mathcal{F} = H$. The usual notation for this would be something like $G = F - \int BH$, but we will just choose to call $F(H)$ the free energy, rather than something like "Gibbs free energy". The type of free energy we're dealing with will be distinguished by context or by its arguments. Note that instead of defining the Legendre transformation with BH , we can also do the transformation with $-\int MH\mu_0$ —the two differ by the background term $\int H^2\mu_0$, which being a constant in every phase won't affect relative free energy difference between phases. In the next little bit we will indeed use $-\int MH\mu_0$, but later on may switch.

In the normal state, we assume that the magnetization vanishes, and hence that $B = \mu_0 H$.⁷¹ Therefore $F_n(B) = V\mu_0 H^2/2$. In the SCing state, $B = 0$, and so $M = -H$, meaning

⁷⁰Well, it doesn't have to be, but we'll take it to be uniform in space. The point is that unlike B which is "dynamical" (meaning that its expectation value can be affected by the dynamical degrees of freedom, not [necessarily] that we're integrating over it), H is completely static and fixed—it's just a device used to generate correlation functions of M .

⁷¹Recall that $\mu_0^{-1}B = H + M$ in SI units. Unfortunately the conventions here aren't very great— H and M are microscopic things determined by computing expectation values, while H is a background field used to generate M correlation functions. Therefore its a little weird to put the μ_0^{-1} only on the B —oh well.

that the magnetic part of the SC free energy is $F_s^{(H)} = V\mu_0 H^2$. The difference in these two free energies defines the thermodynamic critical field via⁷²

$$\frac{\mu_0}{2} VH_c^2 \equiv F_s^{(H)} - F_n. \quad (899)$$

H_c can be expressed in GL theory by equating it to the condensation energy F_s^ψ of the SC, since at a field of H_c we have $F_s^{(H)} + F_s^\psi = F_n$. Writing the ψ part of the GL free energy as

$$F = \int \left(\gamma |D_{\mathbf{A}}\psi|^2 + t|\psi|^2 + \frac{1}{2}u|\psi|^4 + \dots \right), \quad (900)$$

we see that $\psi_0 = \sqrt{-t/u}$, so that the condensation energy is

$$F_s^\psi = -\frac{t^2}{2u} = \frac{t\gamma}{2u\xi^2(T)} \quad (901)$$

where $\xi(T) = \sqrt{-\gamma/t}$ is the LG "coherence" length (better would be "correlation length" or "stiffness length" or similar).

Given this definition of H_c , we can determine what the critical current in an SCing wire is. If we force a current J through a SCing wire, then it will produce a magnetic field parallel to the surface of magnitude $B_\parallel = J\mu_0/(2\pi a)$, where a is the radius of the SCing wire. Therefore the critical current is

$$J_c = \mu_0^{-1} 2\pi a H_c \quad (902)$$

The fact that J_c scales with the circumference of the wire is in keeping with the fact that the current is confined to within a distance λ_L of the surface of the SC by the London equations (this gives a critical current density which is essentially independent of the wire size, as long as $\lambda_L \ll a$).

⁷² Aside: this has all been in SI units. For better or worse, most people use cgs when talking about SCs. The version of this in CGS is as follows. There the magnetic energy is

$$\mathcal{H} \supset \frac{1}{2 \cdot 4\pi} \int B^2, \quad (895)$$

while the relationship between M, B and H is

$$B = H + 4\pi M. \quad (896)$$

The free energy then contains

$$F \supset \int \left(\frac{1}{8\pi} B^2 - MH \right). \quad (897)$$

In the normal state we have $B = H$ and $F_n = H^2/8\pi$, while in the SCing state $M = -H/4\pi$ and so $F_s = H^2/4\pi$, and now the definition of H_c is

$$V \frac{H_c^2}{8\pi} = F_s^{(H)} - F_n. \quad (898)$$

We can now ask what order the SCing phase transition is. This is a problem in A&M, and the strategy is to look at which thermodynamic quantities are discontinuous across the transition. Since we have a background H field but don't have a background field (pressure) for volume or anything else, the operative free energy has a differential of $dF = -(SdT + MdH)$. Right at the critical point the free energies for the SCing and normal states are the same, and hence dF will be the same when computed in either the SC or the normal metal. Therefore at the critical field,

$$S_s dT + M_s dH_c = S_n dT + M_n dH_c \quad (903)$$

Therefore

$$d_T H_c = \frac{S_n - S_s}{M_s - M_n}. \quad (904)$$

Since $M_n = 0$ while $M_s = -H$, we have

$$S_n - S_s = -H_c \partial_T H_c. \quad (905)$$

Since this is a nonzero discontinuity, we expect the SCing transition to be first order at all non-zero fields. If the transition occurs at zero field (by tuning T up to T_c so that $H_c \propto t = 0$), we get $S_n = S_s$, indicating a second-order transition (modulo HLM). The latent heat is just the above multiplied by T . We can find the specific heat jump by

$$C = \partial_T U = \partial_S U \partial_T S = \partial_{\ln T} S, \quad (906)$$

so that

$$C_n - C_s = -T ((\partial_T H_c)^2 + H_c \partial_T^2 H_c), \quad (907)$$

which at zero field gives

$$(C_n - C_s)|_{H_c=0} = -T(\partial_T H_c)^2. \quad (908)$$

Since we can compute this discontinuity within LG theory without having to turn on any EM fields, this lets us get $\partial_T H_c$ without having to actually do calculations with magnetic fields.

H_{c2} —the biggest field for which $|\psi|$ can be non-zero

H_{c2} is best thought of by imagining a process in which one starts at $T < T_c$ ($t < 0$) but with a field high enough to completely kill SCtivity, and then reduces the field to the point where SCing order can form. H_{c2} is identified as the external field for which the LG equations first admit solutions for infinitesimal nonzero values of $|\psi|$.

Since at H_{c2} $|\psi|$ is infinitesimal, we can work with the linearized LG free energy for ψ , namely

$$F = \int (\gamma |D_{\mathbf{A}}\psi|^2 + t|\psi|^2 + \dots), \quad (909)$$

where the \dots are magnetic field terms. Varying with respect to ψ^* , we have

$$(t - \gamma D_A^2)\psi = 0 \implies (\mathbf{p} - q\mathbf{A})^2\psi = \xi^{-2}(T)\psi. \quad (910)$$

Solving this is then just a Landau level problem. Choosing e.g. $\mathbf{A} = (0, Bx, 0)$, we write this suggestively as

$$\frac{2(qB)^2\gamma}{2} \left[\frac{1}{(qB)^2} p_x^2 + (p_y/qB - x)^2 \right] \psi = -t\psi \quad (911)$$

Therefore we have a harmonic oscillator where $k = 2(qB)^2\gamma$ and $1/m = 2\gamma$. Therefore $\omega = 2\gamma qB$, and so we require that

$$E_n = 2\gamma qB(n + 1/2) = -t. \quad (912)$$

As we reduce B , this will first happen at the $n = 0$ level. Therefore we will have a solution at H_{c2} , where

$$H_{c2} = -\frac{t}{\gamma q} = \frac{1}{q\xi^2(T)} = \frac{\Phi_0}{2\pi\xi^2(T)}, \quad (913)$$

with $\Phi_0 = \frac{h}{2e}$ the SCing flux quantum (using that $q = 2e$). This means that we start getting solutions with nonzero ψ when there is half a SCing flux quantum per "correlation area" $A_c = \pi\xi^2(T)$.

We can now use this as a diagnosis for when we expect superconductors to be type II. Imagine reducing the magnetic flux from a very large value. Suppose that $H_{c2} < H_c$. Then the system will "supercool", since the field will fall below H_c without order being able to form (the order has to start somewhere, but it can only get started at the field H_{c2} , where infinitesimal order can form). Once it then reaches H_{c2} there will be an instability towards large $|\psi|$, since converting everything to a SCing state is thermodynamically favorable.

On the other hand, suppose that $H_c < H_{c2}$. Then as the field is reduced just past H_{c2} , parts of the sample will become SCing because of the existence of a solution to the LG equations with nonzero $|\psi|$, but the whole sample won't,⁷³ since it is not thermodynamically favorable to setup the perfect diamagnetic response until H_c is reached.

This means the dividing line between type I and type II occurs at $H_c = H_{c2}$. We can express this condition in a more meaningful way by writing H_{c2} and H_c in terms of the length scales in the system. For the critical field, we use our above expression for H_c to write

$$H_c = \frac{|\psi_0|\sqrt{\mu_0\gamma}}{\xi(T)}, \quad (914)$$

so that the ratio of the fields is

$$H_{c2}/H_c = \frac{1}{\xi(T)q|\psi_0|\sqrt{\mu_0\gamma}}. \quad (915)$$

The London penetration depth can be found from Maxwell's equations and the eom for the current as $\lambda_L^2 = m/(nq^2\mu_0)$, where m and q are the mass and charge of the Cooper pair and $n = |\psi_0|^2$ is the "carrier density". Since $\gamma = 1/2m$ we have

$$\lambda_L = \frac{1}{\sqrt{2\gamma|\psi_0|q^2\mu_0}}, \quad (916)$$

⁷³I guess we expect that the superconducting order parameter will have some sort of nontrivial spatial profile. While this is a bit tricky to extract from the above analysis since the profile of the wavefunctions is gauge dependent, we at least don't expect a uniform order parameter everywhere.

which means that

$$\frac{H_{c2}}{H_c} = \sqrt{2} \frac{\lambda_L}{\xi}, \quad (917)$$

which means that type II superconductors will be those ones for which

$$\frac{\lambda_L}{\xi} > \frac{1}{\sqrt{2}}. \quad (918)$$

H_{c1} —the field for which vortices can enter type II superconductors

We can find H_{c1} by looking for the field at which an inserted vortex has zero free energy. The region of the vortex contributes a free energy difference (now taking the Legendre transform with $F(H) \supset -H \int B$ rather than $-H \int M \mu_0$) of

$$\Delta F = V f_s^\psi - (-H\Phi L + F_v), \quad (919)$$

where V is the volume of the vortex which we assume to be a tube of length L with flux Φ , f_s^ψ is the SCing condensation free energy density, and F_v is the free energy of the vortex.⁷⁴ Therefore vortices can enter the SC when the field satisfies (assuming a vortex cross-section of A)

$$H_{c1} = \frac{1}{\Phi} (F_v/L - f_s^\psi A). \quad (920)$$

Let's calculate F_v/L by ignoring what goes on in the core. That is, we will calculate the energy of the fields for all radii bigger than the correlation length ξ , which we take to be the vortex radius.⁷⁵ Later on we will argue that in fact most of the energy of the vortex comes from the kinetic part contained in the fields outside of the core, rather than from the core itself.

Anyway, this part of the cross-sectional energy of the vortex is (in what follows, ψ will denote the [constant] magnitude of the OP)

$$\mathcal{E} = \int_P \left(\frac{1}{2\mu_0} B^2 + \gamma\psi^2 |D_A \phi|^2 \right), \quad (921)$$

where P (for "punctured plane") is \mathbb{R}^2 minus the vortex core. The current is, as a differential form, $J = 2q\gamma\psi^2(d\phi - qA)$. Therefore we may write

$$\mathcal{E} = \frac{1}{2\mu_0} \int \left(B \wedge \star B + \frac{\mu_0^2}{2\gamma\psi^2 q^2 \mu_0} J \wedge \star J \right). \quad (922)$$

The penetration depth is $\lambda_L^{-2} = 2\gamma\psi^2 q^2 \mu_0$, and so

$$\mathcal{E} = \frac{1}{2\mu_0} \int (B \wedge \star B + \mu_0^2 \lambda_L^2 J \wedge \star J) = \frac{1}{2\mu_0} \int (B \wedge \star B + \lambda_L^2 dB \wedge \star dB), \quad (923)$$

⁷⁴We haven't forgotten about the kinetic energy $B^2/2\mu_0$ in the vortex! That's included in F_v .

⁷⁵Since we are at fields $H < H_c$, we don't want to have extended regions where we replace superconductor with magnetic flux, and hence we expect ψ to dip down to zero at the vortex core (over a distance of ξ), but then to immediately rise back up to ψ_0 —hence the radius of the vortex will be $\approx \xi$.

where we have used Maxwell's equation $\mu_0^{-1} \star dB = J$. Integrating by parts,

$$\mathcal{E} = \frac{1}{2\mu_0} \int_P (B + \lambda_L^2 d^\dagger dB) \wedge \star B - \frac{\lambda_L^2}{2\mu_0} \oint_{\partial P} B \wedge \star dB. \quad (924)$$

The first term can be dealt with using the London equation in the presence of the flux. Since $\oint_C d\phi = q\Phi$ for any curve $C \subset P$ encircling the flux, $d^2\phi = q\Phi\delta^2(r)$. Therefore the London equation is now

$$\star dJ = 2\gamma\psi^2 q^2 (\Phi\delta^2(r) - B) = \frac{1}{\mu_0\lambda_L^2} (\Phi\delta^2(r) - B). \quad (925)$$

Then again using $\mu_0^{-1} \star dB = J$, along with $d^\dagger dB = \square B = -\nabla^2 B$ on account of $d^\dagger B = \star d^2 A = 0$, we get the London equation for the B field:

$$\lambda_L^2 \square B = \Phi\delta^2(r) - B. \quad (926)$$

Now we can write the first term in \mathcal{E} as

$$\mathcal{E} \supset \frac{1}{2\mu_0} \int_P (B + \lambda_L^2 \square B) \wedge \star B = \frac{\Phi}{2\mu_0} \int_P \delta^2(r) = 0, \quad (927)$$

since P excludes the core.

Now all that's left is the \oint term. The integral along $d\theta$ selects out radial derivative, and so we get

$$\mathcal{E} = -\frac{\lambda_L^2}{2\mu_0} 2\pi\xi (B \partial_r B)|_{r=\xi}. \quad (928)$$

We can find $B(r)$ by solving the London equation. Notice that the London equation is precisely the same as the equation of motion obeyed by a massive scalar field (with mass $1/\lambda_L$) in two dimensions with a source current of strength Φ/λ_L^2 at the origin. This is very reasonable, since we know that in 2d the B field is a scalar, and that it gets made massive by the Higgs mechanism with a mass set by the penetration depth.

Now the integral along ∂P takes place at a distance $r \ll \lambda_L$ (remember we are in a type II SC and assuming $\lambda_L \gg \xi$) from the origin, where B can still be accurately described by a massless scalar field. The propagator for B at a separation of r will then have the usual form of $\ln(r/r_{IR})$, where $r_{IR} = \lambda_L$ is the IR cutoff provided by the mass.

To get $B(r)$ we then just need $\langle B \rangle$ in the presence of the delta-function source. Since the 1-point function for a scalar field in the presence of a current is $\langle \phi(x) \rangle_J = \int_y G(x-y)J(y)$, we have

$$B(r) = \Phi G(r) = \frac{1}{2\pi\lambda_L^2} \ln(\lambda_L/r). \quad (929)$$

This means, after some algebra

$$\mathcal{E} = \frac{1}{4\pi\mu_0} \left(\frac{\Phi}{\lambda_L} \right)^2 \ln(\lambda_L/\xi). \quad (930)$$

Since we are interested in the minimal field for which vortices can penetrate the SC, we can take Φ to be equal to one SCing flux quantum, viz $\Phi = 2\pi/q$. We can eliminate Φ from

the above equation by noticing that the product $H_c \lambda_L \xi$ is independent of the parameters that enter the LG theory. Indeed, $H_c = \sqrt{\mu_0 t^2/u}$, $\lambda_L = \sqrt{-u/(2\gamma\mu_0 tq^2)}$ and $\xi = \sqrt{-\gamma/t}$, so that

$$H_c \lambda_L \xi = \frac{1}{\sqrt{2}q}. \quad (931)$$

Therefore $\lambda_L^2 = \frac{1}{2q^2\xi^2 H_c^2}$, so that

$$\mathcal{E} = \frac{2\pi\xi^2 H_c^2}{\mu_0} \ln(\lambda_L/\xi). \quad (932)$$

The condensation energy lost by the core is $-f_s^\psi \pi \xi^2$, and so in the type II limit we have

$$-f_s^\psi \pi \xi^2 = \frac{\pi}{2\mu_0} H_c^2 \xi^2 \ll \mathcal{E}. \quad (933)$$

Therefore we can basically ignore the core contribution to the free energy when computing H_{c1} , at least in the type II limit.

Finally, we can plug this result for \mathcal{E} into the expression for H_{c1} . Writing one of the H_c s as $(\sqrt{2}q\lambda_L\xi)^{-1}$ and then canceling the q with the Φ , we see that

$$H_{c1} = \frac{H_c \xi}{\sqrt{2}\lambda_L} \ln(\lambda_L/\xi). \quad (934)$$

This is exactly the same expression as in Tinkham! Yay! (also, sanity check is that $H_{c1} < H_c$).

H_{c3} —the value of H_{c2} for a superconducting slab

Consider a thin SCing slab of width d , with a magnetic field applied normal to its surface. Starting from high fields, at what point will a nonzero SCing order parameter be a solution to the GL equations? In fact we already know the answer, namely H_{c2} . The reason why the computation is unchanged between the 3d and thin slab cases is because in the 3d case, the wavefunctions we found by solving the linearized LG equations were totally independent of the z coordinate,⁷⁶ and so the entire calculation goes through unchanged for the thin slab case.

Now suppose the field is \parallel to the slab. In this case we expect the critical field H_{c3} to be much higher—the small amount of expelled flux from the diamagnetic response gets pushed to right outside the surface of the SC, which causes a comparatively small amount of increase in the magnetic field free energy outside the SC.

To estimate H_{c3} , let the slab be in the xy plane, extending from $z = -d/2$ to $z = d/2$. We will assume the thin film limit so that $d \ll \lambda_L$. We can take the field in the sample to be $A = (0, -Hz, 0)$, with H the applied field. This ignores the screening from the SCing OP, but this is a higher order effect and can be neglected. Indeed, this simplification misses

⁷⁶Or rather they had momentum k_z ; the lowest energy one has $k_z = 0$ and hence is independent of the z coordinate.

Notational reminder vis-a-vis polarization and susceptibility

a term in the free energy that goes like $\int dz (H - B\mu_0^{-1})^2$. From the London equation we expect this to go like $\int dz H^2(1 - e^{-z^2/\lambda_L^2})^2$, so that the free energy per unit area that we're missing is

$$\delta f \propto \int dz H^2 d^5 \lambda_L^{-4}, \quad (935)$$

which as we will see is suppressed by a factor of $(d/\lambda_L)^2$ relative to the other terms we will be keeping.

We will make the simplifying assumption that ψ is a constant throughout the slab. The LG equations are then, after integrating over z ,

$$\gamma q^2 H^2 \frac{d^3}{24} + td + ud\psi^2 = 0 \implies \psi^2 = \psi_0^2 \left(1 + \frac{\gamma q^2 H^2 d^2}{24t} \right) \quad (936)$$

where $\psi_0^2 = -t/u$ is the MF value of the OP in the absence of the field. Now the thermodynamic critical field is $H_c^2 = \mu_0 t^2/u$, and so

$$\psi^2 = \psi_0^2 \left(1 + \frac{t\gamma q^2 \mu_0 H^2 d^2}{24 H_c^2 u} \right). \quad (937)$$

But then the penetration depth is $\lambda_L^{-2} = q^2 \mu_0 \gamma |\psi_0|^2$, so

$$\psi^2 = \psi_0^2 \left(1 - \frac{d^2 H^2}{24 \lambda_L^2 H_c^2} \right) \implies H_{c3} = \sqrt{24} \frac{\lambda_L}{d} H_c. \quad (938)$$

Therefore as the film gets smaller and smaller compared to λ_L , the film starts to have a nonzero SCing OP at higher and higher fields.



Notational reminder vis-a-vis polarization and susceptibility

Today is basically just recalling some notation about polarization and susceptibilities in basic electron response theory, since I found keeping track of the various definitions and types of susceptibility to be rather confusing.



The definitions of the susceptibility and the dielectric function are (in \mathbf{q} space with arguments suppressed)

$$\phi_a = \varepsilon \phi, \quad \rho_i = \chi \phi_a, \quad (939)$$

where ϕ_a is an applied potential and ρ_i is an induced charge density. Notice how χ multiplies the *applied* potential, not the full potential, while for ε it is the opposite. We can calculate the relationship between χ and ε as follows. We start with Poisson's equations for the full and applied charges:

$$\phi_a = \frac{4\pi}{q^2}(\rho - \rho_i), \quad \phi = \frac{4\pi}{q^2}\rho, \quad (940)$$

where ρ_i is the induced charge (density). Dividing these two equations,

$$\phi_a/\phi = \varepsilon = 1 - \rho_i/\rho. \quad (941)$$

Then we write the ratio of the charges as $\rho_i/\rho = 4\pi\rho_i/(\phi q^2) = 4\pi q^{-2}\chi\phi_a/\phi = 4\pi q^{-2}\chi\varepsilon$. Therefore

$$\varepsilon = 1 - \frac{4\pi}{q^2}\chi\varepsilon \implies \varepsilon = \frac{1}{1 + \frac{4\pi}{q^2}\chi}. \quad (942)$$

We can also define a "susceptibility" for relating the induced charge to the *full* potential, not just the applied one. Annoyingly these are often not distinguished in the literature, and so here we will give it a different name, viz. $\chi_{1\text{PI}}$ —the reason for this will become clear in a sec. We find

$$\varepsilon = 1 - \rho_i/\rho = 1 - \chi_{1\text{PI}}\phi/\rho = 1 - \frac{4\pi}{q^2}\chi_{1\text{PI}}. \quad (943)$$

The reason why we've given it the 1PI moniker can be seen by solving for χ in terms of $\chi_{1\text{PI}}$:

$$\chi = \frac{\chi_{1\text{PI}}}{1 - \frac{4\pi}{q^2}\chi_{1\text{PI}}}. \quad (944)$$

We recognize this as the geometric sum performed when doing the Dyson series thing—hence $\chi_{1\text{PI}}$ is indeed the 1PI part of the "full" susceptibility χ .

The reason for introducing both of these susceptibilities is that there are scenarios in which both of them are the more natural things to calculate.

Thomas-Fermi: When we do TF screening, we most naturally calculate $\chi_{1\text{PI}}$. Recall how this works: in \mathbb{R} space,

$$\rho_i = - \int_{\mathbf{k}} (n_F(\varepsilon_{\mathbf{k}} - \phi) - n_F(\varepsilon_{\mathbf{k}})), \quad (945)$$

where ϕ is the *full* potential, not the applied one. Assuming that the full potential ϕ varies slowly compared to $1/k_F$ (we are assuming ϕ is constant in time, so we will be computing $\chi(0, q)$ at small q), we approximate the integral as

$$\rho_i \approx -e \int_{\varepsilon} \nu(\varepsilon) \beta(\partial_{\mu} n_F) e\phi \approx -e^2 \nu(\varepsilon_F) \phi \implies \chi_{1\text{PI}} \approx -e^2 \nu(\varepsilon_F). \quad (946)$$

Sanity check: the susceptibility is negative, since a positive potential should attract electrons, making the charge density more negative. Also note that it is a constant, i.e. independent of q —this guarantees the divergence of $\varepsilon(0, q)$ as $q \rightarrow 0$, which we know has to be there in a metal. Indeed, this tells us that ($\nu \equiv \nu(\varepsilon_F)$)

$$\varepsilon = 1 + e^2 \nu \frac{4\pi}{q^2} \quad (947)$$

and that

$$\chi = -\frac{e^2 \nu}{1 + \frac{4\pi}{q^2} e^2 \nu}. \quad (948)$$

Since $\nu(\varepsilon_F) \propto \sqrt{\varepsilon_F}$, $\chi_{1\text{PI}} \propto \sqrt{\varepsilon_F}$. Another quick way to see this is as follows: since $n \propto \varepsilon_F^{3/2}$, we have

$$\delta \ln n = \frac{3}{2} \delta \ln \varepsilon_F. \quad (949)$$

Now $\delta \varepsilon_F = -e\phi$, and so

$$\delta \rho = -\frac{3}{2} \rho \frac{e\phi}{\varepsilon_F} \implies \chi_{1\text{PI}} = -\frac{3e\rho}{2\varepsilon_F}. \quad (950)$$

Then since $\rho \sim e\varepsilon_F^{3/2}$, we again confirm that $\chi_{1\text{PI}} \propto e^2 \varepsilon_F^{3/2}$.

Lindhard: By contrast, when we do Lindhard screening, the most natural susceptibility to calculate is the full one χ (although of course in the calculation we just compute $\chi_{1\text{PI}}$ directly). The reason why χ is more natural here is that the Lindhard calculation is the usual linear response one, where $\delta\rho \sim \int \langle [\rho, \rho] \rangle \phi_{ext}$ —since it is the external potential and not the full one that appears here, the response function calculated when doing screening in this way is χ , not $\chi_{1\text{PI}}$. The calculation goes through in the same way.

Finally, since I don't know where else to put it, a comment on the order of limits in the susceptibilities. When one first sets $\omega = 0$ and then takes $q \rightarrow 0$, we get a nonzero χ as calculated by e.g. TF screening. However, if one first sets $q = 0$ at finite ω and only then sends $\omega \rightarrow 0$, then from the $f_{\mathbf{k}-\mathbf{q}} - f_{\mathbf{k}}$ in the denominator of the polarization bubble, we see that $\chi(\omega \rightarrow 0, 0) = 0$. The reason for this is that χ is the response function for the charge density: if we set $q = 0$ then we are examining the response of the system to a spatially uniform potential—but this is not something the system can respond to, since charge must be conserved. So asking for the response at $q = 0$ isn't really meaningful—a $q = 0$ result should always be derived by sending $q \rightarrow 0$ at the end (e.g. after the q^2 from χ has canceled the $1/q^2$ pole of the interaction in the dielectric function).



Today's diary entry is very simple, and was done when preparing for the oral exam. We will consider a gas of electrons confined to a plane and interacting according to a (three-dimensional) Coulomb potential $V(x) = \frac{e^2}{\kappa x}$. We will find the plasmon dispersion, and contrast it to the answer in three dimensions.



In 3d, the plasma frequency can be found by a simple argument where one considers a bunch of electrons oscillating coherently on top of a uniform positive background. Since the electric field from an infinite charged sheet in 3d is independent of the distance away from the sheet and proportional to the charge density of the sheet (times 4π), the classical eom are (cgs!)

$$m\partial_t^2 x = -4\pi nxe^2, \quad (951)$$

where x is the displacement of the block of electrons. This correctly gives the plasma frequency.

In two dimensions the Coulomb potential is still $1/r$, but now we need to look at the electric field sourced by an infinite line charge, which falls off as distance as $1/r$. Since the plasma frequency should be proportional to the 2d density of the electrons and the field falls off as $1/r$, the only way the units can work out is to have

$$\omega^2 \propto \frac{nqe^2}{m}, \quad (952)$$

where now n is the two-dimensional density. This means that the plasmons will be gapless, unlike in three dimensions.

Now we can confirm this with a more careful screening calculation. The polarization is⁷⁷

$$\varepsilon(\omega, \mathbf{q}) = 1 - V_{\mathbf{q}}\chi_{1\text{PI}}(\omega, \mathbf{q}), \quad (955)$$

where $\chi_{1\text{PI}}$ is the 1PI part of the susceptibility and $V_{\mathbf{q}}$ is the Coulomb interaction. The important thing here is that the Fourier transform for V is calculated in two dimensions, despite the fact that V is the $1/r$ three-dimensional Coulomb potential (the Fourier transform happens by expanding the electronic modes in two-dimensional Fourier harmonics—one cannot (obviously) just take the 3d FT'd potential and restrict to in-plane momenta!). We do the Fourier transform with

$$V_q = e^2 \int dx d\theta \frac{xe^{iqx \cos \theta}}{x\kappa} = \frac{2\pi e^2}{q\kappa} \int_0^\infty dx J_0[x] = \frac{2\pi e^2}{\kappa q}. \quad (956)$$

⁷⁷Recall why we're calculating this: the effective interaction at momentum q , as determined by the density-density correlator, goes as V_q/ε_q . Therefore a zero of the dielectric function is a pole in the density-density correlator, i.e. a collective excitation of the medium. Above the plasma frequency ε is positive (and tends to 1 as ω gets big), while below it is negative; hence ω_p marks the frequency above which a metal becomes transparent, since $\varepsilon(\omega_p) = 0$. Note also that $\varepsilon(\omega) = 0$ is a requirement for any longitudinal wave that propagates in a medium where the only charge is the induced (bound) charge, simply because $\mathbf{k} \cdot \mathbf{E} \neq 0$ means that if $\rho = 0$ then $\varepsilon = 0$ on account of

$$\varepsilon \mathbf{k} \cdot \mathbf{E} = \rho_f, \quad (953)$$

where ρ_f is the free charge density.

Actually one can do a bit better, and prove that $\varepsilon = 0$ without making assumptions about ρ_f (although $\varepsilon = 0 \implies \rho_f$). This is done by writing

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \implies i\mathbf{k} \cdot \mathbf{E} = i\sigma^{-1}\mathbf{k} \cdot \mathbf{j} = i\sigma^{-1}\omega\rho = \rho/\varepsilon_0 \implies \sigma = \frac{i\omega}{\varepsilon_0}. \quad (954)$$

Then since $\varepsilon = 1 + i\sigma/(\omega\varepsilon_0)$, we get that $\varepsilon = 0$, and hence $\rho_f = 0$ as well.

We will calculate χ in the low wavelength limit using the standard method: starting from the point at which we've already exactly integrated out the frequency running in the polarization bubble, we have (working at $T = 0$)

$$\begin{aligned}
 \chi(\omega, q) &= - \int_{\mathbf{k}} \frac{f_{\mathbf{k}+\mathbf{q}} - f_{\mathbf{k}}}{\omega - (\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_{\mathbf{k}})} \\
 &\approx - \int_{\mathbf{k}} \frac{\mathbf{q} \cdot \mathbf{v}_{\mathbf{k}} \partial_{\varepsilon} f(\varepsilon_{\mathbf{k}})}{\omega - \mathbf{q} \cdot \mathbf{v}_{\mathbf{k}}} \\
 &\approx - \int_{\mathbf{k}} \frac{(\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q})^2 \partial_{\varepsilon} f(\varepsilon_{\mathbf{k}})}{\omega^2} \\
 &= - \frac{q^2}{4\pi^2 \omega^2} \int d\theta k dk |\mathbf{v}_{\mathbf{k}}|^2 \cos^2 \theta \partial_{\varepsilon} f \\
 &= \frac{q^2 N(0) v_F^2}{\omega^2} \int \frac{d\theta}{2\pi} \cos^2 \theta \\
 &= \frac{q^2 N(0) v_F^2}{2\omega^2}.
 \end{aligned} \tag{957}$$

Therefore the condition that the dielectric function vanish is (adding in a factor of 2 for spin degeneracy which I forgot, so that $N(0) = m/\pi$, and writing stuff in terms of the electron density $n = 2v_F^2 m^2/\pi$)

$$\omega = q \sqrt{\frac{ne^2}{4m\kappa}}, \tag{958}$$

where the $1/4$ is likely not very 靠谱. This is exactly the form we expected: all the same constants appear as in the 3d case, except the density is replaced with the 2d electron density, with a corresponding factor of q appearing to make the units work.



Landau diamagnetism

Today we will briefly discuss Landau diamagnetism from a few different perspectives.



We will be assuming spinless (or spin-polarized) particles in what follows—hence the magnetization that we will derive comes entirely from kinetic effects (think of current loops and stuff).

We will need the degeneracy of Landau levels in what follows, so let's recall how to get that. My favorite way at the time of writing is as follows: the kinetic energy of the particles undergoing the cyclotron motion will be minimized when the AB phase accumulated around the orbit is in $2\pi\mathbb{N}$. We expect the ground state to have orbits which enclose the minimal flux of 2π , and so therefore for a uniform field we have

$$r_o = \sqrt{\frac{2\hbar c}{eB}} = \sqrt{\frac{2\hbar}{m\omega_c}} \implies N = \frac{L^2}{\pi r_o^2} = \frac{L^2 m \omega_c}{2\pi\hbar}. \quad (959)$$

Now the DOS for a free Fermi gas in two dimensions is $\rho(\varepsilon) = L^2 m / (2\pi\hbar^2)$, and so an easy way to remember this degeneracy is by writing

$$N = \rho_{2d} \hbar \omega_c, \quad (960)$$

which makes total sense: we estimate the degeneracy by $\delta N = (dN/d\varepsilon)\delta\varepsilon = \rho\delta\varepsilon$, with $\delta\varepsilon = \hbar\omega_c$ the energy that we expect the electrons to acquire as the result of their quantized orbital motions.

This can of course be confirmed by a few different methods: one is to explicitly solve the Hamiltonian by writing it as

$$\left[-\frac{\hbar^2}{2m} \partial_x^2 - \frac{e^2 B^2}{2mc^2} \left(\frac{c\hbar^2}{eB} k_y - x \right)^2 \right] \psi = \varepsilon_n \psi. \quad (961)$$

The smallest k_y can be is $2\pi\hbar/L_y$, while from the above the largest it can be is $L_x eB/(c\hbar^2)$. Therefore since k_y labels the solutions, we have a degeneracy of (taking $L_x = L_y$)

$$N \frac{L_x eB / (c\hbar^2)}{2\pi\hbar/L_y} = L^2 \frac{eB}{2\pi\hbar c}, \quad (962)$$

which is exactly what we found using the other approach.

tl;dr argument: To get the magnetic response, we need to compute $M = -\partial_B U$ (strictly speaking we can use U instead of F only at $T = 0$ but this is a tl;dr argument anyway). When a B field is turned on, the former continuum of levels gets quantized. Basically what will happen is that all levels within $\pm\hbar\omega_c/2$ of a Landau level $\hbar\omega_c(n+1/2)$ will get sucked into the n th Landau level. Since the occupation number is a monotonically decreasing function of energy ($\theta(\mu - \varepsilon)$ counts as monotonically decreasing, but note that we are *not* making any arguments that are specific to fermions), we will always have that more levels will be sucked up in energy to a Landau level, rather than down. Therefore the B -field part of U will be positive, and the response will be diamagnetic.

For example, in the fermionic case at low T , this is corroborated by the heuristic estimate of the B -dependent part of U : since at low T only the shuffling of the energy levels near the FS contributes to the change in U (since the quantization of the orbits inside the FS doesn't change the total energy of the filled states given that ρ_{2d} for a Fermi gas is independent of ε)

$$U \supset (\delta\varepsilon)\delta N \sim (\delta\varepsilon)^2 \rho(\varepsilon_F) = (\hbar\omega_c/2)^2 \rho(\varepsilon_F), \quad (963)$$

which is close. Slightly more carefully, since we expect a density that exactly fills an integer number of LLs to be at the same energy in the magnetic field as not (equal numbers get sucked up as sucked down, at least to the extent that the occupation number v), the magnetization should be positive and should go to zero when an integer number of LLs are filled, presumably reaching a maximum when a LL is half-filled. This is indeed what happens, as we will see in a sec.

Low T argument for Fermions: A less schematic (but still very schematic) argument for $T \ll T_F$ fermions goes as follows. First, note that when filled parts of the FSea get re-shuffled into LLs, their total energies do not change. This is essentially because in 2d the Fermi gas DOS is constant. As a check, the fermions that go into a given LL with energy $\omega_c(n + 1/2)$ will have energies in

$$\varepsilon \in [\varepsilon_L, \varepsilon_U] = [\omega_c n, \omega_c(n + 1)]. \quad (964)$$

The number of fermions that want to go into the LL is

$$N_n = L^2 \int_{\varepsilon_L}^{\varepsilon_U} \frac{k dk}{2\pi} = L^2 \frac{m}{2\pi} (\varepsilon_U - \varepsilon_L) = L^2 \frac{m\omega_c}{2\pi}, \quad (965)$$

which is exactly what we deried for the LL degeneracy above. Therefore the energy of the filled LL is

$$\mathcal{E} = L^2 \frac{m\omega_c^2}{2\pi} (n + 1/2). \quad (966)$$

On the other hand, the energy of that part of the FSea is

$$L^2 \int_{\varepsilon_L}^{\varepsilon_U} \frac{k dk}{2\pi} \frac{k^2}{2m} = L^2 \frac{m}{4\pi} (\varepsilon_U^2 - \varepsilon_L^2) = \mathcal{E}, \quad (967)$$

as expected.

Anyway, the point of this is that only the LLs very close to the FS (and for $T \ll \omega_c$, only the very top LL) will contribute to the B -dependent part of the energy. Therefore in this limit, the B -dependent part of the energy is

$$\mathcal{E} = \int_{\omega_c n}^{\mu} d\varepsilon \frac{m}{2\pi} (\omega_c(n + 1/2) - \varepsilon) = \frac{m}{2\pi} \left(\omega_c(n + 1/2)(\mu - \omega_c n) - \frac{\mu^2 - \omega_c^2 n^2}{2} \right). \quad (968)$$

If we write $\mu = \omega_c(n + 1) - \delta$, then a bit of algebra (I don't see any reason to write it out) gives

$$\mathcal{E} = \frac{m}{4\pi} \delta(\omega_c - \delta). \quad (969)$$

As expected, this goes to zero at $\delta = 0, \omega_c$ when an integer number of LLs are filled, and is positive (we are in conventions where $\omega_c > 0$). This gives a magnetization of

$$M = -\frac{|e|}{4\pi} \delta. \quad (970)$$

The diamagnetization is therefore maximal right after a LL gets fully filled, in line with the DOS having a 1d $1/\sqrt{\varepsilon}$ divergence at each of the LL energies (seeing this requires remembering about k_z).

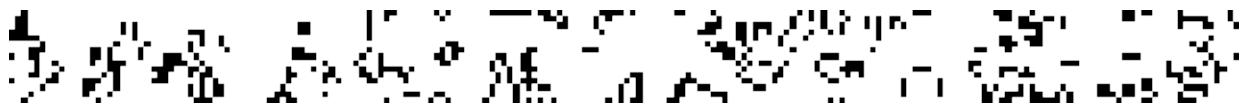
Finally, we note that at $T \gg \omega_c$, where the distribution function for the charged particles is essentially flat over any given Landau level (we are not just talking about fermions!), one can do another calculation. At T high enough that the statistics are Boltzmannian, one still gets a diamagnetic response, this time going as $M \propto -B/T$. This is actually in Pathria and Beale, so I won't write out the details.

The cool thing about this is that it is a completely quantum phenomenon—formally, this can be seen from restoring units and noting that $\chi \propto \hbar^2$. Even better though, one can note that in equilibrium, *no classical system of charged particles can have a nonzero magnetization*.⁷⁸ The easy way to argue this is that in a classical system, the partition function looks like

$$Z \propto \int \prod_i d^d p_i e^{-\beta f(p_i - eA(r_i)/c, \dots)}, \quad (971)$$

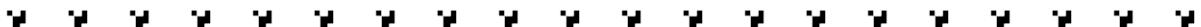
where f is some appropriate function of the thermodynamic background fields and the positions and momenta of the particles in the ensemble. Since we are doing classical mechanics where momenta are continuous and commute with the r_i , we can simply shift $p_i \mapsto p_i + eA(r_i)/c$, eliminating A from Z entirely—hence we must have zero magnetization. This argument also holds when doing semiclassical computations (e.g. take into account QM by putting the electrons in bands, but then ignore QM by determining dynamics with the classical eom), and so nontrivial magnetic responses of systems of electrons are always beyond a semiclassical approach.

Heuristically, one might think of this in the following way: Landau diamagnetism comes from the little loops of current formed by the rotating charged particles, which generate magnetic moments. We can treat the particles as current loops since they are wavelike and delocalized. However in classical mechanics we cannot think of them as loops of current, since the charge is localized to a particular place, and hence unable to produce a legit magnetic moment.



Speed of sound in a Fermi gas

Today is a short one: we will derive the speed of sound in a three-dimensional Fermi gas.



⁷⁸Of course the assumption of equilibrium is essential, since Lenz's law means there is obviously magnetization for out-of-equilibrium things.

The usual way to do this is to write $PV\beta = \ln Z$ and $U = -\partial_\beta \ln Z$, and then to eliminate the chemical potential μ so as to get an equation of state. There's a slightly more pedestrian approach that I like, though.

We start by figuring out the $V = L^3$ dependence of the energy, which will then give us the pressure via $P = -\partial_V U$. The L dependence comes in through the quantization of the momenta $p_i \in 2\pi L^{-1} \mathbb{Z}$ ($\hbar = 1$). Writing then $\int d^3p = \sum_{\{m_x, m_y, m_z\} \in \mathbb{N}^3} (2\pi L^{-1})^3$, we see that the L^{-3} cancels against the $\int d^3r$, and so the energy is

$$U = \sum_{\{m_i\} \in \mathbb{N}^3} \frac{1}{2m} \left(\sum_i m_i^2 \right) \left(\frac{2\pi}{L} \right)^2 n_F(\varepsilon). \quad (972)$$

Taking the derivative then,

$$P = -\partial_V U = -\frac{1}{3L^2} \partial_L U = -\frac{1}{3L^2} \partial_L \sum_{\{m\}} \frac{1}{2m} \left(\frac{2\pi}{L} \right)^2 \sum_i m_i^2 n_F(\varepsilon). \quad (973)$$

The subtle thing here is that the derivative needs to be taken while keeping the other extensive variables, namely N and S , fixed. This means in fact that $n_F(\varepsilon)$ must be in fact fixed under the adiabatic process by which we're changing the size of the box—the temperature / chemical potential will change in such a way to keep the occupation number constant, even as the values of the momenta change (the adiabaticity means that particles won't get "knocked out" of the states they're in during the expansion process, i.e. the "occupation probability" $n_F(\varepsilon)$ won't change). Therefore the derivative only acts on the $1/L^2$ factor, and so

$$P = \frac{2U}{3V}. \quad (974)$$

A quicker way to get here is to use dimensional analysis to realize that $U \sim V k_F^5$ and $N \sim V k_F^3 \implies k_F \sim V^{-1/3}$ at fixed N to write $U \sim V^{1-5/3}$, which tells us that $\partial_V U = 2U/3V$.

Now we can get the speed of sound from the formula

$$c = \sqrt{\partial_\rho P}. \quad (975)$$

The density is

$$\rho = m \int d\varepsilon \sqrt{\varepsilon} \bar{\nu}, \quad (976)$$

where $\bar{\nu} = \nu(\varepsilon)/\sqrt{\varepsilon}$ is the constant part of the DOS. Therefore

$$\rho = \frac{2m}{3} \bar{\nu} \varepsilon_F^{3/2}. \quad (977)$$

Then we use $P = 2U/3V$ to write

$$P = \frac{2}{3} \int d\varepsilon \varepsilon^{3/2} \bar{\nu} = \frac{4}{15} \bar{\nu} \varepsilon_F^{5/2}. \quad (978)$$

Then

$$\partial_\rho P = \frac{1}{\sqrt{\varepsilon_F} m \bar{\nu}} \partial_{\varepsilon_F} \left[\frac{4}{15} \bar{\nu} \varepsilon_F^{5/2} \right] = \frac{2}{3m} \varepsilon_F = \frac{k_F^2}{3m^2}. \quad (979)$$

Therefore the speed of sound is

$$c = \frac{v_F}{\sqrt{3}}, \quad (980)$$

which importantly is slower than the speed of the qps at the FS. Note that sound exists even at zero T an in a non-interacting problem because of the degeneracy pressure caused by Fermi statistics.



Wiedemann–Franz law sanity check

Today we're doing a trivial but instructive problem that came up when studying for the oral exam: checking the WF law for a clean⁷⁹ 1d system.



First we get the electrical conductance.⁸⁰ Our setup will be to consider a 1d wire connected between a source of chemical potential μ_L and a sink at μ_R . Of course, this is just saying that the wire is placed in an electric field. The current is (semiclassicaly, i.e. integrating over all momentum modes near the two wire endpoints)

$$j = 2 \int \frac{dk}{2\pi} j_k = -2e \left(\int_0^\infty \frac{dk}{2\pi} v(k) n_F(\beta(\varepsilon_k - \mu_L)) - \int_0^\infty \frac{dk}{2\pi} v(k) n_F(\beta(\varepsilon_k - \mu_R)) \right), \quad (981)$$

where $v(k) = \hbar^{-1} |\partial_k \varepsilon_k|$ and the factor of 2 is for spin.⁸¹ Doing the integrals with the help of $dk v(k) = d\varepsilon$, we get

$$j = -\frac{2e}{\hbar} (\mu_L - \mu_R) \implies G = \frac{2e^2}{\hbar}, \quad (983)$$

⁷⁹One might say that this is slightly against the spirit of the WF law—one point of the law is that the ratio of conductances is independent of details of the non-cleanness of the system, like the value of τ and so on. But the clean system provides a good sanity check.

⁸⁰Recall that conductance G differs from conductivity in that it is the response function that relates to the gauge potential rather than to the electric field (one could argue that since the current couples to the gauge field microscopically, this is the more natural response function to study). In general, $j_\mu = \int G_{\mu\nu} A^\nu$. In our application this will just read $j = GV$.

⁸¹Annoyingly pedantic footnote about why the velocity appears here: we start from an action like $\int \psi^\dagger (\omega - \varepsilon_k) \psi$, and then perform $\psi \mapsto e^{i\theta} \psi$ to determine j through the usual Noether procedure. j is the thing that contracts with $\nabla \theta$ in the transformed action, and so we need to select out all ways of having one derivative in $\varepsilon_k = \varepsilon(-i\hbar \nabla)$ act on the θ . This term is

$$\int \psi^\dagger \nabla_j \theta \frac{\partial \varepsilon}{\partial (-i\hbar \nabla_j)} \psi, \quad (982)$$

which goes over to the expression written above when we pass to momentum space and take expectation values to get the Fermi function.

where again the conductance is defined in our setting via

$$j = GV = G(\mu_R - \mu_L)/e. \quad (984)$$

Now for the thermal conductance K ⁸² Here we set $\mu_L = \mu_R$ but put the ends of the wire at temperatures T_L, T_R , with $T_L/T_R \approx 1$ (since the thermal conductivity is defined in linear response, we will just work to order $O((T_L - T_R)^1)$).

The heat current is $j_Q = T j_S = j_U - \mu j$. Therefore

$$j_Q = \int_0^\infty \frac{dk}{\pi} v(k)(\varepsilon_k - \mu) [n_F(\beta_R(\varepsilon_k - \mu)) - n_F(\beta_L(\varepsilon_k - \mu))]. \quad (985)$$

Now we write

$$n_F(\beta_R(\varepsilon_k - \mu)) - n_F(\beta_L(\varepsilon_k - \mu)) \approx (\partial_\beta n_F)|_{\beta\xi} (\beta_R - \beta_L) = (\partial_\varepsilon n_F)|_{\beta\xi} \frac{\xi}{\beta} (\beta_R - \beta_L), \quad \xi \equiv \varepsilon - \mu. \quad (986)$$

Then since $\beta^{-1}(\beta_R - \beta_L) = -\Delta T/T$, we have

$$K = -\frac{2}{T} \int_0^\infty \frac{d\varepsilon}{h} \xi^2 \partial_\varepsilon n_F(\beta\xi) = -\frac{2T}{h} \int_{-\beta\mu}^\infty dx x^2 \partial_x \frac{1}{e^x + 1}. \quad (987)$$

Working under the reasonable assumption that $\mu\beta \gg 1$, we can safely extend the lower limit of integration to $-\infty$. Then we use

$$\int_{\mathbb{R}} dx x^2 \partial_x \frac{1}{e^x + 1} = -\pi^2/3 \quad (988)$$

to get

$$K = \frac{2T\pi^2}{3h}. \quad (989)$$

The fact that K/T is constant is a general property of Fermi liquids, and not just limited to one dimension. Anyway, we see that

$$\frac{K}{TG} = \frac{\pi^2}{3e^2}, \quad (990)$$

which is indeed the WF law (we are setting $k_B = 1$ since it's a fake physical constant that only exists because humanity didn't define units of temperature correctly).

Now this check is rather trivial, since we are working in a clean system. More generally, the WF law is expected to hold in the presence of disorder provided that the disorder is elastic. This is because both $(\varepsilon - \mu)\beta$ and $eV\beta$ appear in the distribution function for the electrons in the same way, and so from the usual kinetic theory arguments we expect that if both energy and charge are conserved, the heat and charge transport response functions will be related (the asymmetric extra factor of T relating the conductivities is because the background field for $\varepsilon - \mu$, namely β , also couples to the background field for charge density).



⁸²This differs from the thermal conductivity in the same way that G differs from σ : we have $j = K\Delta T$, so that K couples directly to the background field $T \sim V$ rather than to $\partial_x T \sim E$.

Basic Hall angle stuff

Consider a metal well described by the Drude theory. Assume the electrons have a density n , mass m , a scattering rate τ , and are placed in a weak uniform magnetic field B along the z -direction. We will do several things:

- Calculate the frequency dependent Hall conductivity in such a metal.
- The Hall angle is defined to be between the electric current and electric field. For a (weak) electric field in the xy -plane, and in the dc limit, calculate the Hall angle as a function of B , and other parameters characterizing the electron gas.
- Write down the analogous formulae for the case of two bands with different charge carriers.
- Now consider a circularly polarized electric field $E_y = \pm iE_x$. Find the resulting current, and find the dielectric function. When do wave solutions to Maxwell's equations exist? In the low-frequency limit, what's their dispersion?



We will be working with an approach based on the semiclassical eom. We will therefore always need to be working with background fields that vary incredibly slowly on the scale of the lattice constant. This is because the semiclassical approach works with wavepackets $\psi(\mathbf{r}, \mathbf{k})$, and assumes that they are near eigenstates of the Hamiltonian (whence the semiclassical eoms). In order for this to be true $\psi(\mathbf{r}, \mathbf{k})$ must have a nearly well-defined energy, which means that for a generically dispersing band the \mathbf{k} -space width of the wavepacket must be $\ll 2\pi a^{-1}$. This then means the \mathbb{R} -space width must be $\gg a$. Since semiclassics means treating the wavepackets as particles wrt the external fields, the wavelengths of the fields must be $\gg a$. Furthermore, since we ignore tunneling events between different bands, we need the fields to vary temporally slowly enough such that $\hbar\omega \ll E_g$, where E_g is the minimum energy gap between bands.

In fact we have a further restriction in the context of computing the conductivity: we will always be working with EM fields whose momenta satisfy $\hbar q^{-1} \gg v\tau$, where $v \approx v_F$ is a typical speed of electrons in the metal, i.e. we will assume that electrons experience many scattering events before they travel large enough distances for them to become aware of the spatial modulation of the fields. The reason for making this assumption is that it means the current can be determined just by knowing the local value of \mathbf{E} —one doesn't need to know the details of the electron's past history to describe its present evolution, since these details are presumed to be erased by scattering. Indeed in general, the conductivity is a non-local response function, with $\mathbf{j}(r, t) = \int \sigma(r - r', t - t') \mathbf{E}(r', t') \, d^3r' dt'$. The above assumption means that $\sigma(r - r', t - t') \propto \delta(r - r')$, which means that $\sigma(q, \omega)$ is independent of q (since the δ function becomes a constant after FTing). Therefore in what follows, we will be working locally in space, but non-locally in time (i.e. locally in frequency).

Anyway, we start from the eom:

$$\partial_t \mathbf{k} = e\mathbf{E} - \nu \mathbf{k} + \frac{e}{m} \mathbf{k} \times \mathbf{B}, \quad (991)$$

where e is the actual charge of the electron ($e < 0$) and $\nu \equiv \tau^{-1}$.⁸³ In what follows we will assume that we are near a band minimum or maximum, where we can write

$$\partial_t \mathbf{v}^a = \partial_t \partial_{\mathbf{k}^a} \varepsilon = (\partial_t \mathbf{k}^b) \partial_{\mathbf{k}^b} \partial_{\mathbf{k}^a} \varepsilon = [m^{-1}]^{ab} \partial_t \mathbf{k}^b, \quad (992)$$

where the matrix m^{-1} is assumed not to depend on \mathbf{k} . For simplicity, we will take $m \propto \mathbf{1}$, but the generalization to anisotropic bands is straightforward.

In terms of the current,

$$j^a(-i\omega + \nu) = \varepsilon_0 \omega_p^2 E^a + \frac{e}{m} \varepsilon^{abc} j_b B_c, \quad (993)$$

where

$$\omega_p^2 = \frac{e^2 n}{m \varepsilon_0} \quad (994)$$

is the $q = 0$ plasma frequency. Wolog, we may take $\mathbf{B} = B \hat{\mathbf{z}}$. Then we have

$$(\delta^{ab}(-i\omega + \nu) - \omega_c \varepsilon^{abz}) j^b = \varepsilon_0 \omega_p^2 E^a, \quad (995)$$

where ω_c is of course the cyclotron frequency. The resistivity is therefore

$$\rho(\omega) = \frac{1}{\varepsilon_0 \omega_p^2} \begin{pmatrix} -i\omega + \nu & -\omega_c & \\ \omega_c & -i\omega + \nu & \\ & & -i\omega + \nu \end{pmatrix}. \quad (996)$$

Note that the longitudinal resistivity $\rho_{xx}(\omega)$ is independent of B . The Hall coefficient is $R_H = \rho_{yx}(0)/B$, which means that (remember we are in SI and $e < 0$)

$$R_H = \frac{\omega_c B^{-1}}{\varepsilon_0 \omega_p^2} = \frac{1}{ne} \quad (997)$$

which is the usual result. This means the conductivity tensor is

$$\sigma(\omega) = \frac{\varepsilon_0 \omega_p^2}{\omega_c^2 + (-i\omega + \nu)^2} \begin{pmatrix} -i\omega + \nu & \omega_c & \\ -\omega_c & -i\omega + \nu & \\ & & \frac{\omega_c^2 + (-i\omega + \nu)^2}{-i\omega + \nu} \end{pmatrix}, \quad (998)$$

which is properly antisymmetric and reduces to the expected result when $\omega_c = 0$. Note that in a clean system where $\nu = 0$, the conductivity diverges at $\omega = \omega_c$ since at this frequency we're on resonance, with the electrons being hit with the electric field in the same point of

⁸³A further assumption: to avoid talking about transitions between bands, we need the kinetic energy acquired by a wavepacket between scattering events, viz. $\int dt W = \int dt \mathbf{E} \cdot \mathbf{j} \sim \tau v e^2 E$, to be $\ll E_g$. This is an incredibly reasonable assumption in e.g. semiconductors and stuff.

their orbit every time they circle around. Looking for the location of this peak would be one way to measure the electron mass, for example.

Now we want to find the Hall angle. We determine θ_H by setting $\omega = 0$ and calculating

$$\cos \theta_H = \mathbf{j} \cdot \mathbf{E} / \sqrt{j^2 E^2} = \frac{\mathbf{E}^T \sigma(0) \mathbf{E}}{\sqrt{E^2 \mathbf{E}^T \sigma(0)^T \sigma(0) \mathbf{E}}}. \quad (999)$$

Note that since we're doing linear response, i.e. $j \propto \mathbf{E}$, the Hall angle will be independent of E .

We can take $\mathbf{E} = E\hat{\mathbf{x}}$ wolog. Then this becomes

$$\cos \theta_H = \frac{\sigma_{xx}}{\sqrt{\sigma_{xx}^2 + \sigma_{xy}^2}} = (1 + \sigma_{xy}^2 / \sigma_{xx}^2)^{-1/2} = (1 + \omega_c^2 / \nu^2)^{-1/2}. \quad (1000)$$

Note that the plasma frequency has totally canceled out of the result. For small magnetic fields or short scattering times, so that $\tau \omega_c \ll 1$, this gives

$$\theta_H \approx \omega_c \tau. \quad (1001)$$

Sanity check: in this limit, the electrons experience many scatterings before they can complete a cyclotron orbit; hence the \mathbf{B} field is not very effective at making their motions \perp to \mathbf{E} , and the current is dominated by the force coming from the electric field that accelerates the electrons between collisions.

On the other hand, for large magnetic fields or long scattering times (but still short enough so that $\sqrt{\langle \mathbf{v}^2 \rangle} \tau$ is smaller than any spatial variation of the EM fields), we instead have

$$\theta_H \approx \frac{\pi}{2} - \frac{1}{\omega_c \tau}. \quad (1002)$$

Sanity check: for very long scattering times, the Lorentz force is very effective at deflecting the motion of the electrons away from the electric field. The current becomes directed along $\mathbf{B} \times \mathbf{E}$ in the limit of $\omega_c \nu \rightarrow \infty$, which is precisely the direction of the drift velocity along which the electric field vanishes.

In a finite-sized sample, the Hall conductivity will give us a current in the $\mathbf{E} \times \mathbf{B}$ direction, which will create an electric field in this direction after equilibrium is reached (in equilibrium here we mean that \mathbf{j} along $\mathbf{E} \times \mathbf{B}$ vanishes). For example, let $\mathbf{E} \parallel \hat{\mathbf{x}}$. Then in the DC limit, $\mathbf{j}_y = 0$ means that

$$\mathbf{E}_y = \omega_c \tau \mathbf{E}_x. \quad (1003)$$

The sign is right since for us $e < 0 \implies \omega_c < 0$. Sanity check: when $\omega_c \tau \rightarrow 0$ we don't need an electric field in the y direction, since we already know that in that limit $\theta_H \rightarrow 0$. On the other hand, when $\omega_c \tau \rightarrow -\infty$, we know that the electrons will move off along the y direction super fast, and hence they require an infinitely strong \mathbf{E}_y to maintain equilibrium.

We have been assuming a situation appropriate for free electrons without any kind of potential. When the electrons are put into bands, the basic analysis above holds as long as we are at energies where the electrons execute closed orbits in \mathbf{k} space (as they do for the free electron model). When the orbits are open (and every band must have some open orbits for some choice of μ , modulo the contrived situation where $\varepsilon_{\mathbf{k}}$ is uniform on the boundary

of the BZ) the situation changes, as now the electrons do not execute paths in \mathbf{k} space that tend to direct the current along $\mathbf{B} \times \mathbf{E}$.

Generalizing this to the case of multiple carrier species is straightforward. From the semiclassical eom for each carrier species, we obtain the conductivity $\sigma_a(\omega)$. Since the currents add to produce the total current, so too do the conductivities (it is the σ_a s that add, *not* the ρ_a s). Focusing just on the xy plane then, we have

$$\sigma(\omega) = \sum_a \frac{\varepsilon_0 \omega_{pa}^2}{\omega_{ca}^2 + \alpha_a^2} \begin{pmatrix} \alpha_a & \omega_{ca} \\ -\omega_{ca} & \alpha_a \end{pmatrix}, \quad \alpha_a \equiv -i\omega + \nu_a. \quad (1004)$$

I don't really see the point of writing down the most general formulae, due to the unilluminating algebra involved. For simplicity, we will therefore specialize to the "perfect intrinsic semiconductor" case, where we have one band with $m_1 = m, q_1 = q, \nu_1 = \nu$ and another with $m_2 = m, q_2 = -q, \nu_2 = \nu$.⁸⁴ With these simplifications,

$$\sigma(0) = \frac{q^2/m}{\omega_c^2 + \nu^2} \begin{pmatrix} \nu(n_1 + n_2) & \omega_c(n_1 - n_2) \\ -\omega_c(n_1 - n_2) & \nu(n_1 + n_2) \end{pmatrix}, \quad (1005)$$

where $\omega_c = qB/m$. As expected, the densities of the oppositely charged particles add for the longitudinal conductance, and subtract for the Hall conductance: the particles and holes move in opposite directions under the \mathbf{E} field, which due to their opposite charges produces currents that add. However since the \mathbf{E} field makes them move in opposite directions, they are deflected in *the same* transverse direction by the \mathbf{B} field, and hence their transverse currents subtract from one another.

The Hall angle in this case satisfies

$$\cos \theta_H = \left(1 + \frac{\tau^2 \omega_c^2 (n_1 - n_2)^2}{(n_1 + n_2)^2} \right)^{-1/2}. \quad (1006)$$

In the limit of small $\tau\omega_c$, we have

$$\theta_H \approx \tau\omega_c \frac{n_1 - n_2}{n_1 + n_2}, \quad (1007)$$

which reduces with the right sign to our answer before in the case of $n_2 = 0$. Now as expected, when $n_1 = n_2$ the Hall angle is always zero, regardless of the strength of B . But if we are in the strong field limit with imbalanced carrier concentrations, such that $\omega\tau(n_1 - n_2)/(n_1 + n_2) \gg 1$, then

$$\theta_H \approx \pi/2 - \frac{n_1 + n_2}{\omega_c\tau(n_1 - n_2)}. \quad (1008)$$

Letting $N \equiv n_1 + n_2$ and $\delta \equiv n_1 - n_2$, the resistivity is

$$\rho(0) = \frac{\omega_c^2 + \nu^2}{q^2 m^{-1} (\nu^2 N^2 + \omega_c^2 \delta^2)} \begin{pmatrix} \nu N & -\omega_c \delta \\ \omega_c \delta & \nu N \end{pmatrix}. \quad (1009)$$

⁸⁴Recall that the effective masses of holes are always defined to be positive, i.e. they are defined by $m_h = -\partial_k^2 \varepsilon$ near a band maximum. The eom would be the same whether we put the minus sign on the charge of the hole or on the mass; we choose to do the former.

At high fields, this gives the Hall coefficient

$$R_H = \rho_{yx}(0)/B \xrightarrow{B \rightarrow \infty} \frac{1}{q\delta}. \quad (1010)$$

Sanity check: δ is the effective number of charge carriers, and it appears exactly in R_H in exactly the way we expect it to. Likewise at high fields, we have the longitudinal resistance

$$\rho_{xx}(0) \xrightarrow{B \rightarrow \infty} \frac{mvN}{q^2\delta^2}. \quad (1011)$$

This passes the $n_2 = 0$ sanity check, but note that $\rho_{xx}(0) \rightarrow \infty$ when $n_1 = n_2$ (which is not obvious from looking at the expression for $\sigma(0)$).



Helicons

Today we'll use the conductivity calculated in the previous diary entry on Hall angles to explore what kind of waves can propagate in the metal.



From Maxwell's equations, the dielectric tensor at zero momentum is

$$\varepsilon(\omega) = \mathbf{1} + \frac{i}{\omega\varepsilon_0}\sigma(\omega). \quad (1012)$$

In what follows we will be focusing on \mathbf{E} plane waves that lie within the xy plane. Using our expression for σ , the xy -plane part of the dielectric function is (for electric fields along the z direction, we just get the usual $\varepsilon_z(\omega) = 1 - \omega_p^2/\omega^2$)

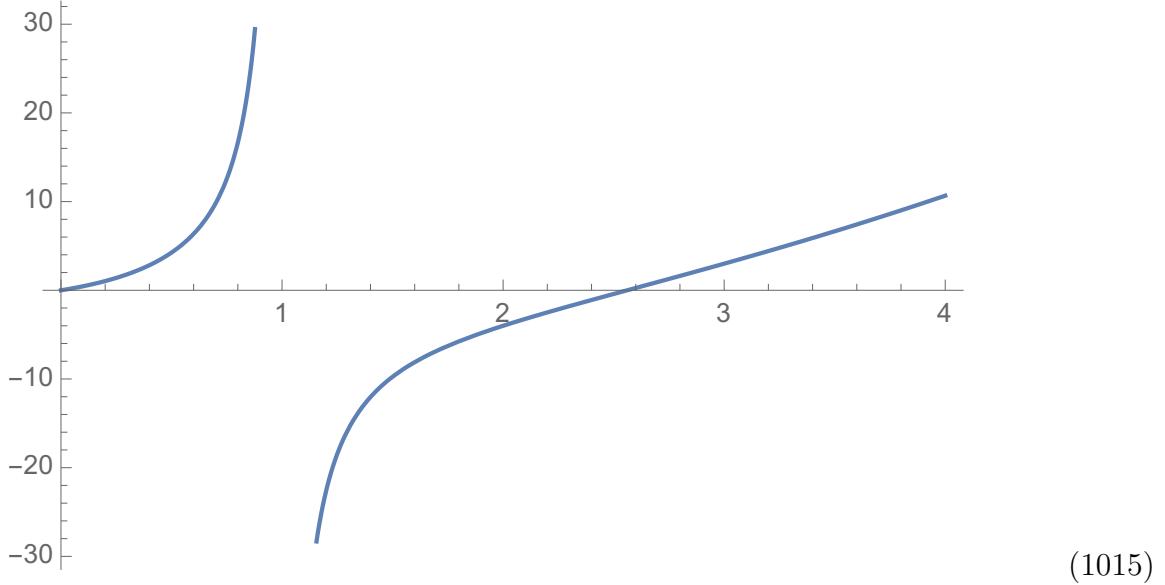
$$\varepsilon(\omega) = \mathbf{1} + i\frac{\omega_p^2}{\omega r} \begin{pmatrix} d & \omega_c \\ -\omega_c & d \end{pmatrix}, \quad d \equiv -i\omega + \nu, \quad r \equiv d^2 + \omega_c^2. \quad (1013)$$

Taking the clean limit $\nu = 0$, we can plot $\omega^2\varepsilon(\omega)$ to see when we have wave solutions to Maxwell's equations for the given circularly polarized fields.⁸⁵ Since the dispersion is $c^2k^2 = \omega^2\varepsilon(\omega)$, the wave solutions are

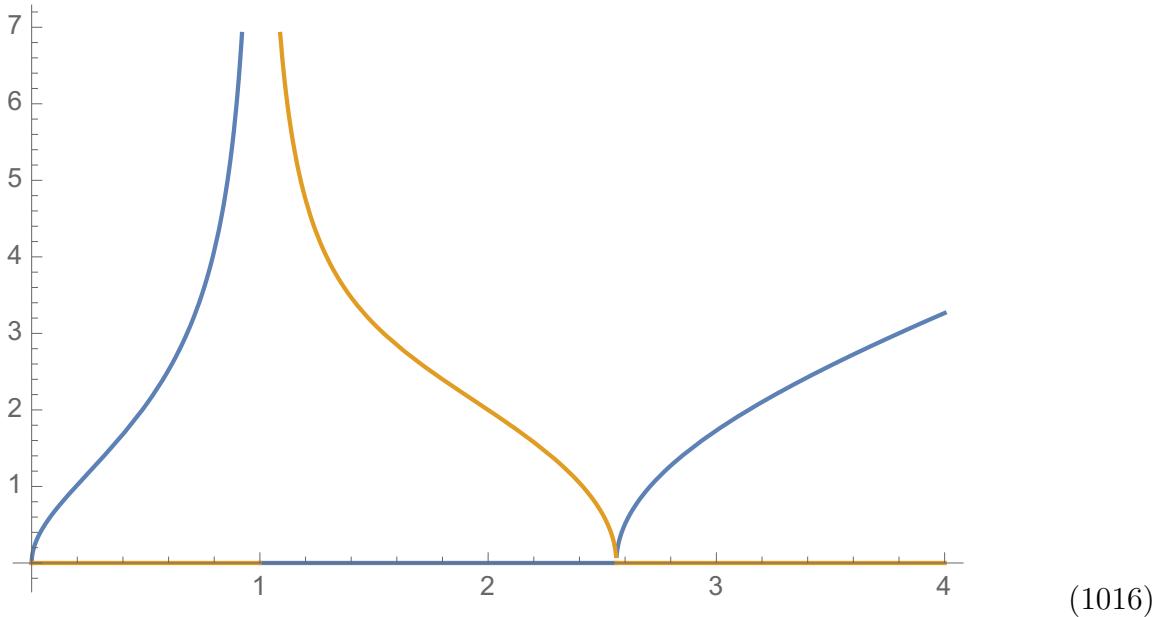
$$q^2c^2 = \omega^2 + \frac{\omega\omega_p^2}{\omega - \omega_c}. \quad (1014)$$

⁸⁵By "plot $\omega^2\varepsilon(\omega)$ ", we mean that we plot the x component of $E^a q_a^2 c^2 = \omega^2 \varepsilon_{ab}(\omega) E^b$, evaluated at $E_x = iE_y$.

we will have solutions when $\omega^2 \varepsilon(\omega) > 0$. Choosing e.g. $\omega_p = 2, \omega_c = 1$, we have

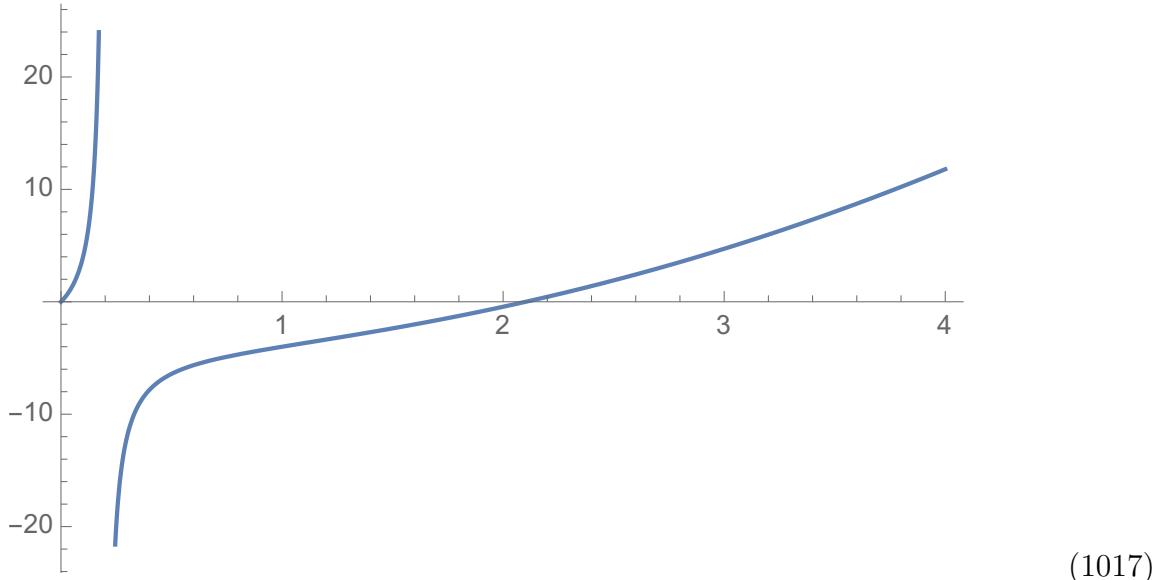


Since $\omega^2 \varepsilon(\omega)$ runs through all positive \mathbb{R} values between 0 and $1 = \omega_c$ and between ≈ 2.6 and ∞ , we will have wave solutions for any k whenever $\omega < \omega_c$ or $\omega \gtrsim 2.6$. I guess a clearer plot is one showing the \mathbb{R} and \mathbb{I} parts of $cq(\omega)$: this looks like



However, in real life situations, we basically always have $\omega_p \gg \omega_c$. In this limit, the allowed boundary for propagation of course moves back to $\omega > \omega_p$: indeed if we make ω_c much

smaller, $\varepsilon(\omega)\omega^2$ looks like



with the upper $\varepsilon(\omega) = 0$ point moving closer to ω_p . Therefore in realistic situations, the allowed frequencies for wave propagation are basically $\omega < \omega_c$ and $\omega > \omega_p$. In the very low $\omega/\omega_c \ll 1$ frequency limit the dispersion becomes quadratic,

$$\omega \approx k^2 \frac{\omega_c c^2}{\omega_p^2}. \quad (1018)$$

These types of waves are apparently known as helicons. I guess the point of this is that helicons afford us a way of getting a wave solution *below* the plasma frequency, which is normally not something that happens because of screening (metals are normally only transparent above ω_p !). What happens here is that the magnetic field forces the electrons to precess around at a frequency of ω_p , and so they are precessing too fast to be able to screen out a wave on frequencies less than ω_c , allowing the wave to propagate freely in the medium.



Surface plasmons

Today we're doing a problem from A&M that came up when reviewing for the oral exam. We will be considering a situation in which two dielectrics with dielectric constants ε_{\leq} occupy the $z < 0$ and $z > 0$ half-spaces, and will show the existence of wave solutions to Maxwell's equations which are localized to the interface (for a certain range of frequencies).



Our ansatz will be one appropriate for waves in both E_x and E_z which propagate in the x direction and which are exponentially localized to the $z = 0$ plane:

$$\mathbf{E} = \begin{cases} (A_x^> e^{i(qx-\omega t)-k>z}, 0, A_z^> e^{i(qx-\omega t)-k>z}), & z > 0 \\ (A_x^< e^{i(qx-\omega t)+k<z}, 0, A_z^< e^{i(qx-\omega t)+k<z}), & z < 0 \end{cases}. \quad (1019)$$

where $k_>, k_< > 0$. Now, we have assumed that the charge density vanishes for $z \neq 0$. Applying this for both half-planes,

$$iqA_x^> = k_>A_z^>, \quad iqA_x^< = -k_<A_z^<. \quad (1020)$$

Now since \mathbf{E}_x is continuous across the interface (do $\oint \mathbf{E} \cdot d\mathbf{l}$ along a narrow rectangle with short sides in the z direction and long sides in the $z = 0$ plane), we have $A_x^> = A_x^< \equiv E_0$. From the above then, we get

$$k_>/k_< = -A_z^</A_z^>. \quad (1021)$$

Furthermore, we also know that $(\epsilon\mathbf{E})_z$ is continuous across the interface (a discontinuity in \mathbf{D} would mean a free charge density on the surface, while we are assuming the only charge density comes from bound charges). This means that

$$\epsilon_>A_z^> = \epsilon_<A_z^< \implies k_>/k_< = -\epsilon_>/\epsilon_<. \quad (1022)$$

Note that this must mean that the two dielectric functions on either side of the interface have opposite signs, if we are to get a legit wave solution.

Now we take curls of Maxwell's equations in the usual way to get the wave equations, which read (in our notation the ϵ_i s are relative dielectric constants, so that the "real" dielectric constants are $\epsilon_0\epsilon$ —in particular, $\epsilon_i = 1$ for vacuum)

$$c^2(q^2 - k_i^2) = \omega^2\epsilon_i, \quad (1023)$$

with $i = >, <$. The two equations coming from this, plus the relation $k_>/k_< = -\epsilon_>/\epsilon_<$, lets us solve for q as a function of ω, ϵ_i . Some algebra gives

$$k_{\leqslant} = \pm \frac{\epsilon_{\leqslant}\omega}{c} \sqrt{\frac{-1}{\epsilon_< + \epsilon_<}}, \quad (1024)$$

with the \pm chosen so that $k_{\leqslant} > 0$ (we only care that the real part of $k_{\leqslant} > 0$; it is totally fine for k_{\leqslant} to have an imaginary part). Putting this back into the wave equation, we get

$$q = \frac{\omega}{c} \sqrt{\frac{\epsilon_>\epsilon_<}{\epsilon_> + \epsilon_<}}. \quad (1025)$$

Let us now specialize to the case where the UHP is vacuum, and the LHP is a clean dielectric, so that

$$\epsilon_> = 1, \quad \epsilon_< = 1 - \omega_p^2/\omega^2. \quad (1026)$$

Then the dispersion is

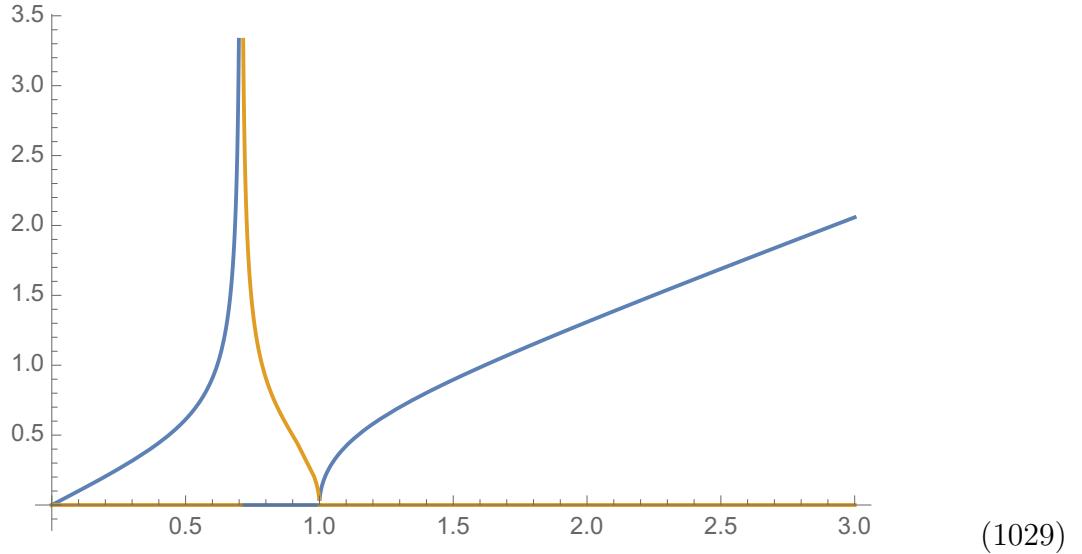
$$qc = \omega \sqrt{\frac{\omega^2 - \omega_p^2}{2\omega^2 - \omega_p^2}} = \omega^2 \varepsilon_{eff}(\omega), \quad \varepsilon_{eff}(\omega) = \sqrt{\frac{1 - \omega_p^2/\omega^2}{2 - \omega_p^2/\omega^2}}. \quad (1027)$$

Therefore at large $qc \gg \omega$, we have a solution at $\omega \rightarrow \omega_p/\sqrt{2}$. The k parameters are

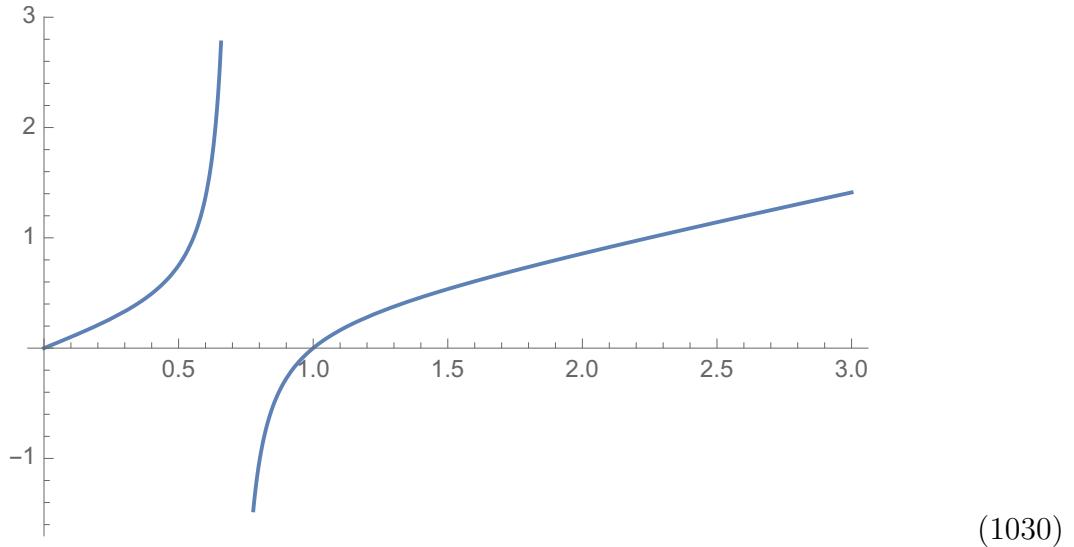
$$k_> = \frac{\omega^2}{c} \sqrt{\frac{1}{\omega_p^2 - 2\omega^2}}, \quad k_< = \frac{1}{c} (\omega_p^2 - \omega^2) \sqrt{\frac{1}{\omega_p^2 - 2\omega^2}}. \quad (1028)$$

Therefore as ω goes to $\omega_p/\sqrt{2}$ from below, both k_{\leq} parameters are positive, and go to infinity in a way such that $k_>/k_< \rightarrow 1$, so that at $\omega_* = \omega_p/\sqrt{2}$ we get a perfectly boundary localized mode. As expected, at $\omega < \omega_*$ we have $k_< > k_>$, so that the fields are more localized inside of the medium. Note that since the 2d localized plasmon frequency $\omega_* < \omega_p$, the boundary localized plasmons have lower energy than the bulk ones.

A plot of qc as a function of ω is shown below, with $\omega_p = 1$. The blue curve is the real part, and the orange curve is the imaginary part:



The corresponding plot of $\omega^2 \varepsilon(\omega)$ is (I guess this is rather superfluous)



Therefore we see that we can always get wave solutions when either $\omega < \omega_*$ or when $\omega > \omega_p$. The $\omega < \omega_*$ solutions are localized surface plasmons, while the $\omega > \omega_p$ solutions are delocalized, since in that case we don't have $k_{\leq} \in \mathbb{R}_{>0}$. We see that the dispersion of the surface plasmons is initially linear, but then saturates once we approach ω_* , so that no matter how large of a wavevector they carry, they always saturate in frequency at ω_* .

The point of making the plots is to make a comparison with the helical modes computed in the last diary entry—they're exactly the same thing! This isn't surprising in hindsight, since both the magnetic field in that problem and the interface in this problem provide ways of confining the motion of the electrons to a plane. In the helicon problem we were given a frequency ω_c that set the lower boundary for the disallowed region where waves can't propagate, while in the present context this frequency just came directly from the plasma frequency and the geometry of the problem.



Today we're doing a third installment in the series "Maxwell's equations, dielectric functions, and all the different types of "ons"". Today's "ons" are polarons, which we will see are basically phononic manifestations of the helicons and surface plasmons discussed previously.



The setting is an ionic crystal, which is basically an insulator with a nontrivial unit cell, such that the atoms in the unit cell have different electric charges. We will focus on the simplest case where there are two atoms per unit cell, each with charge $\pm q$.

The strategy will be to find the dielectric function using the same classical tricks as can be used to get the plasma frequency, and then to examine the consequences for wave propagation in the medium in question.

If r_{\pm} are the coordinates of the two ions in the basis, the equations of motion are

$$m_{\pm} \partial_t^2 r_{\pm} = \mp k(r_+ - r_-) \pm qE. \quad (1031)$$

The first term on the RHS is the restoring force on the spring connecting the ions together, while the second term is the contribution from the microscopic electric field that each of the ions feels. If we subtract these we get, letting $x \equiv r_+ - r_-$,

$$\mu \partial_t^2 x = -kx + qE \implies -\omega^2 x = -\omega_I^2 x + \frac{q}{\mu} E, \quad \omega_I^2 \equiv k/\mu. \quad (1032)$$

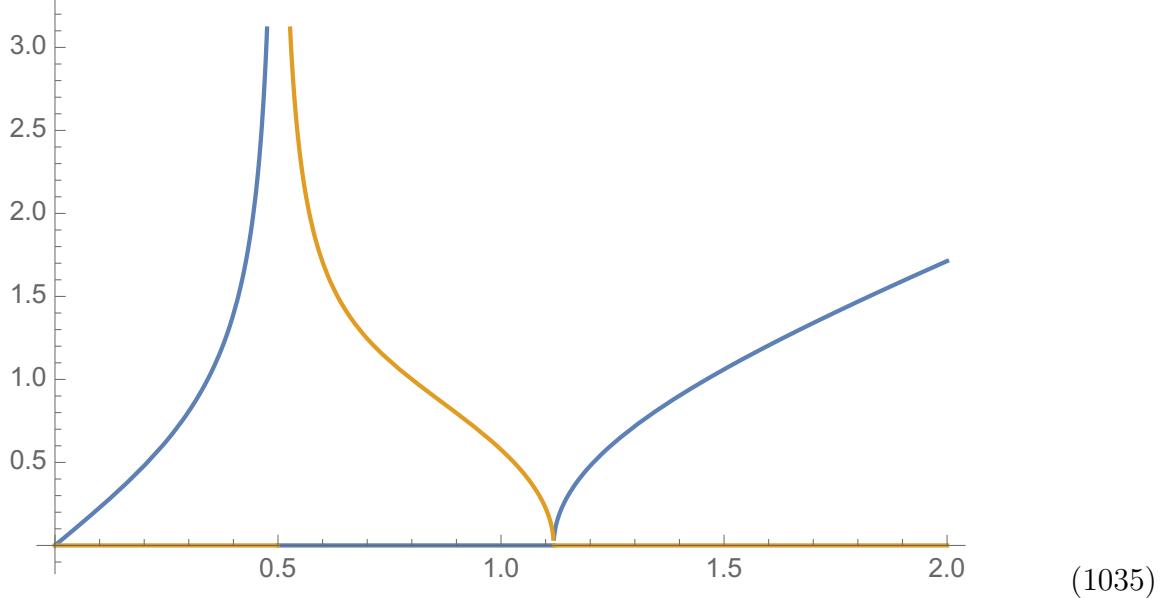
Since $\nabla \cdot \mathbf{P} = -\rho_b$, the polarization is $P = nqx$.⁸⁶ Therefore

$$-\omega^2 P = -\omega_I^2 P + \omega_p^2 \varepsilon_0 E, \quad \omega_p \equiv \frac{nq^2}{\varepsilon_0 \mu}. \quad (1033)$$

Using $\varepsilon = 1 + P/(\varepsilon_0 E)$, we then get⁸⁷

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_I^2}. \quad (1034)$$

When we plot the \mathbb{R} and \mathbb{I} parts of $c^2 q^2$ with e.g. $\omega_I = 1/2$ and $\omega_p = 1$, we get



⁸⁶We want $\nabla \cdot \mathbf{P} = -\rho_b$, so I guess here we think of $x = r_+ - r_-$ and the ∇ as acting on x as $(\nabla \cdot \mathbf{P}) = qn(\delta(y - r_+) - \delta(y - r_-))$. ethan: come back to this; signs are sloppy

⁸⁷We can get away with just calculating $\varepsilon(\omega) = \varepsilon(\omega, q = 0)$ since the q for light will always be tiny compared to the qs relevant for phonon stuff, in lieu of $cq \approx sk \implies q/k = s/c \ll 1$, where k is the phonon wavevector and s the speed of sound.

Look familiar? This is essentially the exact same plot that we made in our discussions of surface plasmons and of helicons. We see that the lower of ω_I set the upper limit for the lower branch of the wave solutions (regardless of ω_p), while ω_p approximately sets the lower bound for the upper branch, at least in the limit of large $\omega_p \gg \omega_I$.

At ω_I we have $\varepsilon \rightarrow \infty$ from below, which since $\varepsilon \rightarrow \infty$ represents a longitudinal mode in the material. At $\omega_* \approx \omega_p$ we have $\varepsilon \rightarrow 0$, which is a transverse mode that at high frequencies disperses as a regular light wave (this is the correct high-frequency limit since at very high frequencies the material is unable to react and becomes transparent).

But hold on, why do the longitudinal modes always have $\varepsilon(\omega) = 0$? An easy argument goes as follows: we suppose there is no free charge, so that $\nabla \cdot \mathbf{D} = 0$. Then assuming an isotropic medium where $\varepsilon(\omega)$ is diagonal, a longitudinal wave will have $\mathbf{D}, \mathbf{E}, \mathbf{P}$ all proportional to \mathbf{k} . Therefore $\mathbf{k} \cdot \mathbf{D} = 0 \implies \mathbf{D} = 0$, which tells us from $\mathbf{D} = \varepsilon \mathbf{E}$ that $\varepsilon = 0$ (assuming wolog that $\mathbf{E} \neq 0$ since otherwise all the fields are zero). This also goes the other way if we assume that the dynamics of the \mathbf{B} field aren't important, so that $\nabla \times \mathbf{E} = 0$. Then since $\mathbf{k} \times \mathbf{E} = 0$ but $\mathbf{E} \neq 0$ (else all the fields are zero), then $\mathbf{E} \perp \mathbf{k}$; hence the mode is longitudinal.

Similarly, we can argue that transverse modes usually have $\varepsilon(\omega) = \infty$. Indeed under the same assumptions above, suppose $\mathbf{k} \cdot \mathbf{D} = 0$ but $\mathbf{D} \neq 0$. Then $\varepsilon \mathbf{k} \cdot \mathbf{E} = 0$. Assuming $\varepsilon \neq 0$, then the mode being transverse means that the previous equation implies $\mathbf{E} = 0$, which in turn from $\mathbf{D} \neq 0$ but $\mathbf{D} = \varepsilon \mathbf{E}$ means that $\varepsilon = \infty$. The same steps also work in reverse.



Josephson junction stuff

Today we are going over basic Josephson junction phenomenology. Part of this will include filling out the details of one of the problems in A&S.



Schematic arguments

To explain the Josephson effects quickly, one could say the following: first, for two SCs L and R , the current from L to R should be proportional to $\langle c_R^\dagger c_R^\dagger c_L c_L \rangle - h.c.$ where $W \sim e^{iq \int_L^R A}$ is needed for gauge invariance. In MF this is something $\propto W e^{i(\phi_L - \phi_R)} - h.c.$, which gives us the expected

$$J_{R \rightarrow L} \propto \sin[q \int A + \Delta\phi] \quad (1036)$$

dependence. Secondly, we expect a voltage difference to cause a change in phase according to

$$H \ni V_L n_L + V_R n_R \implies [\phi_R - \phi_L, H] = V_R[\phi_R, n_R] - V_L[\phi_L, n_L] \propto \Delta V \implies \partial_t(\Delta\phi) \propto \Delta V. \quad (1037)$$

Going one level up in carefulness (or rather loquacity), we can model the system as follows. Let $\psi_{L/R}$ denote wavefunctions for the two superconductors. The amount of current coming from tunneling from L to R will be proportional to ψ_R , with a term in the action like $\psi_L^* \psi_R W$. If we write down separate Schrodinger equations for the two ψ s, then tunneling terms appear as additional contributions to $\partial_t \psi$:

$$i\partial_t \psi_L = E_L \psi_L + W \psi_R, \quad i\partial_t \psi_R = E_R \psi_R + W^* \psi_L. \quad (1038)$$

Again, W is there to make all terms in each equation transform in the same way under gauge transformations. Now we write $\psi_i = \sqrt{\rho_i} e^{i\phi_i}$, so that

$$i\partial_t \rho_L - \rho_L \partial_t \phi_L = E_L \rho_L + \sqrt{\rho_L \rho_R} W e^{i\Delta\phi}, \quad i\partial_t \rho_R - \rho_R \partial_t \phi_R = E_R \rho_R + \sqrt{\rho_L \rho_R} W^* e^{-i\Delta\phi} \quad (1039)$$

Now take the \mathbb{R} and \mathbb{I} parts of each equation, and subtract them to get equations for the current $\partial_t(\rho_R - \rho_L)$ and the phase $\partial_t(\phi_R - \phi_L)$. For the former, we have

$$J_{R \rightarrow L} \propto \sqrt{\rho_L \rho_R} \sin[q \int A + \Delta\phi] \quad (1040)$$

as expected, while for the latter, if we make the approximation that $\rho_L \approx \rho_R$, then

$$\partial_t \Delta\phi \propto E_L - E_R + (\sqrt{\rho_R/\rho_L} - \sqrt{\rho_L/\rho_R}) \cos[q \int A - \Delta\phi] \approx E_L - E_R \propto \Delta V, \quad (1041)$$

so that indeed the voltage sets the time evolution of the phase.

Getting more serious

To get more serious, we can consider the action (what follows are the details for a problem given in A&S)

$$S = \int d\tau \bar{\Psi} \left(\partial_\tau + \xi Z + \bigoplus_a e^{i\phi Z} \Delta_a X + (Z \otimes \tilde{X} \otimes \mathcal{T}) \right) \Psi + S_C, \quad (1042)$$

where

$$S_C = \frac{E_C}{4} \int \left(\bar{\Psi} (Z \otimes \tilde{Z}) \Psi \right)^2. \quad (1043)$$

The notation is as follows: each Ψ has the labels $(\Psi_{\alpha\uparrow}^a, [\Psi_{\alpha\downarrow}^a]^\dagger)^T$, where $a = L, R$ is a flavor index keeping track of what SC the field annihilates particles in, and α is an index that is summed over all free-particle eigenstates of the respective SCs (e.g. a momentum index). Z, X are Pauli matrices in particle-hole space for the Nambu spinors, \tilde{Z}, \tilde{X} are Pauli matrices in L/R flavor space, and the matrix \mathcal{T} is a tunneling matrix between different eigenstates,

i.e. $\mathcal{T}_{\alpha\beta}$ (the \tilde{X} in the \mathcal{T} term means that the \mathcal{T} term couples the two SCs together). The SCing OP on each SC is $\Delta_a e^{i\phi_a}$. Finally, $\bar{\Psi}(Z \otimes \tilde{Z})\Psi$ has alias $N_L - N_R$, so that the S_C term represents a charging energy, which we add phenomenologically onto the mean-field action.

We proceed in the usual way of decoupling the interaction with an auxiliary field. First, we decouple S_C by adding a Gaussian field V with standard deviation $\sigma_V^2 = 2E_C$ to the action, and then shifting $V \mapsto V + iE_C(N_L - N_R)$. The fermion part of the new action is

$$S = \int \bar{\Psi} \left(\partial_\tau + \xi Z + \bigoplus_a e^{i\phi_a Z} \Delta_a X + (Z \otimes \tilde{X} \otimes T) + \frac{iV}{2} Z \otimes \tilde{Z} \right) \Psi. \quad (1044)$$

Now we shift the fields by

$$\Psi^a \mapsto \begin{pmatrix} e^{-i\phi_a/2} \\ e^{i\phi_a/2} \end{pmatrix} \Psi^a, \quad (1045)$$

which shifts the phase dependence of the OPs onto the diagonal in the particle-hole space when the ∂_τ hits it, so that it appears with the V . The structure of this phase in particle-hole space has a Z , since particles and holes get opposite phases. Since the T term is off-diagonal in flavor space, the T terms therefore pick up phases like $e^{i\delta}$, where $\delta \equiv \phi_L - \phi_R$. Taking all of this into account, and integrating out the fermions, we get (the minus sign in front of the Tr is because we're in \mathbb{I} time)

$$S = \frac{1}{4E_C} \int V^2 - \text{Tr} \ln \left[\partial_\tau + \frac{i}{2} (\partial_\tau \phi Z + V Z \otimes \tilde{Z}) + \xi Z + \Delta X + \mathbf{1} \otimes \begin{pmatrix} e^{-i\delta Z/2} \mathcal{T} \\ e^{i\delta Z/2} \mathcal{T} \end{pmatrix} \right], \quad (1046)$$

where $\phi = \phi_L \oplus \phi_R$ is to be viewed as a matrix, and the matrix we've written out explicitly is in flavor space.

As usual, we write the thing in the log in terms of the propagators G_a :

$$\ln \left[(G_L^{-1} \oplus G_R^{-1}) \begin{pmatrix} G_L \mathcal{D}_L Z + \mathbf{1} & G_1 \mathcal{T} e^{-i\delta Z/2} \\ G_2 \mathcal{T} e^{i\delta Z/2} & G_2 \mathcal{D}_R Z + \mathbf{1} \end{pmatrix} \right] \rightarrow \ln \left[\mathbf{1} + \begin{pmatrix} G_L \mathcal{D}_L Z & G_1 \mathcal{T} e^{-i\delta Z/2} \\ G_2 \mathcal{T} e^{i\delta Z/2} & G_2 \mathcal{D}_R Z \end{pmatrix} \right], \quad (1047)$$

where the arrow means ignoring the infinite constant from the first factor in the log. Here we have defined

$$\mathcal{D}_a \equiv \frac{i}{2} (\partial_\tau \phi_a + (-1)^a V), \quad (1048)$$

and the propagators G_a , for each single-particle eigenstate α , are (in PH space, and still in \mathbb{I} time)

$$G_{a\alpha} = \frac{1}{\omega^2 + \xi_\alpha^2 + \Delta^2} \begin{pmatrix} i\omega + \xi_\alpha & \Delta \\ \Delta & i\omega - \xi_\alpha \end{pmatrix}. \quad (1049)$$

As a first pass, we look at the effective action when $\mathcal{T} = 0$. The terms quadratic in the \mathcal{D}_a s are, in the time domain,

$$S[\mathcal{T} = 0] \supset \frac{1}{2} \sum_{a,\alpha} \int d\tau, d\tau' \mathcal{D}_a(\tau) G_{a\alpha}(\tau - \tau') \mathcal{D}_a(\tau') G_{a\alpha}(\tau' - \tau). \quad (1050)$$

In frequency space, we get

$$S[\mathcal{T} = 0] \supset \frac{1}{2} \sum_{\omega\nu,a,\alpha} \mathcal{D}_a(\omega) G_{a\alpha}(\nu + \omega) \mathcal{D}_a(-\omega) G_{a\alpha}(\nu). \quad (1051)$$

Now the Greens function is, after summing over the single-particle states,

$$\sum_{\alpha} G_{\alpha a}(\omega) \approx N(0) \int_{\mathbb{R}} d\xi \frac{i\omega \mathbf{1} + \xi Z}{\xi^2 + \omega^2 + \Delta^2} = i\pi N(0) \frac{\omega}{\sqrt{\omega^2 + \Delta^2}}, \quad (1052)$$

which is localized around zero in the time domain due to the gap caused by Δ . As long as the fluctuations of the \mathcal{D}_a fields are on longer times than Δ^{-1} , we can then treat the $G(\tau - \tau')$ s in the above equation as being proportional to $\delta(\tau - \tau')$ s. Therefore we have

$$S[\mathcal{T} = 0] \propto \sum_a \int d\tau \mathcal{D}_a(\tau)^2. \quad (1053)$$

This is evidently minimized when $\mathcal{D}_a = 0$ for both a . According to the definition of \mathcal{D}_a then, this is minimized when

$$V = -\partial_{\tau}\phi_R = +\partial_{\tau}\phi_L, \quad (1054)$$

which are the Josephson relations for the time evolution of the phase difference.

Now let $\mathcal{T} \neq 0$. We will work at energy scales well below Δ , and hence will impose the above constraint exactly in what follows. The leading nonzero piece of the action is then

$$S \supset \frac{1}{2} \text{Tr} \left[\begin{pmatrix} G_1 \mathcal{T} e^{-i\delta Z/2} \\ G_2 \mathcal{T} e^{i\delta Z/2} \end{pmatrix}^2 \right]. \quad (1055)$$

Writing this out,

$$S \supset \mathcal{T}^2 \sum_{\alpha\beta\omega\nu} \text{Tr}[G_{1\alpha}(\omega + \nu) \mathcal{E}(\nu) G_{2\beta}(\omega) \mathcal{E}^*(\nu)], \quad \mathcal{E}(\nu) \equiv \int d\tau e^{-i\nu\tau - i\delta Z/2}. \quad (1056)$$

Note how we are doing *independent* sums over the eigenstates α, β on the two SCs—we have made a simplifying assumption in which the tunneling matrix elements are independent of the eigenstates they connect (think of the two SCs as two quantum dots).

We can break the tunneling contribution to S into two parts—one coming from the diagonal ($i\omega \mathbf{1} + \xi Z$) parts of the Greens functions, and one coming from the off-diagonal part ΔX . The diagonal part is, replacing the sums over eigenstates with integrals over ξ and approximating the DOS as a constant as usual, (we can pull out the \mathcal{E} s since they commute with Z and since $\mathcal{E}(\nu)\mathcal{E}^*(\nu)$ out of the brackets since $\sum_{\nu} \mathcal{E}(\nu)\mathcal{E}^*(\nu) = T^{-1} \int d\tau \mathcal{E}(\tau)\mathcal{E}^*(\tau) = 1$)

$$S_d = \mathcal{T}^2 N(0)^2 \sum_{\omega\nu} \int_{\mathbb{R}} d\xi_{\alpha} d\xi_{\beta} \text{Tr} \left[\frac{(i\omega + \xi_{\alpha}Z)(i(\omega + \nu) + \xi_{\beta}Z)}{(\omega^2 + \xi_{\alpha}^2 + \Delta^2)((\omega + \nu)^2 + \xi_{\beta}^2 + \Delta^2)} \mathcal{E}(\nu)\mathcal{E}^*(\nu) \right] \quad (1057)$$

The terms with ξ s in the numerator die by oddness, while the others are actans. Therefore

$$S_d = -2\mathcal{T}^2 N(0)^2 \pi^2 \sum_{\omega\nu} \frac{\omega(\omega + \nu)}{\sqrt{\omega^2 + \Delta^2} \sqrt{(\omega + \nu)^2 + \Delta^2}} \text{Tr}[\mathcal{E}(\nu)\mathcal{E}^*(\nu)]. \quad (1058)$$

For $\nu \gg \Delta$ we can do this sum by crudely ignoring Δ ; this gives something proportional to ν/T . However, we are instead interested in the $\nu/\Delta \ll 1$ limit (since we used it to fix $V = -\partial_\tau \phi_R!$), and since we are already dropping ν/Δ from the Josephson relation for the phase difference, we must drop this integral as well.⁸⁸

Therefore the only remaining term is the off-diagonal part, viz. (have to be careful here since the \mathcal{E} 's don't commute with X)

$$S \supset \mathcal{T}^2 N(0)^2 \Delta^2 \sum_{\omega\nu} \int_{\mathbb{R}} d\xi_\alpha d\xi_\beta \text{Tr} \left[\frac{X\mathcal{E}(\nu)X\mathcal{E}^*(\nu)}{(\omega^2 + \xi_\alpha^2 + \Delta^2)((\omega + \nu)^2 + \xi_\beta^2 + \Delta^2)} \right] \quad (1059)$$

Now $X\mathcal{E}(\nu)X\mathcal{E}^*(\nu) = \mathcal{E}(\nu)^2$, and so (again dropping the ν in the denominator on the grounds that we are dropping $\mathcal{O}(\nu/\Delta)$) we do the arctan integrals and get

$$S \supset \mathcal{T}^2 \pi^2 N(0)^2 \Delta^2 \sum_\nu T^{-1} \int dz n_F(z) \text{Tr}[\mathcal{E}(\nu)^2] \frac{1}{\Delta^2 - z^2}. \quad (1060)$$

In the limit where $e^{-\beta\Delta} = 0$, only the $z = -\Delta$ pole contributes (said another way, in this limit we may just send $\sum_\omega \rightarrow T \int d\omega$). Therefore

$$S \supset \frac{1}{2} (\mathcal{T}\pi N(0))^2 \Delta T^{-1} \sum_\nu (e^{i\delta} + e^{-i\delta}) \rightarrow (\mathcal{T}\pi N(0))^2 \Delta \int d\tau \cos \delta. \quad (1061)$$

Remembering the charging part of the action with now $V = -\partial_\tau \phi_R = \partial_\tau \delta/2$, we get

$$S = \frac{1}{16E_c} \int (\partial_\tau \delta)^2 + \gamma \int \cos \delta, \quad \gamma \equiv (\pi \mathcal{T} N(0))^2 \Delta, \quad (1062)$$

which is our final action for the phase difference $\delta = \phi_L - \phi_R$. We can also write this in Hamiltonian form by introducing the conjugate momentum $N = \frac{1}{8}\partial_\tau \delta$ and doing the Legendre transform (alias integrating in N), in terms of which

$$S = \int \left(\frac{i}{2} N \partial_\tau \delta + E_C N^2 + \gamma \cos \delta \right). \quad (1063)$$

Since N is conjugate to δ , it represents the difference of the number operators on the two SCs. Therefore the time derivative of N is the current, which we see is (going to \mathbb{R} time; the unit of charge is included in N , which is really the charge density)

$$J = \gamma \sin \delta. \quad (1064)$$

⁸⁸Since the ν dependence of the summand (excluding the \mathcal{E} s) is trivial, and since $\sum_\nu \mathcal{E}(\nu)\mathcal{E}^*(\nu) = 1$, S_d actually gives us an (infinite) constant. Being independent of δ , which by now is the only dynamical variable left in the action, we will ignore it. Or said another way, we will regulate it by subtracting off the contribution of S_d when $\nu = 0$ —we can also think of this as fixing boundary conditions for $e^{i\delta}$, so that only non-zero-mode fluctuations of \mathcal{E} are allowed, with the $\nu = 0$ part being disallowed by virtue of the fact that it changes the boundary conditions.

Likewise, the eom for the phase difference δ is

$$\partial_t \delta = 2E_C N, \quad (1065)$$

which tells us that the evolution of the phase difference is set by the potential difference across the junction (viz. the product of $E_C \sim \mu_L - \mu_R$ and N). Since $J \propto \sin \delta$ we see that a constant potential difference across the junction, which naively would only induce a DC current, actually induces a current which has an AC part, so that we can get current flowing in the *opposite* direction to the applied potential! Very cool.



Quick derivation of quantum oscillations and non-uniform magnetic fields

In today's diary entry we're going to be providing a quick way of deriving SdH oscillations, and then doing a simple problem in A&M.



Quick derivation of SdH oscillations

The quickest way to argue for quantum oscillations seems to be as follows. Consider a Fermi liquid, which possesses an (emergent) continuous translation symmetry. In the presence of a magnetic field translation symmetry is broken, since the translation generators in the plane normal to the field fail to commute. However, translations by distances of the magnetic unit cell size, viz. $L \sim \sqrt{2\pi/B}$, still commute. Therefore we expect the phenomenology of the system to change each time a magnetic band is completely filled, viz. when $\nu_M \equiv 2\pi\rho/B \in \mathbb{N}$.⁸⁹ Taking the difference in $1/B$ between two adjacent band fillings at fixed ρ gives the usual condition $\Delta(1/B) = 1/2\pi\rho$.

More carefully, consider electrons in a metal moving in the presence of a magnetic field. Because of interference effects, the \mathbb{R} -space electron orbits want to enclose an amount of flux in $\Phi_0 \mathbb{Z}$, with $\Phi_0 = 2\pi\hbar/e$. Therefore the \mathbb{R} -space areas of successive LL orbits will differ by

$$\Delta A_{\mathbb{R}} = \Phi_0/B. \quad (1066)$$

⁸⁹Since quantum oscillations are well-defined when the flux per unit cell is $\ll 2\pi$, we will have $\nu_M \gg 1$.

Thinking semiclassically, we now take the cross product of the \mathbf{k} eom with $\widehat{\mathbf{B}}$: since $\widehat{\mathbf{B}} \times (\mathbf{v} \times \mathbf{B}) = B\mathbf{v} - \mathbf{B}(\widehat{\mathbf{B}} \cdot \mathbf{v}) = B\mathbf{v}_\perp$, with \mathbf{v}_\perp the velocity in the plane normal to \mathbf{B} , we have

$$\hbar \star \dot{\mathbf{k}} = e\mathbf{v}_\perp B \implies \mathbf{r}_\perp(t) - \mathbf{r}_\perp(0) = \star \frac{\hbar}{eB} (\mathbf{k}(t) - \mathbf{k}(0)), \quad (1067)$$

where the \star is in the plane \perp to B . Therefore the \mathbb{R} space orbit in the plane \perp to \mathbf{B} is the same as the \mathbf{k} space orbit, but rotated by $\pi/2$ (from the \star) and scaled by a factor of $[\Phi_0/2\pi B]^2$. Hence the change in the k -space orbit area is

$$\Delta A_k = (\Delta A_{\mathbb{R}}) \frac{(2\pi B)^2}{\Phi_0^2} = (2\pi)^2 \frac{B}{\Phi_0}. \quad (1068)$$

Notice how the areas work out in such a way that $(\Delta A_k)(\Delta A_{\mathbb{R}})/(2\pi)^2 = 1$, as we might expect from phase space reasons. Now the energy levels are indexed by n and k_z , and when $E = \hbar\omega(n + 1/2)$ for some $n \in \mathbb{N}$ we are right at the bottom of a band which disperses along one direction, and hence has a $1/\sqrt{E}$ singularity in the DOS as characteristic of a 1d Fermi gas. Therefore we will see a singularity in the DOS whenever the Landau orbits cross the chemical potential (which we assume to be unchanged by the B field). The effects are most dramatic when this happens simultaneously for many different k_z slices, and hence the extremal orbits are selected out as giving the biggest changes in the DOS.

Oscillations in inhomogenous fields

Now consider the case where the magnetic field is not homogeneous (we will keep its orientation fixed for simplicity, and just consider fluctuations in B). What kinds of variations in B are allowed before the signal is washed out? Since we are working semiclassically, we can still think in terms of a phase space picture, where as we move along in \mathbb{R} space, the k -space areas A_n enclosed by the orbits change slightly.

In order for the variation in B to not screw up the resolution of the oscillations, we evidently need the variation in the energy levels to satisfy

$$\delta E_n \ll E_{n+1} - E_n. \quad (1069)$$

This means that we need

$$\delta E = \partial_B E \delta B \ll \frac{2\pi e B}{\hbar c} \partial_A E \implies \delta \ln B \ll \frac{2\pi e}{\hbar c} \frac{\partial_E B}{\partial_E A} = \frac{2\pi e}{\hbar c} (\partial_B A)^{-1}. \quad (1070)$$

Now from the above discussion we know that A depends linearly on B , and so

$$\delta \ln B \ll \frac{2\pi e}{\hbar c} \frac{1}{(n + \delta) 2\pi e / \hbar c}, \quad (1071)$$

where n is the level at the Fermi energy. Simplifying the RHS, we get the condition for the oscillations to not be washed out by the field variations:

$$\delta \ln B \ll \frac{\Delta A_k}{A}. \quad (1072)$$



Analytic structure of different Greens functions, fluctuation-dissipation, and other topics in linear response theory

Today we will look at the exact difference between the retarded, advanced, and time-ordered correlators, and prove some general results about them with the aid of the spectral representation, including the precise statement of the fluctuation dissipation theorem. This is of course pretty standard material, but I wasn't able to find a textbook where the discussion was really made completely explicit and done without sign errors.⁹⁰



First we will fix conventions for the various types of correlators we usually encounter. This is basic, but I feel like every book has slightly different sign conventions, so I think it's worth fixing conventions here to have as a reference. Our signs for Fourier transforming things will be that spatial momentum integrals always get plus signs and factors of 2π , and that spatial and temporal signs are always different. Therefore e.g.

$$G(t) = \int_{\omega} e^{-i\omega t} G(\omega), \quad G(\mathbf{k}) = \int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} G(\mathbf{x}) \quad (1073)$$

where \int_{ω} has a $1/2\pi$ but $\int_{\mathbf{x}}$ does not.

Response functions

We define the retarded response function for a pair of operators $\mathcal{O}, \mathcal{O}'$ as

$$\chi(t - t') = -i\theta(t - t') \langle [\mathcal{O}(t), \mathcal{O}'(t')]_{\zeta_{\mathcal{O}}} \rangle. \quad (1074)$$

Physically, this measures the expectation value of the operator \mathcal{O} in the presence of a time-varying source that couples to \mathcal{O}' through a term $\int F \mathcal{O}'$, via $\langle \mathcal{O}(t) \rangle_{\int F \mathcal{O}'} = \int dt' \chi(t - t') F(t')$. A few comments about the notation: first, the $[,]_{\zeta_{\mathcal{O}}}$ means a commutator if the \mathcal{O} s are bosonic, and an anticommutator if the \mathcal{O} s are fermionic,⁹¹ but we will sloppily keep this (and other fermionic minus signs) implicit in what follows. Also, the spatial coordinates / momenta in the arguments of all operators will be suppressed in what follows (although I

⁹⁰Unfortunately this is not to say that the following discussion is sign-error-free, although I tried to be careful.

⁹¹This comes from the fact that then F is also fermionic, and this produces a minus sign when the F is moved past the \mathcal{O} .

guess they don't really need to be local operators anyway). We are doing this because the really important structures of the various correlators we'll use all depend only on time / frequency.

The expectation value in the last equation is to be taken in some thermal state. When we say that we are interested in looking at the response function at finite times, we mean that we are interested in looking how operators in the theory respond to the presence of a time-varying source that acts as a weak perturbation on top of the thermal state. As is usual when doing linear response in QFT, the only place that the effects of temperature appear here is in the choice of the state in which to evaluate the expectation value: we set up the state to evaluate the (anti)-commutator in, and then treat the time dependence of the operators involved in the usual Heisenberg way.

The advanced response function is the anti-causal version of the retarded response function, viz.

$$\chi_A(t - t') = +i\theta(t' - t)\langle [\mathcal{O}(t), \mathcal{O}'(t')]_{\zeta_{\mathcal{O}}}\rangle. \quad (1075)$$

Finally, the time-ordered response function is

$$\chi_T(t - t') = -i\langle T\{\mathcal{O}(t)\mathcal{O}'(t')\}\rangle. \quad (1076)$$

Physically, $\chi(t)$ measures the dissipative response to an external force / current that couples to \mathcal{O} : our sign conventions have been chosen such that if $J(t)\mathcal{O}(t)$ (with $\mathcal{O} = \mathcal{O}'$ for simplicity) is added to the Hamiltonian, then \mathcal{O} acquires an expectation value as

$$\langle \mathcal{O}(t) \rangle = \int dt' \chi(t - t') J(t'), \quad (1077)$$

which can be derived using the usual interaction representation calculations. The correlator χ_A is just the anti-causal version of this. On the other hand, the time-ordered correlator is a bit different: χ_T is what we actually compute when doing Feynman diagrams, and so is the central object in QFT (just because in the path integral approach, inserting operators in the path integral to produce correlators automatically given by time-ordered expectation values, essentially just by construction). Thus χ_T is the full 2-point function, and as such we know that it takes the form (now restoring the explicit momentum dependence)

$$\chi_T(\omega, \mathbf{k}) = G(\omega, \mathbf{k}) = \frac{1}{\omega - \varepsilon_{\mathbf{k}} - \Sigma(\omega, \mathbf{k})} \quad (1078)$$

where as usual $\varepsilon_{\mathbf{k}}$ is the dispersion of the free Hamiltonian (the one we diagonalize to form the coherent states).

To gain more insight into the relations between these response functions and to figure out how to write χ, χ_A in a form more similar to the expression for $G(\omega, \mathbf{k})$ written above, we need to use the spectral representation. Using our conventions for Fourier transformations,

the standard manipulations give⁹²⁹³

$$\chi(\omega) = -i \int dt \sum_{ab} e^{-\beta E_a} e^{i(\omega+i\eta)t} (\mathcal{O}_{ab} \mathcal{O}'_{ba} - \zeta_{\mathcal{O}} e^{-itE_{ab}} \mathcal{O}_{ba} \mathcal{O}'_{ab}) \theta(t), \quad (1079)$$

where the sums are over eigenstates of the full interacting Hamiltonian, $E_{ab} = E_a - E_b$, $\mathcal{O}_{ab} = \langle a | \mathcal{O} | b \rangle$, and in this case $\zeta_{\mathcal{O}} = \pm 1$ is the $(-1)^F$ eigenvalue of \mathcal{O} . Doing the integral,

$$\chi(\omega) = \sum_{ab} e^{-\beta E_a} \left(\frac{\mathcal{O}_{ab} \mathcal{O}'_{ba}}{\omega - E_{ba} + i\eta} - \zeta_{\mathcal{O}} \frac{\mathcal{O}_{ba} \mathcal{O}'_{ba}}{\omega + E_{ba} + i\eta} \right). \quad (1080)$$

If one wants the $T = 0$ result one just sets a to be the vacuum; hence we have written it in terms of E_{ba} and not E_{ab} since $E_{ba} > 0$ at $T \rightarrow 0$. We can also write this slightly more compactly as

$$\chi(\omega) = \sum_{ab} \frac{\mathcal{O}_{ab} \mathcal{O}'_{ba}}{\omega - E_{ba} + i\eta} (e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}). \quad (1081)$$

Anyway, we see firstly that the poles of $\chi(\omega)$ are all in the lower half plane. The analyticity in the UHP is then tantamount to causality in these conventions, since

$$\chi(t) = \int_{\omega} e^{-i\omega t} \chi(\omega) \quad (1082)$$

means that when $t < 0$, the contour needs to be closed in the upper half plane—analyticity then tells us that this vanishes, hence providing a sanity check on $\chi(t < 0) = 0$.

To get the advanced propagator, one simply multiplies the by two minus signs (one from the definition (which is why we included it!) and one from the $\theta(-t)$ giving a relative minus sign from the integration over t), and replaces $i\eta$ with $-i\eta$, since the convergence of the Fourier transform of $\chi_A(\omega)$ at $t \rightarrow -\infty$ is now the relevant issue, meaning one must Fourier transform with $e^{i(\omega-i\eta)t}$. Therefore,

$$\chi_A(\omega) = \sum_{ab} \frac{\mathcal{O}_{ab} \mathcal{O}'_{ba}}{\omega - E_{ba} - i\eta} (e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}). \quad (1083)$$

One important reason for choosing our conventions is that in frequency space, the retarded and advanced response functions are simply complex conjugates of one another:

$$\chi(\omega) = \chi_A(\omega)^*. \quad (1084)$$

Finally we need to go through the same song and dance for the time-ordered response function. This time the convergence factors are dictated as

$$\chi_T(\omega) = -i \int dt \sum_{ab} e^{-\beta E_a} (\mathcal{O}_{ab} \mathcal{O}'_{ba} \theta(t) e^{iE_{ba}t+i(\omega+i\eta)t} + \zeta_{\mathcal{O}} \mathcal{O}_{ba} \mathcal{O}'_{ab} \theta(-t) e^{iE_{ba}t+i(\omega-i\eta)t}). \quad (1085)$$

⁹²How to remember the sign conventions here: the sign convention in the Schrodinger equation tells us that $H = +i\hbar d_t$. This is natural since it is the opposite sign to momentum—we can think of this as being in “mostly negative signature conventions” if we like. Now $\mathcal{O}(t) = e^{-t\partial_t} \mathcal{O}(0) e^{t\partial_t}$, since $e^{t\partial_t}$ “drags functions backwards along the time axis by an amount t ”, and so the correct sign is $\mathcal{O}(t) = e^{+iHt} \mathcal{O}(0) e^{-iHt}$.

⁹³Also, notational warning: we will be forgetting to put $1/Z$'s to normalize the thermal Boltzmann weights $e^{-\beta E_a}$. Sorry.

Doing the integrals, we see that since the signs for the $i\eta$ s now track the signs of the E_{ba} s instead of the sign of ω , we get

$$\chi_T(\omega) = \sum_{ab} \mathcal{O}_{ab} \mathcal{O}'_{ba} \left(\frac{e^{-\beta E_a}}{\omega - E_{ba} + i\eta} - \zeta_{\mathcal{O}} \frac{e^{-\beta E_b}}{\omega - E_{ba} - i\eta} \right). \quad (1086)$$

where we dropped the unimportant $i\eta$ in the numerator. The poles of this expression are now at

$$\omega = E_{ba} \pm i\eta. \quad (1087)$$

The $\mp i\eta$ is what distinguishes the time-ordered correlator from the others: it has poles both above and below the \mathbb{R} axis! Also, it's worth noting how different the two representations of the time-ordered correlator look: the spectral representation gives us something with imaginary numbers only appearing through the $i\eta$ factor, while (1078) contains imaginary numbers through the imaginary part of the self energy. The equivalence of these two expressions relies on the fact that a continuum of delta functions can merge to form a branch cut.

Finally, note that our sign conventions are such that all three response functions can be written as special cases of (after re-labeling some indices)

$$\chi(z) = \sum_{ab} \mathcal{O}_{ab} \mathcal{O}'_{ba} \frac{e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}}{z - E_{ba}}, \quad (1088)$$

where we choose the z contour to run slightly above / below the \mathbb{R} axis (retarded / advanced) or along the $R_{-\eta}\mathbb{R}$ axis, where R_θ is a rotation in the \mathbb{C} plane (time-ordered). From this, we see that χ_T and χ_A agree when $\omega < 0$, while χ_T and χ_R agree when $\omega > 0$.

"Imaginary time"

Now we will discuss correlators defined for operators at different points along the thermal circle. The reason why I don't like referring to this as "imaginary time" is that for me, imaginary time is what we get when we Wick-rotate a \mathbb{R} -time path integral, viz. an imaginary time path integral is one where we're integrating over a $(d+1)$ -dimensional manifold $\mathbb{R} \times \Sigma$, where the metric is locally $1 \oplus g_\Sigma$ (viz. the time direction is always \mathbb{R} ; we are not Wick-rotating to something with a temporal S^1). The "imaginary time" coordinate of the thermal circle is just some fake variable that we introduced to write the thermal trace as a path integral, with operators inserted at different points on the S^1 not a priori yielding any physically relevant correlation functions (rather, they produce different states). That said, we know that $\beta \rightarrow \infty$ projects onto the ground state, and since $T = 0$ QFT is basically the statistical mechanics of the ground state, we can at least expect that taking $\beta \rightarrow \infty$ produces something like a Wick-rotated $T = 0$ path integral (after all, we do normally assume in $T = 0$ QFT that the system has the same ∂ cons at $t = \pm\infty$, so we can think about $T = 0$ path integrals as happening on a gigantic S^1 if we wish).

Anyway, let us formally define the operator $\mathcal{O}(\tau)$ as

$$\mathcal{O}(\tau) = e^{\tau H} \mathcal{O}(0) e^{-\tau H}, \quad (1089)$$

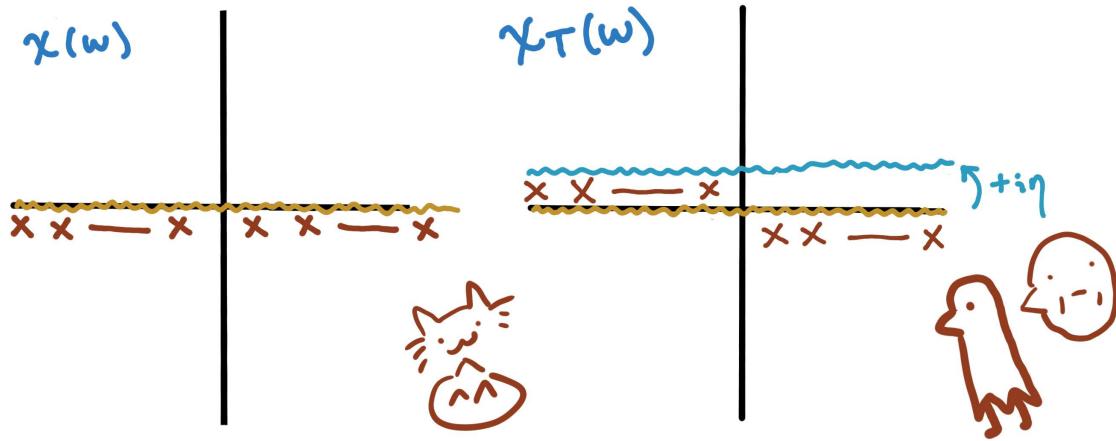


Figure 17: Left: the causal retarded Greens function, which is analytic in the UHP (brown crosses are poles). Changing conventions for the sign in the Fourier transform would move the singularities into the UHP instead. The orange squiggly line indicates the integration contour. Right: the time-ordered Greens function, with the orange squiggle again the integration contour. By pushing the contour up to the blue squiggle we get analyticity in the UHP, and so $\chi_T(\omega + i\eta) = \chi(\omega)$.

which is the same as if $\mathcal{O}(\tau)$ were $\mathcal{O}(0)$ time-evolved by $\tau = it$. We then define the correlator

$$\chi_E(\tau, \tau') = -\langle T[\mathcal{O}(\tau)\mathcal{O}'(\tau')]\rangle, \quad \tau, \tau' \in [0, \beta) \quad (1090)$$

where T orders according to Euclidean time, and includes a sign for statistics, viz.

$$T[\mathcal{O}(\tau)\mathcal{O}'(\tau')] = \mathcal{O}(\tau)\mathcal{O}'(\tau')\Theta(\tau - \tau') + \zeta_{\mathcal{O}}\mathcal{O}'(\tau')\mathcal{O}(\tau)\Theta(\tau' - \tau). \quad (1091)$$

Note that we can assume that $\zeta_{\mathcal{O}} = \zeta_{\mathcal{O}'}$ wolog, since otherwise the correlator vanishes. Note that the τ arguments in the correlator are *defined* to lie within the thermal circle interval. That is, we can consider $\mathcal{O}(\tau)$ for any τ , but the correlator χ_E is only defined on a restricted range of its arguments—we are always inserting the operators along the thermal circle while doing the trace. Again, a priori this isn't a super physical thing to consider—e.g. for $\tau' = 0$, this “correlator” represents the thermal expectation value of \mathcal{O}' in the state given by the density matrix $\rho = e^{-(\beta-\tau)H}\mathcal{O}e^{\tau H}$, which a priori isn't a super interesting density matrix to consider. But let us press ahead all the same.

Now since H is by definition constant along the S^1 (again, since τ is just a fake coordinate that we added in by hand), we have $\chi_E(\tau, \tau') = \chi_E(\tau - \tau')$, and so χ_E is a function defined on $[-\beta, \beta]$. We can then write it as

$$\chi_E(\tau) = -\langle T[\mathcal{O}(\tau)\mathcal{O}'(0)]\rangle, \quad \tau \in [-\beta, \beta]. \quad (1092)$$

If $\tau > 0$, we have (writing e.g. $\mathcal{O}(0) = \mathcal{O}$)

$$\begin{aligned} \chi_E(\tau) &= -\langle \mathcal{O}(\tau)\mathcal{O}' \rangle = -\text{Tr}[e^{-(\beta-\tau)H}\mathcal{O}e^{-\tau H}\mathcal{O}'] \\ &= -\text{Tr}[e^{-\beta H}\mathcal{O}'e^{(\tau-\beta)H}\mathcal{O}e^{-(\tau-\beta)H}] \\ &= -\langle \mathcal{O}'\mathcal{O}(\tau - \beta) \rangle = -\zeta_{\mathcal{O}}\langle T[\mathcal{O}(\tau - \beta)\mathcal{O}'] \rangle \\ &= \zeta_{\mathcal{O}}\chi_E(\tau - \beta). \end{aligned} \quad (1093)$$

Likewise, $\chi_E(\tau + \beta) = \zeta_{\mathcal{O}} \chi_E(\tau)$ if $\tau < 0$. This property is basically the reason why we are focusing on the T -ordered correlator—without the T ordering, the shift by $\pm\beta$ would switch the ordering of the operators, and wouldn’t give us this nice (anti)periodicity property. Also note that in the second line we used the cyclicity of the trace, which holds even when the operators in question are fermionic.⁹⁴ This means that when we FT, instead of getting $\chi_E(\omega)$ for any $\omega \in 2\pi/(2\beta)\mathbb{Z}$, we instead get $\omega \in (\pi\beta^{-1})2\mathbb{Z}$ if $\zeta_{\mathcal{O}} = 1$, and $\omega \in (\pi\beta^{-1})(2\mathbb{Z} + 1)$ if $\zeta_{\mathcal{O}} = -1$. When we FT we then choose to split the $1/(2\beta)$ normalization factor (to ensure that squaring the FT gets us back with the right normalization) asymmetrically by putting a $1/2$ on the $d\tau$ integration and a β^{-1} on the ω sum, so that

$$\chi_E(\omega) = \frac{1}{2} \int_{[-\beta, \beta)} d\tau e^{i\omega\tau} \chi_E(\tau) = \int_{[0, \beta)} d\tau e^{i\omega\tau} \chi_E(\tau), \quad (1094)$$

where we used the restriction on ω and the transformation property of χ_E under a shift by β .

To see why these definitions will be useful, let’s look at the spectral representation of $\chi_E(\omega)$:⁹⁵

$$\begin{aligned} \chi_E(\omega) &= - \sum_{ab} \int_0^\beta d\tau \mathcal{O}_{ab} \mathcal{O}'_{ba} e^{-\beta E_a} e^{\tau(E_{ab} + i\omega)} \\ &= - \sum_{ab} \mathcal{O}_{ab} \mathcal{O}'_{ba} \frac{e^{-\beta E_a}}{i\omega - E_{ba}} (e^{\beta E_{ab} + i\omega\beta} - 1). \end{aligned} \quad (1095)$$

Here comes the slick part: due to the constraints on the allowed values of ω , we can drop the $i\omega\beta$ in the exponent at the cost of a $\zeta_{\mathcal{O}}$ sign, and write

$$\chi_E(\omega) = \sum_{ab} \mathcal{O}_{ab} \mathcal{O}'_{ba} \frac{e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}}{i\omega - E_{ba}}. \quad (1096)$$

Anyway, the point of doing this is that we see that $\chi_E(\omega)$ actually has the same form as $\chi(\omega)$, provided that we rotate the frequency in the complex plane: (this is the reason for defining χ_E with the minus sign in front of the expectation value)

$$\chi_E(-i\omega + \eta) = \chi(\omega). \quad (1097)$$

This means that we can compute the Fourier transform of the finite-temperature *real-time* response function (or time-ordered correlator, by suitably shifting the frequency with a small imaginary part) by analytically continuing the correlator $\chi_E(\omega)$ that we computed by Fourier-transforming a correlator defined on the thermal circle. Note however that the

⁹⁴Consider breaking up the trace into each individual \otimes factor of the Fock space. Then wolog we can consider the traces of the operators $c^\dagger c$ and cc^\dagger , since these are the only types of nonzero terms with that will appear in the traces calculated above for fermionic $\mathcal{O}, \mathcal{O}'$. But the traces are the same: $\text{Tr}[c^\dagger c] = \text{Tr}[cc^\dagger] = 1$. Therefore we can move the cs and $c^\dagger s$ around cyclicly in the trace for every \otimes factor in the Fock space, and hence the trace is actually cyclic, even for fermionic operators.

⁹⁵Note that we are just going straight to $\chi_E(\omega)$ without first looking at $\chi_E(\tau)$. This is because the latter is more annoying due to the “time” ordering. The properties of the allowed frequencies for $\chi_E(\omega)$ mean that when we write $\chi_E(\tau)$ inside an integral, we can reduce the integral to one over only $\tau \in [0, \beta]$. However we should still remember that by itself, $\chi_E(\tau)$ is still defined over the interval $[-\beta, \beta]$, and not just $[0, \beta]$.

analytic continuation that we had to do is slightly subtle, because of the fact that it can only be done in a simple fashion (by sending $\omega \mapsto -i\omega$) at the *end* of the calculation. If we do $\omega \mapsto -i\omega$ at the beginning of the calculation, we get total nonsense (and indeed, remember that we had to assume that ω was a Matsubara frequency midway through the calculation!). Presumably there is some prescription for doing the continuation in general, but the naive way of doing it evidently only works when the only way in which ω appears in the correlator is in the denominator of a simple pole.⁹⁶ The reason why this is useful is that when we compute $\text{Tr}[e^{-\beta H} \mathcal{O}\mathcal{O}']$, we don't need to ever do an explicit sum over states weighted by $e^{-\beta H}$ (which we'd have to do in the \mathbb{R} -time approach), since we can turn the trace into a path integral in the usual way. The motto here is "use the spectral representation of the trace to prove stuff, and the path integral representation to calculate stuff". Anyway, the somewhat amazing thing is that the thermal information about the sum over states weighted by $e^{-\beta H}$ is built into the geometry of the path integral we're doing, and yet we can *still* get dynamical \mathbb{R} -time information at the end of the day by analytically continuing.

I still find it very surprising that this works. I've never seen this discussed well in books, where one often gets the impression that the crucial fact is noticing that $e^{-\beta H}$ is e^{-itH} for $t = -i\tau$. If we were really just analytically continuing a time coordinate from \mathbb{R} time to the thermal circle when computing the correlators, then we would be limited to computing equal-time correlators, since the time coordinate would be rotated away. Furthermore, we are actually never analytically continuing time, only frequency: we can wick rotate a function defined on one axis of the \mathbb{C} plane to a function defined on another axis, but we can't wick rotate a \mathbb{R} time correlator defined on \mathbb{R} to one defined on S^1 , and so the rotation has to be done in frequency space. Actually this is still not very justified: analytic continuations are uniquely determined if you know the function on a dense subset of \mathbb{C} (e.g. a line segment), but not if you just know the function on a copy of $\mathbb{Z} \subset \mathbb{C}$. However, the feeling is that the content of correlators at Matsubara frequencies are usually enough to make the analytic continuation well-defined.

We should stress that we really are computing the *real-time* response function (which contains information about dynamics), despite coming from a starting point, viz. $\text{Tr}[e^{-\beta H} \mathcal{O}\mathcal{O}']$, which looks like it contains no information about dynamics, only thermodynamics (really should be called "thermostatics"). This is kind of amazing because \mathbb{R} time disturbances propagate as waves and never die out, while $i\mathbb{R}$ time has no propagating waves. Where does all this intricate information about quantum coherence and long-time wave propagation show up in the comparatively placid $i\mathbb{R}$ time theory? The answer is that it shows up (presumably) in a very complicated and non-local manner, rendered so by the complications of analytic continuation. A related fact is that while this analytic continuation is possible in principle and in practice for the simplest (free) examples, it becomes essentially impossible when any kind of approximation scheme is introduced; the analytic continuations are fiddly enough that slightly modifying the input data will totally change what once gets for the dynamical correlation functions. So when any kind of approximations need to be made this whole $i\mathbb{R}$ time thing is of questionable use for computing non-static quantities.

⁹⁶Also note that we are assuming that a unique analytic continuation is picked out on all of \mathbb{C} , despite only knowing the value of $\chi_E(\omega_n)$ on a copy of \mathbb{Z} . While in general one needs an uncountable number of values of χ_E to determine a unique continuation, there turns out to be a theorem guaranteeing a unique continuation as long as $\chi_E(\omega_n)$ grows at most exponentially with n , which is true for us.

A brief clarification on this: thermodynamics is done in the assumption that nothing is changing in time, so what exactly do we mean by "real time correlators in a system at temperature T "? If the concept of T is to be defined, shouldn't we be in a situation where nothing is changing with time?⁹⁷ The point here is that for us, T is just a label that goes into determining the state in which we want to compute the expectation value. We create a thermal state with T , but then evolve it forward in \mathbb{R} time, and study \mathbb{R} time fluctuations on top of it (well, small \mathbb{R} -time fluctuations; we're working under the purview of linear response theory)—we're just doing regular QFT, except that instead of the density matrix $|0\rangle\langle 0|$, we're using $e^{-\beta H}$. In particular, all of our time evolution is unitary, and there is no reservoir that our system is coupled to. While this is slightly against the spirit of thermodynamics, wherein the whole point is that we need to couple to a bath to define thermodynamic quantities, our assumption is that the system will "self-thermalize", so that it can act as its own bath. This assumption means that we can get away with doing unitary time evolution in a closed system, and still expect to see the same kinds of effects that we would see in an open system coupled to a bath and evolving in a non-unitary way, provided that we choose the appropriate state with which to take the trace in the path integral.

Fluctuation-dissipation

Now we are in a position to see one precise statement of the fluctuation-dissipation theorem (every book seems to have their own statement for the fluctuation-dissipation theorem; what follows is one possible definition). The analytic structure of the time-ordered and retarded correlators is shown in figure 17. From the picture, we see that if we shift the argument of the time-ordered correlator up slightly in the \mathbb{C} plane, we get the retarded correlator. Hence we have

$$\chi(\omega) = \chi_T(\omega + i\eta). \quad (1098)$$

Likewise,

$$\chi_A(\omega) = \chi_T(\omega - i\eta). \quad (1099)$$

The LHS is dissipation (it is computed from measuring the response to an external driving force, and tells us how transport occurs in the system—for example, if $\mathcal{O} = \mathbf{j}$ is an electric current then χ is the conductivity, telling us how energy of charge carriers dissipate); the RHS is fluctuation (it is computed by computing two-point correlation functions in the path integral). This tells us that fluctuations carry the same information as the dissipative response, but in order to see this one must nudge the frequency of the fluctuations slightly upwards in the \mathbb{C} plane. One could argue that a more apt statement of the theorem would involve only the imaginary part of $\chi(\omega)$ since it is really the imaginary part that controls dissipation⁹⁸. Because of this, one often instead formulates fluctuation-dissipation by relating the non-time-ordered correlator to the imaginary part of $\chi(\omega)$. More on this in a sec.

⁹⁷Also, the definition of $\chi(t)$ comes from adding an explicitly time-dependent background field term to the action—how can something like this, where we have a term that gradually turns on and off over an infinite time, be captured by a trace $\text{Tr}[e^{-\beta H}\mathcal{O}\mathcal{O}']$ that is completely static? The point is that $\chi(t)$ is actually computed within linear response as a commutator evaluated in a state in which the Hamiltonian is *time independent*, so that as long as we're doing linear response, this issue doesn't come up.

⁹⁸In the case where χ is a genuine analytic response function. The conductivity is different, since the \mathbb{R} part of σ is what controls dissipation. More on this later.

We can use these results to prove that the real part of Σ is continuous across the real axis, while the imaginary part flips sign:

$$\Sigma_R(\omega + i\eta) = \Sigma_R(\omega - i\eta), \quad \Sigma_I(\omega + i\eta) = -\Sigma_I(\omega - i\eta). \quad (1100)$$

This is proved using the spectral representation for the correlator. From the spectral representations for the advanced and retarded correlators, we see that they are complex conjugates of one another:

$$\chi(\omega)^* = \chi_A(\omega). \quad (1101)$$

Furthermore, the complex conjugation only acts on the $i\eta$ factor—there are no other complex numbers appearing in the spectral representation. Therefore we can also write the above as

$$\chi(\omega)^* = \chi_T(\omega + i\eta)^* = \chi_T(\omega - i\eta) = \chi_A(\omega). \quad (1102)$$

This reads

$$\left(\frac{1}{\omega - \varepsilon_{\mathbf{k}} - \Sigma(\omega + i\eta, \mathbf{k}) + i\eta} \right)^* = \frac{1}{\omega - \varepsilon_{\mathbf{k}} - \Sigma(\omega - i\eta, \mathbf{k}) - i\eta}, \quad (1103)$$

telling us that

$$\Sigma(\omega + i\eta)^* = \Sigma(\omega - i\eta), \quad (1104)$$

which then proves the claim.

Now we will look at the non-time-ordered correlator (just focusing on a single Hermitian operator \mathcal{O} for simplicity)

$$S(t) = \langle \mathcal{O}(t)\mathcal{O}(0) \rangle. \quad (1105)$$

Note that this is *not* what is computed by a path integral, i.e. the $\langle \rangle$ above does not correspond to a path integral insertion. We aren't using the letter χ here since $S(t)$ has no nice properties vis-a-vis causality.

The difference between $S(\omega)$ and $\chi(z)$ is that when Fourier transformed we get a delta function of frequency, rather than a pole (constants FT to δ functions; θ functions FT to poles). Explicitly, doing the usual steps,

$$S(\omega) = 2\pi \sum_{ab} e^{-\beta E_a} \delta(\omega - E_{ba}) |\mathcal{O}_{ab}|^2, \quad (1106)$$

where the 2π comes from the integral producing the δ function. Now we calculate $\chi_{\mathbb{I}}(\omega)$:

$$\begin{aligned} \chi_{\mathbb{I}}(\omega) &= -\pi \sum_{ab} |\mathcal{O}_{ab}|^2 (e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}) \delta(\omega - E_{ba}) \\ &= \pi \sum_{ab} |\mathcal{O}_{ab}|^2 (e^{-\beta E_a} - \zeta_{\mathcal{O}}) \delta(\omega - E_{ba}) \end{aligned} \quad (1107)$$

where we used the constraint coming from the δ function to re-write $E_{ab} = -\omega$ and pull an $e^{-\beta E_{ab}}$ out of the sum. Comparing these two expressions, we see that

$$\chi_{\mathbb{I}}(\omega) = \frac{1}{2} (e^{\beta\omega} - \zeta_{\mathcal{O}}) S(\omega) = \frac{1}{2n_{\mathcal{O}}(\omega)} S(\omega), \quad (1108)$$

where $n_{\mathcal{O}}$ is the FD or BE distribution, as appropriate. If we had kept track of the \hbar s we would have one in the denominator and one in the exponent, giving a prefactor of $\hbar^{-1}(e^{\hbar\omega\beta} - 1)$, which in the classical limit (for bosonic statistics) reduces to the ω/T response that we expect.

Spectral density

Now let's talk about the spectral density. We will *not* define it as the imaginary part of the time-ordered correlator: the reason for doing this is that as we have seen, $\Sigma_I(\omega)$ has a discontinuity on the \mathbb{R} axis, and so it is better to define the spectral density with respect to a correlator whose integration contour stays consistently to one side of the discontinuity. So with that in mind, we define

$$A(\omega) = -\frac{1}{\pi} \text{Im}[\chi(\omega)]. \quad (1109)$$

From the spectral representation, we conclude that

$$A(\omega) = \sum_{ab} |\langle a | \mathcal{O}(0) | b \rangle|^2 e^{-\beta E_a} (\delta(\omega - E_{ba}) - \delta(\omega + E_{ba})), \quad (1110)$$

which shows why the name “spectral density” is apt. The spectral density is essentially the density of states at frequency ω , but it only counts the states that the operator \mathcal{O} can “see”, because of the matrix element factor. Because of this, the phrase “the spectral density” is a little ambiguous; one always needs to qualify by saying which operator one is computing the spectral density of.

On the other hand, using the above statement of fluctuation-dissipation, we can get an expression for the spectral density in terms of the self-energy: we write

$$A(\omega) = -\frac{1}{\pi} \text{Im}[\chi_T(\omega + i\eta)]. \quad (1111)$$

If $\Sigma_I(\omega) = 0$, then we just get a delta function from the $i\eta$ in the denominator. On the other hand if $\Sigma_I(\omega) \neq 0$, we get a Lorentzian distribution with width determined by Σ_I :

$$A(\omega) = \begin{cases} \delta(\omega - \varepsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k})) & \text{if } \Sigma_I(\omega + i\eta, \mathbf{k}) = 0 \\ -\pi^{-1} \frac{\Sigma_I(\omega + i\eta, \mathbf{k})}{(\omega - \varepsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k}))^2 + \Sigma_I(\omega + i\eta, \mathbf{k})^2} & \text{if } \Sigma_I(\omega + i\eta, \mathbf{k}) \neq 0 \end{cases} \quad (1112)$$

We will see shortly that the $\Sigma_I = 0$ regime is below the threshold for particle production (and so the spectral weight is concentrated in δ function peaks at the locations of the long-lived particles), while the $\Sigma_I \neq 0$ regime is above the threshold (and so the spectral weight is spread out over a broad continuum, made possible by interactions).

Sum rules and KK

This stuff is again standard, but a full 50% of the cond-mat books I own had typos in the signs and stuff, so we'll try to get everything straight (no promises).

The KK relations follow from the analyticity of the causal response function $\chi(z)$ for $z \in UHP$. One can then write, for any $\omega \in \mathbb{R}$,

$$\chi(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} d\omega' \frac{\chi(\omega')}{\omega - \omega' + i\eta}, \quad (1113)$$

since the contour can be closed in the UHP and by causality picks up a contribution solely from the pole in the denominator (a similar expression can be written down for the acausal response).

Now we can use the Dirac identity on the denominator in the integral. The imaginary part is $+i\pi\delta(\omega - \omega')$, which then gives a $\chi(\omega)/2$ on the RHS. Subtracting this off, we then have

$$\chi(\omega) = \frac{1}{\pi i} P \int_{\mathbb{R}} d\omega' \frac{\chi(\omega')}{\omega - \omega'}. \quad (1114)$$

Splitting this up into real and imaginary parts gives the KK relations

$$\chi_{\mathbb{R}}(\omega) = \frac{1}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\chi_{\mathbb{I}}(\omega')}{\omega - \omega'}, \quad \chi_{\mathbb{I}}(\omega) = -\frac{1}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\chi_{\mathbb{R}}(\omega')}{\omega - \omega'}. \quad (1115)$$

One can also use the reality of $\chi(t)$, which implies $\chi_{\mathbb{R}}(-\omega) = \chi_{\mathbb{R}}(\omega)$ and $\chi_{\mathbb{I}}(-\omega) = -\chi_{\mathbb{I}}(\omega)$, to write

$$\chi_{\mathbb{R}}(\omega) = \frac{1}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\omega' \chi_{\mathbb{I}}(\omega')}{\omega^2 - \omega'^2}, \quad \chi_{\mathbb{I}}(\omega) = -\frac{\omega}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\chi_{\mathbb{R}}(\omega')}{\omega^2 - \omega'^2}. \quad (1116)$$

Another thing often brought up in these discussions are sum rules. The most general way of writing down a sum rule is to integrate a causal response function (or a suitably shifted time-ordered response function) along the whole frequency axis, with a convergence factor that requires the integral to be closed in the LHP:

$$\int_{\mathbb{R}} d\omega \chi(\omega) e^{-i\omega\eta} = \int_{\mathbb{R}} d\omega \chi_T(\omega + i\eta) e^{-i\omega\eta} = C, \quad (1117)$$

where C is some constant that doesn't change as we vary the microscopic parameters of the system. The fact that C is a constant just comes from analyticity. Since all of the poles of $\chi(\omega)$ must lie in the LHP by causality, changing microscopic parameters in the theory will never result in a pole crossing from the LHP into the UHP. The only thing that can happen to the poles as we smoothly tune parameters is that they move around within the LHP, that opposite-residue poles collide and annihilate, or that opposite-residue poles are created in pairs. All of these processes don't change the above integral, which therefore is a constant.⁹⁹

Now earlier we showed that $\chi_T(\omega + i\eta) = \chi_T(\omega - i\eta)^*$. We also know that all the poles of $\chi_T(\omega + i\eta)$ lie slightly below the \mathbb{R} axis. The latter fact means that when we do the above integral by a semicircular contour in the LHP, we can deform the circular part of the contour to run just below the \mathbb{R} axis. Using the former fact, this gives

$$\int_{\mathbb{R}} d\omega \chi(\omega) e^{-i\omega\eta} = \int_{\mathbb{R}} d\omega [\chi_T(\omega + i\eta) - \chi_T(\omega - i\eta)] = 2i \int_{\mathbb{R}} d\omega \chi_{\mathbb{I}}(\omega) = 0, \quad (1118)$$

because the reality of $\chi(t)$ means that $\chi_{\mathbb{I}}(\omega) = -\chi_{\mathbb{I}}(-\omega)$.

So wait, wtf are we doing, if we just get zero? The actual application of the sum rule is actually not to legit analytic response functions coming from time-ordered correlation

⁹⁹And this constant is usually not zero! One might think it needed to be zero since the sum of the residues of all the poles for any function on \mathbb{C} vanishes, and after all, all of the poles of $\chi(\omega)$ are contained within the contour enclosing the LHP. However, the function $\chi(\omega)e^{-i\omega\eta}$ is singular when $\eta \rightarrow i\infty$, and so there is indeed singular stuff going on in the UHP (but only when we multiply by $e^{-i\omega\eta}!$) that renders the integral non-zero. Note that we indeed need the $e^{-i\omega\eta}$ to use the contour integral trick, since $\chi(\omega)$ generically won't go to something that satisfies $|\omega\chi(\omega)| \rightarrow 0$ as $|\omega| \rightarrow \infty$. If $\chi(\omega)$ did fall off this fast, then we could compute the integral without the $e^{-i\omega\eta}$ factor and get zero, implying that C would be zero.

functions and stuff, but to things like the conductivity, which are basically i times a legit response function. Indeed, the conductivity satisfies

$$\sigma_T(\omega + i\eta) = -\sigma_T(\omega - i\eta)^*, \quad (1119)$$

essentially because the actual microscopic response function in the theory relates the current and vector potential, and to get to the conductivity one needs to differentiate the vector potential wrt time, which in frequency space brings down an extra factor of i (this is why it is the *real* part of the conductivity that is related to dissipation!). If we apply the sum rule to a function like the conductivity which satisfies (1119), then we instead have

$$\int_{\mathbb{R}} d\omega \sigma(\omega) e^{-i\omega\eta} = \int_{\mathbb{R}} d\omega [\sigma_T(\omega + i\eta) - \sigma_T(\omega - i\eta)] = 2 \int_{\mathbb{R}} d\omega \sigma_{\mathbb{R}}(\omega) = C, \quad (1120)$$

where C will generically be nonzero, since $\sigma_{\mathbb{R}}$ is symmetric in ω (again, the notation is that $\sigma(\omega)$ is a causal response function, and σ_T is time-ordered). This is the usual sum rule.

Self-energies

Now we discuss the meaning of the different parts of the self-energy. As we can see from the above form of the spectral function (1112), Σ_R has the effect of shifting the dispersion for the poles away from the free value of $\varepsilon_{\mathbf{k}}$ (below the threshold the locations of the poles are shifted; above the threshold the center of the Lorentzian is shifted, at least to the extent that Σ_I can be treated as frequency-independent). Therefore we are prompted to define the renormalized energy as the solution to the equation

$$\varepsilon_{\mathbf{k}}^* = \varepsilon_{\mathbf{k}} + \Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k}). \quad (1121)$$

This shift is what will affect the renormalization of things like masses and Fermi velocities.

On the other hand, Σ_I loosely speaking determines the decay rate of the qp in question. One way to argue this is to consider a scenario in which the free theory has a single qp pole, which will be broadened and will have its location shifted slightly by interactions. If we assume the interactions leave the renormalized qp pole still roughly intact and distinguishable from the background incoherent continuum, and if we assume that Σ_I is roughly constant over the frequency range we're interested in, we may expand

$$\Sigma(\omega, \mathbf{k}) \approx i\Sigma_I(\varepsilon_{\mathbf{k}}^*, \mathbf{k}) + \Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k}) + (\omega - \varepsilon_{\mathbf{k}}^*)(\partial_{\omega}\Sigma_R)(\varepsilon_{\mathbf{k}}^*, \mathbf{k}). \quad (1122)$$

Inserting this expansion into the expression for χ in terms of the shifted time-ordered correlator, we obtain

$$\chi(\omega, \mathbf{k}) = \frac{Z_{\mathbf{k}}}{\omega - \varepsilon_{\mathbf{k}}^* - i\Gamma_{\mathbf{k}}} + \dots, \quad (1123)$$

where

$$\Gamma_{\mathbf{k}} \equiv \Sigma_I(\varepsilon_{\mathbf{k}}^* + i\eta, \mathbf{k}) Z_{\mathbf{k}}, \quad Z_{\mathbf{k}} \equiv \frac{1}{1 - (\partial_{\omega}\Sigma_R)(\varepsilon_{\mathbf{k}}^* + i\eta, \mathbf{k})}, \quad (1124)$$

and where the ... represent contributions from the incoherent background, which are of course needed to give a properly normalized spectral density.¹⁰⁰ When this is Fourier transformed, we get an exponential damping on a timescale set by $\Gamma_{\mathbf{k}}$: thus $\Gamma_{\mathbf{k}}$ is a decay rate, and so Σ_I indeed controls the decay rate of the particle in question. In particular, it will vanish below the threshold for particle production, since then there is nothing for the particle to decay into.

One brief comment on Fermi liquids: since $\Sigma_I(\omega, \mathbf{k})$ is determined by the type of interactions in the theory, one might naively expect that it would be roughly constant and nonzero near the FS (since in many metals the bare interaction strength is very large). Now in order for a qp of energy $\varepsilon_{\mathbf{k}}$ to be well-defined, we need

$$1/\varepsilon_{\mathbf{k}} \ll \frac{1}{Z_{\mathbf{k}} \Sigma_I(\omega = \varepsilon_{\mathbf{k}}, \mathbf{k})} \quad (1125)$$

since we need to be able to measure the energy of the qp before it decays. Now if $\Sigma_I(\omega, \mathbf{k})$ is roughly frequency-independent near the FS, then when we take $\varepsilon_{\mathbf{k}} \rightarrow 0$ at the FS, the condition on the well-definedness of the qp will always be violated. This line of thinking would lead us to expect that well-defined qps only exist for large energies, when in fact it's the opposite! The fact that this naive picture is so wrong is the power of FLT and the Pauli exclusion principle.¹⁰¹

A more QFT-centric way to discuss this (at zero temperature) is via the optical theorem, and is a simple consequence of the unitarity of the S -matrix.¹⁰² Writing $S = 1 + iT$ where T represents nontrivial scattering, $S^\dagger S = 1$ means $i(T^\dagger - T) = T^\dagger T$. In the case of the self energy, we are interested in one-particle to one-particle scattering, and so sandwiching this equation with $|\mathbf{k}, \omega\rangle$, we can insert a resolution of $\mathbf{1}$ in the $T^\dagger T$ term to get, using rather schematic but hopefully clear notation,

$$\Sigma(\mathbf{k}, \omega) - \Sigma^*(\mathbf{k}, \omega) = i2\Sigma_I(\mathbf{k}, \omega) \approx -i \sum_a |T_{(\mathbf{k}, \omega) \rightarrow |a\rangle}|^2 \quad (1126)$$

where a runs over all possible final state particles that the incoming particle can decay into while conserving momentum and energy, except the incoming state itself (the trivial free propagation is not part of T ; it is the 1 in $1 + iT$). We have written \approx here because the self-energy is determined only by the 1PI diagrams, whereas the T matrix elements involve

¹⁰⁰We are of course assuming here that $Z_{\mathbf{k}} \neq 0$, so that we are in the domain of FLT. We could run into problems if e.g. Σ_R failed to be analytic, with e.g. $\Sigma_R(\omega) \propto \ln(\omega/\omega_0)$ as in a marginal FL.

¹⁰¹It also means that $\Sigma_I(\omega, \mathbf{k})$ can *not* be roughly independent of ω with \mathbf{k} near the FS and ω small (in fact, we know that it goes like ω^2), and so the assumption of constant Σ_I used to talk about $\Gamma_{\mathbf{k}}$ as a decay rate after Fourier transforming the correlator really isn't all that reasonable.

¹⁰²One caveat before we continue is that since we are in the CMT diary, we are using CMT conventions, and writing the propagator in terms of frequency and Hamiltonian density and so on. In QFT, we would usually write the propagator as e.g. $1/(p^2 - \Sigma(p^2))$. The annoying thing here is that in QFT Σ has the dimensions of p_0^2 , while in most of CMT it has the dimensions of ω . Therefore the discussion that follows should not be taken literally—it is just meant to be suggestive. Were we to do things in the QFT way we'd need to add some extra factors of masses to get dimensions right and remember how the single-particle states are normalized with factors of $1/\sqrt{E}$, and so on. It didn't seem to be worth the trouble to sort all of this out, so please do not take the following formulae too seriously.

all connected diagrams, 1PI or not—in a weakly coupled theory the dominant terms in an expansion in terms of loops or coupling constants all come from 1PI diagrams, but eh, this approximation is still taken a bit crudely.¹⁰³ Anyway, pressing ahead, we see that since squaring $T_{a \rightarrow b}$ gives the amplitude for the given processes $a \rightarrow b$, we see that indeed, $-2\Sigma_I$ determines the decay rate of the incoming particle. We also check that if the frequency ω is below the threshold for producing particles / bound states, there are no nonzero terms in the sum, and so $\Sigma_I(\mathbf{k}, \omega) = 0$ below the threshold frequency (the final states $|a\rangle$ appearing in (1126) cannot be single-particle states below the threshold, since we need to conserve momentum and energy in (1126)).

At the level of perturbation theory, (1126) can be understood by using the cutting rules: everything in the diagrammatics of Feynman rules is real except for the $i\eta$ convergence factors, and they only produce finite imaginary parts when their parent propagators go on shell. In fact, the optical theorem tells us that a given diagram will only produce an imaginary part when *all* particles along a given cut go on-shell—in this scenario these particles define a legit final state for a scattering process, and it is these diagrams which produce the RHS of (1126) above.

Effective masses and the quasiparticle residue in FLT

Finally, we will take a look at the effective mass, and its relation to the quasiparticle residue, within the context of FLT.

There are many different ways to define the effective mass, depending on what situations one is interested in. For example, in semiconductors, where one is interested in the behavior of the band dispersion right near a band minimum / maximum, one characterizes the shape of the band with the effective mass $m_{sc} = (\partial^2 \varepsilon / \partial^2 \mathbf{k})^{-1}$. In our case, since we are mostly interested in FLT (i.e. in metals), we will use a different definition, viz.

$$m \equiv k_F/v_F, \quad v_F \equiv \frac{\partial \varepsilon}{\partial \mathbf{k}}. \quad (1127)$$

Although we have written v_F in notation that suggests it is a scalar, in reality it carries dependence on momentum, since it can vary over different regions of the FS.

The notion of an effective mass defined in this way will apply in scenarios where the spectral function can be well-approximated by a sharp qp peak standing out from a broader incoherent background. We will fix a momentum \mathbf{k} at which a qp state lives, and will take the qp pole to be located at the renormalized frequency $\varepsilon_{\mathbf{k}}^*$, where $\varepsilon_{\mathbf{k}}^*$ is $\varepsilon_{\mathbf{k}}$ renormalized by the real part of the self-energy: as before, this is

$$\varepsilon_{\mathbf{k}}^* = \varepsilon_{\mathbf{k}} + \Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k}). \quad (1128)$$

Therefore the Fermi velocity is

$$v_F = \partial_{\mathbf{k}}(\varepsilon_{\mathbf{k}}^* - \Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k})) = v_F^* - v_F^*(\partial_{\omega}\Sigma_R)(\varepsilon_{\mathbf{k}}^*, \mathbf{k}) - \partial_{\mathbf{k}}\Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k}) = Z_{\mathbf{k}}^{-1}v_F^* - \partial_{\mathbf{k}}\Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k}). \quad (1129)$$

¹⁰³Another point to clarify: the self-energy appears here and not the full propagator (or the 1PI part thereof), since the T matrix elements are computed using *amputated* diagrams, with the external propagators removed (also, since this formula doesn't include the trivial non-interacting process, it can't be related to the full propagator).

This means that

$$v_F^*/v_F = Z_{\mathbf{k}}(1 + v_F^{-1}\partial_{\mathbf{k}}\Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k})) = \frac{1 + v_F^{-1}\partial_{\mathbf{k}}\Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k})}{1 - (\partial_{\omega}\Sigma_R)(\varepsilon_{\mathbf{k}}^*, \mathbf{k})}. \quad (1130)$$

In typical situations where the Fermi momentum is not renormalized¹⁰⁴ the effective masses are consequently related as

$$m^*/m = \frac{1 - (\partial_{\omega}\Sigma_R)(\varepsilon_{\mathbf{k}}^*, \mathbf{k})}{1 + v_F^{-1}\partial_{\mathbf{k}}\Sigma_R(\varepsilon_{\mathbf{k}}^*, \mathbf{k})}. \quad (1131)$$

In situations where the momentum dependence of Σ_R is weak near $\varepsilon_{\mathbf{k}}^*$, we may drop the \mathbf{k} derivative of the self-energy, and in these cases the qp residue determines the renormalized Fermi velocity and mass (recall that v_F and m are really generically \mathbf{k} -dependent; for notation's sake this dependence is remaining implicit)

$$v_F^*/v_F = m/m^* \approx Z_{\mathbf{k}}. \quad (1132)$$

What is the physical meaning of this¹⁰⁵ definition of m ? For simplicity, consider one dimension. In the “filling up momentum levels” Fermi-gas picture of the FL, states near the Fermi level come separated in fixed momentum intervals of $\Delta k = 2\pi/L$. If we linearize $\varepsilon_{\mathbf{k}}$ near k_F , then an energy window of width $\Delta\varepsilon$ contains

$$N_{\Delta\varepsilon} = v_F^{-1} \frac{\Delta\varepsilon}{2\pi/L} = \frac{m\Delta\varepsilon}{2\pi/L} \quad (1135)$$

states. Therefore the bigger the effective mass m , the more states that are contained in a fixed energy window. This is the meaning of this definition of m : larger m has the effect of compressing the energy levels in \mathbf{k} space. Note that since $Z_{\mathbf{k}}$ is always between 0 and 1 in the approximation we’re working in (since it determines the weight of the coherent pole in the spectral weight), the effect of interactions always compresses the energy levels, rather making them more dispersed. This means that this definition of m is the one relevant for thinking about the specific heat: a greater number of states in a fixed energy window means a larger specific heat, and since $N_{\Delta\varepsilon}$ depends linearly on m , we expect $C \propto m$, with interactions having the effect of increasing the specific heat.

One way in which a Fermi liquid can die is if $Z_{\mathbf{k}} \rightarrow 0$ as in the LL, with qps disappearing into a pole-free spectral function. If the approximations stated above about the momentum dependence of Σ_R continue to hold, then a Fermi liquid which dies in this way implies that the Fermi sea “spills over” at the transition with $v_F^* \rightarrow 0$ and $m^* \rightarrow \infty$: a given energy window in \mathbf{k} space now contains a diverging number of states.

¹⁰⁴By Luttinger’s theorem if the shape of the FS is unrenormalized then \mathbf{k}_F is unchanged—nothing I’m aware of prevents the shape of the FS from renormalizing a priori, though.

¹⁰⁵If we use the semiconductor definition, then we instead get

$$\partial_{\mathbf{k}}^2\varepsilon_{\mathbf{k}} = \partial_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^* - \partial_{\mathbf{k}}^2\Sigma_R - 2\partial_{\mathbf{k}}\varepsilon_{\mathbf{k}}^*\partial_{\omega}\partial_{\mathbf{k}}\Sigma_R - \partial_{\mathbf{k}}^2\varepsilon_{\mathbf{k}}^*\partial_{\omega}\Sigma_R - (\partial_{\mathbf{k}}\varepsilon_{\mathbf{k}}^*)^2\partial_{\omega}^2\Sigma_R. \quad (1133)$$

If we drop the momentum derivatives of Σ_R and the second-order frequency derivative, this becomes

$$m^{-1} = (m^*)^{-1}/Z_{\mathbf{k}}, \quad (1134)$$

which in this approximation agrees with the definition used in the main text.

Connection with QFT language

Now we will briefly comment on the relation between $Z_{\mathbf{k}}$ and the factor appearing when doing wavefunction renormalization in the QFT sense. First, consider an approximation in which we neglect the finite width of the qp peak in the spectral function. The retarded Greens function in this approximation is

$$\chi(\omega, \mathbf{k}) \approx \frac{1}{\omega - \varepsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k}) + i\eta}, \quad (1136)$$

where we have dropped the $+i\eta$ in the argument of Σ_R since it's continuous across the \mathbb{R} axis. The imaginary part gives us the spectral weight, and so near the qp pole

$$A(\omega, \mathbf{k}) \approx \delta(\omega - \varepsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k})) = Z_{\mathbf{k}}\delta(\omega - \varepsilon_{\mathbf{k}}^*), \quad (1137)$$

where in the last step we used the usual $\delta(f(\omega)) = |(\partial_{\omega}f)(\varepsilon_{\mathbf{k}}^*)|^{-1}\delta(\omega - \varepsilon_{\mathbf{k}}^*)$, since $\varepsilon_{\mathbf{k}}^*$ is the value of ω that solves the constraint from the first δ function. Therefore from the spectral representation, we conclude that in this approximation,

$$Z_{\mathbf{k}} \approx |\langle \varepsilon_{\mathbf{k}}^* | c_{\mathbf{k}}^\dagger | 0 \rangle|^2, \quad (1138)$$

where $|\varepsilon_{\mathbf{k}}^*\rangle$ is the qp state we're interested in (which is a genuine eigenstate of H since we're pretending that Σ_I vanishes). Thus $Z_{\mathbf{k}}$ is the amount of \mathbf{k} electron contained in the renormalized qp excitation—exactly what we expect from the physical interpretation of the wavefunction renormalization factor.

To expand on this, let us think about how wavefunction renormalization should work in non-relativistic theories. The usual relation between bare and renormalized fields is $\psi_b = Z^{1/2}\psi_r$. In the old-fashioned point of view, the $Z^{1/2}$ term is present to cancel k -dependent divergences; in the RG it is there to ensure that the fields maintain the same kinetic term during the flow. Indeed, when we do RG the self-energy diagrams will generate terms that modify the $\bar{\psi}\omega\psi$ part of the action to

$$\bar{\psi}\omega\psi \mapsto \bar{\psi}(\omega - \Sigma(0, \mathbf{k}) - \omega(\partial_{\omega}\Sigma)(0, \mathbf{k}) - \dots)\psi. \quad (1139)$$

When we do the field renormalization during the RG step, we want to renormalize the fields such that the frequency part has unit coefficient, hence we define

$$\psi' = (Z_{\mathbf{k}})^{-1/2}\psi, \quad Z_{\mathbf{k}} = (1 - (\partial_{\omega}\Sigma)(0, \mathbf{k}))^{-1}. \quad (1140)$$

This is essentially the same thing we do when performing wavefunction renormalization, with the bare and renormalized fields related by $\psi_0 = \sqrt{Z}\psi_R$.



T-dependence of carrier concentration in semiconductors

Today we're doing a slight elaboration on problem 28.6 from A&M. We'll be computing the carrier concentration in a semiconductor as a function of T , and breaking up the behavior into various different regimes.



First, some notation. N_a (N_d) will denote the total number of acceptor (donor) impurities added to the sample. Donor impurities are basically weakly bound hydrogen atoms added to the sample, and acceptors are basically weakly bound antihydrogen atoms. We will write the number of neutral donors (acceptors) as N_d^0 (N_a^0). Donors can be singly ionized but cannot accept additional electrons due to an assumed Coulomb repulsion; we write the number of such ionized donors as $N_d^+ = N_d - N_d^0$. Conversely, acceptors can have their holes singly ionized but cannot accept additional holes (for the same reason); we write the number of ionized acceptors as $N_a^- = N_a - N_a^0$.

The carrier concentrations in the undoped case are straightforward to get. The conduction electrons have number density $n = \int_{\varepsilon_c}^{\infty} d\varepsilon g(\varepsilon)f(\varepsilon)$, with f the Fermi function, ε_c the conduction band edge, and g the density of states for a free Fermi gas with effective mass determined by the curvature of the conduction band. We usually make the approximation that $\varepsilon_c - \mu \gg T$ (justified since semiconductor band gaps don't get much smaller than 0.1eV or so, which is already greater than the ~ 0.02 eV of room temperature), which lets us approximate Fermi by Maxwell-Boltzmann and do the integral. This gives

$$n = 2 \left(\frac{m_c T}{2\pi} \right)^{3/2} e^{-\beta \xi_c} = N_c(T) e^{-\beta \xi_c}, \quad (1141)$$

with $\xi_c \equiv \varepsilon_c - \mu$ and m_c the conduction band effective mass. In the same way, the number of holes in the valence band is¹⁰⁶

$$h = 2 \left(\frac{m_v T}{2\pi} \right)^{3/2} e^{-\beta \xi_v} = H_v(T) e^{-\beta \xi_v}, \quad (1142)$$

with m_v the valence band effective mass and $\xi_v = \mu - \varepsilon_v$. In the undoped case we must have $n = h$ since electrons can't just disappear; hence we can divide the last two equations and get

$$1 = (m_c/m_v)^{3/2} e^{-\beta(\varepsilon_c + \varepsilon_v - 2\mu)} \implies \mu = \frac{\varepsilon_c + \varepsilon_v}{2} + \frac{3}{4} T \ln(m_v/m_c). \quad (1143)$$

Thus at $T = 0$ the chemical potential is right in the middle of the gap. We see that μ at finite T is closer to the band with the smaller effective mass.

Now we consider the doped case. At very high temperatures the majority of carriers will come from ones that are thermally activated to cross the band, since even if

$$\Delta_G \equiv \varepsilon_c - \varepsilon_v \quad (1144)$$

¹⁰⁶Most people use the letter p_v (for positive) but I don't like this because particle also starts with p .

is large compared to T (e.g. $\Delta_G \sim 1\text{eV} \sim 10^4 K$ and $T \sim 10^3 K$), the concentration of impurities is basically always small enough compared to the density of normal atoms that the number of carriers that are excited across the gap is greater than the number of carriers excited from / into the impurity bands. Therefore the number of carriers is basically the same as it is in the intrinsic case. Since in this case we have $n = h$, we have

$$n \approx \sqrt{H_v N_c} e^{-\beta \Delta_G}, \quad (1145)$$

which we obtained by multiplying the expressions for n and h so as to eliminate μ . This is typically the regime present for $T \gtrsim 500\text{K}$.

Now let us assume an n-doped semiconductor and let us reduce the temperature to the point where the majority of carriers come from excited donor atoms.

Our starting point will be the expression for charge conservation:

$$n + N_a^- = h + N_d^+, \quad (1146)$$

where each term represents a contribution to the charge resulting from a finite-energy excitation (conduction electrons, donor holes, etc.).

Since in the n-doped case the dominant charge carriers will be the conduction electrons, we will want to use the above equation to solve for n . We want to get an expression for n in terms of data intrinsic to the system, viz. N_d , N_a , T , and m_i . Now in the n-doped case all the action will be going on in the donor levels and conduction band—the only effect of the acceptor levels will be to eat up electrons from the donor levels, and the valence band will be completely passive. Since the acceptor levels will essentially always be completely filled (not by losing their bound holes to the valence band, but by absorbing electrons from the donor levels), we can approximate $N_a^- \approx N_a$. We can also neglect h , and so we have

$$n \approx N_d^+ - N_a, \quad (1147)$$

and hence all we have to do is find N_d^+ .

This is done as follows. The ratio N_d^0/N_d is the probability that any given donor level is occupied by an electron. Since there are $\uparrow + \downarrow = 2$ ways to put the electron in,

$$N_d^0/N_d = \frac{2e^{-\beta \xi_d}}{1 + 2e^{-\beta \xi_d}} = \frac{1}{1 + \frac{1}{2}e^{\beta \xi_d}}. \quad (1148)$$

Since this is also equal to $1 - N_d^+/N_d$, we see that

$$N_d^+/N_d = \frac{1}{1 + 2e^{-\beta \xi_d}}. \quad (1149)$$

The problem with the RHS is that it involves μ . We can get rid of the $e^{\beta \mu}$ by writing it in terms of

$$n e^{\beta \varepsilon_c} / N_c = e^{\beta \mu}. \quad (1150)$$

Therefore

$$N_d^+ = \frac{N_d}{1 + 2n N_c^{-1} e^{-\beta \Delta}}, \quad \Delta \equiv \varepsilon_c - \varepsilon_d. \quad (1151)$$

Putting this into our charge conservation equation, we then get an equation that's quadratic in n and only contains the input parameters N_a, T, m_c, N_d :

$$n^2 \left(\frac{2}{N_c} e^{\beta\Delta} \right) + n \left(1 + \frac{2N_a}{N_c} e^{\beta\Delta} \right) = N_d - N_a, \quad (1152)$$

which can be solved non-enlighteningly.

After the high- T regime, the next regime we expect is a plateau where the carrier concentration doesn't change appreciably—in this regime T is small enough so that there aren't many carriers coming from jumping the band gap, but T hasn't yet become small enough to change the fact that basically all the donor electrons are ionized into the conduction band. In this regime we can typically ignore the n^2/N_c and $N_a/N_c \ll 1$ terms, and we just get a constant $n = N_d - N_a$, as expected. This plateau extends usually from $150K \lesssim T \lesssim 500K$, ish.

We next come to a regime where a freezeout of the conduction electrons starts to occur. We assume that T is now low enough that a significant fraction of the donor electrons are not ionized up to the conduction band, but that T is still high enough, and that the semiconductor is n-doped enough, so that $N_a \ll n$. In this regime we will have T getting smaller than Δ , so that the exponentials in (1152) start to get large. We assume that $N_a \ll n, n^2 e^{\beta\Delta} \gg n$ sufficiently strongly so that we can drop the linear term in (1152), giving

$$n \approx \sqrt{\frac{(N_d - N_a)N_c}{2}} e^{-\beta\Delta/2}, \quad (1153)$$

so that the carrier concentration now gets cutoff exponentially fast in $\beta\Delta/2$. This regime occurs usually from $\sim 150K$ to maybe $\sim 30K$, ish.

There is a final low- T regime below $T \sim 30K$, where the freeze-out is sharper. In this regime the carrier concentration drops even below N_a . In this case $n^2 \ll nN_a$ and $N_a e^{\beta\Delta}/N_c \gg 1$, and so only the third term on the LHS of (1152) survives. Therefore in this regime (note: I think there is a typo in A&M)

$$n \approx \frac{N_c(N_d - N_a)}{2N_a} e^{-\beta\Delta}, \quad (1154)$$

which gives a faster exponential suppression. This regime continues up to the point where the semiconductor becomes insulating.

This all of course generalizes straightforwardly to a p-type semiconductor. In the p-doped case the charge conservation equation is instead approximated with

$$h + N_d \approx N_a^-, \quad (1155)$$

(since basically all the donor levels have lost their electrons to the acceptor levels and since $n \ll h$) and the problem is now to write N_a^- in terms of stuff like H_v, N_a , and so on. The intermediate step to the analogous quadratic equation for h is

$$N_a^-/N_a = \frac{e^{-\beta\xi_a}}{2 + e^{-\beta\xi_a}} \implies N_a^- = \frac{N_a}{1 + 2e^{\beta\xi_a}}. \quad (1156)$$

Estimate for the size of the depletion region in a p-n junction

This comes from the fact that there is only one way for the acceptor to be “ionized” (i.e. to loose its hole; the associated anti-Hydrogen atom is then empty and in a unique state), but there are two ways for it to be filled, since the hole can be bound with either spin. This leads to the quadratic equation

$$p^2 \left(\frac{2}{H_v} e^{\beta \Delta'} \right) + n \left(1 + \frac{2N_d}{H_v} e^{\beta \Delta'} \right) = N_a - N_d, \quad \Delta' = \varepsilon_a - \varepsilon_v, \quad (1157)$$

which then has all the same T -dependent regimes as the analogous equation for n .



Estimate for the size of the depletion region in a p-n junction

Today we’re doing a rather undergrady problem that’s still fun: estimating the size of the depletion region in a p-n junction using simple electrostatics.



First, recall the setup of a p-n junction. Imagine putting an n-doped SC (the S is for semi, not super) next to a p-doped one. At the boundary, the conduction electrons on the n-doped side will be attracted to the holes on the p-doped side, and will annihilate (as the electrons move farther into the *p*-doped region their energies increase, and they start looking for holes in the valence band to annihilate with, and vice-versa for the holes). The annihilating electron / hole pairs mean that the ionized donors / acceptors on the respective sides of the boundary no longer have their charges screened by charge carriers in their immediate vicinities. This results in a net positive charge on the n-doped side of the boundary (from the ionized donors) and a net negative charge on the p-doped side (from the hole-ionized acceptors), giving an electric field pointing from the n-doped side to the p-doped one. The charged regions constitute the depletion region, so named since all the mobile carriers have pair-annihilated in this region.

There will thus be some equilibrium configuration where the electric potential created by this charge exactly compensates for the difference in the band energies of each band across the junction, giving a constant μ (remember that μ is always constant in equilibrium!).

To find the width of the depletion region, we make the crude approximation where the depletion region is modeled as a plane of thickness w_a with charge density $-N_a$ coming from the acceptors, and a plane of thickness w_d of charge density N_d , from the donors. We then want to solve

$$-\nabla^2 \phi(x) = \rho(x)/\varepsilon \quad (1158)$$

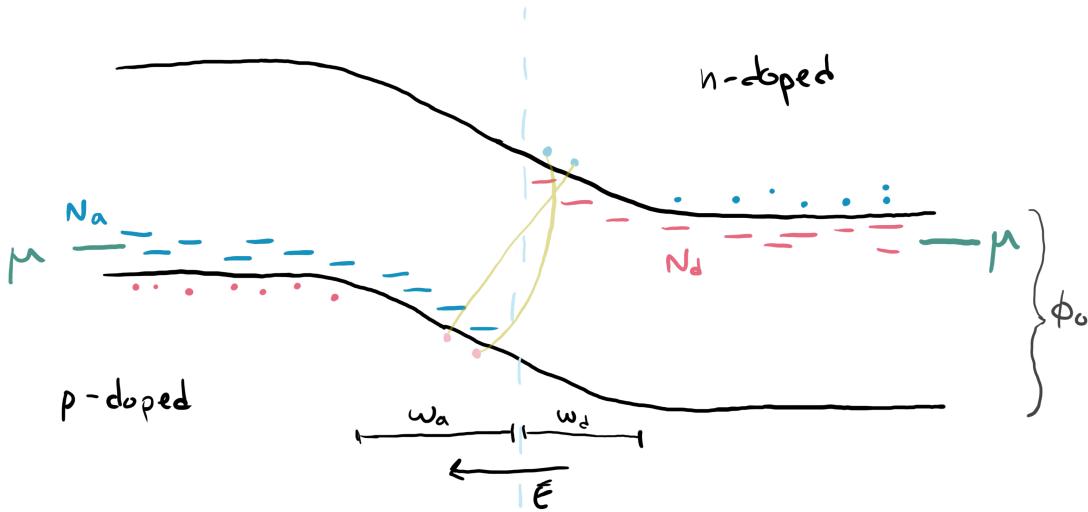


Figure 18: The acceptor levels are marked in blue, since they are always “hole-ionized” in the regime we’re interested and hence negatively charged. In the depletion region the distance between μ and the band edge grows, and the holes in the valence band annihilate with electrons in the conduction band (yellow arrows). This leaves behind unscreened ionic charges, which establish the electric field as shown.

for the region $0 < x < w_a + w_d$, with ε the renormalized dielectric “constant” in the SC (the bandgap isn’t changing across the junction, so taking a constant ε seems reasonable). In keeping with our crude model, the charge distribution is assumed to be

$$\rho(x) = \begin{cases} -N_a & 0 < x < w_a \\ +N_d & w_a < x < w_a + w_d \end{cases} \quad (1159)$$

where $x = 0$ is the left edge of the depletion region.

We then integrate this twice to get the potential, choosing boundary terms for continuity: this gives, for the induced potential set to be $\phi(0) = 0$,

$$\varepsilon\phi(x) = \begin{cases} \frac{N_a x^2}{2} & 0 < x < w_a \\ w_a(N_d + N_a)x - \frac{N_d x^2}{2} - \frac{w_a^2}{2}(N_d + N_a) & w_a < x < w_a + w_d \end{cases} \quad (1160)$$

Now the chemical potential μ is constant in equilibrium. We know that on the p-doped side μ is just above the valence band edge, while on the n-doped side it is just below the conduction band edge. This difference in energy $\phi_0 \equiv \varepsilon_c - \varepsilon_v$ must be compensated for by the induced potential $\phi(x)$ in order for μ to be constant. Therefore we must have that $\phi(w_a + w_d) = \phi_0$. We still need one more condition to solve for the two unknowns w_i ; this comes from overall charge neutrality, which tells us that

$$w_a N_a = w_d N_d, \quad w_d = \frac{N_a}{N_d} w_a \equiv r w_a. \quad (1161)$$

The condition that $\phi(w_a + w_d) = \phi_0$ is

$$w_a^2 \left(\frac{N_d}{2}(1+r)^2 - \frac{N_d}{2}(1+r) \right) = \varepsilon\phi_0 \implies w_a = \sqrt{\frac{2\varepsilon N_d \phi_0}{N_a(N_d + N_a)}}. \quad (1162)$$

Hence we also have

$$w_d = \sqrt{\frac{2\varepsilon N_a \phi_0}{N_d(N_d + N_a)}}. \quad (1163)$$

The induced charge on each of the “plates” of this “capacitor” is

$$|Q_{ind}| = w_a N_a = w_d N_d = w_a = \sqrt{\frac{2\varepsilon N_d N_a \phi_0}{N_d + N_a}}, \quad (1164)$$

which is symmetric in N_a, N_d as required. The capacitance of the junction is

$$C = \sqrt{\frac{2\varepsilon N_d N_a}{N_d + N_a}}. \quad (1165)$$



Boltzmann equation and the relaxion-time approximation

Today we’re doing a couple exercises in Ashcroft and Mermin chapter 16. The name of the game is understanding the Boltzmann equation. We will be setting $\hbar = c = e = k_B = 1$ throughout.



Problem 1: We start with two rather obvious things. First, let H be the expectation value of some conserved single-particle quantity $h_{\mathbf{k}}$. If the 1-particle distribution function is $g_{\mathbf{k}}$ (implicitly a function of \mathbf{x}), then we write

$$H = \int_{\mathbf{k}} h_{\mathbf{k}} g_{\mathbf{k}}. \quad (1166)$$

The change in H due to collisions is

$$d_t H|_c = \int_{\mathbf{k}} h_{\mathbf{k}} \partial_t g_{\mathbf{k}}|_c, \quad (1167)$$

where the $|_c$ stands for the contribution due to collisions. We claim that that $d_t H|_c = 0$ if all the matrix elements $W_{\mathbf{k}\mathbf{k}'}$ for scattering from \mathbf{k} to \mathbf{k}' are nonzero only when $h_{\mathbf{k}} = h'_{\mathbf{k}}$ (duh). Indeed, this follows directly from the usual expression

$$\partial_t g_{\mathbf{k}}|_c = \int_{\mathbf{k}'} (-W_{\mathbf{k}\mathbf{k}'} g_{\mathbf{k}} (1 - g_{\mathbf{k}'}) - (\mathbf{k} \leftrightarrow \mathbf{k}')) \quad (1168)$$

and doing a change of variables in the integral for $d_t H|_c$.

Second, note that the continuity equation follows from the Boltzmann equation. The divergence in the particle current is, using the Boltzmann equation,

$$\nabla \cdot \mathbf{j} = \nabla \cdot \int_{\mathbf{k}} g_{\mathbf{k}} \mathbf{v} = \int_{\mathbf{k}} (-\partial_t g_{\mathbf{k}} - \mathbf{F} \cdot \tilde{\nabla} g_{\mathbf{k}} + d_t g_{\mathbf{k}}|_c), \quad (1169)$$

where \mathbf{F} is the external force and $\tilde{\nabla}$ is a gradient in momentum space. The \mathbf{F} term dies after integrating by parts ($g_{\mathbf{k}}$ will either have no support at infinity, or momentum space will be compact), while the collisional piece also dies by the antisymmetry between \mathbf{k} and \mathbf{k}' in its integral representation written two equations back. Therefore

$$\nabla \cdot \mathbf{j} = -\partial_t \int_{\mathbf{k}} g_{\mathbf{k}} \implies \partial_{\mu} j^{\mu} = 0. \quad (1170)$$

Our next task is to deduce the appropriate continuity equation for the heat flow. We will assume that the collisions happening are energy-conserving, so that the collisional contribution to the change in energy density u is zero:

$$d_t u|_c = \int_{\mathbf{k}} \varepsilon_{\mathbf{k}} d_t g_{\mathbf{k}}|_c = 0. \quad (1171)$$

First we get an expression for dq/dt . From $TdS = dU - \mu dN$, we use the Boltzmann equation to write

$$\dot{q} = d_t \int g_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) = - \int [(\mathbf{v} \cdot \nabla + \mathbf{F} \cdot \tilde{\nabla}) g_{\mathbf{k}} - d_t g_{\mathbf{k}}|_c] (\varepsilon_{\mathbf{k}} - \mu) = - \int [\mathbf{v} \cdot \nabla g_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) - \mathbf{F} \cdot \mathbf{v} g_{\mathbf{k}}]. \quad (1172)$$

In the last step, we have integrated by parts, used the equation of motion for the velocity to write $\tilde{\nabla} \varepsilon_{\mathbf{k}} = \mathbf{v}$, and used that $d_t u|_c = 0$ as well as $d_t \mu|_c = 0$ (since the total particle number is conserved).

On the other hand, consider the divergence of the heat current. This is

$$\nabla \cdot \mathbf{j}_q = \nabla \cdot \int \mathbf{v} g_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) = \int [\mathbf{v} \cdot \nabla g_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) - g_{\mathbf{k}} \nabla \mu \cdot \mathbf{v}]. \quad (1173)$$

We define a vector field \mathcal{E} encapsulating the effective "electric field" acting on the particles by (note that the sign is different from the one in Ashcroft and Mermin; I think they have a typo)

$$\mathcal{E} \equiv \mathbf{F} - \nabla \mu. \quad (1174)$$

Then we can cancel the $\nabla\mu$ piece of \mathbf{j}_q by a term like $-\mathcal{E} \cdot \mathbf{j}$. Indeed, we see that we in fact get a continuity equation for q , modified by a term containing the external forces (we think of $\nabla\mu$ as an external force field):

$$\dot{q} + \nabla \cdot \mathbf{j}_q = \mathcal{E} \cdot \mathbf{j}. \quad (1175)$$

Problem 4: We will go beyond the relaxation-time approximation, but will assume that the collisions are elastic, with the scattering matrix element $W_{\mathbf{k}\mathbf{k}'} \propto \delta(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'}) |\langle \mathbf{k}|U|\mathbf{k}' \rangle|^2$, with U some scattering potential. With this assumption $W_{\mathbf{k}\mathbf{k}'} = W_{\mathbf{k}'\mathbf{k}}$, and so the collisional part of the BE contains only linear terms:

$$d_t g_{\mathbf{k}}|_c = \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} (g_{\mathbf{k}'} - g_{\mathbf{k}}). \quad (1176)$$

We write the distribution function as $g = f + \boldsymbol{\delta} \cdot \mathbf{E}$, and solve the Boltzmann equation to linear order in \mathbf{E} with f the zero-field equilibrium distribution, since we want the conductivity. The term linear in \mathbf{E} gives

$$\mathbf{E} \cdot \mathbf{v} \partial_{\varepsilon} f_{\mathbf{k}} = \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} (\boldsymbol{\delta}_{\mathbf{k}'} - \boldsymbol{\delta}_{\mathbf{k}}) \cdot \mathbf{E}. \quad (1177)$$

Let \mathbf{u} be the solution to the integral equation

$$\mathbf{v}_{\mathbf{k}} = \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} (\mathbf{u}_{\mathbf{k}'} - \mathbf{u}_{\mathbf{k}}). \quad (1178)$$

Then we can solve for $\boldsymbol{\delta}$ as

$$\boldsymbol{\delta}_{\mathbf{k}} = \mathbf{u}_{\mathbf{k}} \partial_{\varepsilon} f_{\mathbf{k}}. \quad (1179)$$

The reason why this works is that $W_{\mathbf{k}\mathbf{k}'}$, being energy conserving, is only nonzero when $\partial_{\varepsilon} f_{\mathbf{k}} = \partial_{\varepsilon} f_{\mathbf{k}'}$, so that we can pull out a $\partial_{\varepsilon} f_{\mathbf{k}}$ from both terms in the integral. The conductivity tensor is then calculated by

$$\mathbf{j} = \int_{\mathbf{k}} (f_{\mathbf{k}} + \boldsymbol{\delta} \cdot \mathbf{E}) \mathbf{v} \implies \sigma = \int_{\mathbf{k}} \partial_{\varepsilon} f_{\mathbf{k}} \mathbf{u}_{\mathbf{k}} \otimes \mathbf{v}_{\mathbf{k}} \quad (1180)$$

where the matrix nature of σ is implicit.

Now we define the metric

$$\langle \alpha, \gamma \rangle = \int_{\mathbf{k}} (-\partial_{\varepsilon} f) \alpha_{\mathbf{k}} \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} (\gamma_{\mathbf{k}} - \gamma_{\mathbf{k}'}). \quad (1181)$$

Again because of the symmetry of $W_{\mathbf{k}\mathbf{k}'}$, by changing \mathbf{k} to \mathbf{k}' in one of the integrals, we see that $\langle \alpha, \gamma \rangle = \langle \gamma, \alpha \rangle$. Replacing $\mathbf{v}_{\mathbf{k}}$ in the formula for the conductance with its integral representation in terms of $\mathbf{u}_{\mathbf{k}}$, we see that the conductivity is

$$\sigma = \langle \mathbf{u}, \otimes \mathbf{u} \rangle. \quad (1182)$$

Let's prove that this does indeed define a norm. First, positivity: note that $-\partial_{\varepsilon} f_{\mathbf{k}}, W_{\mathbf{k}\mathbf{k}'} \geq 0 \forall \mathbf{k}$. So we want to show that

$$\int_{(x,y) \in D} K(x, y) f(x)^2 \geq \int_{(x,y) \in D} K(x, y) f(x) f(y) \quad (1183)$$

for any domain D on which $K(x, y) \geq 0$ and $K(x, y) = K(y, x)$. Indeed, this follows just from using

$$\int_D (f(x) - f(y))^2 K(x, y) \geq 0, \quad (1184)$$

together with the symmetry of $K(x, y)$.

Now we prove the Δ inequality. We can do this by looking at the positivity of the smallest-length vector formed by a linear combination of two functions α, γ . Minimizing $\|\alpha + \lambda\gamma\|^2$ over λ , we find a minimum at $\lambda_* = -\langle \alpha, \gamma \rangle / \|\gamma\|^2$. Then

$$\langle \alpha + \lambda_* \gamma, \alpha + \lambda_* \gamma \rangle \geq 0 \implies \|\alpha\|^2 \|\gamma\|^2 \geq \|\langle \alpha, \gamma \rangle\|^2, \quad (1185)$$

which is what we wanted to show. Since $\sigma_{ii} = \langle \mathbf{u}_i, \mathbf{u}_i \rangle$ and since $\langle \mathbf{u}_x, \gamma \rangle = \int_{\mathbf{k}} (-\partial_{\varepsilon} f) \gamma_{\mathbf{k}} \mathbf{v}_i(\mathbf{k})$ for any function γ , we conclude that

$$\sigma_{ii} \geq \frac{1}{\|\gamma\|^2} \left[\int_{\mathbf{k}} (-\partial_{\varepsilon} f) \mathbf{v}_i(\mathbf{k}) \gamma_{\mathbf{k}} \right]^2. \quad (1186)$$

Now consider a situation where the scattering comes from two sources $W = W_1 + W_2$, with both W_i s symmetric positive scattering matrix elements coming from separate elastic scattering processes. Let us then plug in the choice \mathbf{u}_i for γ in the above inequality, using the norm coming from just W_1 . This gives

$$\sigma_1^{ii} \geq \frac{1}{\|\mathbf{u}_i\|_{W_1}^2} \left[\int_{\mathbf{k}} (-\partial_{\varepsilon} f) \mathbf{v}_i(\mathbf{k}) \mathbf{u}_i(\mathbf{k}) \right]^2 = \frac{(\sigma_{1+2}^{ii})^2}{\|\mathbf{u}_i\|_W^2 - \|\mathbf{u}_i\|_{W_2}^2} = \frac{(\sigma_{1+2}^{ii})^2}{\sigma_{1+2}^{ii} - \|\mathbf{u}_i\|_{W_2}^2}. \quad (1187)$$

We get a similar formula if we replace $1 \leftrightarrow 2$. We then invert both of these inequalities (everything in sight is positive so this is easily done), and add them together. We get

$$\frac{1}{\sigma_1^{ii}} + \frac{1}{\sigma_2^{ii}} \leq \frac{2\sigma_{1+2}^{ii} - \|\mathbf{u}_i\|_{W_1}^2 - \|\mathbf{u}_i\|_{W_2}^2}{(\sigma_{1+2}^{ii})^2}, \quad (1188)$$

which tells us (this seems to give a stronger bound than the one in Ashcroft and Mermin?!)

$$\frac{1}{\sigma_{1+2}^{ii}} \geq \frac{1}{\sigma_1^{ii}} + \frac{1}{\sigma_2^{ii}}. \quad (1189)$$



Practice with the Kubo formula and electrical conductivity

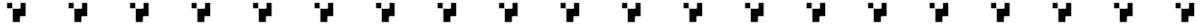
This is from Coleman's many body theory book, chapter 10. It's basically just a sanity check of whether or not we know how to do simple diagramatics to reproduce a Drude-like form for the conductivity in a metal. Our starting point is of course the Kubo formula, viz.

$$\sigma^{ab}(i\nu) = e^2 \frac{T}{\nu} \sum_{\mathbf{q}, \omega} v_{\mathbf{q}}^a v_{\mathbf{q}}^b (G(\mathbf{q}, i\omega + i\nu)G(\mathbf{q}, i\omega) - G(\mathbf{q}, i\omega)^2), \quad (1190)$$

which represents the polarization bubble with the zero-frequency diamagnetic part subtracted off. In this formula, the electron propagators G are

$$G(\mathbf{q}, i\omega) = \frac{1}{i\omega - \epsilon_{\mathbf{q}} - \Sigma(i\omega)}, \quad (1191)$$

for some momentum-independent self energy, the $1/\nu$ comes from $-\partial_t A^i = E^i$, the $v_{\mathbf{q}}^a$'s are (kinematic) velocities, and the second term in parenthesis is the diamagnetic part. We will be finding an expression for the conductivity $\sigma^{ab}(\omega)$ expressed in terms of the analytic properties of the self energy.



Our first task is to switch out the momentum integral in the Kubo formula for an energy integral. Since the Greens functions are rotationally invariant and we are in three dimensions, we can swap out $v^a v^b$ for $v^2/3$. Furthermore, since we are subtracting off the zero frequency part which picks up the finite density of the fermi surface, at frequencies low compared to the Fermi energy we can get away with approximating the density of states as a constant (equal to its value at the FS), and in doing so we can integrate the energy (relative to the FS) from $-\infty$ to $+\infty$. So we can do (not keeping track of spatial volumes)

$$\sum_{\mathbf{q}} v^a v^b(\cdot) \rightarrow \delta^{ab} \int_{\mathbb{R}} d\epsilon N(0) \frac{k_F^2}{3m^2}(\cdot) = \delta^{ab} \int_{\mathbb{R}} d\epsilon \frac{k_F^3}{2\pi^2 m}(\cdot), \quad (1192)$$

since the density of states at the FS is $N(0) = mk_F/(4\pi^2)$. The number density is

$$n = \int d\epsilon N(\epsilon) = \frac{2m}{3 \cdot 2\pi^2} \sqrt{2m\epsilon_F} \epsilon_F = \frac{k_F^3}{3 \cdot 2\pi^2} = \frac{2}{3} N(0) \frac{k_F^2}{m}, \quad (1193)$$

where we integrated up to the FS. Thus the diagonal part of the conductivity is

$$\sigma^{aa}(i\nu) = \frac{ne^2 T}{m\nu} \int_{\mathbb{R}} d\epsilon \sum_{\omega} (G(\epsilon, i\omega + i\nu)G(\epsilon, i\omega) - G(\epsilon, i\omega)^2). \quad (1194)$$

Now we do the Matsubara sum by integrating against the Fermi distribution. Continuing $i\omega$ to z ,

$$\sigma^{aa}(i\nu) = \frac{ne^2}{2\pi im\nu} \int_{\mathbb{R}} d\epsilon \int dz f(z) (G(\epsilon, z + i\nu)G(\epsilon, z) - G(\epsilon, z)^2). \quad (1195)$$

The singularities of the integrand (apart from the ones at the Matsubara frequencies) are branch cuts which run horizontally across the complex plane. For $\alpha \in \mathbb{R}$ and δ infinitesimal, we have branch cuts at $z = \alpha \pm i\delta$ and at $z = \alpha - i\nu \pm i\delta$, which come from the fact that $\Sigma(\alpha + i\delta)^* = \Sigma(\alpha - i\delta)$ (as derived from e.g. $\chi_A^* = \chi_R$), meaning that one gets a discontinuity from $\Sigma_{\mathbb{I}}(z)$ when z crosses the \mathbb{R} axis. Since the sum of the residues of all the poles vanishes, the contour enclosing just the Matsubara frequencies can be computed as the sum of the contours hugging the two branch cuts. The integral hugging the first of these branch cuts gives

$$\begin{aligned} & \frac{1}{\alpha + i\nu - \epsilon - \Sigma(\alpha + i\nu)} \left(\frac{1}{\alpha - i\delta - \epsilon - \Sigma(\alpha - i\delta)} - \frac{1}{\alpha + i\delta - \epsilon - \Sigma(\alpha - i\delta)} \right) \\ & \rightarrow \frac{2\pi i\delta(\epsilon + \Sigma(\alpha - i\delta) - \alpha)}{\alpha + i\nu - \epsilon - \Sigma(\alpha + i\nu)}, \end{aligned} \quad (1196)$$

where we've subtracted in the given order since the counterclockwise path of the contour means that the $\alpha - i\delta$ part is oriented from $-\infty$ to $+\infty$. The branch cut at $z = \alpha - i\nu \pm i\delta$ gives us a similar expression, namely

$$\frac{2\pi i\delta(\epsilon + \Sigma(\alpha + i\delta) - \alpha)}{\alpha - i\nu - \epsilon - \Sigma(\alpha - i\nu)}. \quad (1197)$$

Putting these in and doing the energy integral, we reduce to an integral of the parameter α along the \mathbb{R} line:

$$\sigma^{aa}(i\nu) = \frac{ne^2}{m\nu} \int_{\mathbb{R}} d\alpha f(\alpha) \left(\frac{1}{i\nu - \Sigma(\alpha + i\nu) + \Sigma(\alpha - i\delta)} + \frac{1}{-i\nu + \Sigma(\alpha + i\delta) - \Sigma(\alpha - i\nu)} \right), \quad (1198)$$

where we have dropped $i\delta$'s when they appear in a sum with non-infinitesimal imaginary terms.

Now we analytically continue by rotating the frequency by nearly $\pi/2$ in the \mathbb{C} plane: $i\nu \rightarrow \nu + i\delta$. This gives

$$\sigma^{aa}(\nu + i\delta) = i \frac{ne^2}{m} \int_{\mathbb{R}} d\alpha \frac{f(\alpha)}{\nu} \left(\frac{1}{\nu - \Sigma(\alpha + \nu + i\delta) + \Sigma(\alpha - i\delta)} - \frac{1}{\nu - \Sigma(\alpha + i\delta) + \Sigma(\alpha - \nu - i\delta)} \right). \quad (1199)$$

Now shift α by $-\nu/2$ in the first term and by $+\nu/2$ in the second term:

$$\sigma^{aa}(\nu + i\delta) = i \frac{ne^2}{m} \int_{\mathbb{R}} d\alpha \frac{f(\alpha - \nu/2) - f(\alpha + \nu/2)}{\nu} \frac{1}{\nu - \Sigma(\alpha + \nu/2 + i\delta) + \Sigma(\alpha - \nu/2 - i\delta)}. \quad (1200)$$

Now define the average scattering rate by

$$\bar{\tau}^{-1} = \text{Im} [\Sigma(\alpha - \nu/2 - i\delta) + \Sigma(\alpha + \nu/2 - i\delta)] \quad (1201)$$

and the qp residue by

$$Z^{-1} = 1 - \frac{1}{\nu} \text{Re} [\Sigma(\alpha - \nu/2) - \Sigma(\alpha + \nu/2)]. \quad (1202)$$

Then we have

$$\sigma^{aa}(\nu + i\delta) = \frac{ne^2}{m} \int_{\mathbb{R}} d\alpha \frac{f(\alpha - \nu/2) - f(\alpha + \nu/2)}{\nu} \frac{1}{\bar{\tau}^{-1} - i\nu Z^{-1}}. \quad (1203)$$

This just comes from decomposing the denominator into real and imaginary parts: the sign on one of the Σ 's in τ is flipped since we flipped the sign of the $i\delta$ in its argument, and we've dropped the $i\delta$'s in the self-energy since we're taking the \mathbb{R} part.

Finally, if we can ignore the α dependence of τ and Z , then we can do the frequency integral at low temperatures and get

$$\sigma^{aa}(\nu + i\delta) = \frac{ne^2}{m} \frac{1}{\bar{\tau}^{-1} - i\nu Z^{-1}}, \quad (1204)$$

since the integral of the difference of the Fermi functions just gives ν . This gives us the expected Drude form for the conductivity, and tells us that the frequency part is modified by the real part of the self energy, while the (generically frequency-dependent) "scattering time" $\bar{\tau}$ is determined by the imaginary part (remember that the *real* part of σ is associated with dissipation since σ is basically i times a properly causal response function, so the fact that $\text{Re}[\Sigma]$ contributes to σ_{I} while $\text{Im}[\Sigma]$ contributes to σ_{R} is to be expected). If we set $\nu = 0$ to go to the DC limit then we get the expected

$$\sigma_{DC}^{aa} = \frac{ne^2\tau}{m}, \quad (1205)$$

where $\tau = 2\text{Im}[\Sigma(0 - i\delta)]$ is the effective zero-frequency scattering time.



Hacky way of getting graphene conductivity within Drude approach

Today is a quick one: going through a quick way to get the DC conductivity of graphene.



One way that gets the dependence on the parameters right is the following. First, note that since $\mathbf{j} \cdot \mathbf{E}$ is a power, electrons moving with velocity \mathbf{v} for time τ in a field \mathbf{E} will gain an energy equal to

$$\delta\varepsilon = -\mathbf{v} \cdot \mathbf{E}e\tau. \quad (1206)$$

In graphene, this means a momentum shift of (the momentum shift doesn't change the velocity, so we can do this self-consistently)

$$\delta k = e\tau E \cos\theta. \quad (1207)$$

This will then tilt the circular Fermi surface by an angle $\phi \approx v\delta k$ above the horizontal in $k_x - k_y - \varepsilon$ space, where the FS pivots about the $\star\mathbf{E}$ axis. When projected onto the $k_x - k_y$ plane, this means that the FSea gets displaced by an amount δk . The current is found by counting the net number of displaced states and multiplying by e times their velocities. Neglecting numerical factors that come from integrals over angles, we then have

$$j \sim ev\delta n \sim ev(\delta k)k_F \sim e^2\tau v k_F E \implies \sigma \sim e^2\tau v k_F, \quad (1208)$$

since the net area of the FS that gets displaced is $\sim \delta k k_F$.

An equally hacky way is to use $\sigma = e^2\tau n/m$ for regular metals, and to come up with an appropriate way of defining n/m . The effective mass we'd assign to graphene is k -dependent, e.g.

$$\frac{1}{m_{ii}} = v\partial_{k_i}^2 |\mathbf{k}| = v \frac{(\epsilon_{ij}k_j)^2}{|k|^3}. \quad (1209)$$

Therefore we can estimate n/m by integrating $1/m$ over the occupied part of the Dirac cone:

$$n/m \sim v \int_0^{k_F} dk d\theta \cos^2 \theta \sim v k_F, \quad (1210)$$

where we of course aren't paying attention to numerical constants. This then gives

$$\sigma \sim e^2\tau v k_F, \quad (1211)$$

in agreement with the previous approach.

Suppose we are at half-filling, so that the Fermi energy is at zero. Then the carrier density at $T = 0$ is zero, but becomes nonzero at finite T . What's the conductivity in this case? We can use the above results by assuming that the effect of $T > 0$ is to setup an effective chemical potential of $\mu \sim T$, so that $k_F = T/v$, which gives

$$\sigma \sim e^2\tau T, \quad (1212)$$

which is linear in T . Note that the number of thermally activated carriers goes like $k_F^2 \sim T^2$. So we have the interesting situation where the number of carriers scales like T^2 , and yet the conductivity only scales like T , which is not what would happen if the conductivity just went as $\sigma \sim n$ as in normal Drude theory.



The Ioffe-Larkin rule for resistivity in doped Mott insulators

Today we will be discussing the derivation and physical picture of the Ioffe-Larkin rule, which says that in charge-spin fractionalized MIs, the resistivity is determined by the sum of the resistivities of the spin and charge partons.



The setting for this problem is a Mott insulator which is usefully described by fractionizing the electron operators into charge + spin parts by introducing bosonic and fermionic fields via the usual $c^\dagger = bf^\dagger$ (spin indices will be kept implicit throughout). We will choose the physical $U(1)_{EM}$ field to couple to the boson b with charge -1 so that the bosons keep track of the holons, but assigning the charge to the spinon of course gives physically equivalent results. Working with this representation means adding a dynamical gauge field a , under which we will take b to have charge -1 and f to have charge $+1$. The part of the Hamiltonian involving the coupling to the gauge fields then reads

$$H \supset \sum_{\langle ij \rangle} \left(b_i^\dagger e^{i(A-a)_{ij}} b_j + f_i^\dagger e^{ia_{ij}} f_j \right), \quad (1213)$$

where the signs associated to the orientations of the links are implicitly taken into account in the notation. We then integrate out the bosons and fermions, parametrizing the results with the polarizations Π_b and Π_f , respectively. Since we are interested in the linear response (the conductivity), we only need to keep terms in the effective action to quadratic order in A .

Working at long distances in continuum notation, the quadratic part of the effective action is then

$$S \supset \int d^D x \left((A - a)_\mu \Pi_b^{\mu\nu} (A - a)_\nu + a_\mu \Pi_f^{\mu\nu} a_\nu \right). \quad (1214)$$

We can decompose the two polarizations into transverse and longitudinal projectors as

$$\Pi_\alpha^{\mu\nu} = \Pi_T^{\mu\nu} \pi_\alpha^T + \Pi_L^{\mu\nu} \pi_\alpha^L, \quad (1215)$$

where the $\pi_\alpha^{T/L}$ are functions of frequency and momenta. The (DC) conductivity for each species is defined as $\sigma_\alpha(\omega) = \omega^{-1} \delta_{\mathcal{A},0}^2 Z|_{\mathcal{A}=0,k=0}$, where \mathcal{A} is the gauge field that α couples to and we have assumed rotational invariance and restricted to the diagonal component of the conductivity. Note that this definition requires treating a as a background field and setting it to zero in the calculation, which of course is strictly speaking illegal since a is a dynamical emergent gauge field. We can therefore then just think of σ_α as simply notation if we like, but in terms of expectation values we expect $dA \propto \langle da \rangle$ for weak dA , so in this respect the conductivities σ_α are not entirely unphysical (another way to say this is that we are working under the assumption that MFT for the partons is useful; hence under this assumption we are free to treat a within linear response as well).

In terms of the polarizations, we see that $\sigma_\alpha = \pi_\alpha^T(k=0, \omega)/\omega$ (the $k \rightarrow 0$ limit is taken such that $k_i k_j / k^2 \rightarrow 0$, and since the effective action for the gauge fields won't involve terms

Different types of superexchange interactions

more singular than $k_i k_j / k^2$ the $\pi_\alpha^{T/L}$ can be expanded as a Taylor series in k at small k ; hence $k_i k_j \pi_\alpha^{L/T} / k^2 \rightarrow 0$ when computing σ_α). Note that the physical conductivity then vanishes if either of the polarizations vanish with $\omega \rightarrow 0$ as $\omega^{n>1}$. For Π_b this is obvious, since then there is nothing for A to couple to. But this also happens for Π_f , since in that case we can shift a by A to eliminate A from the effective action. Of course the symmetry here is just a reflection of the fact that we can assign the $U(1)_{EM}$ charge to either b or f without changing the physics.

Now we integrate out a . The transverse part is (now just writing π_α for π_α^T)

$$S \supset \int d^D x A_\mu \Pi_T^{\mu\nu} \left(\pi_b - \frac{\pi_b^2}{\pi_f + \pi_b} \right) A_\nu. \quad (1216)$$

Therefore the physical conductivity is

$$\sigma = \frac{\sigma_b \sigma_f}{\sigma_b + \sigma_f}, \quad (1217)$$

or, in terms of resistivities,

$$\rho = \rho_b + \rho_f, \quad (1218)$$

so that the two species of fields behave as resistors in series, a rather counterintuitive result at first given that independent fields should conduct in parallel. The resolution is of course that the gauge constraint means that they are not really independent, and that at long distances we in fact have $J_b + J_f = 0$; this is ultimately what is responsible for the series resistance.



Different types of superexchange interactions

Today we will explain what different types of superexchange interactions are possible, and where they come from. The original paper by Moriya is a good reference.



First, the setup: consider a lattice of atoms at positions indexed by R , and consider the case when the interactions between the electrons is strong. For concreteness, we will specialize to a Hubbard-like situation where there is a single orbital per site, with a strong Hubbard on-site repulsion U . This is mainly just for notation, and the generalization to multiple levels at each site is straightforward.

Superexchange interactions are the interactions between the effective (pseudo)spin 1/2 moments that form in the low-energy manifold where the site occupancy is (say) one on every site, and which arise from virtual electron hopping between sites. We write the annihilation operator appearing in the Hamiltonian as

$$\hat{\psi}_\alpha(r) = \sum_R \psi_\alpha(r - R) c_{R\alpha}, \quad (1219)$$

where the hat is just to indicate that it's an operator, and where $\alpha = \pm 1/2$ is the pseudospin index. The kinetic part of the Hamiltonian is

$$H_t = t \sum_{\alpha\beta} \sum_{RR'} \int d^d r \psi_\alpha^*(r - R) c_{R\alpha}^\dagger \mathcal{H}(r) \psi_\beta(r - R') c_{R'\beta}, \quad (1220)$$

where the 1-particle Hamiltonian is

$$\mathcal{H}(r) = -\frac{\nabla^2}{2m} + V(r) + \frac{1}{2m^2} (\nabla V) \cdot (-\nabla \times \boldsymbol{\sigma}) \quad (1221)$$

with $V(r)$ the 1-body potential (this is a matrix in pseudospin space of which the $\psi_\alpha(r)$ s are eigenfunctions, in particular the pseudospin label α labels the eigenfunctions). Now given the way in which $SU(2)$ rotations act on the c operators, we know that the terms in H_t can be classified according to either the 1 or 3 representations of $SO(3)$. It is helpful to make this manifest by coming up with new notation as follows (S for singlet and V for vector, and summation on the (pseudo)spin indices is implied)

$$H_t = t \sum_{RR'} \left(S(R - R') c_{R\alpha}^\dagger c_{R'\alpha} + V_i(R - R') c_{R\alpha}^\dagger \sigma_{\alpha\beta}^i c_{R'\beta} \right) \quad (1222)$$

with

$$S(R - R') = \int d^d r \psi_\alpha^*(r - R') \mathcal{H}(r) \psi_\alpha(r - R), \quad V_i(R - R') = \int d^d r \psi_\alpha^*(r - R') \sigma_{\alpha\beta}^i \mathcal{H}(r) \psi_\beta(r - R). \quad (1223)$$

We are working at large U and so the low-energy subspace at half filling has single occupancy on each site, which is violated by H_t . The effective Hamiltonian in the low-energy subspace thus appears at order t^2/U . The usual perturbative analysis gives, to second order in the hopping,

$$H_{eff} \supset \frac{t^2}{U} \sum_{RR'} \left(S(R - R') c_{R\alpha}^\dagger c_{R'\alpha} + V_i(R - R') c_{R\alpha}^\dagger \sigma_{\alpha\beta}^i c_{R'\beta} \right) \left(S(R' - R) c_{R'\lambda}^\dagger c_{R\lambda} + V_i(R' - R) c_{R'\gamma}^\dagger \sigma_{\gamma\lambda}^i c_{R\lambda} \right). \quad (1224)$$

All that we then need to do is to simplify this and write it in the form of a spin-spin interaction between the R and R' sites. The way to do this is to organize the terms in representations of the $SO(3)$ spin symmetry, under which the $\mathbf{S}_R = c_{R\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} c_{R\beta}$ operators transform in the spin 1 representation. Now the above interaction is already organized as the tensor product $(0 \oplus 1) \otimes (0 \oplus 1)$, but unfortunately the 0 and 1 representations arise from tensoring together the spin 1/2s at R and R' together, whereas to get an expression in

terms of the \mathbf{S}_R operators, we need to have the two spin 1/2s at R tensored together, and likewise for the two at R' . In schematic but hopefully clear notation, we currently have

$$\left((1/2)_R^\dagger \otimes (1/2)_{R'} \right) \otimes \left((1/2)_{R'}^\dagger \otimes (1/2)_R \right), \quad (1225)$$

but what we want is

$$\left((1/2)_R^\dagger \otimes (1/2)_R \right) \otimes \left((1/2)_{R'}^\dagger \otimes (1/2)_{R'} \right), \quad (1226)$$

since the 1 representations in each of the big parenthesis are the spin operators for the R and R' lattice sites. Of course these two ways of taking the \otimes are related since there is a natural associativity isomorphism, but tracking down how the decomposition works can get a bit complicated if one tries to write everything out in indicies.

From the above \otimes decomposition, the fact that $1 \otimes 1 = 0 \oplus 1 \oplus 2$, and the fact that we are working in the singly-occupied subspace so that $c_{R\alpha}^\dagger c_{R\alpha} = \mathbf{1}$ for all R (this lets us drop some trace terms in what follows, since they become unimportant constants), we know that we will be able to break this into three terms: the singlet (trace; one-dimensional), the vector (antisymmetric combination; three-dimensional) and the traceless 2-index tensor (symmetric; five-dimensional). Moreover we know that each of these terms can be built only using \mathbf{S}_R and $\mathbf{S}_{R'}$, again by the above \otimes decomposition. From what we've just said about the representations in $0 \oplus 1 \oplus 2$, the only possible structure we can write down is

$$H_{eff} \supset \sum_{RR'} \left(J(R - R') \mathbf{S}_R \cdot \mathbf{S}_{R'} + \mathbf{D}(R - R') \cdot (\mathbf{S}_R \times \mathbf{S}_{R'}) + S_R^i \Gamma_{ij}(R - R') S_{R'}^j \right), \quad (1227)$$

where $\Gamma_{ij} = \Gamma_{ji}$ and $\text{Tr}[\Gamma] = 0$. These terms are known as the (Anderson) superexchange interaction, the DM interaction, and the pseudodipolar interaction, respectively. We can relate them to the \mathbf{S} and \mathbf{V}^i matrices essentially just by using symmetry arguments. The regular superexchange term is a singlet, and it comes from the combination of the two \mathbf{S} singlets: indeed, using $\sigma_{ab}^i \sigma_{cd}^i = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}$ one can see that (up to a constant, since we're in the singly occupied subspace)

$$J(R - R') = J(|R - R'|) = \frac{2t^2}{U} \mathbf{S}^2(R - R'). \quad (1228)$$

The DM term is the vector, and evidently must come from combining the \mathbf{V} vectors with the \mathbf{S} singlets. Indeed, one can check that (the antisymmetry here coming from electron anticommutativity, so that re-arranging the \otimes s picks up signs)

$$D^i(R - R') = \frac{4it^2}{U} (\mathbf{V}^i(R - R') \mathbf{S}(R' - R) - \mathbf{V}^i(R - R') \mathbf{S}(R - R')), \quad (1229)$$

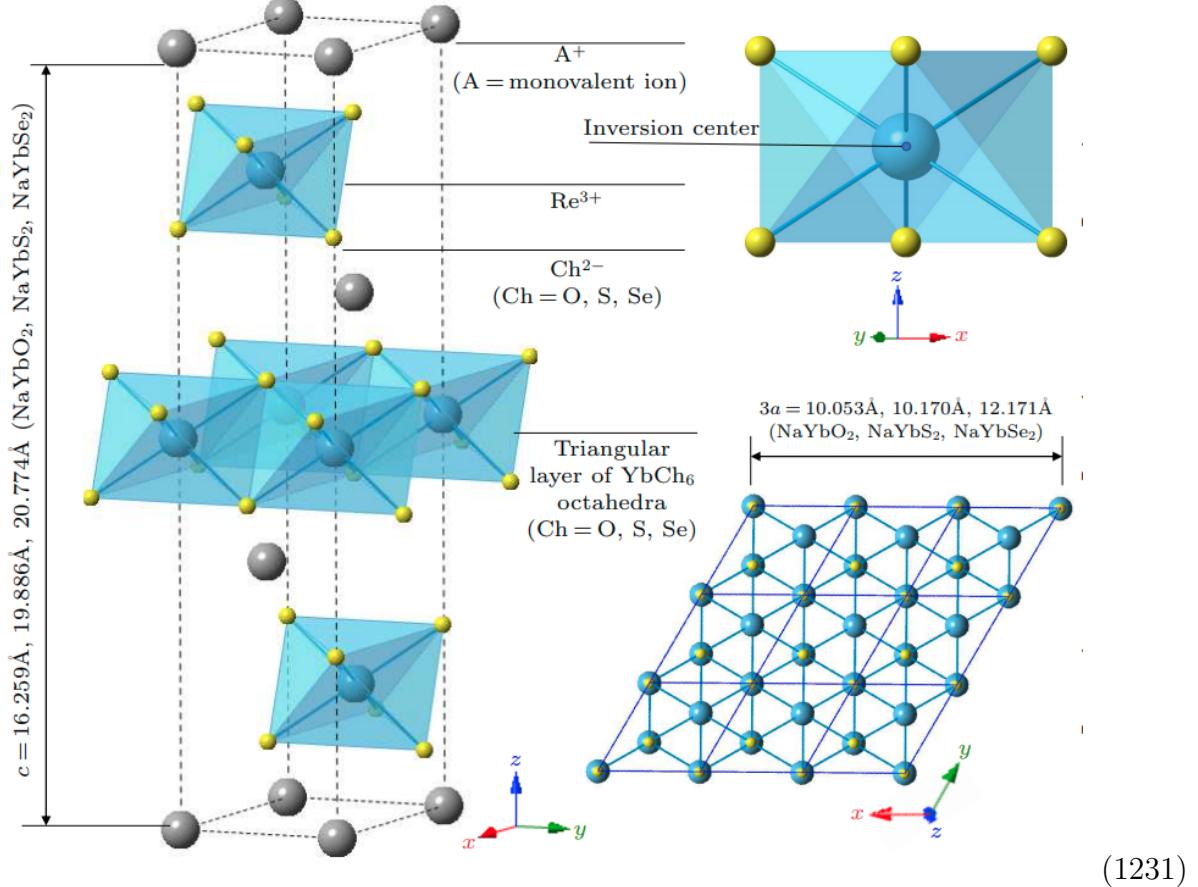
where e.g. the i comes from relating matrix elements of $\boldsymbol{\sigma} \times \boldsymbol{\sigma}$ to those of $\boldsymbol{\sigma}$. Finally, the Γ_{ij} matrix comes from \otimes ing the two \mathbf{V}_i s together and taking out the trace; hence it has the structure

$$\Gamma_{ij} = \frac{4t^2}{U} \left(\mathbf{V}_i(R - R') \mathbf{V}_j(R' - R) + \mathbf{V}_j(R - R') \mathbf{V}_i(R' - R) - \frac{2}{3} \mathbf{V}_k(R - R') \mathbf{V}^k(R' - R) \delta_{ij} \right). \quad (1230)$$

If $SO(3)$ spin symmetry is preserved, then of course the only nonzero term is $J(R - R')$. The vector DM term is also often zero; for example it vanishes whenever there is a symmetry which interchanges R and R' but generates no other signs, e.g. an inversion about the RR' bond center, which sends $R \leftrightarrow R'$ and $S^i \mapsto -S^i$. Even if it does not vanish, the vector \mathbf{D} is usually highly constrained by any symmetries that happen to be present involving the R and R' sites. The pseudodipolar term can also be constrained by symmetry, but generally to a lesser extent.

Application: NaYbSe_2

As an example of how one organizes the various superexchange terms in practice, we consider the material NaYbSe_2 . This material is a quasi 2d triangular lattice insulator with pseudospin $1/2$ degrees of freedom on the Yb atoms, which being rare earth atoms naturally come with substantial SOC that often leads to rather anisotropic spin interactions. The lattice looks like (figure from [10])



Given the strong SOC, the only constraints we can put on the spin interactions are those coming from the symmetry of the lattice. Each bond has an inversion symmetry about its center, which means that the DM interactions must vanish, but in general both other terms are present.

Consider first a bond $\langle RR' \rangle$ along the \hat{x} direction. A π rotation about this bond is a symmetry, and acts as conjugation by $\text{diag}(1, -1, -1)$. Therefore the only possible off-diagonal

term in Γ is $[\Gamma_{\langle RR' \rangle}]_{yz}$; the others are all odd under this rotation. The spin Hamiltonian along this bond $\langle RR' \rangle$ is therefore

$$H_{\langle RR' \rangle \parallel \hat{x}} = \mathbf{S}_R^T \begin{pmatrix} J_x & & \\ & J_y & J_{yz} \\ & J_{yz} & J_z \end{pmatrix} \mathbf{S}_{R'} \quad (1232)$$

with the J_i s some constants parameterizing the model. To get the Hamiltonian along the other bonds, we can just apply $\pm\pi/3$ rotations about the z axis and centered on the site R , by conjugating with $\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \oplus 1$. After some algebra, one finds for a bond $\langle RR' \rangle_\phi$ that makes an angle ϕ with \hat{x} ,

$$H_{\langle RR' \rangle_\phi \parallel R_\phi \hat{x}} = \mathbf{S}_R^T \begin{pmatrix} J_+ + J_- \cos 2\phi & J_- \sin 2\phi & -J_{yz} \sin \phi \\ J_- \sin 2\phi & J_+ - J_- \cos 2\phi & J_{yz} \cos \phi \\ -J_{yz} \sin \phi & J_{yz} \cos \phi & J_z \end{pmatrix} \mathbf{S}_{R'} \quad (1233)$$

where

$$J_\pm \equiv \frac{J_x \pm J_y}{2}. \quad (1234)$$

Notice in particular that while symmetric as required, a rotation by π does not leave H invariant, with the J_{yz} terms picking up minus signs. This means in particular that as we move along a given line of bonds in the lattice, the signs of the J_{yz} terms will alternate. Having a nonzero J_{yz} is however not in contradiction with translational invariance / symmetry of the lattice, because in fact a πz -rotation though a given pseudospin site is not a symmetry of the lattice (due to the tilt of the octahedral cages enclosing each Yb site).

We can isolate the different types of superexchange terms by organizing things according to their $SO(3)$ representations. We already said that the DM interaction is zero, and so we just need to take out the trace from the interaction above. Doing this, we find

$$H_{\langle RR' \rangle_\phi \parallel R_\phi \hat{x}} = J \mathbf{S}_R \cdot \mathbf{S}_{R'} + \mathbf{S}_R^T \Gamma_{\langle RR' \rangle_\phi} \mathbf{S}_{R'}, \quad (1235)$$

where

$$J = \frac{J_x + J_y + J_z}{3}, \quad \Gamma_{\langle RR' \rangle_\phi} = \begin{pmatrix} J_+ + J_- \cos 2\phi - J/3 & J_- \sin 2\phi & -J_{yz} \sin \phi \\ J_- \sin 2\phi & J_+ - J_- \cos 2\phi - J/3 & J_{yz} \cos \phi \\ -J_{yz} \sin \phi & J_{yz} \cos \phi & J_z - J/3 \end{pmatrix} \quad (1236)$$



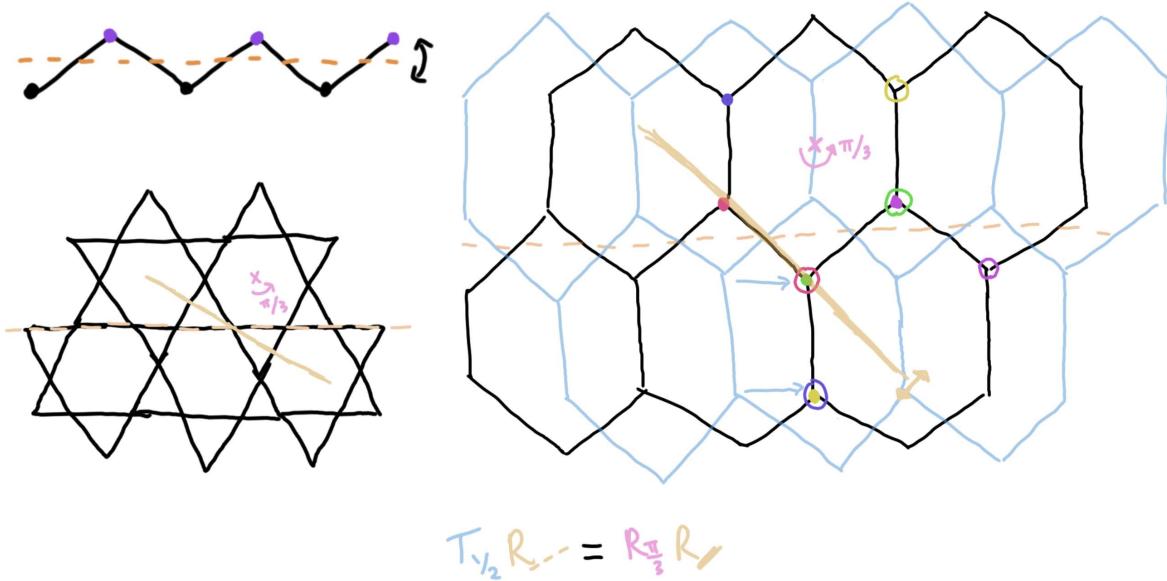


Figure 19: Top left: reflection about the orange-ish dashed line plus a half unit cell translation maps the chain to itself; this type of operation appears in the honeycomb (and therefore in the Kagome as well). Right: see main text. Each solid dot is mapped to the corresponding open circle under the glide symmetry in question (reflection about orange dashed line + half translation in the direction of the blue arrows. The blue lattice is the image of the black one under reflection through the dashed line.

A clarification on non-symmorphic symmetries

Today's entry is a tiny elaboration on some comments made by Sid Parameswaran during his 2019 Les Houches lectures on HOLSM theorems and non-symmorphic symmetries, recorded here for posterity's sake. The basic point was that the presence of a glide / screw symmetry is not the same as the non-symmorphic-ness of the space group.



First, recall the definition of a symmorphic (aka split) space group. We say that G is symmorphic if there exists a point p such that all elements in G consist of a product ft , where f is a symmetry fixing p and t is a translation. A non-symmorphic group is then (duh) a group where not all elements take this form. A more mathematical way to say this is that G is non-symmorphic if it takes the form $G' \rtimes T$, where T is a free factor coming from translation.

We then have a non-symmorphic operation if we can perform a rotation / reflection which does not preserve the lattice, and then get back to the original lattice by acting with a translation through a vector which is left invariant under the reflection / rotation.¹⁰⁷

¹⁰⁷The condition that the vector through which we're translating be left invariant by the reflection / rotation

The content of this entire diary entry is essentially contained in figure 19, which shows an example an example where we have glide / screw symmetry in a symmorphic space group. If we look at the honeycomb lattice, we see that it contains zigzag chains of the sort drawn in the top left of the figure. That means that it has a glide symmetry, which acts e.g. first as reflection along the dotted orange line drawn (giving the blue lattice), and then as a translation by half a unit vector horizontally (note that the unit vector pointing in the direction of the translation is invariant under the reflection, as required of a glide transformation).

Does this mean that the symmetry group of the honeycomb lattice is non-symmorphic, since it contains a glide? No, because the glide is not essential, meaning that it can be described in an alternative way that doesn't make use of the glide. Indeed from the figure we can see that the glide is actually the product of two operations which are themselves symmetries. This is indicated in the figure: doing the glide reflection through the dashed orange line and then translating in the direction of the blue arrows is actually equivalent to first doing a reflection through the solid orange line and then rotating by $\pi/3$ about the center of the hexagon marked with the purple x.



KL mecahnism calculation

Today we're recording the details of a calculation used when deriving the KL mechanism for superconductivity. The result is in the original KL paper, but I couldn't find the derivation anywhere.



Recall how the KL mechanism works: we start off with a purely repulsive interaction in the UV, which for simplicity we will take to be independent of momentum. At second order in the interaction, a momentum-dependent effective vertex is generated. Because of the sharpness of the Fermi surface, the effective vertex is a singular function of momentum at momentum transfer $q = 2k_F$. When decomposed into harmonics l , this ensures (at least in three dimensions, which we will be specializing to) that the effective interaction has harmonics decaying only as a power law in l , rather than exponentially (recall that by normalization any analytic contribution to the effective vertex must decay as $V_l^{an} \sim e^{-l}$). At large enough l , the power law from the singular piece always dominates over the exponentially decaying analytic part (which is built of the bare interaction and vertices at momentum

is important, and it means e.g. that there are no types of screw symmetries in two dimensions, since in that case there are no vectors left invariant under rotations.

transfer away from $2k_F$). Since the singular part of the effective vertex can be negative (as we will show), this gives a mechanism to produce attractive interactions at large l , regardless of the strength / sign / analytic momentum dependence of the bare potential.

The calculation we will do is to estimate the leading l dependence of the effective vertex at large l . As we just said, this will be dominated by the non-analytic part. For spinless fermions, the non-analytic part of the zero-frequency (since we are interested in Cooper pairing) polarization $\Pi(q)$ near momentum transfer $q \lesssim 2k_F$ has the well-known form

$$\Pi(q) \sim ((2k_F)^2 - q^2) \ln |(2k_F)^2 - q^2|, \quad (1237)$$

where \sim means up to constants and analytic terms. Let us write q in terms of the scattering angle θ by $q = 2k_F \sin(\theta/2)$ (draw a picture to see this); henceforth we will also use units where $2k_F = 1$. To find the angular momentum modes of this potential, we integrate $\Pi(q)$ over the sphere against $P_l(\cos \theta)$:

$$\begin{aligned} \Pi_l &\sim \int_0^\pi d\theta \sin \theta P_l(\cos \theta) \Pi(\theta) \\ &\sim \int_{-1}^1 dz P_l(z) (1+z) \ln(1+z), \end{aligned} \quad (1238)$$

where we used $1 - \sin^2(\theta/2) = (1 + \cos(\theta))/2 = (1 + z)/2$ and where \sim means equality up to constant l -independent prefactors, which we will be dropping.

To evaluate this, we use the generating expression for the Legendre polynomials:

$$P_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l} ((z^2 - 1)^l). \quad (1239)$$

We plug this into the expression for Π_l and then integrate by parts twice to kill the log:

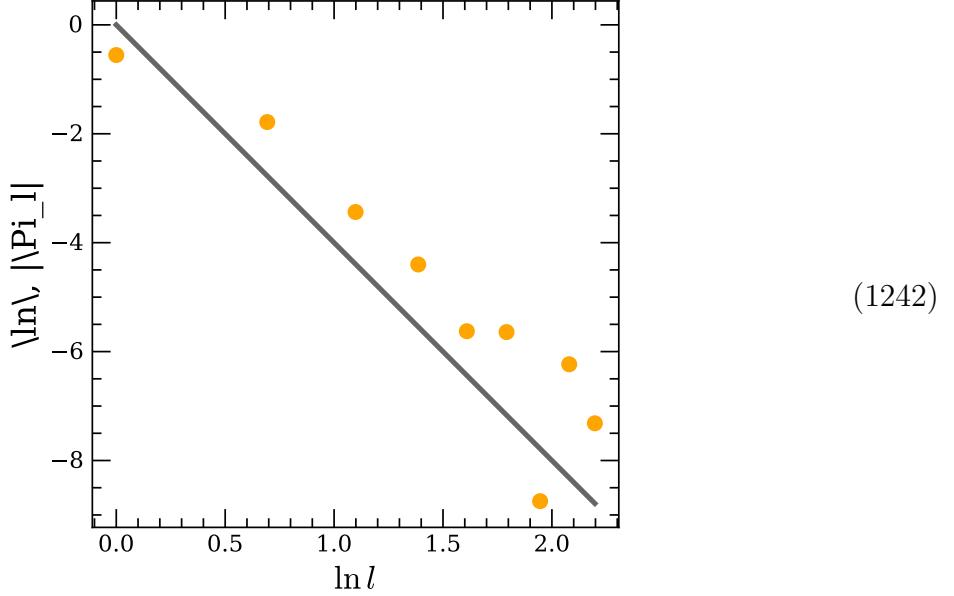
$$\Pi_l \sim \int_{-1}^1 dz \frac{1}{2^l l!} \frac{d^{l-2}}{dz^{l-2}} ((z^2 - 1)^l) \frac{1}{z+1} \quad (1240)$$

From here on, each of the subsequent $l - 2$ integrations by parts produces no minus sign (because of the $1/(z+1)$), and we pick up constants from the differentiation that multiply to $(l-2)!$. The boundary terms are safe despite the pole of $1/(z+1)$ since the $d_z^{l-m}(z^2 - 1)^l$ term goes as $(z^2 - 1)^m$, which is always enough to kill the $1/(z+1)^{m-1}$ pole. Therefore one gets at large l

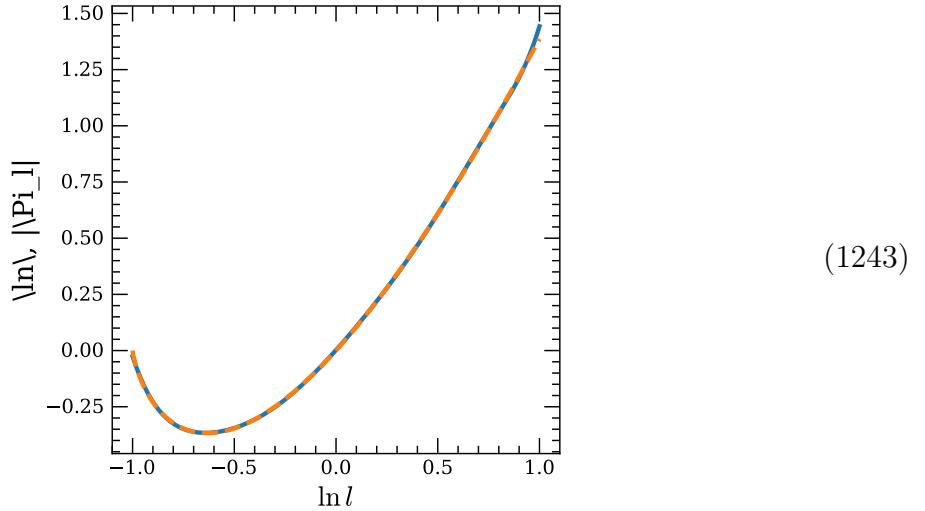
$$\begin{aligned} \Pi_l &\sim \frac{(l-2)!}{2^l l!} \int_{-1}^1 dz \frac{(z^2 - 1)^l}{(z+1)^{l-1}} \\ &\sim \frac{1}{l^2 2^l} \int_{-1}^1 dz (z+1)(z-l)^l \\ &= \frac{1}{l^2 2^l} \int_{-2}^0 du u^l (u+2) \\ &= -\frac{1}{l^2 2^l} \left(\frac{1}{l+1} (-2)^{l+2} + \frac{2(-2)^{l+1}}{l} \right) \\ &\sim -\frac{1}{l^2} (-1)^l \left(\frac{1}{l+1} - \frac{1}{l} \right) \\ &\sim (-1)^l \frac{1}{l^4}. \end{aligned} \quad (1241)$$

Thus the effective interaction has a sign depending on the parity of the harmonic (this is modified if we take into account spin), and indeed decays only polynomially, as $1/l^4$. This is what we wanted to show.

To check that we haven't made any mistakes, we can explicitly find the coefficients numerically. The first 10 coefficients are



with the line being a fit to a l^{-4} power law. To check that this reproduces the correct function, we plot $(1+z) \ln(1+z)$ (dashed) and its approximation with the first ten modes (solid)



The story is different in two dimensions. There the polarization $\Pi(q)$ is actually *independent* of momentum for $q < 2k_F$ (made possible by the fact that the singularity at $2k_F$ is of the form $\sqrt{q^2 - (2k_F)^2}$, which has no real part if $q < 2k_F$). This calculation is done within the context of a Fermi gas, but changing the band structure can only affect the analytic part of $\Pi(q)$. Hence in 2d one must go to three loops to see the KL effect, as shown by Chubukov.



Sound modes in (non)-relativistic superfluids

In this diary entry we are going to recall the differences between the sound modes in relativistic and non-relativistic superfluids. In the former there are both phase and density modes, with the density modes being gapped. In the latter there is only one mode, which is gapless. The radial Higgs mode doesn't really exist in the non-relativistic setting, and is tied in with the gapless phase mode.



Let us first review what happens in the relativistic case. Here the action is

$$S = \int \left(|\partial_t \psi|^2 - |\nabla \psi|^2 + \mu |\psi|^2 - \frac{g}{2} |\psi|^4 \right), \quad (1244)$$

where we take $\mu > 0$. The minimum of the potential is at $|\psi|^2 = \mu/g$. Writing $\psi = \sqrt{\mu/g} + \lambda$, we get the linearized eom

$$\partial_t^2 \lambda = \nabla^2 \lambda - \mu(\lambda + \lambda^*), \quad (1245)$$

together with its complex conjugate. We then write

$$\lambda = A e^{i(\omega t - kx)} + B e^{-i(\omega t - kx)}, \quad (1246)$$

and put this form into the linearized eom. This tells us that

$$\begin{pmatrix} -\omega^2 + k^2 + \mu & \mu \\ \mu & -\omega^2 + k^2 + \mu \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \quad (1247)$$

The spectrum accordingly consists of two modes ω_r, ω_p , with :

$$\omega_r = \sqrt{k^2 + 2\mu}, \quad \omega_p = k. \quad (1248)$$

The former is the massive Higgs mode, and the latter is the massless Goldstone. The eigenvector for the Higgs mode has $A = B = 1$. If we put this into (1246), we get $\lambda(x = 0, t) = 2 \cos(\omega t)$. Thus the Higgs mode corresponds to harmonic motion along the radial direction of the potential. The eigenvector for the phase mode has $A = -B = 1$; in this case we get $\lambda(x = 0, t) = 2i \sin(\omega t)$. Therefore the phase mode corresponds to harmonic motion along the minimum of the potential. This is of course all very trivial, and is reviewed only to make comparison with the non-relativistic case.

Now we look at the non-relativistic case, with

$$S = \int \left(i\psi^* \partial_t \psi - \frac{1}{2m} |\nabla \psi|^2 + \mu |\psi|^2 - \frac{g}{2} |\psi|^4 \right). \quad (1249)$$

The minimum of the potential is the same as before, and the linearized eom is now

$$i\partial_t \lambda = \frac{1}{2m} \nabla^2 \lambda - \mu(\lambda + \lambda^*). \quad (1250)$$

Using the same decomposition (1246) as before, we now need

$$\begin{pmatrix} \omega + k^2/2m + \mu & \mu \\ \mu & -\omega + k^2/2m + \mu \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \quad (1251)$$

Since ω only appears linearly, we will now get only *one* mode. The dispersion is the usual Bogoliubov form:

$$\omega = \sqrt{\frac{k^2}{2m} \left(\frac{k^2}{2m} + \mu \right)}. \quad (1252)$$

Note that now μ sets the actual density of the ψ particles (compared to the SF density in the relativistic case). Thus the above dispersion tells us that we have regular non-relativistic massive particles at short length scales, which gives way to linearly dispersing sound modes at distances long enough so that the particles can be treated hydrodynamically.

Since there is only one mode (which is gapless), there is no notion of a “massive radial mode” like there is in the relativistic case. Rather, the phase and radial modes are mixed together. This mixing happens essentially since now the radial mode is a density mode, and density is conjugate to phase. If we solve for A and B , we find that A/B at small $k \ll \sqrt{2m/\mu}$ is

$$A/B \equiv r \approx -\frac{\mu}{ck + k^2/2m + \mu}, \quad (1253)$$

where

$$\omega \approx ck, \quad c \equiv \sqrt{\frac{\mu}{2m}}. \quad (1254)$$

The eom tell us that the field ψ exhibits harmonic motion about the value $\sqrt{\mu/g}$. Here however the harmonic motion involves both radial and phase oscillations, with

$$\lambda(x=0, t) \propto (1+r) \cos \omega t - i(1-r) \sin \omega t. \quad (1255)$$

Thus we have an elliptical motion in the complex plane. The eccentricity of the ellipse is

$$e = \frac{2}{1+r} \sqrt{r}. \quad (1256)$$

Since $-1 < r < 0$, the ellipse is extended in the direction along the minimum of the potential, as is to be expected.



Perspectives on the Kelvin circulation theorem in two and three dimensions

This diary was inspired by wanting to understand parts of Dam Son's paper on chiral metric hydrodynamics [16]. The setting will be a general hydrodynamic theory of translation-invariant conserved bosons.



The two fundamental hydrodynamic quantities in the present setting are the generators for the two conserved quantities, viz. momentum and particle number. We define the particle number and momentum densities as¹⁰⁸

$$n(x) = \sum_{\alpha} \delta(x_{\alpha} - x), \quad \pi^i(x) = \sum_{\alpha} \delta(x_{\alpha} - x)p_{\alpha}^i, \quad (1257)$$

where the index α labels microscopic particles and where the phase space coordinates are $\{x_{\alpha}, p_{\alpha}\}$. We see that

$$\{n(x), n(y)\} = 0 \quad (1258)$$

while

$$\begin{aligned} \{n(x), \pi^i(y)\} &= \sum_{\alpha} \delta(x_{\alpha} - y) \frac{\partial}{\partial x_{\alpha}^i} \delta(x_{\alpha} - x) \\ &= n(y) \frac{\partial}{\partial y^i} \delta(x - y) \end{aligned} \quad (1259)$$

and

$$\begin{aligned} \{\pi^i(x), \pi^j(y)\} &= \sum_{\alpha} \left(p_{\alpha}^i \delta(x_{\alpha} - y) \frac{\partial}{\partial x_{\alpha}^j} \delta(x_{\alpha} - x) - p_{\alpha}^j \delta(x_{\alpha} - x) \frac{\partial}{\partial x_{\alpha}^i} \delta(x_{\alpha} - y) \right) \\ &= \left(\pi^i(y) \frac{\partial}{\partial y^j} - \pi^j(x) \frac{\partial}{\partial x^i} \right) \delta(x - y) \end{aligned} \quad (1260)$$

The densities n, π^i allow us to generate symmetry actions by integrating them against various functions. For example, define the translation generator

$$Q_{\xi} = - \int_x \xi^k \pi_k. \quad (1261)$$

The claim is that Q_{ξ} generates translations, in the sense that

$$\{Q_{\xi}, X\} = \mathcal{L}_{\xi} X, \quad (1262)$$

¹⁰⁸We could also take the quantum approach of working with commutators, but this is more annoying due to the fact that the ordering of operators matters. This means that e.g. the velocity has to be defined in Weyl quantization like $(\sim jn^{-1} + n^{-1}j)/2$, etc — see Landau's OG paper on HeII. Sometimes working in the QM context is nice since the commutator automatically satisfies the Jacobi identity, which depending on the situation might be a difficult property to satisfy when constructing a PB. Here we will have no such issues however, and it is more convenient to work with PBs from start to finish.

where \mathcal{L}_ξ is the Lie derivative along the vector field ξ . Indeed, using the above Poisson brackets one checks that¹⁰⁹

$$\{Q_\xi, n(y)\} = - \int_x \xi^k \{\pi_k(x), n(y)\} = \int_x \xi^k n(x) \frac{\partial}{\partial x^k} \delta(x - y) = -\xi^i \partial_i n - (\partial_i \xi^i) n \quad (1263)$$

and

$$\begin{aligned} \{Q_\xi, \pi^i(y)\} &= - \int_x \xi^k(x) \left(\pi_k(y) \frac{\partial}{\partial y^i} - \pi_i(x) \frac{\partial}{\partial x^k} \right) \delta(x - y) \\ &= -(\pi_k \partial_i \xi^k + \xi_k \partial_k \pi_i + \pi_i \partial_k \xi^k). \end{aligned} \quad (1264)$$

The last terms proportional to $\nabla \cdot \xi$ in these expressions arise because n, π^i are *densities*, i.e. they implicitly contain a factor of $\sqrt{|g|}$. To recall why this gives $\nabla \cdot \xi$, remember that

$$\begin{aligned} \frac{1}{\sqrt{|g|}} \delta \sqrt{|g|} &= \frac{1}{2} \delta \text{Tr}(\ln g) = \frac{1}{2} g^{ij} (\xi^k \partial_k g_{ij} + \partial^k \xi_i g_{kj} + \partial^k \xi_j g_{ik}) \\ &= \partial_k \xi^k, \end{aligned} \quad (1265)$$

where the last line comes from $\partial_k \text{Tr}g = 0$.

The momentum per particle on the other hand, viz.

$$u^i \equiv \pi^i/n, \quad (1266)$$

transforms as a proper vector, without the $\nabla \cdot \xi$ term in the PB with Q_ξ . The PB with π^i which makes this work is

$$\{\pi^i(x), u^j(y)\} = \left(u^i(y) \frac{\partial}{\partial x^j} + \frac{n(x)}{n(y)} (u^j(x) - u^j(y)) \frac{\partial}{\partial x^i} \right) \delta(x - y). \quad (1267)$$

This is easy to check using the fact that the PB is a derivation, so that we have e.g.

$$\{A(f_1, f_2, \dots), B\} = \sum_n \frac{\partial A}{\partial f_n} \{f_n, B\}. \quad (1268)$$

The fact that u^i transforms as a vector rather than a vector density is one of the reasons that we sometimes prefer to work with u^i rather than π^i .

These PBs can be used to derive the equations of motion for the particle number and momentum densities. The Hamiltonian will be a function of n and π , and we denote

$$v^i \equiv \frac{\partial H}{\partial \pi^i}, \quad \mu \equiv \frac{\partial H}{\partial n}. \quad (1269)$$

We then have the correct continuity equation¹¹⁰ (note that in our conventions $d_t \mathcal{O} = \{\mathcal{O}, H\}$)

$$d_t n(y) = \int_x v_i(x) \{n(y), \pi^i(x)\} = -\partial_i(v^i n) = -\nabla \cdot \mathbf{j}, \quad (1270)$$

¹⁰⁹There are almost certainly several minus signs wrong here, as I am being cavalier about index placement even though co/contra-variant indices appear with different signs in the Lie derivative.

¹¹⁰The equations of motion computed here are not comoving, in the sense that $d_t n$ does not include a term $u^i \partial_i n$.

as well as the correct equation of motion

$$\begin{aligned} d_t \pi_i(y) &= \int_x (v_j \{\pi^i(y), \pi^j(x)\} + \mu \{\pi^i(y), n(x)\}) \\ &= -\partial^j(v_j \pi_i) - \pi^j \partial_i v_j - n \partial_i \mu. \end{aligned} \quad (1271)$$

The eom for the momentum per particle is also seen to be

$$d_t u^i(y) = -u^j \partial_i v_j - v^j \partial_j u^i - \partial_i \mu, \quad (1272)$$

which accordingly lacks the $\partial_j v^j$ term of the π_i eom. As such, we may write

$$d_t u = -d\mu + \mathcal{L}_v u. \quad (1273)$$

To perform a sanity check of these results, consider what happens when we take the Hamiltonian to be (setting the mass of the particles to be $m = 1$)

$$H = \int \left(\frac{\pi^2}{2n} + V(n) \right) \implies \mu = -\frac{u^2}{2} + \partial_n V, \quad v_i = u_i. \quad (1274)$$

The continuity equation then takes the usual form

$$d_t n = -\partial_i(u^i n) \implies D_t n \equiv d_t n + u^i \partial_i n = -n \partial_i u^i, \quad (1275)$$

which is the form that one sees in most hydro contexts. The eom for u^i similarly becomes

$$D_t u^i = -\partial_i \partial_n V. \quad (1276)$$

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We are now in a position to discuss the Kelvin circulation theorem. We will first work in two dimensions, which is a special case due to the fact that the vorticity is a scalar. Higher dimensions will be addressed in the next section.

Let us introduce the vorticity

$$\omega \equiv \star du. \quad (1277)$$

Recalling that we are in two dimensions, we see that ω is a scalar density (u is a proper vector, but the \star turns it into a density). The claim of Kelvin's circulation theorem is then that

$$I_F = \int_x n F(\omega/n) \quad (1278)$$

is a conserved quantity, where F is an *arbitrary* function of its argument (which can be arbitrary since ω/n is a scalar, not a scalar density).

A sufficient condition for I_F to be conserved is that it have zero PB with both n and π^i , since these are the quantities that the Hamiltonian H is built out of. Indeed, this turns out to be true. First, it is easy to see that $\{I_F, n(x)\} = 0$, since $\{n, n\} = 0$ and

$$\begin{aligned} \{\omega(x), n(y)\} &= \varepsilon^{ij} \frac{\partial}{\partial x^i} \{u_j(x), n(y)\} \\ &= -\varepsilon^{ij} \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial x^i} \delta(x - y) \right) \\ &= 0. \end{aligned} \quad (1279)$$

Furthermore since ω is a scalar density, it must have a PB with the momentum of

$$\{\omega(x), \pi^i(y)\} = \omega(y) \frac{\partial}{\partial y^i} \delta(x - y), \quad (1280)$$

which is easy to check. Using this PB and the fact that the PB is a derivation, we then have (using the notation $z \equiv \omega/n$)

$$\begin{aligned} \{I_F, \pi^i(y)\} &= \int_x [F(z(x))\{n(x), \pi^i(y)\} + F'(z(x))\{\omega(x), \pi^i(y)\} - F'(z(x))z(x)\{n(x), \pi^i(y)\}] \\ &= - \int_x (F(z(x))n(y) + F'(z(x))\omega(y) - F'(z(x))z(x)n(y)) \frac{\partial}{\partial x^i} \delta(x - y) \\ &= n\partial_i F + \omega\partial_i F' - n\partial_i(F'z) \\ &= F'(\partial_i\omega - z\partial_i n) - F'n\partial_i z \\ &= 0. \end{aligned} \quad (1281)$$

Therefore $\{I_F, H\} = 0$ and I_F is conserved for any F as claimed.

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In higher dimensions, the usual statement of the circulation theorem is given in terms of the conservation of the integral of the vorticity around closed curves, and is the statement that

$$\mathcal{I}_C = \oint_C u \quad (1282)$$

is conserved for all closed curves C . We don't have the freedom of sticking in arbitrary functions of u into the integrand (since magnetic fields are scalars only in two dimensions).

The statement that \mathcal{I}_C is conserved requires a bit of qualification. Certainly if C is fixed in time this cannot be true — if the curve C encloses a vortex at time t , the vortex may very well move to a location not enclosed by C at some later time. Instead, the statement is that \mathcal{I}_C is conserved provided that we take C to be comoving with the fluid flow. Mathematically, this means that $C = C(t)$, with

$$d_t C = \mathcal{L}_v C. \quad (1283)$$

The proof that \mathcal{I}_C is conserved is then rather trivial:

$$\begin{aligned} d_t \mathcal{I}_C &= \frac{d}{dt} \int \widehat{C} \wedge u \\ &= \int (\mathcal{L}_v \widehat{C} \wedge u + \widehat{C} \wedge (\mathcal{L}_v u - d\mu)) \\ &= \int \mathcal{L}_v (\widehat{C} \wedge u), \end{aligned} \quad (1284)$$

since $d\widehat{C} = 0$. The using Cartan's magic formula,

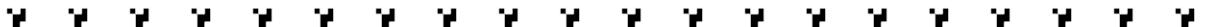
$$\begin{aligned} d_t \mathcal{I}_C &= \int (i_v d + di_v)(\widehat{C} \wedge u) \\ &= \int di_v (\widehat{C} \wedge u) \\ &= 0, \end{aligned} \quad (1285)$$

as $d(\widehat{C} \wedge u) = 0$ on account of $\widehat{C} \wedge u$ being top-dimensional. Note that this proof makes no reference to the fact that we are in three dimensions.



Comments on electromagnetic response in superconductors and metals

Today we are going to be talking about some very simple aspects of response functions in (super)conductors. Everything will be very simple, but this stuff is not talked about in the literature so it's good to have a place where we set stuff straight. (update: a nice discussion of the various limits of the EM response function is in [15])



Let us first review the EM response of a superconductor. Deep in the SCing phase, we have

$$S = \rho \int (d\phi - A)^2, \quad (1286)$$

which gives

$$S_{\text{eff}}[A] \supset \rho A \wedge \star \frac{d^\dagger d}{\square} A = \rho |A_T|^2, \quad (1287)$$

where A_T is the transverse component. If we gauge fix so that $d^\dagger A = 0 \implies A = A_T$, then including the maxwell term gives (if it wasn't already clear, we are ignoring numerical factors)

$$S_{\text{eff}}[A] = A \wedge \star (\square + \rho) A. \quad (1288)$$

The point is of course here that A is made massive: even though we integrated out a massless degree of freedom, all it did was cancel the longitudinal piece of A , and we do not end up with a singular effective action.

The superconducting strength should clearly be measured by something like the mass-squared of A (viz. ρ). The way this is usually done in the condensed matter literature is to define

$$D_s \sim \pi \lim_{\mathbf{q} \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{\delta^2 \mathcal{F}[A]}{\delta A_x^2} \Big|_{A=0}, \quad (1289)$$

where in the derivative we have chosen one particular component of \mathbf{A} for concreteness and where the π is because of the π in the Dirac identity (and where dimensionful quantities relating to the system size have been omitted). The reason for the order of limits here is that intuitively, the superconducting strength should be measured by the phase stiffness, viz. the energy cost to twisted boundary conditions in the phase ϕ . To do this we need to first take $\omega = 0$, and then send $\mathbf{q} \rightarrow 0$.

For example, consider taking this limit for the above S_{eff} in 2+1D. Consider furthermore turning on \mathbf{A} only along the $\hat{\mathbf{x}}$ direction. Then in $\nabla \cdot \mathbf{A} = 0$ gauge, the limit $\mathbf{q} \rightarrow 0$ means that $q_x = 0, q_y \rightarrow 0$. This gives

$$\lim_{\mathbf{q} \rightarrow 0} \lim_{\omega \rightarrow 0} \mathcal{F}[A] \sim \lim_{\mathbf{q} \rightarrow 0} \lim_{\omega \rightarrow 0} \rho \frac{1}{\mathbf{q}^2 + \omega^2} \begin{pmatrix} \mathbf{q}^2 & -\omega q_x & -\omega q_y \\ -\omega q_x & \omega^2 + q_y^2 & -q_x q_y \\ -\omega q_y & -q_x q_y & \omega^2 + q_x^2 \end{pmatrix} = \rho \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}, \quad (1290)$$

telling us that $D_s = \rho$. Higher dimensions are similar, but note that in 1+1D, we instead have

$$\lim_{\mathbf{q} \rightarrow 0} \lim_{\omega \rightarrow 0} \mathcal{F}_{1+1}[A] \sim \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \rho \frac{1}{q^2 + \omega^2} \begin{pmatrix} q^2 & -q\omega \\ -q\omega & \omega^2 \end{pmatrix} = \rho \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad (1291)$$

so that $D_s = 0$ in 1+1D. This is because there is no transverse component in 1+1D, and is of course in agreement with the fact that phase twists in ϕ cost no energy in one dimension (as $\partial\phi \sim 1/L$ means $\int dx (\partial\phi)^2 \rightarrow 0$).

Note that this definition of D_s looks nearly identical to the definition of the Drude weight in a metal, and indeed in superconductors we know that (at least in the idealized case considered here), the Drude and superconducting weights are equal. Nevertheless the definition of the Drude weight is different: to measure the conductivity we need to set $\mathbf{q} = 0$ first, since we need $\mathbf{q} = 0$ to get a net current flow. Therefore we define

$$D_d \sim \pi \lim_{\omega \rightarrow 0} \lim_{\mathbf{q} \rightarrow 0} \frac{\delta^2 \mathcal{F}[A]}{\delta A_x^2} \Big|_{A=0}. \quad (1292)$$

We then see that $D_s = D_d$ for the model of a superconductor above, even in 1+1D. We of course also know that $D_s \neq 0$ for a clean metal.

Note that $D_s = D_d$ means that the response $\mathcal{F}[A]$ is non-singular at small momenta / frequencies. If equality occurs because $D_s = D_d = 0$, then the system does not couple to the gauge fields at all at long distances. In this case, the system is an insulator. On the other hand if $D_s = D_d \neq 0$, then the coupling to the gauge field is nontrivial, but still analytic at small \mathbf{q}, ω . This can only happen if the gauge field is made massive. Therefore we can use D_s, D_d to distinguish between an insulator (both vanishing), a metal (only D_d nonzero) and a superconductor ($D_s = D_d$ both nonzero).

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Lets see how $D_s = 0, D_d \neq 0$ arises in Fermi liquids. At first sight, one might find the bosonized description of Fermi liquids rather curious, since if we can faithfully represent all the IR physics of a FL in terms of phase degrees of freedom which couple to A as $(d\phi - A)$, we might expect to obtain a superconductor, rather than a metal. The geometry of the FS prevents this from happening, though: A couples to phase degrees of freedom which fluctuate in many different directions, and summing over these directions means that we don't just get a projector onto the transverse component like we found above.¹¹¹

We will just examine the simplest case of a circular Fermi surface in 2+1D. Taking the FS to be circular (and with uniform v_F everywhere) means we can combine the chiral modes on

¹¹¹The situation is different in 1+1D as we have already mentioned above.

opposite points of the Fermi surface into non-chiral antipodal pairs. This is nice because it lets us easily couple the FL to A (recall that we cannot just minimally couple A to the chiral modes; this gives the wrong answer for anomaly reasons). Integrating out the fermions,

$$\mathcal{F}[A] = \frac{k_F}{2\pi} \int_0^\pi d\gamma A \wedge \star \Pi_T(\gamma) A, \quad (1293)$$

where $\Pi_T(\gamma)$ is the transverse projector onto the spacetime plane (τ, γ) . Explicitly,

$$\begin{aligned} \mathcal{F}[A] &\supset \frac{k_F}{2\pi v_F} \int_0^\pi d\gamma \frac{A_\tau^2 v_F k_\gamma^2 + A_\gamma^2 \omega^2 - 2A_\tau A_\gamma v_F k_\gamma \omega}{\omega^2 + v_F^2 k_\gamma^2} \\ &= \frac{k_B v_F}{4\pi} \left(\frac{A_\tau^2 \zeta^2 v_F^{-2}}{1 + \zeta^2 + \sqrt{1 + \zeta^2}} + \frac{\mathbf{A}^2}{1 + \sqrt{1 + \zeta^2}} \right), \end{aligned} \quad (1294)$$

where we have defined $\zeta \equiv v_F k / \omega$. We see then that

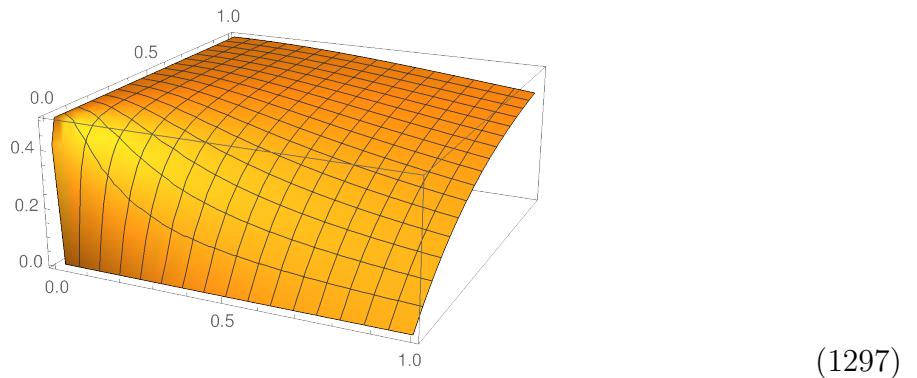
$$\frac{\delta^2 \mathcal{F}[A]}{\delta A_i \delta A_j} = \delta_{ij} \frac{k_F v_F}{2\pi} \frac{1}{1 + \sqrt{1 + \zeta^2}}. \quad (1295)$$

To measure the superconducting weight, we send $\zeta \rightarrow \infty$. Clearly the above expression vanishes in this limit—hence there is no Meissner effect, and the FL is (duh) not a SC.

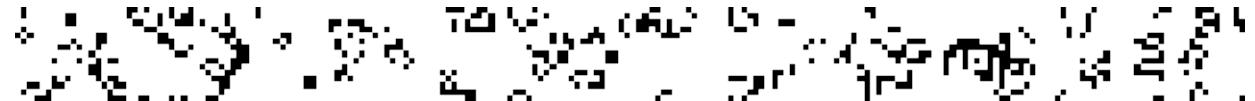
To measure the Drude weight, we send $\zeta \rightarrow 0$. This gives

$$D_d = \frac{k_F v_F}{4} = \frac{\pi (\pi k_F^2 / (2\pi)^2)}{k_F/v} = \frac{\pi n}{m}, \quad (1296)$$

where $m = k_F/v_F$ and $n = (\pi k_F^2)/(2\pi)^2$. This is precisely the Drude weight we expect from free electrons; even the numerical factors are correct. In units where $k_F v_F / 2\pi = 1$, the response function $\frac{\delta^2 \mathcal{F}[A]}{\delta A_i \delta A_j}$ looks like



where the horizontal axis is k . This reproduces what we expect from the Lindard function.



(anisotropic) EBL correlator

Today we are going to be deriving the double-log form of the correlator in a 2+1D theory with Lagrangian

$$\mathcal{L} = \frac{R^2}{4\pi} (a^{-1}(\partial_\tau \phi)^2 + a(\partial_x \partial_y \phi)^2), \quad (1298)$$

where a is a lattice cutoff and ϕ is dimensionless. We also show how the IR divergences are softened if we let the bosons disperse along a third direction. The former result is in the original EBL paper; here we merely provide the details.



In the context of the EBL, we are most interested in equal-time correlation functions, since the row/column symmetry means that correlations of $e^{i\phi}$ are local in space. Therefore we want to know

$$C(\tau) = \frac{2\pi a}{R^2} \int_{\mathbf{k},\omega} \frac{e^{i\tau\omega}}{\omega^2 + a^2 k_x^2 k_y^2}. \quad (1299)$$

To evaluate this correlator we will take ω to run over all of \mathbb{R} , with both k integrals running over $[1/L, 1/a]$. We will be taking the TDL by sending $L \rightarrow \infty$ before we send $a \rightarrow 0$.

The calculation is rather easy: assuming $a \ll \tau \ll L$, the ω integral gives

$$C(\tau) = \frac{1}{4\pi R^2} \int_{\mathbf{k}} \frac{e^{-a|k_x k_y|\tau}}{|k_x k_y|} \equiv \frac{1}{4\pi R^2} I(\tau). \quad (1300)$$

Then

$$\begin{aligned} \partial_\tau I &= 4a \int_{1/L}^{1/a} dk_x dk_y e^{-ak_x k_y \tau} \\ &= 4a \int_{1/L}^{1/a} dk_x \frac{1}{ak_x \tau} (e^{-ak_x \tau/L} - e^{-k_x \tau}) \\ &\equiv \frac{4}{\tau} J(\tau). \end{aligned} \quad (1301)$$

Now

$$\begin{aligned} \partial_\tau J &= \int_{1/L}^{1/a} dk_x \left(e^{-k_x \tau} - \frac{a}{L} e^{-ak\tau/L} \right) \\ &= \frac{1}{\tau} \left(e^{-\tau/L} - e^{-a\tau/L^2} - e^{-\tau/a} + e^{-\tau/L} \right) \\ &\rightarrow \frac{1}{\tau} (1 - 1 - 0 + 1) \\ &\implies J = \ln(\tau/a). \end{aligned} \quad (1302)$$

Therefore

$$I = 4 \ln(\tau a)^2, \quad (1303)$$

and so

$$C(\tau) = \frac{1}{\pi R^2} \ln^2 |\tau a|, \quad (1304)$$

which means that $\langle e^{i\phi(\tau)} e^{-i\phi(0)} \rangle$ decays faster than any power law.

◊ ◊ ◊

Now consider what happens when we add a third dimension, with

$$\mathcal{L} = \frac{R^2}{4\pi} (a^{-2}((\partial_\tau\phi)^2 + (\partial_z\phi)^2) + (\partial_x\partial_y\phi)^2). \quad (1305)$$

Again, consider the correlator at purely timelike separations. We know we will not be getting purely a $\ln|\tau|$ type of thing, since dimensional analysis tells us that the correlator must go like $1/\tau$. To figure out the correlators of $\exp(i\phi)$, we need to know

$$D(\tau) = \frac{2\pi a^2}{R^2} \int_{\mathbf{k},\omega} \frac{e^{i\tau\omega}}{\omega^2 + k_z^2 + (ak_x k_y)^2}. \quad (1306)$$

The integrals work as follows:

$$\begin{aligned} D(\tau) &= \frac{\pi a^2}{R^2} \int_{\mathbf{k}} \frac{e^{-\tau\sqrt{k_z^2 + (ak_x k_y)^2}}}{\sqrt{k_z^2 + (ak_x k_y)^2}} \\ &= \frac{a^2}{2R^2} \int_{k_x, k_y} \int_{a|k_x k_y|}^{\infty} du \frac{e^{-\tau u}}{\sqrt{u^2 - (ak_x k_y)^2}} \\ &= \frac{a^2}{2\pi^2 R^2} \int dk_x dk_y K_0(ak_x k_y \tau) \\ &= \frac{a}{4\pi R^2 \tau} \int_{1/L}^{1/a} d \ln k_x \\ &\sim \frac{a}{4\pi R^2 \tau} \ln(L/a). \end{aligned} \quad (1307)$$

The disturbing part about this is that we have an IR divergent factor which multiplies the τ dependence.

This means that

$$\begin{aligned} \langle e^{i\phi(\tau)} e^{-i\phi(0)} \rangle &\sim \exp(-\langle \phi(a)\phi(0) \rangle + \langle \phi(\tau)\phi(0) \rangle) \\ &\sim (a/L)^{(1-a/\tau)/4\pi R^2} \rightarrow 0 \end{aligned} \quad (1308)$$

Thus it is impossible to order the $e^{i\phi}$ variables. However, derivatives in the x, y directions give us something that is ordered, as

$$\langle e^{ia\partial_x\phi(\tau)} e^{-ia\partial_x\phi(0)} \rangle \sim \exp\left(\frac{a}{8\pi R^2 \tau}\right), \quad (1309)$$

which goes to a constant as $\tau \rightarrow \infty$. They order in the xy plane as well. For example,

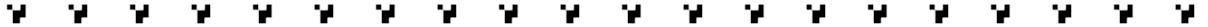
$$\begin{aligned} a^2 \langle \partial_x \phi(x) \partial_x \phi(0) \rangle &\sim a^4 \int dk_x dk_y d\varpi \frac{\varpi k_x^2 e^{ik_x x}}{\varpi^2 + (ak_x k_y)^2} \\ &\sim a \int dk_y d\varpi \frac{|\varpi|^3}{k_y^2} \frac{e^{-|x\varpi/k_y a|}}{|k_y \varpi|} \\ &\sim \frac{a}{|x|}, \end{aligned} \quad (1310)$$

which is the same power law as for the case of timelike separation. One can further check that the above goes as $1/|y|$ when the separation is purely in the y direction as well. Therefore despite the anisotropy of the derivative structure, the ordering has the same falloff in all directions.



Fermi velocity in TBG

Today we're going to be elaborating and doing a tiny generalization for the estimate of the Fermi velocity in TBG first made by BM.



In the original BM paper $w_{AA} = w_{AB}$, but here we will let them be different, defining

$$\alpha \equiv w_{AB}/vk_\theta, \quad \alpha' \equiv w_{AA}/vk_\theta. \quad (1311)$$

As usual, we define the momenta

$$\mathbf{q}_1 = -k_\theta \hat{\mathbf{y}}, \quad \mathbf{q}_2 = R_3(\mathbf{q}_1), \quad \mathbf{q}_3 = R_{-3}(\mathbf{q}_1), \quad (1312)$$

where $R_{\pm 3}(\mathbf{q})$ denotes a rotation of \mathbf{q} by $\pm 2\pi/3$ about $\hat{\mathbf{z}}$. The T matrices are

$$T_1 = k_\theta(\alpha' \mathbf{1} + \alpha X), \quad T_2 = R_3(T_1), \quad T_3 = R_{-3}(T_1). \quad (1313)$$

In what follows it will be helpful to define $\tilde{T}_i = T_i - k_\theta \alpha' \mathbf{1}$.

We proceed by considering the 8-band Hamiltonian at the K point:

$$H_K = \begin{pmatrix} 0 & T_1 & T_2 & T_3 \\ T_1 & \boldsymbol{\sigma} \cdot \mathbf{q}_1 & & \\ T_2 & & \boldsymbol{\sigma} \cdot \mathbf{q}_2 & \\ T_3 & & & \boldsymbol{\sigma} \cdot \mathbf{q}_3 \end{pmatrix} \quad (1314)$$

since for us $T_i^\dagger = T_i$. Following BM, we will define the elements along the diagonal as h_j .

Consider a zero mode of the form $(\psi_0, \psi_1, \psi_2, \psi_3)^T$. We see that we must have

$$T_i \psi_0 = h_i \psi_i, \quad \sum_i T_i \psi_i = 0. \quad (1315)$$

To solve the first equation, we simply take one of the two solutions $\psi_0^1 = (1, 0)^T$ and $\psi_0^2 = (0, 1)^T$, and let $\psi_i = h_i^{-1} T_i \psi_0^a = \frac{1}{k_\theta^2} h_i T_i \psi^a$. We then need to check that the second equation is also satisfied, viz. that

$$\sum_i T_i h_i T_i \psi_0^a = 0. \quad (1316)$$

The $(\alpha')^2$ component of the above holds since $\sum_i h_i = 0$ on account of the h_i being distributed uniformly on the unit circle (in the X, Y plane of S^3). The $\alpha'\alpha$ component holds on account of $\tilde{T}_i h_i = -h_i \tilde{T}_i = Z$, since each \tilde{T}_i is always spaced at an angle of $\pi/2$ from h_i on the unit circle. Finally the α^2 component holds because $h_i \tilde{T}_i = Z$ and $\sum_i \tilde{T}_i = 0$ since the \tilde{T}_i are also uniformly spaced on the unit circle.

Thus there are two zero modes at K (likewise at K'). Since we chose $|\psi_0^i| = 1$, they have norm

$$\begin{aligned} \langle \psi_{zm}^a | \psi_{zm}^a \rangle &= 1 + \sum_i \langle \psi_0^a | (h_i^{-1} T_i)^\dagger h_i^{-1} T_i | \psi_0^a \rangle \\ &= 1 + \frac{1}{k_\theta^2} \sum_i \langle \psi_0^a | T_i^2 | \psi_0^a \rangle \\ &= 1 + 3((\alpha')^2 + \alpha^2) \end{aligned} \quad (1317)$$

If we then properly normalize the zero modes, the Hamiltonian is

$$\begin{aligned} \langle \psi_{zm}^a | H | \psi_{zm}^b \rangle &= \langle \psi_{zm}^a | \boldsymbol{\sigma} \cdot \mathbf{k} \cdot \mathbf{1}_{4 \times 4} | \psi_{zm}^b \rangle \\ &= \frac{1}{1 + 3((\alpha')^2 + \alpha^2)} \langle \psi_0^a | \boldsymbol{\sigma} \cdot \mathbf{k} + k_\theta^{-2} \sum_i T_i h_i \boldsymbol{\sigma} \cdot \mathbf{k} h_i T_i | \psi_0^b \rangle. \end{aligned} \quad (1318)$$

Since as we said above $\tilde{T}_i h_i = Z$, the second term gives a contribution of $-3\alpha^2 \boldsymbol{\sigma} \cdot \mathbf{k}$. Furthermore $\sum_i h_i \boldsymbol{\sigma} \cdot \mathbf{k} h_i = 0$, since the sum of a given vector reflected about three axes evenly spaced along the unit circle vanishes. Finally, the $\sum_i h_i \boldsymbol{\sigma} \cdot \mathbf{k} h_i \tilde{T}_i$ term also vanishes on account of $h_i \tilde{T}_i = Z$ and $\sum_i h_i = 0$. Therefore

$$\langle \psi_{zm}^a | H | \psi_{zm}^b \rangle = \frac{1 - 3\alpha^2}{1 + 3((\alpha')^2 + \alpha^2)} \langle \psi_0^a | \boldsymbol{\sigma} \cdot \mathbf{k} | \psi_0^b \rangle, \quad (1319)$$

and in particular,

$$v_F(\alpha, \alpha') = \frac{1 - 3\alpha^2}{1 + 3((\alpha')^2 + \alpha^2)}. \quad (1320)$$

In the $\alpha' = 0, \alpha$ limits this recovers the expressions in BM and Tarnopolsky+.



Force on a vortex in a 2d SF

Today's entry is the derivation of a well-known fact that seemed reasonable but which I hadn't seen derived explicitly: the force on a vortex in a SF in the presence of a supercurrent (and other vortices).



We start from the action of a SF in 2+1d coupled to a current I . We separate $\phi = \phi_v + \phi_s$ into a vortex and smooth part, replace $d\phi$ with a general 1-form field X , and then enforce $X = d\phi$ with a gauge field a . This gives (deliberately not being too careful about signs and numerical constants)

$$\begin{aligned}\mathcal{L} &= \frac{R^2}{4\pi}|X + d\phi_v|^2 + \frac{1}{2\pi}X \wedge da + (X + d\phi_v) \wedge \star I \\ &\rightarrow \frac{R^2}{4\pi}|X|^2 + X \wedge (da/2\pi + \star I) + \frac{1}{2\pi}a \wedge d^2\phi_v.\end{aligned}\tag{1321}$$

Integrating out X ,

$$\mathcal{L} = \frac{1}{2g^2}f_a^2 + a \wedge (\star J_v - dI/R^2)\tag{1322}$$

where $g^2 = 2\pi R^2$ and $J_v = \star d^2\phi_v$. We therefore get (ignoring the term independent of J_v)

$$-\ln Z = \int_{x,y} \left((\star dI)_\mu(x) \frac{1}{|x-y|} J_v^\mu(y) + \pi R^2 J_v^\mu(x) \frac{1}{|x-y|} J_v^\mu(x) \right). \tag{1323}$$

Since we are interested in 2d SFs at finite temperature, we now set all fields to be static in time. We can then integrate over time in the G_a propagator to get the usual logarithmic interaction. This gives a free energy of

$$F = \int_{\mathbf{x},\mathbf{y}} \ln |\mathbf{x} - \mathbf{y}| ((\star dI)_\mu(\mathbf{x}) J_v^\mu(\mathbf{y}) + \pi R^2 J_v^\mu(\mathbf{x}) J_v^\mu(\mathbf{y})). \tag{1324}$$

We now specialize to a situation where

$$J_v^\mu(\mathbf{x}) = \delta^{\mu,0} \sum_\alpha q_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha). \tag{1325}$$

Neglecting the self-interaction,

$$F = \sum_{\alpha<\beta} 2\pi R^2 q_\alpha q_\beta \ln |\mathbf{x}_\alpha - \mathbf{x}_\beta| + \sum_\alpha q_\alpha \int_{\mathbf{x}} (\star dI)_0(\mathbf{x}) \ln |\mathbf{x} - \mathbf{x}_\alpha|. \tag{1326}$$

To deal with the last term, we use $d^\dagger I = 0$ and $\partial_\tau I = 0$ to write $I^i = \varepsilon^{ij} \partial_j \zeta$. Then

$$(\star dI)_0 = \nabla^2 \zeta. \tag{1327}$$

The ∇^2 can be transferred onto the log to produce a delta function; hence

$$F = \sum_{\alpha<\beta} 2\pi R^2 q_\alpha q_\beta \ln |\mathbf{x}_\alpha - \mathbf{x}_\beta| + \sum_\alpha q_\alpha \int_{\mathbf{x}} \zeta(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_\alpha). \tag{1328}$$

To get the force on a vortex, we differentiate the above free energy with respect to \mathbf{x}_α . The derivative with respect to \mathbf{x}_α can be replaced by one with respect to \mathbf{x} , which can then be transferred onto ς . We then use $\partial_i \varsigma = \varepsilon^{ij} I_j$ to write the force on the vortex α as

$$\mathbf{F}_\alpha = \pi R^2 \sum_{\beta \neq \alpha} q_\alpha q_\beta \frac{\mathbf{x}_\alpha}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^2} + q_\alpha (\mathbf{I} \times \hat{\mathbf{z}}). \quad (1329)$$

We therefore see that the force on a vortex is obtained from a sum of Coulomb forces from other vortices, together with a Lorentz force coming from the superflow.



PV duality for 2d SCs

Today's we will show why a SC in 2+1D can be described by a free $U(1)$ gauge field and a CS term. This is incredibly trivial but still perhaps worth doing.



The statement is that the EFT for a 2d SC can be written down as

$$\mathcal{L} = \frac{1}{2g^2} da \wedge \star da + \frac{k}{2\pi} a \wedge dA, \quad (1330)$$

where A is the background EM field. Of course, this is extremely trivial since running duality on a gives us the usual GL action. Nevertheless it's neat to see how things work out: electric charges for a are evidently magnetic flux tubes in A , and flux tubes for a are electric charges for A .

Indeed, working in Feynman gauge, we integrate out a by doing

$$a \mapsto a - \frac{kg^2}{2\pi} \frac{1}{\square} \star dA, \quad (1331)$$

so that, being cavalier about signs,

$$\begin{aligned} \mathcal{L}_{\text{eff}}[A] &= \frac{k^2 g^2}{4\pi^2} F_A \wedge \star \square^{-1} F_A \\ &= \frac{k^2 g^2}{4\pi^2} A \wedge \star \frac{d^\dagger d}{\square} A \\ &= \frac{k^2 g^2}{4\pi^2} A \wedge \star \Pi_T A, \end{aligned} \quad (1332)$$

so that the transverse component A_T becomes massive — this is the Meissner effect. Evidently we should set $k = 2$ (duh) and take $g^2/(2\pi^2)$ to be equal to the SCing stiffness. Note that the mutual CS term doesn't break \mathcal{T} , since \mathcal{T} is to a what \mathcal{CT} is to A (and vice versa), as usual in PV-type dualities.

As a bonus, let's ask what happens when we add a CS term for a :

$$\mathcal{L} \supset \frac{q}{4\pi} a \wedge da. \quad (1333)$$

It's obvious that this should kill the Meissner effect for A , since quantum Hall systems are not superconductors. However, the propagator for a still has a pole at zero momentum, so perhaps it deserves a check.

The propagator of a was worked out in an earlier diary entry, and in Lorentz gauge is

$$G^{\mu\nu} = \frac{2g^2}{k^2 + m^2} \left(\Pi_T^{\mu\nu} + m \frac{\varepsilon^{\mu\nu\lambda} k_\lambda}{k^2} \right), \quad (1334)$$

where the mass of a is

$$m \equiv \frac{qg^2}{2\pi}. \quad (1335)$$

If we now integrate out a , we get

$$\mathcal{L}_{\text{eff}}[A] = \frac{g^2}{\square + m^2} \left(|F_A|^2 + \frac{m}{\square} F_A \wedge d^\dagger F_A \right), \quad (1336)$$

where the second term is kinda like a CS term for the vector field $\star F_A$.

If we just had the first term, we would obviously not have a superconductor, due to the m^2 in the denominator. For the second term, we can be careful and evaluate the superconducting weight through

$$D_s \sim \lim_{k_y \rightarrow 0} \frac{\delta^2 \mathcal{L}_{\text{eff}}[A]}{\delta A_x \delta A_x} \Big|_{A_0=A_y=0; q_x=\omega=0}. \quad (1337)$$

If we do this, we find that $D_s = 0$ as we could have guessed: the CS term kills the Meissner effect.

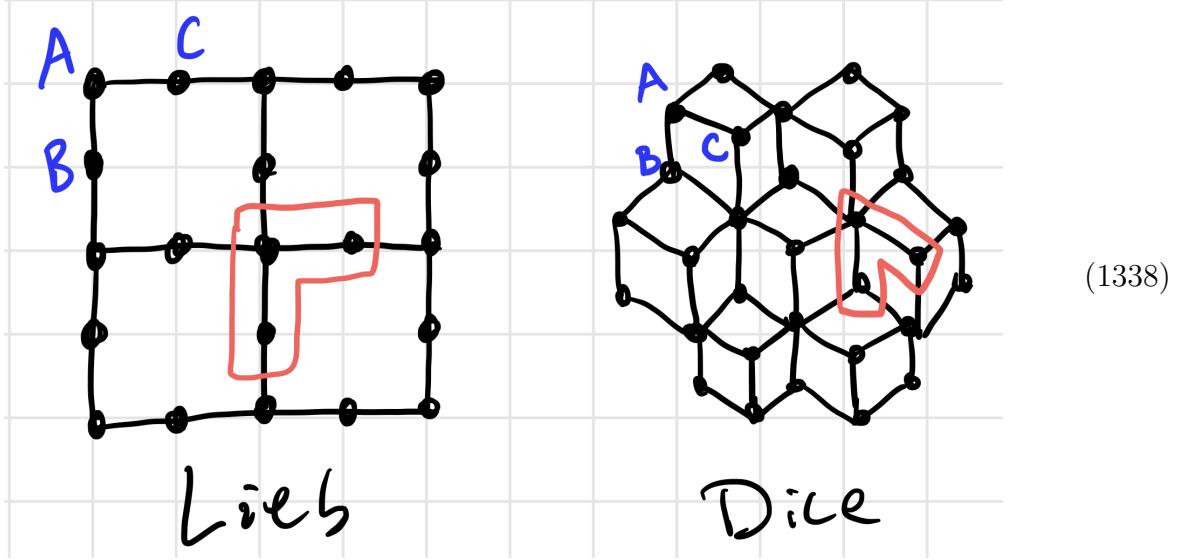


Dice and Lieb lattices

Today we are doing a trivial calculation that I heard about during a talk and wanted to remember: showing that the tight binding models on the Lieb and Dice lattices have bands which are exactly flat.



The Lieb and Dice lattices look like



where the unit cells are drawn in red. The important part about these lattices is that the B and C sites only connect to the A sites of other unit cells. This means that the tight-binding model on e.g. the Lieb lattice looks like (in the basis (A, B, C))

$$H = \begin{pmatrix} \cos(k_y) & \cos(k_x) \\ \cos(k_x) & \cos(k_y) \end{pmatrix}. \quad (1339)$$

We see then that the spectrum is

$$\varepsilon_{fb} = 0, \quad \varepsilon_{\pm} = \pm \sqrt{\cos^2(k_x) + \cos^2(k_y)}, \quad (1340)$$

which gives a nice flat band and Dirac cones. The dice lattice is basically the same.

Apparently the existence of a flat band is a consequence of Lieb's theorem, viz. that a flat band appears for a tight-binding model on any bipartite lattice after removing a single site per unit cell (for the dice lattice, consider adding new sites at the centers of all identically-oriented diamond faces: the resulting lattice is bipartite with unit cells consisting of 4-site long chains stretching along a direction fixed by the choice of diamond face).



suggested in “simple dmrg” by Garrison + Mishmash, and reproducing some of the results in White’s OG paper [?]. For educational purposes this will be done without recourse to TeNPy / iTensor.



The code used for this diary entry was rather primitive, and did not make use of the MPS representation. Recall how the (infinite-size) algorithm works in this case: after fixing a given maximal block dimension χ and initializing by constructing the Hamiltonian for a small chain size,

1. We enlarge the system by adding on a single site, and coupling the single site to the existing system using the nearest-neighbor term of the Hamiltonian.
2. Since we want to compute a wavefunction whose reduced density matrices look like those of the actual infinite-system ground state, we realize the system as a subsystem of a larger chain by constructing a larger Hamiltonian given by stacking a reflected copy of the system’s Hamiltonian on to itself.
3. The exact ground state of this larger composite system is found, and the reduced density matrix ρ_S for the system (viewed as a subsystem of the doubled chain) is computed.
4. We perform an SVD on ρ_S , and form the projector Π_χ onto the subspace spanned by the eigenvectors of ρ with the χ largest eigenvalues.
5. We then project the Hamiltonian of the system into the image of Π_χ , and restrict our attention to this subspace in subsequent steps.
6. We then start over from step 1 until convergence (in e.g. the energy density) is achieved.

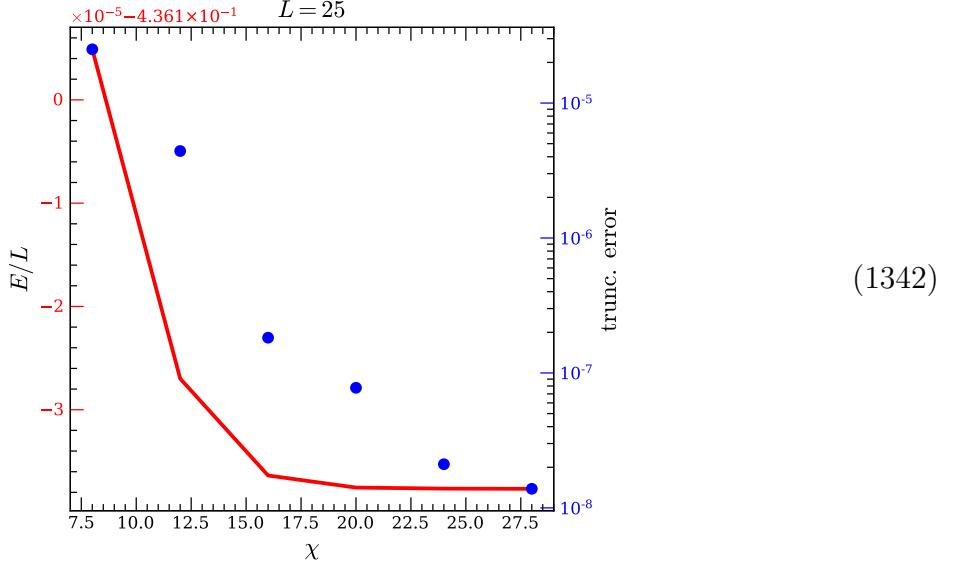
The fact that step 4 is the correct thing to do can be argued as follows. Suppose we have a wavefunction $|\Psi\rangle$ defined on $\mathcal{H}_A \otimes \mathcal{H}_B$, and we want to compress it into a wavefunction defined on $\mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_B$, with $\dim \mathcal{H}_{\tilde{A}} < \dim \mathcal{H}_A$. We can do this compression by acting with a Hermitian projector \mathcal{P} , so that the new wavefunction is $|\tilde{\Psi}\rangle = \mathcal{P}|\Psi\rangle/|\mathcal{P}|\Psi\rangle|$. Then the overlap with the original wavefunction is

$$|\langle \tilde{\Psi} | \Psi \rangle|^2 = \frac{|\text{Tr}[\mathcal{P}\rho_A]|^2}{\text{Tr}[\mathcal{P}\mathcal{P}^\dagger\rho_A]}. \quad (1341)$$

Since the denominator is just $\text{Tr}[\mathcal{P}\rho_A]$ on account of \mathcal{P} being a Hermitian projector, we see that the overlap is maximized provided that we choose \mathcal{P} to project onto the eigensubspace of ρ_A with the $\dim \mathcal{H}_{\tilde{A}}$ largest eigenvalues, as claimed.

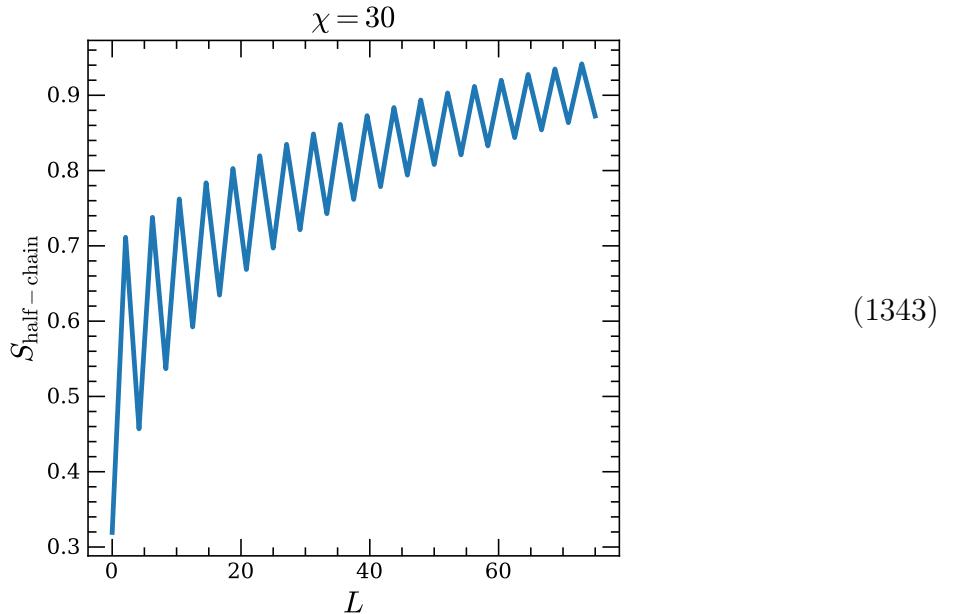
AF Heisenberg chain

Now we show numerical results for the spin 1/2 AF chain. Before calculating anything else, we can check for convergence by plotting the energy density and truncation error for various bond dimensions χ :



For this choice of system size, the energy converges to approximately the right answer when $\chi \gtrsim 20$, at which the truncation error is an acceptable 10^{-7} .

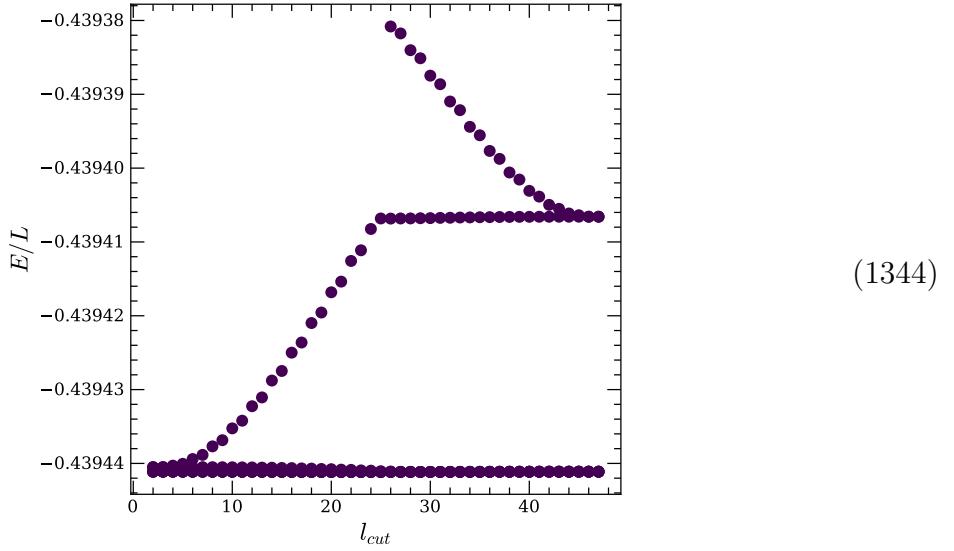
The half-chain entanglement entropy as a function of L is trivial to calculate, since the SVD of the half-chain reduced density matrix is performed as part of the algorithm. We find



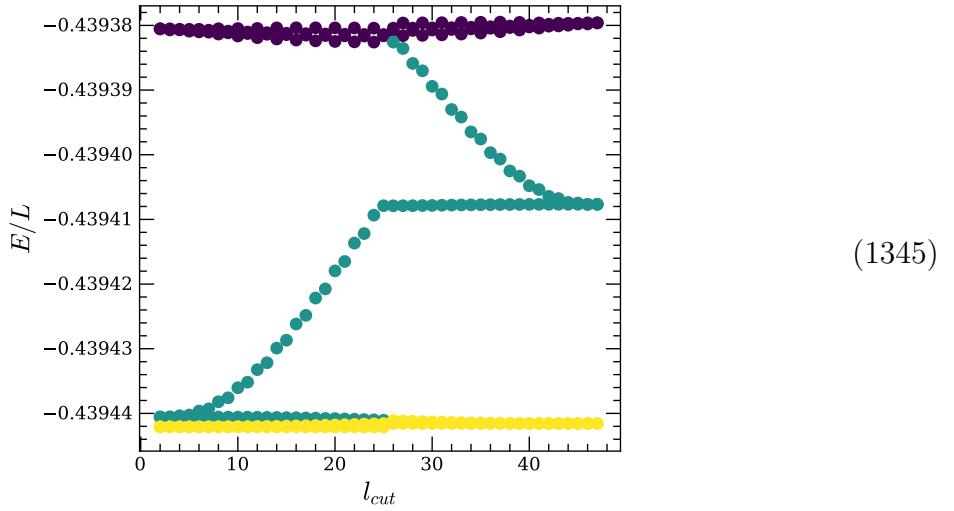
Why the sawtooth feature? This is a finite-size effect coming from the fact that the ground state for small system sizes is approximately a VBS. The edges of the chain like to

form singlets with their neighbors, and if we then let every site form a singlet with one of its nearest neighbors, then a) for $L \in 4\mathbb{Z}$ a bipartition does not cut through any singlets while b) for $L \in 4\mathbb{Z} + 2$ it cuts through a single singlet (L is always even by virtue of the doubling inherent in the infinite-size algorithm). Of course the ground state is not literally this VBS, and the pinning only occurs due to the boundary spins. Therefore S is non-vanishing even when $L \in 4\mathbb{Z}$, and as L gets larger (so that the boundaries are further away from the cut) the sensitivity to $L \bmod 4$ dies off (pretty slowly, though).

We now want to evaluate the central charge of this system using the scaling of S with cut size (for a fixed system length). Since SVDs of the reduced density matrices at each cut size are computed during the course of the finite-size DMRG algorithm, we will switch over to using that algorithm. To know how many sweeps we should do, we can plot E/L as a function of DMRG timestep. For example, taking $L = 50$ and $\chi = 20$, we have



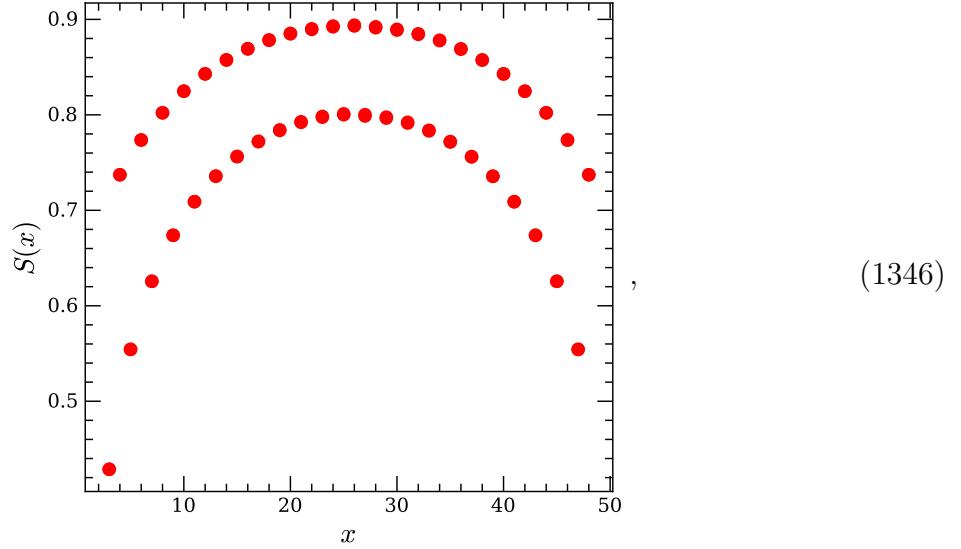
where we have done three sweeps in total. To then establish whether or not the bond dimension we chose is sufficient, we can vary χ at each sweep: letting χ increase from 10 to 30,



where the different colors denote sweeps with different χ . The first value ($\chi = 10$) is obviously

not good enough, but $\chi = 30$ does no better than $\chi = 20$ in the present case.

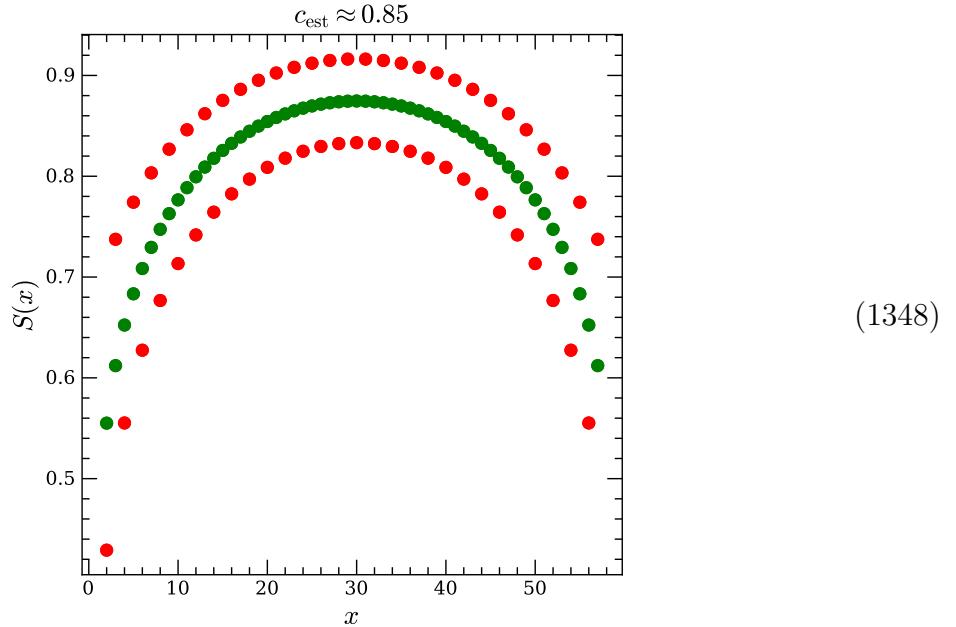
Letting x denote the position of the cut, we get (for a $L = 50$ site chain with maximal bond dimension $\chi = 20$)



with the oscillating behavior occurring for the same reason as discussed above. To extract the central charge, we use

$$S(x) = \frac{c}{6} \ln \left(\frac{L}{\pi} \sin \left(\frac{\pi x}{L} \right) \right) + C, \quad (1347)$$

where C is a constant. Using this transformation, we fit $S(x)$ and find



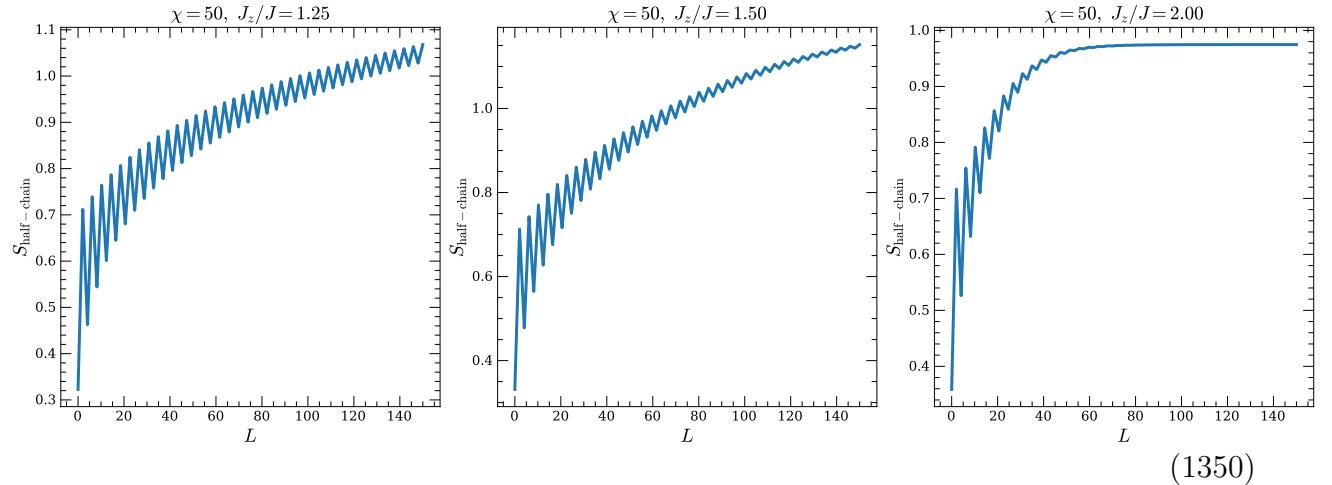
where the green dots are fit to the above formula. The correct answer is $c = 1$, but given the rather strong finite-size effects I suppose this is acceptable for now.

XXZ chain

We will now turn on a $SU(2) \rightsquigarrow U(1)$ anisotropy, so that the Hamiltonian is

$$H = \sum_i \left(\frac{J}{2} (S_i^+ S_{i+1}^- + h.c.) + J_z Z_i Z_{i+1} \right). \quad (1349)$$

Recall that the sign of J is irrelevant (we can multiply both X and Y by -1 on a given sublattice without changing the $\mathfrak{su}(2)$ algebra). If $|J_z/J| > 1$ then we have added a chemical potential to the Heisenberg chain in a way which makes it gapped (with the ground state being an Ising FM or AFM, depending on the sign of J_z). The easiest way to check this in the DMRG algorithm is again to just compute the entanglement entropy. Using the infinite-system algorithm, the half-chain entanglement entropy is



I'm slightly surprised that the gap is barely visible when $J_z/J = 5/4$, but at least the $J_z/J = 2$ plot is obviously gapped by eye.

Correlation functions and excited states

We will now calculate the spin gap, viz. the gap between the ground state and the lowest energy state of the $S^z = 1$ sector (returning to $J_z/J = 1$). This is facilitated by the fact for a theory with a conserved $U(1)$ charge generated by Q , DMRG easily finds the lowest energy states in any given charge sector, as ρ_A is block-diagonal in a basis which diagonalizes $Q|_A$.¹¹² Theoretically there is nothing about the DMRG algorithm that restricts to the lowest energy

¹¹²We will prove that the reduced density matrix ρ_A is always block-diagonal provided that the ground state wavefunction $|\psi\rangle$ on the whole system $A \cup B$ has a definite charge. To see this rather obvious fact formally, we can note that if $|\psi\rangle$ has eigenvalue z under Q , then

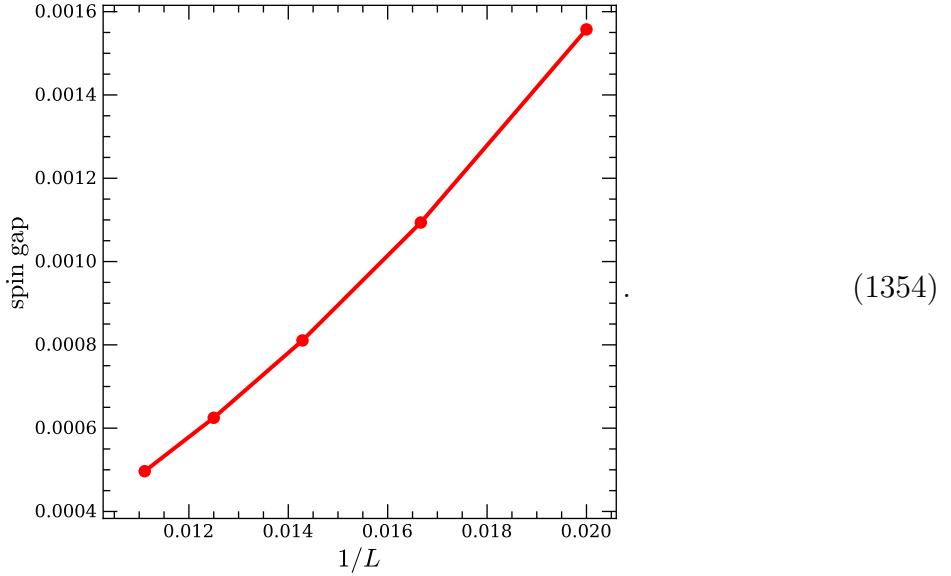
$$|\psi\rangle = \sum_{l \in \mathbb{Z}} \mathcal{P}_l^A \otimes \mathcal{P}_{z-l}^B |\psi\rangle, \quad (1351)$$

where \mathcal{P}_l^A projects onto the subspace with $Q|_A = z$ (with the restriction $Q|_A$ well-defined if the symmetry is non-anomalous). By acting on both sides with $\mathbf{1} = \sum_k \mathbf{1}_A \otimes \mathcal{P}_k^B$, we see that

$$\mathcal{P}_k^A \otimes \mathbf{1}_B |\psi\rangle = \mathcal{P}_k^A \otimes \mathcal{P}_{z-k}^B |\psi\rangle. \quad (1352)$$

states within a given symmetry sector, but in practice excited states (in a given symmetry sector) are harder to obtain since the routine we use when diagonalizing the composite system Hamiltonians is usually an eigensolver specialized for sparse Hamiltonians, which is usually only good at finding the ground state.

For the $J_z/J = 1$ AF chain, we find (doing three sweeps with $\chi = 30$)



This is not as close to $\Delta \propto 1/L$ as we would like, but I believe this is due to our smallish value of χ and the fact that we only did three sweeps (there are also $1/(L \ln L)$ corrections, but I think these should make the spin gap look curved in the opposite way).

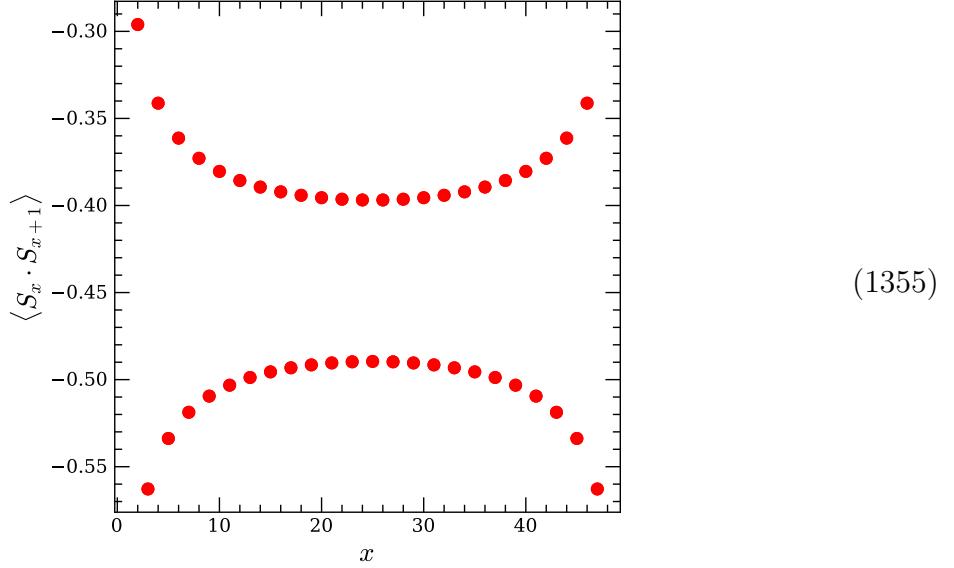
The easiest correlation functions to calculate are $\langle S_x \cdot S_{x+1} \rangle$, since an explicit expression for $S_x \cdot S_{x+1}$ is obtained during the course of the DMRG algorithm anyway (when the link between x and $x + 1$ is the location where the system is cut in two — correlators which straddle the cut are however less accurate). Evaluating the correlator on the last sweep of a

Then

$$\mathcal{P}_k^A \rho_A \mathcal{P}_l^A = \mathcal{P}_k^A \text{Tr}_B [(\mathbf{1}_A \otimes \mathcal{P}_{z-k}^B) |\psi\rangle\langle\psi| (\mathbf{1}_A \otimes \mathcal{P}_{z-l}^B)] \mathcal{P}_l^A \propto \delta_{k,l}, \quad (1353)$$

so that ρ_A is block-diagonal as claimed.

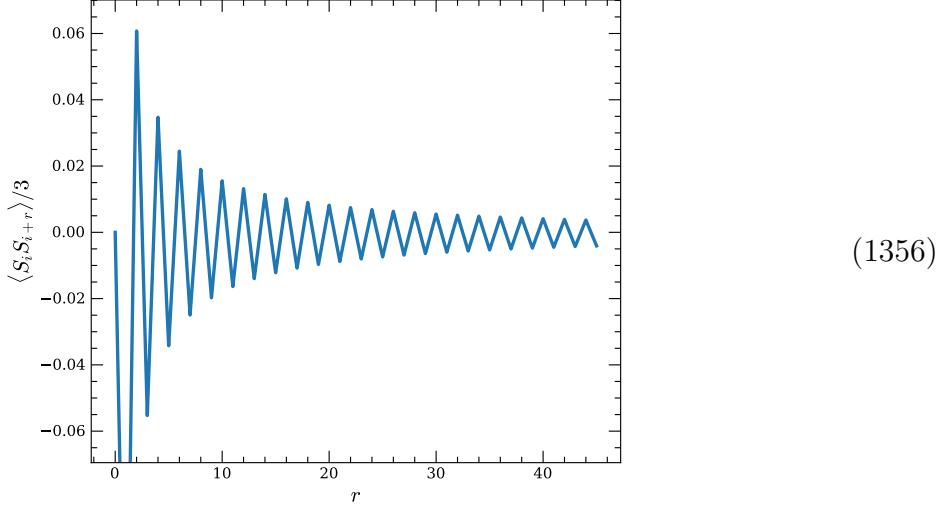
finite-size system with $L = 50$, we find



The oscillating behavior is of course due to the formation of singlets, as discussed above (with the fact that the oscillations get stronger near the system edges being due to the pinning of singlets to the edge sites).

Now we will calculate $C(r) = \langle S_x \cdot S_{x+r} \rangle$, which necessitates keeping track of how all of the operators $S_x \cdot S_{x+r}$ evolve under the basis change and truncation performed at each DMRG step. The easiest way to do this is with the infinite system size algorithm, which we will use to calculate $\langle S_{x_c}^z S_{x_c+r}^z \rangle$, where x_c is the center of the chain. To do this, we evolve the system until it reaches a length L_{warm} . The length L_{warm} will set how close to the boundary we measure correlation functions, with the maximum value of r being set by $L - L_{\text{warm}}$ (with L the final system size). During this warmup phase, we do not keep track of the evolution of the representation of any operators other than the Hamiltonian itself. Once the system size reaches L_{warm} , we begin keeping track of the representations of $S_{L_{\text{warm}}}^z$ and $S_{L_{\text{warm}}}^z S_{L_{\text{warm}}+r}^z$, for all r (note that it is important to store the representations of the products, rather than just of the $S_{L_{\text{warm}}+r}^z$ themselves). When the system size reaches L , we then measure all of the operators $S_{L_{\text{warm}}}^z S_{L_{\text{warm}}+r}^z$ in the ground state of the final Hamiltonian. Doing this for

the AF Heisenberg chain produces



which is well fit by the expected $\sim (-1)^r/r$ of the QLRO'd AF.



Practice with TeNPy

In today's diary entry we will use TeNPy to make a few plots of various quantities obtainable from DMRG in a handful of simple 1d models. This is meant simply as a way of getting familiar with how to use TeNPy.

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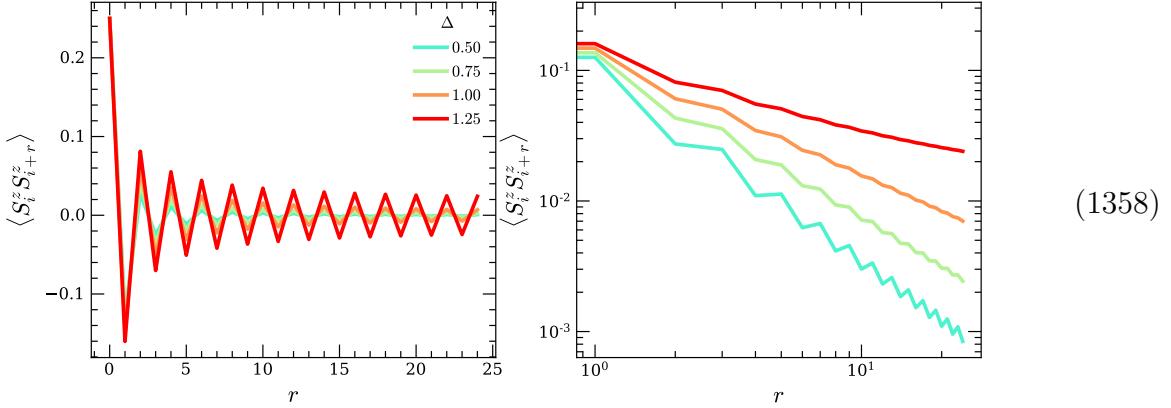
XXZ Chain

As a simple first example we look at the XXZ chain, with Hamiltonian

$$H = \sum_i \left(J(\sigma_i^+ \sigma_{i+1}^- + h.c.) + \frac{\Delta}{2} \sigma_i^z \sigma_{i+1}^z \right) \quad (1357)$$

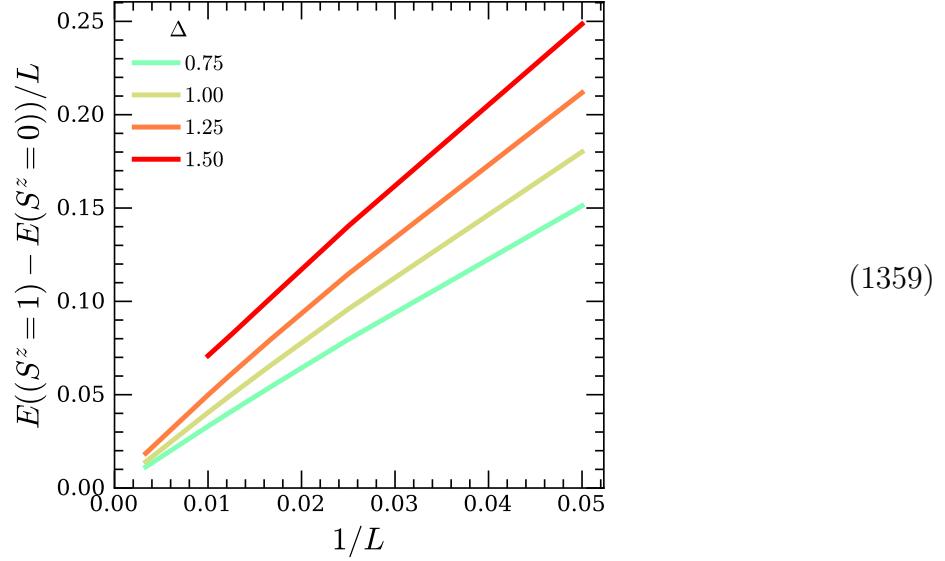
As a first check that we are using tenpy correctly, we can plot the ZZ correlator for the MPS obtained by infinite DMRG. Letting $J = 1$ so that the ground states have AF

correlations, we have



where the plot on the right is really of $\langle |S_i^z S_{i+r}^z| \rangle$. As expected, for $\Delta = 1$ we get a correlation function going as $(-1)^r/r$, while the exponent of the power law envelope is modified for $\Delta < 1$. For $\Delta > 1$ we know that the ground state is an Ising AFM and as such is gapped; this is evident from the non-powerlaw behavior of the red curve in the right panel.

As a further check we can compute the spin gap, viz. the energy difference between the lowest-energy states in the $S_{tot}^z = 1$ and $S_{tot}^z = 0$ sectors. For various values of Δ , we find



where the breakoff of the $\Delta = 1.5$ curve is due to me getting impatient with the simulations. From this plot, it is clear that a) the gap vanishes in the $L \rightarrow \infty$ limit for $\Delta \leq 1$ but is nonzero for $\Delta = 1.5$, while b) the power law in the scaling of the gap with $1/L^\eta$ varies when $\Delta \leq 1$. These facts are of course exactly in accordance with what we know from e.g. bosonization.

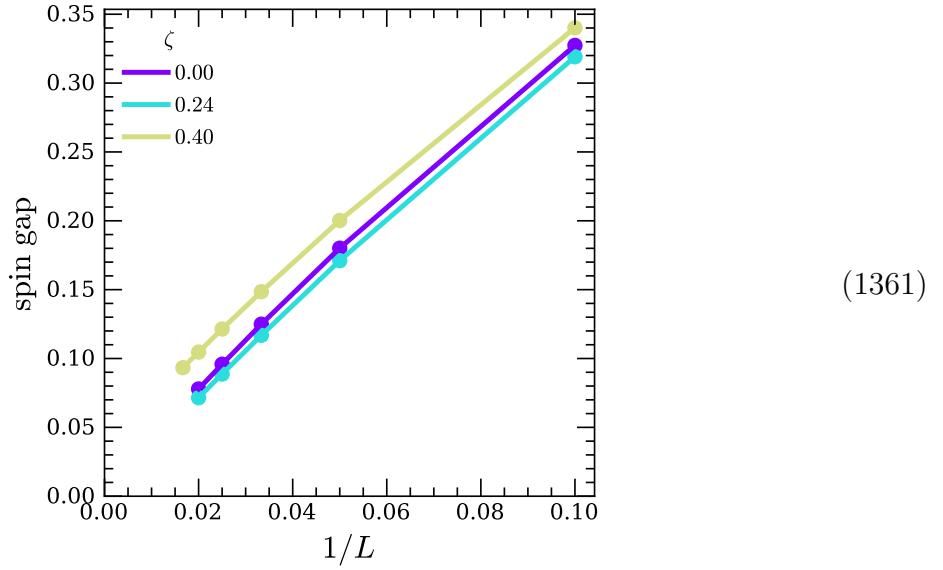
$J_1 - J_2$ model

We now consider a model defined by adding a nnn Heisenberg exchange to the conventional Heisenberg AF:

$$H = \sum_i (J_1 S_i \cdot S_{i+1} + J_2 S_i \cdot S_{i+2}). \quad (1360)$$

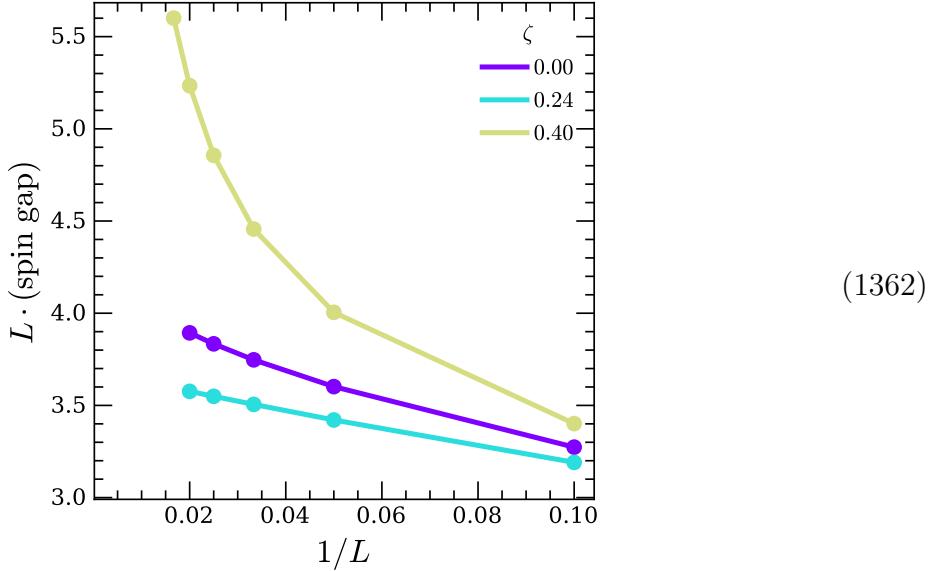
When $\zeta \equiv J_2/J_1 = 1/2$ this is the Majumdar-Ghosh model, whose ground state is a VBS. Recall that this follows from writing $H \propto \sum_i (S_{i-1} + S_i + S_{i+1})^2$: thus H is the sum of quadratic Casimirs for the $(1/2)^{\otimes 3}$ representations formed by the spins on each 3-site cluster, and is therefore minimized by projecting each 3-site cluster onto the spin 1/2 subspace. The critical point separating the VBS phase from the conventional Heisenberg AF is apparently known to occur at $\zeta_* \approx 0.241$. For $\zeta < \zeta_*$ the model is gapless and has $\ln L$ corrections to scaling, when $\zeta > \zeta_*$ the model is gapped, and the critical point at $\zeta = \zeta_*$ is a KT transition where the logarithmic corrections vanish (see e.g. Alexander Chan's masters thesis for references).

We will now check that these facts are born out numerically. Plotting the spin gap (defined above) for a few different values of ζ , we have



Here the $\zeta = 0.4 > \zeta_*$ curve looks gapped, but it is hard to tell exactly. To check this and to more easily see the effects of the logarithmic corrections, we can remake the plot with the

energy gap scaled by L :



Here the $\zeta = 0.4$ curve definitely looks gapped, while the $\zeta \approx \zeta_*$ curve is indeed the closest we get to something which looks independent of $1/L$ (the $\ln L$ corrections manifest themselves as the upturn at large L).

Spin-1 Heisenberg chain

Now we consider the spin-1 Heisenberg chain, building the model ourselves rather than just using the `SpinChainNNN` class provided by TeNPy (which we used in the previous section). The Hamiltonian is

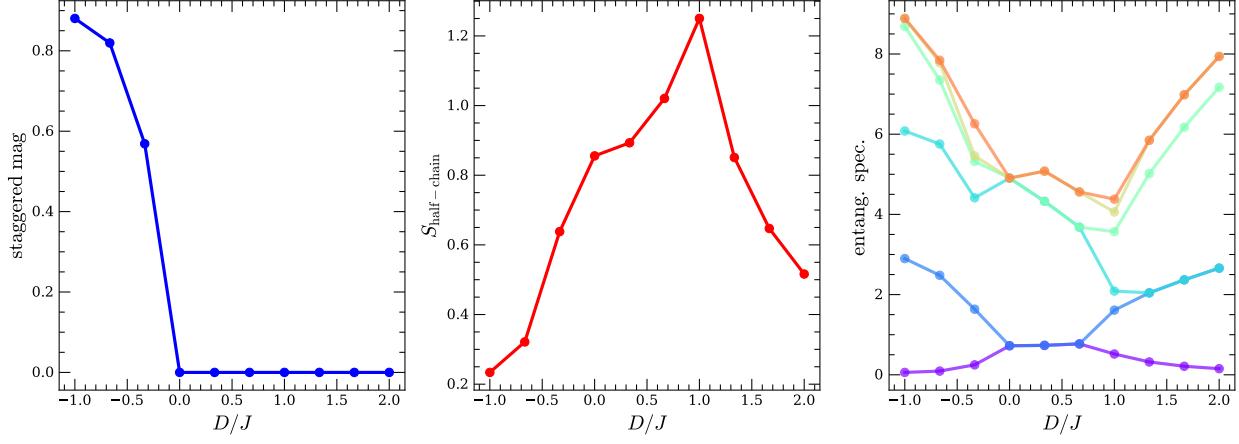
$$H = \sum_i (JS_i \cdot S_{i+1} + D(S_i^z)^2 + \lambda(S_i \cdot S_{i+1})^2). \quad (1363)$$

A trivial product state $\otimes|0\rangle$ occurs for $D \rightarrow \infty$, while when $D = 0$ we are in the Haldane phase (with the AKLT model corresponding to $\lambda = J/3$).

To study the phase diagram of this model we will compute the staggered magnetization (to detect SSB in the AF phase), the half-chain entanglement entropy (whose divergence allows us to locate phase transitions) and the entanglement spectrum (to distinguish between different gapped ground states).

Since we are just interested in performing a sanity check, we will not map out the whole phase diagram in detail, and will just consider the limits in which one of D, λ vanishes.

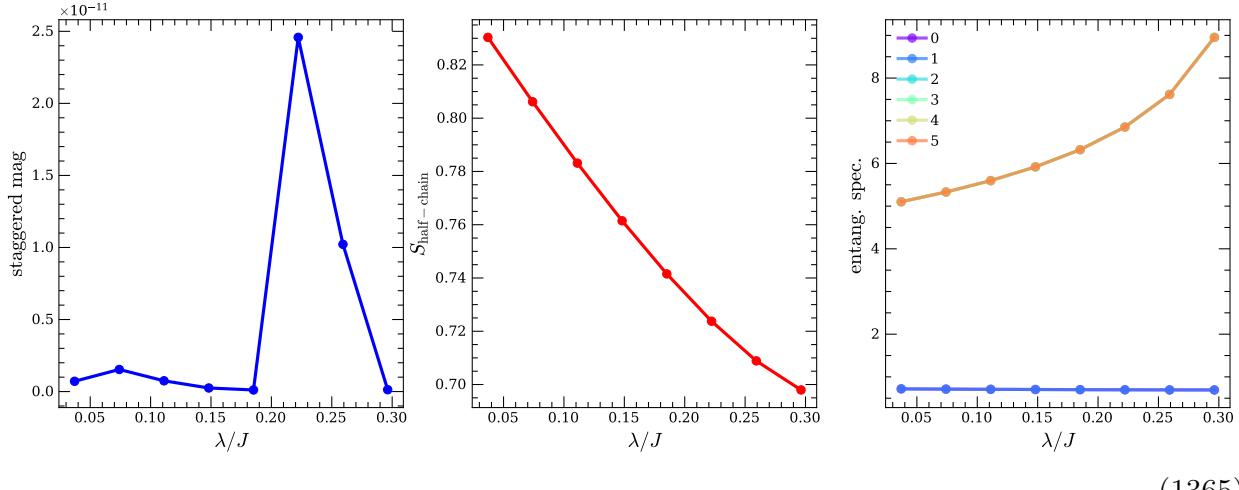
When $\lambda = 0$, we find



(1364)

While we have not used a very high resolution, we can clearly see a phase transition from an AF phase at negative D to one with vanishing staggered magnetization at $D = 0$. From the half-chain entanglement entropy we also see that a phase transition occurs at $D/J \lesssim 1$. By looking at the degeneracy of the entanglement spectrum (only showing the smallest 6 eigenvalues), we see that the phases at intermediate and large D/J are distinguished by the intermediate phase having a doubly degenerate spectrum. This is in keeping with what we know about the Haldane phase (intermediate D/J) and the trivial $\otimes|0\rangle$ paramagnet (large D/J).

Now we take $D = 0$ and vary λ . This apparently gives



(1365)

Note the scale on the left plot: the staggered magnetization is just zero throughout, as expected. The point of making this plot is to demonstrate that the model at $\lambda = 0, D = 0$ (the Haldane-gapped spin-1 AF) is smoothly connected to the model at $\lambda = J/3, D = 0$ (the AKLT model where the ground state is known and a simple MPS representation thereof exists).



Effective dipole hopping in a tilted potential via Schrieffer-Wolff

Today's diary entry contains a short calculation of how one obtains an effective dipole-conserving Hamiltonian from a Bose Hubbard model in a strongly tilted potential. Thanks for Ehud Altman for pointing out that the effective dipole hopping term one generates is nonzero only in the presence of interactions.

* * * * *

Consider the Hamiltonian

$$H = \sum_i \left(-t(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) - \mu n_i + \frac{U}{2} n_i(n_i - 1) + V i n_i \right) \equiv H_t + H_U + H_V, \quad (1366)$$

where H_U includes the chemical potential. Our goal is to perform a rotation into a basis in which the Hamiltonian commutes with the linear potential term $V \sum_i i n_i$ up to some fixed order in $t/V, U/V$. The usual way of rotating the Hamiltonian is through an SW transformation, wherein H becomes

$$e^\Lambda H e^{-\Lambda} = \sum_{k \geq 0} \frac{1}{k!} \text{Ad}_\Lambda^k(H), \quad (1367)$$

where $\text{Ad}_\Lambda(\cdot) = [\Lambda, \cdot]$ and Λ is anti-Hermitian, with Λ chosen so as to eliminate the offending single-particle hopping term $-t(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i)$ at lowest order.

Note that it is already clear that interactions are required for producing a nonzero dipolar hopping term. Indeed, without the interaction term, H is built solely of 2-body terms — we can thus choose Λ to be a 2-body operator, and $\text{Ad}_\Lambda^k(H)$ will consequently always itself be built from 2-body operators, which can only either be purely onsite or dipole non-conserving. In fact if we just take

$$\Lambda = \Lambda_t \equiv \frac{t}{V} \sum_i (b_i^\dagger b_{i+1} - b_{i+1}^\dagger b_i), \quad (1368)$$

it is easy to check that when $U = 0$,

$$[\Lambda_t, H] = [\Lambda_t, H_V] = t \sum_i (b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) = -H_t. \quad (1369)$$

Since this is just the negative of the hopping term, the first order part $\text{Ad}_\Lambda(H_V)$ dutifully kills H_t . Moreover, since $[\Lambda_t, [\Lambda_t, H_t + H_V]] = 0$, the effective Hamiltonian stops at linear order, and we simply obtain $H_{eff} = e^{\Lambda_t}(H_t + H_V)e^{-\Lambda_t} = H_V$, which is purely onsite. This

means that when $U = 0$, no effective dipole hopping terms are generated — there is perfect destructive interference between all putative hopping processes, and no such processes are generated to all orders in perturbation theory.

Let us then bring back a nonzero U . We take

$$\Lambda = \sum_{n=1}^{\infty} \Lambda_n, \quad (1370)$$

where Λ_n is order n in $t/V, U/V$, and we set $\Lambda_1 = \Lambda_t$. We fix the second order term Λ_2 by requiring that it cancel the terms generated when commuting $\Lambda_1 = \Lambda_t$ against H_U . Specifically, we require

$$[\Lambda_2, H_V] = -[\Lambda_t, H_t + H_U]. \quad (1371)$$

Keeping terms to third order in this expansion, we find that H_{eff} becomes

$$H_{eff} = H_V + H_U + [\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2 - \Lambda_t, H_t] + [\Lambda_3, H_V]. \quad (1372)$$

We then need the commutators

$$\begin{aligned} [\Lambda_t, H_t] &= \frac{t^2}{V} \sum_i \left(b_{i+2}^\dagger b_i + 2n_{i+1} + b_i^\dagger b_{i+2} - (i \rightarrow i+1) \right) = 0, \\ [\Lambda_t, H_U] &= \frac{tU}{V} \sum_i \{ b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i, n_{i+1} - n_i \} \end{aligned} \quad (1373)$$

which together determines Λ_2 as (it is easy to guess this using the facts that the terms in Λ_2 must have support on two adjacent sites, consist only of 4-body operators, and must not commute with H_V)

$$\Lambda_2 = -\frac{tU}{V^2} \sum_i \{ b_i^\dagger b_{i+1} - b_{i+1}^\dagger b_i, n_{i+1} - n_i \}. \quad (1374)$$

The effective Hamiltonian to cubic order is then

$$H_{eff} = H_V + H_U + [\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2, H_t] + [\Lambda_3, H_V]. \quad (1375)$$

The commutators $[\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2, H_t]$ will include terms which do not commute with H_V , with Λ_3 chosen to cancel these terms. Now since H_U is purely on-site while Λ_2 is purely off-site and only supported on nearest neighbors, $[\Lambda_2, H_U]$ must consist only of dipole non-conserving terms. However, $[\Lambda_2, H_t]$ may contain terms that commute with H_V , as both Λ_2 and H_t are supported on nearest neighbor sites (allowing us to get e.g. $b_i^\dagger b_{i+1}^2 b_{i+2}^\dagger$). Λ_3 cannot be chosen to cancel these terms, as if $[A, H_V] \neq 0$ then $[H_V, [A, H_V]] \neq 0$.¹¹³ The

¹¹³One can see this by writing a general A as a linear combination of terms of the form $C = \sum_i C_{\mathbf{n}} \prod_{k \in \mathbb{N}} b_{i+k}^{n_k}$, where each $n_k \in \mathbb{Z}$ (we have ignored number operators n_i since they commute with H_V). Since the action of Ad_{H_V} does not change the powers of b_i operators it acts on, the only way for $[H_V, [A, H_V]]$ to vanish is for $[H_V, [C, H_V]] = 0$ for all such operators C appearing in A . But $[H_V, [C, H_V]]$ takes the same form as C , except with $C_{\mathbf{n}}$ replaced with $C_{\mathbf{n}} \text{dip}(\mathbf{n})^2$ with $\text{dip}(\mathbf{n})$ the dipole moment of $\prod_k b_k^{n_k}$; thus this vanishes only if $\text{dip}(\mathbf{n}) = 0 \implies [C, H_V] = 0$.

part of $\frac{1}{2}[\Lambda_2, H_t]$ which commutes with H_V thus unambiguously provides the leading dipole hopping terms. An unilluminating calculation shows that these terms are

$$\begin{aligned} \frac{1}{2}[\Lambda_2, H_t] &\supset -\frac{t^2 U}{V^2} \sum_{i,j} \{[b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i, n_{j+1} - n_j], b_j^\dagger b_{j+1} - b_{j+1}^\dagger b_j\} \\ &\supset -\frac{t^2 U}{V^2} \sum_i b_i^\dagger b_{i+1}^2 b_{i+2} + h.c., \end{aligned} \quad (1376)$$

where \supset denotes those terms which commute with H_V . Thus the effective Hamiltonian to cubic order in $t/V, U/V$ is

$$H_{eff} = \sum_i \left(-\frac{t^2 U}{V^2} b_i^\dagger b_{i+1}^2 b_{i+2} + h.c. - \mu n_i + \frac{U}{2} n_i(n_i - 1) + V i n_i \right) \quad (1377)$$



The nonlinear IV relation in 2d superconductors

In this section we give a derivation of the nonlinear IV relation near the KT transition, as originally derived by Halperin and Nelson. The original paper omitted the calculation, which luckily turns out to be rather simple.



In the presence of a SF velocity \mathbf{v} ,¹¹⁴ the energy of a vortex-antivortex pair separated by a vector \mathbf{r} is

$$E_p(\mathbf{r}) \sim 2\pi K \ln(r) - \mathbf{r} \times \mathbf{v}. \quad (1378)$$

The first term is the gradient energy of the two vortices, while the second term comes from the ‘Lorentz force’ acting on the vortices, as can be understood within the analogy between vortex mechanics and electrodynamics (being as it is dual to the energy of a charge dipole in an electric field). Perhaps more intuitively, it is essentially just a magnus force: on one side of a vortex \mathbf{v} adds to the vortex flow, creating a ‘high pressure’ region, while on the other side it subtracts, creating a ‘low pressure’ region. The difference in pressures is what leads to the above energy.

¹¹⁴Defined in units of stiffness / length, so e.g. $\mathbf{v} = K \nabla \phi$; this is why the Lorentz force term does not involve K .

$E_p(\mathbf{r})$ is minimal when $\mathbf{r} \perp \mathbf{v}$; in what follows we will take $\mathbf{r} \parallel \hat{\mathbf{y}}$ and $\mathbf{v} \parallel \hat{\mathbf{x}}$. Taking a derivative with respect to r , the location of the energy barrier at which the vortex pair can unbind is then¹¹⁵

$$r_* \sim 2\pi K/v. \quad (1379)$$

The rate γ at which unbound vortex pairs are produced from the vacuum per unit area is then

$$\nu \sim f e^{-E_p(r_*)/T} \sim f v^{2\pi K/T}, \quad (1380)$$

where f is a tunneling attempt frequency per unit area, which is some non-universal constant that is presumably independent of v and K .

To calculate the IV relation, we use the Josephson relation for the voltage difference

$$V \sim \frac{d\Delta\phi}{dt} \sim \frac{dv}{dt}. \quad (1381)$$

The decay of the SF velocity is caused by unbound vortices moving across the sample in the direction normal to \mathbf{v} . A vortex which traverses the whole sample decreases the SF velocity by $2\pi/L$, where L is the linear size of the sample in the direction transverse to the current. However, individual vortices will not generically make it all the way across the sample; instead they will only propagate for a mean free path λ . In terms of the MFP, we then have

$$\frac{dv}{dt} \sim -\frac{2\pi}{L}(\nu\lambda L) \sim -\nu\lambda. \quad (1382)$$

To find λ , we argue in the following way.¹¹⁶ First, we calculate the area A_v within which we expect on average to find one unbound vortex. On one hand, a given vortex can exist anywhere within an annulus of radius r_* and thickness λ , so that

$$A_v \sim \lambda r_*. \quad (1383)$$

On the other hand, we also have $A_v = (\nu\tau)^{-1}$, with τ the vortex lifetime. But the vortex lifetime is $\tau = \lambda/s_v$, where s_v is the mean velocity at which the unbound vortices move. If we take the force on an unbound vortex to be $F \sim v - \eta s_v$ where η is some drag coefficient, then $s_v \sim v$. Therefore

$$\tau = \lambda/v \implies A_v = \frac{v}{\nu\lambda}. \quad (1384)$$

Putting these two expressions for A_v together, we get

$$\lambda \sim v/\sqrt{\nu} \implies \frac{dv}{dt} \sim -v\sqrt{\nu}. \quad (1385)$$

¹¹⁵For small currents and temperatures not too close to T_c , the unbinding scale r_* will be much larger than the typical size of bound vortex-antivortex pairs. Thus in what follows we will take K to be its fully renormalized (i.e. long-distance) value, obtained after we have flowed to the point where the vortex fugacity vanishes.

¹¹⁶After I decided to teach this to Levitov's solid state II students, I realized that the following argument is a bit more complicated than it needs to be. The rate for creating unbound vortices is $n_c \propto e^{-E_p(r_*)/T}$, while the rate for annihilating vortices is $n_a \propto n_v^2$, where n_v is the vortex density. In equilibrium we thus have $n_v \sim e^{-E_p(r_*)/2T}$. Since the resistance is proportional to n_v , we know that $V \sim \int dI e^{-E_p(r_*)/2T}$, where the integrand depends on I by virtue of $v \sim I$. This then gives the correct answer.

Using the above expression for ν and $v \sim I$ with I the supercurrent,

$$V \sim I^{1+\pi K/T}. \quad (1386)$$

At the BKT transition we have $\pi K/T = 2$, reproducing the $V \sim I^3$ scaling commonly used to identify the transition temperature.



Phase stiffness and critical currents in nodal superconductors

In today's entry we will calculate the phase stiffness and critical currents in nodal superconductors, working in the approximation where the nodal quasiparticles can be treated as Dirac fermions.



Setup

We will assume a d -wave gap for simplicity, and assume that the gap has nodes at \mathcal{T} -related momenta. For concreteness, consider a d_{xy} order parameter with nodes on the x, y axes. We can decompose the electron annihilation operator as¹¹⁷

$$\psi = \psi_1 e^{iKx} + \psi_2 e^{iKy} + \psi_3 e^{-iKx} + \psi_4 e^{-iKy}, \quad (1387)$$

where the nodes are at momenta $(\pm K, 0), (0, \pm K)$. The most natural way to proceed is to combine nodes at opposite momenta into Dirac fermions. ψ_1 and ψ_3 can hybridize with the order parameter while conserving momentum; as can ψ_2 and ψ_4 . Therefore these Dirac fermions are most naturally written in a Nambu basis as

$$\Psi_+ = \begin{pmatrix} \psi_1 \\ \psi_3^\dagger \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} \psi_2 \\ \psi_4^\dagger \end{pmatrix}. \quad (1388)$$

The $i\mathbb{R}$ time Lagrangian governing the qps is

$$\mathcal{L}_{qp} = \Psi_+^\dagger (-i\omega_n + v_F k_x \sigma^z + v_\Delta k_y \sigma^x) \Psi_+ + \Psi_-^\dagger (-i\omega_n + v_F k_y \sigma^z + v_\Delta k_x \sigma^x) \Psi_-. \quad (1389)$$

If the ψ fermions form a Fermi liquid, we can identify v_F with the Fermi velocity and $v_\Delta \propto \Delta_0$, with Δ_0 the maximum size of the gap. Regardless of the details though, v_Δ will vanish as $\Delta \rightarrow 0$. Note also that since we are in 2+1D, interactions are irrelevant.

¹¹⁷We will be ignoring the physical spin throughout.

We claim that the correct way to couple these fields to a background gauge field is as

$$\mathcal{L}_{qp} = \Psi_+^\dagger (-i\omega_n + v_F(k_x + A_x\sigma^z)\sigma^z + v_\Delta k_y\sigma^x) \Psi_+ + \Psi_-^\dagger (-i\omega_n + v_F(k_y + A_y\sigma^z)\sigma^z + v_\Delta k_x\sigma^x) \Psi_-. \quad (1390)$$

Here the σ^z 's multiply the gauge field due to the fact that we're in a Nambu basis.

Due to the fact that there is no gauge field appearing in the terms proportional to v_Δ , this does not look gauge invariant. This is okay however, as v_Δ is related to the (non-gauge-invariant) order parameter, and so there is more to the gauge transformations of these terms than meets the eye. It may be helpful here to go back to real space. The terms proportional to v_Δ are the low-energy representations of $\int_{\mathbf{x},\mathbf{y}} \Psi_\pm^\dagger(\mathbf{x})\Delta(\mathbf{x},\mathbf{y})\sigma^x\Psi_\pm(\mathbf{y})$, which do not need the background field to be gauge invariant. Expanding these terms in momentum space makes it look like we have a problem, but this is just due to the fact that gauge transformations become complicated in momentum space.¹¹⁸

Stiffness

As a function of a constant vector potential A , the free energy is¹¹⁹

$$\mathcal{F}[A] = \rho_{\mu\nu} A^\mu A^\nu + O(A^3) \quad (1391)$$

At finite T , $\rho_{\mu\nu}$ receives contributions from both the condensate and from the nodal quasiparticles. At low T the T -dependence will come from the nodal quasiparticles.

We now compute the contribution of these qps to $\rho_{\mu\nu}$. At first it may seem like the Ψ_\pm cannot affect $\rho_{\mu\nu}$. Indeed, a nonzero contribution means that the qps have a finite paramagnetic / diamagnetic response, which comes from induced currents. Since the Ψ_\pm do not have a well-defined charge, it may seem like this response must vanish. But this is too fast: since the Ψ_\pm involve fields with both opposite charge *and* opposite momentum, they have well-defined currents, allowing them to have a nontrivial paramagnetic / diamagnetic response. The response is proportional to their density, which is made nonzero in the presence of a field, since the field acts as a chemical potential for the qps, as we saw above.

We can calculate the response by finding the coefficient of $A_i A_j$ in the effective action obtained by integrating out the Ψ_\pm fields. The Ψ_+ fields give the contribution

$$\begin{aligned} \mathcal{F}[A] &\supset \frac{A_x^2 v_F T}{2v_\Delta} \sum_n \int_{\mathbf{k}} \frac{\text{Tr}[(i\omega_n + k_x\sigma^z + k_y\sigma^x)^2]}{(\omega_n^2 + k^2)^2} \\ &= \frac{A_x^2 v_F T}{v_\Delta} \sum_n \int_{\mathbf{k}} \frac{-\omega_n^2 + k^2}{(\omega_n^2 + k^2)^2} \\ &= -\frac{A_x^2 v_F}{4v_\Delta T} \int_{\mathbf{k}} \text{sech}^2(k/2T) \\ &= -c \frac{A_x^2 v_F T}{v_\Delta}, \quad c \equiv \frac{\ln 2}{\pi}. \end{aligned} \quad (1392)$$

¹¹⁸One can also note that adding the naive terms like $\Psi_+^\dagger v_\Delta(k_y + A_y\sigma^z)\sigma^x\Psi_+$ to \mathcal{L}_{qp} gives a Hamiltonian that is imaginary at the nodal points — not good.

¹¹⁹I see no reason for writing m everywhere; hence we are absorbing it into ρ .

The Ψ_- fields give the same contribution but with $x \leftrightarrow y$; hence the quasiparticle contribution to ρ is

$$\rho_{qp}^{ij} = -\delta^{ij} \frac{cv_F}{v_\Delta} T. \quad (1393)$$

The full stiffness is then obtained just by adding on the $T = 0$ diamagnetic part ρ_0 (ignoring the variations of ρ_0 with T , which are subleading for $T/\Delta \ll 1$), giving $\rho^{ij} = \delta^{ij}\rho$, with

$$\rho = \rho_0 - \frac{cv_F}{v_\Delta} T. \quad (1394)$$

Note that the stiffness is *isotropic*, even though the gap is not. This comes from the averaging between the two nodal directions, and the response would remain isotropic upon including more nodes (in a symmetric fashion). The factor of $1/v_\Delta$ is just a reflection of the fact that $1/v_\Delta$ controls the qp DOS at the nodes. This expression also matches with that given in Lee+Wen.

Critical current

Now we will compute the critical current. We will work in a slightly more general setting with pairs of nodes labeled by n at angles $\theta_n, \theta_n + \pi$. The nodes at $\theta_n, \theta_n + \pi$ combine to form a Dirac fermion Ψ_n , which at node n has velocity $v_{F,n}$ ($v_{\Delta,n}$) normal to (along) the gap direction. As we saw above, a constant vector potential \mathbf{A} simply acts as a chemical potential of magnitude $\mathbf{A} \cdot \hat{\theta}_n$ for each Ψ_n . The current is then obtained as

$$\mathbf{j} = \rho_0 \mathbf{A} - \sum_n \frac{1}{v_{\Delta,n}} \hat{\theta}_n \int_{\mathbf{k}} f(k - v_F \mathbf{A} \cdot \hat{\theta}_n), \quad (1395)$$

with f the Fermi function (for $T \ll \Delta_0$, which is the regime we are interested in, non-linear contributions to \mathbf{j} from the superfluid part can be ignored). Note that for a SC with an isotropic gap, the current above would consist of only the first term (up to corrections exponentially small in $e^{-\Delta_0/T}$) until $A v_F = \Delta_0$.

At $T = 0$, the integral is just determined by geometry, with

$$\int_{\mathbf{k}} f(k - v_F \mathbf{A} \cdot \hat{\theta}_n) = \frac{1}{4\pi} (v_F \mathbf{A} \cdot \hat{\theta}_n)^2. \quad (1396)$$

Therefore

$$\mathbf{j}(T = 0) = \rho_0 \mathbf{A} - \sum_n \frac{v_{F,n}^2}{4\pi v_{\Delta,n}} \hat{\theta}_n (\mathbf{A} \cdot \hat{\theta}_n)^2. \quad (1397)$$

Consider e.g. a d_{xy} gap. If we take \mathbf{A} to point along $\hat{\mathbf{x}}$ and let $v_{F,n}^2/v_{\Delta,n} \equiv v$ be the same for all n , then $j_y = 0$ and

$$j_x(T = 0) = \rho_0 A - v \frac{A^2}{4\pi} \implies j_{c,x} = \frac{\pi \rho_0^2}{v}. \quad (1398)$$

On the other hand, consider a $d_{x^2-y^2}$ gap, but keep \mathbf{A} along $\hat{\mathbf{x}}$. Then j_y still vanishes, and we get

$$j_x(T = 0) = \rho_0 A - v \frac{A^2}{4\pi\sqrt{2}} \implies j_{c,x} = \sqrt{2} \frac{\pi \rho_0^2}{v}, \quad (1399)$$

a factor of $\sqrt{2}$ larger than in the previous case, since now the field is a factor of $\sqrt{2}$ less efficient at exciting qps.

For a gap with more nodes, the anisotropy is significantly reduced. For example, for a $d_{xy(x^2-y^2)}$ gap the biggest difference in critical currents at $T = 0$ is between those along the $\hat{\mathbf{x}}$ and $R_{\pi/8}\hat{\mathbf{x}}$ directions, which differ by a factor of only

$$\frac{j_{c,0}}{j_{c,\pi/8}} = \frac{1 + 1/\sqrt{2}}{2(\cos(\pi/8)^3 + \cos(3\pi/8)^3)} \approx 1.01. \quad (1400)$$

Expressions for $0 < T \ll v_F v_c$ where v_c is the critical velocity can be obtained from the Sommerfeld expansion of the Fermi function, giving a dependence of j_c on T^2 . To order T^2 , this gives for a d_{xy} gap

$$j_{c,x} = \frac{\pi\rho_0^2}{v} - v\frac{\pi T^2}{12}. \quad (1401)$$

The quadratic dependence here is essentially due to the particle-hole symmetry of the Dirac fermions at the nodes: to linear order in T fermions moving parallel and antiparallel to \mathbf{j} are excited in equal numbers, meaning that the leading T dependence is T^2 . This crosses over to being linear when $T \sim v_F v_c$. When T is close to T_c (the temperature for which the critical current vanishes), the critical value of A will be small, and we can use the expansion employed above to determine ρ_{qp}^{ij} . This tells us that the critical temperature is

$$T_c = \frac{\pi\rho_0}{v \ln 2}. \quad (1402)$$

Note that $T_c \sim \rho_0 v_\Delta / v_F$. The v_Δ/v_F factor grows with Δ_0 , but ρ_0 is proportional to the electron density, meaning that at low electron densities we can have $T_c/\Delta_0 \ll 1$.

The behavior of the critical current near T_c can be obtained by an expansion in Av_F/T . We need to expand to the leading nonlinear order in Av_F/T in order to derive an expression for j_c . Taking a d_{xy} gap with $\mathbf{A} \parallel \hat{\mathbf{x}}$ for simplicity, we have

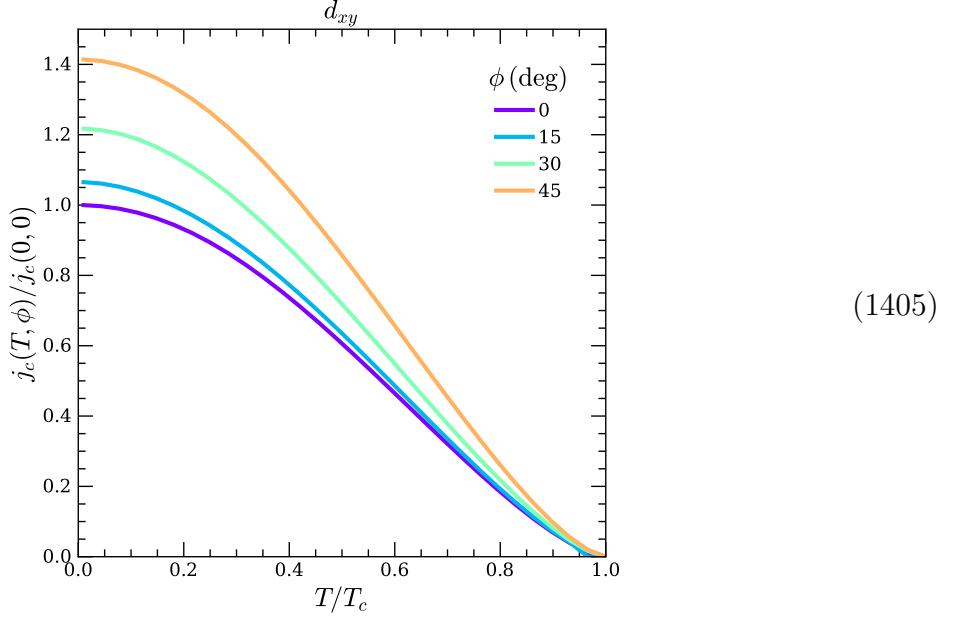
$$j_x \approx \rho(T)A + 2\frac{vv_F^2 A^3}{3!} \int_{\mathbf{k}} \partial_k^3 f(k) = \rho(T)A - \frac{vv_F^2 A^3}{24\pi T}, \quad (1403)$$

where $\rho(T)$ is given in (1394). This gives a critical current of

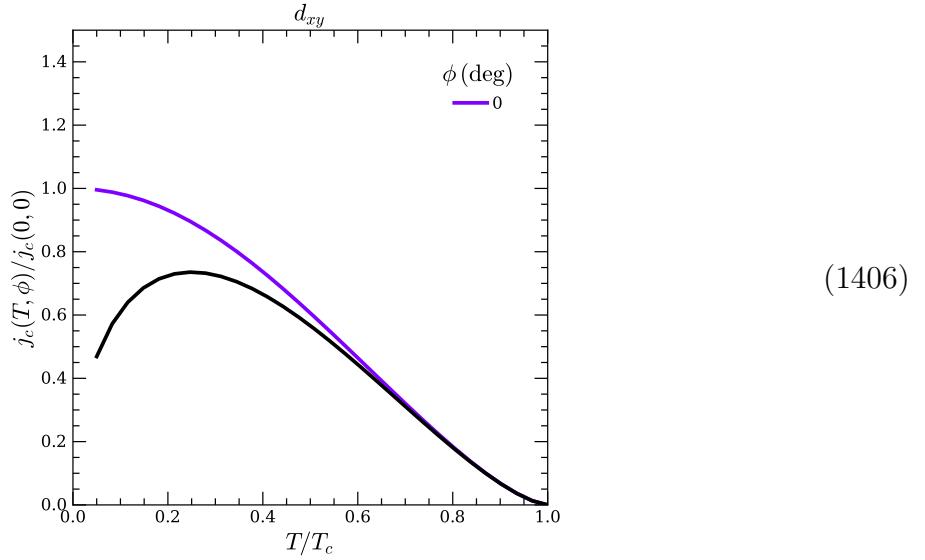
$$j_c(v_F A/T \ll 1) \approx d\rho(T)^{3/2} T^{1/2}, \quad d \equiv \frac{2}{3v_F} \sqrt{\frac{8\pi}{v}}, \quad (1404)$$

which near T_c goes as $(1 - T/T_c)^{3/2}$. This power-law dependence is also what one gets within GL theory, as it must since we are working at $T/T_c \lesssim 1$.

We now illustrate these expectations with some plots.¹²⁰ For d_{xy} symmetry, we have e.g.

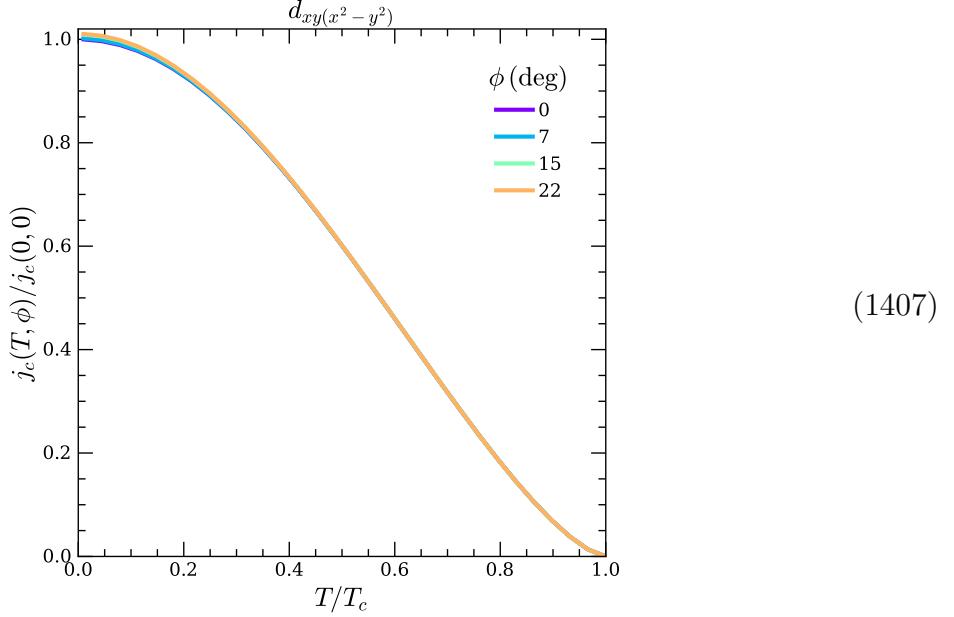


Here ϕ is the angle of the current relative to the x axis (with the plots being periodic in ϕ mod $\pi/4$). The $v_F A/T$ expansion in (1404) works quite well: with the fit in black, we have



¹²⁰One slightly confusing aspect of (1395) is that the current is in general not parallel to the applied field. This is because the qps at node n only flow along $\hat{\theta}_n$, and so if the field is not directed along a high-symmetry direction, the current will not be parallel to the field. Therefore when finding the critical current along a given direction, we have to search over different field directions.

As we said above, the angular dependence is much smaller for a $d_{xy(x^2-y^2)}$ gap:



where now ϕ is periodic mod $\pi/8$.

Comparison with *s*-wave

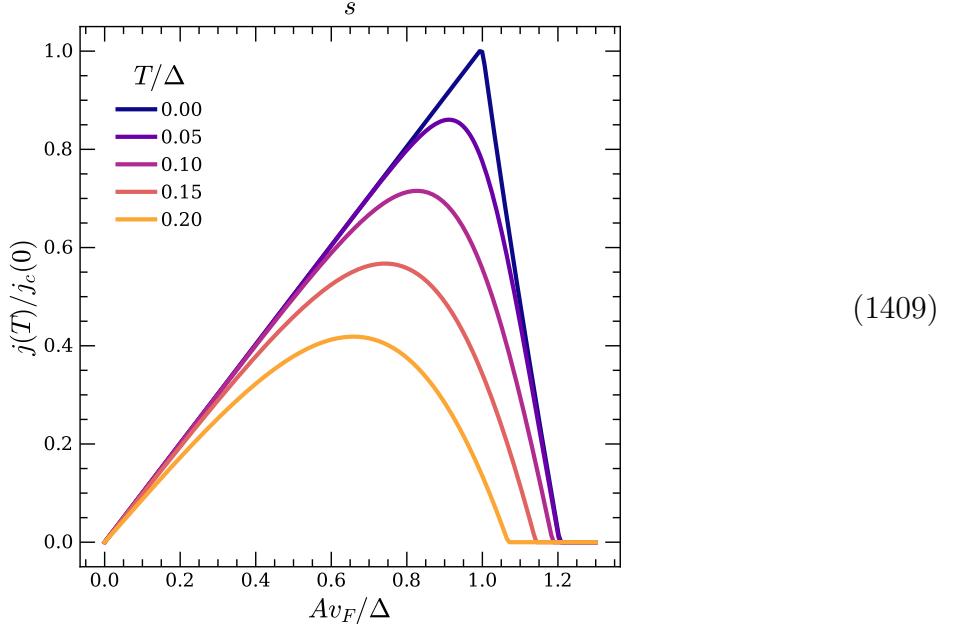
Let's compare this with what we'd expect from a fully-gapped SC. Here of course the relationship between j and A will be very different, since at $T = 0$ j is linear in A until $v_F A \sim \Delta_0$. It is however not so obvious that the functional form of the T -dependence of $j_c(T)$ will be different from the nodal case, since in both instances the T -dependence occurs from the thermal activation of qps, which we might expect to take the same T^2 form in both cases. We will see however that in the *s*-wave case the T -dependence is linear all the way until $T \sim T_c$.

For a uniform *s*-wave gap and with $\mathbf{A} \parallel \hat{\mathbf{x}}$ wolog, we have

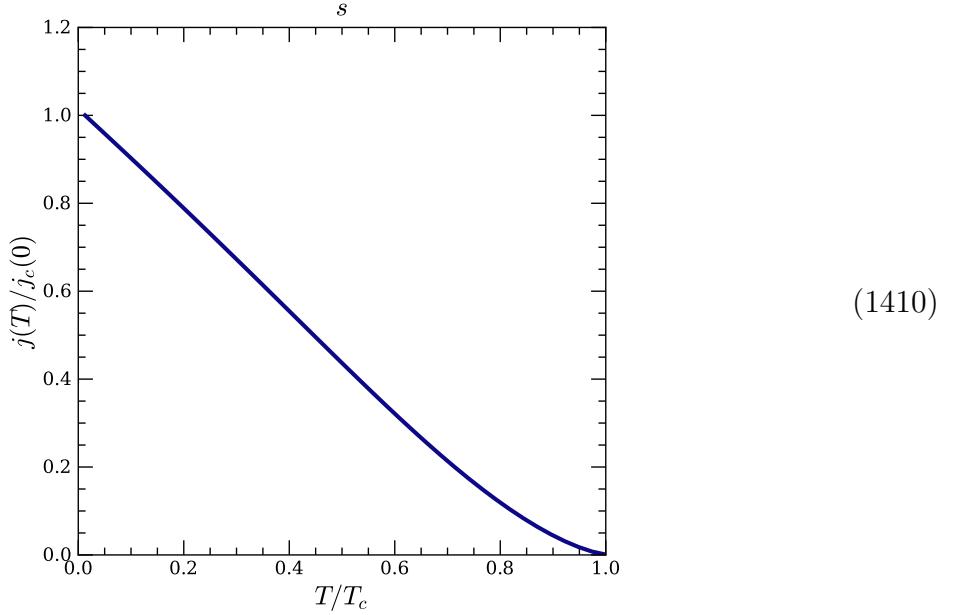
$$j_x = \nu v_F (v_F A) - 2\nu v_F \int d\xi d\theta \cos \theta f \left(\sqrt{\xi^2 + |\Delta|^2} - v_F A \cos \theta \right), \quad (1408)$$

where ν is the DOS at the Fermi level. When $T/\Delta \ll 1$ the qps cannot provide a backflow current until rather large fields $v_F A \sim \Delta$. Until these fields the current is linear in A , and

above these fields the current drops very quickly. For example, a typical plot looks like



We now plot j_c as a function of T :



The distinguishing feature here is the absence of non-linear behavior at low T/T_c . At $T/T_c \lesssim 1$ however, the nodes are washed out and the nodal and s -wave cases look functionally identical (note that we have not taken into account thermal suppression of the gap, i.e. we have neglected the T -dependence of Δ . This is completely justified for the nodal case where we found $\Delta/T_c \gg 1$, and even for the s -wave case where T_c is of order Δ the T -dependence of Δ is subleading).



Spin susceptibility of unitary triplet superconductors

This diary entry contains some simple calculations relating to triplet BCS superconductors that I did when preparing to teach a class on the subject.



Consider first a triplet superconductor with only charge, spin, and momentum quantum numbers. We will write the gap function as

$$\Delta_{\mathbf{k}} = is^y \mathbf{d}_{\mathbf{k}} \cdot \mathbf{s}. \quad (1411)$$

With this notation the different components of \mathbf{d} give pairing in the following channels:

$$d^x \sim |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle, \quad d^y \sim |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle, \quad d^z \sim |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle. \quad (1412)$$

The self-consistent equation for the pairing is

$$\langle c_{ks} c_{-ks'} \rangle = \int_{\mathbf{k}'} V_{kk'}^{ss' \bar{s}\bar{s}'} [\Delta_{\mathbf{k}'}]_{\bar{s}\bar{s}'} \quad (1413)$$

where V is the interaction.

We will mostly consider *unitary* superconductors, where the d-vector is real up to a phase (we will take it to be real wolog). Consider such a superconductor in the presence of a magnetic field \mathbf{B} (we will set $\mu_B = 1$ to declutter the notation). The BdG Hamiltonian is (omitting momenta indices and working in particle-hole \otimes spin space)

$$\mathcal{H} = \begin{pmatrix} \xi + \mathbf{B} \cdot \mathbf{s} & is^y \mathbf{d} \cdot \mathbf{s} \\ -\mathbf{d} \cdot \mathbf{s} is^y & -\xi - \mathbf{B} \cdot \mathbf{s} \end{pmatrix}. \quad (1414)$$

The eigenvalues are easiest to find if we take $\mathbf{B} \parallel \hat{\mathbf{z}}$, done wolog. Then the spectrum is

$$E_{\pm} = \sqrt{|d|^2 + \xi^2 + B^2 \pm 2B\sqrt{(d^z)^2 + \xi^2}}. \quad (1415)$$

Note that

$$\begin{aligned} \mathbf{d} \perp \mathbf{B} &\implies E_{\pm} = \sqrt{(\xi \pm B)^2 + |d|^2} \\ \mathbf{d} \parallel \mathbf{B} &\implies E_{\pm} = \sqrt{\xi^2 + |d|^2} \pm B. \end{aligned} \quad (1416)$$

Thus in the case where $\mathbf{d} \parallel \mathbf{B}$ — where the Cooper pairs look like $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$ — the quasiparticles have well-defined nonzero spin, and the effect of the field is similar to that in

a singlet superconductor (the pair-breaking effects of the field don't care about the minus sign present in the singlet). If $\mathbf{d} \perp \mathbf{B}$ however then the field simply shifts the origin of ξ , rather than acting as a chemical potential for the quasiparticles.

To compute things like the magnetic susceptibility, it is easiest to specify to one of the two limits in (1416). First consider the case when $\mathbf{d} \perp \mathbf{B}$. The Greens function is

$$\frac{1}{i\omega - \mathcal{H}_\perp} = \frac{1}{\omega^2 + E_{s^z}^2} \begin{pmatrix} -i\omega - Bs^z - \xi & -is^y \mathbf{d} \cdot \mathbf{s} \\ \mathbf{d} \cdot \mathbf{s} & is^y \\ \end{pmatrix}. \quad (1417)$$

We can then derive the magnetization as (letting ν be the Fermi-level DOS)

$$\begin{aligned} M_\perp^z &= \int_{\mathbf{k}} \langle n_{\mathbf{k},\uparrow} - n_{\mathbf{k},\downarrow} \rangle \\ &= \nu \int_{\xi} \sum_{\omega} \sum_s s \frac{\xi + sB}{\omega^2 + E_s^2} \\ &= \nu \int_{\xi} \sum_s s(\xi + sB) \frac{f(E_s) - f(-E_s)}{E_s} \\ &= -\nu \int_{\xi} \sum_s s(\xi + sB) \frac{\tanh(E_s/2T)}{E_s}. \end{aligned} \quad (1418)$$

Here we must resist the temptation to shift the integral over ξ by $-sB$ to cancel out the magnetic field. Instead, letting B be small, we may write

$$M_\perp^z = -2\nu B \int_{\xi} \frac{d}{d\xi} \left(\xi \frac{\tanh(E/2T)}{E} \right) = 2\nu B. \quad (1419)$$

where E (no subscript) is the usual $E = \sqrt{\xi^2 + |\mathbf{d}|^2}$. This then gives a susceptibility which (in our present units) exactly matches the conventional Pauli susceptibility χ_P of the normal state. Thus when $\mathbf{d} \perp \mathbf{B}$ (and the quasiparticles lack a well-defined spin), the susceptibility is unchanged in the superconductor:

$$\chi_\perp = \chi_P. \quad (1420)$$

For $\mathbf{d} \parallel \mathbf{B}$ it is easy to check that the low-field susceptibility is the same as in a singlet superconductor, a standard calculation giving

$$\chi_{\parallel} = \chi_P \int_{\xi} \frac{1}{4T \cosh^2(E/2T)}. \quad (1421)$$



Friedel oscillations for noncircular Fermi surfaces

In this entry we will see how the Friedel oscillations created by non-circular Fermi surfaces differ from those created by circular ones.



To do this we start from the equal-time free electron Greens function

$$G(\mathbf{x}) = \int_{\text{Fermi Sea}} d^2k e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (1422)$$

At distances much bigger than the inverse Fermi momentum, the contributions to this integral from the interior of the Fermi sea cancel out, and one is left with a contribution only from the surface. This means we do not have to assume that the thing under study is a Fermi gas – anything with a Fermi surface will do.

We will be interested in the angular average

$$\langle G \rangle_\phi(x) \equiv \int \frac{d\phi}{2\pi} G(x\hat{\phi}). \quad (1423)$$

Since we are only interested in the oscillations, we will drop all unimportant constants and prefactors. Then

$$\begin{aligned} \langle G \rangle_\phi(x) &\sim \int_0^{2\pi} d\theta d\phi \int_0^{k_F(\theta)} dk k e^{ikx \cos(\theta-\phi)} \\ &\sim \int_0^{2\pi} d\theta \int_0^{k_F(\theta)} dk k J_0(kx) \\ &\sim \int_0^{2\pi} d\theta \int_0^{k_F(\theta)} dk \sqrt{k/x} \cos(kx - \pi/4) \\ &\sim \int_0^{2\pi} d\theta \sqrt{k_F(\theta)} \frac{\cos(k_F(\theta)x - 3\pi/4)}{x^{3/2}}, \end{aligned} \quad (1424)$$

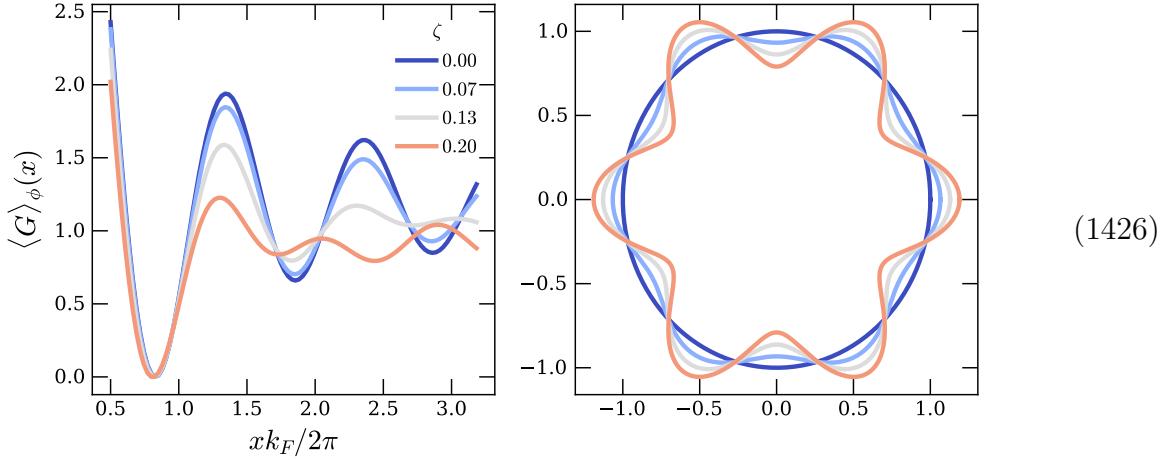
where the integral over k was done assuming $x \gg 1/k_F(\theta)$ for all θ .

Consider as an example a Fermi surface with a C_6 anisotropy. We will write

$$k_F(\theta) = \sqrt{k_F - \zeta^2/2} + \zeta \cos(6\theta), \quad (1425)$$

where ζ controls the degree of anisotropy and k_F determines the area of the Fermi surface,

which the above formula fixes at πk_F^2 . Evaluating the above integral numerically then gives



We thus see that the period of the oscillations can change with a fairly moderate amount of anisotropy.



Basic optical lattice stuff

In this entry we will remind ourselves of some basic facts concerning the tight-binding models that one gets in optical lattices.



The potential is generating by a monochromatic electric field of wavelength k_L that forms a standing wave and oscillates with frequency ω_L . For atoms with two levels carrying different dipole moments and separated by a transition frequency ω_0 , a sinusoidal potential can be generated by taking the detuning Δ and the Rabi frequency Ω of the electric field to satisfy $\omega_L \gg \Delta \gg \Omega$. Letting I be the maximum intensity of the laser light and Γ be the linewidth of the transition, the potential formed in this limit is

$$V_{dip}(x) = \frac{3\pi c^2 \Gamma I}{2\omega_0^3 \Delta} \sin^2(k_L x) \equiv V \sin^2(k_L x). \quad (1427)$$

In the following we will define the *recoil energy* as

$$E_r \equiv \frac{\hbar^2 k_L^2}{2m} = h \times \left(\frac{m_{Rb}}{m}\right) \times \left(\frac{700 \text{ nm}}{a}\right)^2 \times 1.17 \text{ kHz} \quad (1428)$$

where the lattice spacing is $a \equiv 2\pi/(2k_L)$, m is the mass of the atom being loaded into the trap, and m_{Rb} is the mass of an ^{87}Rb atom. We will also define the normalized well height

$$v \equiv \frac{V}{E_r}. \quad (1429)$$

Then

$$H = 4E_r \left(-\frac{\nabla^2}{(2k_L)^2} + \frac{v}{4} \sin^2(k_L x) \right). \quad (1430)$$

The eigenfunctions of this Hamiltonian — or any other single particle Hamiltonian with discrete translation symmetry — are $e^{iqx}u_{nq}(x)$, where

$$u_{nq}(x) = \frac{1}{\sqrt{a}} \sum_G e^{iGx} u_{nq}(G), \quad (1431)$$

with G the reciprocal lattice vectors and a the lattice spacing. With $\sum_G u_{nq}^*(G)u_{n'q'}(G) = \delta_{n,n'}\delta_{q,q'}$, we have the normalization $\int_{UC} dx u_{nq}^*(x)u_{n'q'} = \delta_{n,n'}\delta_{q,q'}$, where the integral is over a unit cell.

We are interested in modeling the physics of the lowest band of this system with a tight-binding model. To do this we employ the Wannier functions

$$|w_{nj}\rangle = \frac{1}{\sqrt{N}} \sum_q e^{iqR_j} |u_{nq}\rangle, \quad (1432)$$

where R_j is the coordinate of the j th lattice site and N is the number of lattice points (equal to the number of q points in the BZ). These functions are normalized as $\langle w_{nj}|w_{n'j'}\rangle = \delta_{n,n'}\delta_{j,j'}$, and in terms of them and the dispersion ε_{nq} , the Hamiltonian reads

$$H = \sum_{n,q} \varepsilon_{nq} |u_{nq}\rangle\langle u_{nq}| = \sum_{j,j'} t_n(R_j - R_{j'}) |w_{nj}\rangle\langle w_{n(j')}|, \quad (1433)$$

where

$$t_n(R) = \frac{1}{N} \sum_q \varepsilon_{nq} e^{iqR}. \quad (1434)$$

Note that to compute the $t_n(R)$, and thus the parameters appearing in the tight-binding Hamiltonian we are looking for, we never need to compute the Bloch functions at any point; it is instead sufficient to simply Fourier transform the spectrum of the band under study.

Focusing on the lowest band ($n = 1$), we define

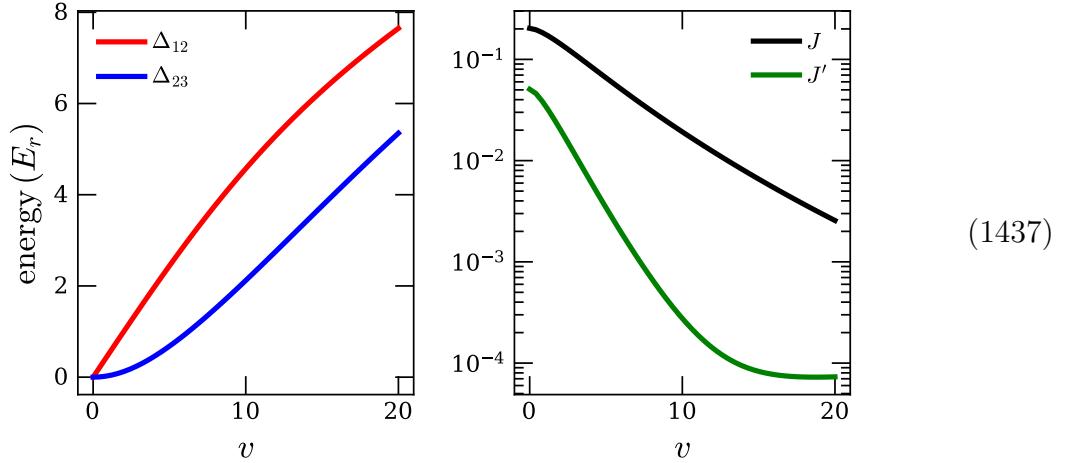
$$J \equiv -t_1(a), \quad J' \equiv -t_1(2a) \quad (1435)$$

as the parameters appearing in the TB Hamiltonian $H = -\sum_j (J b_j^\dagger b_{j+1} + J' b_j^\dagger b_{j+2} + \dots) + h.c.$ Using the explicit solution for the Bloch states in terms of Mathieu functions, one can derive approximate analytic forms for J, J' which are exact in the $v \gg 1$ limit, e.g.

$$\frac{J}{E_r} \approx \frac{4v^{3/4}}{\sqrt{\pi}} e^{-2\sqrt{v}}. \quad (1436)$$

However, since computing the TB parameters numerically is so easy in practice, getting analytic approximations like this is not too important.

On general grounds we expect the bandgaps to rapidly get smaller for larger band indices and to increase linearly with v at large v ; we also expect J (i.e. the bandwidth of the lowest band) to decrease exponentially with v once $v \gtrsim 1$, and J' to do the same but to decrease exponentially with twice the coefficient in the exponent. This is indeed what happens, as shown in the following plot:



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