The best definition of time reversal

In today's diary, we'll briefly go over the reasons for why what the definitions the community at large uses for T and for CT should really be swapped, and will try to elucidate the meaning of CPT in simpler terms. This discussion should be prefaced by mentioning that the philosophy here is that symmetries aren't less of some fundamental thing that a theory is defined by; rather they are more like tools that we use to study a given theory. Consequently, the exact definition of T and CT and so on is rather subjective. We define symmetries in a way that suits our needs for doing calculations, rather than having them handed down to us from on high. That said, I think some identifications of symmetries are more natural than others; hence the present diary entry.

Solution:

"Problems" with the historical definition of T

Conceptually, I think that the most natural definition of a time-reversal symmetry is one which is antiunitary and reverses time, and does nothing else pertaining to other possible internal symmetries. In particular, T should be a part of the Lorentz group, in the sense that its action (modulo the $\mathbb{C}$ conjugation part) should be restricted to act only via an action of the Lorentz group on the Lorentz indices of the fields in the theory.[1]

To see an example of why the historically-used T isn't necessarily part of the Lorentz group and hence fails this criterion, consider fermions coupled to a gauge field A,[2] with the action containing the term $\bar{\psi}\not{A}^a T^a \psi$ for A^a_μ some real vector fields. We will focus on this term for clarity and since it is conceptually the simplest thing to look at (all other irreps of Lorentz can be obtained from $\otimes$s of (s)pinor reps; hence the focus on fermions). Anyway, this term is definitely something that from experience we "expect" to be T-invariant. Since A^a_μ has a vector index the natural action of time reversal would be something like

$$T : A_\mu \mapsto U_T^\dagger A_\mu U_T = (-1)^{\delta_{\mu,0}} A_\mu, \tag{1}$$

for some unitary U_T—note that there is no action on the gauge index of A^a_μ. However, this does not lead to T-invariance in general, because of the $\mathbb{C}$ conjugation. The general condition for the T invariance of this coupling is (since we are physicists, the T^as are Hermitian)

$$U_T^\dagger A^a_\mu (T^a)^* U_T = (-1)^{\delta_{\mu,0}+1} A^a_\mu T^a, \tag{2}$$

which will not be satisfied for a general gauge group and representation.

Consider first the case when the fermions transform in a real representation. Then $(T^a)^* = -T^a$ on account of the group representations being $e^{i\theta^a T^a}$, and so in this case the expected transformation law for T, viz. (1), leaves $\bar{\psi}\not{A}\psi$ invariant.[3]

[1] Terminology reminder: the Lorentz group is not connected, and only the **1** component of the Lorentz group needs to be a symmetry in relativistic QFT. So e.g. reflections are part of the Lorentz group, but may or may not be symmetries.

[2] This field may or may not be dynamical; we are just introducing it in order to be able to talk about the action of symmetries in a convenient way.

[3] In this case there may not be any nontrivial notion of charge conjugation anyway, and so T and CT might not be distinguishable to begin with.

Now consider the case when the fermions are in a pseudoreal representation R, with J the antisymmetric matrix relating R and R^* through conjugation. Then since our generators are Hermitian, $J^\dagger T^a J = -(T^a)^*$. Therefore we see that we can take A to transform with the sign in (1) along with conjugation by J, so that $U_T = J$ when acting on the gauge indices.[4] This means that the action of time reversal actually involves a nontrivial action on the gauge indices, because of the conjugation by J. Therefore the action of T does not just depend on the Lorentz indices of the field in question, which in my opinion is not ideal for a definition of T.

The situation is even worse for fermions in a complex representation. Since there is no longer an isomorphism connecting R and R^*, we won't be able to find a choice of U_T that will work, and we have to map each of the different A^a individually with a different sign: we need to act on the generator index with a transformation via $A^a_\mu \mapsto R^{ab} A^b_\mu$. For example, for the fundamental of $SU(3)$ with the usual basis for T^a, we would need T to map A^a for $a = 2, 5, 7$ with one sign, and A^a for $a = 1, 3, 4, 6, 8$ with the other.

Anyway, the point here is that this definition of T necessitates adding in an action on the gauge indices which cancels the complex conjugation performed by T, except in the case where the fermions transform in a real representation. This means that this T does not act solely on Lorentz indices. In other words, if we write $U_T = U_{T,I} \otimes U_{T,L}$ where the first factor acts on the internal indices and the second factor acts on the Lorentz indices, then $U_{T,I} \neq \mathbf{1}$.[5] In fact the precise requirements for the two factors are that (assuming the Dirac adjoint is used)[6]

$$U^\dagger_{T,I} (T^a)^* U_{T,I} = -T^a, \quad U^\dagger_{T,L} \mathcal{K}[i\gamma^0(-\gamma^0\partial_0 + \gamma^i\partial_i)]\mathcal{K} U_{T,L} = i\not{\partial}. \tag{4}$$

With this transformation, $\not{A}$ transforms in the same way as $i\not{\partial}$, and the fermion action is consequently invariant.

Why CT is better

First, recall what we mean by C: it is a $\mathbb{Z}_2$ outer automorphism of the symmetry group G,[7] which acts on a field ψ in a representation R of G as $\psi \mapsto C\psi$, with $C\psi$ transforming in the dual representation R^*. Sometimes we would write this as $\psi \mapsto \psi^*$ in the case of e.g. a $U(1)$ symmetry for a single complex fermion, but this could always be re-written as $\psi \mapsto C\psi$ where we think of ψ as two real fermions, and of C as acting via the matrix Z.

There are two possible conventions for the action of C: it can dualize the representation of only the *internal* symmetries involved, or it can include a dualization of the spacetime

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[4] We can either have

$$T : A^a_\mu \mapsto (-1)^{\delta_{\mu,0}} J^\dagger A^a_\mu J, \tag{3}$$

or we can strip off the Js and modify the transformation of the fermions ψ under T by an action of J on the flavor indices. Either way, $\bar{\psi}\not{A}\psi$ is invariant.

[5] We are as always ignoring weird things like SUSY where the internal and spacetime dof mix.

[6] Pedantic detail: remember that the sign in front of ∂_0 doesn't come from $U^\dagger_{T,L}\partial_0 U_{T,L} = -\partial_0$, since in the present way of thinking about things we are never *actually* acting on spacetime; the action of symmetries is entirely on the fields, and not on numbers like t (conceptually, I think it's best to always think of the action as being entirely performed through conjugating second-quantized operators). The minus sign instead comes when when change variables $t \mapsto -t$ in the action.

[7] Where pedantically the "full" symmetry group is G', with $G'/\mathbb{Z}_2^C = G$, or $G' \cong G \rtimes \mathbb{Z}_2^C$.

symmetry representation as well. For example, in the former definition, C would map a field which annihilates left-handed neutrinos to a field which annihilates left-handed antineutrinos, while in the latter definition it would map to a field which annihilates right-handed antineutrinos. Here an antiparticle is one whose *internal* symmetry quantum numbers are all the duals of the quantum numbers of the particle in question—therefore in the former definition C sends particles to antiparticles, while in the latter definition it does this plus an action on the Lorentz indices (usually by parity). Now the former definition of C may not even be a legit operation to perform in a given QFT (right-handed neutrinos may not even exist!), while with the second definition, C is always a legit thing to do. Therefore, we will work with the later definition, as we have done throughout most of the diary.

In any case, in the discussion of C, P, and T (or better, C, R, T), C always feels like a bit of a misfit, since it's not part of the Lorentz group. In fact, with our definition of T, we have already seen that T is not always part of the Lorentz group either! However, we will now argue that the product CT always *is* part of the Lorentz group.[8]

If we include the action of C in the definition of time reversal, the problems found above for the case of a ps$\mathbb{R}$ or $\mathbb{C}$ representation go away. Indeed, assuming that the theory possesses a C symmetry and writing the charge conjugation matrix as $C_I \otimes C_L$, we see that C-invariance requires[9]

$$C_L^\dagger \gamma_\mu C_L = -\gamma_\mu^T, \qquad C_I^\dagger T^a C_I = -[T^a]^T. \tag{5}$$

Now recall that we are in conventions where the T^a are Hermitian; this means that the equation for C_I can be written

$$C_I [T^a]^T C_I^\dagger = -T^a \implies C_I [T^a]^* C_I^\dagger = -T^a. \tag{6}$$

But we see that this is the same as the equation for $U_{T,I}$, just with †s in different places! Hence we may in fact set $U_{T,I} = C_I^\dagger$, meaning that when we put C and T together as $CT = (CT)_I \otimes (CT)_L \mathcal{K}$, we have $(CT)_I = U_{T,I} C_I = C_I^\dagger C_I = \mathbf{1}$, so that in fact CT acts only on Lorentz indices as claimed.

1 *Global symmetry that remains after gauging a subgroup*

Suppose we have some fields transforming under a global symmetry group $\mathcal{G}$ (which includes spacetime symmetries, e.g. some pin group if they are fermions), and we gauge a subgroup $G \subset \mathcal{G}$. What is the surviving global symmetry group? Not $\mathcal{G}/G$: G may not be normal, and so it may "take out" more of $\mathcal{G}$ than just itself. Today we will answer the question, and discuss a few illustrative examples.

Solution:

[8]Again, by "part of the Loretnz group", we mean that its action on a field is determined entirely by the Lorentz indices of the field—it acts trivially on all other indices, like flavor indices. It still complex conjugates fields though, so the action is not solely through the action of the Loretnz group.

[9]We are using the same conventions as in the long diary entry on pinors and representation theory, where $\bar\psi M \psi \mapsto \bar\psi C^\dagger M^T C \psi$ under C.

We can find the resulting global symmetry group by examining how various charge operators for the symmetries in $\mathcal{G}$ commute with the generator of gauge transformations in G. If U_θ generates the gauge transformation $\psi \mapsto e^{\theta^a T^a_G}\psi$ for the G gauge group (T^a are the generators of G; the case of G discrete is basically the same) and if e^{iQ_h} is the charge operator for a global symmetry acting with an element $h \in \mathcal{G}$, then we require that

$$e^{iQ_h} U_\theta = U_{\theta'} e^{iQ_h}, \tag{7}$$

since then the action of e^{iQ_h} is well-defined when acting on physical states, for which U_θ acts as $\mathbf{1}$ for all choices of θ.

Let the surviving global symmetry group be denoted by $\mathcal{G}'$. Then the above means that if $h \in \mathcal{H}'$ then for any $g \in G$ we must have $h^{-1}gh = g'$ for some $g' \in G$. The means that h must be in the normalizer of G with respect to $\mathcal{G}$, quotiented by G itself (which is trivially in the normalizer):

$$\mathcal{G}' = N_{\mathcal{G}}(G)/G. \tag{8}$$

That $N_{\mathcal{G}}(G)/G$ is a subgroup[10] of $\mathcal{G}$ is easy to check (easy to see that the normalizer is a subgroup, and $\mathcal{G}$ is by definition normal in $N_{\mathcal{G}}(G)$, so we can consistently take the quotient). The simplest example is of course $\mathcal{G} = H \times G$, for which $N_{\mathcal{G}}(G) = \mathcal{G}$ and hence $\mathcal{G}' = \mathcal{G}/G = H$.

Gauging $U(1) \subset O(2n)$

As a more interesting example, think of fermions and let $\mathcal{G} = O(2n)$ with G the diagonal $U(1)$, a given element of which takes the form $R_\theta^{\oplus n}$, where R_θ is a 2×2 rotation matrix. Now $[R_\theta, x\mathbf{1} + yJ] = 0$ for all x, y, and therefore one can show that $U(n) \subset O(2n)$ commutes with the diagonal $U(1)$ (we embed $U(n)$ into $O(2n)$ by writing each complex entry $z = x + iy$ as $x\mathbf{1} + yJ$), and so we at least have $U(n) \subset N_{O(2n)}(U(1))$.

What about the reflection that extends $SO(2n)$ to $O(2n)$? If we take this reflection to be the generator R of $\mathbb{Z}_2^R$ such that $R = Z \oplus \mathbf{1}_{2n-2}$, then we get something that doesn't commute with the $U(1)$, since conjugation by R does

$$R_\theta^{\oplus n} \mapsto R R_\theta^{\oplus n} R = R_{-\theta} \oplus R_\theta^{n-1} \notin U(1). \tag{9}$$

Hence, $\mathbb{Z}_2^R \notin N_{O(2n)}(U(1))$. However, consider the action of $Z^{\oplus n}$, which reflects every other axis. This performs the action that we would usually associate with charge conjugation, viz. $Z^{\oplus n} R_\theta^{\oplus n} Z^{\oplus n} = R_{-\theta}^{\oplus n} \in U(1)$. Therefore

$$C \equiv Z^{\oplus n} \in N_{O(2n)}(U(1)). \tag{10}$$

Suppose $n \in 2\mathbb{Z}$. Then C has determinant 1, and is in fact part of $SO(2n)$—it is not an outer automorphism extending $SO(2n)$ to $O(2n)$. However, if $n \in 2\mathbb{Z}+1$, $\det Z^{\oplus n} = -1$, and it is a reflection outer automorphism that extends to $O(2n)$. Regardless of whether it is a reflection or not, it is not in $U(n)$, since when represented as a matrix in $O(2n)$, the only diagonal matrix in $U(n)$ is the identity. Now one checks that C is a good moniker for $Z^{\oplus n}$ by noting

[10] We might call this the Weyl group of G in $\mathcal{G}$, or something like that, since the Weyl group in the context of Lie theory is defined as $N_{\mathcal{G}}(T)/T$, where T is a maximal torus.

that $C^\dagger U C = U^*$ for any $U \in U(n)$. Therefore we have at least $N_{O(2n)}(U(1)) \supset U(n) \rtimes \mathbb{Z}_2^C$. In fact the normalizer is exactly $U(n) \rtimes \mathbb{Z}_2^C$: $U(n) \rtimes \mathbb{Z}_2^C$ is a maximal subgroup of $O(2n)$ (which can be worked out from the material in [**?**]), and since the normalizer is a proper subgroup of $O(2n)$, it must be $U(n) \rtimes \mathbb{Z}_2^C$. Taking the quotient by $U(1)$, we then get that the remaining global symmetry is

$$\mathcal{G}' = PSU(n) \rtimes \mathbb{Z}_2^C. \tag{11}$$

In this case the $\mathbb{Z}_2^C$ really is a charge conjugation symmtey, since it corresponds to the group $\mathrm{Out}(PSU(n)) = \mathbb{Z}_2$.[11]

Gauging $SU(2) \subset O(4n)$

As another example relevant for fermions, considering gauging an $SU(2)$ subgroup of $O(4n)$, where the $SU(2)$ subgroup acts in a block-diagonal way, with each block a 4×4 orthogonal matrix. We will write basis vectors in $\mathbb{R}^{4n}$ suggestively as $v \equiv (\chi_1^\uparrow, \eta_1^\uparrow, \chi_1^\downarrow, \eta_1^\downarrow, \chi_2^\uparrow, \dots)^T$, where we think of the variables as Majorana fermions coming from n complex fermions in the fundamental of $SU(2)$; e.g. $\psi_{\sigma i} = \chi_i^\sigma + i\eta_i^\sigma$. The $SU(2)$ we're gauging is then realized as (the subscript g appears when needed to distinguish the gauge group from other groups floating around)

$$SU(2)_g \ni \begin{pmatrix} x\mathbf{1} + yJ & w\mathbf{1} + zJ \\ -w\mathbf{1} + zJ & x\mathbf{1} - yJ \end{pmatrix}^{\oplus n}, \qquad (x,\, y,\, w,\, z) \in S^3. \tag{12}$$

Let's first work out the simple case of $n = 1$. We claim that the normalizer includes $Sp(1) \cong SU(2)$, which is realized as matrices of the form

$$Sp(1) \ni x\mathbf{1} \otimes \mathbf{1} + w\mathbf{1} \otimes J + yJ \otimes X + zJ \otimes Z, \qquad (x,\, y,\, w,\, z) \in S^3. \tag{13}$$

Here the first tensor factors keep track of the way of representing complex numbers with real matrices, while the second factors keep track of the Pauli matrix structure of the group elements. Indeed, the above matrices can be seen to commute with the gauged $SU(2)_g$, which is generated by matrices of the form

$$SU(2)_g \ni x\mathbf{1} \otimes \mathbf{1} + yJ \otimes \mathbf{1} + wZ \otimes J + zX \otimes J, \qquad (x,\, y,\, w,\, z) \in S^3. \tag{14}$$

Here by contrast it is the second tensor factors that keep track of the representation of $1, i$ in terms of real matrices. The antisymmetric form preserved by the $Sp(1)$ in the normalizer is $J \otimes \mathbf{1}$. Therefore the normalizer is at least $N_{O(4)}(SU(2)) = [Sp(1) \times SU(2)]/\mathbb{Z}_2 = SO(4)$, with the quotient coming from $-\mathbf{1}$ being in both groups.[12]

[11] We are tacitly assuming $n \neq 2$.

[12] A rather highbrow way of saying why the $Sp(1)$ commutes with the $SU(2)_g$ is that it uses the isomorphism coming from pseudoreality of the fundamental rep of $SU(2)$ together with complex conjugation to create a trivial action on the $SU(2)_g$. The matrices $\mathbf{1} \otimes \mathbf{1}$ and $\mathbf{1} \otimes J$ in $Sp(1)$ form a $U(1)$ that is the obvious one which commutes with $SU(2)_g$ (the diagonal particle-number $U(1)$ symmetry in the action). The matrices $\mathbf{1} \otimes X, \mathbf{1} \otimes Z$ both have the effect of complex-conjugating the $SU(2)_g$, since they both anti-commute with the way we've chosen to represent the number i in that group, namely as $\mathbf{1} \otimes J$. They can then be combined with the matrix $J \otimes \mathbf{1}$, which is the isomorphism establishing the pseudoreality of $SU(2)$, to produce $J \otimes X$ and $J \otimes Z$, which complete the $Sp(1)$ part of the normalizer.

We claim that the reflection that extends $SO(4)$ to $O(4)$ is not in the normalizer. Abstractly, this is because this reflection generates $\mathrm{Out}(SO(4)) = \mathbb{Z}_2$, which from experience we know exchanges the two $SU(2)$s in $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$—therefore conjugation by the reflection should not be in the normalizer of a single $SU(2)$ factor. Let's check this with an example: the reflection can be taken to be $R = (-Z) \oplus \mathbf{1}$. When we act on the matrix $J \otimes X \in SU(2)_g$ in the gauged $SU(2)$ we get

$$R^{-1}(J \otimes X)R = -(\mathbf{1} \otimes J) \in Sp(1), \tag{15}$$

so that conjugation by the reflection indeed swaps $SU(2)_g$ and $Sp(1)$, meaning that the $Sp(1)$ is the reflected image of the gauge group. Therefore since the normalizer is a proper subgroup of $O(4)$ and contains $SO(4)$, it must be $SO(4)$. Another way of saying this is that since $SO(4)$ is a maximal subgroup of $O(4)$, $SO(4)$ must be the whole normalizer, with the global symmetry group thus being $\mathcal{G}' = PSp(1) = PSU(2)$.

This means "charge conjugation" is already included in the normalizer: indeed, if we let it act as $\chi^\sigma + i\eta^\sigma \mapsto \chi^\sigma - i\eta^\sigma$ then $C = \mathbf{1} \otimes Z$ is already included, and if we let it exchange spins then it is $C = J \otimes Z$, which is also included. But really, the point is that we shouldn't be calling such a thing charge conjugation: $\mathrm{Out}(PSU(2)) = \mathbb{Z}_1$ and $PSU(2)$ isn't a semi-direct product, and so there can't possibly be any type of charge conjugation symmetry remaining in the theory after the gauging occurs.

Now we return to the case of arbitrary n. It is easier in this case to make a change of basis and package the vector of χs and ηs as a $2n \times 2$ matrix with complex entries. Thought of as a n-component column vector V with matrix-valued entries, the ith entry is the "quaternonic fermion" matrix

$$V_i = \begin{pmatrix} \chi_i^\uparrow + i\eta_i^\uparrow & \chi_i^\downarrow + i\eta_i^\downarrow \\ -\chi_i^\downarrow + i\eta_i^\downarrow & \chi_i^\uparrow - i\eta_i^\uparrow \end{pmatrix}. \tag{16}$$

The $SU(2)$ we are gauging is then realized as the right action on the above matrices, with the elements in $SU(2)$ written as 2×2 complex matrices, rather than 4×4 real matrices. The length of the vector v is determined by

$$|v|^2 = \frac{1}{2}\mathrm{Tr}[V^\dagger V], \tag{17}$$

and as such is properly preserved by the $SU(2)$ right action. This form also makes it clear that the length is preserved by a left $SU(2n)$ action, which by construction commutes with the right $SU(2)$. However, the left $SU(2n)$ action is too big: the structure of the is in the above form of V_i needs to be preserved by any putative left action, so that the right $SU(2)$ acts properly. The structure of the is (the $\sqrt{-1}s$, not the flavor labels) is encapsulated in the relation $(\mathbf{1}_n \otimes J)^\dagger V J = V^*$, which needs to be preserved by the left action. Therefore if $U \in SU(2n)$ then we need

$$(\mathbf{1}_n \otimes J)^\dagger U (\mathbf{1}_n \otimes J)(\mathbf{1}_n \otimes J)^\dagger V J = (UV)^* \implies (\mathbf{1}_n \otimes J)^\dagger U (\mathbf{1}_n \otimes J) = U^*, \tag{18}$$

which since U is unitary means $U^T(\mathbf{1} \otimes J)U = \mathbf{1} \otimes J$, and so in fact we must have $U \in Sp(n)$. If this is a bit too slick, one can also make the following (still rather slick) argument: whatever the global symmetry group that remains is, the full symmetry group (gauged $SU(2)$ + global) better have only real representations, since we started off with an $O(4n)$ symmetry. Since we gauged an $SU(2)$ acting in the fundamental, which is pseudoreal, the global symmetry group must also act via a pseudoreal representation, since the $\otimes$ of two ps$\mathbb{R}$ reps is $\mathbb{R}$. Therefore the global symmetry can't be $U(2n)$—$Sp(n)$ works though, since it acts in a ps$\mathbb{R}$ way.

Anyway, we now know that the normalizer is at least $N_{O(4n)} \subset [Sp(n) \times SU(2)]/\mathbb{Z}_2$. Is this the whole normalizer? This is in fact the whole normalizer, since a math fact [**?**] is that $(Sp(n) \times Sp(1))/\mathbb{Z}_2 = PSp(n) \times SU(2)$ is a maximal subgroup of $O(2n)$, and through the same reasoning as in the last example, this must be the full normalizer. Therefore the global symmetry remaining is found by taking a quotient by $SU(2)$, producing

$$\mathcal{G}' = PSp(n). \tag{19}$$

Note that we do not get $U(n) \rtimes \mathbb{Z}_2$ or similar, which we might have naively concluded based on thinking about complex fermions.

Finally, what about charge conjugation? There actually is no real charge conjugation symmetry in this case: $\mathrm{Out}(PSp(n)) = \mathbb{Z}_1$ and so there's no type of charge conjugation that we're missing, and $PSp(n)$ can't be written as a semidirect product involving a $\mathbb{Z}_2$ factor,[13] and so there is no $\mathbb{Z}_2^C$ symmetry hiding in the $PSp(n)$. If we were to think about charge conjugation as sending $\chi_i^\sigma \mapsto \chi_i^\sigma$ and $\eta_i^\sigma \mapsto -\eta_i^\sigma$ then it is already included in $(Sp(n) \times SU(2))/\mathbb{Z}_2$, while if we were to have it acting with J in on the $SU(2)$ factor then it'd already be included in the $Sp(n)$ factor—but either of these actions wouldn't really be a charge conjugation symmetry, since neither of them are outer automorphisms. So, while our un-gauged symmetry group $O(4n)$ includes a charge conjugation symmetry since we can write it as $SO(4n) \rtimes \mathbb{Z}_2^C$, when we gauge $SU(2)$, the existence of a charge conjugation symmetry goes away.

[13]Because $PSp(n)$ is connected.