1 *$SU(N)$ WZW basics*

Consider the $SU(N)$ WZW model on a two-dimensional spacetime. The action is

$$S = S_{kin} + S_{wzw} = \frac{1}{8\pi}\int_X \mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}] + i2\pi \int_{B^3} f^*(\alpha), \tag{1}$$

where g is a map from the spacetime X (which since we will take the fields to be constant at infinity is topologically an S^2) into $SU(N)$, $f : B^3 \to SU(N)$ where $\partial B^3 = X$ is a three-ball which bounds spacetime, and where α is some nontrivial form in $H^\bullet(SU(N);\mathbb{R})$.

If $\omega = g^{-1}dg$ is the Maurer-Cartan form on $SU(N)$ pulled back to X, then the forms

$$\lambda_j = \mathrm{Tr}(\omega^{\wedge j}), \qquad j = 3, 5, \ldots, 2N-1 \tag{2}$$

are classes in $H^\bullet(SU(N);\mathbb{R})$ pulled back to functions on X. We will mainly be interested in $SU(2)$, so we only have the $\mathrm{Tr}(\omega\wedge\omega\wedge\omega)$ form. So, take

$$f^*(\alpha) = C\mathrm{Tr}(\omega\wedge\omega\wedge\omega), \tag{3}$$

where C is a normalization constant to be determined, which will ensure that the periods of $f^*(\alpha)$ lie in $\mathbb{Z}$ (so that S is independent of the choice of the bounding manifold B^3). We will focus on $SU(2)$ because we are interested in a model which lives in two dimensions, so that the WZW term is defined through the pullback of a cohomologically nontrivial 3-form from the group manifold to S^2. For $SU(N)$ the 3-form will always be $\mathrm{Tr}(\omega\wedge\omega\wedge\omega)$, and so there isn't much difference between the different N's. The case of $SU(2)$ is easiest since α is then proportional to the volume form on S^3, which makes things simple.[1]

[1]As a general comment,

$$H^\bullet(SU(N);\mathbb{Z}) \cong \mathbb{Z}[\alpha_3, \alpha_5, \ldots, \alpha_{2N-1}], \tag{4}$$

where the α_i are of degree i and come from traces of odd numbers of wedge products of ωs. This can be

First, show that α indeed defines a nontrivial cohomology class. Then calculate the equations of motion from varying g, and interpret them as current conservation equations. Finally, calculate the value of C.

First let us check that α is closed. By using the cyclic invariance of the trace, we have

$$d\lambda_3 = \mathrm{Tr}[d\omega \wedge \omega \wedge \omega]. \tag{6}$$

Now

$$d\omega = -g^{-1}dg \wedge g^{-1}dg = -\omega \wedge \omega, \tag{7}$$

so

$$d\lambda_3 = -\mathrm{Tr}[\omega \wedge \omega \wedge \omega \wedge \omega]. \tag{8}$$

When we take the trace, we can apply the normal supercommutativity of the wedge product since

$$\mathrm{Tr}(A \wedge B) = \sum_{ij} A_{ij} \wedge B_{ji} = (-1)^{|A||B|} B_{ji} \wedge A_{ij} = (-1)^{|A||B|}\mathrm{Tr}(B \wedge A). \tag{9}$$

Applying this to the above with $A = \omega$ and $B = \omega \wedge \omega \wedge \omega$, we conclude that $d\lambda_3 = 0$. We will confirm that λ_3 is not exact in a little bit, which will then show that α is indeed a nontrivial cohomology class.

Now for the equations of motion. First, let's do the kinetic term. We have

$$\begin{aligned} S_{kin}[e^{iW}g] - S_{kin}[g] &= \frac{1}{8\pi}\int \mathrm{Tr}\left[\partial_\mu(g + iWg)\partial^\mu(g^{-1} - ig^{-1}W)\right] - S_{kin}[g] \\ &= \frac{i}{8\pi}\int \mathrm{Tr}\left[\partial_\mu W g \partial^\mu g^{-1} + W\partial_\mu g \partial^\mu g^{-1} + \partial_\mu g(-g^{-1}\partial^\mu W - \partial^\mu g^{-1} W)\right] \\ &= \frac{i}{8\pi}\int \mathrm{Tr}\left[W((\partial^2 g)g^{-1} - g\partial^2 g^{-1}\right], \end{aligned} \tag{10}$$

where we integrated by parts in the last step.

Now for the wzw term. We first need the variation of the Maurer-Cartan form:

$$\begin{aligned} \delta\omega &= g^{-1}(\mathbf{1} - iW)d[(\mathbf{1} + iW)g] - \omega \\ &= -ig^{-1}Wdg + ig^{-1}(dWg + Wdg), \end{aligned} \tag{11}$$

proved by noting that $SU(2) = S^3$ gives an easy base case, and then using the SES

$$1 \to SU(N-1) \to SU(N) \to S^{2N-1} \to 1. \tag{5}$$

and so the variation in the WZW term is

$$\begin{aligned}S_{wzw}[e^{iW}g] - S[g] &= 6\pi i C \int_{B^3} \mathrm{Tr}\left[(-ig^{-1}Wdg + ig^{-1}(dWg + Wdg)) \wedge \omega \wedge \omega\right] \\ &= -6\pi C \int_{B^3} \mathrm{Tr}[W \wedge dg \wedge dg^{-1}] \\ &= -6\pi C \int_X d^2x\ W \epsilon^{\mu\nu} \partial_\mu g \partial_\nu g^{-1},\end{aligned} \tag{12}$$

where we used the cyclicity of the trace.

In a little bit we will show that the normalization constant C needs to be

$$C = -\frac{1}{24\pi^2}. \tag{13}$$

Putting this in, we arrive at the equations of motion:

$$2i\epsilon^{\mu\nu}\partial_\mu g \partial_\nu g^{-1} + (\partial^2 g)g^{-1} - g\partial^2 g^{-1} = 0. \tag{14}$$

We can also re-write this as

$$i\epsilon^{\mu\nu}\partial_\mu g \partial_\nu g^{-1} + (\partial^2 g)g^{-1} - (\partial_\mu g)g^{-1}(\partial^\mu g)g^{-1} = 0, \tag{15}$$

or finally, as

$$i\epsilon^{\mu\nu}\partial_\mu g \partial_\nu g^{-1} + (\partial^2 g)g^{-1} + \partial_\mu g \partial^\mu g^{-1} = 0. \tag{16}$$

Now define the antiholomorphic current $\bar{J}_\mu$ by

$$\bar{J} = -(\partial_{\bar{z}} g)g^{-1}, \tag{17}$$

where $z = x^0 + ix^1$ (depending on your preferences there may be a different constant out in front). Then we can compute

$$\partial_z \bar{J} = -(\partial^2 g)g^{-1} - \partial_{\bar{z}}\partial_z g^{-1} = -(\partial^2 g)g^{-1} - \partial_\mu g \partial^\mu g^{-1} - i\epsilon^{\mu\nu}\partial_\mu g \partial_\nu g^{-1} = 0, \tag{18}$$

which vanishes by the equations of motion. Similarly, we also define a holomorphic current J, with

$$J = g^{-1}\partial_z g, \qquad \partial_{\bar{z}} J = 0. \tag{19}$$

A few words to help us understand these currents: in keeping with the two currents being "conjugates" of one another, the action of time reversal (sending $g \mapsto g^{-1}$ and $z \mapsto \bar{z}$) exchanges the two currents, since

$$T : J \mapsto g\partial_{\bar{z}} g^{-1} = -(\partial_{\bar{z}} g)g^{-1} = \bar{J}. \tag{20}$$

The currents also couple chirally to the symmetries in the product $SU(2)_L \times SU(2)_R$ (which act on g on the left and right respectively), since under $SU(2)_L$,

$$J \mapsto g^{-1}h^{-1}\partial_z hg = J, \qquad \bar{J} \mapsto -h^{-1}(\partial_{\bar{z}} g)g^{-1}h^{-1} = hJh^{-1}, \tag{21}$$

while similarly under $SU(2)_R$,

$$J \mapsto h^{-1} J h, \qquad \bar{J} \mapsto \bar{J}. \tag{22}$$

Finally, note that the wzw term here is essential for getting the "right" equations of motion (at least, the right ones if we have bosonization in mind as a physical context): without it, the equations of motion would be a conservation of a different kind of current, and would read

$$d^\dagger \omega = 0. \tag{23}$$

Now we check that C is indeed given by $-1/24\pi^2$. We do this by requiring that the action be independent of the exact choice of B^3, modulo elements of $2\pi i\mathbb{Z}$. This will be the case if

$$2\pi i C \int_{M_3} \mathrm{Tr}[\omega \wedge \omega \wedge \omega] \in \mathbb{Z} \tag{24}$$

for all closed 3-manifolds M_3. We need only check this for a particular manifold, like S^3, since as we saw earlier, the integrand is closed. The fact that the integrand is closed also tells us that the above integral will be quantized: we just have to figure out what the correct normalization is. One can also check this by computing the variation of the integrand under $g \mapsto e^{iW} g$: as we saw earlier, the variation is exact, and so the above integral is stationary under any infinitesimal variation of g, meaning that it must be quantized.

Anyway, let's find the normalization coefficient. We parametrize the S^3 by

$$g = e^{ix^3 n_a \sigma^a} = \cos(x^3) + i n_a \sigma^a \sin(x^3), \tag{25}$$

where x^3 is an angular coordinate running from 0 to π and $n : X \to S^2$ is a field which maps spacetime to S^2 (this is the usual way of building a three-sphere out of an S^2 and an S^1). Then we have

$$\omega_\mu \, dx^\mu = g^{-1} \partial_\mu g \, dx^\mu = i n_a \sigma^a \, dx^3 + i \partial_j n_a \sigma^a \, dx^j, \tag{26}$$

where i runs through the spacetime coordinates and we've used $(n_a \sigma^a)^2 = 1$.

Putting this in, the integral is

$$I = 6\pi i C \int_{M_3} d^3x \, \epsilon^{3jk} \sin^2(x^3) \, \mathrm{Tr}[n_a \sigma^a \partial_j n_b \sigma^b \partial_k n_c \sigma^c]. \tag{27}$$

Doing the x^3 integral and taking the trace,

$$I = 6\pi^2 C \int_X d^2x \, \epsilon^{jk} \epsilon^{abc} n_a \partial_j n_b \partial_k n_c. \tag{28}$$

Because of the double-antisymmetrization, this is

$$I = 12\pi^2 C \int_X d^2x \, \epsilon^{abc} n_a \partial_1 n_b \partial_2 n_c. \tag{29}$$

We recognize the $n \cdot \partial_i n \times \partial_j n \, dx^i \wedge dx^j$ as the pullback of the volume form on S^2 to X through the map n (it is just the Jacobian for the map: $\partial_i n \times \partial_j n$ lies perpendicular to the S^2, parallel to n, and has magnitude equal to the local magnification of the mapping $n : X \to S^2$). Thus we have

$$I = 12\pi^2 C \int_{n(X)} \mathrm{vol}_{S^2}. \tag{30}$$

Since this can be non-zero, $\mathrm{Tr}[\omega^3]$ must not be exact and hence it must indeed represent a nontrivial cohomology class.

Since the volume form is closed, the integral is a topological invariant which is of course the winding number. So we have

$$I = 48\pi^3 C w, \qquad w \in \mathbb{Z}. \tag{31}$$

Since we have to consider maps with winding number $w = 1$, we see that we need (the minus sign is just convention)

$$C = -\frac{1}{24\pi^2} \implies 2\pi i C \int_{M_3} \mathrm{Tr}[\omega^3] \in \mathbb{Z}. \tag{32}$$

We also see a little bit of the relation between WZW terms and topological terms: after we integrated over the x^3 coordinate, the WZW term became a topological θ term on the compactified spacetime $X \sim S^2$.

2 σ model in two dimensions

Consider a σ model in two dimensions, where the target space is the sphere S^N. The action, after introducing the Lagrangian multiplier in the usual way, is

$$S = \frac{1}{2\lambda} \int d^2x \left(\partial_\mu \phi \partial^\mu \phi - i\lambda\sigma(\phi^2 - 1) \right). \tag{33}$$

Working in the large N limit, find the two point function for σ, namely $\langle \sigma(-q)\sigma(q) \rangle$. It will help to expand about the equilibrium constant value for σ. The expression should simplify in the large q limit.

The equation of motion for ϕ is

$$\phi \Delta \phi = i\lambda\sigma, \tag{34}$$

where we have used $\phi^2 = 1$. Here, $\Delta = -\partial_\mu \partial^\mu$ is the Hodge Laplacian. From this, we expect on dimensional grounds that

$$\langle \sigma(x)\sigma(y)\rangle \sim \frac{1}{|x-y|^4}. \tag{35}$$

Fourier-transforming, we expect

$$\langle \sigma(-q)\sigma(q)\rangle \sim |q|^2. \tag{36}$$

Let's get the exact propagator. Integrating out ϕ, we have

$$Z = \int \mathcal{D}\sigma \exp\left(-\frac{N}{2}\mathrm{Tr}\ln\left(\frac{\Delta}{\lambda} - i\sigma\right) - \frac{i}{2}\int \sigma\right). \tag{37}$$

Again, Δ is the Hodge Laplacian. Since $N \to \infty$ we can minimize with respect to $\sigma = \mathrm{const}$ to find the equilibrium value of σ. Doing this tells us that

$$\frac{i}{2} = \frac{iN}{2}\int_p \frac{1}{p^2/\lambda - i\sigma}, \tag{38}$$

i.e. that

$$1 = \frac{N}{4\pi}\ln\left(-\Lambda^2/i\lambda\sigma\right), \tag{39}$$

so that the equilibrium value for the mass of ϕ is determined by

$$\sigma = i\frac{M^2}{\lambda}, \qquad M^2 = \Lambda^2 e^{-4\pi/N\lambda}. \tag{40}$$

To get the inverse propagator for σ, we just need to find the term in the effective action quadratic in σ. We vary σ as $\sigma = iM^2/\lambda + \delta\sigma$, and isolate the piece in the effective action quadratic in $\delta\sigma$. This comes from the usual polarization bubble type of Feynman diagram. Let Γ be the effective action for σ that we wrote down above. Then

$$\begin{aligned}\frac{\delta^2\Gamma}{\delta\sigma(-q)\delta\sigma(q)}\Big|_{\sigma=iM^2/\lambda} &= -\frac{\delta^2}{\delta\sigma(-q)\delta\sigma(q)}\frac{N}{2}\mathrm{Tr}\left[\frac{1}{\Delta^2+M^2}i\lambda\delta\sigma\frac{1}{\Delta^2+M^2}i\lambda\delta\sigma\right]\\ &= \frac{\delta^2}{\delta\sigma(-q)\delta\sigma(q)}\frac{N\lambda^2}{2}\int_{x,y} G(x,y)\delta\sigma(x)G(y,x)\delta\sigma(y),\end{aligned} \tag{41}$$

where G is the Greens function for the massive scalar. This gives

$$\begin{aligned}\frac{\delta^2\Gamma}{\delta\sigma(-q)\delta\sigma(q)}\Big|_{\sigma=iM^2/\lambda} &= N\lambda^2\int_p \frac{1}{(p^2+M^2)((p-q)^2+M^2)}\\ &= N\lambda^2\int_p\int_0^1 dx\frac{1}{[p^2+M^2+x(1-x)q^2]^2}\\ &= \frac{N\lambda^2}{4\pi}\int_0^1 dx\frac{1}{M^2+x(1-x)q^2}\\ &= \frac{N\lambda^2}{4\pi}\frac{\ln\left(1+\frac{q(q+\sqrt{4M^2+q^2})}{2M^2}\right)}{q\sqrt{4M^2+q^2}}.\end{aligned} \tag{42}$$

If we take the $q^2 \gg M^2$ limit, things simplify, and the RHS goes to $N\lambda^2 \ln(q^2/M^2)/(4\pi q^2)$. This is the inverse propagator, so the propagator in this limit is thus

$$\langle \sigma(-q)\sigma(q) \rangle = \frac{4\pi}{N\lambda^2} \frac{q^2}{\ln(q^2/M^2)}. \tag{43}$$

The point of this is that the dynamically generated mass gives us a new scale that can appear in the propagator (viz. M), and that even in the deep UV this scale is still present in the two point function, which does not reduce to that of a free field.

3 *Even more on the $O(N)$ and nonlinear σ models in two dimensions*

This problem is a way to try to understand some of the content in Polyakov's book — the goal is to understand what's written there and to fill in the details that are omitted.

Consider the $O(N)$ vector model which maps spacetime into the $N-1$ sphere, so that the symmetry group is $O(N)$. As we have seen earlier, this model admits a saddle point solution in $N \to \infty$ limit: using the saddle point, one can see how dimensional transmutation occurs and defines a mass scale.

Using the saddle point method, what is the propagator for the massive excitations? What kind of divergences arise and what kind of renormalization must be done? After looking at this (you should compute the field strength renormalization), go back and find the exact propagator and beta function to order g^2 (without using large N) by employing a background field method with a Wilsonian-picture momentum integration. Show how to do the renormalization and compare to the saddle point results. Show that as $N \to \infty$, the model describes a free theory.

Now consider the case of the $SU(N)$ nlσm, still in 2 dimensions. Find the propagator and the beta function for g^2. In the large N limit, is the theory described by the saddle point, like the $O(N)$ model is? Why not?

Solution:

The $O(N)$ vector model: The action, with the Lagrange multiplier λ to enforce that the vector n lives on S^{N-1}, is

$$S = \frac{1}{2g^2} \int \left(\partial_\mu n_a \partial^\mu n^a + \lambda(n^2 - 1) \right). \tag{44}$$

Assuming that a massive solution exists and integrating out n,

$$S = -\frac{1}{2g^2}\int \lambda + \frac{N}{2}\ln\det(-\partial^2 + \lambda). \tag{45}$$

We have already seen how to do the saddle point multiple times: it gives the mass

$$\lambda = m^2 = \Lambda^2 \exp\left(-\frac{4\pi}{Ng^2}\right). \tag{46}$$

Let's find the 2-point function for the saddle point solution, which we will use for comparison later. We will primarily be interested in short-distance behavior (or small mass) in the following so for $m \ll r^{-1}$ we find

$$\langle n(r)\cdot n(0)\rangle = \frac{Ng^2}{2\pi}\int dp\, p\frac{e^{ipr}}{p^2+m^2} \approx \frac{Ng^2}{2\pi}\int_0^{1/mr} d\alpha \frac{\alpha e^{i\alpha m r}}{1+\alpha^2} \tag{47}$$

where $\alpha = p/m$ and the N comes from the N components of the n vector. Thus in the small mr limit we get the log which indicates that as $N \to \infty$ the n field is free:

$$\langle n(r)\cdot n(0)\rangle \approx -\frac{Ng^2}{2\pi}\ln(mr) = 1 - \frac{Ng^2}{2\pi}\ln(r/a), \tag{48}$$

where we have used the saddle point expression for m and where $\Lambda = a^{-1}$. Now as stated, this result cannot be correct for small N, since for $N = 2$ it predicts that the propagator for the n field has nontrivial field strength renormalization, which cannot be true since in that case the n field is a map into S^1 and therefore has no interactions and is free. Later on we will see that the correct thing to do is to replace N by $N-2$.

Now we want to consider fluctuations about the saddle point. To this end, define the field ϕ through

$$\lambda = m^2 + \sqrt{\frac{2}{N}}\phi, \tag{49}$$

where the $q = 0$ momentum component of ϕ vanishes. Putting this in the $\mathrm{Tr}\ln$ and expanding to third order gives

$$S_{eff} = S_{eff,\phi=0} - \frac{N}{2}\left(\frac{1}{N}\int_q \phi_q\phi_{-q}\Pi_2(q^2) - \frac{2}{N}\sqrt{\frac{2}{N}}\int_{q,r}\phi_q\phi_r\phi_{-q-r}\Pi_3(q,r) + \dots\right), \tag{50}$$

where the first term is the saddle point, $\Pi_2(q^2)$ is the polarization bubble with two external ϕ legs, $\Pi_3(q^2)$ is the analogous diagram with three external ϕ legs, and so on. In these diagrams, the internal propagators are $G(p) = 1/(p^2+m^2)$. Note that there is no bubble with a single leg, since we are expanding around a saddle point. The $1/\sqrt{N}$ factor in front of the 3-legged bubble tells us that in the large N limit we can only worry about the polarization bubble as for as renormalizability goes. From this expression, we see that the propagator for the ϕ field is

$$G_\phi(q^2) = \frac{1}{\Pi_2(q^2)}. \tag{51}$$

Thus the λ field has gained some dynamics from the n field. Let us look at the large q limit of this expression to see how renormalization should work. We have

$$\Pi_2(q^2) = \int_p \frac{1}{(p^2+m^2)((p-q)^2+m^2)} = \int_p\int_x \frac{1}{(p^2+q^2(x^2+x)+m^2)^2}. \tag{52}$$

We can estimate this in the large q limit by noting that only small x will be important to the integral, so that defining $dy = q^2 dx$,

$$\Pi_2(q^2) \approx \int_p\int_y \frac{q^{-2}}{(p^2+y+m^2)^2} \approx \int_p \frac{q^{-2}}{p^2+m^2}, \tag{53}$$

where the integral over y was done from 0 to q^2. So then

$$\Pi_2(q^2) \approx \frac{1}{2\pi}\int dp \frac{pq^{-2}}{p^2+m^2}. \tag{54}$$

In the $q^2 \gg m^2$ limit then,

$$\Pi_2(q^2) \approx \frac{1}{2\pi q^2}\ln(q^2/m^2), \tag{55}$$

and so the propagator for the ϕ field is

$$G_\phi(q^2) = \frac{2\pi q^2}{\ln(q^2/m^2)}, \tag{56}$$

which diverges as $q \to \infty$, which is bad. This divergence will also appear in the propagator for the n field, which one can see by considering the first 1-loop correction to the n field two-point function.

The reason for this divergence is essentially due to the fact that we haven't renormalized the mass m^2 of the elementary excitations. As Polyakov points out in the book, the divergence arises because we are confining the particles to the sphere with the λ field, which because of the uncertainty relation leads to the above divergences. So we can fix this by adding a mass term for λ:

$$\mathcal{L} \mapsto \frac{1}{2g^2}\left[(\partial n)^2 + \lambda(n^2-1)\right] - \frac{\lambda^2}{4\beta}. \tag{57}$$

Solving the equation of motion for λ gives us a Lagrangian with a term $\beta(n^2-1)/g^4$, and so in the large β limit (small mass for λ) we recover the σ model. When we do the large N expansion with this term added, we just have to add the new mass term into the ϕ propagator:

$$G_\phi(q^2) \mapsto \frac{1}{\Pi_2(q^2)+\frac{1}{\beta N}}, \tag{58}$$

which resolves the power law divergences at large q.

So, we want to do mass renormalization. The way to do it is to ensure that the inverse propagator has no power divergences like the one appearing above for the naive ϕ propagator.

To this end we subtract off the zero momentum part of the self energy to define the mass renormalization, so that the n field Greens function is

$$G_n(q^2) = \frac{1}{q^2 + m^2 + \Sigma(q^2) - \Sigma(0)}. \tag{59}$$

Let us now check that subtracting off $\Sigma(0)$ cancels the power divergences. The first term in $\Sigma(q^2)$ is a diagram with a straight n field line of momentum q that has a ϕ field arc attached to it at momentum p. There are two factors of $\sqrt{2/N}$ coming from the two vertices, and so we can write (we are just interested in the structure of the divergences, so we will take $q^2 \gg m^2$ in what follows)

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N}\int_p \frac{1}{\Pi_2(p^2)}\left(\frac{1}{(p-q)^2} - \frac{1}{p^2}\right). \tag{60}$$

As we can see, if we didn't have the second $\Sigma(0)$ term, the power law divergence of the ϕ field propagator would infect the self energy of the n field, and we would not get something that was logarithmically divergent. So indeed, mass renormalization is the correct procedure for dealing with the power divergence.

Now combine the two fractions in the last integrand into one and then multiply the numerator and denominator by $(p+q)^2$:

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N}\int_p \frac{1}{\Pi_2(p^2)}\left(\frac{(2q\cdot p - q^2)(q^2 + p^2 + 2q\cdot p)}{p^2(p^4 + q^4 - 4(q\cdot p)^2)}\right). \tag{61}$$

Since the denominator is even under $p \mapsto -p$ we can simplify this to

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N}\int_{q<p<\Lambda} dp \frac{p}{2\pi\Pi_2(p^2)}\frac{4(q\cdot p)^2 - p^2 q^2}{p^6(1 + q^4/p^4 - 4(p\cdot q)^2/p^4)} + \text{finite}. \tag{62}$$

We can drop the terms other than the p^6 in the denominator and absorb them into the +finite part. Then using $\int p_\mu p_\nu \to \frac{1}{2}\delta_{\mu\nu}\int p^2$ since we are in two dimensions, we get

$$\Sigma(q^2) - \Sigma(0) = \frac{1}{N}\int_{q<p<\Lambda} dp \frac{2}{p\ln(p^2/m^2))} + \text{finite}. \tag{63}$$

Now $d\ln(\ln(p^2/m^2)) = 2/[p\ln(p^2/m^2)]$, so

$$\Sigma(q^2) - \Sigma(0) = \frac{1}{N}\ln\left(\frac{\ln(\Lambda^2/m^2)}{\ln(q^2/m^2)}\right) + \text{finite}. \tag{64}$$

As promised, we see that the new propagator has no power law divergences, but it does have a weird nested log structure. Thus mass renormalization has led to logarithmic divergences which can subsequently be cleaned up with field strength renormalization for the n field.

Now let us calculate the propagator directly (i.e. not assuming large N but assuming small $g^2(q)$), without using the saddle point approximation, and see to what extent the saddle point results are reproduced. We do this essentially by the background field method. We split up the field into slow and fast components by letting the slow component be some vector $n_0(x)$ and letting the fast components be fluctuations about that vector. Thus the relevant decomposition is

$$n = \sqrt{1-\psi^2}n_0 + \psi_a e^a, \tag{65}$$

where the $\{e^a\}$ are a collection of vectors orthogonal to $n_0(x)$ on the sphere[2] Note that $n_0(x)$ has unit length and is orthogonal to all of the frames e^a. Now since n lies on the sphere, we know that $\partial_\mu n_0(x)$ will lie in the tangent space at x and hence be orthogonal to $n_0(x)$ and expressible in terms of the e^a's. Thus for some 1-form ω^a and some $\mathfrak{o}(N-1)$-valued 1-form A (since $O(N-1)$ is the group which acts on the basis vectors normal to n_0), we may follow Polyakov and write

$$\partial^\mu n_0 = \omega^a_\mu e^a, \qquad \partial_\mu e^a = [A_\mu]^{ab} e_b - \omega^a_\mu n_0. \tag{67}$$

The minus sign in the second term is needed since n_0 being $\perp$ to all the e^a implies that the change of n_0 in the e^a direction is equal to the change of e^a in the $-n_0$ direction (draw a picture to check, or notice that the fact that the whole ensemble of frames rotates rigidly implies $n_0 \cdot \partial_\mu e^a + e^a \cdot \partial_\mu n_0 = 0$). Now we put this into the action for the n field (the $n^2 = 1$ constraint is explicitly built into our parametrization, so no need for Lagrange multipliers). The quadratic parts in the fields are

$$S_{\psi^2} = \frac{1}{2g^2}\int \left[(\delta_{ab}\partial_\mu\psi_b - A^{ab}_\mu\psi_b)^2 + B^a_\mu B^{b\mu}(\psi_a\psi_b - \psi_c\psi^c\delta_{ab}) + B^a_\mu B^{a\mu}\right], \tag{68}$$

where we have used the skew symmetry of A by virtue of the fact that it lives in $\mathfrak{o}(N-1)$. The $B^a_\mu B^\mu_a$ term is the slow $(\partial_\mu n_0)^2$ part.

Let us now compute the β function, the field strength renormalization, and the propagator. Because of the decomposition of the n field we have chosen we will work in the Wilsonian point of view where we change the high energy cutoff by a small amount. Note that A_μ has dimension 1 and appears in the action only in the covariant derivative. A_μ really is a gauge field, since it is a connection on a frame bundle which just tells us how to relate one arbitrary choice for the e^a frame at one point to the arbitrary choice made at another. Thus the only way that A can appear in what follows is in the Maxwell term $F_A \wedge \star F_A$, but since we are in two dimensions this is irrelevant. Thus we can ignore the A field in matters concerning renormalization.

Now let us compute the correlation function

$$C^{ab}_\psi = \langle \psi^a\psi^b - \psi_c\psi^c\delta^{ab}\rangle. \tag{69}$$

As mentioned above, we only need to care about the ω fields. Since the ω fields couple quadratically to the ψ's we have a single tetravalent vertex, and so to one loop order our only diagrams are a ψ loop and a ψ loop with an ω loop glued on. Since the free term for

[2] This is an expansion which is essentially tantamount to going over to Riemann normal coordinates. The action is $\partial_\mu n^a \partial^\mu n_a$, which for our present decomposition becomes schematically

$$\partial_\mu(n_0+\psi)^a\partial^\nu(n_0+\psi)^b(\delta_{ab} - \frac{1}{3}R_{acbd}\psi^c\psi^d), \tag{66}$$

where R_{acbd} is the Riemann curvature tensor for the target space S^{N-1}.

the ψ's is diagonal in a, b, we get

$$C_\psi^{ab} = (\delta^{ab} - (N-1)\delta^{ab}) \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{dp}{2\pi} \frac{g^2}{p}, \tag{70}$$

since there are $N-1$ fields in the $\psi_c \psi^c$ contraction. So then taking $d\Lambda = -\delta\Lambda$ to be infinitesimal,

$$C_\psi^{ab} = \delta^{ab}(N-2)\frac{g^2}{2\pi} d\ln\Lambda. \tag{71}$$

When we expand the exponential of the action in small ψ, do the integral over the fast fields, and then re-exponentiate, we thus get a term that goes as

$$\frac{1}{2g^2} \int B_\mu^a B^{a\mu}(N-2)\frac{g^2}{2\pi} d\ln\Lambda. \tag{72}$$

Therefore the effective charge is

$$g_{eff}^2 = \frac{g^2}{1 + g^2 \frac{N-2}{2\pi} d\ln\Lambda}. \tag{73}$$

Writing $g_{eff}^2 = g^2 + dg^2$ we obtain the beta function

$$\beta(g^2) = \frac{2-N}{2\pi} g^4, \tag{74}$$

which as we have seen several times so far is asymptotically free when the symmetry group $SO(N)$ is non-Abelian (or when the n field maps into a space with positive curvature). We will need the expression for $g^2(p)$, which we can get by integrating the β function from a reference scale μ to p:

$$g^2(p) = \frac{g^2(\mu)}{1 - \frac{2-N}{2\pi} g^2(\mu) \ln(p/\mu)}. \tag{75}$$

Now we want to get the 2-point function for the n fields and compare it to what we got with the saddle point approximation. We have

$$\begin{aligned}
\langle n(r) n(0) \rangle &= \langle n_0(r) n_0(0) \sqrt{(1-\psi^2(r))(1-\psi^2(0))} + \psi_a(r)\psi^a(0) \rangle \\
&\approx \langle n_0(r) n_0(0) (1 - \psi^2(0)) \rangle \\
&= \langle n_0(r) n_0(0) \rangle_{p<\Lambda-\delta\Lambda} (1 - \langle \psi_c(0) \psi^c(0) \rangle_{\Lambda-\delta\Lambda<p<\Lambda}),
\end{aligned} \tag{76}$$

where we dropped the $\psi(r)\psi(0)$ term since the ψ's are rapidly fluctuating, and expanded to quadratic order in ψ. We know the ψ^2 expectation value, since $\langle \psi_c \psi^c \rangle = -C_\psi^{aa} + \langle \psi_a \psi_a \rangle$ (no sum over a). So then

$$\langle n(r) n(0) \rangle = \langle n_0(r) n_0(0) \rangle_{p<\Lambda-\delta\Lambda} \left(1 + \frac{N-1}{2\pi} g^2(\Lambda) d\ln\Lambda \right), \tag{77}$$

since the two-point function for a single ψ component goes as $-d\ln\Lambda$.

From here we can read off the field strength renormalization: in order to ensure that the log divergences in the 2 point functions get canceled to one loop order, we need to have

$$\gamma(g^2) = \frac{N-1}{2\pi} g^2 \tag{78}$$

(the positive sign comes from the fact that the free propagator goes to minus of the log term).

Knowing this, we can finally get the expression for the n-field propagator which is not limited to the large N saddle point. Let the propagator be $G_n(p^2)$. Then we have

$$\frac{\partial (G_n p^2)}{\partial \ln(p/\mu)} = \gamma(g^2(p))(G_n p^2), \tag{79}$$

with μ the RG scale (momentarily going to a continuum RG picture). This equation is most easily proven graphically: the field strength renormalization counter terms appear in the full expression for the propagator in a geometric series of the form $\frac{1}{p^2}\gamma(g^2)\ln(p/\mu)\frac{1}{p^2} + \ldots$, which gives rise to the above equation. We can also write this as

$$\frac{\partial \ln(p^2 G_n)}{\partial \ln(p/\mu)} = \gamma(g^2(p)). \tag{80}$$

Thus to order g^2 we can use this and our knowledge of γ to get

$$\begin{aligned} G_n(p^2) &= \frac{1}{p^2} \exp\left(\frac{N-1}{2\pi} g^2(\mu) \int \frac{d\ln(p/\mu)}{1 + \frac{N-2}{2\pi} g^2(\mu)\ln(p/\mu)} \right) \\ &= \frac{1}{p^2}\left(1 + \frac{N-2}{2\pi} g^2(\mu)\ln(p/\mu)\right)^{\frac{N-1}{N-2}}. \end{aligned} \tag{81}$$

This would look like what we would expect from a free field provided that the term in the big parenthesis was not raised to a power. Thus as $N \to \infty$ we have

$$G_n(p^2; N \to \infty) = \frac{1}{p^2}\left(1 + \frac{N-2}{2\pi} g^2(\mu)\ln(p/\mu)\right), \tag{82}$$

which indeed indicates that the n field becomes free in the $N \to \infty$ limit.

Now we can compare this to the saddle point propagator we derived earlier. We see from the correlation function (48) that the difference between the saddle point answer and the actual result is just a replacement of N with the correct coefficient $N-2$ (and so in particular we get the correct behavior in the Abelian $N = 2$ case).

$SU(N)$ **nlσm:** Now we want to look at the case of the $SU(N)$ nonlinear σ model (here the fields are sections of $SU(N)$ bundles [though we will actually use $U(N)$], unlike the case of the $O(N)$ model where $O(N)$ was the symmetry group, not the target space. Sorry for the bad but standard terminology), which is very similar to the $O(N)$ model technically, but which must give us different answers since as we will see the $SU(N)$ version is *not* described by the saddle point in the large N limit.

To get the propagator we need the β function (so that we can get the effective coupling constant at an arbitrary momentum scale) and the anomalous dimension. We already found the β function back in (**??**). There we had $\lambda = 2g^2$, and we had taken $N = 2$. Getting the β function for general N isn't hard, though: we just have N terms in the trace rather than 2, and so we can translate our old result to

$$\beta(g^2) = -\frac{Ng^4}{4\pi}, \tag{83}$$

which holds to order g^4. Just like the $O(N)$ model we have asymptotic freedom, with it becoming "more" asymptotically free at large N. Integrating this gives the charge at a given momentum scale in terms of the RG scale μ:

$$g^2(p) = \frac{g^2(\mu)}{1 + \frac{N}{4\pi} g^2(\mu) \ln(p/\mu)}. \tag{84}$$

Now we need to know the anomalous dimension. This is also easy to get by looking back at the previous problem we did. We first write $T^a T^a = C_2 \mathbf{1}$ where the T^a are the $SU(N)$ generators. The quadratic casimir here is $C_2 = (N^2-1)/2N$ (check with $C_2 = 3/4$ for $SU(2)$, which works since the generators are $X/2, Y/2, Z/2$). By looking back at the previous problem, we find the anomalous dimension

$$\gamma = \frac{N^2-1}{2\pi N} g^2. \tag{85}$$

Then we can use the earlier formula for $\partial \ln(Gp^2)$ to get the propagator for the $SU(N)$ field:

$$\begin{aligned} G_U(p^2) &= \frac{1}{p^2} \exp\left(\int d\ln(p/\mu)\, \frac{C_2}{\pi} \frac{g^2(\mu)}{1 + \frac{N}{4\pi} g^2(\mu) \ln(p/\mu)} \right) \\ &= \frac{1}{p^2} \left(1 + \frac{N}{4\pi} g^2 \ln(p/\mu) \right)^{4C_2/N}. \end{aligned} \tag{86}$$

When we integrate this over p to get the correlator in real space, the exponent gets another power since we are integrating the above expression in parenthesis against $d\ln p$. In the $N \to \infty$ limit $4C_2/N + 1 \to 3$, so we get

$$G_U(r, N \to \infty) \approx \left(1 - \frac{Ng^2}{4\pi} \ln(r/a) \right)^3. \tag{87}$$

This does *not* go to just a single ln in the $N \to \infty$ limit, and so the $SU(N)$ theory does not look like a free theory as $N \to \infty$ (it actually *does* go to a free theory as $N \to \infty$, but the propagators are still not free and it only becomes free in a way that is hard to see with the present variables).

Given that the propagator is not free, the saddle-point approximation should not be good as $N \to \infty$. Why is this? Proceeding along the lines of the previous analysis, it would seem like we could take the $\mathrm{Tr} \ln$ in the effective action for the Lagrange multiplier and do a saddle point analysis on it, since the coefficient of the $\mathrm{Tr} \ln$ will contain an N.

This is not the case though, essentially because the matrix nature of the principal fields mean that the number of Lagrange multiplier fields grows with N, which destroys the saddle point. To see this, let us write the action as

$$S = \frac{1}{g^2} \int \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger], \tag{88}$$

where $U(x)$ is a $U(N)$-valued field. There is really no difference between this and the $SU(N)$ case since the determinant part (the Abelian $U(1)$) decouples and doesn't affect issues relating to renormalizability.

As with the $O(N)$ model, we enforce that $UU^\dagger = \mathbf{1}$ using a Lagrange multiplier (note the differing factor of 1/2 since we now are working with complex fields):

$$S = \frac{1}{g^2} \int \left(\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \mathrm{Tr}[\lambda(\mathbf{1} - UU^\dagger)] \right). \tag{89}$$

Integrating over the U's, which is possible since the addition of λ loosened the unitary constraint on the U's and turned the integral into a Gaussian, gives

$$S_{eff}[\lambda] = -\frac{1}{g^2} \int \mathrm{Tr}\lambda + N \mathrm{Tr} \ln(-\partial^2 \mathbf{1} + \lambda). \tag{90}$$

Here we get the N from e.g. diagonalizing and seeing that after taking the trace we are left with N complex scalars.

The effective action for λ looks essentially the same as it did for the $O(N)$ vector model, so it is natural to expect that λ gets an expectation value given by some saddle point condition:

$$\langle \lambda \rangle = m^2 \mathbf{1}. \tag{91}$$

Indeed, doing the saddle point analysis by varying with respect to λ_{aa} gives the same answer for $\langle \lambda_{aa}$ as it did for λ in the $O(N)$ vector model, just with slightly different numerical coefficients accounting for the complex nature of the field.

Just as we defined the ϕ fields in the $O(N)$ vector model, we define Φ fields as

$$\lambda = m^2 \mathbf{1} + \frac{1}{\sqrt{N}} \Phi, \tag{92}$$

where the trace of Φ has no $q = 0$ component, i.e. $\int \mathrm{Tr}\Phi = 0$, so that Φ doesn't affect the expectation value. Now put this into the effective action:

$$\begin{aligned} S_{\text{eff}} = &- \int \mathrm{Tr} \left[\mathbf{1} \left(\frac{m^2}{g^2} - p^2 - m^2 \right) \right] - \frac{1}{2} \int_q \Phi_{ab} [\Pi_2]_{ab,cd}(q^2) \Phi_{cd} \\ &+ \frac{1}{3\sqrt{N}} \int_{q,r} [\Pi_3]_{ab,cd,ef}(q,r) \Phi_{ab,q} \Phi_{cd,r} \Phi_{ef,-q-r} + \dots, \end{aligned} \tag{93}$$

where e.g.

$$[\Pi_2]_{ab,cd}(q^2) = \frac{\pi_2(q^2)}{2} [\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad}], \qquad [\Pi_3]_{ab,cd,ed}(q^2) = \frac{\pi_3(q,r)}{6} [\delta_{ae}\delta_{fd}\delta_{cb} + \delta_{af}\delta_{ed}\delta_{bc} + \dots], \tag{94}$$

where the $\ldots$ indicates all ways to pair the indices in Kronecker delta functions so that a does not appear in the same δ as b and likewise for the pairs cd and ef. Here π_2 is the regular polarization bubble (with a fixed color structure) with internal propagators computed using the mass m^2 and π_3 is the three-legged version.

In the $O(N)$ model the saddle point was stable and we got a free theory since the nonlinear part with the Π_3 kernel made an insignificant contribution as N went to infinity, allowing the corrections to the free Φ propagator to vanish. This is not so in the $SU(N)$ model, though, and the saddle-point does not work. We see this by writing the Π_3 term in double-line notation as a sum of three-tentacled amoeba, where the sum is over the various ways to assign matrix indices to the six ends of the lines in the amoeba. The first order correction to the free Φ propagator caused by the Π_3 term thus looks like $N^{-1} \times \mathbb{P}$, where $\mathbb{P}$ is a double-line version of the regular polarization bubble. Importantly, it has a completely internal circular line that when summed over produces a factor of N, cancelling the N^{-1} that came from the two factors of $1/\sqrt{N}$. Thus this contribution can not be ignored as $N \to \infty$, and so the nonlinearities are important even at large N, corroborating our finding that the $SU(N)$ model was not free at large N. Proceeding in the standard way and using the Gauss-Bonnet theorem, one can show that all planar diagrams do not vanish as $N \to \infty$, while diagrams that glue up on higher-genus surfaces become increasingly small.

4 *WZW Miscellanea*

Today's problem is a smattering of little things about the WZW term. The format will be a series of mini-questions. I found these questions listed as exercises in Abanov's lecture notes on topological terms in QFT.

Preliminaries: For a vector n that lives in S^2, define

$$W = \frac{1}{16\pi i}\int_D \mathrm{Tr}[\widehat{n} \wedge d\widehat{n} \wedge d\widehat{n}], \tag{95}$$

where D is a two-disk and $\widehat{n} = n^a\sigma^a$. Show that the variation of W only depends on the values that the n field takes on ∂D. Also show that

$$S_{WZW} = 4\pi S W[n^a] \tag{96}$$

is well-defined as an action on the manifold ∂D provided that $S \in \frac{1}{2}\mathbb{Z}$.

First we rewrite W as

$$W = \frac{1}{16\pi i}\int_D i\epsilon^{abc}\mathrm{Tr}[n^a dn^b \wedge dn^d \sigma^c\sigma^d] = \frac{1}{8\pi}\int_D \epsilon^{\mu\nu} n\cdot(\partial_\mu n \times \partial_\nu n), \tag{97}$$

where we've written out the derivative explicitly since $dn \wedge \times dn$ is bad notation. Now we vary this, sending $n \mapsto n + \delta n$. Since $n^1 = 1$, $\delta n \cdot n = 0$. But since $\partial_\mu n$ is also orthogonal to n, $\partial_\mu n \times \partial_\nu n$ is parallel to n, and so $(\delta n) \cdot (\partial_\mu n \times \partial_\nu n) = 0$. Thus the variation is

$$\delta W = \frac{1}{8\pi} \int_D \epsilon^{abc} n^a (d\delta n^b \wedge dn^c + dn^b \wedge d\delta n^c). \tag{98}$$

Integrating both terms by parts and again using that $(\delta n) \cdot (\partial_\mu n \times \partial_\nu n) = 0$ as well as the supercommutativity of $\wedge$, we get

$$\delta W = \frac{1}{4\pi} \int_{\partial D} dx^\mu \; \epsilon^{abc} \delta n^a \partial_\mu n^b n^c, \tag{99}$$

and so indeed, the variation only cares about the fields on the boundary. This is because W is measuring the area on the target S^2 swept out by the n field. This area only depends on the trajectory of the n field as one moves around the circle ∂D. Indeed, the boundary values of n on ∂D trace out some region on the target S^2, and the values of the field on the interior of D "fill in" the area enclosed by this trajectory in a smooth way. By thinking of this visually, it's clear that changing the way that D "fills in" this region cannot change the total (signed) area swept out by the n field.

Now for the well-definedness of S_{WZW}. The difference in two extensions to the 2-manifold D from the given ∂D is the integral of S_{WZW} over a closed 2-manifold M. Since δW only depended on the values of n on the boundary, since $\partial M = 0$ we know that the values of W evaluated over M must be quantized. All we need to do is check the coefficient. We've got

$$S_{WZW} = S \int_M \epsilon^{abc} \frac{1}{2} n^a (dn^b \wedge dn^c). \tag{100}$$

But the integrand on the RHS is precisely the pullback of one half of the volume form on S^2 by the map $n : M \to S^2$ (the 1/2 because of the antisymmetrization and since the volume form is $\epsilon^{abc} n^a \partial_x n^b \partial_y n^c$ if the coordinates are x, y). So then

$$S_{WZW} = S \int_{n(M) \subset S^2} \text{vol} = 4\pi S w, \tag{101}$$

where $w \in \mathbb{Z}$ is the winding number. Thus we get something well-defined provided that $S \in \frac{1}{2}\mathbb{Z}$. The most suggestive way to write S_{WZW} is probably

$$S_{WZW} = 2S \frac{2\pi}{\Omega_2} \frac{1}{2} \int_D \epsilon^{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n), \tag{102}$$

where $\Omega = 4\pi$. If we let D be a sphere, then the integral gives us an integer multiple of $2\Omega_2$ (where the 2 comes from the antisymmetrization of the derivatives in spacetime—if we used a $\wedge$ for the spacetime derivatives we wouldn't need the 2).

Spin precession: Now add a magnetic field h, so that

$$S[h] = S_{WZW}[n] - S \int dt \; h^a n^a. \tag{103}$$

Assume h is independent of time for simplicity. Derive spin precession in an $SU(2)$ invariant way.

We just compute the equations of motion. To do this, add a Lagrange multiplier to enforce $n^2 = 1$ so that we can vary n freely:

$$S[h, \lambda] = S_{WZW}[n] - S \int dt \; h^a n^a + \frac{1}{2}\lambda(n^2 - 1). \tag{104}$$

We can use our knowledge from the first part to write

$$\delta S = \int dt \; \left(S\epsilon^{abc}\partial_t n^b n^c + \lambda n^a - Sh^a \right) \delta n^a. \tag{105}$$

Taking the term in the parenthesis and dotting it with n,

$$\lambda = Sh^a n^a - S\epsilon^{abc} n^a \partial_t n^b n^c = Sh^a n^a. \tag{106}$$

Thus the equation of motion gives

$$\epsilon^{abc}\partial_t n^b n^c + n^a(h^b n^b) - h^a = 0, \tag{107}$$

or as a vector equation,

$$\partial_t n \times n = h - (h \cdot n)n. \tag{108}$$

Note that as expected, if we set n parallel to h then n doesn't want to move, while if we set n normal to h then n wants to precess about the h axis.

Quantization: Derive the commutation relations for the spin operator from S_{WZW}. Use this to check that the equations of motion you derived earlier correspond to the Heisenberg equations of motion.

To derive the spin commutation relations we should figure out what the symplectic form is. Since we only have the ϕ and θ variables, they will label the phase space. Since the phase space is a sphere, we expect a closed but not exact symplectic form. We want to start from the term

$$S_{WZW} = \frac{S}{2} \int_D \epsilon^{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n). \tag{109}$$

We can proceed in kind of a dumb way by varying this to find the symplectic current and then taking a second variation to find what the symplectic form is, after which the Poisson brackets can be read off. Luckily we already computed the variation in terms of the n field. Plugging in the coordinate representation for n, doing a fair bit of algebra (use Mathematica as it's fairly heinous), and using the equations of motion gives the symplectic current. Taking another variation gives the symplectic form, which if I did the algebra correctly is

$$j = -\delta(S\cos\theta) \wedge \phi \implies \Omega = -\delta(S\cos\theta) \wedge \delta\phi, \tag{110}$$

where the expression is evaluated at some given initial time (symplectic currents are integrated over codimension 1 Cauchy slices, which in our case are just points). Thus from the symplectic form we conclude that the symplectic partner for ϕ is $S\cos\theta$, and so

$$\{\phi, S\cos\theta\} = 1. \tag{111}$$

Thus when we take the Poisson bracket of two quantities we do it as follows:

$$\{X, Y\} = -\frac{1}{S\sin\theta}(\partial_\phi X \partial_\theta Y - \partial_\phi Y \partial_\theta X), \tag{112}$$

which indeed gives $\{\phi, S\cos\theta\} = 1$.

As expected, these expressions only make sense within a coordinate patch, and the symplectic form is not globally exact, since it is the field strength for a monopole configuration on S^2.

Anyway, now we can use this to compute the Poisson bracket of the different n vectors. The relevant ones are $S^x = S\cos\phi\sin\theta,\ S^y = S\sin\phi\sin\theta,\ S^z = S\cos\theta$. For example,

$$\{S^y, S^z\} = -\frac{S}{\sin\theta}(-\cos\phi\sin^2\theta + 0) = S^x. \tag{113}$$

Of course, the general rule is $\{S^a, S^b\} = \epsilon^{abc}S^c$. When we pass to quantum mechanics we change to commutators and add in the i, and as a result derive the spin commutation relations. Note that we did not start from anything involving Pauli matrices to do this.

Now we check the Heisenberg equations of motion. The WZW term doesn't enter into the Hamiltonian since it is linear in time derivatives (but of course it still affects the equations of motion, unlike e.g. a theta term). Then we can compute

$$[H, S^a] = [S^a, S^b]h^b = i\epsilon^{abc}S^c h^b \implies \partial_t S^a = \epsilon^{abc}S^b h^c \implies \partial_t n^a = \epsilon^{abc}n^b h^c. \tag{114}$$

Is this equivalent to the equations of motion we derived earlier? Yes: putting this into the equations of motion implies

$$h^a - (h^b n^b)n^a = \epsilon^{abc}\epsilon^{bde}n^d h^e n^c = (\delta_{cd}\delta_{ae} - \delta_{ad}\delta_{ce})n^d h^e n^c, \tag{115}$$

which is a trivial equality.

Relation to the θ term: Now suppose that the value of n on the 1-manifold is constrained to lie on a circle of constant latitude, with θ a constant. Find S_{WZW} in this case (note that we do not impose this restriction on the extension of n into the disk [and it will in general not be possible to impose such a condition]). Show that making this restriction on θ turns the WZW term into a θ term.

Also, in an AFM chain, show how the WZW term gives rise to a θ term that is nontrivial when $2S$ is odd. Sort of conversely, motivate why an $S = 1$ chain with a θ term placed on an open line has two spin 1/2's at the boundary.

First let's compute δW with this restriction. In spherical coordinates,

$$\delta n = (-\delta\phi\sin\phi\sin\theta,\ \delta\phi\cos\phi\sin\theta,\ 0)^T. \tag{116}$$

Similarly, $\partial_\mu n$ also has no z component. Thus the integrand of δW goes like $n^z(\delta n \times \partial_t n)^z$. But from the form of δn, the cross product vanishes, and so $\delta W = 0$. This means that with this restriction on θ, W is quantized. Thus in order to compute W, we are free to choose a convenient extension of n into the 2-manifold as well as a convenient shape for the 2-manifold, since W cannot change if we change either of these choices. So we will choose the bounding 2-manifold to be the unit disk, with coordinates ϕ, r. We are even allowed to choose the ϕ coordinate of the disk to be an integer multiple of the ϕ coordinate on the S^2 that n maps into, where the integer is the winding number of the spacetime (really just time) S^1 into the $S^1 \subset S^2$ defined by the constant value of θ. If this winding number is w, then with this restriction we can choose n to be

$$n = (-\sin(w\phi)\sin(r\theta),\ \cos(w\phi)\sin(r\theta),\ \cos(r\theta))^T. \tag{117}$$

The first two components reduce to w times the volume form on S^1 at $r = 1$ where the field is constrained to lie on the constant θ circle. This field extension we've chosen looks like a cowlick on the head of a particularly curly-haired person, with the hair all standing straight up at the center of their head.

Putting this into (no Jacobian changing from Cartesian coordinates since our integrand is a differential form)

$$S_{WZW} = S \int_{D^2} drd\phi\ \epsilon^{abc} n^a (\partial_r n^b \partial_\phi n^c) \tag{118}$$

and doing some algebra gives

$$S_{WZW} = 2\pi i S w(\cos\theta - 1), \tag{119}$$

which is just measuring the area on the sphere defined by the line of constant latitude at the given value of θ. So, we get a θ term (where the action is proportional to the volume form on the temporal S^1) with coefficient iSw. As it must, when $\theta = \pi$ (the polar angle of the n vector, not the θ of the θ term—sorry!) this gives us something in $2\pi i\mathbb{Z}$. Also note that when the vector is restricted to lie on the equator, we get a theta term at theta-angle $\theta = \pi$ (relevant for te AFM spin chain).

For the AFM spin chain, the action has a term which is just $\sum_i S_{WZW}[n_i]$, where n_i is the n-field at the site i. Since we expect a staggered configuration to be a good "mean field", we can as a first pass examine $\sum_i S_{WZW}[(-1)^i n_i]$, where now n_i is slowly varying. Since $S_{WZW}[n]$ measures the area on the target S^2 enclosed by the trajectory of the n field, $S_{WZW}[n_j] + S_{WZW}[-n_{j+1}] = S_{WZW}[n_j] - S_{WZW}[n_{j+1}]$ measures a difference in the areas swept out by the two curves along their trajectories. Drawing a picture shows that this area difference is proportional to $\int dt\ \epsilon^{abc} n^a \partial_x n^b \partial_t n^c$, where x is the coordinate along the chain. Integrating this over the length of the chain, we get

$$\frac{1}{2} S \int \epsilon^{abc} n^a dn^b \wedge dn^c. \tag{120}$$

Note that this term now maps the spacetime S^2 into the target S^2, and is defined without reference to a bounding 3-manifold. Since it integrates to $2\pi S w$ where w is the winding number, it is a θ term which only contributes if $2S$ is odd. Thus the WZW terms sum up and interfere with eachother to produce a θ term.

Now suppose we are given the nlσm description of an open AFM chain with the θ term, for $2S$ even. Suppose we already know that the chain is gapped for this choice of S, so that we can focus on the θ term. Now the θ term is the integral of the volume form over the chain. The volume form is closed and is also exact since spacetime has trivial H^2, so the θ

term must reduce to something localized on the ends of the chain. Indeed, plugging in the form for n means the θ term is, after some algebra,

$$\frac{S}{2}\int dxdt\ \sin\theta(\partial_t\theta\partial_x\phi - \partial_x\theta\partial_t\phi). \tag{121}$$

Let's assume that we're working with periodic time. Then we can integrate by parts and write the term as

$$-\frac{S}{2}\int_{\partial C}\cos\theta\partial_t\phi, \tag{122}$$

where the integral is over the ends of the chain. This means that we get WZW terms on the chain ends. In particular if $S=1$ then we get WZW terms with spin 1/2, suggesting that the critical $S=1$ AFM spin chain hosts spin 1/2's at its edges.

5 *WZW action from fermions*

This problem is one of the problems I found listed as an exercise in Abanov's lecture notes on topological terms in QFT.

Consider a fermion in 0+1 dimensions coupled to a vector:

$$S = \int d\tau\ \bar{\psi}(\partial_\tau + Mn^a\sigma^a)\psi, \tag{123}$$

where M is a mass and n maps into S^2. Integrate out the fermion and find the leading two terms in a $1/M$ expansion. To do this, it is helpful to first compute δS and then un-do the variation at the very end.

Integrating over the fermions,

$$\delta S = -\delta\mathrm{Tr}\ln(\partial_\tau + M\widehat{n}), \tag{124}$$

where we have used the notation $\widehat{n} = n^a\sigma^a$. Let $D = \partial_\tau + M\widehat{n}$ be the "covariant derivative". Then

$$\delta S = -\mathrm{Tr}(\delta D D^{-1}) = -\mathrm{Tr}(\delta D D^\dagger(DD^\dagger)^{-1}), \tag{125}$$

which we have written in this way since expanding $DD^\dagger$ in $1/M$ is easier. Now since $\widehat{n}^2 = n^2\mathbf{1} = \mathbf{1}$,

$$DD^\dagger = (\partial_\tau + M\widehat{n})(-\partial_\tau + M\widehat{n}) = -\partial_\tau^2 + M^2 + M\partial_\tau\widehat{n}. \tag{126}$$

We then expand the inverse of this in the following way, writing $G_f = (-\partial_\tau^2 + M^2)^{-1}$ for the free propagator:

$$(DD^\dagger)^{-1} = \frac{G_f}{\mathbf{1} + G_f M\partial_\tau\widehat{n}} = \frac{G_f(\mathbf{1} - G_f M\partial_\tau\widehat{n})}{1 - (G_f M\partial_\tau n)^2}, \tag{127}$$

where the denominator on the RHS is now just a number. To the leading orders in M we can actually just replace the denominator by 1, and so

$$\delta S = -\mathrm{Tr}\left[(M\delta\widehat{n})(-\partial_\tau + M\widehat{n})G_f(\mathbf{1} - G_f M\partial_\tau\widehat{n})\right]. \tag{128}$$

We do the trace by

$$\mathrm{Tr}[\mathcal{O}] = \int d\tau \int \frac{d\omega}{2\pi} e^{i\omega t} \mathrm{Tr}_\sigma[\mathcal{O}] e^{-i\omega t}, \tag{129}$$

where Tr_σ indicates a trace over the spin degrees of freedom. We wrote the trace like this (by inserting a resolution of the identity in frequency space) since the operator $\mathcal{O}$ in question isn't local in time (it involves a bunch of G_f's), but is local in frequency space. Writing this out,

$$\delta S = -\frac{M}{2\pi} \int d\tau d\omega \; \mathrm{Tr}_\sigma \left[\delta\widehat{n}(i\omega + M\widehat{n}) \frac{1}{\omega^2 + M^2} \left(\mathbf{1} - \frac{1}{\omega^2 + M^2} M \partial_\tau \widehat{n} \right) + \delta\widehat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau^2 \widehat{n} \right]. \tag{130}$$

Dropping things that will die after taking the spin trace and things that are odd in frequency,

$$\delta S = \frac{M}{2\pi} \int d\tau d\omega \; \mathrm{Tr}_\sigma \left[\delta\widehat{n} M \widehat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau \widehat{n} - \delta\widehat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau^2 \widehat{n} \right]. \tag{131}$$

The relevant integral is

$$\int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + M^2)^2} = \frac{1}{4M^3}, \tag{132}$$

and so

$$\delta S = \int d\tau \; \mathrm{Tr}_\sigma \left[\frac{1}{4} \delta\widehat{n}\widehat{n} \partial_\tau \widehat{n} - \frac{1}{4M} \delta\widehat{n} \partial_\tau^2 \widehat{n} \right]. \tag{133}$$

Taking the trace and integrating the second term by parts,

$$\delta S = \int d\tau \left(-\frac{i}{2} \epsilon^{abc} \delta n^a \partial_\tau n^b n^c + \frac{1}{2M} \delta(\partial_\tau n)^a \partial_\tau n^a \right). \tag{134}$$

Looking at yesterday's problem, we see that the first term is precisely the variation of the WZW term for a single spin 1/2 in 0+1 dimensions (spin 1/2 because of the prefactor), while the second term is the variation of a kinetic term for the vector. So the effective action is

$$S_{eff} = S_{kin} + S_{WZW} = \frac{1}{4M} \int d\tau \; (\partial_\tau n^a)^2 - \frac{1}{8\pi i} \int_D \mathrm{Tr}[\widehat{n} \wedge d\widehat{n} \wedge d\widehat{n}], \tag{135}$$

where D is any two-disk that bounds the temporal circle.

6 QED_3 as a σ model

The goal of this problem is to work out the details of a description of QED_3 as a σ model that I read about in [**?**].

The starting point is QED_3 with $N = 2$ flavors of Dirac fermions and the square of an $SU(2)$ vector that we want to use as a potential order parameter field. The action is

$$S = \int \left(\bar{\psi} i \not{D}_A \psi + g(\bar{\psi}\sigma^a \psi)^2\right), \tag{136}$$

where σ^a is a Pauli matrix in flavor space (not spin space!) and we've omitted the $F \wedge \star F$ term. Your mission is to show that at long distances,[3] this behaves like an $O(4)$ model with a theta term at $\theta = \pi$.

To show this, first decouple the $g(\bar{\psi}\sigma^a\psi)^2$ term with a 3-component vector N^a. After this is done, treat the vector as having a fixed length, with $N^a = Mn^a$ for n a unit vector in S^2. Working with M large, integrate out the fermions and find the current. There is also a Hopf term hiding in the resulting effective action, which you should find (it will give spin to the skyrmions).

Switching to a $\mathbb{CP}^1$ representation for the vector, show that you get a mixed CS term between the electromagnetic gauge field and the $\mathbb{CP}^1$ gauge field. Show that this in turn reduces to the $O(4)$ model with a θ term at $\theta = \pi$.

First we do the decoupling. We add the term $N^2/2g$ to the action where N is a 3-component vector, and then shift

$$N^a \mapsto N^a + ig\bar{\psi}\sigma^a\psi. \tag{137}$$

We then take $N = Mn$ for n a unit vector (this restriction is supposed to not affect the phase diagram), and so

$$S = \int \bar{\psi}\left(i\not{D}_A + iMn^a\sigma^a\right)\psi. \tag{138}$$

Let's first find the current after integrating out the fermions, which we will see gives electric charge to the solitons. Using the trick of two days ago, we have

$$J^\mu = i\frac{\delta}{\delta A_\mu}\mathrm{Tr}\ln\left(i\not{D}_A + iMn^a\sigma^a\right) = i\mathrm{Tr}\left(\gamma^\mu (D^\dagger D)^{-1} D^\dagger\right), \tag{139}$$

with

$$D = i\not{\partial} + iMn^a\sigma^a. \tag{140}$$

Here γ^μ are the Pauli matrices acting on spin space (we are in Euclidean signature), and we have set $A = 0$ in D since to find the current we can take the functional derivative with respect to J and then set the gauge field to zero. We then expand

$$(D^\dagger D)^{-1} = \frac{1}{-\partial^2 + M - M(\not{\partial}n^a)\sigma^a} = \frac{G_f}{\mathbf{1} - G_f M(\not{\partial}n^a)\sigma^a}, \tag{141}$$

[3] And long distances are really all we can talk about: from the kinetic term, we see that the dimension of ψ is 1, so that $[g] = -1$ and the model is (naively) non-renormalizable. We will be relating this to the $O(4)$ nlσm in three dimensions, which is also non-renormalizable, and we will only be able to make statements about IR physics in the following.

where as usual $G_f = (-\partial^2 + M^2)^{-1}$. To write it in this form, we have to remember that γ^μ and σ^a are both Pauli matrices, but commute since they act on different tensor factors. We expand this as

$$(D^\dagger D)^{-1} \approx G_f(\mathbf{1} + G_f M(\partial\!\!\!/ n^a)\sigma^a)(\mathbf{1} + G_f M(\partial\!\!\!/ n^b)\sigma^b). \tag{142}$$

After putting this into our expression for J^μ and dropping things which get obviously killed by the momentum integration or the spin trace,

$$J^\mu = i\mathrm{Tr}\left(\gamma^\mu \left[2G_f^2 M^2(\partial\!\!\!/ n^a)\sigma^a + G_f^3 M(\partial\!\!\!/ n^a)\sigma^a(\partial\!\!\!/ n^b)\sigma^b\right] iMn^c\sigma^c\right). \tag{143}$$

The first term goes like $(\partial_\mu n^a)n^a$ (with the index structure required if it wants to survive the flavor trace) which dies since n being a unit vector means that it's orthogonal to ∂n. In order to survive the traces over the spin and flavor indices, we need a $\epsilon^{\mu\nu\lambda}\epsilon^{abc}$ index structure. Thus

$$J^\mu = M^3\mathrm{Tr}(\mathbf{1}_{4\times 4})\int d^3x \int \frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+M^2)^3}\epsilon^{\mu\nu\lambda}\epsilon^{abc}n^a\partial_\nu n^b\partial_\lambda n^c, \tag{144}$$

which is starting to look a lot like a skyrmiony thing. Indeed, the integral gives $1/(32\pi|M|^3)$ and so

$$J^\mu = \frac{\mathrm{sgn}(M)}{8\pi}\int \epsilon^{abc}n^a[\star(dn^b\wedge dn^c)]^\mu. \tag{145}$$

Thus, we have shown that skyrmions carry electric charge! The current that couples to electromagnetism is precisely the topological charge density.

There are a few more parts of the effective action that we need to get. One is the kinetic term for n. We get the kinetic term by taking a variation with respect to n^a. Writing the variation of the trace as $\mathrm{Tr}[\delta D D^\dagger(DD^\dagger)^{-1}]$ and expanding, to leading order in the large M expansion the relevant term comes when the $i\partial\!\!\!/$ in $D^\dagger$ hits the term in the expansion for $(DD^\dagger)^{-1}$ that is linear in M. We get

$$\begin{aligned}\delta S &= -\mathrm{Tr}[M^2\delta n^a\sigma^a G_f^2\partial\!\!\!/^2 n^b\sigma^b] + \dots \\ &= -4M^2\int d^3x\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+M^2)^2}\delta n^a\partial_\mu\partial^\mu n^a + \dots.\end{aligned} \tag{146}$$

The integral is $1/8\pi M$, and so after integrating by parts over spacetime and integrating over the variation (we have $(\delta\partial_\mu n^a)\partial^\mu n^a$ which is a total variational derivative) we get

$$\delta S = \frac{M}{4\pi}\int d^3x(\partial_\mu n^a)^2. \tag{147}$$

The last thing in the effective action we need is a topological term, which bestows spin upon the skyrmions. Since it's non-perturbative we have to find a clever way of getting to it. Such a way can be found in e.g. [**?**] (or in the classic paper by Wilczek and Zee), which we now go through. Basically, the idea is to consider a particular field history for a skyrmion and find what the topological term is in that case, and then write down the general topological term by writing down a covariant version of the specific topological term.

We consider a field configuration on e.g. $S^2\times S^1$ with a skyrmion that rotates by 2π along the S^1 factor. This trajectory measures the spin angular momentum of the skyrmion, since the phase acquired during the trajectory is $e^{iS} = e^{2\pi i J}$. If the skyrmion is to be rotated around z axis as we pass around the S^1, then we can consider the configuration where

$$n^a\sigma^a = e^{i\sigma^z\alpha(t)/2} n_0^a \sigma^a e^{-i\sigma^z\alpha(t)/2}, \tag{148}$$

where t is the S^1 coordinate, $\alpha(2\pi) = \alpha(0) + 2\pi$, and n_0 is a static reference vector.

To get at the angular momentum of the skyrmion in a perturbative way, we can perturb in $\partial_t\alpha$ since the evolution is adiabatic. To bring a time derivative of α into the functional determinant, we consider performing the transformation $\psi \mapsto e^{i\alpha(t)\sigma^z/2}\psi$. This is not single-valued though, and it changes the boundary conditions of the fermions around the S^1 (from periodic to antiperiodic or vice-versa). We can fix the boundary conditions by doing a further transformation which only affects the phase by $\psi \mapsto e^{i\alpha(t)/2}\psi$, which cancels out the change in boundary conditions. Doing this transformation kills the exponentials in the $e^{i\sigma^z\alpha(t)/2} n_0^a \sigma^a e^{-i\sigma^z\alpha(t)/2}$ term, and picks up a time derivative term from the Dirac operator. Thus we have to evaluate

$$\operatorname{Tr}\ln\left(i\not{D}_A + \frac{i\gamma^0}{2}(\mathbf{1}+\sigma^z)\partial_t\alpha + iMn_0^a\sigma^a\right). \tag{149}$$

The angular momentum of the skyrmion is obtained by differentiating the effective action with respect to the angular frequency with which the skyrmion rotates (think $E = m\omega^2r^2/2 \implies \partial_\omega E = mvr = J$), and so to get the spin of the skyrmion we need to functionally differentiate the above with respect to $\partial_t\alpha$. This is easy though, since we can just look back at how we computed the variation of the effective action with respect to the gauge field—$\partial_t\alpha$ appears in the functional determinant in the exact same way as A_0, except with an extra $(\mathbf{1}+\sigma^z)/2$ tacked on. Computing the variation in the same way as before and then integrating over the variation gives the topological term (a Hopf term, as we will see shortly) on this manifold:

$$S_H = \frac{i}{2}\int_{S^2\times S^1} \partial_t\alpha J^0, \tag{150}$$

where J^0 is the time component of the topological current given earlier (selected out because of the γ^0 in the trace). Note that since the topological charge is conserved, the Hopf term vanishes if α is topologically trivial, i.e. if it does not wind by some element of $2\pi\mathbb{Z}$ (since if it does not wind we can integrate by parts and get zero).

So, we've found a topological term that is constructed exactly in the same way as the coupling of A_μ to the topological current, except that A_μ is replaced by parts of the n field (in our example it is replaced by $\partial_t\alpha$). Because of the normalization of the topological current and the factor of $1/2$ in front, we see that the above term gives the skyrmions spin $1/2$.

To find the general presentation of the Hopf term, we just have to "covariantize" the particular form of S_H found above. One way to do this is to write it in terms of the $SU(2)$ matrix U which rotates between σ^z and $n^a\sigma^a$: with some algebra one can check that we get

$$S_H = -i\pi\frac{1}{24\pi^2}\int \operatorname{Tr}[U^\dagger dU \wedge U^\dagger dU \wedge U^\dagger dU]. \tag{151}$$

Actually, a slightly cooler presentation of this action is as a linking number a la the usual interpretation of the Hopf invariant. To write it down in this way we need to recast stuff in the $\mathbb{CP}^1$ language.

Let's then switch over to $\mathbb{CP}^1$ variables. The kinetic term for the n vector becomes the $|D_a z|^2$ term (here $a_\mu = -iz^\dagger \partial_\mu z$), where $n^a \sigma^a = 2zz^\dagger - \mathbf{1}$. What about the $A^\mu J_\mu$ term coupling the skyrmions to the electromagnetic field? The topological current J_μ maps to $i \star da/2\pi = i \star (dz^\dagger \wedge dz)/2\pi$ in the $\mathbb{CP}^1$ variables, which one can see either with a fair bit of algebra or with a "what else could it be" argument (the coefficient is fixed by the integrality of the topological charge).[4]

Finally, the Hopf term is just like the $A_\mu J^\mu$ coupling, except it has no A and only involves the n field. This means that in the $\mathbb{CP}^1$ language, it becomes a CS term for the $\mathbb{CP}^1$ gauge field, $a \wedge da$ (this is another way to see that it computes the Hopf invariant). Thus our new action is

$$S = \frac{M}{4\pi} \int |(\partial_\mu - ia_\mu) z|^2 + \frac{i}{2\pi} \int A \wedge da + \frac{1}{2e^2} \int F_A \wedge \star F_A + S_H, \tag{155}$$

where S_H is now the CS term for a.

Since the $\mathbb{CP}^1$ variables were introduced to deal with a theory involving a vector living in S^2, we do not yet have the $O(4)$ model we were promised. To get an $O(4)$ model, we have to access the full $O(4)$ symmetry of the z variables, instead of the $O(3)$ symmetry we get when acting on the quotient $S^3/S^1 = S^2$. Thus to get an $O(4)$ symmetry, we need to "eliminate the $\mathbb{CP}^1$ gauge field a", since it is responsible for quotienting out by S^1.

We can see how this might happen by doing the integral over A and checking what the resulting action for the z fields is. The parts in the action involving A are

$$S_A = \frac{1}{2e^2} \int F_A \wedge \star F_A + \frac{i}{2\pi} \int a \wedge F_A. \tag{156}$$

To integrate over A carefully, we first add an extra field ϕ in the path integral that enforces the exactness of F, and then treat F as an unconstrained field that we path-integrate over independently. The new action for the gauge fields reads

$$S_A = \frac{1}{2e^2} \int F \wedge \star F + \frac{i}{2\pi} \int a \wedge F_A + \frac{i}{2\pi} \int \phi \wedge dF. \tag{157}$$

We can now perform the shift

$$F \mapsto F + \alpha \star a, \tag{158}$$

[4] More carefully, working in a parametrization where

$$z = \begin{pmatrix} \cos(\theta/2) e^{i(\gamma+\phi)/2} \\ \sin(\theta/2) e^{i(\gamma-\phi)/2} \end{pmatrix}. \tag{152}$$

Then (factors of 2 and minus signs?!)

$$da = -i dz^\dagger \wedge dz = -\sin(\theta/2)\cos(\theta/2)\, d\theta \wedge d\phi = \sin(\theta/2)\, d\theta \wedge d\phi, \tag{153}$$

which is the the pull-back of the volume form on S^2, and hence $\star da$ is indeed the topological current. This can also be written as

$$[F_a]_{\mu\nu} = (D_{[\mu} z)^\dagger D_{\nu]} z, \tag{154}$$

which can be checked with a little bit of algebra.

where α is some constant to be determined. A little algebra shows that choosing $\alpha = -ie^2/2\pi$ cancels out the CS coupling of F to a. After doing this shift, we have

$$S_A = \frac{1}{2e^2}\int F \wedge \star F + \frac{i}{2\pi}\int \phi \wedge dF + \frac{e^2}{4\pi^2}\int \phi \wedge d \star a + \frac{e^2}{8\pi}\int a \wedge \star a. \tag{159}$$

If we work in transverse gauge $d^\dagger a = 0$ things become simpler since then $d \star a = 0$ and the extra $\phi \wedge d \star a$ term dies. Then we can do the integral over ϕ to set $F = dA$, and so

$$S_A = \frac{1}{2e^2}\int F_A \wedge F_A + \frac{e^2}{8\pi}\int a \wedge \star a. \tag{160}$$

Thus we have traded the CS coupling between the gauge fields for a mass term for a. The Maxwell field is now totally decoupled, and just gives a constant in the path integral which we can drop. Furthermore, the massiveness of a induced by the electromagnetic field means that we can integrate it out and get

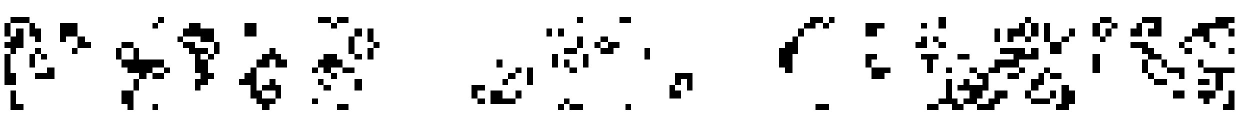
$$S = \frac{M}{4\pi}\int |\partial_\mu z|^2 + S_H + \dots, \tag{161}$$

where $\dots$ include terms that are not $O(4)$ symmetric (note that S_H is $O(4)$ symmetric). In this way of writing it, the $O(4)$ symmetry comes from the fact that we can rotate among the four components of the complex spinor z (recall $|z|^2 = 1$). All the terms in $\dots$ are seemingly less relevant at long distances than the $|\partial_\mu z|^2$ term. Whether the $\dots$ terms can really be ignored or not is a tricky dynamical question that apparently no one knows the answer to. However, it is reasonable to hypothesize (based on the present understanding of the "duality web; more on this is a future diary entry) that these terms are indeed irrelevant, which frees up the full $O(4)$ symmetry of the model.

So accepting this somewhat shaky conclusion, we have what we wanted to find: an $O(4)$ model with a theta term at $\theta = \pi$ (it's a θ term since we started out with a Hopf term for the $n \in S^2$ vector, but after switching to z variables and killing off the gauge fields it becomes a θ term for an $O(4)$ field since we are now mapping $S^3 \to S^3$ rather than $S^3 \to S^3/S^1$). Oh yeah, one final thing: do we have an $O(4)$ symmetry in the formulation in terms of the U's? Yes, it acts as

$$O(4) \cong (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2, \tag{162}$$

with the two $SU(2)$ factors acting on the U matrices on the left and on the right, respectively (this is a symmetry since in S_θ all the U's are sandwiched by $U^\dagger$'s, and vice versa—the quotient by $\mathbb{Z}_2$ avoids double-counting the center $Z(SU(2)) = \mathbb{Z}_2$).



7 β function for general two-dimensional nonlinear σ models and Ricci flow

Consider a nlσm in two dimensions, with action[5]

$$S = \frac{1}{4\pi\alpha} \int_\Sigma d^2\sigma \, \gamma^{\mu\nu} g_{ij}(X) \partial_\nu X^i \partial_\mu X^j. \tag{163}$$

Here $X : \Sigma \to M$ for some Riemannian manifold M with metric g_{ij} and some Riemann surface Σ with metric $\gamma_{\mu\nu}$, and where $g_{ij}(X) = X^* g_{ij}$ is the pullback of the metric on M by X.

Show that the beta function(al) for g_{ij} is, to one loop order, given by the Ricci flow equation

$$\beta_{ij} = \frac{dg_{ij}}{d \ln \mu} = R_{ij} = \frac{1}{2} R g_{ij}, \tag{164}$$

where R_{ij} is the Ricci tensor (of the target space M), R is the Ricci scalar, and we have used

$$R_{ij} = \frac{1}{2} g_{ij} R, \tag{165}$$

which holds in two dimensions (see the previous diary entry on the linear dilaton CFT). This is a weak-coupling result, which you should derive by assuming that the geometry of M varies slowly compared to $\sqrt{\alpha}$. Note that the coupling for the theory is roughly $\sim 1/r$ for r the radius of curvature, so that if $R_{ij} > 0 \implies \beta_{ij} > 0$, then since$r$ increases as we flow to the UV, we see that $R_{ij} > 0$ implies the theory is asymptotically free.

Note that unlike a free boson in two dimensions, here we are letting X be *dimensionful*, with mass dimension -1, and hence $[\alpha] = -2$ so that $\sqrt{\alpha}$ is a length which can be compared with a radius of curvature, which means that weak coupling is when M is "big" compared to the "string scale". This is the more string theory oriented way of writing things down. Alternatively we could let X be dimensionless and write the action as $l^2 \int \partial X^i \partial X^j g_{ij}$, where now the invariant distance is $ds^2 = l^2 g_{ij} dX^i dX^j$, with ds^2 and l both dimensionless. The small parameter in this case is then l^{-1}.

First let's assess the sensibility of such a result. We see that for target manifolds with positive curvature, the theory is asymptotically free, since large radii of curvature translate to weak coupling (we've also derived this earlier for the case where $M = S^n$). This makes perfect sense, since the strength of the coupling at a given point in M is given by the inverse radius of curvature at that point: going to the UV by "zooming in" on a positive curvature region means increasing the radius of curvature locally, which leads to a smaller coupling

[5] In string theory, it is more natural to let X have Greek indices since the target space is spacetime, and to let γ have Roman indices since γ lives on the worldsheet. Here we are thinking of Σ as spacetime, and so the conventions are reversed.

constant, agreeing with asymptotic freedom. Geometrically, this is Ricci flow: positive curvature regions get smaller and negative curvature regions get larger. Also, we could have got the answer by using dimensional analysis to write, to leading order,

$$\beta_{ij} = aR_{ij} + bg_{ij} + cRg_{ij}. \tag{166}$$

However, $b = 0$ since we know that when the target space is flat, the theory is totally conformal and $\beta_{ij} = 0$. Moreover, we know that $R_{ij} = \frac{1}{2}g_{ij}R$ in two dimensions, so that $\beta_{ij} \propto R_{ij}$. Then using remarks similar to those above, we could fix the sign of the coefficient by physical reasoning: we see that these general arguments get us nearly all the way to the answer!

To do things more carefully, we will adopt the background field approach. We will let the "slow" degree of freedom (the background field) be denoted by ϕ, so that the full field is $X^i = \phi^i + \gamma^i$, where γ^i is the "fast" degree of freedom representing fluctuations away from the background field. We want to integrate out the γ^i and see how this changes the coupling constant (the metric) for the slow degrees of freedom. The problem with this is that γ^i is defined as the difference between X^i and ϕ^i, which for a given spacetime coordinate map to different points on the target manifold M. Thus γ^i is not actually a vector, since it transforms non-covariantly under coordinate transformations on M.

Instead of the object γ^i, we can work with vectors by using the following prescription. Let $\lambda^i(s)$ be the geodesic (we assume there is only one) passing between ϕ^i and X^i, with $\lambda^i(0) = \phi^i$ and $\lambda^i(1) = X^i$. Furthermore, define

$$\zeta^i \equiv \frac{d\lambda^i(s)}{ds}\bigg|_{s=0} \tag{167}$$

as the tangent to the geodesic at ϕ^i. ζ^i is of course a vector, and we will use it as the integration variable instead of γ^i. We can expand X^i about the slow field ϕ^i in terms of ζ^i by flowing along the geodesic from ϕ^i with the help of $e^{s\zeta}$:

$$\lambda(s) = e^{s\zeta}\phi = \phi + s\nabla_\zeta X|_\phi + \frac{1}{2}s^2\nabla_\zeta^2 X|_\phi + \dots \tag{168}$$

with X^i obtained by evaluating this at $s = 1$.

We now want to find an explicit expansion for $(e^{s\zeta}\phi)^i$ in terms of ζ^i (i.e. in terms of s). We write

$$(e^{s\zeta}\phi)^i = \phi^i + s\zeta^i + \frac{1}{2}s^2 C_2^i + \frac{1}{3!}s^3 C_3^i + \dots \tag{169}$$

for some as-yet-undetermined coefficients C_n^i. We also need to expand the Christoffel symbols about ϕ:

$$\Gamma^k_{ij}(e^{s\zeta}\phi) = \Gamma^k_{ij}(\phi) + s\partial_l\Gamma^k_{ij}(\phi)\left(s\zeta^i + \frac{1}{2}s^2C_2^i + \frac{1}{3!}s^3C_3^i + \dots\right)^l + \dots \tag{170}$$

Now recall the geodesic equation, which we get by requiring that the covariant derivative of the tangent vector along the geodesic (namely ζ^i) vanish:

$$\frac{d^2\zeta^i}{ds^2} + \Gamma^i_{jk}\frac{d\zeta^j}{ds}\frac{d\zeta^k}{ds} = 0. \tag{171}$$

We plug the above power series into the geodesic equation (after plugging in the expansion of $e^{s\zeta}\phi$ into the series for the Christoffel symbols), and equate powers in s. The s^0 term tells us that

$$C_2^i = -\Gamma^i_{jk}\zeta^j\zeta^k. \tag{172}$$

The next order term is

$$C_3^i = \left(2\Gamma^i_{km}\Gamma^m_{lj} - \frac{1}{3}\partial_{(j}\Gamma^i_{kl)}\right)\zeta^k\zeta^l\zeta^j, \tag{173}$$

and the higher order terms won't be important to us.

So, putting this in for C_2^i, we can write the expansion of $e^\zeta\phi$ about ϕ as

$$X^i = (e^\zeta\phi)^i = \phi^i + \zeta^i - \frac{1}{2}\Gamma^i_{jk}\zeta^j\zeta^k + O(\zeta^3). \tag{174}$$

Now suppose we had chosen coordinates about ϕ^i in which the geodesics were straight lines passing through ϕ. These are Riemann normal coordinates, in which geodesics passing through ϕ are used to construct a coordinate system which is locally $\mathbb{R}^{\dim M}$. In Riemann normal coordinates then, the linear order approximation $\phi^i + s\zeta^i$ is actually exact, since the geodesic is a straight line. So then looking at the series expansion for X^i, we see that in Riemann normal coordinates, all of the C_n^i must vanish. This means in particular that

$$\Gamma^k_{ij}(\phi) = \partial_{(i}\Gamma^j_{kl)}(\phi) = 0 \quad \text{(in Rieman normal coordinates)}. \tag{175}$$

This is consistent with the fact that the normal coordinate system is locally $\mathbb{R}^{\dim M}$. Note that this statement is made at the origin of the coordinate system, and does not hold at other points (Christoffel symbols without an argument will always be assumed to be evaluated at ϕ). One upside is that the Riemann tensor in normal coordinates is (just by definition)

$$R^i{}_{jkl} = \partial_k\Gamma^i_{lj} - \partial_l\Gamma^i_{kj} \quad \text{(in Rieman normal coordinates)}, \tag{176}$$

since the terms quadratic in Γ die. By writing $\frac{1}{3}(R^i{}_{jkl} + R^i{}_{lkj})$ in terms of the Christoffel symbols in this way, and then adding and subtracting $(1/3)\partial_k\Gamma^i_{lj}$ and using that $\partial_{(i}\Gamma^j_{kl)} = 0$, the derivative of the Christoffel symbol at the origin is related to the curvature tensor as (still in Rieman normal coordinates)

$$\frac{1}{3}(R^i{}_{jkl}(\phi) + R^i{}_{lkj}(\phi)) = \frac{1}{3}(3(\partial_k\Gamma^i_{lj})(\phi) - \partial_{(l}\Gamma^i_{jk)}(\phi)) = (\partial_k\Gamma^i_{lj})(\phi). \tag{177}$$

Finally we need to know how to expand more general tensors about the origin ϕ in terms of the ζ's. In normal coordinates, this is just the normal Taylor expansion since we locally are in $\mathbb{R}^{\dim M}$. However, if we write it in terms of covariant things, the expansion will work for any coordinate system. So, we will presently work in normal coordinates and re-write the taylor expansion in terms of covariant derivatives.

Consider a $(2,0)$ tensor T_{ij} (two covariant indices). Since the Christoffel symbols at the origin vanish in normal coordinates, the first order term in the expansion is

$$\partial_k T_{ij}(\phi)\zeta^k = \nabla_k T_{ij}(\phi)\zeta^k. \tag{178}$$

The second order term is trickier though. We have

$$\frac{1}{2}\nabla_k\nabla_l T_{ij}\zeta^k\zeta^l = \frac{1}{2}\nabla_k(\partial_l T_{ij} - \Gamma^m_{il}T_{mj} - \Gamma^m_{jl}T_{im})\zeta^k\zeta^l. \tag{179}$$

After we take the ∇_k and evaluate at ϕ (the origin), all the un-differentiated Christoffel symbols will die. Thus, using our expression for the derivative of the Christoffel symbols in terms of the Riemann curvature tensor,

$$\frac{1}{2}(\nabla_k\nabla_l T_{ij})(\phi)\zeta^k\zeta^l = \frac{1}{2}\left(\partial_k\partial_l T_{ij}(\phi) - \frac{1}{3}({R^m}_{lki} + {R^m}_{ikl})T_{mj}(\phi) - \frac{1}{3}({R^m}_{lkj} + {R^m}_{jkl})T_{im}(\phi)\right)\zeta^k\zeta^l. \tag{180}$$

Now we use the Bianchi identity in the form

$${R^m}_{lki} + {R^m}_{ikl} = -{R^m}_{kil} \tag{181}$$

on the curvature tensors in the above expansion. After solving for $\frac{1}{2}\partial_k\partial_l T_{ij}(\phi)$ in terms of the other covariant stuff, we can finally rewrite the Taylor expansion for T as

$$T_{ij}(e^\zeta\phi) = T_{ij}(\phi) + \nabla_k T_{ij}(\phi)\zeta^k + \frac{1}{2}\zeta^k\zeta^l\left(\nabla_k\nabla_l T_{ij}(\phi) - \frac{1}{3}{R^m}_{kil}T_{mj}(\phi) - \frac{1}{3}{R^m}_{kjl}T_{im}(\phi)\right) + O(\zeta^3). \tag{182}$$

Behold, everything is covariant! Thus, this expansion holds in any coordinate system, not just in Riemann normal coordinates. Also note that if T is taken to be the metric, then $\nabla_k g_{ij} = 0$ means

$$g_{ij}(e^\zeta\phi) = g_{ij}(\phi) - \frac{1}{3}R_{ikjl}\zeta^k\zeta^l + O(\zeta^3). \tag{183}$$

Again, this holds in any coordinate system.

With this preparatory work out of the way, we can start massaging the action. We need to expand both the metric and the derivatives. We know how to do the former; the latter is, in Riemann normal coordinates centered on ϕ^i, (note to self: convince yourself that the derivatives on ϕ don't need to be covariant ones)

$$\begin{aligned}\partial_\mu\left(\phi^i + \zeta^i - \frac{1}{2}\Gamma^i_{jk}\zeta^l\zeta^k\right) &\approx \partial_\mu\phi^i + \partial_\mu\zeta^i - \frac{1}{6}(\partial_\mu\phi^m)(R^i_{lmk} + R^i_{kml})\zeta^k\zeta^l, \\ &= \partial_\mu\phi^i + \nabla_\mu\zeta^i - \frac{1}{3}\partial_\mu\phi^m R^i_{lmk}\zeta^k\zeta^l,\end{aligned} \tag{184}$$

where the covariant derivative acting on ζ is

$$\nabla_\mu\zeta^i = \partial_\mu\zeta^i + \partial_\mu\phi^m\Gamma^i_{mj}\zeta^j. \tag{185}$$

The derivatives on ϕ just convert the derivative into the $i, j, k, \ldots$ index space[6]. Again, we derived this in normal coordinates, but the answer holds in a generic coordinate system since

[6]This is made possible because the Christoffel symbols transform as a $(2,1)$ tensor (two co-vector indices and one vector index) up to an extra term

$$\Gamma^k_{ij} = \partial_a x^k\left[\Gamma^a_{bc}\partial_i x^b\partial_j x^c + \partial_i\partial_j x^a\right], \tag{186}$$

where the transformation is from x^i coordinates to x^a coordinates. This can be checked using algebra and $\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. This means that if we want to only transform one of the bottom indices of the Christoffel symbol, then we can treat Γ as a tensor, essentially since the non-tensorial piece is $\partial_i\partial_b x^a = \partial_i\delta_{b^a} = 0$; this allows us to write $\nabla_\mu\zeta^i$ as above.

it's gauge-invariant under gauge transformations (coordinate transformations) on ζ^i. This is good news for us, since we can use the above expression in the action, the fields of which generically cannot be written in normal coordinates for more than one point in the target manifold (unless of course if $R_{ijkl} = 0$).

Now we put the above into the action, along with the expansion of the metric, and keep everything below third order in ζ. This produces

$$S = \frac{1}{4\pi\alpha} \int \sqrt{\gamma}\gamma^{\mu\nu} \Big[g_{ij}(\phi) \left(\partial_\mu \phi^i \partial_\nu \phi^j + \nabla_\mu \zeta^i \nabla_\nu \zeta^j + 2\nabla_\mu \zeta^i \partial_\nu \phi^j \right) + R_{ijkl}(\phi) \partial_\mu \phi_i \partial_\nu \phi^k \zeta^j \zeta^l \Big] \tag{187}$$

Unfortunately, we cannot do Feynman diagrams with this action. The reason is that the kinetic term for ζ is dependent on ϕ, since the kinetic term involves the full metric $g_{ij}(\phi)$. This makes doing calculations with the ζ propagator essentially impossible.

To fix this, we need to change the metric in the ζ kinetic term to a flat one. We do this by switching over to vielbeins, with $\zeta^i = e^i_a \zeta^a$. The e^i_a are orthonormal frames and $g_{ij} = e^a_i e^b_j \eta_{ab}$, so that the kinetic term is

$$\gamma^{\mu\nu} g_{ij} \partial_\mu \zeta^i \partial_\nu \zeta^j = \eta_{ab} \gamma^{\mu\nu} \partial_\mu \zeta^a \partial_\nu \zeta^b, \tag{188}$$

so that the ζ^a propagator is the usual propagator which can now be used in Feynman diagrams. From now on, we will assume a flat spacetime for simplicity, so that $\gamma^{\mu\nu} = \eta^{\mu\nu}$.

To translate the rest of the action into the vielein formulation, we just need to note how the covariant derivatives change. Let us denote the spin connection (a matrix-valued 1-form on Σ) as $\omega_\mu{}^a{}_b$. The spin connection is built out of the regular affine connection with Γ, plus a term which keeps track of how the basis frames rotate as we travel around Σ, which gives it the properties of an $SO(\dim M)$ gauge field over Σ (in Euclidean signature). This extra term is the analogue of the Mrauer-Cartan form $g^{-1}dg$ in gauge theory, and is written as $e^i_b \partial_\mu e^a_i$. Using the vielbeins to deal with the mixed indices in the Christoffel symbols, the spin connection is

$$\omega_\nu{}^a{}_b = e^a_i e^k_b \Gamma^i{}_{\nu k} - e^a_i \partial_\nu e^i_b = \partial_\nu \phi^j e^a_i e^k_b \Gamma^i{}_{jk} - e^a_i \partial_\nu e^i_b. \tag{189}$$

The minus sign in front of the $e^a_i \partial_\nu e^i_b$ term is chosen so that the vielbein frames are covariantly constant: taking the covariant derivative of the framing e^i_a and using the Christoffel symbols to deal with the lower index and the spin connection to deal with the upper index, we have (recall that covariant indices get a minus sign in the covariant derivative and contravariant indices get a plus sign)

$$D_\mu e^a_j = \partial_\mu e^a_j - \Gamma^k_{ij} e^a_k \partial_\mu \phi^i + \omega_\mu{}^a{}_b e^b_j = 0, \tag{190}$$

which follows from the definition of the spin connection and the definition of the inverse framing $e^i_b e^b_k = \delta^i_k$. In what follows, we will use D_μ to denote covariant derivatives in the vielbein (gauge) formalism, and ∇_μ for covariant derivatives in the regular coordinate basis.

We can use these properties to convert the action into the vielbein formalism. After swapping out the covariant derivatives ∇_μ for the gauge covariant derivatives D_μ, we get

$$S = \frac{1}{4\pi\alpha} \int \eta^{\mu\nu} \Big[g_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j + \eta_{ab} \left(D_\mu \zeta^a D_\nu \zeta^b + 2D_\mu \zeta^a e^b_j \partial_\nu \phi^j \right) + e^j_a e^l_b R_{ijkl}(\phi) \partial_\mu \phi_i \partial_\nu \phi^k \zeta^a \zeta^b \Big]. \tag{191}$$

Now the kinetic term for ζ is nice and ready for doing Feynman diagrams. The relevant vertices are a $\omega\omega\zeta\zeta$ vertex, a $k^\mu\omega\zeta\zeta$ vertex, and a $\zeta\zeta\partial\phi\partial\phi R$ vertex. The gauge field ω won't actually contribute to any β functions, since gauge theory in two dimensions has no divergences[7].

All this means that getting the β function for the metric is incredibly simple. We can do it either by figuring out what counterterms need to be added or by letting the fast fields ζ^i be defined over an energy range from Λ to $\Lambda + d\Lambda$. Adopting this approach, there is only one diagram that contributes to the renormalization of $g_{ij}(\phi)$, namely the one with two outgoing $\partial_\mu\phi^i$ legs joined at a bubble formed from a single ζ^a propagator. This diagram gives

$$(\text{diagram})^{ik} = \int_k e^j_a e^l_b \langle\zeta^a\zeta^b\rangle(k^2) R_{ijkl} = \ln\left(\frac{\Lambda + d\Lambda}{\Lambda}\right)\eta^{ab}e^j_a e^l_b R_{ijkl} = d\ln\Lambda R_{ik}. \tag{193}$$

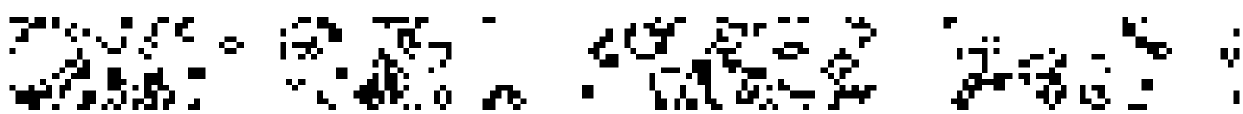
In the last step, we have used (basically, the trace is the same in all coordinates)

$$\eta^{ab}e^j_a e^l_b R_{ijkl} = R^m{}_{ilk}g_{mj}e^j_a e^l_n \eta^{ab} = R^m{}_{ilk}g_{mj}g^{jl} = R^m{}_{imk} = R_{ik}. \tag{194}$$

So finally, we differentiate the effective g_{ij} (the one which includes the above radiative correction) with respect to $\ln\Lambda$ and find that

$$\beta_{ij} = R_{ij} \tag{195}$$

as required.



8 Skrymion energy in $O(3)$ nslm

Today's diary entry is a fast one: calculating the energy of the minimally-charged Skyrmion in the two-dimensional (classical) $O(3)$ nlsm, with energy

$$E = \frac{\rho}{2}\int d^2r\,|\nabla\mathbf{n}|^2, \tag{196}$$

with $|\mathbf{n}| = 1$. I couldn't find the derivation in any freely available stuff online (though I didn't look that hard), but the calculation is fun and quick.

[7]Divergences for gauge theories are two degrees lower than naive power counting suggests, I think roughly since we loose one degree of freedom to the non-dynamical A_0 and another to gauge invariance. One can check e.g. that the two diagrams contributing to the renormalization of the ω propagator give a contribution

$$(\text{two diagrams})^{\mu\nu}(q) \sim \int_k\left(\frac{\delta^{\mu\nu}}{k^2} - \frac{2k^\mu(k-q)^\nu}{k^2(k-q)^2}\right) \sim \int_{k,x}\frac{\delta^{\mu\nu}q^2}{(k^2-\Delta_q)^2}, \tag{192}$$

which indeed has an integrand going as $1/k^3$ rather than the naive $1/k$. Here the μ,ν are the spacetime indices for the external ω legs.

We can do the calculation by thinking about a 2π flux background for the $\mathbb{CP}^1$ representation, but we will instead stick with the n^a variables.

By symmetry, the minimal-charge skrymion solution $\mathbf{n}(\mathbf{r})$ will only be a function of r. Since we are in two dimensions the skyrmion action will satisfy $S[\mathbf{n}(r/\xi)] = S[\mathbf{n}(r)]$ for any ξ; hence we must have a one-parameter family of solutions related by scale transformations. Since the generator of $\pi_2(S^2)$ is the unit map, the skyrmion solution will therefore take the form of a conformal map from S^2 onto the plane.

As is the case with any sigma model, the action can be written as

$$S = \int_{\mathbb{R}^2} \mathrm{Tr}[\mathbf{n}^*(g)\eta], \tag{197}$$

with η the (in this case flat) metric on $\mathbb{R}^2$ and g the metric on the target space (here S^2), with $\mathbf{n}^*(g)$ denoting the pullback by $\mathbf{n} : \mathbb{R}^2 \to S^2$. Since $\mathbf{n}$ must be a conformal map, the pullback metric $\mathbf{n}^*(g)$ must have the conformally flat form

$$[\mathbf{n}^*(g)]_{ij} = \frac{4\xi^2}{(\xi^2+r^2)^2}\delta_{ij}, \tag{198}$$

for some scale parameter ξ. We then have

$$S[\mathbf{n}] = \frac{\rho}{2}\int_0^\infty dr\, 2\pi r \mathrm{Tr}[\mathbf{n}^*(g)] = \frac{\rho}{2}\int_0^\infty dr\, 2\pi r \frac{8}{(1+r^2)^2} = 4\pi\rho. \tag{199}$$

Note that we never had to know the explicit coordinate representation for $\mathbf{n}$!

Most papers I have read quote the skyrmion energy as $8\pi\rho$, but note that this only is consistent if the spin stiffness appears as $\rho|\nabla\mathbf{n}|^2$ in the action, without the factor of $1/2$ that we have in the present setting.

We can check this result by writing out the explicit form of the conformal map. The conformal maps from $S^2 \to \mathbb{R}^2$ correspond to doing stereographic projections with different scale factors ξ. Therefore the explicit coordinate expressions for the components of $\mathbf{n}$ can be obtained from

$$x^i/\xi = \frac{n^i}{1+n^z}, \tag{200}$$

where $i = x, y$. Letting $r^2 = x^2 + y^2$, the fact that $|\mathbf{n}| = 1$ tells us that, after a bit of algebra, the skyrmion solution has components

$$n^z = \frac{\xi^2 - r^2}{\xi^2 + r^2} \implies (n^x,\, n^y) = \frac{2}{1 + r^2/\xi^2}(x/\xi,\, y/\xi). \tag{201}$$

Now we only need to plug this into the action, and see what we get. Since we know all choices for ξ will have the same action, we will now set $\xi = 1$. The Laplacian of the above solution is

$$-\nabla^2\mathbf{n} = \frac{8}{(1+r^2)^3}\left(2r\cos(\phi),\, 2r\sin(\phi),\, 1 - r^2\right). \tag{202}$$

This then gives

$$-\mathbf{n} \cdot \nabla^2 \mathbf{n} = \frac{8}{(1 + r^2)^2}, \tag{203}$$

which then is integrated with the same integral as above. ✓