October 18, 2018

Abstract

1 Problem 1

We'll do the n-state generalization, with Hamiltonian

$$H_P = -J\left[\sum_j \sum_{l=0}^{n-1} Z_j^{-1} Z_{j+1} + g\sum_j \sum_{l=0}^{n-1} X_j\right]. \tag{1}$$

Here Z is the diagonal matrix $Z = \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1})$ for $\omega = e^{2\pi i/n}$, and X is the shift matrix (generator of the alternating part of S_n), which is a "backwards"' permutation taking the lth eigenstate of Z to the $l-1$ eigenstate. The commutation relation is

$$ZX = \omega^{-1} XZ. \tag{2}$$

a) If we think of X as measuring "electric flux", then eigenstates of X are those with definite electric flux. This means that they are uniform superpositions of the eigenstates of the conjugate variable, Z. We thus compute the eigenstates by Fourier transforming those of Z: they are

$$|Q\rangle = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} e^{iQl}|l\rangle, \tag{3}$$

where $|l\rangle$ are the eigenstates of Z. Acting with X does $X : |l\rangle \mapsto |l-1\rangle$, so that

$$X|Q\rangle = e^{iQ}|Q\rangle. \tag{4}$$

The model has an internal $\mathbb{Z}_3$ symmetry rotating the "spins" on every site by $2\pi/n$, which is performed by the operator $\prod_j X_k$.

b) When $g \to \infty$, we must work in a state with zero electric flux. Thus the system is in a unique ground state

$$|GS\rangle_{g\to\infty} = |Q = 0\rangle = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} |l\rangle. \tag{5}$$

c) When $g \to 0$, we need to satisfy the interaction term. Since there is no X in the Hamiltonian we can work in the Z eigenbasis. If site j is in the eigenstate $|l_j\rangle$ and site $j+1$ is in $|l_{j+1}\rangle$, then the interaction term between them is

$$\sum_m \omega^{m(l_{j+1}-l_j)} = \delta_{l_j, j_{j+1}}. \tag{6}$$

Since this term appears with negative coefficient in the Hamiltonian, we want the δ constraint to be satisfied. Thus this forces neighboring sites to be in identical Z eigenstates, and we have n degenerate ground states

$$|GS_l\rangle_{g\to 0} = \bigotimes_j |l_j\rangle \tag{7}$$

(here $l_j \in \mathbb{Z}_n$ is the same for all j).

d) Let $j+$ denote the site $j+1/2$ on the dual lattice and let j_- denote the dual lattice site $j-1/2$. We define dual operators by

$$\widetilde{X}_j+ = Z_j^{-1} Z_{j+1}, \qquad \widetilde{Z}_{j-}^{-1}\widetilde{Z}_{j+} = X_j. \tag{8}$$

The latter equation is satisfied if we write $\widetilde{Z}_{j\pm}$ as the string

$$\widetilde{Z}_{j\pm} = \prod_{j<j_\pm} X_j. \tag{9}$$

Now we want to compute the commutator between $\widetilde{X}_{j+}$ and $\widetilde{Z}_{k+}$. Suppose first that the two dual sites are equal, $j+ = k+$. To move $\widetilde{X}_{j+}$ to the right of $\widetilde{Z}_{j+}$, we need to pass the two Z's in the definition of $\widetilde{X}$ through the chain of X operators created by $\widetilde{Z}$. Since we put the string in $\widetilde{Z}$ "to the left", the Z_{j+1} operator goes through for free, while the Z_j^{-1} operator picks up an ω. Thus we have

$$\widetilde{X}_{j+}\widetilde{Z}_{k+} = \omega \widetilde{Z}_{j+}\widetilde{X}_{j+}. \tag{10}$$

Now suppose that the two dual operators are not at the same site. If the $\widetilde{X}$ is at a site to the left of the $\widetilde{Z}$, it commutes though for free since the operators involved act on different $\otimes$ factors. If it is to the left of the $\widetilde{Z}$, then the Z it contains picks up a factor of ω^{-1}, while the Z^{-1} picks up a factor of ω which cancels the ω^{-1}, and so the two operators commute. Thus the commutation relation is

$$\widetilde{X}_{j+}\widetilde{Z}_{k+} = \omega^{\delta_{j,k}} \widetilde{Z}_{k+}\widetilde{X}_{j+}, \tag{11}$$

which is the same relation as the one the X, Z operators satisfy.

We know that there's a phase transition at some g_c because the ground state degeneracy must change. Note that in terms of the dual variables, the Hamiltonian is

$$H = -J \sum_j \left[g \sum_{l=0}^{n-1} \widetilde{Z}_j^{-l} \widetilde{Z}_{j+1}^{l} + \sum_{l=0}^{n-1} \widetilde{X}_j \right], \tag{12}$$

which is the same as the original Hamiltonian when $g = 1$. Therefore we identify $g_c = 1$ as the self-dual point where the phase transition is likely to happen.

e) We take the mean field ansatz

$$|\Psi(x_1, \ldots, x_n)\rangle = \bigotimes_j |\hat{n}(\{x_i\})\rangle_j, \qquad |\hat{n}(\{x_i\})\rangle = \sum_{i=1}^{n} x_i|0\rangle, \qquad \vec{x} \in S^{n-1}. \quad (13)$$

Here the fact that $\vec{x} \in S^{n-1}$ ensures that $|\Psi\rangle$ is normalized correctly.

What is the variational energy? Let's first look at the X term. Because of the sum over l, this is a projector onto the eigenstate of X with eigenvalue 1. In the basis where Z is diagonal, this is a projector onto the uniform sum $\sum_j |j\rangle$. It is represented by a matrix with 1 in every entry. Thus

$$\langle\Psi(\vec{x})|X|\Psi(\vec{x})\rangle = \left(\sum_j x_j\right)^2. \quad (14)$$

Now for the nearest neighbor term. Indexing the entries of $Z_j^{-l} \otimes Z_{j+1}^l$ as (a, b), we see from $\sum_j \omega^{jk} \propto \delta k, 0$ that the matrix $Z_j^{-l} \otimes Z_{j+1}^l$ is diagonal with entries $(a, b) = \omega^{l(a-b)}$. Summing over l gives a non-zero result only when $a = b$, and so $\sum_l Z_j^{-l} \otimes Z_{j+1}^l$ has entries $(a, b) = \delta_{a,b} n$. Thus

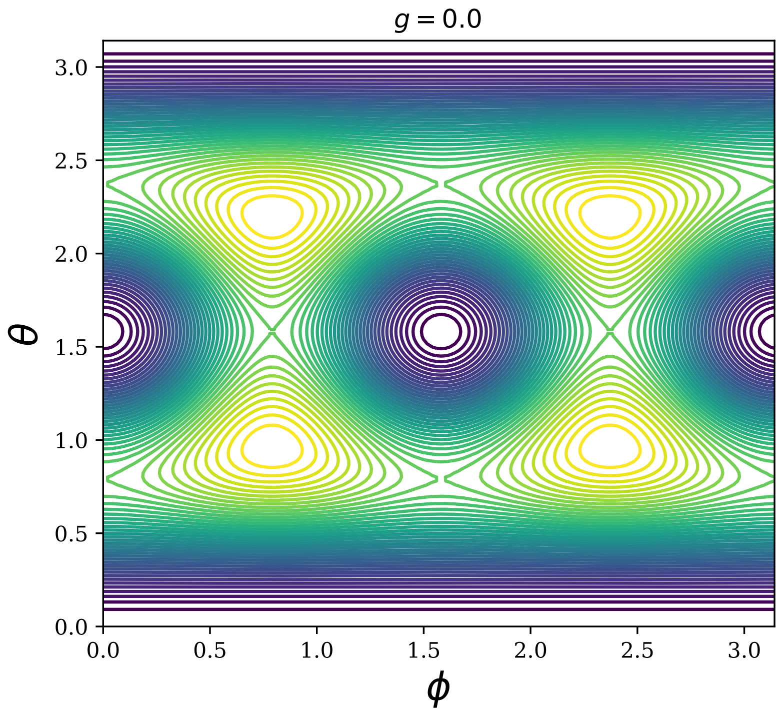
$$\langle\Psi(\vec{x})|_j \otimes \langle\Psi(\vec{x})|_{j+1} \sum_l Z_j^{-l} \otimes Z_{j+1}^l |\Psi(\vec{x})\rangle_j \otimes |\Psi(\vec{x})\rangle_{j+1} = n \sum_i x_i^4. \quad (15)$$

Putting this together, on a length L chain we just get the contribution above for every j, and so the variational energy is

$$E(\vec{x}) = -JN \left[n \sum_i x_i^4 + g \left(\sum_j x_j\right)^2 \right]. \quad (16)$$

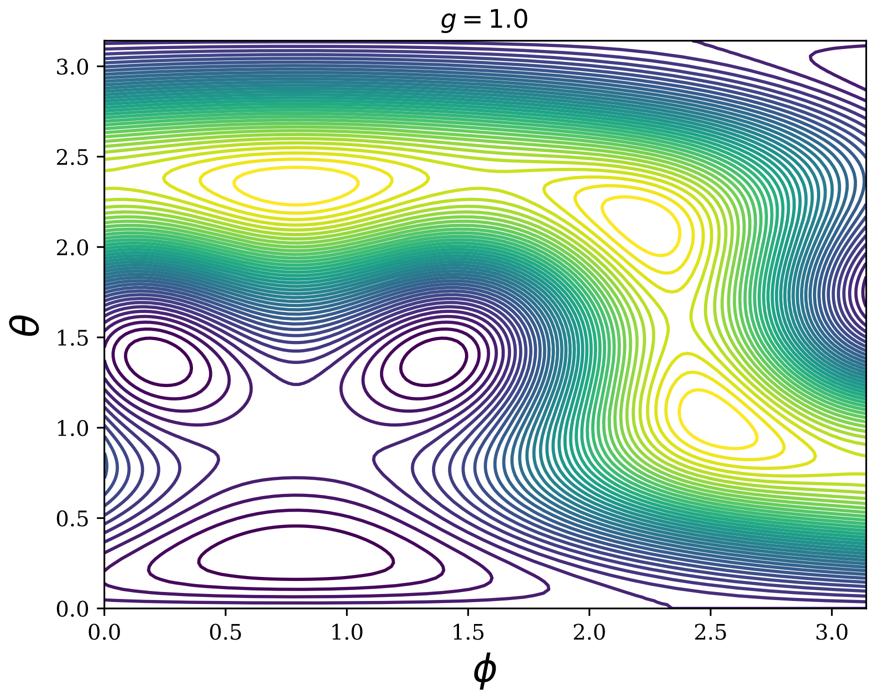
Now we specialize to the case of the 3-state Potts model, parametrizing the coordinates on S^2 by θ, ϕ. Only half of the sphere gives a physically distinct wavefunction, since $|\Psi\rangle$ and $-|\Psi\rangle$ are equivalent minima. We will choose the hemisphere

$(\theta, \phi) \in [0, \pi]^2$. Setting $JN = 1$, when $g = 0$ $E(\theta, \phi)$ looks like

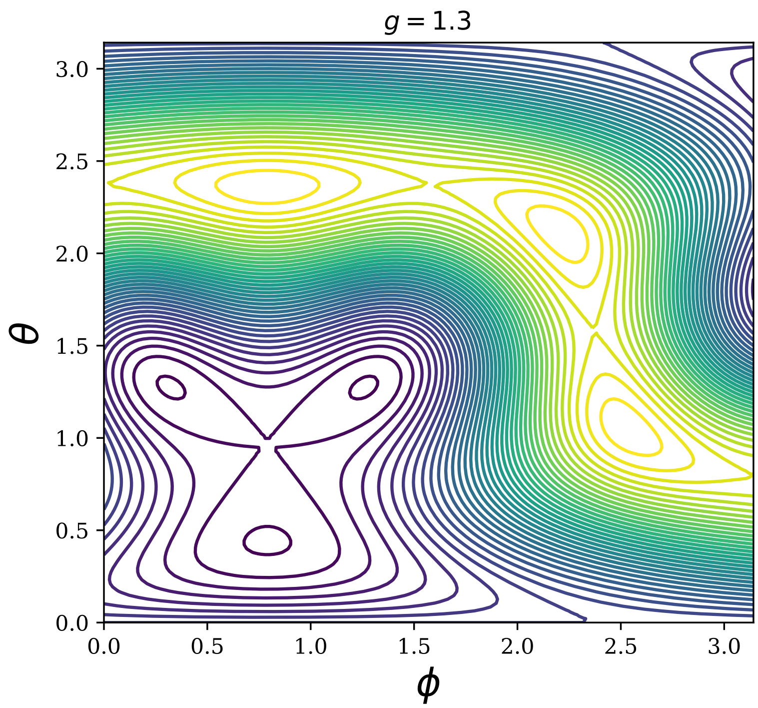


We see that there are three distinct minima, in agreement with what we predicted earlier (there is a single minimum at the top of the figure (the point $(0, 0, 1)$), which is identified with the minimum at the bottom of the figure. Likewise, the two minima at $\phi = 0, \pi$ are identified). They are the eigenstates of the Z operators, and consequently appear at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

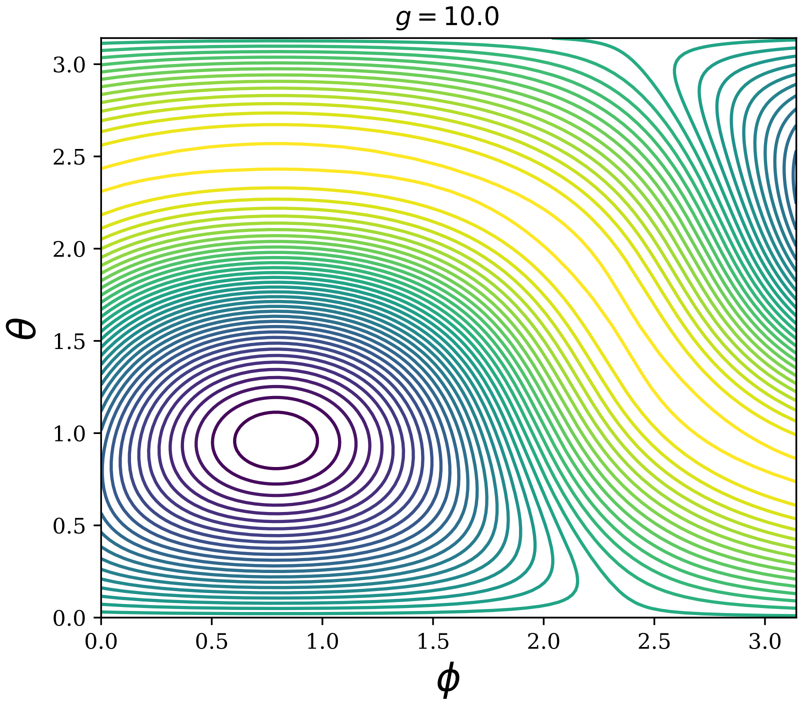
When we set $g = 1$ the plot looks like



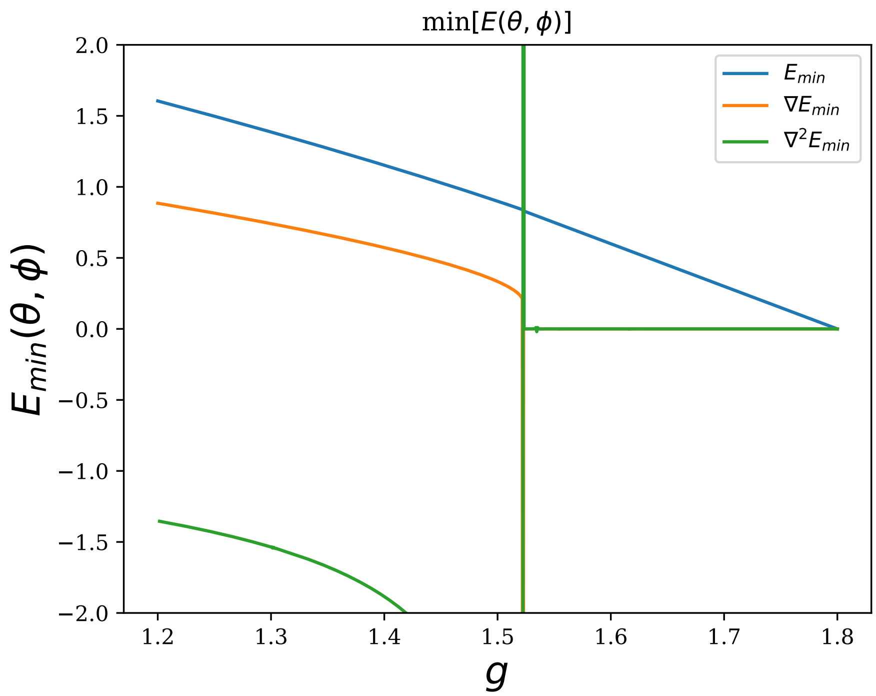
The three minima start to merge as g is increased:



When g is large we need to get a $\otimes$ state from satisfying the $\sum_l X^l$ term. Since this state will be one in the eigenvalue 1 eigenstate of X, it will have coordinates on the sphere $x = y = z$. Thus we expect a single minimum at $\theta = \pi/4, \phi = \pi/4$. Indeed, this is what happens at large g:



f) We now plot $\min[E(\theta, \phi)]$, again with $JN = 1$. The interesting behavior happens around $g = 1.5$:



Here we have subtracted constants from E_{min} and ∇E_{min} so that they both hit 0 at $g = 1.8$. The behavior for $g < 1.2$ and $g > 1.8$ is what you would expect from the figure. We see that we have a second order phase transition at $g \approx 1.5225$, since $\nabla^2 E_{\text{min}}$ is singular there. We know the self-dual point is at $g = 1$, so we see that mean-field theory over-estimates the critical value of g. This is what we expect from a mean-field treatment: the mean-field ansatz neglects fluctuations and assumes a $\otimes$ state, so that it is biased towards ordered states (for us, small g).

2 Problem 2

3 Problem 3

a) Let $\widetilde{\chi} = O\chi$, where O is orthogonal. Since O has real entries and the χ are Hermitian, the $\widetilde{\chi}$ are Hermitian as well. The Clifford algebra relation $\{\chi_\alpha, \chi_\beta\} = 2\delta_{\alpha\beta}$ is also satisfied by the new Majoranas:

$$\{\widetilde{\chi}_\alpha, \widetilde{\chi}_\beta\} = O_{\alpha\gamma}O_{\beta\lambda}\{\chi_\gamma, \chi_\lambda\} = 2O_{\alpha\gamma}O_{\beta\lambda}\delta_{\lambda\gamma} = 2[OO^T]_{\alpha\beta} = 2\delta_{\alpha\beta}. \tag{17}$$

b) The antisymmetric matrix A is of the form $A = a \otimes iY$, where a is symmetric. Thus a can be diagonalized by an orthogonal matrix O, with $D = O^T aO$ diagonal.

Then the matrix $O \otimes iY$ is also orthogonal, and taking $(O \otimes iY)^T A(O \otimes iY)$ will bring A into the form $D \otimes iY$. The statement that we can always do this can be understood in another way: since A is antisymmetric and real, it is in the lie algebra $\mathfrak{so}(4)$. The Cartan subalgebra of $\mathfrak{so}(4)$ is the pre-image of the maximal torus of $SO(4)$ under the exp map. The maximal torus of $SO(2n)$ consists of a direct sum of n 2-by-2 rotation matrices, and so its preimage in the Lie algebra are the set of matrices $D \otimes iY$ for a diagonal n-by-n $\mathbb{R}$ matrix D. The statement that we can bring A into the given form by an orthogonal transformation is just saying that we can always perform an adjoint transformation in the Lie algebra to align a given Lie algebra element with a member of the Cartan subalgebra.

Anyway, applying this to the problem at hand, we just need to diagonalize a with an orthogonal matrix. Mathematica does this and tells us that a matrix diagonalizing A is

$$O = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \otimes iY. \tag{18}$$

Then we find that

$$O^T AO = iY \oplus 0_{2\times 2}. \tag{19}$$