My personal bosonization cheat-sheet

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In these notes we'll be discussing the compact boson in 1+1D and bosonization (both of bosons and fermions). A large part of what follows will be involved with trying to set standards vis-a-vis conventions for the compact boson and the bosonization procedure. The need for doing this stems from the fact that every source in the literature does things differently. First we'll deal just with the boson theory by itself, and later discuss fermions.

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Compact boson

We take a 1+1D compact boson at radius R to be described by the following action:

$$S = \frac{R^2}{4\pi} \int d\phi \wedge \star d\phi. \tag{1}$$

Here by "radius" of the boson, we just mean the coefficient in front of the kinetic term: we are always identifying $\phi \sim \phi + 2\pi$, but not restricting ourselves to kinetic terms with coefficients of e.g. 1/2 (we could say that we are fixing the target space to always be the same S^1, but allowing ourselves to consider different metrics on it). The Euclidean-space propagator for the ϕ fields with this normalization is

$$\langle \phi(x,t)\phi(0)\rangle \sim -\frac{1}{R^2} \ln|x-y|. \tag{2}$$

Therefore if the coefficient in front of the integral is 1/2, we would have $-\frac{1}{2\pi}\ln|x-y|$.

Instead of working with the field ϕ, whose equation of motion gives a spectrum containing both positive and negative momentum modes, it is often helpful to work with chiral "fields" $\phi_\pm$, which have the equations of motion

$$\bar\partial \phi_+ \overset{\text{eom}}{=} 0, \qquad \partial \phi_- \overset{\text{eom}}{=} 0, \qquad \partial = \frac{\partial_t - \partial_x}{2}, \qquad \bar\partial = \frac{\partial_t + \partial_x}{2}. \tag{3}$$

This means that the mode ϕ_+ has a spectrum with modes only for positive momentum (right-moving), while ϕ_- is the opposite. The full field ϕ is $\phi = \phi_+ + \phi_-$, which can be classically split up in this way just because solutions to the wave equation in 1+1D are given by $f(x-t) + g(x+t)$. It is very important to stress the fact that $\phi_\pm$ are only (anti)holomorphic *on the equations of motion*, i.e. we have $\bar\partial\phi_+(x) = \partial\phi_-(x) = 0$ only when inserted in correlation functions, and only when there are no other operators inserted at x in the correlation function, i.e. the usual Ward identity is

$$\langle \partial_\mp \phi_\pm(x) \mathcal{O} \rangle \propto \frac{\delta}{\delta \phi_\pm(x)} \langle \mathcal{O} \rangle. \tag{4}$$

Therefore it is of course not permissible to e.g. take the action $\int d\phi \wedge \star d\phi = \int \partial\phi\bar{\partial}\phi$ and replace it with $\int \partial\phi_+ \bar{\partial}\phi_-$—manipulations like this lead to total nonsense, because the path integral is over all field configurations of $\phi_\pm$, not just (anti-)holomorphic ones.

In terms of the chiral fields, the action is instead (note the $R^2/2\pi$ in front, not $R^2/4\pi$!)

$$S = \frac{R^2}{2\pi} \int \left(-\partial_t \phi_+ \partial_x \phi_+ + \partial_t \phi_- \partial_x \phi_- - (\partial_x \phi_+)^2 - (\partial_x \phi_-)^2 \right). \tag{5}$$

This ensures that the equations of motion are $(\partial_t \pm \partial_x)\phi_\pm \overset{\text{eom}}{=} 0$, as required. One can also check that this gives the correct log propagators: the momentum space propagator is

$$G_\pm(p, \omega; q, \nu) = \frac{\pi}{R^2} \delta(p - q)\delta(\omega - \nu)\Theta(\pm p) \frac{1}{k(k \mp \omega)}, \tag{6}$$

so that

$$G_\pm(x, t) = \frac{\pi}{R^2} \int_{\mathbb{R}} \frac{dk}{2\pi} \Theta(\pm k) \frac{e^{ik(x \mp t) - a|k|}}{k} \sim -\frac{1}{2R^2} \ln \left[\frac{ia}{x \mp t + ia} \right], \tag{7}$$

where in the first step we did the contour integral over ω and in the second step we did the integral by first differentiating wrt x and then re-integrating, choosing the constant term so that $G_\pm(x, \pm t) = 0$, which is a convinient normalization. Of course only derivatives and exponentials of $\phi_\pm$ are actually well-defined fields, but this correlator is still useful for calculating things. The ias are needed for dimensions to work out, with a essentially being a short-distance cutoff used to regulate the theory.[1] The scaling dimensions and conformal spins of the vertex operators $e^{i\alpha\phi_\pm}$ are therefore[2]

$$\Delta_\pm = \pm s_\pm = \frac{\alpha^2}{4R^2}. \tag{9}$$

Recall that these are defined via Now in order for an operator to be local its 2-point function must be single-valued, which means that $s_{\mathcal{O}} \in \frac{1}{2}\mathbb{Z}$ (since $z/\bar{z}$ is charge 2 under rotations). So unless $R^2 = \frac{1}{2n}$ for some $n \in \mathbb{Z}^+$, the $V_\pm =: e^{i\phi_\pm}$: vertex operators are *not* local.[3]

While we have written $\phi = \phi_+ + \phi_-$ and shown that ϕ has the same propagator as that of the original boson action (1), we will now justify the correspondence between the two ways of writing the action more carefully. For this it is helpful to introduce the field

$$\theta \equiv R^2(\phi_- - \phi_+). \tag{10}$$

[1]Sanity check: the propagator for the field $\phi \equiv \phi_+ + \phi_-$ is then

$$G_\phi(x, t) = -\frac{1}{R^2} \ln \sqrt{x^2 - t^2 + ia}, \tag{8}$$

which is the same thing we wrote down earlier, just in $\mathbb{R}$ time.

[2]The scaling dimension can be any nonzero number since the unitary bound on scalars is $\Delta \geq (d-2)/2$, which gives no nontrivial constraint for $d = 2$.

[3]Note that as long as $R^2 = m^2/(2n)$ for $m, n \in \mathbb{Z}^+$, mth powers of $V_\pm$ are local.

The R^2 and the perhaps unexpected minus sign is to make our lives easier in the future. If we re-write the action in terms of these variables, we get

$$S = \frac{1}{4\pi}\int \left(2\partial_x\theta\partial_t\phi - R^2(\partial_x\phi)^2 - R^{-2}(\partial_x\theta)^2\right). \tag{11}$$

The equation of motion for θ is then $\partial_x\theta \overset{\text{eom}}{=} R^2\partial_t\phi$ (this is the reason for the weird sign in the def of θ), while the eom for ϕ is $R^2\partial_x\phi \overset{\text{eom}}{=} \partial_t\theta$. Therefore

$$d\phi \overset{\text{eom}}{=} \frac{1}{R^2}\star d\theta, \tag{12}$$

where we have to remember to use the mixed-signature $\star$. If we then integrate out θ by shifting $\delta\theta = \partial_x^{-1}\partial_t\phi$, we find an action identical to (1) (in the signature $(+,-)$), establishing the correspondence between the two presentations. Also note that the eom are preserved under the duality

$$T:\phi\mapsto\frac{1}{R^2}\theta, \qquad \theta\mapsto R^2\phi, \tag{13}$$

since $\star^2 = \mathbf{1}$ acting on 1-forms in $\mathbb{R}^{1,1}$. We can check that $R=1$ is the right self-dual radius because at this point the exponentials of $\theta\pm\phi$ both have scaling dimension 1; hence they can act as chiral conserved currents and generate the $SU(2)$ that we know to be emergent at the self-dual point.

The commutation relations for the various fields involved all follow from the above actions. In this scheme we have $[\phi_\sigma,\phi_{\sigma'}]\propto\delta_{\sigma\sigma'}$, which is not true in some other conventions. The nonzero commutators for the chiral fields are

$$\pi_\pm = \mp\frac{R^2}{\pi}\partial_x\phi_\pm \implies [\phi_\pm(x),\phi_\pm(y)] = \pm i\frac{\pi}{2R^2}\mathrm{sgn}(x-y). \tag{14}$$

There is a factor of 2 that is a little bit subtle here—from the action we might have guessed that instead $\pi_\pm = \mp(R^2/2\pi)\partial_x\phi_\pm$, but this is not correct. One way to see this is by varying the action: we get[4]

$$\delta S = \int\delta\phi_\pm\left(\mp\frac{R^2}{\pi}\partial_t\partial_x\phi_\pm - \frac{\delta H}{\delta\phi_\pm}\right), \tag{16}$$

which implies from Hamilton's equations that $\pi_\pm = \mp\frac{R^2}{\pi}\partial_x\phi_\pm$ as claimed. We can check this by requiring that $\partial_t\phi_\pm = \mp\partial_x\phi_\pm$ hold as a consequence of $\partial_t\phi_\pm = i[H,\phi_\pm]$ (of course this is just another way of doing the same calculation), which gives

$$\mp\partial_x\phi_\pm = \mp i\frac{R^2}{2\pi}[(\partial_x\phi_\pm)^2,\phi_\pm] = \mp i\frac{R^2}{\pi}[\partial_x\phi_\pm,\phi_\pm]\partial_x\phi_\pm, \tag{17}$$

[4] Pedantic point here: when deriving this we need to integrate by parts. While we can't write $\int\partial_t\phi_\pm\partial_x\phi_\pm = -\int\phi_\pm\partial_t\partial_x\phi_\pm$ since $\phi_\pm$ by itself is not well-defined, we can however write

$$\int\partial_t\delta\phi_\pm\partial_x\phi_\pm = -\int\delta\phi_\pm\partial_t\partial_x\phi_\pm, \tag{15}$$

since $\delta\phi_\pm$, like $\partial\phi_\pm$, is well-defined.

giving us the desired commutator.[5] This factor of 2 is the same factor of 2 as in Chern-Simons theory (where we don't have a Hamiltonian to check) which means in that context that the momentum of the gauge field A is $k\star A/2\pi$ (with the $\star$ taken in space), instead of $k\star A/4\pi$.

The calculated commutators for $\phi_\pm$ tell us that

$$[\phi(x),\theta(y)] = -i\pi\mathrm{sgn}(x-y) \implies \pi_\phi = \frac{1}{2\pi}\partial_x\theta, \tag{19}$$

which agrees with the canonical momentum derived from the action for ϕ and θ we wrote above.

We can use these results to compute commutators of vertex operators. This works by writing, for X, Y Gaussian variables with c-number commutator,[6]

$$e^X \odot e^Y = e^X e^Y e^{-\frac{1}{2}\langle (X+Y)^2\rangle} = e^{X+Y}e^{\frac{1}{2}[X,Y]-\frac{1}{2}\langle (X+Y)^2\rangle} = e^Y e^X e^{[X,Y]-\frac{1}{2}\langle (X+Y)^2\rangle} = e^Y\odot e^X e^{[X,Y]}, \tag{21}$$

where $\odot$ means "operator product" (the colons for normal-ordering look ugly!), so that $e^X\odot e^Y =: e^X :: e^Y :$ and $e^Xe^Y =: e^Xe^Y :$. Basically, $A\odot B$ is used to denote a product that is not fully normal-ordered, with AB denoting a single operator with normal-ordering $: AB :$.

Spectrum of local operators

All of this is fine, but very formal. It is very formal because the fields we've been manipulating, the $\phi_\pm$s and their linear combinations, aren't really well-defined. Indeed, their two-point functions are nonsensical. The fields that are well-defined are of course exponentials and derivatives of the $\phi_\pm$. In fact even exponentials of $\phi_\pm$ are problematic, since as we mentioned they are non-local for generic values of R.

As we said above, when we say that the field ϕ is compact with $\phi\cong\phi+2\pi$, what we really mean is that we restrict ourselves to only considering vertex operators for ϕ of the form $V_{n,0} = e^{in\phi}$, with $n\in\mathbb{Z}$ (the notation will become clear in a sec). That is, we take $\phi\in\mathbb{R}$ (which we did when computing correlators), but impose that all physical operators be invariant under shifting ϕ by 2π. Since the conformal spin of $V_{n,0}$ is $s_{n,0}=0$, $V_{n,0}$ is always non-chiral, and has a well-defined two-point function. Note that this definition of compactness is *not* the same as saying that we restrict ourselves only to combinations of chiral

[5] Yet a third way to check is to write the chiral fields as

$$\phi_\pm(x) = \pm\int_0^{\pm\infty} dp\,\frac{1}{2\pi\sqrt{|p|R^2/\pi}}(\phi_p e^{ipx} + \phi_p^\dagger e^{-ipx}), \tag{18}$$

and then to compute the commutator explicitly (by regulating it with a factor of $e^{-\eta p^2}$; the integral in the commutator then becomes $\propto \mathrm{Erf}((x-y)/\sqrt{\eta})$, which has the correct $i\pi\mathrm{sgn}(x-y)$ limit as $\eta\to 0$).

[6] We need

$$e^X\odot e^Y = e^Xe^Ye^{\langle XY+\frac{1}{2}(X^2+Y^2)\rangle}, \tag{20}$$

which can be proved by writing down the series expansions and doing a bit of algebra (remember that the normal-ordering gets rid of *all* contractions between the two operators; for the vertex operators there are infinitely many such contractions to take into account).

vertex operators of the form $e^{in\phi_\pm}$! We can impose $\phi \cong \phi + 2\pi$, but somewhat confusingly this is not the same as having $\phi_\pm \cong \phi_\pm + 2\pi$, despite $\phi = \phi_+ + \phi_-$.

So, what about vertex operators of θ? The vertex operators $e^{in\theta}$ are also non-chiral and have well-defined correlators. However, they are not generically local with respect to the $V_{n,0}$. We will find the allowed vertex operators for θ by requiring that they create self-consisent field configurations for ϕ.

We can write $e^{i\alpha\theta(x)}$ as $e^{i\alpha\int_C d\theta}$, where C is a path extending from x out to infinity. This operator is only local if correlation functions are independent of the choice of C. From the commutation relations, no θ vertex operators can detect C, but ϕ vertex operators can. When $V_{n,0}$ moves through te curve C, it picks up a phase of $e^{2\pi i\alpha n}$. Hence for $e^{i\alpha\theta}$ to be local, we need $\alpha \in \mathbb{Z}$.[7]

Since the spectrum of the theory is generated by exponentials / derivatives of linear combinations of ϕ, θ (or $\phi_\pm$, either way), the claim is that

$$V_{n,w} = e^{in\phi + iw\theta}, \qquad (n,w) \in \mathbb{Z}^2 \tag{22}$$

generate the full spectrum of local vertex operators. As a check, we compute the OPE

$$\begin{aligned} V_{n,w}(z) \odot V_{n',w'}(w) &= \frac{V_{n+n',w+w'}}{(z-w)^{-(n-R^2w)(n'-R^2w')/2R^2}(\bar{z}-\bar{w})^{-(n+R^2w)(n'+R^2w')/2R^2}} \\ &= V_{n+n',w+w'}|z-w|^{-(nn'R^{-2}+ww'R^2)}\left(\frac{z-w}{\bar{z}-\bar{w}}\right)^{\frac{1}{2}(wn'+w'n)}. \end{aligned} \tag{23}$$

This OPE is evidently only well-defined provided that $wn' + w'n \in \mathbb{Z}$, and so indeed by taking $(n,w) \in \mathbb{Z}^2$, the operators $V_{n,w}$ are always well-defined local operators with sensible OPEs (and since we can have $wn' + w'n$ be the minimal value of 1, they generate all such local vertex operators).[8]

From the above OPE, we read off

$$\Delta_{n,w} = \frac{1}{2}(n^2/R^2 + w^2R^2), \qquad s_{n,w} = -nw, \tag{25}$$

with $s \in \mathbb{Z}$ as required. Note that the spin is independent of R, essentially by construction. T-duality acts as

$$T\,:\, V_{n,w} \mapsto V_{w,n},\, R \mapsto R^{-1}, \tag{26}$$

as expected.

[7]If one tries to think about the $\phi_\pm$ fields as well-defined entities in their own right, madness sets in quickly: $\phi \cong \phi + 2\pi$ means that if the $\phi_\pm$ are legit fields, we should also have e.g. $\phi_+ \cong \phi_+ + 2\pi$. But this then seems to say $\theta \cong \theta + 2\pi R^2$, which is a contradiction unless we are at the self-dual point $R^2 = 1$ (where the $\phi_\pm$ fields produce legit current operators). Therefore in general its best to only work with vertex operators and derivatives of ϕ, θ unless we're at a special point where there's an important chiral symmetry.

[8]Note that while $V_{n,0} \odot V_{m,0} = V_{n+m,0}$, $V_{n,w} \neq V_{n,0} \odot V_{0,w}$: instead, we have

$$V_{n,0} \odot V_{0,w} \sim V_{n,w}(\varepsilon/\bar{\varepsilon})^{nw/2}, \tag{24}$$

where ε is a point-splitting distance (the $\sim$ is because there's another numerical factor coming from combining the exponentials between $e^{in\phi}e^{iw\theta}$ and $e^{in\phi+iw\theta}$). Since this depends on the choice of ε, $V_{n,w}$ cannot be split-up as a product of operators in a well-defined way.

As mentioned above, for some values of R the spectrum includes operators that are genuinely chiral, with $\Delta_{n,w} = \pm s_{n,w}$. For this to be the case, we need to have $n/R = wR$ for some n, w. This means that we must have $R^2 = n/w$, and so we only have chiral operators when $R^2 \in \mathbb{Q}$. Thus only for rational values of R^2 do there exist local operators that are exponentials only of either ϕ_+ or ϕ_-.

Bosonizing bosons

In this subsection we use a cond-mat hydrodynamically-flavored line of reasoning to explain the ubiquity of the compact boson theory in one-dimensional problems.

The basic setting is non-relativistic[9] system of complex bosons with translation and $U(1)_N$ particle number symmetries. The basic UV Hamiltonian is

$$H = \frac{1}{2}\int dx\, \left(\frac{1}{m}|\partial_x\psi|^2 + V\rho^2\right), \tag{27}$$

where ρ is the density operator and V is a hard-core repulsion. The basic strategy is to write down an EFT in the IR whose variables keep track of the densities of the two conserved charges (momentum and $U(1)_N$ charge density). We know that neither of the two symmetries will truly be broken since we are in 1+1D; however we know that both symmetries can be very nearly broken, and in what follows we will use fluctuations about both nearly-ordered states to construct a hydrodynamic EFT.

The limit of weak interactions, where the system is close to a superfluid, is easy to deal with. Indeed we know that we have to get the action of a compact boson in the IR, since this is the action describing the "Goldstone modes" of the "broken" symmetry. This action can just be written down on phenomenological grounds, with the form of the parameters in the action fixed using common sense. But we want to do a bit better, viz. we want to provide a slightly more explicit mapping of the boson operators to the operators appearing in the "Goldstone mode" action, and relate the parameters in this action explicitly to V and the SF density. To this end, we write $\psi = \sqrt{\rho}e^{i\phi}$ and drop the fluctuations in ρ; this gives

$$H = \frac{1}{2}\int dx\, \left(K(\partial_x\phi)^2 + V\rho^2\right) \tag{28}$$

where $K = \rho_0/m = \langle\rho\rangle/m$. Now from $[\rho(x), e^{i\phi(y)}] = \delta(x-y)e^{i\phi(x)}$, we can introduce a field θ such that $\rho = \frac{1}{2\pi}\partial_x\theta$ (this is just for suggestive notation as we pass from H to S). Therefore we can write the action as[10]

$$S = \int \left(\frac{1}{2\pi}\partial_x\theta\partial_t\phi - \frac{K}{2}(\partial_x\phi)^2 - \frac{V}{8\pi^2}(\partial_x\theta)^2\right). \tag{29}$$

[9] The relativistic case is the familiar XY model, which manifestly has the form of a compact boson in the IR.

[10] The time derivative term here is as usual fixed in accordance with the commutation relations that we have imposed. It can also be derived from the non-relativistic term $i\psi^\dagger\partial_t\psi$ present in the action for the original bose fields.

Now integrating out θ,

$$
\begin{aligned}
S &= \frac{1}{2}\int \left(\frac{1}{V}(\partial_t\phi)^2 - K(\partial_x\phi)^2\right) = \frac{R^2}{4\pi}\int\left(\frac{1}{v}(\partial_t\phi)^2 - v(\partial_x\phi)^2\right) \\
R &= \sqrt{2\pi}\left(\frac{K}{V}\right)^{1/4}, \qquad v = \sqrt{KV}.
\end{aligned}
\tag{30}
$$

The near-SF is thus equivalent to a compact boson in the large-radius limit. The extent to which this system fails to be a SF is measured by the correlation function

$$
\langle V_{1,0}(x)V_{-1,0}(y)\rangle \sim \frac{1}{|x-y|^{\frac{1}{2\pi}\sqrt{V/K}}}. \tag{31}
$$

So of course we do not actually have a SF because of the algebraic falloff, but the power of the decay is arbitrarily small in the weakly interacting $K/V \to \infty$ limit.

In the limit of strong (repulsive) interactions, the natural starting point is the (Wigner) crystal. Again from a phenomenological point of view we know that the IR action has to be that of a compact boson (the "Goldstone" for translation); the only nontrivial part is finding the radius of the compact boson in terms of the interaction strength and the lattice spacing of the crystal, and providing a mapping between the boson operators and those of the IR action.

The phenomenological approach works like this: let us introduce a dimensionless field θ which measures the displacements of the atoms from their equilibrium position, with a shift in θ of 2π corresponding to a translation of all atoms in the crystal by one lattice constant a. That is, we let $x = x_0 + a\theta/2\pi$, where x is the position operator and x_0 a series of delta functions at the lattice sites. From this we see that for small fluctuations of the lattice, $\partial_x\theta$ keeps track of the fluctuations of the equilibrium number density through

$$
\rho = \rho_0 + \frac{1}{2\pi}\partial_x\theta. \tag{32}
$$

Then the phenomenological Goldstone action is, taking the IR limit of $\sum_i \left(\frac{1}{2}m(\partial_t\delta x)^2 + \frac{1}{2}Va(\delta\rho)^2\right)$,

$$
S = \frac{1}{2}\int\left[\frac{ma}{\pi^2}(\partial_t\theta)^2 - \frac{V}{4\pi^2}(\partial_x\theta)^2\right] = \frac{R^2}{4\pi}\int\left(\frac{1}{v}(\partial_t\theta)^2 - v(\partial_x\theta)^2\right), \tag{33}
$$

where now $R = \pi^{-1/2}(maV)^{1/4}$ and $v = \sqrt{V/4ma}$ (the latter is $\sqrt{T/\mu}$ since V has dimensions of energy times length). The extent to which this system fails to be a crystal is determined by the correlation functions of the wavevector $k = 2\pi/a$ component of the number density. To be notationally suggestive, we will define the momentum k_F by

$$
2k_F \equiv \frac{2\pi}{a} = 2\pi\rho_0. \tag{34}
$$

Now we can translate the whole crystal by one lattice spacing by changing θ by 2π. Under this change, the $\mathbb{R}$-space density of the $2k_F$ wavevector part of the density, viz. ρ_{2k_F}, has a phase change of 2π, and hence the relation between the two must be[11]

$$\rho_{2k_F}(x) \propto e^{i(2\pi\rho_0 x+\theta)}. \tag{35}$$

Therefore we have

$$\langle \rho_{2k_F}(x)^\dagger \rho_{2k_F}(y)\rangle \sim \frac{1}{|x-y|^{\pi/\sqrt{maV}}}, \tag{36}$$

which is algebraically decaying but with a power that is arbitrarily small in the strongly-interacting $maV \to \infty$ crystalline limit (maV has dimensions of $\hbar^2$ and so is properly dimensionless).

Now we will use a more explicit operator mapping to "derive" the compact boson action. Since we are coming from the starting point of a crystal, we want to do the mapping in a subspace where the density operator ρ is a sum of integer-weight delta functions. This will be the case if the combination $2\pi\rho_0 x+\theta$ is constrained to take values only in $2\pi\mathbb{Z}$, since then $\rho = \partial_x(\rho_0 x + \theta/2\pi)$ will be an appropriate sum of delta functions. Now working explicitly with a discontinuous field like this is of course a pain, and in any case we will eventually want to relax this constraint. Therefore we will incorporate the constraint on ρ by adding in an appropriate delta function:

$$\rho = (\rho_0 + \partial_x\theta/2\pi)\sum_{n\in\mathbb{Z}} e^{in(2\pi\rho_0 x+\theta)}. \tag{37}$$

The commutation relations between $\partial_x\theta$ and ϕ are fixed by $[\rho(x), e^{i\phi(y)}] = \delta(x-y)e^{i\phi(x)} \implies [\partial_x\theta(x)/2\pi, \phi(y)] = -i\delta(x-y)$, which gives us the expected momentum for ϕ.

To write the mapping of the boson field ψ, we need to take $\sqrt{\rho}$. But this is actually straightforward, since the square root of a sum of delta functions is just proportional to the same sum of delta functions. Therefore we can write

$$\psi \sim \sqrt{\rho_0 + \partial_x\theta/2\pi}\sum_{n\in\mathbb{Z}} e^{in(2\pi\rho_0 x+\theta)}e^{i\phi}. \tag{38}$$

Therefore we see how the spectrum of the compact boson theory comes out of the original model: all operators built from polynomials in the ψ fields are manifestly given by vertex operators (plus derivatives of θ, ϕ). Furthermore we expect that for IR questions we can soften the constraint on ρ by dropping most of the terms appearing in the sum which enforce the discreteness constraint, since terms with larger n oscillate more quickly in $\mathbb{R}$-space by an amount given by the UV scale $\rho_0 = 1/a$.

Anyway, now we put this relation into the boson Hamiltonian. The first term $|\partial_x\psi|^2$ is rather complicated—it involves the simple $\rho_0(\partial_x\phi)^2 + \dots$ (where $\dots$ are higher in derivatives and hence irrelevant), but also the complicated $(\partial_x\sqrt{\rho})^2$. However, one sees that all the terms in the expansion of $(\partial_x\sqrt{\rho})^2$ are actually all irrelevant, since they are all of the form $(\partial_x^2\theta)^2$ or $(\partial_x\theta)^2 P(\cos\theta, \sin\theta)$, where Ps are polynomials in various cosines and sines of θ. Therefore we can drop all the terms in the $|\partial_x\psi|^2$ term except for the gradient term for ϕ. Similarly

[11] Note that the $2k_F$ component of the density is related to a vertex operator of θ, while the zero-momentum component is related to $\partial_x\theta$.

the interaction term $V(\delta\rho)^2$ just becomes $\propto V(\partial_x\theta)^2$ after dropping irrelevant terms, and so the Hamiltonian is

$$H = \frac{1}{2}\int dx\,\left(\frac{\rho_0}{m}(\partial_x\phi)^2 + \frac{V}{4\pi^2}(\partial_x\theta)^2\right). \tag{39}$$

If we then integrate out ϕ to get an action just in terms of θ, we get the same compact boson action for θ as above, with the same radius.

In the general case where we are somewhere between a SF and a crystal, we simply map the boson operators with an extrapolation between their images in the two limits:

$$\psi \sim \sqrt{\rho_0 + \partial_x\theta/2\pi}\sum_{n\in\mathbb{Z}} U_n e^{in(2\pi\rho_0 x+\theta)}e^{i\phi}, \tag{40}$$

where the U_n are phenomenological coefficients. When we are close to a SF all the U_n are nearly zero expect for U_0, while when we are close to a crystal the U_n are nearly independent of n.

Bosonizing fermions: formal approach

Now we will discuss a field-theory flavored way of motivating bosonization (of fermions). This has the advantage of being rather clean and easy to work with, but the disadvantage of being slightly subtle once interactions are added and of having the overall appearance of a magic trick (which it is not).

The strategy in the field theory approach is to "rigorously" establish the mapping for the case of free fermions, and then make a rather sketchy argument about the generalization to the interacting case.

The compact boson theory discussed in the last section is, of course, a bosonic theory: the spectrum of operators, viz. $\{V_{n,w}, d\theta, d\phi\}$, are all bosonic (and the derivatives can be obtained from the vertex operators by taking OPEs).[12] To get a fermionic theory, we have to generalize slightly. From the fact that $s_{n,w} = -nw$, we see that all we have to do is to generalize the operator algebra to e.g. include operators either with $n \in \frac{1}{2}\mathbb{Z}$ or $w \in \frac{1}{2}\mathbb{Z}$ (but not both). Taking one of n, w to be fractional effectively attaches a JW string branch cut to the vertex operator, and provides the commutation relations we expect from a fermion. The choices of whether we allow for fractional n or w are related by T-duality, and correspond to whether we want the free fermions to occur at $R = 1/\sqrt{2}$ or $R = \sqrt{2}$. Therefore without loss of generality we can study fermionic theories by looking the theory whose spectrum is generated by the operators $V_{n,\frac{1}{2}w}$, for $n, w \in \mathbb{Z}$.

Unfortunately, it turns out that this notation makes a bunch of formulas that appear later rife with ugly factors of 2. We will therefore introduce the variables

$$\Phi \equiv \phi, \qquad \Theta \equiv \frac{R^2}{2}(\phi_- - \phi_+) = \frac{\theta}{2}, \qquad \pi_\Phi = \frac{1}{\pi}\partial_x\Theta \tag{41}$$

Here Φ is introduced just to make the notation look slightly more visually pleasing. We then define the vertex operators

$$\mathcal{V}_{n,w} = V_{n,w/2} = e^{i(n\Phi + w\Theta)} = e^{i\phi_+(n - R^2w/2) + i\phi_-(n + R^2w/2)}, \tag{42}$$

[12] We are working on $\mathbb{R}^2$ throughout, and hence are not caring about global issues like spin structure dependence.

which, for the free action (1), have scaling dimensions and spins given by

$$\Delta_{n,w} = \frac{1}{2}\left(\frac{n^2}{R^2} + w^2\frac{R^2}{4}\right), \qquad s_{n,w} = -nw/2. \tag{43}$$

The space of vertex operators is thus still obtained from two compact fields, each still with periodicity 2π,[13] except now the vertex operators can be fermionically nonlocal.

From our discussion of the compact boson, we see that at $R = \sqrt{2}$, the chiral fields[14]

$$V_\pm = e^{2i\phi_\pm} = \mathcal{V}_{1,\mp 1} \tag{44}$$

are well-defined in the sense that their two-point functions are single-valued and have the same correlation functions as free fermions (as well as the same self-anti-commutation relations as fermions, so they are only local to the extent that fermions are local). This means that all correlation functions of the $\psi_{L/R}$ fields calculated with the free Dirac action will be identical to those calculated with the vertex operators $e^{i\phi_\pm}$ in a compact boson theory at $R = \sqrt{2}$.

We will thus write the bosonization map *for free fermions* as[15] (note that this mapping *only works at* $R = \sqrt{2}$)

$$\begin{aligned}\mathcal{B}[\psi_R] &= \gamma_R\mathcal{V}_{1,-1} = \frac{\gamma_R}{\sqrt{a}}e^{i(\Phi-\Theta)} = \frac{\gamma_R}{\sqrt{a}}e^{i2\phi_+},\\ \mathcal{B}[\psi_L] &= \gamma_L\mathcal{V}_{1,1} = \frac{\gamma_L}{\sqrt{a}}e^{i(\Phi+\Theta)} = \frac{\gamma_L}{\sqrt{a}}e^{i2\phi_-}\end{aligned} \tag{45}$$

where a is a UV cutoff needed to get the dimensions correct, which until now we have been hiding in the implicit normal-ordering of the vertex operators, and where γ_σ are Klein factors (Majorana fermions) needed so that $\mathcal{B}[\psi_R]$ anticommutes with $\mathcal{B}[\psi_L]$ in our quantization scheme. From now on, the γ_σs and the $(a)^{-1/2}$s will only be written out when needed.

Note that the translation $U(1)_T$ and particle-number $U(1)_N$ symmetries act on the fermions (in the IR) as

$$\begin{aligned}U(1)_T &: \psi_{L/R} \mapsto e^{\mp i\rho\delta x/2}\psi_{L/R}\\ U(1)_N &: \psi_{L/R} \mapsto e^{i\alpha}\psi_{L/R},\end{aligned} \tag{46}$$

where $\rho = 2k_F$ is the density (note that $U(1)_T$ acts axially). Hence on Φ, Θ we have (not writing out as)

$$U(1)_T : \Phi \mapsto \Phi, \quad \Theta \mapsto \Theta - \frac{\rho}{2}\delta x, \qquad U(1)_N : \Phi \mapsto \Phi + \alpha, \quad \Theta \mapsto \Theta. \tag{47}$$

The factor of $1/2$ in the $U(1)_T$ transformation of Θ means that if we translate by $\delta x = 2\pi/\rho$, which is the distance over which we expect to find one fermion, Θ shifts by π, which is

[13] If we like, we could stick with the old ϕ, θ notation and just say that we are increasing the periodicity condition on θ to $\theta \sim \theta + 4\pi\mathbb{Z}$.

[14] The factor of 2 in the exponent is an unavoidable causality of our notation—this seemed like the least annoying place for factors of 2 to live, so we'll just deal with it.

[15] When the coefficient in front of the kinetic term for the action (1) is normalized to be 1/2, which is another popular choice, the fermions are $e^{\pm i\sqrt{4\pi}\phi_\pm}$.

non-trivial. This tells us that $e^{i\Theta}$ counts fermion parity, a conclusion which we will confirm shortly. From the commutation relations above the generators of the two $U(1)$s are

$$Q_T = -\frac{\rho}{2\pi}\int d\Phi, \qquad Q_N = \frac{1}{\pi}\int d\Theta. \tag{48}$$

The mixed anomaly between the two $U(1)$s then can be understood from the commutation relations of the above charge densities; since this is done in another diary entry we won't go in to further detail.

We will find it convenient to define the fields

$$\varphi_R = \Phi - \Theta, \qquad \varphi_L = \Phi + \Theta, \qquad [\varphi_{R/L}(x), \varphi_{R/L}(y)] = \pm\frac{2\pi i}{R^2}\mathrm{sgn}(x-y). \tag{49}$$

At the free fermion radius $R = \sqrt{2}$ we have $\varphi_{R/L} = 2\phi_\pm$, but this is not true for general R (in particular, the $\varphi_{R/L}$ are *not* chiral at generic radii). In terms of these fields then,[16]

$$\mathcal{B}[\psi_{L/R}] = \frac{1}{\sqrt{a}}e^{i\varphi_{L/R}}, \tag{50}$$

The bosonization map is the statement that the two actions

$$\mathcal{B} : \frac{1}{2\pi}\int \bar\psi i\partial\!\!\!/\psi \leftrightarrow \frac{1}{4\pi}\int \sum_{\sigma=L,R}\left((-1)^\sigma \partial_t\varphi_\sigma\partial_x\varphi_\sigma - v\partial_x\varphi_\sigma\partial_x\varphi_\sigma\right), \tag{51}$$

generate the same correlation functions, where $(-1)^\sigma$ is -1 for $\sigma = R$ and $+1$ for $\sigma = L$, and v is the velocity of the dirac fermions. The RHS is the same as (5) since we are at $R = \sqrt{2}$. In terms of the Φ, Θ fields, we may write

$$\mathcal{B} : \frac{1}{2\pi}\int \bar\psi i\partial\!\!\!/\psi \leftrightarrow \frac{1}{2\pi}\int \left(2\partial_t\Phi\partial_x\Theta - v(\partial_x\Phi)^2 - v(\partial_x\Theta)^2\right). \tag{52}$$

The statement here is again that these two actions generate the same correlation functions provided we identify operators using $\mathcal{B}$. That is, since $\psi_{L/R}$ has the same correlation functions as $e^{i\phi_{L/R}} = e^{i\phi_\pm}$ in the free theory, the claim is that

$$\langle \mathcal{O}[\psi]\rangle_{\frac{1}{2\pi}\bar\psi i\partial\!\!\!/\psi} = \langle \mathcal{O}[\mathcal{B}[\psi]]\rangle_{R=\sqrt{2}}, \tag{53}$$

where $\mathcal{O}[\psi]$ is any polynomial of ψ fields at arbitrary positions. The claim is that the spectrum of operators $\mathcal{V}_{n,w}$ (and their derivatives) exhaust all operators in the fermion theory. It's clear that we get all polynomials of the fermions by taking products of the $\mathcal{V}_{1,\pm1}$s—the operators with n and / or w odd have less obvious fermionic counterparts; we will see in a sec that they are related to $(-1)^F$ operators.

[16] The $\pm$ sign in the exponent is convention: many times it is instead written as $e^{\pm i\phi_\pm}$. Changing these conventions, which amounts to mapping $\psi_L \leftrightarrow \psi_L^\dagger$, simply swaps the physical interpretations of the ϕ and θ via T-duality. In the present conventions, Φ is a phase variable (Φ getting a vev is a "SF"), while Θ is a density variable (Θ getting a vev is a "crystal").

Just to make the claim about the matching of correlation functions in the free theory completely explicit, we know that in the fermionic theory we have (looking at e.g. the R fermions wolog)

$$\langle \psi_R(x_1)\dots\psi_R(x_n)\psi_R^\dagger(y_1)\dots\psi_R^\dagger(y_n)\rangle = \det\left(\frac{1}{x_i-y_j}\right). \tag{54}$$

On the other hand, the vertex operators give (with the implicit normal-ordering eliminating the $i=j$ terms)

$$\langle e^{i\varphi_R(x_1)}\dots e^{i\varphi_R(x_n)}e^{-i\varphi_R(y_1)}\dots e^{-i\varphi_R(y_n)}\rangle = \frac{\prod_{i<j<n}(x_i-x_j)\prod_{i<j\leq n}(y_i-y_j)}{\prod_{i<j\leq n}(x_i-y_j)}. \tag{55}$$

This is indeed exactly equal to the determinant; one can show this e.g. by looking at the zeros and the poles: both functions have poles when some x_i equals some y_j, and both have zeros when two x's or two y's are coincident (since then the matrix in the determinant becomes degenerate).

Interactions are dealt with by expanding the exponential $e^{iS_{int}}$ as a bunch of correlation functions:

$$\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\, e^{i(S+S_{int})} = \int \mathcal{D}\phi\,\mathcal{D}\theta\, \exp\left(iS_{R=\sqrt{2}}[\phi,\theta] + i\mathcal{B}[S_{int}]\right). \tag{56}$$

In my opinion what we've done so far is rigorous (for physicists). The formula for $\mathcal{B}[\psi_{L/R}]$ above is useful in that tells us that the parition functions in the two theories above are identical, and furthermore provides us with a may of matching correlation functions in the two free theories.

Bosonization mapping for a few operators

In accordance with the above discussion, we need to figure out how to bosonize products of $\psi_{L/R}$ fields in the free theory. We do this by resolving products of operators by point-splitting in the usual way. We will point-split by displacing the operators in space, since this is most convenient for employing the commutation relations when doing calculations. Shankar's book is an okay reference for some of the following.

We will need to bosonize some operators that take the form of normal-ordered products / derivatives of fermion operators. However, but our bosonization map as written above only works on the constituent fermions themselves, since they are the fields whose correlation functions are matched on the boson side. So in order to map more complicated operators we un-normal-order them and express them in terms of the $\psi_{L/R}$, then bosonize by using the fact that the bosonization map is a homomorphism

$$\mathcal{B}[\mathcal{O}_1\odot\mathcal{O}_2] = \mathcal{B}[\mathcal{O}_1]\odot\mathcal{B}[\mathcal{O}_2] \tag{57}$$

for $\mathcal{O}_i$ any single-fermion operators, and finally re-write things in terms of normal-ordered products to find the image of the given operator under bosonization (also remember that Taylor expansions can only be performed *inside* the normal-ordering symbol, at the very last step).

For example, let us consider the R fermion density. We first need to remember that

$$G_{L/R}(x,t) = \frac{i}{t \mp x + ia}, \tag{58}$$

where the $+ia$ convergence factor usually won't we written. The fact that the $\pm$ sign appears on the x and not the t is important for some calculations, so we will try to keep track of it correctly.[17] The i here is because the propagator comes from inverting $i^2\not{\partial}$, not $i\not{\partial}$.[18] Also note that the $1/2\pi$ in front of the fermion action means there's no 2π in the above propagator. Anyway, we can now write

$$\begin{aligned}(\psi_R^\dagger \psi_R)(z) &= \lim_{\varepsilon\to 0}\left(\psi_R^\dagger(z+\varepsilon)\odot\psi_R(z) - \frac{i}{\varepsilon}\right)\\ &\to \lim_{\varepsilon\to 0}\left(e^{-i\varphi_R(z_\varepsilon)}\odot e^{i\varphi_R(z)} - \frac{i}{\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0}\left(e^{-i\varepsilon\partial_x\varphi_R(z)+\dots}\frac{i}{\varepsilon} - \frac{i}{\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0}\left([1-\varepsilon(i\partial_x\varphi_R)(z)]\frac{i}{\varepsilon} - \frac{i}{\varepsilon}\right)\\ &= \partial_x\varphi_R(z),\end{aligned} \tag{59}$$

where the i when combining the two vertex operators into a single exponential comes from the BCH-formula phase of $e^{\frac{1}{2}[\varphi_R(z+\varepsilon),\varphi_R(z)]} = e^{i\pi\mathrm{sgn}(\varepsilon)/2} = i$, provided that we point-split in the "correct" way (recall we are at $R=\sqrt{2}$). Of course this feels pretty arbitrary (and I'm going to stop paying attention to signs too carefully at this point so as to retain my sanity), and this is one of the reasons why the field theory approach is a bit annoying.

A similar calculation for the L fields gives the opposite sign[19]

$$\mathcal{B}[\psi_{R/L}^\dagger(x)\psi_{R/L}(x)] = \pm\partial_x\varphi_{R/L}(x) = \pm\partial_x\Phi \mp \partial_x\Theta. \tag{60}$$

Therefore the currents map as[20]

$$\mathcal{B}[2\pi j^\mu] = \mathcal{B}[(\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R, \psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L)^\mu] = 2(\partial_x\Theta, \partial_x\Phi)^\mu. \tag{62}$$

[17] The easy way to remember this is that in the Dirac action, the derivatives appear as $\partial_t \pm v\partial_x$. It is checked by Fourier transforming with the Feynman propagator (i.e. with $i\varepsilon\mathrm{sgn}(k)$ in the denominator): $\langle\psi_{L/R}(k)\psi_{L/R}(k)^\dagger\rangle \propto \pm\Theta(k)e^{-k0^+}$, which gives the desired result.

[18] There are various conventions for this, but ours is the one in which $G_R(z-w) = \langle\psi_R(z)\psi_R^\dagger(w)\rangle$, with no factor of i.

[19] There's a minus sign from the fact that φ_Ls have an opposite sign in the commutator when re-combining the exponentials.

[20] Note that the spatial part of the current here is $n_R - n_L$; in other parts of the diary it's been the other way around. Also note that with these conventions, because of the $1/2\pi$ in the fermion action,

$$n_\sigma = \frac{1}{2\pi}\psi_\sigma^\dagger\psi_\sigma. \tag{61}$$

Since $d\Phi \overset{\text{eom}}{=} R^{-2}\star d\theta = 2R^{-2}\star d\Theta$ and we are at $R=\sqrt{2}$, we have (making the replacement on the equations of motion here is exact since both Φ and Θ appear quadtratically in the action — if cosines were added to the action then manipulations like this would not be legit)

$$\mathcal{B}[j] \overset{\text{eom}}{=} \frac{1}{\pi}\star d\Theta \overset{\text{eom}}{=} \frac{1}{\pi} d\Phi. \tag{63}$$

In particular, the density is (this is exact, not just on the eom)

$$\mathcal{B}[j^0] = \frac{1}{\pi}\partial_x\Theta \tag{64}$$

Therefore the operator $e^{i\Theta(x)}$ counts the fermion parity to the left (or right) of x[21]:

$$e^{i\Theta(x)} = e^{i\pi\int^x j^0}. \tag{65}$$

We thus have the physical interpretation of Θ as the field which counts the fermion density, which shifts by $\delta\Theta = \pi$ at the location of a fermion (θ shifts by $\delta\theta = 2\pi$ at each fermion). Therefore in a Euclidean time picture, fermion-odd operators insert π vortices in Θ. Note that the Thirring interaction bosonizes as $\mathcal{B}[j_\mu j^\mu] = -\frac{1}{\pi^2}\partial_x\varphi_R\partial_x\varphi_L$.

The off-diagonal bilinears are easy, since the OPE is trivial:

$$\mathcal{B}[\psi_L(x)\psi_R^\dagger(x)] = e^{i2\Theta(x)}, \qquad \mathcal{B}[\psi_L(x)\psi_R(x)] = e^{i2\Phi(x)}. \tag{66}$$

Finally for the bosonization of kinetic terms for the fermions. Since

$$\langle\psi_R^\dagger(z)(\partial_w\psi_R(w))\rangle = -i\frac{1}{(z-w)^2}, \tag{67}$$

we have, focusing on the ∂_x term for concreteness, (it's better to get rid of the derivative first by point-splitting and then bosonize rather than the other way around)

$$\begin{aligned} \psi_R^\dagger\partial_x\psi_R &= \lim_{\epsilon\to 0}\left(\psi_R^\dagger(z)\frac{\psi_R(z+\varepsilon)-\psi_R(z-\varepsilon)}{2\varepsilon} + \frac{i}{\epsilon^2}\right) \\ &\to \lim_{\epsilon\to 0}\left(e^{-i\varphi_R(z)}\odot\frac{1}{2\varepsilon}\left(e^{i\varphi_R(z+\varepsilon)} - e^{i\varphi_R(z-\varepsilon)}\right) + \frac{i}{\epsilon^2}\right) \end{aligned} \tag{68}$$

The RHS is, remembering the is coming from recombining the exponentials,

$$\frac{i}{2\varepsilon}e^{-i\varphi_R(z)+i\varphi_R(z+\varepsilon)}\frac{1}{-\varepsilon} - \frac{i}{2\varepsilon}e^{-i\varphi_R(z)+i\varphi_R(z-\varepsilon)}\frac{1}{+\varepsilon} + \frac{i}{\varepsilon} \approx \frac{-i}{2\varepsilon^2}\left(2 + i\varepsilon^2\partial_x^2\varphi_R - \varepsilon^2(\partial_x\varphi_R)^2\right) + \frac{i}{\varepsilon^2}, \tag{69}$$

where we have expanded the exponentials to $O(\varepsilon^2)$. Up to total derivatives, this just gives $i\frac{1}{2}(\partial_x\varphi_R)^2$, and therefore

$$\mathcal{B}[\psi_+^\dagger i\partial_x\psi_+] = -\frac{1}{2}(\partial_x\varphi_R)^2. \tag{70}$$

[21] This nice factor-of-2-free result, which vibes nicely with the conventions in part of the CMT literature is our reason for choosing to work with the $V_{n,w/2}$ vertex operators.

This gets us part of the kinetic term. The rest of the kinetic term comes from the other derivative of ψ_+ and the derivatives of ψ_- in the similar way. For $\psi_+^\dagger i\partial_t\psi_+$ we just get $-\partial_t\varphi_R\partial_x\varphi_R/2$, while for the ψ_- terms we get opposite signs on both the ∂_t and ∂_x terms (from the opposite sign in the commutation relations when combining the exponential). One then checks that the two kinetic terms indeed map into one another, and that even the coefficients are correct!

It's easy to get lost in the formalities here, but note that we're still just doing hydrodynamics. The two conserved quantities are again momentum and $U(1)_N$ charge, and as we have seen above, bosonization gives us a way to represent their currents j in terms of free bosons (we have two symmetries but only one current because of the mixed anomaly between the two symmetries). The resulting action is just an EFT formed from the conserved currents. This is of course true in the non-interacting case, but since it really comes from general EFT principles the general idea holds when interactions are turned on as well.

Adding interactions

As a simple example, consider adding the term

$$S_{int} = -\frac{1}{4\pi}\int U_{\alpha\beta}\rho_\alpha\rho_\beta, \tag{71}$$

where $\rho_\alpha = \psi_\alpha^\dagger\psi_\alpha = 2\pi n_\alpha$. The off-diagonal part $\rho_L\rho_R$ is a $j_\mu j^\mu$ Thirring-type interaction (viz. $\frac{\pi}{2}U_{LR}j_\mu j^\mu$), while the forward scattering terms $U_{\sigma\sigma}$ will be seen to renormalize the velocities. Indeed, the bosonized version of this is

$$\begin{aligned} S_b &= \frac{1}{4\pi}\int\left(\sum_\sigma(-1)^\sigma\partial_t\varphi_\sigma\partial_x\varphi_\sigma - \sum_\sigma(\partial_x\varphi_\sigma)^2(v_\sigma+U_{\sigma\sigma}) + 2U_{LR}\partial_x\varphi_L\partial_x\varphi_R\right) \\ &= \frac{1}{4\pi}\int\left(-\partial_t\varphi^T Z\partial_x\varphi - \partial_x\varphi^T\mathcal{H}\partial_x\varphi\right), \qquad \mathcal{H} = \begin{pmatrix} v_R' & -U_{LR} \\ -U_{LR} & v_L'\end{pmatrix}, \end{aligned} \tag{72}$$

where the renormalized velocities are $v_\sigma' \equiv v_\sigma + U_{\sigma\sigma}$ and $\varphi = (\varphi_R, \varphi_L)^T$. We can calculate the OPE of the vertex operators by diagonalizing the Hamiltonian. We will preserve the commutation relations (first term in the action) if we can diagonalize $\mathcal{H}$ with something in $SO(1,1)$, i.e. a matrix of the form $M = \mathbf{1}\cosh\psi + X\sinh\psi$. This can always be done if $\mathcal{H}$ is positive-definite, which we of course assume on physical grounds. A bit of algebra (in the diary entry on correlators in Luttinger liquids) shows that

$$M^T\mathcal{H}M = \begin{pmatrix} -U_{LR}\sinh(2\psi) + v_R'\cosh^2\psi + v_L'\sinh^2\psi & 0 \\ 0 & -U_{LR}\sinh(2\psi) + v_L'\cosh^2\psi + v_R'\sinh^2\psi \end{pmatrix}, \tag{73}$$

provided that

$$\tanh(2\psi) = \frac{U_{LR}}{(v_R'+v_L')/2}. \tag{74}$$

This is always possible if

$$|U_{LR}| < \frac{v_R'+v_L'}{2}. \tag{75}$$

Now on the other hand, the condition that $\mathcal{H}$ be positive-definite can be checked to be that $|U_{LR}| < \sqrt{v'_R v'_L}$. Since the geometric mean is always $\leq$ the arithmetic mean, $\mathcal{H}$ being positive-definite automatically guarantees that there's a boost M diagonalizing it. This stability condition $|U_{LR}| < \sqrt{v'_L v'_R}$ is less obvious on the fermion side (also note that because the condition is only on $|U_{LR}|$, the fermions are [equally] unstable to both attractive *and* repulsive Thirring-type interactions). Also note that when $v_L = v_R$ so that $v'_R = v'_L \equiv v'$, we get (after using e.g. $\cosh(\tanh^{-1}(x)) = (1-x^2)^{-1/2}$)

$$M^T \mathcal{H} M = \widetilde{v}\mathbf{1}, \qquad \widetilde{v} = v'/\gamma, \qquad M = \gamma \begin{pmatrix} 1 & -U_{LR}/v' \\ -U_{LR}/v' & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1 - U_{LR}^2/v'^2}}, \tag{76}$$

which is exactly what we expect from a Lorentz boost.

To deal with the more general situation (just for fun), define

$$\bar{v} \equiv (v'_R + v'_L)/2, \qquad \delta \equiv (v'_R - v'_L)/2. \qquad \gamma \equiv (1 - U_{LR}^2/\bar{v}^2)^{-1/2}. \tag{77}$$

Then we have

$$\cosh^2\psi = \frac{\gamma+1}{2}, \qquad \sinh^2\psi = \frac{\gamma-1}{2}, \qquad \sinh(2\psi) = \gamma U_{LR}/\bar{v}, \tag{78}$$

which allows us to write

$$M^T \mathcal{H} M = \bar{v} \begin{pmatrix} \gamma^{-1} + \delta/\bar{v} & \\ & \gamma^{-1} - \delta/\bar{v} \end{pmatrix} \equiv \begin{pmatrix} \widetilde{v}_R & \\ & \widetilde{v}_L \end{pmatrix}. \tag{79}$$

In terms of the Φ, Θ fields, (sanity check: the correct momentum for Φ is recovered) some algebra gives

$$S_b = \frac{1}{2\pi} \int \left(2\partial_t \Phi \partial_x \Theta - (\partial_x \Phi)^2 (\bar{v} - U_{LR}) - (\partial_x \Theta)^2 (\bar{v} + U_{LR}) + 2\delta \partial_x \Phi \partial_x \Theta \right). \tag{80}$$

From this presentation, the stability bound on $|U_{LR}|$ in the case when $\delta = 0$ is more obvious. We can integrate out Θ since it appears quadratically, using

$$\partial_x \Theta \overset{\text{eom}}{=} \frac{\partial_t \Phi + \delta \partial_x \Phi}{\bar{v} + U_{LR}} \tag{81}$$

which gives, after some algebra

$$S_b = \frac{R^2}{4\pi} \int \left(\frac{1}{\mathsf{v}} (\partial_t \Phi + \delta \partial_x \Phi)^2 + \mathsf{v} (\partial_x \Phi)^2 \right), \qquad R^2 \equiv 2\sqrt{\frac{\bar{v} - U_{LR}}{\bar{v} + U_{LR}}}, \qquad \mathsf{v} \equiv \sqrt{\bar{v}^2 - U_{LR}^2}. \tag{82}$$

The rather unfortunate 2 in the definition of R^2 comes from our use of $\Theta = \theta/2$ i.e. our choice that the Φ, Θ fields are 4π, not 2π periodic. There's essentially no perfect notational choice here, so we'll just live with it.

To get a feel for what the interaction does, consider for simplicity the case where $\delta = 0$, and work in units where $\mathsf{v} = 1$. Then to lowest order in U_{LR}, the action is

$$S_{b; v'_L = v'_R} \approx \frac{R_{eff}^2}{4\pi} \int d\Phi \wedge \star d\Phi, \qquad R_{eff}^2 = 2(1 - U_{LR}/\bar{v}). \tag{83}$$

Therefore adding a small repulsive interaction between the two chiral fermions has the effect of decreasing the boson radius. Sanity check: this means that repulsive interactions increase the scaling dimension of the SF ordering term $\cos\Phi$, and decrease that of the crystal ordering term $\cos\Theta$, while attractive interactions do the opposite. This is exactly what we expect on physical grounds—attractive interactions favor k-space order (SF), while repulsive ones favor $\mathbb{R}$-space order (crystal).

Another thing to note is how T-duality works in terms of the fermionic parameters. Again since we are using $\Theta = \theta/2$, T-duality is no longer just $R \mapsto R^{-1}$, but rather

$$T : R \mapsto \frac{2}{R}, \tag{84}$$

which is still an involution. We see that this is equivalent to $T : U_{LR} \mapsto -U_{LR}$, so that here T-duality acts to change the sign of the interaction. Thus attractive and repulsive interactions are actually exactly equivalent to one another, as we noted above. On the fermion side, we can implement the sign flip of U_{LR} by doing charge conjugation on only a single chirality, e.g.

$$T : \psi_+ \mapsto \psi_+, \qquad \psi_- \mapsto \psi_-^*, \tag{85}$$

which flips the sign of the interaction. One can also check that this action exchanges axial and vector currents on the fermion side, in keeping with the fact that it exchanges the regular and topological currents on the boson side.

Anyway, let us now find the scaling dimensions of the spectrum of vertex operators. The OPEs are straightforward to calculate. Let us label vertex operators $\mathcal{V}_{n,w}$ by the vector $\mathsf{n} = (n, w)^T$. The R/L components of this vector are $((n-w)/2, (n+w)/2)^T = S\mathsf{n}$, where $S = \frac{1}{2}(\mathbf{1} + J)$. We then find the OPE

$$\mathcal{V}_{\mathsf{n}}(x,t) \odot \mathcal{V}_{\mathsf{m}}(0,0) = \mathcal{V}_{\mathsf{n}+\mathsf{m}}(x,t) \frac{1}{(x + \widetilde{v}_R t)^{-\mathsf{n}^T S^T M |R\rangle\langle R| M S \mathsf{m}} (x - \widetilde{v}_L t)^{-\mathsf{n}^T S^T M |L\rangle\langle L| M S \mathsf{m}}} + \dots, \tag{86}$$

where the $\dots$ are less singular. This means that the conformal dimension is, skipping some algebra,

$$\Delta_{\mathsf{n}} = \frac{1}{2}\mathsf{n}^T S^T M^2 S \mathsf{n} = \frac{\gamma}{4}\mathsf{n}^T \begin{pmatrix} 1 + U/\bar{v} & \\ & 1 - U/\bar{v} \end{pmatrix} \mathsf{n} = \frac{\gamma}{4}\left(n^2(1 + U/\bar{v}) + w^2(1 - U/\bar{v})\right). \tag{87}$$

As we saw earlier for the simpler example, repulsive interactions make the Θ (w) vertex operators more relevant, favoring CDW order, while attractive interactions make the Φ (n) operators more relevant, favoring SF order.

The conformal spin is

$$s_{n,w} = \frac{1}{2}\mathsf{n}^T S^T M Z M S \mathsf{n} = \frac{1}{2}\mathsf{n}^T S^T Z S \mathsf{n} = -\frac{1}{4}\mathsf{n}^T X \mathsf{n} = -\frac{nw}{2}. \tag{88}$$

Note in particular that the conformal spin is unchanged by interactions or changes in velocity, as it should be (which is a consequence of the fact that $M \in SO(1,1)$ preserves the commutation relations).

Let us also compute the correlation functions of the UV fermions. In real space, the dominant part of this correlation function is

$$\langle \psi(x)\psi^\dagger(0)\rangle \sim \frac{\cos(k_F x)}{|x|^{1/R^2+R^2/4}}. \tag{89}$$

Fourier transforming, we then have

$$\langle \psi_k \psi_k^\dagger \rangle \sim |k-k_F|^\eta + |k+k_F|^\eta, \tag{90}$$

where

$$\eta \equiv \frac{1}{R^2} + \frac{R^2}{4} - 1. \tag{91}$$

Note that $\eta \geq 0$ for all R^2, with the equality saturated only at $R^2 = 2$ (where the distribution is of course instead a step function).

For nontrivial interactions there are no quasiparticles, in the sense that the UV fermions no longer have overlap with the excitations (hydro modes) which diagonalize the Hamiltonian. One can see this either by the fact that the hydro modes have different quantum numbers than the UV fermions, or by the fact that the fermion-fermion correlators have branch cuts, with no well-defined quasiparticle pole. The cool thing is that even though $\psi_{L/R} \sim e^{i\varphi_{L/R}}$ has power-law correlations with a coefficient that is a continuous function of the interaction parameters, the current still has scaling dimension exactly equal to one and is not renormalized by interactions (as it always bosonizes to $\sim (\partial_x\Theta, \partial_x\Phi)$), in keeping with the Ward identity.

It's worth comparing for a second between the approach above and the treatment in a few field theory texts, e.g. Shankar's QFT in CMT book and Witten's lectures on Abelian bosonization, both of which make it seem like bosonization in the presence of interactions can be done exactly while keeping a Lorentz-invariant structure.

From a field theory point of view, given that $\mathcal{B}[j^0] = \frac{1}{2\pi}\partial_x\Phi$, it is natural to make things covariant by writing $\mathcal{B}[j] = \frac{1}{2\pi}d\Phi$. Now as we saw, this is not the correct way to bosonize—when we point-split correctly the j^1 component of the current maps to something with a ∂_x derivative, namely $\partial_x(\varphi_L - \varphi_R)$. This is consistent with the action of spacetime symmetries since under e.g. time reversal that maps $j \mapsto -j$ (as a form). But if we were to take $\mathcal{B}[j] = \frac{1}{2\pi}d\Phi$ at face value, we'd have

$$S_{\text{free}}[\psi] - \frac{g}{\pi}\int j_\mu j^\mu \quad \overset{?}{\leftrightarrow} \quad \frac{1-g}{8\pi}\int d\phi \wedge \star d\phi. \tag{92}$$

When reading Witten and Shankar one gets the feeling that this relation is exact, but this can't be correct: as we discussed earlier, T-duality on the boson side is the same as the "KW Duality" sending $j \mapsto \star j$ on the Fermion side, which acts as $\psi_L \mapsto \psi_L^\dagger$, $\psi_R \mapsto \psi_R$. This sends $g \mapsto -g$, but is also supposed to do $R^2 \mapsto \frac{2}{R^2}$, which is not compatible with the above equation. Indeed we explicitly saw above, the above is only correct to leading order in g, and hence is a perturbative statement (dimensionally correct since the velocity is being suppressed). The full relation between R and g is non-linear, as seen by the formula above with the square roots and such, and is only derived in the QFT framework by doing a self-consistent point-splitting in e.g. space only.

To summarize: while the field theory way of thinking is slicker and nicer for doing calculations, the cond-mat way of doing this is more rigorous and intuitive; one should learn the cond-mat way but usually calculate things the QFT way (just like in RG).

Relevance of symmetry-breaking perturbations

The two symmetries of the system we've been studying so far are the $U(1)_N$ of particle number conservation, which shifts Φ, and the $U(1)_T$ of translation, which shifts Θ.

If we restrict our attention to actions which are symmetric under both symmetries, the previous subsection covers all possibilities, up to the effects of irrelevant derivative interactions. If we allow ourselves to consider perturbations which break the symmetries though, we can add sines and cosines of integer multiples of the Φ and Θ fields (since we usually don't want to add operators with nonzero spin to the theory, we can restrict our attention to just $\cos(n\Phi)$ and $\cos(m\Theta)$, without any mixed Θ-Φ terms).[22] Furthermore note that to preserve $(-1)^F$ we need $n \in 2\mathbb{Z}$, and since we only want to consider perturbations which are local, we also need to take $m \in 2\mathbb{Z}$.

The minimal perturbations are therefore $\cos(2\Theta), \cos(2\Phi)$. At the free fixed point, the dimensions of these two are in fact equal:

$$\Delta_{SC}^{free} = \Delta_{CDW}^{free} = 1. \tag{93}$$

The fact that $\Delta_{CDW}^{free} < 2$ is the statement of Peierls instability: a translation-breaking potential at wavevector $2k_F$ always drives an instability. Note that the filling with the most relevant $U(1)_T$-symmetric cosine is half-filling with $k_F = \pi/2$, which permits $\cos(4\Theta)$ as a $U(1)_T$-preserving perturbation. For free fermions this is comfortably irrelevant, with a scaling dimension of 4. Note that as expected, the $U(1)_T$-breaking perturbations become more relevant as the strength of the interactions is increased (i.e. as the radius of the boson is decreased) since the interactions favor $\mathbb{R}$-space Wigner-crystal ordering; similarly, getting closer to a superfluid by decreasing the interactions makes the $U(1)_N$-breaking perturbations more relevant.

Bosonization: CMT approach

We can use the results in the subsection on bosonizing bosons to get a much more intuitive, but also less rigorous, derivation of the bosonization formulae outlined in the previous section.

Bosonizing fermions is essentially the same as bosonizing bosons, which we have already discussed from an EFT point of view. In order to have notation that's consistent with the

[22] Note that we might think that adding e.g. $\cos(2\Theta - 2k_F x)$ would be a way to add a symmetry-allowed density modulation, but this in fact vanishes: it comes from a term like $\int dx\, e^{2ik_F x}\psi_L^\dagger \psi_R + h.c.$, but this vanishes because the support of the $\psi_{L/R}$ in momentum space is narrow enough to preclude the $\psi_L^\dagger \psi_R$ term from having the required $2k_F$ momentum transfer to survive integration. The correct density modulation term is instead just $\int dx\, L^\dagger R + h.c.$, which is nonzero but breaks $U(1)_T$.

previous section we will add an extra minus sign in the expression for the bosonzied density operator, so that

$$\rho = \rho_0 - \partial_x \Theta/\pi. \tag{94}$$

In the Wigner crystal limit, we thus tack on the constraint enforcing the discreteness of the lattice by writing ρ as

$$\rho = (\rho_0 - \partial_x \Theta/\pi) \sum_{n \in 2\mathbb{Z}} e^{in(\pi \rho_0 x - \Theta)}. \tag{95}$$

Extending this approach to fermions is very easy—we just add on JW tails to fermionize the operators appearing in H (we add the tails in the same way in both the SF and crystal limits). The JW strings need to count the fermion number to the left (say) of a given fermion, and so they must be given by $(-1)^{\int^x \rho} = e^{i(\pi\rho_0 x - \Theta(x))}$. Then the fermions are given generically by

$$\psi(x) \sim \sqrt{\rho_0 - \partial_x\Theta/\pi} \sum_{n \in 2\mathbb{Z}} U_n e^{i(n+1)(\pi\rho_0 x - \Theta)} e^{i\Phi(x)} \sim \sqrt{k_F + \partial_x \Theta} \sum_{n \in 2\mathbb{Z}+1} U_n e^{in(k_F x - \Theta(x))} e^{i\Phi}, \tag{96}$$

where U_n are some phenomenological constants. Therefore the effect of the JW tails is to shift the sum of the exponents of the Θ vertex operators from $2\mathbb{Z}$ to $2\mathbb{Z}+1$. Including the $\partial_x\Theta$ term in the square root is done to account for situations in which we imagine k_F varying semiclassically throughout space, with the k_F in the square root representing a spatial average of the Fermi momentum. In a situation where k_F is fixed, then by Luttingers theorem there is no zero-momentum modulation in the density (the Fermi sea only sloshes back and forth, it does not pulse in size). In the non-interacting case only $U_{\pm 1}$ are nonzero; larger U_ns come from processes which transfer momenta $2nk_F$, which are only possible in the presence of interactions. Therefore in the free limit, if we write $k_F = 1/a$ as a UV cutoff, we reproduce exactly the formulae (45) motivated through more formal field-theory methods. One can also check that the symmetry actions work out in the same way as in the bosonic case.

Of course, while this method for mapping the fermions agrees in the IR with the previous field theory approach, it also gives us an idea of what happens when we back away from this limit, and has the conceptual advantage that we didn't have to start with respect to a reference free theory with a certain Fermi surface. Indeed, this approach relied only on basic hydrodynamically-flavored reasoning, and at no point did we bring up complicated ways of counting bosonic and fermionic excitations with respect to a Fermi sea, normal ordering prescriptions, etc. etc.