Duals of global symmetries

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Examples of dual symmetries

Our first example will be that of topological $\mathbb{Z}_N$ gauge theory (lattice $\mathbb{Z}_N$ q-form gauge theory with infinitely massive magnetic fields, or deep in the Higgs phase of $U(1)$ gauge theory with charge N $(q-1)$-form matter).

First let's remind ourselves of what symmetries such a topological $\mathbb{Z}_N$ q-form gauge theory possesses. One way to write the action in the continuum is

$$S = \frac{i}{2\pi}\int_X G \wedge (d\phi - nA), \tag{1}$$

where A is a $U(1)$ q-form gauge field (i.e. $\int_{M_{q+1}} F_A \in 2\pi\mathbb{Z}\ \forall$ closed $M_{q+1} \subset X$), G is a $D-q$ form with 2π-quantized periods, and ϕ is a $q-1$ form, again with $d\phi$ having 2π-quantized periods. Integrating out G sets $A = \frac{1}{n}d\phi$, meaning that A is constrained to be flat and that n copies of A are a large gauge transformation.

An alternate way to write this that makes the symmetries manifest is to integrate out ϕ, which sets $G = F_B$ for some $U(1)$ $D-q-1$ form $U(1)$ gauge field B. We usually integrate by parts (for a careful discussion of why this works even though the integrations involve bare gauge fields, see the earlier diary entry on DB cohomology) and write the result in the BF form

$$S = \frac{in}{2\pi}\int B \wedge F_A. \tag{2}$$

We have *two* global symmetries in this theory. One is the q-form global symmetry on A. This is a $\mathbb{Z}_N$ symmetry (*not* a $U(1)$ symmetry), which is most clearly seen when we re-write the integrand as $A \wedge F_B$ (again, see earlier diary entry). We similarly have a $D-q-1$ form $\mathbb{Z}_N$ symmetry (we will use the notation $\mathbb{Z}_N^{(D-q-1)}$) that shifts B. From the action, we see that the Wilson operators for A (those for B) generate the $\mathbb{Z}_N^{(D-q-1)}$ symmetry (the $\mathbb{Z}_N^{(q)}$ symmetry). Note how the "magnetic" symmetry on B is shifted by a degree from the magnetic symmetry in $U(1)$ gauge theory, which is a $U(1)_m^{(D-q-2)}$ symmetry. Formally, this basically comes from the fact that the discrete action is written in terms of the fields B, A which transform under the symmetry as $B \wedge dA$, while in the $U(1)$ case it is $F \wedge \star F = dA \wedge d\widetilde{A}$ with $\widetilde{A}$ the dual gauge field—the extra d in this expression is what shifts the counting of the form of the symmetry. More physically, the offset comes from the fact that the magnetic symmetries in the two theories come from different conservation laws: the magnetic symmetry in the $U(1)$ theory comes from the non-existence of "monopole" events (magnetic flux "lines" are unbreakable), while the magnetic symmetry in the $\mathbb{Z}_N$ theory comes from the non-existence of field strength (t' Hooft "surfaces" are unbreakable).

When the $\mathbb{Z}_N$ gauge theory arises in the context of coupling to something with a $\mathbb{Z}_N^{(q-1)}$ global symmetry, the $\mathbb{Z}_N^{(q)}$ symmetry will be explicitly broken, since the Wilson operators for A will be allowed to terminate on the $(q-1)$-dimensional charged objects. However, the $\mathbb{Z}_N^{(D-q-1)}$ magnetic symmetry will remain, since we haven't added any magnetic matter. Another way to say this is that whenever we couple a theory with a q-form $\mathbb{Z}_N$ symmetry to a q-form $\mathbb{Z}_N$ gauge field, we turn a global q-form symmetry into a local one, but at the same time we add a $\mathbb{Z}_N^{(D-q-1)}$ global symmetry to the theory. This extra symmetry is hidden in the natural variables that we usually write down the theory in, and is generated by the Wilson operators of the q-form gauge field.

Let's elaborate on this for a second: we want to know what happens when a q-form $\mathbb{Z}_N$ field is coupled to a $(q+1)$-form $\mathbb{Z}_N$ field. This is the kind of coupling we need to turn on to gauge the $\mathbb{Z}_N$ q-form symmetry of the $\mathbb{Z}_N$ BF theory, for example. Now if the original field whose symmetry is to be gauged is A and the gauge field is $\mathcal{A}$ (bad notation! 何それ), we have

$$S \supset \frac{i}{2\pi} \int (N F_A \wedge B - \mathcal{A} \wedge B + N \mathcal{A} \wedge F_{\mathcal{B}}) \,, \tag{3}$$

for the appropriate degree fields $\mathcal{A}, \mathcal{B}$. The last term sets $\mathcal{A}$ to be a $\mathbb{Z}_N$ $(q+1)$-form field ($F_{\mathcal{B}}$ is a properly quantized $U(1)$ field strength). The second term is what allows us to gauge the $\mathbb{Z}_N^{(q)}$ global symmetry. Under a "local" action of this symmetry (i.e. a change in higher transition functions such that the cocycle condition on $(q+2)$-fold overlaps of patches fails by an Nth root of unity), $\frac{N}{2\pi} F_A$ changes by an element in $H^{q+1}(X;\mathbb{Z})$. We can write this as $F_A \mapsto F_A + \frac{1}{N} F_\alpha$, where F_α is a properly quantized $U(1)$ field strength. This change is canceled out by the corresponding shift $\mathcal{A} \mapsto \mathcal{A} + F_\alpha$, and so the whole action is gauge-invariant.

There are four symmetries to look at: first, the "electric" symmetry that shifts A is gauged. Second, the "magnetic" symmetry which shifts B is explicitly broken by the coupling to $\mathcal{A}$: shifting $B \mapsto B + \lambda$ for a λ a $(D-q-1)$ form with periods quantized in $\frac{2\pi}{N}$ shifts the action by something in $\frac{2\pi i}{N}\mathbb{Z}$. This means that coupling the q-form theory to the gauge field explicitly breaks the dual magnetic symmetry of the q-form theory. This is clear from the fact that the gauging of the q-form symmetry means that field strength for A is now no longer constrained to vanish; hence t' Hooft operators for the A field are now allowed to be broken. Furthermore, the electric shift symmetry of $\mathcal{A}$ is broken by the presence of B (gauge transformations on $\mathcal{A}$ still act trivially, though). The only remaining global symmetry is the magnetic symmetry of $\mathcal{A}$, which acts by shifting $\mathcal{B}$. Thus the global symmetry of this system is $\mathbb{Z}_N^{(D-q-2)}$.

In general, if we have a chain of a q form $\mathbb{Z}_N$ field coupled to a $q+1$ form coupled to a $\dots$ coupled to a $q+n$ form, only the magnetic symmetry of the $q+n$ form survives as a genuine global symmetry—all the lower symmetries are eliminated by the tower of gauge fields: the electric symmetries are explicitly gauged away, while the magnetic ones are broken by the fact that the gauging procedure allows the fields in all but the top level of the chain to have nonzero field strength.

The $U(1)$ case is similar, but the notation is a lot more concise. Let A be a $U(1)$ q-form field, whose electric $\mathbb{Z}_N^{(q)}$ symmetry is gauged by coupling to a $(q+1)$-form G: ignoring coupling constants, and writing the action in a way that makes the magnetic symmetry of the un-gauged theory manifest,

$$S \supset \int \left[(F_A - G)\wedge\star(F_A - G) + \frac{i}{2\pi}(F_A - G)\wedge d\widetilde{A} + dG\wedge\star dG\right]. \tag{4}$$

Here F_A is a 2-form that is rendered a proper $U(1)$ field strength after $\widetilde{A}$ is integrated out, with the gauge transformations shifting F_A by an exact $(q+1)$-form. We see that unlike in the discrete example, adding G does not kill the magnetic $D-q-2$ form symmetry of the original A theory—again, this is because the magnetic symmetries in the discrete and continuous cases come from different conservation laws. We also have the magnetic symmetry on G, but unfortunately since G is coupled to electric matter we cannot write S in a way that makes this symmetry manifest (since S has to contain the bare G, and not its field strength). Therefore we have the original $\mathbb{Z}_N^{(D-q-2)}$ magnetic symmetry and a new $\mathbb{Z}_N^{(D-q-3)}$ symmetry on $\widetilde{G}$.

Summary: Let's now summarize what happens when we gauge the electric symmetry in both a topological $\mathbb{Z}_N$ q-form gauge theory and in a $U(1)$ q-form gauge theory. In both cases, the electric symmetry is gauged away, and replaced with a new global symmetry:

$$\begin{aligned} \mathbb{Z}_N: &\qquad \text{new } \mathbb{Z}_N^{(D-q-2)} \text{ symmetry}, \quad Q = \int \mathcal{A} \\ U(1): &\qquad \text{new } U(1)^{(D-q-3)} \text{ symmetry}, \quad Q = \int F_G, \end{aligned} \tag{5}$$

where $\mathcal{A}$ and G are the fields we added to do the gauging and Q are the charge operators for the new symmetries. In the discrete case the gauging kills the magnetic symmetry of the original theory, while in the $U(1)$ case it does not.

$\mathbb{Z}_N$ spin models

Now we will take a look at the two-dimensional $\mathbb{Z}_N$ spin model (specializing to two dimensions is just for notation's sake; we will generalize later). From the above, we know that if it indeed maps to a $\mathbb{Z}_N$ gauge theory, then we will have a $\mathbb{Z}_N^{(D-1-1)} = \mathbb{Z}_N^{(0)}$ global symmetry generated by the Wilson line for the gauge field. This is exactly the right symmetry we need to match with the symmetry of the spin model.

The partition function is (hopefully the notation won't get too horrible)

$$Z = \sum_{s_i}\prod_l E(\zeta^{s_i - s_{i+l}}). \tag{6}$$

Here the spin variables on each site are $S_i = \zeta^{s_i}$, where $\zeta = e^{2\pi i/N}$. E is some energy functional that gives the Boltzmann weight for a particular configuration of two neighboring spins. In the product, i and $i+l$ are determined as the sites at ∂l. Fourier transforming,

$$Z = \sum_{s_i}\sum_{a_l}\prod_l \widetilde{E}(\zeta^{a_l})\zeta^{a_l(s_i - s_{i+l})}. \tag{7}$$

The coupling between a and s is schematically $\int a \wedge \star ds = \int s \wedge \star d^\dagger a$, and the sum over all s_i configurations thus implements the delta function $\delta(d^\dagger a)$. The constraint $d^\dagger a = 0$ becomes a flatness constraint on the dual lattice, which we can then deal with in the usual way. On the direct lattice though, a_l is not a gauge field: there is no local symmetry that shifts a_l by something exact. The local symmetry will only exist on the dual lattice.

Now we will show how this is compatible with a gauge theory presentation. The gauge theory we start with is

$$Z_g = \sum_{s_P, \widetilde{a}_L, \widetilde{s}_I} \prod_L \zeta^{s_P (d\widetilde{a})_P} \widetilde{E}(\zeta^{\widetilde{a}_L + (d\widetilde{s})_L}). \tag{8}$$

For notation, we have used the following: fields with a tilde over them are the ones which naturally live on the dual lattice. The links, sites, and plaqettes. of the dual lattice are denoted in capital letters: L, I, P. Here $d\widetilde{a}$ is the (signed) sum of $\widetilde{a}_L$'s around a plquette on the dual lattice. We can equivalently write $(d\widetilde{a})_P$ as $(d\widetilde{a})_i$, where i is at the center of P. In the continuum, the coupling of s and $d\widetilde{a}$ goes to the integral $\int s \wedge \star d\widetilde{a} = \int s \wedge \star d^\dagger a$, where $d\widetilde{a} = d^\dagger a$. The basic rule for mapping to the dual lattice is e.g. $\widetilde{a} = \star a$, with d acting on the dual lattice becoming $\star d$ on the original lattice. Anyway, in this partition function we see that the $s_i = s_P$ variables are Lagrange multipliers which enforce the flatness of the $\widetilde{a}$ gauge field. One should think of s_i (or really, the two-cochains $s_P \in C^2(X^*; \mathbb{Z}_N)$) as being the B in BF theory.

To see that this Z is the same as the spin model Z, we work in unitary gauge: we Hodge decompose $\widetilde{a}_L$, and fix the exact part in the decomposition equal to $-d\widetilde{s}$. This kills off the $\widetilde{s}_I$ variables. Then we can re-write the $sd\widetilde{a}$ term using $\int s \wedge \star d\widetilde{a} = \int s \wedge \star d^\dagger a = \int a \wedge \star ds$ to get

$$Z_g = \sum_{s_i, \widetilde{a}_L} \prod_L \zeta^{\widetilde{a}_L (s_i - s_{i+\star L})} \widetilde{E}(\zeta^{\widetilde{a}_L}) = Z, \tag{9}$$

so that Z_g is the same as the $\mathbb{Z}_N$ spin theory. The dual spins $\widetilde{s}_I$ basically play the role of soaking up the gauge redundancy of the $\widetilde{a}$ field, leaving a field ($\widetilde{a}$ after unitary gauge fixing) that is not invariant under the shift of an exact form, which can then map onto the "momentum" field a_l of the direct lattice.

Now although in the Z_g formulation the sum over s_P sets the $\widetilde{a}$ field to be flat, in general topology prevents us from fixing $\widetilde{a} = d\widetilde{\lambda}$ to be trivial. If $H^1(X) = 0$ however we can choose a gauge in which $\widetilde{a}$ is gauge equivalent to zero. This gives

$$Z_g|_{H^1(X)=0} = \sum_{\widetilde{s}_I} \prod_L \widetilde{E}(\zeta^{\widetilde{s}_I - \widetilde{s}_{I+L}}), \tag{10}$$

which is the Fourier-transformed spin system on the dual lattice. However, in general we need to keep s_P and $\widetilde{a}_L$, and so the true dual of the original $\mathbb{Z}_N$ spin system is a Fourier-transformed dual spin system, living on the dual lattice and coupled to a $\mathbb{Z}_N$ gauge field. We write this duality as

$$\mathbb{Z}_N^{(0)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(0)} \wedge \widetilde{\mathbb{Z}_N}^{(1)}, \tag{11}$$

where the RHS denotes the dual gauge theory. As discussed before, the global symmetries on both sides match: the global $\mathbb{Z}_N^{(0)}$ on the LHS is generated by the Wilson line on the RHS, and acts on the "monopole operators" s_P. Note that if we hadn't kept the original

s_P variables in the Z_g formulation, the origin of the $\mathbb{Z}_N^{(0)}$ symmetry on the RHS would be somewhat hidden.

Let us briefly discuss the operator content of both sides of the duality. As we just mentioned, the s_i operators (or the "vertex operators" ζ^{s_i}) become the "monopoles for the gauge theory. Inserting ζ^{s_x} into the partition function and performing the Fourier transform leads to a coupling like $\sum_i s_i(\delta_{i,x} + (d^\dagger a)_i)$, which sets the divergence of the "momentum" a_l to be a delta function concentrated at x (this is just Gauss' law). When we go to the dual lattice $d^\dagger a$ goes to $d\widetilde{a}$, and so the constraint from s_i leads to $(d\widetilde{a})_P = \delta_{P,\widetilde{x}}$, where $\widetilde{x}$ is the plaquette in the dual lattice associated with x. Thus inserting the original spins, which are the "magnetic" lagrange multipliers in the dual formulation, leads to places where the topologicalness of the dual gauge field is violated.

The dual spins are not gauge invariant by themselves, so we can only insert operators of the form

$$W_\gamma = \zeta^{k\widetilde{s}_I} \prod \zeta^{k\widetilde{a}_L}_{L\in\gamma} \zeta^{k\widetilde{s}_J}, \tag{12}$$

where γ is a path on the dual lattice from I to J and $k \in \mathbb{Z}_N$. When we insert this operator in the gauge theory and perform the gauge fixing by setting the exact part of $\widetilde{a}$ to kill the dual spins, we get

$$\langle W_\gamma \rangle = \sum_{\widetilde{a}^f_L, s_P} \prod_L \widetilde{E}(\zeta^{\widetilde{a}^f_L}) \zeta^{\widetilde{a}^f_L((ds)_L + k\delta_{L\in\gamma})}. \tag{13}$$

Here the superscript on $\widetilde{a}^f_L$ denotes the gauge-fixing. Now un-doing the Fourier transform ($a_l = \star\widetilde{a}^f_L$ is the momentum variable), we get

$$\langle W_\gamma \rangle = \sum_{s_i} \prod_l E(\zeta^{(ds)_L + k(\star\gamma)_L}). \tag{14}$$

Here $\star\gamma$ is γ on the direct lattice, and is obtained by rotating all the segments of γ by $\pi/2$. Thus we see that computing the partition function with the insertion of W_γ is equivalent to computing the partition function with a modified Hamiltonian for the spin model: the Hamiltonian gets modified on all links in $\star\gamma$, i.e. for all links that intersect γ transversely. For example in the case where $N = 2$ we might have an Ising model nearest-neighbor interaction, and the insertion of W_γ would flip the sign of the interaction between ferromagnetic and anti-ferromagnetic on all the links in $\star\gamma$ (of course, provided $k = 1$).

In the symmetry-breaking phase $\langle \zeta^{s_i} \rangle \neq 0$ of the spin model, the Wilson operators $\langle W_\gamma \rangle$ have an exponential decay that goes as $|\min(\gamma)|$, where $\min(\gamma)$ is the shortest path connecting I to J: we have linear confinement. This is because the insertion of W_γ changes the sign of the coupling along γ (it is a disorder operator), so that in the symmetry-breaking phase the value of $\langle \zeta^{s_i} \rangle$ jumps by an amount determined by k upon crossing γ. In order for $\langle \zeta^{s_i} \rangle$ to be well-defined there must be some other line accross which $\langle \zeta^{s_i} \rangle$ jumps by the oppoiste amount. Since the Hamiltonian is not modified on this line, this line will have a tension determined by the strength of the coupling in the Hamiltonian. Note that the line defined by γ itself is not tensionful, since the modification of E along γ means that jumps in $\langle \zeta^{s_i} \rangle$ are not energetically costly there. Thus the minimal energy configuration will be one where $\langle \zeta^{s_i} \rangle$ flips along γ, and then again along the shortest line connecting I and J (again, only the latter line is energetically costly). Thus we get a linearly confining phase. On the other hand, if $\langle \zeta^{s_i} \rangle = 0$ then the W_γ operators have nonzero vevs. This is the Higgs phase of the gauge theory.

How does this work in higher dimensions? Basically, everything goes through in the same way. We first Fourier-transform the partition function by introdcuing "momentum" variables a_l on the links of the original lattice. These are then dual to gauge fields $\widetilde{a} = \star a$, which are cochains in $C^{D-1}(X^*)$. To write the gauge theory version Z_g of the partition function, we add dual "matter fields" $\widetilde{s}$, which are cochains in $C^{D-2}(X^*; \mathbb{Z}_N)$ (i.e. $(D-2)$-form gauge fields on the dual lattice). As before, the spin system partition function is recovered upon fixing the unitary gauge. Everything else goes through in the same way—this is why we've been using differential forms. Thus we get that in D dimensions, the $\mathbb{Z}_N$ spin system is dual to a theory on the dual lattice with a $(D-2)$-form $\mathbb{Z}_N$ gauge field coupled to a $(D-1)$-form $\mathbb{Z}_N$ gauge field. We write this as

$$\mathbb{Z}_N^{(0)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(D-2)} \wedge \widetilde{\mathbb{Z}_N}^{(D-1)}. \tag{15}$$

Again, the global symmetries match. The LHS has a $\mathbb{Z}_N^{(0)}$ symmetry, while the only global symmetry on the RHS[1] is the one generated by the Wilson operators of the $(D-1)$-form gauge field: since the charge operators are $(D-1)$-dimensional, this is a regular zero-form symmetry, which matches with the LHS. Note that in keeping with duality, the $\mathbb{Z}_N$ symmetry on the LHS is an electric symmetry (the symmetry "acts on the fundamental fields"), while on the RHS it is a magnetic symmetry (the symmetry is "generated by the fundamental fields").

Dualizing a $\mathbb{Z}_N$ gauge theory (a non-topological one) is also done in a similar way. For example, in three dimensions the partition function is

$$Z = \sum_{a_l} \prod_p E(\zeta^{da_p}) = \sum_{a_l, \widetilde{b}_L} \prod_p \widetilde{E}(\zeta^{\widetilde{b}_L}) \zeta^{(da_L)\widetilde{b}_L} = \sum_{a_l, \widetilde{b}_L} \prod_p \widetilde{E}(\zeta^{\widetilde{b}_L}) \zeta^{(d^\dagger \widetilde{a})_L \widetilde{b}_L}, \tag{16}$$

where $\widetilde{a}_P = \star a_l$. Following the usual procedure, this is equivalent to (using $\int d^\dagger \widetilde{a} \wedge \star \widetilde{b} = \int \star \widetilde{a} \wedge d\widetilde{b}$)

$$Z = \sum_{\widetilde{b}_L, \widetilde{s}_I, a_P} \prod_L \widetilde{E}(\zeta^{\widetilde{b}_L + (d\widetilde{s})_L}) \zeta^{\widetilde{a}_P (d\widetilde{b})_P}, \tag{17}$$

which we see upon fixing unitary gauge to kill $\widetilde{s}_I$. This is a spin model coupled to a topological $\mathbb{Z}_N$ gauge field, so we have

$$\mathbb{Z}_N^{(1,g)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(0)} \wedge \widetilde{\mathbb{Z}_N}^{(1)}. \tag{18}$$

Here the $\mathbb{Z}_N^{(1,g)}$ means a non-topological $\mathbb{Z}_N$ 1-form gauge theory. Note that the LHS doesn't have the same magnetic symmetry coming from the lagrange multiplier enforcing flatness that the topological gauge theories have. It does have the electric 1-form symmetry though, which is matched with the magnetic 1-form symmetry of the $\widetilde{\mathbb{Z}_N}^{(1)}$ factor on the RHS.

[1] At the risk of repeating myself: the "electric" symmetry of the $\mathbb{Z}_N^{(D-2)}$ factor is gauged, and the "magnetic" symmetry of the $(D-2)$-form field along with the electric symmetry of the $(D-1)$-form field are both explicitly broken by the coupling between the two fields. The only remaining global symmetry is the magnetic symmetry of the $(D-1)$-form field.

What happens if we were to gauge the 1-form symmetry on the LHS? This is done by coupling to a topological 2-form $\mathbb{Z}_N$ gauge theory. The only symmetry left is the magnetic symmetry for the 2-form, which is generated by the Wilson surface and hence is a $\mathbb{Z}_N^{(0)}$ symmetry. Thus we expect that the result is just a $\mathbb{Z}_N$ spin model. Indeed, this is what we get: the partition function is

$$Z = \sum_{a_l, B_p, \lambda_c} \prod_{p,c} E(\zeta^{(da)_p + B_p}) \zeta^{\lambda_c (dB)_c}, \tag{19}$$

where λ_c is a Lagrange multiplier enforcing flatness for the background field B_p. So then Fourier transforming,

$$\begin{aligned} Z &= \sum_{a_l, B_p, \lambda_c, b_p} \prod_{p,c} \widetilde{E}(\zeta^{b_p}) \zeta^{\lambda_c (dB)_c + b_p((da)_p + B_p)} \\ &= \sum_{\widetilde{s}_I, \widetilde{a}_P, \widetilde{B}_L, \widetilde{\lambda}_I, \widetilde{b}_L} \prod_{I,L} \widetilde{E}(\zeta^{\widetilde{b}_L + (d\widetilde{s})_L}) \zeta^{\widetilde{\lambda}_I (d\widetilde{B})_I + \widetilde{b}_L((d^\dagger \widetilde{a})_L + \widetilde{B}_L)}. \end{aligned} \tag{20}$$

We can now integrate out the $\widetilde{B}_L$ field. It appears as $\int \star \widetilde{B} \wedge (\widetilde{b} + d\widetilde{\lambda})$, which sets $\widetilde{b}$ to be exact. Thus $\widetilde{b}$ is pure gauge, and the term $\int \widetilde{b} \wedge \star d^\dagger \widetilde{a} \to \int \widetilde{\lambda} \wedge d(\star d^\dagger \widetilde{a})$ vanishes. Thus since all the fields except $\widetilde{s}$ dissapear and as predicted, we get a $\mathbb{Z}_N^{(0)}$ spin model:

$$Z = \sum_{\widetilde{s}_I} \prod_L \widetilde{E}(\zeta^{(d\widetilde{s})_L}). \tag{21}$$

This is written as

$$\mathbb{Z}_N^{(1,g)} \wedge \mathbb{Z}_N^{(2)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(0)}. \tag{22}$$

The story is similar if we start with a higher-form $\mathbb{Z}_N$ gauge theory, or if we work in more general dimensions. Going through the arguments above, we find

$$\mathbb{Z}_N^{(q,g)} \leftrightarrow \mathbb{Z}_N^{(D-q-2,g)} \wedge \widetilde{\mathbb{Z}_N}^{(D-q-1)}. \tag{23}$$

The q-form electric symmetry on the LHS matches with the $(D-(D-q-1)-1) = q$ form magnetic symmetry on the RHS. As was the case for $q = 1, D = 3$, gauging the electric symmetry on the LHS by coupling to a topological $\mathbb{Z}_N$ $(q+1)$-form field removes the topological gauge field from the RHS of the duality, and we get

$$\mathbb{Z}_N^{(q,g)} \wedge \mathbb{Z}_N^{(q+1)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(D-q-2,g)}. \tag{24}$$

This time the magnetic $D-(q+1)-1$ form symmetry on the LHS matches with the electric $D-q-2$ form symmetry on the RHS.