Skyrmions

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These notes contain some calculations relating to skyrmion physics in quantum Hall ferromagnets and Chern bands.

1 Naive field theory for $C = 1$

In this section we will use an overly-complicated field theory procedure to derive the correct quantum mechanical description for Quantum Hall skyrmions in a system with a weak periodic potential.

We start from the familiar CS GL theory (spin indices suppressed)

$$\mathcal{L} = b^\dagger \left(\partial_\tau - ia_0 + V(x) + \frac{1}{2m^*}\nabla_a^2 \right) b + \frac{g}{2}(\rho - \rho_0)^2 + i\frac{1}{4\pi}(a - A) \wedge d(a - A) \tag{1}$$

where the effective mass (coming from projecting the interactions into the LLL) is

$$m^* \sim \frac{\rho_0}{U} \tag{2}$$

and the background field A satisfies $\star dA = (B, 0, 0) = (2\pi\rho_0, 0, 0)$. We are temporarily ignoring any long-range Coulomb interactions that may be present, and also temporarily ignoring the Zeeman field term.

The goal is to get an effective action in terms of the skyrmion current

$$J = \star \frac{dz^\dagger \wedge dz}{2\pi i}, \tag{3}$$

where $b = \rho z$ and $|z|^2 = 1$.[1] To do this it is helpful to recall that

$$\frac{\rho}{8m^*}(\nabla \mathbf{n})^2 = \frac{\rho}{2m^*}|(\nabla - i\mathbf{a})z|^2 - \frac{m^*}{2\rho}\mathcal{J}^2, \tag{4}$$

where $\mathbf{n} = z^\dagger \boldsymbol{\sigma} z$ (no factor of 1/2) and the number current is

$$\mathcal{J}_i = -i\frac{\rho}{m^*}(z^\dagger \partial_i z - ia_i). \tag{5}$$

[1] In what follows we will allow ourselves to replace ρ with ρ_0 if it appears multiplying a dynamical field, but not if it appears by itself.

This just follows from making a gauge-invariant version of the usual $(\partial n)^2 \sim |\partial z|^2 - |z^\dagger \partial z|^2$ CP^1 model Lagrangian. Then we may write

$$\begin{aligned}\mathcal{L} = \rho(z^\dagger \partial_\tau z - ia_0 + V(x)) + \frac{m^*}{2\rho}\mathcal{J}_i \mathcal{J}^i + i(z^\dagger \partial_i z - ia_i)\mathcal{J}^i + \frac{\rho}{8m^*}(\nabla \mathbf{n})^2 \\ + \frac{g}{2}(\rho - \rho_0)^2 + \frac{i}{4\pi}(a - A) \wedge d(a - A)\end{aligned} \tag{6}$$

where now $\mathcal{J}$ is an independent vector field. From the way it couples to $z^\dagger \partial_\mu z$, we see that it is natural to extend the current to a three-vector by defining $\mathcal{J}_\mu = (\rho, \mathcal{J}_i)$.

We now integrate out a:[2]

$$\mathcal{L} = \mathcal{J}_\mu \xi^\mu + \frac{1}{2\alpha}\left(c\mathcal{J}_0^2 + \frac{1}{c}\mathcal{J}_i \mathcal{J}^i\right) + \frac{i}{2}(2\pi)\mathcal{J} \wedge \frac{1}{d^\dagger}\mathcal{J} + \frac{\rho}{8m^*}(\nabla \mathbf{n})^2, \tag{7}$$

where

$$\xi_\mu = (iz^\dagger \partial_\tau z + V - g\rho_0 + A_0,\ iz^\dagger \partial_i z + A_i) \tag{8}$$

and where we have defined

$$c \equiv \sqrt{\frac{g\rho_0}{m^*}}, \qquad \frac{1}{\alpha} \equiv \sqrt{\frac{m^* g}{\rho_0}}. \tag{9}$$

In what follows we will work in $c = 1$ units (note that α has dimensions of momentum, as g has dimensions of velocity / momentum).

Next up is $\mathcal{J}$. First, note that the phase mode of z ensures that $\mathcal{J}$ is conserved, i.e. that $d^\dagger \mathcal{J} = 0$. Thus we may write

$$\mathcal{J} = \star \frac{d\varsigma}{2\pi} \tag{10}$$

for ς a 1-form field, so that

$$\mathcal{L} = \frac{1}{2\pi}\varsigma \wedge d\xi + \frac{1}{2\widetilde{\alpha}}d\varsigma \wedge \star d\varsigma + \frac{i}{4\pi}\varsigma \wedge d\varsigma, \tag{11}$$

where the $(\nabla \mathbf{n})^2$ term has been ignored and $\widetilde{\alpha} = \alpha(2\pi)^2$. In the transverse gauge, the CS propagator is

$$G_\varsigma^{\mu\nu} = \frac{\widetilde{\alpha}}{k^2 + m^2}\left(\Pi_T^{\mu\nu} + \frac{m}{k^2}\varepsilon^{\mu\nu\lambda}k_\lambda\right), \tag{12}$$

where

$$m \equiv \frac{\widetilde{\alpha}}{2\pi} = 2\pi\alpha. \tag{13}$$

Since we are only interested in the long-wavelength physics we will replace $k^2 + m^2$ in the denominator with just m^2. Then

$$\begin{aligned}\mathcal{L} &= \frac{\alpha}{2m^2}(\star d\xi) \wedge \star\left(\frac{\Box - dd^\dagger}{\Box} + \frac{im}{\Box} \star d\right) \star d\xi \\ &= \frac{\alpha}{2m^2}\left(d\xi \wedge \star d\xi + im\xi \wedge d\xi\right).\end{aligned} \tag{14}$$

[2]Here $\mathcal{J} \wedge \frac{1}{d^\dagger}\mathcal{J}$ is just a pretentious way of writing $\mathcal{J} \wedge \frac{d}{\Box}\mathcal{J}$.

Now

$$d\xi = 2\pi \star J + dA + \partial_i V\, dx^i \wedge dt. \tag{15}$$

Then, ignoring constants, (including the Hall response for A and remembering that $dA = 2\pi\rho_0 dx \wedge dy$)

$$\mathcal{L} = \frac{1}{2\alpha c}\left(J_i J^i + (cJ_0 - \rho_0 - \alpha(V + g\rho_0))^2\right) + \frac{1}{2\pi\alpha c}\varepsilon^{0ij}\partial_i J_j V \; + \frac{\pi}{2} J \wedge \frac{1}{d^\dagger} J + J \wedge \star A. \tag{16}$$

The last term tells us that the skyrmions have unit electric charge, and the second-to-last term tells us that they have fermionic statistics. The last term is also responsible for the WZW term for the ferromagnet. To see this, we use $\star dA = 2\pi\rho_0 dt$ to write

$$\int J \wedge \star A = i\rho_0 \int z^\dagger \partial_t z. \tag{17}$$

At unit filling, showing that the RHS is equal to the usual WZW term is standard.

The first term determines the mass of the skyrmions. Indeed, letting the skyrmion coordinates be R^i, we write $\mathbf{n} = \mathbf{n}(R^j)$, so that

$$\begin{aligned} J &= \frac{1}{8\pi} \star \mathbf{n} \cdot (d\mathbf{n} \wedge d\mathbf{n}) \\ &= \rho_S dt + \frac{1}{8\pi}\varepsilon^{i0j}\partial_t R^k \mathbf{n} \cdot (\partial_k \mathbf{n} \times \partial_j \mathbf{n}) dx^i \\ &= \rho_S dt + \rho_S \partial_t R_i dx^i, \end{aligned} \tag{18}$$

where $\rho_S = \frac{1}{8\pi}\varepsilon^{0ij}\mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n})$ is the skyrmion density for a skyrmion located at $\mathbf{R}$. The J_i^2 term is then

$$\frac{1}{2\alpha c}\int J_i^2 = \frac{1}{2\alpha c}|\partial_t \mathbf{R}|^2 \int \rho_S^2. \tag{19}$$

For concreteness we will take the skyrmions to be a conformal map $\mathbb{R}^2 \to S^2$ with scale parameter ζ, so that

$$\frac{1}{2\alpha c}\int J_i J^i = \frac{1}{2} m_S \int dt\, |\partial_t \mathbf{R}|^2, \tag{20}$$

with the skyrmion mass

$$m_S = \frac{m^*}{3\pi\zeta^2\rho_0}, \tag{21}$$

where the numerical coefficient can be figured out by using

$$n^z = \frac{\zeta^2 - r^2}{\zeta^2 + r^2} \implies (n^x,\, n^y) = \frac{2}{1 + r^2/\zeta^2}(x/\zeta,\, y/\zeta), \tag{22}$$

so that the derivatives are

$$\begin{aligned} \partial_i n^z &= -\frac{4x^i\zeta^2}{(\zeta^2 + r^2)^2} \\ \partial_x n^x &= 2\frac{\zeta^3 + \zeta(y^2 - x^2)}{(\zeta^2 + r^2)^2} \\ \partial_y n^x &= -\frac{4xy\zeta}{(\zeta^2 + r^2)^2} \end{aligned} \tag{23}$$

which gives the topological charge (normalized to $\int Q = 1$)

$$Q(r) = \frac{1}{\pi\zeta^2}\frac{1}{(1+(r/\zeta)^2)^2}. \tag{24}$$

Using this formula,

$$\frac{1}{\zeta^2}\int\left(\frac{1}{4\pi}\mathbf{n}\cdot(\partial_x\mathbf{n}\times\partial_y\mathbf{n})\right)^2 = \frac{2\pi}{\zeta^2}\int_0^\infty dr\, r\frac{1}{(\pi(1+r^2)^2)^2} = \frac{1}{3\pi\zeta^2}. \tag{25}$$

Let us also look at how the potential leads to a dispersion for the skyrmions. The important term to look at is, assuming that the skyrmion solutions are still conformal maps parametrized by ζ,

$$\begin{aligned} S &\supset \int V J_0 \\ &= \int dt \int d^2x \sum_{\mathbf{q}} V_{\mathbf{q}}\, e^{i\mathbf{q}\cdot(\mathbf{x}+\mathbf{R})}\, Q(\mathbf{x}) \\ &= 2\pi\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}} V_{\mathbf{q}} \int dt \int dr\, r\, Q(r) J_0(qr) \\ &= \sum_{\mathbf{q}} e^{i\mathbf{R}\cdot\mathbf{q}} V_{\mathbf{q}} \int dt\, \zeta q K_1(\zeta q). \end{aligned} \tag{26}$$

The Bessel function decays exponential at large arguments, and so as is intuitively obvious, the potential has a small effect on the skyrmion motion in the limit where the skyrmion size is much larger than the characteristic wavelength of the potential. For example, if we take $V(\mathbf{x}) = V_0(\cos(qx) + \cos(qy))$, then we approximately have

$$S \supset \sqrt{\frac{\pi}{2}}\sqrt{q\zeta}e^{-q\zeta}\int dt\, V(\mathbf{R}). \tag{27}$$

If the skyrmions are localized, the exponential dependence disappears. For example, if we let

$$Q(r) = \frac{1}{\zeta^2}e^{-r/\zeta}, \tag{28}$$

then

$$S \supset \frac{1}{\sqrt{(q\zeta)^2+1}}\int dt\, V(\mathbf{R}). \tag{29}$$

The Coulomb energy for a conformal Skyrmion of size ζ is, using the un-screened Coulomb interaction,[3]

$$\begin{aligned} E_C &= e^2\int d^2x\, d^2y\, \frac{1}{|x-y|}Q(x)Q(y) \\ &= e^2\int dq\, (\zeta q)^2 K_1(\zeta q)^2 \\ &= \frac{3\pi^2 e^2}{32\zeta}, \end{aligned} \tag{30}$$

[3]We are taking $U(r) = e^2/r$, so that $U(q) = 2\pi e^2/q$.

where we have used the Fourier transform of Q used above, viz.

$$Q(q) = 2\int dx\, \frac{J_0(\zeta q x)x}{(1+x^2)^2} = \zeta q K_1(\zeta q). \tag{31}$$

For a screened Coulomb interaction, we may take[4]

$$U(q) = 2\pi \frac{e^2}{q}(1 - e^{-qd}), \tag{32}$$

which in $\mathbb{R}$ space is

$$U(x) = \frac{e^2}{x}\left(1 - \frac{x}{\sqrt{x^2+d^2}}\right). \tag{33}$$

In this case

$$E_C = \frac{e^2}{\zeta}\int dq\,(1 - e^{-qd/\zeta})q^2 K_1(q)^2. \tag{34}$$

The integral here is complicated in general. If we expand in d/ζ, then

$$E_C \approx \frac{3\pi^2 e^2}{32\zeta}\left(\frac{2d}{3\zeta} - \frac{45\pi^2 d^2}{1024\zeta^2} + \cdots\right). \tag{35}$$

If the interaction is completely screened down to a δ function, then

$$\begin{aligned} E_C &= e^2 a \int d^2x\, Q(x)^2 \\ &= \frac{ae^2}{3\pi\zeta^2}, \end{aligned} \tag{36}$$

where a is the scattering length.

Therefore the more localized the Coulomb interaction is, the more energy we save by making the skyrmion large.

2 Skrymion energetics

In the absence of the Coulomb interaction and Zeeman field, the skyrmion energy is minimized by conformal maps from the plane to S^2. These conformal maps are parametrized by a single scale factor ζ, and the conformal form of the solution means that integrals involving the topological charge density can be easily performed. After the Zeeman and Coulomb energies are turned on, this allows us to analytically find the value of ζ which minimizes the skyrmion energy within this class of variational ansatz.

However, the solutions obtained in this way always have a topological charge density which decays algebraically away from the skyrmion center. This means that the skyrmion bandwidth will always be exponentially small in ζ/a.

[4] Taking $\tanh(dq)/q$ also works, but integrating it against stuff is more difficult.

A rotationally-invariant skyrmion solution is specified by a function $\theta(r)$ where $\theta(\infty) = 0$ and $\theta(0) = \pi$. Some algebra then shows that

$$Q = \frac{1}{4\pi r}\partial_r \cos\theta. \tag{37}$$

The skyrmion energy is

$$E_S = \pi\rho_s \int rdr\, \left((\partial_r\theta)^2 + \frac{1}{r^2}(\sin\theta)^2\right) + 2\pi\mu_B gB \int rdr\,(1-\cos(\theta)) + \int \frac{d^2q}{(2\pi)^2}\, U(\mathbf{q})|Q_{\mathbf{q}}|^2. \tag{38}$$

2.1 Solutions from the EOM

There are two strategies one can take to figure out the optimal form of $\theta(r)$. The first is to just solve the Euler-Lagrange equations numerically. This works when we approximate $U(\mathbf{q}) = U_0$ as a contact interaction, which is allowable when the screening length is less than the skyrmion radius. The equation of motion is

$$\partial_r^2\theta\left(-r - \frac{2u}{r}\sin^2\theta\right) - 2(\partial_r\theta)^2\frac{u\sin\theta\,\cos\theta}{r} + \partial_r\theta\left(-1 + \frac{2u\sin^2\theta}{r^2}\right) + \frac{\sin\theta\,\cos\theta}{r} + br\sin\theta = 0 \tag{39}$$

where

$$u = \frac{U_0}{16\pi^2\rho_s}, \qquad b = \frac{\mu_B gB}{\rho_s}. \tag{40}$$

To solve this numerically, it helps to have a good guess for the solution. We therefore use as a guess the optimal conformal solution, which we can evaluate analytically. For a conformal skyrmion with scale parameter ζ the topological charge is

$$Q(r) = \frac{1}{\pi\zeta^2}\frac{1}{(1+(r/\zeta)^2)^2}. \tag{41}$$

Using $Q(r) = \partial_r\cos\theta/4\pi r$, this gives

$$\cos(\theta) = 1 - \frac{2}{1+(r/\zeta)^2}, \tag{42}$$

so that the spins have turned halfway over at $r = \zeta$. The energy of the conformal solution is then

$$E_S(\zeta) = 4\pi\rho_s + 2\mu_B gB \int d^2r\,\frac{1}{1+(r/\zeta)^2} + \frac{U_0}{\pi^2\zeta^4}\int d^2r\,\frac{1}{(1+(r/\zeta)^2)^4}. \tag{43}$$

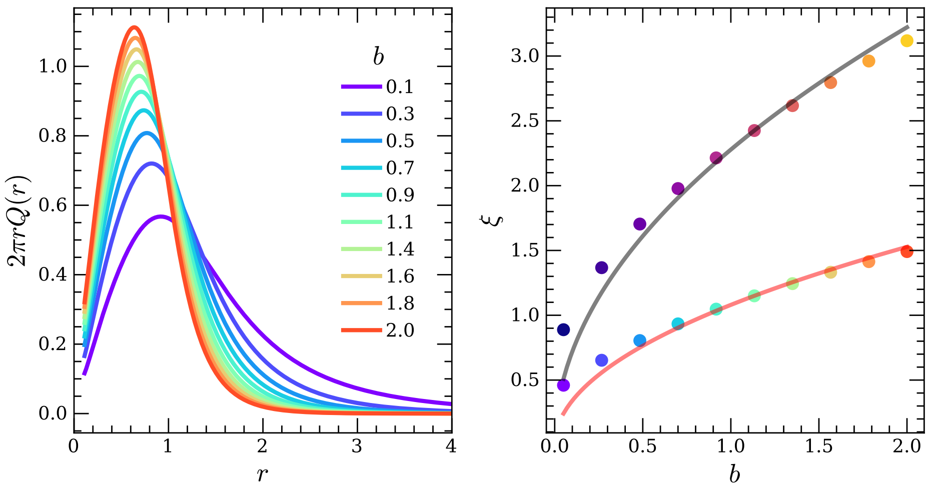
The second term diverges, telling us that the conformal solution does not work in the presence of a magnetic field — this is evident because after explicitly breaking $SO(3)$ spin symmetry the spin fluctuations have a finite correlation length, and so skyrmions whose spin textures have algebraic decay out to infinity should be ruled out. In order to minimize E_S analytically, we need the cutoff of the integral to scale with ζ. Since we are just looking for a guess to use when solving things numerically we will simply cutoff the integral at ζ (not counting U_0 we don't have enough length scales to modify this by something dimensionless). Therefore

$$E_S(\zeta) \approx 4\pi\rho_s + 2\pi\mu_B g B\zeta^2 \ln 2 + \frac{U_0}{3\pi\zeta^2} \implies \zeta_* = \left(\frac{u}{6\pi^2 b \ln 2}\right)^{1/4}. \tag{44}$$

Now the EL equation only involves the two parameters u, b. u has dimensions of length squared, while b has dimensions of inverse length squared. On physical grounds we know that the skyrmion size should vanish when $u \to 0, b > 0$, while it should diverge for $b \to 0, u > 0$. In fact the correlation length for the exponential behavior at large r is a function only of $1/\sqrt{b}$, with u only affecting the scale at which the short-distance behavior crosses over to exponential. Empirically, solving the EL equations gives the large r behavior

$$\theta(r) \sim e^{-\alpha_\theta\sqrt{b}\theta r}, \qquad Q(r) \sim e^{-\alpha_Q\sqrt{b}r}, \qquad \alpha_\theta \approx 1, \qquad \alpha_Q \approx 2.27 \tag{45}$$

The plot which tells us this is



(46)

Here we fix $u = 1$ throughout and vary b. The quantity ξ on the RHS appears in either $\theta \sim e^{-r/\xi}$ (rainbow dots) or $Q \sim e^{-r/\xi}$ (plasma dots); the solid lines are fits to $\sqrt{b}$. The numeric routine has a hard time solving the ODE at small r.

This behavior can be understood quasianalytically by noting that at large r, where $\theta \ll 1$, the EOM is just

$$\partial_r\theta^2 = b\theta, \tag{47}$$

giving $\theta \sim e^{-\sqrt{b}r}$ at large r, as claimed. We can also solve the EOM at large u and small r, which gives $\theta = \pi + O(r^2/u)$. Thus it is reasonable to look for a solution that starts out more or less constant over a core distance determined by $\sqrt{u}$, and then starts an exponential decay with a decay length set by $1/\sqrt{b}$.

2.2 Solutions from variational ansatz

To get a better understanding for the nature of the solutions, as well as to examine the effects of having a finite-ranged Coulomb interaction, we now turn to looking at variational ansatze.

The simplest ansatz which is in rough accordance with the EOM is to take

$$2\pi r Q(r) = \frac{re^{-r/\xi}}{\xi^2} \implies \cos\theta = 1 - 2\left(\frac{r}{\xi} + 1\right) e^{-r/\xi}. \tag{48}$$

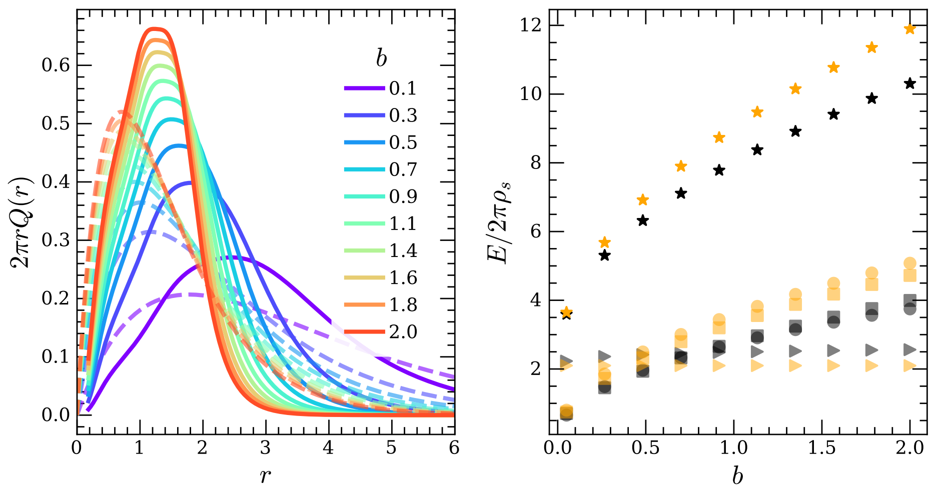
Since the ansatz contains only a single parameter, the energy can be optimized analytically, giving an optimal value of

$$\xi_* = \left(\frac{u}{3b}\right)^{1/4} \tag{49}$$

so that

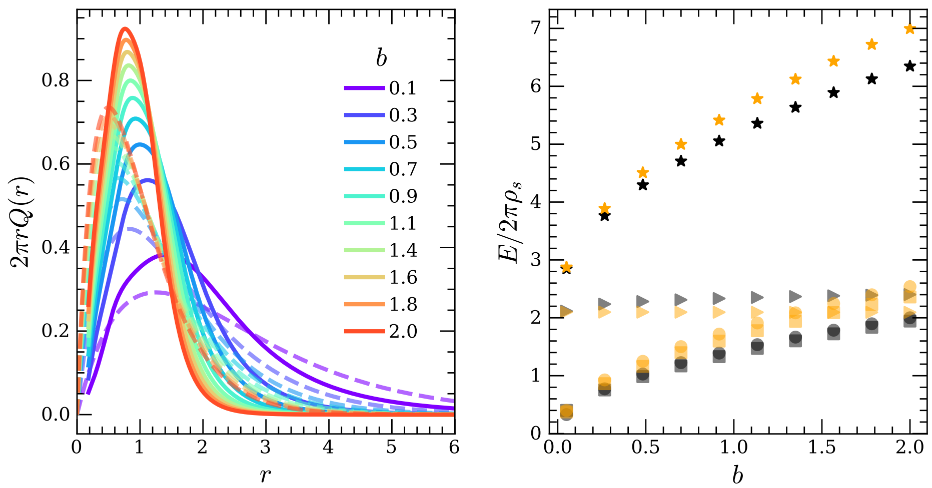
$$\begin{aligned} \frac{E_S(\xi_*)}{2\pi\rho_s} &= c + 3b\xi_*^2 + u\xi_*^{-2} \\ &= c + 3\sqrt{ub}, \end{aligned} \tag{50}$$

with $c \approx 2.1$. The $(u/b)^{1/4}$ dependence is fixed by the fact that the ansatz obeys scaling, as discussed earlier. Note that the Coulomb and magnetic contributions to the energy are equal. The elastic contribution is larger than the conformal solution by about a factor of 1/10. We can evaluate this ansatz with, at $u = 2$,



(51)

and at $u = 1/2$,



(52)

Here the dashed lines are the ansatz. In the RHSs the yellow markers are the ansatz energy (star = total, circle = Coulomb, triangle = elastic, square = magnetic), and the black markers are the energy from solving the EOM. Note that as expected, the tails of the ansatz are significantly longer at larger b, since the correlation length for the ansatz only scales as $1/b^{1/4}$.

An ansatz that seems to be widely adopted in the skyrmion community is

$$\tan(\theta/2) = \frac{r_s}{r} e^{-(r-r_s)/\xi}, \tag{53}$$

where r_s is the skyrmion radius and ξ determines the skyrmion thickness. When $\xi \to \infty$ we recover the conformal solution. A bit of work shows that the topological charge is

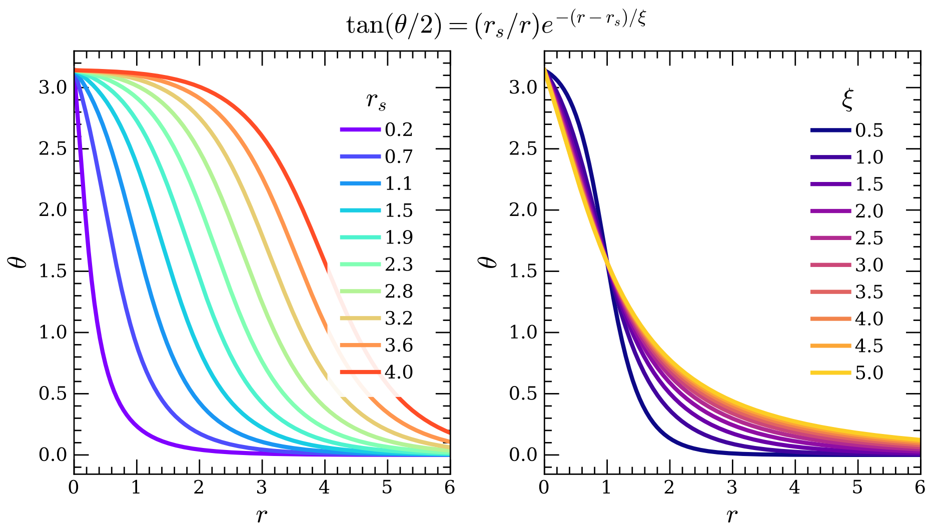
$$Q(r) = \frac{r_s^2}{\pi} \frac{1 + r/\xi}{(r^2 e^{(r-r_s)/\xi} + e^{-(r-r_s)/\xi} r_s^2)^2}. \tag{54}$$

To check,

$$\int d^2r\, Q(r) = \int_0^\infty dr\, \frac{2\eta^2(r + r^2)}{(r^2 e^{r-\eta} + \eta^2 e^{\eta - r})^2} = 1, \tag{55}$$

with $\eta = r_s/\xi$ and the middle integral curiously being independent of η.

$\theta(r)$ and $Q(r)$ for these solutions look like



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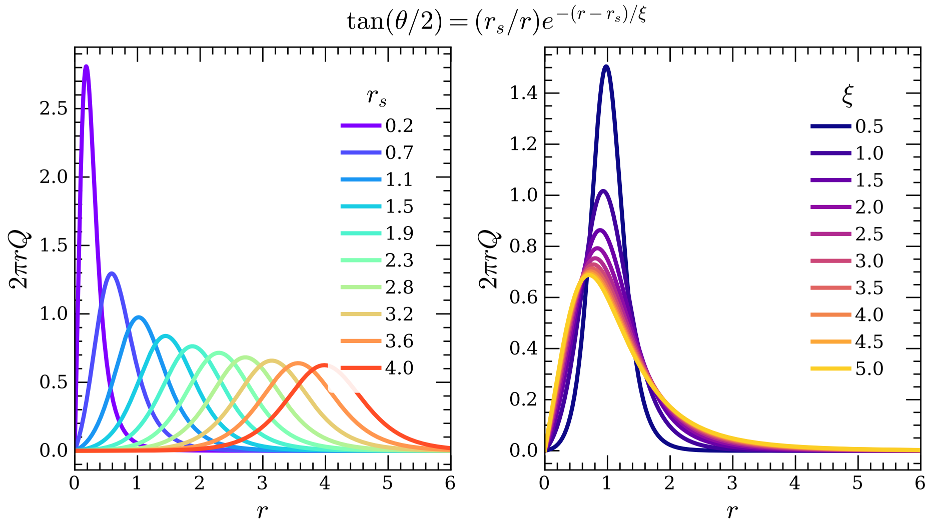

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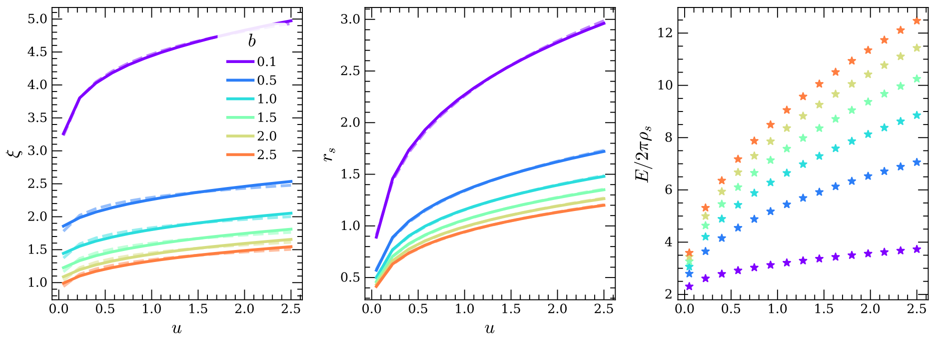
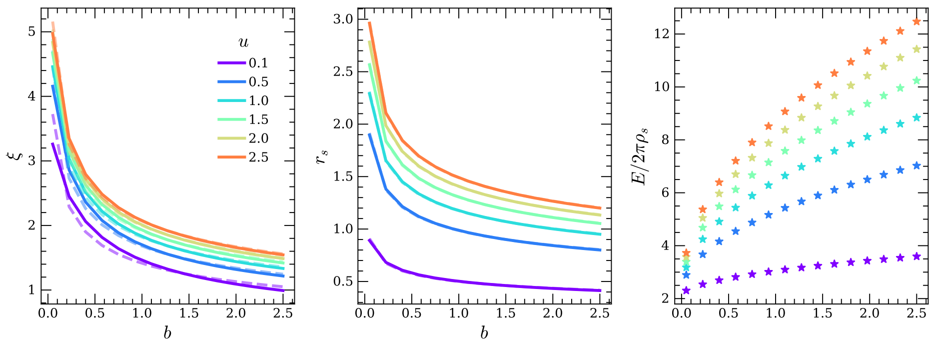
where in the plot where r_s is varied, $\xi = 1$, and vice versa.

We now use gradient descent to find the optimal values of r_s, ξ. As a function of u for various values of b,



(58)

where the dashed lines are fits to power laws. As a function of b for different u:



(59)

For simplicity, we will fit ξ, r_s to power laws in u, b. This is evidently a very good approximation for r_s, and a not-perfect but still acceptable one for ξ. We will write

$$\xi = \xi_0 u^{1/2-\alpha} b^{-\alpha}, \qquad r_s = r_{s,0} u^{1/2-\beta} b^{-\beta}. \tag{60}$$

We know from the EOM that the exact solution should have $\alpha = 1/2$. This is not completely in accordance with the above fit, but presumably this is due to inaccuracies coming from the numerical fitting and the singularities at small r. On the other hand, r_s is very well fit to $\beta = 1/4$. Here $\xi_0 \approx 2, r_{s,0} \approx 1$.

Note that the magnetic energy of the skyrmion can be estimated as

$$E_M \sim r_s^2 B \sim \sqrt{B}, \tag{61}$$

which grows *sublinearly* in B. This should be contrasted with the magnetic energy of a particle-hole excitation, which varies linearly with B. Also note that for small b the magnetic and Coulomb energies are only a small fraction of the elastic energy, meaning that the field and self-interaction will not have a large importance for determining whether skyrmions or particles are favored at small nonzero doping.

2.2.1 Dealing with finite-ranged Coulomb interaction

A form usually taken for a double-gate-screened potential is (taking $e = 1$)

$$U(q) = \frac{1}{2\varepsilon q} \tanh(qd). \tag{62}$$

Sometimes people also use

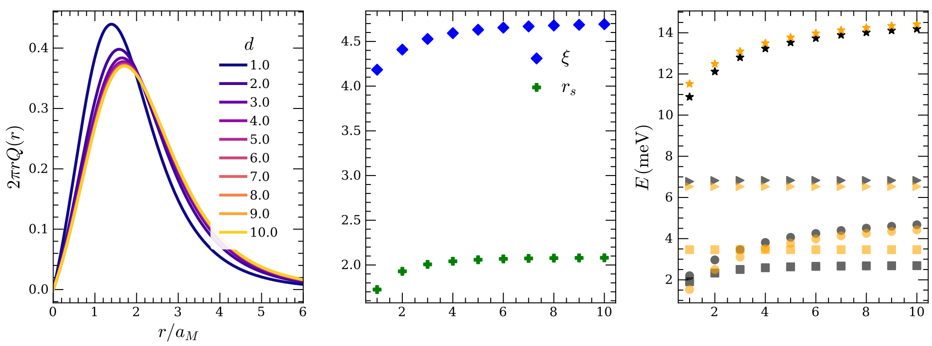
$$U_1(q) = \frac{1}{2\varepsilon q}(1 - e^{-qd}) \implies U(x) = \frac{1}{\pi \varepsilon x}\left(1 - \frac{Ux}{\sqrt{x^2 + d^2}}\right). \tag{63}$$

Note that $U_1(q)$ is power-law decaying in space because of its non-analyticity at $q = 0$; $U(q)$ on the other hand is nice and smooth everywhere, and decays exponentially in real space. Finally, we may use the Coulomb interaction

$$U_c(q) = \frac{1}{2\varepsilon q}, \tag{64}$$

which is just the $d \to \infty$ limit of $U(q)$.

As an example, for realistic values of U and B we can compute



(65)

We see that ξ, r_s essentially saturate to the values they take in the unscreened case by the time $d \sim 10$. The final panel shows the comparison between the approximate minimum of the two-parameter ansatz (black), and the exact minimum of the one-parameter ansatz (orange).

2.3 Estimate of the bandwidth

As we saw above, the bandwidth for the skyrmions is determined by the fourier weights of the topological charge density. Analytic calculation is possible for the single-parameter ansatze.

Conformal solution: Here we have

$$\begin{aligned}\int d^2r\, e^{i\mathbf{r}\cdot\mathbf{q}} Q(r) &= 2\pi \int dr\, r J_0(qr) \frac{1}{\pi(1+(r/\xi)^2)^2} \\ &= \frac{q\xi}{2} K_1(q\xi) \\ &\sim \sqrt{q\xi} e^{-q\xi}.\end{aligned} \tag{66}$$

The exponential decay here means that the bandwidth is very small.

One-parameter ansatz: For $Q(r) = e^{-r/\xi}/(2\pi\xi^2)$,

$$\int d^2r\, e^{i\mathbf{r}\cdot\mathbf{q}} Q(r) = \frac{1}{(1+(q\xi)^2)^{3/2}}, \tag{67}$$

which only decays as $(q\xi)^{-3}$.

Two-parameter ansatz:

Here $Q(r)$ cannot be Fourier transformed analytically.

For $\xi \gg q^{-1}$

2.4 Computing the spin stiffness

The projected density and spin operators are

$$\rho_{\mathbf{q}} = \sum_{\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q}} c_{\mathbf{p}} \lambda_{\mathbf{q},\mathbf{p}}, \qquad S^i_{\mathbf{q}} = \frac{1}{2}\sum_{\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q}} s^i c_{\mathbf{p}} \lambda_{\mathbf{q},\mathbf{p}}. \tag{68}$$

Here the form factor $\lambda_{\mathbf{q},\mathbf{p}}$ measures the modulation of the wavefunction upon moving from $\mathbf{q}$ to $\mathbf{p}+\mathbf{q}$. Here and in the following, we will be using notation in which $\mathbf{p}, \mathbf{p}'$ are always summed over the BZ, with e.g. $\sum_{\mathbf{p}} \equiv \frac{1}{\sqrt{N_M}} \sum_{\mathbf{p}\in BZ}$ with N_M the number of Moire cells, while $\mathbf{q}, \mathbf{k}$ are always summed over all of momentum space, so that e.g. $\sum_{\mathbf{k}} = \frac{1}{\sqrt{N_g}} \sum_{\mathbf{k}\in BZ_g}$, where N_g is the number of graphene unit cells and BZ_g is the graphene BZ.

To evaluate the commutators, we use

$$[c^\dagger A c, c^\dagger B c] = c^\dagger [A,B] c. \tag{69}$$

Using this, we find the identities

$$\begin{aligned}[][\rho_{\mathbf{q}}, \rho_{\mathbf{q}'}] &= \sum_{\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q}+\mathbf{q}'} c_{\mathbf{p}} \Lambda_{\mathbf{q},\mathbf{q}',\mathbf{p}} \\ [S^i_{\mathbf{q}}, \rho_{\mathbf{q}'}] &= \frac{1}{2}\sum_{\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q}+\mathbf{q}'} s^i c_{\mathbf{p}} \Lambda_{\mathbf{q},\mathbf{q}',\mathbf{p}} \\ [S^i_{\mathbf{q}}, S^j_{\mathbf{q}'}] &= \frac{i}{4}\varepsilon^{ijk} \sum_{\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q}+\mathbf{q}'} s^k c_{\mathbf{p}} (\lambda_{\mathbf{q},\mathbf{p}+\mathbf{q}'}\lambda_{\mathbf{q}',\mathbf{p}} + \lambda_{\mathbf{q}',\mathbf{p}+\mathbf{q}}\lambda_{\mathbf{q},\mathbf{p}}) \\ [S^i_{\mathbf{q}}, \sum_{\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q}'} s^j c_{\mathbf{p}}] &= \frac{i}{2}\varepsilon^{ijk} \sum_{\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q}+\mathbf{q}'} s^k c_{\mathbf{p}} (\lambda_{\mathbf{q},\mathbf{p}+\mathbf{q}'} + \lambda_{\mathbf{q},\mathbf{p}})\end{aligned} \tag{70}$$

where

$$\Lambda_{\mathbf{q},\mathbf{q}',\mathbf{p}} \equiv (\lambda_{\mathbf{q},\mathbf{p}+\mathbf{q}'}\lambda_{\mathbf{q}',\mathbf{p}} - \lambda_{\mathbf{q}',\mathbf{p}+\mathbf{q}}\lambda_{\mathbf{q},\mathbf{p}}). \tag{71}$$

Here $\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}}$ measures the differences in the form factors accrued upon going from $\mathbf{p}$ to $\mathbf{p}+\mathbf{k}+\mathbf{q}$ along the paths $\mathbf{p} \to \mathbf{p}+\mathbf{k} \to \mathbf{p}+\mathbf{k}+\mathbf{q}$ and $\mathbf{p} \to \mathbf{p}+\mathbf{q} \to \mathbf{p}+\mathbf{k}+\mathbf{q}$ (looks like the ordering here is different to that in [1]). Thus when $\mathbf{k},\mathbf{q}$ are small, $\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}}$ can be expressed in terms of the Berry curvature. Also note that $\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}} = 0$ if either of $\mathbf{q},\mathbf{k}$ are zero.

2.4.1 No magnetic field

We will assume that the interaction satisfies $U_{\mathbf{k}} = U_{-\mathbf{k}}$, which will be the case for the examples of interest. Then using the notation in [1],

$$[S^i_{\mathbf{q}}, H_I] = \frac{1}{2}\sum_{\mathbf{k},\mathbf{p}} V_{\mathbf{k}}\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}}\{c^\dagger_{\mathbf{p}+\mathbf{q}+\mathbf{k}}s^i c_{\mathbf{p}},\ \rho_{-\mathbf{k}}\}. \tag{72}$$

Since

$$\rho_{-\mathbf{k}}|\Omega\rangle \propto \delta_{\mathbf{k},\mathbf{0}}|\Omega\rangle \tag{73}$$

for the ground states we are considering, the ground-state expectation value of the above first-order term vanishes.

The second order term is, using (73) and the fact that $\rho_{\mathbf{0}}$ commutes with everything,

$$\langle[S^i_{\mathbf{q}},[S^j_{\mathbf{q}'},H_I]]\rangle = \sum_{\mathbf{k}} V_{\mathbf{k}}\langle[S^j_{\mathbf{q}'},\rho_{-\mathbf{k}}][S^i_{\mathbf{q}},\rho_{\mathbf{k}}] + [S^i_{\mathbf{q}},\rho_{-\mathbf{k}}][S^j_{\mathbf{q}'},\rho_{\mathbf{k}}]\rangle \tag{74}$$

Using the commutators computed above,

$$\begin{aligned}
\langle[S^i_{\mathbf{q}},[S^j_{\mathbf{q}'},H_I]]\rangle &= \frac{1}{4}\sum_{\mathbf{k},\mathbf{p},\mathbf{p}'} V_{\mathbf{k}}\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}}\Lambda_{\mathbf{q}',-\mathbf{k},\mathbf{p}'}\langle c^\dagger_{\mathbf{p}'+\mathbf{q}'-\mathbf{k}}s^j c_{\mathbf{p}'}c^\dagger_{\mathbf{p}+\mathbf{q}+\mathbf{k}}s^i c_{\mathbf{p}} + c^\dagger_{\mathbf{p}'+\mathbf{q}-\mathbf{k}}s^i c_{\mathbf{p}'}c^\dagger_{\mathbf{p}+\mathbf{q}'+\mathbf{k}}s^j c_{\mathbf{p}}\rangle \\
&= \frac{\delta_{\mathbf{q},-\mathbf{q}'}}{4}\sum_{\mathbf{k},\mathbf{p}} V_{\mathbf{k}}\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}}\Lambda_{-\mathbf{q},-\mathbf{k},\mathbf{p}+\mathbf{q}+\mathbf{k}}(s^j_{\uparrow\downarrow}s^i_{\downarrow\uparrow} + s^i_{\uparrow\downarrow}s^j_{\downarrow\uparrow}) \\
&= \frac{\delta_{\mathbf{q},-\mathbf{q}'}\delta^{ij}}{2}\sum_{\mathbf{k},\mathbf{p}} V_{\mathbf{k}}\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}}\Lambda_{-\mathbf{q},-\mathbf{k},\mathbf{p}+\mathbf{q}+\mathbf{k}}.
\end{aligned} \tag{75}$$

The spin stiffness is then found by extracting the $O(q^2)$ term in the expansion of this expression at small q.

We are interested in the spin stiffness, so we expand this to order q^2. As in [1], we will further assume that the absolute values of the form factors $\lambda_{\mathbf{q},\mathbf{p}}$ decay rapidly in q (think of e.g. the LLL where $\lambda_{\mathbf{q},\mathbf{p}} \sim e^{-q^2}$) and are independent of the basepoint $\mathbf{p}$. Writing[5]

$$\lambda_{\mathbf{q},\mathbf{p}} = F(\mathbf{q})e^{i\arg(\lambda_{\mathbf{q},\mathbf{p}})} \tag{77}$$

[5] A simplifying assumption which lets us do slightly more is to take

$$\lambda_{\mathbf{q},\mathbf{p}} = F(\mathbf{q})e^{i\mathcal{B}_{\mathbf{p}}\mathbf{q}\wedge\mathbf{p}/2} \tag{76}$$

for $F(\mathbf{q})$ rapidly decaying with q, we expand

$$\arg(\lambda_{\mathbf{q},\mathbf{p}}) \approx \arctan\left(-i\frac{\mathbf{q}\cdot\langle\partial u_{\mathbf{p}}|u_{\mathbf{p}}\rangle}{1+O(q^2)}\right) \approx \mathbf{q}\cdot\mathcal{A}_{\mathbf{p}}. \tag{78}$$

Thus

$$\begin{aligned}\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}} &\approx F(\mathbf{k})F(\mathbf{q})\left((1+\mathbf{k}\cdot\mathcal{A}_{\mathbf{p}})(1+\mathbf{q}\cdot\mathcal{A}_{\mathbf{k}+\mathbf{p}})-(1+\mathbf{q}\cdot\mathcal{A}_{\mathbf{p}})(1+\mathbf{k}\cdot\mathcal{A}_{\mathbf{q}+\mathbf{p}})\right)\\ &\approx F(\mathbf{k})F(\mathbf{q})q_ik_j\mathcal{B}^{ij}_{\mathbf{p}}\\ &= F(\mathbf{k})F(\mathbf{q})\frac{\mathbf{q}\wedge\mathbf{k}}{2}\mathcal{B}_{\mathbf{p}},\end{aligned} \tag{79}$$

with $\mathcal{B}$ the Berry curvature. Hence, keeping only terms to quadratic order in q,

$$\langle[S^i_{\mathbf{q}},[S^j_{\mathbf{q}'},H_I]]\rangle \approx \frac{\delta_{\mathbf{q},-\mathbf{q}'}\delta^{ij}}{8}\sum_{\mathbf{k},\mathbf{p}}V_{\mathbf{k}}F(\mathbf{k})^2(\mathbf{q}\wedge\mathbf{k})^2\mathcal{B}^2_{\mathbf{p}}. \tag{80}$$

Therefore the spin stiffness in this limit simplifies to

$$\rho_s \approx \frac{1}{16A_M}\sum_{\mathbf{p}}\mathcal{B}^2_{\mathbf{p}}\sum_{\mathbf{k}}V_{\mathbf{k}}k^2F(\mathbf{k})^2, \tag{81}$$

where the area of the Moire unit cell A_M enters upon performing the Fourier transform and writing the sum over Moire lattice sites (in real space, the S^i operators are defined on the Moire [not graphene] sites) as an integral.

As a sanity check, consider the case of the QHE in the LLL, where $V_{\mathbf{k}} = V(k)$. Here we have

$$F(\mathbf{k})^2 = e^{-k^2l_B^2/2}, \qquad \mathcal{B}_{\mathbf{p}} = 1. \tag{82}$$

Therefore

$$\rho_{s,LLL} = \frac{1}{32\pi^2}\int dk\, k^3V(k)e^{-k^2l_B^2/2}, \tag{83}$$

which is the correct answer [?].

As an example, consider the ansatz

$$\lambda_{\mathbf{q},\mathbf{p}} = F(q)e^{i\mathcal{B}_{\mathbf{p}}\mathbf{q}\wedge\mathbf{p}/2}, \tag{84}$$

which for $F(q)\sim e^{-q^2}$ would give the form factors for the LLL. For this ansatz,

$$\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}} = 2iF(q)F(k)\sin\left(\frac{\mathbf{q}\wedge\mathbf{k}}{2}\right)e^{i\mathcal{B}_{\mathbf{p}}(\mathbf{q}+\mathbf{k})\wedge\mathbf{p}/2} \tag{85}$$

3 Spin stiffness

In this appendix we derive an expression for the spin stiffness in terms of the form factors $\lambda_{\mathbf{k},\mathbf{p}}$, Berry connection, and quantum metric. Our result will be slightly more general than that appearing in [1, 3, 4], but will reduce to the expression given in these references in a certain approximation scheme.

3.1 Single mode approximation

In the single-mode approximation, the spin stiffness is obtained by computing the energy cost of a spin flip, viz.

$$\delta E = \frac{\langle S^+_{-\mathbf{q}} H S^-_{\mathbf{q}} \rangle}{|\langle S^+_{-\mathbf{q}} S^-_{\mathbf{q}} \rangle|} - \langle H \rangle, \tag{86}$$

where the expectation value is evaluated in the state where all spins are polarized in the $\uparrow$ direction, and where the (band-projected) spin-flip operator is

$$S^-_{\mathbf{q}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{p}} \lambda_{\mathbf{q},\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q},\downarrow} c_{\mathbf{p},\uparrow}, \tag{87}$$

where N is the number of (Moire) unit cells, the sum over $\mathbf{p}$ (as well as all sums to follow) runs over the (Moire) BZ, and where the $c_{\mathbf{p},\sigma}$ operators are second quantized operators for the band in question.

We will decompose $H = H_0 + H_I$ into one- and two-body terms, and look at their contributions to δE separately. We will start with the contribution from the interaction, δE_I.

As a reminder, the interaction Hamiltonian is

$$H_I = \frac{1}{2} \int_{\mathbf{k}} V_{\mathbf{k}} : \rho_{\mathbf{k}} \rho_{-\mathbf{k}} :, \qquad \rho_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{p},\sigma} \lambda_{\mathbf{q},\mathbf{p}} c^\dagger_{\mathbf{p}+\mathbf{q},\sigma} c_{\mathbf{p},\sigma}, \tag{88}$$

where the sum over $\mathbf{p}$ (as well as all sums over $\mathbf{p}$ to follow) runs over the mini BZ, and we have written the sum over $\mathbf{k}$ as an integral $\int_{\mathbf{k}} \equiv \int \frac{d^2k}{(2\pi)^2}$ to indicate that it runs over all of $\mathbb{R}^2$ in the continuum limit (we could discretize it as $\int_{\mathbf{k}} \to \frac{1}{A_M} \sum_{\mathbf{k} \in mBZ} \sum_{\mathbf{G}}$, with $\mathbf{G}$ in the Moire reciprocal lattice). Computing the appropriate expectation value is a straightforward application of Wick's theorem, and yields [1, 3]

$$\delta E_I = \frac{1}{2} \int_{\mathbf{k}} \left[\frac{\sum_{\mathbf{p}} V_{\mathbf{k}} (|\lambda_{\mathbf{q},\mathbf{p}}|^2 |\lambda_{\mathbf{k},\mathbf{p}}|^2 - \lambda_{-\mathbf{q},\mathbf{p}} \lambda_{-\mathbf{k},\mathbf{p}-\mathbf{q}} \lambda_{\mathbf{q},\mathbf{p}-\mathbf{k}-\mathbf{q}} \lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}})}{\sum_{\mathbf{p}'} |\lambda_{-\mathbf{q},\mathbf{p}'}|^2} + (\mathbf{k} \leftrightarrow -\mathbf{k}) \right]. \tag{89}$$

Here the first terms in the numerators come from the Fock terms involving either the particle or hole created by $S^-_{\mathbf{q}}$ (positive since they involve a loss of exchange energy), while the second terms come from the (attractive) interaction between the two.

The spin stiffness is obtained by extracting the $O(q^2)$ contribution to δE in the limit of small q, and so we need to expand this expression to order q^2. This can be done with the help of expressions like

$$\lambda_{-\mathbf{q},\mathbf{p}} = 1 - i q^j \mathcal{A}^j_{\mathbf{p}} + \frac{q^i q^j}{2} (i \partial_i \mathcal{A}^j_{\mathbf{p}} - \mathrm{Re}\left[\langle \partial_i u(\mathbf{p}) | \partial_j u(\mathbf{p}) \rangle \right]) + \cdots, \tag{90}$$

where the $\cdots$ are higher order in q and we have used the shorthand $\partial_i \equiv \frac{\partial}{\partial k^i}$. Multiplying this by its complex conjugate, we see that

$$|\lambda_{-\mathbf{q},\mathbf{p}}|^2 = 1 - q^i q^j g^{ij}_{\mathbf{p}} + \cdots \tag{91}$$

We can use this to write the denominator of (89) to order q^2 as

$$\sum_{\mathbf{p}'} \lambda_{-\mathbf{q},\mathbf{p}'}\lambda_{\mathbf{q},\mathbf{p}'-\mathbf{q}} = \sum_{\mathbf{p}'}(1 - q^i q^j g^{ij}_{\mathbf{p}'}). \tag{92}$$

Similarly, the first set of terms in the numerator of (89) evaluate to

$$\sum_{\mathbf{p}} |\lambda_{\mathbf{q},\mathbf{p}}|^2|\lambda_{\mathbf{k},\mathbf{p}}|^2 + (\mathbf{k} \leftrightarrow -\mathbf{k}) = \sum_{\mathbf{p}}(1 - q^i q^j g^{ij}_{\mathbf{p}})(|\lambda_{\mathbf{k},\mathbf{p}}|^2 + |\lambda_{-\mathbf{k},\mathbf{p}}|^2). \tag{93}$$

The remaining two terms in (89) are slightly more involved, but can be worked out in a similar fashion. After this is done, we find that to order q^2,

$$\delta E_I = \frac{1}{2N}\int_{\mathbf{k}}\sum_{\mathbf{p}} V_{\mathbf{k}}\Big(|\lambda_{-\mathbf{k},\mathbf{p}}|^2 q^i q^j(\mathcal{A}^i_{\mathbf{p}} - \mathcal{A}^i_{\mathbf{p}-\mathbf{k}})(\mathcal{A}^j_{\mathbf{p}} - \mathcal{A}^j_{\mathbf{p}-\mathbf{k}}) - 2q^i(\mathcal{A}^i_{\mathbf{p}-\mathbf{k}} - \mathcal{A}^i_{\mathbf{p}})\mathrm{Im}[\lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}} q^j\partial_j\lambda_{-\mathbf{k},\mathbf{p}}] \\ - q^i q^j\,\mathrm{Re}\,[\lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}}\partial_i\partial_j\lambda_{-\mathbf{k},\mathbf{p}}]\Big). \tag{94}$$

The contributions of single body terms to the spin stiffness can be worked out more easily. For a general term

$$H_0 = \frac{1}{N}\sum_{\mathbf{p}}\sum_{\mathbf{G}} \mathcal{O}^{\alpha\beta}_{\mathbf{p},\mathbf{G}} c^\dagger_{\mathbf{p}+\mathbf{G},\alpha} c_{\mathbf{p},\beta}, \tag{95}$$

we get, as usual to order q^2,

$$\delta E_0 = \frac{1}{N}\sum_{\mathbf{p}}(\mathcal{O}^{\uparrow\uparrow}_{\mathbf{p},\mathbf{0}} - \mathcal{O}^{\downarrow\downarrow}_{\mathbf{p},\mathbf{0}})(1 - q_i q_j \bar{g}^{ij}_{\mathbf{p}}) \tag{96}$$

where we have defined $\bar{g}^{ij}_{\mathbf{p}} = g^{ij}_{\mathbf{p}} - \frac{1}{N}\sum_{\mathbf{p}'} g^{ij}_{\mathbf{p}'}$. Therefore single-particle dispersion and potential terms do not affect ρ_s, while the coupling to the magnetic field does not modify the spin stiffness if the field is uniform in space (since then $\mathcal{O}^{\sigma\sigma}_{\mathbf{p},\mathbf{0}}$ is independent of $\mathbf{p}$, giving a contribution proportional to $\sum_{\mathbf{p}} \bar{g}^{ij}_{\mathbf{p}} = 0$). Since we will only be interested in uniform fields, we can thus ignore the single-particle terms in what follows.

The spin stiffness tensor is obtained from δE via $\delta E = 2A\rho^{ij}_s q_i q_j$, where A is the size of the moire unit cell (so that the charge density is $1/A$). From (94), we then obtain

$$\rho^{ij}_s = \frac{1}{4NA}\int_{\mathbf{k}}\sum_{\mathbf{p}} V_{\mathbf{k}}\Big(|\lambda_{-\mathbf{k},\mathbf{p}}|^2(\mathcal{A}^i_{\mathbf{p}} - \mathcal{A}^i_{\mathbf{p}-\mathbf{k}})(\mathcal{A}^j_{\mathbf{p}} - \mathcal{A}^j_{\mathbf{p}-\mathbf{k}}) - 2(\mathcal{A}^i_{\mathbf{p}-\mathbf{k}} - \mathcal{A}^i_{\mathbf{p}})\mathrm{Im}[\lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}}\partial^j\lambda_{-\mathbf{k},\mathbf{p}}] \\ - \mathrm{Re}\,[\lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}}\partial^i\partial^j\lambda_{-\mathbf{k},\mathbf{p}}]\Big). \tag{97}$$

Note that we have not made any approximations in deriving this expression.

To simplify (97), we may follow [1, 3] and assume that $|\lambda_{-\mathbf{k},\mathbf{p}}|$ decreases fast enough with k to make an expansion in small k (we will work to order k^2) warranted for terms which multiply factors of $|\lambda_{-\mathbf{k},\mathbf{p}}|$. In this approximation scheme, we may write

$$\begin{aligned}(\mathcal{A}^i_{\mathbf{p}-\mathbf{k}} - \mathcal{A}^i_{\mathbf{p}})\mathrm{Im}[\lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}}\partial^j\lambda_{-\mathbf{k},\mathbf{p}}] &= -|\lambda_{-\mathbf{k},\mathbf{p}}|^2 k_l\partial^l\mathcal{A}^i_{\mathbf{p}}\partial^j\arg(\lambda_{-\mathbf{k},\mathbf{p}}) \\ &\approx |\lambda_{-\mathbf{k},\mathbf{p}}|^2 k_l k_m\partial^l\mathcal{A}^i_{\mathbf{p}}\partial^j\mathcal{A}^m_{\mathbf{p}}\end{aligned} \tag{98}$$

and

$$\begin{aligned}\mathrm{Re}\,[\lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}}\partial^i\partial^j\lambda_{-\mathbf{k},\mathbf{p}}] &= -|\lambda_{-\mathbf{k},\mathbf{p}}|^2\partial_i\arg(\lambda_{-\mathbf{k},\mathbf{p}})\partial_j\arg(\lambda_{-\mathbf{k},\mathbf{p}}) + |\lambda_{-\mathbf{k},\mathbf{p}}|\partial_i\partial_j|\lambda_{-\mathbf{k},\mathbf{p}}| \\ &\approx |\lambda_{-\mathbf{k},\mathbf{p}}|^2\left(k_l k_m\partial^i\mathcal{A}^l_{\mathbf{p}}\partial^j\mathcal{A}^m_{\mathbf{p}} - \frac{1}{2}k_l k_m\partial^i\partial^j g^{lm}_{\mathbf{p}}\right).\end{aligned} \tag{99}$$

This then gives

$$\rho_s^{ij} = \frac{1}{4NA}\int_{\mathbf{k}}\sum_{\mathbf{p}} V_{\mathbf{k}}|\lambda_{-\mathbf{k},\mathbf{p}}|^2 k^l k^m\left(\varepsilon^{il}\varepsilon^{jm}\mathcal{B}_{\mathbf{p}}^2 - \frac{1}{2}\partial^i\partial^j g^{lm}_{\mathbf{p}}\right) \tag{100}$$

where the Berry curvature is $\partial^i\mathcal{A}^j_{\mathbf{p}} - \partial^j\mathcal{A}^i_{\mathbf{p}} = \varepsilon^{ij}\mathcal{B}_{\mathbf{p}}$. Since the Berry curvature and metric are gauge-invariant so is ρ_s^{ij}, as required.

Finally, if the interaction and form factors are rotation invariant to a good approximation, this then becomes

$$\rho_s^{ij} = \frac{1}{8NA}\int_{\mathbf{k}}\sum_{\mathbf{p}} V_{\mathbf{k}}|\lambda_{-\mathbf{k},\mathbf{p}}|^2 k^2\left(\delta^{ij}\mathcal{B}_{\mathbf{p}}^2 - \frac{1}{2}\partial^i\partial^j\mathrm{Tr}[g_{\mathbf{p}}]\right). \tag{101}$$

If we ignore the second term involving the metric, this agrees with [3, 4, 1].

The energy cost of a particle-hole pair can be computed by evaluating δE in the $q\to\infty$ limit, which we do by sending $q\to\infty$ and averaging over unit vectors $\mathbf{q}/q$. This gives approximately

$$\begin{aligned}\delta E_{ph} &\approx \frac{1}{N}\int_{\mathbf{k}}\sum_{\mathbf{p}} V_{\mathbf{k}}|\lambda_{-\mathbf{k},\mathbf{p}}|^2(1-\langle\cos(\arg[\lambda_{-\mathbf{q},\mathbf{p}}\lambda_{-\mathbf{k},\mathbf{p}-\mathbf{q}}\lambda_{\mathbf{q},\mathbf{p}-\mathbf{k}-\mathbf{q}}\lambda_{\mathbf{k},\mathbf{p}-\mathbf{k}}])\rangle_{\hat{\mathbf{q}}}) + \mu_B g B \\ &= \frac{1}{N}\int_{\mathbf{k}}\sum_{\mathbf{p}} V_{\mathbf{k}}|\lambda_{-\mathbf{k},\mathbf{p}}|^2 + \mu_B g B.\end{aligned} \tag{102}$$

As a check that our factors of 2 are correct, we can compute the spin stiffness and particle-hole pair energy in the context of a LLL, where $\mathcal{B}_{\mathbf{p}} = A/2\pi$ is constant, $V_{\mathbf{k}}\propto 1/k$ is the Coulomb interaction, the metric is constant, and $|\lambda_{-\mathbf{k},\mathbf{p}}|^2 = e^{-k^2 l_B^2/2}$. If we further set $g\mu_B = 0$ we must then find $8\pi\rho_s/\delta E_{ph} = 1/2$ [4, 2]. Indeed, in units where $\mathcal{B}_{\mathbf{p}} = 1$ the area of the unit cell is $A = 2\pi l_B^2$ (as in $l_B = 1$ units the density is $n = 1/2\pi$), and so

$$\frac{8\pi\rho_s}{\delta E_{ph}} = \frac{\pi\int_0^\infty dk\,k^2 e^{-k^2 l_B^2/2}}{A\int_0^\infty dk'\,e^{-(k')^2 l_B^2/2}} = \frac{A l_B^{-2}}{4\pi} = \frac{1}{2}. \tag{103}$$

3.1.1 Exactness of the SMA

The SMA is exact when $S^-_{\mathbf{q}}|\Omega\rangle$ is an eigenstate of H, which is true if $[H, S^-_{\mathbf{q}}]|\Omega\rangle$ is proportional to $S^-_{\mathbf{q}}|\Omega\rangle$. We compute

$$[H, S^-_{\mathbf{q}}]|\Omega\rangle = -\sum_{\mathbf{k},\mathbf{p}} V_{\mathbf{k}}(\Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}}\lambda_{-\mathbf{k},\mathbf{p}+\mathbf{q}+\mathbf{k}} - \Lambda_{\mathbf{q},\mathbf{k},\mathbf{p}-\mathbf{k}}\lambda_{-\mathbf{k},\mathbf{p}})c^\dagger_{\mathbf{p}+\mathbf{q},\downarrow}c_{\mathbf{p},\uparrow}|\Omega\rangle, \tag{104}$$

which in general is not proportional to $S^-_{\mathbf{q}}|\Omega\rangle$. In the case of (84), we have

$$[H, S^-_{\mathbf{q}}]|\Omega\rangle = -\sum_{\mathbf{k},\mathbf{p}} 2iV_{\mathbf{k}}F(k)^2\sin(\mathbf{q}\wedge\mathbf{k}/2)\Big(e^{i[\mathbf{k}\wedge\mathbf{p}(\mathcal{B}_{\mathbf{p}}-\mathcal{B}_{\mathbf{p+q+k}})+\mathcal{B}_{\mathbf{p+q+k}}\mathbf{q}\wedge\mathbf{k}]/2} \\ - e^{i[\mathbf{q}\wedge\mathbf{p}(\mathcal{B}_{\mathbf{p-k}}-\mathcal{B}_{\mathbf{p}})+\mathbf{k}\wedge\mathbf{p}(\mathcal{B}_{\mathbf{p-k}}-\mathcal{B}_{\mathbf{p}})-\mathbf{q}\wedge\mathbf{k}\mathcal{B}_{\mathbf{p-k}}]/2}\Big)F(q)e^{i\mathbf{q}\wedge\mathbf{p}\mathcal{B}_{\mathbf{p}}/2}c^\dagger_{\mathbf{p+q},\downarrow}c_{\mathbf{p},\uparrow}|\Omega\rangle \tag{105}$$

We therefore see that even if the form factors satisfy this simple ansatz, the SMA is exact only when the Berry curvature is uniform in the BZ.

3.2 Bethe-Salpeter equation

The SMA makes an assumption that single magnon states can be generated by the application of a single raising / lowering operator acting on the polarized ground state. This is however too restrictive of an ansatz, and a more correct treatment is to take a variational approach. In this approach the magnon states are assumed to be created by a linear combination of spin waves at different momenta:

$$\sum_{\mathbf{p}} z^{\mathbf{p}}_{\mathbf{q}}\, c^\dagger_{\mathbf{q+p},\downarrow}c_{\mathbf{p},\uparrow}|\Omega\rangle, \tag{106}$$

where $|\Omega\rangle$ is the spin-polarized ground state and the sum over $\mathbf{p}$ runs over the BZ.

We then vary the Schrodinger equation with respect to $(z^{\mathbf{p+q}}_{\mathbf{q}})^*$, giving

$$E_{\mathbf{q}}z^{\mathbf{p}}_{\mathbf{q}} = \sum_{\mathbf{p}'}\langle c^\dagger_{\mathbf{p},\uparrow}c_{\mathbf{p+q},\downarrow}Hc^\dagger_{\mathbf{q+p}',\downarrow}c_{\mathbf{p}',\uparrow}\rangle z^{\mathbf{p}'}_{\mathbf{q}}. \tag{107}$$

Computing the expectation value, we then just need to solve the eigenvalue problem[6][7]

$$E_{\mathbf{q}}z^{\mathbf{p}}_{\mathbf{q}} = (\varepsilon_{\mathbf{p+q}}-\varepsilon_{\mathbf{p}})z^{\mathbf{p}}_{\mathbf{q}} + \sum_{\mathbf{G}}\sum_{\mathbf{p}'}\left(V_{\mathbf{G}}\lambda_{\mathbf{G},\mathbf{p}'}(\lambda_{\mathbf{G},\mathbf{p+q}}-\lambda_{\mathbf{G},\mathbf{p}})z^{\mathbf{p}}_{\mathbf{q}} + V_{\mathbf{G}+\mathbf{p}'}|\lambda_{\mathbf{G}+\mathbf{p}',\mathbf{p}}|^2\right)z^{\mathbf{p}}_{\mathbf{q}} \\ - \sum_{\mathbf{G}}\sum_{\mathbf{p}'}V_{\mathbf{p}-\mathbf{p}'+\mathbf{G}}\lambda_{\mathbf{p}-\mathbf{p}'+\mathbf{G},\mathbf{p}'+\mathbf{q}}\lambda_{-\mathbf{p}+\mathbf{p}'-\mathbf{G},\mathbf{p}}z^{\mathbf{p}'}_{\mathbf{q}} \tag{108}$$

or equivalently (which makes it more obvious that when $\mathbf{q}=0$ one has a zero-energy solution with $z^{\mathbf{p}}_{\mathbf{0}}=1$)

$$E_{\mathbf{q}}z^{\mathbf{p}}_{\mathbf{q}} = (\varepsilon_{\mathbf{p+q}}-\varepsilon_{\mathbf{p}})z^{\mathbf{p}}_{\mathbf{q}} + \sum_{\mathbf{G}}\sum_{\mathbf{p}'}\left(V_{\mathbf{G}}\lambda_{\mathbf{G},\mathbf{p}'}(\lambda_{\mathbf{G},\mathbf{p+q}}-\lambda_{\mathbf{G},\mathbf{p}})z^{\mathbf{p}}_{\mathbf{q}} + V_{\mathbf{p}'+\mathbf{G}}|\lambda_{\mathbf{p}'+\mathbf{G},\mathbf{p}}|^2\right)z^{\mathbf{p}}_{\mathbf{q}} \\ - \sum_{\mathbf{G}}\sum_{\mathbf{p}'}V_{\mathbf{p}'+\mathbf{G}}\lambda^*_{\mathbf{p}'+\mathbf{G},\mathbf{p+q}}\lambda_{\mathbf{p}'+\mathbf{G},\mathbf{p}}z^{\mathbf{p}'+\mathbf{p}}_{\mathbf{q}} \tag{109}$$

[6] We have ignored the Zeeman contribution to the energy, since for a uniform field this just modifies $E_{\mathbf{q}}$ by a $\mathbf{q}$-independent constant gap.

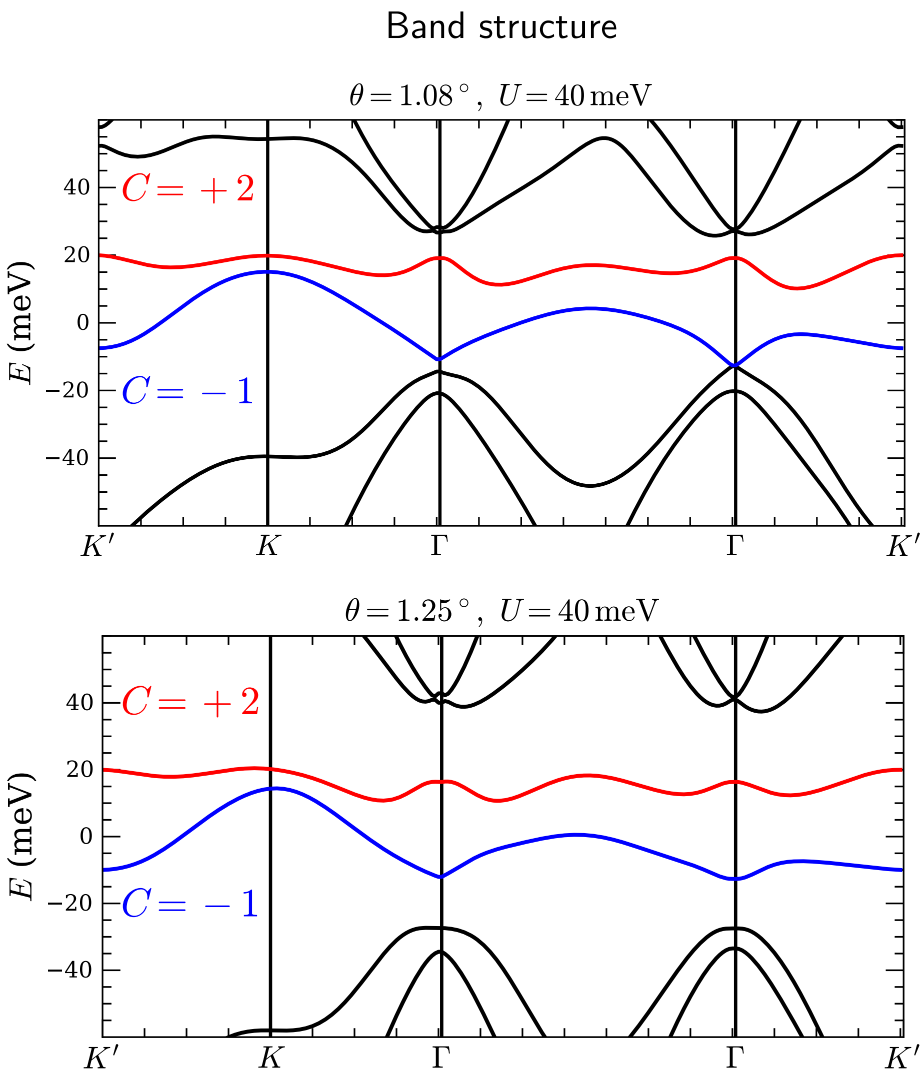
[7] Here we are choosing a periodic gauge where $\lambda_{\mathbf{q},\mathbf{k+G}} = \lambda_{\mathbf{q},\mathbf{k}}$. This is because this is more natural to do when doing numerics as we are only calculating λ in the BZ. But a consequence of this is that λ will not be smooth; hence trying to expand λ in derivatives and stuff won't work without introducing coordinate charts.

where for simplicity we have assumed the interaction satisfies $V_{\mathbf{k}} = V_{-\mathbf{k}}$. Since the spin wave creation operator only involves a sum over the BZ, $z_{\mathbf{q}}^{\mathbf{p}'+\mathbf{p}}$ really means $z_{\mathbf{q}}^{P(\mathbf{p}'+\mathbf{p})}$. Here the first term is the kinetic energy of the particle-hole pair, the second term is the Hartree energy of the particle and hole (which is nonzero only because $\langle \rho_{\mathbf{G}} \rangle \neq 0$), the third term is the Fock energy of the hole, and the last term is the (attractive) interaction energy between the particle and hole. Since the form factors usually decay rather quickly as a function of momentum transfer, the sums are often restricted to the single $\mathbf{G} = \mathbf{0}$ term.

As a sanity check, one can check that this reduces to the expected quantum Hall result in the limit where the form factors take the quantum Hall form and $\varepsilon_{\mathbf{k}} = 0$.

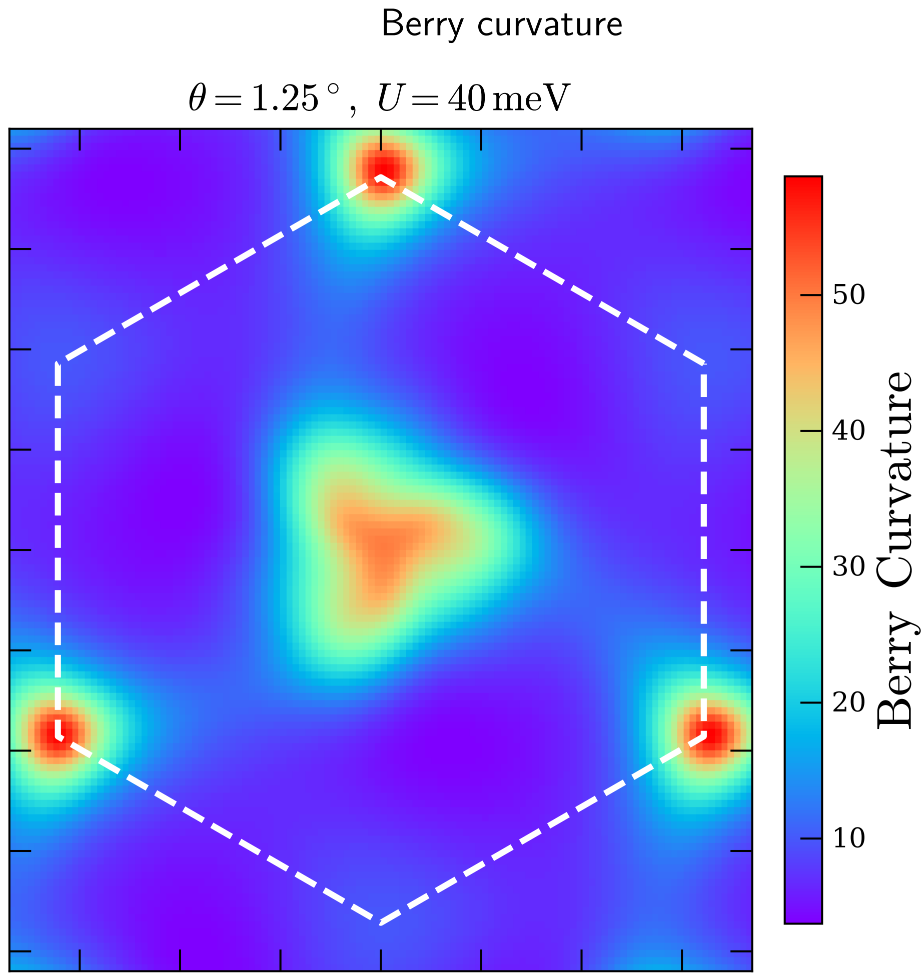
4 Application: twisted mono-bilayer graphene

There are a host of graphene-based Moire materials that offer easily accessed $C = 2$ bands. Here we run some numerics for one of them — twisted mono-bilayer graphene.



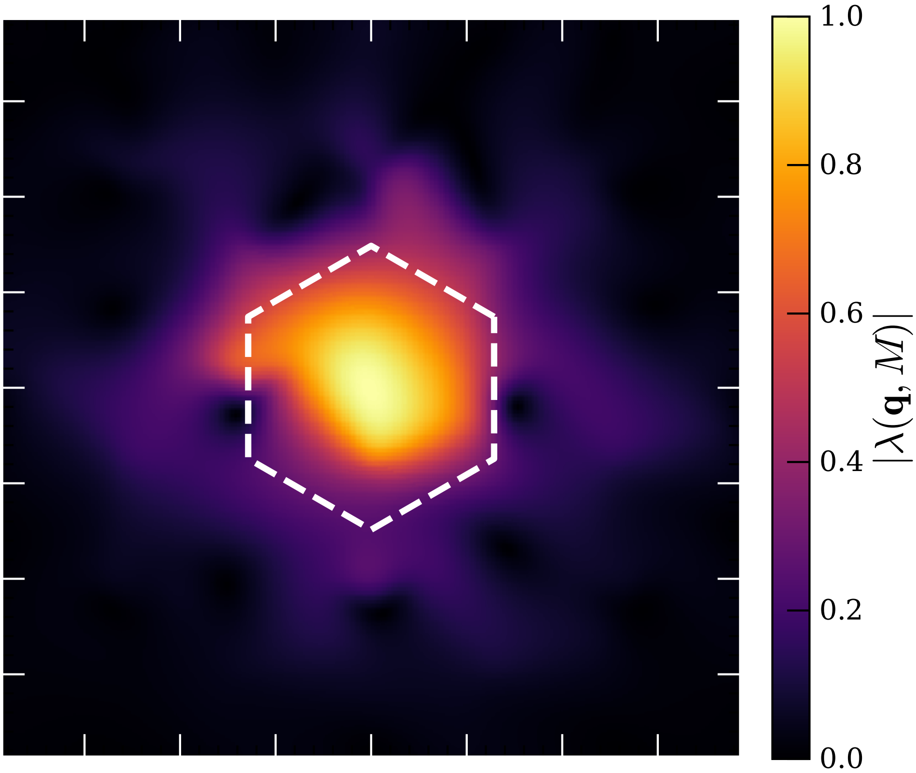
(110)

(111)

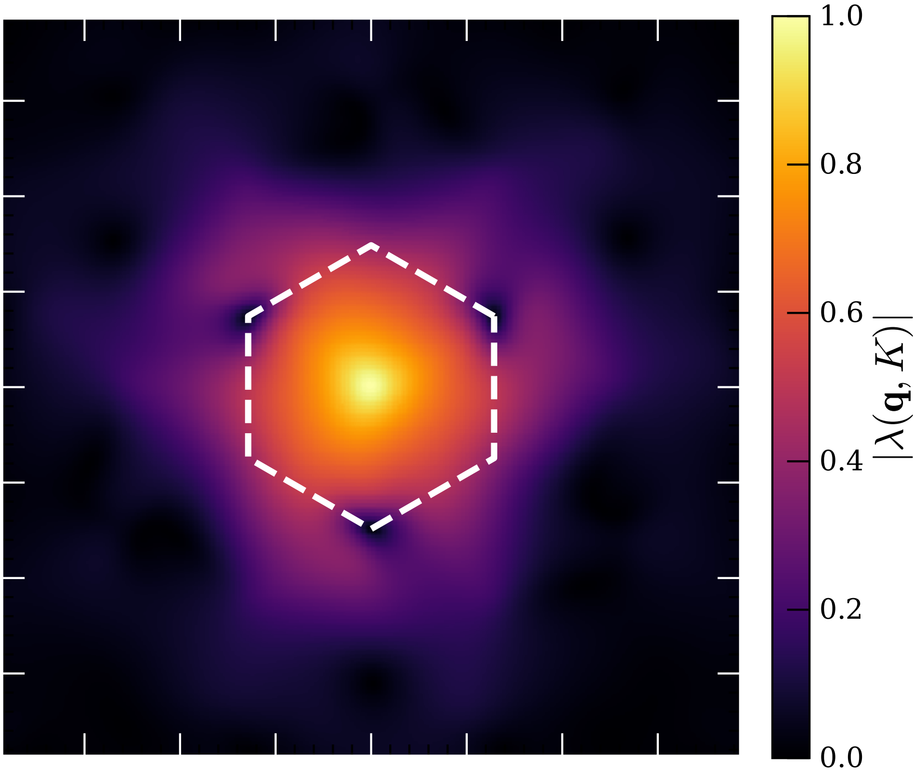

Berry curvature
$\theta = 1.25^\circ,\ U = 40\,\mathrm{meV}$
50
40
30
20
10
Berry Curvature

(112)

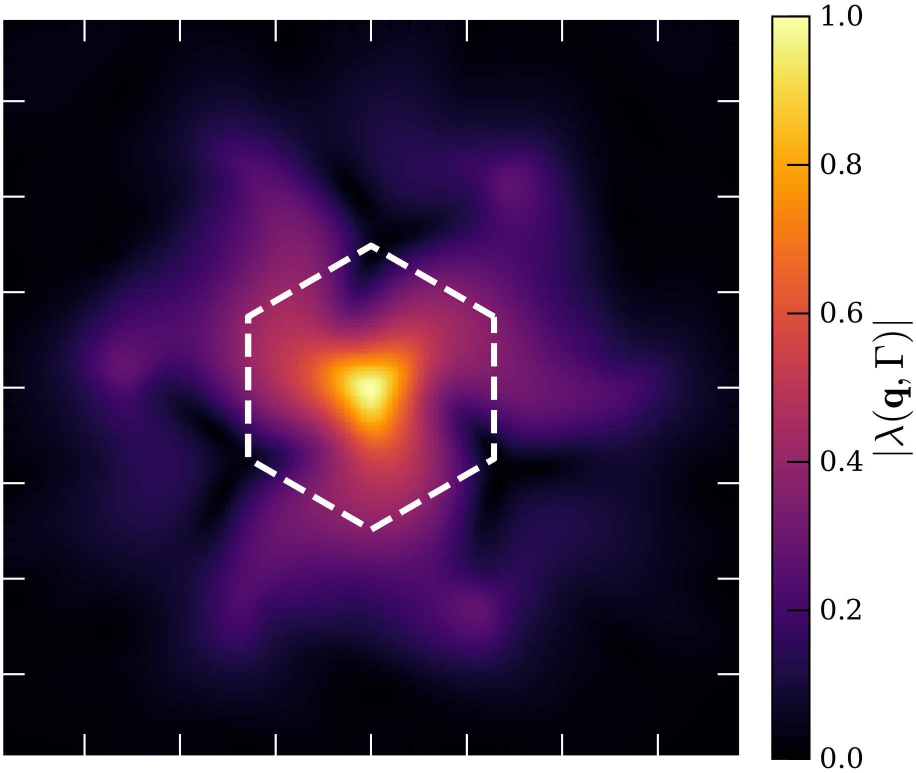
Form factors



(113)



(114)



(115)

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