Parton analysis of a single spin

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To examine exact what sort of approximations are made when doing partons and mean field theory,in this note we will take a look at what happens when we use this technology to compute the free energy of a single $SU(2)$ spin in 0+1D. By comparing with the exact answer (0+1D is the only dimension in which this is really possible), we will be able to see which aspects of the thermodynamics are missed by the partons at the mean field level. Of course to what extent the 0+1D problem really lets us build intuition for higher dimensions is debatable, but at the very least we can actually make concrete statements.

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Preliminaries about the parton approach applied to spin systems

In these notes we will mostly be interested in studying $SU(2)$ spins in the representation s. If we want to fractionalize the spin operator in terms of Schwinger bosons, we write[1]

$$S_\alpha^\beta = b_\alpha^\dagger b^\beta - s\delta_\alpha^\beta, \tag{1}$$

where we are constrained to work in the subspace in which

$$b_\alpha^\dagger b^\alpha = 2s, \tag{2}$$

which ensures that the S are properly traceless and cuts down the Hilbert space to the correct dimension of $2s+1$. Here the spin components are

$$S^i = \frac{1}{2}\mathrm{Tr}[S\sigma^i], \tag{3}$$

and with this one checks that $\mathbf{S}\cdot\mathbf{S} = S(S+1)\mathbf{1}$ when acting on the constrained Hilbert space: indeed, using $\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2(\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{2}\delta_{\alpha\beta}\delta_{\gamma\delta})$,

$$\mathbf{S}\cdot\mathbf{S} = \frac{1}{2}S_\alpha^\beta S_\gamma^\delta\left(\delta_\alpha^\delta\delta_\beta^\gamma - \frac{1}{2}\delta_\alpha^\beta\delta_\gamma^\delta\right) = \frac{1}{2}\mathrm{Tr}[S^2] = \frac{1}{2}\left(-4s^2 + 2s^2 + b_\alpha^\dagger(1 + b_\beta^\dagger b^\beta)b^\alpha\right) = s(s+1)\checkmark \tag{4}$$

where we have been careful to only use $b_\alpha^\dagger b^\alpha = 2s$ when acting on $\mathcal{H}_{phys}$: this is *not* an operator indentity, and so in particular we cannot replace the $b_\beta^\dagger b^\beta$ in the inner parenthesis in the fourth term above with $2s$.

[1] All lower indicies are row indicies.

Note also that we could also fractionalize the spin into fermions. In this case, instead of the normalization condition (2), the s-dependence enters by introducing a flavor index $a = 1, \ldots, 2s$, with

$$S_\alpha^\beta = c^\dagger_{\alpha a} c^{\beta a} - s\delta_\alpha^\beta, \tag{5}$$

where we are required to work in the $c^\dagger_{\alpha a} c^{\alpha b} = \delta_a^b$ subspace. Checking the commutation relations and $\mathbf{S} \cdot \mathbf{S} = s(s+1)\mathbf{1}$ on $\mathcal{H}_{phys}$ is done similarly to the bosonic case.[2]

The difference in representation here actually is actually just the $N = 2$ case of a general relation between representations of $SU(N)$ spins: $SU(N)$ operators acting on a Hilbert space transforming in a representation whose Young tableaux has n columns and m rows can be represented either by m flavors[3] of bosons satisfying $b^\dagger_{\alpha a} b^{\alpha c} = n\delta_a^c$, or by n flavors of fermions satisfying $c^\dagger_{\alpha a} c^{\alpha c} = m\delta_a^c$—see Ref. [?] for a good discussion of this. The equivalence between these two representations essentially boils down to the fact that columns in Young tableaux represent antisymmetrization while rows represent symmetrization; hence exchanging bosons with fermions swaps rows and columns.

In a given situation, how do we know whether we should use bosonic or fermionic partons? Consider for example the $SU(2)$ Heisenberg Hamiltonian on a bipartite lattice. Then we have

$$H \sim -J \sum_{\langle ij \rangle} a^\dagger_{i\alpha} a_{i\beta} a^\dagger_{j\beta} a_{j\alpha} \sim -J \sum_{\langle ij \rangle} (-1)^\eta a^\dagger_{i\alpha} a_{j\alpha} a^\dagger_{j\beta} a_{i\beta} \equiv -J \sum_{\langle ij \rangle} B^\dagger_{ij} B_{ij} \tag{6}$$

where $\sim$ denotes equality up to constants and $(-1)^\eta$ is -1 for fermions and $+1$ for bosons.

If we have a ferromagnet and the as are bosons, then H is proportional to a sum of negative-definite terms. This is good, because it allows us to decouple the interaction by integrating in a bosonic HS field (we need to integrate in something like $\int \mathcal{D}A_{ij}\, e^{-A_{ij}^2}$; for $A_{ij} \in \mathbb{R}$ the decoupling only works if the interaction in H has a minus sign), and it also tells us physically what the system likes to do: it likes to have links in the $+1$ eigenstate of the "triplet operator" $B_{ij} = a^\dagger_{i\uparrow} a_{j\uparrow} + a^\dagger_{i\downarrow} a_{j\downarrow}$. The same is true if we have an antiferromagnet and the as are fermions.

If we e.g. have an antiferromagnet and want to use bosons, this construction does not work. Instead, we perform a PH transformation on the B sublattice by sending $\mathbf{S}_B \mapsto J^\dagger \mathbf{S}_B J$ (with $J = -iY$ as usual). This has the effect of sending $\boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}^*$ on the B sublattice, so that

$$H \sim +J \sum_{\langle ij \rangle} a^\dagger_{i\alpha} a_{i\beta} a^\dagger_{j\alpha} a_{j\beta} \sim J \sum_{\langle ij \rangle} a^\dagger_{i\alpha} a^\dagger_{j\alpha} a_{j\beta} a_{i\beta} \equiv J \sum_{\langle ij \rangle} A^\dagger_{ij} A_{ij}, \tag{7}$$

where we have used that the matrix elements of $\otimes$s of generators for the fundamental and its conjugate are given as

$$\sigma^a_{\alpha\beta} (\sigma^a)^*_{\gamma\delta} = 2\left(\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{2}\delta_{\alpha\beta}\delta_{\gamma\delta}\right). \tag{8}$$

The interaction can then be decoupled with a $\mathbb{R}$ HS field if $J < 0$. Note that in the original unrotated basis the operator A_{ij} is $A_{ij} = a_{i\uparrow} a_{j\downarrow} - a_{i\downarrow} a_{j\uparrow}$, which gives us a concerete way of understanding why anitferromagets like spins to be locked up in singlets.

[2]The constrained Hilbert space for the fermions seems a bit wonkier though, and when realizing the constraints through Lagrange multipliers it seems to be easier to use the bosonic representation.

[3]Meaning that the flavor index runs from 1 to m; adding in the physical $SU(N)$ index gives a total of Nm bosons.

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Bosonic partons

Now we will look at the single-spin path integral, first starting in the Schwinger boson representation. We will choose the magnetic field to point along the Z direction, so that the field term is (in our normalization of H) $H\mathrm{Tr}[SZ]/2 = Hb^\dagger Zb/2$. Writing the gauge field as $\lambda - ia$ with λ the saddle point, the partition function is

$$\begin{aligned} Z &= \int \mathcal{D}b(\tau)\,\mathcal{D}a(\tau)\,\exp\left(-\int d\tau\,\left[b^\dagger(\partial_\tau + \lambda - ia - HZ/2)b - 2s(\lambda - ia)\right]\right) \\ &= \int \mathcal{D}a(\tau)\,e^{-S_{eff}[a]}, \end{aligned} \tag{9}$$

with

$$S_{eff}[a(\tau)] = \sum_{\sigma=\pm1}\int d\tau\,\ln(\partial_\tau + \lambda - ia - \sigma H/2) - 2\bar{\lambda}s + 2is\int d\tau\,a, \tag{10}$$

where $\bar{\lambda} \equiv \lambda/T$. Note that the time derivative appears here as ∂_τ and *not* $2s\partial_\tau$, which we might have naively thought would be the case if we thought about the WZW term arising when quantizing a particle on a sphere. The mean-field equation has $\bar{a} = 0$ with λ determined by the constraint on the total density:

$$2s = \sum_{\sigma=\pm1} n_B(\lambda + \sigma h/2), \tag{11}$$

where in this section $h \equiv H/T$. Defining $q \equiv e^{\bar{\lambda}}$, we rewrite the above as a quadratic equation for q in terms of s, h:

$$(l-1)q^2 - 2lq\cosh(h/2) + l + 1 = 0, \qquad l \equiv 2s+1. \tag{12}$$

The solution is

$$q = \frac{1}{1-1/l}\left(\cosh(h/2) + \sqrt{l^{-2} + \sinh^2(h/2)}\right). \tag{13}$$

At large h this is $e^{h/2}/(1-1/l)$, while at $h = 0$ it is $(l+1)/(l-1)$. When evaluated on the mean-field solution, the free energy is

$$f_{MF}/T = \ln[(1 - e^{-h/2}/q)(1 - e^{h/2}/q)] - 2s\bar{\lambda}. \tag{14}$$

Therefore we can write the mean-field partition function as

$$Z_{MF} = \sum_{n,m=0}^{\infty} \exp\left(-\bar{\lambda}(n+m-2s) + h(n-m)/2\right). \tag{15}$$

Since this is a product of up-spin and down-spin partition functions, it receives contributions from an infinite number of Boltzmann weights, with terms proportional to $e^{kh/2}$ for all $k \in \mathbb{Z}$.

The mean-field approximation becomes exact in the large-N limit, where we add a flavor index $a = 1, \dots, N$ to the Schwinger bosons. To generalize what we have done so far to

$N > 1$, we need only replace S_{eff} with NS_{eff}. In the following we will study the theory at different values of N, which allows us to interpolate between the exact $N = 1$ solution and the MF approximation.

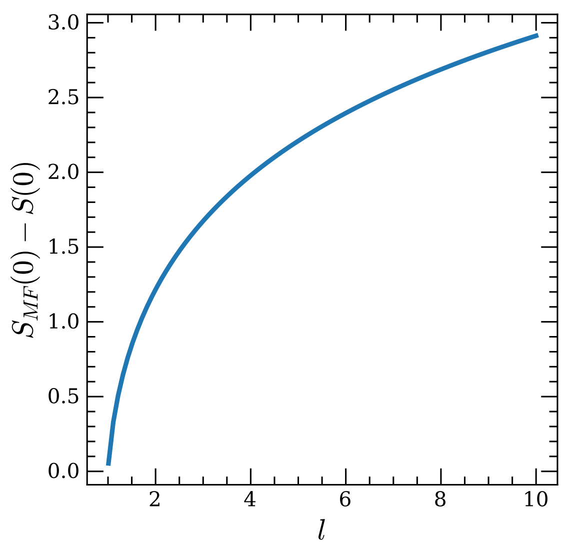
The mean-field and exact ($N = 1$) entropies are

$$\begin{aligned} S_{MF,s} &= -\ln\left(1 + \frac{1}{q^2} - \frac{2}{q}\cosh(h/2)\right) + \frac{2q\ln(q)\cosh(h/2) - qh\sinh(h/2) - 2\ln(q)}{1 + q^2 - 2q\cosh(h/2)}, \\ S_{1,s} &= \ln\left(\sum_{n=0}^{2s} e^{h(m-s)}\right) - \frac{\sum_{m=0}^{2s}(m-s)he^{h(m-s)}}{\sum_{n=0}^{2s} e^{h(n-s)}} \end{aligned} \tag{16}$$

with $S_{MF,s} \equiv \lim_{N\to\infty} S_{N,s}/N$. To see the "problems" with S_{MF}, consider first the limit $h \to 0$. In this limit, $q \to (l+1)/(l-1)$, and after some algebra we find

$$S_{MF,s}(h=0) = \ln\left(\frac{(1+l)^2}{4}\right) + (l-1)\ln\left(\frac{l+1}{l-1}\right). \tag{17}$$

Of course, the exact answer is $S(h=0) = \ln(l)$. The difference between these two functions looks like



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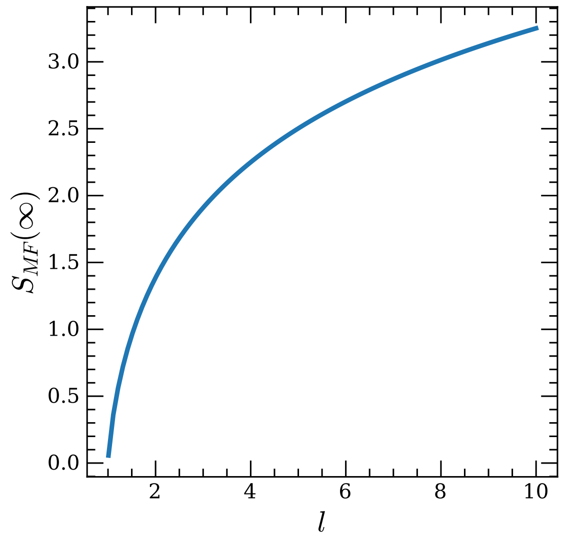
$S_{MF,s}$ also gives unphysical results at large fields. Here we have

$$S_{MF,s}(h \gg 1) \approx \ln(l) + (l-1)\ln\left(\frac{l}{l-1}\right) + \mathcal{O}(e^{-h}), \tag{19}$$

which looks almost the same as $S_{MF,s}(0) - S_{1,s}(0)$:



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The fact that we get a constant as $h \to \infty$ is unphysical. Here by "unphysical" we mean "not a feature of the $N = 1$ model". In fact this $h \to \infty$ entropy is a completely physical feature of the $N > 1$ models, in that it is present even after exactly integrating out the gauge field. This is just due to the fact that taking $N > 1$ gives $N > 1$ degenerate flavors of bosons in the large h limit, which has nonzero entropy (which cannot be taken care of within the $N > 1$ models without breaking the large-N structure of the effective action). To better understand these issues, we now turn to calculating the exact partition function at general N.

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The exact partition function is obtained by integrating over the 'fluctuations' of the gauge field. The most convenient choice of gauge is one in which $\partial_\tau a = 0$, so that $a(\tau) = a$ is just a constant. Since in this gauge a is a constant, to obtain the full partition function we need only replace $\bar\lambda \mapsto \bar\lambda - i\bar a$ in Z_{MF}, and then integrate over a.[4] With $N = 1$, we expand the terms in the log appearing in f_{MF} in a power series and get

$$Z_{1,s} = \int d\bar a \sum_{n,m=0}^{\infty} \exp\left(-(\bar\lambda - i\bar a)(n + m - 2s) + h(n - m)/2\right). \tag{21}$$

Note that the integrand is invariant under $\bar a \mapsto \bar a + 2\pi$, as required. The integral over the zero mode a then cuts down the infinite number of summands appearing in Z_{MF} to just those allowed by spin quantization and doing the integral gives the required result, viz.

$$Z_{1,s} = \sum_{n=0}^{2s} e^{h(n-s)}. \tag{22}$$

[4]Technically we also need to add the $\det(-\partial_\tau^2)$ from the gauge-fixing; however since this is h-independent it will be ignored.

For larger N, the combinatorics is slightly more complicated. For general N, we have

$$Z_{N,s} = \int d\bar{a} \left[\sum_{n,m=0}^{\infty} \exp\left(i\bar{a}(n+m) + h(n-m)/2\right) \right]^N e^{-i2sN\bar{a}}. \tag{23}$$

We then need to select out the terms in the brackets whose total power of $e^{i\bar{a}}$ is $2sN$ and determine their h dependence. The combinatorics works out as follows. We can think about selecting terms from the expansion by looking at the number of ways to have a total occupancy of $2sN$ in a system with $2N$ flavors of bosons. Here the bosons are divided into two groups A and B, with A the first N flavors and B the second N flavors. Each A boson comes with a factor of $r \equiv e^{h/2}$, and each B boson comes with a factor of r^{-1}. Each allowed configuration of $2sN$ bosons then comes with a factor of $r^{i-(2sN-i)}$, with $i \in \mathbb{Z}_{2sN}$ representing the number of A bosons. The number of ways to give the N bosons in A total filling i is $\binom{N+i-1}{N-1}$, with the corresponding combinatorial factor for the B bosons being $\binom{N+2sN-i-1}{N-1}$. Therefore the partition function is

$$Z_{N,s} = \sum_{i=0}^{2sN} r^{2i-2sN} \binom{N+i-1}{N-1} \binom{N+2sN-i-1}{N-1}. \tag{24}$$

For $s = 1/2$ and small N, some more explicit expressions are

$$\begin{aligned} Z_{2,1/2} &= 3(r^2 + r^{-2}) + 4 \\ Z_{3,1/2} &= 10(r^3 + r^{-3}) + 18(r + r^{-1}) \\ Z_{4,1/2} &= 35(r^4 + r^{-4}) + 80(r^2 + r^{-2}) + 100. \end{aligned} \tag{25}$$

Entropies are then obtained from the above expressions by using $S = -f/T - \frac{hr}{2}\partial_r(f/T)$. As an example, one finds

$$S_{2,1/2} = \ln(3(r^2 + r^{-2}) + 4) - 3h \left(\frac{r^2 + r^{-2}}{3(r^2 + r^{-2}) + 4} \right). \tag{26}$$

In particular, the *exact* $h \to \infty$ entropy is non-zero for all $N > 1$. Indeed, as $h \to \infty$ all the bosons which create spins not parallel to the field disappear from the thermodynamics, but this still leaves N degenerate species of bosons which do not get projected out. The $h \to \infty$ entropy is then just the log of the number of ways to have a total occupation number of $2sN$ in a system of N bosons, which is $\binom{lN-1}{N-1}$ with $l = 2s + 1$ as before. Therefore

$$S_{N,s}(h \to \infty) = \ln[(lN-1)!] - \ln[((l-1)N)!] - \ln[(N-1)!], \tag{27}$$

and at large N,

$$S_{N\to\infty,s}(h \to \infty) \approx lN \ln\left(\frac{lN-1}{(l-1)N}\right) + N \ln\left(\frac{(l-1)N}{N-1}\right) + \ln\left(\frac{N-1}{lN-1}\right) \tag{28}$$

so that

$$S_{MF,s}(h \to \infty) = l \ln l - (l-1)\ln(l-1), \tag{29}$$

which agrees with the result above.

The entropy in the other limit of $h = 0$ is obtained by counting the number of ways for $2N$ flavors of bosons to have an occupation number of $2sN$, so that

$$S_{N,s}(h=0) = \ln[(lN + N - 1)!] - \ln[((l-1)N)!] - \ln[(2N-1)!], \tag{30}$$

which at large N gives

$$S_{N\to\infty,s}(h=0)/N \approx Nl \ln\left(\frac{(l+1)N-1}{(l-1)N}\right) + N\ln\left(\frac{((l+1)N-1)(l-1)N}{(2N-1)^2}\right), \tag{31}$$

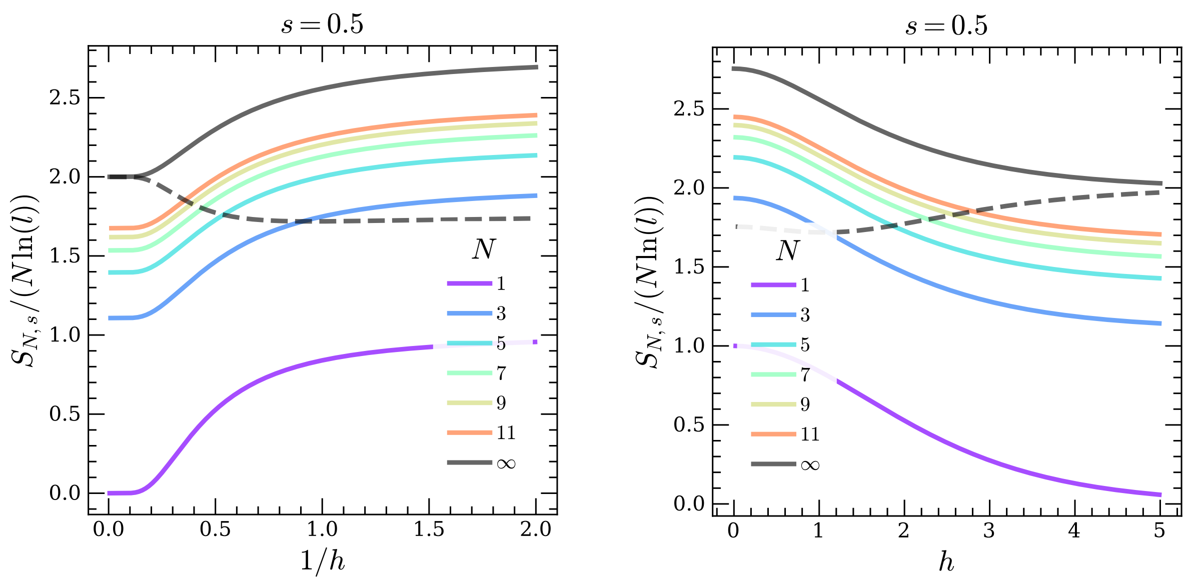
which agrees with the MF result derived above.

These results can also be derived in a potentially clearer way without ever bringing in the notion of N. It is easy to show that a single species of boson in 0+1D with $\langle b^\dagger b\rangle = n$ has entropy

$$S(n) = \ln(1+n) + n\ln(1+1/n). \tag{32}$$

At $h \to \infty$ mean field (in the $N = 1$ model of physical interest) we have a single species of boson with $n = 2s$, while at $h = 0$ we have two species of bosons each with $n = s$. Plugging these values into the above equation then produces the $N \to \infty$ results above.

In pictures, the entropy as a function of N for $s = 1/2$ is



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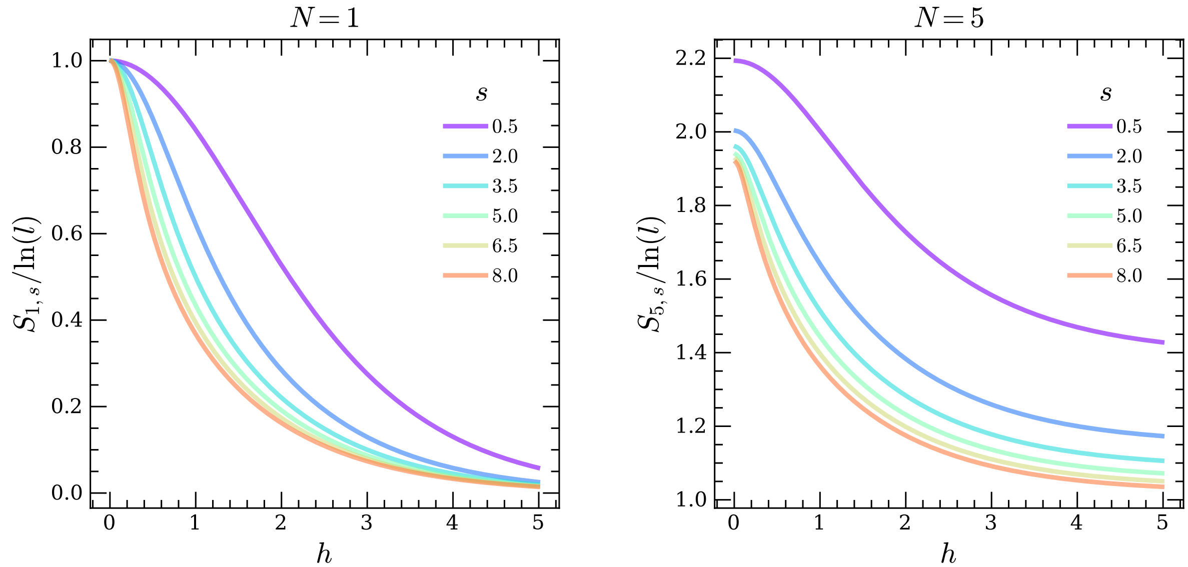
where the dashed line is the difference between the mean field and $N = 1$ results. For

$N = 1, 5$ as a function of s, we get



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Let us also briefly discuss the magnetization. Unlike the entropy, the magnetization at $N > 1$ provides a good estimate of the $N = 1$ magnetization, and its large- and small-h limits are physical, and the unphysical N-fold degeneracy of each σ^z eigenvalue can be completely accounted for by dividing the magnetization by N.

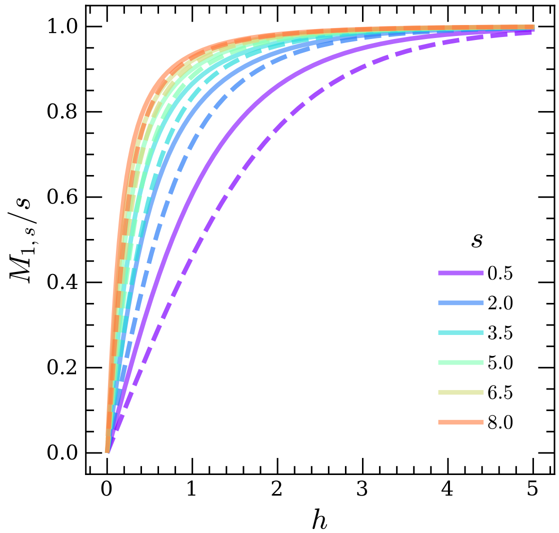
The mean-field and $N = 1$ magnetizations are calculated as

$$\begin{aligned} M_{MF,s} &= \frac{1}{2}\left(\frac{1}{qe^{-h/2}-1} - \frac{1}{qe^{h/2}-1}\right), \\ M_{1,s} &= \frac{\sum_{m=0}^{2s}(m-s)e^{h(m-s)}}{\sum_{n=0}^{2s} e^{h(n-s)}}. \end{aligned} \tag{35}$$

For $h \gg 1$, the mean field solution gives

$$M_{MF}(h \gg 1) \approx \frac{1}{2}\frac{1}{\frac{1}{1-1/l}-1} + \mathcal{O}(e^{-h}) = s + \mathcal{O}(e^{-h}), \tag{36}$$

which is the expected large-field limit. Plotting M_{MF} as solid lines and M as dashed lines,



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so that M_{MF} indeed does better at large s, as expected. Also note that $M_{MF} > M$ for all h at fixed s; this is in agreement with the fact that including fluctuations about the saddle point will reduce the magnetization.

Fermionic partons

Suppose now that we break apart the spin into fermions. The structure of the Fock space for fermions means that the mean-field solution will not have any entropy in the $h \to \infty$ limit, unlike the case with bosons. The disadvantage is that the mean-field magnetization and the field-dependence of the mean-field entropy at intermediate fields are worse than the bosonic results.

To work with $s > 1/2$ this requires introducing a flavor index (even at $N = 1$), and the constraints become rather awkward. Therefore we will specialize to $s = 1/2$ for now.

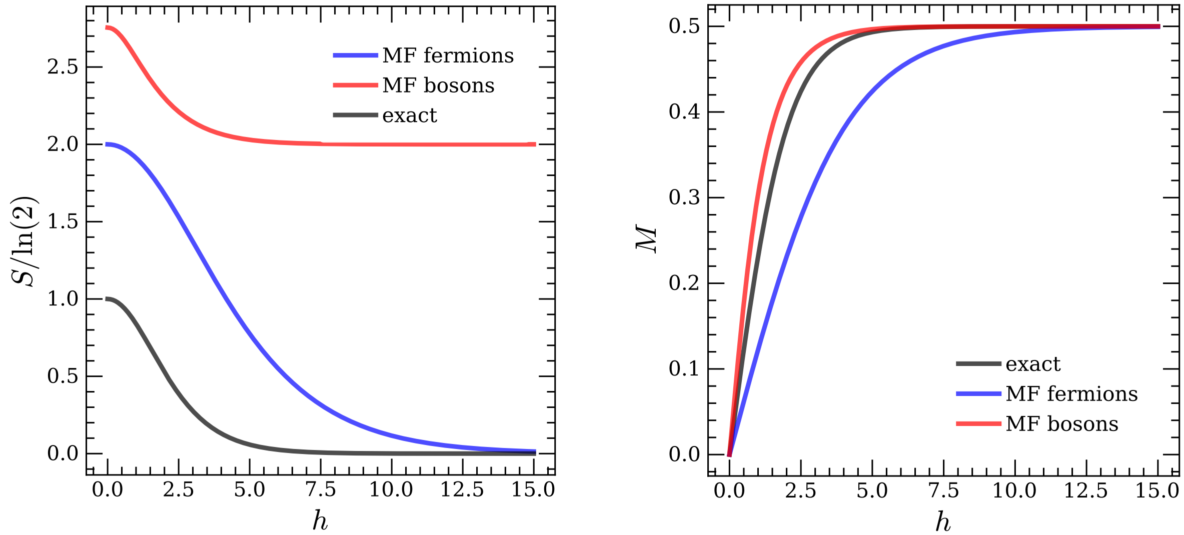
The mean-field equation, viz. $1 = n_F(\lambda + h/2) + n_F(\lambda - h/2)$, is very simple; since $n_F(x) + n_F(-x) = 1$ we just have $\bar{\lambda} = 0$. The mean field free energy, entropy, and magnetization are

$$\begin{aligned} f_{MF} &= -T\ln(2) - T\ln(1+\cosh(h/2)) \\ S_{MF} &= \ln(2) + \ln(1+\cosh(h/2)) - \frac{h}{2}\frac{\sinh(h/2)}{1+\cosh(h/2)} \\ M_{MF} &= \frac{\sinh(h/2)}{2(1+\cosh(h/2))}. \end{aligned} \tag{38}$$

The exact partition function is recovered by integrating over a as in the previous section:

$$Z = \int da\, e^{-ia}(1+e^{-h/2+ia})(1+e^{h/2+ia}) = 2\cosh(h/2) \qquad \checkmark \tag{39}$$

These functions look like



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While the fermions don't suffer from the same $h \to \infty$ entropy that the bosons do, if this part is subtracted off, the bosons do a better job at capturing the functional form of $S(h)$.