Note on one-axis twisting

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Today I attended a nice talk by Norm Yao on spin squeezing. In his talk he briefly introduced the "one axis twisting" model as a simple model for understanding spin squeezing. In this note we will try to derive the formulae he quoted on his slides for the squeezing rate. I have made no attempt to dig through the literature and hence won't provide citations to the original sources. Any mistakes in the following discussion are due to me not understanding the talk fully.

Simplifying a bit, the goal of quantum metrology is to use (an appropriate type of) quantum entanglement to perform a measurement of an external field which is more precise than what could be achieved with classical methods. The most common setting is where one aims to measure the strength of a magnetic field; if this field is represented by the Hamiltonian $\sum_i Z_i$ then the GHZ state provides an optimal state for performing the sensing, basically due to the fact that the GHZ state involves a coherent superposition of states with macroscopically different values of the quantum number we want to sense (viz. S^z).

One challenge with this is that the GHZ state is hard to prepare. To this end we might look for states which are good for sensing (i.e. involve superpositions of states with relatively large differences in magnetic moments), but which are still easy to prepare. For us, a state of N qubits will count as being easy to prepare if it is obtained by quenching a product state by a simple Hamiltonian and evolving for time poly(N) with some reasonably small polynomial.

The OG example of such a state is apparently the one constructed by quenching a state with maximal spin along the $\mathbf{a}$ direction by the Hamiltonian $\frac{1}{N}(\mathbf{S}_T \cdot \mathbf{b})^2$, where $\mathbf{b} \cdot \mathbf{a} = 0$ ($\mathbf{S}_T$ is the total spin operator of the N qubits). For concreteness, we will consider performing a quench on the state $|+\rangle^{\otimes N}$ with the Hamiltonian

$$H = \frac{1}{N}\sum_{i,j} Z_i Z_j = \frac{1}{N} Z_T^2, \tag{1}$$

where $Z_T = \sum_i Z_i$ is the "total S^z spin" (in quotes because we are not including factors of $1/2$).

Thinking back to when we learned about squeezed states in optics, it is perhaps reasonable that $|\psi(t)\rangle = e^{-itH}|+\rangle^{\otimes N}$ is a squeezed state of some form (recall that photonic squeezed states are created using squeezing operators like $e^{\zeta a^2 - \zeta^* a^{\dagger 2}}$ or by including interactions like this into a Hamiltonian; the resemblance between the non-linearities here and the Z_T^2 Hamiltonian should be apparent). That $|\psi(t)\rangle$ is indeed squeezed can be 'proven' with a simple physical argument. At $t = 0$, the total spin $\mathbf{S}$ is prepared in a 'cloud' spread out near the point $\hat{\mathbf{x}}$ on the Bloch sphere (with radius $\sim L/2$). Evolution with H makes the cloud of spin precess around the $\hat{\mathbf{z}}$ direction *with a rate that depends on the z-component of the spin.* Thus the part of the cloud lying at positive z will precess one way around $\hat{\mathbf{z}}$, while the part at negative z will precess the opposite way. The precession is faster for larger $|z|$, and the result is a dynamics which 'shears' the could, distending it by wrapping it around the sphere. Pictorially, this shearing process then leads to a squeezing of the cloud, and getting a squeezed spin state for long enough times is then quite reasonable (this was the logic given in Norm's talk).

Now we make this intuition more precise. We will aim to compute a figure of merit introduced in the talk, which adapted to the present notation is

$$\xi^2(t) \equiv N \frac{\min_\phi \operatorname{var}(Z_T(t)\cos\phi + Y_T(t)\sin\phi)}{\langle X_T(t)\rangle^2}. \tag{2}$$

Here the minimum over ϕ acts to select out the "most squeezed" direction of the spin in the plane normal to the initial magnetization vector $\parallel \hat{\mathbf{x}}$. On one of Norm's slides he claimed that the minimum value ξ_s^2 of $\xi^2(t)$ is $\xi_s^2 \sim N^{-2/3}$, and that this minimum is reached at a time $t \sim N^{1/3}$. These results are what we will aim to reproduce below.

To calculate the variance appearing in ξ^2, we start off with the useful relation

$$S_T^s(t) = e^{itZ_T^2/N} S_T^s e^{-itZ_T^2/N} = S_T^s e^{\frac{it}{N}[(Z_T+2s)^2 - Z_T^2]} = S_T^s e^{s\frac{4it}{N}Z_T + \frac{4it}{N}} = e^{s\frac{4it}{N}Z_T - \frac{4it}{N}} S_T^s \tag{3}$$

together with $Z_T(t) = Z_T$. Thus defining $\lambda \equiv 4t/N$, we have

$$\begin{aligned} \langle X_T(t)\rangle &= \sum_s \langle e^{is\lambda Z_T - i\lambda} S_T^s\rangle \\ &= N\sum_s \langle +|S^s|+\rangle (\langle +|\cos(\lambda Z)|+\rangle)^{N-1} \\ &= N\cos(\lambda)^{N-1} \end{aligned} \tag{4}$$

where the second line follows from an explicit expansion of S_T^s. On the other hand we obviously have

$$\langle Y_T(t)\rangle = \langle Z_T(t)\rangle = 0. \tag{5}$$

Now for the two point functions. We start with

$$\begin{aligned} \langle S_T^s(t) S_T^{-s}(t)\rangle &= \frac{1}{4}\sum_{j,k} \langle +|(X+sJ)_j (X-sJ)_k|+\rangle \\ &= \frac{1}{4}\sum_{j,k}\langle +|2\delta_{j,k} + (1-\delta_{j,k})X_j X_k|+\rangle \\ &= \frac{N(N+1)}{4}. \end{aligned} \tag{6}$$

The other combination is

$$\begin{aligned}\langle S_T^s(t)^2\rangle &= \langle S_T^s e^{2i\lambda Z_T} S_T^s\rangle \\ &= \cos(2\lambda)^{N-2} N(N-1)\langle +|S^s e^{2i\lambda Z}|+\rangle\langle +|e^{2i\lambda Z} S^s|+\rangle \\ &= \frac{N(N-1)}{4}\cos(2\lambda)^{N-2}.\end{aligned} \tag{7}$$

The ZZ correlators are of course $\langle Z_T(t)^2\rangle = N$ since the initial state has $\sqrt{N}$ fluctuations in both Z_T and Y_T, and Z_T commutes with the Hamiltonian. The last correlator we will need is

$$\begin{aligned}\langle Z_T(t) S_T^s(t)\rangle &= \langle Z_T e^{i\lambda s Z_T - i\lambda} S_T^s\rangle \\ &= \sum_{j,k}\left(\delta_{j,k}\cos(\lambda)^{N-1}\frac{s}{2} + (1-\delta_{j,k})\frac{\cos(\lambda)^{N-2}}{2}\langle +|Z e^{i\lambda s Z}|+\rangle\right) \\ &= \frac{s}{2}\left(N\cos(\lambda)^{N-1} + iN(N-1)\cos(\lambda)^{N-2}\sin(\lambda)\right).\end{aligned} \tag{8}$$

Define the operator $W_\phi \equiv \cos(\phi)Z_T + \sin(\phi)Y_T$. Since $\langle W_\phi(t)\rangle = 0$ for all t, $\mathrm{var}(W_\phi(t))^2 = \langle W_\phi(t)^2\rangle$, which we compute via

$$\begin{aligned}\langle W_\phi(t)^2\rangle &= \cos(\phi)^2\langle Z_T(t)^2\rangle + \sin(\phi)^2\langle Y_T(t)^2\rangle + \cos(\phi)\sin(\phi)\langle\{Z_T(t), Y_T(t)\}\rangle \\ &= \cos(\phi)^2 N + \sin(\phi)^2\sum_s\langle S_T^s(t)S_T^{-s}(t) - S_T^s(t)S_T^s(t)\rangle + 2\mathrm{Im}\langle Z_T(t)(S_T^+(t) - S_T^-(t))\rangle \\ &= \cos(\phi)^2 N + \sin(\phi)^2\left(N^2\frac{1-\cos(2\lambda)^{N-2}}{2} + N\frac{1+\cos(2\lambda)^{N-2}}{2}\right) \\ &\quad + 2\sin(\phi)\cos(\phi)N(N-1)\cos(\lambda)^{N-2}\sin(\lambda).\end{aligned} \tag{9}$$

Note that the RHS is equal to N at $t = 0$, as required by the isotropic (in the yz plane) $\sqrt{N}$ fluctuations present in the initial state.

To simplify the above expression, we bring all of the trig functions involving ϕ into a common form. One of the intermediate steps along the way involves writing $\cos(2\phi)\alpha + \sin(2\phi)\beta = \sqrt{\alpha^2+\beta^2}\cos(2\phi - 2\theta)$, where $\theta = \arctan(\beta/\alpha)$. Some unilluminating algebra along these lines leads to

$$\langle W_\phi(t)^2\rangle = \frac{1}{4}\left(N^2(1-\cos(2\lambda)^{N-2}) + N(3+\cos(2\lambda)^{N-2})\right) + \sqrt{\Xi^2+\Upsilon^2}\cos(2\phi - 2\theta), \tag{10}$$

where we have defined

$$\begin{aligned}\theta &\equiv \arctan(\Xi/\Upsilon) \\ \Xi &\equiv N(N-1)\cos(\lambda)^{N-2}\sin(\lambda) \\ \Upsilon &\equiv N(N-1)\frac{\cos(2\lambda)^{N-2}-1}{4}.\end{aligned} \tag{11}$$

The most squeezed direction is clearly the direction defined by $\phi = \theta - \pi/2$ (while $\phi = \theta$ is accordingly the least squeezed). We now need to evaluate the squeezing factor ξ^2 in the case where $\phi = \theta - \pi/2$, and find the time t at which ξ^2 is maximized.

By either plotting $\xi^2(\lambda)$ or by recognizing that we have a $\langle X_T(t)\rangle^2 = N^2\cos(\lambda)^{2N-2}$ in the denominator, one sees that the maximum of ξ^2 is attained at small $\lambda \ll 1$.

Simply performing a series expansion of ξ^2 in small λ doesn't work however, as doing so yields a squeezing time $t_s \equiv \text{argmin}_t(\xi^2)$ which scales as $t_s = \Theta(N^0)$ and a gives a squeezing parameter $\xi_s^2 \equiv \xi^2(t = t_s)$ of $\xi_s^2 = \Theta(N^{-1/2})$ — which is has the same scaling with N as $\xi^2(0)$!

Playing around with the plots of $\xi^2(t)$ more carefully (by rescaling t by N) shows that t_s is not small (and in fact grows with N), but that t_s/N always is. Thus we can find the minimum by expanding in $\lambda \ll 1$ while at the same time taking $\lambda N \gg 1$. Doing so yields

$$\langle W_\phi(t)^2\rangle \sim N\left(\frac{A}{(N\lambda)^2} + B\lambda^2(N\lambda)^2\right), \tag{12}$$

where (don't quote me on this) $A = 1, B = 384$ (really?). Thus

$$\xi_s^2 \sim \frac{A}{(N\lambda)^2} + B\lambda^2(N\lambda)^2, \tag{13}$$

which is minimized when $\lambda = (2B/A)^{-1/6}N^{-2/3}$; writing this in terms of t gives

$$t_s = \gamma N^{1/3}, \qquad \gamma \equiv \frac{A^{1/6}}{4(2B)^{1/6}}, \tag{14}$$

which correctly gives the $N^{1/3}$ scaling quoted on Norm's slide. By plugging this back in, we see that

$$\xi_s = \gamma' N^{-2/3}, \qquad \gamma' \equiv \frac{A}{\gamma^2} + B\gamma^4, \tag{15}$$

which also correctly gives the claimed $N^{-2/3}$ scaling (and hence does better than the $N^{-1/2}$ that we get in the unsqueezed $t = 0$ state).