Consider the Hamiltonian

$$H = \sum_i \left(-t(b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) - \mu n_i + \frac{U}{2} n_i(n_i - 1) + V i n_i \right) \equiv H_t + H_U + H_V, \quad (1)$$

where H_U includes the chemical potential. Our goal is to perform a rotation into a basis in which the Hamiltonian commutes with the linear potential term $V \sum_i i n_i$ up to some fixed order in $t/V, U/V$. The usual way of rotating the Hamiltonian is through an SW transformation, wherein H becomes

$$e^\Lambda H e^{-\Lambda} = \sum_{k \geq 0} \frac{1}{k!} \mathrm{Ad}_\Lambda^k(H), \quad (2)$$

where $\mathrm{Ad}_\Lambda(\cdot) = [\Lambda, \cdot]$ and Λ is anti-Hermitian.

Note that it is already clear that interactions are required for producing a nonzero dipolar hopping term. Indeed, without the interaction term, H is built solely of 2-body terms — we can thus choose Λ to be a 2-body operator, and $\mathrm{Ad}_\Lambda^k(H)$ will consequently always itself be built from 2-body operators, which can only either be purely onsite or dipole non-conserving. In fact if we just take

$$\Lambda = \Lambda_t \equiv \frac{t}{V} \sum_i (b_i^\dagger b_{i+1} - b_{i+1}^\dagger b_i), \quad (3)$$

it is easy to check that when $U = 0$,

$$[\Lambda_t, H] = [\Lambda_t, H_V] = t \sum_i (b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) = -H_t. \quad (4)$$

Since this is just the negative of the hopping term, the first order part $\mathrm{Ad}_\Lambda(H_V)$ dutifully kills H_t. Moreover, since $[\Lambda_t, [\Lambda_t, H_t + H_V]] = 0$, the effective Hamiltonian stops at linear order, and we simply obtain $H_{eff} = e^{\Lambda_t}(H_t + H_V)e^{-\Lambda_t} = H_V$, which is purely onsite. This means that when $U = 0$, no effective dipole hopping terms are generated — there is perfect destructive interference between all putative hopping processes, and no such processes are generated to all orders in perturbation theory.

Let us then bring back a nonzero U. We take

$$\Lambda = \sum_{n=1}^{\infty} \Lambda_n, \quad (5)$$

where Λ_n is order n in $t/V, U/V$, and we set $\Lambda_1 = \Lambda_t$. We fix the second order term Λ_2 by requiring that it cancel the terms generated when commuting $\Lambda_1 = \Lambda_t$ against H_U. Specifically, we require

$$[\Lambda_2, H_V] = -[\Lambda_t, H_t + H_U]. \quad (6)$$

Keeping terms to third order in this expansion, we find that H_{eff} becomes

$$H_{eff} = H_V + H_U + [\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2 - \Lambda_t, H_t] + [\Lambda_3, H_V]. \quad (7)$$

We then need the commutators

$$
\begin{aligned}
[\Lambda_t, H_t] &= \frac{t^2}{V}\sum_i \Big(b^\dagger_{i+2}b_i + 2n_{i+1} + b^\dagger_i b_{i+2} - (i\to i+1)\Big) = 0,\\
[\Lambda_t, H_U] &= \frac{tU}{V}\sum_i \{b^\dagger_i b_{i+1} + b^\dagger_{i+1}b_i, n_{i+1}-n_i\}
\end{aligned}
\tag{8}
$$

which together determines Λ_2 as (it is easy to guess this using the facts that the terms in Λ_2 must have support on two adjacent sites, consist only of 4-body operators, and must not commute with H_V)

$$
\Lambda_2 = -\frac{tU}{V^2}\sum_i \{b^\dagger_i b_{i+1} - b^\dagger_{i+1}b_i, n_{i+1}-n_i\}. \tag{9}
$$

The effective Hamiltonian to cubic order is then

$$
H_{eff} = H_V + H_U + [\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2, H_t] + [\Lambda_3, H_V]. \tag{10}
$$

The commutators $[\Lambda_2, H_U] + \frac{1}{2}[\Lambda_2, H_t]$ will include terms which do not commute with H_V, with Λ_3 chosen to cancel these terms. Now since H_U is purely on-site while Λ_2 is purely off-site and only supported on nearest neighbors, $[\Lambda_2, H_U]$ must consist only of dipole non-conserving terms. However, $[\Lambda_2, H_t]$ may contain terms that commute with H_V, as both Λ_2 and H_t are supported on nearest neighbor sites (allowing us to get e.g. $b^\dagger_i b^2_{i+1} b^\dagger_{i+2}$). Λ_3 cannot be chosen to cancel these terms, as if $[A, H_V] \neq 0$ then $[H_V, [A, H_V]] \neq 0$. [1] The part of $\frac{1}{2}[\Lambda_2, H_t]$ which commutes with H_V thus unambiguously provides the leading dipole hopping terms. An unilluminating calculation shows that these terms are

$$
\begin{aligned}
\frac{1}{2}[\Lambda_2, H_t] &\supset -\frac{t^2U}{V^2}\sum_{i,j}\{[b^\dagger_i b_{i+1} + b^\dagger_{i+1}b_i, n_{j+1}-n_j], b^\dagger_j b_{j+1} - b^\dagger_{j+1}b_j\}\\
&\supset -\frac{t^2U}{V^2}\sum_i b^\dagger_i b^2_{i+1} b_{i+2} + h.c,
\end{aligned}
\tag{11}
$$

where $\supset$ denotes those terms which commute with H_V. Thus the effective Hamiltonian to cubic order in $t/V, U/V$ is

$$
H_{eff} = \sum_i \left(-\frac{t^2U}{V^2} b^\dagger_i b^2_{i+1} b_{i+2} + h.c. - \mu n_i + \frac{U}{2}n_i(n_i-1) + V i n_i\right) \tag{12}
$$

[1] One can see this by writing a general A as a linear combination of terms of the form $C = \sum_i C_{\mathbf{n}} \prod_{k\in\mathbb{N}} b^{n_k}_{i+k}$, where each $n_k \in \mathbb{Z}$ (we have ignored number operators n_i since they commute with H_V). Since the action of Ad_{H_V} does not change the powers of b_i operators it acts on, the only way for $[H_V, [A, H_V]]$ to vanish is for $[H_V, [C, H_V]] = 0$ for all such operators C appearing in A. But $[H_V, [C, H_V]]$ takes the same form as C, except with $C_{\mathbf{n}}$ replaced with $C_{\mathbf{n}}\mathrm{dip}(\mathbf{n})^2$ with $\mathrm{dip}(\mathbf{n})$ the dipole moment of $\prod_k b^{n_k}_k$; thus this vanishes only if $\mathrm{dip}(\mathbf{n}) = 0 \implies [C, H_V] = 0$.