Fermions, bundles, and Spin_G structures

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The goal in these notes is to write down the right words explaining what bundles are relevant when dealing with fermions coupled to a gauge field. We will be rather mathematically pedantic, since this is designed to serve as a reference for me to come back to. Ref [1] seems to be a good reference in the mathematical physics literature, and looking at Nakahara can be helpful as a refresher for some of the math prerequisites.

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Preliminaries

First we review what kind of bundles regular fermions (not coupled to any particular gauge field) are associated with. Let X be spacetime,[1] $n = \dim X$, and let TX denote the tangent bundle of X. The fibers of the tangent space are acted on by $GL(n;\mathbb{R})$, which we can (with the metric) reduce to an action of $SO(n)$. Let LX, the frame bundle, be the principal $SO(n)$ bundle (we are assuming X is oriented) associated[2] to the tangent bundle (i.e. the principal bundle obtained from TX's transition functions).

Spinors are constructed with the help of the "square root" of the frame bundle, namely a principal $\mathrm{Spin}(n)$ bundle that we will write as SLX. Forming SLX is done, if possible, by

[1] In these notes we will be working in Euclidean signature — thus there are no reality conditions on fermions, and so our spinors will always live in complex vector spaces.

[2] The word "associated" used to describe a bundle can mean a few different things. First, suppose we are given a principal G bundle $\pi : P_G \to X$, and a vector space V which carries an action of G via some representation $R : G \to \mathrm{Aut}(X)$. Then the vector bundle E associated to P_G is denoted

$$E = P_G \times_R V, \tag{1}$$

and consists of pairs $(u, v) \in P_G \times V$ modulo the equivalence relation

$$(u, v) \sim (gu, R_g^{-1} v), \tag{2}$$

where gu is the action of g on the fiber. This is basically like a $\otimes$ of P_G and V. Here the inverse of the representation is chosen so that $(gu, v) \sim (u, R_g v)$. Anyway, we can use this to see that as we go from patch to patch, the vector space V gets acted on by $R(t_{\alpha\beta})$, where $t_{\alpha\beta}$ are the transition functions. Thus the transition functions for E are given by $R(t_{\alpha\beta})$.

Another notion of an associated bundle is the principal bundle associated to a vector bundle. If the fibers of the vector bundle $\pi : F \to X$ are $\mathbb{K}^n$ then we have transition functions valued in $G \subset GL(n;\mathbb{K})$ (the subgroup G depends on how much extra structure our manifold has, like orientability or a metric, etc), and these transition functions can be used to construct a principal G bundle over X, associated to the vector bundle F.

lifting the transition functions on the frame bundle from $SO(n)$-valued functions to $\mathrm{Spin}(n)$-valued functions; this can be done provided the cocycle conditions for the transition functions in LX hold in a particular way (more on this in a sec).

Suppose that SLX is well-defined. Then we can then form the spinor bundle S, which is the associated vector bundle made from the principal bundle SLX, the vector space $\mathbb{C}^{2^{\lfloor n/2 \rfloor}}$, and the spinor action of $\mathrm{Spin}(n)$ on $\mathbb{C}^{2^{\lfloor n/2 \rfloor}}$.[3][4] To save on writing, we will define the vector space that spinor fields live in as

$$\Delta_n \equiv \mathbb{C}^{2^{\lfloor n/2 \rfloor}}. \tag{4}$$

In our notation,

$$S = SLX \times_{1/2} \Delta_n, \tag{5}$$

where $1/2$ is the spinor representation. Fermion fields are sections of S.[5]

To address when the spin frame bundle SLX exists, we need to ask the following question: when is it possible to take the square root of a principal $SO(n)$-bundle? Roughly speaking, the square root of $G/\mathbb{Z}_2$ is G, since $\mathbf{1} \in G/\mathbb{Z}_2$ is the image of $\sqrt{\mathbf{1}} = \{\mathbf{1}, -\mathbf{1}\} \in G$. This means that we can form the bundle "$SLX = \sqrt{LX}$" when the structure group in LX can be lifted to $\mathrm{Spin}(n)$ without causing the cocycle condition to fail by $\pm\mathbf{1} \in \mathrm{Spin}(n)$ at the triple overlaps. The second SW class makes an appearance here through

$$(\delta g^{\mathrm{Spin}(n)})_{\alpha\beta\gamma} = (-1)^{w_2(LX)_{\alpha\beta\gamma}} \mathbf{1}, \tag{7}$$

where the $g^{\mathrm{Spin}(n)}_{\alpha\beta}$ are spin lifts of the transition functions in LX. Here the notation $w_2(LX)$ is a bit sloppy, since the SW classes are defined only for vector bundles. What we really mean is $w_2(V)$, where V is the vector bundle associated to LX via $V = LX \times_1 \mathbb{R}^n$, with 1 the fundamental (vector) representation. V is isomorphic to the tangent bundle TX,[6] and so

[3] Here $\lfloor n/2 \rfloor$ is the dimension of the spinor representation, which is what it is for the following reason. The spin group is generated by all elements in the Clifford algebra

$$\mathrm{Spin}(n) \ni \gamma_0^{j_0} \cdots \gamma_n^{j_n}, \qquad \sum_i j_i \in 2\mathbb{Z}. \tag{3}$$

There are 2^{n-1} possible products, and so the dimension of the spin group is 2^{n-1}. This means that it must be represented by matrices of dimension $d \geq 2^{(n-1)/2}$. When $n \in 2\mathbb{Z}$ we cannot have an equality. However when $n \in 2\mathbb{Z}$, the spinor representation is reducible, since we can project onto ± 1 eigenstates of $\bar{\gamma}$. In this case, the reducibility of the representation into two $d/2 \times d/2$ blocks means that $2(d/2)^2 = 2^{n-1}$, so that when d is even, we have $d^2 = 2^n$, and so $d = n/2$. Combining the odd and even case with the floor function, we get the stated result.

[4] if the representation of $\mathrm{Spin}(n)$ is real, then we can take the vector space to be real instead; we will ignore this possibility for simplicity

[5] In even dimensions, when the spinor representation is reducible, we can also form the bundles

$$S_\pm = SLX \times_{(1/2)_\pm} \Delta_n^\pm, \qquad \Delta_n^\pm = \mathbb{C}^{2^{\lfloor n/2 \rfloor - 1}}, \tag{6}$$

where $(1/2)_\pm$ is the representation of $\mathrm{Spin}(n)$ on the positive / negative chirality reducible component. Sections of these bundles are Weyl fermions.

[6] The isomorphism is given by

$$V = LX \times_1 \mathbb{R}^n \ni (u, v) = ((p, g), v) \mapsto (p, [R_1(g)](v)) \in TX, \tag{8}$$

$w_2(LX)$ is just another way of writing $w_2(TX)$. From the Cech 3-cochain one can construct a class in $H^2(X;\mathbb{Z}_2)$; more on this in the next section.

So roughly speaking, $SLX = \sqrt{LX}$. What about the associated vector bundles? We can use our knowledge of the representations of the spin group to conclude, sgain roughly speaking, that S is the "square root" of the associated bundle $V = LX \times_1 \mathbb{C}^n$ (with the 1 the vector representation; note to self: since our spinors are living in $\mathbb{C}$ I wrote $\mathbb{C}^n$, but should this be $\mathbb{R}^n$?), in the sense that

$$S \otimes S \supset V, \tag{9}$$

with the tensor product of associated bundles being performed by tensoring both the vector spaces and the representations. In the special case of $n = 2$ we have $\mathrm{Spin}(2) = U(1)$ and this is easy to understand, since we are just tensoring two line bundles together.[7]

The Chern class and w_2

The above discussion showed how $w_2(TX)$ appeared from a Cechian point of view. Later on we will need to consider $w_2(E)$ for other types of vector bundles E. In the case where E is a complex vector bundle (which will be relevant to us when discussing fermions coupled to gauge fields), $w_2(E)$ has a very simple relation with $c_1(E)$, which we now describe.

In the following, fix G to be some compact Lie group, containing $\mathbb{Z}_2$ in its center. Further let E be the complex vector bundle associated to a principal $G/\mathbb{Z}_2$ bundle over X, with a given representation of $G/\mathbb{Z}_2$, and with transition functions

$$g_{\alpha\beta} = e^{i2\pi\Lambda_{\alpha\beta}} \in GL(n;\mathbb{C}). \tag{10}$$

Since E is a legit $G/\mathbb{Z}_2$ associated bundle, we have

$$\delta\Lambda \in H^2(X;\mathbb{Z}), \tag{11}$$

where we have identified the Cech 3-cocycle $\delta\Lambda$ with a simplicial 2-cohomology class (more on this in a sec; $\delta\Lambda$ is not exact in $H^2(X;\mathbb{Z})$ since Λ isn't a $\mathbb{Z}$-valued cochain).

Taking the square root $E^{1/2}$ of E by passing to an associated G bundle means taking the square root of all of E's transition functions. Suppose first that $\delta\Lambda \in 2H^2(X;\mathbb{Z})$. If this is true, then the transition functions of $E^{1/2}$ will fail the cocycle condition by

$$(\delta g^{E^{1/2}})_{\alpha\beta\gamma} \in \mathbf{1}e^{2\pi i\mathbb{Z}}, \tag{12}$$

which is still acceptable. If however the $\delta\Lambda$ is just a class in $H^2(X;\mathbb{Z})$, then the cocycle condition can fail in $E^{1/2}$ by $\pm\mathbf{1}$ on each patch; this is not acceptable, and the bundle $E^{1/2}$ does not exist.

This means that the square-root-ibility of E is determined by how the transition functions fail the cocycle condition. But we know that if $(\delta\Lambda)_{\alpha\beta\gamma} = n_{\alpha\beta\gamma}$, then the first Chern class of E is just the class in $H^2(X;\mathbb{Z})$ which counts the $n_{\alpha\beta\gamma}$. Now from our above discussion, we see that in this case, $w_2(E)_{\alpha\beta\gamma}$ is precisely the class which counts $n_{\alpha\beta\gamma}$ mod 2. Thus $w_2(E)$ is the mod-2 reduction of c_1,[8] and so the nontriviality of $w_2(E) = c_1(E) \mod 2$ means an

where $u \in LX$, p is a basepoint in X, $g \in SO(n)$, and $v \in \mathbb{R}^n$. Note that as required, both (u,v) and $(hu, [R_1^{-1}(h)]v)$ get mapped to the same point in TX.

[7] In this case the spinor bundle S can be thought of as containing an action in the "charge 1/2 representation" of $U(1)$ on $\mathbb{R}^2$ (or $\mathbb{C}^2$?).

[8] this is only true for complex vector bundles!

obstruction to consistently defining a bundle with square root transition functions.

Relating this to trivializing the 2-skeleton

How does this cocycle-centric definition of w_2 relate to other definitions of it as a SW class? For example, we know that $w_2(TX) \neq 0$ means that there is an obstruction to a extending a trivialization of a $SO(n)$ principal bundle from the 1-skeleton of X to the 2-skeleton[9]: how is this connected to the failure of the square roots of the transition functions to be closed in Cech cohomology? Now one way (I'm not sure if this is a really general statement) to go between a skeleton and a Cechian patch-covering of a manifold is to associate patches to each node of the 0-skeleton, such that the patches are chosen to be the convex regions that extend slightly beyond the halfway point of each of the 1-cells emanating from the 0-cell they are centered on. Choosing the patches this way means that we can associate to each 1-cell a 2-fold overlap of patches; this is nice because we can thinking about moving between 0-cells on the 0-skeleton as moving between patches, applying a transition function when we move across each 1-cell. Anyway, this also means that each 2-cell is associated to a triple overlap of patches, and so on: in this construction, each k-cell in the k-skeleton is associated to a $(k+1)$-fold overlap of patches.

Most importantly for us, each 2-cell corresponds to a triple overlap of patches. Now, what prevents us from extending a trivialization of a principal $SO(n)$ bundle from the 1-skeleton into the 2-skeleton? Such an extension will not be possible at a given 2-cell if the framing winds by the nontrivial element in $\pi_1(SO(d))$ around the boundary of that 2-cell. But this exactly translates to the condition that the square root of the transition functions on the triple patch overlap at the center of that 2-cell fail the cocycle condition by $-\mathbf{1}$: the framing winds by an odd multiple of 2π around the 2-cell, which corresponds to $-\mathbf{1}$ in $\mathrm{Spin}(n)$. Thus the Cechian way of thinking about $w_2(TX)$ and the skeleton way of thinking about $w_2(TX)$ are the same: the nontriviality of either indicates that we won't be able to take the square root of our principal $SO(n)$ bundle. The same applies to other groups G and $G/\mathbb{Z}_2$, where the nontriviality of $\pi_1(G/\mathbb{Z}_2)$ prevents the trivialization of a G bundle to be extended into the 2-skeleton.

Fermions with a gauge field

Now let the fermions be coupled to a gauge field A, with gauge group G. In general, if we have a field coupled to a background field for $G \times H$, then we will have a principal $G \times H$ bundle, and the vector space used to construct the associated bundle of which the field is a section will transform in a representation of $G \times H$. What representation we choose is up to us, but it will always be expressible as a direct sum of $\otimes$'s between an irrep of G and one of

[9]Here by k-skeleton, we mean the k-th dimensional part of a cell complex.

H, since the irreps of $G \times H$ are constructed as the $\otimes$ of irreps of G with irreps of H.[10] An example where we have a principal $G \times H$ bundle but use a reducible representation for the action on the vector space is when we are considering fermions in e.g. four dimensions, and making use of the decomposition $\mathrm{Spin}(4) = SU(2) \times SU(2)$. Here we do not want to take the $\otimes$ of two spin 1/2 irreps; rather, we want to take the representation

$$(1/2)_L \otimes \mathbf{1}_R \oplus \mathbf{1}_L \otimes (1/2)_R, \tag{15}$$

which is reducible. In this case, the associated bundle we get is a direct sum of two associated $SU(2)$ bundles, rather than a tensor product:

$$S = (SLX \times_{(1/2)_+} \mathbb{C}^{2^2}) \oplus (SLX \times_{(1/2)_-} \mathbb{C}^{2^2}). \tag{16}$$

Anyway, back to fermions coupled to a gauge field for an internal symmetry. We start out with a principal $G \times SO(n)$ bundle. If $w_2(TX) = 0$ then we can lift the $SO(n)$ factor to $\mathrm{Spin}(n)$, and we get the spinor bundle which we will write as

$$S_G = P_{G\times \mathrm{Spin}(n)} \times_{R_G \otimes 1/2} \Delta_n^G, \qquad \Delta_n^G = \mathbb{C}^{(\dim R_G)2^{\lfloor n/2 \rfloor -1}}. \tag{17}$$

The spinors are sections of this bundle. We will also write this as

$$S_G = (P_G \times_{R_G} \Delta^G) \otimes (P_{\mathrm{Spin}(n)} \times_{1/2} \Delta_n), \qquad \Delta^G = \mathbb{C}^{\dim R_G}. \tag{18}$$

Spin_G structures

There is another option which we use to construct spinors, that can be employed even when $w_2(TX) \neq 0$ in cohomology. This option is available to us when $G = \widetilde{G}/\mathbb{Z}_2$ for some Lie group $\widetilde{G}$, such that $\mathrm{Spin}(n)$ and $\widetilde{G}$ share a common central $\mathbb{Z}_2$ factor in the way they act on the fermions: this is possible if they act on the fermions in a tensor product representation of the spinor representation and a representation of $\widetilde{G}$ which includes this $\mathbb{Z}_2$ factor. For example if $G = SO(3)$, we might take the fermions to transform in the $(1/2)_{\mathrm{Spin}(n)} \otimes (1/2)_{SU(2)}$ representation, since the fundamental of $SU(2)$ has a $-\mathbf{1}$ factor. In this case, the group that couples to the fermions is really $(\widetilde{G} \times \mathrm{Spin}(n))/\mathbb{Z}_2$, and the transition functions in the full bundle are blind to the quotiented $\mathbb{Z}_2$ factor.

[10] That is, every rep of $G \times H$ can be written as

$$(G \times H \to \mathrm{Aut}(V)) \ni R = \bigoplus_i \rho_i^g \otimes \rho_i^h, \tag{13}$$

where ρ_i^g, ρ_i^h are irreps of G and H, respectively. Furthermore, each factor in the direct sum is irreducible (as an irrep of $G \times H$). To prove this, we use Peter-Weyl: irreps of a compact group are the same as L^2 functions on that group (which exist because of the assumed compactness condition). A basis for these L^2 functions are precisely the characters. Now $L^2(G \times H) = L^2(G) \otimes L^2(H)$, which since the bases for the L^2 functions are provided by the characters, is the same thing as saying

$$\chi_{\rho^g}(g)\chi_{\rho^h}(h) = \chi_{\rho^g \otimes \rho^h}(g \times h), \tag{14}$$

which is true because $\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\mathrm{Tr}[B]$. Since the L^2 functions of the product group is the $\otimes$ of the individual L^2's, using Peter-Weyl proves the claim.

Anyway, while we might not be able to construct a bundle $SLX \times_{1/2} \Delta_n$ because the cocycle condition fails as $\delta g^{\mathrm{Spin}(n)} = -\mathbf{1}$ at some points, we still may be able to form the bundle

$$E = P_{(\widetilde{G}\times \mathrm{Spin}(n))/\mathbb{Z}_2} \times_{R_{\widetilde{G}}\otimes 1/2} \Delta_n^{\widetilde{G}}. \tag{19}$$

Here the bundle does *not* split as a tensor product of vector bundles:

$$E \neq (P_{\widetilde{G}} \times_{R_{\widetilde{G}}} \Delta^{\widetilde{G}}) \otimes (SLX \times_{1/2} \Delta_n), \tag{20}$$

since the latter factor does not exist. If E is to be well-defined, since the latter factor in (20) is not well-defined, the principal bundle $P_{\widetilde{G}}$ must also not be well-defined, in a compensating way: the transition functions in G must fail to lift to $\widetilde{G}$-valued transition functions in a way that cancels the ill-defined-ness of the latter tensor factor.

Now the transition functions of E are given by the matrices

$$g^E_{\alpha\beta} = R_{\widetilde{G}}(g^{\widetilde{G}}_{\alpha\beta}) \otimes R_{1/2}(g^{\mathrm{Spin}(n)}_{\alpha\beta}). \tag{21}$$

This means that if we choose our bundles such that cocycle conditions in each of the factors in (20) fail in the same way, the transition functions above will satisfy the cocycle condition. From what we saw earlier, the condition for the transition functions in the SLX and $P_{\widetilde{G}}$ factors to not be closed in Cech cohomology in the same way is given by

$$w_2(TX) = w_2(E_G), \tag{22}$$

where E_G is the vector bundle associated to the principal bundle P_G in the representation R_G.[11] Such an E is called (idk if this is standard?) a Spin_G bundle. Remember that for this to work, we also need to impose the condition that the representation $R_G \otimes 1/2$ that the fermions transform under is such that the representation R_G includes the $\mathbb{Z}_2$ factor corresponding to $-\mathbf{1}$ in G (i.e., if $G = SU(2)$, we must choose a half-odd-integer spin representation).

[11] Since E_G is a complex vector bundle, let's be pedantic and elaborate on exactly what we mean by this. Since we have been working with complex spinors, E_G is formed by

$$E_G = P_G \times_{R_G} \mathbb{C}^{\dim R_G}. \tag{23}$$

Then we really mean

$$w_2(E_G) = w_2([E_G]_{\mathbb{R}}) = w_2([P_G \times_{R_G} \mathbb{C}^{\dim R_G}]_{\mathbb{R}}), \tag{24}$$

where the subscript denotes the realification, which is accomplished by taking $\mathbb{C}^{\dim R_G} \to \mathbb{R}^{2\dim R_G}$ and realifying the G action by

$$R_G \to [R_G]_{\mathbb{R}} : G \to GL(2\dim(R_G); \mathbb{R}) \tag{25}$$

via the inclusion $GL(n;\mathbb{C}) \to GL(2n;\mathbb{R})$.

Now as we have seen, the SW classes of the realification of a complex vector bundle E are the mod-2 reduction of that vector bundle's Chern classes:

$$c_j(E) \xrightarrow{reduction\, mod\, 2} w_{2j}(E_{\mathbb{R}}), \tag{26}$$

and so we can equivalently write

$$w_2(E_G) = c_1(E_G) \mod 2. \tag{27}$$

The simplest case is when $G = U(1)$. Suppose that $w_2(TX) \neq 0$, so that SLX does not exist. Suppose also that the line bundle $L = P_{U(1)}$ is such that

$$w_2(L) = [c_2(L)]_2 = w_2(TX). \tag{28}$$

Then a fermion field of charge q will be a section of the spinc bundle

$$E = P_{[U(1)\times \mathrm{Spin}(n)]/\mathbb{Z}_2} \times_{q/2\otimes 1/2} (U(1) \otimes \Delta_n) \text{ “} = \text{” } (\sqrt{L} \times_{q/2} U(1)) \otimes (SLX \times_{1/2} \Delta_n), \tag{29}$$

where strictly speaking the second way of writing things is schematic, since neither tensor factor makes sense. Here in order for the representation q of $U(1)$ to include the central $\mathbb{Z}_2$, we also need to take $q \in (2\mathbb{Z}+1)$, so that the element -1 in $U(1)$ is represented nontrivially in the transition functions.

When $G = U(1)$, the classification of such Spin_G structures works just in the same way as for Spin structures, but up a dimension: $\mathrm{Spin}_{U(1)}$ structures, alias $\mathrm{Spin}_{\mathbb{C}}$ structures, are obstructed by $w_3(TX)$, and they are in (non-canonical) bijection with elements in $H^2(X;\mathbb{Z})$, which correspond to different large instantons that can be inserted into the $U(1)$ factor of $\mathrm{Spin}(n) \times U(1)$.

References

[1] S. Avis and C. Isham. Generalized spin structures on four dimensional space-times. *Communications in Mathematical Physics*, 72(2):103–118, 1980.