Things I learned while studying for the CMT qual

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December 1, 2019

Band theory

Fermi liquids and surfaces

Stat mech and phase transitions

Transport

Experimental things

Topological stuff and spts

Spin chains and one-dimensional physics

Soft condensed matter physics

Superfluidity, superconductivity, and condensates

Two dimensional lattice models

Diagrammatics and Greens functions

Magnetism

Miscellaneous

1   *Dipole moments and ferromagnetism*

A friend of mine explained to me last night that she thought that dipolar interactions between electronic magnetic moments were responsible for ferromagnetism. In what follows we'll provide an estimate that shows why this is unreasonable.

Solution:

First, recall how we get the field for a magnetic dipole with dipole moment $\boldsymbol{\mu}$: we imagine two magnetic charges separated by a small distance $\mathbf{d}$, with $\mathbf{d}$ pointing from the negative charge (at the origin) to the positive charge (so that $\boldsymbol{\mu} \propto \mathbf{d}$). Then we determine the field with

$$-\frac{\mathbf{r}}{r^3} + \frac{\mathbf{r}-\mathbf{d}}{|\mathbf{r}-\mathbf{d}|^3} \approx -\frac{\mathbf{r}}{r^3} + \frac{\mathbf{r}-\mathbf{d}}{r^3}(1 + 3\mathbf{r}\cdot\mathbf{d}/r^2) = -\frac{1}{r^3}(\mathbf{d} - 3(\hat{\mathbf{r}}\cdot\mathbf{d})\hat{\mathbf{r}}). \tag{1}$$

This means that the Hamiltonian for two magnetic dipoles $\boldsymbol{\mu}_i$ is (SI units, sorry)

$$H = -\frac{\mu_0}{4\pi r^3}(3(\hat{\mathbf{r}}\cdot\boldsymbol{\mu}_1)(\hat{\mathbf{r}}\cdot\boldsymbol{\mu}_2) - \boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2), \tag{2}$$

with $\mathbf{r}$ the vector pointing between the two dipoles. Note that this favors antiferromagnetism; this can be checked just by drawing the field lines. So regardless, dipolar interactions definitely can't be responsible for ferromagnetism. Considering two electrons with magnetic moments pointing in the directions $\hat{\boldsymbol{\ell}}_i$,

$$H = -\frac{\mu_0}{4\pi r^3}g_J^2\left(\frac{e\hbar}{2m_e}\right)^2(3\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\ell}}_1\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\ell}}_2 - \hat{\boldsymbol{\ell}}_1\cdot\hat{\boldsymbol{\ell}}_2). \tag{3}$$

Let's check units: in SI, μ_0 is a microhenry per meter, i.e. $\mathrm{A}^{-2}\,\mathrm{kg}\,\mathrm{m}\,\mathrm{s}^{-2}$. So then the expression on the RHS has units of $C^{-2}\,\mathrm{s}^2\,\mathrm{kg}^2\,\mathrm{m}\,\mathrm{s}^{-2}\,\mathrm{m}^{-3}\,C^2\,\mathrm{J}^2\,\mathrm{s}^2\,\mathrm{kg}^{-2}$, which reduces to just Joules, as required.

We can now plug in numbers for all of the stuff above: for a separation of $r = 2\mathring{A}$, we get a dipolar interaction energy of 10^{-4}eV, ish. Now a Kelvin is about $10^{-4}\,$eV, and so the dipolar interaction could be important below about 1K. Since the dipolar interaction favors antiferromagnetism, in order to determine its possible importance we should look at the ordering temperature (Neel temperature) for some antiferromagnets. According to Wikipedia typical T_Ns are of the order of a few hundred K. So the dipolar energy is lower than the Neel temperature by a factor of about

$$T_{dip}/T_N \sim 10^{-2}\text{--}10^{-3}, \tag{4}$$

indicating that dipolar interactions can't be playing a role in setting the AF ordering (for materials with lower T_N, we can't make definitive statements). Note however that dipolar interactions can play a significant role when you have a bunch of aligned moments in a ferromagnet—the AF dipole-dipole interaction is the driving force behind the creation of magnetic domains at low temperatures.

Anyway, evidently magnetism must be caused by electronic exchange. Recall the heuristic for this: if the electron orbitals are diffuse so that their overlap is large, then we get a ferromagnetic exchange interaction in order for the potential energy of the electrons to be minimized. This applies to multiple orbitals on the same site, or to large orbitals on different sites that hybridize significantly, so that the Coulomb interaction is largely unscreened. A good place to read about this is in "Interacting Electrons and Quantum Magnetism"

by Auerbach which I will follow for a sec. We can make the heuristic picture precise by considering an interacting Hamiltonian

$$H = \int_{x,y} \psi_\alpha^*(x)\psi_\beta^*(y)K(x-y)\psi_\beta(y)\psi_\alpha(x), \tag{5}$$

which conserves spin locally ($K(x-y)$ is some interaction kernel, e.g. $e^2/|x-y|$). Suppose the field operators are decomposed into two single-particle operators $\psi_\alpha(x) = \sum_{a=1,2} \lambda(x)_a c_{a\alpha}$, with the two λ_i orthonormal (think of e.g. two orbitals in a single atom). We will assume that the two orbitals carry distinct quantum numbers under some symmetry preserved by the Hamiltonian (e.g. they could have different angular momenta), so that in the expansion for H in terms of the constituents of the field operators, there will no terms of the form $c_{a\alpha}^\dagger c_{a\beta}^\dagger c_{a\alpha} c_{b\beta}$ with $a \neq b$. Then the part of the interaction which determines the spin exchange interaction is

$$H \supset H_J = \sum_{a \neq b, \alpha\beta} \int_{x,y} \lambda_a^*(x)\lambda_b^*(y)K(x-y)\lambda_a(y)\lambda_b(x)\, c_{a\alpha}^\dagger c_{b\beta}^\dagger c_{a\beta} c_{b\alpha} \equiv J c_{a\alpha}^\dagger c_{b\beta}^\dagger c_{a\beta} c_{b\alpha}. \tag{6}$$

The reason that this determines the spin exchange is because

$$\frac{1}{2}\sum_{\mu=0}^{3}(c_1^\dagger \sigma_\mu c_1)(c_2^\dagger \sigma_\mu c_2) = \sum_{\alpha\beta} c_{1\alpha}^\dagger c_{1\beta} c_{2\beta}^\dagger c_{2\alpha} \implies H_J = -\frac{J}{2}(\rho_1\rho_2 + 4S_1 \cdot S_2). \tag{7}$$

where the density operators are of course $\rho_a = n_{a\uparrow} + n_{a\downarrow}$. Therefore we get a term in H like $-JS_1 \cdot S_2$, and so positive (negative) J is (anti)ferromagnetic. If the interaction is a completely screened delta function $K = \delta(x-y)$, then we have $J = \int_x |\lambda_1(x)\lambda_2(x)|^2$, which is manifestly positive and hence the interaction is ferromagnetic. Likewise, suppose the interaction kernel is the unscreened $K(r) \sim e^2/r$. As an operator $(\hat{K}\psi)(x) = \int_{x,y} K(x-y)\psi(y)$, it is positive-definite: the eigenfunctions are of course $\psi(x) = e^{iqx}$, with eigenvalues $4\pi e^2/q^2 > 0$. Hence the parameter J, which is $\langle \phi_a^* \phi_b | \hat{K} | \phi_a^* \phi_b \rangle$, must be positive, leading to a ferromagnetic exchange, as expected. This is the pedantic reason why Hund's rules are a thing.

How do we get an antiferromagnetic interaction, then?

On the other hand, if the electron orbitals have a comparatively small overlap, then an antiferromagnetic exchange interaction is induced, which allows the electrons to lower their energy by hopping.

2 Chemistry of strong interactions

Where on the periodic table do we expect to find compounds with strong electronic interactions? With weak ones?

Solution:

3 Interaction strength in electron gas

Today we're doing some problems from chapter 7 of Coleman's many-body theory book. We consider a perturbative treatment of the electron gas in three dimensions. To first order, there are two interaction contributions to the ground state energy: one comes from a direct interaction (two bubbles connected by an interaction line), and the other comes from the exchange interaction (a bubble with an interaction line dividing it in two). The former contribution only selects out the $\mathbf{q} = 0$ component of the interaction, and hence is a uniform charging energy that is canceled by the ionic background (in field theory language, all tadpoles are canceled by a simple mass counterterm). Therefore the ground state energy density is the kinetic part and the exchange interaction: remembering to account for spin degeneracy,

$$E/V = 2 \int_{\mathbf{k}} \frac{k^2}{2m} f_{\mathbf{k}} - \int_{\mathbf{k},\mathbf{k}'} \frac{e^2}{\epsilon_0 |\mathbf{k} - \mathbf{k}'|^2} f_{\mathbf{k}} f_{\mathbf{k}'} \tag{8}$$

with the minus sign coming from the fermion bubble or from the antisymmetrization in the Slater determinants used as the ground state wavefunctions, whichever you prefer. In what follows we will calculate this and compare the relative contributions of the kinetic and potential terms.

Solution:

The kinetic term contributes

$$2 \int_{\mathbf{k}} \frac{k^2}{2m} f_{\mathbf{k}} = \frac{3}{5} \left[\frac{k_F^5}{6\pi^2 m} \right] = \frac{3}{5} \left[\frac{k_F^3}{3\pi^2} \epsilon_F \right] = \frac{3\rho\epsilon_F}{5}, \tag{9}$$

since the density is

$$\rho = \frac{4}{3}\pi k_F^3 \frac{2}{(2\pi)^3}. \tag{10}$$

Writing the momenta as $\mathbf{k} = \mathbf{x}k_F$, the interaction term I is

$$I = \frac{8\pi^2 e^2 k_F^4}{\epsilon_0 (2\pi)^6} \int dx\, dy\, d\cos\theta \frac{x^2 y^2}{x^2 + y^2 - 2xy\cos\theta}, \tag{11}$$

with the x, y integrals going from 0 to 1 and the cos integral going from -1 to 1. A little algebra shows that the prefactor is $3e^2 k_F \rho/(8\pi^2 \epsilon_0)$, while the integral is

$$\int dx\, dy\, d\cos\theta \frac{x^2 y^2}{x^2 + y^2 - 2xy\cos\theta} = \int dx\, dy\, (-xy/2) \ln \left| \frac{(x^2+y^2)/2xy - 1}{(x^2+y^2)/2xy + 1} \right|$$
$$= \frac{1}{2} \int dx\, dy\, xy \ln \left| \frac{x+y}{x-y} \right| = \frac{1}{4}, \tag{12}$$

where the integral was done with Mathematica and gives $1/2$.

Adding up the two contributions, we see that the energy per particle is (factor of two compared to Coleman?!)

$$\frac{E}{\rho V} = \frac{3}{5}\epsilon_F - \frac{3e^2 k_F}{32\pi^2 \epsilon_0}. \tag{13}$$

5

We define the average separation of electrons in a Fermi gas by

$$R_e = \left[\frac{3}{4\pi\rho} \right]^{1/3}. \tag{14}$$

We further take $r_s = R_e/a_B$ with $a_B = 4\pi\epsilon_0/(me^2)$ as a dimensionless measure of the electron density. Some easy algebra shows that $k_F^{-1} = \beta r_s a_B$, with $\beta = (4/9\pi)^{1/3}$. Recalling the Rydberg energy $R_Y = 1/(2ma_B^2) = 13.6\text{eV}$, we can then re-write the parameters in the expression for the energy density as

$$\frac{E}{\rho V} = 3R_Y \left[\frac{1}{5\beta^2 r_s^2} - \frac{1}{2\pi\beta r_s} \right], \tag{15}$$

which reveals that βr_s can be taken as a expansion parameter in the expression for the ground-state energy of the interacting electron gas (note that as long as r_s is sufficiently small, the above expression is positive). The point of this is that the interactions are *weaker when r_s is smaller*, meaning that the *high-density* limit is the weakly-interacting limit—perhaps contrary to what might naively expect, with higher density meaning that the electrons are closer together and hence have higher Coulomb energies. This calculation shows however that the increasing kinetic energy of the electrons mandated by the increasing of k_F with density more than compensates for the growing Coulomb interactions, with kinetic energy dominating at higher densities.

4 $\sigma(\omega)$ in metals and SCs

What is the ω dependence of $\sigma(\omega)$ in the Drude model? What is it in a SC? What is weight of the delta function in limit of large SC gap, and what physical quantities depend on the weight of the delta function?

Solution:

First let's recall how things work in metals under the purview of the Drude model. The conductivity just comes from the equation of motion of free electrons in an electric field, with an appropriate dissipative term $-\mathbf{k}/\tau$ to account for the scattering. If we want to be pedantic, we can use the Boltzmann equation with the collisional term $\partial_t g_{\mathbf{k}}|_{coll} = (g_{\mathbf{k}}^0 - g_{\mathbf{k}})/\tau$ with $g_{\mathbf{k}}^0$ the equilibrium distribution: the time derivative of the expectation value of the total momentum $\mathbf{K}$ is

$$\partial_t \mathbf{K} = \int_{\mathbf{k}} \partial_t g_{\mathbf{k}} \mathbf{k} = \int_{\mathbf{k}} (e\mathbf{E}(\partial_{\mathbf{k}} g_{\mathbf{k}})\mathbf{k} + \tau^{-1}(g_{\mathbf{k}}^0 - g_{\mathbf{k}})\mathbf{k}) = -e\mathbf{E} - \frac{\mathbf{K}}{\tau}, \tag{16}$$

where we used that $\mathbf{K} = 0$ in the equilibrium distribution. The point of writing the equation of motion like this is that it stresses that $\mathbf{K}$ isn't supposed to be thought of as the momentum of a particular electron, but rather a "hydrodynamic" momentum label attached to the whole

distribution of electrons as a whole. This then gives (I guess we are using conventions where $e = |e|$)

$$(\partial_t + 1/\tau)\mathbf{J} = \frac{e^2 n}{m}\mathbf{E} \implies \sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \qquad \sigma_0 \equiv ne^2\tau/m, \tag{17}$$

with n the number density. $\omega = 0$ correctly reproduces the DC result, while at large ω we get the τ-independent $\omega(w \to \infty) = ine^2/(m\omega)$. The large ω limit is generally the reliable one in the relaxion time approximation (where $\partial_t g_{\mathbf{k}}|_{coll}$ has the form given above, with the scattering matrix elements $W_{\mathbf{k}\to\mathbf{k}'}$ independent of the distribution function). This is because if the timescale ω^{-1} of the background fields is much shorter than the timescale τ it takes for electrons to scatter, then scattering events won't be super important in the response, and so our crude treatment of them won't be such a disaster. Also note that $\sigma_\mathbb{R}$ and $\sigma_\mathbb{I}$ each separately have ω integrals that are independent of τ_s, in accordance with the sum rules.

Anyway, this form of σ lets us find the dielectric properties of the medium. We start with the time derivative maxwells equations:

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \qquad \mu_0^{-1}\nabla \times \mathbf{B} = \epsilon_0 \partial_t \mathbf{E} + \mathbf{J}. \tag{18}$$

Taking $\nabla\times$ of the latter to get the wave equation and using the conductivity gives the dispersion (the same for both electric and magnetic waves)

$$k^2 = \omega^2/c^2 + i\mu_0\omega\sigma(\omega). \tag{19}$$

We can rewrite this in terms of the dielectric function as

$$k^2 = \omega^2\mu_0\epsilon(\omega), \qquad \epsilon(\omega) = \epsilon_0 + \frac{i\sigma(\omega)}{\omega}. \tag{20}$$

In the large frequency regime, which is where we said the relaxion time approximation is usually legit, the point $\epsilon = 0$ is reached when $\omega = \omega_p$ with the plasma frequency [1]

$$\omega_p \equiv \sqrt{\frac{e^2 n}{m\epsilon_0}}. \tag{23}$$

ω_p is the roughly the point at which the metal switches from being shiny to being transparent (the criterion for transparency is that $\epsilon_\mathbb{R} > 0$, while for shinyness it is $\epsilon_\mathbb{R} < 0$).

Now we come to the case of a superconductor. A first approximate model is to treat the superconductor in a two-fluid scheme, with the lifetime τ_s of the superfluid component going

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[1] Recall that the plasma frequency can also be found in a more pedestrian way by considering the jiggling of a coherent block of electrons on top of a positively charged ionic background. To do this, use that the electric field a distance d away from an infinite charged slab of charge density ρ is $2\pi/\epsilon$ (indep. of d).

Also recall that since the plasma frequency is an instability of the motions of the charges in the metal, it can also be found by examining the charge distribution more directly, by Fourier transforming the continuity equation:

$$d^\dagger j = 0 \implies i\omega\rho(\omega) = \nabla \cdot \mathbf{J}. \tag{21}$$

Taking the divergence of both sides,

$$i\omega\epsilon_0 = \frac{\sigma_0}{1 - i\omega\tau}, \tag{22}$$

which at large $\omega\tau$ tells us that $\omega = \omega_p$.

to infinity. At low frequencies $\omega < 2\Delta$ (and at $T = 0$), the SCing and normal contributions to the conductivity are respectively

$$\sigma_s(\omega) = \frac{ne^2\tau_s/m}{1 + i\omega\tau_s}, \qquad \sigma_n(\omega) \approx ne^2\tau_n/m, \tag{24}$$

where we assumed that $\omega\tau_n \ll 1$. As $\tau_s \to \infty$, and writing $\sigma_s = \sigma_s^{\mathbb{R}} + i\sigma_s^{\mathbb{I}}$, we find

$$\sigma_s^{\mathbb{R}}(\tau_s \to \infty) = \frac{\pi ne^2}{m}\delta(\omega), \tag{25}$$

which is checked by doing the integral and getting the arctan. This gives what we expect at low frequencies, namely a δ function concentrated on the DC response (plus a broader dissipative part from the normal fluid component). The strength of this pole is related to various quantities of interest. For example, consider the penetration depth. Recall that this is obtained from Maxwell's equations in the static limit and the London equation $\nabla \times \mathbf{J} = -(ne^2/m)\mathbf{B} \implies \nabla^2\mathbf{B} = (ne^2\mu_0/m)\mathbf{B} \implies \lambda_L = \sqrt{m/(ne^2\mu_0)}$, so that the weight of the $\delta(\omega)$ DC response pole is $\propto \lambda_L^{-2}$.

The imaginary part of the conductivity is

$$\sigma_s^{\mathbb{I}}(\tau_s \to \infty) = -\frac{ne^2}{m\omega}, \tag{26}$$

which is in accordance with the sum rule $\sigma^{\mathbb{I}} = -\omega/\pi P \int d\omega' \, \sigma^{\mathbb{R}}(\omega')/(\omega^2 - \omega'^2)$.

When $\omega \geq 2\Delta$ we expect the $\mathbb{R}$ part of the conductivity to be nonzero, since $\sigma^{\mathbb{R}}$ is related to dissipation in the SC in response to external fields, which becomes possible when the electric fields are strong enough to excite quasiparticles across the gap. The rough behavior of $\sigma_s^{\mathbb{R}}$ in this domain can be estimated just by knowledge of the density of states $\rho(\omega)$, which goes as $N(0)\omega/\sqrt{\omega^2 - \Delta^2}$ when $\omega \geq 2\Delta$. To eliminate dependence on non-universal parameters one usually divides $\sigma_s^{\mathbb{R}}(\omega)$ by $\sigma_n^{\mathbb{R}}(\omega)$, with the later the conductivity one calculates from a normal metal.[2] This gives (letting $\Delta \in \mathbb{R}$ wolog and working in "intrinsic semiconductor conventions" where the zero of energy is right in the middle of the gap)

$$\begin{aligned}
\frac{\sigma_s^{\mathbb{R}}(\omega > 2\Delta)}{\sigma_n^{\mathbb{R}}(\omega > 2\Delta)} &= \frac{1}{\omega}\int_{\mathbb{R}} dE \, \frac{|E(E+\omega) + \Delta^2|}{\sqrt{E^2 - \Delta^2}\sqrt{(E+\omega)^2 - \Delta^2}}\theta(-E-\Delta)\theta(E+\omega-\Delta) \\
&= \frac{1}{\omega}\int_{\Delta-\omega}^{-\Delta} dE \, \frac{|E(E+\omega) + \Delta^2|}{\sqrt{E^2 - \Delta^2}\sqrt{(E+\omega)^2 - \Delta^2}}.
\end{aligned} \tag{28}$$

Here the $+\Delta^2$ in the numerator is a little subtle, and comes from coherence factors stemming from the nature of the BCS wavefunction (basically, matrix elements connecting two qp states

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[2]Following the same steps, this is just

$$\sigma_n^{\mathbb{R}} \propto \int_{\mathbb{R}} dE \, (f(E) - f(E+\omega))N(0)^2 = \omega \tag{27}$$

at $T = 0$.

are added coherently with their particle-hole conjugates; see Tinkham's book for the details). When we plot this ratio as a function of ω from $\omega = 2\Delta$ up, we see that $\sigma_s^{\mathbb{R}}(2\Delta) = 0$ but that $\partial_\omega \sigma_s^{\mathbb{R}}|_{2\Delta} > 0$, and that $\sigma_s^{\mathbb{R}} \to \sigma_n^{\mathbb{R}}$ as $\omega \to \infty$, as expected:

$$\tag{29}$$
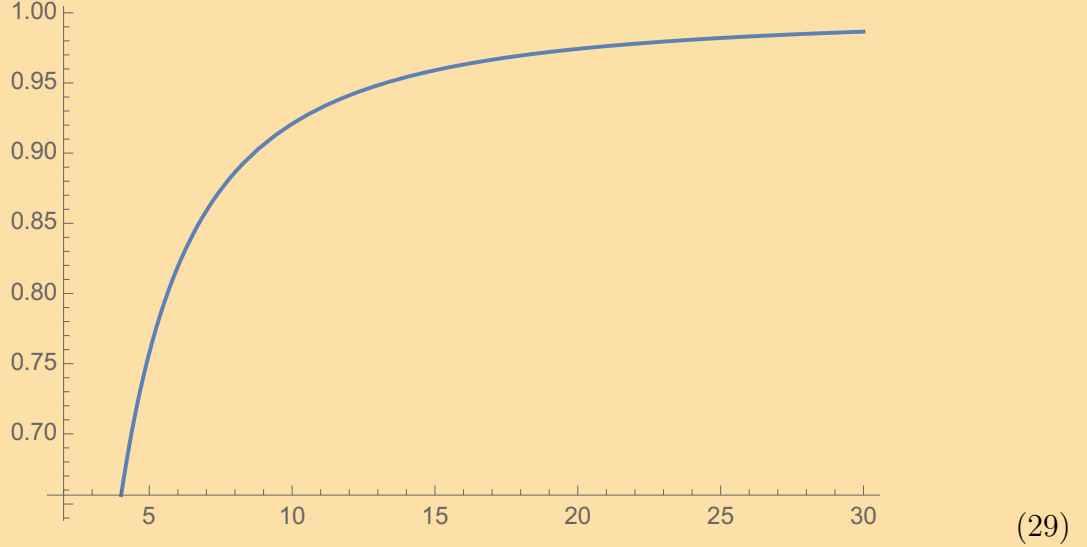
Miscellaneous comment: how would these results be affected by the presence of disorder? In a superconductor, non-magnetic impurities shouldn't really change anything: the coherence length $\xi_0 \sim \hbar v_F/kT_c$ (the UV cutoff for GL theory) is typically big enough that the effects of the impurities average out on scales $\gtrsim$ than ξ_0, which doesn't affect the SCing DOS or charge transport significantly. Magnetic impurities are a different story though, since they can break Cooper pairs, and so we expect some finite concentration of them to destroy SCvity all together.

5 DOS in a SC

What's the DOS $\rho(\omega)$ in a BCS SC?

Solution:

We just want the functional form of ρ, and so in the following will be very fast and loose with numerical factors. We will get the DOS through the imaginary part of the retarded Greens function, which is tractable since we'll be working within the purview of MFT. We are interested in the DOS at a given frequency, and so we need to sum $\rho(\omega, q)$ over all momenta q (we want the $\mathbb{R}$-space local DOS at finite ω). Therefore we will start by computing the time-ordered Greens function

$$G_T(\omega, 0) = \int_q \text{Tr} \begin{pmatrix} \omega + \xi & \Delta \\ \Delta^* & \omega - \xi \end{pmatrix}^{-1}, \tag{30}$$

where we are in $\mathbb{R}$ time and as usual ξ is the energy measured wrt the chemical potential. Note how the signs on the ξs are swapped due to the choice of the Nambu spinors, but the

signs of the ωs are the same since the action is $\int \Psi^\dagger (i\partial_t - H)\Psi$ when we do the path integral over fields which are eigenstates of H.

Inverting and taking the trace,

$$G_T(\omega, 0) = \int_q \frac{\omega}{-\omega^2 + \xi^2 + |\Delta|^2} \tag{31}$$

where as warned we are not keeping track of overall prefactors. Changing to an integral over ξ and extending the integral to one over $\mathbb{R}$ as usual, we have

$$G_T(\omega, 0) \approx N(0) \int_{\xi \in \mathbb{R}} \frac{\omega}{-\omega^2 + \xi^2 + |\Delta|^2} \approx \frac{N(0)\omega}{\sqrt{-\omega^2 + \xi^2}}, \tag{32}$$

since the integral is an arctan. To get the causal response $G_+(\omega, 0)$, we need to evaluate this guy at $\omega + i\eta$ with $\eta \to 0^+$, and then we need to take the imaginary part of G_+ to get the DOS. The $i\eta$ in the numerator doesn't make a difference in the limit $\eta \to 0$ (easy to check), and neither does the $i\eta$ in the denominator. Therefore we just pick up the imaginary part of $(-\omega^2 + |\Delta|^2)^{-1/2}$, which gives

$$\rho(\omega) \propto \begin{cases} 0 & \omega < |\Delta| \\ \frac{N(0)\omega}{\sqrt{\omega^2 - |\Delta|^2}} & \omega > |\Delta| \end{cases} \tag{33}$$

For $\Delta = 1$ this looks like

$$\tag{34}$$
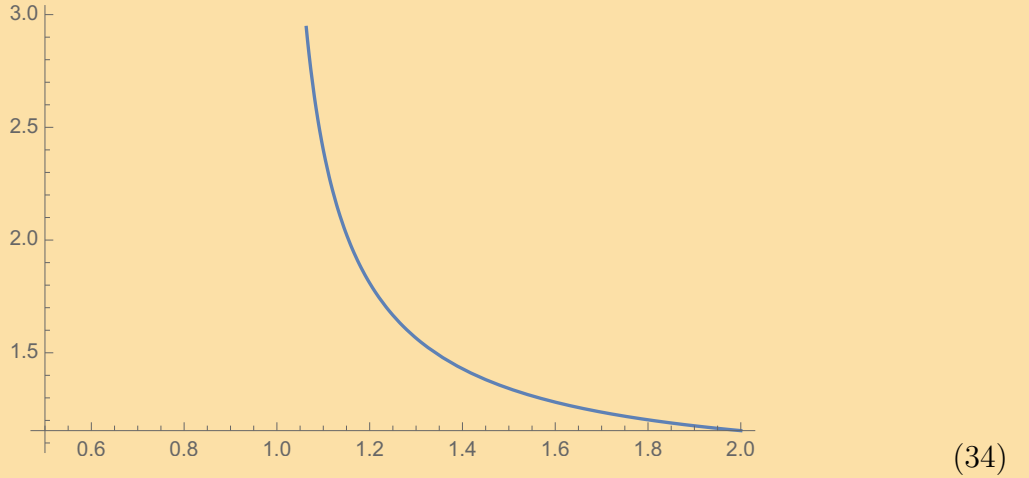
The spectral weight gets pushed up to the gap edge, and then falls back down to the free fermion value of $N(0)$ at higher frequencies.

6 SC specific heat and $\Delta(T)$

Today is simple: calculating the specific heat in an ordinary SC. Along the way we'll recall how to get $\Delta(T)$ near T_c. We're mostly just interested in the T dependence, and so largely we will neglect keeping track of numerical factors.

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Solution:

The usual way to proceed is via BCS theory. We start from

$$C = \partial_{\ln T} S = -\partial_{\ln \beta} S = \beta \partial_\beta \int_{\mathbf{k}} \left((1 - f_{\mathbf{k}}) \ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}} \ln f_{\mathbf{k}} \right). \tag{35}$$

Remember that we have to include the holes here because they specify information distinct from the particles, e.g. just knowing that you have a particle a certain distance above the FS doesn't tell you how much energy the state you're looking at has; you need to know where in the FSea the hole is located as well. Taking the derivative,

$$C = -\beta \int_{\mathbf{k}} \partial_\beta f_{\mathbf{k}} \ln \frac{1 - f_{\mathbf{k}}}{f_{\mathbf{k}}} = -\beta^2 \int_{\mathbf{k}} (\partial_\beta f_{\mathbf{k}}) E_{\mathbf{k}} \tag{36}$$

In BCS theory, $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$, with $\Delta \in \mathbb{R}$ wolog. The unusual part about this energy is that it itself is temperature dependent. Therefore the specific heat gets two contributions: one from how the qps redistribute among the energy levels as T changes, and one from how the energy levels themselves change with T:

$$C = \beta^2 \int_{\mathbf{k}} \frac{1}{\cosh^2(\beta E_{\mathbf{k}}/2)} (E_{\mathbf{k}}^2 + \beta \partial_\beta \Delta) \approx \beta^2 \nu \int_\xi d\xi \frac{1}{\cosh^2[\beta \sqrt{\xi^2 + \Delta^2}/2]} (\xi^2 + \Delta^2 + \Delta \partial_{\ln \beta} \Delta), \tag{37}$$

with ν as usual the DOS at the FS.

First consider the $T > T_c$ regime, where $\Delta = 0$. Then we get

$$C(T > T_c) = \beta^2 \nu \int_\xi \frac{d\xi \, \xi^2}{\cosh^2[\beta \xi/2]} = \beta^{-1} \nu \int_x \frac{dx \, x^2}{\cosh^2[x/2]} \propto \nu T, \tag{38}$$

which is the usual Fermi liquid result. Now consider what happens when $T \ll T_c$. In this regime Δ is approximately constant (to be substantiated below), and we have

$$C(T \ll T_c) \approx \beta^2 \nu \int_\xi d\xi \frac{\xi^2 + \Delta^2}{\cosh^2[\beta \sqrt{\xi^2 + \Delta^2}/2]}. \tag{39}$$

Since there is now no region of ξ in which the argument of the $\cosh^2$ is small, the integral will be exponentially small, and will go as

$$C(T \ll T_c) \propto P(T) e^{-\Delta/T}, \tag{40}$$

where $P(T)$ is some polynomial in T^3—the important part is the exponential decay, as expected of a gapped system.

Finally we examine the regime $T \lesssim T_c$ and look for the discontinuity in C across T_c. Here Δ^2 is small, but $\partial_{\ln \beta} \Delta$ is not. Indeed, we will show in a sec that

$$\Delta(T) \sim \sqrt{(T_c - T)T_c} = T_c \sqrt{-t} \tag{41}$$

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[3]I guess it goes like $a\beta + b\beta^{-1}$.

for $T \lesssim T_c$. Therefore $\partial_{\ln \beta} \Delta = -T \partial_T \Delta \implies (\Delta \partial_{\ln \beta} \Delta)|_{T_c} \sim T_c^2$. So right below T_c, we have

$$C(T = T_c - \epsilon) \approx \frac{\nu}{T_c^2} \int_\xi d\xi \frac{\xi^2 + T_c^2}{\cosh^2[\beta\xi/2]} \sim \nu T_c \int dx \frac{x^2 + 1}{\cosh^2[x/2]}. \tag{42}$$

Therefore $C_{SC}(T_c - \epsilon)/C_{\text{normal metal}}(T_c)$ is some $O(1)$ number independent of T_c that can be computed by actually doing the integrals over x. So, the overall picture is that as we decrease T starting from above T_c, we have a linear decrease, then a discontinuous (but finite) jump at T_c, and then an exponential decay down to zero below T_c.

Our claim about the jump in C relied on knowing that $\Delta(T) \sim T_c\sqrt{-t}$ near T_c. This is established by solving the gap equation

$$\frac{1}{g\nu} = \int_{-\Lambda}^{\Lambda} d\xi \sum_\omega \frac{1}{(i\omega - E(\xi))(-i\omega - E(\xi))}, \tag{43}$$

where Λ is the UV cutoff, g the interaction strength, and the propagators here come from the 1-loop bubble encountered when getting the quadratic part of the action for Δ after doing mean field theory (the opposite signs on the ωs is because it's a BCS-channel bubble). The sum is over $\omega \in 2\pi T \mathbb{Z}$, and we treat this in the usual way by introducing $z = i\omega$, multiplying by $f(z)$, and integrating z along a contour encircling the $i\mathbb{R}$ axis (with f the Fermi function). There are two poles not on the $i\mathbb{R}$ axis, viz at $z = \pm E(\xi)$, and so we get

$$\frac{1}{g\nu} = \int_{-\Lambda}^{\Lambda} d\xi \frac{f(E(\xi)) - f(E(\xi))}{2E(\xi)} = \int_0^{\Lambda} d\xi \frac{\tanh(E(\xi)\beta/2)}{E(\xi)}. \tag{44}$$

The integral here is actually not that straightforward. Getting the critical temperature is easy, at least: setting $\Delta = 0$, we have

$$\frac{1}{g\nu} = \int_0^{\Lambda/2T_c} dx \frac{\tanh(x)}{x} \approx \ln(\Lambda/2T_c) \implies T_c \sim \Lambda e^{-1/g\nu}, \tag{45}$$

which is the usual dimensional transmutation result (here we used $\Lambda/2T_c \gg 1$).

Now we'd like to get $\Delta(T)$ near T_c. Unfortunately even though the integral above for T_c is dominated by large x, i.e. large ξ, we cannot just do an expansion in Δ/ξ in the gap equation integral—when we're trying to get the T dependence of $\Delta(T)$, the small ξ regions actually do contribute to the integral. So we will split up the integral into a small ξ regime and a large ξ regime. Taking the variation of the gap equation wrt Δ and setting $\tanh(x > 1) \approx 1$, we have

$$\frac{1}{g\nu} \approx \ln(\Lambda/2T) + \Delta\delta_\Delta \left(\int_0^1 dx \frac{\tanh\sqrt{x^2 + d^2}}{\sqrt{x^2 + d^2}} + \int_1^{\Lambda/2T} dx \frac{1}{\sqrt{x^2 + d^2}} \right), \tag{46}$$

where $x \equiv \xi/2T$, $d \equiv \Delta/2T$. We will crudely treat the last integral as $\ln(\Lambda/2T)$ which dies under the δ_Δ. The first integral is roughly $\int_0^1 dx(y - y^3/3)/y$ with y the argument of tanh; hence

$$\frac{1}{g\nu} \sim \ln(\Lambda/2T) + \Delta\delta_\Delta \int_0^1 dx(1 - (x^2 + d^2)/3) \implies \frac{1}{g\nu} \sim -\ln(2T_c(1 + t)/\Lambda) - \Delta^2/T_c^2, \tag{47}$$

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where we took t small and dropped $O(\Delta^2 t)$. Expanding the log then tells us that

$$\Delta \sim T_c\sqrt{-t} \tag{48}$$

as claimed earlier.

7 Getting the Landau-Ginzburg coefficients microscopically

Today we're going to show how the coefficients in the LG action are obtained from a microscopic action, and will confirm that indeed the ϕ^2 coefficient is $\nu(T - T_c)/T_c$, with ν the FS DOS. Doing this calculation from the microscopic action is somewhat against the EFT philosophy of which I am a strong subscriber to, but I think it's still useful as a consistency check.

Solution:

The way this works is by now tediously straightforward and barely worth a diary entry, so we'll be rather telegraphic. We start by decoupling the microscopic 4-fermion interaction $g\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow$ in the BCS channel with a bosonic field Δ. Then after integrating out the fermions, we have

$$S[\Delta] = \int \frac{1}{g}|\Delta|^2 + \mathrm{Tr}\ln\begin{pmatrix} i\omega - \xi & \Delta \\ \Delta^* & i\omega + \xi \end{pmatrix}^{-1} \tag{49}$$

Separating out the part of the log that doesn't involve Δ and dropping it as usual, this is

$$S[\Delta] = \int_{\mathbf{k}} \left(\frac{1}{g}|\Delta_{\mathbf{k}}|^2 - T\sum_\omega \mathrm{Tr}\ln\left[\mathbf{1} + \begin{pmatrix} \frac{1}{i\omega-\xi_{\mathbf{k}}} & \\ & \frac{1}{i\omega+\xi_{\mathbf{k}}} \end{pmatrix}\begin{pmatrix} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & \end{pmatrix}\right]\right) \tag{50}$$

Since we are interested in the parameters in LG theory, we can expand this in powers of Δ. Furthermore if we just want to get a flavor of how the coefficients are determined by computing the mass term in the LG action, we can content ourselves with a quadratic expansion in Δ, and furthermore may set the momenta of all Δs to zero. This then gives (the structure of the particle-particle propagators in the bubble comes from the off-diagonal-ness of the Δ matrix in the log, and we have absorbed the minus sign from the expansion of the log into the denominator)

$$S[\Delta] \supset \nu \int_\xi |\Delta|^2 \left(\frac{1}{g} - T\sum_\omega \frac{1}{(i\omega - \xi)(-i\omega - \xi)}\right). \tag{51}$$

In a previous diary entry we recalled how to do this sum: multiply by $f(z)/T$ with $z = i\omega$ and f the Fermi distribution, and then deform the contour to pick up the poles at $z = \pm\xi$. This then gives

$$S[\Delta] \supset \nu \int_\xi |\Delta|^2 \left(\frac{1}{g} - \frac{f(\xi) - f(-\xi)}{2\xi}\right). \tag{52}$$

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Now we write $f(\xi) - f(-\xi) = 1 - 2f(\xi)$ and expand $f(\xi)$ about T_c. Since

$$\partial_T f(\xi) = -T^{-2}\partial_\beta f(\xi) = -T^{-2}(\partial_\xi f(\xi))(\xi/\beta) = -T^{-1}\partial_\xi f(\xi), \tag{53}$$

we have after some algebra

$$S[\Delta] \supset 2\nu \int_0^\Lambda d\xi |\Delta|^2 \left(\frac{1}{g} - \frac{1 - 2f(\xi; T_c)}{2\xi} - t\partial_\xi f(\xi; T_c) \right) \tag{54}$$

with $t = (T - T_c)/T_c$ the reduced temperature.

The first two terms cancel by the definition of T_c, while the third just integrates to νt since $\int_0^\Lambda d\xi \, \partial_\xi f(\xi) = -1/2$. Therefore the quadratic term in the LG action is in fact

$$S_{LG} \supset \int t\nu |\Delta|^2, \tag{55}$$

with ν playing the role of a "large N parameter" as is typical in Fermi liquid physics.

Higher orders of $|\Delta|^2$ in the expansion come with coefficients of the form (from now on we will drop unimportant numerical factors)

$$S[\Delta] \supset \sum_n T \sum_w \int_\xi \nu \frac{|\Delta|^{2n}}{[(i\omega - \xi)(i\omega + \xi)]^n}, \tag{56}$$

where there is only a single frequency / momentum sum since we've taken Δ to be constant, and there are only even terms due to the off-diagonal-ness of the Δ matrix in the log. Also note that when $n \in 2\mathbb{Z}$, the $|\Delta|^{2n}$ term has a positive coefficient—therefore e.g. unlike the $|\Delta|^2$ term, the $|\Delta|^4$ term is guaranteed to be positive, meaning that regardless of the sign of the $|\Delta|^2$ term, we can generically stop at order $|\Delta|^4$ in the expansion and get a stable theory.

To find the T dependence of this term, we do something a bit slick. If we do the frequency sum first as usual, we get an integral that includes stuff like $\int d\xi \xi^{-2n} f(\xi)$, which has a bad IR divergence. So instead, we integrate over ξ first. Extending the upper limit to ∞ and dropping constants (in the form of Γ functions) we get

$$S[\Delta] \supset \sum_n \nu \oint dz \frac{f(z)|\Delta|^{2n}}{z^{2n-1}}, \tag{57}$$

where the contour on z encloses all the poles except for the order $2n - 1$ pole at $z = 0$.[4] The residue of the (non-Matsubara) pole at $z = 0$ comes from taking $2n - 2$ derivatives of the Fermi function in the numerator, and so we get

$$S[\Delta] \supset \sum_n \nu (\partial_\xi^{2n-2} f(\xi)|_{\xi=0} |\Delta|^{2n}. \tag{58}$$

Each derivative of the Fermi function brings down one power of T^{-1}, and so

$$S[\Delta] \supset \sum_n \nu T^2 |\Delta/T|^{2n}. \tag{59}$$

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[4]This is a little bit subtle since this pole lies on top of one of the Matsubara frequencies—I think this is the reason for the IR divergence that comes up if you do the $\oint dz$ first. Being physicists, we will just charge ahead.

Note that this power counting is consistent with our more careful evaluation of the $n = 1$ bubble, in terms of the powers of T appearing in the nth order expression. Also note that unlike the quadratic term, all of the higher order terms do not vanish near T_c—it is only the quadratic term that generically controls what phase the system is in.

8   Basic spin waves stuff

Today we are doing an elaboration on two problems in Simon's solid state book and one from A&M. We'll be deriving and commenting on the spectrum of spin waves in (anti)-ferromagnets.

Solution:

Note: in this journal entry only, X, Y, Z will denote spin operators, i.e. $1/2$ times the appropriate Pauli matrix.

Ferromagnets

The Hamiltonian is

$$H = -\sum_{\langle \mathbf{rr'} \rangle} J(\mathbf{r} - \mathbf{r'}) S_{\mathbf{r}}^a S_{\mathbf{r'}}^a \tag{60}$$

with $J(\mathbf{r}) \geq 0$ (n.b. we will not assume short-ranged J).

First we will do the "quantum" approach, whereby we start from the state $\otimes| \uparrow\rangle$ (which is a *provably exact* ground state) and look at the excitations on top of it that carry minimal S^z. These excitations are obtained by flipping a single spin. In $\mathbb{R}$ space, we then define the states

$$|\mathbf{r}\rangle = \frac{1}{\sqrt{2S}} S_{\mathbf{r}}^- |\otimes \uparrow\rangle, \tag{61}$$

where $|\otimes \uparrow\rangle$ is the all $\uparrow$ ground state and the prefactor is for normalization. Since the $XX + YY$ term means that H isn't diagonal in this basis, we need to Fourier transform. Hence define

$$|\mathbf{k}\rangle = \frac{1}{\sqrt{N}} \sum_{\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} |\mathbf{r}\rangle \tag{62}$$

where $N = L^d$ is the number of sites. In these states, we have $\langle Z_{\mathbf{r}}\rangle_{\mathbf{k}} = S - 1/N$, so that the $\delta Z = 1$ of the spin flip is spread out uniformly over the system.

We want to compute $E_{\mathbf{k}} - E_0$. The difference in the ZZ part of the interaction is, using translation invariance and orthogonality to do two of the sums,

$$E_{\mathbf{k}} - E_0 \supset -\sum_{\mathbf{r}} \sum_{\mathbf{R}} J(\mathbf{R}) \left(\langle \mathbf{r}|Z_0 Z_{\mathbf{R}}|\mathbf{r}\rangle - \langle \uparrow |Z_0 Z_{\mathbf{R}}| \uparrow\rangle \right). \tag{63}$$

The difference comes when $\mathbf{r}$ is 0 or $\mathbf{R}$ (we are assuming $J(0) = 0$ wolog since this would just add a trivial constant to the Hamiltonian), with an energy difference of $S(S-1) - S^2 = -S$. Hence

$$E_\mathbf{k} - E_0 \supset 2S \sum_\mathbf{R} J(\mathbf{R}). \tag{64}$$

The other part contributing to the energy difference is the hopping term, which has zero vev in the ground state. Hence

$$E_\mathbf{k} - E_0 \supset -\frac{1}{N} \sum_{\mathbf{R}\mathbf{R}'\mathbf{r}\mathbf{r}'} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} J(\mathbf{R} - \mathbf{R}') \langle \mathbf{r}' | \frac{S_\mathbf{R}^+ S_{\mathbf{R}'}^- + S_{\mathbf{R}'}^+ S_\mathbf{R}^-}{2} | \mathbf{r} \rangle$$

$$= -\frac{1}{2NS} \sum_{\mathbf{R}\mathbf{R}'\mathbf{r}\mathbf{r}'} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} J(\mathbf{R} - \mathbf{R}') \langle \uparrow | \frac{\delta_{\mathbf{R}\mathbf{r}}\delta_{\mathbf{R}'\mathbf{r}'} S_\mathbf{R}^+ S_\mathbf{R}^- S_{\mathbf{R}'}^+ S_{\mathbf{R}'}^- + (\mathbf{R} \leftrightarrow \mathbf{R}')}{2} | \uparrow \rangle \tag{65}$$

$$= -2S \sum_\mathbf{R} J(\mathbf{R}) \cos(\mathbf{R} \cdot \mathbf{k}).$$

Therefore if we add a constant to H so that the ground state is at zero energy, we have

$$E_\mathbf{k} = 2S \sum_\mathbf{R} J(\mathbf{R})(1 - \cos(\mathbf{R} \cdot \mathbf{k})). \tag{66}$$

The spin waves are therefore gapless, but perhaps counter to our naive expectations based on symmetry breaking, they are *quadratically* dispersing near zero momentum.

All this was for the isotropic chain. What happens when we break the $SO(3)$ (and it is $SO(3)$ not $SU(2)$, since the operators in the Hamiltonian transform adjointly under $SU(2)$!) symmetry by generalizing to

$$H = -\sum_{\langle \mathbf{R}\mathbf{R}' \rangle} [(J(\mathbf{R} - \mathbf{R}') + \Delta(\mathbf{R} - \mathbf{R}'))Z_\mathbf{R} Z_{\mathbf{R}'} + J(\mathbf{R} - \mathbf{R}')(X_\mathbf{R} X_{\mathbf{R}'} + Y_\mathbf{R} Y_{\mathbf{R}'})]\,? \tag{67}$$

The answer is fairly obvious if $\Delta > 0$, which we will assume, since then the symmetry is explicitly broken down to $U(1)$, with $\otimes | \uparrow \rangle$ remaining the ground state. Since the symmetry is broken explicitly down to $U(1)$ (while $U(1)$ is presumed to be unbroken in our trial spin wave vacuum state), there should be no Goldstones, with the spectrum totally gapped. This is checked by running the calculation above for $E_\mathbf{k}$—the only change from the anisotropy is a $2S \sum_\mathbf{R} \Delta(\mathbf{R})$ from the ZZ term, and so

$$\delta E_\mathbf{k} = 2S \sum_\mathbf{R} \Delta(\mathbf{R}), \tag{68}$$

which gaps the spectrum. Note that while the spin wave spectrum just gets shifted up in energy by a constant, this is of course not a trivial shift in the Hamiltonian by a constant—it only looks like this when we look at the ground state or the single spin wave states.

The "classical" way of doing this is to take the eoms and expand them about a reference ordered state. For simplicity, we will assume only nearest-neighbor interactions—general long-ranged Js are harder to deal with in this approach. From the Heisenberg eoms, we find

$$i\hbar d_t S_i^a = [S_i^a, H] = -Ji \sum_j \epsilon_{abc} S_i^c S_j^b, \tag{69}$$

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where the sum is over the neighbors of i. To get spin waves, we use the ansatz that

$$S^a = \begin{pmatrix} S + O(\epsilon^2) \\ x \\ y \end{pmatrix}, \tag{70}$$

where x, y are of order ϵ, where $\epsilon = 1/N$ with N the chain length. Putting this ansatz in and keeping only the terms linear in x, y, we have

$$d_t x_i = -JS \sum_j (y_j - y_i)$$
$$d_t y_i = -JS \sum_j (x_i - x_j) \tag{71}$$

Now we can add the first to i times the second to get

$$d_t S_i^+ = iJS \sum_j (S_j^+ - S_i^+). \tag{72}$$

The fact that we get only one of equation of motion for the spin waves despite them naively having two degrees of freedom (spontaneously breaking $SO(3)$ down to $U(1)$ should give us two goldstone modes since the coset space is locally S^2) is due to the fact that the two putative Goldstone modes are actually canonically conjugate variables, and hence are not independent. Said another way, the way in which the x and y components of the spin wave time evolve are not independent from one another: they differ by a $\pi/2$ phase shift,[5] so that the operators $S^\pm$ remain well-defined under time evolution.

Anyway, from the S^+ eom we see that on a cubic lattice,

$$\omega = 2JS \left(d - \sum_j \cos k_j \right). \tag{74}$$

This gives us the same gapless quadratic dispersion (with a mass determined by $1/4JS$) as the "quantum" computation above.

How does the anisotropy result in a mass term here? If we add the term

$$H \supset -\Delta \sum_{\langle ij \rangle} Z_i Z_j \qquad \Delta > 0 \tag{75}$$

then we see that the eom for S^+ is now

$$d_t S^+ = iJS \sum_j (S_j^+ - S_i^+(1 - \Delta)). \tag{76}$$

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[5]The equations of motion in Fourier space look like

$$\begin{pmatrix} \omega & \lambda \\ -\lambda & \omega \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0. \tag{73}$$

One finds that $\lambda/\omega = \pm i$, which means that $X = \pm iY$; hence the phase shift.

Therefore the dispersion is now

$$\omega = 2JS\left(d - \sum_j \cos k_j + \Delta\right),$$ (77)

which is gapped by $2JS\Delta$, as expected.

Antiferromagnets

With AFMs the situation is trickier—in the FM case we started by introducing minimal excitations on top of a symmetry-broken state, which was an exact ground state of the system. For AFMs in general dimensions we don't even know what the AFM ground state is, and so we lack a suitable starting point.

The equations of motion approach is still useful though, if we make the assumption that we can describe the spin waves as perturbations away from a Neel-ordered reference state. Therefore we take the same equations of motion as above, but insert the ansatz

$$\mathbf{S}_i = \begin{pmatrix} x_i \\ y_i \\ (-1)^i S \end{pmatrix},$$ (78)

where $(-1)^i$ takes on different signs on the two sublattices. Putting this into the eom and using that $(-1)^{i+j} = -1$ if i, j are nearest neighbors, we find

$$d_t x_i = (-1)^i JS \sum_j (-y_i - y_j)$$
$$d_t y_i = (-1)^i JS \sum_j (x_i + x_j).$$ (79)

Adding i times the second one to the first to get an eom for the raising operator,

$$d_t S_i^+ = i(-1)^i JS \sum_j (S_i^+ + S_j^+).$$ (80)

There are two notational ways one can deal with solving problems like this, where the lattice has an enlarged unit cell in the ansatz under conisderation. One is to solve it with momenta variables as if there was no enlarged unit cell, and then after computing the spectrum to realize that the periodicity of the momenta variables you're working with needs to be changed to account for the different unit cell. The other is to work in coordinates where each unit cell defines a single point on the lattice, with hoppings in between sites of a unit cell picking up no momentum phases. We will use the first approach. Therefore we write the above in Fourier space as

$$\omega \begin{pmatrix} S_A^+ \\ S_B^+ \end{pmatrix} = \begin{pmatrix} 2dJS & -2JS\sum_j \cos k_j \\ 2JS\sum_j \cos k_j & -2dJS \end{pmatrix} \begin{pmatrix} S_A^+ \\ S_B^+ \end{pmatrix}$$ (81)

where now A denotes a sublattice, not a lattice site. Diagonalizing, the dispersion is

$$\omega = \pm 2JS \sqrt{d^2 - \left(\sum_{i=1}^{d} \cos(k_i)\right)^2}. \tag{82}$$

In one dimension, this reduces to the simple

$$\omega = \pm 2JS |\sin(k)|. \tag{83}$$

In constrast to the ferromagnet, we have the expected number of Goldstones, namely 2, and as in the usual scenario, they are linearly dispersing near zero momenta. One can check in a similar way that explicitly breaking the symmetry by adding e.g. a ZZ term gaps the modes.

9   A QFT-flavored view of thermodynamic potentials

In my opinion discussions of the various thermodynamic[6] "potentials" (free energy, Gibbs free energy, Helmholtz, enthalpy, ad nauseum) are usually much more confused that they need to be. For some reason most textbooks assume that defining a jillion different functions and then calculating partial derivatives in a jillion different ways will teach people something— didn't work for me, in any case. Today we'll try to explain a common framework for thinking about these different functions.

Solution:

Thermodynamics

To derive thermodynamics, we need to start with an axiom regarding entropy / Boltzmann weights / etc. in order to get going. As far as I can think of, there are two reasonably good choices. One is a "bath-centric" point of view. Here we imagine our system X as being in contact with a bath B, such that the full system $X \cup B$ is closed. The full system can then be treated micro-canonically, with an entropy that is simply $S = \ln \Omega$, with Ω the number of microstates.[7] Since we are working in a closed system, there is no real notion of an interesting probability distribution here—we are assuming the ergodic hypothesis holds so that each mircostate makes an equal contribution to S. Thermodynamics is then done on X by e.g. maximizing S subject to the constraint that e.g. the energy of the subsystem X has a certain value, with equilibrium being defined through $\partial_{E_X} S_X = \partial_{E_B} S_B$. After tracing out the bath, one can then do thermodynamics in X. The point of this approach is that one is always secretly working in a closed system, with entropy always being calculated by $\ln \Omega$.

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[6]Even the name of the field sucks! Why thermo*dynamics*, when the whole point is that the theory can only answer questions about static stuff that's in equilibrium?

[7]We will always be setting $k = 1$.

The second point of view, which we will adopt, basically never deals explicitly with a bath per se. Instead, the starting point is to declare that the right thing to do is to maximize the shannon entropy (or its quantum equivalent; here we will be using classical language)

$$S = \langle I \rangle = -\sum_k p_k \ln p_k, \tag{84}$$

where I is the information. Recall why this function is singled out as the right function to use: it is positive, maximal on uniform distributions, doesn't count states with $p_k = 0$, vanishes if the state of the system is certain with some $p_k = 1$, behaves correctly under super-imposing two systems ($\{p_k\} \to \{p_k q_l\}$ gives $S(\{p_k, q_l\}) = S(\{p_k\}) + S(\{q_l\})$), and so on.

The variables that act as input to a thermodynamic theory in this philosophy are the conserved quantities that we use to label the equilibrium states: U, V, N, M, etc. These extensive variables are all expectation values of operators that to each state k assign quantities U_k, V_k, etc, so that e.g. $U = \sum_k p_k U_k$.

These conserved quantities are what make thermodynamics, the study of the maximization of (84), nontrivial. They act as constraints on the optimization of (84), which if unconstrained would be maximized by all of the p_k being constant.

The problem is then to maximize (84) subject to $\sum_k p_k \mathcal{O}_k^i$ for each conserved quantity $\mathcal{O}^i$. There are two paths forward. One is to solve the constraints first, i.e. to solve for the p_k as a function of the $\mathcal{O}^i = \sum_k p_k \mathcal{O}_k^i$, and then try to maximize S with respect to these expectation values. This is basically impossible, though. Instead, it is better to use Lagrange multipliers. This way we can first do an *unconstrained* minimization (which in contrast is easy—just take derivatives and set them equal to 0), and then later solve the constraint. Therefore we want to instead maximize

$$\mathscr{S} = -\sum_k p_k \ln p_k + \sum_i \lambda_i \left(\sum_k p_k \mathcal{O}_k^i - \mathcal{O}^i \right) + \alpha \left(\sum_k p_k - 1 \right), \tag{85}$$

where the (unconstrained) maximization is carried out with respect to the p_ks and the λ_is as well as the last Lagrange multiplier α, which lets the p_k be fully unconstrained by being a Lagrange multiplier for the normalization of the p_ks.

The extremization is now easy to do (we get a maximum by convexity); we get

$$p_k = \frac{1}{Z} \exp\left(-\sum_i \lambda_i \mathcal{O}_k^i \right). \tag{86}$$

Therefore we have derived Boltzmann weights, rather than starting for them as an axiom. The point then is that as long as we write the probabilities in the above Boltzmann weight form, *we have already maximized the entropy*—there is no further extremization to be done in order to find equilibrium. The (maximized) entropy is evidently

$$S = -\sum_k \frac{e^{-\sum_i \lambda_i \mathcal{O}_k^i}}{Z} \left(-\sum_i \lambda_i \mathcal{O}_k^i - \ln Z \right) = \sum_i \lambda_i \mathcal{O}^i + \ln Z. \tag{87}$$

To make contact with conventional notation, we call the Lagrange multiplier for energy β, and the Lagrange multipliers for the other variables things like $p\beta, \mu\beta, H\beta$, and so on.[8] If we then define a free energy $\mathcal{F} \equiv -\beta^{-1} \ln Z$, we have

$$\mathcal{F} = U - \beta^{-1} S + \sum_i (\lambda_i \beta^{-1}) \mathcal{O}^i, \tag{88}$$

where we've separated out the energy U from the list of Lagrange multipliers due to the (somewhat historical) special role played by temperature and energy (of course, the terms in the sum would typically be written $-pV + \mu N - MH + \dots$). The fact that we've singled out U here is because in what follows we will always invert $S(U, V, N, \dots)$ and think about S, and not U, as an extensive variable, so that $U = U(S, V, N, \dots)$. This inversion can be done since we expect S to be monotonically increasing as a function of U, and is done because U is a more readily measurable quantity than S.

Now we're not really done, because a full solution to the problem would be to write the Lagrange multipliers λ_i in terms of the expectation values $\mathcal{O}^i$, which we took as inputs to the theory. However, doing this would require exactly computing the partition function and then doing some seriously heinous algebra to invert the resulting mess and get expressions for the λ_i.

However, now we can change our viewpoint a little, and regard the Lagrange multipliers as the independent variables. From the definition of $\mathcal{F}$, we see that functionally differentiating $\mathcal{F}$ with respect to these Lagrange multipliers (since I think like a QFTheorist, I'm imagining that the sum over all states k has a product over e.g. all spatial points, so that we get statistical weights like $e^{-\beta \int d^d x\, U - MH + \cdots}$; hence functional differentiation is meaningful, and we can take the Lagrange multipliers / expectation values to be non-uniform in space) generates correlation functions of their associated conserved quantities. Therefore we see that the "intensive" variables $\beta, p, H, \dots$ are nothing more than background fields—a background β field generates correlation functions of energy, a background H field generates correlation functions of magnetization, and so on.

These background fields / Lagrange multipliers are the way in which the "bath" manifests itself in this way of thinking about things. This is just as in regular QFT, where coupling the theory to background fields can be thought of as coupling the theory to a bath which can act as a source for particles, currents, etc.

The process of switching from using a background field (Lagrange multiplier) as an independent variable to using its corresponding intesive variable (expectation value of a conserved quantity) is done via a Legendre transform.

One particular Legendre transform is the one that swaps out all of the background fields as independent variables for all of the expectation values of conserved quantities. This is the Legendre transformation that relates U and F. For economy of notation, consider the case when energy is the only conserved quantity. Then we will have $\mathcal{F} = \mathcal{F}(\beta)$ and

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[8]Another complaint: who decided that β should multiply all the terms in the exponential of the Boltzmann weight? It'd seem to be much better if it only appeared in front of the energy. And furthermore, why do we work with T more often than β? Shouldn't we then also work with $1/p$ instead?

$U = U(S) = U[S(\beta)]$. We calculate

$$\partial_\beta \mathcal{F} = \beta^{-2} \ln Z - \beta^{-1} \frac{\partial_\beta Z}{Z} = \sum_k \left(\beta^{-2} p_k \ln Z + \beta^{-1} \frac{p_k U_k}{Z} \right)$$
$$= -\sum_k \beta^{-2} \left(-p_k \ln Z + p_k \ln(p_k Z) \right) = \beta^{-2} S. \tag{89}$$

Therefore

$$\partial_T \mathcal{F} = -S. \tag{90}$$

Then applying this to $\mathcal{F} = U - TS$,

$$-S = \partial_T S \partial_S U - S - T \partial_T S \implies \partial_S U = T. \tag{91}$$

So we see that $\mathcal{F}$ and U are not just any two functions satisfying $\mathcal{F} = U - TS$; rather they are constrained by the relations

$$\partial_T \mathcal{F} = -S, \qquad \partial_S U = T. \tag{92}$$

The relation between the potentials $\mathcal{F}$ and U is therefore better expressed by writing

$$\mathcal{F} = \min_S [U - TS], \qquad U = \min_T [F + TS], \tag{93}$$

which explicitly shows that U and $\mathcal{F}$ are Legendre transform pairs with respect to the conjugate variables S and T.[9] When there are more conserved quantities beyond just energy, there are accordingly more variables appearing, and more ways of taking Legendre transforms—but the basic game is still the same.[10]

Let's now briefly recall why Legendre transforms come up when solving constrained optimization problems. We will use the example of L and H, for simplicity. The problem is to extremize

$$\min_{x,v} \int dt \, L(x,v)|_{v=\partial_t x}, \tag{94}$$

where we need to put in the constraint that $v = \partial_t x$. If we directly solve the constraint first and then optimize we get the Euler-Lagrange equations, which are second order and sometimes hard to solve. In the Hamiltonian approach, we first optimize over an unconstrained system with a Lagrange multiplier p, and solve the constraint at the end: therefore the problem is to minimize

$$\min_{p,x,v} \int dt \, (L(x,v) + p(\partial_t x - v)) = \min_{p,x} \int dt \, (p\partial_t x - H(x,p)), \tag{95}$$

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[9]Again, this whole rather arbitrary procedure of working with T instead of the Lagrange multiplier β, and attaching β's to all the conserved quantities in the Boltzmann weight rather than just the energy, is done so that we end up getting equations with dependent variables such that it is U, and not S, which appears as a thermodynamic potential with a free energy Legendre transform partner.

[10]Here we have defined "the" free energy to be a function only of the Lagrange multiplier background fields, but we could instead do some Legendre transformations and write it as a function of extensive quantities. For example, the Helmholtz free energy is normally $F = U - TS$, which is a function of extensive variables like $V, N, \ldots$ but differs from U by the fact that it depends on T, and not S. When we say "free energy" in the text, we will generally one which is the "full" Legendre transform of U, which only depends on intensive quantities.

where we have defined

$$H(p, x) = \min_v(vp - L). \tag{96}$$

Notice that the minimization over p and x in (95) is unconstrained—this is the primary advantage of the Hamiltonian approach.

Maybe it helps to make explicit the analogies between Lagrangian mechanics, thermodynamics, and effective actions (as usual, the ϕ in the effective actions column denotes an expectation value in the presence of a current). The analogies in the first two columns are helpful for thinking about things physically since they give a nice field theory-flavored way of looking at thermodynamics, while the last column on mechanics is just included for building mathematical familiarity of what sorts of objects go where in Legendre transforms.

Thermo	Effective actions	Mechanics
T	J	p
S	$-\phi$	v
$\mathcal{F}(T)$	$W[J]$	$H(x, p)$
$U(S)$	$\Gamma[\phi]$	$L(x, v)$
$\mathcal{F} = \min_S[U - TS]$	$W = \min_\phi[\Gamma + \int J\phi]$	$H = \min_v[vp - L]$
$U = \min_T[\mathcal{F} + TS]$	$\Gamma = \min_J[W - \int J\phi]$	$L = \min_p[\partial_t xp - H]$
$\partial_T\mathcal{F} = -S$	$\delta_J W = \phi$	$\partial_p H = v$
$\partial_S U = T$	$\delta_\phi\Gamma = -J$	$\partial_v L = p$

$$\tag{97}$$

The most interesting things to compute here are usually second derivatives of the generating functionals $\mathcal{F}, U, W, \Gamma$. By the Legendre transform pairing, the first derivative of a functional of the background fields / intensive variables (like $\mathcal{F}$) will be an argument of its Legendre pair (like U), i.e. it will be an expectation value / extensive variable. Therefore for any thermodynamic potential $F[X]$ which is "dual" to a potential $\widetilde{F}[\widetilde{X}]$, we will have

$$\partial_X^2 F[X] = \partial_X \widetilde{X}. \tag{98}$$

Therefore second derivatives of thermodynamic potentials are response functions.[11]

Anyway, the point to keep in mind is that "all of physics is computing path integrals with background fields". This is basically all of what thermodynamics is, and the panoply of thermodynamic potentials that appear in textbooks are there just because in different situations one has different conserved quantities and different background fields to which they can couple. Even given a fixed set of conserved quantities, we can still choose to solve the constraints and write our thermodynamic potentials in terms of the expectation values of the extensive quantities rather than the background fields, with the process of switching out the independent variables being given by the Legendre transform.

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[11]Recall that response functions appear as the kernels that determine the expectation value of a variable X in the presence of its Legendre-dual source field $\widetilde{X}$ as (schematically) $\langle X \rangle = \int \chi \widetilde{X}$. χ is then $\partial_{\widetilde{X}}\langle X \rangle = \partial_{\widetilde{X}}^2 F$, so that response functions always come from second derivatives of thermodynamic potentials (of course we don't need to take both derivatives with respect to the same variable, we can also consider how $\langle X \rangle$ changes under the variation of background fields other than $\widetilde{X}$.

10 (Most of) the different critical fields for type I and type II superconductors

Today we will explain the criterion for distinguishing between type I and type II SCs, and will give derivations for the values of the various critical fields $H_{c1}, H_c,$ and H_{c2}.

Solution:

H_c—the thermodynamic critical field

When we think about the electrodynamics and thermodynamics of SCs, we are usually interested in situations where the SC is placed in an external field.

The field H, being a background field, is constant everywhere[12] regardless of what phase the system is in, and its purpose in life is to generate correlation functions of the magnetization M through functional differentation of $\ln Z$.

Let's now define the thermodynamic critical field H_c. One makes the assumption that inside of the material, the free energy at fixed B contains the term

$$\mathcal{F}(B) \supset \frac{1}{2\mu_0} \int d^d x \, B^2, \tag{99}$$

and that $\mathcal{F}(B)$ has no other B dependence. Now in the context of thinking about magnetism in SCs, we would rather work at fixed H than at fixed B. Therefore we define the fixed-H free energy by doing a Legendre transform as

$$F(H) = \mathcal{F}(B) - \int BH, \tag{100}$$

where we have $\partial_H F = -B$, $\partial_B \mathcal{F} = H$. The usual notation for this would be something like $G = F - \int BH$, but we will just choose to call $F(H)$ the free energy, rather than something like "Gibbs free energy". The type of free energy we're dealing with will be distinguished by context or by its arguments. Note that instead of defining the Legendre transformation with BH, we can also do the transformation with $-\int MH\mu_0$—the two differ by the background term $\int H^2 \mu_0$, which being a constant in every phase won't affect relative free energy difference between phases. In the next little bit we will indeed use $-\int MH\mu_0$, but later on may switch.

In the normal state, we assume that the magnetization vanishes, and hence that $B = \mu_0 H$.[13] Therefore $F_n(B) = V\mu_0 H^2/2$. In the SCing state, $B = 0$, and so $M = -H$, meaning

---

[12]Well, it doesn't have to be, but we'll take it to be uniform in space. The point is that unlike B which is "dynamical" (meaning that its expectation value can be affected by the dynamical degrees of freedom, not [necessarily] that we're integrating over it), H is completely static and fixed—it's just a device used to generate correlation functions of M.

[13]Recall that $\mu_0^{-1} B = H + M$ in SI units. Unfortunately the conventions here aren't very great—H and M are microscopic things determined by computing expectation values, while H is a background field used to generate M correlation functions. Therefore its a little weird to put the μ_0^{-1} only on the B—oh well.

that the magnetic part of the SC free energy is $F_s^{(H)} = V\mu_0 H^2$. The difference in these two free energies defines the thermodynamic critical field via[14]

$$\frac{\mu_0}{2} V H_c^2 \equiv F_s^{(H)} - F_n. \tag{105}$$

H_c can be expressed in GL theory by equating it to the condensation energy F_s^{ψ} of the SC, since at a field of H_c we have $F_s^{(H)} + F_s^{\psi} = F_n$. Writing the ψ part of the GL free energy as

$$F = \int \left(\gamma |D_{\mathbf{A}}\psi|^2 + t|\psi|^2 + \frac{1}{2} u|\psi|^4 + \dots \right), \tag{106}$$

we see that $\psi_0 = \sqrt{-t/u}$, so that the condensation energy is

$$F_s^{\psi} = -\frac{t^2}{2u} = \frac{t\gamma}{2u\xi^2(T)} \tag{107}$$

where $\xi(T) = \sqrt{-\gamma/t}$ is the LG "coherence" length (better would be "correlation length" or "stiffness length" or similar).

Given this definition of H_c, we can determine what the critical current in an SCing wire is. If we force a current J through a SCing wire, then it will produce a magnetic field parallel to the surface of magnitude $B_{\parallel} = J\mu_0/(2\pi a)$, where a is the radius of the SCing wire. Therefore the critical current is

$$J_c = \mu_0^{-1} 2\pi a H_c \tag{108}$$

The fact that J_c scales with the circumference of the wire is in keeping with the fact that the current is confined to within a distance λ_L of the surface of the SC by the London equations (this gives a critical current density which is essentially independent of the wire size, as long as $\lambda_L \ll a$).

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[14]Aside: this has all been in SI units. For better or worse, most people use cgs when talking about SCs. The version of this in CGS is as follows. There the magnetic energy is

$$\mathcal{H} \supset \frac{1}{2 \cdot 4\pi} \int B^2, \tag{101}$$

while the relationship between M, B and H is

$$B = H + 4\pi M. \tag{102}$$

The free energy then contains

$$F \supset \int \left(\frac{1}{8\pi} B^2 - MH \right). \tag{103}$$

In the normal state we have $B = H$ and $F_n = H^2/8\pi$, while in the SCing state $M = -H/4\pi$ and so $F_s = H^2/4\pi$, and now the definition of H_c is

$$V \frac{H_c^2}{8\pi} = F_s^{(H)} - F_n. \tag{104}$$

We can now ask what order the SCing phase transition is. This is a problem in A&M, and the strategy is to look at which thermodynamic quantities are discontinuous across the transition. Since we have a background H field but don't have a background field (pressure) for volume or anything else, the operative free energy has a differential of $dF = -(SdT + MdH)$. Right at the critical point the free energies for the SCing and normal states are the same, and hence dF will be the same when computed in either the SC or the normal metal. Therefore at the critical field,

$$S_s dT + M_s dH_c = S_n dT + M_n dH_c \tag{109}$$

Therefore

$$d_T H_c = \frac{S_n - S_s}{M_s - M_n}. \tag{110}$$

Since $M_n = 0$ while $M_s = -H$, we have

$$S_n - S_s = -H_c \partial_T H_c. \tag{111}$$

Since this is a nonzero discontinuity, we expect the SCing transition to be first order at all non-zero fields. If the transition occurs at zero field (by tuning T up to T_c so that $H_c \propto t = 0$), we get $S_n = S_s$, indicating a second-order transition (modulo HLM). The latent heat is just the above multiplied by T. We can find the specific heat jump by

$$C = \partial_T U = \partial_S U \partial_T S = \partial_{\ln T} S, \tag{112}$$

so that

$$C_n - C_s = -T \left((\partial_T H_c)^2 + H_c \partial_T^2 H_c \right), \tag{113}$$

which at zero field gives

$$(C_n - C_s)|_{H_c = 0} = -T(\partial_T H_c)^2. \tag{114}$$

Since we can compute this discontinuity within LG theory without having to turn on any EM fields, this lets us get $\partial_T H_c$ without having to actually do calculations with magnetic fields.

H_{c2}—the biggest field for which $|\psi|$ can be non-zero

H_{c2} is best thought of by imagining a process in which one starts at $T < T_c$ ($t < 0$) but with a field high enough to completely kill SCtivity, and then reduces the field to the point where SCing order can form. H_{c2} is identified as the external field for which the LG equations first admit solutions for infinitesimal nonzero values of $|\psi|$.

Since at H_{c2} $|\psi|$ is infinitesimal, we can work with the linearized LG free energy for ψ, namely

$$F = \int \left(\gamma |D_{\mathbf{A}} \psi|^2 + t |\psi|^2 + \ldots \right), \tag{115}$$

where the ... are magnetic field terms. Varying with respect to ψ^*, we have

$$(t - \gamma D_A^2)\psi = 0 \implies (\mathbf{p} - q\mathbf{A})^2 \psi = \xi^{-2}(T)\psi. \tag{116}$$

Solving this is then just a Landau level problem. Choosing e.g. $\mathbf{A} = (0, Bx, 0)$, we write this suggestively as

$$\frac{2(qB)^2\gamma}{2}\left[\frac{1}{(qB)^2}p_x^2 + (p_y/qB - x)^2\right]\psi = -t\psi \tag{117}$$

Therefore we have a harmonic oscillator where $k = 2(qB)^2\gamma$ and $1/m = 2\gamma$. Therefore $\omega = 2\gamma qB$, and so we require that

$$E_n = 2\gamma qB(n + 1/2) = -t. \tag{118}$$

As we reduce B, this will first happen at the $n = 0$ level. Therefore we will have a solution at H_{c2}, where

$$H_{c2} = -\frac{t}{\gamma q} = \frac{1}{q\xi^2(T)} = \frac{\Phi_0}{2\pi\xi^2(T)}, \tag{119}$$

with $\Phi_0 = \frac{h}{2e}$ the SCing flux quantum (using that $q = 2e$. This means that we start getting solutions with nonzero ψ when there is half a SCing flux quantum per "correlation area" $A_c = \pi\xi^2(T)$.

We can now use this as a diagnosis for when we expect superconductors to be type II. Imagine reducing the magnetic flux from a very large value. Suppose that $H_{c2} < H_c$. Then the system will "supercool", since the field will fall below H_c without order being able to form (the order has to start somewhere, but it can only get started at the field H_{c2}, where infinitesimal order can form). Once it then reaches H_{c2} there will be an instability towards large $|\psi|$, since converting everything to a SCing state is thermodynamically favorable.

On the other hand, suppose that $H_c < H_{c2}$. Then as the field is reduced just past H_{c2}, parts of the sample will become SCing because of the existence of a solution to the LG equations with nonzero $|\psi|$, but the whole sample won't,[15] since it is not thermodynamically favorable to setup the perfect diamagnetic response until H_c is reached.

This means the dividing line between type I and type II occurs at $H_c = H_{c2}$. We can express this condition in a more meaningful way by writing H_{c2} and H_c in terms of the length scales in the system. For the critical field, we use our above expression for H_c to write

$$H_c = \frac{|\psi_0|\sqrt{\mu_0\gamma}}{\xi(T)}, \tag{120}$$

so that the ratio of the fields is

$$H_{c2}/H_c = \frac{1}{\xi(T)q|\psi_0|\sqrt{\mu_0\gamma}}. \tag{121}$$

The London penetration depth can be found from Maxwell's equations and the eom for the current as $\lambda_L^2 = m/(nq^2\mu_0)$, where m and q are the mass and charge of the Cooper pair and $n = |\psi_0|^2$ is the "carrier density". Since $\gamma = 1/2m$ we have

$$\lambda_L = \frac{1}{\sqrt{2\gamma|\psi_0|q^2\mu_0}}, \tag{122}$$

---

[15]I guess we expect that the superconducting order parameter will have some sort of nontrivial spatial profile. While this is a bit tricky to extract from the above analysis since the profile of the wavefunctions is gauge dependent, we at least don't expect a uniform order parameter everywhere.

which means that

$$\frac{H_{c2}}{H_c} = \sqrt{2}\frac{\lambda_L}{\xi}, \tag{123}$$

which means that type II superconductors will be those ones for which

$$\frac{\lambda_L}{\xi} > \frac{1}{\sqrt{2}}. \tag{124}$$

H_{c1}—the field for which vortices can enter type II superconductors

We can find H_{c1} by looking for the field at which an inserted vortex has zero free energy. The region of the vortex contributes a free energy difference (now taking the Legendre transform with $F(H) \supset -H \int B$ rather than $-H \int M\mu_0$) of

$$\Delta F = V f_s^\psi - (-H\Phi L + F_v), \tag{125}$$

where V is the volume of the vortex which we assume to be a tube of length L with flux Φ, f_s^ψ is the SCing condensation free energy density, and F_v is the free energy of the vortex.[16] Therefore vortices can enter the SC when the field satisfies (assuming a vortex cross-section of A)

$$H_{c1} = \frac{1}{\Phi}\left(F_v/L - f_s^\psi A\right). \tag{126}$$

Let's calculate F_v/L by ignoring what goes on in the core. That is, we will calculate the energy of the fields for all radii bigger than the correlation length ξ, which we take to be the vortex radius.[17] Later on we will argue that in fact most of the energy of the vortex comes from the kinetic part contained in the fields outside of the core, rather than from the core itself.

Anyway, this part of the cross-sectional energy of the vortex is (in what follows, ψ will denote the [constant] magnitude of the OP)

$$\mathcal{E} = \int_P \left(\frac{1}{2\mu_0}B^2 + \gamma\psi^2|D_{\mathbf{A}}\phi|^2\right), \tag{127}$$

where P (for "punctured plane") is $\mathbb{R}^2$ minus the vortex core. The current is, as a differential form, $J = 2q\gamma\psi^2(d\phi - qA)$. Therefore we may write

$$\mathcal{E} = \frac{1}{2\mu_0}\int\left(B \wedge \star B + \frac{\mu_0^2}{2\gamma\psi^2 q^2\mu_0}J \wedge \star J\right). \tag{128}$$

The penetration depth is $\lambda_L^{-2} = 2\gamma\psi^2 q^2\mu_0$, and so

$$\mathcal{E} = \frac{1}{2\mu_0}\int\left(B \wedge \star B + \mu_0^2\lambda_L^2 J \wedge \star J\right) = \frac{1}{2\mu_0}\int\left(B \wedge \star B + \lambda_L^2 dB \wedge \star dB\right), \tag{129}$$

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[16]We haven't forgotten about the kinetic energy $B^2/2\mu_0$ in the vortex! That's included in F_v.

[17]Since we are at fields $H < H_c$, we don't want to have extended regions where we replace superconductor with magnetic flux, and hence expect ψ to dip down to zero at the vortex core (over a distance of ξ), but then to immediately rise back up to ψ_0—hence the radius of the vortex will be $\approx \xi$.

where we have used Maxwell's equation $\mu_0^{-1} \star dB = J$. Integrating by parts,

$$\mathcal{E} = \frac{1}{2\mu_0} \int_P (B + \lambda_L^2 d^\dagger dB) \wedge \star B - \frac{\lambda_L^2}{2\mu_0} \oint_{\partial P} B \wedge \star dB. \tag{130}$$

The first term can be dealt with using the London equation in the presence of the flux. Since $\oint_C d\phi = q\Phi$ for any curve $C \subset P$ encircling the flux, $d^2\phi = q\Phi\delta^2(r)$. Therefore the London equation is now

$$\star dJ = 2\gamma\psi^2 q^2 (\Phi\delta^2(r) - B) = \frac{1}{\mu_0\lambda_L^2}(\Phi\delta^2(r) - B). \tag{131}$$

Then again using $\mu_0^{-1} \star dB = J$, along with $d^\dagger dB = \Box B = -\nabla^2 B$ on account of $d^\dagger B = \star d^2 A = 0$, we get the London equation for the B field:

$$\lambda_L^2 \Box B = \Phi\delta^2(r) - B. \tag{132}$$

Now we can write the first term in $\mathcal{E}$ as

$$\mathcal{E} \supset \frac{1}{2\mu_0} \int_P (B + \lambda_L^2 \Box B) \wedge \star B = \frac{\Phi}{2\mu_0} \int_P \delta^2(r) = 0, \tag{133}$$

since P excludes the core.

Now all that's left is the $\oint$ term. The integral along $d\theta$ selects out radial derivative, and so we get

$$\mathcal{E} = -\frac{\lambda_L^2}{2\mu_0} 2\pi\xi(B\partial_r B)|_{r=\xi}. \tag{134}$$

We can find $B(r)$ by solving the London equation. Notice that the London equation is precisely the same as the equation of motion obeyed by a massive scalar field (with mass $1/\lambda_L$) in two dimensions with a source current of strength Φ/λ_L^2 at the origin. This is very reasonable, since we know that in 2d the B field is a scalar, and that it gets made massive by the Higgs mechanism with a mass set by the penetration depth.

Now the integral along ∂P takes place at a distance $r \ll \lambda_L$ (remember we are in a type II SC and assuming $\lambda_L \gg \xi$) from the origin, where B can still be accurately described by a massless scalar field. The propagator for B at a separation of r will then have the usual form of $\ln(r/r_{IR})$, where $r_{IR} = \lambda_L$ is the IR cutoff provided by the mass.

To get $B(r)$ we then just need $\langle B \rangle$ in the presence of the delta-function source. Since the 1-point function for a scalar field in the presence of a current is $\langle \phi(x) \rangle_J = \int_y G(x-y)J(y)$, we have

$$B(r) = \Phi G(r) = \frac{1}{2\pi\lambda_L^2} \ln(\lambda_L/r). \tag{135}$$

This means, after some algebra

$$\mathcal{E} = \frac{1}{4\pi\mu_0}\left(\frac{\Phi}{\lambda_L}\right)^2 \ln(\lambda_L/\xi). \tag{136}$$

Since we are interested in the minimal field for which vortices can penetrate the SC, we can take Φ to be equal to one SCing flux quantum, viz $\Phi = 2\pi/q$. We can eliminate Φ from

the above equation by noticing that the product $H_c \lambda_L \xi$ is independent of the parameters that enter the LG theory. Indeed, $H_c = \sqrt{\mu_0 t^2 / u}$, $\lambda_L = \sqrt{-u/(2\gamma \mu_0 t q^2)}$ and $\xi = \sqrt{-\gamma/t}$, so that

$$H_c \lambda_L \xi = \frac{1}{\sqrt{2}q}. \tag{137}$$

Therefore $\lambda_L^2 = \frac{1}{2q^2 \xi^2 H_c^2}$, so that

$$\mathcal{E} = \frac{2\pi \xi^2 H_c^2}{\mu_0} \ln(\lambda_L / \xi). \tag{138}$$

The condensation energy lost by the core is $-f_s^\psi \pi \xi^2$, and so in the type II limit we have

$$-f_s^\psi \pi \xi^2 = \frac{\pi}{2\mu_0} H_c^2 \xi^2 \ll \mathcal{E}. \tag{139}$$

Therefore we can basically the core contribution to the free energy when computing H_{c1}, at least in the type II limit.

Finally, we can plug this result for $\mathcal{E}$ into the expression for H_{c1}. Writing one of the H_cs as $(\sqrt{2}q\lambda_L \xi)^{-1}$ and then canceling the q with the Φ, we see that

$$H_{c1} = \frac{H_c \xi}{\sqrt{2}\lambda_L} \ln(\lambda_L / \xi). \tag{140}$$

This is exactly the same expression as in Tinkham! Yay! (also, sanity check is that $H_{c1} < H_c$).

H_{c3}—the value of H_{c2} for a superconducting slab

Consider a thin SCing slab of width d, with a magnetic field applied normal to its surface. Starting from high fields, at what point will a nonzero SCing order parameter be a solution to the GL equations? In fact we already know the answer, namely H_{c2}. The reason why the computation is unchanged between the 3d and thin slab cases is because in the 3d case, the wavefunctions we found by solving the linearized LG equations were totally independent of the z coordinate,[18] and so the entire calculation goes through unchanged for the thin slab case.

Now suppose the field is $\parallel$ to the slab. In this case we expect the critical field H_{c3} to be much higher—the small amount of expelled flux from the diamagnetic response gets pushed to right outside the surface of the SC, which causes a comparatively small amount of increase in the magnetic field free energy outside the SC.

To estimate H_{c3}, let the slab be in the xy plane, extending from $z = -d/2$ to $z = d/2$. We will assume the thin film limit so that $d \ll \lambda_L$. We can take the field in the sample to be $A = (0, -Hz, 0)$, with H the applied field. This ignores the screening from the SCing OP, but this is a higher order effect and can be neglected. Indeed, this simplification misses

---

[18]Or rather they had momentum k_z; the lowest energy one has $k_z = 0$ and hence is independent of the z coordinate.

a term in the free energy that goes like $\int dz\,(H - B\mu_0^{-1})^2$. From the London equation we expect this to go like $\int dz\,H^2(1 - e^{-z^2/\lambda_L^2})^2$, so that the free energy per unit area that we're missing is

$$\delta f \propto \int dz\, H^2 d^5 \lambda_L^{-4}, \tag{141}$$

which as we will see is suppressed by a factor of $(d/\lambda_L)^2$ relative to the other terms we will be keeping.

We will make the simplifying assumption that ψ is a constant throughout the slab. The LG equations are then, after integrating over z,

$$\gamma q^2 H^2 \frac{d^3}{24} + td + ud\psi^2 = 0 \implies \psi^2 = \psi_0^2\left(1 + \frac{\gamma q^2 H^2 d^2}{24t}\right) \tag{142}$$

where $\psi_0^2 = -t/u$ is the MF value of the OP in the absence of the field. Now the thermodynamic critical field is $H_c^2 = \mu_0 t^2/u$, and so

$$\psi^2 = \psi_0^2\left(1 + \frac{t\gamma q^2 \mu_0 H^2 d^2}{24 H_c^2 u}\right). \tag{143}$$

But then the penetration depth is $\lambda_L^{-2} = q^2 \mu_0 \gamma |\psi_0|^2$, so

$$\psi^2 = \psi_0^2\left(1 - \frac{d^2 H^2}{24\lambda_L^2 H_c^2}\right) \implies H_{c3} = \sqrt{24}\frac{\lambda_L}{d} H_c. \tag{144}$$

Therefore as the film gets smaller and smaller compared to λ_L, the film starts to have a nonzero SCing OP at higher and higher fields.

11 Mean field theory for AFMs and Ferrimagnets

This is an elaboration on an exercise in A& M. The goal is to determine the analogue of the Curie-Weiss law $\chi \sim 1/(T - T_c)$ for AFMs.

Solution:

The starting point is a Hamiltonian of the form

$$\mathcal{H} = -J_A \sum_{\langle ij\rangle_{AA}} \mathbf{S}_i \cdot \mathbf{S}_j - J_B \sum_{\langle ij\rangle_{BB}} \mathbf{S}_i \cdot \mathbf{S}_j - J_3 \sum_{\langle ij\rangle_{AB}} \mathbf{S}_i \cdot \mathbf{S}_j - \mu g \sum_{i\in A,B} \mathbf{S}_i \cdot \mathbf{H}, \tag{145}$$

where A, B label two bipartite sublattices and we assume that the spins on the two sublattices couple to the external field H in the same way. To get an AFM, we need to assume that $J_3 < 0$ and that $|J_3|$ is sufficiently large. Depending on the relative sizes of J_A, J_B, we can either get something that is AFMic or ferrimagnetic.

The MFT works by basically doing a separate MF for each sublattice. The magnetizations M_A, M_B for each sublattice are then computed by setting them equal to the magnetization

M_0 experienced in a non-interacting Hamiltonian with an effective magnetic field set by the epxectation values of the other spins in the various interaction terms: we get

$$
\begin{aligned}
M_A &= M_0[(H + \lambda_A M_A + \lambda_3 M_B)/T] \\
M_B &= M_0[(H + \lambda_B M_B + \lambda_3 M_A)/T],
\end{aligned}
\tag{146}
$$

where $\lambda_i \equiv J_i/(\mu g)^2$.[19] The susceptibility is then

$$
\chi_A = (\partial_{H_{eff;A}} M_0)(\partial_H H_{eff;A}), \qquad \chi_B = (\partial_{H_{eff;B}} M_0)(\partial_H H_{eff;B}).
\tag{147}
$$

When evaluated at zero external field and above T_c,[20] the derivative of M_0 will just be given by the derivative of the usual tanh formula, which above T_c (where T_c is such that when $T > T_c$ we have $M_A = M_B = 0$ so that $H_{eff;A/B} = 0$) gives a zero-field result for $S = 1/2$ of $\chi_0 = (g\mu)^2/4T \equiv \alpha T^{-1}$. Then the zero-field susceptibility above T_c is[21]

$$
\chi = \chi_A + \chi_B = \chi_0(2 + \lambda_A \chi_A + \lambda_B \chi_B + \lambda_3 \chi) \implies \chi = \frac{2 + \lambda_A \chi_A + \lambda_B \chi_B}{\chi_0^{-1} - \lambda_3}.
\tag{148}
$$

Solving the two equations for χ_A, χ_B and plutting them into this, some algebra gives

$$
\chi = \frac{\alpha^2(\lambda_A + \lambda_B) - 2\alpha T - 2\alpha^2 \lambda_3}{\alpha^2 \lambda_3^2 - \alpha^2 \lambda_A \lambda_B + \alpha T(\lambda_A + \lambda_B) - T^2}.
\tag{149}
$$

As a sanity check, we can set $\lambda_A = \lambda_B \equiv \lambda$ and take $\lambda, \lambda_3 > 0$: then we get a bunch of identical ferromagnetically coupled atoms. Do we get the usual Curie-Weiss law? Yes: in this case

$$
\chi = \frac{2}{\chi_0^{-1} - (\lambda_3 + \lambda)},
\tag{150}
$$

so that $J_3 + J_A$ determines T_c. If $\lambda_3 < -\lambda < 0$ we expect AF order. In this case we see that T_c becomes negative, and that MF thus predicts that AFMs do *not* display a diverging susceptibility at any finite T, unlike FMs.

12 Super basic symbol pushing with polarization and 1PI vs full susceptibilities

Today is basically just recalling some notation about polarization and susceptibilities in basic electron response theory.

Solution:

---

[19]The magnetization is $M_i = \mu g \langle \mathbf{S}_i \rangle$, so that the terms in the Hamiltonian go over to e.g. $J(\mu g)^{-1}\mathbf{M} \cdot \mathbf{S} = \mu g \mathbf{H}_{eff} \cdot \mathbf{S}$, where $H_{eff} = MJ/(\mu g)^2$.

[20]We will always be above T_c because $T < T_c$ is pretty hard to treat in MFT: the onset of order means that the derivatives of M_0 above have to be evaluated in a nonzero magnetic field (due to the spontaneous magnetization), which then needs to be solved for numerically in a self-consistent way.

[21]Sanity check: χs have units of $(\mu g)^2/T = J/(\lambda T)$, so that $\lambda \chi$s are properly dimensionless.

The definitions of the susceptibility and the dielectric function are (in $\mathbf{q}$ space with arguments suppressed)

$$\phi_a = \epsilon\phi, \qquad \rho_i = \chi\phi_a, \tag{151}$$

where ϕ_a is an applied potential and ρ_i is an induced charge density. Notice how χ multiplies the *applied* potential, not the full potential, while for ϵ it is the opposite. We can calculate the relationship between χ and ϵ as follows. We start with Poisson's equations for the full and applied charges:

$$\phi_a = \frac{4\pi}{q^2}(\rho - \rho_i), \qquad \phi = \frac{4\pi}{q^2}\rho. \tag{152}$$

Dividing these two equations,

$$\phi_a/\phi = \epsilon = 1 - \rho_i/\rho. \tag{153}$$

Then we write the ratio of the charges as $\rho_i/\rho = 4\pi\rho_i/(\phi q^2) = 4\pi q^{-2}\chi\phi_a/\phi = 4\pi q^{-2}\chi\epsilon$. Therefore

$$\epsilon = 1 - \frac{4\pi}{q^2}\chi\epsilon \implies \epsilon = \frac{1}{1 + \frac{4\pi}{q^2}\chi}. \tag{154}$$

We can also define a "susceptibility" for relating the induced charge to the *full* potential, not just the applied one. Annoyingly these are often not distinguished in the literature, and so here we will give it a different name, viz. χ_{1PI}—the reason for this will become clear in a sec. We find

$$\epsilon = 1 - \rho_i/\rho = 1 - \chi_{1PI}\phi/\rho = 1 - \frac{4\pi}{q^2}\chi_{1PI}. \tag{155}$$

The reason why we've given it the 1PI moniker can be seen by solving for χ in terms of χ_{1PI}:

$$\chi = \frac{\chi_{1PI}}{1 - \frac{4\pi}{q^2}\chi_{1PI}}. \tag{156}$$

We recognize this as the geometric sum performed when doing the Dyson series thing—hence χ_{1PI} is indeed the 1PI part of the "full" susceptibility χ.

The reason for introducing both of these susceptibilities is that there are scenarios in which both of them are the more natural things to calculate.

Thomas-Fermi: When we do TF screening, we most naturally calculate χ_{1PI}. Recall how this works: in $\mathbb{R}$ space,

$$\rho_i = -\int_{\mathbf{k}} (n_F(\epsilon_{\mathbf{k}} - \phi) - n_F(\epsilon_{\mathbf{k}})), \tag{157}$$

where ϕ is the *full* potential, not the applied one. Assuming that the full potential ϕ varies slowly compared to $1/k_F$ (we are assuming ϕ is constant in time, so we will be computing $\chi(0, q)$ at small q), we approximate the integral as

$$\rho_i \approx -e\int_{\epsilon} \nu(\epsilon)\beta(\partial_\mu n_F)e\phi \approx -e^2\nu(\epsilon_F)\phi \implies \chi_{1PI} \approx -e^2\nu(\epsilon_F). \tag{158}$$

Sanity check: the susceptibility is negative, since a positive potential should attract electrons, making the charge density more negative. Also note that it is a constant, i.e. independent

of q—this guarantees the divergence of $\epsilon(0, q)$ as $q \to \infty$, which we know has to be there in a metal. Indeed, this tells us that $(\nu \equiv \nu(\epsilon_F))$

$$\epsilon = 1 + e^2 \nu \frac{4\pi}{q^2} \tag{159}$$

and that

$$\chi = -\frac{e^2 \nu}{1 + \frac{4\pi}{q^2} e^2 \nu}. \tag{160}$$

Since $\nu(\epsilon_F) \propto \sqrt{\epsilon_F}$, $\chi_{1\text{PI}} \propto \sqrt{\epsilon_F}$. Another quick way to see this is as follows: since $n \propto \epsilon_F^{3/2}$, we have

$$\delta \ln n = \frac{3}{2} \delta \ln \epsilon_F. \tag{161}$$

Now $\delta \epsilon_F = -e\phi$, and so

$$\delta \rho = -\frac{3}{2} \rho \frac{e\phi}{\epsilon_F} \implies \chi_{1\text{PI}} = -\frac{3e\rho}{2\epsilon_F}. \tag{162}$$

Then since $\rho \sim e\epsilon_F^{3/2}$, we again confirm that $\chi_{1\text{PI}} \propto e^2 \epsilon_F^{3/2}$.

Lindhard: By contrast, when we do Lindhard screening, the most natural susceptibility to calculate is the full one χ (although of course in the calculation we just compute $\chi_{1\text{PI}}$ directly). The reason why χ is more natural here is that the Lindhard calculation is the usual linear response one, where $\delta \rho \sim \int \langle [\rho, \rho] \rangle \phi_{ext}$—since it is the external potential and not the full one that appears here, the response function calculated when doing screening in this way is χ, not $\chi_{1\text{PI}}$. The calculation goes through in the same way.

Finally, since I don't know where else to put it, a comment on the order of limits in the susceptibilities. When one first sets $\omega = 0$ and then takes $q \to 0$, we get a nonzero χ as calculated by e.g. TF screening. However, if one first sets $q = 0$ at finite ω and only then sends $\omega \to 0$, then from the $f_{\mathbf{k}-\mathbf{q}} - f_{\mathbf{k}}$ in the denominator of the polarization bubble, we see that $\chi(\omega \to 0, 0) = 0$. The reason for this is that χ is the response function for the charge density: if we set $q = 0$ then we are examining the response of the system to a spatially uniform potential—but this is not something the system can respond to, since charge must be conserved. So asking for the response at $q = 0$ isn't really meaningful—a $q = 0$ result should always be derived by sending $q \to 0$ at the end (e.g. after the q^2 from χ has canceled the $1/q^2$ pole of the interaction in the dielectric function).

13 Plasmons in a 2DEG

Consider a gas of electrons confined to a plane and interacting according to a (three-dimensional) Coulomb potential

$$V(x) = \frac{e^2}{\kappa x}. \tag{163}$$

Find the plasmon dispersion and contrast it to the 3d case.

Solution:

In 3d, the plasma frequency can be found by a simple argument where one considers a bunch of electrons oscillating coherently on top of a uniform positive background. Since the electric field from an infinite charged sheet in 3d is independent of the distance away from the sheet and proportional to the charge density of the sheet (times 4π), the classical eom are (cgs!)

$$m\partial_t^2 x = -4\pi n x e^2, \tag{164}$$

where x is the displacement of the block of electrons. This correctly gives the plasma frequency.

In two dimensions, this argument isn't going to work. Here the Coulomb potential is still $1/r$, but now we need to look at the electric field sourced by an infinite line charge, which falls off as distance as $1/r$. Since the plasma frequency should be proportional to the 2d density of the electrons and the field falls off as $1/r$, the only way the units can work out is to have

$$\omega^2 \propto \frac{nqe^2}{m}, \tag{165}$$

where now n is the two-dimensional density. This means that the plasmons will be gapless, unlike in three dimensions.

Now we can confirm this with a more careful screening calculation. The polarization is[22]

$$\epsilon(\omega, \mathbf{q}) = 1 - V_\mathbf{q} \chi_{1PI}(\omega, \mathbf{q}), \tag{168}$$

where χ_{1PI} is the 1PI part of the susceptibility and $V_\mathbf{q}$ is the Coulomb interaction. The important thing here is that the Fourier transform for V is calculated in two dimensions, despite the fact that V is the $1/r$ three-dimensional Coulomb potential (the Fourier transform happens by expanding the electronic modes in two-dimensional Fourier harmonics—one cannot just take the 3d FT'd potential and restrict to in-plane momenta!). We do the Fourier transform with

$$V_q = e^2 \int dx\, d\theta\, \frac{x e^{iqx\cos\theta}}{x\kappa} = \frac{2\pi e^2}{q\kappa} \int_0^\infty dx\, J_0[x] = \frac{2\pi e^2}{\kappa q}. \tag{169}$$

---

[22]Recall why we're calculating this: the effective interaction at momentum q, as determined by the density-density correlator, goes as V_q/ϵ_q. Therefore a zero of the dielectric function is a pole in the density-density correlator, i.e. a collective excitation of the medium. Above the plasma frequency ϵ is positive (and tends to 1 as ω gets big), while below it is negative; hence ω_p marks the frequency above which a metal becomes transparent, since $\epsilon(\omega_p) = 0$. Note also that $\epsilon(\omega) = 0$ is a requirement for any longitudinal wave that propagates in a medium where the only charge is the induced (bound) charge, simply because $\mathbf{k} \cdot \mathbf{E} \neq 0$ means that if $\rho = 0$ then $\epsilon = 0$ on account of

$$\epsilon \mathbf{k} \cdot \mathbf{E} = \rho_f, \tag{166}$$

where ρ_f is the free charge density.

Actually one can do a bit better, and prove that $\epsilon = 0$ without making assumptions about ρ_f (although $\epsilon = 0 \implies \rho_f$). This is done by writing

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \implies i\mathbf{k} \cdot \mathbf{E} = i\sigma^{-1}\mathbf{k} \cdot \mathbf{j} = i\sigma^{-1}\omega\rho = \rho/\epsilon_0 \implies \sigma = \frac{i\omega}{\epsilon_0}. \tag{167}$$

Then since $\epsilon = 1 + i\sigma/(\omega\epsilon_0)$, we get that $\epsilon = 0$, and hence $\rho_f = 0$ as well.

We will calculate χ in the low wavelength limit using the standard method: starting from the point at which we've already exactly integrated out the frequency running in the polarization bubble, we have (working at $T = 0$)

$$
\begin{aligned}
\chi(\omega, q) &= -\int_{\mathbf{k}} \frac{f_{\mathbf{k}+\mathbf{q}} - f_{\mathbf{k}}}{\omega - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})} \\
&\approx -\int_{\mathbf{k}} \frac{\mathbf{q} \cdot \mathbf{v_k} \partial_\epsilon f(\epsilon_{\mathbf{k}})}{\omega - \mathbf{q} \cdot \mathbf{v_k}} \\
&\approx -\int_{\mathbf{k}} \frac{(\mathbf{v_k} \cdot \mathbf{q})^2 \partial_\epsilon f(\epsilon_{\mathbf{k}})}{\omega^2} \\
&= -\frac{q^2}{4\pi^2\omega^2} \int d\theta \, k \, dk \, |\mathbf{v_k}|^2 \cos^2\theta \partial_\epsilon f \\
&= \frac{q^2 N(0) v_F^2}{\omega^2} \int \frac{d\theta}{2\pi} \cos^2\theta \\
&= \frac{q^2 N(0) v_F^2}{2\omega^2}.
\end{aligned}
\tag{170}
$$

Therefore the condition that the dielectric function vanish is (adding in a factor of 2 for spin degeneracy which I forgot, so that $N(0) = m/\pi$, and writing stuff in terms of the electron density $n = 2v_F^2 m^2/\pi$)

$$
\omega = q\sqrt{\frac{ne^2}{4m\kappa}},
\tag{171}
$$

where the $1/4$ should probably not be unduly trusted. This is exactly the form we expected: all the same constants appear as in the 3d case, expect the density is replaced with the 2d electron density, with a corresponding factor of q appearing to make the units work.

14   Specific heats of stuff

Today we're going to derive specific heats for a bunch of different systems that come up often in cond-mat.

Solution:

Preliminary remark: since we are interested in calculating the specific heat at constant density, particle number and volume will be held fixed, which means that μ will *not* be an independent variable when taking derivatives, but rather will be a function of N and T and so on, complicating things. First we'll do stuff without chemical potentials.

Free classical stuff

We start with the most trivial example: non-interacting classical systems. Here we assume a Hamiltonian like $H = \sum_i p_i^2/2m + \sum_{i,j} u_{i-j} r_i r_j$, so that if we are working in the canonical

ensemble,

$$\ln Z = \ln \left[\frac{1}{N!} \left(\int d^d r d^d p \, e^{-\beta H} \right)^N \right] \tag{172}$$

We then re-scale the dof by $p' = \sqrt{\beta} p$, $r' = \sqrt{\beta} r$, which makes the T-dependence of $\ln Z$ manifest:

$$\ln Z = -Nd \ln \beta + \ldots, \tag{173}$$

where $\ldots$ is T-independent. Therefore in such systems U always has a T-dependence of dNT, meaning that

$$C = dN, \tag{174}$$

which is precisely the equipartition theorem. In fact we can make a slightly more general statement: if the single-particle partition function goes as T^α, then we will have $\ln Z \sim N\alpha \ln T$ and hence $U \sim N\alpha \implies C = N\alpha$. Therefore all such classical systems have a constant heat capacity. We get around this in e.g. quantum mechanics since the single-particle partition function $\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = (1 - e^{-\beta\hbar\omega})^{-1}$ doesn't scale as a power of T.

However, the single-particle harmonic oscillator can also be imagined as an indefinite-N classical system, modulo distinguishability issues.

What about in the GCE? There we just have, for a classical system,

$$U = \sum_\epsilon \epsilon e^{-\beta(\epsilon - \mu)}. \tag{175}$$

To find C, we need to know if the derivative is taken at constant μ or at constant $e^{\beta\mu}$. $\ldots$

Heisenberg and Ising models

We'll do the high T limit first. We expand the energy U as a series in $\beta J \ll 1$,

$$U = \frac{1}{Z} \sum_{\{s_i\}} \sum_{\langle ij \rangle} J s_i s_j \left(1 - \beta J \sum_{\langle ij \rangle} s_i s_j + \frac{\beta^2 J^2}{2} \sum_{\langle ij \rangle \langle kl \rangle} s_i s_j s_k s_l + \ldots \right). \tag{176}$$

The first contribution is the one where $s_i s_j$s from the first sum multiply the same pair of spins in the sum proportional to β. Therefore to lowest order,

$$U \approx -\frac{J^2 \beta}{Z} \sum_{\{s_i\}} \sum_{\langle ij \rangle}. \tag{177}$$

When we differentiate this wrt T the term with a derivative of Z^{-1} will die by oddness of the sum, and so the derivative just hits the β. Therefore

$$C(T) \approx 2^N (N/2)(J/T)^2, \tag{178}$$

where N is the number of sites. Therefore we get a decaying power law with exponent 2.[23]

---

[23]This makes sense cause in the Ising model the energy starts to saturate at high temperatures, in contrast to e.g. harmonic oscillators, where the energy can be made arbitrarily big by heating the system sufficiently.

The exact same story goes through for the Heisenberg model: there we're taking a trace, and the trace (to lowest order in β) projects onto the same product of spins on identical links as before (the structure only becomes different when you go to higher powers in the expansion, since then you can have e.g. $XYZ \propto \mathbf{1}$, which is traceful). Therefore we again get $C(T) \sim T^{-2}$ at large T.

Free things without chemical potentials

Let the things be described by a distribution function $\rho(\beta\epsilon)$ (the absence of a chemical potential means that the things are not conserved), and let the dynamical exponent z be such that $\epsilon = \alpha k^z$. Then

$$U = \int d^d k \, \rho(\beta k^z \alpha) \alpha k^z. \tag{179}$$

Then letting $x \equiv \beta k^z \alpha$, we have

$$U \sim \int_0^\infty dx \, \rho(x) \left(\frac{x}{\beta\alpha} \right)^{d/z+1}, \tag{180}$$

where we've ignored factors relating to the volume of spheres. Since x is dimensionless, we get

$$U \sim (\beta\alpha)^{-1-d/z} \implies C(T) \sim \alpha^{-1-d/z} T^{d/z}. \tag{181}$$

When talking about e.g. phonons, this T dependence is appropriate at low temperatures, with $T/\omega_D \ll 1$ where ω_D is the Debeye frequency, set by the finite size of the BZ. As T approaches ω_D $C(T)$ will start to freeze out (as with any system where the energies of the modes are bounded from above), eventually reaching the constant $C \sim N$ at $T \gtrsim \omega_D$ in accordance with equipartition. This is checked just by fixing ω_D as the upper bound on the integral, but needs to be done numerically to get the full profile.

One comment here: is there anything quantum about the above heat capacities? We often talk about the Debeye T^3 scaling of the specific heat as being quantum, but is that really true? I think the answer is basically no. The discreteness of the oscillator states is not really a quantum thing, because we can equally well think of the oscillator modes as simply being filled with multiple classical particles, rather than a single quantum particle (this is possible since we're only talking about harmonic degrees of freedom). The counting is different in the two cases due to indistinguishability, but I don't think that has a significant effect on things like the T-dependence of $C(T)$. For example, consider a model of phonons where the phonons are treated as classical particles with a dispersion $\omega = ck$. Then the only thing that is different in the calculation of the specific heat is the distribution function (MB instead of BE), but this difference doesn't affect the T-scaling of the integral, at least as far as we may send $\omega_D \to \infty$. Admittedly having $\omega = ck$ for a classical particle is a bit weird (at least non-relativistically), but the point is that the only real quantum-ness here is in the statistics that determines the distribution function, which has little bearing on the scaling of $C(T)$.

Free fermions with a chemical potential

This is actually a surprisingly hard example to do in full generality! The presence of the chemical potential actually complicates our ability to solve things exactly.

One limit where we can of course solve things exactly is when $\mu\beta \gg 1$—in this case, only states with energies close to μ contribute, and we can essentially take $\mu(T) = \mu(0)$.[24] The classic example is that of the Fermi liquid at $T \ll T_F$. Intuitively, U (relative to the ground state) will be given by the number of particles excited to deviate from the $T = 0$ distribution times their average energies. Now a typical thermally excited fermion comes from within a distance β^{-1} of $\mu \approx \epsilon_F$, and acquires a typical energy of β^{-1}; hence $U \propto \beta^{-2} \implies C(T) \propto T$. Of course this can be checked by doing

$$U \sim \int_0^\infty d\epsilon\, N(0) \frac{\epsilon}{1 + e^{\beta(\epsilon-\mu)}} \sim \int_{-\infty}^\infty dx\, N(0) T^2 \frac{x}{1 + e^x} \implies C(T) \sim N(0)T. \qquad (183)$$

The simplifications coming from $\mu\beta \gg 1$ were that we could approximate the density of states by the constant $N(0)$, and that we could take $\mu \to \epsilon_F$ to be T-independent and big enough to extend the integral over $\beta(\epsilon - \mu)$ to be over the whole $\mathbb{R}$ line.[25]

Now in the low density limit we know that we need to get $C \sim T^{d/z}$, and so evidently there must be some rather complicated crossover as T changes form between the small $\mu\beta$ and large $\mu\beta$ regimes. Basically, the complications stem from doing an integral from $-\beta\mu$ to ∞, and expanding the result in terms of powers of $\beta\mu$. Since the analysis looks rather involved, we'll leave it at that.

Gasses with internal degrees of freedom

Let the typical energy spacing between the levels coming from the internal dof be Δ. If $\beta\Delta \gg 1$ or $\beta\Delta \ll 1$, the internal levels makes no contribution to the specific heat. Intuitively this is clear since for $\beta\Delta \gg 1$ the internal dof are frozen out, while for $\beta\Delta \ll 1$ their energies are completely saturated, so that U no longer changes with T. More precisely, if $\beta\Delta \ll 1$ then the internal dof contribute to the partition function a factor

$$Z_I \approx N_{dof}, \qquad (184)$$

which affects things like the entropy and chemical potential, but not the specific heat due to the trivial T-dependence. For $\beta\Delta \gg 1$, we can project onto the internal energy level of lowest energy—since the internal dofs are then frozen out, we should get $C_I = 0$. Indeed,

$$C = -\partial_T \partial_\beta \ln Z = \beta^{-2} \partial_\beta^2 \ln Z \implies C_I = \partial_T \partial_\beta(\beta\Delta) = 0. \qquad (185)$$

---

[24]I think one thing that is easy to forget after just reading cond-mat stuff is that the chemical potential in this setting is not the Fermi energy, nor is it T-independent. Indeed, we have

$$N \approx \int d^d\mathbf{k}\, \rho[\beta(\epsilon_{\mathbf{k}} - \mu)] \qquad (182)$$

for some distribution ρ. When we change β but hold N fixed, μ must evidently also change to keep the LHS invariant. This is basically another way of saying that when we write the distribution function in terms of μ, μ is a Lagrange multiplier that we haven't yet solved the constraint to determine—it will generally be a function of all the expectation values of the conserved quantities in the system.

[25]This is of course a good assumption in most metals, since the corrections to μ from ϵ_F are of order T^2/ϵ_F^2, which is $O(10^{-4})$ in most metals.

Therefore the contribution to C from the internal dof will have some nontrivial behavior as a function of T, with a local maximum near the order of $T \sim \Delta$ and dying back down to zero at $T = 0$ and $T \to \infty$.
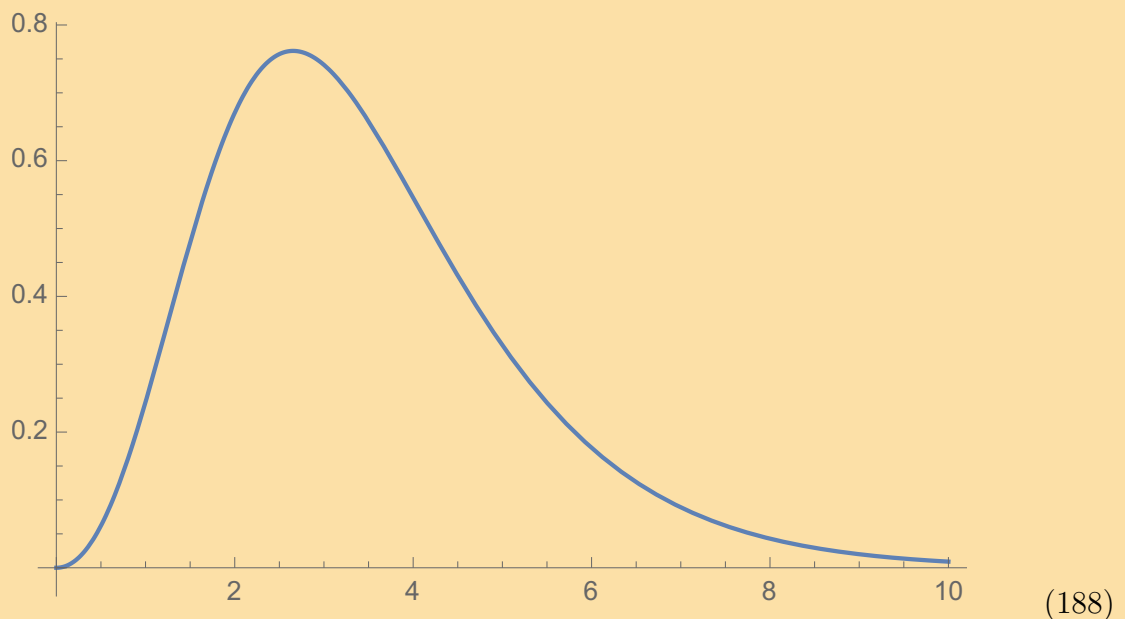
A simple example is a two-level system with a gap Δ. Since $e^{-\epsilon\beta}$ factors don't contribute to the specific heat, we may take the lower level to have energy 0 and the upper level to have energy Δ. Therefore the internal dof partition function is

$$Z_I = g_0 + g_1 e^{-\beta\Delta}. \tag{186}$$

The specific heat is then easy to calculate—some algebra gives

$$C(T) = \frac{(\beta\Delta)^2}{2 + (g_0/g_1)e^{-\beta\Delta} + (g_1/g_0)e^{\beta\Delta}}. \tag{187}$$

This looks like (shown is $C(\beta\Delta)$ against $\beta\Delta$).

$$\tag{188}$$
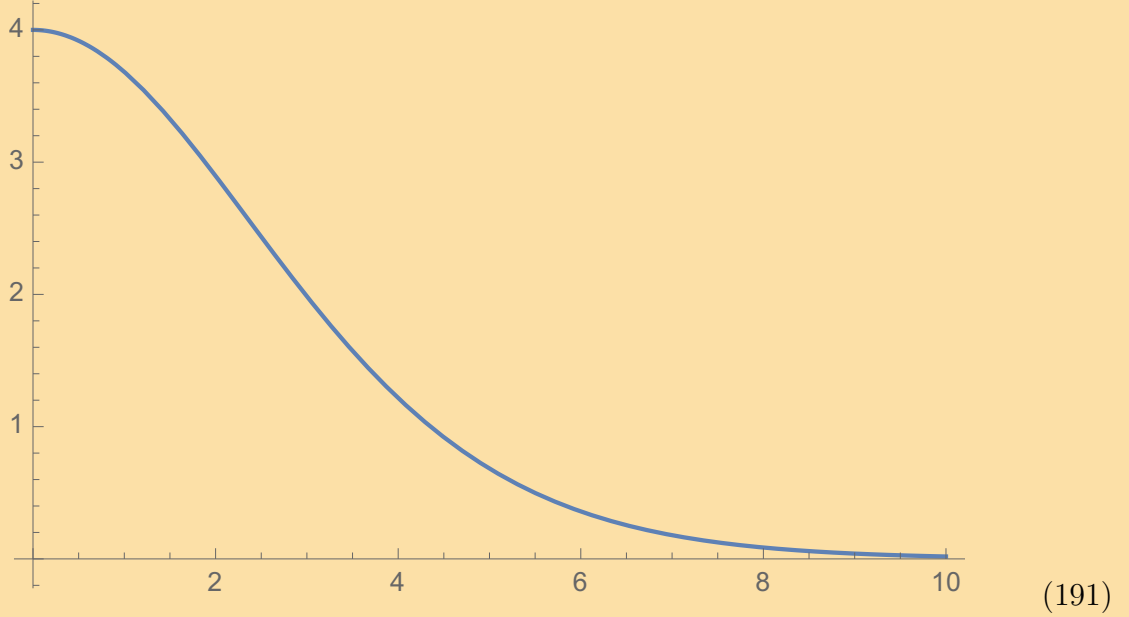
which is the shape we expect.

Another example is the specific heat coming from internal vibrational modes, which we treat as harmonic oscillators. The internal partition function is then

$$Z_I = \frac{2}{\sinh(\omega\beta/2)}. \tag{189}$$

After some algebra, we get

$$C = (\beta\omega)^2 \sinh^{-2}(\omega\beta/2). \tag{190}$$

Again, plotted as a function of $\beta\omega$, this looks like

$$(191)$$

The important thing here is that at high T, we get a non-zero constant, in accordance with the equipartition theorem. As T drops below ω, $C \to 0$ as expected from freezeout.

Yet another example is rigid rotors coming from rotational dof. The parition function is

$$Z_I = \sum_{l \in \mathbb{N}} (2l+1)e^{-\beta l(l+1)J}, \tag{192}$$

with $J = 1/2I$. When $\beta J \ll 1$, we can turn this into an integral. This yields

$$\begin{aligned} Z_I &\approx \int dl \, (2l+1)e^{-\beta Jl(l+1)} \\ &= \int_{1/2}^{\infty} dl \, 2l e^{-(l^2-1/4)\beta J} \\ &= \int_{\beta J/4}^{\infty} du \, \frac{1}{\beta J} e^{\beta J/4} e^{-u} \\ &= (\beta J)^{-1}. \end{aligned} \tag{193}$$

Anytime we get a partition function that's linear in T, we get a specific heat capacity that's constant—again, this is in accordance with the equipartition theorem. On the other hand, when $\beta J \gg 1$, we can keep only the first two terms, which are all that's needed to get a nontrivial T dependence:

$$Z_I \approx 1 + 3e^{-2\beta J}, \tag{194}$$

which is in the same form as that of the two-level system we considered earlier, and gives

$$C \approx 12(\beta J)^2 e^{-2\beta J}, \tag{195}$$

where we have expanded the sinh at large βJ.

15 Clarification of chemical potential confusions

Doing the Legendre transform, we introduce

$$\mathcal{F} = \min_N [F - \mu N] \tag{196}$$

This just means that $\mathcal{F} = F - \mu N$, subject to the constraint that $\partial_N F = \mu$. This consequently ensures that

$$\partial_\mu \mathcal{F} = \partial_N F \partial_\mu N - N - \mu \partial_\mu N = -N, \tag{197}$$

as desired. Of course, the explicit formula for $\mathcal{F}$ is

$$\mathcal{F} = -\beta^{-1} \ln \mathcal{Z}, \tag{198}$$

where

$$\mathcal{Z} = \prod_\alpha \sum_{\{n_\alpha\}} e^{-n_\alpha \beta (\epsilon_\alpha - \mu)}, \tag{199}$$

where the product is over all energy levels. Seeing why this prescription gives $\mathcal{F} = F - \mu N$ directly from the Boltzmann weights is a little tricky algebra-wise. A quicker way to see it directly is to just calculate $\partial_\mu \mathcal{F}$ and check that it gives N (as required by the Legendre transform)

$$\partial_\mu \mathcal{F} = -\frac{1}{\beta} \partial_\mu \sum_\alpha \ln \left[\sum_{\{n_\alpha\}} e^{-n_\alpha \beta (\epsilon_\alpha - \mu)} \right] = \sum_\alpha \frac{\sum_{\{n_\alpha\}} n_\alpha e^{-\beta n_\alpha (\epsilon_\alpha - \mu)}}{\sum_{\{n_\alpha\}} e^{-\beta n_\alpha (\epsilon_\alpha - \mu)}} = N, \tag{200}$$

and therefore $\mathcal{F} = F - \mu N + C$, where C is a constant.

Some books make a point of only discussing the fugacity $z = e^{\beta \mu}$. This is a good idea, since dealing with the chemical potential directly is cumbersome on account of the fact that the Lagrange multiplier fixing the expectation value of N is *not* μ, but rather z (or $\beta\mu$ if you prefer). The reason for often focusing on μ is because of the β^{-1} in the definition of the free energy, which means that the free energies are more naturally thought of as functions of μ than z. But really we'd all probably be better off just working with $-\ln Z$ all of the time, at least from the point of view of computing things.

An example of why z is the more natural variable comes from trying to compute the expectation value of the energy, U, in the GCE. Here the log of the partition function is

$$\ln \mathcal{Z} = \zeta \sum_\epsilon \ln[1 + \zeta z e^{-\beta \epsilon}], \tag{201}$$

where $\zeta = \pm 1$ according to statistics. Obviously, taking $\partial_\beta \ln \mathcal{Z}$ at fixed μ does not give U—rather, we need to take the derivative wrt β at fixed z, not fixed μ. This means that the more natural variables to work with are the real Lagrange multipliers, namely β and z.

Sign of μ

Recall that the chemical potentials for classical gases and bosons are negative, i.e. always less that the lowest energy level of the system—this is just to ensure that e.g. the Boltzmann weights $e^{-\beta\xi} = e^{-\beta\epsilon}z$ are less than 1 and that occupation numbers $(e^{\beta\xi}-1)^{-1} = (z^{-1}e^{\beta\epsilon}-1)^{-1}$ are positive. Therefore μ is *not* a measure of the amount of energy that it costs to add a particle to the system, which is never negative. Instead, μ is the *free* energy that it costs to add a particle to the system (or rather, to change the expectation value of the particle number by 1). Adding a particle will increase the energy, but it will also increase S at fixed T, since heuristically the system now has more ways to occupy its energy levels. Now μ could be defined as the energy cost of a particle while working at fixed S—but to fix S after adding a particle would only be possible if that particle had negative energy; hence $\mu < 0$. The situation for fermions can be different because of Pauli exclusion. Adding a particle to a Fermi gas with relatively high density at low T will cost a large energy $\sim \epsilon_F$ (okay, ϵ_F is probably only defined at $T = 0$, but you know what I mean). However, it will not come with a correspondingly large change in entropy, since Pauli exclusion means that the number of additional ways you have to arrange your occupation numbers doesn't become much greater—you still just have to fill up the FSea. Therefore for fermions the change in energy can be larger than the change in TS, meaning that low T fermions can have $\mu > 0$. Since turning up T decreases the free energy cost for added particles, we will have $\mu(T) < \mu(0)$ for all T, with μ being expanded like $\mu(T)/\mu(0) = 1 - a(T/T_F)^2 - b(T/T_F)^4 - \ldots$, with $a, b, \cdots > 0$. Eventually at high enough T_*, for which

$$E_F^{3/2} \sim \int_0^\infty d\epsilon \frac{\sqrt{\epsilon}}{e^{\epsilon/T_*} + 1}, \tag{202}$$

we'll get $\mu < 0$; this happens at around $T_* = T_F$.

Classical limit

What is the classical limit? It is not $\beta \to 0$, but rather $\beta\mu \to -\infty$, i.e. $z \to 0$.[26] Actually more precisely, it is $\beta(\epsilon - \mu) \to \infty$—the important thing here is that $\langle n_\epsilon \rangle \to 0$.

One can see this rather explicitly in the case of the regular classical ideal gas. When we deal with the partition function we break up phase space into volume elements of size $2\pi\hbar$, which is introduced to make Z properly dimensionless. In three dimensions then, we need

$$N = \frac{V}{(2\pi\hbar)^3} \int d^3p \, e^{-\beta(p^2/2m - \mu)} \implies e^{-\beta\mu} = \frac{v}{(2\pi\hbar)^3}(2\pi m\beta^{-1})^{3/2}, \tag{203}$$

where $v = V/N$ is the specific volume. Therefore

$$\beta\mu = \frac{3}{2} \ln\left[\frac{2\pi\hbar^2}{v^{2/3}mT}\right]. \tag{204}$$

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[26]Thinking of it as $T \to \infty$ isn't good—T isn't dimensionless, and we are totally within our rights to consider low T classical systems. Plus just sending $T \to \infty$ would mean $\beta\mu \to 0$, which as we will see is definitely not the right classical limit.

Therefore in the dilute limit $v \to \infty$, or in the high-T limit, or in the small phase-space-cell limit $\hbar \to 0$, we always get $\mu \to \infty$; since all of these limits are ones in which we expect to recover classical behavior, we see that indeed $\mu\beta \to -\infty$. This is another reason why BECs and metals are very much not classical.

As another check that the appropriate limit is $\mu\beta \to -\infty$, we can require that the partition functions of FD and BE gasses reduce to the Boltzmann gas partition function in the limit. Recall that

$$\ln Z_\zeta = -\zeta \sum_\epsilon \ln[1 - \zeta e^{-\beta(\epsilon-\mu)}], \tag{205}$$

where ζ is $+1$ for bosons and -1 for fermions. On the other hand, a classical gas is

$$\ln Z_{MB} = \ln\left[\sum_N \frac{1}{N!} e^{\beta\mu N} \left(\sum_\epsilon e^{-\beta\epsilon}\right)^N\right] = \sum_\epsilon e^{-\beta(\epsilon-\mu)}. \tag{206}$$

We see that $\ln Z_\zeta \to \ln Z_{MB}$ when we take $\beta\mu \to -\infty$, but that e.g. setting $\beta(\epsilon - \mu) \to 0$, which would be the ($T \to \infty \implies$ classical) thing to do, doesn't give the right answer.

16   Landau diamagnetism

Today we will briefly discuss Landau diamagnetism from a few different perspectives.

Solution:

Note that we will be assuming spinless (or spin-polarized) particles in what follows—hence the magnetization that we will derive comes entirely from kinetic effects (think of current loops and stuff).

We will need the degeneracy of Landau levels in what follows, so let's recall how to get that. Here's a slick way that I haven't seen in textbooks: the kinetic energy of the particles undergoing the cyclotron motion will be minimized when the AB phase accumulated around the orbit is in $2\pi\mathbb{N}$. We expect the ground state to have the minimal flux of 2π, and so therefore we require that[27]

$$\frac{e}{\hbar c} \oint d\mathbf{x} \cdot \mathbf{A} = 2\pi \tag{207}$$

which for a uniform magnetic field implies that the radius of the orbit must satisfy

$$r_o = \sqrt{\frac{2\hbar c}{eB}} = \sqrt{\frac{2\hbar}{m\omega_c}}. \tag{208}$$

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[27]Here we're just saying that the solution to the eom should be to have ψ be covariantly constant. This is a little bit slick since we know that the spin of the electron (note: is this the real reason?) means there's an extra $+\pi/2$ when doing Bohr-Sommerfeld, which we need to not think about in the present context.

This then gives a degeneracy of

$$N = \frac{L^2}{\pi r_o^2} = \frac{L^2 m \omega_c}{2\pi \hbar}. \tag{209}$$

Now the DOS for a free Fermi gas in two dimensions is $\rho(\epsilon) = L^2 m/(2\pi\hbar^2)$, and so an easy way to remember this degeneracy is by writing

$$N = \rho_{2d} \hbar \omega_c, \tag{210}$$

which makes total sense: we estimate the degeneracy by $\delta N = (dN/d\epsilon)\delta\epsilon = \rho\delta\epsilon$, with $\delta\epsilon = \hbar\omega_c$ the energy that we expect the electrons to acquire as the result of their quantized orbital motions.

This can of course be confirmed by a few different methods: one is to explicitly solve the Hamiltonian by writing it as

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 - \frac{e^2 B^2}{2mc^2}\left(\frac{c\hbar^2}{eB}k_y - x \right)^2 \right]\psi = \epsilon_n \psi. \tag{211}$$

The smallest k_y can be is $2\pi\hbar/L_y$, while from the above the largest it can be is $L_x eB/(c\hbar^2)$. Therefore since k_y labels the solutions, we have a degeneracy of (taking $L_x = L_y$)

$$N\frac{L_x eB/(c\hbar^2)}{2\pi\hbar/L_y} = L^2 \frac{eB}{2\pi\hbar c}, \tag{212}$$

which is exactly what we found using the other approach.

tl;dr argument: To get the magnetic response, we need to compute $M = -\partial_B U$ (strictly speaking we can use U instead of F only at $T = 0$ but this is a tl;dr argument anyway). When a B field is turned on, the former continuum of levels gets quantized. Basically what will happen is that all levels within $\pm\hbar\omega_c/2$ of a Landau level $\hbar\omega_c(n+1/2)$ will get sucked into the nth Landau level. Since the occupation number is a monotonically decreasing function of energy ($\theta(\mu - \epsilon)$ counts as monotonically decreasing, but note that we are *not* making any arguments that are specific to fermions), we will always have that more levels will be sucked up in energy to a Landau level, rather than down. Therefore the B-field part of U will be positive, and the response will be diamagnetic.

For example, in the fermionic case at low T, this is corroborated by the heuristic estimate of the B-dependent part of U: since at low T only the shuffling of the energy levels near the FS contributes to the change in U (since the quantization of the orbits inside the FS doesn't change the total energy of the filled states given that ρ_{2d} for a Fermi gas is independent of ϵ)

$$U \supset (\delta\epsilon)\delta N \sim (\delta\epsilon)^2\rho(\epsilon_F) = (\hbar\omega_c/2)^2\rho(\epsilon_F), \tag{213}$$

which is close. Slightly more carefully, since we expect a density that exactly fills an integer number of LLs to be at the same energy in the magnetic field as not (equal numbers get sucked up as sucked down, at least to the extent that the occupation number v), the magnetization should be positive and should go to zero when an integer number of LLs are filled, presumably

reaching a maximum when a LL is half-filled. This is indeed what happens, as we will see in a sec.

Low T argument for Fermions: A less schematic (but still very schematic) argument for $T \ll T_F$ fermions goes as follows. First, note that when filled parts of the FSea get re-shuffled into LLs, their total energies do not change. This is essentially because in 2d the Fermi gas DOS is constant. As a check, the fermions that go into a given LL with energy $\omega_c(n + 1/2)$ will have energies in

$$\epsilon \in [\epsilon_L, \epsilon_U] = [\omega_c n, \omega_c(n + 1)]. \tag{214}$$

The number of fermions that want to go into the LL is

$$N_n = L^2 \int_{\epsilon_L}^{\epsilon_U} \frac{k dk}{2\pi} = L^2 \frac{m}{2\pi}(\epsilon_U - \epsilon_L) = L^2 \frac{m\omega_c}{2\pi}, \tag{215}$$

which is exactly what we deried for the LL degeneracy above. Therefore the energy of the filled LL is

$$\mathcal{E} = L^2 \frac{m\omega^2}{2\pi}(n + 1/2). \tag{216}$$

On the other hand, the energy of that part of the FSea is

$$L^2 \int_{\epsilon_L}^{\epsilon_U} \frac{k dk}{2\pi} \frac{k^2}{2m} = L^2 \frac{m}{4\pi}(\epsilon_U^2 - \epsilon_L^2) = \mathcal{E}, \tag{217}$$

as expected.

Anyway, the point of this is that only the LLs very close to the FS (and for $T \ll \omega_c$, only the very top LL) will contribute to the B-dependent part of the energy. Therefore in this limit, the B-dependent part of the energy is

$$\mathcal{E} = \int_{\omega_c n}^{\mu} d\epsilon \frac{m}{2\pi}(\omega_c(n + 1/2) - \epsilon) = \frac{m}{2\pi}\left(\omega_c(n + 1/2)(\mu - \omega_c n) - \frac{\mu^2 - \omega_c^2 n^2}{2}\right). \tag{218}$$

If we write $\mu = \omega_c(n + 1) - \delta$, then a bit of algebra (I don't see any reason to write it out) gives

$$\mathcal{E} = \frac{m}{4\pi}\delta(\omega_c - \delta). \tag{219}$$

As expected, this goes to zero at $\delta = 0, \omega_c$ when an integer number of LLs are filled, and is positive (we are in conventions where $\omega_c > 0$). This gives a magnetization of

$$M = -\frac{|e|}{4\pi}\delta. \tag{220}$$

The diamagnetization is therefore maximal right after a LL gets fully filled, in line with the DOS having a 1d $1/\sqrt{\epsilon}$ divergence at each of the LL energies (seeing this requires remembering about k_z).

Finally, we note that at $T \gg \omega_c$, where the distribution function for the charged particles is essentially flat over any given Landau level (we are not just talking about fermions!), one can do another calculation. At T high enough that the statistics are Boltzmannian, one still

gets a diamagnetic response, this time going as $M \propto -B/T$. This is actually in Pathria and Beale, so I won't write out the details.

The cool thing about this is that it is a completely quantum phenomenon—formally, this can be seen from restoring units and noting that $\chi \propto \hbar^2$. Even better though, one can note that in equilibrium, *no classical system of charged particles can have a nonzero magnetization.*[28] The easy way to argue this is that in a classical system, the partition function looks like

$$Z \propto \int \prod_i d^d p_i \, e^{-\beta f(p_i - eA(r_i)/c, \dots)}, \tag{221}$$

where f is some appropriate function of the thermodynamic background fields and the positions and momenta of the particles in the ensemble. Since we are doing classical mechanics where momenta are continuous and commute with the r_i, we can simply shift $p_i \mapsto p_i + eA(r_i)/c$, eliminating A from Z entirely—hence we must have zero magnetization. This argument also holds when doing semiclassical computations (e.g. take into account QM by putting the electrons in bands, but then ignore QM by determining dynamics with the classical eom), and so nontrivial magnetic responses of systems of electrons are always beyond a semiclassical approach.

Heuristically, one might think of this in the following way: Landau diamagnetism comes from the little loops of current formed by the rotating charged particles, which generate magnetic moments. We can treat the particles as current loops since they are wavelike and delocalized. However in classical mechanics we cannot think of them as loops of current, since the charge is localized to a particular place, and hence unable to produce a legit magnetic moment.

17 Speed of sound in a Fermi gas

What's the speed of sound in a three-dimensional Fermi gas?

Solution:

The usual way to do this is to write $PV\beta = \ln Z$ and $U = -\partial_\beta \ln Z$, and then to eliminate the chemical potential μ so as to get an equation of state. There's a slightly more pedestrian approach that I like, though.

We start by figuring out the $V = L^3$ dependence of the energy, which will then give us the pressure via $P = -\partial_V U$. The L dependence comes in through the quantization of the momenta $p_i \in 2\pi L^{-1} \mathbb{Z}$ ($\hbar = 1$). Writing then $\int d^3 p = \sum_{\{m_x, m_y, m_z\} \in \mathbb{N}^3} (2\pi L^{-1})^3$, we see that the L^{-3} cancels against the $\int d^3 r$, and so the energy is

$$U = \sum_{\{m_i\} \in \mathbb{N}^3} \frac{1}{2m} \left(\sum_i m_i^2 \right) \left(\frac{2\pi}{L} \right)^2 n_F(\epsilon). \tag{222}$$

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[28] Of course the assumption of equilibrium is essential, since Lenz's law means there is obviously magnetization for out-of-equilibrium things.

Taking the derivative then,

$$P = -\partial_V U = -\frac{1}{3L^2}\partial_L U = -\frac{1}{3L^2}\partial_L \sum_{\{m\}} \frac{1}{2m} \left(\frac{2\pi}{L}\right)^2 \sum_i m_i^2 n_F(\epsilon). \tag{223}$$

The subtle thing here is that the derivative needs to be taken while keeping the other extensive variables, namely N and S, fixed. This means in fact that $n_F(\epsilon)$ must be in fact fixed under the adiabatic process by which we're changing the size of the box—the temperature / chemical potential will change in such a way to keep the occupation number constant, even as the values of the momenta change (the adiabaticity means that particles won't get "knocked out" of the states they're in during the expansion process, i.e. the "occupation probability" $n_F(\epsilon)$ won't change). Therefore the derivative only acts on the $1/L^2$ factor, and so

$$P = \frac{2U}{3V}. \tag{224}$$

A quicker way to get here is to use dimensional analysis to realize that $U \sim Vk_F^5$ and $N \sim Vk_F^3 \implies k_F \sim V^{-1/3}$ at fixed N to write $U \sim V^{1-5/3}$, which tells us that $\partial_V U = 2U/3V$.

Now we can get the speed of sound from the formula

$$c = \sqrt{\partial_\rho P}. \tag{225}$$

The density is

$$\rho = m \int d\epsilon \sqrt{\epsilon}\bar{\nu}, \tag{226}$$

where $\bar{\nu} = \nu(\epsilon)/\sqrt{\epsilon}$ is the constant part of the DOS. Therefore

$$\rho = \frac{2m}{3}\bar{\nu}\epsilon_F^{3/2}. \tag{227}$$

Then we use $P = 2U/3V$ to write

$$P = \frac{2}{3}\int d\epsilon\, \epsilon^{3/2}\bar{\nu} = \frac{4}{15}\bar{\nu}\epsilon_F^{5/2}. \tag{228}$$

Then

$$\partial_\rho P = \frac{1}{\sqrt{\epsilon_F}m\bar{\nu}}\partial_{\epsilon_F}\left[\frac{4}{15}\bar{\nu}\epsilon_F^{5/2}\right] = \frac{2}{3m}\epsilon_F = \frac{k_F^2}{3m^2}. \tag{229}$$

Therefore the speed of sound is

$$c = \frac{v_F}{\sqrt{3}}, \tag{230}$$

which importantly is slower than the speed of the qps at the FS. Note that sound exists even at zero T an in a non-interacting problem because of the degeneracy pressure caused by Fermi statistics.

18 Wiedemann–Franz law sanity check

Today we're doing a rather trivial but instructive problem from Simon's intro to solid state book: checking the WF law for a clean[29] 1d system.

Solution:

First we get the electrical conductance.[30] Our setup will be to consider a 1d wire connected between a source of chemical potential μ_L and a sink at μ_R. Of course, this is just saying that the wire is placed in an electric field. The current is

$$j = 2 \int \frac{dk}{2\pi} j_k = -2e \left(\int_0^\infty \frac{dk}{2\pi} v(k) n_F(\beta(\epsilon_k - \mu_L)) - \int_0^\infty \frac{dk}{2\pi} v(k) n_F(\beta(\epsilon_k - \mu_R)) \right), \quad (231)$$

where $v(k) = \hbar^{-1}|\partial_k \epsilon_k|$ and the factor of 2 is for spin.[31] Doing the integrals with the help of $dk\, v(k) = d\epsilon$, we get

$$j = -\frac{2e}{h}(\mu_L - \mu_R) \implies G = \frac{2e^2}{h}, \quad (233)$$

where again the conductance is defined in our setting via

$$j = GV = G(\mu_R - \mu_L)/e. \quad (234)$$

Now for the thermal conductance K.[32] Here we set $\mu_L = \mu_R$ but put the ends of the wire at temperatures T_L, T_R, with $T_L/T_R \approx 1$ (since the thermal conductivity is defined in linear response, we will just work to order $O((T_L - T_R)^1)$).

The heat current is $j_Q = T j_S = j_U - \mu j$. Therefore

$$j_Q = \int_0^\infty \frac{dk}{\pi} v(k)(\epsilon_k - \mu) \left[n_F(\beta_R(\epsilon_k - \mu)) - n_F(\beta_L(\epsilon_k - \mu)) \right]. \quad (235)$$

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[29]One might say that this is slightly against the spirit of the WF law—one point of the law is that the ratio of conductances is independent of details of the non-cleanness of the system, like the value of τ and so on. But the clean system provides a good sanity check.

[30]Recall that conductance G differs from conductivity in that it is the response function that relates to the gauge potential rather than to the electric field (one could argue that since the current couples to the gauge field microscopically, this is the more natural response function to study). In general, $j_\mu = \int G_{\mu\nu} A^\nu$. In our application this will just read $j = GV$.

[31]Annoyingly pedantic footnote about why the velocity appears here: we start from an action like $\int \psi^\dagger(\omega - \epsilon_k)\psi$, and then perform $\psi \mapsto e^{i\theta}\psi$ to determine j through the usual Noether procedure. j is the thing that contracts with $\nabla\theta$ in the transformed action, and so we need to select out all ways of having one derivative in $\epsilon_k = \epsilon(-i\hbar\nabla)$ act on the θ. This term is

$$\int \psi^\dagger \nabla_j \theta \frac{\partial \epsilon}{\partial(-i\hbar\nabla_j)} \psi, \quad (232)$$

which goes over to the expression written above when we pass to momentum space and take expectation values to get the Fermi function.

[32]This differs from the thermal conductivity in the same way that G differs from σ: we have $j = K\Delta T$, so that K couples directly to the background field $T \sim V$ rather than to $\partial_x T \sim E$.

Now we write

$$n_F(\beta_R(\epsilon_k - \mu)) - n_F(\beta_L(\epsilon_k - \mu)) \approx (\partial_\beta n_F|)_{\beta\xi}(\beta_R - \beta_L) = (\partial_\epsilon n_F)|_{\beta\xi}\frac{\xi}{\beta}(\beta_R - \beta_L), \qquad \xi \equiv \epsilon - \mu. \tag{236}$$

Then since $\beta^{-1}(\beta_R - \beta_L) = -\Delta T/T$, we have

$$K = -\frac{2}{T}\int_0^\infty \frac{d\epsilon}{h}\xi^2 \partial_\epsilon n_F(\beta\xi) = -\frac{2T}{h}\int_{-\beta\mu}^\infty dx\, x^2 \partial_x \frac{1}{e^x + 1}. \tag{237}$$

Working under the reasonable assumption that $\mu\beta \gg 1$, we can safely extend the lower limit of integration to $-\infty$. Then we use

$$\int_\mathbb{R} dx\, x^2 \partial_x \frac{1}{e^x + 1} = -\pi^2/3 \tag{238}$$

to get

$$K = \frac{2T\pi^2}{3h}. \tag{239}$$

Therefore

$$\frac{K}{TG} = \frac{\pi^2}{3e^2}, \tag{240}$$

which is indeed the WF law (we are setting $k_B = 1$ since it's a fake physical constant that only exists because humanity didn't define units of temperature correctly).

Now this check is rather trivial, since we are working in a clean system. More generally, the WF law is expected to hold in the presence of disorder provided that the disorder is elastic. This is because both $(\epsilon - \mu)\beta$ and $eV\beta$ appear in the distribution function for the electrons in the same way, and so from the usual kinetic theory arguments we expect that if both energy and charge are conserved, the heat and charge transport response functions will be related (the asymmetric extra factor of T relating the conductivities is because the background field for $\epsilon - \mu$, namely β, also couples to the background field for charge density).

19 What makes quantum gasses quantum

The difference between a quantum bose gas and a gas of classical particles lies in the indistinguishability of the particles in the quantum case. Today we will review why this is far from just a bookkeeping effect, and indeed in responsible for very interaction-y features in non-interacting quantum gasses.

Solution:

First, we will look at things in the canonical ensemble, where seeing the effects of the state counting are clearer. We would like to calculate the coordinate-space matrix elements

of the density matrix for a free gas of n quantum particles. Since the density matrix is diagonal in momentum space, we have

$$\langle r_1, \ldots, r_n | \rho | r_1', \ldots, r_n' \rangle = \sum_{\mathbf{K}} \langle r_1, \ldots, r_n | \Psi_{\mathbf{K}} \rangle \exp\left(-\sum_i \frac{\beta k_i^2}{2m}\right) \langle \Psi_{\mathbf{K}} | r_1', \ldots, r_n' \rangle. \qquad (241)$$

Here $\mathbf{K}$ is a composite label which sums over all distinct vectors $\mathbf{K} = (\mathbf{k}_1, \ldots, \mathbf{k}_n)$. By distinct, we mean that we regard e.g. $\mathbf{K} = (\mathbf{p}, \mathbf{q}) \sim (\mathbf{q}, \mathbf{p})$; since the particles are indistinguishable we shouldn't distinguish eigenstates based on which particle gets which momentum, but rather just on the un-ordered set $\mathbf{K}$ (this is just the same as counting states in the harmonic oscillator—there is only one state for a given occupation number of the levels).

The many-body[33] wavefunctions $\Psi_{\mathbf{K}}$ are given as appropriately (anti)symmetrized products of free plane waves: with $\zeta = \pm 1$ depending on the statistics of the particles in question,

$$\Psi_{\mathbf{K}} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \zeta^\sigma \bigotimes_i \psi(\mathbf{k}_{\sigma(i)}), \qquad (242)$$

where $\langle \mathbf{r} | \psi(\mathbf{k}_{\sigma(i)}) \rangle = e^{i\mathbf{k}\cdot\mathbf{r}}$. The $1/\sqrt{n!}$ ensures normalization (the combinatorics doesn't quite work when some of the k_i's are equal, but these choices of $\mathbf{K}$ contribute a set of measure zero to the integral over $\mathbf{K}$ and so we will ignore this subtlety). Putting this in,

$$\langle r_1, \ldots, r_n | \rho | r_1', \ldots, r_n' \rangle = \frac{1}{n!} \sum_{\mathbf{K}} \sum_{\sigma, \omega \in S_n} \zeta^{\sigma+\omega} \exp\left(i \sum_i [\mathbf{r}_i \cdot \mathbf{k}_{\sigma(i)} - \mathbf{r}_i' \cdot \mathbf{k}_{\omega(i)}] - \beta \sum_i \frac{k_i^2}{2m}\right) \qquad (243)$$

Now instead of summing over all distinct $\mathbf{K}$, we can instead sum over all of the $\mathbf{k}_i$ independently (i.e. counting both $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{q}, \mathbf{p})$). Since we are already symmetrizing over the $\mathbf{k}_i$, the only thing that this will do is to give us an answer which is too big by a factor of $n!$. So then

$$\langle r_1, \ldots, r_n | \rho | r_1', \ldots, r_n' \rangle = \frac{1}{n!^2} \int \prod_i \frac{d^d k_i}{(2\pi)^d} \sum_{\sigma, \omega \in S_n} \zeta^{\sigma+\omega} \exp\left(i \sum_i [\mathbf{r}_i \cdot \mathbf{k}_{\sigma(i)} - \mathbf{r}_i' \cdot \mathbf{k}_{\omega(i)}] - \beta \sum_i \frac{k_i^2}{2m}\right) \qquad (244)$$

Now the only thing that matters for this integral is the relative difference between $\sigma(i)$ and $\omega(i)$. Therefore we may fix $\omega = \mathbf{1}$ and multiply by $n!$. We may further transfer the permutation onto the $\mathbf{r}_i$ index, so that the exponent looks like $\mathbf{k}_i \cdot (\mathbf{r}_{\sigma(i)} - \mathbf{r}_i') - \ldots$. We can then do the integrals over k_i, and get

$$\langle r_1, \ldots, r_n | \rho | r_1', \ldots, r_n' \rangle = \frac{1}{n!} \lambda^{-dn} \sum_{\sigma \in S_n} \zeta^\sigma \exp\left(-\pi \sum_i \frac{|\mathbf{r}_i - \mathbf{r}_{\sigma(i)}'|^2}{\lambda^2}\right), \qquad (245)$$

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[33]It may seem weird to use the adjective "many-body" when we're talking about a free system, but in fact it is entirely appropriate—the whole point of quantum statistical mechanics is that the effect of the statistics of the particles depends on how many particles there are, so that having many free bodies is not equivalent to the problem of just one (the partition function does *not* factor as a product of single-particle partition functions, even in the free case).

where the thermal length is

$$\lambda \equiv \sqrt{\frac{h^2}{2\pi mT}}. \tag{246}$$

When $h \to 0$ only the trivial permutation in S_n contributes, and we get the correct classical partition function (with the correct Gibbs $1/n!$ factor—recall that the h surviving in the thermal length also appears in classical stat mech in order to make Z dimensionless).

To understand the effects of the permutations, consider the two-pody density matrix $\rho(0, x)$ for a one-dimensional system. This matrix element is plotted below for the case of the three different statistics ($\zeta = \pm 1$ and then the $h \to 0$ limit)

$$\tag{247}$$
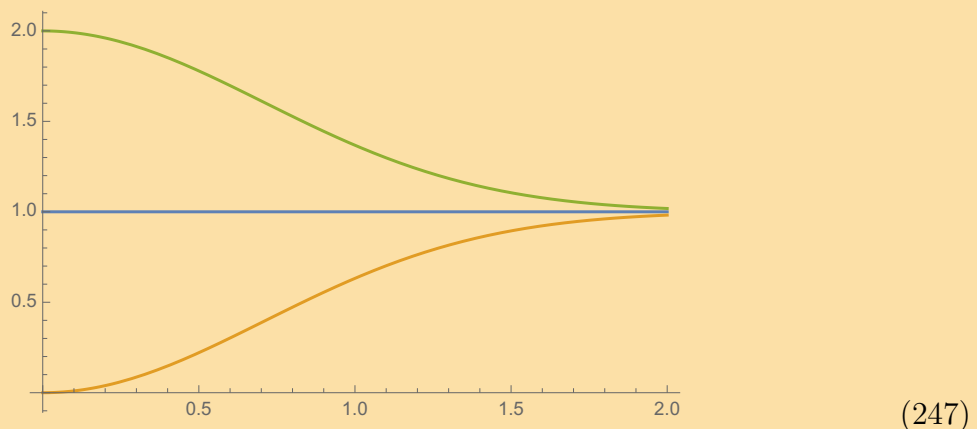
We see that even though they are non-interacting, bose particles tend to clump together more than expected classically, while fermions tend to avoid each other.

This result is perhaps best understood within the framework of the path integral. We know that we can represent the $\mathbb{R}$-space matrix elements density matrix as a path integral over a strip of length β, with fixed ∂ conditions. In quantum mechanics, loosely speaking the uncertainty principle means that we can't track the trajectories of each of the particles along the strip exactly, and indistinguishability means our boundary conditions can't distinguish between trajectories which differ by particles braiding past each other and swapping endpoints at the boundary. Therefore we must sum over all possible worldline topologies for the particles, keeping the endpoints fixed. Now the amplitude for a particle to propagate from r to r' along a time interval of length β is precisely

$$Z(r, r'; \beta) = \sqrt{\frac{2\pi m}{h^2 \beta}} e^{-\pi (r-r')^2/\lambda^2}, \tag{248}$$

which is of course exactly the factor that appears above. This can be derived by doing the trace over momentum states as we have done, or by solving the appropriate diffusion equation

$$\partial_\beta Z(r, r'; \beta) = D\partial_r^2 Z(r, r'; \beta), \tag{249}$$

which just comes from writing the density matrix as $e^{-\beta H}$.

How does this manifest itself in the GCE when particle numbers are only kept track of statistically? ...

20 Basic Hall angle stuff

Consider a metal well described by the Drude theory. Assume the electrons have a density n, mass m, a scattering rate τ, and are placed in a weak uniform magnetic field B along the z-direction. Do several things:

- Calculate the frequency dependent Hall conductivity in such a metal.

- The Hall angle is defined to be between the electric current and electric field. For a (weak) electric field in the xy-plane, and in the dc limit, calculate the Hall angle as a function of B, and other parameters characterizing the electron gas.

- Write down the analogous formulae for the case of two bands with different charge carriers.

- Now consider a circularly polarized electric field $E_y = \pm i E_x$. Find the resulting current, and find the dielectric function. When do wave solutions to Maxwell's equations exist? In the low-frequency limit, what's their dispersion?

Solution:

We will be working with an approach based on the semiclassical eom. We will therefore always need to be working with background fields that vary incredibly slowly on the scale of the lattice constant. This is because the semiclassic approach works with wavepackets $\psi(\mathbf{r}, \mathbf{k})$, and assumes that they are near eigenstates of the Hamiltonian (whence the semiclassical eoms). In order for this to be true $\psi(\mathbf{r}, \mathbf{k})$ must have a nearly well-defined energy, which means that for a generically dispersing band the $\mathbf{k}$-space width of the wavepacket must be $\ll 2\pi a^{-1}$. This then means the $\mathbb{R}$-space width must be $\gg a$. Since semiclassics means treating the wavepackets as particles wrt the external fields, the wavelengths of the fields must be $\ggg a$. Furthermore, since we ignore tunneling events between different bands, we need the fields to vary temporally slowly enough such that $\hbar\omega \ll E_g$, where E_g is the minimum energy gap between bands.

In fact we have a further restriction in the context of computing the conductivity: we will always be working with EM fields whose momenta satisfy $\hbar q^{-1} \gg v\tau$, where $v \approx v_F$ is a typical speed of electrons in the metal, i.e. we will assume that electrons experience many scattering events before they travel large enough distances for them to become aware of the spatial modulation of the fields. The reason for making this assumption is that it means the current can be determined just by knowing the local value of $\mathbf{E}$—one doesn't need to know the details of the electron's past history to describe its present evolution, since these details are presumed to be erased by scattering. Indeed in general, the conductivity is a non-local response function, with $\mathbf{j}(r, t) = \int \sigma(r - r', t - t') \mathbf{E}(r', t')$. The above assumption means that $\sigma(r - r', t - t') \propto \delta(r - r')$, which means that $\sigma(q, \omega)$ is independent of q (since the δ function becomes a constant after FTing). Therefore in what follows, we will be working locally in space, but non-locally in time (i.e, locally in frequency).

Anyway, we start from the eom:

$$\partial_t \mathbf{k} = e\mathbf{E} - \nu\mathbf{k} + \frac{e}{m}\mathbf{k} \times \mathbf{B}, \tag{250}$$

where e is the actual charge of the electron ($e < 0$) and $\nu \equiv \tau^{-1}$.[34] In what follows we will assume that we are near a band minimum or maximum, where we can write

$$\partial_t \mathbf{v}^a = \partial_t \partial_{\mathbf{k}^a}\epsilon = (\partial_t \mathbf{k}^b)\partial_{\mathbf{k}^b}\partial_{\mathbf{k}^a}\epsilon = [m^{-1}]^{ab}\partial_t \mathbf{k}^b, \tag{251}$$

where the matrix m^{-1} is assumed not to depend on $\mathbf{k}$. For simplicity, we will take $m \propto \mathbf{1}$, but the generalization to anisotropic bands is straightforward.

In terms of the current,

$$j^a(-i\omega + \nu) = \epsilon_0\omega_p^2 E^a + \frac{e}{m}\epsilon^{abc}j_b B_c, \tag{252}$$

where

$$\omega_p^2 = \frac{e^2 n}{m\epsilon_0} \tag{253}$$

is the $q = 0$ plasma frequency. Wolog, we may take $\mathbf{B} = B\widehat{\mathbf{z}}$. Then we have

$$\left(\delta^{ab}(-i\omega + \nu) - \omega_c\epsilon^{abz}\right)j^b = \epsilon_0\omega_p^2 E^a, \tag{254}$$

where ω_c is of course the cyclotron frequency. The resistivity is therefore

$$\rho(\omega) = \frac{1}{\epsilon_0\omega_p^2}\begin{pmatrix} -i\omega + \nu & -\omega_c & \\ \omega_c & -i\omega + \nu & \\ & & -i\omega + \nu \end{pmatrix}. \tag{255}$$

Note that the longintudinal resistivity $\rho_{xx}(\omega)$ is independent of B. The Hall coefficient is $R_H = \rho_{yx}(0)/B$, which means that (remember we are in SI and $e < 0$)

$$R_H = \frac{\omega_c B^{-1}}{\epsilon_0\omega_p^2} = \frac{1}{ne} \tag{256}$$

which is the usual result. This means the conductivity tensor is

$$\sigma(\omega) = \frac{\epsilon_0\omega_p^2}{\omega_c^2 + (-i\omega + \nu)^2}\begin{pmatrix} -i\omega + \nu & \omega_c & \\ -\omega_c & -i\omega + \nu & \\ & & \frac{\omega_c^2 + (-i\omega + \nu)^2}{-i\omega + \nu} \end{pmatrix}, \tag{257}$$

which is properly antisymmetric and reduces to the expected result when $\omega_c = 0$. Note that in a clean system where $\nu = 0$, the conductivity diverges at $\omega = \omega_c$ since at this frequency we're on resonance, with the electrons being hit with the electric field in the same point of

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[34]A further assumption: to avoid talking about transitions between bands, we need the kinetic energy acquired by a wavepacket between scattering events, viz. $\int dt\, W = \int dt\, \mathbf{E} \cdot \mathbf{j} \sim \tau \nu e^2 E$, to be $\ll E_g$. This is an incredibly reasonable assumption in e.g. semiconductors and stuff.

their orbit every time they circle around. Looking for the location of this peak would be one way to measure the electron mass, for example.

Now we want to find the Hall angle. We determine θ_H by setting $\omega = 0$ and calculating

$$\cos\theta_H = \mathbf{j} \cdot \mathbf{E} / \sqrt{j^2 E^2} = \frac{\mathbf{E}^T \sigma(0) \mathbf{E}}{\sqrt{E^2 \mathbf{E}^T \sigma(0)^T \sigma(0) \mathbf{E}}}. \tag{258}$$

Note that since we're doing linear response, i.e. $j \propto \mathbf{E}$, the Hall angle will be independent of E.

We can take $\mathbf{E} = E\hat{\mathbf{x}}$ wolog. Then this becomes

$$\cos\theta_H = \frac{\sigma_{xx}}{\sqrt{\sigma_{xx}^2 + \sigma_{xy}^2}} = (1 + \sigma_{xy}^2 / \sigma_{xx}^2)^{-1/2} = (1 + \omega_c^2 / \nu^2)^{-1/2}. \tag{259}$$

Note that the plasma frequency has totally canceled out of the result. For small magnetic fields or short scattering times, so that $\tau\omega_c \ll 1$, this gives

$$\theta_H \approx \omega_c \tau. \tag{260}$$

Sanity check: in this limit, the electrons experience many scatterings before they can complete a cyclotron orbit; hence the $\mathbf{B}$ field is not very effective at making their motions $\perp$ to $\mathbf{E}$, and the current is dominated by the force coming from the electric field that accelerates the electrons between collisions.

On the other hand, for large magnetic fields or long scattering times (but still short enough so that $\sqrt{\langle \mathbf{v}^2 \rangle}\tau$ is smaller than any spatial variation of the EM fields), we instead have

$$\theta_H \approx \frac{\pi}{2} - \frac{1}{\omega_c \tau}. \tag{261}$$

Sanity check: for very long scattering times, the Lorentz force is very effective at deflecting the motion of the electrons away from the electric field. The current becomes directed along $\mathbf{B} \times \mathbf{E}$ in the limit of $\omega_c \nu \to \infty$, which is precisely the direction of the drift velocity along which the electric field vanishes.

In a finite-sized sample, the Hall conductivity will give us a current in the $\mathbf{E} \times \mathbf{B}$ direction, which will create an electric field in this direction after equilibrium is reached (in equilibrium here we mean that $\mathbf{j}$ along $\mathbf{E} \times \mathbf{B}$ vanishes). For example, let $\mathbf{E} \parallel \hat{\mathbf{x}}$. Then in the DC limit, $\mathbf{j}_y = 0$ means that

$$\mathbf{E}_y = \omega_c \tau \mathbf{E}_x. \tag{262}$$

The sign is right since for us $e < 0 \implies \omega_c < 0$. Sanity check: when $\omega_c \tau \to 0$ we don't need an electric field in the y direction, since we already know that in that limit $\theta_H \to 0$. On the other hand, when $\omega_c \tau \to -\infty$, we know that the electrons will move off along the y direction super fast, and hence they require an infinitely strong $\mathbf{E}_y$ to maintain equilibrium.

We have been assuming a situation appropriate for free electrons without any kind of potential. When the electrons are put into bands, the basic analysis above holds as long as we are at energies where the electrons execute closed orbits in $\mathbf{k}$ space (as they do for the free electron model). When the orbits are open (and every band must have some open orbits for some choice of μ, modulo the contrived situation where $\epsilon_{\mathbf{k}}$ is uniform on the boundary

of the BZ) the situation changes, as now the electrons do not execute paths in $\mathbf{k}$ space that tend to direct the current along $\mathbf{B} \times \mathbf{E}$.

Generalizing this to the case of multiple carrier species is straightforward. From the semiclassical eom for each carrier species, we obtain the conductivity $\sigma_a(\omega)$. Since the currents add to produce the total current, so too do the conductivities (it is the σ_as that add, *not* the ρ_as). Focusing just on the xy plane then, we have

$$\sigma(\omega) = \sum_a \frac{\epsilon_0 \omega_{pa}^2}{\omega_{ca}^2 + \alpha_a^2} \begin{pmatrix} \alpha_a & \omega_{ca} \\ -\omega_{ca} & \alpha_a \end{pmatrix}, \qquad \alpha_a \equiv -i\omega + \nu_a. \tag{263}$$

I don't really see the point of writing down the most general formulae, due to the unillumi-nating algebra involved. For simplicity, we will therefore specialize to the "perfect intrinsic semiconductor" case, where we have one band with $m_1 = m, q_1 = q, \nu_1 = \nu$ and another with $m_2 = m, q_2 = -q, \nu_2 = \nu$.[35] With these simplifications,

$$\sigma(0) = \frac{q^2/m}{\omega_c^2 + \nu^2} \begin{pmatrix} \nu(n_1 + n_2) & \omega_c(n_1 - n_2) \\ -\omega_c(n_1 - n_2) & \nu(n_1 + n_2) \end{pmatrix}, \tag{264}$$

where $\omega_c = qB/m$. As expected, the densities of the oppositely charged particles add for the longitudinal conductance, and subtract for the Hall conductance: the particles and holes move in opposite directions under the $\mathbf{E}$ field, which due to their opposite charges produces currents that add. However since the $\mathbf{E}$ field makes them move in opposite directions, they are deflected in *the same* transverse direction by the $\mathbf{B}$ field, and hence their transverse currents subtract from one another.

The Hall angle in this case satisfies

$$\cos \theta_H = \left(1 + \frac{\tau^2 \omega_c^2 (n_1 - n - 2)^2}{(n_1 + n_2)^2} \right)^{-1/2}. \tag{265}$$

In the limit of small $\tau \omega_c$, we have

$$\theta_H \approx \tau \omega_c \frac{n_1 - n_2}{n_1 + n_2}, \tag{266}$$

which reduces with the right sign to our answer before in the case of $n_2 = 0$. Now as expected, when $n_1 = n_2$ the Hall angle is always zero, regardless of the strength of B. But if we are in the strong field limit with imbalenced carrier concentrations, such that $\omega \tau (n_1 - n_2)/(n_1 + n_2) \gg 1$, then

$$\theta_H \approx \pi/2 - \frac{n_1 + n_2}{\omega_c \tau (n_1 - n_2)}. \tag{267}$$

Letting $N \equiv n_1 + n_2$ and $\delta \equiv n_1 - n_2$, the resistivity is

$$\rho(0) = \frac{\omega_c^2 + \nu^2}{q^2 m^{-1}(\nu^2 N^2 + \omega_c^2 \delta^2)} \begin{pmatrix} \nu N & -\omega_c \delta \\ \omega_c \delta & \nu N \end{pmatrix}. \tag{268}$$

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[35]Recall that the effective masses of holes are always defined to be positive, i.e. they are defined by $m_h = -\partial_k^2 \epsilon$ near a band maximum. The eom would be the same whether we put the minus sign on the charge of the hole or on the mass; we choose to do the former.

At high fields, this gives the Hall coefficient

$$R_H = \rho_{yx}(0)/B \xrightarrow{B \to \infty} \frac{1}{q\delta}.$$

(269)

Sanity check: δ is the effective number of charge carriers, and it appears exactly in R_H in exactly the way we expect it to. Likewise at high fields, we have the longitudinal resistance

$$\rho_{xx}(0) \xrightarrow{B \to \infty} \frac{m\nu N}{q^2 \delta^2}.$$

(270)

This passes the $n_2 = 0$ sanity check, but note that $\rho_{xx}(0) \to \infty$ when $n_1 = n_2$ (which is not obvious from looking at the expression for $\sigma(0)$).

21 Helicons

Today we'll use the conductivity calculated in the previous diary entry on Hall angles to explore what kind of waves can propagate in the metal.

Solution: From Maxwell's equations, the dielectric tensor at zero momentum is

$$\epsilon(\omega) = \mathbf{1} + \frac{i}{\omega \epsilon_0} \sigma(\omega).$$

(271)

In what follows we will be focusing on $\mathbf{E}$ planes that line within the xy plane. Using our expression for σ, the xy-plane part of the dielectric function is (for electric fields along the z direction, we just get the usual $\epsilon_z(\omega) = 1 - \omega_p^2/\omega^2$)

$$\epsilon(\omega) = \mathbf{1} + i\frac{\omega_p^2}{\omega r} \begin{pmatrix} d & \omega_c \\ -\omega_c & d \end{pmatrix}, \qquad d \equiv -i\omega + \nu, \qquad r \equiv d^2 + \omega_c^2.$$

(272)

Taking the clean limit $\nu = 0$, we can plot $\omega^2 \epsilon(\omega)$ to see when we have wave solutions to Maxwell's equations for the given circularly polarized fields.[36] Since the dispersion is $c^2 k^2 = \omega^2 \epsilon(\omega)$, the wave solutions are

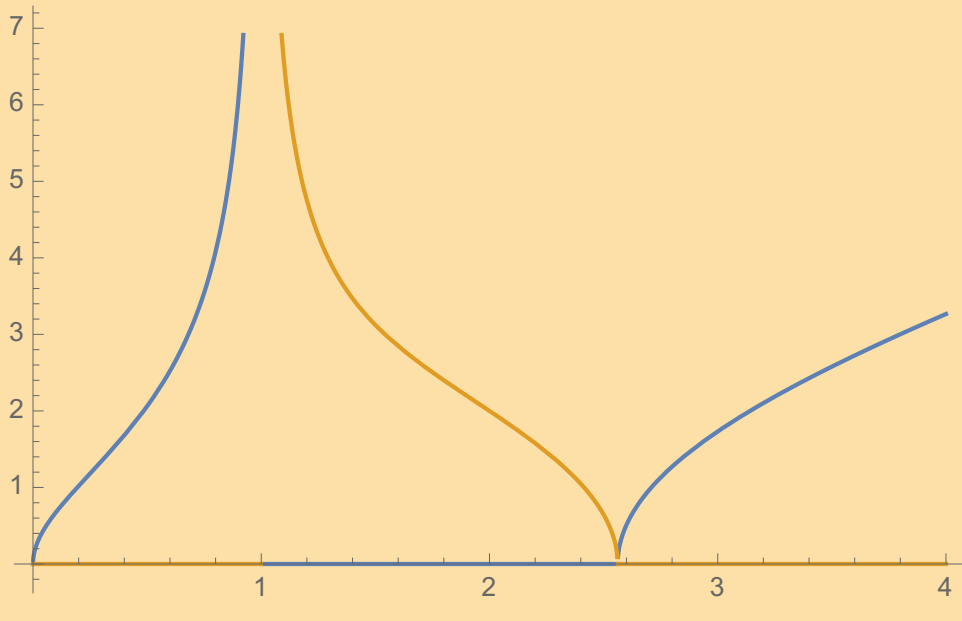$$q^2 c^2 = \omega^2 + \frac{\omega \omega_p^2}{\omega - \omega_c}.$$

(273)

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[36]By "plot $\omega^2 \epsilon(\omega)$", we mean that we plot the x component of $E^a q_a^2 c^2 = \omega^2 \epsilon_{ab}(\omega) E^b$, evaluated at $E_x = iE_y$.

we will have solutions when $\omega^2 \epsilon(\omega) > 0$. Choosing e.g. $\omega_p = 2, \omega_c = 1$, we have
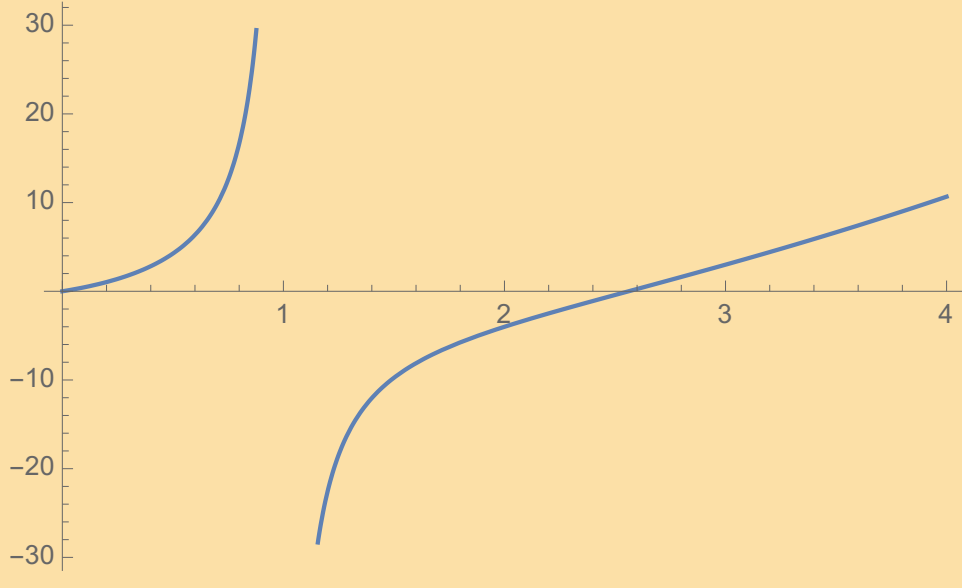
$$(274)$$

Since $\omega^2 \epsilon(\omega)$ runs through all positive $\mathbb{R}$ values between 0 and $1 = \omega_c$ and between ≈ 2.6 and ∞, we will have wave solutions for any k whenever $\omega < \omega_c$ or $\omega \gtrsim 2.6$. I guess a more clearer plot is one showing the $\mathbb{R}$ and $\mathbb{I}$ parts of $cq(\omega)$: this looks like

$$(275)$$

However, in real life situations, we basically always have $\omega_p \gg \omega_c$. In this limit, the allowed boundary for propagation of course moves back to $\omega > \omega_p$: indeed if we make ω_c much

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smaller, $\epsilon(\omega)\omega^2$ looks like
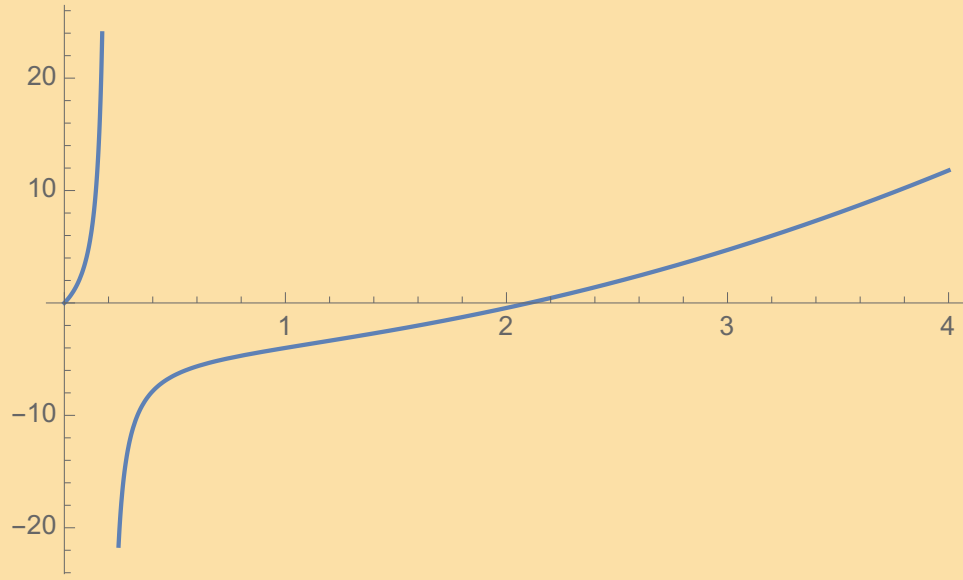
$$(276)$$

with the upper $\epsilon(\omega) = 0$ point moving closer to ω_p. Therefore in realistic situations, the allowed frequencies for wave propagation are basically $\omega < \omega_c$ and $\omega > \omega_p$. In the very low $\omega/\omega_c \ll 1$ frequency limit the dispersion becomes quadratic,

$$\omega \approx k^2 \frac{\omega_c c^2}{\omega_p^2}. \tag{277}$$

These types of waves are apparently known as helicons. I guess the point of this is that helicons afford us a way of getting a wave solution *below* the plasma frequency, which is normally not something that happens because of screening (metals are normally only transparent above ω_p!). What happens here is that the magnetic field forces the electrons to precess around at a frequency of ω_p, and so they are precessing too fast to be able to screen out a wave on frequencies less than ω_c, allowing the wave to propagate freely in the medium.

22 Surface plasmons

Today we're doing a problem from A&M. We will be considering a situation in which two dielectrics with dielectric constants $\epsilon_\lessgtr$ occupy the $z < 0$ and $z > 0$ half-spaces, and will show the existence of wave solutions to Maxwell's equations which are localized to the interface (for a certain range of frequencies).

Solution:

Our ansatz will be that appropriate for waves in both E_x and E_z which propagate in the x direction and which are exponentially localized to the $z = 0$ plane:

$$\mathbf{E} = \begin{cases} (A_x^> e^{i(qx-\omega t)-k_> z}, 0, A_z^> e^{i(qx-\omega t)-k_> z}), & z > 0 \\ (A_x^< e^{i(qx-\omega t)+k_< z}, 0, A_z^< e^{i(qx-\omega t)+k_< z}), & z < 0 \end{cases}. \tag{278}$$

where $k_>, k_< > 0$. Now, we have assumed that the charge density vanishes for $z \neq 0$. Applying this for both half-planes,

$$iqA_x^> = k_> A_z^>, \qquad iqA_x^< = -k_< A_z^<. \tag{279}$$

Now since $\mathbf{E}_x$ is continuous across the interface (do $\oint \mathbf{E} \cdot dl$ along a narrow rectangle with short sides in the z direction and long sides in the $z = 0$ plane), we have $A_x^> = A_x^< \equiv E_0$. From the above then, we get

$$k_>/k_< = -A_z^</A_z^>. \tag{280}$$

Furthermore, we also know that $(\epsilon \mathbf{E})_z$ is continuous across the interface (a discontinuity in $\mathbf{D}$ would mean a free charge density on the surface, while we are assuming the only charge density comes from bound charges). This means that

$$\epsilon_> A_z^> = \epsilon_< A_z^< \implies k_>/k_< = -\epsilon_>/\epsilon_<. \tag{281}$$

Note that this must mean that the two dielectric functions on either side of the interface have opposite signs, if we are to get a legit wave solution.

Now we take curls of Maxwell's equations in the usual way to get the wave equations, which read (in our notation the ϵ_is are relative dielectric constants, so that the "real" dielectric constants are $\epsilon_0 \epsilon$—in particular, $\epsilon_i = 1$ for vacuum)

$$c^2(q^2 - k_i^2) = \omega^2 \epsilon_i, \tag{282}$$

with $i =>, <$. The two equations coming from this, plus the relation $k_>/k_< = -\epsilon_>/\epsilon <$, lets us solve for q as a function of ω, ϵ_i. Some algebra gives

$$k_\lessgtr = \pm \frac{\epsilon_\lessgtr \omega}{c} \sqrt{\frac{-1}{\epsilon_< + \epsilon_<}}, \tag{283}$$

with the $\pm$ chosen so that $k_\lessgtr > 0$ (we only care that the real part of $k_\lessgtr > 0$; it is totally fine for $k_\lessgtr$ to have an imaginary part). Putting this back into the wave equation, we get

$$q = \frac{\omega}{c} \sqrt{\frac{\epsilon_> \epsilon_<}{\epsilon_> + \epsilon_<}}. \tag{284}$$

Let us now specialize to the case where the UHP is vacuum, and the LHP is a clean dielectric, so that

$$\epsilon_> = 1, \qquad \epsilon_< = 1 - \omega_p^2/\omega^2. \tag{285}$$

Then the dispersion is

$$qc = \omega \sqrt{\frac{\omega^2 - \omega_p^2}{2\omega^2 - \omega_p^2}} = \omega^2 \epsilon_{eff}(\omega), \qquad \epsilon_{eff}(\omega) = \sqrt{\frac{1 - \omega_p^2/\omega^2}{2 - \omega_p^2/\omega^2}}. \tag{286}$$

Therefore at large $qc \gg \omega$, we have a solution at $\omega \to \omega_p/\sqrt{2}$. The k parameters are

$$k_> = \frac{\omega^2}{c} \sqrt{\frac{1}{\omega_p^2 - 2\omega^2}}, \qquad k_< = \frac{1}{c}(\omega_p^2 - \omega^2) \sqrt{\frac{1}{\omega_p^2 - 2\omega^2}}. \tag{287}$$

Therefore as ω goes to $\omega_p/\sqrt{2}$ from below, both $k_{\lessgtr}$ parameters are positive, and go to infinity in a way such that $k_>/k_< \to 1$, so that at $\omega_* = \omega_p/\sqrt{2}$ we get a perfectly boundary localized mode. As expected, at $\omega < \omega_*$ we have $k_< > k_>$, so that the fields are more localized inside of the medium. Note that since the 2d localized plasmon frequency $\omega_* < \omega_p$, the boundary localized plasmons have lower energy than the bulk ones.

A plot of qc as a function of ω is shown below, with $\omega_p = 1$. The blue curve is the real part, and the orange curve is the imaginary part:
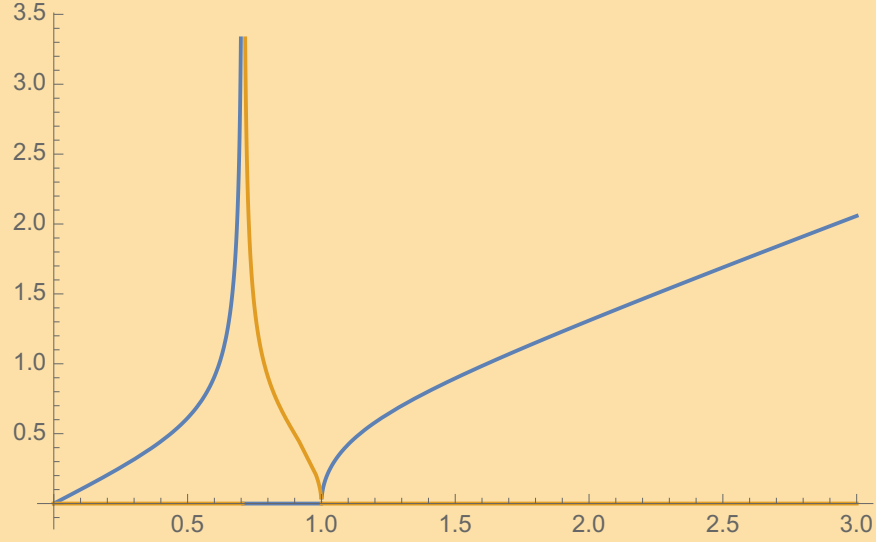
$$(288)$$

The corresponding plot of $\omega^2 \epsilon(\omega)$ is (I guess this is rather superfluous)
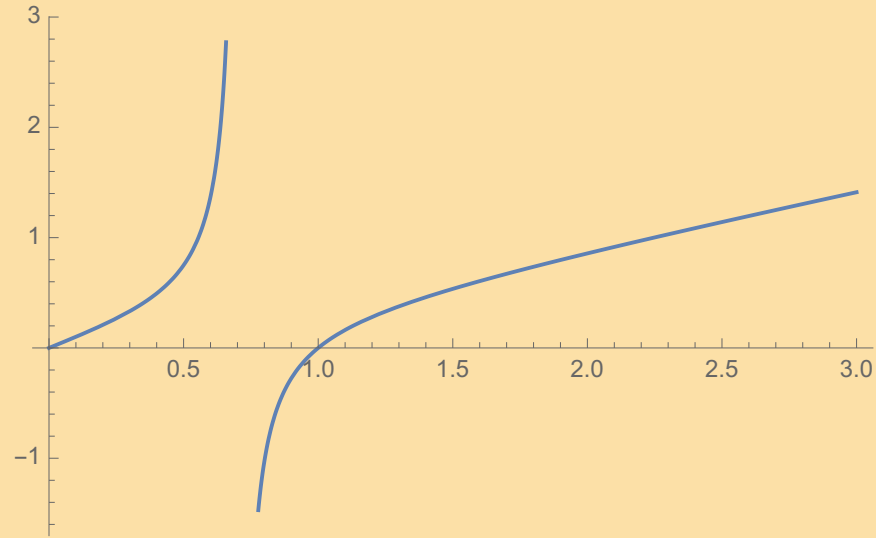
$$(289)$$

Therefore we see that we can always get wave solutions when either $\omega < \omega_*$ or when $\omega > \omega_p$. The $\omega < \omega_*$ solutions are localized surface plasmons, while the $\omega > \omega_p$ solutions are delocalized, since in that case we don't have $k_{\lessgtr} \in \mathbb{R}_{>0}$. We see that the dispersion of the surface plasmons is initially linear, but then saturates once we approach ω_*, so that no matter how large of a wavevector they carry, they always saturate in frequency at ω_*.

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The point of making the plots is to make a comparison with the helical modes computed in the last diary entry—they're exactly the same thing! This isn't surprising in hindsight, since both the magnetic field in that problem and the interface in this problem provide ways of confining the motion of the electrons to a plane. In the helicon problem we were given a frequency ω_c that set the lower boundary for the disallowed region where waves can't propagate, while in the present context this frequency just came directly from the plasma frequency and the geometry of the problem.

23 Phonons in ionic crystals / metals and polaritons, etc.

Today we're doing a third installment in the series "Maxwell's equations, dielectric functions, and all the different types of "ons"". Today's "ons" are polarons, which we will see are basically phononic manifestations of the helicons and surface plasmons discussed previously.

Solution:

The setting is an ionic crystal, which is basically an insulator with a nontrivial unit cell, such that the atoms in the unit cell have different electric charges. We will focus on the simplest case where there are two atoms per unit cell, each with charge $\pm q$.

The strategy will be to find the dielectric function using the same classical tricks as can be used to get the plasma frequency, and then to examine the consequences for wave propagation in the medium in question.

If $r_\pm$ are the coordinates of the two ions in the basis, the equations of motion are

$$m_\pm \partial_t^2 r_\pm = \mp k(r_+ - r_-) \pm qE. \tag{290}$$

The first term on the RHS is the restoring force on the spring connecting the ions together, while the second term is the contribution from the microscopic electric field that each of the ions feels. If we subtract these we get, letting $x \equiv r_+ - r_-$,

$$\mu \partial_t^2 x = -kx + qE \implies -\omega^2 x = -\omega_I^2 x + \frac{q}{\mu}E, \qquad \omega_I^2 \equiv k/\mu. \tag{291}$$

Since $\nabla \cdot \mathbf{P} = -\rho_b$, the polarization is $P = nqx$.[37] Therefore

$$-\omega^2 P = -\omega_I^2 P + \omega_p^2 \epsilon_0 E, \qquad \omega_p \equiv \frac{nq^2}{\epsilon_0 \mu}. \tag{292}$$

Using $\epsilon = 1 + P/(\epsilon_0 E)$, we then get[38]

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_I^2}. \tag{293}$$

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[37] We want $\nabla \cdot \mathbf{P} = -\rho_b$, so I guess here we think of $x = r_+ - r_-$ and the ∇ as acting on x as $(\nabla \cdot \mathbf{P}) = qn(\delta(y - r_+) - \delta(y - r_-))$. *come back to this; signs are sloppy*

[38] We can get away with just calculating $\epsilon(\omega) = \epsilon(\omega, q = 0)$ since the q for light will always be tiny compared to the qs relevant for phonon stuff, in lieu of $cq \approx sk \implies q/k = s/c \ll 1$, where k is the phonon wavevector and s the speed of sound.

When we plot the $\mathbb{R}$ and $\mathbb{I}$ parts of c^2q^2 with e.g. $\omega_I = 1/2$ and $\omega_p = 1$, we get

$$(294)$$
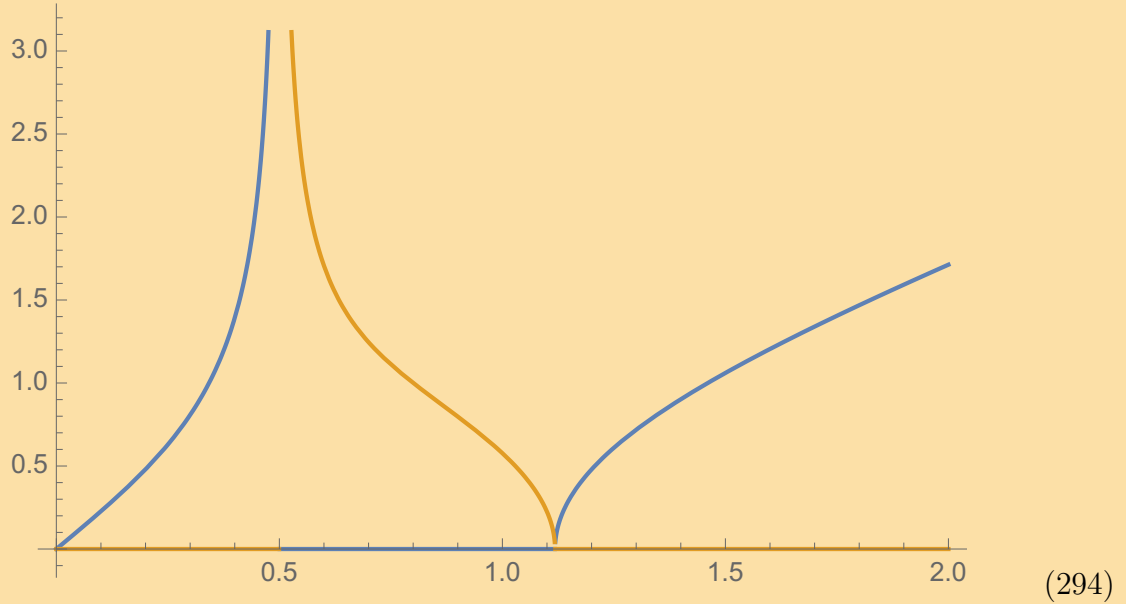
Look familiar? This is essentially the exact same plot that we made in our discussions of surface plasmons and of helicons. We see that the lower of ω_I set the upper limit for the lower branch of the wave solutions (regardless of ω_p), while ω_p approximately sets the lower bound for the upper branch, at least in the limit of large $\omega_p \gg \omega_I$.

At ω_I we have $\epsilon \to \infty$ from below, which since $\epsilon \to \infty$ represents a longitudinal mode in the material. At $\omega_* \approx \omega_p$ we have $\epsilon \to 0$, which is a transverse mode that at high frequencies disperses as a regular light wave (this is the correct high-frequency limit since at very high frequencies the material is unable to react and becomes transparent).

But hold on, why do the longitudinal modes always have $\epsilon(\omega) = 0$? An easy argument goes as follows: we suppose there is no free charge, so that $\nabla \cdot \mathbf{D} = 0$. Then assuming an isotropic medium where $\epsilon(\omega)$ is diagonal, a longitudinal wave will have $\mathbf{D}, \mathbf{E}, \mathbf{P}$ all proportional to $\mathbf{k}$. Therefore $\mathbf{k} \cdot \mathbf{D} = 0 \implies \mathbf{D} = 0$, which tells us from $\mathbf{D} = \epsilon \mathbf{E}$ that $\epsilon = 0$ (assuming wolog that $\mathbf{E} \neq 0$ since otherwise all the fields are zero). This also goes the other way if we assume that the dynamics of the $\mathbf{B}$ field aren't important, so that $\nabla \times \mathbf{E} = 0$. Then since $\mathbf{k} \times \mathbf{E} = 0$ but $\mathbf{E} \neq 0$ (else all the fields are zero), then $\mathbf{E} \perp \mathbf{k}$; hence the mode is longitudinal.

Similarly, we can argue that transverse modes usually have $\epsilon(\omega) = \infty$. Indeed under the same assumptions above, suppose $\mathbf{k} \cdot \mathbf{D} = 0$ but $\mathbf{D} \neq 0$. Then $\epsilon \mathbf{k} \cdot \mathbf{E} = 0$. Assuming $\epsilon \neq 0$, then the mode being transverse means that the previous equation implies $\mathbf{E} = 0$, which in turn from $\mathbf{D} \neq 0$ but $\mathbf{D} = \epsilon \mathbf{E}$ means that $\epsilon = \infty$. The same steps also work in reverse.

24 Pierls instability

Today we're doing an elaboration on a problem from A&S.

Solution:

The first way to look for the instability is to consider a modulation of the hopping term in the Hamiltonian, and to then do a variational treatment of the exact energies of the system. Since we don't want to diagonalize dimension > 2 matrices by hand, we will really only want to employ this strategy in the case of half-filling, where the momentum connecting the Fermi points is π so that the trial lattice deformation has momentum π and alternates in sign on each site—this will give us two bands in the new BZ, which is tractable.

Since we've already talked about the SSH chain in this diary entry, we will be brief. We model the alternating hopping in the SSH chain by a small elastic deformation of the lattice, which costs energy according to a potential term in the Hamiltonian. We write

$$H = -t \sum_j (1 + u_j) \left(c_j^\dagger c_{j+1} + h.c. \right) + \frac{k}{2}(u_{j+1} - u_j)^2 \right), \tag{295}$$

where the u_js measure the displacement of the lattice sites from their equilibrium positions (if $u_j > 0$ then sites j and $j+1$ are closer together, and hence their hopping matrix element is larger).

25   Josephson junction stuff

Schematic arguments

To explain the Josephson effects in the quickest way possible, one could say the following: first, for two SCs L and R, the current from L to R should be proportional to $\langle c_R^\dagger c_R^\dagger c_L c_L P - h.c. \rangle$ where $P \sim e^{iq \int_L^R A}$ is needed for gauge invariance. In MF this is something $\propto P e^{i(\phi_L - \phi_R)} - h.c.$, which gives us the expected

$$J_{R \to L} \propto \sin[q \int A + \Delta\phi] \tag{296}$$

dependence. Secondly, we expect a voltage difference to cause a change in phase according to

$$H \ni V_L n_L + V_R n_R \implies [\phi_R - \phi_L, H] = V_R[\phi_R, n_R] - V_L[\phi_L, n_L] \propto \Delta V \implies \partial_t(\Delta\phi) \propto \Delta V. \tag{297}$$

Going one level up in carefulness, we can model the system as follows. The amount of current coming from tunneling from L to R will be proportional to ψ_R, with a term in the action like $\psi_L \psi_R$. If we write down separate Schrodinger equations for the two ψs, then then tunneling terms appear as additional contributions to $\partial_t \psi$:

$$i\partial_t \psi_L = E_L \psi_L + P \psi_R, \qquad i\partial_t \psi_R = E_R \psi_R + P^* \psi_L. \tag{298}$$

Again, P is there to make all terms in each equation transform in the same way under gauge transformations. Now we write $\psi_i = \sqrt{\rho_i} e^{i\phi_i}$, so that

$$i\partial_t \rho_L - \rho_L \partial_t \phi_L = E_L \rho_L + \sqrt{\rho_L \rho_R} P e^{i\Delta\phi}, \qquad i\partial_t \rho_R - \rho_R \partial_t \phi_R = E_R \rho_R + \sqrt{\rho_L \rho_R} P^* e^{-i\Delta\phi} \tag{299}$$

Now take the $\mathbb{R}$ and $\mathbb{I}$ parts of each equation, and subtract them to get equations for the current $\partial_t(\rho_R - \rho_L)$ and the phase $\partial_t(\phi_R - \phi_L)$. For the former, we have

$$J_{R \to L} \propto \sqrt{\rho_L \rho_R} \sin[q \int A + \Delta\phi] \tag{300}$$

as expected, while for the latter, if we make the approximation that $\rho_L \approx \rho_R$, then

$$\partial_t \Delta\phi \propto E_L - E_R + (\sqrt{\rho_R/\rho_L} - \sqrt{\rho_L/\rho_R}) \cos[q \int A - \Delta\phi] \approx E_L - E_R \propto \Delta V, \tag{301}$$

so that indeed the voltage sets the time evolution of the phase.

Getting more serious

To get more serious, we can consider the action (what follows are the details for a problem in A&S)

$$S = \int d\tau \, \bar{\Psi} \left(\partial_\tau + \xi Z + \bigoplus_a e^{i\phi Z} \Delta_a X + (Z \otimes \widetilde{X} \otimes \mathcal{T}) \right) \Psi + S_C, \tag{302}$$

where

$$S_C = \frac{E_C}{4} \int \left(\bar{\Psi}(Z \otimes \widetilde{Z})\Psi \right)^2. \tag{303}$$

The notation is as follows: each Ψ has the labels $(\Psi^a_{\alpha\uparrow}, [\Psi^a_{\alpha\downarrow}]^\dagger)^T$, where $a = L, R$ is a flavor index keeping track of what SC the field annihilates particles in, and α is an index that is summed over all free-particle eigenstates of the respective SCs (e.g. a momentum index). Z, X are Pauli matrices in particle-hole space for the Nambu spinors, $\widetilde{Z}, \widetilde{X}$ are Pauli matrices in L/R flavor space, and the matrix $\mathcal{T}$ is a tunneling matrix between different eigenstates, i.e. $\mathcal{T}_{\alpha\beta}$ (the $\widetilde{X}$ in the $\mathcal{T}$ term means that the $\mathcal{T}$ term couples the two SCs together). The SCing OP on each SC is $\Delta_a e^{i\phi_a}$. Finally, $\bar{\Psi}(Z \otimes \widetilde{Z})\Psi$ has alias $N_L - N_R$, so that the S_C term represents a charging energy.

First, we decouple S_C by adding a field $V^2/4E_C$ to the action, and then shifting $V \mapsto V + iE_C(N_L - N_R)$. The fermion part of the new action is

$$S = \int \bar{\Psi} \left(\partial_\tau + \xi Z + \bigoplus_a e^{i\phi_a Z} \Delta_a X + (Z \otimes \widetilde{X} \otimes T) + \frac{iV}{2} Z \otimes \widetilde{Z} \right) \Psi. \tag{304}$$

Now we shift the fields by

$$\Psi^a \mapsto \begin{pmatrix} & e^{-i\phi_a/2} \\ e^{i\phi_a/2} & \end{pmatrix} \Psi^a, \tag{305}$$

which shifts the phase dependence of the OPs onto the diagonal in the particle-hole space when the ∂_τ hits it, so that it appears with the V. The structure of this phase in particle-hole space has a Z, since particles and holes get opposite phases. Since the T term is off-diagonal in flavor space, the T terms therefore pick up phases like $e^{i\delta}$, where $\delta \equiv \phi_L - \phi_R$. Taking all

of this into account, and integrating out the fermions, we get (the minus sign in front of the Tr is because we're in $\mathbb{I}$ time)

$$S = \frac{1}{4E_C} \int V^2 - \text{Tr} \ln \left[\partial_\tau + \frac{i}{2}(\partial_\tau \phi Z + VZ \otimes \tilde{Z}) + \xi Z + \Delta X + \mathbf{1} \otimes \begin{pmatrix} & e^{-i\delta Z/2}\mathcal{T} \\ e^{i\delta Z/2}\mathcal{T} & \end{pmatrix} \right], \tag{306}$$

where $\phi = \phi_L \oplus \phi_R$ is to be viewed as a matrix, and the matrix we've written out explicitly is in flavor space.

As usual, we write the thing in the log in terms of the propagators G_a:

$$\ln \left[(G_L^{-1} \oplus G_R^{-1}) \begin{pmatrix} G_L \mathcal{D}_L Z + \mathbf{1} & G_1 \mathcal{T} e^{-i\delta Z/2} \\ G_2 \mathcal{T} e^{i\delta Z/2} & G_2 \mathcal{D}_R Z + \mathbf{1} \end{pmatrix} \right] \to \ln \left[\mathbf{1} + \begin{pmatrix} G_L \mathcal{D}_L Z & G_1 \mathcal{T} e^{-i\delta Z/2} \\ G_2 \mathcal{T} e^{i\delta Z/2} & G_2 \mathcal{D}_R Z \end{pmatrix} \right], \tag{307}$$

where the arrow means ignoring the infinite constant from the first factor in the log. Here we have defined

$$\mathcal{D}_a \equiv \frac{i}{2}(\partial_\tau \phi_a + (-1)^a V), \tag{308}$$

and the propagators G_a, for each single-particle eigenstate α, are (in PH space, and still in $\mathbb{I}$ time)

$$G_{a\alpha} = \frac{1}{\omega^2 + \xi_\alpha^2 + \Delta^2} \begin{pmatrix} i\omega + \xi_\alpha & \Delta \\ \Delta & i\omega - \xi_\alpha \end{pmatrix}. \tag{309}$$

As a first pass, we look at the effective action when $\mathcal{T} = 0$. The terms quadratic in the $\mathcal{D}_a$s are, in the time domain,

$$S[\mathcal{T} = 0] \supset \frac{1}{2} \sum_{a,\alpha} \int d\tau, d\tau' \, \mathcal{D}_a(\tau) G_{a\alpha}(\tau - \tau') \mathcal{D}_a(\tau') G_{a\alpha}(\tau' - \tau). \tag{310}$$

In frequency space, we get

$$S[\mathcal{T} = 0] \supset \frac{1}{2} \sum_{\omega\nu,a,\alpha} \mathcal{D}_a(\omega) G_{a\alpha}(\nu + \omega) \mathcal{D}_a(-\omega) G_{a\alpha}(\nu). \tag{311}$$

Now the Greens function is, after summing over the single-particle states,

$$\sum_\alpha G_{\alpha a}(\omega) \approx N(0) \int_{\mathbb{R}} d\xi \frac{i\omega \mathbf{1} + \xi Z}{\xi^2 + \omega^2 + \Delta^2} = i\pi \frac{\omega}{\sqrt{\omega^2 + \Delta^2}}, \tag{312}$$

which is localized around zero in the time domain due to the gap caused by Δ. As long as the fluctuations of the $\mathcal{D}_a$ fields are on longer times that Δ^{-1}, we can then treat the $G(\tau - \tau')$s in the above equation as being proportional to $\delta(\tau - \tau')$s. Therefore we have

$$S[\mathcal{T} = 0] \propto \sum_a \int d\tau \, \mathcal{D}_a(\tau)^2. \tag{313}$$

This is evidently minimized when $\mathcal{D}_a = 0$ for both a. According to the definition of $\mathcal{D}_a$ then, this is minimized when

$$V = -\partial_\tau \phi_R = +\partial_\tau \phi_L, \tag{314}$$

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which are the Josephson relations for the time evolution of the phase difference.

We will work at energy scales well below Δ, and hence will impose the above constraint exactly in what follows. The leading nonzero piece of the action is then

$$S \supset \frac{1}{2}\text{Tr}\left[\left(\begin{matrix} & G_1\mathcal{T}e^{-i\delta Z/2} \\ G_2\mathcal{T}e^{i\delta Z/2} & \end{matrix}\right)^2\right]. \tag{315}$$

Writing this out,

$$S \supset \mathcal{T}^2 \sum_{\alpha\beta\omega\nu} \text{Tr}[G_{1\alpha}(\omega+\nu)\mathcal{E}(\nu)G_{2\beta}(\omega)\mathcal{E}^*(\nu)], \qquad \mathcal{E}(\nu) \equiv \int d\tau\, e^{-i\nu-i\delta Z/2}. \tag{316}$$

Note how we are doing *independent* sums over the eigenstates α, β on the two SCs—we have made a simplifying assumption in which the tunneling matrix elements are independent of the eigenstates they connect.

We can break the tunneling contribution to S into two parts—one coming from the diagonal $(i\omega\mathbf{1} + \xi Z)$ parts of the Greens functions, and one coming from the off-diagonal part ΔX. The diagonal part is, replacing the sums over eigenstates with integrals over ξ and approximating the DOS as a constant as usual, (we can pull out the $\mathcal{E}$s since they commute with Z and since $\mathcal{E}(\nu)\mathcal{E}^*(\nu)$ out of the brackets since $\sum_\nu \mathcal{E}(\nu)\mathcal{E}^*(\nu) = T^{-1}\int d\tau\, \mathcal{E}(\tau)\mathcal{E}^*(\tau) = 1$)

$$S_d = \mathcal{T}^2 N(0)^2 \sum_{\omega\nu} \int_{\mathbb{R}} d\xi_\alpha\, d\xi_\beta\, \text{Tr}\left[\frac{(i\omega + \xi_\alpha Z)(i(\omega+\nu) + \xi_\beta Z)}{(\omega^2 + \xi_\alpha^2 + \Delta^2)((\omega+\nu)^2 + \xi_\beta^2 + \Delta^2)}\mathcal{E}(\nu)\mathcal{E}^*(\nu)\right] \tag{317}$$

The terms with ξs in the numerator die by oddness, while the others are actans. Therefore

$$S_d = -2\mathcal{T}^2 N(0)^2\pi^2 \sum_{\omega\nu} \frac{\omega(\omega+\nu)}{\sqrt{\omega^2+\Delta^2}\sqrt{(\omega+\nu)^2+\Delta^2}}\text{Tr}[\mathcal{E}(\nu)\mathcal{E}^*(\nu)]. \tag{318}$$

For $\nu \gg \Delta$ we can do this by crudely ignoring Δ; this gives something proportional to ν/T. However, we are instead interested in the $\nu/\Delta \ll 1$ limit (since we used it to fix $V = -\partial_\tau\phi_R$!), and since we are already dropping ν/Δ from the Josephson relation for the phase difference, we must drop this integral as well.[39]

Therefore the only remaining term is the off-diagonal part, viz. (have to be careful here since the $\mathcal{E}$'s don't commute with X)

$$S \supset \mathcal{T}^2 N(0)^2\Delta^2 \sum_{\omega\nu} \int_{\mathbb{R}} d\xi_\alpha\, d\xi_\beta\, \text{Tr}\left[\frac{X\mathcal{E}(\nu)X\mathcal{E}^*(\nu)}{(\omega^2 + \xi_\alpha^2 + \Delta^2)((\omega+\nu)^2 + \xi_\beta^2 + \Delta^2)}\right] \tag{319}$$

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[39]Since the ν dependence of the summand (excluding the $\mathcal{E}$s) is trivial, and since $\sum_\nu \mathcal{E}(\nu)\mathcal{E}^*(\nu) = 1$, S_d actually gives us an (infinite) constant. Being independent of δ, which by now is the only dynamical variable left in the action, we will ignore it. Or said another way, we will regulate it by subtracting off the contribution of S_d when $\nu = 0$—we can also think of this as fixing boundary conditions for $e^{i\delta}$, so that only non-zero-mode fluctuations of $\mathcal{E}$ are allowed, with the $\nu = 0$ part being dissallowed by virtue of the fact that it changes the boundary conditions.

Now $X\mathcal{E}(\nu)X\mathcal{E}^*(\nu) = \mathcal{E}(\nu)^2$, and so (again dropping the ν in the denominator on the grounds that we are dropping $\mathcal{O}(\nu/\Delta)$) we do the arctan integrals and get

$$S \supset \mathcal{T}^2\pi^2 N(0)^2\Delta^2 \sum_\nu T^{-1} \int dz\, n_F(z)\mathrm{Tr}[\mathcal{E}(\nu)^2]\frac{1}{\Delta^2 - z^2}. \tag{320}$$

In the limit where $e^{-\beta\Delta} = 0$, only the $z = -\Delta$ pole contributes (said another way, in this limit we may just send $\sum_\omega \to T\int d\omega$). Therefore

$$S \supset \frac{1}{2}(\mathcal{T}\pi N(0))^2\Delta T^{-1} \sum_\nu (e^{i\delta} + e^{-i\delta}) \to (\mathcal{T}\pi N(0))^2\Delta \int d\tau\, \cos\delta. \tag{321}$$

Remembering the charging part of the action with now $V = -\partial_\tau\phi_R = \partial_\tau\delta/2$, we get

$$S = \frac{1}{16E_c} \int (\partial_\tau\delta)^2 + \gamma \int \cos\delta, \qquad \gamma \equiv (\pi\mathcal{T}N(0))^2\Delta, \tag{322}$$

which is our final action for the phase difference $\delta = \phi_L - \phi_R$. We can also write this in Hamiltonian form by introducing the conjugate momentum $N = 1/8\partial_\tau\delta$ and doing the Legendre transform (alias integrating in N), in terms of which (the i is because we're in $\mathbb{I}$ time; the symplectic part of the action always has an i, regardless of signature)

$$S = \int \left(\frac{i}{2}N\partial_\tau\delta + E_C N^2 + \gamma\cos\delta\right). \tag{323}$$

Since N is conjugate to δ, it represents the difference of the number operators on the two SCs. Therefore the time derivative of N is the current, which we see is (going to $\mathbb{R}$ time; the unit of charge is included in N, which is really the charge density)

$$J = \sin\delta. \tag{324}$$

Likewise, the eom for the phase difference δ is

$$\partial_t\delta = 2E_C N, \tag{325}$$

which tells us that the evolution of the phase difference is set by the potential difference across the junction (viz. the product of $E_C \sim \mu_L - \mu_R$ and N).

26   Basic JW applications

Solution:

Now consider the XY model with a magnetic field:

$$H = -\sum_{\langle ij\rangle}(X_iX_j + Y_iY_j) - h\sum_i Z_i. \tag{326}$$

When acting on a spin-polarized state like $\otimes|\uparrow\rangle$, we see that c_k inserts a spin wave of momentum k (plus an unimportant JW tail).[40] The fact that the spin wave operators are fermionic makes sense: you can't lower the spin twice.

After doing JW, we get (there's actually some nontrivial algebra here needed to show that all the quartic fermion terms die—I won't write it out cause we've done this a million times)

$$H = \sum_i (c_i^\dagger c_{i+1} + h.c. + h c_i^\dagger c_i). \tag{327}$$

Therefore the spectrum is just

$$\epsilon_k = 2\cos(k) - h. \tag{328}$$

As expected this is gapped at large $|h|$, while for $-2 < h < 2$ it is gapless, with the ground state obtained by filling up the $\cos(k)$ band. Therefore the ground state for this range of h is obtained by creating a bunch of spin waves on top of e.g. $\otimes|\uparrow\rangle$. At low T the heat capacity will change from being exponentially suppressed, to varying as $T^{d/z} = T^{1/2}$ right at the critical point, to the free fermion $C \propto T$ at intermediate h, and then finally will reverse this pattern back to an exponential suppression.

The JW tails aren't totally unimportant, and we need to remember them when talking about correlators of spin operators and the like. For example, we can consider what happens when we do neutron scattering. A neutron with momentum p hitting the system will create a state like

$$|\psi'\rangle = \sum_j e^{ijp} S_j^- |0\rangle, \tag{329}$$

where $|0\rangle$ is the ground state. Now $\sum_j e^{ijp} c_j |0\rangle$ is an eigenstate with energy ϵ_p, but $|\psi'\rangle$ is *not* an eigenstate. Indeed,

$$|\psi'\rangle = \sum_j e^{ijp} \sum_q e^{-iqj} c_q \prod_{l<j} (c_l^\dagger c_l - 1/2)|0\rangle. \tag{330}$$

If we select out only the factors of $1/2$ from the product, we indeed get an ϵ_p eigenstate. But all the other terms mean that $|\psi'\rangle$ isn't an eigenstate; instead of destroying a single electron of momentum p, it does basically all possible things that it can do while changing the electron number by -1. The very non-eigenstate-ness of this means that the neutron scattering spectrum will not just be a boring $\cos(p)$ like ϵ_p is, but will rather be a broad continuum.

The other usual example is the TFIM, which lacks $U(1)$ symmetry:

$$H = -\sum_{\langle ij\rangle} X_i X_j - h\sum_i Z_i. \tag{331}$$

Since there is no $U(1)$, it will behoove us to write things in terms of $\mathbb{R}$ fermions, rather than $\mathbb{C}$ fermions, the latter of which are really only natural when one has a $U(1)$. One then fermionizes with fermions λ_j, η_j, where λ_j goes to X_j (plus a tail) and η_j goes to Y_j (plus

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[40]Remember that we JWize with e.g. $c_j = \sigma_j^- \prod_{k>j} Z_k$, or $\sigma_j^- = c_j \prod_{k>j} (-1)_j^F$.

a tail). Up to constants, this is then (after some algebra to deal with the tails on the first term)

$$H \sim - \sum_{\langle ij \rangle} \lambda_i \eta_j - h \sum_i \lambda_i \eta_i. \tag{332}$$

Now the Hamiltonian in momemtum space is a real antisymmetric matrix, in keeping with the reality of the Majoranas. Therefore when diagonalized it is complex, and so the diagonalization strictly speaking is done by rewriting the two Majoranas in terms of $\mathbb{C}$ fermion operators. Since we've already done this before, I won't go into any detail.

27 *Quantum oscillations and non-uniform magnetic fields*

Today we're doing an elaboration on a problem in A&M. We will go over the derivation of the frequency of quantum oscillations in a metal, and then show to what extent inhomogeneities in the magnetic field are allowed before the oscillations get washed out.

Solution:

First let us recall the classical equations of motion for a qp in a metal. We will write the action as

$$S = \int dt \left(\dot{q}^i (\hbar k_i - e A_i / c) - H(k) \right), \tag{333}$$

where $k_i = p_i + e A_i / \hbar c$ is the kinetic momentum, and p_i the canonical momentum. The kinematic momentum is the one we want to use to describe the qp's phase space trajectory: the equations of motion are

$$\dot{q}_i = \hbar^{-1} \frac{\partial H}{\partial k^i} \equiv v_i(k), \qquad \hbar \dot{k}_i = \frac{e}{c} \left(-\partial_i A_j + \partial_j A_i \right) \dot{q}^j = -\frac{e}{c} v^j(k) F_{ij} \implies \hbar \dot{\mathbf{k}} = -\frac{e}{c} \mathbf{v}(\mathbf{k}) \times \mathbf{B}, \tag{334}$$

since H is independent of the q_i when we use the kinematic momentum. Since $H(k)$ is independent of A_i, the magnetic field only affects the eom + commutators, and not the form of the energy. Thus, our strategy will be to put the qps in the band structure determined by H, and then to examine the effects of the magnetic field by looking at how the magnetic field affects motion within the bands.

The eom for $\mathbf{k}$ tells us that energy is conserved (duh, no work is done by the magnetic field), and that the component of the momentum along the magnetic field is conserved. Let the $\mathbf{B}$ field be in the $\hat{z}$ direction. Then in the space of the $\mathbf{k}$s, a qp with a fixed k_z and fixed energy E will move along a trajectory determined by the intersection of an iso-energy surface and the plane of all points with z-coordinate k_z. If the iso-energy surface is closed, provided that the intersection of these two surfaces is non-empty, it will generically be diffeomorphic to a circle. Therefore, assuming the iso-energy surface is closed, the semiclassical trajectory will be closed in $\mathbf{k}$ space.[41] We can then compute the time taken for the qp to execute one semi-classical orbit:

$$T = \oint dk^i \frac{1}{dk^i / dt} = \frac{c \hbar^2}{eB} \oint dk_\parallel \partial_E k_\perp, \tag{335}$$

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[41]No reason for it to be closed in real space, though!

where the $\parallel$ and $\perp$ indicate directions tangent and normal to the contour (both of them are in the xy plane on account of $\dot{k}_z = 0$), and we have assumed that B is constant (here E is the energy). The point here is that the integral computes $\partial_E A$, where A is the area in $\mathbf{k}$ space enclosed by the contour. So we conclude that

$$T = \frac{c\hbar^2 \partial_E A}{eB}, \tag{336}$$

for constant B situations.

One small caveat: we are talking about trajectories in the space of kinematic momenta rather casually, but of course strictly speaking this is a bit loose, since the kinematic momenta do not all commute among themselves. The fact that $[k_i, k_j] = -i\hbar eB/c$ means that the notion of a qp being at a definite point in the k_x-k_y plane only makes sense when we look at areas larger than the momentum-space area on the RHS of the commutator, viz. $A_m = eB/c$. So our semiclassical description relies on the assumption that the wavevector of the qp satisfies

$$\hbar^2 k^2 \gg \frac{\hbar eB}{c} = \hbar m\omega_c. \tag{337}$$

Of course, this is (very strongly) satisfied for any reasonable choice of B and any k near k_F for any reasonable Fermi energy E_F (the estimate for the energy here is written in terms of ω_c, which determines the energy of the circular orbits in the case when the dispersion is quadratic, $E = k^2/2m$. In general the orbits won't be circular, but ω_c still gives the appropriate estimate).

So far, except for the last paragraph, we haven't really done any quantum mechanics. Quantum mechanics comes in when we use Bohr's quantization rule to determine the spacings of the energy levels. It tells us that if the energy levels are E_n, we have[42]

$$E_{n+1} - E_n = 2\pi\hbar/T = \frac{2\pi eB}{c\hbar \partial_E A}. \tag{339}$$

The usual motivation for this comes from thinking about particles forming standing waves along their classical trajectories—here this logic is a bit stretched since the qps don't execute periodic motion in $\mathbb{R}$ space. They almost do, though: taking the cross product of the $\mathbf{k}$ equation of motion with $\hat{\mathbf{B}}$, one can check that in the xy plane in $\mathbb{R}$ space, the qps undergo periodic motion with a trajectory that's basically a rescaled copy of their $\mathbf{k}$-space trajectory. The $\mathbb{R}$ space trajectory isn't periodic because of the possibility that $k_z \neq 0$, but this is okay: k_z just behaves as a flavor label with respect to the problem in the xy plane, and so we can treat it as such and just focus on the xy plane motion. Doing this, we can demand that the

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[42]Miscellaneous comment: we will be working at zero temperature, for simplicity. Adding a temperature T will smear out the FS on a scale kT; this will make the effects of the magnetic field washed out if $kT \gtrsim \hbar\omega_c$, and so we will need to be at temperatures lower than this. Similarly we will be ignoring interaction effects; if interactions lead to a qp lifetime τ, then we will similarly need $\hbar/\tau \ll \hbar\omega_c$; else the qps can't be thought of as living in a single energy level. Therefore the approximations needed for this discussion to work are

$$kT, \hbar/\tau \ll \hbar\omega_c \ll E_F. \tag{338}$$

qps form standing waves along their trajectories in the xy plane, justifying the application of the correspondence principle.[43]

We will be focused on qps near the FS. By the time we get out to the FS in momentum space, we will have filled up many of the E_ns, since the difference in energies between the levels is $\approx \hbar\omega_c$ (the result for the $H(k) \propto k^2$ case), which we have taken to be much less than the Fermi energy. Therefore we may approximate $\partial_E A$ as a discrete difference, and so

$$\Delta A = A_{n+1} - A_n = \frac{2\pi eB}{\hbar c}. \tag{340}$$

This means that

$$A_n = (n + \delta)\frac{2\pi eB}{\hbar c}, \tag{341}$$

where δ is a constant. This constant is usually assumed to be independent of B and k_z, a fact which can be checked in the $H(k) \propto k^2$ case.[44]

Thus, the picture is the following: we draw closed loops in the k_x-k_y plane, and then take a cartesian product of these loops with $\mathbb{R}$ to form things diffeomorphic to cylinders. We then intersect these tube-like objects with the Fermi sea, and the resulting intersection areas then determine the semiclassical trajecotires of the qps. Note that since the Fermi surface projected into the k_x-k_y plane and the loops we have drawn in the k_x-k_y plane are both iso-energy contours, we will never have loops that cross or exit the Fermi sea, and so the motion is well-defined.

The density of states will be high at points where the tubes intersect the FS. Generically these intersection regions are one-dimensional, but as the size of the tubes becomes equal to the size of the FS, the intersection regions degenerate and become nearly two-dimensional. Therefore we will have peaks in the qp dos whenever a level E_n at a given k_z hits the FS. If B_1 is a field strength at which this happens (i.e., for which there is some n such that at a fixed k_z, we have $E_n = E_F$) then $B_2 > B_1$ will also be such a field strength, provided that

$$(n + \delta)\frac{2\pi eB_1}{\hbar c} = ((n-1) + \delta)\frac{2\pi eB_2}{\hbar c} \implies \frac{1}{B_2} - \frac{1}{B_1} \equiv \Delta_{1/B} = \frac{2\pi e}{\hbar c A(E_F)}. \tag{342}$$

Therefore the dos will exhibit oscillations in the field with a "frequency" given by the RHS of the above. By altering the orientation of $\mathbf{B}$, this allows us to map out the FS, by fitting observations to a family of guesses about $A(E_F)$ in the planes $\perp$ to $\mathbf{B}$.

The tl;dr version

I'm writing this subsection after coming back to this after a gap of O(6 months), and just wanted to clarify here how simple the problem really is. Because of interference effects,

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[43] Again, don't confuse $\mathbf{k}$ with a velocity for the qps! If one does this, one would say that a trajectory that e.g. always stays at $k_x, k_y > 0$ could not possibly be closed in $\mathbb{R}$ space. But it is; the velocity is $\partial_{\mathbf{k}}E/\hbar$, and is not directly related to $\mathbf{k}$.

[44] In this case, we know that the k_x-k_y plane contribution to the energy is $\hbar\omega_c(n + 1/2)$. This means that the radius of the circular motion in $\mathbf{k}$ space is $\sqrt{2m\hbar\omega_c(n + 1/2)}$, which when squared and multiplied by π exactly gives the formula for A_n above, with $\delta = 1/2$.

the $\mathbb{R}$-space electron orbits want to enclose an amount of flux in $\Phi_0\mathbb{Z}$, with $\Phi_0 = 2\pi\hbar/e$.[45]
Therefore the $\mathbb{R}$-space areas of successive LL orbits will differ by

$$\Delta A_{\mathbb{R}} = \Phi_0/B. \tag{343}$$

Now take the cross product of the $\mathbf{k}$ eom with $\widehat{\mathbf{B}}$: since $\widehat{\mathbf{B}} \times (\mathbf{v} \times \mathbf{B}) = B\mathbf{v} - \mathbf{B}(\widehat{\mathbf{B}} \cdot \mathbf{v}) = B\mathbf{v}_\perp$, with $\mathbf{v}_\perp$ the velocity in the plane normal to $\mathbf{B}$, we have

$$\hbar\dot{\mathbf{k}} = e\mathbf{v}_\perp B \implies \mathbf{r}_\perp(t) - \mathbf{r}_\perp(0) = \frac{\hbar}{eB}(\mathbf{k}(t) - \mathbf{k}(0)). \tag{344}$$

Therefore the $\mathbb{R}$ space orbit in the plane $\perp$ to $\mathbf{B}$ is the same as the $\mathbf{k}$ space orbit, but rotated by $\pi/2$ and scaled by a factor of $[\Phi_0/2\pi B]^2$. Hence

$$\Delta A = (\Delta A_{\mathbb{R}})\frac{(2\pi B)^2}{\Phi_0^2} = (2\pi)^2\frac{B}{\Phi_0}. \tag{345}$$

Notice how the areas work out in such a way that $(\Delta A)(\Delta A_{\mathbb{R}})/(2\pi)^2 = 1$, as we might expect from phase space reasons. Now the energy levels are indexed by n and k_z, and when $E = \hbar\omega(n + 1/2)$ for some $n \in \mathbb{N}$ we are right at the bottom of a band which disperses along one direction, and hence has a $1/\sqrt{E}$ singularity in the DOS as characteristic of a 1d Fermi gas. Therefore we will see a singularity in the DOS whenever the Landau orbits cross the chemical potential (which we assume to be unchanged by the B field). The effects are most dramatic when this happens simultaneously for many different k_z slices, and hence the extremal orbits are selected out as giving the biggest changes in the DOS.

Brief comment on inhomogenous fields

Now consider the case where the magnetic field is not homogeneous (we will keep its orientation fixed for simplicity, and just consider fluctuations in B). What kinds of variations in B are allowed before the signal is washed out? Since we are working semiclassically, we can still think in terms of a phase space picture, where as we move along in $\mathbb{R}$ space, the areas enclosed by the orbits A_n change slightly.

In order for the variation in B to not screw up the resolution of the oscillations, we evidently need the variation in the energy levels to satisfy

$$\delta E_n \ll E_{n+1} - E_n. \tag{346}$$

This means that we need

$$\delta E = \partial_B E \delta B \ll \frac{2\pi eB}{\hbar c}\partial_A E \implies \delta\ln B \ll \frac{2\pi e}{\hbar c}\frac{\partial_E B}{\partial_E A} = \frac{2\pi e}{\hbar c}(\partial_B A)^{-1}. \tag{347}$$

Now from the above discussion we know that A depends linearly on B, and so

$$\delta\ln B \ll \frac{2\pi e}{\hbar c}\frac{1}{(n + \delta)2\pi e/\hbar c}, \tag{348}$$

where n is the level at the Fermi energy. Simplifying the RHS, we get the condition that

$$\delta\ln B \ll \frac{\Delta A}{A}. \tag{349}$$

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[45]Yes, I am committing the sin of switching between SI and CGS mid-diary entry.

28 Why the quantum critical fan is fan-shaped (FINISH THIS YOU IMBECILE)

Solution:

Another way to think of this is in terms of the gap of $\Delta \sim 1/\xi \sim t^\nu$ that the (former) critical fluctuations have away from the critical point. When we flow to the IR at $T = 0$, we end up at energies below Δ, and can't "see" the critical fluctuations. Once we increase T such that $T > \Delta$ however, the critical fluctuations at Δ are thermally activated, and we have a chance to see their effects on low-energy physics. This reasoning basically assumes that the effect of non-zero t is solely to displace the spectrum at criticality upwards by opening a gap, while preserving the features of the critical modes.

29 Analytic structure of different Greens functions and fluctuation-dissipation

Need to go back and distinguish between bosonic and fermionic statistics! The difference matters!
Today we will look at the exact difference between the retarded, advanced, and time-ordered correlators, and prove some general results about them with the aid of the spectral representation, including the precise statement of the fluctuation dissipation theorem. This is of course pretty standard material, but I wasn't able to find a textbook where the discussion was really made completely explicit and done without sign errors. This diary entry is going in the cmt diary and not the qft diary, since we'll be focusing on Greens functions in the form that they take when working in coherent state quantization, and most of our motivation is cmt-related.

Solution:

First we will fix conventions for the various types of correlators we usually encounter. This is basic, but I feel like every book has slightly different sign conventions, so I think it's worth fixing conventions here to have as a reference. Our signs for Fourier transforming things will be that spatial momentum integrals always get plus signs and factors of 2π, and that spatial and temporal signs are always different. Therefore e.g.

$$G(t) = \int_\omega e^{-i\omega t} G(\omega), \qquad G(\mathbf{k}) = \int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} G(\mathbf{x}) \tag{350}$$

where $\int_\omega$ has a $1/2\pi$ but $\int_{\mathbf{x}}$ does not. Let these conventions be henceforth remembered and cherished for generations, amen.

Response functions

We define the retarded response function for a pair of operators $\mathcal{O}, \mathcal{O}'$ as

$$\chi(t - t') = -i\theta(t - t')\langle[\mathcal{O}(t), \mathcal{O}'(t')]_{\zeta_\mathcal{O}}\rangle. \tag{351}$$

Physically, this measures the expectation value of the operator $\mathcal{O}$ in the presence of a time-varying source that couples to $\mathcal{O}'$ though a term $\int F\mathcal{O}'$, via $\langle\mathcal{O}(t)\rangle_{\int F\mathcal{O}'} = \int dt' \, \chi(t-t')F(t')$. A few comments about the notation: first, the $[,]_{\zeta_\mathcal{O}}$ means a commutator if the $\mathcal{O}$s are bosonic, and an anticommutator if the $\mathcal{O}$s are fermionic,[46] but we will sloppily keep this (and other fermionic minus signs) implicit in what follows. Also, the spatial coordinates in the arguments of all operators will be suppressed (although I guess they don't really need to be local operators anyway). We will be suppressing spatial coordinate and spatial momentum indices in a lot of the following discussion—when they are not present explicitly, they are present implicitly. We are doing this because the really important structures of the various correlators we'll use all depend only on time / frequency. This means that basically all functions that are written below seemingly only as a function of ω, are also functions of $\mathbf{k}$.

The expectation value in the last equation is to be taken in some thermal state. When we say that we are interested in looking at the response function at finite times, we mean that we are interested in looking how operators in the theory respond to the presence of a time-varying source that acts as a weak perturbation on top of the thermal state. As is usual in QFT, the only place that the effects of temperature appear here is in the choice of the state in which to evaluate the expectation value: we set up the state to evaluate the (anti)-commutator in, and then treat the time dependence of the operators involved in the usual Heisenberg way. The time evolution is totally unitary, and the system is not coupled to a bath—the idea is that once prepared in a thermal state,

The advanced response function is the anti-causal version of the retarded response function, viz.

$$\chi_A(t - t') = +i\theta(t' - t)\langle[\mathcal{O}(t), \mathcal{O}'(t')]_{\zeta_\mathcal{O}}\rangle. \tag{352}$$

Finally, the time-ordered response function is

$$\chi_T(t - t') = -i\langle T\{\mathcal{O}(t)\mathcal{O}'(t')\}\rangle. \tag{353}$$

Physically, $\chi(t)$ measures the dissipative response to an external force / current that couples to $\mathcal{O}$: our sign conventions have been chosen such that if $H(t) = J(t)\mathcal{O}(t)$ (with $\mathcal{O} = \mathcal{O}'$ for simplicity) is added to the Hamiltonian, then $\mathcal{O}$ acquires an expectation value as

$$\langle\mathcal{O}(t)\rangle = \int dt' \, \chi(t - t')J(t'), \tag{354}$$

which can be derived using the usual interaction representation calculations. The correlator χ_A is just the anti-causal version of this. On the other hand, the time-ordered correlator is a bit different: χ_T is what we actually compute when doing Feynman diagrams, and so is

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[46]This comes from the fact that then F is also fermionic, and this produces a minus sign when the F is moved past the $\mathcal{O}$.

the central object in QFT (just because in the path integral approach, inserting operators in the path integral to produce correlators automatically given by time-ordered expectation values, essentially just by construction). Thus χ_T is the full 2-point function, and as such we know that it takes the form (now restoring the explicit momentum dependence)

$$\chi_T(\omega, \mathbf{k}) = G(\omega, \mathbf{k}) = \frac{1}{\omega - \epsilon_{\mathbf{k}} - \Sigma(\omega, \mathbf{k})} \tag{355}$$

where as usual $\epsilon_{\mathbf{k}}$ is the dispersion of the free Hamiltonian (the one we diagonalize to form the coherent states).

To gain more insight into the relations between these response functions and to figure out how to write χ, χ_A in a form more similar to the expression for $G(\omega, \mathbf{k})$ written above, we need to use the spectral representation. Using our conventions for Fourier transformations, the standard manipulations give[47][48]

$$\chi(\omega) = -i \int dt \sum_{ab} e^{-\beta E_a} e^{i(\omega + i\eta)t} \left(e^{itE_{ab}} \mathcal{O}_{ab} \mathcal{O}'_{ba} - \zeta_{\mathcal{O}} e^{-itE_{ab}} \mathcal{O}_{ba} \mathcal{O}'_{ab} \right) \theta(t), \tag{356}$$

where the sums are over eigenstates of the full interacting Hamiltonian, $E_{ab} = E_a - E_b$, $\mathcal{O}_{ab} = \langle a | \mathcal{O} | b \rangle$, and in this case $\zeta_{\mathcal{O}} = \pm 1$, with the upper sign for bosons. Doing the integral,

$$\chi(\omega) = \sum_{ab} e^{-\beta E_a} \left(\frac{\mathcal{O}_{ab} \mathcal{O}'_{ba}}{\omega - E_{ba} + i\eta} - \zeta_{\mathcal{O}} \frac{\mathcal{O}_{ba} \mathcal{O}'_{ba}}{\omega + E_{ba} + i\eta} \right). \tag{357}$$

If one wants the $T = 0$ result one just sets a to be the vacuum; hence we have written it in terms of E_{ba} and not E_{ab} since $E_{ba} > 0$ at $T \to 0$. We can also write this slightly more compactly as

$$\chi(\omega) = \sum_{ab} \frac{\mathcal{O}_{ab} \mathcal{O}'_{ba}}{\omega - E_{ba} + i\eta} (e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}). \tag{358}$$

Anyway, we see firstly that the poles of $\chi(\omega)$ are all in the lower half plane. The analyticity in the UHP is then tantamount to causality in these convetions, since

$$\chi(t) = \int_\omega e^{-i\omega t} \chi(\omega) \tag{359}$$

means that when $t < 0$, the contour needs to be closed in the upper half plane—analyticity then tells us that this vanishes, hence providing a sanity check on $\chi(t < 0) = 0$.

To get the advanced propagator, one simply multiplies the by two minus signs (one from the definition (which is why we included it!) and one from the $\theta(-t)$ giving a relative minus

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[47]Something that I just thought of to remember the sign in the Heisenberg evolution: the sign convention in the Schrodinger equation tells us that $H = +i\hbar d_t$. This is natural since it is the opposite sign to momentum—we can think of this as being in "mostly negative signature conventions" if we like. Now $\mathcal{O}(t) = e^{-t\partial_t} \mathcal{O}(0) e^{t\partial_t}$, since $e^{t\partial_t}$ "drags functions backwards along the time axis by an amount t", and so the correct sign is $\mathcal{O}(t) = e^{+iHt} \mathcal{O}(0) e^{-iHt}$.

[48]Also, notational warning: we will be forgetting to put $1/Z$'s to normalize the thermal Boltzmann weights $e^{-\beta E_a}$. Sorry.

sign from the integration over t), and replaces $i\eta$ with $-i\eta$, since the convergence of the Fourier transform of $\chi_A(\omega)$ at $t \to -\infty$ is now the relevant issue, meaning one must Fourier transform with $e^{i(\omega-i\eta)t}$. Therefore,

$$\chi_A(\omega) = \sum_{ab} \frac{\mathcal{O}_{ab}\mathcal{O}'_{ba}}{\omega - E_{ba} - i\eta} (e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}). \tag{360}$$

One important reason for choosing our conventions is that in frequency space, the retarded and advanced response functions are simply complex conjugates of one another:

$$\chi(\omega) = \chi_A(\omega)^*. \tag{361}$$

Finally we need to go through the same song and dance for the time-ordered response function. This time the convergence factors are dictated as

$$\chi_T(\omega) = -i \int dt \sum_{ab} e^{-\beta E_a} \left(\mathcal{O}_{ab}\mathcal{O}'_{ba}\theta(t)e^{iE_{ba}t+i(\omega+i\eta)t} + \zeta_{\mathcal{O}}\mathcal{O}_{ba}\mathcal{O}'_{ab}\theta(-t)e^{iE_{ba}t+i(\omega-i\eta)t} \right). \tag{362}$$

Doing the integrals, we see that since the signs for the $i\eta$s now track the signs of the E_{ba}s instead of the sign of ω, we get

$$\chi_T(\omega) = \sum_{ab} \mathcal{O}_{ab}\mathcal{O}'_{ba} \left(\frac{e^{-\beta E_a}}{\omega - E_{ba} + i\eta} - \zeta_{\mathcal{O}} \frac{e^{-\beta E_b}}{\omega - E_{ba} - i\eta} \right). \tag{363}$$

where we dropped the unimportant $i\eta$ in the numerator. The poles of this expression are now at

$$\omega = E_{ba} \pm i\eta. \tag{364}$$

The $\mp i\eta$ is what distinguishes the time-ordered correlator from the others: it has poles both above and below the $\mathbb{R}$ axis! Also, it's worth noting how different the two representations of the time-ordered correlator look: the spectral representation gives us something that is an infinite sum of terms with only ω^2s in them and with imaginary numbers only appearing through the $i\eta$ factor, while (355) is a single term linear in ω and containing imaginary numbers through the imaginary part of the self energy. The equivalence of these two expressions relies on the fact that a continuum of delta functions can merge to form a branch cut.

Finally, note that our sign conventions are such that all three response functions can be written as special cases of (after re-labeling some indices)

$$\chi(z) = \sum_{ab} \mathcal{O}_{ab}\mathcal{O}'_{ba} \frac{e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}}{z - E_{ba}}$$

where we choose the z contour to run slightly above / below the $\mathbb{R}$ axis (retarded / advanced) or along the $R_{-\eta}\mathbb{R}$ axis, where R_θ is a rotation in the $\mathbb{C}$ plane (time-ordered). From this, we see that χ_T and χ_A agree when $\omega < 0$, while χ_T and χ_R agree when $\omega > 0$.
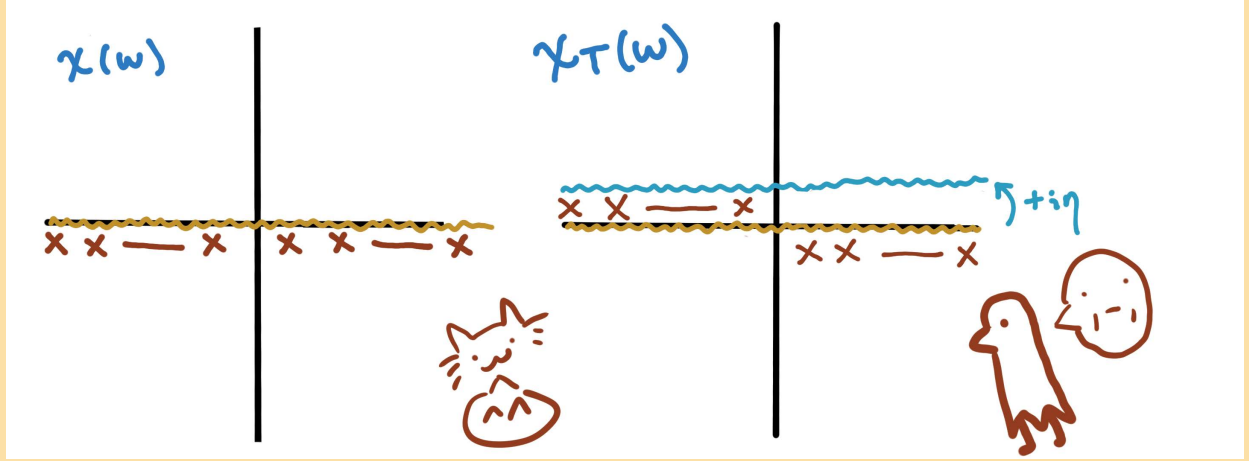
Figure 1: Left: the causal retarded Greens function, which is analytic in the UHP (brown crosses are poles). The orange squiggley line indicates the integration contour. Right: the time-ordered Greens function, with the orange squiggle again the integration contour. By pushing the contour up to the blue squiggle we get analyticity in the UHP, and so $\chi_T(\omega + i\eta) = \chi(\omega)$.

"Imaginary time"

Now we will discuss correlators defined for operators at different points along the thermal circle. The reason why I don't like referring to this as "imaginary time" is that for me, imaginary time is what we get when we Wick-rotate a $\mathbb{R}$-time path integral, viz. an imaginary time path integral is one where we're integrating over a $(d+1)$-dimensional manifold $\mathbb{R} \times \Sigma$, where the metric is locally $1 \oplus g_\Sigma$ (viz. the time direction is always $\mathbb{R}$; we are not Wick-rotating to something with a temporal S^1). The "imaginary time" coordinate of the thermal circle is just some fake variable that we introduced to write the thermal trace as a path integral, with operators inserted at different points on the S^1 not a priori yielding any physically relevant correlation functions (rather, they produce different states). That said, we know that $\beta \to \infty$ projects onto the ground state, and since $T = 0$ QFT is basically the statistical mechanics of the ground state, we can at least expect that taking $\beta \to \infty$ produces something like a Wick-rotated $T = 0$ path integral (after all, we do normally assume in $T = 0$ QFT that the system has the same ∂ conds at $t = \pm\infty$, so we can think about $T = 0$ path integrals as happening on a gigantic S^1 if we wish).

Anyway, let us formally define the operator $\mathcal{O}(\tau)$ as

$$\mathcal{O}(\tau) = e^{\tau H} \mathcal{O}(0) e^{-\tau H}, \tag{365}$$

which is the same as if $\mathcal{O}(\tau)$ were $\mathcal{O}(0)$ time-evolved by $\tau = it$. We then define the correlator

$$\chi_E(\tau, \tau') = -\langle T[\mathcal{O}(\tau)\mathcal{O}'(\tau')]\rangle, \qquad \tau, \tau' \in [0, \beta) \tag{366}$$

where T orders according to Euclidean time, and includes a sign for statistics, viz.

$$T[\mathcal{O}(\tau)\mathcal{O}'(\tau')] = \mathcal{O}(\tau)\mathcal{O}'(\tau')\Theta(\tau - \tau') + \zeta_\mathcal{O}\mathcal{O}'(\tau')\mathcal{O}(\tau)\Theta(\tau' - \tau). \tag{367}$$

Note that we can assume that $\zeta_{\mathcal{O}} = \zeta_{\mathcal{O}'}$ wolog, since otherwise the correlator vaishes. Note that the τ arguments in the correlator are *defined* to lie within the thermal circle interval. That is, we can consider $\mathcal{O}(\tau)$ for any τ, but the correlator χ_E is only defined on a restricted range of its arguments—we are always inserting the operators along the thermal circle while doing the trace. Again, a priori this isn't a super physical thing to consider—e.g. for $\tau' = 0$, this "correlator" represents the thermal expectation value of $\mathcal{O}'$ in the state given by the density matrix $\rho = e^{-(\beta-\tau)H}\mathcal{O}e^{\tau H}$, which a priori isn't a super interesting density matrix to consider. But let us press ahead all the same.

Now since H is by definition constant along the S^1 (again, since τ is just a fake coordinate that we added in by hand), we have $\chi_E(\tau, \tau') = \chi_E(\tau - \tau')$, and so χ_E is a function defined on $[-\beta, \beta)$. We can then write it as

$$\chi_E(\tau) = -\langle T[\mathcal{O}(\tau)\mathcal{O}'(0)]\rangle, \qquad \tau \in [-\beta, \beta). \tag{368}$$

If $\tau > 0$, we have (writing e.g. $\mathcal{O}(0) = \mathcal{O}$)

$$\begin{aligned}
\chi_E(\tau) &= -\langle \mathcal{O}(\tau)\mathcal{O}'\rangle = -\mathrm{Tr}[e^{-(\beta-\tau)H}\mathcal{O}e^{-\tau H}\mathcal{O}'] \\
&= -\mathrm{Tr}[e^{-\beta H}\mathcal{O}'e^{(\tau-\beta)H}\mathcal{O}e^{-(\tau-\beta)H}] \\
&= -\langle\mathcal{O}'\mathcal{O}(\tau-\beta)\rangle = -\zeta_{\mathcal{O}}\langle T[\mathcal{O}(\tau-\beta)\mathcal{O}']\rangle \\
&= \zeta_{\mathcal{O}}\chi_E(\tau-\beta).
\end{aligned} \tag{369}$$

Likewise, $\chi_E(\tau + \beta) = \zeta_{\mathcal{O}}\chi_E(\tau)$ if $\tau < 0$. This property is basically the reason why we are focusing on the T-ordered correlator—without the T ordering, the shift by $\pm\beta$ would switch the ordering of the operators, and wouldn't give us this nice (anti)periodicity property. Also note that in the second line we used the cyclicity of the trace, which holds even when the operators in question are fermionic.[49] This means that when we FT, instead of getting $\chi_E(\omega)$ for any $\omega \in 2\pi/(2\beta)\mathbb{Z}$, we instead get $\omega \in (\pi\beta^{-1})2\mathbb{Z}$ if $\zeta_{\mathcal{O}} = 1$, and $\omega \in (\pi\beta^{-1})(2\mathbb{Z} + 1)$ if $\zeta_{\mathcal{O}} = -1$. When we FT we then choose to split the $1/(2\beta)$ normalization factor (to ensure that squaring the FT gets us back with the right normalization) asymmetrically by putting a $1/2$ on the $d\tau$ integration and a β^{-1} on the ω sum, so that

$$\chi_E(\omega) = \frac{1}{2}\int_{[-\beta,\beta)} d\tau\, e^{i\omega\tau}\chi_E(\tau) = \int_{[0,\beta)} d\tau\, e^{i\omega\tau}\chi_E(\tau), \tag{370}$$

where we used the restriction on ω and the transformation property of χ_E under a shift by β.

To see why these definitions will be useful, let's look at the spectral representation of

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[49]Consider breaking up the trace into each individual $\otimes$ factor of the Fock space. Then wolog we can consider the traces of the operators $c^\dagger c$ and $cc^\dagger$, since these are the only types of nonzero terms with that will appear in the traces calculated above for fermionic $\mathcal{O}, \mathcal{O}'$. But the traces are the same: $\mathrm{Tr}[c^\dagger c] = \mathrm{Tr}[cc^\dagger] = 1$. Therefore we can move the cs and $c^\dagger$s around cyclicly in the trace for every $\otimes$ factor in the Fock space, and hence the trace is actually cyclic, even for fermionic operators.

$\chi_E(\omega)$:[50]

$$\chi_E(\omega) = -\sum_{ab} \int_0^\beta d\tau \, \mathcal{O}_{ab}\mathcal{O}'_{ba} e^{-\beta E_a} e^{\tau(E_{ab}+i\omega)}$$

$$= -\sum_{ab} \mathcal{O}_{ab}\mathcal{O}'_{ba} \frac{e^{-\beta E_a}}{i\omega - E_{ba}} (e^{\beta E_{ab}+i\omega\beta} - 1). \tag{371}$$

Here comes the slick part: due to the constraints on the allowed values of ω, we can drop the $i\omega\beta$ in the exponent at the cost of a $\zeta_\mathcal{O}$ sign, and write

$$\chi_E(\omega) = \sum_{ab} \mathcal{O}_{ab}\mathcal{O}'_{ba} \frac{e^{-\beta E_a} - \zeta_\mathcal{O} e^{-\beta E_b}}{i\omega - E_{ba}}. \tag{372}$$

Anyway, the point of doing this is that we see that $\chi_E(\omega)$ actually has the same form as $\chi(\omega)$, provided that we rotate the frequency in the complex plane: (this is the reason for defining χ_E with the minus sign in front of the expectation value)

$$\chi_E(-i\omega + \eta) = \chi(\omega). \tag{373}$$

This means that we can compute the Fourier transform of the finite-temperature *real-time* response function (or time-ordered correlator, by suitably shifting the frequency with a small imaginary part) by analytically continuing the correlator $\chi_E(\omega)$ that we computed by Fourier-transforming a correlator defined on the thermal circle. Note however that the analytic continuation that we had to do is slightly subtle, because of the fact that it can only be done in a simple fashion (by sending $\omega \mapsto -i\omega$) at the *end* of the calculation. If we do $\omega \mapsto -i\omega$ at the beginning of the calculation, we get total nonsense (and indeed, remember that we had to assume that ω was a Matsubara frequency midway through the calculation!). Presumably there is some prescription for doing the continuation in general, but the naive way of doing it evidently only works when the only way in which ω appears in the correlator is in the denominator of a simple pole.[51] The reason why this is useful is that when we compute $\text{Tr}[e^{-\beta H}\mathcal{O}\mathcal{O}']$, we don't need to ever do an explicit sum over states weighted by $e^{-\beta H}$ (which we'd have to do in the $\mathbb{R}$-time approach), since we can turn the trace into a path integral in the usual way. The motto here is "use the spectral representation of the trace to prove stuff, and the path integral representation to calculate stuff". Anyway, the somewhat amazing thing is that the thermal information about the sum over states weighted by $e^{-\beta H}$ is built into the geometry of the path integral we're doing, and yet we can *still* get dynamical $\mathbb{R}$-time information at the end of the day by analytically continuing.

I still find it very surprising that this works. I've never seen this discussed well in books, where one often gets the impression that the crucial fact is noticing that $e^{-\beta H}$ is e^{-itH} for

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[50]Note that we are just going straight to $\chi_E(\omega)$ without first looking at $\chi_E(\tau)$. This is because the latter is more annoying due to the "time" ordering. The properties of the allowed frequencies for $\chi_E(\omega)$ mean that when we write $\chi_E(\tau)$ inside an integral, we can reduce the integral to one over only $\tau \in [0, \beta)$. However we should still remember that by itself, $\chi_E(\tau)$ is still defined over the interval $[-\beta, \beta)$, and not just $[0, \beta)$.

[51]Also note that we are assuming that a unique analytic continuation is picked out on all of $\mathbb{C}$, despite only knowing the value of $\chi_E(\omega_n)$ on a copy of $\mathbb{Z}$. While in general one needs an uncountable number of values of χ_E to determine a unique continuation, there turns out to be a theorem guaranteeing a unique continuation as long as $\chi_E(\omega_n)$ grows at most exponentially with n, which is true for us.

$t = -i\tau$. If we were really just analytically continuing a time coordinate from $\mathbb{R}$ time to the thermal circle when computing the correlators, then we would be limited to computing equal-time correlators, since the time coordinate would be rotated away. Furthermore, we are actually never analytically continuing time, only frequency (we can wick rotate a function defined on one axis of the $\mathbb{C}$ plane [well for us, just a $\mathbb{Z}$ subset of one axis, which we then extend to the whole axis] to a function defined on another axis, but we can't wick rotate a $\mathbb{R}$ time correlator defined on $\mathbb{R}$ to one defined on S^1). Also we should stress that we really are computing the *real-time* response function (which contains information about dynamics), despite coming from a starting point, viz. $\text{Tr}[e^{-\beta H}\mathcal{O}\mathcal{O}']$, which looks like it contains no information about dynamics, only thermodynamics (really should be called "thermostatics").

A brief clarification on this: thermodyanmics is done in the assumption that nothing is changing in time, so what exactly do we mean by "real time correlators in a system at temperature T"? If the concept of T is to be defined, shouldn't we be in a situation where nothing is changing with time?[52] The point here is that for us, T is just a label that goes into determining the state in which we want to compute the expectation value. We create a thermal state with T, but then evolve it forward in $\mathbb{R}$ time, and study $\mathbb{R}$ time fluctuations on top of it (well, small $\mathbb{R}$-time fluctuations; we're working under the purview of linear response theory)—we're just doing regular QFT, except that instead of the density matrix $|0\rangle\langle 0|$, we're using $e^{-\beta H}$. In particular, all of our time evolution is unitary, and there is no reservoir that our system is coupled to. While this is slightly against the spirit of thermodynamics, wherein the whole point is that we need to couple to a bath to define thermodynamic quantities, our assumption is that the system will "self-thermalize", so that it can act as its own bath. This assumption means that we can get away with doing unitary time evolution in a closed system, and still expect to see the same kinds of effects that we would see in an open system coupled to a bath and evolving in a non-unitary way, provided that we choose the appropriate state with which to take the trace in the path integral.

Fluctuation-dissipation

Now we are in a position to see one precise statement of the fluctuation-dissipation theorem (every book seems to have their own statement for the fluctuation-dissipation theorem; what follows is one possible definition). The analytic structure of the time-ordered and retarded correlators is shown in figure 1. From the picture, we see that if we shift the argument of the time-ordered correlator up slightly in the $\mathbb{C}$ plane, we get the retarded correlator. Hence we have

$$\chi(\omega) = \chi_T(\omega + i\eta). \tag{374}$$

Likewise,

$$\chi_A(\omega) = \chi_T(\omega - i\eta). \tag{375}$$

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[52]Also, the definition of $\chi(t)$ comes from adding an explicitly time-dependent background field term to the action—how can something like this, where we have a term that gradually turns on and off over an infinite time, be captured by a trace $\text{Tr}[e^{-\beta H}\mathcal{O}\mathcal{O}']$ that is completely static? The points is that $\chi(t)$ is actually computed within linear response as a commutator evaluated in a state in which the Hamiltonian is *time independent*, so that as long as we're doing linear response, this issue doesn't come up.

The LHS is dissipation (it is computed from measuring the response to an external driving force, and tells us how transport occurs in the system—for example, if $\mathcal{O} = \mathbf{j}$ is an electric current then χ is the conductivity, telling us how energy of charge carriers dissipate); the RHS is fluctuation (it is computed by computing two-point correlation functions in the path integral). This tells us that fluctuations carry the same in formation as the dissipative response, but in order to see this one must nudge the frequency of the fluctuations slightly upwards in the $\mathbb{C}$ plane. One could argue that a more apt statement of the theorem would involve only the imaginary part of $\chi(\omega)$ since it is really the imaginary part that controls dissipation[53]. Because of this, one often instead formulates fluctuation-dissipation by relating the non-time-ordered correlator to the imaginary part of $\chi(\omega)$. More on this in a sec.

We can use these results to prove that the real part of Σ is continuous across the real axis, while the imaginary part flips sign:

$$\Sigma_R(\omega + i\eta) = \Sigma_R(\omega - i\eta), \qquad \Sigma_I(\omega + i\eta) = -\Sigma_I(\omega - i\eta). \tag{376}$$

This is proved using the spectral representation for the correlator. From the spectral representations for the advanced and retarded correlators, we see that they are complex conjugates of one another:

$$\chi(\omega)^* = \chi_A(\omega). \tag{377}$$

Furthermore, the complex conjugation only acts on the $i\eta$ factor—there are no other complex numbers appearing in the spectral representation. Therefore we can also write the above as

$$\chi(\omega)^* = \chi_T(\omega + i\eta)^* = \chi_T(\omega - i\eta) = \chi_A(\omega). \tag{378}$$

This reads

$$\left(\frac{1}{\omega - \epsilon_\mathbf{k} - \Sigma(\omega + i\eta, \mathbf{k}) + i\eta} \right)^* = \frac{1}{\omega - \epsilon_\mathbf{k} - \Sigma(\omega - i\eta, \mathbf{k}) - i\eta}, \tag{379}$$

telling us that

$$\Sigma(\omega + i\eta)^* = \Sigma(\omega - i\eta), \tag{380}$$

which then proves the claim.

Now we will look at the non-time-ordered correlator (just focusing on a single Hermitian operator $\mathcal{O}$ for simplicity)

$$S(t) = \langle \mathcal{O}(t)\mathcal{O}(0) \rangle. \tag{381}$$

Note that this is *not* what is computed by a path integral, i.e. the $\langle\rangle$ above does not correspond to a path integral insertion. We aren't using the letter χ here since $S(t)$ has no nice properties vis-a-vis causality.

The difference between $S(\omega)$ and $\chi(z)$ is that when Fourier transformed we get a delta function of frequency, rather than a pole (constants FT to δ functions; θ functions FT to poles). Explicitly, doing the usual steps,

$$S(\omega) = 2\pi \sum_{ab} e^{-\beta E_a} \delta(\omega - E_{ba}) |\mathcal{O}_{ab}|^2, \tag{382}$$

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[53]In the case where χ is a genuine analytic response function. The conductivity is different, since the $\mathbb{R}$ part of σ is what controls dissipation. More on this later.

where the 2π comes from the integral producing the δ function. Now we calculate $\chi_{\mathbb{I}}(\omega)$:

$$
\begin{aligned}
\chi_{\mathbb{I}}(\omega) &= -\pi \sum_{ab} |\mathcal{O}_{ab}|^2 (e^{-\beta E_a} - \zeta_{\mathcal{O}} e^{-\beta E_b}) \delta(\omega - E_{ba}) \\
&= \pi \sum_{ab} |\mathcal{O}_{ab}|^2 (e^{-\beta E_a} - \zeta_{\mathcal{O}}) \delta(\omega - E_{ba})
\end{aligned}
\tag{383}
$$

where we used the constraint coming from the δ function to re-write $E_{ab} = -\omega$ and pull an $e^{-\beta E_{ab}}$ out of the sum. Comparing these two expressions, we see that

$$
\chi_{\mathbb{I}}(\omega) = \frac{1}{2}(e^{\beta\omega} - \zeta_{\mathcal{O}}) S(\omega) = \frac{1}{2n_{\mathcal{O}}(\omega)} S(\omega),
\tag{384}
$$

where $n_{\mathcal{O}}$ is the FD or BE distribution, as appropriate. If we had kept track of the $\hbar$s we would have one in the denominator and one in the exponent, giving a prefactor of $\hbar^{-1}(e^{\hbar\omega\beta} - 1)$, which in the classical limit (for bosonic statistcs) reduces to the ω/T response that we expect.

Spectral density

Now let's talk about the spectral density. We will *not* define it as the imaginary part of the time-ordered correlator: the reason for doing this is that as we have seen, $\Sigma_I(\omega)$ has a discontinuity on the $\mathbb{R}$ axis, and so it is better to define the spectral density with respect to a correlator whose integration contour stays consistently to one side of the discontinuity. So with that in mind, we define

$$
A(\omega) = -\frac{1}{\pi} \text{Im}[\chi(\omega)].
\tag{385}
$$

From the spectral representation, we conclude that

$$
A(\omega) = \sum_{ab} |\langle a|\mathcal{O}(0)|b\rangle|^2 e^{-\beta E_a} (\delta(\omega - E_{ba}) - \delta(\omega + E_{ba})),
\tag{386}
$$

which shows why the name "spectral density" is apt.

On the other hand, using the above statement of fluctuation-dissipation, we can get an expression for the spectral density in terms of the self-energy: we write

$$
A(\omega) = -\frac{1}{\pi} \text{Im}[\chi_T(\omega + i\eta)].
\tag{387}
$$

If $\Sigma_I(\omega) = 0$, then we just get a delta function from the $i\eta$ in the denominator. On the other hand if $\Sigma_I(\omega) \neq 0$, we get a Lorentizian distribution with width determined by Σ_I:

$$
A(\omega) = \begin{cases} \delta(\omega - \epsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k})) & \text{if} \quad \Sigma_I(\omega + i\eta, \mathbf{k}) = 0 \\ -\pi^{-1} \dfrac{\Sigma_I(\omega + i\eta, \mathbf{k})}{(\omega - \epsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k}))^2 + \Sigma_I(\omega + i\eta, \mathbf{k})^2} & \text{if} \quad \Sigma_I(\omega + i\eta, \mathbf{k}) \neq 0 \end{cases}
\tag{388}
$$

We will see shortly that the $\Sigma_I = 0$ regime is below the threshold for particle production (and so the spectral weight is concentrated in δ function peaks at the locations of the long-lived particles), while the $\Sigma_I \neq 0$ regime is above the threshold (and so the spectral weight is spread out over a broad continuum, made possible by interactions).

Sum rules and KK

This stuff is again standard, but a full 50% of the cond-mat books I own had typos in the signs and stuff, so we'll try to get everything straight.

The KK relations follow from the analyticity of the causal response function $\chi(z)$ for $z \in UHP$. One can then write, for any $\omega \in \mathbb{R}$,

$$\chi(\omega) = \frac{1}{2\pi i} \int_{\mathbb{R}} d\omega' \frac{\chi(\omega')}{\omega - (\omega' + i\eta)}, \tag{389}$$

since the contour can be closed in the UHP and by causality picks up a contribution solely from the pole in the denominator (a similar expression can be written down for the acausal response).

Now we can use the Dirac identity on the denominator in the integral. The imaginary part is $+i\pi\delta(\omega - \omega')$, which then gives a $\chi(\omega)/2$ on the RHS. Subtracting this off, we then have

$$\chi(\omega) = \frac{1}{\pi i} P \int_{\mathbb{R}} d\omega' \frac{\chi(\omega')}{\omega - \omega'}. \tag{390}$$

Splitting this up into real and imaginary parts gives the KK relations

$$\chi_{\mathbb{R}}(\omega) = \frac{1}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\chi_{\mathbb{I}}(\omega')}{\omega - \omega'}, \qquad \chi_{\mathbb{I}}(\omega) = -\frac{1}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\chi_{\mathbb{R}}(\omega')}{\omega - \omega'}. \tag{391}$$

One can also use the reality of $\chi(t)$, which implies $\chi_{\mathbb{R}}(-\omega) = \chi_{\mathbb{R}}(\omega)$ and $\chi_{\mathbb{I}}(-\omega) = -\chi_{\mathbb{I}}(\omega)$, to write

$$\chi_{\mathbb{R}}(\omega) = \frac{1}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\omega' \chi_{\mathbb{I}}(\omega')}{\omega^2 - \omega'^2}, \qquad \chi_{\mathbb{I}}(\omega) = -\frac{\omega}{\pi} P \int_{\mathbb{R}} d\omega' \frac{\chi_{\mathbb{R}}(\omega')}{\omega^2 - \omega'^2}. \tag{392}$$

Another thing often brought up in these discussions are sum rules. The most general way of writing down a sum rule is to integrate a causal response function (or a suitably shifted time-ordered response function) along the whole frequency axis, with a convergence factor that requires the integral to be closed in the LHP:

$$\int_{\mathbb{R}} d\omega \, \chi(\omega) e^{-i\omega\eta} = \int_{\mathbb{R}} d\omega \, \chi_T(\omega + i\eta) e^{-i\omega\eta} = C, \tag{393}$$

where C is some constant that doesn't change as we vary the microscopic parameters of the system. The fact that C is a constant just comes from analyticity. Since all of the poles of $\chi(\omega)$ must lie in the LHP by causality, changing microscopic parameters in the theory will never result in a pole crossing from the LHP into the UHP. The only thing that can happen to the poles as we smoothly tune parameters is that they move around within the LHP, that opposite-residue poles collide and annihilate, or that opposite-residue poles are created in pairs. All of these processes don't change the above integral, which therefore is a constant.[54]

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[54] And this constant is usually not zero! One might think it needed to be zero since the sum of the residues of all the poles for any function on $\mathbb{C}$ vanishes, and after all, all of the poles of $\chi(\omega)$ are contained within the contour enclosing the LHP. However, the function $\chi(\omega)e^{-i\omega\eta}$ is singular when $\eta \to i\infty$, and so there is indeed singular stuff going on in the UHP (but only when we multiply by $e^{-i\omega\eta}$!) that renders the integral non-zero. Note that we indeed need the $e^{-i\omega\eta}$ to use the contour integral trick, since $\chi(\omega)$ generically won't go to something that satisfies $|\omega\chi(\omega)| \to 0$ as $|\omega| \to \infty$. If $\chi(\omega)$ did fall off this fast, then we could compute the integral without the $e^{-i\omega\eta}$ factor and get zero, implying that C would be zero.

Now earlier we showed that $\chi_T(\omega + i\eta) = \chi_T(\omega - i\eta)^*$. We also know that all the poles of $\chi_T(\omega + i\eta)$ lie slightly below the $\mathbb{R}$ axis. The latter fact means that when we do the above integral by a semicircular contour in the LHP, we can deform the circular part of the contour to run just below the $\mathbb{R}$ axis. Using the former fact, this gives

$$\int_{\mathbb{R}} d\omega\, \chi(\omega) e^{-i\omega\eta} = \int_{\mathbb{R}} d\omega [\chi_T(\omega + i\eta) - \chi_T(\omega - i\eta)] = 2i \int_{\mathbb{R}} d\omega\, \chi_{\mathbb{I}}(\omega) = 0, \tag{394}$$

because the reality of $\chi(t)$ means that $\chi_{\mathbb{I}}(\omega) = -\chi_{\mathbb{I}}(-\omega)$.

So wait, wtf are we doing, if we just get zero? The actual application of the sum rule is actually not to legit analytic response functions coming from time-ordered correlation functions and stuff, but to things like the conductivity, which are basically "i times a legit response function". Indeed, the conductivity satisfies

$$\sigma_T(\omega + i\eta) = -\sigma_T(\omega - i\eta)^*, \tag{395}$$

essentially because the actual microscopic response function in the theory relates the current and vector potential, and to get to the conductivity one needs to differentiate the vector potential wrt time, which in frequency space brings down an extra factor of i (this is why it is the *real* part of the conductivity that is related to dissipation!). If we apply the sum rule to a function like the conductivity which satisfies (395), then we instead have

$$\int_{\mathbb{R}} d\omega\, \sigma(\omega) e^{-i\omega\eta} = \int_{\mathbb{R}} d\omega [\sigma_T(\omega + i\eta) - \sigma_T(\omega - i\eta)] = 2 \int_{\mathbb{R}} d\omega\, \sigma_{\mathbb{R}}(\omega) = C, \tag{396}$$

where C will generically be nonzero, since $\sigma_{\mathbb{R}}$ is symmetric in ω (again, the notation is that $\sigma(\omega)$ is a causal response function, and σ_T is time-ordered). This is the usual sum rule.

Self-energies

Now we discuss the meaning of the different parts of the self-energy. As we can see from the above form of the spectral function (388), Σ_R has the affect of shifting the dispersion for the poles away from the free value of $\epsilon_{\mathbf{k}}$ (below the threshold the locations of the poles are shifted; above the threshold the center of the Lorentizan is shifted, at least to the extent that Σ_I can be treated as frequency-independent). Therefore we are prompted to define the renormalized energy as the solution to the equation

$$\epsilon_{\mathbf{k}}^* = \epsilon_{\mathbf{k}} + \Sigma_R(\epsilon_{\mathbf{k}}^*, \mathbf{k}). \tag{397}$$

This shift is what will affect the renormalization of things like masses and Fermi velocities.

On the other hand, Σ_I loosely speaking determines the decay rate of the qp in question. One way to argue this is to consider a scenario in which the free theory has a single qp pole, which which be broadened and will have its location shifted slightly by interactions. If we assume the interactions leave the renormalized qp pole still roughly intact and distinguishable from the background incoherent continuum, and if we assume that Σ_I is roughly constant over the frequency range we're interested in, we may expand

$$\Sigma(\omega, \mathbf{k}) \approx i\Sigma_I(\epsilon_{\mathbf{k}}^*, \mathbf{k}) + \Sigma_R(\epsilon_{\mathbf{k}}^*, \mathbf{k}) + (\omega - \epsilon_{\mathbf{k}}^*)(\partial_\omega \Sigma_R)(\epsilon_{\mathbf{k}}^*, \mathbf{k}). \tag{398}$$

Inserting this expansion into the expression for χ in terms of the shifted time-ordered correlator, we obtain

$$\chi(\omega, \mathbf{k}) = \frac{Z_\mathbf{k}}{\omega - \epsilon_\mathbf{k}^* - i\Gamma_\mathbf{k}} + \dots, \tag{399}$$

where

$$\Gamma_\mathbf{k} \equiv \Sigma_I(\epsilon_\mathbf{k}^* + i\eta, \mathbf{k})Z_\mathbf{k}, \qquad Z_\mathbf{k} \equiv \frac{1}{1 - (\partial_\omega \Sigma_R)(\epsilon_\mathbf{k}^* + i\eta, \mathbf{k})}, \tag{400}$$

and where the ... represent contributions from the incoherent background, which are of course needed to give a properly normalized spectral density.[55] When this is Fourier transformed, we get an exponential damping on a timescale set by $\Gamma_\mathbf{k}$: thus $\Gamma_\mathbf{k}$ is a decay rate, and so Σ_I indeed controls the decay rate of the particle in question. In particular, it will vanish below the threshold for particle production, since then there is nothing for the particle to decay into.

One brief comment on Fermi liquids: since $\Sigma_I(\omega, \mathbf{k})$ is determined by the type of interactions in the theory, one might naively expect that it would be roughly constant and nonzero near the FS (since in many metals the interaction strength is comparable or [much] greater than the Fermi energy). Now in order for a qp of energy $\epsilon_\mathbf{k}$ to be well-defined, we need

$$1/\epsilon_\mathbf{k} \ll \frac{1}{Z_\mathbf{k}\Sigma_I(\omega = \epsilon_\mathbf{k}, \mathbf{k})} \tag{401}$$

since we need to be able to measure the energy of the qp before it decays. Now if $\Sigma_I(\omega, \mathbf{k})$ is roughly frequency-independent near the FS, then when we take $\epsilon_\mathbf{k} \to 0$ at the FS, the condition on the well-definedness of the qp will always be violated. This line of thinking would lead us to expect that well-defined qps only exist for large energies, when in fact it's the opposite! The fact that this naive picture is so wrong is the power of FLT and the Pauli exclusion principle.[56]

A more QFT-centric way to discuss this (at zero temperature) is via the optical theorem, and is a simple consequence of the unitarity of the S-matrix.[57] Writing $S = 1 + iT$ where T represents nontrivial scattering, $S^\dagger S = 1$ means $i(T^\dagger - T) = T^\dagger T$. In the case of the self energy, we are interested in one-particle to one-particle scattering, and so sandwiching this equation with $|\mathbf{k}, \omega\rangle$, we can insert a resolution of $\mathbf{1}$ in the $T^\dagger T$ term to get, using rather

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[55]We are of course assuming here that $Z_\mathbf{k} \neq 0$, so that we are in the domain of FLT. We could run into problems if e.g. Σ_R failed to be analytic, with e.g. $\Sigma_R(\omega) \propto \ln(\omega/\omega_0)$ as in a marginal FL.

[56]It also means that $\Sigma_I(\omega, \mathbf{k})$ can *not* be roughly independent of ω with $\mathbf{k}$ near the FS and ω small (in fact, we know that it goes like ω^2), and so the assumption of constant Σ_I used to talk about $\Gamma_\mathbf{k}$ as a decay rate after Fourier transforming the correlator really isn't all that reasonable.

[57]One caveat before we continue is that since we are in the CMT diary, we are using CMT conventions, and writing the propagator in terms of frequency and Hamiltonian density and so on. In QFT, we would usually write the propagator as e.g. $1/(p^2 - \Sigma(p^2))$. The annoying thing here is that in QFT Σ has the dimensions of p_0^2, while in most of CMT it has the dimensions of ω. Therefore the discussion that follows should not be taken literally—it is just meant to be suggestive. Were we to do things in the QFT way we'd need to add some extra factors of masses to get dimensions right and remember how the single-particle states are normalized with factors of $1/\sqrt{E}$, and so on. It didn't seem to be worth the trouble to sort all of this out, so please do not take the following formulae too seriously.

schematic but hopefully clear notation,

$$\Sigma(\mathbf{k}, \omega) - \Sigma^*(\mathbf{k}, \omega) = i2\Sigma_I(\mathbf{k}, \omega) \approx -i \sum_a |T_{(\mathbf{k},\omega) \to |a\rangle}|^2 \tag{402}$$

where a runs over all possible final state particles that the incoming particle can decay into while conserving momentum and energy, except the incoming state itself (the trivial free propagation is not part of T; it is the 1 in $1 + iT$). We have written $\approx$ here because the self-energy is determined only by the 1PI diagrams, whereas the T matrix elements involve all connected diagrams, 1PI or not—in a weakly coupled theory the dominant terms in an expansion in terms of loops or coupling constants all come from 1PI diagrams, but eh, this approximation is still taken a bit crudely.[58] Anyway, pressing ahead, we see that since squaring $T_{a \to b}$ gives the amplitude for the given processes $a \to b$, we see that indeed, $-2\Sigma_I$ determines the decay rate of the incoming particle. We also check that if the frequency ω is below the threshold for producing particles / bound states, there are no nonzero terms in the sum, and so $\Sigma_I(\mathbf{k}, \omega) = 0$ below the threshold frequency (the final states $|a\rangle$ appearing in (402) cannot be single-particle states below the threshold, since we need to conserve momentum and energy in (402)).

At the level of perturbation theory, (402) can be understood by using the cutting rules: everything in the diagrammatics of Feynman rules is real except for the $i\eta$ convergence factors, and they only produce finite imaginary parts when their parent propagators go on shell. In fact, the optical theorem tells us that a given diagram will only produce an imaginary part when *all* particles along a given cut go on-shell—in this scenario these particles define a legit final state for a scattering process, and it is these diagrams which produce the RHS of (402) above.

Effective masses and the quasiparticle residue in FLT

Finally, we will take a look at the effective mass, and its relation to the quasiparticle residue, within the context of FLT.

There are many different ways to define the effective mass, depending on what situations one is interested in. For example, in semiconductors, where one is interested in the behavior of the band dispersion right near a band minimum / maximum, one characterizes the shape of the band with the effective mass $m_{sc} = (\partial^2 \epsilon / \partial^2 \mathbf{k})^{-1}$. In our case, since we are mostly interested in FLT (i.e. in metals), we will use a different definition, viz.

$$m \equiv k_F / v_F, \qquad v_F \equiv \frac{\partial \epsilon}{\partial \mathbf{k}}. \tag{403}$$

Although we have written v_F in notation that suggests it is a scalar, in reality it carries dependence on momentum, since it can vary over different regions of the FS.

The notion of an effective mass defined in this way will apply in scenarios where the spectral function can be well-approximated by a sharp qp peak standing out from a broader

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[58] Another point to clarify: the self-energy appears here and not the full propagator (or the 1PI part thereof), since the T matrix elements are computed using *amputated* diagrams, with the external propagators removed (also, since this formula doesn't include the trivial non-interacting process, it can't be related to the full propagator).

incoherent background. We will fix a momentum $\mathbf{k}$ at which a qp state lives, and will take the qp pole to be located at the renormalized frequency $\epsilon_\mathbf{k}^*$, where $\epsilon_\mathbf{k}^*$ is ϵ_k renormalized by the real part of the self-energy: as before, this is

$$\epsilon_\mathbf{k}^* = \epsilon_\mathbf{k} + \Sigma_R(\epsilon_\mathbf{k}^*, \mathbf{k}). \tag{404}$$

Therefore the Fermi velocity is

$$v_F = \partial_\mathbf{k}(\epsilon_\mathbf{k}^* - \Sigma_R(\epsilon_\mathbf{k}^*, \mathbf{k}) = v_F^* - v_F^*(\partial_\omega \Sigma_R)(\epsilon_\mathbf{k}^*, \mathbf{k}) - \partial_\mathbf{k}\Sigma_R(\epsilon_\mathbf{k}^*, \mathbf{k}) = Z_\mathbf{k}^{-1}v_F^* - \partial_\mathbf{k}\Sigma_R(\epsilon_\mathbf{k}^*, \mathbf{k}). \tag{405}$$

This means that

$$v_F^*/v_F = Z_\mathbf{k}(1 + v_F^{-1}\partial_\mathbf{k}\Sigma_R(\epsilon_\mathbf{k}^*, \mathbf{k})) = \frac{1 + v_F^{-1}\partial_\mathbf{k}\Sigma_R(\epsilon_\mathbf{k}^*, \mathbf{k})}{1 - (\partial_\omega \Sigma_R)(\epsilon_\mathbf{k}^*, \mathbf{k})}. \tag{406}$$

In typical situations where the Fermi momentum is not renormalized[59] the effective masses are consequently related as

$$m^*/m = \frac{1 - (\partial_\omega \Sigma_R)(\epsilon_\mathbf{k}^*, \mathbf{k})}{1 + v_F^{-1}\partial_\mathbf{k}\Sigma_R(\epsilon_\mathbf{k}^*, \mathbf{k})}. \tag{407}$$

In situations where the momentum dependence of Σ_R is weak near $\epsilon_\mathbf{k}^*$, we may drop the $\mathbf{k}$ derivative of the self-energy, and in these cases the qp residue determines the renormalized Fermi velocity and mass (recall that v_F and m are really generically $\mathbf{k}$-dependent; for notation's sake this dependence is remaining implicit)

$$v_F^*/v_F = m/m^* \approx Z_\mathbf{k}. \tag{408}$$

When are we justified in dropping the $\partial_\mathbf{k}\Sigma_R$ term? ...

Unfortunately this assumption is not valid even in the simplest example of a self-energy, viz. the Hartree-Fock approximation. In this approximation the interaction is treated as completely static, and hence the self-energy will have no ω dependence, since the electrons can never change their frequency as they travel through the 1PI diagrams in Σ.

What is the physical meaning of this[60] definition of m? For simplicity, consider one dimension. In the "filling up momentum levels" Fermi-gas picture of the FL, states near the Fermi level come separated in fixed momentum intervals of $\Delta k = 2\pi/L$. If we linearize $\epsilon_\mathbf{k}$ near k_F, then an energy window of width $\Delta\epsilon$ contains

$$N_{\Delta\epsilon} = v_F^{-1}\frac{\Delta\epsilon}{2\pi/L} = \frac{m\Delta\epsilon}{2\pi/L} \tag{411}$$

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[59] By Luttinger's theorem if the shape of the FS is unrenormalized then $\mathbf{k}_F$ is unchanged—nothing I'm aware of prevents the shape of the FS from renormalizing a priori, though.

[60] If we use the semiconductor definition, then we instead get

$$\partial_\mathbf{k}^2\epsilon_\mathbf{k} = \partial_\mathbf{k}^2\epsilon_\mathbf{k}^* - \partial_\mathbf{k}^2\Sigma_R - 2\partial_\mathbf{k}\epsilon_\mathbf{k}^*\partial_\omega\partial_\mathbf{k}\Sigma_R - \partial_\mathbf{k}^2\epsilon_\mathbf{k}^*\partial_\omega\Sigma_R - (\partial_\mathbf{k}\epsilon_\epsilon^*)^2\partial_\omega^2\Sigma_R. \tag{409}$$

If we drop the momentum derivatives of Σ_R and the second-order frequency derivative, this becomes

$$m^{-1} = (m^*)^{-1}/Z_\mathbf{k}, \tag{410}$$

which in this approximation agrees with the definition used in the main text.

states. Therefore the bigger the effective mass m, the more states that are contained in a fixed energy window. This is the meaning of this definition of m: larger m has the effect of compressing the energy levels in $\mathbf{k}$ space. Note that since $Z_{\mathbf{k}}$ is always between 0 and 1 in the approximation we're working in (since it determines the weight of the coherent pole in the spectral weight), the effect of interactions always compresses the energy levels, rather making them more dispersed. This means that this definition of m is the one relevant for thinking about the specific heat: a greater number of states in a fixed energy window means a larger specific heat, and since $N_{\Delta\epsilon}$ depends linearly on m, we expect $C \propto m$, with interactions having the effect of increasing the specific heat.

One way in which a Fermi liquid can die is if $Z_{\mathbf{k}} \to 0$ as in the LL, with qps disappearing into a pole-free spectral function. If the approximations stated above about the momentum dependence of Σ_R continue to hold, then a Fermi liquid which dies in this way implies that the Fermi sea "spills over" at the transition with $v_F^* \to 0$ and $m^* \to \infty$: a given energy window in $\mathbf{k}$ space now contains a diverging number of states.

Now we will comment on the relation between $Z_{\mathbf{k}}$ and the factor appearing when doing wavefunction renormalization in the QFT sense. First, consider an approximation in which we neglect the finite width of the qp peak in the spectral function. The retarded Greens function in this approximation is

$$\chi(\omega, \mathbf{k}) \approx \frac{1}{\omega - \epsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k}) + i\eta}, \tag{412}$$

where we have dropped the $+i\eta$ in the argument of Σ_R since it's continuous across the $\mathbb{R}$ axis. The imaginary part gives us the spectral weight, and so near the qp pole

$$A(\omega, \mathbf{k}) \approx \delta(\omega - \epsilon_{\mathbf{k}} - \Sigma_R(\omega, \mathbf{k})) = Z_{\mathbf{k}}\delta(\omega - \epsilon_{\mathbf{k}}^*), \tag{413}$$

where in the last step we used the usual $\delta(f(\omega)) = |(\partial_\omega f)(\epsilon_{\mathbf{k}}^*)|^{-1}\delta(\omega - \epsilon_{\mathbf{k}}^*)$, since $\epsilon_{\mathbf{k}}^*$ is the value of ω that solves the constraint from the first δ function. Therefore from the spectral representation, we conclude that in this approximation,

$$Z_{\mathbf{k}} \approx |\langle \epsilon_{\mathbf{k}}^* | c_{\mathbf{k}}^\dagger | 0 \rangle|^2, \tag{414}$$

where $|\epsilon_{\mathbf{k}}^*\rangle$ is the qp state we're interested in (which is a genuine eigenstate of H since we're pretending that Σ_I vanishes). Thus $Z_{\mathbf{k}}$ is the amount of $\mathbf{k}$ electron contained in the renormalized qp excitation—exactly what we expect from the physical interpretation of the wavefunction renormalization factor.

To expand on this, let us think about how wavefunction renormalization should work in non-relativistic theories. The usual relation between bare and renormalized fields is $\psi_b = Z^{1/2}\psi_r$. In the old-fashioned point of view, the $Z^{1/2}$ term is present to cancel k-dependent divergences; in the RG it is there to ensure that the fields maintain the same kinetic term during the flow.

First, why is it called Z? When we do RG, the self-energy diagrams will generate terms that modify the $\bar{\psi}\omega\psi$ part of the action to[61]

$$\bar{\psi}\omega\psi \mapsto \bar{\psi}(\omega - \Sigma(0, \mathbf{k}) - \omega(\partial_\omega\Sigma)(0, \mathbf{k}) - \dots)\psi. \tag{415}$$

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[61]We are expanding about zero frequency for concreteness;

When we do the field renormalization during the RG step, we want to renormalize the fields such that the frequency part has unit coefficient, hence we define

$$\psi' = (Z_{\mathbf{k}})^{-1/2}\psi, \qquad Z_{\mathbf{k}} = (1 - (\partial_\omega\Sigma)(0, \mathbf{k}))^{-1}. \tag{416}$$

This is essentially the same thing we do when performing wavefunction renormalization, with the bare and renormalized fields related by $\psi_0 = \sqrt{Z}\psi_R$.

30 *T-dependence of carrier concentration in semiconductors*

Today we're doing a slight elaboration on problem 28.6 from A&M. We'll be computing the carrier concentration in a semiconductor as a function of T, and breaking up the behavior into various different regimes.

Solution:

First, some notation. N_a (N_d) will denote the total number of acceptor (donor) impurities added to the sample. Donor impurities are basically weakly bound hydrogen atoms added to the sample, and acceptors are basically weakly bound antihydrogen atoms. We will write the number of neutral donors (acceptors) as N_d^0 (N_a^0). Donors can be singly ionized but cannot accept additional electrons due to an assumed Coulomb repulsion; we the number of such ionized donors as $N_d^+ = N_d - N_d^0$. Conversely, acceptors can have their holes singly ionized but cannot accept additional holes (for the same reason); we write the number of ionized acceptors as $N_a^- = N_a - N_a^0$.

The carrier concentrations in the undoped case are straightforward to get. The conduction electrons have number density $n = \int_{\epsilon_c}^\infty d\epsilon\, g(\epsilon)f(\epsilon)$, with f the Fermi function, ϵ_c the conduction band edge, and g the density of states for a free Fermi gas with effective mass determined by the curvature of the conduction band. We usually make the approximation that $\epsilon_c - \mu \gg T$ (justified since semiconductor band gaps don't get much smaller than 0.1eV or so, which is already greater than the $\sim$ 0.02eV of room temperature), which lets us approximate Fermi by Maxwell-Boltzmann and do the integral. This gives

$$n = 2\left(\frac{m_c T}{2\pi}\right)^{3/2} e^{-\beta\xi_c} = N_c(T)e^{-\beta\xi_c}, \tag{417}$$

with $\xi_c \equiv \epsilon_c - \mu$ and m_c the conduction band effective mass. In the same way, the number of holes in the valence band is[62]

$$h = 2\left(\frac{m_v T}{2\pi}\right)^{3/2} e^{-\beta\xi_v} = H_v(T)e^{-\beta\xi_v}, \tag{418}$$

with m_v the valence band effective mass and $\xi_v = \mu - \epsilon_v$. In the undoped case we must have $n = h$ since electrons can't just disappear; hence we can divide the last two equations and get

$$1 = (m_c/m_v)^{3/2}e^{-\beta(\epsilon_c + \epsilon_v - 2\mu)} \implies \mu = \frac{\epsilon_c + \epsilon_v}{2} + \frac{3}{4}T\ln(m_v/m_c). \tag{419}$$

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[62]Most people use the letter p_v (for positive) but I don't like this because particle also starts with p.

Thus at $T = 0$ the chemical potential is right in the middle of the gap. We see that μ at finite T is closer to the band with the smaller effective mass.

Now we consider the doped case. At very high temperatures the majority of carriers will come from ones that are thermally activated to cross the band, since even if

$$\Delta_G \equiv \epsilon_c - \epsilon_v \tag{420}$$

is large compared to T (e.g. $\Delta_G \sim 1\text{eV} \sim 10^4 K$ and $T \sim 10^3 K$), the concentration of impurities is basically always small enough compared to the density of normal atoms that the number of carriers that are excited across the gap is greater than the number of carriers excited from / into the impurity bands. Therefore the number of carriers is basically the same as it is in the intrinsic case. Since in this case we have $n = h$, we have

$$n \approx \sqrt{H_v N_c} e^{-\beta \Delta_G}, \tag{421}$$

which we obtained by multiplying the expressions for n and h so as to eliminate μ. This is typically the regime present for $T \gtrsim 500\text{K}$.

Now let us assume an n-doped semiconductor and let us reduce the temperature to the point where the majority of carriers come from excited donor atoms.

Our starting point will be the expression for charge conservation:

$$n + N_a^- = h + N_d^+, \tag{422}$$

where each term represents a contribution to the charge resulting from a finite-energy excitation (conduction electrons, donor holes, etc.).

Since in the n-doped case the dominant charge carriers will be the conduction electrons, we will want to use the above equation to solve for n. We want to get an expression for n in terms of data intrinsic to the system, viz. N_d, N_a, T, and m_i. Now in the n-doped case all the action will be going on in the donor levels and conduction band—the only affect of the acceptor levels will be to eat up electrons from the donor levels, and the valence band will be completely passive. Since the acceptor levels will essentially always be completely filled (not by losing their bound holes to the valence band, but by absorbing electrons from the donor levels), we can approximate $N_a^- \approx N_a$. We can also neglect h, and so we have

$$n \approx N_d^+ - N_a, \tag{423}$$

and hence all we have to do is find N_d^+.

This is done as follows. The ratio N_d^0/N_d is the probability that any given donor level is occupied by an electron. Since there are $\uparrow + \downarrow = 2$ ways to put the electron in,

$$N_d^0/N_d = \frac{2e^{-\beta \xi_d}}{1 + 2e^{-\beta \xi_d}} = \frac{1}{1 + \frac{1}{2}e^{\beta \xi_d}}. \tag{424}$$

Since this is also equal to $1 - N_d^+/N_d$, we see that

$$N_d^+/N_d = \frac{1}{1 + 2e^{-\beta \xi_d}}. \tag{425}$$

The problem with the RHS is that it involves μ. We can get rid of the $e^{\beta\mu}$ by writing it in terms of

$$ne^{\beta\epsilon_c}/N_c = e^{\beta\mu}. \tag{426}$$

Therefore

$$N_d^+ = \frac{N_d}{1 + 2nN_c^{-1}e^{-\beta\Delta}}, \qquad \Delta \equiv \epsilon_c - \epsilon_d. \tag{427}$$

Putting this into our charge conservation equation, we then get an equation that's quadratic in n and only contains the input parameters N_a, T, m_c, N_d:

$$n^2\left(\frac{2}{N_c}e^{\beta\Delta}\right) + n\left(1 + \frac{2N_a}{N_c}e^{\beta\Delta}\right) = N_d - N_a, \tag{428}$$

which can be solved non-enlighteningly.

After the high-T regime, the next regime we expect is a plateau where the carrier concentration doesn't change appreciably—in this regime T is small enough so that there aren't many carriers coming from jumping the band gap, but T hasn't yet become small enough to change the fact that basically all the donor electrons are ionized into the conduction band. In this regime we can typically ignore the n^2/N_c and $N_a/N_c \ll 1$ terms, and we just get a constant $n = N_d - N_a$, as expected. This plateau extends usually from $150K \lesssim T \lesssim 500K$, ish.

We next come to a regime where a freezeout of the conduction electrons starts to occur. We assume that T is now low enough that a significant fraction of the donor electrons are not ionized up to the conduction band, but that T is still high enough, and that the semiconductor is n-doped enough, so that $N_a \ll n$. In this regime we will have T getting smaller than Δ, so that the exponentials in (428) start to get large. We assume that $N_a \ll n$, $n^2e^{\beta\Delta} \gg n$ sufficiently strongly so that we can drop the linear term in (428), giving

$$n \approx \sqrt{\frac{(N_d - N_a)N_c}{2}}e^{-\beta\Delta/2}, \tag{429}$$

so that the carrier concentration now gets cutoff exponentially fast in $\beta\Delta/2$. This regime occurs usually from $\sim 150K$ to maybe ~ 30K, ish.

There is a final low-T regime below $T \sim 30K$, where the freeze-out is sharper. In this regime the carrier concentration drops even below N_a. In this case $n^2 \ll nN_a$ and $N_ae^{\beta\Delta}/N_c \gg 1$, and so only the third term on the LHS of (428) survives. Therefore in this regime (note: I think there is a typo in A&M)

$$n \approx \frac{N_c(N_d - N_a)}{2N_a}e^{-\beta\Delta}, \tag{430}$$

which gives a faster exponential suppression. This regime continues up to the point where the semiconductor becomes insulating.

This all of course generalizes straightforwardly to a p-type semiconductor. In the p-doped case the charge conservation equation is instead approximated with

$$h + N_d \approx N_a^-, \tag{431}$$

(since basically all the donor levels have lost their electrons to the acceptor levels and since $n \ll h$) and the problem is now to write N_a^- in terms of stuff like H_v, N_a, and so on. The intermediate step to the analogous quadratic equation for h is

$$N_a^-/N_a = \frac{e^{-\beta \xi_a}}{2 + e^{-\beta \xi_a}} \implies N_a^- = \frac{N_a}{1 + 2e^{\beta \xi_a}}. \tag{432}$$

This comes from the fact that there is only one way for the acceptor to be "ionized" (i.e. to loose its hole; the associated anti-Hydrogen atom is then empty and in a unique state), but there are two ways for it to be filled, since the hole can be bound with either spin. This leads to the quadratic equation

$$p^2 \left(\frac{2}{H_v} e^{\beta \Delta'} \right) + n \left(1 + \frac{2N_d}{H_v} e^{\beta \Delta'} \right) = N_a - N_d, \qquad \Delta' = \epsilon_a - \epsilon_v, \tag{433}$$

which then has all the same T-dependent regimes as the analogous equation for n.

31 Estimate for the size of the depletion region in a p-n junction

Today we're doing a rather undergrady problem that's still fun: estimating the size of the depletion region in a p-n junction using simple electrostatics.

Solution:

First, recall the setup of a p-n junction. Imagine putting an n-doped SC (the S is for semi, not super) next to a p-doped one. At the boundary, the conduction electrons on the n-doped side will be attracted to the holes on the p-doped side, and will annihilate (as the electrons move farther into the p-doped region their energies increase, and they start looking for holes in the valence band to annihilate with, and vice-versa for the holes). The annihilating electron / hole pairs mean that the ionized donors / acceptors on the respective sides of the boundary no longer have their charges screened by charge carriers in their immediate vicinities. This results in a net positive charge on the n-doped side of the boundary (from the ionized donors) and a net negative charge on the p-doped side (from the hole-ionized acceptors), giving an electric field pointing from the n-doped side to the p-doped one. The charged regions constitute the depletion region, so named since all the mobile carriers have pair-annihilated in this region.

There will thus be some equilibrium configuration where the electric potential created by this charge exactly compensates for the difference in the band energies of each band across the junction, giving a constant μ (remember that μ is always constant in equilibrium!).

To find the width of the depletion region, we make the crude approximation where the depletion region is modeled as a plane of thickness w_a with charge density $-N_a$ coming from the acceptors, and a plane of thickness w_d of charge density N_d, from the donors. We then want to solve

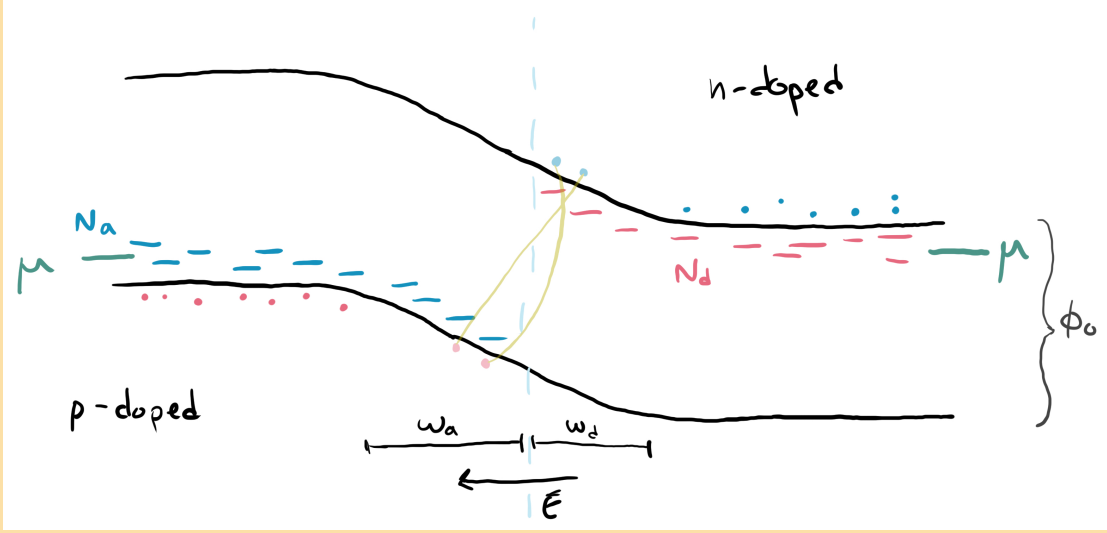$$-\nabla^2 \phi(x) = \rho(x)/\epsilon \tag{434}$$

Figure 2: The acceptor levels are marked in blue, since they are always "hole-ionized" in the regime we're interested and hence negatively charged. In the depletion region the distance between μ and the band edge grows, and the holes in the valence band annihilate with electrons in the conduction band (yellow arrows). This leaves behind unscreened ionic charges, which establish the electric field as shown.

for the region $0 < x < w_a + w_d$, with ϵ the renormalized dielectric "constant" in the SC (the bandgap isn't changing across the junction, so taking a constant ϵ seems reasonable). In keeping with our crude model, the charge distribution is assumed to be

$$\rho(x) = \begin{cases} -N_a & 0 < x < w_a \\ +N_d & w_a < x < w_a + w_d \end{cases} \tag{435}$$

where $x = 0$ is the left edge of the depletion region.

We then integrate this twice to get the potential, choosing boundary terms for continuity: this gives, for the induced potential set to be $\phi(0) = 0$,

$$\epsilon\phi(x) = \begin{cases} \frac{N_a x^2}{2} & 0 < x < w_a \\ w_a(N_d + N_a)x - \frac{N_d x^2}{2} - \frac{w_a^2}{2}(N_d + N_a) & w_a < x < w_a + w_d \end{cases} \tag{436}$$

Now the chemical potential μ is constant in equilibrium. We know that on the p-doped side μ is just above the valence band edge, while on the n-doped side it is just below the conduction band edge. This difference in energy $\phi_0 \equiv \epsilon_c - \epsilon_v$ must be compensated for by the induced potential $\phi(x)$ in order for μ to be constant. Therefore we must have that $\phi(w_a + w_d) = \phi_0$. We still need one more condition to solve for the two unknowns w_i; this comes from overall charge neutrality, which tells us that

$$w_a N_a = w_d N_d, \qquad w_d = \frac{N_a}{N_d} w_a \equiv r w_a. \tag{437}$$

The condition that $\phi(w_a + w_d) = \phi_0$ is

$$w_a^2 \left(\frac{N_d}{2}(1+r)^2 - \frac{N_d}{2}(1+r) \right) = \epsilon\phi_0 \implies w_a = \sqrt{\frac{2\epsilon N_d \phi_0}{N_a(N_d + N_a)}}. \tag{438}$$

Hence we also have

$$w_d = \sqrt{\frac{2\epsilon N_a \phi_0}{N_d(N_d + N_a)}}. \tag{439}$$

The induced charge on each of the "plates" of this "capacitor" is

$$|Q_{ind}| = w_a N_a = w_d N_d = w_a = \sqrt{\frac{2\epsilon N_d N_a \phi_0}{N_d + N_a}}, \tag{440}$$

which is symmetric in N_a, N_d as required. The capacitance of the junction is

$$C = \sqrt{\frac{2\epsilon N_d N_a}{N_d + N_a}}. \tag{441}$$

32 Ginzburg criterion for fluctuations at second-order phase transitions

Today we're looking at the Ginzburg criterion, which is a way of estimating how close one has to get to a second-order phase transition in order for fluctuations of the order parameter to be "important".

Solution:

The starting point is of course

$$H = \int \left(\frac{K}{2}|\nabla\phi|^2 + \frac{t}{2}|\phi|^2 + u|\phi|^4 \right) \tag{442}$$

where ϕ is some *dimensionless n*-component field (always broken up into real parts; the $||^2$s mean dot products). Near the phase transition, we split up the field into a longitudinal part φ along the ordering direction $\hat{\mathbf{x}}_1$, and transverse components ζ_i as

$$\phi = (v + \varphi)\hat{\mathbf{x}}_1 + \sum_{i>1}^{n} \zeta_i \hat{\mathbf{x}}_i, \tag{443}$$

with v the magnitude of the vev of ϕ. We then put this expression into F, and expand to quadratic order in all of the fluctuation fields. This gives

$$H \approx H_0 + \frac{K}{2} \int \left(\varphi[-\nabla^2 + \xi_l^{-2}]\varphi + \zeta_i[-\nabla^2 + \xi_t^{-2}]\zeta^i \right), \tag{444}$$

where $F_0 = F|_{\phi=v\hat{\mathbf{x}}_1}$ and we have defined the two parameters

$$\xi_t^{-2} \equiv \frac{t + 4uv^2}{K}, \qquad \xi_l^{-2} \equiv \frac{t + 12uv^2}{K}, \tag{445}$$

which set the masses for the two types of fluctuations in the theory. Note that the value of v is $v = \sqrt{-t/(4u)}$ for $t < 0$ and $v = 0$ for $t > 0$, and hence the longitudinal modes are massless below the phase transition as expected, with $\xi_l(t < 0) = \infty$, and are massive above the transition, with $\xi_l(t > 0) = \sqrt{K/t}$. In contrast, the longitudinal mode has $\xi_l(t < 0) = \sqrt{-K/2t}$[63] and $\xi_l(t > 0) = \xi_t(t > 0)$ (since there is no difference between longitudinal and transverse in the absence of ordering).

Doing the integral of course gives the free energy

$$F = -\frac{t^2}{16u}\theta(-t) + \frac{1}{2}\int_{\mathbf{q}} \left(\ln[K(q^2 + \xi_l^{-2})] + (n-1)\ln[K(q^2 + \xi_t^{-2})] \right). \tag{446}$$

To diagnose the relative importance of the fluctuations, we will look at their affect on thermodynamical variables. Actually we will just look at the (singular part of the) specific heat, since it gives us an idea of what degrees of freedom are getting turned on as we pass through the critical point. We then just have to take the above and differentiate twice with respect to t. The only t-dependence is in the lengths ξ_i, and the first F_0 term in the front, provided that $t < 0$. We then find the singular contribution to C to be

$$C = -\partial_t^2 F = \begin{cases} 0 + \dfrac{n}{2K^2}\displaystyle\int_{\mathbf{q}} \dfrac{1}{(|\mathbf{q}|^2 + \xi_l^{-2})^2} & t > 0 \\[4mm] \dfrac{1}{8u} + \dfrac{2}{K^2}\displaystyle\int_{\mathbf{q}} \dfrac{1}{(|\mathbf{q}|^2 + \xi_l^{-2})^2} & t < 0 \end{cases} \tag{447}$$

We just care about how this scales with the correlation length, and so we don't need to do the integral very carefully. When $d > 4$ the integral is UV divergent and diverges in the usual MF way. When $d < 4$, dimensional analysis tells us that the integrals are ξ_l^{4-d}. We can then separate the contribution to the change in C across the transition into two parts: one from the ordering,

$$\delta C_0 \sim 1/u, \tag{448}$$

and one from the fluctuations,

$$C_{fluct} \sim K^{-2}\xi_l^{4-d}. \tag{449}$$

In the above we have restricted to $d < 4$, since when $d > 4$ the fluctuation contribution gives us a UV divergent term that is just a constant across the transition, and so the form of the discontinuity across the transition is unchanged by fluctuations in $d > 4$ dimensions.

The "importance" (in the thermodynamic sense) of fluctuations for $d < 4$ can be diagnosed by comparing the two specific heat discontinuities; fluctuations will become strong when $C_{fluct} \gtrsim C_0$. Since $\xi_l \sim \sqrt{-K/t}$, fluctuations will be important provided that

$$\delta C_0 \lesssim C_{fluct} \implies t \lesssim \frac{u^{2/(4-d)}}{K^{d/(4-d)}}, \tag{450}$$

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[63]Note that this is precisely the same as the coherence length in LG theory (the typical length scale of fluctuations in the magnitude of the order parameter).

which is the Ginzburg criterion. The parameters K and u will of course depend on the microscopics, and the message here is that even below the upper critical dimension where MFT should fail, it may happen that fluctuations are sufficiently weak due to the values of K, u that it is experimentally infeasible to tune t close enough to 0 to be able to see their effects. This is what happens for many type I superconductors, where for $d = 3$ we would need to tune t to something like $t < (T_c/E_F)^4 \sim 10^{-16}$, which is out of the question. A sanity check is that the better the SC, the smaller we have to tune t to see fluctuations, since stronger order implies larger K.[64] The fact that t needs to be so small in order to see the thermodynamic effects of fluctuations is why type I superconductors are so well described by mean field theory: we can essentially always set $|\phi| = v$ to be a constant, except when t is (often prohibitively) small.

33   *Boltzmann equation and the relaxion-time approximation*

Today we're doing all of the exercises in Ashcroft and Mermin chapter 16. The name of the game is understanding the Boltzmann equation. We will be setting $\hbar = c = e = k_B = 1$ throughout.

Solution:

Problem 1: We start with two rather obvious things. First, let H be some conserved quantity carried by each individual particle. If the 1-particle distribution function is $g_{\mathbf{k}}$ (implicitly a function of $\mathbf{x}$), then we write

$$H = \int_{\mathbf{k}} h_{\mathbf{k}} g_{\mathbf{k}}. \tag{451}$$

The change in H due to collisions is

$$d_t H|_c = \int_{\mathbf{k}} h_{\mathbf{k}} \partial_t g_{\mathbf{k}}|_c, \tag{452}$$

where the $|_c$ stands for the contribution due to collisions. We claim that that $d_t H|_c = 0$ if all the matrix elements $W_{\mathbf{kk'}}$ for scattering from $\mathbf{k}$ to $\mathbf{k'}$ are nonzero only when $h_{\mathbf{k}} = h'_{\mathbf{k}}$. Indeed, this follows directly from the usual expression

$$\partial_t g_{\mathbf{k}}|_c = \int_{\mathbf{k'}} (-W_{\mathbf{kk'}} g_{\mathbf{k}}(1 - g_{\mathbf{k'}}) - (\mathbf{k} \leftrightarrow \mathbf{k'})) \tag{453}$$

and doing a change of variables in the integral for $d_t H|_c$.

Second, note that the continuity equation follows from the Boltzmann equation. The divergence in the particle current is, using the Boltzmann equation,

$$\nabla \cdot \mathbf{j} = \nabla \cdot \int_{\mathbf{k}} g_{\mathbf{k}} \mathbf{v} = \int_{\mathbf{k}} (-\partial_t g_{\mathbf{k}} - \mathbf{F} \cdot \widetilde{\nabla} g_{\mathbf{k}} + d_t g_{\mathbf{k}}|_c), \tag{454}$$

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[64]e.g. in three dimensions $K \sim 1/\xi_0$, with ξ_0 the ($T = 0$) microscopic coherence length of the SC, which serves as the UV cutoff for the continuum theory and which is smaller for better superconductors, with $\xi_0 \sim \hbar v_F/kT_c$.

where $\mathbf{F}$ is the external force and $\widetilde{\nabla}$ is a gradient in momentum space. The $\mathbf{F}$ term dies after integrating by parts ($g_\mathbf{k}$ will either have no support at infinity, or momentum space will be compact), while the collisional piece also dies by the antisymmetry between $\mathbf{k}$ and $\mathbf{k}'$ in its integral representation. Therefore

$$\nabla \cdot \mathbf{j} = -\partial_t \int_\mathbf{k} g_\mathbf{k} \implies \partial_\mu j^\mu = 0. \tag{455}$$

Our next task is to deduce the appropriate continuity equation for the heat flow. We will assume that the collisions happening are energy-conserving, so that the collisional contribution to the change in energy density u is zero:

$$d_t u|_c = \int_\mathbf{k} \epsilon_\mathbf{k} d_t g_\mathbf{k}|_c = 0. \tag{456}$$

First we get an expression for dq/dt. From $TdS = dU - \mu dN$, we have

$$\dot{q} = d_t \int g_\mathbf{k}(\epsilon_\mathbf{k} - \mu) = -\int \left[(\mathbf{v} \cdot \nabla + \mathbf{F} \cdot \widetilde{\nabla})g_\mathbf{k} - d_t g_\mathbf{k}|_c \right] (\epsilon_\mathbf{k} - \mu) = -\int \left[\mathbf{v} \cdot \nabla g_\mathbf{k}(\epsilon_\mathbf{k} - \mu) - \mathbf{F} \cdot \mathbf{v} g_\mathbf{k} \right]. \tag{457}$$

In the last step, we have integrated by parts, used the equation of motion for the velocity, and used that $d_t u|_c = d_t \mu|_c = 0$.

On the other hand, consider the divergence of the heat current. This is

$$\nabla \cdot \mathbf{j}_q = \nabla \cdot \int \mathbf{v} g_\mathbf{k}(\epsilon_\mathbf{k} - \mu) = \int \left[\mathbf{v} \cdot \nabla g_\mathbf{k}(\epsilon_\mathbf{k} - \mu) - g_\mathbf{k} \nabla \mu \cdot \mathbf{v} \right]. \tag{458}$$

We define a vector field $\mathcal{E}$ by (note that the sign is different from the one in Ashcroft and Mermin; I think they have a typo)

$$\mathcal{E} \equiv \mathbf{F} - \nabla \mu. \tag{459}$$

Then we can cancel the $\nabla \mu$ piece of $\mathbf{j}_q$ by a term like $-\mathcal{E} \cdot \mathbf{j}$. Indeed, we see that we in fact get a continuity equation for q, modified by a term containing the external forces (we think of $\nabla \mu$ as an external force field):

$$\dot{q} + \nabla \cdot \mathbf{j}_q = \mathcal{E} \cdot \mathbf{j}. \tag{460}$$

$$g_\mathbf{k} = g_0 + \frac{\partial g_0}{\partial \epsilon_\mathbf{k}} \mathbf{v} \cdot \left[\mathbf{E} - \frac{\nabla T}{T}(\epsilon_\mathbf{k} - \mu) \right] \tau(\mathbf{k}). \tag{461}$$

Problem 4: Now we go back to just a uniform electric field. We will go beyond the relaxation-time approximation, but will assume that the collisions are elastic, with the scattering matrix element $W_{\mathbf{k}\mathbf{k}'} \propto \delta(\epsilon_\mathbf{k} - \epsilon_{\mathbf{k}'})|\langle \mathbf{k}|U|\mathbf{k}'\rangle|^2$, with U some scattering potential. With this assumption $W_{\mathbf{k}\mathbf{k}'} = W_{\mathbf{k}'\mathbf{k}}$, and so the collisional part of the BE contains only linear terms:

$$d_t g_\mathbf{k}|_c = \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}(g_{\mathbf{k}'} - g_\mathbf{k}). \tag{462}$$

We write the distribution function as $g = f + \boldsymbol{\delta} \cdot \mathbf{E}$, and solve the Boltzmann equation to linear order in $\mathbf{E}$, since we want the conductivity. The term linear in $\mathbf{E}$ gives

$$\mathbf{E} \cdot \mathbf{v} \partial_\epsilon f_\mathbf{k} = \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}(\boldsymbol{\delta}_{\mathbf{k}'} - \boldsymbol{\delta}_\mathbf{k}) \cdot \mathbf{E}. \tag{463}$$

Let $\mathbf{u}$ be the solution to the integral equation

$$\mathbf{v}_\mathbf{k} = \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}(\mathbf{u}_{\mathbf{k}'} - \mathbf{u}_\mathbf{k}). \tag{464}$$

Then we can solve for $\boldsymbol{\delta}$ as

$$\boldsymbol{\delta}_\mathbf{k} = \mathbf{u}_\mathbf{k} \partial_\epsilon f_\mathbf{k}. \tag{465}$$

The reason why this works is that $W_{\mathbf{k}\mathbf{k}'}$, being energy conserving, means that it is only nonzero when $\partial_\epsilon f_\mathbf{k} = \partial_\epsilon f_{\mathbf{k}'}$, so that we can pull out a $\partial_\epsilon f_\mathbf{k}$ from both terms in the integral. The conductivity is then calculated by

$$\mathbf{j} = \int_\mathbf{k} (f_\mathbf{k} + \boldsymbol{\delta} \cdot \mathbf{E}) \mathbf{v} \implies \sigma = \int_\mathbf{k} \partial_\epsilon f_\mathbf{k}\, \mathbf{u}_\mathbf{k} \otimes \mathbf{v}_\mathbf{k}. \tag{466}$$

Now we define the metric

$$\langle \alpha, \gamma \rangle = \int_\mathbf{k} (-\partial_\epsilon f) \alpha_\mathbf{k} \int_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}(\gamma_\mathbf{k} - \gamma_{\mathbf{k}'}). \tag{467}$$

Again because of the symmetry of $W_{\mathbf{k}\mathbf{k}'}$, by changing $\mathbf{k}$ to $\mathbf{k}'$ in one of the integrals, we see that $\langle \alpha, \gamma \rangle = \langle \gamma, \alpha \rangle$. Replacing $\mathbf{v}_\mathbf{k}$ in the formula for the conductance with its integral representation in terms of $\mathbf{u}_\mathbf{k}$, we see that the conductivity is

$$\sigma = \langle \mathbf{u}, \otimes \mathbf{u} \rangle. \tag{468}$$

Let's prove that this does indeed define a norm. First, positivity: note that $-\partial_\epsilon f_\mathbf{k}, W_{\mathbf{k}\mathbf{k}'} \geq 0 \,\forall\, \mathbf{k}$. So we want to show that

$$\int_{(x,y) \in D} K(x,y) f(x)^2 \geq \int_{(x,y) \in D} K(x,y) f(x) f(y) \tag{469}$$

for any domain D on which $K(x,y) \geq 0$ and $K(x,y) = K(y,x)$. Indeed, this follows just from using

$$\int_D (f(x) - f(y))^2 K(x,y) \geq 0, \tag{470}$$

together with the symmetry of $K(x,y)$.

Now we prove the $\triangle$ inequality. We can do this by looking at the positivity of the smallest-length vector formed by a linear combination of two functions α, γ. Minimizing $||\alpha + \lambda\gamma||^2$ over λ, we find a minimum at $\lambda_* = -\langle \alpha, \gamma \rangle / ||\gamma||^2$. Then

$$\langle \alpha + \lambda_* \gamma, \alpha + \lambda_* \gamma \rangle \geq 0 \implies ||\alpha||^2 ||\gamma||^2 \geq ||\langle \alpha, \gamma \rangle||^2, \tag{471}$$

which is what we wanted to show. Since $\sigma_{ii} = \langle \mathbf{u}_i, \mathbf{u}_i \rangle$ and since $\langle \mathbf{u}_x, \gamma \rangle = \int_{\mathbf{k}} (-\partial_\epsilon f) \gamma_{\mathbf{k}} \mathbf{v}_i(\mathbf{k})$ for any function γ, we conclude that

$$\sigma_{ii} \geq \frac{1}{||\gamma||^2} \left[\int_{\mathbf{k}} (-\partial_\epsilon f) \mathbf{v}_i(\mathbf{k}) \gamma_{\mathbf{k}} \right]^2 . \tag{472}$$

Now consider a situation where the scattering comes from two sources $W = W_1 + W_2$, with both W_is symmetric positive scattering matrix elements coming from separate elastic scattering processes. Let us then plug in the choice $\mathbf{u}_i$ for γ in the above inequality, using the norm coming from just W_1. This gives

$$\sigma_1^{ii} \geq \frac{1}{||\mathbf{u}_i||_{W_1}^2} \left[\int_{\mathbf{k}} (-\partial_\epsilon f) \mathbf{v}^i(\mathbf{k}) \mathbf{u}^i(\mathbf{k}) \right]^2 = \frac{(\sigma_{1+2}^{ii})^2}{||\mathbf{u}_i||_W^2 - ||\mathbf{u}_i||_{W_2}^2} = \frac{(\sigma_{1+2}^{ii})^2}{\sigma_{1+2}^{ii} - ||\mathbf{u}_i||_{W_2}^2} . \tag{473}$$

We get a similar formula if we replace $1 \leftrightarrow 2$. We then invert both of these inequalities (everything in sight is positive so this is easily done), and add them together. We get

$$\frac{1}{\sigma_1^{ii}} + \frac{1}{\sigma_2^{ii}} \leq \frac{2\sigma_{1+2}^{ii} - ||\mathbf{u}_i||_{W_1}^2 - ||\mathbf{u}_i||_{W_2}^2}{(\sigma_{1+2}^{ii})^2} , \tag{474}$$

which tells us (this seems to give a stronger bound than the one in Ashcroft and Mermin?!)

$$\frac{1}{\sigma_{1+2}^{ii}} \geq \frac{1}{\sigma_1^{ii}} + \frac{1}{\sigma_2^{ii}} . \tag{475}$$

34 *Practice with the Kubo formula and electrical conductivity*

This is from Coleman's many body theory book, chapter 10. Find the electrical conductivity in a metal with Feynman diagrams by restricting your attention to the polarization bubble diagram. Your starting point should be the Kubo formula for the conductance, viz.

$$\sigma^{ab}(i\nu) = e^2 \frac{T}{\nu} \sum_{\mathbf{q},\omega} v_{\mathbf{q}}^a v_{\mathbf{q}}^b \left(G(\mathbf{q}, i\omega + i\nu) G(\mathbf{q}, i\omega) - G(\mathbf{q}, i\omega)^2 \right) , \tag{476}$$

which represents the polarization bubble with the zero-frequency diamagnetic part subtracted off. In this formula, the electron propagators G are

$$G(\mathbf{q}, i\omega) = \frac{1}{i\omega - \epsilon_{\mathbf{q}} - \Sigma(i\omega)} , \tag{477}$$

for some momentum-independent self energy, the $1/\nu$ comes from $-\partial_t A^i = E^i$, the $v_{\mathbf{q}}^a$'s are (kinematic) velocities, and the second term in parenthesis is the diamagnetic part.

Solution:

Our first task is to switch out the momentum integral for an energy integral. Since the Greens functions are rotationally invariant and we are in three dimensions, we can swap out $v^a v^b$ for $v^2/3$. Furthermore, since we are subtracting off the zero frequency part which picks up the finite density of the fermi surface, at frequencies low compared to the Fermi energy we can potentially get away with crudely approximating the density of states as a constant (equal to its value at the FS), and in doing so we can integrate the energy (relative to the FS) from $-\infty$ to $+\infty$. So we can do (not keeping track of spatial volumes)

$$\sum_{\mathbf{q}} v^a v^b \to \delta^{ab} \int_{\mathbb{R}} d\epsilon \; N(0) \frac{k_F^2}{3m^2} = \delta^{ab} \int_{\mathbb{R}} d\epsilon \; \frac{k_F^3}{2\pi^2 m}, \tag{478}$$

since the density of states at the FS is $N(0) = mk_F/(4\pi^2)$. The number density is

$$n = \int d\epsilon N(\epsilon) = \frac{2m}{3 \cdot 2\pi^2} \sqrt{2m\epsilon_F}\epsilon_F = \frac{k_F^3}{3 \cdot 2\pi^2} = \frac{2}{3} N(0) \frac{k_F^2}{m}, \tag{479}$$

where we integrated up to the FS. Thus the diagonal part of the conductivity is

$$\sigma^{aa}(i\nu) = \frac{ne^2 T}{m\nu} \int_{\mathbb{R}} d\epsilon \sum_{\omega} \left(G(\epsilon, i\omega + i\nu)G(\epsilon, i\omega) - G(\epsilon, i\omega)^2 \right). \tag{480}$$

Now we do the Matsubara sum by integrating against the Fermi distribution. Continuing $i\omega$ to z,

$$\sigma^{aa}(i\nu) = \frac{ne^2}{2\pi i m\nu} \int_{\mathbb{R}} d\epsilon \int dz \; f(z) \left(G(\epsilon, z + i\nu)G(\epsilon, z) - G(\epsilon, z)^2 \right). \tag{481}$$

The singularities of the summand are branch cuts which run horizontally across the complex plane. For $\alpha \in \mathbb{R}$ and δ infinitesimal, we have branch cuts at $z = \alpha \pm i\delta$ and at $z = \alpha - i\nu \pm i\delta$. The first of these gives

$$\frac{1}{\alpha + i\nu - \epsilon - \Sigma(\alpha + i\nu)} \left(\frac{1}{\alpha - i\delta - \epsilon - \Sigma(\alpha - i\delta)} - \frac{1}{\alpha + i\delta - \epsilon - \Sigma(\alpha - i\delta)} \right)$$
$$\to \frac{2\pi i \delta(\epsilon + \Sigma(\alpha - i\delta) - \alpha)}{\alpha + i\nu - \epsilon - \Sigma(\alpha + i\nu)}, \tag{482}$$

where we've subtracted in the given order since the counterclockwise path of the contour means that the $\alpha - i\delta$ part is oriented from $-\infty$ to $+\infty$. The branch cut at $z = \alpha - i\nu \pm i\delta$ gives us a similar expression, namely

$$\frac{2\pi i \delta(\epsilon + \Sigma(\alpha + i\delta) - \alpha)}{\alpha - i\nu - \epsilon - \Sigma(\alpha - i\nu)}. \tag{483}$$

Putting these in and doing the energy integral, we reduce to an integral of the parameter α along the $\mathbb{R}$ line:

$$\sigma^{aa}(i\nu) = \frac{ne^2}{m\nu} \int_{\mathbb{R}} d\alpha \; f(\alpha) \left(\frac{1}{i\nu - \Sigma(\alpha + i\nu) + \Sigma(\alpha - i\delta)} + \frac{1}{-i\nu + \Sigma(\alpha + i\delta) - \Sigma(\alpha - i\nu)} \right), \tag{484}$$

where we have dropped $i\delta$'s when they appear in a sum with other imaginary things.

Now we analytically continue by taking $i\nu \to \nu + i\delta$. This gives

$$\sigma^{aa}(\nu+i\delta) = i\frac{ne^2}{m}\int_{\mathbb{R}}d\alpha\frac{f(\alpha)}{\nu}\left(\frac{1}{\nu - \Sigma(\alpha+\nu+i\delta) + \Sigma(\alpha-i\delta)} - \frac{1}{\nu - \Sigma(\alpha+i\delta) + \Sigma(\alpha-\nu-i\delta)}\right).$$
(485)

Now shift α by $-\nu/2$ in the first term and by $+\nu/2$ in the second term:

$$\sigma^{aa}(\nu+i\delta) = i\frac{ne^2}{m}\int_{\mathbb{R}}d\alpha\frac{f(\alpha-\nu/2) - f(\alpha+\nu/2)}{\nu}\frac{1}{\nu - \Sigma(\alpha+\nu/2+i\delta) + \Sigma(\alpha-\nu/2-i\delta)}.$$
(486)

Now define the average scattering rate by

$$\bar{\tau}^{-1} = \text{Im}\left[\Sigma(\alpha-\nu/2-i\delta) + \Sigma(\alpha+\nu/2-i\delta)\right]$$
(487)

and the wavefunction renormalization by

$$Z^{-1} = 1 - \frac{1}{\nu}\text{Re}\left[\Sigma(\alpha-\nu/2) - \Sigma(\alpha+\nu/2)\right].$$
(488)

Then we have

$$\sigma^{aa}(\nu+i\delta) = \frac{ne^2}{m}\int_{\mathbb{R}}d\alpha\frac{f(\alpha-\nu/2) - f(\alpha+\nu/2)}{\nu}\frac{1}{\bar{\tau}^{-1} - i\nu Z^{-1}}.$$
(489)

This just comes from decomposing the denominator into real and imaginary parts: the sign on one of the Σ's in τ is flipped since we flipped the sign of the $i\delta$ in its argument, and we've dropped the $i\delta$'s in the self-energy since we're taking the $\mathbb{R}$ part.

Finally, if we can ignore the α dependence of τ and Z, then we can do the frequency integral at low temperatures and get

$$\sigma^{aa}(\nu+i\delta) = \frac{ne^2}{m}\frac{1}{\bar{\tau}^{-1} - i\nu Z^{-1}},$$
(490)

since the integral of the difference of the Fermi functions just gives ν. This gives us the expected Drude form for the conductivity, and tells us that the frequency part is modified by the real part of the self energy, while the (frequency-dependent!) "scattering time" $\bar{\tau}$ is determined by the imaginary part (remember that the *real* part of σ is associated with dissipation, so the fact that $\text{Re}[\Sigma]$ contributes to $\sigma_{\mathbb{I}}$ while $\text{Im}[\Sigma]$ contributes to $\sigma_{\mathbb{R}}$ is to be expected). If we set $\nu = 0$ to go to the DC limit then we get

$$\sigma^{aa}_{DC} = \frac{ne^2\tau}{m},$$
(491)

where $\tau = 2\text{Im}[\Sigma(0-i\delta)]$ is the effective zero-frequency scattering time.

35 Efficiency of the Otto cycle

I can't believe I'm actually doing this—the only point is for preparing for the qual.

Solution:

36 Transfer matrices and quantum-to-classical heuristics

Solution:

see e.g. Mattis's stat mech book for a good discussion of this.

37 2D diatomic gas

Find the specific heat for a 2D diatomic gas as a function of temperature. Estimate transition temperatures if the gas molecule are modeled as springs of length L and spring constant k, with masses m.

38 Hacky way of getting graphene conductivity within Drude approach

Solution:

One way that gets the dependence on the parameters right is the following. First, note that since $\mathbf{j} \cdot \mathbf{E}$ is a power, electrons moving with velocity $\mathbf{v}$ for time τ in a field $\mathbf{E}$ will gain an energy equal to

$$\delta\epsilon = -\mathbf{v} \cdot \mathbf{E} e\tau. \tag{492}$$

In graphene, this means a momentum shift of (the momentum shift doesn't change the velocity, so we can do this self-consistently)

$$\delta k = e\tau E \cos\theta. \tag{493}$$

This will then tilt the circular Fermi surface by an angle $\phi \approx v\delta k$ above the horizontal in $k_x - k_y - \epsilon$ space, where the FS pivots about the $\star\mathbf{E}$ axis. When projected onto the $k_x - k_y$ plane, this means that the FSea gets displaced by an amount δk. The current is found by counting the net number of displaced states and multiplying by e times their velocities. Neglecting numerical factors that come from integrals over angles, we then have

$$j \sim ev\delta n \sim ev(\delta k)k_F \sim e^2\tau v k_F E \implies \sigma \sim e^2\tau v k_F, \tag{494}$$

since the net area of the FS that gets displaced is $\sim \delta k k_F$.

An equally hacky way is to use $\sigma = e^2\tau n/m$ for regular metals, and to come up with an appropriate way of defining n/m. The effective mass we'd assign to graphene is k-dependent, e.g.

$$\frac{1}{m_{ii}} = v\partial_{k_i}^2|\mathbf{k}| = v\frac{(\epsilon_{ij}k_j)^2}{|k|^3}. \tag{495}$$

Therefore we can estimate n/m by integrating $1/m$ over the occupied part of the Dirac cone:

$$n/m \sim v \int_0^{k_F} dk\, d\theta\, \cos^2\theta \sim v k_F, \qquad (496)$$

where we of course aren't paying attention to numerical constants. This then gives

$$\sigma \sim e^2 \tau v k_F, \qquad (497)$$

in agreement with the previous approach.

Suppose we are at half-filling, so that the Fermi energy is at zero. Then the carrier density at $T = 0$ is zero, but becomes nonzero at finite T. What's the conductivity in this case? We can use the above results by assuming that the effect of $T > 0$ is to setup an effective chemical potential of $\mu \sim T$, so that $k_F = T/v$, which gives

$$\sigma \sim e^2 \tau T, \qquad (498)$$

which is linear in T. Note that the number of thermally activated carriers goes like $k_F^2 \sim T^2$. So we have the interesting situation where the number of carriers scales like T^2, and yet the conductivity only scales like T, which is not what would happen if the conductivity just went as $\sigma \sim n$ as in normal Drude theory.