Bosonization and dualities

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October 5, 2020

1 *Setting conventions for the compact boson and bosonizaton* ✓

Today we'll be trying to set a standard vis-a-vis conventions for the compact boson field theory in 1+1D and the bosonization procedure which relates it to fermionic theories. First we'll deal just with the boson theory by itself, and later discuss the fermionic side.

Compact boson

The following action is taken to describe a compact boson at radius R:

$$S = \frac{R^2}{4\pi} \int d^2x \, d\phi \wedge \star d\phi. \tag{1}$$

Here by "radius" of the boson, we just mean the coefficient in front of the kinetic term: we are always identifying $\phi \sim \phi + 2\pi$, but not restricting ourselves to kinetic terms with coefficients of e.g. $1/2$ (we could say that we are fixing the target space to always be the same S^1, but allowing ourselves to consider different metrics on it). The Euclidean-space propagator for the ϕ fields with this normalization is

$$\langle \phi(x,t)\phi(0)\rangle \sim -\frac{1}{R^2} \ln|x-y|, \tag{2}$$

for the action above we have $-\frac{1}{2}\ln|x-y|$, while if the coefficient of the kinetic term were $1/2$ so that $R=\sqrt{2\pi}$, we would have the more familiar $-\frac{1}{2\pi}\ln|x-y|$.

Instead of working with the field ϕ, whose equation of motion gives a spectrum containing both positive and negative momentum modes, it is often helpful to work with chiral "fields" $\phi_\pm$, which have the equations of motion

$$\bar\partial\phi_+ \overset{\text{eom}}{=} 0, \qquad \partial\phi_- \overset{\text{eom}}{=} 0, \qquad \partial = \frac{\partial_t - \partial_x}{2}, \qquad \bar\partial = \frac{\partial_t + \partial_x}{2}. \tag{3}$$

This means that the mode ϕ_+ has a spectrum with modes only for positive momentum (right-moving), while ϕ_- is the opposite (unfortunately, this means in the standard conventions that ϕ_+ has *negative* chirality—fml). The full field ϕ is $\phi = \phi_+ + \phi_-$, which can be classically split up in this way just because solutions to the wave equation in 1+1D are given by $f(x-t)+g(x+t)$. It is very important to stress the fact that $\phi_\pm$ are only (anti)holomorphic *on the equations of motion*, i.e. we have $\bar\partial\phi_+(x) = \partial\phi_-(x) = 0$ only when inserted in correlation functions, and only when there are no other operators inserted at x in the correlation function: the usual Ward identity is

$$\langle \partial_\mp \phi_\pm(x)\mathcal{O}\rangle \propto \frac{\delta}{\delta\phi_\pm(x)}\langle\mathcal{O}\rangle. \tag{4}$$

Therefore it is *not* permissible to e.g. take the action $\int d\phi\wedge\star d\phi = \int \partial\phi\bar\partial\phi$ and replace it with $\int \partial\phi_+\bar\partial\phi_-$—manipulations like this lead to total nonsense.

In terms of the chiral fields, the action is instead (note the $R^2/2\pi$ in front, not $R^2/4\pi$!)

$$S = \frac{R^2}{2\pi}\int \left(-\partial_t\phi_+\partial_x\phi_+ + \partial_t\phi_-\partial_x\phi_- - (\partial_x\phi_+)^2 - (\partial_x\phi_-)^2\right). \tag{5}$$

This ensures that the equation of motions are $(\partial_t \pm \partial_x)\phi_\pm \overset{\text{eom}}{=} 0$, as required. One can also check that this gives the correct log propagators: the momentum space propagator is

$$G_\pm(p,\omega;q,\nu) = \frac{\pi}{R^2}\delta(p-q)\delta(\omega-\nu)\Theta(\pm p)\frac{1}{k(k\mp\omega)}, \tag{6}$$

(note the $\Theta(\pm p)$ that we basically put there by hand) so that

$$G_\pm(x,t) = \frac{\pi}{R^2}\int_{\mathbb{R}}\frac{dk}{2\pi}\Theta(\pm k)\frac{e^{ik(x\mp t)-a|k|}}{k} \sim -\frac{1}{2R^2}\ln\left[\frac{ia}{x\mp t+ia}\right], \tag{7}$$

where in the first step we did the contour integral over ω and in the second step we did the integral by first differentiating wrt x and then re-integrating, choosing the constant term so that $G_\pm(x,\pm t) = 0$, which is a convinient normalization. Of course, this is a pretty sick-looking correlator, but it is still useful for calculating things. The ias are needed for dimensions to work out, with a essentially being a short-distance cutoff used to regulate the theory.[1] The scaling dimensions and conformal spins of the vertex operators $e^{i\alpha\phi_\pm}$ are therefore[2]

$$\Delta_\pm = \pm s_\pm = \frac{\alpha^2}{4R^2}. \tag{9}$$

Recall that these are defined via

$$\langle\mathcal{O}^\dagger(z)\mathcal{O}(w)\rangle = \frac{1}{(z\bar z)^\Delta(z/\bar z)^s}. \tag{10}$$

[1]Sanity check: the propagator for the field $\phi \equiv \phi_+ + \phi_-$ is then

$$G_\phi(x,t) = -\frac{1}{R^2}\ln\sqrt{x^2-t^2+ia}, \tag{8}$$

which is the same thing we wrote down earlier, just in $\mathbb{R}$ time.

[2]The scaling dimension can be any nonzero number since the unitary bound on scalars is $\Delta \geq (d-2)/2$, which gives no nontrivial constraint for $d = 2$.

Now in order for an operator to be local its 2-point function must be single-valued, which means that $s_{\mathcal{O}} \in \frac{1}{2}\mathbb{Z}$ (since $z/\bar{z}$ is charge 2 under rotations). So unless $R^2 = \frac{1}{2n}$ for some $n \in \mathbb{Z}^+$, the $V_\pm =: e^{i\phi_\pm}$: vertex operators are *not* local, and in my opinion should only be thought of as calculational devices used to calculate propagators.[3] There are further problems that arise if one tries to think about the $\phi_\pm$ fields as well-defined entities in their own right, with madness really seting in quickly. For example, if we take $\phi \cong \phi + 2\pi$ literally, then if the $\phi_\pm$ are legit fields, we should also have e.g. $\phi_+ \cong \phi_+ + 2\pi$, which then says that $\theta \cong \theta + 2\pi$, which is in general a contradiction. Therefore its best to only work with vertex operators and derivatives of the fields when at all possible.

While we have written $\phi = \phi_+ + \phi_-$ and shown that ϕ has the same propagator as that of the original boson action (1), we will now justify the correspondence between the two ways of writing the action more carefully. For this it is helpful to introduce the field

$$\theta \equiv R^2(\phi_- - \phi_+). \tag{11}$$

The R^2 and perhaps unexpected minus sign is to make our lives easier in the future. If we re-write the action in terms of these variables, we get

$$S = \frac{1}{4\pi}\int \left(2\partial_x\theta\partial_t\phi - R^2(\partial_x\phi)^2 - R^{-2}(\partial_x\theta)^2\right). \tag{12}$$

The equation of motion for θ is then $\partial_x\theta \overset{\text{eom}}{=} R^2\partial_t\phi$ (this is the reason for the weird sign in the def of θ), while the eom for ϕ is $R^2\partial_x\phi \overset{\text{eom}}{=} \partial_t\theta$. Therefore

$$d\phi \overset{\text{eom}}{=} \frac{1}{R^2}\star d\theta, \tag{13}$$

where we have to remember to use the mixed-signature $\star$. If we then integrate out θ by shifting $\delta\theta = \partial_x^{-1}\partial_t\phi$, we find an action identical to (1) (in the signature $(+,-)$), establishing the correspondence between the two presentations. Also note that the eom are preserved under the duality

$$T : \phi \mapsto \frac{1}{R^2}\theta, \qquad \theta \mapsto R^2\phi, \tag{14}$$

since $\star^2 = \mathbf{1}$ acting on 1-forms in $\mathbb{R}^{1,1}$.

The commutation relations for the various fields involved all follow from the above actions. In this scheme we have $[\phi_\sigma, \phi_{\sigma'}] \propto \delta_{\sigma\sigma'}$, which is not true in some other conventions. The nonzero commutators for the chiral fields are

$$\pi_\pm = \mp\frac{R^2}{\pi}\partial_x\phi_\pm \implies [\phi_\pm(x), \phi_\pm(y)] = \pm i\frac{\pi}{2R^2}\mathrm{sgn}(x-y). \tag{15}$$

There is a factor of 2 that is a little bit subtle here—from the action we might have guessed[4] that instead $\pi_\pm = \mp(R^2/2\pi)\partial_x\phi_\pm$, but this is not correct. One way to see this is by varying

[3] Note that as long as $R^2 = m^2/(2n)$ for $m, n \in \mathbb{Z}^+$, mth powers of $V_\pm$ are local.

[4] If we assigned $\pi_\pm$ to $\delta\mathcal{L}/(\delta\partial_t\phi_\pm)$ and naively took the derivative.

the action: we get[5]

$$\delta S = \int \delta\phi_\pm \left(\mp \frac{R^2}{\pi} \partial_t \partial_x \phi_\pm - \frac{\delta H}{\delta \phi_\pm} \right), \tag{17}$$

which implies from Hamilton's equations that $\pi_\pm = \mp \frac{R^2}{\pi} \partial_x \phi_\pm$ as claimed. We can check this by requiring that $\partial_t \phi_\pm = \mp \partial_x \phi_\pm$ hold as a consequence of $\partial_t \phi_\pm = i[H, \phi_\pm]$ (of course this is just another way of doing the same calculation), which gives

$$\mp \partial_x \phi_\pm = \mp i \frac{R^2}{2\pi} [(\partial_x \phi_\pm)^2, \phi_\pm] = \mp i \frac{R^2}{\pi} [\partial_x \phi_\pm, \phi_\pm] \partial_x \phi_\pm, \tag{18}$$

giving us the desired commutator.[6] This factor of 2 is the same factor of 2 as in Chern-Simons theory (where we don't have a Hamiltonian to check) which means in that context that the momentum of the gauge field A is $k \star A/2\pi$ (with the $\star$ taken in space), instead of $k \star A/4\pi$.

The calculated commutators for $\phi_\pm$ tell us that

$$[\phi(x), \theta(y)] = -i\pi \mathrm{sgn}(x-y) \implies \pi_\phi = \frac{1}{2\pi} \partial_x \theta, \tag{20}$$

which agrees with the canonical momentum derived from the action for ϕ and θ we wrote above.

We can use these results to compute commutators of vertex operators. This works by writing, for X, Y Gaussian variables with c-number commutator,[7]

$$e^X \odot e^Y = e^X e^Y e^{-\frac{1}{2}\langle (X+Y)^2 \rangle} = e^{X+Y} e^{\frac{1}{2}[X,Y] - \frac{1}{2}\langle (X+Y)^2 \rangle} = e^Y e^X e^{[X,Y] - \frac{1}{2}\langle (X+Y)^2 \rangle} = e^Y \odot e^X e^{[X,Y]}, \tag{22}$$

where $\odot$ means "operator product" (the colons for normal-ordering look ugly!), so that $e^X \odot e^Y =: e^X :: e^Y :$ and $e^X e^Y =: e^X e^Y :$. Basically, $A \odot B$ is used to denote a product that is not fully normal-ordered, with AB denoting a single operator with normal-ordering $: AB :$.

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[5] Pedantic point here: when deriving this we need to integrate by parts. While we can't write $\int \partial_t \phi_\pm \partial_x \phi_\pm = -\int \phi_\pm \partial_t \partial_x \phi_\pm$ since $\phi_\pm$ by itself is not well-defined, we can however write

$$\int \partial_t \delta\phi_\pm \partial_x \phi_\pm = -\int \delta\phi_\pm \partial_t \partial_x \phi_\pm, \tag{16}$$

since $\delta\phi_\pm$, like $\partial\phi_\pm$, is well-defined.

[6] Yet a third way to check is to write the chiral fields as

$$\phi_\pm(x) = \pm \int_0^{\pm\infty} dp \, \frac{1}{2\pi\sqrt{|p|R^2/\pi}} (\phi_p e^{ipx} + \phi_p^\dagger e^{-ipx}), \tag{19}$$

and then to compute the commutator explicitly (by regulating it with a factor of $e^{-\eta p^2}$; the integral in the commutator then becomes $\propto \mathrm{Erf}((x-y)/\sqrt{\eta})$, which has the correct $i\pi \mathrm{sgn}(x-y)$ limit as $\eta \to 0$).

[7] We need

$$e^X \odot e^Y = e^X e^Y e^{\langle XY + \frac{1}{2}(X^2+Y^2) \rangle}, \tag{21}$$

which can be proved by writing down the series expansions and doing a bit of algebra (remember that the normal-ordering gets rid of *all* contractions between the two operators; for the vertex operators there are infinitely many such contractions to take into account).

Spectrum of local operators

All of this is fine, but very formal. It is very formal because the fields we've been manipulating, the $\phi_\pm$s and their linear combinations, aren't really well-defined. Indeed, their two-point functions are nonsensical. The fields that are well-defined are of course exponentials and derivatives of the $\phi_\pm$. In fact even exponentials of $\phi_\pm$ are problematic, since as we mentioned they are non-local for generic values of R.

When we say that the field ϕ is compact with $\phi \cong \phi + 2\pi$, what we really mean is that we restrict ourselves to only considering vertex operators for ϕ of the form $V_{n,0} = e^{in\phi}$, with $n \in \mathbb{Z}$ (the notation will become clear in a sec). That is, we take $\phi \in \mathbb{R}$ (which we did when computing correlators), but impose that all physical operators be invariant under shifting ϕ by 2π. Since the conformal spin of $V_{n,0}$ is $s_{n,0} = 0$, $V_{n,0}$ is always non-chiral, and has a well-defined two-point function. Note that this definition of compactness is *not* the same as saying that we restrict only chiral vertex operators of the form $e^{in\phi_\pm}$! We can impose $\phi \cong \phi + 2\pi$, but somewhat confusingly this is *not* the same as imposing $\phi_\pm \cong \phi_\pm + 2\pi$, despite $\phi = \phi_+ + \phi_-$. Again, it is really best to take all the fields involved here, $\phi_\pm$ and ϕ, θ, as being $\mathbb{R}$-valued, and to just place constraints on the types of vertex operators that appear in the spectrum.

So, what about vertex operators of θ? The vertex operators $e^{in\theta}$ are also non-chiral and have well-defined correlators. However, they are not generically local with respect to the $V_{n,0}$. We will find the allowed vertex operators for θ by requiring that they create self-consisent field configurations for ϕ.

We can write $e^{i\alpha\theta(x)}$ as $e^{i\alpha\int_C d\theta}$, where C is a path extending from x out to infinity. This operator is only local if correlation functions are independent of the choice of C. From the commutation relations, no θ vertex operators can detect C, but ϕ vertex operators can. When $V_{n,0}$ moves through te curve C, it picks up a phase of $e^{2\pi i\alpha n}$. Hence for $e^{i\alpha\theta}$ to be local, we need $\alpha \in \mathbb{Z}$.

Since the spectrum of the theory is generated by exponentials / derivatives of linear combinations of ϕ, θ (or $\phi_\pm$, either way), the claim is that

$$V_{n,w} = e^{in\phi + iw\theta}, \qquad (n, w) \in \mathbb{Z}^2 \tag{23}$$

generate the full spectrum of local vertex operators. As a check, we compute the OPE

$$\begin{aligned} V_{n,w}(z) \odot V_{n',w'}(w) &= \frac{V_{n+n',w+w'}}{(z-w)^{-(n-R^2w)(n'-R^2w')/2R^2}(\bar{z}-\bar{w})^{-(n+R^2w)(n'+R^2w')/2R^2}} \\ &= V_{n+n',w+w'}|z-w|^{-(nn'R^{-2}+ww'R^2)}\left(\frac{z-w}{\bar{z}-\bar{w}}\right)^{\frac{1}{2}(wn'+w'n)}. \end{aligned} \tag{24}$$

This OPE is evidently only well-defined provided that $wn' + w'n \in \mathbb{Z}$, and so indeed by taking $(n, w) \in \mathbb{Z}^2$, the operators $V_{n,w}$ are always well-defined local operators with sensible OPEs (and since we can have $wn' + w'n$ be the minimal value of 1, they generate all such local vertex operators).[8] If we like, we can say that in this theory, both ϕ and θ are to be treated as 2π-periodic variables.

[8] Note that while $V_{n,0} \odot V_{m,0} = V_{n+m,0}$, $V_{n,w} \neq V_{n,0} \odot V_{0,w}$: instead, we have

$$V_{n,0} \odot V_{0,w} \sim V_{n,w}(\varepsilon/\bar{\varepsilon})^{nw/2}, \tag{25}$$

From the above OPE, we read off

$$\Delta_{n,w} = \frac{1}{2}(n^2/R^2 + w^2R^2), \qquad s_{n,w} = -nw, \tag{26}$$

with $s \in \mathbb{Z}$ as required. Note that the spin is independent of R, essentially by construction. T-duality acts as $V_{n,w} \mapsto V_{w,n}$, $R \mapsto R^{-1}$, as expected.

As mentioned above, for some values of R the spectrum includes operators that are genuinely chiral, with $\Delta_{n,w} = \pm s_{n,w}$. For this to be the case, we need to have $n/R = wR$ for some n, w. This means that we must have $R^2 = n/w$, and so we only have chiral operators when $R^2 \in \mathbb{Q}$. Thus only for rational values of R^2 do there exist local operators that are exponentials only of either ϕ_+ or ϕ_-.

Bosonizing bosons, aka where the compact boson action comes from

ethan: note: this whole strategy seems just like the usual expansion about a symmetry-broken state that gets used when we're near the lower critical dimension. So there won't be an interacting fixed point but there would be if we were in $2+\varepsilon$ dimensions. Should expand on this. Near the upper critical dimension... as d goes to the lower critical dimension T_c vanishes, and so near the lower cd we are prompted to expand about a putative symmetry-broken state. This is exactly what we do when doing RG for σ models in 1+1D; see other diary entries for details.

In this subsection we use a cond-mat hydrodynamically-flavored line of reasoning to explain the ubiquity of the compact boson theory in one-dimensional problems. *ethan: need to elaborate on this! The whole thing is hydro! So, should first start with a discussion of the two $U(1)$ symmetries, and then describe how the EFT approach works*

The basic starting point is that of a system of bosons interacting with a hard-core repulsion:

$$H = \frac{1}{2}\int dx\, \left(\frac{1}{m}|\partial_x\psi|^2 + V\rho^2\right), \tag{27}$$

where ρ is the density operator.

The limit of weak interactions, where the system is close to a superfluid, is easy to deal with. Indeed we know that we have to get the action of a compact boson in the IR, since this is the action describing the "Goldstone modes" of the "broken" symmetry. This action can just be written down on phenomenological grounds, with the form of the parameters in the action fixed using common sense. But we want to do a bit better, viz. we want to provide a slightly more explicit mapping of the boson operators to the operators appearing in the "Goldstone mode" action, and relate the parameters in this action explicitly to V and the SF density. To this end, we write $\psi = \sqrt{\rho}e^{i\phi}$ and drop the fluctuations in ρ; this gives

$$H = \frac{1}{2}\int dx\, \left(K(\partial_x\phi)^2 + V\rho^2\right) \tag{28}$$

where $K = \rho_0/m = \langle\rho\rangle/m$. Now from $[\rho(x), e^{i\phi(y)}] = \delta(x-y)e^{i\phi(x)}$, we can introduce a field θ such that $\rho = \frac{1}{2\pi}\partial_x\theta$ (this is just for suggestive notation as we pass from H to S; we will

where ε is a point-splitting distance (the $\sim$ is because there's another numerical factor coming from combining the exponentials between $e^{in\phi}e^{iw\theta}$ and $e^{in\phi+iw\theta}$). Since this depends on the choice of ε, $V_{n,w}$ cannot be split-up as a product of operators in a well-defined way.

just be integrating θ out in any case — but writing it as $\partial_x\theta$ shows us that the fluctuations in ρ can indeed be neglected since they are irrelevant under RG). Therefore we can write the action as[9]

$$S = \int \left(\frac{1}{2\pi}\partial_x\theta\partial_t\phi - \frac{K}{2}(\partial_x\phi)^2 - \frac{V}{8\pi^2}(\partial_x\theta)^2 \right). \tag{29}$$

Now integrating out θ,

$$\begin{aligned} S &= \frac{1}{2}\int \left(\frac{1}{V}(\partial_t\phi)^2 - K(\partial_x\phi)^2 \right) = \frac{R^2}{4\pi}\int \left(\frac{1}{v}(\partial_t\phi)^2 - v(\partial_x\phi)^2 \right) \\ R &= \sqrt{2\pi}\left(\frac{K}{V}\right)^{1/4}, \qquad v = \sqrt{KV}. \end{aligned} \tag{30}$$

The near-SF is thus equivalent to a compact boson in the large-radius limit. The extent to which this system fails to be a SF is measured by the correlation function

$$\langle V_{1,0}(x)V_{-1,0}(y)\rangle \sim \frac{1}{|x-y|^{\frac{1}{2\pi}\sqrt{V/K}}}. \tag{31}$$

So of course we do not actually have a SF because of the algebraic falloff, but the power of the decay is arbitrarily small in the weakly interacting $K/V \to \infty$ limit.

In the limit of strong (repulsive) interactions, the natural starting point is the (Wigner) crystal. Again from a phenomenological point of view we know that the IR action has to be that of a compact boson (the "Goldstone" for translation); the only nontrivial part is finding the radius of the compact boson in terms of the interaction strength and the lattice spacing of the crystal, and providing a mapping between the boson operators and those of the IR action.

The phenomenological approach works like this: let us introduce a dimensionless field θ which measures the displacements of the atoms from their equilibrium position, with a shift in θ of 2π corresponding to a translation of all atoms in the crystal by one lattice constant a. That is, we let $x = x_0 + a\theta/2\pi$, where x is the position operator and x_0 a series of delta functions at the lattice sites. From this we see that for small fluctuations of the lattice, $\partial_x\theta$ keeps track of the fluctuations of the equilibrium number density through

$$\rho = \rho_0 + \frac{1}{2\pi}\partial_x\theta. \tag{32}$$

Then the phenomenological Goldstone action is, taking the IR limit of $\sum_i \left(\frac{1}{2}m(\partial_t\delta x)^2 + \frac{1}{2}Va(\delta\rho)^2\right)$,

$$S = \frac{1}{2}\int \left[\frac{ma}{\pi^2}(\partial_t\theta)^2 - \frac{V}{4\pi^2}(\partial_x\theta)^2 \right] = \frac{R^2}{4\pi}\int \left(\frac{1}{v}(\partial_t\theta)^2 - v(\partial_x\theta)^2 \right), \tag{33}$$

where now $R = \pi^{-1/2}(maV)^{1/4}$ and $v = \sqrt{V/4ma}$ (the latter is $\sqrt{T/\mu}$ since V has dimensions of energy times length). The extent to which this system fails to be a crystal is determined

[9] The time derivative term here is as usual fixed in accordance with the commutation relations that we have imposed. It can also be derived from the non-relativistic term $i\psi^\dagger\partial_t\psi$ present in the action for the original bose fields.

by the correlation functions of the wavevector $k = 2\pi/a$ component of the number density. To be notationally suggestive, we will define the momentum k_F by

$$2k_F \equiv \frac{2\pi}{a} = 2\pi\rho_0. \tag{34}$$

Now we can translate the whole crystal by one lattice spacing by changing θ by 2π. Under this change, the $\mathbb{R}$-space density of the $2k_F$ wavevector part of the density, viz. ρ_{2k_F}, has a phase change of 2π, and hence the relation between the two must be[10]

$$\rho_{2k_F}(x) \propto e^{i(2\pi\rho_0 x+\theta)}. \tag{35}$$

Therefore we have

$$\langle \rho_{2k_F}(x)^\dagger \rho_{2k_F}(y)\rangle \sim \frac{1}{|x-y|^{\pi/\sqrt{maV}}}, \tag{36}$$

which is algebraically decaying but with a power that is arbitrarily small in the strongly-interacting $maV \to \infty$ crystalline limit (maV has dimensions of $\hbar^2$ and so is properly dimensionless).

Now we will use a more explicit operator mapping to derive the compact boson action. Since we are coming from the starting point of a crystal, we want to do the mapping in a subspace where the density operator ρ is a sum of integer-weight delta functions. This will be the case if the combination $2\pi\rho_0 x+\theta$ is constrained to take values only in $2\pi\mathbb{Z}$, since then $\rho = \partial_x(\rho_0 x + \theta/2\pi)$ will be an appropriate sum of delta functions. Now working explicitly with a discontinuous field like this is of course a pain, and in any case we will eventually want to relax this constraint. Therefore we will incorporate the constraint on ρ by adding in an appropriate delta function:

$$\rho = (\rho_0 + \partial_x\theta/2\pi)\sum_{n\in\mathbb{Z}} e^{in(2\pi\rho_0 x+\theta)}. \tag{37}$$

The commutation relations between $\partial_x\theta$ and ϕ are fixed by $[\rho(x), e^{i\phi(y)}] = \delta(x-y)e^{i\phi(x)} \implies [\partial_x\theta(x)/2\pi, \phi(y)] = -i\delta(x-y)$, which gives us the expected momentum for ϕ.

To write the mapping of the boson field ψ, we need to take $\sqrt{\rho}$. But this is actually straightforward, since the square root of a sum of delta functions is just proportional to the same sum of delta functions. Therefore we can write

$$\psi \sim \sqrt{\rho_0 + \partial_x\theta/2\pi}\sum_{n\in\mathbb{Z}} e^{in(2\pi\rho_0 x+\theta)}e^{i\phi}. \tag{38}$$

Therefore we see how the spectrum of the compact boson theory comes out of the original model: all operators built from polynomials in the ψ fields are manifestly given by vertex operators (plus derivatives of θ, ϕ). Furthermore we expect that for IR questions we can soften the constraint on ρ by dropping most of the terms appearing in the sum which enforce the discreteness constraint, since terms with larger n oscillate more quickly in $\mathbb{R}$-space by an amount given by the UV scale $\rho_0 = 1/a$.

[10] Note that the $2k_F$ component of the density is related to a vertex operator of θ, while the zero-momentum component is related to $\partial_x\theta$.

Anyway, now we put this relation into the boson Hamiltonian. The first term $|\partial_x\psi|^2$ is rather complicated—it involves the simple $\rho_0(\partial_x\phi)^2+\dots$ (where $\dots$ are higher in derivatives and hence irrelevant), but also the complicated $(\partial_x\sqrt{\rho})^2$. However, one sees that all the terms in the expansion of $(\partial_x\sqrt{\rho})^2$ are actually all irrelevant, since they are all of the form $(\partial_x^2\theta)^2$ or $(\partial_x\theta)^2P(\cos\theta,\sin\theta)$, where Ps are polynomials in various cosines and sines of θ. Therefore we can drop all the terms in the $|\partial_x\psi|^2$ term except for the gradient term for ϕ. Similarly the interaction term $V(\delta\rho)^2$ just becomes $\propto V(\partial_x\theta)^2$ after dropping irrelevant terms, and so the Hamiltonian is

$$H=\frac{1}{2}\int dx\,\left(\frac{\rho_0}{m}(\partial_x\phi)^2+\frac{V}{4\pi^2}(\partial_x\theta)^2\right). \tag{39}$$

If we then integrate out ϕ to get an action just in terms of θ, we get the same compact boson action for θ as above, with the same radius.

In the general case where we are somewhere between a SF and a crystal, we simply map the boson operators with an extrapolation between their images in the two limits:

$$\psi\sim\sqrt{\rho_0+\partial_x\theta/2\pi}\sum_{n\in\mathbb{Z}}U_n e^{in(2\pi\rho_0 x+\theta)}e^{i\phi}, \tag{40}$$

where the U_n are phenomenological coefficients. When we are close to a SF all the U_n are nearly zero expect for U_0, while when we are close to a crystal the U_n are nearly independent of n.

Bosonization: field theory approach

First we will discuss a field-theory flavored way of motivating bosonization. This has the advantage of being rather clean and easy to work with, but the disadvantage of being slightly subtle once interactions are added and of having the overall appearance of a magic trick (and bosonization is not magic).

The strategy in the field theory approach is to "rigorously" establish the mapping for the case of free fermions, and then make a rather sketchy argument about the generalization to the interacting case.

The compact boson theory discussed in the last section is, of course, a bosonic theory: the spectrum of operators, viz. $\{V_{n,w},d\theta,d\phi\}$, are all bosonic (and the derivatives can be obtained from the vertex operators by taking OPEs).[11] To get a fermionic theory, we have to generalize slightly. From the fact that $s_{n,w}=-nw$, we see that all we have to do is to generalize the operator algebra to e.g. include operators either with $n\in\frac{1}{2}\mathbb{Z}$ or $w\in\frac{1}{2}\mathbb{Z}$ (but not both). Taking one of n,w to be fractional effectively attaches a JW string branch cut to the vertex operator, and provides the commutation relations we expect from a fermion. The choices of whether we allow for fractional n or w are related by T-duality, and correspond to whether we want the free fermions to occur at $R=1/\sqrt{2}$ or $R=\sqrt{2}$. Therefore we can study fermionic theories by looking the theory whose spectrum is generated by the operators $V_{n,\frac{1}{2}w}$, for $n,w\in\mathbb{Z}$.

[11] We are working on $\mathbb{R}^2$ throughout, and hence are not caring about global issues like spin structure dependence.

Unfortunately, it turns out that this notation makes a bunch of formulas that appear later rife with ugly factors of 2. We will therefore introduce the variables

$$\Phi \equiv \phi, \qquad \Theta \equiv \frac{R^2}{2}(\phi_- - \phi_+) = \frac{\theta}{2}, \qquad \pi_\Phi = \frac{1}{\pi}\partial_x \Theta \tag{41}$$

Here Φ is introduced just to make the notation look slightly more visually pleasing. We then define the vertex operators

$$\mathcal{V}_{n,w} = V_{n,w/2} = e^{i(n\Phi + w\Theta)} = e^{i\phi_+(n - R^2 w/2) + i\phi_-(n + R^2 w/2)}, \tag{42}$$

which, for the free action (1), have scaling dimensions and spins given by

$$\Delta_{n,w} = \frac{1}{2}\left(\frac{n^2}{R^2} + w^2 \frac{R^2}{4}\right), \qquad s_{n,w} = -nw/2. \tag{43}$$

The space of vertex operators is thus still obtained from two compact fields, each still with periodicity 2π,[12] except now the vertex operators can be fermionically nonlocal.

From our discussion of the compact boson, we see that at $R = \sqrt{2}$, the chiral fields[13]

$$V_\pm = e^{2i\phi_\pm} = \mathcal{V}_{1,\mp 1} \tag{44}$$

are well-defined in the sense that their two-point functions are single-valued and have the same correlation functions as free fermions (as well as the same self-anti-commutation relations as fermions, so they are only local to the extent that fermions are local). This means that all correlation functions of the $\psi_{L/R}$ fields calculated with the free Dirac action will be identical to those calculated with the vertex operators $e^{i\phi_\pm}$ in a compact boson theory at $R = \sqrt{2}$.

We will thus write the bosonization map *for free fermions* as[14]

$$\begin{aligned} \mathcal{B}[\psi_R] &= \gamma_R \mathcal{V}_{1,-1} = \frac{\gamma_R}{\sqrt{a}} e^{i(\Phi - \Theta)} = \frac{\gamma_R}{\sqrt{a}} e^{i2\phi_+}, \\ \mathcal{B}[\psi_L] &= \gamma_L \mathcal{V}_{1,1} = \frac{\gamma_L}{\sqrt{a}} e^{i(\Phi + \Theta)} = \frac{\gamma_L}{\sqrt{a}} e^{i2\phi_-} \end{aligned} \tag{45}$$

where a is a UV cutoff needed to get the dimensions correct, which until now we have been hiding in the implicit normal-ordering of the vertex operators, and where γ_σ are Klein factors (Majorana fermions) needed so that $\mathcal{B}[\psi_R]$ anticommutes with $\mathcal{B}[\psi_L]$ in our quantization scheme. From now on, the γ_σs and the $(\pi a)^{-1/2}$s will only be written out when needed.

Note that the translation $U(1)_T$ and particle-number $U(1)_N$ symmetries act on the fermions as $U(1)_T : \psi_{L/R} \mapsto e^{\mp i\rho\delta x/2}\psi_{L/R}$ and $U(1)_N : \psi_{L/R} \mapsto e^{i\alpha}\psi_{L/R}$, where $\rho = 2k_F$ is the density. Hence on Φ, Θ we have (not writing out as)

$$U(1)_T : \Phi \mapsto \Phi, \quad \Theta \mapsto \Theta - \frac{\rho}{2}\delta x, \qquad U(1)_N : \Phi \mapsto \Phi + \alpha, \quad \Theta \mapsto \Theta. \tag{46}$$

[12] If we like, we could stick with the old ϕ, θ notation and just say that we are increasing the periodicity condition on θ to $\theta \sim \theta + 4\pi\mathbb{Z}$.

[13] The factor of 2 in the exponent is an unavoidable causality of our notation—this seemed like the least annoying place for factors of 2 to live, so we'll just deal with it.

[14] When the coefficient in front of the kinetic term for the action (1) is normalized to be 1/2, which is another popular choice, the fermions are $e^{\pm i\sqrt{4\pi}\phi_\pm}$.

The factor of 1/2 in the $U(1)_T$ transformation of Θ means that if we translate by $\delta x = 2\pi/\rho$, which is the distance over which we expect to find one fermion, Θ shifts by π, which is non-trivial. This tells us that $e^{i\Theta}$ counts fermion parity, a conclusion which we will confirm shortly. From the commutation relations above the generators of the two $U(1)$s are

$$Q_T = -\frac{\rho}{2\pi}\int d\Phi, \qquad Q_N = \frac{1}{\pi}\int d\Theta. \tag{47}$$

The mixed anomaly between the two $U(1)$s then can be understood from the commutation relations of the above charge densities; since this is done in another diary entry we won't go in to further detail.

We will find it convenient to define the fields

$$\varphi_R = \Phi - \Theta, \qquad \varphi_L = \Phi + \Theta, \qquad [\varphi_{R/L}(x), \varphi_{R/L}(y)] = \pm\frac{2\pi i}{R^2}\mathrm{sgn}(x-y). \tag{48}$$

At the free fermion radius $R = \sqrt{2}$ we have $\varphi_{R/L} = 2\phi_\pm$, but this is not true for general R (in particular, the $\varphi_{R/L}$ are *not* chiral at generic radii). In terms of these fields then,[15]

$$\mathcal{B}[\psi_{L/R}] = \frac{1}{\sqrt{a}}e^{i\varphi_{L/R}}, \tag{49}$$

The bosonization map is the statement that the two actions

$$\mathcal{B} : \frac{1}{2\pi}\int \bar\psi i\partial\!\!\!/\psi \leftrightarrow \frac{1}{4\pi}\int \sum_{\sigma=L,R}\left((-1)^\sigma \partial_t\varphi_\sigma\partial_x\varphi_\sigma - v\partial_x\varphi_\sigma\partial_x\varphi_\sigma\right), \tag{50}$$

generate the same correlation functions, where $(-1)^\sigma$ is -1 for $\sigma = R$ and $+1$ for $\sigma = L$, and v is the velocity of the dirac fermions. The RHS is the same as (5) since we are at $R = \sqrt{2}$. In terms of the Φ, Θ fields, we may write

$$\mathcal{B} : \frac{1}{2\pi}\int \bar\psi i\partial\!\!\!/\psi \leftrightarrow \frac{1}{2\pi}\int \left(2\partial_t\Phi\partial_x\Theta - v(\partial_x\Phi)^2 - v(\partial_x\Theta)^2\right). \tag{51}$$

The statement here is again that these two actions generate the same correlation functions provided we identify operators using $\mathcal{B}$. That is, since $\psi_{L/R}$ has the same correlation functions as $e^{i\phi_{L/R}} = e^{i\phi_\pm}$ in the free theory, the claim is that

$$\langle \mathcal{O}[\psi]\rangle_{\frac{1}{2\pi}\bar\psi i\partial\!\!\!/\psi} = \langle \mathcal{O}[\mathcal{B}[\psi]]\rangle_{R=\sqrt{2}}, \tag{52}$$

where $\mathcal{O}[\psi]$ is any polynomial of ψ fields at arbitrary positions. The claim is that the spectrum of operators $\mathcal{V}_{n,w}$ (and their derivatives) exhaust all operators in the fermion theory. It's clear that we get all polynomials of the fermions by taking products of the $\mathcal{V}_{1,\pm1}$s—the

[15] The $\pm$ sign in the exponent is convention: many times it is instead written as $e^{\pm i\phi_\pm}$. Changing these conventions, which amounts to mapping $\psi_L \leftrightarrow \psi_L^\dagger$, simply swaps the physical interpretations of the ϕ and θ via T-duality. In the present conventions, Φ is a phase variable (Φ getting a vev is a "SF"), while Θ is a density variable (Θ getting a vev is a "crystal").

operators with n and / or w odd have less obvious fermionic counterparts; we will see in a sec that they are related to $(-1)^F$ operators.

Just to make the claim about the matching of correlation functions in the free theory completely explicit, we know that in the fermionic theory we have (looking at e.g. the R fermions wolog)

$$\langle \psi_R(x_1) \dots \psi_R(x_n) \psi_R^\dagger(y_1) \dots \psi_R^\dagger(y_n) \rangle = \det \left(\frac{1}{x_i - y_j} \right). \tag{53}$$

On the other hand, the vertex operators give (with the implicit normal-ordering eliminating the $i = j$ terms)

$$\langle e^{i\varphi_R(x_1)} \dots e^{i\varphi_R(x_n)} e^{-i\varphi_R(y_1)} \dots e^{-i\varphi_R(y_n)} \rangle = \frac{\prod_{i<j<n}(x_i - x_j) \prod_{i<j\leq n}(y_i - y_j)}{\prod_{i<j\leq n}(x_i - y_j)}. \tag{54}$$

This is indeed exactly equal to the determinant; one can show this e.g. by looking at the zeros and the poles: both functions have poles when some x_i equals some y_j, and both have zeros when two x's or two y's are coincident (since then the matrix in the determinant becomes degenerate).

Interactions are dealt with by expanding the exponential $e^{iS_{int}}$ as a bunch of correlation functions:

$$\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, e^{i(S+S_{int})} = \int \mathcal{D}\phi\, \mathcal{D}\theta\, \exp\left(iS_{R=\sqrt{2}}[\phi, \theta] + i\mathcal{B}[S_{int}] \right). \tag{55}$$

In my opinion what we've done so far is rigorous (for physicists). The formula for $\mathcal{B}[\psi_{L/R}]$ above is useful in that tells us that the parition functions in the two theories above are identical, and furthermore provides us with a may of matching correlation functions in the two free theories.

Bosonization dictionary

In accordance with the above discussion, we need to figure out how to bosonize products of $\psi_{L/R}$ fields in the free theory. We do this by resolving products of operators by point-splitting in the usual way. We will point-split by displacing the operators in space, since this is most convenient for employing the commutation relations when doing calculations.

We will need to bosonize some operators that take the form of normal-ordered products / derivatives of fermion operators. However, but our bosonization map as written above only works on the constituent fermions themselves, since they are the fields whose correlation functions are matched on the boson side. So in order to map more complicated operators we un-normal-order them and express them in terms of the $\psi_{L/R}$, then bosonize by using the fact that the bosonization map is a homomorphism

$$\mathcal{B}[\mathcal{O}_1 \odot \mathcal{O}_2] = \mathcal{B}[\mathcal{O}_1] \odot \mathcal{B}[\mathcal{O}_2] \tag{56}$$

for $\mathcal{O}_i$ any single-fermion operators, and finally re-write things in terms of normal-ordered products to find the image of the given operator under bosonization (also remember that Taylor expansions can only be performed *inside* the normal-ordering symbol, at the very last step).

For example, let us consider the R fermion density. We first need to remember that

$$G_{L/R}(x,t) = \frac{i}{t \mp x + ia}, \tag{57}$$

where the $+ia$ convergence factor usually won't we written. The fact that the $\pm$ sign appears on the x and not the t is important for some calculations, so we will try to keep track of it correctly.[16] give The i here because the propagator comes from inverting $i^2\partial\!\!\!/$, not $i\partial\!\!\!/$.[17] Also note that the $1/2\pi$ in front of the fermion action means there's no 2π in the above propagator. Anyway, we can now write

$$\begin{aligned}(\psi_R^\dagger \psi_R)(z) &= \lim_{\varepsilon\to 0}\left(\psi_R^\dagger(z+\varepsilon)\odot\psi_R(z) - \frac{i}{\varepsilon}\right) \\ &\to \lim_{\varepsilon\to 0}\left(e^{-i\varphi_R(z_\varepsilon)}\odot e^{i\varphi_R(z)} - \frac{i}{\varepsilon}\right) \\ &= \lim_{\varepsilon\to 0}\left(e^{-i\varepsilon\partial_x\varphi_R(z)+\dots}\frac{i}{\varepsilon} - \frac{i}{\varepsilon}\right) \\ &= \lim_{\varepsilon\to 0}\left([1-\varepsilon(i\partial_x\varphi_R)(z)]\frac{i}{\varepsilon} - \frac{i}{\varepsilon}\right) \\ &= \partial_x\varphi_R(z),\end{aligned} \tag{58}$$

where the i when combining the two vertex operators into a single exponential comes from the BCH-formula phase of $e^{\frac{1}{2}[\varphi_R(z+\varepsilon),\varphi_R(z)]} = e^{i\pi\mathrm{sgn}(\varepsilon)/2} = i$, provided that we point-split in the "correct" way (recall we are at $R=\sqrt{2}$). Of course this feels pretty arbitrary (and I'm going to stop paying attention to signs too carefully at this point so as to retain my sanity), and this is one of the reasons why the field theory approach is a bit annoying.

A similar calculation for the L fields gives the opposite sign[18]

$$\mathcal{B}[\psi_{R/L}^\dagger(x)\psi_{R/L}(x)] = \pm\partial_x\varphi_{R/L}(x) = \pm\partial_x\Phi \mp \partial_x\Theta. \tag{59}$$

Therefore the currents map as[19]

$$\mathcal{B}[2\pi j^\mu] = \mathcal{B}[(\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R, \psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L)^\mu] = 2(\partial_x\Theta, \partial_x\Phi)^\mu. \tag{61}$$

[16]The easy way to remember this is that in the Dirac action, the derivatives appear as $\partial_t \pm v\partial_x$. It is checked by Fourier transforming with the Feynman propagator (i.e. with $i\varepsilon\mathrm{sgn}(k)$ in the denominator): $\langle\psi_{L/R}(k)\psi_{L/R}(k)^\dagger\rangle \propto \pm\Theta(k)e^{-k0^+}$, which gives the desired result.

[17]There are various conventions for this, but ours is the one in which $G_R(z-w) = \langle\psi_R(z)\psi_R^\dagger(w)\rangle$, with no factor of i.

[18]There's a minus sign from the fact that φ_Ls have an opposite sign in the commutator when re-combining the exponentials.

[19]Note that the spatial part of the current here is $n_R - n_L$; in other parts of the diary it's been the other way around. Also note that with these conventions, because of the $1/2\pi$ in the fermion action,

$$n_\sigma = \frac{1}{2\pi}\psi_\sigma^\dagger\psi_\sigma. \tag{60}$$

Since $d\Phi \overset{\text{eom}}{=} R^{-2} \star d\theta = 2R^{-2} \star d\Theta$ and we are at $R = \sqrt{2}$, we have (making the replacement on the equations of motion here is exact since both Φ and Θ appear quadtratically in the action — if cosines were added to the action then manipulations like this would not be legit)

$$\mathcal{B}[j] \overset{\text{eom}}{=} \frac{1}{\pi} \star d\Theta \overset{\text{eom}}{=} \frac{1}{\pi} d\Phi. \tag{62}$$

In particular, the density is (this is exact, not just on the eom)

$$\mathcal{B}[j^0] = \frac{1}{\pi}\partial_x \Theta \tag{63}$$

Therefore the operator $e^{i\Theta(x)}$ counts the fermion parity to the left (or right) of x[20]:

$$e^{i\Theta(x)} = e^{i\pi \int^x j^0}. \tag{64}$$

We thus have the physical interpretation of Θ as the field which counts the fermion density, which shifts by $\delta\Theta = \pi$ at the location of a fermion (θ shifts by $\delta\theta = 2\pi$ at each fermion). Therefore in a Euclidean time picture, fermion-odd operators insert π vortices in Θ. Note that the Thirring interaction bosonizes as $\mathcal{B}[j_\mu j^\mu] = -\frac{1}{\pi^2}\partial_x\varphi_R\partial_x\varphi_L$.

The off-diagonal bilinears are easy, since the OPE is trivial:

$$\mathcal{B}[\psi_L(x)\psi_R^\dagger(x)] = e^{i2\Theta(x)}, \qquad \mathcal{B}[\psi_L(x)\psi_R(x)] = e^{i2\Phi(x)}. \tag{65}$$

Finally for the bosonization of kinetic terms for the fermions. Since

$$\langle \psi_R^\dagger(z)(\partial_w \psi_R(w))\rangle = -i\frac{1}{(z-w)^2}, \tag{66}$$

we have, focusing on the ∂_x term for concreteness, (it's better to get rid of the derivative first by point-splitting and then bosonize rather than the other way around)

$$\begin{aligned} \psi_R^\dagger \partial_x \psi_R &= \lim_{\epsilon\to 0}\left(\psi_R^\dagger(z)\frac{\psi_R(z+\varepsilon)-\psi_R(z-\varepsilon)}{2\varepsilon} + \frac{i}{\epsilon^2}\right) \\ &\to \lim_{\epsilon\to 0}\left(e^{-i\varphi_R(z)} \odot \frac{1}{2\varepsilon}\left(e^{i\varphi_R(z+\varepsilon)} - e^{i\varphi_R(z-\varepsilon)}\right) + \frac{i}{\epsilon^2}\right) \end{aligned} \tag{67}$$

The RHS is, remembering the is coming from recombining the exponentials,

$$\frac{i}{2\varepsilon}e^{-i\varphi_R(z)+i\varphi_R(z+\varepsilon)}\frac{1}{-\varepsilon} - \frac{i}{2\varepsilon}e^{-i\varphi_R(z)+i\varphi_R(z-\varepsilon)}\frac{1}{+\varepsilon} + \frac{i}{\varepsilon} \approx \frac{-i}{2\varepsilon^2}\left(2 + i\varepsilon^2\partial_x^2\varphi_R - \varepsilon^2(\partial_x\varphi_R)^2\right) + \frac{i}{\varepsilon^2}, \tag{68}$$

where we have expanded the exponentials to $O(\varepsilon^2)$. Up to total derivatives, this just gives $i\frac{1}{2}(\partial_x\varphi_R)^2$, and therefore

$$\mathcal{B}[\psi_+^\dagger i\partial_x\psi_+] = -\frac{1}{2}(\partial_x\varphi_R)^2. \tag{69}$$

[20]This nice factor-of-2-free result, which vibes nicely with the conventions in part of the CMT literature is our reason for choosing to work with the $V_{n,w/2}$ vertex operators.

This gets us part of the kinetic term. The rest of the kinetic term comes from the other derivative of ψ_+ and the derivatives of ψ_- in the similar way. For $\psi_+^\dagger i\partial_t\psi_+$ we just get $-\partial_t\varphi_R\partial_x\varphi_R/2$, while for the ψ_- terms the ∂_t term has opposite sign (from the opposite sign in the commutation relations when combining the exponential) while the ∂_x term has the same sign (since another minus sign in the fermion action cancels this sign). One then checks that the two kinetic terms indeed map into one another, and that even the coefficients are correct!

Adding interactions

As a simple example, consider adding the term

$$S_{int} = -\frac{1}{4\pi}\int U_{\alpha\beta}\rho_\alpha\rho_\beta, \tag{70}$$

where $\rho_\alpha = \psi_\alpha^\dagger\psi_\alpha = 2\pi n_\alpha$. The off-diagonal part $\rho_L\rho_R$ is a $j_\mu j^\mu$ Thirring-type interaction (viz. $\frac{\pi}{2}U_{LR}j_\mu j^\mu$), while the forward scattering terms $U_{\sigma\sigma}$ will be seen to renormalize the velocities. Indeed, the bosonized version of this is

$$\begin{aligned} S_b &= \frac{1}{4\pi}\int\left(\sum_\sigma(-1)^\sigma\partial_t\varphi_\sigma\partial_x\varphi_\sigma - \sum_\sigma(\partial_x\varphi_\sigma)^2(v_\sigma+U_{\sigma\sigma}) + 2U_{LR}\partial_x\varphi_L\partial_x\varphi_R\right) \\ &= \frac{1}{4\pi}\int\left(-\partial_t\varphi^T Z\partial_x\varphi - \partial_x\varphi^T\mathcal{H}\partial_x\varphi\right), \qquad \mathcal{H} = \begin{pmatrix} v'_R & -U_{LR} \\ -U_{LR} & v'_L\end{pmatrix}, \end{aligned} \tag{71}$$

where the renormalized velocities are $v'_\sigma \equiv v_\sigma + U_{\sigma\sigma}$ and $\varphi = (\varphi_R, \varphi_L)^T$. We can calculate the OPE of the vertex operators by diagonalizing the Hamiltonian. We will preserve the commutation relations (first term in the action) if we can diagonalize $\mathcal{H}$ with something in $SO(1,1)$, i.e. a matrix of the form $M = \mathbf{1}\cosh\psi + X\sinh\psi$. This can always be done if $\mathcal{H}$ is positive-definite, which we of course assume on physical grounds. A bit of algebra (in the diary entry on correlators in Luttinger liquids) shows that

$$M^T\mathcal{H}M = \begin{pmatrix} -U_{LR}\sinh(2\psi) + v'_R\cosh^2\psi + v'_L\sinh^2\psi & 0 \\ 0 & -U_{LR}\sinh(2\psi) + v'_L\cosh^2\psi + v'_R\sinh^2\psi\end{pmatrix}, \tag{72}$$

provided that

$$\tanh(2\psi) = \frac{U_{LR}}{(v'_R + v'_L)/2}. \tag{73}$$

This is always possible if

$$|U_{LR}| < \frac{v'_R + v'_L}{2}. \tag{74}$$

Now on the other hand, the condition that $\mathcal{H}$ be positive-definite can be checked to be that $|U_{LR}| < \sqrt{v'_R v'_L}$. Since the geometric mean is always $\leq$ the arithmetic mean, $\mathcal{H}$ being positive-definite automatically guarantees that there's a boost M diagonalizing it. This stability condition (that $|U_{LR}| < \sqrt{v'_L v'_R}$ is less obvious on the fermion side (also note that because the condition is only on $|U_{LR}|$, the fermions are [equally] unstable to both attractive *and* repulsive Thirring-type interactions). Also note that when $v_L = v_R$ so that $v'_R = v'_L \equiv v'$, we get (after using e.g. $\cosh(\tanh^{-1}(x)) = (1-x^2)^{-1/2}$)

$$M^T \mathcal{H} M = \widetilde{v}\mathbf{1}, \qquad \widetilde{v} = v'/\gamma, \qquad M = \gamma \begin{pmatrix} 1 & -U_{LR}/v' \\ -U_{LR}/v' & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1 - U_{LR}^2/v'^2}}, \tag{75}$$

which is exactly what we expect from a Lorentz boost.

To deal with the more general situation (just for fun), define

$$\bar{v} \equiv (v'_R + v'_L)/2, \qquad \delta \equiv (v'_R - v'_L)/2. \qquad \gamma \equiv (1 - U_{LR}^2/\bar{v}^2)^{-1/2}. \tag{76}$$

Then we have

$$\cosh^2\psi = \frac{\gamma+1}{2}, \qquad \sinh^2\psi = \frac{\gamma-1}{2}, \qquad \sinh(2\psi) = \gamma U_{LR}/\bar{v}, \tag{77}$$

which allows us to write

$$M^T \mathcal{H} M = \bar{v} \begin{pmatrix} \gamma^{-1} + \delta/\bar{v} & \\ & \gamma^{-1} - \delta/\bar{v} \end{pmatrix} \equiv \begin{pmatrix} \widetilde{v}_R & \\ & \widetilde{v}_L \end{pmatrix}. \tag{78}$$

In terms of the Φ, Θ fields, (sanity check: the correct momentum for Φ is recovered) some algebra gives

$$S_b = \frac{1}{2\pi} \int \left(2\partial_t\Phi\partial_x\Theta - (\partial_x\Phi)^2(\bar{v} - U_{LR}) - (\partial_x\Theta)^2(\bar{v} + U_{LR}) + 2\delta\partial_x\Phi\partial_x\Theta \right). \tag{79}$$

From this presentation, the stability bound on $|U_{LR}|$ in the case when $\delta = 0$ is more obvious. We can integrate out Θ since it appears quadratically, using

$$\partial_x\Theta \overset{\text{eom}}{=} \frac{\partial_t\Phi + \delta\partial_x\Phi}{\bar{v} + U_{LR}} \tag{80}$$

which gives, after some algebra

$$S_b = \frac{R^2}{4\pi} \int \left(\frac{1}{\mathsf{v}}(\partial_t\Phi + \delta\partial_x\Phi)^2 + \mathsf{v}(\partial_x\Phi)^2 \right), \qquad R^2 \equiv 2\sqrt{\frac{\bar{v} - U_{LR}}{\bar{v} + U_{LR}}}, \qquad \mathsf{v} \equiv \sqrt{\bar{v}^2 - U_{LR}^2}. \tag{81}$$

The rather unfortunate 2 in the definition of R^2 comes from our use of $\Theta = \theta/2$ i.e. our choice that the Φ, Θ fields are 4π, not 2π periodic. There's essentially no perfect notational choice here, so we'll just live with it.

To get a feel for what the interaction does, consider for simplicity the case where $\delta = 0$, and work in units where $\mathsf{v} = 1$. Then to lowest order in U_{LR}, the action is

$$S_{b;v'_L = v'_R} \approx \frac{R_{eff}^2}{4\pi} \int d\Phi \wedge \star d\Phi, \qquad R_{eff}^2 = 2(1 - U_{LR}/\bar{v}). \tag{82}$$

Therefore adding a small repulsive interaction between the two chiral fermions has the effect of decreasing the boson radius. Sanity check: this means that repulsive interactions increase the scaling dimension of the SF ordering term $\cos\Phi$, and decrease that of the crystal ordering term $\cos\Theta$, while attractive interactions do the opposite. This is exactly what we expect on physical grounds—attractive interactions favor k-space order (SF), while repulsive ones favor $\mathbb{R}$-space order (crystal).

Another thing to note is how T-duality works in terms of the fermionic parameters. Again since we are using $\Theta = \theta/2$, T-duality is no longer just $R \mapsto R^{-1}$, but rather

$$T : R \mapsto \frac{2}{R}, \tag{83}$$

which is still an involution. We see that this is equivalent to $T : U_{LR} \mapsto -U_{LR}$, so that here T-duality acts to change the sign of the interaction. Thus attractive and repulsive interactions are actually exactly equivalent to one another! On the fermion side, we can implement the sign flip of U_{LR} by doing charge conjugation on only a single chirality, e.g.

$$T : \psi_+ \mapsto \psi_+, \qquad \psi_- \mapsto \psi_-^*, \tag{84}$$

which flips the sign of the interaction. One can also check that this action exchanges axial and vector currents on the fermion side, in keeping with the fact that it exchanges the regular and topological currents on the boson side.

Anyway, let us now find the scaling dimensions of the spectrum of vertex operators. The OPEs are straightforward to calculate. Let us label vertex operators $\mathcal{V}_{n,w}$ by the vector $\mathsf{n} = (n, w)^T$. The R/L components of this vector are $((n-w)/2, (n+w)/2)^T = S\mathsf{n}$, where $S = \frac{1}{2}(\mathbf{1} + J)$. We then find the OPE

$$\mathcal{V}_{\mathsf{n}}(x,t) \odot \mathcal{V}_{\mathsf{m}}(0,0) = \mathcal{V}_{\mathsf{n}+\mathsf{m}}(x,t) \frac{1}{(x + \widetilde{v}_R t)^{-\mathsf{n}^T S^T M |R\rangle\langle R| M S \mathsf{m}} (x - \widetilde{v}_L t)^{-\mathsf{n}^T S^T M |L\rangle\langle L| M S \mathsf{m}}} + \dots, \tag{85}$$

where the ... are less singular. This means that the conformal dimension is, skipping some algebra,

$$\Delta_{\mathsf{n}} = \frac{1}{2}\mathsf{n}^T S^T M^2 S \mathsf{n} = \frac{\gamma}{4}\mathsf{n}^T \begin{pmatrix} 1 + U/\bar{v} & \\ & 1 - U/\bar{v} \end{pmatrix} \mathsf{n} = \frac{\gamma}{4}\left(n^2(1 + U/\bar{v}) + w^2(1 - U/\bar{v})\right). \tag{86}$$

As we saw earlier for the simpler example, repulsive interactions make the Θ (w) vertex operators more relevant, favoring CDW order, while attractive interactions make the Φ (n) operators more relevant, favoring SF order.

The conformal spin is

$$s_{n,w} = \frac{1}{2}\mathsf{n}^T S^T M Z M S \mathsf{n} = \frac{1}{2}\mathsf{n}^T S^T Z S \mathsf{n} = -\frac{1}{4}\mathsf{n}^T X \mathsf{n} = -\frac{nw}{2}. \tag{87}$$

Note in particular that the conformal spin is unchanged by interactions or changes in velocity, as it should be (which is a consequence of the fact that $M \in SO(1,1)$ preserves the commutation relations). ethan: *Write down the correlators and anomalous dimensions explicitly for the fermions, and show the ward identity by showing that the currents pick up no anomalous dimensions. This is nontrivial and made possible by the normal ordering when bosonizing: the currents map to total derivatives which* *just have straight up $1/x$ correlations. So even though the individual fermions are renormalized the currents are not* ethan: *Also talk about why the orthogonality catastrophe / nfl-ness is caused by the fact that the IR modes have different quantum numbers than the UV fermions*

This theory can then be re-written in the form of (1) at $R = \sqrt{2}$. However, once we add interactions, the mapping to the simple form (1) will not be so simple, and the $\varphi_{L/R}$ fields will *not* be the chiral components of a field with the kinetic term of the form (1)–indeed, we know that adding interactions will change R, but that for generic R the chiral components $\phi_\pm$ do not give us vertex operators with fermionic conformal spins[21] (and as we said, the $\phi_\pm$ aren't really well-defined at generic R). This is why we've used the notation $\varphi_{L/R}$ for bosonization. Our notation is such that the vertex operators $V_{L/R} = e^{i\varphi_{L/R}}$ are always well-defined local operators with conformal spin 1/2. The $\varphi_{L/R}$ will *not* generically correspond to the (anti)holomorphic components of a single boson field.

Relevance of symmetry-breaking perturbations

The two symmetries of the system we've been studying so far are the $U(1)_N$ of fermion number conservation, which shifts Φ, and the $U(1)_T$ of translation, which shifts Θ. Θ was introduced as a field which keeps track of the fermion density, but from the expressions for the bosonization of $\psi_{L/R}$ we see that for $x \mapsto x + \lambda$, the action on the IR fields is $\Theta \mapsto \Theta + k_F\lambda$. Therefore translation indeed acts as a $U(1)$ for generic fillings (recall that the filling determines k_F non-perturbatively via Luttinger's theorem), although for rational fillings it will act as a discrete subgroup of $U(1)$, and more symmetry-allowed operators will appear.

If we restrict our attention to actions which are symmetric under both symmetries, the previous subsection covers all possibilities, up to the effects of irrelevant derivative interactions. If we allow ourselves to consider perturbations which break the symmetries though, we can add sines and cosines of integer multiples of the Φ and Θ fields (since we usually don't want to add operators with nonzero spin to the theory, we can restrict our attention to just $\cos(n\Phi)$ and $\cos(m\Theta)$, without any mixed Θ-Φ terms).[22] Furthermore note that to preserve $(-1)^F$ we need $n \in 2\mathbb{Z}$, and since we only want to consider perturbations which are local, we also need to take $m \in 2\mathbb{Z}$.

The minimal perturbations are therefore $\cos(2\Theta), \cos(2\Phi)$. At the free fixed point, the dimensions of these two are in fact equal:

$$\Delta_{SC}^{free} = \Delta_{CDW}^{free} = 1. \tag{88}$$

The fact that $\Delta_{CDW}^{free} < 2$ is the statement of Peierls instability: a translation-breaking[23]

[21] The image of a fermion under bosonization $\mathcal{B}[\psi_\pm]$ must always have conformal spin $\pm 1/2$—the interactions can change the scaling dimension, but the way that $\langle \mathcal{O}^\dagger(x)\mathcal{O}(y)\rangle$ transforms under rotations can't change as interactions are added (if $\mathcal{O}(x)$ is a local operator then its spin is fixed to be in $\frac{1}{2}\mathbb{Z}$ and hence can't change smoothly).

[22] Note that we might think that adding e.g. $\cos(2\Theta - 2k_F x)$ would be a way to add a symmetry-allowed density modulation, but this in fact vanishes: it comes from a term like $\int dx\, e^{2ik_F x}\psi_L^\dagger\psi_R + h.c.$, but this vanishes because the support of the $\psi_{L/R}$ in momentum space is narrow enough to preclude the $\psi_L^\dagger\psi_R$ term from having the required $2k_F$ momentum transfer to survive integration. The correct density modulation term is instead just $\int dx\, L^\dagger R + h.c.$, which is nonzero but breaks $U(1)_T$.

[23] And $\cos(2\Theta)$ is always $U(1)_T$ breaking at any non-trivial filling.

potential at wavevector $2k_F$ always drives an instability. Note that the filling with the most relevant $U(1)_T$-symmetric cosine is half-filling with $k_F = \pi/2$, which permits $\cos(4\Theta)$ as a $U(1)_T$-preserving perturbation. For free fermions this is comfortably irrelevant, with a scaling dimension of 4. Note that as expected, the $U(1)_T$-breaking perturbations become more relevant as the strength of the interactions is increased (i.e. as the radius of the boson is decreased), since the interactions favor $\mathbb{R}$-space ordering; similarly, getting closer to a superfluid by decreasing the interactions makes the $U(1)_N$-breaking perturbations more relevant.

Bosonization: CMT approach

We can use the results in the subsection on bosonizing bosons to get a much more intuitive, but also less clean, derivation of the bosonization formulae outlined in the previous section. This tactic is more of an explanation of why bosonization works, rather than how it works.

According to our (perhaps patently perverse) predilection for writing the vertex operators as $V_{n,w}$ with $n, w \in \mathbb{Z}$, we will again be taking the discussion in the bosonizing bosons section and making the notational change of $\Theta = \theta/2$. In fact in order to have notation that's consistent with the previous section, we will add an extra minus sign, so that the density operator is expressed as

$$\rho = \rho_0 - \partial_x\Theta/\pi. \tag{89}$$

In the Wigner crystal limit, we thus tack on the constraint enforcing the discreteness of the lattice by writing ρ as

$$\rho = (\rho_0 - \partial_x\Theta/\pi) \sum_{n\in 2\mathbb{Z}} e^{in(\pi\rho_0 x - \Theta)}. \tag{90}$$

Extending this approach to fermions is very easy—we just add on JW tails to fermionize the operators appearing in H (we add the tails in the same way in both the SF and crystal limits). The JW strings need to count the fermion number to the left (say) of a given fermion, and so they must be given by $(-1)^{\int^x \rho} = e^{i(\pi\rho_0 x - \Theta(x))}$. Then the fermions are given generically by

$$\psi(x) \sim \sqrt{\rho_0 - \partial_x\Theta/\pi} \sum_{n\in 2\mathbb{Z}} U_n e^{i(n+1)(\pi\rho_0 x - \Theta)} e^{i\Phi(x)} \sim \sqrt{k_F + \partial_x\Theta} \sum_{n\in 2\mathbb{Z}+1} U_n e^{in(k_F x - \Theta(x))} e^{i\Phi}, \tag{91}$$

where U_n are some phenomenological constants. Therefore the effect of the JW tails is to shift the sum of the exponents of the Θ vertex operators from $2\mathbb{Z}$ to $2\mathbb{Z}+1$. Including the $\partial_x\Theta$ term in the square root is done to account for situations in which we imagine k_F varying semiclassically throughout space, with the k_F in the square root representing a spatial average of the Fermi momentum. In a situation where k_F is fixed, then by Luttingers theorem there is no zero-momentum modulation in the density (the Fermi sea only sloshes back and forth, it does not pulse in size). In this case we can remove the $\partial_x\Theta$ in the exponent. In the non-interacting case only $U_{\pm 1}$ are nonzero; larger U_ns come from processes which transfer momenta $2nk_F$, which are only possible in the presence of interactions. Therefore in the free limit, if we write $k_F = 1/a$ as a UV cutoff, we reproduce exactly the formulae (45) motivated through more formal field-theory methods. Of course, while this method for mapping the fermions agrees in the deep IR with the previous field theory approach, it also gives us an idea of what happens when we back away from this limit, and has the conceptual advantage that we didn't have to start with respect to a reference free theory with a certain Fermi surface. Indeed, this approach relied only on basic hydrodynamically-flavored reasoning, and at no point did we bring up complicated ways of counting bosonic and fermionic excitations with respect to a Fermi sea, normal ordering prescriptions, etc. etc.

In terms of symmetries, we also see that since the Φ, Θ fields are slowly fluctuating on the scale of k_F^{-1}, we can write $\psi(x+\delta)$ as $\psi(x)$, but with $\Theta(x)$ replaced with $\Theta(x) - 2\delta k_F$. Hence spatial translations act as shifts in Θ. Note that unless k_F is some nice rational value, all cosines of Θ will break translation symmetry. We also see that the $U(1)$ particle number symmetry acts as shifts in Φ.

How not to bosonize

Reading Shankar and Witten's lectures on Abelian bosonization (both very QFT-centric) left me very confused, so here let me just clear up a few things. The tl;dr is that they are doing things perturbatively, although they discuss things in a way that makes it seem like their mappings are exact.

I think when doing bosonization, it's best not to blindly be led around by QFT intuition. From a field theory point of view, given that $\mathcal{B}[j^0] = \frac{1}{2\pi}\partial_x \Phi$, it is natural to make things covariant by writing $\mathcal{B}[j] = \frac{1}{2\pi} d\Phi$. Now as we saw, this is not the correct way to bosonize—when we point-split correctly the j^1 component of the current maps to something with a ∂_x derivative, namely $\partial_x(\varphi_L - \varphi_R)$. This is consistent with the action of spacetime symmetries since under e.g. time reversal that maps $j \mapsto -j$ (as a form). But if we were to take $\mathcal{B}[j] = \frac{1}{2\pi} d\Phi$ at face value, we'd have

$$S_{\text{free}}[\psi] - \frac{g}{\pi}\int j_\mu j^\mu \quad \overset{?}{\leftrightarrow} \quad \frac{1-g}{8\pi}\int d\phi \wedge \star d\phi. \tag{92}$$

When reading Witten and Shankar one gets the feeling that this relation is exact, but this can't be correct: as we discussed earlier, T-duality on the boson side is the same as the "KW Duality" sending $j \mapsto \star j$ on the Fermion side, which acts as $\psi_L \mapsto \psi_L^\dagger$, $\psi_R \mapsto \psi_R$. This sends $g \mapsto -g$, but is also supposed to do $R^2 \mapsto \frac{2}{R^2}$, which is not compatible with the above equation. Indeed we explicitly saw above, the above is only correct to leading order in g, and hence is a perturbative statement (dimensionally correct since the velocity is being suppressed). The full relation between R and g is non-linear, as seen by the formula above with the square roots and such, and is only derived in the QFT framework by doing a self-consistent point-splitting in space only.

Anyway, to summarize: while the field theory way of thinking is slicker and nicer for doing calculations, the cond-mat way of doing this is more rigorous and intuitive; one should learn the cond-mat way but do things the QFT way (just like in RG).

2 *Two dimensional bosonization, the Schwinger model, and θ angles* ✓

This is an elaboration on one of the problems in Quantum Fields & Strings, a Course for Mathematicians, Vol II. Consider two flavors of massive fermions in two dimensions coupled to a $U(1)$ gauge field with a θ term, in Euclidean signature:

$$S = \frac{1}{2\pi}\int \left(\sum_i \bar{\psi}_i \not{D}_A \psi_i + \bar{\psi}_i m_i \psi_i\right) + \frac{1}{2e^2}\int F \wedge \star F + \frac{i\theta}{2\pi}\int F. \tag{93}$$

The factor of $1/2\pi$ in front of the fermions is there so that the fermion correlators will have no annoying prefactors. Here the mass term is complex, so that in the representation where $\gamma^0 = X$ and $\psi_i = (\psi_{i,+}, \psi_{i,-})^T$,

$$\bar{\psi}_i m_i \psi_i = m\psi^\dagger_{i,+}\psi_{i,-} + m^*\psi^\dagger_{i,-}\psi_{i,+}. \tag{94}$$

We will discuss this theory and its bosonization, show that the θ term can be eliminated when either of the fermions are massless, and describe what happens in various limits for the massive case.

Let us first do a brief recapitulation of our chosen bosonization conventions for a single flavor of Dirac fermion in this setting. We will be in $i\mathbb{R}$ time, with gamma matrices $\gamma^0 = X, \gamma^1 = Y$. Thus the Dirac operator is

$$\not{D} = \begin{pmatrix} 0 & \partial_+ \\ \partial_- & 0 \end{pmatrix}, \qquad \partial_\pm = \frac{1}{2}(\partial_0 \mp i\partial_1). \tag{95}$$

Note the "reversed" signs! This is so that ∂_+ is a "left-moving" derivative and ∂_- is a "right-moving" derivative. The classical eom are $\partial_+\psi_- = \partial_-\psi_+ = 0$, so that classically ψ_- is right-moving and ψ_+ is left-moving, since ∂_- kills things that are only a function of $t + ix$ (which we regard as left-moving), while ∂_+ kills things that are only a function of $t - ix$ (which are right-moving). This ensures that the chirality operator Z counts left-movers minus right-movers, so the helpful thing to remember is that $+$ things have positive chirality and hence move left (counterclockwise), while $-$ things have negative chirality and thus move right.

The correlator of the fermions is

$$D_{\alpha\beta}(x-y) = \delta_{\alpha\beta}\frac{x-y}{|x-y|^2}, \tag{96}$$

where x and y are complex coordinates in the plane. One can verify this by computing the derivative explicitly, or by noting that it is derivative of the free boson propagator (since if

G is the free boson propagator, then schematically $\partial^2 G = \delta \implies D = \partial G$ satisfies $\partial D = \delta$). The factor of $1/2\pi$ in the Lagrangian was chosen so that no factors of 2π appear in the fermion propagator.

We will use the bosonization rules worked out in the diary entry that does an overview of how bosonization works, viz.

$$\mathcal{B}[\psi_{L/R}] = \frac{1}{\sqrt{a}} e^{i\varphi_\pm}, \tag{97}$$

where the actions for both fields are

$$\mathcal{B} : \frac{1}{2\pi} \int \bar{\psi} i \not{\partial} \psi \leftrightarrow \frac{1}{4\pi} \int \sum_{\sigma=\pm} \left(i\sigma \partial_\tau \varphi_\sigma \partial_x \varphi_\sigma - \partial_x \varphi_\sigma \partial_x \varphi_\sigma \right). \tag{98}$$

In terms of the Φ, Θ fields defined by $\varphi_+ = \Phi + \Theta$, $\varphi_- = \Phi - \Theta$, we may write

$$\mathcal{B} : \frac{1}{2\pi} \int \bar{\psi} i \not{\partial} \psi \leftrightarrow \frac{1}{2\pi} \int \left(2i \partial_\tau \Phi \partial_x \Theta - (\partial_x \Phi)^2 - (\partial_x \Theta)^2 \right). \tag{99}$$

The fermion currents map as

$$\mathcal{B}[2\pi j^\mu] = \mathcal{B}[(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-, \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-)^\mu] = 2(\partial_x \Theta, \partial_x \Phi)^\mu. \tag{100}$$

Very importantly for the present problem, the mappings above mean that we have the mapping of the mass term to vertex operators. There are a few ways of writing the mass term. If we take $m_\mathbb{C} \in \mathbb{C}$, then we can write the mass term as $\bar{\psi} m_\mathbb{C} \psi$; what we really mean by this is

$$\bar{\psi} m_\mathbb{C} \psi = m_\mathbb{C} \psi_+^\dagger \psi_- + m_\mathbb{C}^* \psi_-^\dagger \psi_+. \tag{101}$$

We can also equivalently take two real parameters $m, m_5 \in \mathbb{R}$, and write

$$\bar{\psi} m_\mathbb{C} \psi = m \bar{\psi} \psi + m_5 \bar{\psi} \bar{\gamma} \psi = m(\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+) + m_5 (\psi_+^\dagger \psi_- - \psi_-^\dagger \psi_+). \tag{102}$$

Either way, using the bosonization mapping we see that

$$\bar{\psi} m_\mathbb{C} \psi \mapsto m_\mathbb{C} e^{-i\Theta} + m_\mathbb{C}^* e^{i\Theta} = 2\,\mathrm{Re}\,[m_\mathbb{C}] \cos \Theta + 2\mathrm{Im}[m_\mathbb{C}] \sin \Theta. \tag{103}$$

In particular, when m is real, we get a sine-Gordon $\cos \Theta$ interaction (even though the fermions are free!).

When we add a gauge field, not a lot changes. The Dirac operator gets upgraded to

$$\not{D}_A = \begin{pmatrix} 0 & \partial_+ + iA_+ \\ \partial_- + iA_- & 0 \end{pmatrix}, \quad A_\pm = A_0 \mp iA_1. \tag{104}$$

This means that the coupling between A and the fermion current is

$$S \supset \int (A_+ J_- + A_- J_+) dz \wedge d\bar{z}, \tag{105}$$

where $J_\pm = \psi_\pm^\dagger \psi_\pm / 2\pi$. This is the same as $A \wedge \star J$, since the metric in $z, \bar{z}$ coordinates is off-diagonal.

When we do the mapping now, the kinetic term goes to

$$\mathcal{B} \, : \, \frac{1}{2\pi}\int \bar{\psi}\not{D}_A\psi \mapsto S_0[\Theta,\Phi] + \frac{i}{2\pi}\int dx\, d\tau\, \left(A_+\partial_x(\Phi-\Theta) + A_-\partial_x(\Phi+\Theta)\right). \tag{106}$$

Let us now integrate out Φ; we will be able to do this even in the presence of the mass term since Φ still appears quadratically. We get

$$\mathcal{B} \, : \, \frac{1}{2\pi}\int \bar{\psi}\not{D}_A\psi \mapsto \frac{1}{2\pi}\int \left(d\Theta\wedge\star d\Theta + 2i(A_+\partial_-\Theta - A_-\partial_+\Theta)dz\wedge d\bar{z}\right) \tag{107}$$

We can integrate the last two terms by parts, which yields $\frac{i}{\pi}\Theta F_A$, so that 2Θ contributes to the θ angle.[24] This expression makes sense even though Θ is not a legit zero-form since the quantization condition on F_A means that such ambiguities only affect the action by something in $4\pi\mathbb{Z}$.

<diversion>

One quick comment on this: the term $\frac{1}{\pi}\Theta F_A$ is obviously not invariant under shifting Θ by a constant, but naively its counterpart $A_+\partial_-\Theta - A_-\partial_+\Theta$ is invariant under the shift, since Θ only appears under a derivative sign. Actually this is conclusion is wrong, and the latter expression *does* change under shifting Θ by a constant. The proper way to understand this is by using differential cohomology, and interpreting the term as $A_+\star\Theta - A_-\star\Theta$ where $\star$ is the product operation in Deligne-Bellison cohomology. Since the ΘF_A term is not invariant under the shift symmetry of Θ and since the shift symmetry of Θ is the chiral symmetry on the fermion side, this non-invariance brings out the chiral anomaly from the integration measure and makes it more explicit though the non-invariance of the action.

</diversion>

Now consider making the fermions massive with a mass term $m\bar{\psi}\psi + m_5\bar{\psi}\bar{\gamma}\psi$. The full action is now

$$\mathcal{B} \, : \, \frac{1}{2\pi}\int \bar{\psi}i\not{D}_A\psi \mapsto \frac{1}{2\pi}\int \left(d\Theta\wedge\star d\Theta + 4\pi(m\cos\Theta + m_5\sin\Theta) + 2\Theta F_A\right). \tag{108}$$

We can now generalize to the multi-species case given in the introduction to this diary entry. The bosonic theory is (adding in a kinetic term for the gauge field, which we assume to come with a θ term)

$$S = \frac{1}{4\pi}\int \sum_i R_i^2 d\Theta\wedge\star d\Theta + \sum_i (m_i e^{i\Theta_i} + m_i^* e^{-i\Theta}) + \frac{1}{2e^2}\int F_A\wedge\star F_A + \frac{\theta/2+\Theta_1+\Theta_2}{\pi}\int F_A. \tag{109}$$

Here we have also left open for the two bosons to have different radii, which can be modified away from the free-fermion value of $R=\sqrt{2}$ by Thirring current-current interactions for the fermions.

[24] The fact that it is 2Θ and not Θ can be checked from the chiral anomaly: under $\psi\mapsto e^{i\bar{\gamma}\alpha}\psi$ we have $\delta\Theta=\alpha$, while at the same time we know that the action needs to shift by $d^\dagger j_A = \frac{\alpha}{\pi}\int F_A$ — hence the factor of 2 is correct.

If at least one of the $m_i = 0$, one of the boson fields has no $e^{i\Theta_i}$ interaction, and so we can perform the shift $\Theta_i \mapsto \Theta_i - \theta/2$ to eliminate the θ dependence from the action. Hence the parameter θ is meaningful only when *both* fermions are massive.

Consider when both fermion masses are small, $m_i \to 0$. Also for simplicity, let $R_1 = R_2 = R$. Since when $m = 0$ the theory is quadratic, we can just look at the equations of motion and because of the masslessness, we can ignore the θ dependence. Now define the bosonic fields

$$\xi_\pm = \Theta_1 \pm \Theta_2, \tag{110}$$

so that the action is (now in $\mathbb{R}$ time)

$$S = \frac{R^2}{8\pi}\int \sum_{a=\pm} d\xi_a \wedge \star d\xi_a + \frac{1}{2e^2}\int F_A \wedge \star F_A + \frac{\xi_+}{\pi}\int F_A. \tag{111}$$

Since ξ_- doesn't see the gauge field, the eom for ξ_- gives a regular massless wave equation. The equations of motion for A and for ξ_+ are

$$\frac{1}{e^2} d^\dagger F = \frac{1}{\pi} \star d\xi_+, \tag{112}$$

and

$$\frac{R^2}{4\pi}\Box\xi_+ = \frac{1}{\pi}\star F \tag{113}$$

We then plug the first equation into the second by solving for F; this gives $\star F = \frac{e^2}{\pi}\xi_+$ and so we get a massive wave equation for ξ_+:

$$(\Box - m_\varphi^2)\xi_+ = 0, \qquad m_\varphi = \frac{2e}{R\sqrt{\pi}}. \tag{114}$$

So we get one massive and one massless scalar[25] (which is rather interesting as Θ_1, Θ_2 entered the original action completely decoupled from one another!). Note that if we only were bosonizing a single fermion, then we would only have one boson, and the whole theory would be gapped. This is the bosonization way of seeing the main point of the Schwinger model: gauge invariance is not enough to guarantee massless states in two dimensions, since you can get confining gauge fields to do the job.

What happens when one of the fermion masses, say m_1, goes to ∞? In perturbation theory, I don't think we can see that anything happens, other than e.g. the Maxwell term getting corrected by factor that goes to zero as $m_1 \to \infty$. However, on the bosonic side, we can argue as follows: first, perform a chiral rotation so that $m_1 \in \mathbb{R}^{<0}$. This shifts the θ

25 Another way of getting to this result is to take $d^\dagger A = 0$ gauge from the outset, which allows us to write $A^\mu = \varepsilon^{\mu\nu}\partial_\nu\varphi$ for a scalar φ, with $F_A \wedge \star F_A$ becoming $\varphi\Box^2\varphi$. The coupling $\xi_+\Box\varphi$ can then be eliminated by shifting ξ_+, which when done produces a term $\varphi\Box\varphi$. Therefore the φ term has a propagator like $(\partial^2 + \partial^4/e^2)^{-1}$, which leads to a q-space correlation function like (this is just schematic)

$$\frac{1}{q^2 + q^4/e^2} \sim \frac{1}{q^2} - \frac{q^2+e^2}{,} \tag{115}$$

which indeed splits up into massive and massless parts.

term by $\pi - \arg(m)$. The mass is now real, and we get a $m_1 \cos(\Theta_1)$ potential for the bosons. When $m_i \to -\infty$ we can take $\Theta_1 \to 0$, and Θ_1 disappears form the theory. Thus the effect of the heavy fermions is to shift

$$\theta \mapsto \theta + \pi - \arg(m). \tag{116}$$

ethan: *come back and elaborate*

3 *Functional bosonization* ✓

This is from Altland and Simons (but there are some typos in the problem so don't worry to much about reproducing their results — also, our notation will deviate a bit form theirs, and will unfortunately also deviate from other diary entries).

Consider fermions in two dimensions, with action

$$S = -\int \bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\int \rho^T \mathcal{G}\rho - \int (\psi_\sigma^\dagger J_\sigma + J_\sigma^\dagger \psi_\sigma), \tag{117}$$

where $\mathcal{G}$ is an interaction matrix, $\rho = (\rho_+,\ \rho_-)^T$ are the densities, and $\sigma \in \{\pm\}$ are the left- and right-moving indices (the currents J_σ are just there to generate correlation functions—they are not the fermion currents coming from the $U(1)$ particle number symmetries). Our plan is to get to the usual bosonization result in a slightly different way.

First, we will decouple the interactions using a bosonic doublet. It will turn out to be convenient to do a Hodge decomposition on this doublet, since the longitudinal and transverse parts of the decomposition will match with the vector and chiral currents and so this decomposition will behave nicely with respect to the holomorphic / antiholomorphic decomposition of the boson field.

We will then integrate out the fermions and find the effective action for the components of the Hodge decomposition of the Hubbard-Stratonovich field, eventually obtaining the usual interacting Luttinger liquid action.

This diary entry was written a fair amount of time before the diary entry on the standardization of bosonization conventions, and in retrospect I think the approach outlined below is conceptually rather murky. Nevertheless, I've decided to keep it for posterity's sake.

First we decouple the interaction, using a bosonic field $\boldsymbol{\phi} = (\phi_+,\ \phi_-)$. We let $\boldsymbol{\phi}$ appear in the action as $\frac{1}{2}\boldsymbol{\phi}^T\mathcal{G}^{-1}\boldsymbol{\phi}$, and then shift $\boldsymbol{\phi} \mapsto \boldsymbol{\phi} + i\mathcal{G}\rho$. This leaves us with a $i\rho^T\boldsymbol{\phi}$ coupling, which is just like that of a gauge field. To write the coupling in a covariant form, let ϕ_μ be the "vector" field with components

$$\phi_0 = \frac{1}{2}(\phi_+ + \phi_-), \qquad \phi_1 = \frac{1}{2i}(\phi_+ - \phi_-). \tag{118}$$

In a representation where the gamma matrices are $\gamma^0 = X, \gamma^1 = Y$, we then have

$$Z = \int \mathcal{D}\phi\mathcal{D}\psi \exp\left(-\frac{1}{2}\int \phi^T \mathcal{G}^{-1}\phi + \int \bar{\psi}\not{D}_\phi\psi - S_{src}[J,\psi]\right), \tag{119}$$

where

$$\not{D}_\phi = \gamma^\mu(\partial_\mu - i\phi_\mu). \tag{120}$$

We now do a Hodge decomposition on the 1-form ϕ (without the bold font ϕ means the 1-form $\phi_\mu dx^\mu$; with the bold font it means the 2-component scalar (ϕ_+, ϕ_-)—admittedly not the best notation). We will write it as

$$\phi = d\xi + id^\dagger \star \eta. \tag{121}$$

The i is just for convenience, and we have written $\star\eta$ since we'd rather work with zero-forms than two-forms. The Hodge decomposition plays nicely with the chiral nature of the fermions, with ξ relating to the vector current and η to the chiral current. We see this by considering

$$\not{\partial}(e^{i\xi+i\eta Z}\psi) = e^{i\xi+i\eta Z}\not{\partial}\psi + (-\partial_0\xi - i\partial_1\eta X - \partial_1\xi Y + i\partial_0\eta Y)e^{i\xi+i\eta Z}\psi = \gamma^\mu(\partial_\mu + i\phi_\mu)e^{i\xi+i\eta Z}\psi. \tag{122}$$

Thus (if we ignore what happens to $\mathcal{D}\psi$), we can eliminate the coupling to the background field ϕ through a phase rotation by ξ and a chiral rotation by η.

Now we can integrate out the fermions. We expand the $\mathrm{Tr}\ln$ to second order in ϕ, producing the usual polarization bubble. The effective action then has a term (remembering the -1 from the fermion loop)

$$S_{eff} \supset -\frac{1}{2}\int_{q,p} \boldsymbol{\phi}_q^T \frac{1}{\not{p}(\not{p}-\not{q})}\boldsymbol{\phi}_{-q}. \tag{123}$$

Because ϕ_μ couples to the fermions as a gauge field would, the integration kernel is diagonal in the spin indices. The propagator for the ψ's is

$$D_\psi(p)_{\sigma\sigma'} = \delta_{\sigma\sigma'}\frac{i}{\nu + i\sigma p}, \tag{124}$$

which comes from inverting ∂_σ. Note that I am being lazy and not distinguishing between two-momenta and their spatial components: that is, I am writing $q = (\nu, q)$. Sorry not sorry.

We now write the σ part of the above integral as

$$\frac{1}{2}\int_{q,p} \boldsymbol{\phi}_{q,\sigma}^T \frac{1}{\nu + i\sigma p}\frac{1}{\nu + i\sigma p + \omega + i\sigma q}\boldsymbol{\phi}_{-q,\sigma}, \tag{125}$$

where $q = (\omega, q)$. It is helpful to recast this as

$$\frac{1}{2}\int_q \frac{1}{\omega + i\sigma q}\int_p \boldsymbol{\phi}^T_{q,\sigma}\left(\frac{1}{\nu + i\sigma p} - \frac{1}{\nu + i\sigma p + \omega + i\sigma q}\right)\boldsymbol{\phi}_{-q,\sigma}. \tag{126}$$

This looks like it might be zero after doing a contour integral and closing it in either the upper half plane or the lower one (depending on σ), but on the other hand, it's $\sim \int_p d^2p\, p^{-2}$ which is divergent (I think there are some subtle things going on as the pole at $-i\sigma p$ gets pushed to ∞). As suggested in the book, we do a somewhat suspect thing and close the integrals in the plane where they give a non-zero answer by the residue theorem. If $\sigma p > 1$ then the pole lies in the lower half plane and we we get a clockwise integral, giving $-2\pi i$, while if $\sigma p < 1$ then we get a counterclockwise integral, giving $2\pi i$. So then

$$-\frac{i}{2}\int_q\int_p \frac{dp}{2\pi}\ \boldsymbol{\phi}^T_{q,\sigma}\left(\mathrm{sgn}(\sigma p) - \mathrm{sgn}(\sigma(p+q))\right)\boldsymbol{\phi}_{-q,\sigma}. \tag{127}$$

Now we introduce an explicit cutoff for the spatial momentum integration. Luckily as long as we take $\Lambda \to \infty$, the answer doesn't depend on the exact value for Λ. So we get

$$-\frac{1}{4\pi}\int_q \boldsymbol{\phi}^T_{q,\sigma}\frac{-i\sigma q}{\omega + i\sigma q}\boldsymbol{\phi}_{-q,\sigma}. \tag{128}$$

Recapitulating, the effective action for the boson fields is

$$S_{eff}[\boldsymbol{\phi}] = \frac{1}{2}\int_q \boldsymbol{\phi}^T_q\left(\mathcal{G}^{-1} + G_{\boldsymbol{\phi}}(q)\right)\boldsymbol{\phi}_{-q}, \tag{129}$$

where

$$G_{\boldsymbol{\phi}}(q)_{\sigma\sigma'} = \frac{\delta_{\sigma\sigma'}}{2\pi}\frac{i\sigma q}{\omega + i\sigma q}. \tag{130}$$

Now we look at the source term. Before we integrated out the fermions but after we introduced the ϕ fields, we can perform a shift on ψ to eliminate the linear coupling between the fermions and the sources (we have to do this after adding in the ϕ fields since we don't want to mess with the density-density ψ interactions). The source term is then

$$S_{src} = \int_{x,x'} J^\dagger(x) G_{\psi,\phi}(x,x') J(x'), \tag{131}$$

where $G_{\psi,\phi}(x,x')$ is the propagator for the fermions in the background ϕ field.

Using the Hodge decomposition of the ϕ field,

$$S_{src} = \int_{x,x'} J^\dagger(x) e^{-i(\xi+\eta Z)(x)} G_\psi(x,x') e^{i(\xi+\eta Z)(x')} J(x'). \tag{132}$$

Now we can play a trick by representing the fermion propagator with a bosonic doublet of fields. First of all, the actual expression for $G_\psi(x,0)$ is (just invert $\partial_\pm$)

$$[G_\psi(x,0)]_{\sigma\sigma'} = \delta_{\sigma\sigma'}\int_{q,\omega}\frac{e^{-i(xq+\omega\tau)}}{i\sigma q + \omega}. \tag{133}$$

Taking $x > 0$ wolog, the integrand is analytic in the lower half-plane, and so we close the contour for $q \to -i\infty$. Thus we get zero if $\sigma\omega > 0$ and get a residue of $-2\pi\sigma \exp(\omega(\sigma x - i\tau))$ otherwise, so that

$$[G_\psi(x,0)]_{\sigma\sigma'} = -\sigma\delta_{\sigma\sigma'} \int_\omega \Theta(-\sigma\omega) e^{\omega(\sigma x - i\tau)}. \tag{134}$$

Doing the integral,

$$[G_\psi(x,0)]_{\sigma\sigma'} = \frac{\delta_{\sigma\sigma'}}{2\pi} \frac{1}{i\sigma x + \tau}. \tag{135}$$

Now we want to reproduce this with bosons. Consider a doublet of bosons which possesses the following free action:

$$S_\varphi = \frac{1}{2} \int_q \varphi_q^T K_\varphi(q) \varphi_{-q}, \tag{136}$$

where $\varphi = (\varphi_+, \ \varphi_-)^T$ and

$$K_\varphi(q) = \begin{pmatrix} q^2 + iq\omega & 0 \\ 0 & q^2 - iq\omega \end{pmatrix}. \tag{137}$$

One can think of the components of φ as holomorphic and antiholomorphic modes. The Greens function for these guys is

$$[G_\varphi(x,0)]_{\sigma\sigma'} = \delta_{\sigma\sigma'} \int_{q,\omega} \frac{e^{-i(xq+\omega\tau)}}{q^2 + i\sigma q\omega}. \tag{138}$$

The poles of the momentum integral are at $q = -i\sigma\omega$, and the integrand is analytic for $q \to -i\infty$. The residue at the pole is $2\pi/\sigma\omega$ (since the contour is closed clockwise), so we get

$$[G_\varphi(x,0)]_{\sigma\sigma'} = \sigma\delta_{\sigma\sigma'} \int_\omega \Theta(\sigma\omega) \frac{e^{-\omega(\sigma x + i\tau)}}{\omega}. \tag{139}$$

To do the integral, we impose a small frequency cutoff at a^{-1} (or at $-a^{-1}$, depending on σ), where a is the lattice spacing. The integral can then be expanded in small a. We will get a bunch of constants, which will be normal-ordered away. The surviving piece gives a log, and so

$$[G_\varphi(x,0)]_{\sigma\sigma'} = -\ln\left(\frac{i\sigma x + \tau}{a}\right). \tag{140}$$

The whole point of going through this is that the correlators of the vertex operators for the φ fields reproduce the form of the fermion correlators:

$$\langle e^{i\phi_\sigma(x,\tau)} e^{-i\phi_{\sigma'}(0,0)} \rangle = \delta_{\sigma\sigma'} \frac{a}{i\sigma x + \tau}, \tag{141}$$

where the expectation value is over the free φ action. This is exactly equal to the fermion correlator but for a factor of $a/2\pi$. Thus if we absorb this factor into the sources J, we can write the source term as

$$S_{src} = \int_{x,x'} J^\dagger(x) e^{-i(\xi+\eta Z)(x)} \langle e^{i\phi(x)} e^{-i\phi(x')} \rangle e^{i(\xi+\eta Z)(x')} J(x'). \tag{142}$$

Now we play a cute trick: the vertex operators have a Gaussian distribution (since they map to ψ, which has just a free action), and so we can pull the expectation value out of the exponential in $e^{-S_{src}}$, and realize the expectation value by integrating over the φ fields. Thus we just need to take the square root of the above integral, exponentiate it, and path integrate over φ. The parition function is then

$$Z = \int \mathcal{D}\xi \mathcal{D}\eta \int \mathcal{D}\varphi e^{-S_{eff}[\xi,\eta]-S_\varphi[\varphi]} \exp\left(-\int (J^\dagger e^{-i(\xi+\eta Z)} e^{i\varphi} + e^{-i\varphi} e^{i(\xi+\eta Z)} J)\right), \tag{143}$$

with $S_\varphi[\varphi]$ the free action for the φ fields and $S_{eff}[\xi,\eta]$ is the effective action for ϕ that we derived earlier.

This representation of the source term tells us that it would be nice if we had a decomposition of φ to a form like $\xi + \eta Z$. This is easily done by writing

$$\varphi_\pm = \Phi \pm \Theta. \tag{144}$$

One can check that the action S_φ becomes, in this representation,

$$S_\varphi = \frac{1}{2}\int_q (\Phi_q,\ \Theta_q)\widetilde{K}_q \begin{pmatrix} \Phi_{-q} \\ \Theta_{-q} \end{pmatrix}, \qquad \widetilde{K}_q = \begin{pmatrix} q^2 & iq\omega \\ iq\omega & q^2 \end{pmatrix}. \tag{145}$$

With this representation, we can eliminate the ξ, η fields from the source term by shifting the Φ and Θ fields. The sum of the free actions then becomes, after the shift (letting Ψ denote the $(\Phi,\ \Theta)$ doublet and letting Ξ denote the $(\xi,\ \eta)$ doublet)

$$S_{eff}[\xi,\eta] + \frac{1}{2}\int_q \left(\Psi_q^T \widetilde{K}_q \Psi_{-q} + \Xi_q^T \widetilde{K}_q \Psi_{-q} + \Psi_q^T \widetilde{K}_q \Xi_{-q} + \Xi_q^T \widetilde{K}_q \Xi_{-q}\right). \tag{146}$$

Here a miracle occurs. We change basis from the ϕ field to its Hodge representation Ξ by way of the matrix (I think this is listed incorrectly in the book?)

$$\phi_q = U_q \Xi_q, \qquad U_q = \begin{pmatrix} q - i\omega & q - i\omega \\ -q - i\omega & q + i\omega \end{pmatrix}. \tag{147}$$

The miracle is that

$$\widetilde{K}_q = -U_q^T G_\phi(q) U_{-q}. \tag{148}$$

This means that the $\Xi^T \widetilde{K} \Xi$ term in the last integral we wrote actually cancels with one of the terms in $S_{eff}[\xi,\eta]$ after we complete the switch from the ϕ representation to the (ξ,η) representation.

Recapitulating, the action (without the source term) is

$$S = \frac{1}{2}\int_q \left(\Psi_q^T \widetilde{K}_q \Psi_{-q} + \Xi_q^T \widetilde{K}_q \Psi_{-q} + \Psi_q^T \widetilde{K}_q \Xi_{-q} + \Xi_q^T U_q^\dagger \mathcal{G}_q^{-1} U_{-q} \Xi_q\right). \tag{149}$$

Since now only Ψ appears in the source term, we want to integrate out Ξ, which we can now do happily. We integrate it out to get

$$S = \frac{1}{2}\int_q \Psi_q^T \left(\widetilde{K}_q - \widetilde{K}_q [U_q^\dagger \mathcal{G}_q^{-1} U_{-q}]^{-1} \widetilde{K}_q\right) \Psi_{-q}. \tag{150}$$

To write this out explicitly, let us write $\mathcal{G}$ as (following the notation in the book now)

$$\mathcal{G} = g_4\mathbf{1} + g_2 X. \tag{151}$$

We know the explicit form for all the matrices in the above action, and so we can just multiply them out and see what we get. Our final bosonized form for the complete partition function is then

$$Z = \int \mathcal{D}\Theta\mathcal{D}\Phi \exp\left(-S_0 - S_{src}\right), \tag{152}$$

where

$$S_0 = \frac{1}{2\pi}\int_q (\Phi_q,\ \Theta_q)\begin{pmatrix} q^2(1+2\pi(g_4-g_2)) & iq\omega \\ iq\omega & q^2(1+2\pi(g_4+g_2)) \end{pmatrix}\begin{pmatrix}\Phi_{-q} \\ \Theta_{-q}\end{pmatrix}, \tag{153}$$

and

$$S_{src} = \int \left(J_\sigma^\dagger e^{i(\Phi+\sigma\Theta)} + e^{-i(\Phi+\sigma\Theta)} J_\sigma\right). \tag{154}$$

All done!

4 *Another look at currents and operator splitting applied to bosonization* ✓

Today is a fast one. Consider a free Dirac fermion in two dimensions. We will be identifying the currents (vector and chiral) and computing their commutators, being careful to do the point splitting of the operators that constitute the currents. We will find e.g. for the vector current that

$$[j_\mu(x), j_\nu(y)] = C\partial_x\delta(x-y), \tag{155}$$

where C is some constant that depends on how one normalizes the currents.

The regular fermion vector currents are $j^\mu = \bar\psi\gamma^\mu\psi$, so that in Euclidean signature with $\gamma^0 = X, \gamma^1 = Y$ we have for $\psi = (\psi_L, \psi_R)^T$,

$$j_R \equiv \frac{1}{2}(j_0 + ij_1) = \psi_R^\dagger\psi_R, \qquad j_L \equiv \frac{1}{2}(j_0 - ij_1) = \psi_L^\dagger\psi_L. \tag{156}$$

We will choose the coefficient in front of the action to be such that

$$\langle\psi_L^\dagger(z,\bar z)\psi_L(w,\bar w)\rangle = \frac{1}{z-w}, \qquad \langle\psi_R^\dagger(z,\bar z)\psi_R(w,\bar w)\rangle = \frac{1}{\bar z-\bar w}. \tag{157}$$

Since we get this from $\frac{1}{\pi}\bar\partial z^{-1} = \delta(z,\bar z)$, we want the coefficient in front of the action to be $g = 1/2\pi$.

Anyway, now we can compute the (equal-time) commutator of the currents. Of course, j_R and j_L commute. We then have (as usual, the following is to be understood in the OPE sense of having an implicit expectation value)

$$\begin{aligned}[j_R(x), j_R(y)] &= [:(\psi_R^\dagger\psi_R)(x):, :(\psi_R^\dagger\psi_R)(y):] \\ &= \lim_{\epsilon,\eta\to 0}[:\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon):, :\psi_R^\dagger(y+\eta)\psi_R(y-\eta):]. \\ &\sim i\lim_{\epsilon,\eta\to 0}\Big(\frac{1}{x+\epsilon-y+\eta}\delta(x-y-\epsilon-\eta) - \frac{1}{x+\epsilon-y+\eta}\psi_R^\dagger(y+\eta)\psi_R(x-\epsilon) \\ &\quad - \frac{1}{y+\eta-x+\epsilon}\psi_R^\dagger(x+\epsilon)\psi_R(y-\eta) - \frac{1}{(y+\eta-x+\epsilon)(x+\epsilon-y+\eta)}\Big) \\ &\quad - (x\leftrightarrow y, \epsilon\leftrightarrow\eta),\end{aligned} \tag{158}$$

where the factor of i came from the i in $1/(\bar z-\bar w)$ evaluated at $\bar z = -ix, \bar w = -iy$. There may be other factors of 2 somewhere but I'm not going to worry about them too much. The most singular term is symmetric under the interchange $x\leftrightarrow y, \epsilon\leftrightarrow\eta$ and so it dies, while the two middle terms are also symmetric under the interchange. So

$$\begin{aligned}[j_R(x), j_R(y)] &\sim i\lim_{\epsilon,\eta\to 0}\Big(\frac{1}{x+\epsilon-y+\eta}\delta(x-y-\epsilon-\eta) - \frac{1}{y+\eta-x+\epsilon}\delta(y-x-\eta-\epsilon)\Big) \\ &= i\lim_{\epsilon,\eta\to 0}\left(\frac{\delta(x-y-[\epsilon+\eta]) - \delta(x-y+[\epsilon+\eta])}{\eta+\epsilon}\right) = -i\partial_x\delta(x-y).\end{aligned} \tag{159}$$

When we compute the commutator for j_L the only thing changes is that we have a $-i$ up front instead of a $+i$ by virtue of the ψ_L 2-point function being $1/(z-w)$, so

$$[j_R(x), j_R(y)] \sim i\partial_x\delta(x-y). \tag{160}$$

Now we can go and rewrite this in terms of the spacetime components of the current. Since $j_0 = j_R + j_L, j_1 = -i(j_R - j_L)$ and the $j_{R,L}$ commutators are opposite in sign, we have

$$[j_0(x), j_0(y)] = [j_1(x), j_1(y)] = 0, \qquad [j_0(x), j_1(y)] \sim -2\partial_x\delta(x-y). \tag{161}$$

We could probably have chosen a smarter normalization for the currents so that this dumb factor of 2 wasn't there, but too late. Actually from now on I think it'll be good to go over into real time. Doing this means we need to multiply j_1 or j_0 by i, depending on the signature we want (since we need to change either of the γ matrices to i times itself in order to get the right Clifford algebra relations with the new metric—for definiteness we will let the real-time γ matrices be $\gamma^0 = X, \gamma^1 = iY$), and so in real time we have

$$[j_0(x), j_1(y)] \sim -2i\partial_x\delta(x-y). \tag{162}$$

Now consider a free boson ϕ, and define the currents

$$\mathcal{J}_0 = \sqrt{2}\partial_x\phi, \qquad \mathcal{J}_1 = -\sqrt{2}\partial_t\phi, \tag{163}$$

so that $\mathcal{J}_\mu = \sqrt{2}\epsilon_{\mu\nu}\partial^\nu\phi$ is equal to (the dumb factor of $\sqrt{2}$ times) the topological current, which if ϕ is smooth is trivially conserved. Since the conservation of the fermion current will fail at the locations of certain operator insertions, this tells us that these operator insertions create a topological singularity in the dual ϕ field (so that ϕ is not integrable). Anyway, since $\partial_t\phi$ is the momentum we have

$$[\mathcal{J}_\mu(x), \mathcal{J}_\nu(y)] = -2i\epsilon_{\mu\nu}\partial_x\delta(x-y) = [j_\mu(x), j_\nu(y)]. \tag{164}$$

Thus we have found a way to represent the fermion current as the topological current of a boson. Note in particular that under bosonization,

$$-j_\mu j^\mu \mapsto -2\epsilon_{\mu\nu}\partial^\nu\phi\epsilon^{\mu\sigma}\partial_\sigma\phi = 2\partial_\mu\phi\partial^\mu\phi, \tag{165}$$

so that the current bilinear for the fermions becomes the free kinetic term for the boson. This is actually not surprising if we think about the Sugawara construction for the stress tensor: since in models of current algebras we have $T \sim \sum_a : J^a J^a :$, it's not that crazy to think that the current bilinear will bosonize to $\partial\phi\bar{\partial}\phi$, since this is a similar sort of object to the stress tensor.

Anyway, what about the chiral current for the fermions? In two dimensions for our choice of γ matrices we have

$$\gamma^0\gamma^5 = XZ = -iY = \epsilon^{01}iYg_{11}, \tag{166}$$

since for us, $g_{11} = -1$ and $\epsilon^{P(\mu\nu)} = \mathrm{sgn}(P),\ \epsilon_{\mu\nu} = -\epsilon^{\mu\nu}$). We also have

$$\gamma^1\gamma^5 = iYZ = \epsilon^{10}Xg_{00}. \tag{167}$$

Putting these together means that

$$\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu. \tag{168}$$

So this means that the chiral current is related to the vector current via

$$j^{\mu 5} = \epsilon^{\mu\nu}j_\nu, \tag{169}$$

which means that the bosonic avatar of the chiral current is

$$\mathcal{J}^5_\mu = \sqrt{2}\epsilon^{\mu\nu}\epsilon_{\nu\sigma}\partial^\sigma\phi = \sqrt{2}\partial_\mu\phi, \tag{170}$$

which is conserved by virtue of the free boson's equation of motion.

5 *Bosonizing the spin* 1/2 *chain and the sine-Gordon model* ✓

Today we will bosonize the $SU(2)$-symmetric spin 1/2 AFM spin chain (Heisenberg model) using *Abelian* bosonization. This is of course in books / the literature, but I wanted to go through it at least once on my own.

The spin chain is described by (setting the prefactor $J = 1$ for simplicity)

$$H = \sum_{j,a} S^a_j S^a_{j+1}. \tag{171}$$

The grand strategy is to write things in terms of spinless fermions by way of a Jordan-Wigner transformation, and then do bosonization on these fermions.[26]

Anyway, we will work in a basis where S^z is diagonalized. The appropriate strings are built out of $\prod_{i<j}(1-2c_i^\dagger c_i) = \prod_{i<j}(-1)^F_i$, and the raising / lowering operators are

$$S^+_j = (-1)^j \prod_{i<j}(-1)^F_i c_j, \qquad S^-_j = (-1)^j \prod_{i<j}(-1)^F_i c_j^\dagger. \tag{172}$$

The raising and lowering operators have been staggered by a factor of $(-1)^j$ since we anticipate expanding round a staggered spin configuration (this of course doesn't affect the operator algebra). One then uses $[S^+_j, S^-_j] = Z$ to get

$$S^z_j = -\frac{1}{2}(-1)^F_j. \tag{173}$$

Checking that all these operators commute with one another as they should is straightforward.

Now we need to plug these into the Hamiltonian to get H as a function of the c operators. This is straightforward and produces (rescaling J by a factor of 4)

$$H = J\sum_j \left[(-1)^F_j(-1)^F_{j+1} - 2(c^\dagger_{j+1}c_j + c^\dagger_j c_{j+1})\right]. \tag{174}$$

Note that the presence of the S^zS^z term is responsible for giving us the interactions: if the Hamiltonian only had $U(1)$ symmetry (instead of $SU(2)$), we would have free fermions. Also note that if we had a very strong anisotropy for the S^zS^z term we would need $(-1)^F_j = -(-1)^F_{j+1}$, so that the fermion occupancy on the chain would be staggered. This is the CDW state.

[26] Hold on you may say, to do bosonization we will need something with central charge $c = 1$, i.e. a two-component fermion. So don't we need spinful fermions? We do not, since the two "spin" components will come from the left- and right-moving fermion excitations around the two Fermi points.

Now we want to go over to continuum fermions. We build a Dirac fermion out of the the fermionic excitations around each of the Fermi points at $k_F = \pm\pi/2$ (since the hopping term gives a $\cos k$ dispersion for $-\pi/2 \le k \le \pi/2$). Let the lattice spacing be a, and define continuum fields L, R via

$$c_j = \sqrt{a}(R_j e^{ik_F j} + L_j e^{-ik_F j}) = \sqrt{a}(i^j R_j + i^{-j} L_j). \tag{175}$$

The $\sqrt{a}$ here is to get the dimensions right: we want the continuum fermions to have mass dimension $[L] = [R] = 1/2$, while the c_j fermions are dimensionless by virtue of $\{c_i, c_j^\dagger\} = \delta_{ij}$. One of the hopping terms is then

$$c_j^\dagger c_{j+1} \approx a(i^{-j} R_j^\dagger + i^j L_j^\dagger)(i^{j+1} R_j + i^{j+1} a\partial_x R_j + i^{-j-1} L_j + i^{-j-1} a\partial_x L_j). \tag{176}$$

Higher derivative terms get suppressed by higher powers of a. When we sum over j, terms that oscillate with j will die—they represent high energy events that hop fermions between the two Fermi points, with operators like $L_j^\dagger \partial R_j$, and so we can get rid of them. The expression above has $iR^\dagger R$ and $iL^\dagger L$ terms, but when we add the Hermitian conjugate these terms die, and so

$$a^{-1} \sum_j (c_j^\dagger c_{j+1} + h.c.) = -2i \int dx\, (R^\dagger \partial_x R - L^\dagger \partial_x L), \tag{177}$$

which gives us the free Dirac fermion.

Now for the interaction term. Since $k_F = \pi/2$, we are at half filling for the spinless fermions. Thus $\langle c_j^\dagger c_j \rangle = 1/2$, and so

$$-(-1)_j^F = 2c_j^\dagger c_j - 1 = 2 : c_j^\dagger c_j := 2a : (i^{-j} R_j^\dagger + i^j L_j^\dagger)(i^j R_j + i^{-j} L_j) : \tag{178}$$

The interaction term then becomes, in the continuum variables,

$$\begin{aligned} a^{-1} \sum_j (-1)_j^F (-1)_{j+1}^F \approx \int dx\ &: \big(R^\dagger R + L^\dagger L + (-1)^j (L^\dagger R + R^\dagger L)\big) : \\ &\times : \big(R^\dagger R + L^\dagger L - (-1)^j (L^\dagger R + R^\dagger L)\big) : \end{aligned} \tag{179}$$

Here we have dropped all derivatives, since they all contain an extra factor of a that make them comparatively small (remember that we are assuming the L, R vary slowly over the lattice scale). We can drop the terms that go as $(-1)^j$, but we still get an Umklapp term from the $(-1)^{2j} = 1$ term. From yesterday's problem, we recall the currents $j_0 = R^\dagger R + L^\dagger L$, $j_1 = L^\dagger L - R^\dagger R$ (the sign of j_1 is dictated by our choice of $\gamma^1 = -iY$, which we did so that $\gamma^5 = Z$ and not $-Z$). We then have the current bilinear $(j_\mu)^2 = 2[(R^\dagger R)^2 + (L^\dagger L)^2]$. Note that I have stopped indicating the normal ordering, for notation's sake. However, it is important to remember that it is there, so that e.g. $(R^\dagger R)^2$ really means $(: R^\dagger R :)^2$. When we do bosonization, we will need to be careful to only bosonize things that have been normal-ordered.

After some algebra, we can then write the interaction term as (there are some factors of 2 that we've absorbed into a, all we care about is the relative factor between the different terms)

$$H_I = \int dx\ \left(j_\mu j^\mu - 2\left[(L^\dagger R)^2 + (R^\dagger L)^2\right]\right). \tag{180}$$

This means that the full action in terms of the fermions becomes ($\Psi = (L, R)^T$)

$$S = \frac{1}{2\pi}\int dxdt\, \left(i\bar{\Psi}\partial\!\!\!/\Psi - j_\mu j^\mu + 2\left[(L^\dagger R)^2 + (R^\dagger L)^2\right]\right). \tag{181}$$

I think the $1/2\pi$ factor in front will be the most convenient for avoiding gross $\sqrt{4\pi}$'s and stuff, but I'm not sure. We'll see how it goes.

Now let us bosonize. We will roughly follow the normalization conventions in Witten's lectures in Quantum Fields and Strings part II. With these conventions, the Dirac action is bosonized to

$$\mathcal{B}(S_D) = \frac{1}{8\pi}\int \partial_\mu\phi\partial^\mu\phi, \tag{182}$$

which is a compact boson at radius $R = 1/\sqrt{2}$. In terms of the holomorphic / anti-holomorphic components of the boson, the mapping is

$$\mathcal{B}(L) = e^{i\phi_+}, \qquad \mathcal{B}(R) = e^{-i\phi_-}, \tag{183}$$

which one can check reproduces the correct scaling dimensions (the vertex operators $e^{\pm i\phi_\pm}$ have a two-point function that goes like $1/(x-y)^{1/2R^2}$, which is what we want since $R = 1/\sqrt{2}$). If we were to be a bit more careful, we should probably write this as $\mathcal{B}(L) = \frac{1}{\sqrt{2\pi a}}e^{i\phi_+}$ where a is the short-distance cutoff (and we should probably be writing $\sim$ instead of $=$). This ensures that the bosonized fermion has the same dimension of the fermion, and still produces the right correlators since when we are remembering to include the cutoff the propagator for the boson goes like $\ln|r/a|$ instead of just $\ln|r|$. Anyway, we'll suppress the cutoff dependence in what follows.

We can then conclude that

$$\mathcal{B}([LR^\dagger]^2) = e^{2i\phi}, \tag{184}$$

which means that the extra interaction term in the fermionic version of S maps to a sum of vertex operators. The freeness of the bosonized theory is thus ruined by the extra interacting part in S. Anyway, we can now map the full action over to bosons:

$$\mathcal{B}(S) = \int dxdt\, \left[\frac{(1+1)}{8\pi}\partial_\mu\phi\partial^\mu\phi + \frac{1}{\pi}\cos 2\phi\right]. \tag{185}$$

The only part of this that is questionable is second number 1 in $1+1$; this comes from the bosonization of the Thirring-type interaction using today's bosonization conventions. This 1 can (but won't, sorry) be checked by checking the relevance of the $\cos 2\phi$ interaction, by rescaling the fields so that the factor in front of the kinetic term is $1/2$: we let $\varphi = \phi/\sqrt{2\pi}$, and the interaction cosine becomes $\cos 2\phi = \cos(\beta\varphi)$, with $\beta = \sqrt{8\pi}$.

Finally, a side comment on adding interactions to the fermion Hamiltonian. The interactions that are both tractable and interesting are the current-current terms, which can be written in terms of densities as products like $\rho_L\rho_R$ and which bosonize to the free term. Consider on the other hand an interaction I like

$$H \ni I = \int d^2x\, d^2y\, \rho(x)V(x-y)\rho(y), \tag{186}$$

where $V(x-y)$ is taken to be a contact interaction $\delta(x-y)$. Using

$$\rho = \rho_L + \rho_R + (\psi_R^\dagger \psi_L e^{-2ik_F x} + h.c.), \tag{187}$$

we see that (up to an Umklapp term)), the density-density contact term becomes precisely $\rho_L^2 + \rho_R^2$. Since we are thinking in terms of Hamiltonians we do the point-splitting in space, and so this bosonizes to $(\partial_x \phi_L)^2 + (\partial_x \phi_R)^2$, which just renormalizes the speed of light for the bosons, as we expect from a contact term. So the $\rho_{R/L}^2$ terms are less interesting than the $\rho_L \rho_R$ ones.

6 *Canonical momenta and commutation relations in bosonization*

We will find the CCRs for the free chiral components of the compact boson by using a bosonization approach. This result has already been derived in previous diary entries, but here we will use a different approach, viz. by using the known commutators of fermion density operators.

First we need to know where the R density goes (we are working in $\mathbb{R}$ time for this problem!)

$$\begin{aligned} \mathcal{B}[: n_R(x) :] &= \mathcal{B}[\psi_R^\dagger(x+\epsilon)\psi_R(x) - i\epsilon^{-1}] \\ &= e^{i\phi_R(x+\epsilon)} e^{-i\phi_R(\epsilon)} - \frac{i}{\epsilon} \\ &= : e^{i\phi_R(x+\epsilon)} e^{-i\phi_R(\epsilon)} : \frac{i}{\epsilon} - \frac{i}{\epsilon} \\ &= -\partial_x \phi_R(x). \end{aligned} \tag{188}$$

The overall sign here at the end isn't really important and may be incorrect anyway. Like everything in bosonization, the prefactors and minus signs are an unmitigated mess. In the above, we have used the fact that the (equal-time) correlator for the R (L) fermions is $1/(-ix)$ $(1/ix)$. In our conventions we will have $\mathcal{B}[\psi_L(x)] = e^{i\phi_L(x)}$ (this sign is different than in some past diary entries, and is chosen so that $\int \partial_x \theta$ is the thing measuring the fermion number), and thus $\mathcal{B}[: n_L(x) :] = \partial_x \phi_L(x)$. Thus $\mathcal{B}[: n_R(q) :] = q\phi_R(q)$, and the same for $n_L(q)$ but with a minus sign. In fact this bosonization relation for the fermion densities is probably a more natural place to start than the relation for the individual fermion operators themselves—the physical hydrodynamic fields θ, ϕ (or rather, their derivatives) are related to the densities of the fermions, and are $(-1)^F$ even.

Anyway, the L and R fermion densities commute, so the ϕ_L and ϕ_R fields must also commute.[27] We can determine the appropriate commutation relations for the $\phi_{L/R}$ using the commutation relations of the density operators. We have[28]

$$\rho_R(q) = \int_p \psi_p^\dagger \psi_{p-q} = \int_p \left(: \psi_p^\dagger \psi_{p-q} : + \langle \psi_p^\dagger \psi_{p-q} \rangle\right) = \int_p \left(: \psi_p^\dagger \psi_{p-q} : + \delta_{q,0}\theta(-p)\right). \tag{189}$$

In the last step we used that the right-movers are only occupied in the ground state if their momentum is negative (relative to k_F). Now (momentarily dropping R subscripts on ψs)

$$\begin{aligned}
[\rho_R(q), \rho_R(p)] &= \int_{l,k} \left(\delta_{l-q,k}\psi_l^\dagger \psi_{k-p} - \delta_{l,k-p}\psi_k^\dagger \psi_{l-q}\right) \\
&= \int_l \left(\psi_l^\dagger \psi_{l-q-p} - \psi_{l+p}^\dagger \psi_{l-q}\right) \\
&= \int_l \left(: \psi_l^\dagger \psi_{l-q-p} : - : \psi_{l+p}^\dagger \psi_{l-q} : + \delta_{q,-p}\theta(-l) - \delta_{p,-q}\theta(-l-p)\right) \\
&= \delta_{p,-q} \int_l (\theta(-l) - \theta(-l-p)) \\
&= \delta_{p,-q} p.
\end{aligned} \tag{190}$$

Here the units are right since ρ_q is dimensionless, and we take $\int_q \delta_{q,0} = 1$ so that the dimension of the δ and p cancel. Thus in $\mathbb{R}$ space we get

$$[\rho_R(x), \rho_R(y)] = \int_p p e^{ip(x-y)} = -i\partial_x \delta(x-y). \tag{191}$$

Bosonizing, this means that

$$[\partial_x \phi_R(x), \partial_y \phi_R(y)] = i\partial_x \delta(x-y) \implies [\phi_R(x), \partial_y \phi_R(y)] = i\delta(x-y). \tag{192}$$

Thus the canonical momentum conjugate to $\phi_R(x)$ is actually just its derivative, $\partial_x \phi_R(x)$. We can also write this as

$$[\phi_R(x), \phi_R(y)] = \frac{i}{2}\mathrm{sgn}(x-y). \tag{193}$$

This commutation relation is what allows the vertex operators $e^{i\phi_R(x)}$ to be fermionic.

[27] Thus the identification of the single fermions with the vertex operators is not strictly correct—we need the Klein factors to get the statistics straight. Some sources (e.g. Shankar) work with different wonky conventions where the mixed commutator of the $\phi_{L/R}$ is nonzero so that the Klein factors can be done away with, but this isn't conceptually ideal—thinking from a CFT perspective, it's always best to have the L and R sectors be completely decoupled.

[28] Recall that normal-ordering is done so that all the operators which annihilate the ground state are placed to the right. This is *not* the same as placing all annihilation operators to the right: for a system with a finite density of fermions, an annihilation operator with $k > k_F$ will kill the ground state, while a creation operator with $k < k_F$ will also kill the ground state. Thus normal ordering acts nontrivially on things like $\psi^\dagger(x)\psi(x)$, since each of the operators involved involves a sum of many different operators, some of which need to be moved to the right, and some of which do not.

When we repeat this procedure for the ψ_L's, the only thing that is different is the expectation value $\langle \psi^\dagger_{L,p}\psi_{L,p-q}\rangle = \delta_{q,0}\theta(p)$, which gives a minus sign so that the momentum conjugate to $\phi_L(x)$ is $-\partial_x\phi_L(x)$, and we have

$$[\phi_L(x), \phi_L(y)] = -\frac{i}{2}\mathrm{sgn}(x-y). \tag{194}$$

Now let's see if we recover the Lagrangian. We know from an earlier diary entry that the Hamiltonian is (ignoring constant prefactors)

$$H = \int dx \left[(\partial_x\phi_L)^2 + (\partial_x\phi_R)^2\right]. \tag{195}$$

Thus the action should be

$$S = \int dx\, dt\, \left(\partial_x\phi_R(\partial_t - \partial_x)\phi_R - \partial_x\phi_L(\partial_t + \partial_x)\phi_L\right). \tag{196}$$

This looks rather mysterious since it does not have any terms quadratic in time derivatives, even though we know that the action is $\partial\phi\bar\partial\phi$, which is quadratic in time derivatives. Furthermore in this presentation the ϕ_L and ϕ_R are decoupled, whereas in the $\partial\phi\bar\partial\phi$ presentation they are not if we just substitute in $\phi = \phi_L + \phi_R$. Of course the way to make sense of this is to introduce $\phi = \phi_L + \phi_R, \theta = \phi_L - \phi_R$. Then after some algebra and integrating by parts, we get (again not writing constant prefactors)

$$S = \int dx\, dt\, \left(-(\partial_x\phi)^2 - (\partial_x\theta)^2 + 2\partial_x\phi\partial_t\theta\right). \tag{197}$$

Now the equation of motion for θ says that $\partial_x^2\theta = \partial_x\partial_t\phi$, so that $\partial_x\theta = \partial_t\phi$ (this is just the usual $d\theta = \star d\phi$ thing), and so the action goes to (again after integrating by parts)

$$S = \int dx\, dt\, \left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right], \tag{198}$$

which is finally what we expect from $\int \partial\phi\bar\partial\phi$ (in our signature $\partial = -\partial_x + \partial_t, \bar\partial = \partial_x + \partial_t$).

Finally, we note that since $\partial_x\phi \leftrightarrow n_R - n_L$ and $\partial_x\theta \leftrightarrow n_R + n_L$, we have that $\theta(x)$ counts the total fermion number (relative to $-\infty$), while $\phi(x)$ counts the net chirality (these statements are dependent on the sign conventions made for the bosonization mapping). Inserting $e^{i\theta}$ creates a vortex in ϕ, around which $\oint \partial\phi = 2\pi$. This means that inserting $e^{i\theta}$ at a given time changes the chiral charge, since $\oint \partial_x\phi$ takes on different values before and after the insertion. This jives with the fact that the vertex operator for θ bosonizes to

$$e^{i\theta(x)} \leftrightarrow \psi_L^\dagger(x)\psi_R(x), \tag{199}$$

which is indeed a scattering operator that indeed changes the net value of $n_R - n_L$. Likewise, a vertex operator for ϕ creates a vortex for θ, which means that it must change the fermion number $n_L + n_R$. This in turn jives with the fact that it bosonizes to $e^{i\phi} \leftrightarrow \psi_L^\dagger\psi_R^\dagger$. Relatedly, we can kind of motivate why in this formulation spatial translations map to shifts in θ by constants. The vertex operator $e^{i\theta}$ shifts a right-moving fermion to a left-moving one, and thus shifts the total momentum. Since it shifts the momentum, it should not commute with the momentum $\int dx\, T_{01}$, which is $\sim \int dx\, \partial_x \phi$. In terms of the holomorphic and antiholomorphic fields, we then use the commutation relations of the $\phi_{L/R}$ to conclude tht spatial translations do $\phi_L \mapsto \phi_L + c, \phi_R \mapsto \phi_R - c$. Again, this is expected from the fermion side, by using the usual decomposition $\psi(x) = e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x)$.

7 *Chiral almost-symmetry-breaking in two dimensions via bosonization* ✓

Today we're looking at an awesome paper by Witten [**?**]. The goal is to explore the exact relation between QLRO and masslessness in two dimensions. We will make an attempt to work with simplified normalization conventions so as to avoid a lot of the tedious numerical factors present in Witten's paper.

The model is

$$S = \frac{1}{8\pi}\int d\sigma \wedge \star d\sigma + \frac{1}{2\pi}\int i\bar{\psi}\partial\!\!\!/\psi + \frac{g}{2}\int \bar{\psi}(\cos\sigma + i\bar{\gamma}\sin\sigma)\psi, \tag{200}$$

where $\bar{\gamma} = Z$ is the chirality operator. Note that we can write the last interaction term as $\bar{\psi}e^{i\bar{\gamma}\sigma}\psi$, and that this interaction is marginal since the radius of the σ boson is $R_\sigma = 1/\sqrt{2}$, giving the dimension of $e^{i\bar{\gamma}\sigma}$ to be $1/(2R_\sigma)^2 = 1$.

The chiral symmetry of interest is the $U(1)$ symmetry

$$U(1)_A : \psi \mapsto e^{i\alpha\bar{\gamma}/2}\psi, \qquad \sigma \mapsto \sigma + \alpha, \tag{201}$$

which is a symmetry of the action since $\bar{\psi} \mapsto \bar{\psi}e^{+i\alpha\bar{\gamma}}$. A naive guess would be that for large coupling, we get spontaneous symmetry breaking of $U(1)_A$ by virtue of σ acquiring a vev, thereby leading to a fermion mass term. If this happened, the spectrum would contain a Goldstone boson and massive fermions. But of course we are in two dimensions, so this SSB scenario is impossible.

Instead, we will show that the spectrum of the model is a massless boson and a massive fermion as expected from SSB, but that the symmetry is unbroken. We will show the latter by demonstrating that all the symmetry-breaking Greens functions vanish.

We first bosonize $\psi = (\psi_+, \psi_-)^T$, with the conventions

$$\mathcal{B}[\psi_\pm] = e^{\pm i\phi_\pm}. \tag{202}$$

Then the interaction between the fermions and σ is dealt with by using

$$\mathcal{B}[\bar{\psi}\psi] = -\mathcal{B}[\psi_- \psi_+^\dagger + \psi_+ \psi_-^\dagger] = -(e^{-i\phi} + e^{i\phi}) = -2\cos\phi \tag{203}$$

where the minus sign arises due to today's bosonization conventions. Likewise,

$$\mathcal{B}[\bar{\psi}\bar{\gamma}\psi] = \mathcal{B}[\psi_- \psi_+^\dagger - \psi_+ \psi_-^\dagger] = -2i\sin\phi. \tag{204}$$

Thus the action becomes, using $\cos\sigma\cos\phi - \sin\sigma\sin\phi = \cos(\phi+\sigma)$,

$$S = \frac{1}{8\pi}\int (d\sigma \wedge \star d\sigma + d\phi \wedge \star d\phi) - g\int \cos(\phi+\sigma). \tag{205}$$

Note that the interacting part only involves the combination $\phi + \sigma$[29] (this scenario in which one of the two fields decouples and becomes free is exactly what happens in the Schwinger model, where the spectrum contains a decoupled free boson and an interacting massive one). This means that we should define new variables

$$\lambda \equiv \phi + \sigma, \qquad \gamma \equiv \phi - \sigma. \tag{206}$$

The action is then written suggestively as

$$S = \frac{1}{8\pi}\int \left(\frac{1}{2} d\gamma \wedge \star d\gamma + \frac{1}{2} d\lambda \wedge \star d\lambda\right) - g\int \cos(\lambda). \tag{207}$$

Thus we have produced a theory consisting of a free boson and an interacting boson. Since the interaction is marginal, we expect that the spectrum should contain a free γ boson, and a massive fermion (the kink / antikink soliton for λ). We expect the fermion to be massive here because the marginal cosine will be pushed towards marginal relevance by loop corrections. This happens because the effective action obtained from the momentum-shell scheme will contain a term like $-\frac{g^2}{2}\int d^2x\, d^2y\, \cos(\lambda(x) - \lambda(y))$, which when expanded contains a term proportional to $+g^2(\partial\lambda)^2$. Therefore the $\cos(\lambda)$ leads to a renormalization which increases the radius of λ, and since the dimension of $\cos(\lambda)$ goes as $1/R^2$, an increased radius decreases the dimension of the cosine, making it marginally relevant.[30]

To write the theory in terms of the physical variables then, we will fermionize λ with a fermion η such that

$$\eta_\pm \leftrightarrow e^{\pm i\lambda_\pm}, \tag{209}$$

[29]This is again of course still relevant, since it has dimension $2/(2R_\sigma^2) = 2$.

[30]In the XY model with an added vortex term the situation is opposite, with the cosine tipped towards irrelevance. That is, consider (still in $i\mathbb{R}$ time)

$$S = \frac{1}{2\pi}\int \left(\partial_t\phi\partial_x\theta + (\partial_x\phi)^2 + (\partial_x\theta)^2 + g\cos\theta\right). \tag{208}$$

Here the radius of ϕ is 2, so that the cosine is marginal. By the same argument, the RG at one-loop generates a term in the action which schematically $+g^2(\partial\theta)^2$, thus decreasing the "standard deviation" of the θ field. When θ is integrated out this then *decreases* the radius of ϕ, which in turn makes $\cos(\theta)$ less relevant, since the dimension of $\cos(\theta)$ goes as R^2. Therefore cosines of the dual field become marginally irrelevant while those of the primary field become marginally relevant.

which will turn the cosine into a fermion mass term. It might feel a bit silly to be going back to fermion variables, but we needed this intermediate purely bosonic step in order to decouple the fields into the massive and massless parts. Since the coefficient of the λ kinetic term is not $1/8\pi$ (the radius of λ is $1/2$ rather than $1/\sqrt{2}$), we will not get simply the free Dirac kinetic term — the extra $-1/16\pi$ in the kinetic term for λ will lead to a current-current interaction in the new action. If we look back at our diary entry on general bosonization conventions, we see that the relation between the boson radius and the Thirring interaction strength U looks like $R^2 = \frac{1}{2}\sqrt{(1-U)/(1+U)}$, hence in our case we get an interaction at $U = 3/5$. Thus in these variables the action becomes

$$S = \frac{1}{2\pi}\int \left(\bar{\eta} i\partial\!\!\!/ \eta + 2\pi U j_\mu j^\mu\right) + \frac{g}{2}\int \bar{\eta}\eta + \frac{1}{16\pi}\int d\gamma \wedge \star d\gamma. \tag{210}$$

So we have finally arrived at what we wanted: a theory with a free decoupled boson and a massive (but interacting; the solitons are not free) fermion. However, there is *no* SSB (as there must not be, in accordance with the CMW theorem), even though the fermion in the spectrum is massive. We can check this by computing the chirality-nonconserving Greens functions that would be nonzero in the case of SSB, like $\langle \psi_\pm \psi_\mp^\dagger \rangle$. To do this we need

$$\psi_\pm \leftrightarrow e^{\pm i\phi_\pm} = e^{\pm i\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)_\pm}. \tag{211}$$

This means that e.g.

$$\langle \psi_\pm(x)\psi_\mp^\dagger(0)\rangle \leftrightarrow \langle e^{\pm i(\lambda/2)_\pm(x)} e^{\pm i(\lambda/2)_\mp(0)}\rangle \langle e^{\pm i(\gamma/2)_\pm(x)} e^{\pm i(\gamma/2)_\mp(0)}\rangle. \tag{212}$$

In order to find out what this is, we need to know the correlation functions of the chiral vertex operators for the rescaled field $\lambda/2$ (note to self: should come back and just write things in terms of the fields and their duals rather than the chiral components, which is much more confusing). This is slightly nontrivial since multiplicative rescaling of the fields does not preserve the holomorphic / antiholomorphic decomposition of the fields, i.e. we have

$$(a\phi')_\pm \neq a \cdot \phi_\pm, \quad a \in \mathbb{C}, \tag{213}$$

essentially because $\phi_\pm$ do not commute with eachother and since momenta and position get scaled oppositely in order to preserve the CCR. A little digression on this since it's interesting: the holomorphic and antiholomorphic parts of the field are defined by (this is a non-local definition as it must be)

$$\phi_\pm(x) = \frac{1}{2}\left(\phi(x) \pm \int_{-\infty}^{x} dx'\, \Pi(x')\right), \tag{214}$$

where $\Pi = \partial_x \theta$ is the momentum (which leads to the standard $\theta = \phi_+ - \phi_-$). The nonlocal nature of this definition is needed to ensure that the chiral vertex operators anticommute with one another and thus have a chance to become fermions[31] (passing one $e^{i\phi_\pm}$ around

[31] Here by "become fermions" we mean in the weak sense, where the vertex operators behave like fermions when inserted into correlation functions. They aren't *really* fermions in the constructive sense since they do not change the fermion number. To have the vertex operators be fermions in the strong sense of them being equal to fermion fields as operators, we need Klein factors. The Klein factors basically just change the parity of the fermion vacuum by subtracting a fermion from the Dirac sea and then re-arranging all the existing fermions to create the new vacuum state. In any case, for now we will only need bosonization in the weak sense of matching correlation functions, so these subulties won't come up.

another encircles the latter in a $\Pi(x')$ string, which after being wiggled straight must pass over the former vertex operator, which gives the interaction needed for fermionic statistics). Rescaling this, we get the non-homogenous transformation

$$(a\phi)_\pm(x) = \frac{1}{2}\left(a\phi(x) \pm a^{-1}\int_{-\infty}^{x} dx'\,\Pi(x')\right), \qquad a \in \mathbb{C}. \tag{215}$$

Solving for $\phi(x)$ and the momentum integral in terms of the original (unscaled) field, we see that under rescaling the chiral components of the fields get mixed by a "boost"

$$\begin{pmatrix}(a\phi)_+\\(a\phi)_-\end{pmatrix} = \begin{pmatrix}\cosh(\ln a) & \sinh(\ln a)\\ \sinh(\ln a) & \cosh(\ln a)\end{pmatrix}\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}. \tag{216}$$

This Bogoliubov transformation is consistent with the map $\phi \mapsto a\phi, \theta \mapsto a^{-1}\theta$, required in order to preserve the CCR (note that in accordance with yesterday's diary entry, the CCR here are $[\phi_\pm(x), \partial_x\phi_\pm(y)] \propto i\delta(x-y)$: if we just imposed the CCR on θ, ϕ then we could also get away with having $\phi_\pm, \partial_x\phi_\pm$ commute but $\phi_\pm, \partial_x\phi_\mp$ not commute).

Now one may wonder whether it is $\widetilde{\phi}_\pm \equiv (a\phi)_\pm$ which is left / right moving, or whether it is $\phi_\pm$. Thinking about this is perhaps easier in the Hamiltonian formulation. Here since $:\psi_L^\dagger\partial_x\psi_L:$ gets mapped to $(\partial_x\phi_+)^2$ while the same for ψ_R gets mapped to $(\partial_x\phi_-)^2$, the free fermion Hamiltonian is bosonized to

$$H \propto \int dx\ \left[(\partial_x\phi_+)^2 + (\partial\phi_-)^2\right] \tag{217}$$

Now the re-scaling of the ϕ field is accomplished by adding in interactions for the fermions of the form $j_\mu j^\mu$. In terms of fermions this is proportional to $\rho_L\rho_R$, which bosonizes to $(\partial_x\phi_+)(\partial_x\phi_-)$. Thus adding this interaction term to the Hamiltonian means that the bosonized Hamiltonian now has a cross term between $\partial_x\phi_+$ and $\partial_x\phi_-$. We can re-write the Hamiltonian without the cross term, but this requires making a field re-definition. Since only the Hamiltonian with the new re-defined fields separates into a sum of left- and right-moving parts, only the new re-defined fields are holomorphic / anti-holomorphic. This makes sense since adding the $j_\mu j^\mu$ interaction mixes the left- and right-moving fermions, and so the bosonized images of the original fermions should not be purely left- or right-moving.

Anyway, applying this to the problem at hand, we find

$$\langle\psi_+(x)\psi_-^\dagger(0)\rangle \sim \langle e^{i(5\lambda_+(x)/4-3\lambda_-(x)/4)}e^{i(5\lambda_-/4(0)-3\lambda_+(0)/4)}\rangle\langle e^{i(5\gamma_+(x)/4-3\gamma_-(x)/4)}e^{i(5\gamma_-/4(0)-3\gamma_+(0)/4)}\rangle. \tag{218}$$

Now we don't need to know what the correlator of the λ vertex operators are (this is tricky since λ is not free), but we do know what the γ vertex operator correlator is, since γ is free. In a previous diary entry we saw that in order for the correlator to be nonzero, we had to have "charge neutrality" for both of chiral fields $\gamma_\pm$ in the vertex operators if the correlator was to be non-zero, otherwise the correlator vanishes because of infrared effects. Since in this case the correlators are not charge-neutral $(5/4 - 3/4 \neq 0)$, we get

$$\langle\psi_+(x)\psi_-^\dagger(0)\rangle = 0. \tag{219}$$

By checking the other relevant Greens functions, we see that indeed, the symmetry is unbroken. That the cause of the correlators being zero is coming from infrared effects (the condition of charge neturality) is comforting, since we know infrared effects are behind the CMW theorem which is what prevents the symmetry breaking from happening.

8 *More on the massless Schwinger model and its spectrum*

Applications of bosonization in 1+1D is becoming a bit of a tired topic in this diary, but today we have something slightly new: an alternate way of deriving confinement in the (massless) Schwinger model. We will be somewhat fast and loose with factors of i and signs relating to squares of $\star$ (which squares to -1 on 1-forms in 1+1D in Euclidean time).

In 1+1D, we can fix $d^\dagger A = 0$ gauge and write $A = \star d\phi$ for ϕ a scalar. The kinetic term for the gauge field is then (here $\Box = -\partial_\mu \partial^\mu$ is positive-definite)

$$\int F_A \wedge \star F_A = \int d\star d\phi \wedge d^\dagger d\phi = \int d\phi \wedge \star\Box d\phi = \int \phi\Box^2\phi. \tag{220}$$

The fermions then couple to the gauge field via the term

$$S \ni \frac{1}{2\pi}\int i\bar{\psi}\gamma^\mu \epsilon^{\mu\nu}\partial_\nu\phi\psi. \tag{221}$$

Now consider performing a chiral rotation by $\psi \mapsto e^{-i\bar{\gamma}\phi}$, where $\bar{\gamma} = -i\gamma_1\gamma_2 = -iXY = Z$. Then the kinetic term changes by

$$\delta(\bar{\psi}\not{\partial}\psi) = -i\bar{\psi}\gamma^\mu\bar{\gamma}\partial_\mu\phi\psi = i\bar{\psi}(\epsilon^{\mu\nu}\partial_\nu\phi)\gamma^\mu\psi, \tag{222}$$

which is just what is needed to kill the coupling of ψ to ϕ. Of course when we do this the chiral anomaly comes into play, and gives us a term $\frac{1}{2\pi}\int(\phi d\star d\phi) = -\frac{1}{2\pi}\int\phi\Box\phi$. Lastly, we bosonize the fermions with a compact field θ, sticking to Witten's conventions in QFTs and strings. Putting all the pieces together, the action is

$$S = \frac{1}{2}\int\left(\frac{1}{4\pi}|d\theta|^2 + \frac{1}{\pi}|d\phi|^2 + \frac{1}{e^2}\phi\Box^2\phi\right). \tag{223}$$

The ϕ propagator is evidently

$$G^\phi(p) = \frac{1}{p^2/\pi + p^4/e^2} = \frac{\pi}{p^2} - \frac{\pi}{p^2 + e^2/\pi}. \tag{224}$$

Therefore we have two modes: one massive with mass $m^2 \equiv e^2/\pi$, and one massless. The gauge field propagator is then

$$G^A_{\mu\nu}(p) = \epsilon^{\mu\alpha}\epsilon^{\nu\beta}p_\alpha p_\beta\langle\phi(p)\phi(-p)\rangle = \frac{\varepsilon^{\mu\alpha}\varepsilon^{\nu\beta}p_\alpha p_\beta}{p^2}\frac{\pi m^2}{p^2+m^2} = (\delta^{\mu\nu} - p^\mu p^\nu/p^2)\frac{\pi m^2}{p^2+m^2}, \quad (225)$$

and so the gauge field is rendered massive by its coupling to the fermions.

It turns out that the massless θ particle and the massless mode of ϕ cancel each other out, leaving behind only a massive mode. To see this, we first take $\theta \mapsto \theta + 2\phi$, which eliminates the $|d\phi|^2$ term and gives us a $\theta\Box\phi/\pi$ term. Then we take $\phi \mapsto \phi - m^2\Box^{-1}\theta$, which kills the mutual ϕ, θ coupling. This gives an effective action for θ which is

$$S = \frac{1}{8\pi}\int \theta(\Box + m^2)\theta. \quad (226)$$

This tells us that the theory consists of a single massive pseudo-scalar boson (it's "pseudo-scalar" since $\psi^\dagger_\pm\psi_\mp \to e^{\pm i\theta}$ means $P : \theta \mapsto -\theta$, at least in the simple case with a Pin^+ structure, where P acts as $X = \gamma^0$).

One important thing to realize is that the expectation value of the chiral fermion bilinear is nonzero:

$$\langle\psi^\dagger_L\psi_R\rangle = \frac{1}{a}e^{-\langle\theta^2\rangle/2} = e^{-G^\theta(x=0;m)/2} \approx \frac{1}{a}e^{\ln(ma)} = m, \quad (227)$$

where a is a short-distance cutoff.[32] Many people say that the fact that this is non-zero indicates that we have chiral symmetry breaking. This would not contradict the CMW theorem, since in this case we have long-ranged interactions, provided by the gauge field (the absence of a Goldstone is also okay—the massless part of the gauge field was eaten by the term that we had to add to the action in accordance with the chiral anomaly). But this line of reasoning is not really correct, since the anomaly means that *we never actually had chiral symmetry in the first place, so there is nothing to break.*

Now let us look at correlators of the bilinears $\sigma \equiv \bar\psi\psi$.[33] First, as a check of asymptotic freedom, we can compute the 2-point functions $\langle\sigma_s(x)\sigma_{s'}(0)\rangle$ at $x \to 0$, where $\sigma_\pm \equiv \psi^\dagger_\pm\psi_\mp$ and where s, s' are signs. If $s = s'$, then we find

$$\langle\sigma_\pm(x)\sigma_\pm(0)\rangle_{x\to 0} = \langle\sigma^2_\pm(0)\rangle e^{G^\theta(x\to 0)} \approx m^2 e^{-\ln(x^2/m^2)} = \frac{1}{x^2}. \quad (229)$$

On the other hand if $s = -s'$ then the sign in the exponent switches, and we get

$$\langle\sigma_\pm(x)\sigma_\mp(0)\rangle \approx x^2 m^4 \to 0. \quad (230)$$

[32] With the current conventions,

$$G^\theta(0; m) = 4\pi\int_0^{a^{-1}} \frac{dp}{2\pi}\frac{p}{p^2+m^2} = \ln(a^{-2}/m^2 + 1) \approx -2\ln(am). \quad (228)$$

[33] Chirally-invariant correlators of two fermions, i.e. $\langle\psi^\dagger_\pm(x)\psi_\pm(0)\rangle$ are hard since the mass term for θ screws up a holomorphic / anti-holomorphic decomposition for θ, and means that the only correlators that are easy to compute are those of $\sigma_\pm = \psi^\dagger_\pm\psi_\mp$ ($\sigma = \sigma_+ + \sigma_-$), since $\sigma_\pm$ bosonizes to $e^{\pm i\theta}$.

Note that these results are exactly in accordance with what we'd get from free field theory; hence the model is asymptotically free. More complicated correlators are those involving the scalar σ. Using the usual manipulations for expectation values of exponentials, we get

$$\langle\sigma(x)\sigma(0)\rangle = \langle\sigma^2(0)\rangle 4\cosh(G^\theta(x;m)). \tag{231}$$

Now besides the scalar σ, we also have the pseudo-scalar $\widetilde{\sigma}_\pm \equiv \bar{\psi}\bar{\gamma}\psi = i(\psi_+^\dagger\psi_- - \psi_-^\dagger\psi_+)$. The minus sign turns the cosh into a sinh:

$$\langle\widetilde{\sigma}(x)\widetilde{\sigma}(0)\rangle = \langle\widetilde{\sigma}^2(0)\rangle 4\sinh(G^\theta(x;m)). \tag{232}$$

To evaluate these expressions, we expand the hyperbolic functions in powers of G^θ and then go to momentum space. Therefore we need to evaluate integrals like

$$\int\prod_i^k\left(\frac{d^2p_i}{4\pi^2}\frac{1}{p_i^2+m^2}\right)\frac{1}{\left(q-\sum_i p_i\right)^2+m^2} \tag{233}$$

for some fixed q^2. Terms with k even will appear in the expansion of the σ correlator, and terms with k odd will appear in the expansion of the $\widetilde{\sigma}$ correlator.

First, note that only the expansion of the $\sinh(G^\theta)$ will give an isolated pole in momentum space (only for $k=0$ in the above equation, which is the first term in the expansion of sinh, will we get a simple pole at $q^2=-m^2$; all other singularities are part of branch cuts, and are not isolated). This confirms the result that the boson in the spectrum is a pseudo-scalar, since there is a simple pole only in the correlation function of the pseudo-scalar $\widetilde{\sigma}$ field.

In general, I think it is true that the successive terms in the $\langle\sigma(x)\sigma(0)\rangle$ correlation function contribute branch cut singularities at $q^2=-(2n)^2m^2, n\in\mathbb{Z}$, while the successive terms in the $\langle\widetilde{\sigma}(x)\sigma(0)\rangle$ correlation function contribute branch cut singularities at $q^2 = -(2n+1)m^2, n\in\mathbb{Z}$. The first singular contribution to $\langle\sigma(x)\sigma(0)\rangle$ is determined by the integral (ignoring 2πs)

$$\begin{aligned}
I &= \int_p\frac{1}{(p^2+m^2)((p-q)^2+m^2)}\\
&= \int_x\int_p\frac{1}{[x((p-q)^2+m^2)+(1-x)(p^2+m^2)]^2}\\
&= \int_x\int_p\frac{1}{(p^2+m^2+q^2(x-x^2))^2}\\
&= \frac{1}{2}\int_x\frac{1}{m^2+q^2(x-x^2)}\\
&= \frac{1}{q\beta}\ln\left(1+\frac{q}{2m^2}(q+\beta)\right), \qquad \beta\equiv\sqrt{4m^2+q^2}.
\end{aligned} \tag{234}$$

This is singular precisely when $q=2im$, indicating the contribution of particle production involving a particle with mass m to the Greens function. In fact we can already see the singularity before we do the x integral: taking $q^2=-\lambda m^2$, the denominator vanishes when

$$x=\frac{1}{2}\pm\sqrt{\lambda^2-4\lambda}, \tag{235}$$

which tells us that we have singularities as soon as $\lambda > 4$: this gives us a branch point starting at $q = 2im$, as found above.

Now we turn to the leading term in the expansion for the $\widetilde{\sigma}$ correlator. With two integrals, things are much more heinous:

$$\begin{aligned} I &= \int_{p,k} \frac{1}{(p^2+m^2)(k^2+m^2)((q-p-k)^2+m^2)} \\ &= 2\int_{x,y}\int_{p,k} \frac{1}{\left(x(p^2+m^2)+y(k^2+m^2)+z((q-p-k)^2+m^2)\right)^3} \qquad z \equiv 1-x-y \\ &= 2\int_{x,y}\int_{p,k} \frac{1}{(m^2+p^2+k^2+z[-2q\cdot(p+k)+q^2+2p\cdot k])^3}. \end{aligned} \tag{236}$$

Now we need to eliminate the dot product between q and $p+k$. Consider shifting $\delta p = \delta k = \alpha q$. Then the terms involving q in the denominator become

$$2q\cdot(p+k)[\alpha - z + \alpha z] + q^2(z + 2\alpha^2 z + 2\alpha^2) \implies \alpha = \frac{z}{1+z}. \tag{237}$$

Then we shift $\delta p = -zk$, ending up with

$$\begin{aligned} I &= 2\int_{x,y}\int_{p,k} \frac{1}{(m^2+p^2+k^2(1-z^2)+\gamma q^2)^3} \qquad \gamma \equiv (1+3z^2)/(1+z) \\ &= \int_{x,y} \frac{1}{2(1-z^2)(m^2+\gamma q^2)} \end{aligned} \tag{238}$$

The integral can't be done analytically, but if we look at when the denominator vanishes, we can check that it does so at $q^2 = -9m^2$, which is exactly what we'd expect for a contribution coming from a 3-particle intermediate state. (Note to self: come back and work this out more carefully sometime)