Intersection theory on the lattice and the Pontryagin square's purpose in life

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The purpose of today's diary entry is to just make a note of the Pontryagin square and its properties that can be used as a reference. Thanks to Ryan Thorngren for some inspiring discussions in this regard.

First, some opening / notational remarks: we will let $\Delta_{i_1\dots i_{k+1}}$ denote a specific k-simplex in the lattice (alias CW complex) X under consideration (we will always be working with a triangulation). For a given numbering of each vertex in X, our convention is that the Δs are always listed with the sequence $i_1 \dots i_{k+1}$ monotonically increasing.

The cup product of a k-cochain a and an l-cochain b evaluated on the simplex $\Delta_{i_1\dots i_{k+l+1}}$ is

$$(a \cup b)_{\Delta_{i_1\dots i_{k+l+1}}} = a(\Delta_{i_1\dots l_{k+1}})b(\Delta_{i_{k+1}\dots i_{k+l+1}}). \tag{1}$$

Note that there is *no* kind of (anti)symmetrization on the RHS. If we were naively trying to write down a discrete analogue of a wedge product, we might have included a sum of some sorts (of sum sorts?) on the RHS—indeed, without the sum, the cup product is obviously not supercommutative on cochains, which seems to be worrying for its purported use as a discrete wedge product. But we will see later on that this lack of supercommutativity is essential for all the geometric properties of the cup product.

More notation: if $a \in C^\bullet(X;R)$ is any (co)chain, then $\widetilde{a} \in C_\bullet(X^\vee;R)$ will denote its Poincare dual.[1] Furthermore if C is any chain or submanifold, then C' will denote a "pushoff" of C, which is a copy of C displaced from C by a small amount in a direction determined by some choice of framing (not sure if this is standard terminology).

[1]Note that we are using the approach to Poincare duality wherein we have an isomorphism

$$C^k(X;R) \cong C_{n-k}(X^\vee;R), \tag{2}$$

with $X^\vee$ the dual lattice. At first sight, this is rather different, even conceptually, from the standard approach where one establishes $H^k(X;R) \cong H_{n-k}(X;R)$ by sending a k-cochain ϕ to $\widetilde{\phi} = [X] \cap \phi$, with $[X]$ the fundamental class. Not only does the latter approach remain on the same lattice, but it also involves extra minus signs coming from the fundamental class. That these two approaches are actually equivalent for the purposes of doing calculations in cohomology and intersection theory and so on is not obvious; see Ryan Thorngren's thesis for a great discussion of this.

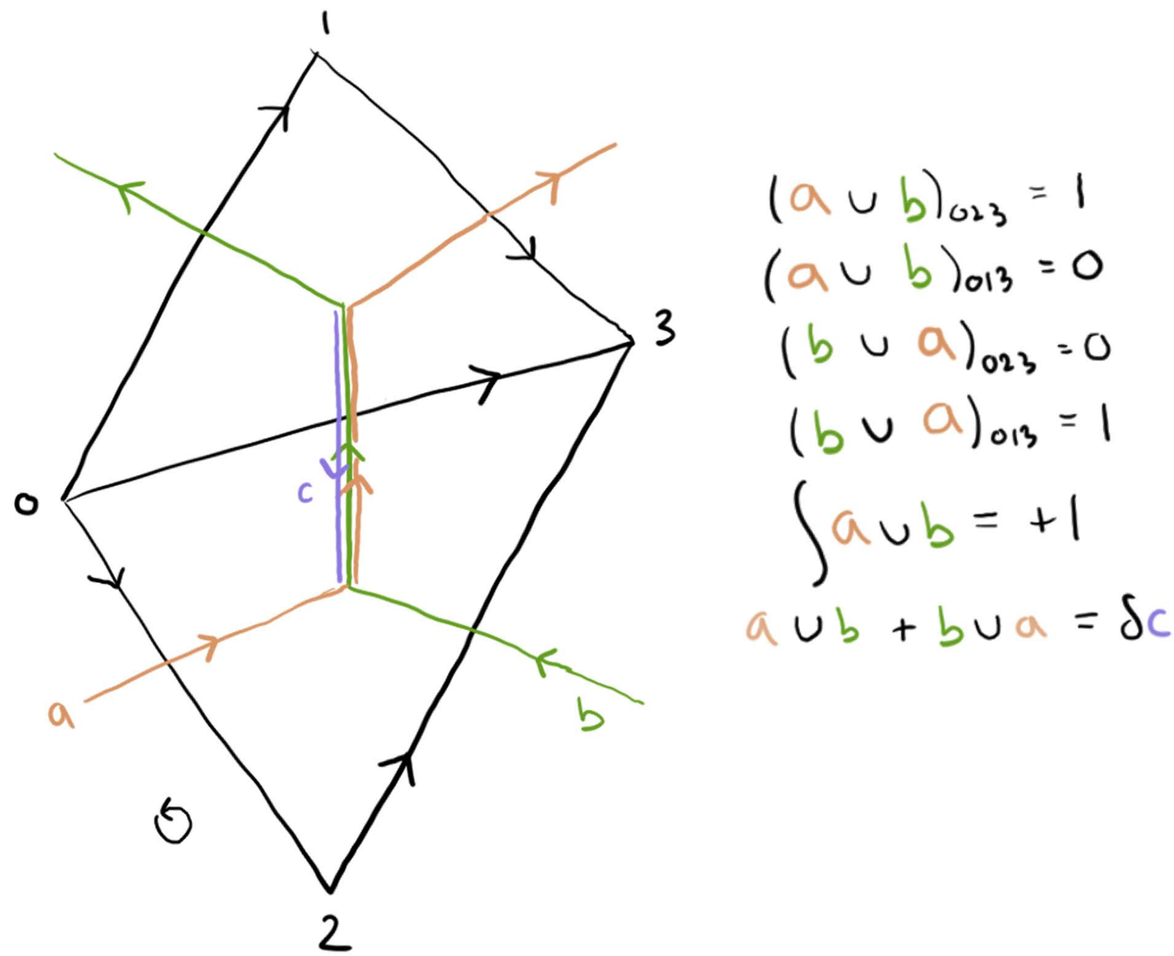

Figure 1: Calculation of the cup product for two 1-cochains a and b in two dimensions; see text for description.

Cup products and intersections

A good reference for stuff in this subsection is Ryan Thorngren's thesis. Results on higher cup products originally come from [**?**], I believe.

As we mentioned above, a very important fact following from the definition (1) is that unlike $\wedge$, $\cup$ is not supercommutative. Instead,

$$a \cup b - (-1)^{|a||b|} b \cup a = (-1)^{|a|+|b|+1} \delta(a \cup_1 b) + (-1)^{|a|+|b|} \delta a \cup_1 b + (-1)^{|b|} a \cup_1 \delta b, \tag{3}$$

Here $\cup_1$ is a degree -1 operation that is needed to allow the cup product to geometrically be dual to the intersection product.[2] It won't be important for us to know the exact CW complex description of $\cup_1$; we only point out that the way it appears in the above formula means that $\cup$ is still supercommutative when acting on equivalence classes of cocycles, but not supercommutative on the cochain level, even when cupping two cocycles.

[2] $\cup_1$ is also not supercommutative, in the same way: the general supercommutativity formula for $\cup_i$ is found by replacing $\cup$ in the above formula with $\cup_i$ and $\cup_1$ with $\cup_{i+1}$.

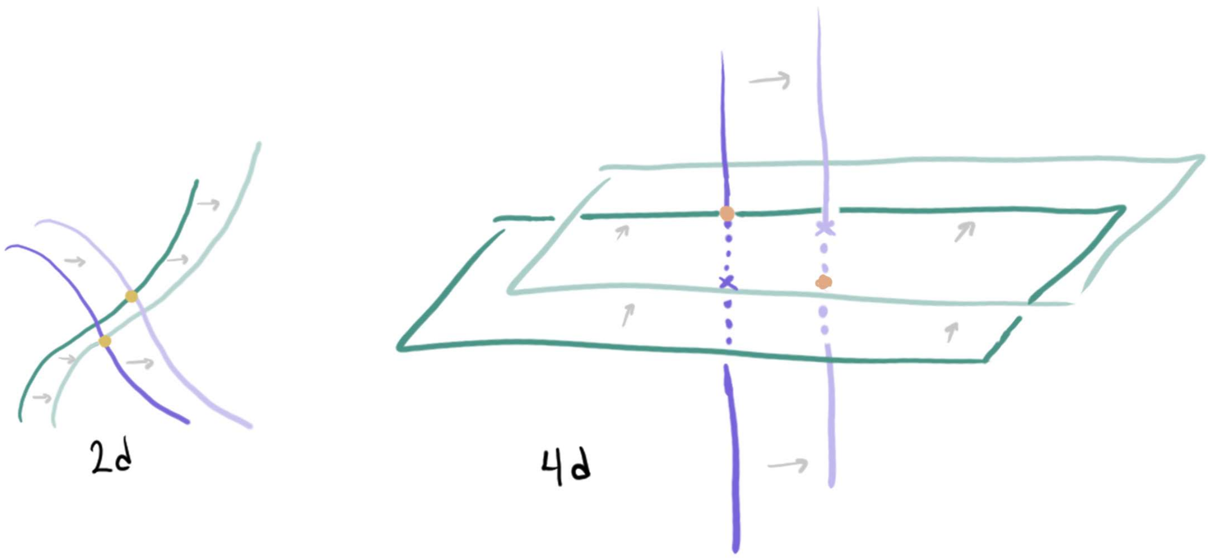

Figure 2: A figure showing the doubling of the explicit self-intersection points for lines in 2D and surfaces in 4D. On the left, the union of the purple and green dark lines is supposed to be a single 1-manifold C that has an explicit self-intersection (retrospectively they should have been the same color; the point is that dark stuff is one manifold and light stuff is a pushoff). The union of the light green and purple lines is a pushoff C'; one sees that it intersects C in two points. The right figure is basically the same thing: the union of the dark green plane + dark purple line (really a plane; the fourth dimension is suppressed) constitute a surface S with an explicit self-intersection. The pushoff S' again intersects S in two points.

The geometric meaning of the cup product is that it is dual to the intersection product via Poincare duality. That is, $a \cup b = \widetilde{a} \cap \widetilde{b}$, with $\cap$ the signed intersection number. When we want to define $a \cup a$ geometrically, we have to make use of a pushoff $\widetilde{a}'$ of $\widetilde{a}$, so that $a \cup a = \widetilde{a} \cap \widetilde{a}'$. We say "a pushoff" because the topological invariance of the intersection number means we just need to choose any pushoff—in fact, any other chain that is in the same homology class as $\widetilde{a}$ can be used for $\widetilde{a}'$ when computing $a \cup a$. This method of defining $a \cup a$ is of course familiar from the regularization procedure used in CS theory.

To get warmed up, we can consider figure 1, which shows the calculation of the cup product of two cochains whose Poincare duals are shown as orange and green lines on the dual lattice (for example, the cochain a assigns the number 1 to each link crossed by the orange lines; the orientation of the link needed to assign $+1$ and not -1 is determined with reference to a global orientation [the black corkscrew]).[3]

[3] For completeness, let's recall how the signs of transverse intersections are determined in general, for two chains / submanifolds N and M of complementary dimensions. One does this by comparing the orientations of the intersecting chains with the orientation from the ambient space X (coming from the fundamental class). Since the intersections we're dealing with are transverse, we can always choose a framing $\mathcal{F}_X$ of X and coordinate systems on N and M such that each basis vector e_i of $\mathcal{F}_X$ is parallel to some basis vector of $\mathcal{F}_N$ or $\mathcal{F}_M$. Let the volume form on some manifold Y be given by $V_Y = e_{i_1} \wedge \cdots \wedge e_{i_Y}$, where in the subscript

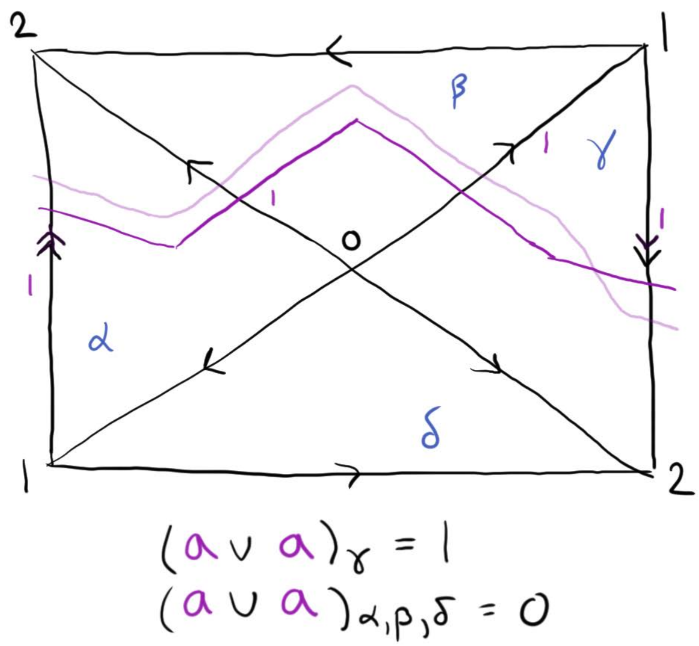

Figure 3: An example of how the $\cup$ product counts the intersection number of a framing SI. Here a is a 1-chain on the mobius strip (the left and right edges are identified as shown), and is Poincare dual to the dark purple line $\widetilde{a}$. The letters $\alpha, \beta, \gamma, \delta$ indicate the four different 2-cells in the triangulation. The light purple line is a pushoff of the dark purple line, and the point of the figure is to show that $\int a \cup a = \widetilde{a} \cap \widetilde{a}'$ (working with mod 2 coefficients as is appropriate for an unorientable manifold). Note that to correctly count explicit SIs, the Pontryagin square would need to be used instead of the $\cup$ product.

The Poincare duals of a and b intersect transversely,[4] and so we expect the cup product to assign $+1$ to this configuration. Indeed, we find $\int a \cup b = +1$ for this region of the CW complex. While $a \cup b \neq -b \cup a$, their difference is a total derivative, as indicated in the figure. Consequently, $\int a \cup b = -\int b \cup a$ (remember that the integral is done using signs coming from capping with the fundamental class!). This figure also lets us see that reversing the branching structure (flipping the direction of the arrows) reverses the cup product, so that if Δ^R is the cell Δ with reversed branching structure, then

$$(a \cup b)_\Delta = (b \cup a)_{\Delta^R}. \tag{5}$$

A baby example testing the claim that $a \cup a$ is dual to the self-intersection number of $\widetilde{a}$ is shown in figure 3, where we compute $a \cup a$ for a such that $\widetilde{a}$ is the nontrivial cycle of the mobius band. In order to not get $a \cup a = 0$ (since a is a 1-cochain), we need to work with $\mathbb{Z}_2$ coefficients. By looking at the figure, we see that indeed $a \cup a = \widetilde{a} \cap \widetilde{a}' = 1 \mod 2$.

The Pontryagin square is all about self-intersections (SIs), so let's talk about them for a little bit. As mentioned above, the self-intersection of a chain / submanifold C is computed as $C \cap C'$ for some pushoff C'. There are two types of SI that a given chain can have. The first type, which we'll call "explicit", is where the chain intersects itself on account of it being immersed but not embedded in the ambient manifold (e.g. the Klein bottle in $\mathbb{R}^3$ intersects itself explicitly). Figure 2 shows an example of an explicit SI in 2d (left) and 4d (right)—the dark colors represent two parts of a single self-intersecting manifold; the light colors are pushoffs.

The second type, which we'll call a "framing" SI, is one where the chain is embedded, but is such that its pushoffs intersect it due to the way it's framed, with its normal bundle having a nontrivial Euler class (the curve on the mobius band in fig. 3 has a framing SI number of 1). These types of framing SI always appear in pairs for curves immersed in spin 2-manifolds and surfaces immersed spin 4-manifolds, so we usually have to go to non-spin manifolds to get a chain which has a minimal framing SI.

The Pontryagin square

The purpose of the Pontryagin square is to allow one to do intersection theory with torsion. Algebraically, the Pontryagin square provides us with a squaring operation on cohomology classes in $H^k(X; \mathbb{Z}_m)$:

$$P : H^k(X; \mathbb{Z}_m) \times H^k(X; \mathbb{Z}_m) \to H^{2k}(X; \mathbb{Z}_n), \tag{6}$$

Y denotes $\dim Y$. To compute the sign of $N \cap M$, we simply compute the sign σ such that

$$\sigma V_N \wedge V_M = V_X. \tag{4}$$

By the supercommutativity of the wedge product, the intersection number is also supercommutative.

[4]Well, not quite. The fact that the lattice forces the duals to be degenerate along a link is the origin of all this non-supercommutativity and higher cup product business. However, in the discussion that follows, when we talk about chains intersecting, we will always be thinking in the continuum, where the intersections can always be wiggled to be transverse and well-defined. We thus always keep in the back of our mind that the "transverse intersections" talked about below really degenerate when put on the lattice. The point of the properties of the cup product is to ensure that this continuum geometric picture for intersections always works out when we restrict to the lattice and calculate with $\cup$s.

where we will determine what n should be momentarily. The defining algebraic feature of P is that it factors in the way one expects a squaring operation to factor, namely

$$P(a+b) = P(a) + P(b) + 2a \cup b. \tag{7}$$

The defining geometric feature of P (of course this is just another way of formulating its algebraic properties) is that it is dual to the self-intersection pairing. We know from the above that the cup product is dual to the intersection pairing when the coefficient group is $\mathbb{Z}$, and so the difference between $P(a)$ and $a \cup a$ must come from torsion effects.

The whole framework of intersection theory described above is equipped to deal with integer linear combinations of chains, and not $\mathbb{Z}_m$-valued linear combinations. Therefore, in order to do intersection theory with $\mathbb{Z}_m$-valued chains, we will first lift them to $\mathbb{Z}$-valued chains, and then later reduce back to torsionful coefficients after calculating intersection numbers. We will write the integer lift of a chain w as $w_{\mathbb{Z}}$. One is then tempted to conclude that a cohomology operation that is dual to the self-intersection pairing could be constructed by taking the $\cup$ of the integer lifts, and then reducing mod m:

$$\widetilde{a} \cap \widetilde{a}' \stackrel{?}{\to} [a_{\mathbb{Z}} \cup a_{\mathbb{Z}}]_m, \tag{8}$$

where $[]_m$ is reduction mod m.

The reason why $[a_{\mathbb{Z}} \cup a_{\mathbb{Z}}]_m$ isn't the appropriate intersection pairing dual in the torsionful case can be illustrated by considering the case of $\mathbb{Z}_2$ coefficients. From looking at figure 2, we see that the intersection number for an explicit SI is always even.[5] Therefore the cup product on $\mathbb{Z}_2$ cochains, which maps into $H^{2k}(X; \mathbb{Z}_2)$, is completely blind to explicit SIs. But of course an explicit SI should still be able to be kept track of, if we want a cohomology operation that is truly dual to the self-intersection. This motivates us to look for a cup product-like operation that maps into $H^{2k}(X; \mathbb{Z}_4)$, since then explicit SIs could be properly counted mod 2.[6]

More generally, consider the case of $\mathbb{Z}_m$ coefficients. If $m \in 2\mathbb{Z}$ then we still have the problem that explicit SIs get doubled-counted by $\cup$, so that $\cup$ can't tell the difference between $m/2$ explicit SIs and 0 explicit SIs (we want it to only fail to distinguish between m and 0). In order to properly count these SIs, we need to look for a cohomology operation that maps into $\mathbb{Z}_{2m}$ coefficients. On the other hand if $m \in 2\mathbb{Z}+1$, then since 2 generates $\mathbb{Z}_m$, a minimal explicit SI is m-torsion, as desired. Therefore in the case where m is odd, we can get away with using just $[a_{\mathbb{Z}} \cup a_{\mathbb{Z}}]_m$.

[5] This means that in a precise sense, framing SIs give us SIs that are "smaller" than the more obvious explicit SIs.

[6] The mod 2 case can also be understood in terms of orientations—this is the subject of a nice math overflow answer pertaining to $P(w)$ by Kevin Walker. An intersection number defined for a $\mathbb{Z}_2$-valued chain needs to be well-defined under local changes in the chain's orientation. In figure 2, we can see what happens under local orientation changes for explicit SIs: we make a local change of the immersed manifold that reverses only the orientation of one part of the manifold at the intersection point (say, we just reverse the orientation of the green part—this is possible because we are changing the orientation of the immersed manifold, and not of the ambient background manifold). This changes the SI number from ± 2 to ∓ 2. On the other hand, framing SI numbers don't change at all, since both the immersed manifold and its pushoff change orientation. Therefore the SI number is well-defined mod 4 in general.

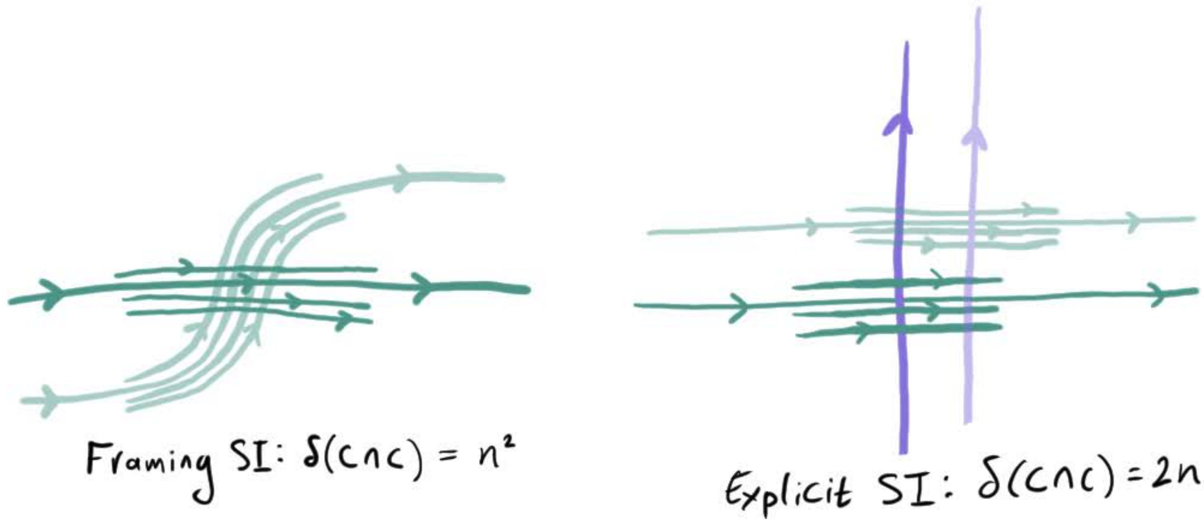

Figure 4: Illustrating the well-definedness of the intersection number for a $\mathbb{Z}_n$-valued chain C (in the figure, $n = 3$). Left: framing SI. Adding a chain in $C_\bullet(X; n\mathbb{Z})$ (the segments of three parallel lines in the figure) to C results in a change of SI number of $\delta(C \cap C) = n^2$ (should really be $\delta(C \cap C')$). Right: explicit SI; the dark green and dark purple lines are two parts of the same chain C; the lighter versions are C'. The change in SI number is now $\delta(C \cap C) = 2n$.

Now we ask whether such a cohomology operation can be well-defined on $\mathbb{Z}_m$ cochains. Consider sending

$$a_\mathbb{Z} \mapsto a_\mathbb{Z} + mC, \tag{9}$$

where $C \in C^k(X; \mathbb{Z})$ is any cochain (not necessarily closed!). The effect of doing this on the SI of a is shown in figure 4. Under this shift, a minimal framing SI changes in value by something in $m^2\mathbb{Z}$, while a minimal explicit SI changes by something in $2m\mathbb{Z}$. Therefore the intersection number is well-defined in $\mathbb{Z}_{\gcd(2m,m^2)}$, which is equal to $\mathbb{Z}_m$ if m is odd, and $\mathbb{Z}_{2m}$ if m is even. We also see that the intersection number of a sum of cochains satisfies the square rule (7)—the self-intersection of $a + b$ contains the SI of a, the SI of b, and if a and b intersect, an explicit SI between a and b that comes with a factor of 2.

Given the above considerations, we are lead to define $P(w)$ for any $w \in H^\bullet(X; \mathbb{Z}_m)$ by

$$P(w) = \begin{cases} [w_\mathbb{Z} \cup w_\mathbb{Z} + w_\mathbb{Z} \cup_1 \delta w_\mathbb{Z}]_{2m} & m \in 2\mathbb{Z} \\ [w_\mathbb{Z} \cup w_\mathbb{Z}]_m & m \in 2\mathbb{Z}+1 \end{cases} \tag{10}$$

which is the appropriate cohomology dual of the mod m intersection pairing. The mysterious $w_\mathbb{Z} \cup_1 \delta_1 w_\mathbb{Z}$ (remember that $w_\mathbb{Z}$ is only closed mod m, so this does not need to vanish mod $2m$) in the m even case (it's not there for the odd case since it dies when reduced mod m) is needed to ensure that $P(w)$ is well-defined and obeys (7), which we just argued is obeyed by the self intersection number. Indeed, for $m \in 2\mathbb{Z}$, we calculate

$$\begin{aligned} P(w+v) = [&w_\mathbb{Z}^{\cup 2} + v_\mathbb{Z}^{\cup 2} + w_\mathbb{Z} \cup_1 \delta w_\mathbb{Z} + v_\mathbb{Z} \cup_1 \delta v_\mathbb{Z} + (1-(-1)^{|w|}) w \cup v \\ &- \delta(v \cup_1 w) + (1+(-1)^{|w|}) v \cup_1 \delta w + \{w, \delta v\}_{\cup_1}]_{2m}, \end{aligned} \tag{11}$$

where the asymmetric nature of the last two terms comes from the way we wrote the supercommutativity law. The first line in (11) contains the answer we want, and happily all the terms in the second line vanish mod $2m$ after we mod out by total coboundaries, since the the $\cup_1$ product is supercommutative up to things in $C^\bullet(X; m\mathbb{Z})$, and because terms that are valued in $m^2\mathbb{Z}$ die on account of $m^2 \in 2m\mathbb{Z}$ if m is even. Therefore we indeed have (7). That (7) is satisfied in the case of m odd is easy to check, since in that case we just reduce mod m and all the $\cup_1$ terms coming from the supercommutativity rule die.

Applying this result, we can then check that $P(w)$ is indeed well-defined on $H^\bullet(X; \mathbb{Z}_m)$: first, under $w \mapsto w + mc$, we have

$$P(w) \mapsto P(w) + P(mc) + [2mw \cup c]_n, \tag{12}$$

with $n = m$ or $2m$ as appropriate. The last term dies because of the $2m$, and the second term dies because of the m^2. Finally, one needs to verify that $P(w)$ is closed in $C^{2|w|}(X; \mathbb{Z}_n)$ if w is closed in $C^{|w|}(X; \mathbb{Z}_m)$; this is again shown through the fact that the $\cup_i$s are supercommutative modulo m. Therefore, we have shown that $P(w)$ obeys all the relations we expect of the cohomology dual to the intersection pairing.

The point of all this is to show that there is a precise way to translate the simple geometric intuition one has from thinking about intersection theory into a dual algebraic formalism one can use on cochains. To quote *The Wild World of Four-Manifolds*, "think with intersections, prove with cup products"!

Finally we can talk about the meaning of the discrete term that often appears in topological actions, viz.

$$S_{top} \supset \frac{p}{2n} \int P(w_2). \tag{13}$$

In order for this to be well-defined, we evidently must have $p \in \mathbb{Z}_{2n}$ if n is even, or $p \in 2\mathbb{Z}_{2n}$ if n is odd.

Consider a w_2 surface which has a minimal explicit self-intersection—the picture here would be e.g. a pair of (discrete) magnetic flux lines propagating and then linking at some point in time. The minimal phase that S_{top} can assign to this SI is $e^{2\pi i/n}$, so that n explicit SIs always give a trivial phase.

We could have been forgiven for thinking that S_{top} would always assign nth roots of unity to w_2 surface intersections, but this is only true for explicit SIs: framing SIs instead can contribute a phase that is only half this value, namely $e^{\pi i/n}$. For $n \in 2\mathbb{Z} + 1$ there is not a sharp distinction between the two types of SIs vis-a-vis how they appear in S_{top} since both 1 and 2 generate $\mathbb{Z}_n$, but for $n \in 2\mathbb{Z}$ (and for non-spin manifolds, where the framing SIs can come in odd numbers) the distinction is important. Therefore even though the characteristic class w_2 is $\mathbb{Z}_n$-valued, the Pontryagin square provides us with a way (in certain circumstances), to get a well-defined topological action which gives phases that are $2n$-th roots of unity, and it does this by constructing a correct torsionful cohomology dual of the intersection pairing.