Wannier-Stark and Landau-Zener

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Energy bands in optical lattices

The potential is generating by a monochromatic electric field of wavelength k_L that forms a standing wave and oscillates with frequency ω_L. For atoms with two levels carrying different dipole moments and separated by a transition frequency ω_0, a sinusoidal potential can be generated by taking the detuning Δ and the Rabi frequency Ω of the electric field to satisfy $\omega_L \gg \Delta \gg \Omega$. Letting I be the maximum intensity of the laser light and Γ be the linewidth of the transition, the potential formed in this limit is

$$V_{dip}(x) = \frac{3\pi c^2 \Gamma I}{2\omega_0^3 \Delta} \sin^2(k_L x) \equiv V \sin^2(k_L x). \tag{1}$$

In the following we will define the *recoil energy* as

$$E_r \equiv \frac{\hbar^2 k_L^2}{2m} = h \times \left(\frac{m_{Rb}}{m}\right) \times \left(\frac{700\,\mathrm{nm}}{a}\right)^2 \times 1.17\,\mathrm{kHz} \tag{2}$$

where the lattice spacing is $a \equiv 2\pi/(2k_L)$, m is the mass of the atom being loaded into the trap, and m_{Rb} is the mass of an ^{87}Rb atom. We will also define the normalized well height

$$v \equiv \frac{V}{E_r}. \tag{3}$$

Then

$$H = 4E_r \left(-\frac{\nabla^2}{(2k_L)^2} + \frac{v}{4}\sin^2(k_L x) \right). \tag{4}$$

The eigenfunctions of this Hamiltonian — or any other single particle Hamiltonian with discrete translation symmetry — are $e^{iqx}u_{nq}(x)$, where

$$u_{nq}(x) = \frac{1}{\sqrt{a}} \sum_G e^{iGx} u_{nq}(G), \tag{5}$$

with G the reciprocal lattice vectors and a the lattice spacing. With $\sum_G u^*_{nq}(G) u_{n'q'}(G) = \delta_{n,n'}\delta_{q,q'}$, we have the normalization $\int_{UC} dx\, u^*_{nq}(x) u_{n'q'} = \delta_{n,n'}\delta_{q,q'}$, where the integral is over a unit cell.

We are interested in modeling the physics of the lowest band of this system with a tight-binding model. To do this we employ the Wannier functions

$$|w_{nj}\rangle = \frac{1}{\sqrt{N}} \sum_q e^{iqR_j} |u_{nq}\rangle, \tag{6}$$

where R_j is the coordinate of the jth lattice site and N is the number of lattice points (equal to the number of q points in the BZ). These functions are normalized as $\langle w_{nj}|w_{n'j'}\rangle = \delta_{n,n'}\delta_{j,j'}$, and in terms of them and the dispersion ε_{nq}, the Hamiltonian reads

$$H = \sum_{n,q} \varepsilon_{nq}|u_{nq}\rangle\langle u_{nq}| = \sum_{j,j'} t_n(R_j - R_{j'})|w_{nj}\rangle\langle w_{nj'}|, \tag{7}$$

where

$$t_n(R) = \frac{1}{N}\sum_q \varepsilon_{nq} e^{iqR}. \tag{8}$$

Note that to compute the $t_n(R)$, and thus the parameters appearing in the tight-binding Hamiltonian we are looking for, we never need to compute the Bloch functions at any point; it is instead sufficient to simply Fourier transform the spectrum of the band under study.

Focusing on the lowest band ($n = 1$), we define

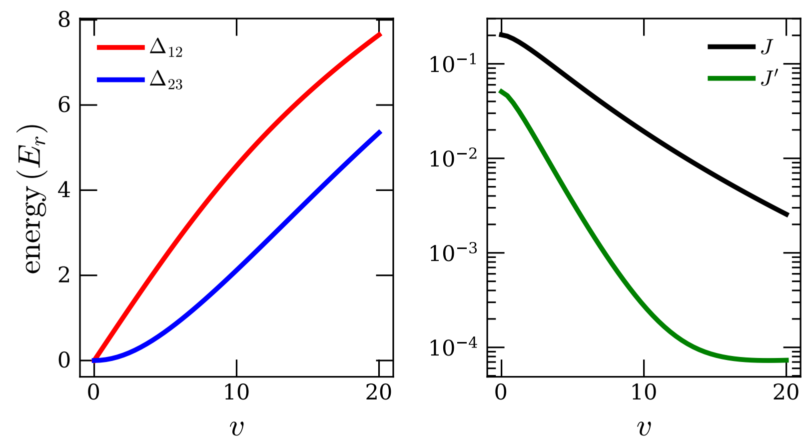
$$J \equiv -t_1(a), \qquad J' \equiv -t_1(2a) \tag{9}$$

as the parameters appearing in the TB Hamiltonian $H = -\sum_j (J b_j^\dagger b_{j+1} + J' b_j^\dagger b_{j+2} + \cdots) + h.c.$ Using the explicit solution for the Bloch states in terms of Mathieu functions, one can derive approximate analytic forms for J, J' which are exact in the $v \gg 1$ limit, e.g.

$$\frac{J}{E_r} \approx \frac{4v^{3/4}}{\sqrt{\pi}} e^{-2\sqrt{v}}. \tag{10}$$

However, since computing the TB parameters numerically is so easy in practice, getting analytic approximations like this is not too important.

On general grounds we expect the bandgaps to rapidly get smaller for larger band indices and to increase linearly with v at large v; we also expect J (i.e. the bandwidth of the lowest band) to decrease exponentially with v once $v \gtrsim 1$, and J' to do the same but to decrease exponentially with twice the coefficient in the exponent. This is indeed what happens, as shown in the following plot:



(11)

Adding a tilt

One straightforward way to add linear tilt in the optical lattice is to use magnetic field gradients, in which case the tilt energy is simply determined by the Zeeman splitting. Another slightly less obvious way — which was the method employed in the original studies of Bloch oscillations in optical lattices — is to construct an accelerating lattice by linearly varying the relative frequency of the two laser beams that form the standing wave. Indeed, letting the two waves be E_L and E_R, their average frequency be $(\omega_L + \omega_R)/2 = \omega$, and letting $r = d(\omega_L - \omega_R)/dt$ be the rate at which the frequency is swept, the full electric field is

$$E_L + E_R = Ee^{i(k_L x + (\omega + rt/2)t)} + Ee^{i(-k_L x + (\omega - rt/2)t)} = 2Ee^{i\omega t}\cos(k_L x + rt^2/2), \tag{12}$$

where we took $E_{L/R}$ to have equal intensities. The lattice potential thus accelerates, and if we go into a frame where the lattice is at rest, this acceleration gives a constant force on the atoms of magnitude

$$F = \frac{rm}{k_L}. \tag{13}$$

Free particles in a tilted potential undergo Bloch oscillations. A particle initially localized at the origin evolves in time to produce a density at site j of

$$\rho_j(t) = \left| \mathcal{J}_j\left(\frac{4J}{\Delta}\sin(\Delta t/2)\right)\right|^2. \tag{14}$$

The Bessel function starts decaying when $|j| \gtrsim 4J/\Delta$, which can thus serve as an estimate of the linear size of the region the particle is localized to.

The interaction U comes predominantly from s-wave elastic scattering between the Rb atoms, which in real space reads $U(r) = (4\pi\hbar^2 a_s/m)\delta(r)$. We can then extract an approximate value for the U appearing in (**??**)

The tunneling J can be obtained from the bandwidth W as $J = W/4$

$$(\sigma/a)^2 \sim \frac{\frac{\hbar^2(1/a)^2}{2m}}{\hbar\omega_c} \tag{15}$$

$$(|A\rangle - |C\rangle)|BCABCA\cdots\rangle \tag{16}$$

$$\sigma^2 = \hbar \int_0^{\pi/a} \frac{q\,dq}{2\pi} \sum_n \frac{2\pi k_B T}{\hbar} \frac{\mathcal{E}_L + \mathcal{E}_T + \rho_m \omega_n^2}{(\mathcal{E}_L + \rho_m\omega_n^2)(\mathcal{E}_T + \rho_m\omega_n^2) + \rho_m^2\omega_n^2\omega_c^2} \tag{17}$$

where

$$\mathcal{E}_L = dq, \qquad \mathcal{E}_T = cq^2 + dq, \qquad \omega_n = 2\pi k_B T n/\hbar. \tag{18}$$

$$H_{TL} + \sum_{i,\alpha} h_i^\alpha |\alpha\rangle\langle\alpha| \tag{19}$$

References