Decoupled fixed points in stacks of 1+1d and 2+1d $\mathbb{Z}_N$ critical points

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In these notes we will examine the stability (or lack thereof) of weakly-coupled arrays / stacks of 1+1d / 2+1d CFTs. The CFTs will be taken to be the critical points of $\mathbb{Z}_N$ clock models, with Hamiltonian

$$H = -g \sum_i \mathsf{x}_i - h \sum_{\langle ij \rangle} \mathsf{z}_i^\dagger \mathsf{z}_j + h.c, \tag{1}$$

where x, z are $\mathbb{Z}_N$ Paulis.

We will be interested in addressing the stability of the decoupled fixed points with respect to couplings which *preserve the independent $\mathbb{Z}_N$ symmetries on each wire / layer.*

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1 1+1d stacks of critical wires

1.1 $N = 2$

The decoupled critical point we are interested in is described by an array of critical Ising* CFTs.[1]

The most relevant gauge-invariant couplings between the chains are formed by coupling the energy operators on two different chains together. If we label the chains by w and assume translation symmetry, the appropriate perturbation to the decoupled fixed point can be written in imaginary time as

$$\delta S = \sum_{w \neq w'} C_{w-w'} \int dz\, d\tau\, \varepsilon_w \varepsilon_{w'}, \tag{2}$$

where ε_w is the energy operator on the chain w. Note that since there is no ε^2 operator in the 1d Ising CFT, we only sum over $w' \neq w$.

Since the critical fluctuations are only in the z-τ spacetime plane, the relevance of δS is obtained by comparing the scaling dimension of $\varepsilon_w \varepsilon_{w'}$ to 2. Since the dimension of the

[1] Here as is conventional, the $*$ superscript signifies the fact that only operators neutral under the global $\mathbb{Z}_2$ symmetry are to be considered.

energy operator in the 1d Ising model is 1, the couplings $C_{w,w'}$ are all marginal at tree level. To one-loop order, the β functions of the C_w couplings are consequently

$$\beta_{C_{w-w'}} = -\sum_{w''\neq w,w'} C_{w-w''}C_{w''-w'}, \tag{3}$$

where we have absorbed a positive scheme-dependent multiplicative factor into the definition of $C_{w-w'}$. If we define $C_0 \equiv 0$, we can rewrite this as a sum over all w'' as

$$\beta_{C_{w-w'}} = -\sum_{w''} C_{w-w''}C_{w''-w'}(1-\delta_{w,w'}), \tag{4}$$

with the delta function ensuring that C_0 remains zero under the flow. Fourier transforming in the coordinates transverse to the chains, we have

$$\beta_{C_{\mathbf{q}}} = -C_{\mathbf{q}}^2 + \frac{1}{L^2}\sum_{\mathbf{p}} C_{\mathbf{p}}^2, \tag{5}$$

where L is the linear size of the array.

Evidently the only fixed points (5) are those where $|C_{\mathbf{q}}|$ is a constant, independent of $\mathbf{q}$. The matrix whose eigenvalues determine the stability of each of these fixed points is

$$\frac{\partial \beta_{C_{\mathbf{q}}}}{\partial C_{\mathbf{p}}} = -2C_{\mathbf{p}}\delta_{\mathbf{q},\mathbf{p}} + \frac{2}{L^2}C_{\mathbf{p}}. \tag{6}$$

Since the trace of this matrix vanishes due to $C_0 = 0$, it must either contain positive eigenvalues (implying the instability of the fixed point) or vanish identically. In the latter case, it is easy to verify that the decoupled $C_{\mathbf{q}} = 0$ fixed point is not stable, and hence none of the fixed points are. Thus the perturbation (2) generically drives the transition first order (although since the flow can be slow, the first order behavior can be made rather weak).

If we are willing to explicitly break translation invariance, one might imagine other possibilities for the flow. For example, suppose that the dominant interchain interactions in the UV are those which couple the chains together into separate pairs. If we label the chains by their x, y coordinates as $w = (x, y)$, we may for example at first consider only the term

$$\delta S = C_{pair}\sum_{(x,y)} \int dz\, d\tau\, \varepsilon_{2x,y}\varepsilon_{2x+1,y}, \tag{7}$$

where we have now labeled the wires by their transverse coordinates (x, y).

In the absence of couplings between pairs of chains, C_{pair} is exactly marginal, with its strength parametrizing a manifold of fixed points with different scaling dimensions — each pair of chains then resides on the critical line of the Ashkin-Teller (AT) model. If we then proceed along the critical line from this starting point we may expect a stable phase upon re-introducing coupling between pairs of chains, since the scaling dimensions of the operators in the AT model vary continuously along the critical line. This turns out not to be the case, however: the operators $\varepsilon_{2x,y} \pm \varepsilon_{2x+1,y}$ have different scaling dimensions except at the point $C_{pair} = 0$, and except at this point, one of them always has scaling dimension less than 1 (see

Kadanoff's paper). Therefore there always exists a coupling between pairs of AT* models with scaling dimension less than 2, and as such this family of decoupled fixed points is also not stable. Therefore, while we do not have a rigorous proof in the case of non-translation-invariant couplings, the $N = 2$ case does not seem to provide us with an example of a stable critical point.

1.2 $N = 3$

Now consider $N = 3$. In this case the putative decoupled critical point corresponds to an array of $\mathbb{Z}_3$ clock models. The most relevant gauge-invariant perturbations between the chains are again energy-energy couplings like those in (2). The scaling dimension of the energy operator at the critical point is known to be $\Delta_\varepsilon = 4/5$, and like in the $N = 2$ case there no ε^2 primary field. The beta function of the energy-energy coupling is therefore

$$\beta_{C_{w-w'}} = \frac{2}{5}C_{w-w'} - \sum_{w''\neq w,w'} C_{w-w''}C_{w''-w'}, \tag{8}$$

where we have assumed $w \neq w'$. Fourier transforming as before,

$$\beta_{C_{\mathbf{q}}} = \frac{2}{5}C_{\mathbf{q}} - C_{\mathbf{q}}^2 + \frac{1}{L^2}\sum_{\mathbf{p}} C_{\mathbf{p}}^2. \tag{9}$$

The trace of $\partial\beta_{C_{\mathbf{q}}}/\partial C_{\mathbf{p}}$ is positive on account of $\sum_{\mathbf{q}} C_{\mathbf{q}} = 0$, and as such every fixed point of (9) is unstable. Therefore (at least with translation-invariant couplings), all of the transitions in the $N = 3$ case are first order.

1.3 $N = 4$

Next up is $N = 4$. This case is dealt with by realizing that the $\mathbb{Z}_4$ clock model is actually equivalent to a decoupled stack of two Ising chains. To see this, we write the $\mathbb{Z}_4$ clock matrix z (omitting the z superscripts) in terms of two $\mathbb{Z}_2$ Pauli matrices $\mathsf{z}_1 \equiv \sigma^z \otimes \mathbf{1}, \mathsf{z}_2 \equiv \mathbf{1} \otimes \sigma^z$ as[2]

$$\mathsf{z} = \frac{1}{\sqrt{2}}(e^{i\pi/4}\mathsf{z}_1 + e^{-i\pi/4}\mathsf{z}_2). \tag{10}$$

The real matrix $\mathsf{x}+\mathsf{x}^\dagger$ also has a simple representation in terms of $\mathsf{x}_1 \equiv \sigma^x \otimes \mathbf{1}$ and $\mathsf{x}_2 = \mathbf{1} \otimes \sigma^x$, with

$$\mathsf{x} + \mathsf{x}^\dagger = \mathsf{x}_1 + \mathsf{x}_2. \tag{11}$$

It is then easy to check that in terms of the $\mathbb{Z}_2$ variables, the $\mathbb{Z}_4$ clock model Hamiltonian splits as

$$H = H_1 + H_2, \tag{12}$$

where H_1 (H_2) is an Ising chain Hamiltonian for the z_1 (z_2) variables. The $\mathbb{Z}_4$ symmetry acts in the $\mathbb{Z}_2$ representation as

$$\mathbb{Z}_4 \,:\, \mathsf{z}_1 \to \mathsf{z}_2, \qquad \mathsf{z}_2 \to \mathsf{z}_1\mathsf{x}_1. \tag{13}$$

[2] This actually gives $\mathsf{z} = \mathrm{diag}(1, i, -i, -1)$, but we will find this nonstandard basis slightly more convenient to work in.

The natural decoupled critical point to consider for this system is then an array of pairs of critical Ising models, similar to the theory mentioned in the discussion of the $N = 2$ case. Let $\varepsilon_1, \varepsilon_2$ be the energy operators for the two Ising models in each pair. Since the operator $\varepsilon_1\varepsilon_2$ is exactly marginal, we can consider critical points located along the full AT critical line. The difference with the $N = 2$ scenario considered above is that only operators invariant under the $\mathbb{Z}_4$ symmetry on each pair are gauge invariant (while for the $\mathbb{Z}_2$ case the gauge invariant ones were those neutral under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting on each pair). The $\mathbb{Z}_4$ symmetry acts on the energy operators of each chain in a $\mathbb{Z}_2$ fashion as $\varepsilon_1 \leftrightarrow \varepsilon_2$. Thus $\varepsilon_1 + \varepsilon_2$ is gauge invariant, while $\varepsilon_1 - \varepsilon_2$ is not. Similarly, couplings between single ε_i operators on different chains, such as $\varepsilon_{1,w}\varepsilon_{1,w'}$, are also not gauge invariant.

The most relevant gauge-invariant coupling between the chains is therefore the term

$$\delta S = \sum_{w \neq w'} C_{w,w'} \int dz\, d\tau\, (\varepsilon_{1,w} + \varepsilon_{2,w})(\varepsilon_{1,w'} + \varepsilon_{2,w'}). \tag{14}$$

As long as the parameter of the AT critical line is such that $\Delta_{\varepsilon_1+\varepsilon_2} > 1$, all allowed interchain couplings will be irrelevant. In this case we therefore obtain a stable continuous phase transition of XY* type, with non-universal critical exponents.

1.4 $N > 4$

For $N > 4$, the situation changes. In this case the 1d chains are more conveniently dealt with using a continuum XY* description. We do this by writing the z^z and x^z variables as exponentials of slowly-varying fields $\Phi_w(z)$ and $\Theta_w(z)$, so that in the IR near the phase transition we have the approximate identifications

$$\begin{aligned} \mathsf{z}^z_{(w,z)} &\sim \exp\left(i\Phi_w(z)\right), \\ \mathsf{x}^z_{(w,z)} &\sim \exp\left(i\frac{1}{N}\partial_z\Theta_w(z)\right) = \exp\left(i\frac{2\pi}{N}\pi_{\Phi_w}(z)\right), \end{aligned} \tag{15}$$

where $\pi_{\Phi_w} = \frac{1}{2\pi}\partial_z\Theta_w$ is the momentum conjugate to Φ_w, and where a suitable continuum limit is taken in the $\hat{\mathbf{z}}$ direction, assuming a continuous transition. The gauge transformations (**??**) act on Φ_w as

$$\Phi_w \mapsto \Phi_w + \frac{2\pi}{N} f_w, \qquad f_w \in \mathbb{Z}_N. \tag{16}$$

The $\mathbb{Z}_N$ nature of the problem can be accounted for with a $\mathbb{Z}_N$ anisotropy term $\cos(N\Phi_w)$. The most general action for the putative fixed point can then be written as $S = S_0 + S_{\partial\partial} + S_I$, where

$$\begin{aligned} S_0 = \frac{1}{4\pi}\sum_w \int dz\, dt\, \big(&2\partial_\tau\Phi_w\partial_z\Theta_w \\ &+ g_0\partial_z\Theta_w\partial_z\Theta_w + h_0\partial_z\Phi_w\partial_z\Phi_w\big) \end{aligned} \tag{17}$$

represents the free decoupled fixed point,

$$\begin{aligned} S_{\partial\partial} = \frac{1}{4\pi}\sum_{w\neq w'} \int dz\, dt\, \big(&g_{w-w'}\partial_z\Theta_w\partial_z\Theta_{w'} \\ &+ h_{w-w'}\partial_z\Phi_w\partial_z\Phi_{w'}\big), \end{aligned} \tag{18}$$

represents marginal derivative-derivative couplings between the chains, and where S_I contains the $\mathbb{Z}_N$ anisotropy and further cosines in Φ_w, Θ_w allowed by gauge invariance, viz.

$$\begin{aligned} S_I = \Lambda^2 \int dz\, d\tau\, \Big[& u \sum_w \cos(N\Phi_w) \\ & + \sum_{\{\alpha_w\}} s_\alpha \cos\Big(N \sum_w \alpha_w \Phi_w\Big) \\ & + \sum_{\{\beta_w\}} r_\beta \cos\Big(\sum_w \beta_w \Theta_w\Big)\Big], \end{aligned} \tag{19}$$

where the sums are over all integer-valued functions α_w, β_w, and where Λ is a cutoff for the z component of the momentum modes of Φ_w, Θ_w. Despite the fact that we naively need $u \to \infty$ to enforce the $\mathbb{Z}_N$ nature of the problem, u can in fact be treated perturbatively.[3]

Consider first the decoupled limit where the terms in $S_{\partial\partial}$ vanish. The phase transition is driven by changing the coefficient of the field term $(\mathsf{z}_i^z)^\dagger \mathsf{z}_{i+1}^z \sim (\partial_z \Phi_w)^2$ in the Hamiltonian; this coefficient controls the scaling dimensions of the Φ_w, Θ_w vertex operators. In this limit, the scaling dimensions of the most relevant gauge-invariant cosines are

$$\Delta_{\cos(N\Phi_w)} = \frac{N^2}{2R_0^2}, \qquad \Delta_{\cos(\Theta_w)} = \frac{R_0^2}{2}, \tag{20}$$

where we have defined

$$R_0^2 \equiv \sqrt{h_0/g_0}. \tag{21}$$

When the field term is small, so that the lineons are not condensed, R_0^2 is also small. Here the $\cos(\Theta_w)$ operators are relevant, pinning the values of the Θ_w fields. This regime occurs for $R_0^2 < N^2/4$. In the limit of large field where the lineons are condensed R_0^2 is large, and here the $\cos(N\Phi_w)$ term is relevant. This happens for $R_0^2 > 4$.

When $N > 4$, we see that there is an intermediate range of fields with

$$4 < R_0^2 < N^2/4 \tag{22}$$

where none of the cosines are relevant. This gives us a massless phase in between the normal and condensed phases, with the phase transitions on either side being of KT type. [4]

To determine whether or not this physics occurs over a finite region of parameter space, we need to consider how the relevance of the above interactions is modified by the $S_{\partial\partial}$ term. Assuming that the couplings in $S_{\partial\partial}$ are translation invariant, this is done by Fourier transforming in the x, y directions transverse to the chains. Doing this gives, after integrating out the Θ_ws,

$$\begin{aligned} S_0 + S_{\partial\partial} = \frac{1}{L} \sum_k \frac{R_k^2}{4\pi} \int dz\, d\tau\, \big(& v_k^{-1} \partial_\tau \Phi_k \partial_\tau \Phi_{-k} \\ & + v_k \partial_z \Phi_k \partial_z \Phi_{-k}\big) \end{aligned} \tag{23}$$

[3]This can be shown by requiring that the known physics of the XY model be recovered in the $N \to \infty$ limit; see Elitzur's paper.

[4]Note that both phase transitions must necessarily be of the same character since they are related by the self-duality of the $\mathbb{Z}_N$ clock model.

where we Fourier transform with e.g.

$$g_k = \frac{1}{L} \sum_w e^{-iwk} g_w \tag{24}$$

with L^2 the number of wires in the array, and where we have defined

$$R_k^2 \equiv \sqrt{h_k/g_k}, \qquad v_k \equiv \sqrt{h_k g_k}. \tag{25}$$

The scaling dimensions of $\cos(N \sum_w \alpha_w \Phi_w)$ and $\cos(\sum_w \beta_w \Theta_w)$ at this fixed point are respectively

$$\Delta_\alpha = \frac{N^2}{2} \sum_k \frac{|\alpha_k|^2}{R_k^2}, \qquad \Delta_\beta = \frac{1}{2} \sum_k |\beta_k|^2 R_k^2. \tag{26}$$

The expression for Δ_α implies in particular that

$$\Delta_u = \frac{N^2}{2} \sum_k \frac{1}{R_k^2}. \tag{27}$$

We are interested in small perturbations about the decoupled fixed point which are finite ranged in the directions transverse to the chains. We can therefore write

$$g_k = g_0 \frac{1 + \delta g_k}{L}, \qquad h_k = h_0 \frac{1 + \delta h_k}{L}, \tag{28}$$

where $1/L \ll \delta g_k, \delta h_k \ll 1$. To leading order in $\delta g_k, \delta h_k$, the scaling dimensions $\Delta_\alpha, \Delta_\beta$ are now

$$\begin{aligned} \Delta_\alpha &= \frac{N^2 R_0^2}{4} \sum_k |\alpha_k|^2 (2 + \delta g_k - \delta h_k), \\ \Delta_\beta &= \frac{1}{4 R_0^2} \sum_k |\beta_k|^2 (2 + \delta h_k - \delta g_k). \end{aligned} \tag{29}$$

Consider e.g. Δ_α. We have

$$\begin{aligned} \Delta_\alpha &\geq \frac{N^2 R_0^2}{4} (2 + \min_k [\delta g_k - \delta h_k]) \sum_w \alpha_w^2 \\ &= \Delta_\alpha(\delta g_k, \delta h_k = 0) \left(1 + \frac{1}{2} \min_k [\delta g_k - \delta h_k]\right). \end{aligned} \tag{30}$$

Similar considerations apply to Δ_β. Therefore as long as the $\delta g_k, \delta h_k$ are small, the terms in $S_{\partial\partial}$ do not change the existence of an intermediate massless phase for $N > 4$. Since the massless phase is still described by a collection of decoupled XY* models, the phase transitions on either side of the massless phase are still expected to be of KT type.

2 2+1d stacks of critical layers

2.1 $N = 2$

The decoupled fixed point is a sum of Ising models, and so we need to examine the fixed points of

$$S = \int d^{4-\varepsilon}x \left(\frac{1}{2} \sum_i \left[(\partial\phi_i)^2 + \frac{t_i \Lambda^2}{2} \phi_i^2 \right] + \frac{\Lambda^\varepsilon}{8} \sum_{i,j} \phi_i^2 \phi_j^2 g_{ij} \right) \tag{31}$$

where again terms that mix derivatives between layers are ruled out by $\mathbb{Z}_2$ gauge invariance on each layer. We are therefore dealing with an $O(N)$ model at $N = L$ (with L the system size), with couplings restricted to those that are invariant under a $\mathbb{Z}_2^L$ action of

$$\mathbb{Z}_2^L : \phi_i \mapsto f_i \phi_i, \tag{32}$$

where each $f_i \in \pm 1$ can be chosen independently. The matrix g_{ij} can be taken to be a symmetric matrix, and for stability reasons we may take it to be positive definite.[5]

Now enumerating all the RG fixed points for general L is essentially impossible (the symmetry group $\mathbb{Z}_2^L$ possesses too many quartic invariants). In fact even in the case of small L like $L = 4$, there are many fixed points—looking at the literature, it's actually unclear if there are any values of $N > 2$ for which the fixed points have been completely classified.

However, if we restrict our attention only to *stable* fixed points (i.e. stable with respect to all quartic perturbations respecting $\mathbb{Z}_2^L$), we will see that the situation becomes much more tractable. In what follows we will first do some direct calculations for the simplest choices of couplings, and then turn to some more powerful general statements.

First, we will assume that the g_{ij} couplings are translation-invariant and symmetric. In accordance with this, we will use the notation $g_{ij} = g_{i-j} = g_{j-i}$. Now from past experience we know that $L = 2$ critical Ising models coupled by their energy operators flows to the (relativistic) XY model, while a stack of $L = 3$ flows to the $O(3)$-symmetric fixed point; both fixed points are stable.

To find out what happens for general L, we need (at least) the 1-loop beta functions.[6] These are calculated in the usual way: the counterterms approach gives (no real need to keep track of the mass term since in dim reg t doesn't appear in the beta functions of the quartic couplings), with the notation $g_{i-j} \equiv g_{ij}$,

$$\beta_{g_j} = \varepsilon g_j - 2g_j^2 - 2g_0 g_j - \frac{1}{2} \sum_k g_k g_{j-k}. \tag{33}$$

Note how g_0 is singled out; this occurs because of the different symmetry factors for vertices that involve four identical indices. If we prefer, in Fourier space this is

$$\beta_{g_p} = \varepsilon g_p - \frac{2}{2\pi} \int_q g_q g_{p-q} - g_p \frac{2}{2\pi} \int_q g_q - \frac{1}{2} g_p^2. \tag{34}$$

[5] The potential is stable as long as g_{ij} is positive semi-definite. However, if $\det g = 0$, then there exists a free direction in field space—by a change of basis, we get a decoupled flavor which does not interact. Therefore if such a g_{ij} is a fixed point it will for sure be unstable; since we are only interested in stable fixed points, degenerate g_{ij}s can be ignored.

[6] It will turn out that the 1-loop answers will be good enough for assessing stability within the context of small ε for all $L \neq 4$; for $N = 4$ there's a cancellation which necessitates going to two-loop.

The matrix determing the scaling dimensions at a given fixed point is accordingly

$$\mathcal{B}_{jl} \equiv \frac{\partial \beta_{g_j}}{\partial g_l}\bigg|_{g_j=g_j^*} = \delta_{jl}(\varepsilon - 4g_j^* - 2g_0^*) - 2g_j^*\delta_{l0} - g_{j-l}^* \tag{35}$$

and in Fourier space,

$$\mathcal{B}_{pq} = \delta_{pq}\left(1 - 2g_{j=0}^* - g_p^*\right) - 4g_{p-q}^* - 2g_p^*. \tag{36}$$

First, we obviously have the usual $g_0 \neq 0, g_{j\neq 0} = 0\ \forall\ j$ Ising$^{\oplus L}$ fixed point; here $g_0^* = \frac{2}{9}\varepsilon$, g_0 is irrelevant with $y_{g_0} = -\varepsilon$, and all of the $g_{j\neq 0}$s are equally relevant with eigenvalue $y_{g_{j\neq 0}} = \varepsilon/3$.

Because of the last term in (33), it is impossible to have a fixed point with strictly finite-ranged couplings if any of the $g_{j>0}$ are non-zero, at least if we restrict to positive couplings. The solutions with maximally long-ranged coupling between the layers are those where the couplings are symmetric under the action of S_L: these are the cubic fixed points, with symmetry group $G = \mathbb{Z}_2^L \rtimes S_L$. Since G has only two quartic invariants, in this case there are only two distinct β functions to solve. Letting $h \equiv g_{j>0}$, the fixed points are determined by

$$\varepsilon g_0 = \frac{9}{2}g_0^2 + \frac{L-1}{2}h^2, \qquad \varepsilon h = 3g_0 h + \frac{L+2}{2}h^2. \tag{37}$$

The fixed points are thus

$$(g_0^*, h^*) = \begin{cases} \mathcal{C}\ : & \left(\dfrac{2\varepsilon}{9}(1-1/L),\, \dfrac{2\varepsilon}{3L}\right) \\ \mathcal{S}\ : & \left(\dfrac{2\varepsilon}{8+L},\, \dfrac{2\varepsilon}{8+L}\right) \end{cases} \tag{38}$$

$\mathcal{S}$ is the usual $O(L)$-symmetric fixed point, while the cubic fixed point $\mathcal{C}$ goes over to Ising$^{\oplus L}$, up to $O(1/L)$ corrections. In this case the matrix determining the scaling dimensions is

$$\mathcal{B} = \begin{pmatrix} \varepsilon - 9g_0^* & -h^* & -h^* & \dots \\ -3h^* & \varepsilon - 4h^* - 3g_0^* & -h^* & \dots \\ -3h^* & -h^* & \varepsilon - 4h^* - 3g_0^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{39}$$

Diagonalizing this, we find three different eigenvalues (now temporarily writing $g = g_0^*$ and $h = h^*$ for notation's sake):

$$\mathrm{Spec}(\mathcal{B}) = \begin{cases} \lambda_{L-2} = \varepsilon - 3g - 3h & m = L-2, \\ \lambda_\pm = \varepsilon - 6g - \dfrac{L+2}{2}h \pm \dfrac{1}{2}\sqrt{36g^2 - (24+12L)gh + (L^2+16L-8)h^2} & m = 1 \end{cases} \tag{40}$$

where m denotes the multiplicity. The $\mathcal{S}$ fixed point is unstable: the eigenvalues of $\mathcal{B}$ are

$$\mathrm{Spec}(\mathcal{B})|_{\mathcal{S}}\ :\ \lambda_{L-2} = \lambda_+ = \frac{\varepsilon(L-4)}{8+L}, \qquad \lambda_- = -\varepsilon, \tag{41}$$

so that as $L \to \infty$ there are $L-1$ relevant parameters, with eigenvalues $y = +\varepsilon$. As a sanity check, these match exactly with the scaling dimensions for the $O(N)$-symmetric fixed point found in Cardy's book.

At $\mathcal{C}$, $\mathcal{B}$ has eigenvalues

$$\mathrm{Spec}(\mathcal{B})|_{\mathcal{C}} \, : \, \lambda_{L-2} = \frac{\varepsilon(L-4)}{3L}, \qquad \lambda_+ = \frac{\varepsilon(4-L)}{3L}, \qquad \lambda_- = -\varepsilon. \tag{42}$$

So this is only barely more stable than the $\mathcal{S}$ fixed point: we still have $L-2$ relevant parameters at large L, although the relevant deformations are less relevant than the ones at the $\mathcal{S}$ fixed point ($y \to \varepsilon/3$ versus $y \to \varepsilon$). Note that the scaling dimensions of the majority of the couplings have only changed with respect to the decoupled fixed point by an amount $-4\varepsilon/3L$, which vanishes as $L \to \infty$. This makes sense because $\mathcal{C}$ moves closer to the decoupled fixed point as $L \to \infty$, but it also means that we still have a long way to flow before we reach stability.

Anyway, at this point, attempting to brute-force solve the β functions and look for the stable points quickly gets out of hand. Additionally, restricting ourselves to translationally-invariant couplings is a bit fine-tuned—I don't see any reason a priori why translationally-non-invariant couplings would necessarily yield only unstable fixed points. For example, a tetragonally-symmetric solution with non-zero couplings $g_{1,2} = g_{3,4} = g_{5,6} = \ldots$ exists, although it turns out to be unstable. Another example is that one could consider first turning on only an energy-energy coupling between pairs of layers: since two Ising models coupled by their energy operators flow to the XY fixed point, this would give us a stack of XY CFTs. Further energy-energy couplings between these stacks are (barely) irrelevant, since in the 3d XY model we have

$$\Delta_\varepsilon \approx 1.51 > 3/2. \tag{43}$$

However, the dimension of the lightest charge-2 operator (think ϕ^2 instead of $|\phi|^2$) in the 3d XY model is

$$\Delta_t \approx 1.23 < 3/2. \tag{44}$$

Since we only have a $\mathbb{Z}_2$ symmetry, t-t couplings between XY planes are allowed, and we again we do not get a stable fixed point. While these examples don't work, it's not clear that we can't find something that does—therefore a more general analysis is needed.

We could attempt an analysis of the β functions in the general case, [7] but this turns out not to be necessary, as long as we are only interested in addressing the question of stability. In fact, we can use some more general results about the beta functions in ϕ^4 theories to prove that *no stable fixed points exist* for any $L > 4$.

[7] For posterity's sake, in this case the CPT approach gives (with $u = g_{ii}$)

$$\begin{aligned}
\beta_u &= \varepsilon u - 36u^2 - 4\sum_k g_{1k} g_{1k} - 4ut \\
\beta_{g_{ij}} &= \varepsilon g_{ij} - 12u g_{ij} - 8\sum_k g_{ik} g_{kj} - 4 g_{ij} t \\
\beta_t &= 2t - 2t^2 - 6ut - \sum_j g_{1j} t - 48u^2 - \sum_j g_{1j} g_{1j}.
\end{aligned} \tag{45}$$

The most concise way of going about the following is to formulate the RG flows as gradient flows on a certain space of symmetric tensors. I learned about this from Zinn-Justin's QFT and critical phenomena book—in what follows we will derive the results that are important for us. For this, let us momentarily step back to the more general case of a L-component theory with interactions g_{ijkl}, where g_{ijkl} is a symmetric tensor that satisfies $v^i v^j v^k v^l g_{ijkl} > 0$ for any real vector $v \in \mathbb{R}^L$.[8] The β function is (re-scaling away numerical factors from the loop integration as usual)

$$\beta_{ijkl} = \varepsilon g_{ijkl} - \sum_{mn} (g_{ijmn} g_{mnkl} + g_{ikmn} g_{mnjl} + g_{ilmn} g_{mnkj}). \tag{46}$$

The three quadratic terms are just the s, t, u channels (the ways of partitioning the $ijkl$ indices into unordered pairs).This simple form is why we've momentarily generalized away from the case with $\mathbb{Z}_2^L$-symmetric couplings—when we keep the couplings arbitrary the symmetry factors are much simpler to keep track of; if we worked directly with the $\mathbb{Z}_2^L$-symmetric couplings we'd have lots more δs floating around.

Anyway, the point of using this notation is that we can write

$$\beta_{ijkl} = \frac{\delta}{\delta g_{ijkl}} \mathcal{U}(g), \qquad \mathcal{U}(g) = \frac{\varepsilon}{2} \sum_{ijkl} g_{ijkl}^2 - \sum_{ijklmn} g_{ijkl} g_{klmn} g_{mnij}. \tag{47}$$

To save space, we will use the multiindex notation $\beta_I = \delta_I \mathcal{U}(g)$, where $g_{ijkl} = g_I$. One technical point that Zinn-Justin glosses over: since we are working only with symmetric tensors, we need to use the variational derivative instead of the partial derivative, since we are in a constrained space where the components of g_{ijkl} are not independent variables—it makes no sense to take the derivative with respect to g_{ijkl} while keeping g_{ijlk} fixed ($\partial_I \mathcal{U}(g) \neq \beta_I$, the difference coming in combinatorial factors from the symmetry). We can still use the partial derivative, but we have to compensate for the fact that the partial derivative is operating in a bigger space by dividing out by the symmetry factor of the tensor in question. Therefore we can write

$$\frac{\delta}{\delta g_I} = M^{IJ} \frac{\partial}{\partial g^J}, \tag{48}$$

where the metric is

$$M^{IJ} = \delta^{IJ} \frac{1}{N_I}, \tag{49}$$

with N_I the number of distinct permutations of I (e.g. $N_{1234} = 4!$, $N_{1223} = \binom{4}{2}$, etc.). In what follows we will use ∂s, and raise / lower indices with this metric, so that e.g. $\beta^I = M^{IJ} \partial_J \mathcal{U}(g)$. It then follows that $\mathcal{U}(g)$ is an RG monotone: letting t be RG time,

$$\frac{d\mathcal{U}(g)}{dt} = \beta^J \partial_J \mathcal{U}(g) = M^{JK} \partial_K \mathcal{U}(g) \partial_J \mathcal{U}(g) > 0, \tag{50}$$

since the metric is positive definite. We will now show that if there is a stable fixed point, it is the *unique* maximum of $\mathcal{U}(g)$ in coupling constant space. Hence if $\mathcal{U}(g) = \mathcal{U}(g')$ with $g \neq g'$, then neither g nor g' can be stable fixed points.

[8]The $\mathbb{Z}_2^L$-symmetric case of interest would be $g_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl} g_{ik} + \delta_{ik}\delta_{jl} g_{ij} + \delta_{il}\delta_{jk} g_{ij})$.

So, suppose g_1, g_2 are two distinct fixed points. We know that $\mathcal{U}$ is an RG monotone, and so we might be able to get an idea about the relative stability of these two fixed points by comparing $\mathcal{U}$ between them. To this end let $g(\lambda) = g_1\lambda + g_2(1-\lambda)$ be an interpolating path of couplings (this is not generically an RG flow, so $\mathcal{U}$ needn't be monotonic in λ). $\mathcal{U}$ varies along the path as

$$\partial_\lambda \mathcal{U}(g(\lambda)) = (g_1^I - g_2^I)\beta_I \equiv \Delta^I \beta_I. \tag{51}$$

Since g_1, g_2 are fixed points, $\partial_\lambda \mathcal{U}(g)|_{\lambda=0,1} = 0$. Since β is quadratic in the g_Is and vanishes at the endpoints,$\beta_I = C_I\lambda(1-\lambda)$ for some λ-independent C_I. Then another derivative wrt λ gives

$$\partial_\lambda^2 \mathcal{U}(g(\lambda)) = \Delta^I \Delta^J \partial_I \partial_J \mathcal{U}(g(\lambda)) = \Delta^I C_I (1-2\lambda). \tag{52}$$

If we evaluate this at the endpoints of the interpolation, we get

$$\Delta^I \mathcal{B}_{IJ}(g_1)\Delta^J = \Delta^I C_I, \qquad \Delta^I \mathcal{B}_{IJ}(g_2)\Delta^J = -\Delta^I C_I, \tag{53}$$

where again $\mathcal{B}_{IJ} = \partial_I \partial_J \mathcal{U}$ determines the scaling dimensions at a fixed point. Therefore

$$\Delta^I(\mathcal{B}_{IJ}(g_1) + \mathcal{B}_{IJ}(g_2))\Delta^J = 0, \tag{54}$$

which since $\Delta^I \neq 0$ means that it is impossible for $\mathcal{B}(g)$ to be positive-definite at both g_1 and g_2—hence at least one of the two fixed points must be unstable, meaning that only at most one stable fixed point exists.

A corollary of this is the following. Let $g_I = R_I^J g_J$, with R a representation of some group element of $O(L)$ acting in the vector$^{\otimes 4}$ representation. Such $O(L)$ transformations map fixed points to fixed points, since β_I transforms covariantly. Therefore from the above result, any fixed point g_I acted on non-trivially by $O(L)$ *must* be unstable, and so the only stable fixed point is the $O(L)$ symmetric one, where $g_{ijkl} \propto (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Since we know the $O(L)$-symmetric is unstable for $L > 4$ by the calculation we did above, we thus conclude that there are no stable fixed points for $L > 4$.

This result was proved by looking at the behavior of $\mathcal{U}(g)$ with respect to arbitrary variations in the space of symmetric tensors g_I. Depending on the application though, we may only be interested in variations that preserve some symmetry group $G \subset O(L)$. Making this restriction can eliminate relevant directions in coupling constant space, and result in symmetry-protected stability.[9] However, since $\lambda g_1 + (1-\lambda)g_2$ is G-invariant if both g_i are, the above result means that there is at most one G-stable fixed point for each G. Furthermore, from the previous paragraph, if g_I is a G-stable fixed point and there exists some group element in $O(L)$ which maps g_I to any G-invariant coupling distinct from g_I itself, then g_I must be unstable.

Now we can finally apply this to the problem at hand to show that there are no $\mathbb{Z}_2^L$-stable fixed points for $L > 4$. Indeed, consider the action of $\sigma \in S_L \subset O(L)$ on a given $\mathbb{Z}_2^L$-invariant coupling g_{ij}. This maps $g_{ij} \mapsto g_{\sigma(i)\sigma(j)}$, which is of course also $\mathbb{Z}_2^L$ symmetric. Therefore a

[9]A dumb example is $G = O(L)$, then the $O(L)$-symmetric fixed point is obviously stable. From the examples we worked out above, we also know that $G = \mathbb{Z}_2^L \rtimes S_N$ (where the two independent couplings were $g_{j>0}$ and g_0) is big enough to protect a stable fixed point.

necessary condition for a g_{ij} to give a $\mathbb{Z}_2^L$-stable fixed point is for $g_{\sigma(i)\sigma(j)} = g_{ij}$ for all $\sigma \in S_L$, which means that g_{ij} must have the form $g_{ij} = g_0\delta_{ij} + h$. But we have already solved this case and shown that there are no stable fixed points with this restricted class of couplings when $L > 4$. Therefore we conclude that as long as $L > 4$, the theory is unstable towards a fluctuation-induced first-order transition.[10]

2.2 $N > 2$

For $N > 2$ the putative theory near the critical point on each plane is given by a field ψ which is a coarse-grained version of the z spins, and is described by the action

$$S = \int d\tau\, d^2x \left(i\psi^*\partial_\tau\psi + \frac{1}{2m}|\nabla\psi|^2 + t|\psi|^2 + \frac{u}{4}|\psi|^4 \right. \\ \left. + \frac{g\Lambda^{4-N}}{N!}(\psi^N + (\psi^*)^N) + \cdots \right), \tag{55}$$

where the $\cdots$ are less relevant terms which are needed for stability of the potential if $N > 4$. Here the first term, which leads to non-relativistic $z = 2$ scaling, is included instead of $|\partial_\tau\psi|^2$ since we are not requiring that our theory also possess any form of particle-hole symmetry. $d = 2$ is thus the upper critical dimension, allowing the RG flow to be accurately computed through a perturbative analysis of the action (55).

The non-relativistic nature of the theory means that the 1-loop computation of the β functions is exact, giving

$$\beta_u = -u^2/2, \qquad \beta_g = (4 - N)g - ug/2. \tag{56}$$

The anisotropy is therefore relevant at $N = 3$, marginally irrelevant at $N = 4$, and irrelevant for $N > 4$. When $N = 3$ Ginzburg-Landau theory predicts a first-order transition due to the shape of the potential for ψ, and so this case can be ignored. For $N \geq 4$ however, we obtain a nontrivial critical theory describable by the non-relativistic XY* model.

We now need to know the stability of the decoupled fixed point for a stack of non-relativistic XY* models. The most relevant gauge-invariant couplings are those which couple the energy operators on different layers; in the present notation they read

$$\delta S = \sum_{i,j} g_{ij} \int d\tau\, d^2x\, |\psi_i|^2|\psi_j|^2. \tag{57}$$

The beta function for g_{ij} is similarly exactly computable from the 1-loop term, which gives

$$\beta_{g_{ij}} = -\frac{1}{2}(g_{ij})^2. \tag{58}$$

An important difference relative to the case of 1d Ising chains examined earlier in (3) is that due to the non-relativistic nature of the theory, none of the g_{ij}s mix with each other. As a result, the decoupled fixed point is stable provided that all of the g_{ij}s are positive. Note

[10] To build confidence, I also checked this by numerically searching for $\mathbb{Z}_2^L$-symmetric fixed points of the β functions for $L \leq 10$ (and of course found no stable ones).

that getting a stable fixed point does not require fine-tuning: we just need to that $g_{ij} > 0$ for all i, j; we do not require that each of the g_{ij} be tuned to any one particular value.

Stability even for negative interlayer couplings can be obtained by imposing an additional particle-hole symmetry in the UV. In terms of the UV spins, this symmetry is realized by conjugating the spins with the matrix

$$C = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & \iddots & \\ & 1 & & \end{pmatrix}, \qquad C\mathsf{z}C = \mathsf{z}^\dagger, \quad C\mathsf{x}C = \mathsf{x}^\dagger. \tag{59}$$

In terms of the coarse-grained field ψ this sends $C : \psi \to \psi^*$, under which the $i\psi^*\partial_\tau\psi$ term is odd.

In this case, the field theory on each layer instead has relativistic $z = 1$ scaling. Calculating the relevance of the $\mathbb{Z}_N$ anisotropy using the ε expansion[11] indicates that the anisotropy is irrelevant as long as $N > 3$. Therefore while again we expect a first order transition when $N = 3$, for $N > 3$ (or if the estimation from the ε expansion is too crude, $N > N_c$ with N_c not too much larger than 3) we have a potential critical fixed point given by a decoupled stack of relativistic $z = 1$ XY* models. The most relevant gauge-invariant coupling between the layers is again the energy-energy coupling. Since as we said above $\Delta_\varepsilon \approx 1.51 > 3/2$, the decoupled fixed point is (barely) stable.

[11] Letting C^c_{ab} denote the OPE coefficient of the operators with couplings a, b, c, we find

$$\begin{aligned} &C^u_{uu} = 20 \quad C^u_{gg} = \frac{N!N(N-1)}{4} \quad C^u_{ut} = 4 \\ &C^g_{ug} = N(N-1) \quad C^g_{tg} = N \\ &C^t_{tt} = 2 \quad C^t_{ut} = 4 \quad C^t_{gg} = N^2(N-1)! \quad C^t_{uu} = 80. \end{aligned} \tag{60}$$

If we then re-scale g by $g \mapsto g/\sqrt{N!}$, the beta functions in this scheme are (at $\varepsilon = 1$),

$$\begin{aligned} \beta_u &= u - 10u^2 - \frac{N(N-1)}{8}g^2 - 4ut \\ \beta_g &= g - guN(N-1) - Ngt \\ \beta_t &= 2t - t^2 - 4ut - \frac{N}{2}g^2 - 40u^2. \end{aligned} \tag{61}$$

If we look at the $g_* = 0$ solution to the β functions, we find a quadratic equation for the scaling dimension of the anisotropy in terms of N, solving this determines the critical value of N_c to be $N_c \approx 3.8$ within this approximation.