Transformation properties of spinors and pinors in QFT

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In these notes we will discuss how spacetime symmetries, in particular spacetime reflections, act on fermions in QFT. We will try to be as general as possible, covering both even and odd spacetime dimensions, real and imaginary time, and different choices of signature. However, we will stay within the context of free fermions, and won't discuss the transformation properties of any other fields they may couple to, or any other internal global symmetries that they might have. The point of this diary is to build up a fermion cheat-sheet that I can refer to later—the number of different conventions one can choose when dealing with fermions is huge, and so having a reference where the conventions are fixed is very helpful. This is likely all in the literature somewhere, but I wasn't able to find it all in one place (Witten's "fermions and path integrals" is definitely a good place to start, as is chapter 8 of Zinn-Justin).

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A word on notation: in this diary entry we will as usual be setting $J = -iY$, and will let $\mathcal{K}$ denote complex conjugation. $d = s + t$ will denote the dimension of spacetime, with s the number of spacelike-signature indices (positive signs in $\eta_{\mu\nu}$) and t the number of timelike-signature indices (negative signs in $\eta_{\mu\nu}$). We will assume throughout that there is only one time coordinate, so that the signature is either $(+,-,-,\dots)$, $(-,+,+,\dots)$, $(+,+,+,\dots)$, or $(-,-,-,\dots)$.

The business of working out the action of reflections on fermions is a mess because knowing just the representation theory of the spin groups is not sufficient — the spin groups of course only contain orientation-preserving group elements. The appropriate extensions of the spin groups needed to discuss reflections are the pin groups, and so statements like "the eigenvalue of a spinor under parity / time reversal" are totally meaningless: to talk about orientation-reversing symmetries, one must talk about their action on pinors, not spinors. The representation theory of the pin groups is rather gross, and the aim for this diary entry is to carefully go through and systematically list the different ways in which fermions can transform under reflections. Since charge conjugation often gets mixed into this business, we include a discussion of it as well.

Before we can talk about the pin groups and objects (pinors) which transform under them, we need to review a bit of representation theory.

Representation theory prelude

Suppose the representation $R : G \to \mathrm{Aut}(V)$ that the fermions transform in is isomorphic to its complex conjugate $\bar{R}$ (here we will assume that V is a complex vector space; when we discuss real fermions we will do so explicitly). Then there exists some map $\mathcal{J}$ that takes R to $\bar{R}$; because of the conjugation, $\mathcal{J}$ must be anti-linear. Since $\mathcal{J}^2$ is a linear map taking R to itself, by Schur's lemma we must have $\mathcal{J}^2 \propto \mathbf{1}$, and wolog we can re-scale $\mathcal{J}$ so that the constant of proportionality is ± 1 (but the sign cannot be eliminated with a rescaling by i due to the anti-linear nature of $\mathcal{J}$).

Suppose that $\mathcal{J}^2 = +\mathbf{1}$. In this case, we say that R is a *real* representation. Although we have been representing the group action as automorphisms on a complex vector space V, if the representation is real, we can also restrict the representation to a map $G \to \mathrm{Aut}(V_{\mathbb{R}})$, with $V_{\mathbb{R}}$ a real vector space (another diary entry on Pointryagin classes explains the connection between this and the fact that $\mathcal{J}^2 = +\mathbf{1}$).

Now suppose that $\mathcal{J}^2 = -\mathbf{1}$. In this case, we say that R is a *pseudoreal* or *quaternionic* representation; the reason for the former moniker is because the relations $\mathcal{J}^2 = (i\mathbf{1})^2 = -\mathbf{1}$ and $\mathcal{J}(i\mathbf{1}) = -(i\mathbf{1})\mathcal{J}$ give us a quaternionic structure formed from $i = i\mathbf{1}, j = \mathcal{J}, k = i\mathcal{J}$. In the quaternionic case, we cannot restrict the representation to an action on $\mathrm{Aut}(V_{\mathbb{R}})$.

Note that if R is a real representation and $\widetilde{R}$ is a pseudoreal representation, $R \otimes R$ and $\widetilde{R} \otimes \widetilde{R}$ are both real, while $R \otimes \widetilde{R}$ is pseudoreal (just by looking at the multiplication rule for the $\otimes$ of the $\mathcal{J}$s in question).

If R is real, we have a chance to define a theory with a single Majorana fermion, since then we can form a singlet under R by using a bilinear of a single field (if R is ps$\mathbb{R}$ it cannot be represented by purely real matrices, and therefore the reality condition of Majoranas cannot be preserved by the group action).

Finally, if $R \not\cong \bar{R}$, the representation is complex. In this case there is no invariant bilinear form that pairs two fields transforming in R, and so to construct an action we need to use two fields: one transforming in $\bar{R}$ and one transforming in R (usually called ψ and $\bar{\psi}$). We cannot define single Majorana fermions since we can't make a singlet using a bilinear of a single field (we can write actions in terms of Majorana fermions, but they need to mix flavors between a Majoranna in R and a different Majorana in $\bar{R}$).

One annoying thing is is that the character of the representations of $\mathrm{Spin}(d,0) \equiv \mathrm{Spin}(d)$ and $\mathrm{Spin}(d-1,1)$ are not the same (although thankfully $\mathrm{Spin}(d-1,1)$ and $\mathrm{Spin}(1,d-1)$ have isomorphic representations). For posterity's sake, fixing the spacetime dimension as d, the character of the spin representations as represented over $\mathbb{C}$ are as follows: (in even dimensions, we list the character of the chiral reducible representation $S_\pm$)

d	Euclidean Time	Real Time
1	$\mathbb{R}$	$\mathbb{R}$
2	$\mathbb{C}$	$\mathbb{R}$
3	$\mathbb{H}$	$\mathbb{R}$
4	$\mathbb{H}$	$\mathbb{C}$
5	$\mathbb{H}$	$\mathbb{H}$

(1)

These are easy enough to check: in real time, the first three entries follow from using the matrix $J = -iY$ for the negative-signature coordinate and X, Z for the others, while e.g. the complexity of the $d = 4$ can be verified by computations we will do later and the $d = 5$ entry is from $\mathrm{Spin}(4,1) = Sp(1,1)$. The Euclidean time entries follow from $\mathrm{Spin}(1) = \mathbb{Z}_2, \mathrm{Spin}(2) = U(1), \mathrm{Spin}(3) = SU(2), \mathrm{Spin}(4) = SU(2) \times SU(2)$, and $\mathrm{Spin}(5) = Sp(2)$. Note how Spin(d) has the same type of spinor representation as Spin($[d+2]-1,1$). The pattern can be continued up to higher d by using Bott periodicity (so that only d mod 8 is relevant).

Now from the above, we see that the representation theory of the spin group strongly depends on our choice of signature. Thus if G is the full symmetry group, the type of representation that G acts on the fermions with will depend on the choice of signature. This does not mean that the symmetries in the Lorentzian and Euclidean theories are different, it just means that the way in which the symmetries act is dependent on the choice of signature: this is true for the action of the spin group and for the action of the pin group (for example, a $\mathbb{R}$ time theory with T symmetry will continue to an $i\mathbb{R}$ time theory with T symmetry, but T will act differently in the $i\mathbb{R}$ time theory). For physics, we should only draw conclusions based on representation theory when working in real time.

Even when we restrict to $\mathbb{R}$ time, the signature matters for reflections. While $\mathrm{Spin}(1,d-1)\cong\mathrm{Spin}(d-1,1)$, and $O(1,d-1)\cong O(d-1,1)$, unfortunately $\mathrm{Pin}(1,d-1)\ncong\mathrm{Pin}(1,d-1)$, and $\mathrm{Pin}(0,d)\ncong\mathrm{Pin}(d,0)$. The fact that pin groups in different signatures aren't isomorphic holds in even the simplest case of 0+1 dimensions, where $\mathrm{Pin}(1,0)\cong\mathbb{Z}_2^2$, while $\mathrm{Pin}(0,1)\cong\mathbb{Z}_4$ (in this case both Spin groups are trivial, and both orthogonal groups are $\mathbb{Z}_2$). Therefore the choice of signature actually is physical in a way—a choice needs to be made in order to determine the algebra obeyed by spacetime reflections, and not all choices of algebra and signature are mutually consistent.

Generalities on representations of Clifford algebras

Let $\mathcal{C}(s,t)$ denote the Clifford algebra generated by the γ matrices γ_μ, $\mu\in\mathbb{Z}_{s+t}$, with

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}, \tag{2}$$

where η has s positive diagonal entries and t negative ones. The γ^μ can always be built out of tensor products of Pauli matrices (with possible factors of i), and so for us will always be unitary.

We will let $\mathcal{C}_\pm(d)$ represent the elements in $\mathcal{C}(s,t)$ that are even / odd under $\gamma_\mu\mapsto-\gamma_\mu\,\forall\,\mu$, respectively. $\mathcal{C}(s,t)$ splits as

$$\mathcal{C}(s,t)\cong\mathcal{C}_+(s,t)\oplus\mathcal{C}_-(s,t). \tag{3}$$

$\mathcal{C}_+(s,t)$ forms a subalgebra of $\mathcal{C}(s,t)$, which we will see is related to Spin(s,t). The representation theory of $\mathcal{C}(s,t)$ depends on the parity of d, and is sorted out by using the matrix

$$\bar\gamma\equiv i^{(d^2-d)/2+t}\prod_\mu\gamma_\mu. \tag{4}$$

The dumb factor out front is to ensure that $\bar\gamma^2=\mathbf{1}$ and $\bar\gamma^\dagger=\bar\gamma$ in every dimension and signature, so that (in even dimensions) it can be employed in the projectors $(\mathbf{1}\pm\bar\gamma)/2$.[1]

If $d\in2\mathbb{Z}+1$, $\bar\gamma$ commutes with all the γ_μ, and so it will be $\pm\mathbf{1}$. The map $\gamma_\mu\mapsto-\gamma_\mu\,\forall\,\mu$ changes the sign of $\bar\gamma$ and preserves the $\mathcal{C}(s,t)$ anticommutaion relations: thus we get two distinct representations, differing my the signs of the γ matrices. When $d\in2\mathbb{Z}$ we only have

[1]Note that in other diary entries $\bar\gamma$ doesn't have the prefactor.

one irreducible representation, since the map which changes the sign of all the γ_μ can be obtained by conjugating with $\bar{\gamma}$: $\bar{\gamma}^\dagger \gamma_\mu \bar{\gamma} = -\gamma_\mu$.

$\mathrm{Spin}(s,t)$ is defined as the elements in $\mathcal{C}_+(s,t)$ of unit norm. Since $\mathrm{Spin}(s,t) \subset \mathcal{C}_+(s,t)$, the two distinct representations of $\mathcal{C}(s,t)$ when $d \in 2\mathbb{Z}+1$ are indistinguishable in $\mathrm{Spin}(s,t)$, and in fact the spinor representation of $\mathrm{Spin}(s,t)$ is irreducible. When $d \in 2\mathbb{Z}$, we can form chiral projectors with $\bar{\gamma} \in \mathcal{C}_+(d)$, which commutes with everything in $\mathrm{Spin}(s,t)$. This means we can decompose the representation matrices of $\mathrm{Spin}(s,t)$ in a form which is block-diagonal in the ± 1 eigenspaces of $\bar{\gamma}$, meaning that the spinor representation of $\mathrm{Spin}(s,t)$ is reducible, with the spinor bundle splitting as $S_+ \oplus S_-$. Including spacetime reflections (elements in $\mathcal{C}_-(s,t)$) mixes sections of S_+ with those of S_- since the reflections all anticommute with $\bar{\gamma}$, and leaves us with only one irreducible representation.

The group $\mathrm{Pin}(s,t)$ is defined as the elements of $\mathcal{C}(s,t)$ of unit norm. We will be interested in a representation (the pinor representation) of $\mathrm{Pin}(s,t)$ on $\mathcal{C}(s,t)$, since this representation is what will allow us to determine how spacetime symmetries act on the fields (pinors) in the Lagrangian. This action is determined by the homomorphism

$$\Omega : \mathrm{Pin}(s,t) \to O(s,t) \tag{5}$$

defined for every $\Lambda \in \mathrm{Pin}(s,t)$ by

$$\Lambda^{-1}\gamma_\mu \Lambda = R_{\mu\nu}\gamma^\nu, \tag{6}$$

where $R_{\mu\nu} \in O(s,t)$. This transformation law is what allows $\bar{\psi}\partial\!\!\!/\psi$, with $\bar{\psi}, \psi$ two pinor fields, to be invariant under the action of Lorentz transformations[2], since it means that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector provided that $\bar{\psi}$ transforms inversely to ψ. The fact that $R_{\mu\nu} \in O(s,t)$ is required can be seen from requiring the anticommutation relations of the Clifford generators to be invariant under the action of $\mathrm{Pin}(s,t)$. From applying the action of $\mathrm{Pin}(s,t)$ on $\eta^{\mu\nu}$, we find

$$2\eta_{\mu\nu} = \{\gamma_\mu, \gamma_\nu\} \mapsto \Lambda^{-1}\{\gamma_\mu,\gamma_\nu\}\Lambda = R_{\mu\lambda}R_{\nu\sigma}\{\gamma^\lambda,\gamma^\sigma\} = 2R_{\mu\lambda}\eta^{\lambda\sigma}[R^T]_{\sigma\nu} \implies [R^T R]_{\mu\nu} = \eta_{\mu\nu}. \tag{7}$$

The homomorphism Ω is obviously not injective, since both Λ and $-\Lambda$ are associated with the same matrix R (this is why the Pin groups are $\mathbb{Z}_2$ extensions of the orthogonal groups). Whether or not Ω is surjective actually depends on whether d is even or odd. Indeed, consider the action on $\bar{\gamma}$. Then

$$\Lambda^{-1}\bar{\gamma}\Lambda = i^{(d^2-d)/2+t}R_{1\mu_1}R_{2\mu_2}\cdots R_{d\mu_d}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_d} = R_{1\mu_1}\cdots R_{d\mu_d}\epsilon^{\mu_1\dots\mu_d}\bar{\gamma} = (\det R)\bar{\gamma}. \tag{8}$$

Here we have used that if any $\mu_i = \mu_j$ but $i \neq j$, then we get a product $R_{i\mu_j}R_{k\mu_j}$ for $i \neq k$, which vanishes by the orthogonality of R. The transformation rule

$$\Lambda^{-1}\bar{\gamma}\Lambda = (\det R)\bar{\gamma} \tag{9}$$

is why $\bar{\gamma}$ is a pseudoscalar.

[2]here $\bar{\psi}$ is a copinor so that $\psi \mapsto \Lambda\psi \implies \bar{\psi} \mapsto \bar{\psi}\Lambda^{-1}$. More on this later.

Now in odd dimensions, $\bar{\gamma}$ commutes with all of $\mathcal{C}(s,t)$, and so in odd dimensions we have $\bar{\gamma} = (\det R)\bar{\gamma} \implies \det R = 1$. Thus in odd dimensions we can only generate an action of $SO(s,t)$. In even dimensions $\bar{\gamma}$ anticommutes with $\mathcal{C}_-(s,t)$, and so if we take Λ to be generated by something in $\mathcal{C}_-(s,t)$, we can pick up matrices with $\det R = -1$, and we get the full $O(s,t)$ algebra. This is basically coming from the fact that unlike in even d, in odd d the matrix $-\mathbf{1}$ is the generator of the $\det R = -1$ part of $O(d)$, which is central and so $O(d) = SO(d) \times \mathbb{Z}_2$: the action of $\mathrm{Pin}(s,t)$ is unable to generate the decoupled $\mathbb{Z}_2$ factor. We can "fix" this and make Ω surjective by defining it instead through

$$(-1)^{s(\Lambda)}\Lambda^{-1}\gamma_\mu\Lambda = R_{\mu\nu}\gamma^\nu, \quad d \in 2\mathbb{Z}+1 \tag{10}$$

where $s(\Lambda) = 0$ if $\Lambda \in \mathcal{C}_+(s,t)$ and $s(\Lambda) = 1$ if $\Lambda \in \mathcal{C}_-(s,t)$.

Now let's look at how the various reflections in $O(s,t)$ are realized. First consider $P = 1 \oplus (-\mathbf{1}_{d-1})$, which acts as parity. We see that this is generated by $\Lambda_P = \gamma_0$. Now for $R_0 = -1 \oplus \mathbf{1}_{d-1}$, which reverses time: this is accomplished with $\Lambda_T = \prod_j \gamma_j$, where the product is over spatial indices. In general, $\Lambda = \gamma_\mu$ reflects all the axes of spacetime except for μ, and so a reflection about the axis μ can be performed with $\Lambda_{R_\mu} = \prod_{\nu\neq\mu}\gamma_\nu$. In even dimensions, we will find it slightly more convenient to write

$$\Lambda_{R_\mu} \equiv \gamma_\mu\bar{\gamma}, \tag{11}$$

which differs from $\prod_{\nu\neq\mu}\gamma_\nu$ by at most a c-number (from the potential power of i in $\bar{\gamma}$), and is easier to work with. In odd dimensions, the $\bar{\gamma}$ will be omitted—the reason for this will be explained in a sec.

The different choices of signature affect what reflections square to, because the choice of signature affects the types of pin group representations that exist. For example, for $d \in 2\mathbb{Z}$,

$$\Lambda_{R_\mu}^2 = \gamma_\mu\bar{\gamma}\gamma_\mu\bar{\gamma} = -\gamma_\mu^2 = -\eta_{\mu\mu} \tag{12}$$

holds for all reflections about a single axis. When we talk about parity — reflection of all the spatial indices — we have to make a choice between simplicity and consistency of notation. We will opt for the former and define

$$\Lambda_P \equiv \gamma_0 \implies \Lambda_P^2 = \eta_{00}. \tag{13}$$

This is particularly nice since it means that spatial reflections and parity square to the same thing in real time, and lets the reversal of the time coordinate always square to the negative of Λ_P^2.[3]

[3] The reason why this is a less consistent choice is that ideally, we would have defined

$$\Lambda_P = \prod_j \Lambda_{R_j}, \tag{14}$$

which after some combinatorial shenanigans is

$$\Lambda_P = i^{d^2/2-2d-d+1+d/2-t}\eta_{00}\gamma_0, \tag{15}$$

which is disgusting. It squares to, when d is even,

$$\Lambda_P^2 = (-1)^{1-t+d/2}\eta_{00}, \tag{16}$$

which is really not very pretty. Hence we have used the simpler $\Lambda_P = \gamma_0$ in the main text.

Action of $\mathrm{Pin}(s,t)$ on fermions

We now elaborate on the general procedure for determining how C, P, and T act on fermion fields. One technical comment first: none of C, P, or T are connected to the identity in the Lorentz group (and C isn't part of it at all). Thus their actions are only really defined up to arbitrary phases which cancel out in Lorentz-invariant quantities. There are certain canonical choices to make and in what follows we will try to make them, but it is important to keep this ambiguity in mind.

In this section, we will be somewhat abstract and field-theory-centric, and will make use of the technology introduced in the previous section. The free fermion action, in real time, is

$$S = \int d^{d-1}x\, dt\, \bar{\psi} i \partial\!\!\!/ \psi. \tag{17}$$

In this expression, ψ is a pinor, and $\bar{\psi}$ is a copinor. This means that the action of $\mathrm{Pin}(s,t)$ is represented on the pinors via

$$\mathrm{Pin}(s,t) \ni g : \psi \mapsto \Lambda_g \psi, \quad \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_g. \tag{18}$$

We will *almost* be setting the barred representation to be the inverse of Λ_g representation, so that $\bar{\Lambda}_g$ is almost Λ_g^{-1}. There are some subtleties involved with reflections in odd dimensions, though, which means this relation won't always hold.

Now in general, Λ_g and $\bar{\Lambda}_g$ will be distinct representations of $\mathrm{Pin}(s,t)$, since the pinor representation may be complex. This means that $\psi, \bar{\psi}$ are generally *distinct* fields in the path integral. If the representation of $\mathrm{Pin}(s,t)$ the pinor ψ transforms under is isomorphic to its dual through some isomorphism $\mathcal{J}$, then we can use $\mathcal{J}$ to relate ψ and $\bar{\psi}$: this is the case where ψ is a Majorana, and because there is only one variable being integrated over in the path integral, the resulting partition function is a Pfaffian, not a determinant. In what follows though, we will simply define $\bar{\psi}$ as a copinor which transforms in the way defined above. When we are trying to be most general we will avoid writing the adjoint as $\bar{\psi} = \psi^\dagger\gamma_0$, since this will not be true if the fermions are Majorana and just adds more clutter. If the fermion is not Majorana then ψ and $\psi^\dagger$ are independent anyway, but $\bar{\psi}, \psi$ is a conceptually nicer set of independent fields to work with than $\psi^\dagger, \psi$.

Another general point worth mentioning is that in even dimensions, the notion of chirality only makes sense for spinors, not pinors (there are Weyl spinors, but not Weyl pinors). Mathematically, this is because $\mathrm{Pin}(s,t)$ has only a single irrep when $s+t$ is even, in contrast to the spin group $\mathrm{Spin}(s,t)$ which splits as $S_\pm$. Physically, this is just because reflections have determinant -1 and mix chiralities, so something that has a definite transformation rule under reflections cannot have a definite chirality.

Reflections

First for reflections / parity. First consider a reflection R_μ about the coordinate x^μ (if $\mu = 0$ we are reversing the flow of time, but not doing anything antilinear—the full antilinear time reversal will be discussed later). When $d \in 2\mathbb{Z}$, the appropriate action is to take

$\psi \mapsto \Lambda_{R_\mu}\psi = \gamma_\mu\bar{\gamma}\psi$. Indeed, the invariance of the kinetic term is demonstrated by

$$\bar{\gamma}^\dagger\gamma_\mu^\dagger(-\partial_\mu\gamma^\mu + \sum_{\nu\neq\mu}\partial_\nu\gamma^\nu)\bar{\gamma}\gamma_\mu = \not{\partial}. \tag{19}$$

This works because when $d \in 2\mathbb{Z}$, $\bar{\gamma}$ anticommutes with all of the γ_μ.

When $d \in 2\mathbb{Z}+1$, we need something different: as we saw above, the homomorphism $\Omega : \mathrm{Pin}(s,t) \to O(s,t)$ is not surjective, and we cannot generate things with odd determinant. The solution to this is to twist the action of $\mathrm{Pin}(s,t)$ on $\bar{\psi}$ by a minus sign. So, we should do something like $\psi \mapsto \Lambda_{R_\mu}\psi$ and $\bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_{R_\mu} = -\bar{\psi}\Lambda_{R_\mu}^{-1}$. Said differently, recall that when d is odd, there are two distinct $\mathrm{Pin}(s,t)$ representations, which differ by the map $\gamma_\mu \mapsto -\gamma_\mu$. That is, if one representation is represented by the matrices $\Lambda_g, g \in \mathrm{Pin}(s,t)$, then the other is represented by the matrices $(\det\Lambda_g)\Lambda_g$, where $\det\Lambda_g$ is the function which splits apart $\mathcal{C}_+(s,t)$ and $\mathcal{C}_-(s,t)$.

If the pinor ψ transforms in one representation, the copinor $\bar{\psi}$ is taken to transform in the other representation, so that it picks up an extra minus sign when acted on by orientation-reversing elements on $\mathrm{Pin}(s,t)$. When d is odd we will thus take $\Lambda_{R_\mu} = \gamma_\mu$ and $\bar{\Lambda}_{R_\mu} = -\gamma_\mu$.[4]

From a slightly different point of view, the difference between odd and even dimensions can be understood in the following way (this paragraph will be in Euclidean signature, for simplicity). When reflections are included, we can consider pinors that are ± 1 eigenpinors under reflection (we can choose ± 1 wolog since $\gamma_\mu^2 = \mathbf{1}$ for all μ). These pinors will be sections of two pinor bundles, that we will denote as $\mathcal{P}$ ($+1$ eigenvalue) and $\mathcal{P}'$ (-1 eigenvalue). These two pinor bundles are related via $\mathcal{P} = \mathcal{P}' \otimes \varepsilon$, where ε is the orientation bundle; basically this is because sections of pinor bundles are glued with reflections along orientation-reversing transition functions, and so changing the signs of these transition functions by tensoring with ε is equivalent to sending $\gamma_\mu \mapsto -\gamma_\mu$ (see e.g. Witten's path integrals and fermions paper). Anyway, in even dimensions, the action of a reflection, $\gamma_\mu\bar{\gamma}, x^\mu \mapsto -x^\mu$, commutes with the Dirac operator $i\not{\partial}$. This means that $i\not{\partial} : \mathcal{P} \to \mathcal{P}$, $\mathcal{P}' \to \mathcal{P}'$, and the Dirac operator is self-adjoint, giving us a determinant.[5] In odd dimensions though, reflections *anticommute* with $i\not{\partial}$, and there is no way to ameliorate this with a factor of $\bar{\gamma}$. Thus in odd dimensions $i\not{\partial} : \mathcal{P} \to \mathcal{P}'$, $\mathcal{P}' \to \mathcal{P}$, and sections of $\mathcal{P}$ must get paired with sections of $\mathcal{P}'$ in the action: the Dirac operator is not self-adjoint, and the parition function is a Pfaffian rather than a determinant. This is just another way of saying that in odd dimensions, the invariant pairing is constructed between pinors transforming in the two distinct representations of $\mathrm{Pin}(s,t)$.

Summarizing, a reflection about the x^μ axis acts as

$$\begin{aligned} &R_\mu : \psi \mapsto \Lambda_{R_\mu}\psi = \gamma_\mu\bar{\gamma}\psi, \qquad \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_{R_\mu} = \bar{\psi}(\gamma_\mu\bar{\gamma})^{-1} \qquad d \in 2\mathbb{Z}, \\ &R_\mu : \psi \mapsto \Lambda_{R_\mu}\psi = \gamma_\mu\psi, \qquad \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_{R_\mu} = -\bar{\psi}\gamma_\mu^{-1} \qquad d \in 2\mathbb{Z}+1. \end{aligned} \tag{20}$$

Note that if we take the copinor to be given by the Dirac adjoint $\bar{\psi} = \psi^\dagger\gamma^0$ and the reflection to be about a spatial axis, the $(-1)^d$ factor for the $\bar{\Lambda}_{R_j}$ transformation is picked up when moving the $\gamma_j\bar{\gamma}$ through the γ^0.

[4]The reason for the absence of the $\bar{\gamma}$ here is just because we want the $\bar{\Lambda}_{R_\mu}$ matrix to be obtained from the $\Lambda_{R_\mu}^\dagger$ matrix through the substitution $\gamma_\mu \mapsto -\gamma_\mu$, which interchanges the representations, and since $\bar{\gamma} \mapsto -\bar{\gamma}$ under this map, we would not get the right minus sign if we included the $\bar{\gamma}$.

[5]In Witten's paper he does something different: the $\bar{\gamma}$ is not included in his reflection action, but he modifies his Dirac operator by a factor of $\bar{\gamma}$ in a compensating way.

An important consequence of the minus sign in the copinor transformation in odd dimensions is that for $d \in 2\mathbb{Z}+1$, the Dirac mass is always odd under spacetime reflections:

$$d \in 2\mathbb{Z}+1 \implies R_\mu : \bar{\psi}\psi \mapsto -\bar{\psi}\psi. \tag{21}$$

The fact that there is no reflection-invariant mass in odd dimensions is what allows the "parity anomaly" (bad name) to be a thing (although this does not tell us that the Dirac mass will be odd under the full antiunitary time-reversal when d is odd).

Reflection P of all the spatial axes is easy: (we are calling it P even though when d is odd it has determinant 1; sorry) in the conventions defined in the previous section,

$$P : \psi \mapsto \Lambda_P \psi = \gamma_0 \psi, \qquad \bar{\psi} \mapsto \bar{\psi}\bar{\Lambda}_P = \bar{\psi}\gamma_0^{-1}. \tag{22}$$

There are no complications in the odd d case since if d is odd P is an element of $\mathrm{Spin}(s,t)$ and is represented identically by both pinor representations of $\mathrm{Pin}(s,t)$.

Now one thing to note here is that we can always work with different reflection / parity operators defined as $\widetilde{\Lambda}_{R_\mu} = \Lambda_{R_\mu}\bar{\gamma}$—this actually seems to be the slightly more popular choice among the few physics papers where these issues are discussed. If d is odd then $\bar{\gamma}$ is central and this modification obviously does nothing, provided we modify the transformation of the copinor in the same way. Thus, if d is odd, this doesn't give us anything new. If d is even though, we can then take the copinor to transform under the matrix $-(\widetilde{\Lambda}_{R_\mu})^{-1}$; this minus sign is naturally generated if the copinor is defined though $\psi^\dagger\gamma_0$. Thus this gives us an alternative way to represent reflections in even dimensions:

$$R_\mu : \psi \mapsto \widetilde{\Lambda}_{R_\mu}\psi = \gamma_\mu \psi, \qquad \bar{\psi} \mapsto -\bar{\psi}\gamma_\mu^{-1} \qquad d \in 2\mathbb{Z}. \tag{23}$$

One then checks that the minus sign in the copinor transformation ensures that e.g. $\bar{\psi}\gamma_\mu\psi$ transforms appropriately as a vector under reflections.

The difference between this alternate reflection and the previous one is simply that in this representation, the reflection squares to something different:

$$(\widetilde{\Lambda}_{R_\mu})^2 = \eta_{\mu\mu} = -\Lambda_{R_\mu}^2. \tag{24}$$

In keeping with this change, we can also define an alternate parity operator $\widetilde{\Lambda}_P = \gamma_0\bar{\gamma}$, so that

$$(\widetilde{\Lambda}_P)^2 = -\eta_{00}. \tag{25}$$

Thus in the tilde representation, parity and reflection square to the same thing in real time.

Time reversal

Now we come to time reversal. We will define the time reversal operation in the conventional way, which complex conjugates scalars but which does not conjugate dynamical fields—that is, we will define time reversal to act as the thing which historically has been called T, and not CT (so that e.g. magnetic fields are odd under our T). There are some reasons to do it the other way (they are motivated by issues that come up when we depart from the context of free fermions; there's a separate diary entry on this), but we will stick with the traditional definition for now.

The full antiunitary time reversal operation is much messier than the spatial reflections: its antiunitary nature means that while it involves the reversal of the time coordinate (which is a transformation in $\mathrm{Pin}(s,t)$), it itself does not act on the (co)pinors through a linear representation of $\mathrm{Pin}(s,t)$. Furthermore because it is antiunitary, it cares about the inner product structure on the Hilbert space in question, and therefore details of how $\bar\psi$ is defined will affect the transformation law.

Anyway, first consider the element R_0 of $\mathrm{Pin}(s,t)$ which reverses time, but which does not act with complex conjugation. By the same argument we used for spatial reflections, this acts as

$$\begin{aligned} R_0 : \psi \mapsto \Lambda_{R_0}\psi = \gamma_0\bar\gamma\psi, \quad \bar\psi \mapsto \bar\psi\bar\Lambda_{R_0} = \bar\psi(\gamma_0\bar\gamma)^{-1} \qquad & d \in 2\mathbb{Z} \\ R_0 : \psi \mapsto \Lambda_{R_0}\psi = \gamma_0\psi, \quad \bar\psi \mapsto \bar\psi\bar\Lambda_{R_0} = -\bar\psi\gamma_0^{-1} \qquad & d \in 2\mathbb{Z}+1. \end{aligned} \tag{26}$$

Similarly to the above discussion of spatial refections, in even dimensions we could also define the transformation through $\widetilde{\Lambda}_{R_0} = \bar\gamma\Lambda_{R_0}$, provided that we also took the copinor to transform as $-[\widetilde{\Lambda}_{R_0}]^{-1}$.

Now suppose we let time reversal act on pinors by

$$T : \psi \mapsto \mathcal{K}\Lambda_{R_0}\mathcal{U}\psi, \qquad \bar\psi \mapsto \bar\psi\bar{\mathcal{U}}\bar\Lambda_{R_0}\mathcal{K}, \tag{27}$$

where $\mathcal{U}$,$\bar{\mathcal{U}}$ are unitaries to be determined. $\mathcal{U}$ and $\bar{\mathcal{U}}$ are independent unitaries, in keeping with the fact that $\bar\psi$ and ψ are independent fields. That said, $\bar{\mathcal{U}}$ will differ from $\mathcal{U}^\dagger$ only by potential minus signs relating to the signature and choice of γ_μ matrices. In the tilde convention for reflections in even dimensions, the time reversal action would instead be

$$\widetilde{T} : \psi \mapsto \mathcal{K}\gamma_0\mathcal{U}\psi, \qquad \bar\psi \mapsto -\bar\psi\bar{\mathcal{U}}\gamma_0^{-1}\mathcal{K} \quad (d \in 2\mathbb{Z}). \tag{28}$$

Requiring that $\bar\psi i\partial\!\!\!/\psi$ be T-invariant means that

$$-\partial\!\!\!/ = \bar{\mathcal{U}}\bar\Lambda_{R_0}(-\partial^0\gamma_0^* + \partial^j\gamma_j^*)\Lambda_{R_0}\mathcal{U} = \bar{\mathcal{U}}\partial\!\!\!/^*\mathcal{U}, \tag{29}$$

which tells us that

$$\bar{\mathcal{U}}\gamma_\mu^*\mathcal{U} = -\gamma_\mu, \quad \forall\,\mu. \tag{30}$$

If all the γ matrices were Hermitian with then we could take $\bar{\mathcal{U}} = \mathcal{U}^\dagger$ with $\mathcal{U}$ the charge conjugation matrix (to be discussed in a sec); unfortunately in $\mathbb{R}$ time this is never the case. The exact transformation properties of various fermion bilinears depends annoyingly on details like the signature and choice of γ matrices, and so we defer such calculations to the examples section. One thing to note however is that if the pinor representation is real then one possible choice of $\bar{\mathcal{U}}$,$\mathcal{U}$ is always $\bar{\mathcal{U}} = -\mathbf{1}, \mathcal{U} = +\mathbf{1}$.

Euclidean time

We now briefly comment on what happens in Euclidean time. In Euclidean time the action is *not* Hermitian, and so we will always avoid writing the copinor field as something dependent on $\psi^\dagger$, since this is needlessly confusing—the spin representation in the Euclidean case is already unitary so the γ_0 isn't needed to get the right transformation rule.

We write the Lagrangian as[6] [7]

$$\mathcal{L}_E = \chi_E^\dagger(\partial\!\!\!/ + m)\psi_E, \tag{34}$$

where $\chi_E^\dagger$ is an independent field (not related to ψ_E by Hermitian conjugation). In even dimensions, the $\mathrm{Pin}(d)$[8] action $\psi \mapsto \Lambda_g\psi$ acts on χ as $\chi^\dagger \mapsto \chi^\dagger\Lambda_g^\dagger$. With this one checks that $\mathcal{L}_E$ is invariant under $\mathrm{Pin}(d)$, since the γ_E^μ matrices are all Hermitian. If d is odd, we have two different $\mathrm{Pin}(d)$ representations, differing by $\gamma_\mu \mapsto -\gamma_\mu$. In this case, we take χ_E and ψ_E to transform in opposite $\mathrm{Pin}(d)$ representations, so that $\chi_E^\dagger \mapsto \chi_E^\dagger\Lambda_g^\dagger \det\Lambda_g$, allowing the Lagrangian to be reflection-invariant.

Action of charge conjugation

Charge conjugation is not a spacetime symmetry, and therefore is a little out of place here. Furthermore, generic QFTs do not even always come equipped with some invariant notion of a $\mathbb{Z}_2$ charge conjugation transformation (for example, it may be subsumed by another global symmetry group), meaning that C really should not be elevated to the same status as spacetime reflections, as it often is.

However, in the free fermion context we're discussing, a notion of charge conjugation always exists. Charge conjugation, in this context, is a unitary operator that relates ψ and

[6]Some comments on Euclidean time Lagrangians: consider first complex fermions, where a Lorentz invariant bilinear is formed (in $\mathbb{R}$ time) by adding on γ^0 to $\psi^\dagger$ in $\psi^\dagger\psi$ to get $\bar\psi\psi$. Now in $\mathbb{R}$ time the operator $\gamma^0 iD\!\!\!/_A$ is Hermitian, regardless of signature, and so the action is Hermitian as required (we treat $\psi^\dagger$ and ψ as the (independent) integration variables). Now when we go to Euclidean time, we might be tempted to just naively do (independent of signature)

$$\gamma^0 iD\!\!\!/_A \xrightarrow{t\mapsto it} \gamma^0\partial\!\!\!/, \tag{31}$$

and write the action as $\int d\tau\, d^{D-1}x\,\bar\psi\partial\!\!\!/\psi$. However, in Euclidean time, while $\partial\!\!\!/\psi$ transforms as a spinor under $\mathrm{Spin}(D)$, $\bar\psi\psi$ is not invariant under $\mathrm{Spin}(D)$, because now the γ^0 in $\bar\psi$ screws stuff up. So, we should really write the Euclidean action as

$$S_E = \int d\tau\, d^{D-1}x\,\chi^\dagger\partial\!\!\!/\psi, \tag{32}$$

where if ψ transforms in a representation R of $\mathrm{Spin}(D)$, then $\chi^\dagger$ transforms in the representation $\bar R$. This does not actually lead to a doubling of the number of fields, since while $\chi^\dagger$ and ψ are independent, so too were $\psi^\dagger$ and ψ in the real time picture (notice that there is no $\psi^\dagger$ or χ field appearing in S_E). One thing to note however is that in Euclidean time, the action is no longer Hermitian (this is okay; recall e.g. theta angles).

[7]Another choice would be to write

$$\mathcal{L}_E = \chi_E^\dagger\bar\gamma(\partial\!\!\!/ + m)\psi_E, \tag{33}$$

but then the reflections act as $\Lambda_{R_j} = \gamma_j$ (without the $\bar\gamma$), which is different from the $\mathbb{R}$ time case, so we avoid this.

[8]Here by $\mathrm{Pin}(d)$ we really mean $\mathrm{Pin}(d,0)$. We take reflections to act as $\Lambda_{R_j} = \gamma_j\bar\gamma$ and $\bar\Lambda_{R_j} = (\gamma_j\bar\gamma)^{-1}$ if d is even and as $\Lambda_{R_j} = \gamma_j$, $\bar\Lambda_{R_j} = -\gamma_j$ if d is odd. Just as in the indefinite signature case, were the squares of the reflections were fixed but could be changed by taking $\mathrm{Pin}(s,t) \to \mathrm{Pin}(t,s)$, here we can get different squares for reflections by taking $\mathrm{Pin}(d,0) \to \mathrm{Pin}(0,d)$. This amounts to taking $\gamma_\mu \to i\gamma_\mu$, and so switches the signs in the squares of the reflection matrices, both for d even and d odd.

$\bar{\psi}$. There are some circumstances where we will take it to related ψ and $\psi^\dagger$, but I think the relation to $\bar{\psi}$ is more natural from the point of view of representation theory. We take C to act as

$$C : \psi \mapsto \bar{\psi} C^\dagger, \qquad \bar{\psi} \mapsto -C\psi. \tag{35}$$

The reason for the minus sign is to ensure that the Dirac mass is always C-even:

$$\bar{\psi}\psi \mapsto -C_{\alpha\beta}\psi_\beta \bar{\psi}_\lambda [C^\dagger]_{\lambda\alpha} = +\bar{\psi} C^\dagger C \psi = \bar{\psi}\psi. \tag{36}$$

The above conventions mean that a general bilinear $\bar{\psi} M \psi$ then transforms nicely as

$$\bar{\psi} M \psi \mapsto \bar{\psi}(C^\dagger M^T C)\psi. \tag{37}$$

In order for the free term to be invariant, we need

$$i\partial\!\!\!/ = (C^T i\partial\!\!\!/ [C^\dagger]^T)^T = C^\dagger [i\partial\!\!\!/]^T C, \implies C^\dagger \gamma_\mu^T C = -\gamma_\mu, \tag{38}$$

since $\partial_\mu^T = -\partial_\mu$. Since $-\gamma_\mu^T$ and γ_μ obey the same algebra they must be unitarily equivalent, and hence such a matrix C always exists. Exact expressions for C depend on the choice of signature and dimension, though.

As we saw, the Diarc mass is always C invariant. When $d \in 2\mathbb{Z}$ we also have a chiral mass, which transforms as

$$\bar{\psi}\bar{\gamma}\psi \mapsto \bar{\psi}(C^\dagger \bar{\gamma}^T C)\psi. \tag{39}$$

Now

$$C^\dagger \bar{\gamma}^T C = \prod_{\mu=d,\ldots,1} C^\dagger \gamma_\mu^T C = \prod_{\mu=d,\ldots,1} \gamma_\mu = (-1)^{((d-1)^2+(d-1))/2}\bar{\gamma}. \tag{40}$$

So, we see that for $d \in 4\mathbb{Z}$ charge conjugation respects chirality and the chiral mass is C-even, while for $d \in 4\mathbb{Z}+2$ charge conjugation exchanges chiral components and the chiral mass is C-odd.

The vector current, since it lacks the ∂_μ of the free term which gives a minus sign when transposed, is (as expected) odd under C:

$$C : \bar{\psi}\gamma^\mu\psi = j_V^\mu \mapsto -j_V^\mu. \tag{41}$$

On the other hand, from the above charge-conjugate of $\bar{\gamma}$, we calculate

$$C : \bar{\psi}\gamma^\mu\bar{\gamma}\psi = j_A^\mu \mapsto \begin{cases} +j_A^\mu & d \in 4\mathbb{Z} \\ -j_A^\mu & d \in 4\mathbb{Z}+2 \end{cases} \tag{42}$$

Majorana fermions

Majorana fermions χ are self-adjoint fermions; this means the co(s)pinor $\bar{\chi}$ is *not* independent from χ. To define a Lorentz-invariant kinetic term with Majorana spinors, we then need to find a matrix that intertwines the spinor representation of $\mathrm{Spin}(s,t)$ with its dual. That is, if Λ_g is a representation matrix of $\mathrm{Spin}(s,t)$ (so that $\Lambda_g^{-1}\gamma^\mu\Lambda_g = R^{\mu\nu}\gamma_\nu$ for $R \in SO(s,t)$), we need to find a matrix $\mathcal{C}$ such that

$$\Lambda_g^T \mathcal{C} = \mathcal{C}\Lambda_g^{-1}. \tag{43}$$

If such a matrix exists, then given a spinor χ we can define a cospinor $\bar{\chi}$ which is linearly dependent on χ, with the two related via $\bar{\chi} \propto \chi^T \mathcal{C}$. The properties of $\mathcal{C}$ then guarantee that $\chi^T \mathcal{C} \chi$ is invariant under $\mathrm{Spin}(s,t)$ transformations.

Recall that the representation of any $g \in \mathrm{Spin}(s,t)$ can be written as

$$\Lambda_g = \exp\left(\frac{1}{8}[\gamma_\mu, \gamma_\nu]\theta_g^{\mu\nu}\right) \implies \Lambda_g^T = \exp\left(-\frac{1}{8}[\gamma_\mu^T, \gamma_\nu^T]\theta_g^{\mu\nu}\right), \tag{44}$$

by the antisymmetry of $\theta_g^{\mu\nu}$. We see that $\mathcal{C}$ will do the job provided that

$$\gamma_\mu^T \mathcal{C} = -\mathcal{C}\gamma_\mu. \tag{45}$$

But this is exactly what charge conjugation does, since C is unitary and satisfies $C^\dagger \gamma_\mu^T C = -\gamma_\mu$. Thus we can take $\mathcal{C} = C$.

Now in order for the theory to be nontrivial, we need $\bar{\chi} i \not{\partial} \chi \neq 0$, i.e. we need $C\not{\partial}$ to be an antisymmetric matrix. Now its transpose is

$$(C\not{\partial})^T = -\gamma_\mu^T \partial^\mu C^T. \tag{46}$$

Transposing $\gamma_\mu^T C = -C\gamma_\mu$ tells us that $\gamma_\mu^T C^T = -C^T\gamma_\mu$, so

$$(C\not{\partial})^T = +C^T \not{\partial}, \tag{47}$$

and so if the action is to be nontrivial we need a charge conjugation which is antisymmetric: $C^T = -C$ (the antisymmetry also implies the existence of a nontrivial mass term for the Majoranas). Fortunately, this property is satisfied if $C^2 = \mathbf{1}$ when acting on fermions, since

$$C^2 : \psi_\alpha \mapsto [C(\bar{\psi}_\gamma [C^\dagger]_{\gamma\lambda})]_\alpha = -\psi_\rho C_{\gamma\rho}[C^\dagger]_{\gamma\lambda} = -C^T C^\dagger \psi, \tag{48}$$

and so $-C^T C^\dagger = \mathbf{1} \implies C = -C^T$.

Finally, in $\mathbb{R}$ time (and in $\mathbb{R}$ time only!), we need the action to be Hermitian. So far we have only used $\bar{\chi} \propto \chi^T C$. Now fix the proportionality constant as

$$\bar{\chi} = \lambda_\chi \chi^T C. \tag{49}$$

Now since $C\not{\partial}$ is antisymmetric, in order to have a Hermitian action, we need $\lambda_\chi i\not{\partial}$ to be purely imaginary. Evidently this is only possible if the γ matrices are either all real (in which case we take $\lambda_\chi = 1$) or all imaginary (in which case we take $\lambda_\chi = i$).[9] Having imaginary γ matrices is okay from a representation-theory point of view since the matrices representing $\mathrm{Spin}(s,t)$ are still real. The existence of such all-real or all-imaginary γ matrices holds in real time in all the dimensions we're usually interested in (2, 3, and 4). Again, remember that we are imposing the reality condition on Majorana spinors in $\mathbb{R}$ time (when we Wick-rotate to $i\mathbb{R}$ time, there is no Hermiticity requirement, and no reality condition for Majoranas).

The above shows how to define Majorana spinors. What about Majorana pinors? Evidently we need to find an intertwiner between the pinor representation of $\mathrm{Pin}(s,t)$ and its

[9] Another way of saying why Majoranas work when all the γ_μ are imaginary is that the equation $(i\not{D} - m)\psi = 0$ implies $(i\not{D} - m)\psi^* = 0$ if all the γ matrices are imaginary.

dual. This is more restrictive, since it relies on having a choice of signature and dimension such that the pinor representation of $\mathrm{Pin}(s,t)$ is real.[10] Said differently, the fact that we have only one field appearing in the action, unlike the two fields appearing in the complex case, means that we have less freedom in choosing transformation properties to make the free kinetic term symmetric. As an example of a constraint coming from the Majorana condition, consider the reflection of all spatial indices. This is usually represented in $\mathrm{Pin}(s,t)$ by $\chi \mapsto \gamma_0\chi$. But we see that $\chi^T C \partial\!\!\!/ \chi$ is only invariant under this action if $\gamma_0^2 = -\mathbf{1}$, since invariance implies

$$P : \chi^T C \partial\!\!\!/ \chi \mapsto \chi^T \gamma_0^T C \gamma_0 \partial\!\!\!/ \chi \implies \gamma_0^T C \gamma_0 = C \implies C^\dagger \gamma_0^T C = \gamma_0^{-1} \implies -\gamma_0 = \gamma_0^{-1}. \quad (50)$$

So, whether or not we can define a Majorana pinor depends on the signature (or better, on the Pin structure) that we choose! For example, the constraint that $\gamma_0^2 = -\mathbf{1}$ means we can only have Majorana pinors with this definition if they are "Kramers doublets" under parity.

Comment on CPT

The discussion of CPT is often slightly more confusing than it needs to be, since the CPT theorem is actually saying something very simple. In fact, instead of the CPT theorem, it should really be called the CRT theorem, where R is a reflection about any one of the coordinate axes—this is the appropriate operation to replace P with in odd dimensions, and for conceptual clarity it's helpful to use it in even dimensions as well.

In fact, even the moniker CRT is a bit confusing, since not all systems come equipped with a natural notion of charge conjugation. The C is present because in the usual (historical) conventions, T does not $\dagger$ the fields that it acts on, and so the action of C is needed to provide this. However, in systems where the global symmetry group doesn't contain a semi-direct product with a $\mathbb{Z}_2$ factor (that acts as an outer automorphism on the group it is $\rtimes$'d with), there is no real invariant notion of charge conjugation, and it is more natural to just define T as an anitlinear operation that Hermitian conjugates fields—in this definition, the CPT theorem is really just the RT theorem, since this definition has a T that acts as CT, in the cases where C is defined.[11]

Why then should we have an RT theorem? In $\mathbb{R}$ time thinking, the reason is elaborated on in a separate diary entry. The answer in imaginary time is simpler: in Euclidean signature both R and T are ordinary reflections, and so RT is just a π rotation. In particular, RT is in the component of the Lorentz group which is connected to $\mathbf{1}$, and so RT must be a symmetry of any relativistic theory. Note that $(RT)^2$ is a 2π rotation, so that as an operator acting on states of spin $j \in \frac{1}{2}\mathbb{Z}$, we have

$$(RT)^2 = (-1)^{2j}. \quad (51)$$

[10] Actually, one might argue that a purely imaginary representation of the γ matrices would again work, since the overall factor of i could just be treated as a "coupling constant" in the action. I think I remember finding some path paper where this was referred to as a "pseudo-Majorana" representation, or similar.

[11] In these conventions, the T that appears in CPT sends e.g. $E \mapsto E, B \mapsto -B$ in EM, while the T that appears in RT sends $E \mapsto -E, B \mapsto B$.

Examples

We now look at examples in low dimensions. We first look at $i\mathbb{R}$ time, and then at $\mathbb{R}$ time, in both mostly-positive and mostly-negative signatures. We will find the character $(\mathbb{R}, \mathbb{H}, \mathbb{C})$ of the spinor and pinor representations for each signature choice (the representation theory of $\mathrm{Spin}(s,t)$ is the same as that of $\mathrm{Spin}(t,s)$, but this is not true for $\mathrm{Pin}(s,t)$). For the $\mathbb{R}$ time examples, we find the action of time reversal and spatial reflections. We will also consider mass terms: in odd dimensions the only Lorentz scalar is $\bar{\psi}\psi$ ($\bar{\gamma} \propto \mathbf{1}$), and so there is only one type of mass term. In even dimensions $\bar{\gamma}$ is nontrivial, and we also have the chiral mass $\bar{\psi}\bar{\gamma}\psi$.

Two dimensions

Euclidean time: In Euclidean signature, we can take our γ matrices to be X and Z. This means that $\mathrm{Spin}(2)$ consists of unit-norm linear combinations of $\mathbf{1}$ and $XZ = J$, so that $\mathrm{Spin}(2) \cong U(1)$. The splitting $S_+ \oplus S_-$ in this case corresponds to spinors that transform as $e^{i\theta}$ or $e^{-i\theta}$. Another basis we commonly use is to take $\gamma^0 = Y, \gamma^1 = X$. Then $\mathrm{Spin}(2)$ has generators $\mathbf{1}, iZ$, the diagonality of which make it clear that the representation is reducible, with the splitting $S_+ \oplus S_-$ being a splitting into left- and right-moving spinors. Each $S_\pm$ spinor representation is complex, and the full spinor representation is the sum $-1 \oplus 1$, which is real. Because of the complexity of $S_\pm$, we can't define chiral Majorana spinors (alias Majorana-Weyl spinors): with the X, Z choice of γ matrices the charge conjugation matrix is $C = Y$, and the resulting term $\chi^T C \partial\!\!\!/ \chi$ mixes S_+ and S_- components; with the X, Y choice the generators of $\mathrm{Spin}(2)$ are not real, which is incompatible with the Majorana reality condition. However, while we can't define chiral Majorana spinors, we can define Majorana pinors, since the representation $-1 \oplus 1$, which becomes irreducible when we lift to $\mathrm{Pin}(2)$, is real.

Real time: In real time with signature $(-,+)$,we may take

$$\gamma^0 = J, \qquad \gamma^1 = X, \qquad \bar{\gamma} = Z. \tag{52}$$

Therefore $\mathrm{Spin}(1,1)$ is generated by diagonal elements $\mathbf{1}$ and Z, which makes the $S_+ \oplus S_-$ decomposition manifest. This time though, the representation is real, since both generators of $\mathrm{Spin}(1,1)$ square to $\mathbf{1}$. Therefore chiral majorana spinors may be defined: the charge conjugation matrix is still $C = Y$, but now the free term $\chi^T C \partial\!\!\!/ \chi$ does not mix the $S_\pm$ components. Of course, Majorana pinors also exist. The fact that chiral Majorana spinors can be defined only in real time is not something to be bothered about—the reality conditions of the Majoranas are fixed in real time, not in imaginary time, because the reality condition is essentially the statement about $\{\chi_i, \chi_j\}$, and the commutator is defined in real time within the Hamiltonian framework. Therefore, to determine whether a given theory admits Majoranas, one needs to do the analysis in $\mathbb{R}$ time, and not worry about what happens when one continues to $i\mathbb{R}$ time.

Now because the charge conjugation matrix is proportional to γ_0, we may write the action of C more transparently as $C : \psi \to \psi^\dagger$ in the $\mathbb{C}$ case, or $C : \chi \to \chi$ in the Majorana case. The point of writing it like this is to demonstrate that C does not act on the Loretnz indices of the theory. In the $\mathbb{C}$ fermion case ψ transforms with charge 1 under a $U(1)$ symmetry and hence is in a complex irrep of the full symmetry group; in this case C acts nontrivially. However in the Majorana case C really does act trivially, as required by the reality of the representation the Majoranas transform in.

Now for time reversal. The equation $\bar{\mathcal{U}}\gamma_\mu^*\mathcal{U} = -\gamma_\mu$ means that we can set $\bar{\mathcal{U}} = \mathcal{U} = Z$ or $\bar{\mathcal{U}} = -\mathbf{1}, \mathcal{U} = \mathbf{1}$. We will choose the former option; thus in the regular (un-tilded) convention, we have

$$T : \psi \mapsto J\psi, \qquad \bar{\psi} \mapsto \bar{\psi}J^{-1}. \tag{53}$$

This gives $T^2 = (-1)^F$. We may also use the tilded definition of T including $\widetilde{\Lambda}_{R_0}$; this gives

$$\widetilde{T} : \psi \mapsto X\psi, \qquad \bar{\psi} \mapsto -\bar{\psi}X, \tag{54}$$

and so in this case we get a different time-reversal algebra, viz. $\widetilde{T}^2 = \mathbf{1}$. Note that we can also swap T with $\widetilde{T}$ by taking the alternate choice of $\bar{\mathcal{U}} = -\mathbf{1}, \mathcal{U} = \mathbf{1}$. Likewise, for parity, we have either

$$P : \psi \mapsto J\psi, \qquad \bar{\psi} \mapsto \bar{\psi}J^{-1} \tag{55}$$

with $P^2 = (-1)^F$, or

$$\widetilde{P} : \psi \mapsto X\psi, \qquad \bar{\psi} \mapsto -\bar{\psi}X, \tag{56}$$

with $P^2 = \mathbf{1}$. One can also compute,[12] using $CT : \psi \mapsto i\bar{\psi}$, $C\widetilde{T} : \psi \mapsto -i\bar{\psi}Z$,

$$(CT)^2 = \mathbf{1}, \qquad (C\widetilde{T})^2 = (-1)^F, \tag{58}$$

so that the algebra obeyed by T and $\widetilde{T}$ is swapped when one tacks on charge conjugation.

The two Hermitian mass terms we can add are the Dirac mass $im\bar{\psi}\psi$ or the chiral mass $m\bar{\psi}Z\psi$. Both mass terms are odd under T, but they are both even under $\widetilde{T}$:

$$\begin{aligned} T &: i\bar{\psi}\psi \mapsto -i\bar{\psi}\psi, \quad \bar{\psi}\bar{\gamma}\psi \mapsto -\bar{\psi}\bar{\gamma}\psi, \\ \widetilde{T} &: i\bar{\psi}\psi \mapsto i\bar{\psi}\psi, \quad \bar{\psi}\bar{\gamma}\psi \mapsto \bar{\psi}\bar{\gamma}\psi. \end{aligned} \tag{59}$$

For parity, we see that the Dirac mass is even under P but odd under $\widetilde{P}$. On the other hand, the chiral mass is odd under P, but even under $\widetilde{P}$. The Dirac mass is even under C as it should be, while the chiral mass is odd under C, consistent with the fact that C does not respect chirality when $d \in 4\mathbb{Z} + 2$. Therefore

$$\begin{aligned} CT &: i\bar{\psi}\psi \mapsto -i\bar{\psi}\psi, \quad \bar{\psi}\bar{\gamma}\psi \mapsto \bar{\psi}\bar{\gamma}\psi \\ C\widetilde{T} &: i\bar{\psi}\psi \mapsto i\bar{\psi}\psi, \quad \bar{\psi}\bar{\gamma}\psi \mapsto -\bar{\psi}\bar{\gamma}\psi \end{aligned} \tag{60}$$

If we took the signature to be $(+,-)$, then we could use the same γ matrices, just with their indices exchanged, and so the full structure is the same, just with the unitary matrices associated with time reversal and parity switched: in both cases, we are looking at the representation theory of $\mathrm{Pin}(1,1)$.

[12] Useful identities for this are

$$C : U\psi \mapsto \bar{\psi}C^\dagger U^T, \qquad \bar{\psi}U \mapsto -\psi C^T U, \tag{57}$$

for any unitary U.

Three dimensions

Euclidean time: In Euclidean signature, we can take our γ matrices to just be the Pauli matrices; this is just because $\text{Spin}(3) = SU(2)$. Consider then the antilinear map $\mathcal{J} = J\mathcal{K}$: this anticommutes with every γ matrix, and so it commutes with all products of an even number of γ matrices—thus, the spinor representation is pseudoreal.

If we pass to $\text{Pin}(3,0)$, $\mathcal{J}$ is no longer an invariant form, since it anticommutes with each individual γ matrix. Thus the pinor representations of $\text{Pin}^+(3)$ are complex. We say representations because there are two, since we are in an odd dimension (recall that the representation matrices Λ_g of the two representations differ by a factor of $\det \Lambda_g$: $\Lambda_g' = (\det \Lambda_g)\,\Lambda_g$). From our general discussion, we know that these two representations should give an invariant pairing. Indeed they do: if ψ is a pinor transforming under Λ_g and ψ' is a pinor transforming under Λ', then the antisymmetry of the fermions means that

$$\text{Pin}(3,0) \ni g : \psi'_\alpha J^{\alpha\beta}\psi_\beta \mapsto [\Lambda'_g]_{\beta\alpha}\psi'_\alpha J^{\beta\gamma}[\Lambda_g]_{\gamma\lambda}\psi_\lambda = (\det \Lambda_g) J_{\alpha\beta}\Lambda_g^{\alpha\lambda}\Lambda_g^{\beta\rho}\psi'_\lambda\psi_\rho = (\det g)^2 J^{\alpha\beta}\psi'_\alpha\psi_\beta, \tag{61}$$

which is invariant since $\det(\Lambda_g)^2 = 1$. Now since $\text{Pin}(3,0)$ is compact, this invariant pairing gives us an isomorphism between the Λ' representation and the complex conjugate of the Λ representation. To find this isomorphism, we look for a unitary U such that

$$U^\dagger \Lambda_g^* U = (\det \Lambda_g)\Lambda_g = \Lambda_g'. \tag{62}$$

Indeed, such an isomorphism is provided by taking $U = Y$.

Real time: In $\mathbb{R}$ time with signature $(-,+,+)$, we will take the γ matrices to be

$$\gamma_0 = J, \qquad \gamma_1 = X, \qquad \gamma_2 = Z. \tag{63}$$

They are all real, and so evidently the spinor representation of $\text{Spin}(2,1)$ and the pinor representations of $\text{Pin}(2,1)$ are all real—both spinor and pinor Majoranas can be defined. Since d is odd, $\text{Pin}(2,1)$ has two irreducible representations. Just as in the $\text{Pin}(3,0)$ case, a fermion ψ transforming in one and a fermion ψ' transforming in the other have an invariant antisymmetric pairing provided by J. However, since $\text{Pin}(2,1)$ is *not compact*, this invariant pairing doesn't imply that the Λ_g representation is related to the complex conjugate of the Λ_g' representation through a unitary transformation. Indeed, now that J is included among the γ matrices there is no unitary U such that $U^\dagger\gamma_\mu^* U = -\gamma_\mu$, so we can not relate the two representations by (62). The non-existence of such a unitary means that we will not be able to choose $\bar{\mathcal{U}} = \mathcal{U}^\dagger$ when figuring out the time-reversal transformation.

Anyway, now for C, R, T. The charge conjugation matrix is $C = Y$, as in two dimensions. Also as in two dimensions we may write $C : \psi \to \psi^\dagger, \chi \to \chi$, which doesn't involve an action on the Lorentz indices; again this is made possible by the reality of the pinor representation of $\text{Pin}(2,1)$.

For T, we need $\bar{\mathcal{U}}\gamma_\mu\mathcal{U} = -\gamma_\mu$. The only solution to this is to take $\bar{\mathcal{U}} = -\mathbf{1}, \mathcal{U} = \mathbf{1}$. Therefore up to a sign, the action of T is fixed to be (the minus signs from the definition of the barred copinor representation and the the $\bar{\mathcal{U}} = -\mathbf{1}$ cancel)

$$T : \psi \mapsto J\psi, \qquad \bar{\psi} \mapsto \bar{\psi}J^{-1}, \tag{64}$$

with $T^2 = (-1)^F$. Note that unlike in two dimensions, we *cannot* choose something that gives an algebra with $T^2 = \mathbf{1}$. From the choice of signature, we see that spatial reflections satisfy $R^2 = +\mathbf{1}$. CT satisfies $CT : \psi \mapsto i\bar{\psi}$; one then finds $(CT)^2 = +\mathbf{1}$. The Dirac mass is $i\bar{\psi}\psi$, which we see is odd under T, CT, and spatial reflections.

In $\mathbb{R}$ time with $(+,-,-)$ signature, we may choose the γ matrices to be purely imaginary:

$$\gamma_0 = Y, \quad \gamma_1 = iX, \qquad \gamma_2 = iZ. \tag{65}$$

This means that Majorana spinors can be defined, but strictly speaking, that Majorana pinors cannot be (although see previous comment on this). Charge conjugation is unchanged. For time reversal, we now need $\bar{\mathcal{U}}\gamma_\mu\mathcal{U} = \gamma_\mu$, which works only if $\bar{\mathcal{U}} = \mathcal{U} = \mathbf{1}$ (up to a sign). Therefore

$$T : \psi \mapsto Y\psi, \qquad \bar{\psi} \mapsto -\bar{\psi}Y. \tag{66}$$

As with mostly negative signature, we have $T^2 = (-1)^F$. Reflections now satisfy $R^2 = (-1)^F$. We now have $CT : \psi \mapsto -\bar{\psi}$, and we find $(CT)^2 = +\mathbf{1}$, which is different from the mostly positive signature. The Dirac mass is $\bar{\psi}\psi$ (no i!), which is still odd under T, CT, and spatial reflections. Therefore in three dimensions, no matter what signature we choose, the Dirac mass is always odd under T and under reflections; this enables the "parity anomaly" (bad terminology) to occur.

Four dimensions

Euclidean time: In Euclidean signature, we can take

$$\gamma^0 = X \otimes \mathbf{1}, \quad \gamma^1 = Y \otimes \mathbf{1}, \quad \gamma^2 = Z \otimes X, \quad \gamma^3 = Z \otimes Z. \tag{67}$$

Consider the antilinear map

$$\mathcal{J} = (X \otimes J)\mathcal{K}. \tag{68}$$

We see that $\mathcal{J}^2 = -\mathbf{1}$ and $\mathcal{J}$ commutes with all products of an even number of γ matrices, and so the spinor representation of Spin(4) is quaternionic, just like for Spin(3) (this is to be expected since Spin(4) $= SU(2)^2$). In fact, $\mathcal{J}$ actually commutes with every individual γ matrix, implying that the pinor representation of $\text{Pin}^+(4)$ is also pseudoreal.

Real time: In $\mathbb{R}$ time with $(-,+,+,+)$ signature, we may take

$$\gamma^0 = iX \otimes \mathbf{1}, \quad \gamma^1 = Y \otimes X, \quad \gamma^2 = Y \otimes Y, \quad \gamma^3 = Y \otimes Z, \qquad \bar{\gamma} = Z \otimes \mathbf{1}. \tag{69}$$

Note that the product of any two of these is block diagonal—this shows us explicitly how Spin(3, 1) acts reducibly on $S_+ \oplus S_-$. Note that the antilinear map $\mathcal{J} = \mathcal{K}(J \otimes J)$ commutes with all the γ matrices and satisfies $\mathcal{J}^2 = +\mathbf{1}$; therefore the pinor representation must actually be real, in spite of the fact that the above matrices contain i. The full $S_+ \oplus S_-$ spinor representation is of course real since $S_+ \cong S_-^*$, but by looking at a few products of two γ matrices, we can check that the chiral spinor reps acting on $S_\pm$ are in fact complex.[13] Therefore we may have Majorana pinors, but not chiral Majorana spinors.

[13] The reality of the pinor representation doesn't imply the reality of the $S_\pm$, since $S_+^* \cong S_-$, which means $S_\pm$ can be complex while keeping $S_+ \oplus S_-$ real, as we have seen.

A more convenient choice for doing calculations is to take

$$\gamma^0 = J\otimes X, \quad \gamma^1 = J\otimes J, \quad \gamma^2 = Z\otimes \mathbf{1}, \quad \gamma^3 = X\otimes \mathbf{1}, \qquad \bar{\gamma} = iJ\otimes Z. \tag{70}$$

Note that all of the γ^μ are real—the reality of the pinor representation is then manifest, but the complexity of $S_\pm$ is harder to see, since $\bar{\gamma}$ is no longer diagonal.

Charge conjugation, in the way presented above, can be chosen to act with $C = i\gamma^0$; the i is so that $C^2 = \mathbf{1}$. This can be written more simply (for either the Dirac adjoint or in the Majorana case) as $C : \psi \mapsto \psi^\dagger$, so that C doesn't act on the Lorentz index of ψ (as before, this is possible due to the reality of the pinor representation).

For time reversal, we need $\bar{\mathcal{U}}\gamma_\mu\mathcal{U} = -\gamma_\mu$; hence as in two dimensions we may choose either $\bar{\mathcal{U}} = \mathcal{U} = \bar{\gamma}$, or $\bar{\mathcal{U}} = -\mathbf{1}, \mathcal{U} = \mathbf{1}$. Opting for the former choice, we have

$$T : \psi \mapsto (J\otimes X)\psi, \quad \bar{\psi} \mapsto \bar{\psi}(J\otimes X)^{-1}, \tag{71}$$

with $T^2 = (-1)^F$. Selecting the other choice for $\mathcal{U}$, for which $\psi \mapsto \gamma_0\bar{\gamma}\psi$ and $\bar{\psi} \mapsto -\bar{\psi}\bar{\gamma}\gamma_0^{-1}$, also gives $T^2 = (-1)^F$, by the the imaginarity of $\bar{\gamma}$. If we choose the tilde transformation rule then we would still have the same two choices—as in two dimensions, changing our choices for $\mathcal{U}$,$\bar{\mathcal{U}}$ is equivalent to changing our choice of the regular or tilded representation. Therefore there are really only two choices for the action of T; the other one we define to be $\widetilde{T}$:

$$\widetilde{T} : \psi \mapsto -(\mathbf{1}\otimes Y)\psi, \qquad \bar{\psi} \mapsto \bar{\psi}(\mathbf{1}\otimes Y), \tag{72}$$

which as we mentioned has $\widetilde{T}^2 = (-1)^F$.

Because of our signature choice, spatial reflections square to $\gamma_j\bar{\gamma}\gamma_j\bar{\gamma} = (-1)^F$ in the regular representation, or $\gamma_j^2 = +\mathbf{1}$ in the tilde representation. Similarly, we can define parity to act as either γ^0 or $\gamma^0\bar{\gamma}$; in the first case $P^2 = (-1)^F$ while in the second case $P^2 = \mathbf{1}$. Thus we see that time reversal and spatial reflections are fundamentally different for this dimension and signature: the former always squares to $(-1)^F$, while the latter can square to either $(-1)^F$ or $+\mathbf{1}$. When we add in charge conjugation, we find

$$CT : \psi \mapsto i\bar{\psi}, \qquad C\widetilde{T} : \psi \mapsto -\bar{\psi}(J\otimes Z), \tag{73}$$

which tells us that

$$(CT)^2 = \mathbf{1}, \qquad (C\widetilde{T})^2 = (-1)^F. \tag{74}$$

The two mass terms are the Dirac mass $im\bar{\psi}\psi$ and the chiral mass $m\bar{\psi}\bar{\gamma}\psi$. Under the T transformation,

$$\begin{aligned} T &: i\bar{\psi}\psi \mapsto -i\bar{\psi}(J\otimes X)^{-1}(J\otimes X)\psi = -i\bar{\psi}\psi, \\ T &: \bar{\psi}\bar{\gamma}\psi \mapsto \bar{\psi}(J\otimes X)^{-1}(-\bar{\gamma})(J\otimes X)\psi = \bar{\psi}\bar{\gamma}\psi. \end{aligned} \tag{75}$$

Therefore under T, the Dirac mass is odd, while the chiral mass is even. On the other hand, $\widetilde{T}$ satisfies

$$\begin{aligned} \widetilde{T} &: i\bar{\psi}\psi \mapsto -i\bar{\psi}(-\mathbf{1}\otimes Y)(\mathbf{1}\otimes Y)\psi = +i\bar{\psi}\psi, \\ \widetilde{T} &: \bar{\psi}\bar{\gamma}\psi \mapsto -\bar{\psi}(\mathbf{1}\otimes Y)\bar{\gamma}^*(\mathbf{1}\otimes Y)\psi = -\bar{\psi}\bar{\gamma}\psi, \end{aligned} \tag{76}$$

so that the Dirac mass is T-even while the chiral mass is T-odd. Therefore even though the two time reversals square to $(-1)^F$, they are distinguished by their actions on fermion bilinears. Since C preserves chirality when $d \in 4\mathbb{Z}$, the transformation properties of the masses under CT and $C\widetilde{T}$ are the same as under T and $\widetilde{T}$, respectively.

In $\mathbb{R}$ time with $(+,-,-,-)$ signature, we can choose e.g.

$$\gamma^0 = X \otimes \mathbf{1}, \quad \gamma^1 = J \otimes X, \quad \gamma^2 = J \otimes Y, \quad \gamma^3 = J \otimes Z, \qquad \bar{\gamma} = Z \otimes \mathbf{1}, \tag{77}$$

which makes the $S_+ \oplus S_-$ splitting manifest. We see that $\mathcal{J} = (X \otimes J)\mathcal{K}$ commutes with each γ matrix and squares to $-\mathbf{1}$; therefore the pinor representation is pseudoreal. Since the representation theory of $\mathrm{Spin}(s,t)$ is the same as that of $\mathrm{Spin}(t,s)$, we know from the previous bit that the chiral spinor representations $S_\pm$ are still complex. Therefore in this signature we can't define Majorana pinors or chiral Majorana spinors.

We will instead prefer the representation

$$\gamma^0 = X \otimes Y, \quad \gamma^1 = i\mathbf{1} \otimes Z, \quad \gamma^2 = J \otimes Y, \quad \gamma^3 = i\mathbf{1} \otimes X, \qquad \bar{\gamma} = Z \otimes Y. \tag{78}$$

This gives us a purely imaginary representation, with γ^0 antisymmetric and the others symmetric. The antilinear map establishing pseudoreality of the pinor representation is now $\mathcal{J} = (Z \otimes J)\mathcal{K}$. All products of two γ matrices are real, and so again, we can have non-chiral Majorana spinors, but not chiral ones, nor Majorana pinors.

The charge conjugation matrix is again $C = X \otimes Y$. For time reversal, we need $\bar{\mathcal{U}}\gamma_\mu \mathcal{U} = \gamma_\mu$, and so we can either set $\bar{\mathcal{U}} = \mathcal{U} = \mathbf{1}$ or $\bar{\mathcal{U}} = -\bar{\gamma}, \mathcal{U} = \bar{\gamma}$. As in the other signature, the choice between these options is equivalent to a choice between the regular and tilded reflection representations. If we choose the latter for defining T, then we have

$$T : \psi \mapsto (X \otimes Y)\psi, \quad \bar{\psi} \mapsto -\bar{\psi}(X \otimes Y). \tag{79}$$

On the other hand, if we choose $\bar{\mathcal{U}} = \mathcal{U} = \mathbf{1}$ for defining $\widetilde{T}$, then we have

$$\widetilde{T} : \psi \mapsto (J \otimes \mathbf{1})\psi, \quad \bar{\psi} \mapsto -\bar{\psi}(J \otimes \mathbf{1}). \tag{80}$$

Reflections again square to either $(-1)^F$ or as $\mathbf{1}$, depending on the choice of regular or tilde representation. It is straightforward to check that the algebra obeyed by $T, \widetilde{T}, CT$, and $C\widetilde{T}$, as well as the transformation properties of the Dirac and chiral masses under these symmetries, is the same as for the case of mostly positive signature, provided that we stick with the above definitions of T and $\widetilde{T}$.