Winter physics diary / problem list

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December 24, 2018

Preface: Disclaimer: I only know that my answers are correct for $\sim 2/3$ of the problems, and of course even correct answers are bound to have a nonzero number of typos / mistakes.

A few problems are listed in multiple section headings as appropriate.

1 Condensed matter problems

2 Conformal field theory problems

3 Math problems

4 (QFT $\setminus$ CFT) problems

5 Quantum information problems

6 Miscellanea

7 November 22 — MFT for Mott insulators and SFs in two dimensions

Today's entry is pretty basic but is something that I hadn't worked out before. Show that in the 1+1 dimensional Fermi-Hubbard model at half filling, mean field theory predicts a CDW state for arbitrarily small repulsive interactions, and an SCing state for arbitrarily small attractive interactions. Of course both of these conclusions are incorrect, with a gapless phase surviving for finite interactions. Explain why mean field theory goes down in this way and why there is actually a gapless phase.

Solution:

First, a note that I'm writing this super late and without paper at hand, so there will likely be numerical mistakes. The Hamiltonian is the usual

$$H = -\frac{1}{2}\sum_j (c_j^\dagger c_{j+1} + h.c.) + U\sum_j (n_j - 1/2)(n_{j+1} - 1/2). \tag{1}$$

First let's examine the CDW state, which we expect to occur for some large enough U. We take as our MFT ansatz

$$n_j = \frac{1}{2}(1 + \Delta(-1)^j) + :\delta n_j:, \tag{2}$$

where $:\delta n_j:$ has zero expectation value and is assumed to be small. Putting this into H and dropping the term quadratic in δn_j,

$$H = -\frac{1}{2}\sum_j (c_j^\dagger c_{j+1} + h.c.) - U\sum_j \left(\frac{\Delta^2}{4} + \Delta(-1)^j :\delta n_j:\right). \tag{3}$$

Now going to momentum space,

$$H \to \sum_{0\le k<\pi} \Psi_k^\dagger \mathcal{H}_k \Psi_k - \Delta^2 UV/4, \qquad \mathcal{H}_k = \begin{pmatrix} \cos k & -\Delta U \\ -\Delta U & -\cos(k)\end{pmatrix}, \qquad \Psi_k = (c_k,\, c_{k+\pi})^T. \tag{4}$$

The eigenvalues of $\mathcal{H}_k$ are $E_\pm(k) = \pm\sqrt{\cos^2 k + \Delta^2 U^2}$, and so getting the ground state by filling all of the $E_-(k)$ modes means the ground state energy is

$$\mathcal{E}_0 \to -\int dk\ \sqrt{\cos^2 k + \Delta^2 U^2} - VU\Delta^2/4. \tag{5}$$

Minimizing this (and normalizing by $V = 1$ for simplicity),

$$\frac{1}{4U} = \int \frac{dk}{\sqrt{\cos^2 k + U^2\Delta^2}}, \tag{6}$$

which has a solution for arbitrarily small positive U since $\cos^2(k)$ vanishes at $k = \pm\pi/2$. We can approximate (ignoring numerical constants so that the important part of the integral is $\int_k dk/\sqrt{k^2 + U^2\Delta^2} \to \ln(k^2 + U^2\Delta^2)|_0^\Lambda \implies \ln(\Lambda^2/U^2\Delta^2) \approx 1/U$)

$$\Delta \approx \frac{\Lambda}{U} e^{-1/U}. \tag{7}$$

Does this solution for Δ give a lower energy than the $\Delta = 0$ state? Indeed it does: the energy with Δ turned on is schematically

$$\mathcal{E}_\Delta = \alpha\Delta^2 - \beta\int_{k,\Delta}\frac{U^2\Delta}{\sqrt{\cos^2 k + U^2\Delta^2}} \approx \alpha\Delta^2 - \gamma\int_\Delta \Delta\ln(\Lambda^2/U^2\Delta^2) = \Delta^2(\lambda + \omega\ln\Delta), \tag{8}$$

where all the greek letters are *positive* numerical constants that we don't care about. Now when $\Delta = 0$ we get $\mathcal{E}_\Delta \to 0$, but because of the logarithm, we see that we can always choose Δ to be small enough so that $\lambda + \omega\ln\Delta < 0$, meaning that $\mathcal{E}_\Delta$ is most negative when $\Delta \neq 0$. Thus MFT predicts a CDW at any finite $U > 0$.

We can also see the CDW instability by looking at what happens when we add a perturbation H_Δ to H_0, where H_0 is the hopping term and H_Δ carries momentum π, e.g.

$$H = H_0 + H_\Delta, \qquad H_\Delta = \Delta\sum_j(-1)^j c_j^\dagger c_j \to \Delta\int dx\,(R^\dagger L + L^\dagger R), \tag{9}$$

where $c_j = e^{-ik_F j}R + e^{ik_F j}L$ (we are setting the Fermi velocity to unity). We now go to second-order perturbation theory, with the second order correction to the energy being (the first-order correction vanishes, since in the ground state $|0\rangle$ where all states below $k_F = \pi/2$ are filled the matrix element $\langle 0|H_\Delta|0\rangle$ vanishes)

$$E_2 = \sum_{l\neq 0}\frac{|\langle l|H_\Delta|0\rangle|^2}{\mathcal{E}_0 - \mathcal{E}_l}, \tag{10}$$

where l run over the non-groundstate eigenstates of the free Hamiltonian and $\mathcal{E}_l$ are the free Hamiltonian energies. Now since the energy denominator will be small only when $|l\rangle$ is a state with excitations near the Fermi surface, we can take $H_0 \approx \int dx\,(iR^\dagger\partial_x R - iL^\dagger\partial_x L)$ for our purposes, so that the energies of the $|l\rangle$ states are linear in momentum. Now the mass term H_Δ has momentum transfer π, so that it moves a hole just below the left Fermi point to a particle just above the right Fermi point, and vice versa. Thus when the relevant matrix element $\langle l|H_\Delta|0\rangle$ is non-zero, $\mathcal{E}_l \sim l$, where l is a momentum. So rather schematically we see that the second order correction to the energy is

$$E_2 \sim \Delta\int_0^\Lambda dk\frac{1}{k} \to \infty, \tag{11}$$

so that E_2 diverges logarithmically, Here Λ is some unimportant upper cutoff (note that we have a divergence here from the *small* momentum modes right by the Fermi surface).

What happens to the CDW instability away from half-filling? Let us more generally write the interaction term as $\sum_j U(n_j - \nu)(n_{j+1} - \nu)$. This adds to the MF Hamiltonian (still assuming a CDW at momentum π since this is where the divergence in the susceptibility is) a term $\sum_j U\delta\nu n_j$, where $\delta\nu = \nu - 1/2$. The negative-energy branch of the spectrum is then $E_- = U\delta\nu - \sqrt{U^2(\delta\nu)^2 + \cos^2 k + \Delta^2U^2}$, which in turn leads to the gap equation (assuming $\Delta \neq 0$)

$$\frac{1}{4U} = \int\frac{dk}{\sqrt{U^2(\delta\nu)^2 + \cos^2 k + \Delta^2U^2}}. \tag{12}$$

Does this lead to an instability for arbitrarily small U? Because of the $U^2(\delta\nu)^2$ term in the denominator, the integrand can no longer be made arbitrarily large by making Δ appropriately small. If we make the same approximation to the integral as before, and drop unimportant numerical factors, the self-consistent equation ends up reading (I think)

$$\Delta^2 \sim -(\delta\nu)^2 + (\Lambda/U)^2 e^{-1/U}. \tag{13}$$

Now for a fixed cutoff Λ, a small enough U will lead to $\Delta^2 < 0$, which is a contradiction. Thus away from half-filling the system is not immediately unstable to a CDW, as expected.

The same analysis can be applied for the SCing instability, which also comes about from a diverging susceptibility. Consider adding the term

$$H_\Delta = \Delta \sum_j (c_j c_{j+1} + h.c.) \to 2\Delta \int dx\, (iLR - iR^\dagger L^\dagger) + \dots, \tag{14}$$

where $\dots$ denotes less relevant operators / ones that oscillate rapidly. The matrix element $\langle 0|H_\Delta|l\rangle$ will be non-zero if $|l\rangle$ contains a pair of particles or holes with opposite momenta $k, -k$. The energy of such a pair is linear in k, and so we again get a (logarithmically) divergent second-order correction to the energy in the same way as before.

Doing MFT for the SCing instability works in essentially the same way as for the CDW instability; we just replace $c_j c_{j+1} = \Delta + (c_j c_{j+1} - \Delta)$, and drop terms quadratic in the deviation about Δ. The symmetry $\cos(k) = \cos(-k)$ is now what allows us to derive the instability by defining the spinor $\Psi_k = (c_k,\, c^\dagger_{-k})$, with the associated Hamiltonian

$$\mathcal{H}_k = \begin{pmatrix} \cos k - \mu & e^{ik}\Delta U \\ e^{-ik}\Delta U & -[\cos(k) - \mu] \end{pmatrix}, \tag{15}$$

where we took $\Delta \in \mathbb{R}$ for simplicity and where μ is a chemical potential determined by the filling. For a filling fraction of ν, we have $\mu = 2\nu U$. Going through the analysis yields the exact same gap equation as in the CDW case, expect with $\cos(k) \mapsto \zeta(k) \equiv \cos(k) - \mu$ and $U \mapsto -U$. As long as the chemical potential is not so large that $\zeta(k)$ the gap equation is solved in the same way as in the CDW case, by linearizing $\zeta(k)$ about its vanishing point. Thus as the SCing instability only relies on the $k \mapsto -k$ TRS symmetry of the diagonal part of the Hamiltonian, it is insensitive to the precise value of the filling ν (at the technical level, the difference is that the chemical potential caused by $\nu \neq 1/2$ shows up in $\mathcal{H}_k$ as proportional to $\mathbf{1}$ in the CDW case, but proportional to Z in the SC case).

8 *December 1 — Topological robustness of Fermi surfaces*

Today's entry is something that somehow escaped my attention until now but is actually pretty cool: explain why the existence of a fermi surface is protected by a topological invariant relating to the winding of a fermion Greens function. I read about this in Volovik's book.

Solution:

For a Fermi "surface" of codimension $p+1$ in (ω, k) space (we will always write k for both $|\vec{k}|$ and $\vec{k}$, hopefully the distinction will be clear from context), a topological invariant is obtained by integrating a p-form around an S^p that links the Fermi surface. The p-form is constructed by taking the trace of wedge products of the Maurer-Cartan form

$$\omega_{\mathcal{G}} = \mathcal{G} d\mathcal{G}^{-1}, \qquad \mathcal{G}(\omega, k) = \frac{1}{i\omega - \mathcal{H}(k)}, \tag{16}$$

where $\mathcal{H}$ is the Hamiltonian density. Note that $\mathcal{G}(\omega, k)$ is defined as the full interacting Greens function at *imaginary* frequency. The nontriviality of the topological invariant will come from the properties of the singularities of $\mathcal{G}$ in ω, k space. Since we want to characterize the robustness of the Fermi surface, we want the only singularities in $\mathcal{G}$ to come at the Fermi surface, where $\mathcal{H}(k)$ and ω vanish. If we were at $\mathbb{R}$ frequencies this would not be the case, since then we'd have singularities in $\mathcal{G}$ whenever we had quasiparticles going on-shell. The topological invariant is constructed via

$$N = \frac{1}{\mathcal{N}} \oint_{S^p} \mathrm{Tr}[\omega_{\mathcal{G}}^p], \tag{17}$$

with $\mathcal{N}$ an (imaginary) normalization constant[1].

The simplest example of this is a prototypical codimension-2 Fermi surface (i.e. an S^2 in three spatial dimensions) for a single free fermion. In this case we have

$$N = \frac{1}{2\pi i} \oint_C dz^\mu \, \frac{1}{i\omega - \mathcal{H}(k)} \partial_\mu (i\omega - \mathcal{H}(k)). \tag{18}$$

Here z^μ is a stand-in for the coordinate along the contour. We will choose the contour to be a small circle of radius R linking the Fermi sphere in the (ω, k_x) plane. Taking R to be small, we can expand $\mathcal{H}(k)$ about the FS as $\mathcal{H}(k) \approx v_F k$, where now k is measured relative to the FS. Wolog we can set $v_F = 1$, so that the invariant is

$$N = -\frac{1}{2\pi i} \oint dz^\mu \, \frac{i\omega + k}{\omega^2 + k^2} \partial_\mu (i\omega - k). \tag{19}$$

Parametrizing the contour in the (ω, k_x) plane by $(R\cos\theta, R\sin\theta)$, we have

$$\begin{aligned} N &= -\frac{1}{2\pi i} \oint d\theta \, (-\sin\theta, \cos\theta)_j (i, -1)^j [i\cos\theta + \sin\theta] \\ &= -\frac{1}{2\pi i} \oint d\theta \, (-i\sin^2\theta - i\cos^2\theta) = 1. \end{aligned} \tag{20}$$

Thus the winding around the Fermi surface is nontrivial, and the FS is topologically protected (N is independent of the contour since $\omega_{\mathcal{G}}$ is closed: $d\mathcal{G} \wedge d\mathcal{G}^{-1} = 0$).

How do we get $N = -1$? We just have to change the sign of $\mathcal{H}(k)$. Indeed, if we do this then we have

$$N = -\frac{1}{2\pi i} \oint dz^\mu \, \frac{i\omega - k}{\omega^2 + k^2} \partial_\mu (i\omega + k) = -\frac{1}{2\pi i} \oint d\theta \, (-\sin\theta, \cos\theta)_j (i, 1)^j [i\cos\theta - \sin\theta] = -1. \tag{21}$$

Of course, we know of many examples where Fermi surfaces can be destroyed by various types of instabilities, like the CDW or SCing instabilities. Evidently for this to happen, we need to

[1] In fact, this is only the topological invariant when p is odd, since ω^p defines a nontrivial class in $H^\bullet(X)$ only for odd p: when $p \in 2\mathbb{Z}$ $\mathrm{Tr}(\omega^p) = -d\mathrm{Tr}(\omega^{p-1})$ is exact (there may be an omitted combinatorial prefactor). This may be related to the fact that Fermi surfaces of even codimension (in full $\omega, \vec{k}$ space) are stable (like regular Fermi surfaces in spatial dimension $d = 3$), while those of odd dimension are unstable (e.g. Fermi lines in $d = 3$, which are codimension 3).

have some way of changing the Greens function holonomy we calculated above so that it becomes trivial. Indeed, imagine starting from a non-interacting problem, and then slowly turning on interactions which potentially lead to an instability of the Fermi surface. Assuming that turning on the interactions adiabatically is not a singular process, since $N \neq 0$ when the interactions are turned off and since $N \in \mathbb{Z}$, it cannot become zero when the interactions are slowly switched on (having $N = 0$ is required if we want to get an instability of the FS which leads to a gapped phase, since in a gapped phase $\mathcal{G}$ has no singularities).

Since we have seen that taking $\mathcal{H}(k) \mapsto -\mathcal{H}(k)$ changes the sign of N, we see that one potential way to create an instability is to find some way of pairing up Fermi surfaces of particles ($N = 1$) and Fermi surfaces of holes ($N = -1$) to create a Fermi surface with net zero winding number, which then has no obstruction to being destroyed by an instability that creates a gap.

As simple examples, both the CDW and SCing instabilities in 1+1 dimensions are created in this way. In each case, we write the Hamiltonian (as we would do in e.g. a MFT analysis) as a bilinear in terms of the spinors

$$\Psi_{CDW} = (c_k,\, c_{k+\pi})^T, \qquad \Psi_{SC} = (c_k,\, c^\dagger_{-k})^T. \tag{22}$$

The crucial property of both of these spinors is that the Hamiltonian matrices they are associated with have a diagonal component proportional to Z (the Pauli matrix), e.g. $\cos(k)Z$ (because e.g. in the CDW case we have $\cos(k+\pi) = -\cos(k)$; in the SCing case the minus sign is due to the mixed creation / annihilation nature of Ψ_{SC} and the symmetry $\cos(-k) = \cos(k)$). More generally, suppose we can write the Hamiltonian as a bilinear in spinors such that after diagonalizing $\mathcal{H}(k)^2$, we have $\mathcal{H}(k) = E(k)Z \approx v_F k Z$. Again setting $v_F = 1$, the index is then

$$\begin{aligned} N &= -\frac{1}{2\pi i}\mathrm{Tr}\left[\oint dz^\mu \begin{pmatrix} \frac{1}{i\omega - k} & 0 \\ 0 & \frac{1}{i\omega + k}\end{pmatrix} \partial_\mu \begin{pmatrix} i\omega - k & 0 \\ 0 & i\omega + k\end{pmatrix}\right] \\ &= -\frac{1}{2\pi i}\oint dz^\mu\, \frac{1}{\omega^2 + k^2}\left((i\omega - k)\partial_\mu(i\omega + k) + (i\omega + k)\partial_\mu(i\omega - k)\right) \\ &= 0. \end{aligned} \tag{23}$$

So in this case, the index when interactions are switched off is vanishing, which means that introducing interactions adiabatically does have the possibility of creating a gap and leading to an instability of the FS.

Finally, we re-iterate that this quantization is a very robust result, and does not change when e.g. the strength of interactions are slowly changed. For example, as long as our propagator is of the form $Z/(i\omega - v_F k)$, the quantization of N is quite insensitive to the exact values of Z and v_F: in FL theory changing the strength of interactions only changes the residue Z and renormalizes the Fermi velocity, neither of which change N.

9 *December 5 — Large N matrix model quantum mechanics and eigenvalue distributions*

This is a problem from a pset assigned in Hong Liu's class on AdS / CFT. We consider a matrix model with partition function

$$Z = \int \mathcal{D}M\, \exp\left(-\frac{N}{g}\mathrm{Tr}[V(M)]\right), \tag{24}$$

[2] Sending $\mathcal{H} \mapsto U^\dagger \mathcal{H} U$ for unitary U leaves $\mathrm{Tr}(\omega_{\mathcal{G}})$ invariant, which is a relatively quick thing to check.

where $V(M)$ is a polynomial potential (so that the action is a function only of the eigenvalues of M), and the integral over M runs over all $N \times N$ Hermitian matrices. We will eventually specialize to $V(x) = x^2/2 + x^4$. We will also denote the eigenvalue density by

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i), \tag{25}$$

where $\{\lambda_i\}$ are the eigenvalues. Taking $N \to \infty$, we will assume $\rho(\lambda)$ approaches a continuous function supported on some interval $I \subset \mathbb{R}$.

Do several things: a) Find an expression for Z to leading order in the $N \to \infty$ limit. b) Define the complex function

$$F(\xi) = \int_I d\lambda \, \frac{\rho(\lambda)}{\xi - \lambda}. \tag{26}$$

Discuss the analytic properties of $F(\xi)$. c) Show that $F(x) \in \mathbb{R}$ if $\in \mathbb{R} \setminus I$, and find $F(x \to \infty)$. d) Use the previous results to determine the form of F. e) Find $\rho(\lambda)$ explicitly. Finally, f) find the leading non-analytic behavior of the free energy near the critical point $g_c = -1/48$.

a) First, we use the Vandermonde determinant (see entry in the summer physics diary) to turn the measure $\mathcal{D}M$ into an integral over the eigenvalues, weighted by an exponential involving a term like $\ln|\lambda_i - \lambda_j|$, which contributes to the action. To get the saddle point equation, we vary the action with respect to a particular eigenvalue λ_i and set the result to zero: this gives

$$\frac{N}{g} V'(\lambda_i) = 2 \sum_{j:j<i} \frac{1}{\lambda_i - \lambda_j} - 2 \sum_{j:j>i} \frac{1}{\lambda_j - \lambda_i} = 2 \sum_{j:j \neq i} \frac{1}{\lambda_i - \lambda_j}. \tag{27}$$

Turning the sum into an integral using the eigenvalue distribution $\rho(\lambda)$, we get

$$\frac{N}{2g} V'(\lambda) = P \int d\mu \, \frac{\rho(\mu)}{\lambda - \mu}, \tag{28}$$

where the principal value has been taken since the sum avoids terms with $i = j$ where $\lambda_i - \lambda_j = 0$.

The saddle point value for the partition function is just obtained by evaluating the matrix exponential on the saddle point distribution of eigenvalues. So the free energy is

$$\mathcal{F} = -\ln Z \approx \frac{N}{g} \sum_i V(\lambda_i) - \ln \prod_{i \neq j} |\lambda_i - \lambda_j| \to \frac{N^2}{g} \int d\lambda \, \rho(\lambda) V(\lambda) - N^2 P \int d\lambda d\mu \, \rho(\lambda)\rho(\mu) \ln|\lambda - \mu|, \tag{29}$$

where λ is a distribution of eigenvalues satisfying the saddle point equation. Here the logarithm comes from putting the vandermonde determinant in the exponential.

b) Now define the complex function

$$F(\xi) = \int_I d\lambda \frac{\rho(\lambda)}{\xi - \lambda}, \qquad \xi \in \mathbb{C}. \tag{30}$$

Here $I = \text{supp}(\rho) \subset \mathbb{R}$ is assumed to be a union of intervals in $\mathbb{R}$. Using the Dirac identity, we can take $\xi = \mu - i\epsilon$ for μ real and send $\epsilon \to 0$ to get

$$F(\mu - i\epsilon) = i\pi \int d\lambda \, \rho(\lambda)\delta(\mu - \lambda) + P \int d\lambda \frac{\rho(\lambda)}{\mu - \lambda}. \tag{31}$$

Here we have used the equation of motion to replace the principal part of the integral above with the derivative of $V(\mu)$. Thus we see that

$$\mathrm{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu), \qquad \mathrm{Re}[F(\mu - i\epsilon)] = \frac{V'(\mu)}{2g}, \qquad \mu \in I. \tag{32}$$

Note that these properties hold only for $\mu \in I$: if $\mu \in \mathbb{R} \setminus I$ then

$$\mathrm{Im}[F(\mu - i\epsilon)] = 0, \qquad \mathrm{Re}[F(\mu - i\epsilon)] = \int d\lambda \frac{\rho(\lambda)}{\mu - \lambda}, \qquad \mu \in \mathbb{R} \setminus I. \tag{33}$$

In particular, the real part of $F(\mu - i\epsilon)$ needn't be related to $V'(\mu)$ if $\mu \notin I$, since the saddle-point equation relating $V'(\mu)$ to the principal part of the relevant integral was derived under the assumption that $\mu \in I$.

Note that $F(\xi)$ is analytic everywhere, and on $I \subset \mathbb{R}$ it has a branch cut ($\rho(\lambda)$ is assumed to be well-behaved in the $N \to \infty$ limit). We see that across the branch cut at $\mu \in I$, $F(\xi)$ changes by $2\pi i\rho(\mu)$.

c) Note that if we take $\xi \in \mathbb{R}$ and send $\xi \to \infty$, we have

$$F(\xi \to \infty) = \int_I d\lambda\, \frac{\rho(\lambda)}{\xi}(1 + \lambda/\xi + \dots) = \frac{1}{\xi} + O(\xi^{-2}), \tag{34}$$

where we have used the normalization of $\rho(\lambda)$. Note that $\mathrm{Re}[F(\mu - i\epsilon)]$ does not go to $V'(\mu)/2g$ for $\mu \notin I$ (unless V is logarithmic, which we will assume to not be the case).

d,e) We can use this information to find out what $F(\xi)$ is. In the following we will assume for simplicity that I is a single connected interval centered on zero, so that $I = [-a, a]$ for some $a \in \mathbb{R}$. This will be the case if we have a potential $V(\Lambda)$ with a unique minimum at 0, like $V(\Lambda) = \frac{1}{2}\Lambda^2 + \Lambda^4$. We will determine a self-consistently using the constraints we've derived on F.

Since we know that $F(\xi)$ is analytic but has a branch cut at $I = [-a, a]$ on the $\mathbb{R}$ axis, we expect that $\sqrt{\xi^2 - a^2} = \sqrt{\xi - a}\sqrt{\xi + a}$ will show up in $F(\xi)$ in order to give us the right branch cut structure, and in order to make $\mathrm{Im}[F(\mu - i\epsilon)]$ nonzero only when $\mu \in I$. Since $F(\xi)$ is analytic, we expect $F(\xi) = g(\xi) + f(\xi)\sqrt{\xi^2 - a^2}$, where f, g are some polynomials in ξ with positive powers and real coefficients (as $\mathrm{Im}[F(\mu - i\epsilon)] = 0$ if $\mu \notin I$).

The requirement that the real part of $F(\mu - i\epsilon)$ go to $V'(\mu)/2g$ when $\mu \in I$ tells us that $g(\xi) = V'(\xi)/2g$. We can then get $f(\xi)$ by requiring $F(\xi \to \infty) \to 1/\xi + O(1/\xi^2)$:

$$F(\xi \to \infty) \approx \frac{V'(\xi)}{2g} + f(\xi)\left(\xi - \frac{a^2}{2\xi} + O(\xi^{-3})\right). \tag{35}$$

Thus

$$f(\xi) = \frac{1}{\xi^2 - a^2/2} + \frac{\xi V'(\xi)}{2g(a^2/2 - \xi^2)}, \tag{36}$$

with a^2 to be determined by requiring $f(\xi)$ to be a $\mathbb{R}$ polynomial with positive powers. We know that $\deg(f) = \deg(V') - 1$, which again follows from our knowledge of $F(\xi \to \infty)$.

We will now specialize to the case

$$V(\lambda) = \frac{1}{2}\lambda^2 + \lambda^4. \tag{37}$$

So then $\xi V'(\xi) = \xi^2 + 4\xi^4$, and

$$f(\xi) = \frac{1}{2g(\xi^2 - a/2)}(2g - \xi^2 - 4\xi^4). \tag{38}$$

Since $V'(\lambda)$ is third order, we know that $f(\xi)$ will be second order, which allows us to stop at the leading order expansion for the square root for now. Writing $f(\xi) = A + B\xi + C\xi^2$ we see that $B = 0$, $C = -2/g$, and

$$\frac{1}{2g} + A = \frac{ca^2}{2}, \tag{39}$$

so that

$$f(\xi) = -\frac{1}{2g}(1 + 2a^2 + 4\xi^2). \tag{40}$$

This then determines the eigenvalue distribution to be, using $\mathrm{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu)$ for $\mu \in [-a, a]$,

$$\rho(\mu) = -\frac{1}{\pi} f(\mu)\sqrt{a^2 - \mu^2} = \frac{1}{2\pi g}(1 + 2a^2 + 4\mu^2)\sqrt{a^2 - \mu^2}. \tag{41}$$

As an aside, we can recover the Wigner distribution by looking at $V(\lambda) = \frac{1}{2}\lambda^2$. In this case, since we know that the degree of $f(\xi)$ is two less than the degree of $V(\xi)$, $f(\xi)$ must be a constant. Working it out and solving for a^2 in the manner described below gives (I think)

$$\rho(\mu)|_{V(\lambda)=\lambda^2/2} = \frac{1}{2\pi g}\sqrt{4g - \mu^2}, \tag{42}$$

which is the famous Wigner distribution.

Anyway, now we return to the quartic potential (37). To get a, we need to get the $1/\xi$ piece of $F(\xi \to \infty)$, which requires expanding the square root to include the $-a^4/8\xi^4$ term. Setting the coefficient of the $1/\xi$ piece to 1 means that

$$3a^4 + a^2 - 4g = 0 \implies a^2 = \frac{1}{6}\left(-1 + \sqrt{1 + 48g}\right). \tag{43}$$

Recapitulating, we have shown that

$$F(\xi) = \frac{1}{2g}\left(\xi + 4\xi^3 - \left[4\xi^2 + \frac{2}{3} + \frac{1}{3}\sqrt{1 + 48g}\right]\sqrt{\xi^2 + \frac{1}{6} - \frac{1}{6}\sqrt{1 + 48g}}\right). \tag{44}$$

e) We can now get an explicit expression for the free energy

$$\mathcal{F}/N^2 \approx \frac{1}{g}\int d\lambda\, \rho(\lambda)V(\lambda) - P\int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\ln|\lambda - \mu|. \tag{45}$$

The second term in the free energy with the ln is hard to integrate, but we have another option: we can integrate the equations of motion to obtain

$$\frac{V(\lambda) - V(0)}{2g} = P\int d\mu\, \rho(\mu)\left[\ln|\lambda - \mu| - \ln|\mu|\right], \tag{46}$$

which means that (since $V(0) = 0$)

$$P\int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\ln|\lambda - \mu| = \frac{1}{2g}\int d\lambda\, \rho(\lambda)V(\lambda) + P\int d\mu\, \rho(\mu)\ln|\mu|. \tag{47}$$

Putting this into the second integral in the expression for the free energy,

$$\mathcal{F}/N^2 \approx \int d\lambda\, \rho(\lambda)\left(\frac{V(\lambda)}{2g} - \ln|\lambda|\right). \tag{48}$$

The first term is

$$\frac{1}{2g}\int_{-a}^{a} d\lambda\, \rho(\lambda)(\lambda^2/2+\lambda^4) = -\frac{a^4}{128g^2}(2+10a^2+9a^4), \tag{49}$$

while the second term is

$$-2\int_0^a d\lambda\, \rho(\lambda)\ln\lambda = \frac{a^2}{16g}\left(2+a^2(3+6\ln 4)+\ln 16 - 4(1+3a^2)\ln a\right). \tag{50}$$

Now we add these two together, and carry out an expansion in small $\epsilon = g - g_c = g + 1/48$. Here the critical point $g_c = -1/48$ is the coupling at which the free energy becomes singular. This point is supposed to mark the phase transition where "complicated" Feynman diagrams dominate and the Feynman diagrams go over to form a "continuum geometry", or something like that. Hold on, you may say: at $g \to g_c$ $a^2 < 0$, but doesn't a^2 always have to be positive, since the eigenvalues we're integrating over must always be real? So, isn't $g < 0$ already ruled out? I guess the philosophy here is that the important thing to look at is really the singular behavior of the partition function: we used the WKB approximation to get the partition function, and while within this approximation $g < 0$ strictly speaking doesn't make sense, after getting our expression for $\mathcal{F}$ we can just work with it directly, forgetting about where it came from.

Anyway, doing the expansion with Mathematica, we find that the free energy has an (imaginary) constant part, a term proportional to ϵ, one proportional to ϵ^2, and then one proportional to $\epsilon^{5/2}$, which is the leading singular part. So, the leading non-analytic behavior of $\mathcal{F}$ is a 5/2 power dependence on the distance from the critical point (n.b. the cancellation of the $\epsilon^{1/2}, \epsilon^{3/2}$ terms is nontrivial!).

10 *December 6 — RG for fermions a la Shankar*

I had to do a mini final project for a class Ashvin Vishwanath taught at Harvard; my topic was on understanding Shankar's RG approach to systems with Fermi surfaces. So, today's diary entry is a bit of cheat: just a coping-and-pasting of the notes I wrote up.

11 *December 7 — Quantization in AdS*

This is a problem taken from a pset assigned in Hong Liu's holography class. The problem is as follows: consider a scalar field in AdS_{d+1}, with action

$$S = -\frac{1}{2}\int d^{d+1}x\, \sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2). \tag{51}$$

Here Greek indices run over time, $d-1$ "space" dimensions, and the radial coordinate z (sorry). Near the boundary at $z = 0$, we have

$$\phi(z\to 0, x) = A(x)z^{d/2-\nu} + B(x)z^{d/2+\nu}, \qquad \nu \equiv \sqrt{d^2/4 + m^2R^2}, \tag{52}$$

where R is the AdS radius, and A, B are some functions of the "spacetime coordinates".

a) Define an inner product for wavefunctions and show its time independence. b) What is the condition on ν for a given ϕ to be normalizable? c) Find the stress tensor and show that it is covariantly conserved. d) Define the energy E as the integral of $\sqrt{-g}g^{tt}T_{tt}$ over a given Cauchy slice, and find the explicit form of $\partial_t E$. e) When does the energy flux at the boundary vanish? f) Show that E is finite if the chosen wavefunction is normalizable, and infinite otherwise.

a) We define the inner product as

$$\langle\phi,\psi\rangle_{\Sigma_t} = -i\int_{\Sigma_t} dz\, d\vec{x}\,\sqrt{-g}g^{tt}(\phi^*\partial_t\psi - \partial_t\phi^*\psi), \tag{53}$$

where Σ_t is any Cauchy surface, and the g^{tt} is required to make the integral invariant under rescaling of t. Taking the difference of the inner products at different times, we have

$$\langle\phi,\psi\rangle_{\Sigma_{t'}} - \langle\phi,\psi\rangle_{\Sigma_t} = -i\int_M d^d x_\perp^\mu(\phi^*\nabla_\mu\psi - \nabla_\mu\phi^*\psi), \tag{54}$$

where M is the timelike boundary of the spacetime volume bounded by the two Cauchy slices, located at $z=0$. Here we have used the fact that the integral over the bounded volume vanishes, on account of

$$\nabla_\mu(\phi^*\nabla^\mu\psi - \nabla^\mu\phi^*\psi)\sqrt{-g} = 0, \tag{55}$$

by virtue of the equations of motion, viz. $\nabla^2 = m^2$ when acting on ϕ and ψ. The integral on the RHS of (54) vanishes if ϕ and ψ are properly normalized at infinity, and so the inner product is time-independent.

b) ϕ has the asymptotic expansion

$$\phi(z\to 0) = A(x)z^{d/2-\nu} + B(x)z^{d/2+\nu}. \tag{56}$$

Now since $\sqrt{-g}\propto z^{-(d+1)}$ and $g^{tt}\propto z^2$, we have

$$\langle\phi,\phi\rangle_\Sigma \sim -i\int_\Sigma dz\, d\vec{x}\,(AA'z^{1-2\nu} + BB'z^{2\nu+1} + 2AB'z). \tag{57}$$

To get something finite, we need the total power of z to be greater than -1. This is always satisfied by the B mode and so the B mode is always normalizable. For the A mode to be normalizable we need $1-2\nu > -1$, so the A mode is normalizable only if

$$0 \le \nu < 1. \tag{58}$$

c) There are two terms that contribute to the stress tensor: the Lagrangian density and the $\sqrt{-g}$ in the measure. The variation of the former wrt the metric is simple, while the latter is found by using

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta e^{\mathrm{Tr}\ln g} = -\frac{\sqrt{-g}}{2}\delta\mathrm{Tr}\ln g \implies \frac{\delta}{\delta g_{\mu\nu}}\sqrt{-g} = -\frac{\sqrt{-g}}{2}g^{\mu\nu}. \tag{59}$$

Since the variation of the Lagrangian density is just $\sqrt{-g}\partial^\mu\phi\partial^\nu\phi$, we have

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\left(\sqrt{-g}\partial^\mu\phi\partial^\nu\phi - \frac{\sqrt{-g}g^{\mu\nu}}{2}(\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2)\right) \\ &= \nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}(\nabla_\lambda\phi\nabla^\lambda\phi + m^2\phi^2), \end{aligned} \tag{60}$$

where in the last step we have replaced ordinary derivatives with covariant derivatives since they act in the same way on ϕ, which is a scalar.

Now let's verify that T is covariantly conserved. This is straightforward since the metric is covariantly constant, meaning that we don't have to worry about the differences between raised and lowered indices, and that the covariant derivatives pass straight through the $g^{\mu\nu}$:

$$\begin{aligned}\nabla_\mu T^{\mu\nu} &= \nabla^2\phi\nabla^\nu\phi + \nabla_\mu\phi\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\nabla_\mu\nabla^\lambda\phi\nabla_\lambda\phi - m^2\phi\nabla^\nu\phi \\ &= \nabla^\nu\phi(\nabla^2\phi - m^2\phi) = 0,\end{aligned} \tag{61}$$

again by virtue of the equations of motion. Thus the stress tensor is covariantly conserved.

d) Define the energy as

$$E = \int_{\Sigma_t} dz\, d\vec{x}\, \sqrt{-g} g^{tt} T_{tt}. \tag{62}$$

The g^{tt} here is needed so that under time rescalings it cancels the rescaling of T_{tt}, so that the whole action rescales like $\sqrt{-g} \mapsto \lambda\sqrt{-g}$ under $t \mapsto \lambda^{-1}t$, which is appropriate for an energy. Another way to write $\sqrt{-g}g^{tt}$ would be $\sqrt{-g_t}n^t$, where g_t is the induced metric on Σ_t and $n^t = (0, z/R, 0, \ldots, 0)$ is the temporal unit vector in coordinates $(z, t, \vec{x})$. Under time rescalings $\sqrt{-g}g^{tt}$ transforms as a vector; as does $\sqrt{-g_t}n^t$ since $\sqrt{-g_t}$ is a scalar under time rescalings (as the metric on Σ_t involves no dt^2 piece).

We can write the time derivative of E as

$$\begin{aligned}\partial_t E &= \frac{1}{\delta t}\left(\int_{V_{\delta t}} d^d x\, dz\, \sqrt{-g}\nabla^\mu T_{t\mu} - \int_{M_{\delta t}} d^d x\, \sqrt{-g_{M_{\delta t}}} n^z T_{tz}\right) \\ &= -\int_{\Sigma_t} d\vec{x}\, \sqrt{g_{\partial\Sigma_t}} n^t n^z T_{tz},\end{aligned} \tag{63}$$

where the various manifolds are defined straightforwardly: $V_{\delta t}$ is the spacetime volume sandwiched between Σ_t, and $\Sigma_{t+\delta t}$, and $M_{\delta t}$ is the timelike component of its boundary. In the last step, we have used the fact that the time integral in the second term on the first line just produces a factor of δt.

e) First, we need

$$n^t = (0, z/R, \vec{0}), \quad n^z = (z/R, 0, \vec{0}), \tag{64}$$

which comes from e.g. $n^z n_z = 1$. Now the induced metric on ∂_{Σ_t} has determinant $g_{\partial\Sigma_t} = +(R/z)^{2d-2}$, and so

$$F|_{z=0} = \partial_t E \sim \int_{\Sigma_t} d\vec{x}\, z^{-d+3} T_{tz} = \int_{\Sigma_t} d\vec{x}\, z^{-d+3} \nabla_t\phi\nabla_z\phi. \tag{65}$$

Now suppose $\phi \sim z^\omega$. Then

$$F|_{z=0} \sim z^{-d+3+2\omega-1}, \tag{66}$$

and so if the flux is to be zero we must have

$$F|_{z=0} = 0 \implies \omega > d/2 - 1. \tag{67}$$

Now for the Bz^Δ mode this is always true, since $\Delta = d/2 + \nu$. For the $Az^{d-\Delta}$ mode, this condition reads $\nu < 1$, which is precisely the condition that the A mode be normalizable. So we see that (non)normalizable modes have (non)zero energy flux at infinity.

f) The part of T_{tt} with the smallest power of z is the $(\nabla_t\phi)^2$ part. Again, suppose $\phi \sim z^\omega$. Then the contribution to E with the smallest power of z is

$$E \sim \int_{\Sigma_t} dz\, d\vec{x}\, z^{-d-1} z^2 z^{2\omega} \sim z^{-d+2+2\omega}. \tag{68}$$

If the energy is to be finite, we need $2(1+\omega) > d$. For the A mode we have $\omega = d/2+\nu$, and the energy is always finite. For the B mode we have $\omega = d/2-\nu$, and

$$E < \infty \implies \nu < 1. \tag{69}$$

Of course this is the same condition on ν for the B mode to be normalizable. So, normalizable modes, in either quantization scheme, are the ones with finite E.

12 *December 8 — AdS Propagators*

This is another problem from Hong's holography class. a) How is Lorentzian AdS different from Euclidean AdS? We will use the latter spacetime in what follows. b) Let ϕ be a massive scalar field, and find the buik-to-∂ propagator $K(z,x;x')$. c) Find a relation between K and the bulk-to-bulk propagator G in terms of the limit of G as one of its arguments approaches the boundary. d) Write down a general boundary correlation function in terms of a limit of a bulk correlation function.

a) From the $1/z^2$ dependence of the metric, we see that in Euclidean AdS, the distances in the x coordinates vanish at $z=\infty$, and so $z=\infty$ is just a single point, unlike in Lorentzian AdS. Another way of seeing this is to recognize that Euclidean AdS is the Poincare disk, with $z\to\infty$ corresponding to the single point at the center of the disk.

b) We want to get the boundary-to-bulk propagator. Using the equations of motion, the propagator K at $z=\infty$ needs to satisfy

$$(\partial_M(\sqrt{-g}g^{MN}\partial_N\phi) - m^2\sqrt{-g})K = 0. \tag{70}$$

Since $z=\infty$ is a single point in the bulk, in the $z\to\infty$ limit $K(x,z;x')$ can only depend on z (not x since all x are the same at $z=\infty$, and not x' by rotational invariance of the Poincare disk). Thus, putting in the z dependence of the metric, we have

$$\left[\partial_z\left((R/z)^{d+1}(z/R)^2\partial_z\right) - m^2(R/z)^{d+1}\right]K(z\to\infty) = 0. \tag{71}$$

Assuming a power-law $K(z)\propto z^\alpha$, we have

$$(1-d)\alpha + \alpha(\alpha-1) - m^2R^2 = 0 \implies \alpha = \frac{d}{2} \pm \sqrt{d^2/4+m^2R^2}. \tag{72}$$

We will see later that the requirement that K go to a δ function at the $z=0$ boundary requires us to select out the larger root (which we denote as Δ), and so

$$K(z\to\infty) = Cz^\Delta, \tag{73}$$

for some $C\in\mathbb{R}$.

Now we can use the homogeneity of AdS (despite how it looks when drawn as a Poincare disk, no point is special) to get $K(x,z;0)$: we first perform the transformation

$$z \mapsto \frac{z}{z^2+x^2}, \quad x^\mu \mapsto \frac{x^\mu}{z^2+x^2} \tag{74}$$

on $K(z \to \infty$. We then use translation invariance in the x^μ directions (rotational invariance of the Poincare disk) to get the Poisson form

$$K(x, z; x') = C \left(\frac{z}{z^2 + (x - x')^2} \right)^\Delta . \tag{75}$$

Here we see that the power of Δ gives us a δ function when $x = x'$. This power is also correct since we have

$$K(z \to 0, x; x') = z^{d-\Delta} \delta^d (x - x'). \tag{76}$$

In standard quantization, ϕ has z^Δ scaling, so that $\int d^d x' \, K(z, x; x') \phi_0(x')$ has the correct scaling.

c) We can relate the bulk-to-bulk propagator to K by using the bulk-to-boundary map and one of Greens identities, namely

$$\int_M d^{d+1} x \sqrt{-g} \left(\phi_1 G^{-1} \phi_2 - \phi_2 G^{-1} \phi_1 \right) = \int_{\partial M} d^d x \sqrt{-g_\partial} \left(\phi_1 n^\mu \partial_\mu \phi_2 - \phi_2 n^\mu \partial_\mu \phi_2 \right), \tag{77}$$

where g_∂ is the induced metric on the boundary, G is the bulk propagator, and n^μ is the unit normal on the boundary. The trick is then to employ this identity with $\phi_1 = K(z, x; x'), \phi_2 = G(z, x; z'', x'')$. Now since the LHS is over the bulk and since $(\nabla^2 - m^2) = G^{-1}$ annihilates K in the bulk (the only place it doesn't annihilate K is at coincident points on the boundary), the LHS is

$$LHS = \int_M d^{d+1} x \sqrt{-g} K(z, x; x') (\nabla^2 - m^2) G(z, x; z'', x'') = K(x'', z''; x'), \tag{78}$$

by definition of G. On the other hand, since $\sqrt{-g_\partial} = (R/z)^d$ and $n^\mu = z$, the RHS is

$$RHS = \int_{\partial M} d^d x \, z^{-d+1} K(z, x; x') \partial_z^{\leftrightarrow} G(z, x; z'', x''), \tag{79}$$

where $\partial^{\leftrightarrow}$ denotes the antisymmetrized derivative. Here we have dropped the R dependence since it will cancel out in the end.

Using the asymptotic $z \to 0$ form for K as written above, we can explicitly take the derivative with respect to z and get

$$RHS = \int_{\partial M} d^d x \, z^{-d+1} \delta(x - x') \left(z^{d-\Delta} \partial_z G(z, x; z'', x'') - (d - \Delta) z^{d-\Delta-1} G(z, x; z'', x'') \right). \tag{80}$$

Now since $G(z, x; z'', x'')$ is normalizable, we know that it has the same $z \to 0$ scaling as the bulk normalizable mode, namely z^Δ. Thus

$$RHS = z^{-d+1} (z^{d-\Delta} \Delta z^{-1} - (d - \Delta) z^{d-\Delta-1}) G(z, x; z'', x''), \tag{81}$$

where we are implicitly taking the $z \to 0$ limit. In the notation we used in class, $\Delta = d/2 + \nu$, and so

$$RHS = \lim_{z \to 0} (2\Delta - d) z^{-\Delta} G(z, x; z'', x'') = 2\nu z^{-\Delta} G(z, x; z'', x''). \tag{82}$$

Setting this equal to LHS and moving the $2\nu z^{-\Delta}$ over to the other side and re-labeling some dummy variables, we get

$$\lim_{z \to 0} G(z, x; z', x') = \frac{z'^\Delta}{2\nu} K(z, x; x'). \tag{83}$$

d) Let ϕ_i be the bulk scalar dual to a boundary operator $\mathcal{O}_i$. The correlation function for a product of $\mathcal{O}_i$'s at various points on the boundary can be determined by computing all Feynman

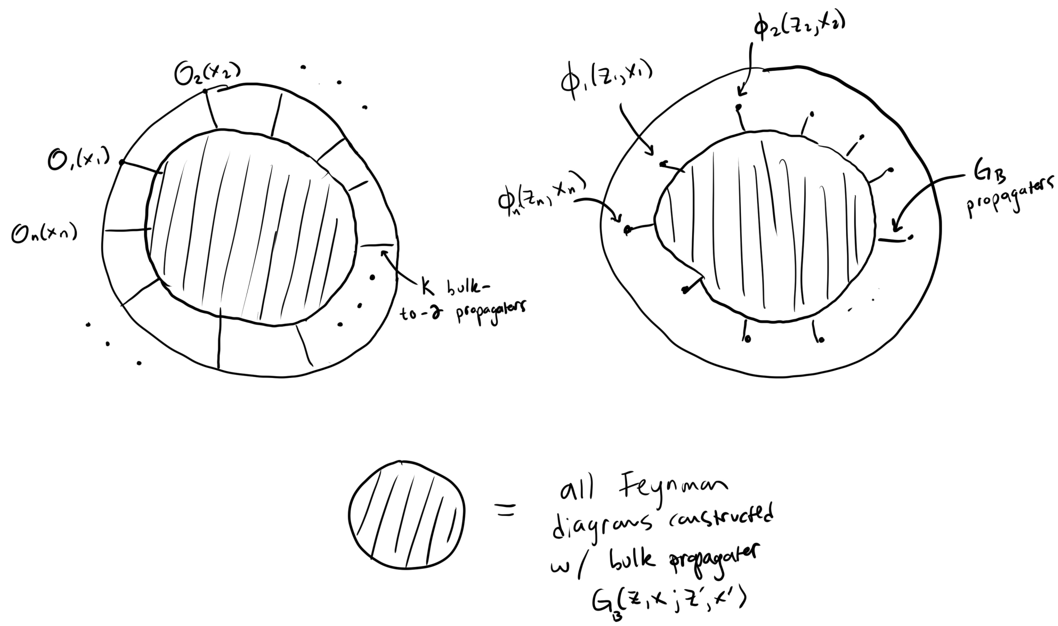

Figure 1: One of the ugliest figures I've ever made. Here, the shaded blob stands for all possible Feynman diagrams constructed from the bulk propagator.

diagrams in the bulk that have external legs on the boundary. Thus, a correlation function of n $\mathcal{O}_i$'s involves n K propagators (which connect the boundary $\mathcal{O}_i$'s to the part of the Feynman diagrams that live in the bulk, plus a bunch of G propagators which constitute the bulk part of the Feynman diagrams. On the other hand, we can consider the same class of Feynman diagrams, but with the external legs all made up of G propagators which terminate at points that have some small value of z. This is a bulk correlation function of ϕ_i fields. Taking the $z \to 0$ limit then gets us back to the correlation function of the $\mathcal{O}_i$'s. So, the only difference between the two correlation functions is whether we use K or G for the external legs. If we use G's, then we need to take the $z \to 0$ limit for one of G's arguments—luckily the previous part told us how to do this. So, using our result from c, we have

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{CFT} = \lim_{\{z_i\} \to 0} \prod_i (2\nu_i z_i^{-\Delta_i}) \langle \phi_1(z_1, x_1) \cdots \phi_n(z_n, x_n) \rangle. \tag{84}$$

All of this is illustrated in figure 1.

13 *December 9 — (Massive) Vectors in AdS*

Yep, another problem from Hong's holography class. Consider a massive vector field in AdS:

$$S = -\int d^{d+1}x \sqrt{-g} \left(\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} m^2 A_M A^M \right), \tag{85}$$

where M, N run over (z, x^μ). a) When $m^2 = 0$, find the asymptotic behavior of A_μ at the boundary $z \to 0$. b) What is the scaling dimension of the boundary current J^μ corresponding to the bulk gauge field A? c) When $m \neq 0$, what is the asymptotic behavior of A_M at the boundary? d) Now what is the scaling dimension of J^μ? e) What happens to A_z when $m \neq 0$? f) What are the differences between the massive and massless cases?

a) The equation of motion when $m = 0$ is just

$$\partial_M(\sqrt{-g} F^{MN}) = 0 \implies \partial_M(z^{-d-1} g^{ML} g^{NO} \partial_{[L} A_{O]}) = 0, \quad \forall N. \tag{86}$$

Let us assume the asymptotic behavior $A_\mu(z \to 0) \sim z^\Delta$, and work in a gauge where $A_z = 0$. The only derivative in the equations of motion we then care about in the $z \to 0$ limit is the one with $L = z$, and so setting $N = \mu$ we have

$$\partial_z(z^{-d-1+2+2} \partial_z z^\Delta) = 0 \implies (-d+3)\Delta + \Delta(\Delta - 1) = 0, \tag{87}$$

and so we have two options: $\Delta = 0$ or $\Delta = d - 2$. Thus we can write

$$A_\mu(z \to 0) = a_\mu(x) + b_\mu(x) z^{d-2}. \tag{88}$$

The a_μ is the non-renormalizable piece, while the b_μ part is renormalizable.

b) The non-renormalizable piece a_μ is the part that is relevant for computing the scaling dimension of the current in the CFT dual to A_μ, since a_μ is the part which we interpret as a change in boundary conditions. To determine the scaling dimension of J^μ, we can look at the boundary integral

$$\int_{\partial AdS} d^d x \, a_\mu J^\mu. \tag{89}$$

Now we consider performing the isometry $z \mapsto \lambda z$, $x^\mu \mapsto \lambda x^\mu$. a_μ is independent of z but it carries a covector index, so it transforms with a factor of λ. Writing $J^\mu(x/\lambda) = \lambda^{\Delta_J} J^\mu(x)$, we have

$$\int_{\partial AdS} d^d x\, a_\mu J^\mu \mapsto \lambda^{-d+1+\Delta_J} \int_{\partial AdS} d^d x\, a_\mu J^\mu. \tag{90}$$

Since we need this term to be invariant, we find that $\Delta_J = d-1$, as expected of a current in a d-dimensional CFT.

Note that A_μ and A^μ have different scaling behaviors since they differ by e.g. $g^{\mu\mu}$, which scales as z^2. To determine the dimension of J^μ we need to integrate it against something with a covariant index, so it is the scaling of A_μ, not A^μ, which is needed.

c) When the vector field is massive, the equation of motion becomes

$$\partial_M(\sqrt{-g}F^{MN}) - \sqrt{-g}m^2 A^N = 0, \qquad \forall N. \tag{91}$$

Again, let $A_\mu \sim z^\Delta$ near the boundary. Then we have

$$\partial_z(z^{-d-1}\partial_z A_\mu g^{zz} g^{\mu\nu}) - z^{-d-1} m^2 A_\mu g^{\mu\nu} = 0, \tag{92}$$

so that

$$\Delta\partial_z(z^{-d+3+\Delta-1}) - z^{-d+1+\Delta} R^2 m^2 = 0 \implies \Delta(-d+2+\Delta) - R^2m^2 = 0, \tag{93}$$

where the R^2 comes from the inverse metric factors. There are thus two possible choices for the scaling behavior of A_μ which are compatible with the equations of motion, and we can write

$$A_\mu = a_\mu z^{\Delta_+} + b_\mu z^{\Delta_-}, \qquad \Delta_\pm = 1 - \frac{d}{2} \pm \sqrt{(d-2)^2/4 + m^2R^2}. \tag{94}$$

Sorry for the profusion of Δ's! It's just entrenched as a theme by this point and there's no going back.

d) In standard quantization, the non-renormalizable part will be the Δ_+ piece. Looking at the boundary term $\int_{\partial AdS} b_\mu z^{\Delta_+} J^\mu$ and performing the re-scaling of x and z tells us that $-d+1+\Delta_J - \Delta_+ = 0$, and so in this case J^μ has scaling dimension

$$\Delta_J = d - 1 + \Delta_+ = \frac{d}{2} + \sqrt{(d-2)^2/4 + m^2R^2}. \tag{95}$$

Sanity check: when $m = 0$ we recover $\Delta_J = d-1$, as required.

e) When $m \neq 0$, we can no longer use gauge invariance to fix $A_z = 0$. The z component of the equations of motion reads, focusing only on the z-dependence of A_z,

$$\partial_z(\sqrt{-g}[g^{zz}]^2 \partial_z A_z) - \sqrt{-g}m^2 g^{zz} A_z = 0. \tag{96}$$

Since all the components of the metric have the same z dependence, the z dependence of A_z is fixed in the same way as that of the A_μ.

f) In the massless case, we have gauge invariance under $A \mapsto A + d\chi$. Accordingly, the boundary J^μ operator must be divergenceless, and so it should be thought of as a conserved current. Since we integrate conserved currents over codimension 1 manifolds to get numbers, we need the dimension of J^μ to be $d-1$. By contrast, when $m \neq 0$, there is no gauge invariance, and J^μ is not a conserved current; hence its scaling dimension is not fixed at $d-1$.

14 December 9 — Wilson Loop Vevs in $\mathcal{N}=4$ SYM using AdS/CFT

Yep, another problem from a pset in Hong's holography class. This time, we're computing Wilson loops in $\mathcal{N}=4$ SYM with holography. The problem is as follows: by evaluating the saddle-point of the NG action corresponding to a geometry in which two quarks have been inserted a distance L apart in the boundary CFT, find the potential energy $V(L)$ coming from the interaction between the two quarks (the relevant Wilson loop here is a rectangle of sides L, t, where $t \gg L$). How does the result behave in the high temperature and low-temperature limits?

We can compute Wilson loop vevs in $\mathcal{N}=4$ SYM in the limit $g_s \to 0$ (no sum over different topologies) and $\alpha' \to 0$ (when we can use the saddle-point solution to the string path integral). On the CFT side this limit is nontrivial since it corresponds to $N, \lambda \to \infty$.

We just need to compute the classical string action, since

$$\langle W(C)\rangle = Z_{str}[\partial\Sigma = C] \approx e^{iS_{cl}[\partial\Sigma=C]}, \tag{97}$$

where Σ is the string worldsheet.

At finite T (here T is temperature, not the temporal length of the Wilson loop, which we will write as t), the appropriate bulk geometry to use is an AdS-Schwarzchild black hole at temperature T. The metric is

$$ds^2 = \frac{R^2}{z^2}\left(-(1-\bar{z}^d)dt^2 + d\vec{x}^2 + \frac{1}{1-\bar{z}^d}dz^2\right), \tag{98}$$

where we've defined

$$\bar{z} \equiv z/z_0, \qquad T = \frac{d}{4\pi z_0}. \tag{99}$$

z_0 is the location of the horizon, which is closer to the $z=0$ boundary at larger temperatures.

Let us choose the contour C to run in the $x^1 - t$ plane. We can then parametrize the worldsheet with coordinates $(\tau, \sigma) = (t, x^1)$. We are interested in the energy of two quarks a distance L apart. If the temporal length T of the curve C is much larger than L, then the shape of Σ is determined by a function $z(\sigma) = z(x^1)$, with boundary conditions $z(\pm L/2) = 0$. We will use the NG action (rather than the Polyakov action) to compute S_{cl}, since we aren't ever going to need to quantize anything. The induced metric on the worldsheet is then determined by

$$ds_w^2 = \frac{R^2}{z^2}\left(-dt^2(1-\bar{z}^d) + d\sigma^2\left[1 + \frac{z'^2}{1-\bar{z}^d}\right]\right), \tag{100}$$

with $z' = \partial_\sigma z$.

The NG action is (again, assuming $t \gg L$ so that the Lagrangian on the classical solution can be treated as time-independent)

$$S_{NG} = -\frac{R^2 t}{\pi\alpha'}\int_0^{L/2}\frac{d\sigma}{z^2}\sqrt{1-\bar{z}^d + z'^2}, \tag{101}$$

where we pulled out the R^4/z^4 from the determinant of the induced metric and used the symmetry $z(\sigma) = z(-\sigma)$ that must be satisfied by the classical solution.

We can eliminate the z' inside the square root by using the equations of motion. Since $\mathcal{L}$ is independent of σ, we have

$$-z'\frac{\partial\mathcal{L}}{\partial z'} + \mathcal{L} = c, \tag{102}$$

where c is a constant. For us, this is

$$\frac{z'^2}{z^2\sqrt{1+z'^2-\bar{z}^d}} = \frac{\sqrt{1+z'^2-\bar{z}^d}}{z^2} + c, \tag{103}$$

or

$$\frac{1-\bar{z}^d}{z^2\sqrt{1+z'^2-\bar{z}^d}} = c. \tag{104}$$

We can get c by noticing that at $\sigma = 0$, $z' = 0$ by symmetry. So, let $z_* \equiv z(0)$. Then

$$c = \frac{\sqrt{1-\bar{z}_*^d}}{z_*^2}. \tag{105}$$

The energy of the Wilson line configuration is then computed as

$$E(L) = \frac{\sqrt{\lambda}}{\pi}\int_0^{z_*} \frac{dz}{z^2 z'}\sqrt{1+z'^2-\bar{z}^d}, \tag{106}$$

since $\lambda = R^4/\alpha'^2$. Solving for z' in terms of z and c, we have

$$z' = \sqrt{(1-\bar{z}^d)\left(\frac{1-\bar{z}^d}{c^2 z^4} - 1\right)}. \tag{107}$$

Putting this into the integral and doing some housekeeping, we get

$$E(L) = \frac{\sqrt{\lambda}}{\pi}\int_0^{z_*} \frac{dz}{z^2\sqrt{1-c^2 z^4/(1-\bar{z}^d)}}. \tag{108}$$

Now $E(L)$ has a $z \to 0$ divergence, but this just corresponds to the diverging mass of the two quarks. Recalling that the quark mass goes as the inverse of their z-coordinates, we expect the quark mass to show up as a $1/\epsilon$ divergence if we cut the integral off below at ϵ.

Now in order to get $E(L)$, we need an expression for z_* in terms of L. We can get an integral equation which gets us part way there by solving for z' and integrating from $\sigma = -L/2$ to $\sigma = 0$:

$$\frac{L}{2} = \int_0^{z_*} dz \left[(1-\bar{z}^d)\left(\frac{1-\bar{z}^d}{c^2 z^4} - 1\right)\right]^{-1/2}. \tag{109}$$

To see what's happening here more clearly, there are two limits we can take. The first is the $T \to 0$ limit (or equivalently, the small L limit). In this limit we can send $\bar{z} \to 0$ and $c^2 \to z_*^{-2}$, since when $T = 0$ the horizon is pushed to $z_0 = \infty$. In this limit, our integral equation determining z_* is

$$\frac{L}{2} = \int_0^{z_*} dz \frac{1}{\sqrt{z_*^4/z^4 - 1}} = z_* \alpha, \tag{110}$$

where $\alpha \approx 0.6$ is determined in terms of Elliptic integrals. This means that at $T = 0$, the maximal $z(\sigma)$ value on the worldsheet extends a distance into the bulk which grows linearly with L.

Now we can calculate $V(L)$ in this limit, by using $E(L) = 2M + V(L)$ and subtracting off the divergent mass piece:

$$V(L) = \frac{\sqrt{\lambda}}{\pi z_*}\left(\int_{\epsilon/z_*}^1 \frac{dx}{x^2\sqrt{1-x^4}} - \frac{1}{\epsilon}\right). \tag{111}$$

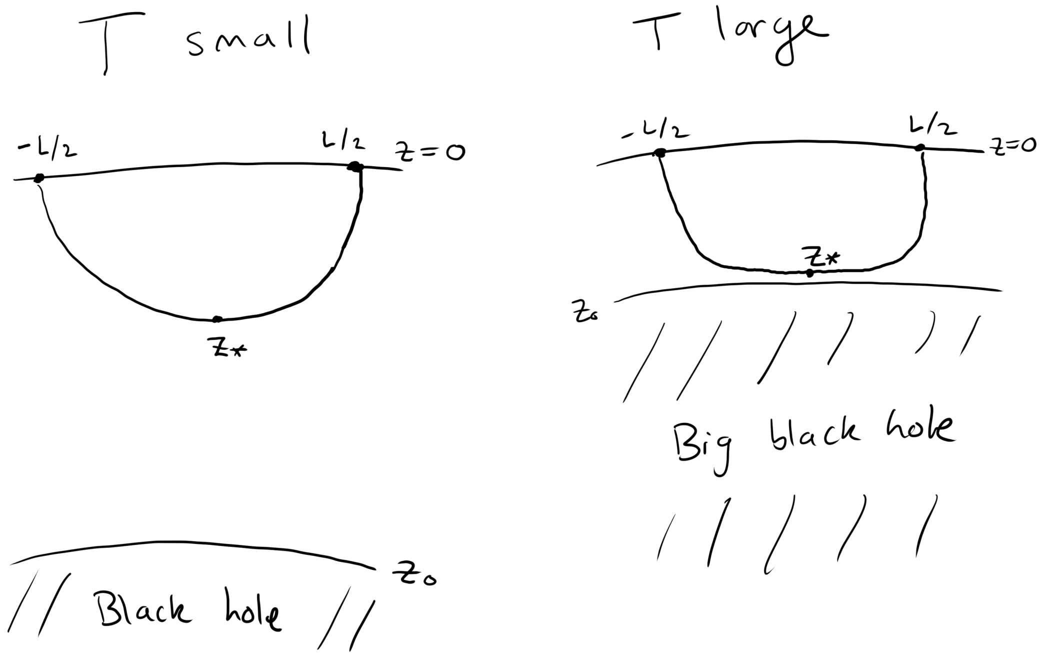

Figure 2: Another one of the ugliest figures I've ever made; explanation is in the text.

The integral can be evaluated in terms of hypergeometric functions. Doing this, sending $\epsilon \to 0$, and using our expression for z_*, we get

$$V(L) = -\frac{2\alpha\sqrt{\lambda/\pi}\,\Gamma(3/4)}{L\Gamma(1/4)}. \tag{112}$$

The most important features here are the $1/L$ dependence (from scale invariance in the CFT), and the interesting $\sqrt{\lambda}$ coupling dependence.

Now we can look at what happens at higher T, or equivalently, at Wilson lines that have L large enough so that z_* approaches the horizon at z_0. By looking at (109), we see that z_* is monotonically increasing with L (this is a bit gross to show, but it ultimately comes down to $\partial_{z_*} c < 0$).

As we keep increasing L, there reaches a point $L_{screening}$ where (109) has no solution. Looking back, we see that this must mean that $c = 0$: this is when z_* "disappears" behind the black hole horizon. For $L > L_{screening}$, we no longer can have a Wilson line connecting the two quarks: the worldsheet ends up splitting apart, and terminating on the black hole. In the way we've been doing things, this corresponds to the trivial solution $V(L) = 0$, and the quarks are fully screened. The crossover between the $1/L$ dependence of the potential and the fully screened potential can be found numerically, but I'll be content with this simple understanding of the two limits.