Orthogonality and completeness for projective representations

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In these notes we will extend a few results about orthogonality and completeness familiar from the theory of linear representations to the case where the representations are projective. I am sure these results (if correct) are in the math literature somewhere, but a quick google didn't turn up anything useful, prompting me to write up the following. Arovas's nice notes on group representation theory are a good reference for recalling how things work in the linear case.

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Let D_μ, D_ν be two unitary irreducible projective[1] representations of a finite group G. We will first prove

Proposition 1. *Let ω be the factor set of two projective irreps μ, ν, and suppose that ω satisfies*

$$\omega(h,g)\omega(g^{-1},h^{-1}) = \omega(h^{-1},h) \qquad \forall g,h \in G. \tag{1}$$

Then the usual orthogonality theorem holds, viz.

$$\int_g [D_\mu(g)]_{ij}[D_\nu(g^{-1})]_{kl} = \delta_{il}\delta_{jk}\delta_{\mu\nu}d_\mu, \tag{2}$$

where d_μ is the dimension of the irrep μ and we have denoted

$$\int_g = \frac{1}{|G|}\sum_{g\in G}. \tag{3}$$

Proof. The proof proceeds essentially as in the non-projective case. We start by finding an intertwiner between μ and ν. This works by averaging over the group, as usual: letting $\mathcal{M}$ be an arbitrary nonzero $d_\mu \times d_\nu$ matrix, we define

$$\mathcal{S} \equiv \int_g D_\mu(g)\mathcal{M}D_\nu(g^{-1}). \tag{4}$$

[1] We will always be interested in the case where the projectivity is a phase, i.e. we will always be interested in the group $H^2(G, U(1))$ (aka the linear representations of extensions of G by $U(1)$).

Picking an arbitrary $h \in G$,

$$\begin{aligned} D_\mu(h)\mathcal{S} &= \int_g D_\mu(h) D_\mu(g) \mathcal{M} D_\nu(g^{-1}) \\ &= \int_g \omega(h,g) D_\mu(hg) \mathcal{M} D_\nu(g^{-1}), \end{aligned} \tag{5}$$

while

$$\begin{aligned} \mathcal{S} D_\nu(h) &= \int_g \omega(g^{-1},h) D_\mu(g) \mathcal{M} D_\nu(g^{-1}h) \\ &= \int_g \omega(g^{-1}h^{-1},h) D_\mu(hg) \mathcal{M} D_\nu(g^{-1}). \end{aligned} \tag{6}$$

We therefore see that if

$$\omega(h,g) = \omega(g^{-1}h^{-1},h) \tag{7}$$

for all h, g, then

$$D_\mu(h)\mathcal{S} = \mathcal{S} D_\nu(h) \qquad \forall\, h \in G. \tag{8}$$

Since $\mathcal{S} \neq 0$ and D_μ, D_ν are assumed irreducible, by Schur's lemma this shows that $\mu = \nu$ (i.e. that D_μ, D_ν are isomorphic).

The above condition on ω can be re-written using $\delta\omega = 1$ and $\omega(g,1) = 1$ in the slightly more symmetric way

$$\omega(h,g)\omega(g^{-1},h^{-1}) = \omega(h^{-1},h) \qquad \forall g, h \in G, \tag{9}$$

as written above. Note that in order for this to be true we must have in particular

$$\omega(g^{-1},g) = 1 \qquad \forall g \in G. \tag{10}$$

</proof>

Note also that a stronger statement might exist, and that I've just missed it.

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We can also make a rather strong statement about the dimensions of projective representations:

Proposition 2. *If G is Abelian, all of the irreducible projective representations with a given factor set ω have the same dimension.*

Proof. Let D_1, D_2 be two irreps with factor set ω, and let $\overline{D}_1$ be the conjugate representation of D_1. Since the factor set of D_1 is $\omega^* = \omega^{-1}$ on account of $\omega \in U(1)$, the representation $\overline{D}_1 \otimes D_2$ is a linear (not projective) representation of G. Since G is assumed Abelian, all of its representations are one-dimensional. Therefore there exists a one-dimensional representation $\mathcal{D}$ of G such that

$$\mathrm{Hom}_G(\mathcal{D}, \overline{D}_1 \otimes D_2) \neq 0. \tag{11}$$

This then means that
$$\mathrm{Hom}_G(\mathcal{D}\otimes D_1, D_2) \neq 0. \tag{12}$$
But as $\mathcal{D}\otimes D_1$ and D_2 are irreducible, by Schur's lemma they must be isomorphic. As $\mathcal{D}$ is one-dimensional, the result follows.

</proof>

This is very different from linear representations, where we e.g. always have the trivial representation (which of course is not projective). This result, when combined with the completeness relation we will prove momentarily, can be used to put strong constraints on the number of projective irreps with a given factor set.

To discuss the completeness relation, we need to define the regular representation:

Definition 1. For a given factor set ω there is a notion of a regular projective representation R, defined explicitly as
$$[R(k)]_{gh} = \omega(g,k)\delta_{h,gk}. \tag{13}$$

Let's check that this definition makes sense. First, the form of the delta function is fixed by requiring that R reduce to the conventional regular representation when ω is trivial. The placement of the arguments of the factor set is seen after some experimentation (idk if there's a better way of deriving it) to be the unique combination that gives R the required projectivity. We check this as:
$$\begin{aligned}\sum_h [R(k)]_{gh}[R(k')]_{hl} &= \sum_h \omega(g,k)\omega(h,k')\delta_{h,gk}\delta_{l,hk'} \\ &= \frac{\omega(g,k)\omega(gk,k')}{\omega(g,kk')}[R(kk')]_{gl} \\ &= \omega(k,k')[R(kk')]_{gl} \quad \checkmark\end{aligned} \tag{14}$$
where in the last line we have used $\delta\omega = 1$.

As is befitting of a regular representation, we have
$$\mathrm{Tr}[R(g)] = |G|\delta_{g,1}, \tag{15}$$
on account of $\omega(g,1) = 1$ for all g.

In the proof of the next theorem we will also see that $R(g)$ decomposes as a $\oplus$ of irreps in the way we expect from linear representations.

Theorem 1. *The usual sum-of-squares law holds for projective representations, viz.*
$$|G| = \sum_{\mu\in\mathrm{Rep}_{\mathrm{G}}^{\omega}} d_\mu^2, \tag{16}$$
where the sum is over all irreps with a given factor set.

Proof. As in the linear case, the proof proceeds by using the decomposition of the regular representation into irreps. First, we use that for any two (not necessarily irreducible) projective representations μ,ν, the inner product of the characters counts the dimension of the intertwining hom space:
$$\int_g \chi_\mu(g)^*\chi_\nu(g) = \dim[\mathrm{Hom}_G(D_\mu, D_\nu)]. \tag{17}$$

This follows from noting that $\overline{D}_\mu \otimes D_\nu$ is a *linear* representation of G. Therefore

$$\Pi_{\bar{\mu}\otimes\nu} = \int_g (\overline{D}_\mu \otimes D_\nu)(g) \tag{18}$$

is a G-equivariant projection (note that if ω is nontrivial then Π_μ is not a projector). In particular, its trace counts the dimension of the space $(V_\mu^* \otimes V_\nu)^G$ (as usual, V_μ are the vector spaces on which the $D_\mu(g)$ act). If we then convert this into a hom space by conjugating the V_μ^*, we see that

$$\mathrm{Tr}[\Pi_{\bar{\mu}\otimes\nu}] = \dim[\mathrm{Hom}_G(D_\mu, D_\nu)], \tag{19}$$

and the statement (17) follows.

We can then use this to show that R splits in the same way that it does for linear representations, viz.

$$R = \bigoplus_{\mu \in \mathrm{Rep}_G^\omega} D_\mu^{\oplus d_\mu}. \tag{20}$$

Indeed, we have

$$\begin{aligned}\dim[\mathrm{Hom}_G(R, D_\mu)] &= \int_g \chi_R(g)^* \chi_\mu(g) \\ &= |G| \int_g \delta_{g,1} \chi_\mu(g) \\ &= d_\mu,\end{aligned} \tag{21}$$

and hence (20) follows.

Now all that is left to do is to take the trace of the decomposition (20) evaluated on the identity:

$$\begin{aligned}\mathrm{Tr}[R(1)] &= \mathrm{Tr}\left[\bigoplus_{\mu \in \mathrm{Rep}_G^\omega} D_\mu^{\oplus d_\mu}\right] \\ &= \sum_\mu d_\mu^2.\end{aligned} \tag{22}$$

Since the LHS is $|G|$, we are done.

</proof>

By combining the above theorem with the previous proposition, we conclude that if G is Abelian then $|G| = N_{irr} d^2$, where d is the dimension of any projective irrep (provided one exists). Therefore we conclude

Proposition 3. *If G is Abelian, then nontrivial projective representations of G can exist only if $|G|$ is a square.*

This result is reasonable given

$$H^2(\mathbb{Z}_q, U(1)) = H^3(\mathbb{Z}_q, \mathbb{Z}) = \mathbb{Z}_1 \tag{23}$$

and[2]

$$H^2(\mathbb{Z}_q^2, U(1)) = \mathbb{Z}_q. \tag{26}$$

Example 1. As an example, consider projective representations of the group $\mathbb{Z}_q^2$. All physicists know one such projective representation, viz.

$$D((1,0)) = Z, \qquad D((0,1)) = X, \tag{27}$$

where $ZX = \zeta_q XZ$ are the $\mathbb{Z}_q$ Pauli matrices. The factor set here is

$$\omega(g,h) = \zeta_q^{g\wedge h}, \tag{28}$$

where $g \wedge h = g_1 h_2 - g_2 h_1$. We then check that

$$\omega(g^{-1}h^{-1}, h) = \zeta_q^{(q-g-h)\wedge h} = \zeta_q^{-g\wedge h} = \zeta_q^{h\wedge g} = \omega(h,g), \tag{29}$$

and as such this rep obeys the conditions needed for the orthogonality relation (2).

We can then use the orthogonality relation to give an alternate proof that this projective rep is in fact irreducible. Indeed, since $X^a Z^b$ is traceless unless a, b are both zero mod q, we have

$$\int_g |\chi(g)|^2 = \int_g \delta_{g,(0,0)} q^2 = 1. \tag{30}$$

Furthermore, completeness tells us that this is in fact the *only* projective irrep. Indeed, its dimension is q and the order of the group is q^2; hence the sum in (16) must be saturated by a single term.

[2] This of course comes from

$$\bigoplus_{p=0}^{n} H^p(G,M) \otimes H^{n-p}(G',M') \to H^n(G\times G', M\otimes M') \to \bigoplus_{p=0}^{n+1} \mathrm{Tor}^{\mathbb{Z}}[H^p(G,M), H^{n-p+1}(G',M')] \tag{24}$$

and

$$H^p(\mathbb{Z}_q, \mathbb{Z}) = \begin{cases} \mathbb{Z}_n & p \in 2\mathbb{Z}^> \\ \mathbb{Z}_1 & p \in 2\mathbb{Z}^> + 1 \\ \mathbb{Z} & p = 0 \end{cases} \tag{25}$$

The LHS of the SES is trivial for $G = G' = \mathbb{Z}_q$ and $M = M' = \mathbb{Z}$, while the RHS is nontrivial at only for the $p = 2$ summand.