Zero mode stuff

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April 30, 2019

On the plane / torus

On the plane / torus, we will work in Landau gauge, where $A_x = 0$ and A_y is a function of x only. Then $\not{D}_A \psi = 0$ reads

$$(\partial_x - k_y + A_y)\psi_L = 0, \qquad (\partial_x + k_y - A_y)\psi_R = 0, \tag{1}$$

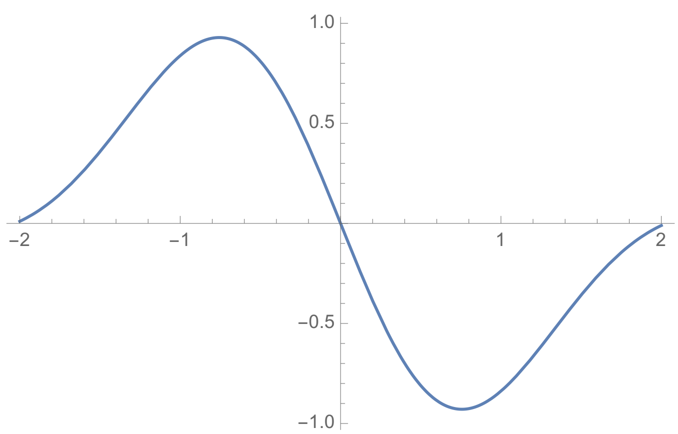
where k_y is the y component of the momentum. For a uniform $U(1)$ flux, we will take $A_y = Bx$, where B is the flux density. For other gauge groups G with $\pi_1(G) \neq 0$, we will take our uniform flux state to be $A_y = A_y^a T^a = BxT^a$ for some particular generator T^a.

Now let us take $G = SO(3)$, with $A_y = BxT^3$. After diagonalizing T^3 from $Y \oplus 0$ to $T^3 = Z \oplus 0$ and writing the spinors in flavor space as (f_1, f_2, f_3) with each f_i a two-component spinor, this gives a left-handed zero mode $(f_L, 0, 0)$ and a right-handed zero mode $(0, f_R, 0)$.

We would like to know whether these zero modes still exist after we perturb with some field strength that does not point uniformly in one direction in flavor space. To do this, consider as an example adding the connection $\widetilde{A} = \widetilde{A}_y dy$, where

$$\widetilde{A}_y = \widetilde{A}_y^2 T^2 = \epsilon \cos(x) e^{-x^2/2} T^2. \tag{2}$$

This has a field strength which has zero integral over the plane[1], and so it is topologically trivial. The field strength as a function of x looks like



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[1]Or torus. If we are on the torus, we take it to be big enough that the usual $e^{-(x-k_y/B)^2 B/2}$ zero mode wavefunctions have support only within width δx which is small compared to the size of the torus, so that the fact that the above wavefunctions are technically speaking not smooth over the torus doesn't really matter.

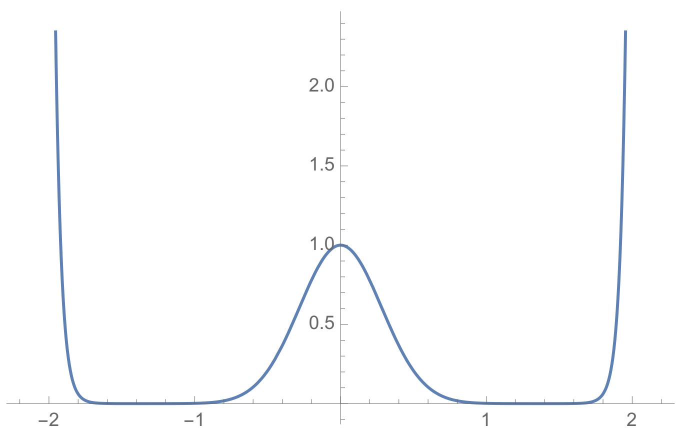
After diagonalizing T^3, the matrix T^2 in flavor space becomes

$$T^2 = \begin{pmatrix} & & i \\ & & \\ -i & & \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} & & i \\ & & i \\ -i & -i & \end{pmatrix}. \tag{4}$$

Now since the fermions are not in a complex representation, we have ind $\not{D}_A = 0$ and we know that there will always be as many left zero modes as right zero modes.[2] Therefore to see whether adding the $\widetilde{A}_y$ term to the connection does anything to $\ker i\not{D}_A$, we can focus wolog on a certain chirality, which we will take to be L for definiteness. Therefore we are interested in whether we can find normalizable solutions to the following equations (setting $k_y = 0$ for simplicity)

$$\begin{aligned} (\partial_x + Bx)f_1 + i\frac{\widetilde{A}_y}{\sqrt{2}} f_3 &= 0 \\ (\partial_x - Bx)f_2 + i\frac{\widetilde{A}_y}{\sqrt{2}} f_3 &= 0 \\ \partial_x f_3 - i\frac{\widetilde{A}_y}{\sqrt{2}}(f_1 + f_2) &= 0. \end{aligned} \tag{6}$$

If $\widetilde{A}_y = 0$ then we just take $f_2 = f_3 = 0$, and let f_1 be the usual harmonic oscillator solution. However, if $\widetilde{A}_y \neq 0$, this is not possible: the last equation means that either f_2 or f_3 must be nonzero if f_1 is nonzero, and the second equation then ensures that in fact $f_2 \neq 0$. Now f_2 is the mode that doesn't have a normalizable solution when $\widetilde{A}_y = 0$, and so we might expect that the $\widetilde{A}_y$ coupling ruins the normalizability of the solution. Indeed, this is what appears to happen: using the form of $\widetilde{A}_y$ above with $B = 2\pi$ and $\epsilon = .1\sqrt{2}$, a plot of the magnitude $\sum_i f_i^* f_i$ as a function of x shows a divergence:



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[2] An operator that provides the (pseudo)real structure here is $\mathcal{K}(Y \otimes \mathbf{1})$, where the first tensor factor is for the spin indices and the second is for the gauge indices, and where $\mathcal{K}$ is complex conjugation. Indeed, using X and Y as the γ matrices, in a basis where the gauge generator matrices are purely imaginary and antisymmetric,

$$[i\not{D}_A, \mathcal{K}(Y \otimes \mathbf{1})] = 0, \tag{5}$$

and so $\mathcal{K}(Y \otimes \mathbf{1})$ provides a way to take a zero mode of a certain chirality and construct another zero mode with opposite chirality.

There of course may be something I've missed, or some tricky choice of initial conditions (the above plot was for $f_1(0) = 1, f_2(0) = f_3(0) = 0$; modifying the latter two to be nonzero makes the divergence worse) that allow this divergence to be avoided, but for now it seems to be a generic consequence of taking $\widetilde{A} \neq 0$.

On the sphere

First let's figure out how to get the proper expression for the Dirac equation on the sphere. In what follows, we will be using veilbeins, since that's the only method we have for dealing with fermions on curved spaces.[3] Recall that the veilbeins are found by taking the square root of the metric:

$$g_{\mu\nu} = e^a_\mu e^a_\nu, \qquad e^a_\mu e^b_\nu g^{\mu\nu} = \delta^{ab}. \tag{8}$$

Since we don't want to constantly be phantoming when writing stuff out, our convention will be that, when viewing the veilbeins as a matrix, the greek (spacetime) letter will always denote the row index of the matrix, and the roman (internal space) letter will always denote the column index. When we break apart the metric like this, we incur a gauge redundancy, since the transformation $e^a_\mu \mapsto [O]^a_b e^b_\mu$ for $O \in O(s,t)$ leaves the splitting $g_{\mu\nu} = e^a_\mu e^a_\nu$ invariant (in what follows we will only be concerned with 2+0 dimensions, so that the relevant "gauge group" is $O(2)$).

The Dirac operator is (roman indices can be raised / lowered with impunity)

$$\not{D}_A = \gamma_a e^{\mu a}(\partial_\mu + i(\omega_\mu + A_\mu)), \tag{9}$$

where ω, A are the spin and gauge connections, with

$${\omega_\mu}^a{}_b = e^a_\nu \partial_\mu e^\nu_b + e^a_\nu \Gamma^\nu_{\mu\lambda} e^\lambda_b. \tag{10}$$

We've tried to take a sign convention that is maximally simple; ours differs from the conventions in many other places though so be careful. The spin connection is needed to ensure that $\not{D}_A\psi$ transforms covariantly under local $O(2)$ gauge rotations of the coordinate frames.[4]

Since the generators of Spin(d) are $-i[\gamma_a, \gamma_b]/4$, the spin connection is, quite generally,

$$\omega_\mu = \frac{1}{2}\omega^{ab}_\mu \Sigma_{ab}, \qquad \Sigma_{ab} = \frac{-i}{4}[\gamma_a, \gamma_b]. \tag{15}$$

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[3]This is because fermion actions need γ^a matrices to be defined, which represent Clifford algebras. We want to represent a Clifford algebra with the relation $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ (or maybe η^{ab}), and definitely don't want to have the anticommutator be equal to $g^{ab}(x)$; this would be a mess. Thus we need veilbeins to switch between spacetime and a frame in which the Clifford generators can be defined.

[4]It is also needed to ensure that the (real-time) action is Hermitian! The added spin-connection term is actually not Hermitian, and this compensates for the non-Hermiticity of $\bar{\psi}(i\not{\partial})\psi$ when working on a curved manifold. Indeed, under Hermitian conjugation,

$$\dagger : i\bar{\psi}e^{a\mu}\gamma_a\partial_\mu\psi \mapsto i\bar{\psi}e^{a\mu}\gamma_a\partial_\mu\psi + i\bar{\psi}(\partial_\mu e^{a\mu})\gamma_a\psi. \tag{11}$$

Now let's look at the spin connection part. For simplicity, we will work in Riemann normal coordinates around a certain point p, where the Christoffel symbols (but not their derivatives) can be chosen to vanish.

The veilbeins for spherical coordinates on the unit S^2 are easy to write down:

$$e^a_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \sin\theta \end{pmatrix}^a_{\ \mu}, \qquad e^{\mu a} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-1}\theta \end{pmatrix}^{\mu a}. \tag{16}$$

Here the fact that the tetrads are the "square root of the metric" is made manifest. Of course, there are infinitely many other choices, related by local $O(2)$ transformations.

To get the spin connection, we will need to know that the nonzero Christoffel symbols on the sphere are

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta. \tag{17}$$

Then we calculate

$$\omega^{ab}_\theta = 0, \qquad \omega^{ab}_\phi = \cos\theta J^{ab}. \tag{18}$$

Then, if we adopt the gamma matrices $\gamma^1 = X, \gamma^2 = Y$ (this is the best choice since it makes the splitting $S = S_+ \oplus S_-$ manifest), we get

$$\omega_\theta = 0, \qquad \omega_\phi = \frac{1}{2}\cos\theta J^{ab}\Sigma_{ab} = -\frac{\cos\theta}{2}Z. \tag{19}$$

Now we need to make a choice for the gauge connection. For a $U(1)$ monopole of flux n we will make the choice

$$A^{N/S} = n\frac{\pm 1 - \cos\theta}{2}d\phi, \tag{20}$$

which gives $\int_{S^2} F = 2\pi n$. Note how similar the forms of the gauge and spin connections are! For a gauge group with $\pi_1(G) \neq 0$, the simplest choice for a monopole field is the above but

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The spin connection part of the Lagrangian density at this point is then

$$\begin{aligned}\mathcal{L} \ni -\frac{1}{2}\sum_{a,b\neq c}\psi^\dagger\gamma^0\gamma_a e^{a\mu}\omega^{bc}_\mu\Sigma_{bc}\psi &= \frac{i}{2}\sum_{a,b\neq c}\psi^\dagger\gamma^0\gamma_a e^{a\mu}(e^b_\nu\partial_\mu e^{c\nu})\gamma_b\gamma_c\psi \\ = \sum_{a\neq b\neq c}\frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_a\gamma_b\gamma_c\psi &+ \sum_{a\neq c}\frac{i}{4}\bar\psi e^{a\mu}(e^a_\nu\partial_\mu e^{\nu c} - e^c_\nu\partial_\mu e^{\nu a})\gamma_c\psi \\ = \sum_{a\neq b\neq c}\frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_a\gamma_b\gamma_c\psi &+ \frac{i}{2}\sum_c\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi\end{aligned} \tag{12}$$

The first term here is Hermitian: using $\gamma_a^\dagger\gamma_0^\dagger = \gamma_0\gamma_a$ (we are in $\mathbb{R}$ time, remember), we have

$$\dagger : \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\psi \mapsto -\frac{i}{4}\psi^\dagger e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}(\gamma_c^\dagger\gamma_b^\dagger\gamma_a^\dagger)\gamma_0^\dagger = \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\gamma_c\gamma_b\gamma_a = \frac{i}{4}\bar\psi e^{a\mu}e^b_\nu\partial_\mu e^{\nu c}\psi, \tag{13}$$

since here $a \neq b \neq c$.

However, the second term at the end of (12) is actually anti-Hermitian:

$$\dagger : \frac{i}{2}\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi \mapsto -\frac{i}{2}\bar\psi\partial_\nu e^{\nu c}\gamma_c\psi. \tag{14}$$

However, when we add in the $+i\bar\psi(\partial_\mu e^{a\mu})\gamma_a\psi$ from (11), we see that it combines with the RHS of the above equation to yield the LHS, giving an action that is Hermitian. Thus the second term at the end of (12) is a counterterm that ensures that the full action is Hermitian.

with a T^a tacked on, where T^a is a particular (Hermitian) generator of $\mathfrak{g}$. Putting the two connections together, the covariant derivatives are

$$\nabla_\theta = \partial_\theta, \qquad \nabla_\phi = \partial_\phi - \frac{iZ \otimes \mathbf{1}}{2}\cos\theta + in\frac{\pm 1 - \cos\theta}{2}\mathbf{1} \otimes T^a, \tag{21}$$

where the first tensor factor is the spin indices and the second is the gauge indices (we won't bother to explicitly write the $\otimes$ in what follows).

We can now finally write down the expression for $i\not{D}_A\psi = 0$, which is, for a uniform monopole field,

$$\not{D}_A\psi^{(N/S)} = \left[X\left(\partial_\theta + \frac{\cot\theta}{2}\right) + Y\csc\theta\left(\partial_\phi + in\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\right]\psi^{(N/S)} = 0 \tag{22}$$

or written out in chiral components,

$$\begin{aligned} \left(\partial_\theta + \frac{\cot\theta}{2} - i\csc\theta\partial_\phi + n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\psi_R^{(N/S)} &= 0 \\ \left(\partial_\theta + \frac{\cot\theta}{2} + i\csc\theta\partial_\phi - n\csc\theta\left(\frac{\pm 1 - \cos\theta}{2}\right)T^a\right)\psi_L^{(N/S)} &= 0 \end{aligned} \tag{23}$$

In what follows, we will take $n = 1$ for concreteness. Then in the $U(1)$ case, we see that we get a single R zero mode, $\psi_R = e^{-i\phi/2}$ (the reason we get an R zero mode and not an L one is because of our sign conventions for the covariant derivative). Also note that this zero mode actually has spin zero; to see this one needs to properly calculate the angular momentum generators, which I won't go into here.

Now for $SO(3)$, if we let the field strength point in the T^3 direction, we see that we get a single L and a single R zero mode, as expected. Do these zero modes survive when a perturbation is added? Let us add the potential

$$\widetilde{A} = \widetilde{A}^2_\phi T^2\, d\phi = \epsilon\sin(2\theta)T^2\, d\phi. \tag{24}$$

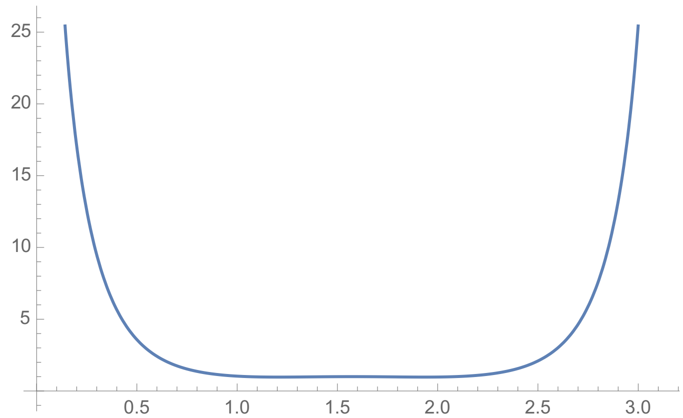
This is well-defined on the sphere since $\widetilde{A}(\theta = 0, \pi) = 0$, and it is topologically trivial since $\int_{S^2} d\widetilde{A} = 0$. As mentioned before, since the zero modes for real gauge groups always come in left-right pairs, we can focus on a single handedness (we will look at R) wolog. We therefore want to find normalizable solutions to (working on the N coordinate patch)

$$\begin{aligned} \left(\partial_\theta - i\csc\theta\partial_\phi + \frac{1}{2}\csc\theta\right)f_1 + i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}f_3 &= 0 \\ \left(\partial_\theta + \cot\theta - i\csc\theta\partial_\phi - \frac{1}{2}\csc\theta\right)f_2 + i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}f_3 &= 0 \\ \left(\partial_\theta + \frac{1}{2}\cot\theta - i\csc\theta\partial_\phi\right)f_3 - i\epsilon\frac{\csc(\theta)\sin(2\theta)}{\sqrt{2}}(f_1 + f_2) &= 0 \end{aligned} \tag{25}$$

When $\epsilon = 0$ we just take $f_1 = e^{-i\phi/2}, f_2 = f_3 = 0$.[5] When $\epsilon \neq 0$ the coupling between the different modes kicks in, and as in the planar case we seem to run into normalizability

[5] Here the f_3 mode has no normalizable solution when $\epsilon = 0$, since taking the ϕ dependence to be trivial

problems caused by the troublesome modes f_2, f_3 being forced to be nonzero. For example, set $\epsilon = \sqrt{2}$. The natural choices for the ϕ dependence of the three modes is $f_1 \propto e^{-i\phi/2}$, $f_2 \propto e^{i\phi/2}$, and with f_3 having no ϕ dependence. With these assignments of ϕ dependence, the volume-element-normalized magnitude $\sum_i f_i^* f_i \sin\theta$ as a function of θ looks like



(28)

So, it blows up at the poles, and we don't get a legit zero mode solution. This seems to be the generic behavior for any $\epsilon \neq 0$.

means

$$f_3 \propto \frac{1}{\sqrt{\sin\theta}}, \tag{26}$$

which is not acceptable (it integrates to something finite because of the $\sin\theta$ in the measure, but it is not differentible). Similarly the f_2 mode has to be zero, since otherwise we have

$$f_2 \propto e^{i\phi/2}\csc(\theta), \tag{27}$$

which is also no good.