

Physics diary

Ethan Lake

June 28, 2019

These notes contain a physics diary of sorts that I kept during 2018, starting in the early summer. There's a mix of textbook problems, elaborations on calculations done in various papers, and original mini research projects. Disclaimer: I only know that my answers are correct for $\sim 2/3$ of the problems, and of course even correct in spirit answers are bound to have a nonzero number of typos / mistakes.

A few problems are listed in multiple section headings as appropriate.

1 Condensed matter problems

17 May 1 — Global properties of KW duality	48
18 May 2 — Staggered Hubbard model at half-filling and criticality on Mott lobes	51
20 May 4 — Dispersion of excitations in \mathbb{Z}_2 gauge theory	56
28 May 11 — Mixed t Hooft anomaly for the compact boson / Luttinger liquid	70
29 May 12 — More on mixed t Hooft anomalies for free fields	74
30 May 13 — The simplest t Hooft anomaly	78
31 May 14 — TIs and fermions in three dimensions	82
32 May 15 — Simple example of Anderson localization	84
34 May 17 — Energies of dislocations and disclinations in two dimensional solids unfin- ished	94
36 May 19 — Functional RG equations	98
37 May 20 — Practice with transfer matrix	100
38 May 21 — Basic setup of elasticity theory	102
39 May 22 — Classical $O(n)$ model in one dimension	106
45 May 28 — Fun with two dimensional bosonization, the Schwinger model, and θ angles	124

48	<i>May 31 — RG for the one dimensional Ising model</i>	138
49	<i>June 1 — $SU(N)$ WZW basics</i>	140
50	<i>June 2 — RG in the $SU(N)$ WZW model</i>	143
51	<i>June 3 — Functional bosonization</i>	146
55	<i>June 8 — Landau parameters and mass renormalization with magnetization</i>	167
59	<i>June 12 — RPA basics</i>	182
60	<i>June 13 — Photon-atom interactions and stimulated emission</i>	184
61	<i>June 14 — Practice with the Kubo formula and electrical conductivity</i>	186
62	<i>June 15 — AFM spin waves and itinerant magnetism</i>	188
63	<i>June 16 — Kondo problem intro</i>	190
64	<i>June 17 — More about the Kondo problem and doing RG with the SW transformation</i>	193
65	<i>June 18 — Beta function in the Kondo problem to 2-loop order</i>	196
67	<i>June 20 — Slave bosons for the Hubbard model unfinished</i>	200
68	<i>June 21 — WZW Miscellanea</i>	202
72	<i>June 25 — QED_3 as a σ model</i>	212
73	<i>June 26 — Number of zero modes allowed by magnetic fluxes in two dimensions</i>	217
74	<i>June 27 — Lasagna models and how to make SPTs</i>	219
95	<i>July 18 — Bosonizing the spin 1/2 chain and the sine-Gordon model</i>	288
96	<i>July 19 — The spin 1/2 chain and the LSM theorem</i>	291
99	<i>August 3 — unfinished Thirring model beta function</i>	299
113	<i>August 16 — Duality in the Ising model</i>	342
126	<i>October 17 — The n-state Potts model</i>	405
127	<i>October 18 — Symmetries of the Hubbard model</i>	410
129	<i>November 3 — SF Mott insulator transition with 2d compact boson</i>	416
130	<i>November 4 — Weakly interacting bosons</i>	418

131	<i>November 5 — Tower of states</i>	421
134	<i>November 8 — Bosonization, the XY chain, and different types of Mott insulators as SPTs</i>	426
137	<i>November 29 — Yet more on the $SU(2)$ point of the $c = 1$ CFT</i>	438
138	<i>November 30 — Specific heats for fermions and bosons</i>	440
139	<i>December 1 — Mass terms in bosonization and FQH from coupled wires</i>	442
140	<i>December 2 — MFT for Mott insulators and SFs in two dimensions</i>	443
141	<i>December 3 — Topological robustness of Fermi surfaces</i>	446
146	<i>December 12 — RG for fermions a la Shankar</i>	458

2 Conformal field theory problems

75	<i>June 29 — More WZW things</i>	224
78	<i>July 2 — Magic at the $SU(2) \times SU(2)$ radius</i>	235
80	<i>July 4 — T is not a conformal primary when $c \neq 0$, and the Hamiltonian on the cylinder</i>	243
81	<i>July 5 — Linear dilaton CFT</i>	245
82	<i>July 6 — Bosonization and torus partition functions</i>	249
83	<i>July 7 — Vertex correlators</i>	254
84	<i>July 7 — Vertex correlators II</i>	258
85	<i>July 8 — Vacuum energy and boundary conditions</i>	259
86	<i>July 9 — Fermion partition functions on the torus: the functional approach</i>	261
87	<i>July 10 — Orbifolding basics</i>	264
88	<i>July 11 — Cardy formula</i>	267
89	<i>July 12 — Alternate route to WZW central charge</i>	268
90	<i>July 13 — Free fermion representation of $\widehat{\mathfrak{so}}(N)_g$ current algebra</i>	273
91	<i>July 14 — More free fermion representations of current algebras (unfinished)</i>	274
92	<i>July 15 — Two more interpretations of the central charge</i>	275

94	<i>July 17 — Another look at currents and operator splitting applied to bosonization</i>	285
102	<i>August 6 — β functions from the OPE and perturbations of CFTs</i>	306
113	<i>August 16 — Duality in the Ising model</i>	342

3 Math problems

23	<i>May 7 — Torsion-free first cohomology</i>	62
26	<i>May 10 — Orientability and the fundamental group</i>	67
27	<i>May 10 — Fundamental groups and homotopies into S^1</i>	68
30	<i>May 13 — The simplest t Hooft anomaly</i>	78
35	<i>May 18 — Cohomology of finite groups with $U(1)$ and \mathbb{Z} coefficients</i>	96
40	<i>May 23 — Path integrals for Majorana fermions in one dimension and spin structures</i>	108
42	<i>May 25 — BF theory basics</i>	115
43	<i>May 26 — Writhe and linking numbers</i>	120
46	<i>May 29 — Careful explanation of CS level quantization</i>	130
52	<i>June 4 — The Hopf map</i>	151
53	<i>June 5 — Reminder about Stiefel-Whitney classes</i>	153
66	<i>June 19 — Level repulsion for kindergarteners</i>	199
79	<i>July 3 — Verlinde formula for finite groups</i>	238
97	<i>August 1 — WZ consistency conditions and descent equations</i>	295
103	<i>August 6 — Orbifolding orbifolds and gauging higher form discrete symmetries</i>	311
114	<i>August 17 and 18 — Let's learn about Lie algebras and weights</i>	349
117	<i>August 25 — Compendium of various things relating to Chern classes / characters and (fractional) instanton numbers</i>	374
132	<i>November 6 — Vandermonde determinant and matrix integrals</i>	423

4 (QFT \ CFT) problems

7	<i>April 20 — The Gross-Neveu model</i>	8
----------	---	---

8	<i>April 21 — The Coleman-Weinberg potential</i>	10
9	<i>April 22 — The $\mathbb{C}P^N$ model and emergent electromagnetism</i>	17
10	<i>April 23 — ϕ^4 theory coupled to fermions and a natural relation</i>	22
11	<i>April 24 — The chiral anomaly in 2 dimensions and the fermion bubble</i>	24
13	<i>April 26 — Dynamic generation of topological photon mass and domain wall anomalies</i>	30
14	<i>April 27 - 28 — β function for sine-Gordon in 2+1 dimensions at finite temperature and the KT transition for $U(1)$ gauge theory</i>	33
15	<i>April 29 — β function for scalars coupled to a non-Abelian gauge field</i>	43
16	<i>April 30 — Checking the chiral anomaly for non-Abelian gauge theory with operator relations</i>	46
19	<i>May 3 — Pion decay</i>	54
21	<i>May 5 — Charge quantization and the weak mixing angle confusion</i>	57
22	<i>May 6 — Anomaly constraints in $SU(3) \times SU(2) \times U(1)$</i>	59
24	<i>May 8 — Chiral anomaly via Pauli-Villars</i>	62
25	<i>May 9 — Trace anomaly via Pauli-Villars</i>	64
28	<i>May 11 — Mixed t Hooft anomaly for the compact boson / Luttinger liquid</i>	70
29	<i>May 12 — More on mixed t Hooft anomalies for free fields</i>	74
30	<i>May 13 — The simplest t Hooft anomaly</i>	78
31	<i>May 14 — TIs and fermions in three dimensions</i>	82
36	<i>May 19 — Functional RG equations</i>	98
40	<i>May 23 — Path integrals for Majorana fermions in one dimension and spin structures</i>	108
41	<i>May 24 — Anomalous dimension in ϕ^4 theory</i>	114
42	<i>May 25 — BF theory basics</i>	115
44	<i>May 27 — σ model in two dimensions</i>	122
45	<i>May 28 — Fun with two dimensional bosonization, the Schwinger model, and θ angles</i>	124
46	<i>May 29 — Careful explanation of CS level quantization</i>	130

47	<i>May 30 — RG and the nonlinear σ model</i>	135
49	<i>June 1 — $SU(N)$ WZW basics</i>	140
50	<i>June 2 — RG in the $SU(N)$ WZW model</i>	143
51	<i>June 3 — Functional bosonization</i>	146
54	<i>June 6 & 7 — Even more on the $O(N)$ and nonlinear σ models in two dimensions</i>	159
65	<i>June 18 — Beta function in the Kondo problem to 2-loop order</i>	196
68	<i>June 21 — WZW Miscellanea</i>	202
69	<i>June 22 — WZW action from fermions</i>	206
70	<i>June 23 — Fractional charge of fermion mass solitons in two dimensions</i>	208
71	<i>June 24 — Fermion nonconservation and the ABJ anomaly</i>	210
72	<i>June 25 — QED_3 as a σ model</i>	212
75	<i>June 29 — More WZW things</i>	224
76	<i>June 30 — Quantization for Chern-Simons</i>	228
77	<i>July 1 — Chern-Simons Propagator</i>	233
93	<i>July 16 — β function for general two-dimensional nonlinear σ models and Ricci flow</i>	280
95	<i>July 18 — Bosonizing the spin 1/2 chain and the sine-Gordon model</i>	288
97	<i>August 1 — WZ consistency conditions and descent equations</i>	295
98	<i>August 2 — Asymptotic symmetry in an RG flow</i>	297
100	<i>August 4 — Canonical momenta and commutation relations in bosonization</i>	300
101	<i>August 5 — Chiral almost-symmetry-breaking in two dimensions</i>	302
103	<i>August 6 — Orbifolding orbifolds and gauging higher form discrete symmetries</i>	311
104	<i>August 7 — Cooler proof of Goldstone's theorem for p-form symmetries</i>	316
105	<i>August 8 — Some stuff about gauging higher symmetries</i>	318
106	<i>August 9 — Maxwell in two dimensions</i>	320
107	<i>August 10 — Anomalies and current OPEs</i>	322

108	<i>August 11 — Subsystem symmetries and their Elitzur's theorem</i>	326
109	<i>August 12 — Alternate approach to Wilson line expectation values in Chern-Simons</i>	327
110	<i>August 13 — Comments on discrete gauge theories, dualities, and global symmetries</i>	329
111	<i>August 14 — The Witten effect</i>	335
112	<i>August 15 — More on the Schwinger model, its phases, and the theta term</i>	340
113	<i>August 16 — Duality in the Ising model</i>	342
114	<i>August 17 and 18 — Let's learn about Lie algebras and weights</i>	349
115	<i>August 19 — Allowed spectrum of charges in Abelian and non-Abelian gauge theory</i>	359
116	<i>August 20, 21 — Higher symmetries in non-Abelian gauge theories</i>	366
117	<i>August 25 — Compendium of various things relating to Chern classes / characters and (fractional) instanton numbers</i>	374
118	<i>August 26 — Dyon spin, statistics, and statistical transmutation from θ angles</i>	384
119	<i>August 30 — unfinished More on discrete θ angles</i>	394
120	<i>September 2 — Basics of large N Yang-Mills</i>	395
121	<i>September 2 — unfinished More on gauging higher symmetries</i>	399
122	<i>September 5 — Inequivalent gauge fields with same field strength in non-Abelian gauge theory (unfinished)</i>	399
128	<i>October 19 — Procession of Mercury's orbit from EFT</i>	414
136	<i>November 12 — Large N matrix model quantum mechanics, eigenvalue distributions,</i>	434
142	<i>December 7 — Quantization in AdS</i>	448
143	<i>December 8 — AdS Propagators</i>	451
144	<i>December 9 — (Massive) Vectors in AdS</i>	454
145	<i>December 9 — Wilson Loop Vevs in $\mathcal{N} = 4$ SYM using AdS/CFT</i>	456

5 Quantum information problems

33	<i>May 16 — Information theory things and modular flow in mega-simple qubit systems</i>	85
56	<i>June 9 — Algebraic entanglement entropy reminder</i>	170

57	<i>June 10 — Entanglement for a harmonic oscillator</i>	174
58	<i>June 11 — Intro to thermofield dynamics</i>	177

6 Miscellanea

123	<i>September 30 — Acceleration along Killing vectors</i>	400
124	<i>October 15 — Kerr-Newmann metric</i>	401
125	<i>October 16 — Basic black hole thermodynamics</i>	403
128	<i>October 19 — Precession of Mercury's orbit from EFT</i>	414
132	<i>November 6 — Vandermonde determinant and matrix integrals</i>	423
133	<i>November 7 — Baby example of RN metric and D branes</i>	425
135	<i>November 9 — Learning about AdS space</i>	432
136	<i>November 12 — Large N matrix model quantum mechanics, eigenvalue distributions,</i>	434

7 April 20 — The Gross-Neveu model

Consider the Gross-Neveu model, which is a theory of N Dirac fermions in 2 spacetime dimensions, interacting through a quartic term:

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - g^2(\bar{\psi}_i\psi^i)^2, \quad (1)$$

with $i = 1, \dots, N$. Show that a dynamical mass term (which spontaneously breaks the chiral symmetry) for the fermions is generated by dimensional transmutation at one-loop order. Note that the chiral symmetry here is just a discrete \mathbb{Z}_2 symmetry which acts as,

$$\psi_i \mapsto \bar{\gamma}\psi_i, \quad (2)$$

and sends $\bar{\psi}_i\psi^i \mapsto -\bar{\psi}_i\psi^i$ (here $\bar{\gamma}$ has alias γ^5). In the limit $N \rightarrow \infty$, Ng^2 fixed, show that the one-loop result is exact, resulting in a mass term which is non-analytic in g^2 , namely $m_{\text{eff}} = \Lambda e^{-\pi/(Ng^2)}$ (think of QCD and BCS theory).

Solution: In two dimensions, we write represent the Clifford algebra with $\gamma^0 = iY, \gamma^1 = X$, so that the chirality operator is $\bar{\gamma} = \gamma^0\gamma^1 = Z$. In the absence of a $\bar{\psi}_i\psi^i$ mass term, we have the \mathbb{Z}_2 chiral symmetry $\psi \mapsto \bar{\gamma}\psi$, while a mass term breaks the symmetry explicitly.

To start, we decouple the fermion interaction in the usual way. Letting the decoupling HS field be called σ , the decoupled partition function is

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp \left(i \int (i\bar{\psi}\not{\partial}\psi - \sigma\bar{\psi}\psi - \frac{1}{2g^2}\sigma^2) \right). \quad (3)$$

Now we integrate out the fermions to get an effective action for σ . Since there are N copies of fermions we get $\det^N(i\cancel{\partial} - \sigma)$, and so

$$Z = \int \mathcal{D}\sigma e^{iS_{\text{eff}}[\sigma]}, \quad (4)$$

where

$$S_{\text{eff}}[\sigma] = -iN \ln \det(i\cancel{\partial} - \sigma) - \frac{1}{2g^2} \int \sigma^2. \quad (5)$$

To evaluate the $\ln \det$, we go to momentum space in which $i\cancel{\partial} - \sigma$ is block-diagonalized, where each block is labeled by a different value of the momentum. One can think of the operator as becoming $\bigoplus_k (\cancel{k} - \sigma)$. Since $\det(\bigoplus_\alpha A_\alpha) = \det \prod_\alpha (A_\alpha \otimes \mathbf{1}_{\bar{\alpha}}) = \prod_\alpha \det A_\alpha$, we write

$$\det(i\cancel{\partial} - \sigma) = \prod_k \det(\cancel{k} - \sigma). \quad (6)$$

Evaluating the determinant with the form of the γ matrices given above gives

$$\ln \det(i\cancel{\partial} - \sigma) = iV \int_k \ln(\sigma^2 + k^2), \quad (7)$$

where we've rotated into Euclidean space, and V is the volume of the two-dimensional space-time. We do the integral with the “replica trick”, i.e. by using $\ln \rho = -\partial_n \rho^{-n}|_{n \rightarrow 0}$. Doing the integral (see e.g. P&S), one gets

$$\begin{aligned} \ln \det(i\cancel{\partial} - \sigma) &= -iV \lim_{n \rightarrow 0} \partial_n \int_k \frac{1}{(\sigma^2 + k^2)^n} \\ &= -iV \Gamma(-d/2) \left(\frac{\sigma^2}{4\pi} \right)^{d/2}, \end{aligned} \quad (8)$$

where we need to take the $d \rightarrow 2$ limit. For $d = 2 + \epsilon$ we can use the expansion

$$\Gamma(-d/2) \approx \frac{2}{\epsilon} + \gamma. \quad (9)$$

Then taking $\epsilon \rightarrow 0$, we get

$$\ln \det(i\cancel{\partial} - \sigma) = -iV \frac{\sigma^2}{4\pi} \left(\frac{2}{\epsilon} + \gamma - \ln(4\pi) + \ln(\sigma^2/\Lambda^2) \right) \quad (10)$$

for some renormalization scale Λ . We choose the counterterm for $1/g^2$ in the original Lagrangian to kill the $1/\epsilon$ terms and the constants. Then the effective potential is

$$\begin{aligned} V_{\text{eff}}[\sigma] &= -\frac{1}{V} \int [\mathcal{L} + \mathcal{L}_{ct} - iN \ln \det(i\cancel{\partial} - \sigma)] \\ &= \frac{1}{2g^2} \sigma^2 + \frac{\sigma^2}{4\pi} N \ln(\sigma^2/\Lambda^2). \end{aligned} \quad (11)$$

With this renormalization scheme we can calculate the β function, which we do by requiring that the physical effective potential be independent of the renormalization scale Λ . We get

$$\beta(g) = \frac{dg}{d \ln \Lambda} = -\frac{N}{2\pi} g^3, \quad (12)$$

which tells us that the theory is asymptotically free: as we go to higher Λ scales, the theory becomes increasingly weakly coupled.

In any case, we can now minimize the effective potential to see if σ gets a vev. Doing this yields

$$\sigma^2 = \Lambda^2 e^{-\frac{2\pi}{g^2 N}} \neq 0. \quad (13)$$

Thus at 1-loop order, σ gets a vev and leads to the chiral symmetry being spontaneously broken as a result of the fermion condensate which forms. Note that the expression for the mass of the fermions has exactly the same form as the expression for the mass gap in a superconductor: it is non-analytic in the coupling, and linearly proportional to the cutoff (debeye frequency). Also note that we have two solutions for σ , consistent with the fact that the symmetry that is being spontaneously broken is a \mathbb{Z}_2 symmetry.

This result is exact in the large N (but fixed $g^2 N$) limit, because upon sending $N \rightarrow \infty$ we see that the saddle point becomes a better and better approximation due to the factor of N in front of the $\ln \det$ term. One can also check that higher-loop corrections to the effective action are suppressed in powers of $1/\sqrt{N}$.

We can also check this by doing a saddle point analysis on the effective action for σ . In the large N limit, we expect this to be exact. Varying the effective action with respect to σ and setting the result equal to zero gives

$$\frac{\sigma(x)}{g^2} = -iN \frac{\delta}{\delta\sigma(x)} \ln \det(i\partial^\mu - \sigma). \quad (14)$$

We use $\delta \ln A = A^{-1} \delta A$ and take the determinant over the spin indices explicitly to get

$$\frac{\sigma(x)}{g^2} = -2iN \int_{k=0}^{\Lambda} \langle x | \frac{\sigma}{k_\mu k^\mu + \sigma^2} | x \rangle \quad (15)$$

where Λ is a cutoff. Going to Euclidean signature to do the integral over momentum gives

$$\frac{\sigma}{g^2} = \frac{N}{2\pi} \ln \left(\frac{\Lambda^2 + \sigma^2}{\sigma^2} \right), \quad (16)$$

which when solved leads to exactly the same vev for σ that we derived using the minimal subtraction renormalization scheme.

8 April 21 — The Coleman-Weinberg potential

This is one of the final projects in P&S. Consider scalar QED in four dimensions with a Mexican hat potential:

$$\mathcal{L} = -\frac{1}{2}F \wedge \star F + |D_A \phi|^2 + \frac{1}{2}\mu^2|\phi|^2 - \frac{\lambda}{6}|\phi|^4, \quad (17)$$

where our sign conventions are $(D_A \phi)_\mu = (\partial_\mu - ieA_\mu)\phi$. There are several things to do:

- Compute the effective potential for ϕ to one-loop order. You should find that even at small values of $\mu^2 < 0$, where classically there is only one minimum, spontaneous symmetry breaking occurs for small e and small λ .
- Find the β functions for e and for λ .
- Use these to evaluate the ratio m_σ^2/m_A^2 , where m_σ is the mass of the fluctuations about the minimum of the potential and m_A is the mass of the Higgsed photon. Show that after taking into account the RG flow, there is still a symmetry-breaking vev for ϕ even when μ^2 is negative, provided that $|\mu|$ is small.

This problem is pretty long, and it would be rather onerous to type out all the algebra so, I'll be a bit laconic in places.

Effective potential: To get the effective potential, we need to integrate out the fluctuations in the ϕ field, as well as integrate out the gauge field. Following P&S we parametrize ϕ as

$$\phi = v + \frac{1}{\sqrt{2}}(\sigma + i\pi), \quad (18)$$

where σ, π are real scalars that we will integrate out.

To one-loop order, the effective action is determined by the terms second order in the fluctuations. The relevant parts of the action are

$$S = \frac{1}{2} \int \left((\partial_\mu \pi \partial^\mu \pi + \partial_\mu \sigma \partial^\mu \sigma + e^2 A^2 (\pi^2 + \sigma^2) + 2e^2 v^2 A^2 + 2\sqrt{2} e^2 A^2 v \sigma - (\lambda v^2/3 - \mu^2) \pi^2 - (\lambda v^2 - \mu^2) \sigma^2) \right). \quad (19)$$

The one-loop diagrams contributing to $V_{\text{eff}}(\phi)$ all have the same form: the $2n$ -th order contribution to the effective potential is a graph with $2n$ external ϕ legs and a single loop connecting them (remember that the n -th order contribution to Γ are the n -point 1PI diagrams). Because of the interactions in the action, this loop can be a π loop, a σ loop, or an A loop, but there are no graphs with different types of fields running around the loop. This means that for the purpose of getting V_{eff} , we can ignore the interactions between the fields and just treat them as separate free fields.

In the transverse gauge $d^\dagger A = 0$, the gauge field contribution is

$$\frac{1}{2} A_\mu (g^{\mu\nu} \partial^2 + 2e^2 v^2) A_\nu. \quad (20)$$

This is a massive vector field, which due to the transverse constraint behaves just like three scalars, giving us a factor of

$$\int \mathcal{D}A \rightarrow [\det(\partial^2 + 2e^2 v^2)]^{-3/2}. \quad (21)$$

The σ and π contributions give us

$$\int \mathcal{D}\sigma \rightarrow [\det(\partial^2 - (\mu^2 - \lambda v^2))]^{-1/2}, \quad \int \mathcal{D}\pi \rightarrow [\det(\partial^2 - (\mu^2 - \lambda v^2/3))]^{-1/2}. \quad (22)$$

The effective action is thus the original Mexican hat for ϕ , plus the three $\ln \det$ terms. These are calculated through dimensional regularization in the regular way, see e.g. the appendix of P&S for the integrals. We get

$$\ln \det(\partial^2 + \alpha^2) = iV \int_k \ln(k^2 + \alpha^2) = -\frac{i\alpha^2 V}{2(4\pi)^2} \left(\frac{2}{\epsilon} - \ln \alpha^2 + \dots \right), \quad (23)$$

where the \dots are constants, $d = 4 - \epsilon$, and V is the spacetime volume. We choose the counterterms in the original Lagrangian to kill off the divergent parts, using the minimal subtraction scheme. The cutoff-dependent term that each $\ln \det$ produces is

$$\ln \det(\partial^2 + \alpha^2) \rightarrow \frac{iV\alpha^4}{2(4\pi)^2} \ln(\alpha^2/\Lambda^2) \quad (24)$$

where Λ^2 is the cutoff. We choose the counterterms to cancel the $2/\epsilon + \dots$ divergence at all scales, and to cancel the logarithmic cutoff-dependent term at the renormalization scale $M^2 < \Lambda^2$. Thus the counterterms contain logarithms of the form $-\ln(\alpha^2/M^2)$. Adding up the contributions from the three fields, and using $\Gamma[\phi]/V = -V_{\text{eff}}$ for a uniform expectation value of ϕ , we get

$$V_{\text{eff}} = -\mu^2\phi^2 + \frac{\lambda\phi^4}{6} + \frac{1}{4(4\pi)^2} \left(\ln[(-\mu^2 + \lambda\phi^2)/M^2](-\mu^2 + \lambda\phi^2)^2 + \ln[(-\mu^2 + \lambda\phi^2/3)/M^2](-\mu^2 + \lambda\phi^2/3)^2 + 3\ln[2e^2\phi^2/M^2](2e^2\phi^2)^2 \right). \quad (25)$$

Working at the classical critical point $\mu^2 = 0$, let us simplify the potential in the region $\lambda \sim e^4$ very small, which we will see is a region through which RG flows always pass. We get

$$V_{\text{eff}} \approx \frac{\lambda\phi^4}{6} + \frac{3e^4\phi^4}{16\pi^2} \ln\left(\frac{2e^2\phi^2}{M^2}\right). \quad (26)$$

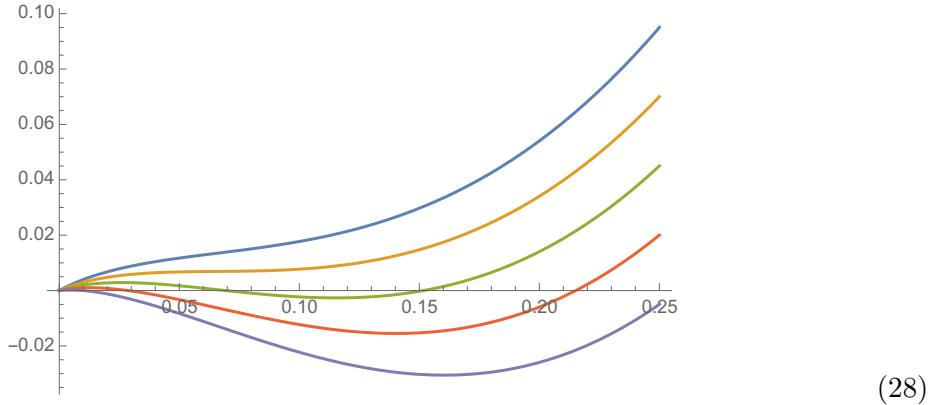
Minimizing this, we find the following vev for ϕ^2 :

$$\phi^2 = \frac{M^2}{2e^2} \exp\left(-\frac{8\pi^2\lambda}{9e^4} - \frac{1}{2}\right). \quad (27)$$

Thus the vev for ϕ is nonzero even when $\mu^2 = 0$, which could not happen classically. For $\mu^2 \neq 0$, we can check by plotting the effective potential for different values of μ^2 (at fixed $\lambda \sim e^4$ and at fixed M^2), that a nontrivial minimum in V_{eff} survives even when $\mu^2 < 0$.¹ Note also that we can solve the above equation for λ as a function of e at a given renormalization scale (which we usually take to be $M^2 = 2e^2\phi^2$ for convenience). Thus we may trade two dimensionless parameters of the model (e and λ) for one dimensionless and one dimensionfull parameter — another example of dimensional transmutation.

¹When doing this, one finds regions of parameter space where for certain values of ϕ , the arguments of the logarithms become negative—this is not a pathology, and indicates that in that case, there is no field configuration with expectation value ϕ : trying to force the expectation value of the field to be ϕ leads to an unstable state. This is like if we try to force the electric field in some region of space to be greater than $\sqrt{2m_e c^2}$: such an expectation value is not stable, since the vacuum will nucleate pairs to screen the electric field. This quantum effect is exactly what the imaginary parts of the effective potential capture.

Since the minimum of the potential becomes nonzero when the coefficient of the quadratic term is still positive, this provides us with an example of a (weakly) first order transition. This can be seen clearly by making a plot; choosing values of e^2 , M^2 , and λ so that the plot looks nice, the effective potential $V_{\text{eff}}(\phi)$ looks like



where the curves are shown for large-ish positive mass squared ($\mu^2 < 0$; blue) down to zero mass squared $\mu^2 = 0$; purple). Here we can explicitly see the first-order nature of the transition. Since the first-order nature of the transition is caused by the logarithm, which is there to account for the 1-loop fluctuations about the classical potential, this is an example of a “fluctuation-induced first-order transition”.

β functions: Now for the β functions. We can go back to working with A and ϕ , which is much easier since it reduces the number of diagrams we have to compute. This is allowed since we only care about β_e and β_λ . e and λ are both dimensionless in this problem, and so they cannot depend on the dimensionful parameter m .

The beta function for e is easier, so we turn to that first. We get β_e by looking at one-loop corrections to the photon propagator, i.e. by examining how charge renormalization occurs.

There are two one-loop diagrams to evaluate: the polarization bubble and a diagram where a ϕ loop intersects the A propagator line at a single vertex. The latter graph is independent of the photon momentum p^2 , and as such won’t contribute to the beta function, since it will be completely killed by the δ_e counterterm at any RG scale M . The polarization bubble is

$$\Pi^{\mu\nu}(p^2) = -e^2 \int_q \frac{i}{q^2} \frac{i}{(q-p)^2} (2q-p)^\mu (2p-q)^\nu, \quad (29)$$

where q is the momentum flowing in the ϕ loop. The momenta in the numerator come from Fourier-transforming the vertex $eA^\mu(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger)$. We evaluate the bubble in the usual way using Feynman parameters and massaging the resulting expression into an integral that we can look up in the appendix of P&S (here \int_x means $\int_0^1 dx$)

$$\begin{aligned} \Pi^{\mu\nu}(p^2) &= e^2 \int_{q,x} \frac{(2q-p)^\mu (2q-p)^\nu}{(q^2 + p^2 x - 2xp \cdot q)^2} \\ &= e^2 \int_{q,x} \frac{(2q + p(2x-1))^\mu (2q + p(2x-1))^\nu}{(q^2 + p^2(x+x^2))^2} \\ &= e^2 \int_{q,x} \frac{g^{\mu\nu} q^2 + p^\mu p^\nu (2x-1)^2}{(q^2 - \Delta)^2}, \end{aligned} \quad (30)$$

where $\Delta = (x^2 + x)(-p^2)$ and where we shifted integration over q to simplify the denominator in the second step. Looking up the integrals and doing dimensional regularization, and then doing the integral over x , we get

$$\Pi^{\mu\nu}(p^2) = \frac{ie^2}{3(4\pi)^2} \left(\frac{1}{\epsilon} - \ln(-p^2/\Lambda^2) + \dots \right) (p^2 g^{\mu\nu} - p^\mu p^\nu), \quad (31)$$

where \dots are irrelevant constants and the momentum dependence is required by our choice of gauge.

We then figure out the charge renormalization by setting the propagator equal to a sum over all numbers sequential polarization bubbles in the usual way:

$$e^2(M^2 = -p^2) D^{\mu\nu}(p^2) = e^2 D^{\mu\nu}(p^2) + e^2 D^{\mu\alpha}(p^2) \frac{e^2}{3(4\pi)^2} [\ln(-p^2/\Lambda^2)(p^2 g_{\alpha\gamma} - p_\alpha p_\gamma)] D^{\gamma\nu}(p^2) + \dots, \quad (32)$$

where \dots contains $n > 1$ polarization bubbles, e.g. $D\Pi D\Pi D$, $D\Pi D\Pi D\Pi D$, and so on. This becomes a geometric series in the \ln factor, since in our gauge $d^\dagger d \square^{-2} d^\dagger d = d^\dagger \square^{-1} d$, i.e. $(p^2 g_{\mu\alpha} - p_\mu p_\alpha)(g_\nu^\alpha/p^2 - p^\alpha p_\nu/p^4) = g^{\mu\nu} - p^\mu p^\nu/p^2$. Summing the geometric series, we get

$$e^2(M^2) = \frac{e^2}{1 - \frac{e^2}{3(4\pi)^2} \ln(M^2/\Lambda^2)}. \quad (33)$$

From this, we calculate the beta function

$$\beta_e = \frac{de(M)}{d \ln M} = \frac{e^3(M)}{48\pi^2}. \quad (34)$$

Now for β_λ , which is more of a pain. Since there are many 1-loop diagrams to compute if we want to get β_λ directly, we take a different approach using the CS equation. First, we find the field-strength renormalization of ϕ .

To one-loop order the ϕ propagator correction due to ϕ itself is killed by the mass counterterm (this is true because we are taking $m^2 = 0$ in the Lagrangian, see P&S chapter 12), so we only need to worry about the gauge field contribution. There are two relevant diagrams: one with a straight ϕ line and a A bubble meeting it at a single vertex, and one with a polarization bubble consisting of one A line and one ϕ line. The former diagram gives a contribution independent of the external ϕ momentum, so we can ignore it in what follows (since we'll be setting the external momentum equal to the RG scale, and then differentiating wrt that scale). The bubble diagram gives

$$\begin{aligned} \text{bubble}^{\mu\nu} &= (-ie)^2 \int_q \frac{1}{(p-q)^2} \left(\frac{g^{\mu\nu}}{p^2} + \frac{p^\mu p^\nu}{p^4} \right) (2p-q)^\mu (2p-q)^\nu \\ &= I_1 + I_2, \end{aligned} \quad (35)$$

where I_1 is the diagonal part and I_2 the $p^\mu p^\nu$ part. The former is

$$\begin{aligned}
I_1 &= -e^2 \int_q \frac{(2p-q)^2}{q^2(p-q)^2} \\
&= -e^2 \int_{q,x} \frac{(p-q)^2}{(xq^2 + (1-x)(q+p)^2)^2} \\
&= -e^2 \int_{q,x} \frac{(p-q+xp)^2}{(q^2 + (x^2+x)p^2)^2} \\
&= -e^2 \int_{q,x} \frac{(1+x)^2 p^2 + q^2}{(q^2 - \Delta)^2}, \quad \Delta = -(x+x^2)p^2.
\end{aligned} \tag{36}$$

Using the integrals in the appendix of P&S, and keeping only the logarithmic part, we get

$$I_1 \sim -\frac{2ie^2}{(4\pi)^2} \ln(-p^2/\Lambda^2). \tag{37}$$

For I_2 , we use the Feynman trick with $(A^2 B)^{-1} = -\partial_A(AB)^{-1}$ to write

$$I_2 = -e^2 \int_{q,x} \frac{2x(p^2 - q^2)^2}{(x(p+q)^2 + (1-x)q^2)^3}. \tag{38}$$

We only care about divergent parts, so keeping only these, and dropping the odd-in-momentum terms that die under the integration,

$$I_2 = -e^2 \int_{q,x} \frac{2x((x^2 - 2)p^2 q^2 + q^4)}{(q^2 + x(1+x)p^2)^3}. \tag{39}$$

Once again we turn to the P&S appendix for the integrals. Keeping only the logarithmic parts, we get

$$I_2 = \frac{ie^2}{4\pi^2} p^2 \ln(-p^2/\Lambda^2), \tag{40}$$

where the value of the numerical prefactor shouldn't be trusted up to a factor of 5 or so.

We now know the wavefunction renormalization γ , at least to one-loop order, since we now know what the δ_Z counterterm for ϕ should be. Using (see P&S chapter 12)

$$\frac{\partial \delta_Z}{\partial \ln M} = 2\gamma, \tag{41}$$

we get

$$\gamma = -\frac{3e^2}{(4\pi)^2}, \tag{42}$$

where again the numerical factor probably shouldn't be trusted since I wasn't being too careful.

Now that we have the wavefunction renormalization we can find β_λ using our knowledge of the effective potential. Just like the connected n-point functions, we can get constraints on the RG flow by evaluating $V_{\text{eff}}(\phi)$ at different renormalization scales. The effective potentials

evaluated at two different scales M, M' are related by how the ϕ field scales under RG, and so taking $M' = M + \delta M$, we obtain

$$(M\partial_M + \beta_\lambda\partial_\lambda + \beta_e\partial_e - \gamma\phi\partial_\phi)V_{\text{eff}} = 0 \quad (43)$$

(see chapter 13 of P&S, the minus sign in front of γ is because the effective potential is related to inverse propagators rather than propagators like the free energy). We know the effective potential and β_e , so plugging in and doing some algebra we get

$$\beta_\lambda = \frac{1}{24\pi^2} (-18e^2\lambda + 54e^4 + 5\lambda^2). \quad (44)$$

In terms of the time $t = -\ln M$ along the RG flow, we have

$$d_t e = -\frac{e^3}{48\pi^2}, \quad d_t \lambda = -\frac{1}{24\pi^2} (-18e^2\lambda + 54e^4 + 5\lambda^2). \quad (45)$$

Note that $d_t e$ is always negative and that $d_t \lambda$ has a negative term proportional to e^4 , so that we flow to smaller values of the couplings very quickly. Also note that $d_t \lambda < -(9e^2/2 - 2\lambda)^2$, so that $d_t \lambda$ is negative definite and the flow to small λ, e is inevitable.

We already know $e(M)$ from our computation of β_e , and we can get λ for $\lambda \ll e^2$ (a regime to which the RG flow will always pass through) by solving the β function by integrating from M_0 to M :

$$\lambda(M) = e^4(M) \left(\frac{\lambda(M_0)}{e^4(M_0)} + \frac{54}{24\pi^2} \ln(M/M_0) \right). \quad (46)$$

Additionally, from the transformation

$$\phi \mapsto (1 - (\delta M/M))\phi \quad (47)$$

under RG, we get $d\phi = -d\ln M \gamma\phi$ so that

$$\phi(M) = \phi(M_0) \left(\frac{M}{M_0} \right)^{-\gamma}. \quad (48)$$

Using these expressions for e, λ, ϕ at the scale M , we can plug them into V_{eff} to figure out the effective potential at any given scale.

The mass (of the oscillatory mode about the potential minimum) is found by taking the second derivative of V_{eff} with respect to ϕ , and evaluating the result at $\langle\phi\rangle$ (which we will continue to lazily just write as ϕ). To simplify the resulting expression, we choose to evaluate it at the RG scale set by the vev, namely $M^2 = 2e^2\phi^2$. Taking the derivatives then gives

$$m_\sigma^2 = 3\phi^2 \left(\frac{2\lambda}{3} - \frac{e^4}{4\pi^2} \right). \quad (49)$$

The value of ϕ is still determined by the same exponential form as in our pre-RG analysis, so that this scale for M , we have

$$\lambda = \frac{9e^4}{8\pi^2}. \quad (50)$$

This lets us write the mass more simply as

$$m_\sigma^2 = \frac{3e^4\phi^2}{2\pi^2}. \quad (51)$$

The gauge boson mass is read off from the Lagrangian as $m_A^2 = 2e^2\phi^2$, and so we determine that the ratio of the masses is

$$\frac{m_\sigma^2}{m_A^2} = \frac{3e^2}{4\pi^2}. \quad (52)$$

Finally, we look what happens if we take $\mu^2 < 0$, i.e. if we give ϕ a positive mass. Classically this would preclude symmetry breaking, but in quantum field theory this is no longer the case, because of the logarithm generated by the fluctuations. We add $m^2\phi^2$ to V_{eff} , find the second derivative of V_{eff} to get m_σ^2 as a function of ϕ , and then minimize V_{eff} to find ϕ in terms of e, λ, m , which then allows us to solve for m_σ^2 . Mathematica gives the unilluminating

$$\begin{aligned} \phi &= \sqrt{\frac{2}{3}} \frac{m\pi}{e^2} \frac{2}{\sqrt{\text{ProductLog}(-16m^2\pi^2/(3e^2M^2))}} \\ &= \frac{M}{\sqrt{2}e} \exp\left(\frac{1}{2}\text{ProductLog}\left[\frac{-16m^2\pi^2}{3e^2M^2}\right]\right). \end{aligned} \quad (53)$$

One can check that this reduces to our old result when $m = 0$, and that the symmetry-breaking minimum of the potential disappears at some $m_c^2 \sim M^2e^2 > 0$, so that even with the RG analysis included, there is still symmetry breaking for $m^2 > 0$ where classically there would be none.

9 April 22 — The $\mathbb{C}P^N$ model and emergent electromagnetism

The $\mathbb{C}P^N$ model is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \left(|\partial_\mu z|^2 - |z^\dagger \partial_\mu z|^2 \right), \quad (54)$$

where z is a $N + 1$ component field, subject to the relations

$$|z|^2 = 1, \quad (z_1, \dots, z_{N+1}) \sim (e^{i\alpha(x)} z_1, \dots, e^{i\alpha(x)} z_{N+1}). \quad (55)$$

Working in two spacetime dimensions, show the invariance of \mathcal{L} under local $U(1)$ gauge transformations, and as a bonus, show the equivalence between the $N = 1$ case and the $O(3)$ nlsm. Then show that one can realize the $|z|^2 = 1$ constraint and the gauge symmetry by introducing extra fields A, λ , such that

$$Z = \int \mathcal{D}A \mathcal{D}\sigma \mathcal{D}z \exp \left(\frac{i}{g^2} \int \left((D_A z)^\dagger D_A z - \lambda(|z|^2 - 1) \right) \right). \quad (56)$$

Write down A in terms of z . Now integrate out z . Show that in the approximation where A, λ are constants, one obtains the superconductor-like formula for the mass $\lambda = m^2$

$$\lambda = \Lambda e^{-2\pi/(g^2 N)}. \quad (57)$$

Then, working in the limit of small A , show that at long distances, the effective action for the gauge field is a Maxwell term. Find an expression for the effective charge in terms of N and the expectation value of λ .

Solution

First, a few general comments on $\mathbb{C}P^N$, which is the space of complex lines through the origin in \mathbb{C}^{N+1} . More often we see $\mathbb{C}P^N$ written as the space $(z_1, \dots, z_{N+1})/\sim$, where $(z_1, \dots, z_{N+1}) \sim (\lambda z_1, \dots, \lambda z_{N+1})$, $\lambda \in \mathbb{C}$, and $(z_1, \dots, z_{N+1}) \neq 0$. If we take

$$\lambda = \frac{1}{\sum_{i=1}^{N+1} |z_i|^2} \quad (58)$$

and multiply through, we get the conditions written in the problem statement, which shows us that²

$$\mathbb{C}P^N = S^{2N+1}/U(1). \quad (59)$$

For example, $\mathbb{C}P^1 = S^3/U(1)$, as realized via the Hopf map.

A way to construct $\mathbb{C}P^N$ as a quotient which is often more useful is

$$\mathbb{C}P^N \cong \frac{SU(N+1)}{U(N)}, \quad (60)$$

where the $U(N)$ factor is embedded via the map

$$U(N) \ni u \mapsto u \oplus \frac{1}{\det u} \in SU(N+1), \quad (61)$$

which ensures that the image of $U(N)$ in $SU(N+1)$ has determinant 1. Alternatively, one might write this as

$$\mathbb{C}P^N \cong \frac{SU(N+1)}{S[U(N) \times U(1)]}. \quad (62)$$

To prove this, we note that $SU(N+1)$ acts transitively on $\mathbb{C}P^N$: it acts transitively on the unit sphere in $S^{2N+1} \subset \mathbb{C}^{N+1}$ and therefore acts transitively on $\mathbb{C}P^N$, which as we saw is a quotient of the aforementioned sphere by $U(1)$. Thus we can identify

$$\mathbb{C}P^N \cong SU(N+1)/\text{Stab}_x, \quad (63)$$

where $\text{Stab}_x \subset SU(N+1)$ is the stabilizer subgroup of an arbitrary point $x \in S_{\mathbb{C}}^N$. This is basically the first isomorphism theorem: the image of a point x under the group action is isomorphic to the group modulo the kernel of the action, which is Stab_x . Anyway, to figure out Stab_x , we can just pick a convenient $x \in \mathbb{C}P^N$, e.g. $x = \{(e^{i\theta}, 0, \dots, 0) \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$.

²A similar thing occurs for $\mathbb{R}P^N$: each line through the origin in \mathbb{R}^{N+1} can be identified with a pair of points on the unit sphere S^N , and so $\mathbb{R}P^N \cong S^N/\mathbb{Z}_2$. Random stream-of-consciousness fact: $\mathbb{R}P^N$ is non-orientable when N is even, and orientable when N is odd. To see this, realize that the antipodal identification in the \mathbb{Z}_2 quotient of S^N is performed by sending a vector $\mathbb{R}^{N+1} \ni v \mapsto -v$. For N odd the map $v \mapsto -v$ is equivalent to a rotation, since it has determinant 1, while for even N it is an inversion, since it has determinant -1 . Thus the S^N in the definition of $\mathbb{R}P^N$ is identified with itself in an even / odd way depending on N , and the claim follows.

This is clearly fixed under the $U(N)$ action on the last N coordinates, and also fixed under the $U(1)$ action on the first coordinate. We can combine these two actions to get something in $SU(N+1)$ if we shift the first coordinate by \det^{-1} of the matrix acting on the last N coordinates, and so $\text{Stab}_x = U(N)$, with the embedding into $SU(N+1)$ as written above. This then proves the claim.

Symmetries and alternate path integral representation

Showing the gauge invariance of \mathcal{L} is straightforward after making use of $|z|^2 = 1$ and $\partial|z|^2 = 0 \implies z^\dagger \partial z = -z \partial z^\dagger$. One also notes that $|z^2| = 1$ means that

$$A_\mu \equiv \frac{1}{2i}(z^\dagger \partial_\mu z - z \partial_\mu z^\dagger) \quad (64)$$

transforms under a gauge transformation as $A \mapsto A + d\alpha$, meaning that $\partial_\mu - iA_\mu$ is an appropriate covariant derivative. One is then led to guess that an alternate representation of the theory can be written as

$$Z = \int \mathcal{D}A \mathcal{D}\lambda \mathcal{D}z \exp \left(\frac{i}{g^2} \int \left((D_A z)^\dagger D_A z - \lambda(|z^2| - 1) \right) \right). \quad (65)$$

One confirms this by integrating out λ to get the sphere constraint on z , and then shifting A by

$$A \mapsto A + iz\partial z^\dagger, \quad (66)$$

which eliminates the coupling between A and z , and results in an action like $|A|^2 + |\partial z|^2 - |z\partial z^\dagger|^2$. Integrating out A then produces the original $\mathbb{C}P^1$ Lagrangian.

A brief digression on the non-linear sigma model: the claim is that the $\mathbb{C}P^1$ model is the same as the $O(3)$ nlsm, through the Hopf map $S^3 \rightarrow S^2$:

$$n^i = z^\dagger \sigma^i z. \quad (67)$$

Here the RHS is a vector since the z 's are in the fundamental of $SU(2)$, and $1/2 \otimes (1/2)^* \cong 1/2 \otimes 1/2 = 0 \oplus 1$: choosing $\mathbf{1}$ instead of σ^i would project onto 0, while choosing σ^i gives us the vector (adjoint) rep. n^i is also normalized properly.³

Now the z spinors live in $S^3 \cong SU(2)$, but because of the $U(1)$ redundancy the target space for the $\mathbb{C}P^1$ model is actually $SU(2)/U(1) = S^2$, the same as the $O(3)$ nlsm. As a sanity check, we can check the global symmetries on each side. The $\mathbb{C}P^1$ has a global $U(2)$ symmetry but a local $U(1)$ symmetry, so the actual global symmetry is⁴

$$PU(2) = U(2)/U(1) = PSU(2) = SU(2)/\mathbb{Z}_2 = SO(3). \quad (69)$$

³The matrix $\sigma^\mu |z\rangle\langle z|\sigma^\mu$ is Hermitian and has trace 3 (since $\text{Tr}[|z\rangle\langle z|] = 1$), so we can write it as $3\mathbf{1}/2 + c_i \sigma^i$. Also, since $n^i n_i$ is rotation-invariant, we can perform an $SU(2)$ rotation on the z s to any convenient spinor: we will choose $|z\rangle = (1, 0)^T$. Then $|z\rangle\langle z| = \mathbf{1}/2 + Z/2$, and conjugating by σ^j and summing over j tells us that $c_z = -1/2$, and so

$$n^i n_i = \langle z | (3\mathbf{1}/2 - Z/2) | z \rangle = 1 \quad (68)$$

as required.

⁴To get this, we have used the second isomorphism theorem: if H, N are subgroups of G with N normal, then $(NH)/N \cong H/(H \cap N)$. This just says that one can't "cancel the N s" in $(NH)/N$ since if $N \cap H \neq 0$ then some elements of H will also be killed by the quotient. We have used the special case where $N = SU(n)$, $H = Z(U(n)) = U(1)$ to show that $PU(n) = (SU(n)U(1))/U(1) \cong SU(n)/(U(1) \cap SU(n)) = PSU(n)$.

This matches the global symmetry of the nlsm if we pretend the global symmetry is $SO(3)$ instead of $O(3)$. We can get the full symmetry by taking into account the reflection that extends $SO(3)$ to $O(3)$, which is represented by the matrix $-\mathbf{1}$ and hence sends $n \mapsto -n$. On the $\mathbb{C}P^1$ side, this reflection is implemented by charge conjugation, which is unitary and acts through the invariant antisymmetric form of $SU(2)$:

$$\mathbb{Z}_2^C : z \mapsto Jz^*, i \mapsto i, \quad J = -iY. \quad (70)$$

Indeed, n is odd under this symmetry:

$$\mathbb{Z}_2^C : z^\dagger \sigma^j z \mapsto z^\dagger (J^T \sigma^j J)^T z = -z^\dagger (J[\sigma^j]^T J) z = z^\dagger (\sigma^j J^2) z = -z^\dagger \sigma^j z. \quad (71)$$

One similarly checks that the $\mathbb{C}P^1$ Lagrangian is invariant under \mathbb{Z}_2^C , and that $A \mapsto -A$, which is how a gauge field is expected to transform. Now $O(3) = SO(3) \times \mathbb{Z}_2^C$, since \mathbb{Z}_2^C is represented by the central element $-\mathbf{1}$. This means that \mathbb{Z}_2^C should commute with the $PSU(2)$ symmetry on the $\mathbb{C}P^1$ side, which it does:

$$JU^* = UJ \implies J(Uz)^* = UJz^*, \quad \forall U \in PSU(2), \quad (72)$$

where the first equality holds since $J\mathcal{K}$ with \mathcal{K} complex conjugation is the antilinear automorphism from the spin 1/2 representation of $SU(2)$ to itself, which is the manifestation of that representation's pseudoreality.

Anyway, sadly actually substituting the representation for n above into $\partial_\mu n^i \partial^\mu n^i$ and showing that you get the $\mathbb{C}P^1$ Lagrangian is pretty messy, so I won't write it down (there's got to be a better way...).

Effective action and emergent electromagnetism

We now integrate out z to get

$$Z = \int \mathcal{D}A \mathcal{D}\lambda \exp \left(-N \ln \det(-D_A^2 - \lambda) + \frac{i}{g^2} \int \lambda \right). \quad (73)$$

Note that the D_A^2 is really D_A^2 , and not $|D_A|^2$. As a first step, consider the case where $A = 0$ and where we approximate λ by a constant. In the large N limit, we can figure out what this constant is by taking the saddle point of the effective action with respect to λ . This produces

$$\frac{i}{Ng^2} = \text{Tr} \frac{1}{\partial^2 + \lambda}, \quad (74)$$

where the trace is to be carried out in momentum space (the factor of the spacetime volume has canceled on both sides). We rewrite this as

$$\frac{1}{Ng^2} = \int_k \frac{1}{k^2 + \lambda}, \quad (75)$$

where the integral is over Euclidean momenta. Doing the integral up to a cutoff Λ gives a mass $m^2 = \lambda$ of

$$m = \Lambda e^{-2\pi/(g^2 N)}, \quad (76)$$

which is the usual dimensional transmutation result: even though the theory was conformally invariant at the classical level, quantum effects produce a non-zero beta function through the Λ dependence of m . The fact that we get a massive theory is also interesting because there is no obvious symmetric mass term we can write down in the UV (e.g. $m^2 \sum_i |z_i|^2 = m^2$ doesn't work because of the constraint).

Note that if the dimension of spacetime were not two, then the presence of the cutoff would make the integral on the RHS of (75) bounded for arbitrarily small λ , and so if Ng^2 were small enough, there would be no solution to the mean-field equation for λ . This would indicate spontaneous symmetry breaking and a Higgsing of the gauge field, and we'd have to go back and expand about the correct vacuum (however, since we're focusing on two dimensions, this need not concern us).

Now we will take λ to be a constant and expand about small A , deriving the effective action for A at one-loop order. We write the $\ln \det$ as

$$-N\text{Tr} \ln(-D_A^2 - \lambda) = -N\text{Tr} \ln \left[(-\partial^2 - \lambda) \left(1 + \frac{A^2 + 2iA\partial + i(\partial A)}{-\partial^2 - \lambda} \right) \right], \quad (77)$$

where the ∂ is understood to act on the z 's. The overall factor of $(-\partial^2 - \lambda)$ in the logarithm is an unimportant constant, so we drop it. We now expand the logarithm to second order. The first order contribution yields a term like $G_z A^2$ (G_z is the z propagator), plus things which vanish after integration. Diagrammatically, the $G_z A^2$ term is a straight z propagator line with a gauge bubble attached to it at a single point. To $O(A^2)$, the next diagram that contributes is a z line with a gauge polarization bubble. Reading the Feynman rules off from the interaction vertex $(2iA\partial + i(\partial A))^2$, and combining this with the $G_z A^2$ term, we get (still in Minkowski space)

$$-N\text{Tr} \ln(-D_A^2 - \lambda) \approx -N \int_{p,q} A^\mu(p) A^\nu(-p) \left(\frac{g_{\mu\nu}}{q^2 - \lambda} + \frac{1}{2} \frac{(2q+p)_\mu(2(q+p)-p)_\nu}{(q^2 - \lambda)((q+p)^2 - \lambda)} \right). \quad (78)$$

We are interested in the IR properties of this action, so we will take the small p limit. By gauge invariant we already know the form of the effective action, but we need to go through the details to figure out what the effective electric charge is. Since the first term in the integrand has no p -dependence, we focus on the second term, since this is the term that will produce the Maxwell term. The integral is done with the usual Feynman parameters: we use the Feynman parameters to simplify the denominator, shift the q momentum to eliminate the $q \cdot p$ term in the denominator, and then drop integrals which vanish because their integrands are odd. This yields (here $\Delta = -((x^2 + x)p^2 - \lambda)$)

$$\begin{aligned} \int_{p,q} A^\mu(p) A^\nu(-p) \frac{1}{2} \frac{(2q+p)_\mu(2q+p)_\nu}{(q^2 - \lambda)((q+p)^2 - \lambda)} &= \frac{1}{2} \int_{p,q} A^\mu(p) A^\nu(-p) \frac{(2(q-xp)+p)_\mu(2(q-xp)+p)_\nu}{(q^2 - \Delta)^2} \\ &= \frac{1}{2} \int_{p,q} A^\mu(p) A^\nu(-p) \frac{4q_\mu q_\nu + p_\mu p_\nu(1-2x)^2}{(q^2 - \Delta)^2}. \end{aligned} \quad (79)$$

We can now look up the integrals. To get a nice answer, we need to focus on small p (i.e. $O(p^2)$), so that e.g.

$$\frac{p_\mu p_\nu}{\Delta} \approx \frac{p_\mu p_\nu}{\lambda}, \quad \ln \Delta \approx \ln \lambda - \frac{(x^2 + x)p^2}{\lambda}. \quad (80)$$

Working in this approximation, the p -dependent parts of the effective action for A become

$$-iN \frac{1}{12\pi\lambda} \int_p A^\mu(p) A^\nu(-p) (g_{\mu\nu} p^2 - p_\mu p_\nu), \quad (81)$$

and so we have generated an emergent Maxwell theory with effective charge **where is g^2 ?**

$$\frac{1}{e_{\text{eff}}^2} = \frac{N}{6\pi\lambda}. \quad (82)$$

What are the dynamics of this theory? Just like the Schwinger model, it confines, with the spectrum consisting of z - \bar{z} bound states.

10 April 23 — ϕ^4 theory coupled to fermions and a natural relation

This is from P&S. We consider a nlsm with symmetry group $O(2)$ coupled to fermions by using the nlsm field to create a varying chiral mass term for the fermions:

$$\mathcal{L} = \frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + i\bar{\psi}\not{\partial}\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi, \quad (83)$$

where $i = 1, 2$.

Find the classical value for the fermion mass, assuming SSB occurs. Then show that this receives corrections at one-loop order, but that these corrections are *finite*. **Solution:**

First, note that while the fermion mass term means that the full $O(2) \times U(1)$ symmetry is broken, the diagonal subgroup consisting of the rotations

$$\phi \mapsto R_\theta\phi, \quad \psi \mapsto e^{-i\gamma^5\theta/2}\psi \quad (84)$$

is still a symmetry of the theory (checking this is straightforward). Secondly, assume that SSB occurs for ϕ . Wolog we can let $\langle\phi\rangle = (v, 0)^T$, so that the classical value of the fermion mass becomes

$$m_f = gv. \quad (85)$$

Now we want to find quantum corrections to this. This is most easily accomplished when we write e.g. $\phi = (v + \sigma, \pi)$, where $v = \mu/\sqrt{\lambda}$ is the classical vev of ϕ in the SSB state. σ is massive and π is massless, and the Feynman rules are derived from the relevant terms

$$\mathcal{L} \supset -ig\bar{\psi}\gamma^5\pi\psi - g\bar{\psi}\sigma\psi - \lambda v\pi^2\sigma. \quad (86)$$

There are further interactions like $-\frac{1}{2}\lambda\sigma^2\pi^2$, but these won't play a role in this problem.

To examine the corrections to m_f , we need to consider three things: the counterterm for g (which affects the mass due to $m_f = gv$) and the corrections to the propagator which come from polarization bubble diagrams with ψ and either π or σ .

The polarization bubble with σ for a ψ fermion with momentum p is (this diagram is a matrix with spinor indices for the ψ spins on the ends of the bubble)

$$\begin{aligned}\sigma \text{ bubble} &= (ig)^2 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \\ &= -g^2 \int_{q,x} \frac{x\not{p} + m_f}{(q^2 - \Delta_\sigma)^2} \\ &= -g^2 \int_x (x\not{p} + m_f) \frac{i}{16\pi^2} \left(\frac{2}{\epsilon} - \gamma \right) (1 + \epsilon \ln \sqrt{4\pi/\Delta_\sigma}),\end{aligned}\tag{87}$$

where $d = 4 - \epsilon$ and Δ_σ is built out of x, p , and m_σ . The diagram involving π is similarly

$$\begin{aligned}\pi \text{ bubble} &= -(ig)^2 \int_q \frac{\gamma^5}{(p-q)^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \\ &= +g^2 \int_{q,x} \frac{-x\not{p} + m_f}{(q^2 - \Delta_\pi)^2} \\ &= -g^2 \int_x (x\not{p} - m_f) \frac{i}{16\pi^2} \left(\frac{2}{\epsilon} - \gamma \right) (1 + \epsilon \ln \sqrt{4\pi/\Delta_\pi}),\end{aligned}\tag{88}$$

where Δ_π is built out of x and p . Adding these together and sending $\epsilon \rightarrow 0$ we get

$$\sigma \text{ bubble} + \pi \text{ bubble} = -g^2 \int_x x\not{p} (\text{divergent stuff}) - g^2 m_f \int_x \frac{i}{16\pi^2} \ln(\Delta_\pi/\Delta_\sigma).\tag{89}$$

The coefficient of the m_f term is finite, and so these diagrams lead to a finite correction to the fermion mass (the correction is non-zero since $\Delta_\sigma \neq \Delta_\pi$ due to σ being massive and π being massless). The field strength renormalization term needed because of the $x\not{p}$ term is infinite, but the mass correction is finite.

Now we need to look at the correction to the coupling g which also affects the renormalized fermion mass: if the correction to m_f is to be finite then the counterterm δ_g better be finite as well.

We will work in an RG scheme suggested in P&S, where the $\psi\psi\pi$ vertex receives no radiative corrections when the π particle carries away zero momentum. To work out δ_g , the counterterm needed to ensure that the vertex has no radiative corrections, we need to compute four diagrams (one-loop corrections to the $\psi\psi\pi$ vertex). Two of the diagrams have only one ψ propagator in the loop (the other two guys in the loop are a π and a σ) and are finite:

$$\text{two finite diagrams} \sim g^2 (-\lambda v) \int_q \frac{1}{(p-q)^2} \frac{1}{(p-q)^2 - m_\sigma^2} \frac{1}{q^2 - m_f^2} (2m_f),\tag{90}$$

which goes like $\int dq q^3 q^{-6}$ and contains no divergences (the \not{q} 's in the numerator cancel since they anticommute with γ^5).

The two diagrams with two fermion propagators in the loop go as $\int dq q^3 q^{-4}$ and do contain a $1/\epsilon$ divergence. The one with an internal π propagator is

$$\begin{aligned}\pi\psi\psi \text{ loop} &= (ig)^3 i^3 \int_q \frac{1}{(p-q)^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \\ &= -g^3 \int_q \frac{1}{(p-q)^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2}.\end{aligned}\tag{91}$$

On the other hand, the one with an internal σ propagator is

$$\begin{aligned}\sigma\psi\psi \text{ loop} &= g^2(ig)i^3 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \\ &= +g^3 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2}.\end{aligned}\tag{92}$$

These two diagrams would entirely cancel if not for the fact that the second diagram has an m_σ in one of the propagators. When we evaluate this in dimensional regularization however, the only dependence on m_σ of the second diagram comes from the $\epsilon \ln \sqrt{4\pi/\Delta_\sigma}$ term, which is not divergent. Thus all the divergent parts are independent of m_σ , and hence cancel between the two diagrams, with their sum looking something like $g^3 \int_x \ln(\Delta_\pi/\Delta_\sigma)$. So, recapitulating, the counterterm δ_g is finite, as are the polarization bubble contributions to the renormalization of m_f , so that m_f only receives finite corrections.

11 April 24 — The chiral anomaly in 2 dimensions and the fermion bubble

For massless fermions coupled to a $U(1)$ gauge field in two dimensions, compute the two-dimensional analogue of the triangle diagram and derive the divergence of the chiral current. Use dimensional regularization.

The relevant diagram to compute when examining $\partial_\mu j_5^\mu$ is a fermion bubble with one insertion of j_5 and one gauge field leg. Thus to get $\partial_\mu j_5^\mu$ we need to compute the polarization bubble $\Pi_{\mu\nu}(q^2)$ with a $\bar{\gamma}$ inserted in the trace, and then contract it with a q_μ and a A_ν . For external momentum q , and making sure to order the matrices in the numerator correctly, the graph gives (the i s from the propagators kill the minus sign from the fermion loop)

$$(\text{bubble graph})(q^2) = e^2 \int_p \text{Tr} \left[\gamma^\mu \bar{\gamma} \frac{i\cancel{p}}{p^2} \gamma^\nu \frac{\cancel{p} + \not{q}}{(p+q)^2} \right].\tag{93}$$

Now use

$$\cancel{p}\gamma^\nu = 2p^\nu - \gamma^\nu \cancel{p}, \quad \gamma^\mu \bar{\gamma} = \epsilon_{\mu\nu\rho\sigma} \gamma^\nu \gamma^\rho,\tag{94}$$

where we are in \mathbb{R} time and have chosen X, iY as γ matrices. We also need

$$\text{Tr}[\gamma^\mu \gamma^\nu] = 2g^{\mu\nu}, \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma] = 2(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})\tag{95}$$

to simplify the trace. One gets

$$(\text{bubble graph})(q^2) = 2e^2 \int_p \frac{p^\nu(p+q)^\mu + p^\mu(p+q)^\nu - g^{\mu\nu}p^\sigma(p+q)_\sigma}{p^2(p+q)^2},\tag{96}$$

which we evaluate with Feynman parameters in the usual way. We then use dimensional regularization to do the integrals and renormalize with minimal subtraction, yielding

$$i\Pi^{\mu\nu}(q^2) = \frac{ie^2}{\pi} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right),\tag{97}$$

which is in the form required by gauge invariance. The divergence of the axial current to one-loop order is then

$$\begin{aligned}\partial_\mu j_5^\mu &\rightarrow q_\mu j^\mu = q_\mu \epsilon^{\mu\nu} \frac{e^2}{\pi} \left(g_{\nu\lambda} - \frac{q_\nu q_\lambda}{q^2} \right) A^\lambda \\ &= \frac{e^2}{\pi} q_\mu \epsilon^{\mu\lambda} A_\lambda.\end{aligned}\tag{98}$$

In \mathbb{R} space then,

$$\partial_\mu j_5^\mu = \frac{e^2}{2\pi} F_{\mu\nu} \epsilon^{\mu\nu} = 2e^2 c_1,\tag{99}$$

where c_1 is the first Chern class. The fact that the divergence of the axial current is *twice* the first Chern class is essential, since without the factor of 2 we'd conclude that the A_μ background field could lead to a non-conservation of fermion number (as c_1 is an integral class). This factor of two comes from the fact that each charge-1 chiral fermion contributes $e^2 c_1$ to the anomaly, and a single Dirac fermion has two such chiral fermions. As another sanity check, we see that if we didn't have the ϵ symbol (corresponding to computing the divergence of the vector current), we'd get a divergence of $q_\mu (g^{\mu\lambda} - q^\mu q^\lambda / q^2) A_\lambda \rightarrow d^\dagger \square^{-1} d^\dagger dA = \square^{-1} d^\dagger d^\dagger dA = 0$.

Note that we also could have regulated by imposing a hard cutoff in momentum space. The $q^\mu q^\nu$ term would be unchanged since it has no UV divergences, but the $g^{\mu\nu} q^2$ piece would pick up a $\ln \Lambda$ divergence, which we would regulate away. The form of the resulting Π would then imply that gauge invariance is violated (the divergence of the regular vector current $\bar{\psi} \gamma^\mu \psi$ is non-zero). Thus we may either preserve gauge invariance or chiral symmetry, but not both.

12 April 25 — Topological defects in axion electrodynamics and anomaly inflow

Consider QED coupled chirally to a scalar with a $U(1)$ -breaking potential:

$$\mathcal{L} = -\frac{1}{2} F \wedge \star F + i\bar{\psi} \not{D}_A \psi - \frac{1}{2} (\partial\sigma)^2 + \bar{\psi} (\text{Re}(\sigma) + i\bar{\gamma} \text{Im}(\sigma)) \psi - V(\sigma),\tag{100}$$

where $V(\sigma)$ has a classical minimum at $m \neq 0$. When the scalar gets a vev, it induces a mass term for the fermions, which in addition to the regular $m\bar{\psi}\psi$ mass term also has a chiral part.

Show that in the approximation where $\sigma = m e^{i\theta}$, the fermions are responsible for generating a spacetime-dependent theta angle for the gauge field. Consider a field configuration with a topological string defect in θ , and show that the action for such a field configuration suffers from a gauge anomaly by computing the divergence of the gauge current (this can be done with a one-loop calculation). By solving the Dirac equation on the string defect, show that the defect hosts chiral zero modes whose own gauge anomaly renders the full theory gauge invariant.

Solution:

The anomaly: Throughout we will assume we are working in the symmetry-broken phase, where σ gets a vev, thereby giving a mass to the fermions. We assume the potential $V(\sigma)$ is such that the classical vacua form an S^1 , and thus it makes sense to consider defect codimension-2 objects (strings) around which the phase of σ winds by something in $2\pi\mathbb{Z}$.

A schematic argument for why the fermions generate a θ term is as follows: for $\sigma = me^{i\theta}$, we can eliminate the $\bar{\gamma}$ coupling in \mathcal{L} by performing the shift

$$\psi \mapsto e^{-i\theta\bar{\gamma}/2}\psi \quad (101)$$

in the path integral. Then because of the chiral anomaly, the action should shift by

$$S \mapsto S + \frac{1}{8\pi^2} \int \theta F \wedge F, \quad (102)$$

thus generating a (spacetime-dependent) θ angle (the shift is the integral of θ against the second Chern character and not twice the second Chern character since we've rotated by $\theta/2$). This is subtle for defect string configurations though, since θ is not well-defined on its own, which makes doing the usual Fujikawa method kind of tricky. It will turn out that the right answer is the integrated-by-parts version of this, namely

$$\frac{1}{8\pi^2} \int d\theta \wedge F \wedge A. \quad (103)$$

To get this, we compute the divergence of the gauge current in the limit of large m and obtain the above expression. To leading order the divergence in the gauge current is caused by a diagram consisting of a single ψ loop with an external J_μ source, an A_ν gauge field leg, and an external σ leg. To evaluate this diagram and get a non-zero answer, we need to correctly take into account the fermion mass. This is done most easily by choosing a particular symmetry-breaking state to evaluate the diagram in. Let $\sigma = \sigma_1 + i\sigma_2$ with σ_i real. We will take the state where the vev of σ is real, with $i\sigma_2$ the Goldstone. We thus write $\langle\sigma\rangle = \langle\sigma_1\rangle = m$ for the fermion mass, and so the chiral coupling between σ and the fermions gives the fermions a mass and leaves them with a $\bar{\psi}i\bar{\gamma}\sigma\psi$ coupling. In this minimum, the contribution to the gauge current from this diagram on scales much larger than m^{-1} is

$$J^\mu(q+k) = e \int_p \text{Tr} \left[\gamma^\mu \frac{\not{p}+m}{p^2-m^2} \gamma^\nu \frac{\not{p}-\not{k}+m}{(p-q)^2-m^2} (i\bar{\gamma}\sigma_2(k)) \frac{\not{p}-\not{k}-\not{k}+m}{(p-q-k)^2-m^2} \right] A_\nu(q). \quad (104)$$

In this expression, q is the momentum of the external photon, and k is the momentum of the axion field (the σ field, but in the chosen symmetry-breaking state only the imaginary part σ_2 is fluctuating).

We can simplify this a fair bit by noting that terms with an odd number of γ matrices will vanish after being traced over, since $\text{Tr}(\gamma^\mu) = 0$ for all μ . In order to survive the trace with $\bar{\gamma}$, we need exactly four compensating γ matrices, so only terms linear in m survive in the numerator. We then use (in minkowski, mostly negative signature)

$$\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \bar{\gamma}] = -4i\epsilon^{\mu\nu\lambda\sigma}. \quad (105)$$

Any numerator odd in the momentum p which is being traced out will vanish by symmetry, and because of the ϵ symbol in the above trace, no terms involving \not{p} will survive. Thus the

only term which survives is

$$J^\mu(q+k) = 4mee^{\mu\nu\lambda\sigma} \int_p \frac{\sigma_2(k)q_\lambda(q_\sigma + k_\sigma)}{(p^2 - m^2)((p+q)^2 - m^2)((p+q+k)^2 - m^2)} A_\nu(q). \quad (106)$$

We then simplify the denominator with Feynman parameters, and shift p to complete the square (which happily doesn't affect the numerator). This gives

$$J^\mu(q+k) = 4mee^{\mu\nu\lambda\sigma} \int_{x,y} \sigma_2(k)q_\lambda(q_\sigma + k_\sigma) A_\nu(q) \int_p \frac{1}{(p^2 - \Delta)^3}, \quad (107)$$

where Δ is a gross function of q, k , and the Feynman parameters x, y . We can then do the integral no problem, and we get

$$J^\mu(q+k) = -\frac{imee^{\mu\nu\lambda\sigma}\sigma_2(k)q_\lambda(q_\sigma + k_\sigma)}{8\pi^2} \int_{x,y} \frac{1}{\Delta}. \quad (108)$$

Now we have been assuming that the fermion mass m is much bigger than the momentum scales of interest, namely q and k . With this approximation Δ turns out to simply be m^2 . We can also drop the quadratic in q term, since the gauge field is assumed to be smooth so that $[\partial_\mu, \partial_\nu]A = 0$, and so we get

$$J^\mu(q+k) = -\frac{iee^{\mu\nu\lambda\sigma}\sigma_2(k)q_\lambda k_\sigma}{4\pi^2 m} A_\nu(q). \quad (109)$$

In real space, we could write this as

$$J^\mu = -\frac{iee^{\mu\nu\lambda\sigma}F_{\nu\lambda}\sigma_1\partial_\sigma\sigma_2}{8\pi^2\sigma_1}. \quad (110)$$

Of course, this expression for J^μ is not general since we've chosen a particular vacuum to evaluate the diagram in. Getting the general expression is easy though: we just write down the appropriate expression which reduces to the above answer when $\langle\sigma\rangle$ is real and which is invariant under the chiral rotation $(\sigma_1, \sigma_2)^T \mapsto R_\alpha(\sigma_1, \sigma_2)^T$ (where $R_\alpha \in O(2)$ is a rotation by α under which $\psi \mapsto e^{-i\alpha\bar{\gamma}/2}$). Such a rotation preserves the matrix $iY\partial$, and so we write the general J^μ as

$$J^\mu = \frac{iee^{\mu\nu\lambda\sigma}F_{\nu\lambda}}{8\pi^2|\sigma|^2} (\sigma_2\partial_\sigma\sigma_1 - \sigma_1\partial_\sigma\sigma_2), \quad (111)$$

which indeed reduces to our previous expression when we choose the $\langle\sigma\rangle = \langle\sigma_1\rangle$ vacuum. One can also check that if we go back and work about the $\langle\sigma\rangle = i\langle\sigma_2\rangle$ vacuum with an $i\bar{\gamma}\sigma_2$ mass term for the fermions and a $\bar{\psi}\psi\sigma_1$ interaction vertex, then we get the other term in the above equation. Finally, parametrizing $\sigma = me^{i\theta}$, we find that the gauge current is

$$J^\mu = \frac{e}{8\pi^2} e^{\mu\nu\lambda\sigma} \partial_\nu \theta F_{\lambda\theta}, \quad (112)$$

which ends up being independent of the fermion mass and matches the $d\theta \wedge F \wedge A$ answer we had guessed earlier for the shift in the action, since $J \propto \star(d\theta \wedge F)$. Since $dF = 0$, The divergence in the gauge current is

$$d^\dagger J = \frac{e}{8\pi^2} \star (d^2\theta \wedge F). \quad (113)$$

This is not zero, since θ is not a well-defined function when we consider a string defect configuration, only $d\theta$ is. Since

$$\int_{\gamma} d\theta = 2\pi \quad (114)$$

for any curve γ which links the defect (assuming the defect has 2π winding),

$$d^2\theta = 2\pi \hat{s}, \quad (115)$$

where s is a 2-chain parametrizing the string and \hat{s} is its Poincare dual. This works because

$$\int_{\gamma} d\theta = \int_D d^2\theta = 2\pi \int_D \hat{s} = 2\pi \text{int}(D, s) = 2\pi, \quad (116)$$

where D is a disc bounded by γ and $\text{int}(D, s) = 1$ is the intersection number. For example, if s lies along the z -axis in space and doesn't move in time, $d\theta = \frac{1}{r}\hat{\phi}$ in cylindrical coordinates. Thus the divergence in the gauge current

$$d^\dagger J = \frac{e}{4\pi} \star (\hat{s} \wedge F) \quad (117)$$

is non-zero only on the string defect, and so under gauge transformations $A \mapsto A - d\alpha$, the action shifts as

$$S \mapsto S + \frac{e^2}{4\pi} \int_s F\alpha, \quad (118)$$

where the integral is over the string defect. In order for this theory to make sense, this anomaly needs to be canceled by something living on the string.

Anomaly cancellation: We expect that an anomaly of the above form will be canceled by a gauge anomaly from chiral fermions living on the string, and this is indeed the case. Suppose the string lives along the z axis, and work in cylindrical coordinates. The Dirac equation is

$$(iD_A + m(r)e^{i\bar{\gamma}\theta})\psi = 0, \quad (119)$$

where $m(r)$ goes to zero at $r = 0$ and goes to the minimum of $V(\sigma)$ at $r \rightarrow \infty$. Since D_A anticommutes with $\bar{\gamma}$, we can write

$$iD_A \psi_{\mp} + m(r)e^{\pm\theta}\psi_{\pm} = 0, \quad (120)$$

where ψ_{\pm} are eigenspinors of $\bar{\gamma}$. For simplicity, let us take only A_z, A_t to be non-zero (we just want to know the form of the fermion solution for a particular gauge field configuration). This still allows for non-zero field strength on the string (i.e. non-zero E_z), which gives us an F for which the anomaly is non-zero. Our ansatz for ψ_- is

$$\psi_- = \eta f(r), \quad (121)$$

where η is a zero-mode of the Dirac operator restricted to the string:

$$i(\gamma^t \partial_t + \gamma^z \partial_z - ie(A_t + A_z))\eta = 0. \quad (122)$$

Since the string worldsheet is two-dimensional, we can assign the zero-mode η a definite parity, and thus it has the potential to contribute to a cancellation of the gauge anomaly (we will determine its parity shortly). Thus we need to solve

$$\eta i \not{D}_A^\perp f(r) = -m(r)e^{i\theta}\psi_+, \quad (123)$$

where \not{D}_A^\perp is the Dirac operator on the coordinates orthogonal to the string. Solving this is easy since A is zero for these coordinates. Thus we have

$$i(\cos\theta\gamma^x + \sin\theta\gamma^y)\eta\partial_r f(r) = -m(r)e^{i\theta}\psi_+. \quad (124)$$

We can take care of the $m(r)$ factor with an $f(r)$ which is exponentially localized to the string. We write $f(r) = \exp(-\int_0^r dr' m(r'))$, thus

$$i(\cos\theta\gamma^x + \sin\theta\gamma^y)\eta \exp\left(-\int_0^r dr' m(r')\right) = e^{i\theta}\psi_+. \quad (125)$$

We can cleverly re-write this as

$$i\gamma^x e^{i\theta\bar{\gamma}_{\text{ext}}}\eta \exp\left(-\int_0^r dr' m(r')\right) = e^{i\theta}\psi_+, \quad (126)$$

where $\bar{\gamma}_{\text{ext}} = -i\gamma^x\gamma^y$ is the chirality operator on the components orthogonal to the string. To write the Dirac equation in this form, we have taken η to be an eigenspinor of $\bar{\gamma}_{\text{ext}}$, this is possible since $\bar{\gamma}$ and $\bar{\gamma}_{\text{ext}}$ commute. Thus we can solve the problem by choosing η to be the positive-chirality eigenstate of $\bar{\gamma}_{\text{ext}}$. Since η came from ψ_- which is a negative chirality eigenstate of $\bar{\gamma}$, the fact that $\bar{\gamma}_{\text{ext}}\eta = +\eta$ implies $\bar{\gamma}_{\text{int}}\eta = -\eta$, so that η is a negative-chirality zero-mode on the string. Since this also determines ψ_+ , we have

$$\psi_- = \eta \exp\left(-\int_0^r dr' m(r')\right), \quad \psi_+ = i\gamma^x\eta\psi_-, \quad (127)$$

where again, $\bar{\gamma}_{\text{int}}\eta = -\eta$. Since γ^x commutes with $\bar{\gamma}_{\text{int}}$, we importantly have that the chirality of the zero mode components of both the ψ_+ and ψ_- is *the same* (while the chirality of the components orthogonal to the string are opposite).

Recapitulating, the presence of the string leads to a pair of zero modes propagating along the string with the same chirality. Since they are chiral with negative chirality, we can write their contribution to the Lagrangian as

$$i\bar{\eta} \not{D}_A^s \frac{(1 - \bar{\gamma}_{\text{int}})}{2} \eta, \quad (128)$$

where \not{D}_A^s is the Dirac operator restricted to the string. Thus we have a coupling between the gauge field and the chiral current, which leads to a gauge anomaly. When we do a gauge transformation, this shifts by the usual gauge anomaly in two dimensions, determined by the index theorem. For $A \mapsto A - d\alpha$, the action for the string zero modes changes by

$$\delta S_s = - \int_s \frac{e^2}{4\pi} F\alpha, \quad (129)$$

which exactly cancels the anomaly in the gauge current caused by the current-fermion-gauge-field diagram we computed earlier! Thus anomaly inflow from the bulk onto the string defect renders the whole theory self-consistent and anomaly-free.

A brief digression on terminology: the $\int F \wedge F$ thing we usually see for the Abelian instanton number isn't the second Chern class, although sometimes people say it is. Recall that the Chern classes are obtained from looking at the different order terms in the expansion of

$$\det \left(\mathbf{1} + \frac{iF}{2\pi} \right). \quad (130)$$

By using $\det A = \exp(\text{Tr} \ln A)$ and expanding the $\ln A$ in the exponent, we get

$$\det \left(\mathbf{1} + \frac{iF}{2\pi} \right) = 1 + \frac{i}{2\pi} \text{Tr} F + \frac{1}{8\pi^2} (\text{Tr}(F \wedge F) - \text{Tr}(F) \wedge \text{Tr}(F)) + \dots \quad (131)$$

For non-abelian theories we usually choose F to be anti-Hermitian and traceless, and so the second Chern class is

$$c_2 = \frac{1}{8\pi^2} \text{Tr}(F \wedge F). \quad (132)$$

However when the gauge group is $U(1)$ then $\text{Tr} F = F$, and so $c_2 = 0$. The abelian instanton number is instead related to the second Chern character, namely

$$\text{ch}_2 = \frac{1}{2}(c_1 \wedge c_1 - 2c_2). \quad (133)$$

(For the definition, recall that the total Chern character is $\text{ch} = \det \exp(-iF/2\pi)$). For $U(1)$ we then get $\text{ch}_2 = c_1^2/2$. So the (integral of the) second Chern character for $U(1)$ bundles is

$$\int \text{ch}_2 = \frac{1}{8\pi^2} \int F \wedge F. \quad (134)$$

On spin manifolds the intersection form is even; applying this to the integral class $F/2\pi$ tells us that on spin manifolds, $\int c_1^2 \in \mathbb{Z}$.

13 April 26 —Dynamic generation of topological photon mass and domain wall anomalies

Consider QED₃, namely

$$S = \int d^3x \left(i\bar{\psi} \not{D}_A \psi + m(x) \bar{\psi} \psi - \frac{1}{2} F \wedge \star F \right). \quad (135)$$

This problem has two parts. First, show that radiative corrections from the fermions induce a Chern-Simons term and hence a topological mass for the photon (at momentum scales smaller than the fermion mass).

Second, consider a domain wall where $m(x)$ changes sign. Show that such an object hosts chiral fermions with a gauge anomaly, and that the anomaly is canceled by the dynamically generated CS terms away from the domain wall.

Solution:

Chern-Simons terms: To find the CS term induced for the gauge field, we just have to compute the one-loop contribution to the effective action for the gauge fields after integrating out the fermions. The relevant integral is

$$\text{bubble}^{\mu\nu} = (-1)i^2(-ie)^2 \int_p \text{Tr} \left[\gamma^\mu \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2} \gamma^\nu \frac{\not{p} + m}{p^2 - m^2} \right]. \quad (136)$$

After doing Feynman parameters to simplify the denominator there are two contributions to the integral in the large m limit: one proportional to $m\gamma^\mu \not{q} \gamma^\nu$, and another which contains terms like $g^{\mu\nu}m^2$ which do not depend on $\text{sgn}(m)$ (the others vanish under $\not{p} \rightarrow -\not{p}$ or under the trace). The latter terms will get renormalized away when we regularize a la PV, so we will ignore them in what follows. In mostly-negative signature \mathbb{R} -time, our γ matrices are

$$\gamma^0 = X, \quad \gamma^1 = iY, \quad \gamma^2 = iZ. \quad (137)$$

Thus they satisfy

$$\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\lambda] = -2i\epsilon^{\mu\nu\lambda}. \quad (138)$$

Using this, we get (there are terms in the numerator involving the Feynman parameter and \not{q} coming from the shift in p , but these end up cancelling due to the spin sum)

$$\text{bubble}^{\mu\nu} = 2ie^2 \epsilon^{\mu\lambda\nu} \int_{p,x} \frac{q_\lambda m}{(p^2 - \Delta)^2}, \quad (139)$$

where Δ is a function of m, q and the Feynman parameter x . Doing the integral,

$$\begin{aligned} \text{bubble}^{\mu\nu} &= -2ei\epsilon^{\mu\nu\lambda} q_\lambda m \frac{i}{(4\pi)^{3/2}} \Gamma(1/2) \Delta^{-1/2} \\ &= \text{sgn}(m) \frac{e^2}{4\pi} \epsilon^{\mu\lambda\nu} q_\lambda, \end{aligned} \quad (140)$$

where we have taken the long-wavelength limit where $m^2 \gg q^2$ so that $\Delta \rightarrow m^2$. This diagram appears in the effective action for the gauge field with a coefficient of $-1/2$ since it is the quadratic term in the expansion of the $\ln \det$, and so it thus gives us the CS term at level $-\text{sgn}(m)/2$. The CS term violates parity (i.e. either reflection or time-reversal) and so our theory violates parity if $m \neq 0$ —but we already knew this, since fermion mass terms are parity-odd in odd dimensions (this provides another derivation of the oddness of m : under T the CS level is odd, so that m must be odd as well). The fractional level here arises since we didn't properly regulate the theory using e.g. PV regularization. If we did and chose e.g. the mass of the PV field to be large and positive, we would get a level of $k = [\text{sgn}(m) - 1]/2 \in \{0, -1\}$, which is well-defined. We could also have chosen the PV field to have a large negative mass, in which case the level would be valued in $\{0, 1\}$. We have a freedom of changing the sign of m and also the sign of the mass of the PV regulators, but the relative sign between the two masses is physical and determines what CS level we get (the two choices are related by time reversal). Another comment is that since the IR CS level is quantized, our calculation must be one-loop exact (diagrams with l loops scale with the

coupling constant as $(e^2)^{l-1}$, so if $l \neq 1$ made a contribution we could tune e^2 continuously and get a non-integral level).

Generalizing slightly to N_f fermions of masses m_i , we have

$$k_{IR} = k_{UV} - \sum_i \frac{\text{sgn}(m_i)}{2}, \quad (141)$$

where k_{UV} is the level of the sum of the PV regulator fields. In particular if $N_f \in (2\mathbb{Z}+1)$, we always have a fractional CS level in the UV. Also note that when $m_i = 0$, if $N_f \in (2\mathbb{Z}+1)$ the UV theory always breaks parity symmetry, since we have an odd number of PV fields—this is the parity anomaly (if $N_f \in 2\mathbb{Z}$, we could choose $N_f/2$ positive-mass PV fields and $N_f/2$ negative-mass PV fields, and the effective CS level would be zero). Note that for odd N_f parity is broken in the UV; it is not an infrared effect associated with the CS level generated by integrating out the fermions.

Domain wall and anomaly cancellation: Now we consider a domain wall where $m(x)$ changes sign. For concreteness, let z be the direction normal to the domain wall. Then the CS terms generated by the fermions are not gauge-invariant, since under $A \mapsto A - d\alpha$ the action changes as

$$S \mapsto S + \frac{2e^2}{4\pi} \int_w \alpha F, \quad (142)$$

where \int_w is an integral over the domain wall and we have chosen the mass to be positive on the $z > 0$ side of the domain wall wolog. This gauge-non-invariance must be canceled by something living on the wall.

Indeed it is; let's solve the Dirac equation to get the relevant anomaly-cancelling zero modes. We have, say on the $z > 0$ side of the domain wall,

$$i(\not{D}_A^w + \gamma^2(\partial_z - ieA_z))\psi = -m(z)\psi. \quad (143)$$

We choose an ansatz where $\psi = \eta(x^0, x^1)f(z = x^2)$, with η a zero mode of the Dirac operator restricted to the wall. We can choose it to have definite chirality under $\bar{\gamma}_w = i\gamma^0\gamma^1$ since the wall is two-dimensional. Let $\bar{\gamma}_w\eta = c_\eta\eta$. We can match the $m(z)$ on the RHS with the usual exponential factor, so we take

$$\psi = i\eta \exp\left(-\int_0^z dz'(m(z') - ieA_z)\right). \quad (144)$$

For this to work, we need

$$i\gamma^2\eta = \eta. \quad (145)$$

But we have

$$i\gamma^2\eta = -i\gamma^0\gamma^1\eta = -\bar{\gamma}_w\eta = -c_\eta\eta, \quad (146)$$

and so if we choose η to be of negative chirality on the wall, we'll get a solution to the Dirac equation. A similar story gets told if we focus on $z < 0$: the two minus signs from $m(-z) = -m(z)$ and $\partial_{-z} = -\partial_z$ cancel out, and in the end we get two chiral zero modes on the domain wall, propagating in the same direction, with action

$$S_w = 2 \int_w i\bar{\eta}\not{D}_A\eta. \quad (147)$$

The fact that the two modes propagate in the same direction along the wall (and so propagate with opposite handedness in their respective half-planes) is because a region with $m(z) > 0$ is essentially the time-reversed version of a region with $m(z) < 0$, since the Dirac mass is odd under time reversal in three dimensions.

Since the wall zero modes are chiral and coupled to the gauge field, they will have a gauge anomaly. Under $A \mapsto A - d\alpha$, the action thus shifts as

$$S \mapsto S - \frac{e^2}{2\pi} \int_w \alpha F. \quad (148)$$

This cancels the gauge anomaly coming from the bulk CS terms, and so the full action is well-defined.

14 April 27 - 28 — β function for sine-Gordon in 2+1 dimensions at finite temperature and the KT transition for $U(1)$ gauge theory

Our diary entry for these two days will be to understand the analysis of the KT transition that occurs in $U(1)$ gauge theory in three dimensions as studied in [?]. Actually, my original motivation for looking at this came from reading [?], which argued that there exists a KT-type confinement transition in parity-invariant QED_3 at finite temperature. More recently, similar types of transitions were discussed briefly in [?], and all these sorts of transitions can be understood elegantly in terms of higher symmetries—this was why I was originally interested, but we won't really talk about higher symmetries in what follows.

An outline of this diary entry: first, we spend a few pages motivating why we expect there to be a KT-type transition in the $U(1)$ gauge theory and explaining how we can think of the critical properties of the gauge theory in terms of the XY model in two dimensions. We then will go through the RG calculation which shows the existence of the KT transition explicitly. This is technically interesting mostly since we will be at finite temperature.

Confinement phase transitions in gauge theories as regular phase transitions in scalar theories

Recall that the most useful diagnostic of confinement at finite temperature is the Polyakov loop

$$P(x) = \text{Tr} \exp \left(i \int dt A_0(x, t) \right), \quad (149)$$

where we are working in the convention where the gauge coupling appears in the action as $\frac{1}{2g^2} F \wedge \star F$. In what follows we will only be working with $U(1)$ theories, and so we can drop the Tr. If the vev of the Polyakov loop vanishes then the free energy of a single charge is infinite, and the theory confines. On the other hand, if the Polyakov loop is allowed to develop a vev, then we have symmetry breaking and the free energy of an external (non-dynamical) charged source is finite — this is the deconfined phase. Of course, we will be working with pure gauge theory throughout, since adding dynamical matter means that screening occurs, preventing

the Polyakov loop from being a useful diagnostic of confinement (since it always has a finite vev after regularization, assuming we add a Wilson loop which is well-defined, i.e. is taken in a legit representation of $U(1)$ so that the charge of the Wilson loop is in \mathbb{Z}).

To come up with an effective action for the order parameter $P(x)$, we need to integrate out the other degrees of freedom, namely the spatial components of the gauge fields. We can argue that this procedure will always give us a local theory as follows: at $T \rightarrow 0$ we will assume the gauge coupling g^2 is such that the theory confines. This will be the case in the example of interest to us, since $U(1)$ gauge theory is confining at $T = 0$ in two spatial dimensions. Thus the spatial Wilson loops obey an area law, and the correlations between the spatial gauge fields are short-ranged, and so we can integrate them out while keeping the theory local.

At $T \rightarrow \infty$, we get what is effectively a zero-temperature version of the gauge theory in one less dimension, at coupling $g_{d-1}^2 = Tg_d^2$, plus a field coming from the now-compactified A_0 ⁵. Since the lower dimensional theory is confining if the $T \rightarrow 0$ theory is (going to lower dimensions makes confinement stronger), the spatial gauge fields again have exponentially decaying correlations and can be integrated out. Note that the fact that the spatial gauge fields have area-law decaying correlations does *not* mean that the full theory (confined spatial gauge fields + compactified A_0 degree of freedom) is confining! Confinement is only probed by measuring the free energy of charged sources (captured by $P(x)$), and not by measuring the spatial correlations of the gauge fields. So confinement is diagnosed with temporal Wilson loops in the $T = 0$ theory, which become things related to the compactified A_0 field in the $T \rightarrow \infty$ theory—they are in principle very different objects than the spatial Wilson loops.

So, as was done in [?], we can then conjecture that at all temperatures, we can always integrate out the spatial gauge fields, producing an effective action for the Polyakov loop $P(x)$. Thus this theory can actually be described by a scalar field spin model in $d - 1$ dimensions. In particular, it can be described with a global 0-form symmetry equal to the 1-form symmetry of the Polyakov loops in the gauge theory (namely, the center of the gauge group). For $U(1)$ gauge theory the dual spin model is thus an XY model, while for \mathbb{Z}_N gauge theory (obtained by Higgsing a charge N scalar with the term $\lambda \cos(\partial_\mu \theta - iNA)$, taking λ to be large, and working in unitary gauge) we get an XY model with cosine-type interactions.

Let's look at the high-temperature and low-temperature limits of the gauge theory and see how they behave in the spin model. At $T \rightarrow 0$ we are in a confining phase, with a linear confining potential between free charges. Thus a two-point function of Polyakov loops goes as

$$\langle P(x)P^*(0) \rangle \sim e^{-\alpha|x|/T}, \quad (150)$$

where α is the string tension — this is an area law. This then corresponds to the disordered phase of the associated spin model, where the spin model correlation length is read off to be

$$\xi = \frac{T}{\alpha}. \quad (151)$$

So, we should think of the confining phase as the *disordered* phase.

⁵The coupling comes from setting a gauge in which A_0 is constant in time, and then doing the integral along the thermal circle. The relevant term in the action is then $(\beta/2g^2) \int \nabla A_0 \cdot \nabla A_0$, and so the effective coupling is $g_{\text{eff}} = \sqrt{T}g$.

Now we take $T \rightarrow \infty$. Generically we would expect the gauge theory to become deconfined—let's see why we would expect this. As $T \rightarrow \infty$, the thermal circle shrinks to become very small. Intuitively this means that nonzero momentum modes in the compactified direction all have very high energy, and so we can focus on field configurations that are constant in time. This means that the value of the Polyakov loop is approximately constant throughout space, meaning that the Polyakov loops have long range correlation functions, implying SSB of the 1-form symmetry and deconfinement.

More precisely, we can see this by thinking about the Wilson action for gauge theory, which looks something like

$$S \sim \sum_{\text{plaquettes}} (\beta_t E^2 + \beta_s B^2), \quad (152)$$

where the E^2 and B^2 terms are the electric and magnetic fields at the plaquettes, realized by taking the product of Wilson lines around the edges of the (temporal and spatial, respectively) plaquettes. The ratio $\sqrt{\beta_t/\beta_s}$ represents the ratio of the spatial lattice spacing to the temporal (thermal circle) lattice spacing, and so fixing the number of lattice sites, we see that when we take T to be large, $\sqrt{\beta_t/\beta_s} \gg 1$. This means that at high T , configurations with nonzero electric flux (a definite value of the 1-form charge) are suppressed: this is deconfinement, and the electric field is “screened” (although we don't actually have charges).

Now, consider two Polyakov loops separated by one spatial lattice spacing: $P(x)$ and $P(x + \hat{i})$. We can write

$$P(x) = \exp \left(i \int_{\text{strip}} F \right) P(x + \hat{i}), \quad (153)$$

where the integral is over the thin cylinder enclosed by the two loops. Since F on this cylinder is the electric field strength and the electric field gets frozen out at high T , we have $P(x) \approx P(x + \hat{i})$, and thus the Polyakov loop must be slowly varying in space. This means that the free energy of two external sources separated by a distance x does not grow linearly with their separation: instead, we have the scaling

$$\langle P(x) P^*(0) \rangle \sim \text{const.} \quad (154)$$

Thus at high T we are in the deconfined phase, and the system can screen charges. In the spin model, we see that this corresponds to symmetry breaking, and so the deconfined phase maps onto the ordered phase of the spin model.

It's worth emphasizing that the *high-temperature* phase of the gauge theory corresponds to the *ordered* phase of the spin model. Normally we think of going to higher temperatures as restoring symmetry, but with this example it's the opposite (and indeed, higher-form symmetries like this generically prefer to be broken at high temperature rather than low temperature).

However, this symmetry-breaking argument is only true in large enough dimensions. Indeed, we have argued that in the case of $U(1)$ gauge theory in 2+1D at high T (which is what we will be most interested in), we can map the problem onto an XY model in two dimensions (zero temperature), which cannot actually have a symmetry-breaking phase by virtue of the Mermin-Wagner theorem. This means that at high T , this theory is actually never really deconfined at high temperature. Instead of a constant two-point function, we get

$$\langle P(x) P^*(0) \rangle \sim \frac{1}{|x|^\eta}, \quad (155)$$

where η is some constant that depends on the gauge coupling. We might call this “quasi-long-ranged-deconfinement”, but we do not have a genuinely deconfined phase at high T . Neither is it deconfined at low temperature because of the effects of monopoles a la Polyakov (this is also true in the continuum without monopoles), and so actually we never have a genuine deconfined phase. However, based on our experience with the XY model, we can guess that there will be some sort of KT-type transition, where the theory always is confined but the characteristics of how it is confined change with the coupling (or with T). This is corroborated by looking at how the two-point function of the Polyakov loop scales: it goes from an exponential decay at low T to a power-law at high T , and thus there must be some kind of phase transition in between. We have good reason to expect that it will be a KT transition, because of our experience with the XY model.

Getting to the sine-Gordon action

First we want to get to a sine-Gordon action starting from the gauge theory, which will make the RG analysis easier. This is standard stuff; see e.g. the discussion in [?]. The Polyakov loops, which let us build intuition for what to expect, will actually not play a role in what follows: the dual field will still be a scalar, but it will be related to A through Hodge duality as $d\phi \sim \star F$, and not by tracing A over a circle.

We start from the usual $U(1)$ gauge theory action at coupling g , and then dualize to a compact scalar field ϕ in the usual way. We will be working on a lattice, but will use continuum notation — hopefully this will not be unduly confusing.

If we are working on a manifold with trivial first cohomology⁶ and the gauge bundle is trivial, then the action in terms of the scalar ϕ is

$$S = -\frac{g^2}{2(2\pi)^2} \int d\phi \wedge \star d\phi + \int \phi \wedge q. \quad (156)$$

Here q parametrizes the locations of the instantons (which we will refer to as monopoles since we are in 2+1D): it is a three-form defined on the cubes of the spacetime lattice and is equal to

$$q = \sigma(x) d^3x, \quad (157)$$

where $\sigma(x)$ is the monopole charge at each cube. Thus we have a situation quite similar to the XY model:

$$\text{spin waves and vortices} \sim \text{gauge fields and monopoles} \quad (158)$$

To get the partition function, we just need to know how to sum over the configurations of monopole charges. We can sum over all possible charge configurations and assume that only charge $\sigma = \pm 1$ monopoles contribute (valid in the limit where the monopole fugacity, which we have to set by hand, is small) to get

$$Z = \left\langle \sum_{N=1}^{\infty} \sum_{\{\sigma_j\} \in \mathbb{Z}_2^N} y^N \frac{1}{(N/2)!^2} \binom{N}{N/2}^{-1} \prod_{i=1}^N \int d^D r_i \exp \left(i \sum_{j=1}^N \sigma_j \phi(r_j) \right) \right\rangle \quad (159)$$

⁶This won’t be true for us since we’ll be on a thermal cylinder. However, the zero modes that we would have to account for actually get killed off by a choice of boundary conditions on the ends of the cylinder at spatial infinity (which we must set non-symmetrically since we are interested in SSB), and so we don’t have to care about them in what follows.

where the expectation value is taken with respect to the free ϕ action and we've defined the dimensionfull variable

$$y \equiv \frac{1}{a^D} e^{-\mu}, \quad (160)$$

with $e^{-\mu}$ the monopole fugacity and a a short-distance cutoff. Some comments are in order: first, the expectation value will vanish for any configurations that have non-zero net charge. Neutral charge configurations will have $N/2$ positive and negative charges. Since the positive charges are indistinct from one another (as are the negative charges), we are required to put in the $[(N/2)!]^{-2}$ term. Furthermore, we need to mod out by the number of ways to partition N charges into a neutral configuration, which is the reason for the $\binom{N}{N/2}$ factor. Expanding out the exponential and then re-exponentiating, we get the sine-Gordon action

$$S = \int d^D x \left(\frac{\alpha}{2} d\phi \wedge \star d\phi - 2y \cos \phi \right), \quad (161)$$

where we've defined

$$\alpha \equiv (g/2\pi)^2. \quad (162)$$

Spatial RG procedure

Having obtained the sine-Gordon model, we now want to do an RG analysis and see whether we get a BKT-like RG flow. This is different from the standard analysis since we will work in arbitrary dimensions (for now), and since we're at finite temperature. Because of non-zero temperature, we will do RG in space, but not in the thermal direction. This anisotropic RG procedure is a bit awkward, but lets us capture the physics of the RG flow correctly.

We will follow [?] pretty closely for now. As usual, we split up ϕ as

$$\phi = \varphi + \chi, \quad (163)$$

where φ is the low-momentum ($p < \Lambda'$) part and χ is the high-momentum ($\Lambda' < p < \Lambda$) part. The $(\partial\phi)^2$ part splits into $(\partial\varphi)^2 + (\partial\chi)^2$ since $(\partial\varphi)^2$ is block diagonal in momentum space, and so

$$Z' = \int D\varphi \exp \left(-\frac{\alpha}{2} \int \partial_\mu \varphi \partial^\mu \varphi \right) Z'[\varphi], \quad (164)$$

where

$$Z'[\varphi] = \int D\chi \exp \left(-\frac{\alpha}{2} \int \partial_\mu \chi \partial^\mu \chi \right) \exp \left(2y \int \cos(\varphi + \chi) \right). \quad (165)$$

The strategy now is to work at weak coupling, where the monopole fugacity y is small (when the field is fluctuating slowly, adding in a monopole is costly since one has to twist the value of the field at a large number of lattice sites, while while the field is rapidly fluctuating, this twisting is sort of already going on because of the fluctuations). Now, we

know that $\ln Z[J]$ is the generating function for correlation functions with disconnected pieces subtracted. For example, we schematically have

$$\left. \frac{\delta^2}{\delta J(x)\delta J(y)} \right|_{J=0} \ln Z[J] = \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle, \quad (166)$$

where $Z[J]$ includes an $\exp(\int \phi J)$ coupling. Taking n variational derivatives of $\ln Z$ produces n -point correlation functions with their disconnected pieces subtracted off. This allows us to easily write down a series expansion for $\ln Z'$ in terms of y , which appears as a current for the $\int \cos$ term. Thus

$$\ln Z' = 1 + 2y \int d^d x \langle \cos \phi \rangle + \frac{(2y)^2}{2} \int d^d x d^d y (\langle \cos \varphi(x) \cos \varphi(y) \rangle - \langle \cos \varphi(x) \rangle \langle \cos \varphi(y) \rangle) + \dots, \quad (167)$$

where the expectation values are taken with respect to the free χ action.

To compute the expectation value $\langle \cos \rangle$, we have to compute $\langle e^{\pm i\chi(x)} \rangle$. We can do this by completing the square in the action in the usual way. Since the expectation value is independent of x , we get

$$\langle \cos(\varphi + \chi) \rangle = \exp\left(-\frac{1}{2}D(0)\right) \cos \varphi. \quad (168)$$

where D is the scalar propagator for χ , $D(x-z) = \langle \chi(x)\chi(y) \rangle$. Note that in our conventions D is dimensionless, and so $D \neq \nabla^{-2}$. Instead,

$$D(x-z) = \frac{1}{2\alpha} \nabla^{-2}(x-z). \quad (169)$$

Since the mass dimension of α is $[\alpha] = D-2$, the dimension of ∇^{-2} must also be $D-2$. This works out since we want $\nabla_x^2 \nabla^{-2}(x-z) = \delta(x-z)$ to be a delta function we can integrate on a D -manifold, and so the mass dimension of the delta function must be D . Also note that D is the propagator for χ , which lives at high momentum. So when defining the real-space D , we have to define it as an integral of $D(k)$ only over a momentum shell $\Lambda' < k < \Lambda$.

When we do the mixed expectation values $\langle \cos \cos \rangle$, things are only marginally more complicated: now we need to compute $\langle \exp(i[\chi(x) \pm \chi(z)]) \rangle$. We still do this by completing the square, but now we get a factor like $D(x-z)$ instead of $D(0)$, so that

$$\langle \exp(i[\chi(x) \pm \chi(y)]) \rangle = \exp(-D(0) \mp D(x-z)). \quad (170)$$

Putting this in and using $\cos a \cos b = (\cos(a+b) + \cos(a-b))/2$, the order y^2 term in $\ln Z'$ is

$$\int d^D x d^D z y^2 e^{-D(0)} \left[(e^{-D(x-z)} - 1) \cos(\varphi(x) + \varphi(z)) + (e^{D(x-z)} - 1) \cos(\varphi(x) - \varphi(z)) \right]. \quad (171)$$

Now we need to make some approximations to make this more tractable. If we are in a big enough dimension⁷, $e^{\pm D(r)} - 1 \rightarrow 0$ for large r . So we may be able to get away with an

⁷For us, this big enough dimension is three: one might worry about three becoming two after compactification at high temperature, but we will be assuming that the monopole fugacity is small enough that the average separation of monopoles is much larger than the length scale set by the thermal circle, so that for the purposes of figuring out the behavior of $D(r)$, we can think of working in \mathbb{R}^3 .

expansion in small $|x - z|$. So define the relative coordinate r by

$$\varphi(z) = \varphi(x) + \partial_\mu \varphi(x) r^\mu, \quad (172)$$

and boldly substitute this in. This generates two terms, one of which goes like $\cos(2\varphi(x) + \partial_\mu \varphi(x) r^\mu)$ and thus will be irrelevant compared to $\cos(\varphi(x))$ (and so we will drop it). The other term we expand in small r^μ , and after dropping an (infinite) φ -independent constant and integrating by parts, we get

$$\int -d^D x d^D r y^2 e^{-D(0)} (e^{D(x-z)} - 1) (\partial_\mu \varphi(x) r^\mu)^2. \quad (173)$$

Putting this into the full action, we obtain

$$Z = \int D\varphi \exp \left(\frac{\alpha}{2} \int d^D x \varphi \partial_\mu K^{\mu\nu} \partial_\nu \varphi + 2y e^{-\frac{1}{2}D(0)} \int d^D x \cos \varphi \right), \quad (174)$$

where

$$K^{\mu\nu} = \delta^{\mu\nu} + \frac{y^2 e^{-D(0)}}{\alpha} \int d^D r (e^{D(r)} - 1) r^\mu r^\nu. \quad (175)$$

Note that K is diagonal. We should now re-scale so that the kinetic term has the same form as before we did the momentum-shell decomposition. Since $K^x \equiv K^{xx} = K^{yy} \neq K^t \equiv K^{tt}$, the scaling is anisotropic. This is of course expected since we are at finite T , and the propagator D is anisotropic. So define

$$\phi = \sqrt{K^x} \varphi \quad (176)$$

and then scale the momentum as $p' = \Lambda p / \Lambda'$ so that Λ is again made the momentum cutoff. When we scale the momentum, we are only scaling the *spatial* momentum, and leaving the temperature fixed. This produces

$$Z = \int D\phi \exp \left(\frac{\alpha}{2} \int d^D x \phi (\partial_i \partial^i + \gamma^2 \partial_t^2) \phi + 2y e^{-\frac{1}{2}D(0)} \int d^D x \cos(\phi / \sqrt{K^x}) \right), \quad (177)$$

where

$$\gamma^2 = (\Lambda / \Lambda')^2 K_t / K_x. \quad (178)$$

Note that the mass dimension of y is $[y] = D$, and so the transformation of y cancels out the transformation of the integration measure in the second term. Also note that our RG step has changed the form of the $\cos \phi$ mass term, since there is now a factor of $1/\sqrt{K_x}$ inside the cosine. This means that we should go back to the original action and instead write the term as

$$\cos(\phi) \mapsto \cos(\sigma \phi), \quad (179)$$

and then examine the RG flow of the dimensionless variable σ . This is the same as looking at the RG flow for the radius of the scalar.

These transformations change the form of the χ propagator D , which is the same as the propagator for ϕ but integrated over a different range of momenta. Namely the coefficient of the time derivative part changes, since we are only scaling in space and not time. Thus we need to go back to our original action and instead write

$$L = \frac{\alpha}{2} \phi (\partial_i \partial^i + \gamma^2 \partial_t \partial^t) \phi, \quad (180)$$

and then examine how γ flows under RG. This is essentially an anisotropic field strength renormalization procedure. The correct propagator for the high-momentum modes is thus (note the prefactor!)

$$D(x) = \frac{1}{2\alpha} \int_{\Lambda'}^{\Lambda} \frac{d^d p}{(2\pi)^d} T \sum_{n \in \mathbb{Z}} \frac{e^{ip \cdot x + i2\pi T n t}}{p^2 + (\gamma 2\pi T n)^2}. \quad (181)$$

Getting the β functions

So, all told we have three things that we need to keep track of under RG: the monopole fugacity (alias y), the period of the mass term (alias σ , alias the compactification radius for the boson), and the “effective temperature” or anisotropy between space and the thermal circle (alias γ ; T is held fixed).

To do the RG, we let $\Lambda' = \Lambda - \delta\Lambda$ for small $\delta\Lambda$. Then for $dD = D|_{\Lambda'=\Lambda-\delta\Lambda} - D|_{\Lambda'=\Lambda}$, we get

$$dD(x, t) = \frac{1}{2\pi} \delta\Lambda \frac{\Lambda}{2\alpha(2\pi)^2 T \gamma^2} J_0(\Lambda x) \sum_n \frac{e^{i2\pi n T t}}{(\Lambda/2\pi T \gamma)^2 + n^2} \quad (182)$$

in $d = 2$ spatial dimensions and

$$dD(x, t) = \frac{1}{(2\pi)^2} \delta\Lambda \frac{\Lambda^2}{2\alpha(2\pi)^2 T \gamma^2} \frac{\sin \Lambda x}{\Lambda x} \sum_n \frac{e^{i2\pi n T t}}{(\Lambda/2\pi T \gamma)^2 + n^2} \quad (183)$$

in $d = 3$.

Now we use the summation

$$\sum_{n \in \mathbb{Z}} \frac{e^{ina}}{n^2 + b^2} = \frac{\pi}{b} \frac{\sinh(ab) + \sinh[(2\pi - a)b]}{\cosh(2\pi b) - 1}, \quad (184)$$

which is valid for $0 \leq a \leq 2\pi$ and arbitrary b (which holds for us since t is runs from 0 to $1/T$ and we are using $a = 2\pi T t$).

This gives (with $b = \Lambda/2\pi T \gamma$)

$$dD(x, t) = \frac{1}{8\pi\alpha} \frac{\delta\Lambda}{\gamma} J_0(\Lambda x) \frac{\sinh(\Lambda t/\gamma) + \sinh[(\Lambda/T - \Lambda t)/\gamma]}{\cosh(\Lambda/T \gamma) - 1} \quad (185)$$

in two spatial dimensions. We can simplify this using

$$\frac{\sinh x + \sinh(y - x)}{\cosh y - 1} = \frac{e^x + e^{-x+y}}{e^y - 1}, \quad (186)$$

so that

$$dD(x, t) = \frac{\delta\Lambda}{8\pi\alpha\gamma} J_0(\Lambda x) \frac{e^{\Lambda t/\gamma} + e^{-\Lambda(t-1/T)/\gamma}}{e^{\Lambda/\gamma T} - 1}. \quad (187)$$

Likewise in three dimensions,

$$dD(x, t) = \frac{\delta\Lambda\Lambda}{16\pi^2\alpha\gamma} \frac{\sin(\Lambda x)}{\Lambda x} \frac{e^{\Lambda t/\gamma} + e^{-\Lambda(t-1/T)/\gamma}}{e^{\Lambda/\gamma T} - 1}. \quad (188)$$

To get the RG equations, we need to compute dK^x , $d\gamma$, and dy . In two dimensions, the first is (one α^{-1} from the definition of $K^{\mu\nu}$ and one from the propagator)

$$dK^x = \frac{y^2}{\alpha} \int d^2 r dD(r) x^2 = \frac{\delta\Lambda}{4\alpha^2} y^2 \Lambda^{-5} I_3, \quad I_n \equiv \int dr r^n J_0(r), \quad (189)$$

while the second is obtained from

$$d\gamma = \frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x). \quad (190)$$

and by using

$$dK^t = \delta\Lambda \frac{y^2 \gamma^2 I_1}{4\alpha^2 \Lambda^5} \left(4 + (\Lambda/T\gamma)^2 - \frac{2\Lambda}{T\gamma} \coth(\Lambda/2T\gamma) \right). \quad (191)$$

Getting dy is easy: we see that $2y$ is replaced by $2ye^{-\frac{1}{2}D(0)}$ under rescaling, so that under an infinitesimal rescaling, we have

$$dy = -\frac{1}{2}dD(0). \quad (192)$$

In three dimensions, we have the similar

$$dK^x = \frac{\delta\Lambda}{16\alpha^2} y^2 \Lambda^{-5} \tilde{I}_3, \quad \tilde{I}_n \equiv \int dr r^n \sin r \quad (193)$$

and

$$dK^t = \delta\Lambda \frac{y^2 \gamma^2 \tilde{I}_1}{4\pi\alpha^2 \Lambda^5} \left(4 + (\Lambda/T\gamma)^2 - \frac{2\Lambda}{T\gamma} \coth(\Lambda/2T\gamma) \right). \quad (194)$$

$d\gamma$ is essentially the same:

$$d\gamma = \frac{3}{2} \frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x). \quad (195)$$

Of course, the I integrals are infinite, but this won't be too much of a problem (this came from the fact that we're doing RG with a hard momentum cutoff instead of something better like using the CS equation).

RG: two spatial dimensions

Our dimensionless RG parameters in two spatial dimensions are (recall that the mass dimension of y is the dimension of spacetime)

$$b \equiv \sigma \frac{T}{4\pi^2 \alpha} = \frac{T}{g^2}, \quad \tau \equiv \frac{\Lambda}{T\gamma}, \quad m \equiv \frac{2y}{\Lambda^2 T}. \quad (196)$$

The definition of τ is used just because the ratio $\Lambda/T\gamma$ is common, and we have defined the dimensionless m (for “monopole”) which is related to the instanton fugacity. The parameter $b = T/g^2$ is the coefficient that appears in front of E^2 in the gauge theory Hamiltonian and so b^{-1} gives the “temporal coupling” of the gauge theory. When b becomes very large the electric field gets frozen out and pinned to a constant value throughout space, since as the radius of the thermal circle becomes small, non-zero-mode fluctuations of the electric flux are suppressed.

Now

$$d\gamma = \frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x), \quad (197)$$

and so

$$\begin{aligned} d\ln\tau &= 2d\ln\Lambda - \frac{1}{2}(dK^t - dK^x) \\ &= d\ln\Lambda(2 + b^2m^2f(\tau, \gamma)), \end{aligned} \quad (198)$$

where $f(\tau, \gamma)$ is a function which as $\tau \rightarrow 0$ goes to $-I_3$ (a constant).

From our expression for dK^x we also get

$$d\ln b = d\ln\Lambda \frac{y^2 I_3}{8\Lambda^4\alpha^2} = d\ln\Lambda \frac{m^2 b^2 \pi^4 I_3}{2}. \quad (199)$$

Finally, for dm we have

$$\begin{aligned} d\ln m &= -2d\ln\Lambda + \frac{d\Lambda}{16\pi\alpha\gamma} \coth(\Lambda/2\gamma T) \\ &= d\ln\Lambda \left(\frac{1}{4}\tau\pi b \coth(\tau/2) - 2 \right), \end{aligned} \quad (200)$$

where the first term comes from dy and the second from the factor of Λ^{-2} in the definition of m and we used

$$dD(0, 0) = \frac{\delta\Lambda}{8\pi\alpha\gamma} \coth(\Lambda/2\gamma T). \quad (201)$$

Absorbing the some unsightly prefactors by scaling $m \rightarrow m\pi^2\sqrt{I_3/2}$ and defining $dt = -d\ln\Lambda$ as time along the RG flow with $t = -\infty$ when no momenta have been integrated out and $t = \infty$ when all momenta have been integrated out, we can rewrite the RG equations as

$$\begin{aligned} db &= -m^2 b^3 dt, \\ dm &= -m \left(\frac{\pi}{4} b \tau \coth(\tau/2) - 2 \right) dt \\ d\tau &= -\tau dt \left(2 - m^2 b^2 \tilde{f}(\tau, \gamma) \right), \end{aligned} \quad (202)$$

where $\tilde{f}(\tau, \gamma)$ is a rather gross function that at $\tau \rightarrow 0$ goes to ⁸. For small enough b^2m^2 , $\ln\tau$ monotonically decreases along the flow, so that $\tau \rightarrow 0$ as $t \rightarrow \infty$. This is intuitive: if we zoom out of the cylinder, the cylinder becomes “longer” (we see more of the cylinder), but since we are holding the radius of the circle fixed, it effectively becomes “skinnier”, which means high “temperature”, which means low τ .

Now it is but a hop and a skip to KT! In fact, we essentially already have the KT RG equations, just with an extra parameter τ related to the fact that we’re at finite temperature. However, if we are at weak enough coupling, and are just limited in the deep IR, we can take $\tau \rightarrow 0$. Heuristically, we can think that as we zoom out in real space, the aspect ratio of the cylinder changes and the thermal circle appears to shrink, thus taking us to higher temperatures (recall that $\tau \rightarrow 0$ as $T \rightarrow \infty$). That this happens is not a forgone conclusion

⁸For posterity’s sake, $\tilde{f}(\tau, \gamma) = 1 - I_3^{-1}\gamma^2(4 + \tau^2 - 2\tau \coth(\tau/2))$.

because of the fact that γ scales under RG as well, but at weak coupling this turns out to be the case. Thus we may send $\tau \rightarrow 0$ and write

$$\begin{aligned}\frac{db}{dt} &= -m^2 b^3, \\ \frac{dm}{dt} &= -m \left(\frac{\pi}{2} b - 2 \right)\end{aligned}\tag{203}$$

as our RG equations. This is essentially the same as in the regular KT analysis, except that b appears in the first equation as b^3 instead of b^2 and some of the numerical coefficients are different by various factors of π and 2. The difference in the numerical factors doesn't affect the existence of a KT transition, and the differing power of b only changes the transition quantitatively, not qualitatively (it changes the time that trajectories spend near the $y = 0$ fixed point).

To analyze our version of the KT equations, we measure the distance to the critical point with the variable x :

$$b = \frac{4}{\pi}(1-x).\tag{204}$$

To lowest order in x , then,

$$\frac{dx}{dt} \approx \frac{16}{\pi^2} m^2, \quad \frac{dm}{dt} \approx 2mx.\tag{205}$$

The fact that we have b^3 instead of b^2 only appears at order x^3 , which is irrelevant for our discussion. We then use the above to write

$$\frac{dx^2}{dm^2} = \frac{8}{\pi^2} \implies m^2 = \frac{\pi^2}{8}(x^2 + C),\tag{206}$$

which is the familiar KT hyperbola (the only difference from regular KT is the number on the RHS). The integration constant C measures the distance from the critical trajectory, since when $C = 0$ we get the critical line. Using the RG equation for x , we see that

$$\frac{dx}{dt} = 2(x^2 + C),\tag{207}$$

which is exactly the same as the KT result. Since x is monotonically increasing, the system continues to get floppier and floppier (smaller spin stiffness) as we go to larger distances, since monopoles become more and more important at large distances. Proceeding from here, one can find the non-analyticities signifying the phase transition in the usual way.

15 April 29 — β function for scalars coupled to a non-Abelian gauge field

This is from P&S, chapter 16. Consider a non-Abelian gauge theory coupled to a scalar field:

$$\mathcal{L} = -\frac{1}{2g^2} \text{Tr}[F \wedge \star F] - (D_\mu \phi)^\dagger D_\mu \phi - V(\phi),\tag{208}$$

with the sign convention $D_\mu = \partial\mu - iA_\mu^a t^a$. Here t is taken in the adjoint representation when acting on A and is taken in some representation r when acting on ϕ . First, write down the Feynman rules for the theory.

Next, compute the β function for g to lowest order. To speed up things, you may take advantage of the fact that for Yang-Mills with fermions, the β function is

$$\beta_g = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3} C_2(G) - \frac{n_f}{2} C(r) \right), \quad (209)$$

where $t^a t^a = C_2(r) \mathbf{1}$ for the representation r (with $r = G$ indicating the fundamental) and $\text{Tr}[t^a t^b] = C(r)$.

Solution:

First let's get the Feynman rules. We can just do this by inspection: we have the usual 3-point and 4-point gross gauge boson vertices from the kinetic term, plus the $(D_\mu \phi)^\dagger D_\mu \phi$ term. The relevant parts of this are

$$g^2 A_\mu^a A_a^\mu \phi^\dagger \phi, \quad ig A^\mu (\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger). \quad (210)$$

The former term gives us a two-scalar-two- A interaction vertex. Since there are two different ways of attaching the gauge boson propagators onto the vertex, this vertex produces the Feynman rule

$$A^a A^b \phi^\dagger \phi \rightarrow ig^2 \{t^a, t^b\}. \quad (211)$$

For the trivalent vertex, we fix our conventions so that the derivative acts on the incoming ϕ propagator with momentum p , so that

$$A_\mu^a \phi^\dagger \phi \rightarrow -ig(p + p')_\mu t^a. \quad (212)$$

To get β_g , we will use the CS equation on the gauge boson + two-scalars vertex (the three-point function between a gauge boson and two scalars). To do this, we need to know the counterterm δ_3 for this vertex, as well as the wavefunction renormalization counterterms δ_ϕ, δ_e for the scalar and the photon. Then varying the three-point function with respect to the RG scale, we get (to lowest order in g)

$$2\gamma_\phi + \gamma_A + \partial_{\ln M} \left(-i\delta_g - ig \sum_i (\dots - \delta_{Z_i}) \right) - i\beta_g = 0, \quad (213)$$

where the \dots are independent of the RG scale, the sum is over the external legs and comes from the external propagator corrections, and δ_g is the counterterm for the vertex we are considering. By varying the two-point functions for the gauge field and the scalar with respect to M , we get (as usual to lowest order)

$$\gamma_i = \frac{1}{2} \partial_{\ln M} \delta_{Z_i}, \quad (214)$$

so that, if we denote $g\delta_v = \delta_g$ for the vertex counterterm,

$$\beta_g = g\partial_{\ln M} (-\delta_1 + \delta_\phi + \delta_A/2). \quad (215)$$

In Abelian gauge theory the first two terms would have canceled for Ward identity reasons, but now they do not. Instead, they provide some gauge-variant divergent part which cancels with a gauge-variant divergent part of the δ_A correction for the gauge field (coming from ghosts and the gauge field self-interaction). The gauge-variant part of δ_A doesn't depend on the matter content, meaning that $-\delta_1 + \delta_\phi$ actually does not depend on the matter content either! See P&S chapter 16.5 for some more discussion about this.

This means that the only dependence of the β function on the chosen matter fields comes from their contribution to δ_A , which is very easy to compute since we only have to do a single polarization bubble integral (the $\phi^2 A^2$ vertex diagram doesn't contribute to the β function since it is momentum-independent). The diagram is (we can ignore a potential mass for the scalar since it can't contribute to the β function by dimensional analysis)

$$(\text{polarization bubble})^{\mu\nu}(q) = -\text{Tr}[t^a t^b] g^2 \int_p \frac{(2p+q)^\mu (2p+q)^\nu}{p^2(p+q)^2}. \quad (216)$$

We know the form of the bubble will have a momentum dependence dictated by being properly transverse, so we can just compute the coefficient of e.g. the q^2 piece. This gives

$$C(r) g^2 \delta^{ab} \int_p \frac{4p^\mu p^\nu}{(p^2 - \Delta)^2} = \frac{1}{3} \frac{g^2 C(r)}{(4\pi)^2} q^2 \ln(-q^2/\Lambda^2). \quad (217)$$

This thus determines the counterterm for the wavefunction renormalization of A to be

$$\delta_A = \frac{1}{3} \frac{g^2 C(r)}{(4\pi)^2} \ln(M^2/\Lambda^2). \quad (218)$$

Its contribute to the beta function is thus

$$\beta_g = \frac{1}{2} g \partial_{\ln M} \delta_A = \frac{g^3 C(r)}{3(4\pi)^2}. \quad (219)$$

Now the only difference between the β function for the theory coupled to fermions and one coupled to scalars is the different values they contribute to δ_A by way of the polarization bubble. Both bubbles have the same color trace, but the fermions have an extra trace over spin indices. This ends up contributing an extra factor of 4 (see similar calculations done earlier), and so fermions in a representation t^a contribute four times as much to β_g as scalars in the same representation do. Thus we only have to turn to P&S chapter 16 and observe that the β function for fermions coupled to a non-Abelian gauge field is

$$\beta_g = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right). \quad (220)$$

Thus we deduce that if the field is coupled to scalars instead, the beta function is

$$\beta_g = -\frac{g^3}{(4\pi)^2} \left(\frac{11}{3} C_2(G) - \frac{1}{3} n_f C(r) \right). \quad (221)$$

Another method to calculate the β function is the background field method, where we integrate out fluctuations of the gauge field and express the effective action in terms of a new effective charge, which is more cmt-y. This is essentially in P&S, so I won't write it out here.

16 April 30 — Checking the chiral anomaly for non-Abelian gauge theory with operator relations

Consider a non-Abelian gauge theory coupled to Dirac fermions in four dimensions. Find the terms in the divergence of the chiral vector current which are quadratic and cubic in the gauge field, and show that they match with the usual $F \wedge F$ form for the anomaly. Do the calculation by explicitly computing the divergence of j_5^μ and splitting the fermion two point function as

$$\bar{\psi}(x - \epsilon/2) P \exp\left(i \int_{x-\epsilon/2}^{x+\epsilon/2} A_\mu dx^\mu\right) \psi(x + \epsilon/2), \quad (222)$$

and taking $\epsilon^\mu \rightarrow 0$.

Solution:

The chiral vector current comes from the symmetry $\psi \mapsto \exp(-\gamma^5 \alpha) \psi$, where by γ^5 we really mean $\gamma^5 \otimes \mathbf{1}_G$, where $\mathbf{1}_G$ is the identity matrix for the representation of the gauge group that the fermions live in. From the $i\bar{\psi}D\psi$ part of the action (with the convention $D = \partial + ieA$), we get

$$\partial_\mu \langle j_5^\mu \rangle = \partial_\mu \langle \bar{\psi}(x - \epsilon/2) \gamma^\mu \gamma^5 W \psi(x + \epsilon/2) \rangle, \quad (223)$$

where we've written W for the Wilson line connecting the two fermions. We take the derivatives and use the equations of motion

$$\not{D}\psi = ieA^a t^a \psi, \quad \partial_\mu \bar{\psi} \gamma^\mu = -ie\bar{\psi} A^a t^a \quad (224)$$

on the terms containing $\partial_\mu \psi$ and $\partial_\mu \bar{\psi}$. This produces

$$\partial_\mu \langle j_5^\mu \rangle = ie \langle \bar{\psi}(x + \epsilon/2) \gamma^\mu \gamma^5 (\partial_\mu A_\nu^a t^a \epsilon^\nu W - A_\mu^a(x + \epsilon/2) t^a W + W A_\mu^a(x - \epsilon/2) t^a) \psi(x - \epsilon/2) \rangle, \quad (225)$$

where the first term comes from the derivative of the Wilson line. We now need to move all the Wilson lines to stand to the right of all the t^a matrices. To this end, we expand the Wilson line to first order in ϵ and write

$$W t^a \approx \left(1 + ie \int A^b t^b\right) t^a = t^a + ie \int A^b (t^a t^b - [t^a, t^b]) \approx t^a W + e A_\nu^b \epsilon^\nu f^{abc} t^c. \quad (226)$$

Then to first order in ϵ ,

$$\partial_\mu \langle j_5^\mu \rangle = ie \langle \bar{\psi}(x + \epsilon/2) \gamma^\mu \gamma^5 ((\partial_\mu A_\nu^a - \partial_\nu A_\mu) t^a \epsilon^\nu + A_\mu^a A_\nu^b \epsilon^\nu f^{cab} t^c) \psi(x - \epsilon/2) \rangle, \quad (227)$$

which we obtained by doing some re-arranging of the gamma matrices and expanding the A 's about x (two terms coming from expanding the Wilson line cancel). We will see momentarily that the leading singularity in the fermion contraction will go as $1/\epsilon$, which will justify our first-order expansion of the terms in the previous equations.

Now we need to contract the fermions. To leading order, the fermion contraction is just a propagator connecting $x - \epsilon/2$ with $x + \epsilon/2$. However, this gives us something like $\text{Tr}[\gamma^5 \gamma^\mu \gamma^\nu]$, which is always zero. In order to get something nonzero, we are going to need

four gamma matrices to give a nonzero trace when they get hit by the γ^5 . The required four gamma matrices come from a process in which the background A field interacts once with the fermion line connecting the two points, forming a T -shaped Feynman diagram (see the picture in chapter 19 of P&S). The Feynman diagram for the $\partial_{[\mu}A_{\nu]}$ term gives

$$\partial_\mu \langle j_5^\mu \rangle = (ie)^2 \int_{p,q} \text{Tr} \left[\frac{\not{p} + \not{q}}{(p+q)^2} \gamma^5 \gamma^\mu \partial_{[\mu} A_{\nu]}^a t^a \epsilon^\nu A_\sigma^b(p) t^b \gamma^\sigma \frac{\not{q}}{q^2} e^{iq\epsilon - ip(x-\epsilon/2)} \right] + \dots \quad (228)$$

Note that here that the brackets simply mean $\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (there is no $1/2!$ normalization). In our signature $\text{Tr}[\gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho] = -4i\epsilon^{\mu\nu\sigma\rho}$ and $\text{Tr}[t^a t^b] = C(r)\delta^{ab}$ (where r is the representation the fermions are in, e.g. $C(N) = 1/2$ for the fundamental of $SU(N)$), so

$$\text{relevant diagram} = 4e^2 C(r) i \epsilon^\nu \epsilon^{\mu\lambda\sigma\rho} \partial_{[\mu} A_{\nu]}^a \int_{p,q} \frac{(q+p)_\lambda A_\sigma^a(p) q_\rho}{(p+q)^2 q^2} e^{iq\epsilon - ip(x-\epsilon/2)}. \quad (229)$$

Since we are sending $\epsilon \rightarrow 0$, the large q limit is what will be relevant, so we can take the denominator to just be q^4 . This is essentially because we are computing the divergences coming from splitting the product $\bar{\psi}\psi$, which come from high-momentum UV physics.⁹ Then the two integrals factor as a product: one is just $i\partial_\lambda A_\sigma^a$, while the other goes to

$$\int_q \frac{e^{iq\epsilon}}{q^4} = -i\partial_{\epsilon_\rho} \left(\frac{i}{16\pi^2} \ln(1/\epsilon^2) \right) = -\frac{1}{8\pi^2} \frac{\epsilon_\rho}{\epsilon^2}, \quad (230)$$

where the other factor of i came from Wick rotation. This has a $1/\epsilon$ divergence, as promised earlier. The integral over p gave us a $\partial_\lambda A_\sigma$ term, and due to the antisymmetry of the ϵ tensor up front we can turn this into a $\partial_{[\lambda} A_{\sigma]}$ at the cost of a factor of $1/2$. This means that after contracting the fermions, the first term in (227) gives us

$$-\frac{C(r)}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) (\partial_\lambda A_\sigma^a - \partial_\sigma A_\lambda^a). \quad (231)$$

Now we need to contract out the fermions in the other term that came from commuting the t^a through the Wilson line. Tracing out the spin and gauge indices using the rules described earlier, we get (the Feynman diagram looks the same, just with a different interaction vertex in between the fermions)

$$\text{relevant diagram} = 4iC(r) e^3 \epsilon^{\lambda\mu\sigma\rho} A_\mu^a A_\nu^b f^{cab} \epsilon^\nu \int_{p,q} \frac{(q+p)_\lambda}{(q+p)^2} A_\sigma^c(p) \frac{q_\rho}{q^2} e^{iq\epsilon - ip(x-\epsilon/2)}. \quad (232)$$

We do the integrals in the same way as before, and this term adds to the previous one to give us the expression for the divergence of the chiral current up to terms cubic in the gauge field.

⁹Wait, but isn't QCD a finite theory? Why are we working with divergent quantities? The reason that we still have to deal with infinities in the calculation is that we wrote down a UV action with a finite value for the bare coupling, when in reality, the coupling flows to 0 in the deep UV. Of course we can't do calculations with zero coupling, and so we have to morally multiply it by infinity to get something which we can work with. One thing I'm currently not sure of is whether there is a way to do perturbative computations like this in QCD without running into any infinities.

Recapitulating, we have

$$\partial_\mu \langle j_5^\mu \rangle = -\frac{C(r)e^2}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} \left((\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial_\lambda A_\sigma^a - \partial_\sigma A_\lambda^a) + e(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(f^{abc} A_\lambda^b A_\sigma^c) \right) + \dots, \quad (233)$$

where \dots is the second term with $\mu, \nu \leftrightarrow \lambda, \sigma$ and also the A^4 terms that we didn't compute. This is exactly what we get from writing down the familiar $F^a \wedge F^a$ formula obtained with Fujikawa's method. In the common case where the gauge group is $SU(N)$ and the fermions are in the fundamental, the $C(r)$ becomes a $1/2$ and we get the $F^a \wedge F^a$ term with a $1/(32\pi^2)$ in front.

17 May 1 — Global properties of KW duality

Consider the usual KW duality for a one-dimensional transverse field Ising model, with Hamiltonian chosen as

$$H = - \sum_i (Z_i Z_{i+1} + h X_i). \quad (234)$$

As usually formulated the duality only makes sense locally, since it maps something degenerate to something non-degenerate and vice versa. Show how to minimally modify the statement of the duality so that the mapping is exact. Do so for both an open chain and a circle. Also consider the Majorana fermion representation, and figure out where the ground states map.

Solution:

The usual statement of the operator mapping for the duality is (dual operators are in mathcal)

$$X_i = \mathcal{Z}_i \mathcal{Z}_{i+1}, \quad \mathcal{X}_i = Z_i Z_{i+1}, \quad (235)$$

where the indices on the dual operators are thought of as being displaced by half a lattice spacing relative to the original operators. Let's first consider a closed chain. These theories can't be actually dual, since the global symmetry generated by $\prod_i X_i$ maps to $\prod_i (\mathcal{Z}_i)^2 = \mathbf{1}$ on the other side. Similarly, the putative global symmetry of the dual theory actually doesn't exist, since it would be generated by $\prod_i \mathcal{X}_i$, but this is also equal to $\mathbf{1}$. So, the dual theory has no information about the global symmetry of the X, Z theory. This is to be expected since \mathcal{X} measures the presence of a domain wall, and since there must always be an even number of domain walls, $\prod_i \mathcal{X}_i = \mathbf{1}$ (here we are taking periodic boundary conditions for the spins: the spins are not fermions and so must have periodic boundary conditions. To talk about "anti-periodic boundary conditions" on spins, we should really do so by introducing a gauge field while keeping the spins periodic).

Related problems come when we try to split the operator mapping and find expressions for individual Z and \mathcal{Z} operators. We would usually write

$$\mathcal{Z}_i = \mathcal{Z}_0 \prod_{j < i} X_j, \quad (236)$$

but we run into a problem when we run around the full length of a chain since the periodic ∂ conditions imply $\prod_i X_i = \mathbf{1}$, which isn't true. (we could also take the chain of operators to run off in the other direction—these two choices are related by the global \mathbb{Z}_2 symmetry.)

To get the full \mathbb{Z}_2 symmetry in the dual theory and fix these issues, we introduce a \mathbb{Z}_2 gauge field A defined on the links of the dual lattice. The operator mapping is upgraded to (writing out the 1/2s in the indices for once)

$$X_i \mapsto \mathcal{Z}_{i-1/2} A_i \mathcal{Z}_{i+1/2}. \quad (237)$$

Products of X operators then become two charges connected with a Wilson line:

$$X_i X_{i+1} \dots X_{i+n} \mapsto \mathcal{Z}_{i-1/2} W_{i,i+n} \mathcal{Z}_{i+n+1/2}, \quad (238)$$

where $W_{i,i+n}$ is the product of the gauge field variables along the path. In particular, the global symmetry in the original model is mapped to a Wilson line spanning the whole system in the dual theory:

$$\prod_i X_i \mapsto \text{hol}(A). \quad (239)$$

Hopefully the notation $\text{hol}(A)$ is clear. If our ground states are $\otimes |\uparrow\rangle \pm \otimes |\downarrow\rangle$ then we are in a basis which diagonalizes $\text{hol}(A)$, and so the single \mathbb{Z}_2 degree of freedom we added by gauging (viz. $\text{hol}(A)$) makes up for the lost \mathbb{Z}_2 symmetry. Likewise if our ground states are $\otimes |\uparrow\rangle$ and $\otimes |\downarrow\rangle$ then the operator $\text{hol}(A)$ takes us between ground states, and so the ground states are labeled by the \mathbb{Z}_2 electric flux. If one wants to gauge-fix, we can choose A so that $A_i = 0$ for all sites on the original lattice except for one, where \mathbb{Z}_2 degree of freedom gets trapped. Finally, gauge invariance in the dual model means that the operator \mathcal{X}_i is equivalent to $E_l E_{l+1}$, where the variables E_l (defined on the links of the dual lattice) are conjugate to the A_i . So, we can write the dual Hamiltonian as

$$\mathcal{H} = - \sum_i (h \mathcal{Z}_i A_l \mathcal{Z}_{i+1} + E_l E_{l+1}). \quad (240)$$

Hopefully the notation here is clear: the *is* and *ls* are dual lattice indices — I just didn't want to write the $\pm 1/2$ s everywhere.

Let's now see in more detail how the gauge field fixes the issues about ground state degeneracy. When $h \gg 1$ the original theory has the unique ground state $\otimes |+\rangle$. The dual coupling is thus very small, and so the gauge field degree of freedom gets frozen out (if we gauge-fix so that A only appears in one term $\mathcal{Z}_0 A \mathcal{Z}_1$, then the ground state has $A = 1$: if A were non-trivial we would need $\mathcal{Z}_0 = -\mathcal{Z}_1$ [recall $h \gg 1$ means diagonalizing \mathcal{Z}], but then this would mean that some $\mathcal{Z}_j = -\mathcal{Z}_{j+1}$ for $j \neq 0$, which because of the term $-h \mathcal{Z}_j \mathcal{Z}_{j+1} \subset H$ is very energetically costly). Note that the dual theory no longer has a \mathbb{Z}_2 symmetry being broken in this phase, since this symmetry has been gauged, and so the $g \gg 1$ phase has a non-degenerate ground state on both sides of the duality (the operator which performs the symmetry action is $\prod_i X_i \mapsto \prod_i (E_l E_{l+1}) = \mathbf{1}$, which indeed acts trivially). Note that if we were on a manifold with boundary or on an infinite line, boundary conditions would come into play and things would be slightly more involved.

Now consider $h \ll 1$. In the original spin model, we have a \mathbb{Z}_2 ground state degeneracy. In the dual model, we see that we also have a \mathbb{Z}_2 degeneracy, with the electric flux taking

on values $E_l = \pm 1$. The \mathbb{Z}_2 symmetry in the original model is generated by $\prod_i X_i$, which maps to $\text{hol}(A)$ as we have said. Does this generate the \mathbb{Z}_2 symmetry that takes us between different electric flux sectors? Of course; inserting a Wilson line changes the electric flux between $+1$ and -1 .

Thus a consistent picture emerges: by adding a \mathbb{Z}_2 gauge field to the dual theory, we eliminate one \mathbb{Z}_2 symmetry, which is spontaneously broken when $h \gg 1$, creating an unphysical degeneracy. Upon gauging, this symmetry is replaced with another \mathbb{Z}_2 symmetry generated by the Wilson lines, which is spontaneously broken in the $h \ll 1$ phase, giving a \mathbb{Z}_2 ground state degeneracy which matches that of the original spin model.

We can also look at open chains. There are several ways of dealing with things. We can map a chain with L sites to one with $L - 1$ sites provided we fix boundary conditions on the first chain, or we can do the reverse, or we can do things slightly weirdly by mapping an L site chain to another L site chain, but then the Hamiltonians have single Z operators and dangling spins and stuff gets annoying. Perhaps most covariantly, we can map an L -site chain to an $L + 1$ site chain, and fix the dual spins on both dual boundaries to be $|\uparrow\rangle$ (this is okay since $\mathcal{X}_0, \mathcal{X}_L \notin \tilde{H}$). This cuts down the Hilbert space to $\mathbb{C}^{2(L-1)}$, and we get the extra degree of freedom back by adding a \mathbb{Z}_2 gauge field so that $\prod_i A_i$ becomes a physical variable. Since the boundary conditions are fixed, gauge transformations cannot change the sign of the $\mathcal{Z}_0 A_{1/2} \mathcal{Z}_1$ or $\mathcal{Z}_L A_{L+1/2} \mathcal{Z}_{L+1}$ terms, and the gauge field carries one degree of freedom even though the topology of the chain is trivial.

Finally we make a few statements about fermions. We define Majorana operators as (carefully putting the tails to the left)

$$\eta_i = \prod_{j < i} X_j Y_i, \quad \gamma_i = \prod_{j < i} X_j Z_i. \quad (241)$$

The Hamiltonian is then

$$H = -i \sum_i (\eta_i \gamma_{i+1} + h \gamma_i \eta_i). \quad (242)$$

Since these are fermions, they can hop around on the sheets of $SU(2)$ as they go around the circle and we can let them have either periodic or anti-periodic boundary conditions (non-bounding or bounding spin structures). Acting on the ground state wavefunction,

$$\eta_L |\psi\rangle = \prod_i X_i \eta_0 |\psi\rangle = -\eta_0 \prod_i X_i |\psi\rangle = \mp \eta_0 |\psi\rangle. \quad (243)$$

So, we have

$$|\psi\rangle = \otimes |\uparrow\rangle - \otimes |\downarrow\rangle \implies \eta_L = \eta_0, NB, \quad |\psi\rangle = \otimes |\uparrow\rangle + \otimes |\downarrow\rangle \implies \eta_L = -\eta_0, B. \quad (244)$$

Thus the antisymmetric ground state, which gets lifted to higher energy after splitting in a finite chain, is the one with non-bounding spin structure (and the Majorana zero mode). The fermion parity of these states is determined by $(-1)^{f_i} = 2c_i^\dagger c_i - 1 = i\eta_i \gamma_i$, so that for the whole system, the total fermion parity is the same as the global \mathbb{Z}_2 symmetry: $(-1)^F = \prod_i X_i$. This gives the bounding spin structure even parity and the non-bounding one odd parity (from the MZM).

So, the two different ground states of the TFIM, which are in the same Hilbert space and related by a global \mathbb{Z}_2 , map to two different Majorana fermion theories in two different

Hilbert spaces, distinguished by their spin structures (and as a consequence, by their fermion parities as well).

18 May 2 — Staggered Hubbard model at half-filling and criticality on Mott lobes

This is a problem from a pset in a class on quantum matter taught by Sachdev in spring 2018.

Consider the boson Hubbard model defined on a bipartite (hypercubic) lattice, with differing chemical potentials on the two sublattices:

$$H = \sum_i \left((-\mu + \epsilon(-1)^{|i|}) b_i^\dagger b_i + \frac{U}{2} b_i^\dagger b_i (b_i^\dagger b_i - 1) \right) - v \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_j^\dagger b_i), \quad (245)$$

where $|i| = 1$ on the sublattice A and $|i| = 2$ on the sublattice B . Take $U > 2\epsilon$, and work in the regime where lattice is at half-filling (one boson per unit cell). The goal is to find a field theory description of this model that reduces to the WF CFT at tip of the Mott “lobe”.

Do several things: first, find the imaginary-time correlator $\chi_\alpha(\tau) = \langle Tb(\tau)b^\dagger(0) \rangle$ at $v = 0$ (and zero temperature), where α is the sublattice index and the b 's are taken to act on a single site. Next, decouple the hopping term in H with a HS transformation, and integrate out the bosons, stopping at one-loop order. Determine the mass variable in the field theory expanded about the critical point in terms of the χ_α 's and v (and express the critical value of v in terms of the χ_α 's). This is most easily accomplished by going over to frequency + momentum space.

Solution:

Since $U > 2\epsilon$, at half-filling (where the A sublattice is filled), new bosons which are added will go into filling up the B sublattice, since $-\mu + \epsilon < -\mu - \epsilon + U$. Also since we have assumed half-filling, we know that $\epsilon > \mu$.

When $v = 0$, we get the boson correlator (at zero temperature) by the usual spectral decomposition (just focusing on a single site wolog and dropping the α sublattice index on the b 's):

$$\begin{aligned} \chi_\alpha(\tau) &= \langle T(b(\tau)b^\dagger(0)) \rangle \\ &= \sum_n \left(\Theta(\tau) e^{-\tau E_{n0}} \langle 0 | b | n \rangle \langle n | b^\dagger | 0 \rangle + \Theta(-\tau) e^{\tau E_{n0}} \langle 0 | b^\dagger | n \rangle \langle n | b | 0 \rangle \right), \end{aligned} \quad (246)$$

where E_{n0} is the energy difference between $|n\rangle$ and $|0\rangle$ on the α sublattice. In frequency space,

$$\chi_\alpha(\omega) = \sum_n \left(-\frac{\langle 0 | b | n \rangle \langle n | b^\dagger | 0 \rangle}{\omega - E_{n0}} + \frac{\langle 0 | b^\dagger | n \rangle \langle n | b | 0 \rangle}{\omega + E_{n0}} \right) \quad (247)$$

At zero temperature, we get only the $n = \pm 1$ terms. For $\alpha = A$ we have $E_{10} = U - \epsilon - \mu$ and $E_{-10} = \mu + \epsilon$, while for $\alpha = B$ we have $E_{10} = \epsilon - \mu$ and $E_{-10} = 0$. So we write (using

$\langle b^\dagger b \rangle = 1$ for the A sublattice and $\langle b^\dagger b \rangle = 0$ for the B sublattice)

$$\begin{aligned}\chi_A(\omega) &= \frac{2}{-i\omega + U - \mu - \epsilon} + \frac{1}{i\omega + \mu + \epsilon}, \\ \chi_B(\omega) &= \frac{1}{i\omega + \mu - \epsilon}.\end{aligned}\tag{248}$$

Writing

$$\chi_\alpha(\omega) = -r_\alpha + iK_{1\alpha}\omega - K_{2\alpha}\omega^2,\tag{249}$$

we expand in small ω and get

$$r_A = \frac{2}{\mu + \epsilon - U} - \frac{1}{\mu + \epsilon}, \quad K_{1A} = \frac{2}{(U - \mu - \epsilon)^2} - \frac{1}{(\mu + \epsilon)^2}, \quad K_{2A} = \frac{2}{(U - \mu - \epsilon)^3} + \frac{1}{(\mu + \epsilon)^3},\tag{250}$$

as well as

$$r_B = \frac{1}{\epsilon - \mu}, \quad K_{1B} = \frac{1}{(\mu - \epsilon)^2}, \quad K_{2B} = \frac{1}{(\mu - \epsilon)^3}.\tag{251}$$

Thus we can verify that

$$\partial_\mu r_A = -K_{1A}, \quad \partial_\mu r_B = -K_{1B}.\tag{252}$$

When we turn this into a mean field theory, these relations will be essential for guaranteeing a $U(1)$ gauge invariance that will appear in our description.

We now do mean field theory, by introducing a HS field Ψ to decouple the inter-site hopping (not the interactions!). Doing this in the usual way, we get

$$L = \sum_i \left(b_i^\dagger \partial_\tau b_i - (\mu - \epsilon(-1)^{|i|}) b_i^\dagger b_i + \frac{U}{2} : n_i n_i : + b_i^\dagger \Psi_i + b_i \Psi_i^\dagger \right) + \sum_{ij} \frac{1}{v} \Psi_i^\dagger A_{ij}^{-1} \Psi_j.\tag{253}$$

Here A_{ij} is the adjacency matrix, with entries which are equal to one if i and j are nearest neighbors and zero otherwise. Note that in this parametrization we have the time-dependent (but not space-dependent) gauge transformation

$$b_i \mapsto e^{i\phi(\tau)}, \quad \Psi \mapsto e^{i\phi(\tau)} \Psi, \quad \mu \mapsto +i\partial_\tau \phi(\tau).\tag{254}$$

We get the effective action for the mean fields Ψ by integrating out the bosons. For the Feynman rules, we see that we can convert Ψ s into b bosons, and we can draw four-point boson vertices. The two-point interactions for Ψ in its effective potential are then determined just by the boson propagators at $v = 0$, which we already know. The four-point boson vertex also gives us a Ψ^4 interaction, while there are no odd n -point interactions (one- or three-point) since they aren't invariant under the gauge transformation (similarly, the gauge invariance means that Ψ s and Ψ^\dagger s appear in equal numbers in each term). Thus the relevant terms are

$$L = \sum_{ij} \frac{1}{v} \Psi_i^\dagger A_{ij}^{-1} \Psi_j + \sum_{i \in A} \chi_A(\omega) |\Psi_i(\omega)|^2 + \sum_{i \in B} \chi_B(\omega) |\Psi_i(\omega)|^2 + \sum_i u |\Psi|^4.\tag{255}$$

Now we go to momentum space to simplify the first term. Since there are two sublattices, and each lattice gets identified into a point in the BZ, A_{ij}^{-1} in band space becomes a two-by-two off-diagonal matrix proportional to σ^x . For a square lattice the entries are proportional

to $\cos(k_x) + \cos(k_y)$ since we can hop in the $\pm x$ and $\pm y$ directions, and for more general coordination numbers we get some function of various cosines that goes as $1/Z$, where Z is the coordination number. Thus the quadratic part of the Lagrangian is (sticking with the square lattice for simplicity)

$$L = \sum_k (\Psi_A^\dagger, \Psi_B^\dagger) \begin{pmatrix} \chi_A(\omega) & (\cos(k_x) + \cos(k_y))/4v \\ (\cos(k_x) + \cos(k_y))/4v & \chi_B(\omega) \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}. \quad (256)$$

The phase transition will happen when Ψ gets a vev, that is when the above matrix acquires a negative eigenvalue at $\omega, k = 0$. This after all is how the effective mass is defined, by the spacetime-independent eigenvectors of the second derivative of the effective potential with respect to Ψ . When such an eigenvector has a negative eigenvalue the mass squared is negative, the system condenses, and we enter the SF phase. The lower eigenvalue at $\omega = 0, k = 0$ is (sending $4 \rightarrow Z$ for a more general lattice)

$$\lambda_- = \frac{1}{2} \left(\chi_A(0) + \chi_B(0) - \sqrt{(\chi_A - \chi_B)^2 + 4/(Z^2 v^2)} \right). \quad (257)$$

Thus we get a condensate when

$$\frac{1}{Z^2 v^2} = \chi_A(0)\chi_B(0) = r_A r_B. \quad (258)$$

Near the phase transition, we can focus on just the lower eigenmode, and so the effective Lagrangian is approximately

$$L = K_1 \Psi_i^\dagger \partial_\tau \Psi_i + K_2 |\partial_\tau \Psi|^2 + b |\nabla \Psi|^2 + \tilde{r} |\Psi|^2 + u |\Psi|^4, \quad (259)$$

where I'm not writing out all the coefficients explicitly. The linear time derivative comes from the $K_{1\alpha}$ terms in $\chi_\alpha(\omega)$, the quadratic time derivative comes from the $K_{2\alpha}$ terms, and so on. The most important one is the mass \tilde{r} . We define the parameter η to measure where we are in relation to the critical point:

$$\eta \equiv \frac{1}{v^2 4^2} - r_A r_B. \quad (260)$$

Expanding the ω and k independent piece of the eigenvalue of the matrix written in (256), a bit of algebra gives something like

$$\tilde{r} \approx -\frac{\eta}{r_A + r_B}. \quad (261)$$

Note that this effective action needs to retain the invariance under the space-independent but time-dependent gauge transformations written down previously, which is only possible since relations like (252) are obeyed. In particular, we see that when \tilde{r} is independent of μ (on the tip of the Mott lobe), the gauge invariance tells us that we must have $K_1 = 0$. At the tips of the Mott lobes then, the theory gains a relativistic invariance and the critical point is governed by the Wilson Fischer CFT.

19 May 3 — Pion decay

This is essentially P&S, problem 19.2. Given the effective Lagrangian

$$\mathcal{L} = \frac{4G_F}{\sqrt{2}} (\bar{l}_L \gamma^\mu \nu_L) (\bar{u}_L \gamma_\mu d_L) + h.c., \quad (262)$$

find the amplitude for the decay $\pi^+ \rightarrow l^+ \nu$, and compute the decay rate. Remember that the vector gauge currents are conserved, but the chiral currents are not. Parametrize them by

$$\langle 0 | j^{\mu 5a}(x) | \pi^b(p) \rangle = -ip^\mu f_\pi \delta^{ab} e^{-ipx}, \quad (263)$$

where f_π is a constant.

Solution:

We get the matrix element for the decay by writing

$$i\mathcal{M}(2\pi)^4 \delta(q+k-p) = i \int d^4x \frac{4G_F}{\sqrt{2}} \langle l, \nu | (\bar{l}_L \gamma^\mu \nu_L) (\bar{u}_L \gamma_\mu d_L) | \pi^+(p) \rangle, \quad (264)$$

where k, q are the momenta of the lepton l and the neutrino. We can also write this as

$$i\mathcal{M}(2\pi)^4 \delta(q+k-p) = i\bar{u}(q) \gamma^\mu \frac{1-\gamma^5}{2} v(k) \int d^4x e^{ix(q+k)} \frac{4G_F}{\sqrt{2}} \langle 0 | (\bar{u}_L \gamma_\mu d_L) | \pi^+(p) \rangle, \quad (265)$$

where \bar{u}, v are the usual spinors which we use to build the plane wave solutions to the Dirac equation.

It's helpful to first rewrite the matrix element a bit by using the currents

$$j^{\mu a} = \bar{Q} \gamma^\mu \tau^a Q, \quad j^{\mu 5a} = \bar{Q} \gamma^\mu \gamma^5 \tau^a Q, \quad (266)$$

where $Q = (u, d)$ and $a = x, y, z$ is the $SU(2)$ index. First consider the combination

$$j^{\mu 1} + ij^{\mu 2}. \quad (267)$$

This choice of $SU(2)$ index structure projects onto $\bar{u}d$ currents (there is no $1/2$ needed since the generators are $\tau^a = \sigma^a/2$). We can then select out the left handed components by projecting with $(1 - \gamma^5)$, so that

$$\bar{u}_L \gamma^\mu d_L = \frac{1}{2} (j^{\mu x} + ij^{\mu y} - j^{\mu 5x} - ij^{\mu 5y}). \quad (268)$$

The first two currents are conserved (they are gauge currents), so when we expand the $\bar{u}\gamma d$ term with the currents, these terms die. Using the parametrization written above for the matrix elements of the chiral currents and the fact that π^+ only has x and y $SU(2)$ indices (a π^+ is an up and an anti-down, so actually it is built as $\sigma^x - i\sigma^y$ in our basis), we get

$$i\mathcal{M}(2\pi)^4 \delta(q+k-p) = i\bar{u}(q) \gamma^\mu \frac{1-\gamma^5}{2} v(k) \int d^4x e^{ix(q+k)} \frac{4G_F}{2} (-ip_\mu f_\pi e^{-ipx}), \quad (269)$$

and so

$$i\mathcal{M} = G_F f_\pi \bar{u}(q) \not{p} (1 - \gamma^5) v(k). \quad (270)$$

Now we can get the decay rate. To square the matrix element \mathcal{M} , we need the spin sums

$$\sum_s u^s(p) \bar{u}^s(p) = \not{p} + m, \quad \sum_s v^s(p) \bar{v}^s(p) = \not{p} - m. \quad (271)$$

Using the Dirac equation and momentum conservation, we have

$$\bar{u}(q) \not{p} = \bar{u}(q) (\not{q} + \not{k}) \approx \bar{u}(q) \not{k}, \quad (272)$$

since the neutrino is essentially massless. Then we use the Dirac equation for the lepton to write

$$\not{k} (1 - \gamma^5) v(k) = (1 + \gamma^5) \not{k} v(k) = -(1 + \gamma^5) m_l v(k). \quad (273)$$

So,

$$i\mathcal{M} = -G_F f_\pi m_l \bar{u}(q) (1 + \gamma^5) v(k). \quad (274)$$

Squaring it and taking the trace with the help of the spin sums, this term becomes

$$G_F^2 f_\pi^2 m_l^2 \text{Tr}[\not{q} (1 + \gamma^5) (\not{k} - m_l) (1 - \gamma^5)]. \quad (275)$$

The terms odd in γ matrices will die, so the m_l piece doesn't contribute. Then since $2\text{Tr}[\not{q} \not{k}] = 2(q_\mu k_\nu) \text{Tr}[g^{\mu\nu} \mathbf{1}] = 8(q \cdot k)$,

$$|\mathcal{M}|^2 = 8G_F^2 f_\pi^2 m_l^2 (E_\nu E_l + k^2), \quad (276)$$

where we went to the rest frame where $\mathbf{p} = 0$. Looking up the formula for the decay rate in P&S (and using $E_{cm} = m_\pi$), we get

$$\gamma(l^+ \nu \rightarrow \pi^+) = 8G_F^2 f_\pi^2 \int \frac{d\Omega}{16\pi^2} \frac{1}{2m_\pi^2} |k| m_l^2 (E_\nu E_l + k^2). \quad (277)$$

Taking the equation $m_\pi = E_\nu + E_l$ and raising it to the fourth power gives, after some algebra,

$$k^2 = \left(\frac{m_\pi^2 - m_l^2}{2m_\pi} \right)^2 \implies E_\nu = \frac{m_\pi^2 - m_l^2}{2m_\pi}. \quad (278)$$

Then since $E_l^2 = k^2 + m_l^2$,

$$E_l = \frac{m_\pi^2 + m_l^2}{2m_\pi}. \quad (279)$$

Now we can plug these into the formula for Γ . Since $\int d\Omega = 4\pi$, this produces after some algebra,

$$\gamma(l^+ \nu \rightarrow \pi^+) = \frac{m_l^2 m_\pi}{4\pi} G_F^2 f_\pi^2 (1 - m_l^2/m_\pi^2)^2. \quad (280)$$

Thus the ratio of the decay rates for $\mu^+ \nu$ decay and $e^+ \nu$ decay is

$$\frac{\gamma(e^+ \nu \rightarrow \pi^+)}{\gamma(\mu^+ \nu \rightarrow \pi^+)} = \frac{m_e^2}{m_\mu^2} \frac{(1 - m_e^2/m_\pi^2)^2}{(1 - m_\mu^2/m_\pi^2)^2} \ll 1. \quad (281)$$

Plugging in numbers with the Fermi constant set at $G_F = \sqrt{2}g^2/(8m_W^2)$ and $\tau_\pi \approx 2.6 \times 10^{-8}$ sec, we get

$$f_\pi = \sqrt{\frac{4\pi}{\tau_\pi m_\mu^2 m_\pi}} \frac{1}{G_F(1 - m_\mu^2/m_\pi^2)} \approx 100 \text{ Mev.} \quad (282)$$

We've used just the decay rate to the muon since the contribution of the electron decay channel is suppressed by a factor of $m_e^2/m_\mu^2 \ll 1$ as we saw above.

20 May 4 — Dispersion of excitations in \mathbb{Z}_2 gauge theory

This is another problem from a pset assigned in Sachdev's spring 2018 class on quantum matter.

Consider \mathbb{Z}_2 gauge theory on a square lattice in two dimensions:

$$H = -K \sum_{\square} \prod_{l \in \square} Z_l - g \sum_l X_l. \quad (283)$$

Find the dispersion of the lowest-energy excitations to order g in the deconfined phase ($K \gg g$) and to order K^2 in the confined phase ($g \gg K$).

Solution:

First consider the deconfined phase, where the minimal excitations are magnetic fluxes which violate a single Wilson-loop plaquette term. The electric field operator X_l creates a pair of magnetic fluxes on the two plaquettes whose boundaries include l , and so the electric operators are responsible for giving the fluxes dynamics. Let $|r\rangle$ represent a state with a π flux at plaquette r (of course, technically the fluxes can only come in pairs). Then to first order, the energy relative to the ground state is

$$E_k = \langle k | -K \sum_{\square} \prod_{l \in \square} Z_l - g \sum_l X_l | k \rangle = \frac{1}{V} \sum_{r,s} \langle r | 2K\delta_{r,s} - e^{ik(r-s)} g \sum_l X_l | s \rangle. \quad (284)$$

At first order in g , the only hopping processes will move the flux excitation to an adjacent plaquette, and so

$$E_k = 2K - 2g(\cos(k_x) + \cos(k_y)). \quad (285)$$

Now for the trickier confined case. It is easiest to get the dispersion by first mapping to the dual transverse field Ising model (technically this mapping only works locally), with the operator mapping

$$\prod_{l \in \square} Z_l \mapsto \mathcal{X}_i, \quad X_{\langle ij \rangle} \mapsto \mathcal{Z}_i \mathcal{Z}_j, \quad (286)$$

so that

$$H = -g \sum_{\langle ij \rangle} \mathcal{Z}_i \mathcal{Z}_j - K \sum_i \mathcal{X}_i. \quad (287)$$

Let $|i\rangle$ denote a state with a spin flip (in the \mathcal{Z} basis) at site i , which are the minimal excitations of the dual Ising model when $g \gg K$. In the gauge theory, a spin flip maps to a

configuration where the electric field terms $-gX_l$ on all the links dual to the links emanating from i are violated — this corresponds to a small electric flux loop encircling the dual vertex i . Note that we couldn't have a smaller excitation that e.g. just violated a single $-gX_l$ term on one link, since that would violate gauge invariance (the electric flux lines, which are tensionfull and constitute the excitations in the system, must form closed loops).

Anyway, let's find the dispersion. Let $|r\rangle$ denote the state in the Ising model with a spin-flip at site r (choosing the spin flip to be measured relative to the global vacuum state $\otimes|\uparrow\rangle$), let $|r,s\rangle$ be the state with spin-flips at sites r and s , and so on. We find the dispersion using the effective Hamiltonian, which is designed to be an effective Hamiltonian to a given order in K/g for the eigenstates $|0\rangle$, $|r\rangle$, $|r,s\rangle$, etc. of the $K = 0$ Hamiltonian. To $O(K^2/g)$, the diagonal pieces are (after subtracting off the ground-state energy)

$$\begin{aligned}\langle r|H_{eff}|r\rangle &= \langle r|H|r\rangle + \langle r|H|0\rangle\langle 0|H|r\rangle \frac{1}{E_r - E_0} + \sum_s \langle r|H|r,s\rangle\langle r,s|H|r\rangle \frac{1}{E_r - E_{r,s}}, \\ &= -2Vg + 4g + \frac{K^2}{4g} + \frac{K^2(V-5)}{4g-8g} + \frac{4K^2}{4g-6g} \\ &= -(2g + K^2/4g)V + 4g - 2K^2/4g.\end{aligned}\tag{288}$$

where V is the number of sites and $2V$ is the number of links. On the other hand, if $r \neq s$ then

$$\langle r|H_{eff}|s\rangle = \frac{1}{2}\langle r|H|0\rangle\langle 0|H|s\rangle \left(\frac{1}{E_r - E_0} + \frac{1}{E_s - E_0} \right) + \frac{1}{2}\langle r|H|r,s\rangle\langle r,s|H|s\rangle \left(\frac{1}{E_r - E_{r,s}} + \frac{1}{E_s - E_{r,s}} \right).\tag{289}$$

If r and s are not nearest neighbors, then the first term in parenthesis is $1/2g^2$ while the second is $-1/2g$, and so we get zero. If r and s are nearest neighbors, then the second term is instead $-1/g$, and so we get

$$\langle r|H_{eff}|s\rangle = \left(\frac{K^2}{4g} - \frac{K^2}{2g} \right) \delta_{\langle rs \rangle} = -\frac{K^2}{4g} \delta_{\langle rs \rangle},\tag{290}$$

where $\delta_{\langle rs \rangle}$ is 1 if r, s are nearest neighbors and zero else.

Now we can get the dispersion by going to Fourier space. We see that, dropping the volume-dependent part and the ground state energy $-Vg$ of the $K = 0$ Hamiltonian, the energy of a single spin flip is

$$E_\downarrow(k) = 4g - \frac{K^2}{2g} (2 + \cos(k_x) + \cos(k_y)).\tag{291}$$

As we mentioned before, this represents the dispersion of a small loop of electric flux.

21 May 5 — Charge quantization and the weak mixing angle confusion

In various places (Ryder's QFT book, stack exchange, etc.) it is often stated that if the weak mixing angle is irrational, then electric charge is not quantized since the $U(1)$ of electromag-

netism is embedded non-compactly in the full $SU(2) \times U(1)_Y$ electroweak gauge group. Show that it is wrong (assuming that the full gauge group is $SU(2) \times U(1)_Y$ and not $SU(2) \times \mathbb{R}$).

Solution:

First let's show why the $U(1)$ of electromagnetism is compact, and afterwards explain why it is often stated otherwise in the literature. We choose the conventions where the covariant derivative and hypercharge gauge transformations look like

$$D_\mu \psi = (\partial_\mu - igW_\mu^a T^a - ig'B_\mu)\psi, \quad \psi \mapsto e^{i\alpha(x)}\psi, \quad B_\mu \mapsto B_\mu + \frac{1}{g'}\partial_\mu\alpha. \quad (292)$$

Note that the transformation is not $e^{ig'\alpha(x)}$! The Higgs gets a vev which we choose to be in the \downarrow direction. Thus the $U(1)_e$ symmetry which we will identify with electromagnetism corresponds to rotations

$$\phi \mapsto e^{i\alpha T^3} e^{i\alpha/2}\phi, \quad (293)$$

which leaves the vev of the Higgs invariant (the second factor is a hypercharge rotation). Note that in this convention, the Higgs has charge 1/2. We choose it to have charge 1/2 since the generators for the $SU(2)$ are $T^a = \sigma^a/2$. This $U(1)_e$ is obviously compact, since the rotations about the 3 axis are compact, the $U(1)_Y$ hypercharge was assumed to be compact, and in the above action we have $\alpha \sim \alpha + 4\pi$.

In this formulation, what are the volumes of the two $U(1)$ factors? In general, we have the representation q of $U(1)$, which acts as

$$q : \mathbb{R}/(2\pi/q\mathbb{Z}) \rightarrow U(1)_{2\pi}, \quad x \mapsto e^{iqx}, \quad (294)$$

where by $U(1)_{2\pi}$ we just mean the complex numbers of norm 1. Thus if our minimally charged field carries a representation q under $U(1)$, we are identifying $2\pi/q$ with 0, and so the gauge group is a circle with “volume” $\text{vol}(G) = 2\pi/q$. Since for $SU(2)$ rotations we identify $\alpha \sim \alpha + 4\pi$, the $U(1)_{T^3}$ rotations come from the representation

$$1/2 : \mathbb{R}/(4\pi\mathbb{Z}) \rightarrow U(1)_{2\pi}, \quad x \mapsto e^{-ix/2}, \quad (295)$$

and so $\text{vol}(U(1)_{T^3}) = 4\pi$. Since we know we have quarks with charge 1/6 (the left handed ones), the field with minimal hypercharge has charge 1/6, and so $\text{vol}(U(1)_Y) = 12\pi$. Note that we could rescale things so that the minimal hypercharge is 1, decreasing the volume of $U(1)_Y$ by a factor of 6, but then we'd also have to change the charge of the Higgs under T^3 rotations, which is less preferable since the normalization of these comes from their embedding in the $SU(2)$ factor. Anyway, if we did this, we see that the aspect ratio of the torus $U(1)_{T^3} \times U(1)_Y$ would be preserved by the rescaling. Since $U(1)_e$ is embedded compactly within this torus (one can think of it as a path wrapping the $U(1)_{T^3}$ cycle three times and the $U(1)_Y$ cycle once, since the ratio of the volumes of the two $U(1)$ s is 3), it is embedded compactly no matter what our conventions regarding charge normalization are.

So, why do people say that $U(1)_e$ is non-compactly embedded inside $SU(2) \times U(1)_Y$? The argument is as follows: suppose we stick with the convention where the gauge couplings appear in the exponentials of the gauge transformations:

$$\phi \mapsto e^{ig\alpha T^3} e^{ig'\beta/2}\phi. \quad (296)$$

ϕ is left invariant if we take $\beta = \alpha g/g'$. We then form the torus $U(1) \times U(1)$, and embed the $\beta = \alpha g/g'$ curve inside. If g/g' is irrational (which it generically is) then the $\beta = \alpha g/g'$ curve is dense in the torus $U(1) \times U(1)$, and hence electromagnetism is actually \mathbb{R}_e , a real line embedded in the full gauge group. Thus charge is not necessarily quantized, Polyakov monopoles cannot exist, and so on.

The problem with this is that the lengths of the sides of the torus are not 2π ! Indeed, in the gauge transformations above, we are not identifying $\alpha \sim \alpha + 2\pi$ or $\beta \sim \beta + 2\pi$: the volumes of the gauge groups are not 2π , they are $2\pi/g$ and $2\pi/g'$. Thus the aspect ratio of the torus should actually be g/g' , and the curve $\beta = \alpha g/g'$ is compactly embedded within such a torus.

There is another way to see that the quantization (or lack thereof) cannot possibly depend on the ratio of the gauge couplings: we just re-define $gW \mapsto W$ and $g'B \mapsto B$. Then the gauge couplings only appear in the kinetic terms for the gauge fields and cannot possibly affect the periodicity with which the fields transform. The gauge couplings only tell us about the dynamics of the gauge fields and how they couple to the matter fields: they have nothing to do with how the matter fields transform under the symmetry, or what representation of the symmetry group they carry.

22 May 6 — Anomaly constraints in $SU(3) \times SU(2) \times U(1)$

Consider a gauge theory with gauge group $G = SU(3) \times SU(2) \times U(1)$, and consider k massless fermions coupled to G in various different representations. Assume that there is at least one fermion charged under each factor of the gauge group, and that none of the fermions are completely neutral. Note that without loss of generality, we can take all the k fermions to be left handed. Assume that no two fermions exist which are conjugate, in the sense that one carries the representation (R, S, q) under $SU(3) \otimes SU(2) \otimes U(1)$ and the other carries $(\bar{R}, \bar{S}, -q)$. Also assume that there is at least one field with nontrivial charge under each factor (like the left-handed quark doublet in the SM).

Find the minimal value of k such that all anomalies (including the gravitational anomaly) are cancelled. How does this change if you assume the fields are all in either the trivial or fundamental of $SU(3)$ and $SU(2)$? Are there any other combinations of 5 fields besides the standard model ones that are anomaly free with these assumptions?

Solution:

Let's start with the case $k = 2$. The only option we have is the pair

$$(R, S, q), \quad (R, S, -q). \tag{297}$$

The $U(1)^3$, $SU(2) \times U(1)^2$, and $SU(3) \times U(1)^2$ anomalies are zero, and the $SU(2)^3$ anomaly is zero since $SU(2)$ has no complex representations. Thus for this to work, we need a trivial $SU(3)^3$ anomaly from the R 's. Unfortunately the only non-anomalous representations of $SU(3)$ are the real ones, which are of the form (i, i) (where the two indices label the values

of the different diagonal generators). Since all the representations of $SU(2)$ are either real or pseudo-real, we in fact have that the conjugate of (R, S, q) is isomorphic to $(R, S, -q)$, which is a contradiction.

What about $k = 3$? After playing around a bit, we can write down the following:

$$(\mathbf{6}, \mathbf{3}, 1), \quad (\mathbf{6}, \mathbf{3}, -1), \quad (\bar{\mathbf{15}}', \mathbf{3}, 0), \quad (298)$$

where **6** is the six-dimensional $SU(3)$ representation $(2, 0)$, $\bar{\mathbf{15}}'$ is the 15-dimensional $SU(3)$ irrep $(1, 2)$, and **3** is the vector representation (spin one) of $SU(2)$. The anomaly coefficients of these $SU(3)$ irreps are $A(\mathbf{6}) = 7$ and $A(\bar{\mathbf{15}}') = -14$ (see e.g. Cutler & Kephart 2000), and so the $SU(3)$ anomaly vanishes:

$$A_{SU(3)^3}((\mathbf{6}, \mathbf{3}, 1) \oplus (\mathbf{6}, \mathbf{3}, -1) \oplus (\bar{\mathbf{15}}', \mathbf{3}, 0)) = 7 \times 3 + 7 \times 3 - 14 \times 3 = 0. \quad (299)$$

The $U(1)^3$ anomaly, $U(1)$ -graviton anomaly, and the mixed $U(1) \times SU(2)^2$, $U(1) \times SU(3)^2$ anomalies are all zero since we have a symmetric coupling between 1 and -1 hypercharges to the rest of the representations. Since $\bar{\mathbf{6}} \not\cong \mathbf{6}$, there are no problems there. Finally, the global $SU(2)$ anomaly is also zero, since it requires the Dynkin index

$$t_2(S) = d(S) \frac{d^2(S) - 1}{12} \quad (300)$$

to be integral, which it is: $t_2(\mathbf{3} \oplus \mathbf{3} \oplus \mathbf{3}) = 3 \times 2$ (this amounts to requiring that $\text{Tr}(T_3^2)$ be integral rather than half-integral, where we define $\text{Tr}(T_3^2) = 1/2$ for the fundamental representation. Essentially, we need there to be an even number of $SU(2)$ doublet fermions). So, evidently these three fields do the job.

This however made use of some big $SU(3)$ representations — what if we restrict ourselves to the trivial and fundamental representations? Cancellation of the global $SU(2)$ anomaly means that we need an even number of $SU(2)$ doublets. Can we do it with three fields? This would mean two doublets, and cancellation of the $SU(2)^2 \times U(1)$ anomaly would mean that they would have opposite hypercharges, with the third field having zero hypercharge. Then cancellation of the $SU(3)^2 \times U(1)$ anomaly would mean that both doublets are in the fundamental of $SU(3)$, but then the $SU(3)^3$ anomaly is non-zero. So at least four fields are needed.

For four fields, we either need two $SU(2)$ doublets or four for cancellation of the global $SU(2)$ anomaly. If we have two then they must have opposite hypercharges, and then must both be in **3** under $SU(3)$ by our assumptions about not having conjugate fields. But then the remaining two fields cannot possibly cancel the $SU(3)^3$ anomaly, so this doesn't work. So, can we have four $SU(2)$ doublets? Yes we can; consider the quadruplet

$$(\mathbf{3}, \mathbf{2}, a), \quad (\bar{\mathbf{3}}, \mathbf{2}, -a), \quad (\mathbf{3}, \mathbf{2}, b), \quad (\bar{\mathbf{3}}, \mathbf{2}, -b), \quad (301)$$

where a, b are subject only to the constraint that $a \neq \pm b$. One sees that all the anomalies cancel.

Now we assume that there are five fields, as in the Standard Model. We can either have two or four $SU(2)$ doublets. Suppose first that there are four $SU(2)$ doublets. Then since we need at least one field charged under everything and we need the $SU(3)^3$ anomaly to vanish, we can either have two of these doublets in the fundamental of $SU(3)$ and two in

the anti-fundamental, or one in the fundamental, one in the anti-fundamental, and two in the trivial. Consider first the former situation. Then the vanishing of the $SU(2)^2 \times U(1)$ anomaly in this case means that

$$\sum_{\text{doublets}} q_i = 0, \quad (302)$$

where q_i is the hypercharge. But then the $U(1)$ -graviton anomaly implies that $q_5 = 0$, where q_5 is the hypercharge of the last field. But then the last field is completely trivial, and we get the same answer as the $k = 4$ case. So, now consider the case where one of the doublets is in the fundamental of $SU(3)$ and the other is in the anti-fundamental. Then the $SU(3)^2 \times U(1)$ anomaly means that the hypercharges of the two fields coupled to $SU(3)$ are equal. Let their hypercharges be a , and let the charges of the other two $SU(2)$ doublets be b, c . Then we need $6a + c + d = 0$. Thus the $U(1)$ -graviton anomaly reads $12a + 2c + 2d + q_5 = 0$, and so again we conclude that $q_5 = 0$ and the last field is completely decoupled.

So, we can assume wolog that there are only two $SU(2)$ doublets. Suppose they are both charged under $SU(3)$. They cannot both be in the same $SU(3)$ irrep, since then there is no way to cancel the $SU(3)^3$ anomaly with the other $SU(2)$ -neutral fields (the dimensions of the other fields are too small). So, one must be in **3** with the other in **̄3**. But then cancellation of the $SU(2)^2 \times U(1)$ anomaly means that the two doublets have opposite hypercharges, which gives a contradiction since we now have two fields that are conjugate to one another.

This means we can without loss of generality take the two doublets to be of the form

$$(\mathbf{3}, \mathbf{2}, a), \quad (\mathbf{0}, \mathbf{2}, b). \quad (303)$$

Cancellation of the $SU(3)^3$ anomaly means that the other three fields have to be of the form

$$(\bar{\mathbf{3}}, \mathbf{0}, c), \quad (\bar{\mathbf{3}}, \mathbf{0}, d), \quad (\mathbf{0}, \mathbf{0}, e), \quad (304)$$

which is starting to look increasingly like the Standard Model. Now one just needs to solve for the $U(1)$ hypercharges. We then have to solve the constraints

$$\begin{aligned} 6a + 2b + 3c + 3d + e &= 0 & 6a^3 + 2b^3 + 3c^3 + 3d^3 + e^3 &= 0, \\ 2a - c - d &= 0, & 3a + b &= 0. \end{aligned} \quad (305)$$

I'm not going to write out all the algebra, which is straightforward. One finds that there are two solutions. One has all of the hypercharges equal to zero except for $c = -d$. Again, this leaves us with a completely decoupled field, and we are back to the case of $k = 4$ (and it violates our assumptions about there being at least one field charged under everything). The other solution is the Standard Model. So, given these assumptions, the SM is unique.

What other sorts of possible matter content can we have? There are many options, since there are many semisimple Lie groups with real or pseudo-real representations. There are also a few groups with complex representations that always have zero anomaly coefficients, like $SO(4n+2)$ with $n > 2$ (see Weinberg II). In particular, any gauge group that doesn't include factors of $SU(n)$ with $n \geq 3$ or factors of $U(1)$ will be automatically anomaly-free, regardless of the field content. If we have the group $SU(3) \times G$ where G is anomaly-free, then under the above restrictions we can have two fields $(R, g), (\bar{R}, g)$ where g is any complex irrep of G , or we can have $(R, g), (\bar{R}, h)$, where g, h are two distinct real representations of the same dimension. We can do a similar thing if the group is instead $U(1) \times G$. At this

point, making concrete statements about what is allowed and what is not allowed has to be done on an increasingly case-by-case basis.

23 May 7 — Torsion-free first cohomology

Prove that $H^1(X; \mathbb{Z})$ is torsion-free for any X (this holds in both singular cohomology and in group cohomology). Of course, $H_1(X; \mathbb{Z})$ is not torsion-free (e.g. the Klein bottle).

Solution:

This is just a quick application of the universal coefficient theorem, which tells us that for any Abelian coefficient group G ,

$$0 \rightarrow \text{Ext}(H_0(X; \mathbb{Z}), G) \rightarrow H^1(X; G) \rightarrow \text{Hom}(H_1(X; \mathbb{Z}), G) \rightarrow 0. \quad (306)$$

Since $H_0(X; \mathbb{Z})$ is always free, the Ext vanishes and we have

$$H^1(X; G) \cong \text{Hom}(H_1(X; \mathbb{Z}), G). \quad (307)$$

Thus in particular if $G = \mathbb{Z}$, none of the elements in $\text{Hom}(H_1(X; \mathbb{Z}), G)$ can have torsion, since if $kh(c) = 0$ for some $k \in \mathbb{Z}, c \in H_1(X; \mathbb{Z}), h \in \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z})$, then $h(c) = 0$ (more generally, $\text{Hom}(H(X), \mathbb{Z})$ is isomorphic to the free part of $H(X)$, provided $H(X)$ is finitely generated). If we were to take $p \geq 1$ cohomology groups, things would change, since the torsion part of $H^p(X; \mathbb{Z})$ is sensitive to the torsion part of $H_{p-1}(X; \mathbb{Z})$, which can be non-zero.

24 May 8 — Chiral anomaly via Pauli-Villars

This is the first logical half of P&S problem 19.4. Derive the chiral anomaly for QED in four dimensions using a PV regulator, i.e. by adding a regulator fermion Ψ with mass M to the theory and showing that $\langle \partial_\mu j^{5\mu} \rangle$ is non-zero in the $M \rightarrow \infty$ limit. As usual, treat the gauge field as a background field. The relevant integral you will need to do should be UV -finite, even before the $M \rightarrow \infty$ limit is taken. To start, show that the anomaly is given by the limit

$$\lim_{M \rightarrow \infty} (\langle p, k | 2iM \bar{\Psi} \gamma^5 \Psi | 0 \rangle), \quad (308)$$

where p, k represent two photons as in the usual triangle diagrams.

Solution:

We will do the regularization by adding the massive fermion to the Lagrangian with the term $-\bar{\Psi}(i\cancel{D}_A - M)\Psi$. Consider now calculating the anomaly with your favorite method, e.g. by splitting up a two-point fermion operator and connecting the two fermions with a Wilson line, as we did in an earlier problem. This calculation will receive contributions from both the original ψ fermion and the regulator fermion Ψ . As we saw earlier, we will end up doing an integral like

$$\int_{k,q} e^{ik \cdot \eta - iq \cdot x} \frac{\epsilon^{\mu\nu\lambda\sigma}(k+q)_\nu A_\lambda k_\sigma}{(k^2 - m^2)((k+q)^2 - m^2)}, \quad (309)$$

where η is an infinitesimal distance between two fermion operators which we are point-splitting and m is either 0 or M . The hope is that since we are sending $\eta \rightarrow 0$, we can get away with doing this before sending $M \rightarrow \infty$, and expand the integral at large k (viz. larger than M), getting a divergent contribution which is independent of M (and only the divergent contribution matters for the anomaly, since this integral is multiplied by an infinitesimal term proportional to η coming from the expansion of the operators in the OPE). So, this will result in the contributions from the two fermions cancelling, even though one is massive and the other isn't. The reason that we might expect this to work is basically that the anomaly is sensitive to UV physics (it comes from doing an OPE of two fermions defined at the same point), and so if we are allowed to take $M \rightarrow \infty$ at the end of the calculation in the spirit of PV regularization, the anomalous term should be independent of M .

Of course, we haven't actually gotten rid of the anomaly, since the Ψ fermion could still contribute to it directly. The chiral current for the massive fermion is

$$j^{5\mu} = \bar{\Psi}\gamma^\mu\gamma^5\Psi. \quad (310)$$

We then take the divergence and use the Dirac equation for Ψ , so that

$$\langle\partial_\mu j^{5\mu}\rangle = \langle 2iM\bar{\Psi}\gamma^5\Psi\rangle. \quad (311)$$

Since this vev can only come from loops we'd naively expect it to be zero after we send $M \rightarrow \infty$, but as we will see this is not the case.

The first contributions to this expectation value come from triangle-diagram-like diagrams, with two outgoing photons radiating from a Ψ loop and with one insertion of $2iM\bar{\Psi}\gamma^5\Psi$ on the Ψ loop. These diagrams are derived in the usual way by adding $\int 2i\omega M\bar{\Psi}\gamma^5\Psi$ to the action, integrating out the fermions, expanding the $\ln \det$, and then differentiating with respect to ω by selecting out only those diagrams with a single insertion of $2iM\bar{\Psi}\gamma^5\Psi$. The relevant diagram (a fermion bubble with two photon legs and an insertion of $M\bar{\Psi}\gamma^5\Psi$) gives us

$$\begin{aligned} \langle p, k | 2iM\bar{\Psi}\gamma^5\Psi | 0 \rangle &= 2iMe^2i^3 \int_q \text{Tr} \left[\gamma^\nu \frac{1}{\not{q}-M} \gamma^\mu \frac{1}{\not{q}-\not{p}-M} \gamma^5 \frac{1}{\not{q}+\not{k}-M} \right] \epsilon_\mu^*(p) \epsilon_\nu^*(k) \\ &= -8iM^2e^2 \int_q \epsilon^{\nu\lambda\mu\sigma} \frac{q_\lambda(q-p)_\sigma - q_\lambda(q+k)_\sigma + (q-p)_\lambda(q+k)_\sigma}{(q^2 - M^2)((q-p)^2 - M^2)((q+k)^2 - M^2)} \epsilon_\mu^*(p) \epsilon_\nu^*(k) \\ &= -8iM^2e^2 \int_q \int_{x,y} 2 \frac{q^2 g_{\lambda\sigma}/4 - p_\lambda k_\sigma}{(q^2 - M^2 + \dots)^3} \epsilon_\mu^*(p) \epsilon_\nu^*(k), \end{aligned} \quad (312)$$

where in the last step the 2 comes from passing to Feynman parameters and the \dots are a standin for terms involving the Feynman parameters and the photon momenta. We do the

shift to simplify the denominator in the usual way, but we don't have to really keep track of what happens since the q^2 term in the numerator dies by antisymmetry anyway. So we get

$$\langle p, k | 2iM\bar{\Psi}\gamma^5\Psi | 0 \rangle = 16iM^2e^2 \int_q \int_{x,y} \epsilon^{\mu\nu\lambda\sigma} p_\lambda k_\sigma \frac{\epsilon_\mu^*(p)\epsilon_\nu^*(k)}{(q^2 - \Delta)^3}, \quad (313)$$

where $\Delta = M^2 + \dots$. This integral is perfectly UV finite, even without the $M \rightarrow \infty$ limit. We get

$$\int_q \frac{1}{(q^2 - \Delta)^3} = -\frac{i}{2(4\pi)^2} \Delta^{-1}. \quad (314)$$

Taking now $M \rightarrow \infty$ the integral over the Feynman parameters just gives 1 since $\Delta \rightarrow M^2$, and so we get

$$\langle k, p | \partial_\mu j^{5\mu} | 0 \rangle = \frac{e^2}{2\pi^2} \epsilon^{\mu\nu\lambda\sigma} \epsilon_\mu^*(p) \epsilon_\nu^*(k) p_\lambda k_\sigma, \quad (315)$$

which is the expected result. Evidently the somewhat dubious arguments we made in the beginning about terms cancelling worked out this time.

25 May 9 — Trace anomaly via Pauli-Villars

This is the second logical half of the P&S problem that we started yesterday. Reproduce the trace anomaly for *QED* in four dimensions with a PV regularization scheme (you will also need to dimensionally-regularize one integral). In a similar spirit to the last problem, you should show that the anomaly is computed by taking the $M \rightarrow \infty$ limit of

$$\langle M\bar{\Psi}\Psi \rangle. \quad (316)$$

Solution:

Recall that the generator of dilations is $D^\mu = T^{\mu\nu}x_\nu$, where $T^{\mu\nu}$ is the “improved” (symmetric, gauge-invariant) stress tensor. Now $\partial_\mu D^\mu = T^\mu_\mu$, so we get a trace anomaly when $\langle T^\mu_\mu \rangle$ is “unexpectedly” nonzero.

Let's first write down what we expect to get for $\langle T^\mu_\mu \rangle$. In four dimensions e is dimensionless, but it contributes to the trace anomaly because of renormalization, which introduces mass scales into the theory. Performing a dilation $x \mapsto e^{-\lambda}x$ for λ infinitesimal, the gauge coupling changes as (apologies for the profusion of e 's — should change to g , but too late)

$$e \mapsto e + \partial_M e \delta M = e + \partial_M e (e^\lambda M - M) \implies \delta e = \beta(e). \quad (317)$$

Performing this change of coordinates in the path integral, by the usual procedure we get

$$\langle \partial_\mu D^\mu \rangle = \langle T^\mu_\mu \rangle = \lambda \beta(e) \partial_e \mathcal{L} = \frac{\beta(e)}{2e^3} F_{\lambda\sigma} F^{\lambda\sigma} = \frac{1}{24\pi^2} F_{\lambda\sigma} F^{\lambda\sigma}, \quad (318)$$

where in the last step we've used the first-order result for the β function in QED.

The goal now is to see how we can reproduce this with an explicit calculation. To find $\langle T_{\mu}^{\mu} \rangle$, we can formally add $\int \eta T_{\mu}^{\mu}$ to the action, integrate out the fermions, and then differentiate with respect to η . To do this we need the energy-momentum tensor for QED. Now we know the Hamiltonian is

$$H = \frac{1}{2e^2}(E^2 + B^2) - i\bar{\psi}(\gamma^i \partial_i - m)\psi, \quad (319)$$

since when going from the Lagrangian to the Hamiltonian the ∂_t part in the Dirac operator gets cancelled. So then we just need to write down a more covariant expression of μ, ν that reduces to the above when $\mu = \nu = 0$. Such an expression is

$$T^{\mu\nu} = -\frac{1}{e^2}F^{\mu\sigma}F_{\sigma}^{\nu} + \frac{1}{4e^2}g^{\mu\nu}F^2 + i\bar{\psi}(\gamma^{\{\mu}D_A^{\nu\}})\psi - g^{\mu\nu}\bar{\psi}(iD_A - m)\psi. \quad (320)$$

When this gets stuck in diagrams, it gives us source counterterms for the gauge field propagator, the fermion propagator, and the electron / photon vertex. If we hadn't added the regulator fermion, we would calculate the anomaly by computing $\langle T_{\mu}^{\mu} \rangle$ perturbatively in dimensional regularization, as outlined in P&S chapter 19. Using $g_{\mu}^{\mu} = g^{\mu\nu}g_{\nu\mu} = d$ in dimensional regularization, we have, for a massless fermion,

$$\langle T_{\mu}^{\mu} \rangle = F^2 \left(\frac{d}{4} - 1 \right) + (1-d)\bar{\psi}iD_A\psi. \quad (321)$$

The second term ends up not contributing to diagrams because of the equations of motion. However the first term does — it acts as a counterterm in photon propagators, and when these are stuck onto the legs of polarization bubbles with fermions running in the loop, the $(d/4 - 1)$ is rendered finite when it hits a divergent part of a $\Gamma(\epsilon/2)$ coming from the fermion integration. When we add in the regulator fermion, we get contributions to the polarization bubble from both the original fermion and the regulator. One can fairly quickly check (I won't write it out since it's similar to the calculation we'll do below) that the relevant one-loop contribution to the trace anomaly due to the regulator fermions comes from the integral

$$\frac{\epsilon}{4} \int_q \frac{f(k, p)}{(q^2 - \Delta)^2}, \quad (322)$$

where p, k are external photon momenta and Δ goes to M^2 in the large M limit. In four dimensions, we get something like

$$\epsilon (\Gamma(\epsilon/2) - \ln \Delta + \dots), \quad (323)$$

and so when we take $\epsilon \rightarrow 0$, we get something which is independent of M ! Thus, the regulator fermions exactly cancel the contribution to anomalous terms in the polarization bubbles that the massless fermions make! This is similar to the reasoning in the last problem — we're still arguing that we can get away with waiting until the very end of the calculation to take $M \rightarrow \infty$.

Of course, this doesn't eliminate the anomaly, since the regulator fermions contribute to the trace of T explicitly through their mass term. So, we have

$$\langle T_{\mu}^{\mu} \rangle = \langle M\bar{\Psi}\Psi \rangle. \quad (324)$$

Our task is to evaluate this in the limit $M \rightarrow \infty$ and see whether it is zero or not.

As usual, the first diagram that shows up is a triangle-type diagram with two photon legs and an insertion of the relevant operator (here $M\bar{\Psi}\Psi$) on the fermion loop. We get

$$\langle p, k | M\bar{\Psi}\Psi | 0 \rangle = iM e^2 \int_q \text{Tr} \left[\gamma^\nu \frac{1}{q - M} \gamma^\mu \frac{1}{q - p - M} \frac{1}{q + k - M} \right] \epsilon_\mu^*(p) \epsilon_\nu^*(k) \quad (325)$$

Only terms with even numbers of γ matrices contribute. Their contributions can be found using standard rules for tracing out products of γ matrices. Note that we cannot at this stage ignore linear in q terms in the numerators, since we haven't Feynman-ized the denominator yet to make it a function of just q^2 .

We need two Feynman parameters to do the integral. After a bit of algebra, one sees that the appropriate shift in momentum needed to eliminate the linear-in- q terms in the denominator is

$$q \mapsto q + kx - py, \quad (326)$$

where x, y are the Feynman parameters. The numerator, before the shift, is

$$4g^{\mu\nu}M^3 + M\text{Tr} [(\not{p} + \not{q})\gamma^\mu \not{q} \gamma^\nu + (\not{p} + \not{q})\gamma^\mu \gamma^\nu (\not{q} - \not{k}) + \gamma^\mu \not{q} \gamma^\nu (\not{q} - \not{k})], \quad (327)$$

where we have dropped things with an odd number of gamma matrices. The possible terms that we get after the shift which have an even number of qs go like

$$g^{\mu\nu}, \quad g^{\mu\nu}q^2, \quad q^\mu q^\nu, \quad k^2 g^{\mu\nu}, \quad p^2 g^{\mu\nu}, \quad k^\mu p^\nu, \quad k^\nu p^\mu, \quad p^\mu p^\nu, \quad k^\mu k^\nu. \quad (328)$$

The k^2 and p^2 terms are zero since we're dealing with photons. Anything with a p^μ or a k^ν is zero, since these momenta will be contracted with the polarizations and will thus be killed. To do the trace over the surviving terms, we use

$$\begin{aligned} \text{Tr}[\not{r}\gamma^\mu \not{s}\gamma^\nu] &= 4(r^\mu s^\nu + r^\nu s^\mu - r \cdot s g^{\mu\nu}), \\ \text{Tr}[\not{r}\gamma^\mu \gamma^\nu \not{s}] &= 4(r^\mu s^\nu - r^\nu s^\mu + r \cdot s g^{\mu\nu}). \end{aligned} \quad (329)$$

I won't write out all the algebra for expediency's sake, since it is straightforward. One can check that the surviving $p^\nu k^\mu$ term appears as

$$4Mp^\nu k^\mu (1 - 4xy), \quad (330)$$

and that the $p \cdot k$ appears with the same coefficient but with opposite sign (as dictated by gauge invariance).

What of the M^3 term in the numerator? This turns out to get killed by other things in the trace. We have

$$\begin{aligned} \text{Tr} [\not{q}\gamma^\mu \not{q}\gamma^\nu + \not{q}\gamma^\mu \gamma^\nu \not{q} + \gamma^\mu \not{q}\gamma^\nu \not{q}] &= 16q^\mu q^\nu - 4g^{\mu\nu}q^2 \rightarrow g^{\mu\nu} \left(\frac{16}{4 - \epsilon} - 4 \right) q^2, \\ &= g^{\mu\nu} \epsilon q^2 \end{aligned} \quad (331)$$

where we've anticipated the use of dimensional regularization in the integral. This term is integrated with a denominator of $(q^2 - \Delta)^3$, where as usual Δ is a function of x, y, p, k . This term then cancels the M^3 term:

$$\begin{aligned} \int_{q,x,y} \frac{4M^3 + \epsilon M q^2}{(q^2 - \Delta)^3} &= \frac{-4iM^3}{32\pi^2 \Delta} + \frac{Mi\epsilon}{16\pi^2} (\Gamma(\epsilon/2) + \text{finite}) \\ &= \frac{i}{8\pi^2} (-M^3/\Delta + M) \rightarrow 0, \end{aligned} \quad (332)$$

since $\Delta \rightarrow M^2$ in the $M \rightarrow \infty$ limit. So, we have

$$\langle p, k | M \bar{\Psi} \Psi | 0 \rangle = -2e^2 Mi \int_{q,x,y} \frac{4(4xy - 1)(p \cdot kg^{\mu\nu} - p^\nu k^\mu)}{(q^2 - \Delta)^3} \epsilon_\mu^*(p) \epsilon_\nu^*(k). \quad (333)$$

The integral over the Feynman parameters is done over the range $\int_0^1 \int_0^{1-x} dx dy$ since the integral is over the face of a 3-simplex in \mathbb{R}^3 . Since we can set $\Delta \rightarrow M^2$ independent of x, y in the $M \rightarrow \infty$ limit, the integral over the Feynman parameters just yields $2/3$. So we get

$$\begin{aligned} \langle T^\mu_{\mu} \rangle &= -\frac{16e^2 Mi}{3} \int_q \frac{p \cdot kg^{\mu\nu} - p^\nu k^\mu}{(q^2 - M^2)^3} \epsilon_\mu^*(p) \epsilon_\nu^*(k) \\ &= \frac{e^2}{6\pi^2} (p \cdot k \epsilon^*(p) \cdot \epsilon^*(k) - p \cdot \epsilon^*(k) k \cdot \epsilon^*(p)). \end{aligned} \quad (334)$$

Is this a sensible answer? Yes! We see that this is exactly equal to what we expected from the argument given earlier relating the trace anomaly to the beta function for the gauge coupling (note to self — did we drop a factor of two somewhere?)

26 May 10 — Orientability and the fundamental group

Show that if M is a connected manifold such that $\pi_1(M)$ has no subgroups of index 2, then M is orientable.

Solution:

We will prove the contrapositive, namely that if M is non-orientable then $\pi_1(M)$ must have an index 2 subgroup.

Intuitively, the reason for this is as follows. Suppose M is non-orientable, and consider curves in $\pi_1(M)$. Since M is non-orientable it should have a cycle $c \in C_1(M; \mathbb{Z})$ around which the orientation cannot be consistently defined, since the orientation bundle of M must be nontrivial. Consider going around c twice. Then $c \times c$ in $\pi_1(M)$ looks like two paths next to each other, both traveling in the same direction. However since the orientation cannot be defined consistently along c , swapping the direction that one of the paths travels in does not change what $c \times c$ is in $\pi_1(M)$. But then we have two copies of the same path traveling in opposite directions, and so we can “annihilate” these paths, producing the trivial element in $\pi_1(M)$. Thus we expect $\pi_1(M)$ to have some sort of \mathbb{Z}_2 character coming from loops of this sort.

Now for the more precise argument. If M is non-orientable then it has two-sheeted double cover \mathcal{M} , which is a covering space $p : \mathcal{M} \rightarrow M$ for M . Let the \mathbb{Z}_2 fiber at a given point $x \in M$ be denoted by F , and consider the monodromy action of the fundamental group based at x on the fiber,

$$\pi_1(M, x) \times F \rightarrow F. \quad (335)$$

The action is defined by lifting up elements in $\pi_1(M, x)$ into \mathcal{M} , and taking $h \cdot x = h(1)$ for h any element of $\pi_1(M, x)$ lifted up into \mathcal{M} . Thus the monodromy action permutes the (two) elements of F . This action is always transitive, since if x, x' are two distinct points in F , the path in \mathcal{M} connecting them projects down into $\pi_1(M, x)$, and gives us a lift with which to have an action that permutes x and x' .

So, we have a homomorphism

$$f : \pi_1(M) \rightarrow \mathbb{Z}_2 \quad (336)$$

defined by the monodromy action on the fibers. Now $\ker(f)$ is a subgroup of $\pi_1(M)$, and so since $\text{im}(f) = \mathbb{Z}_2$, we have

$$[\pi_1(M) : \ker(f)] = \dim \text{im}(f) = 2. \quad (337)$$

Thus $\ker(h)$ provides us with the desired index-two subgroup.

27 May 10 — Fundamental groups and homotopies into S^1

Suppose X is path connected, with $\pi_1(X)$ finite. Show that all maps from X into S^1 are nullhomotopic.

Solution:

We will show this by using the covering spaces \tilde{X} and $\tilde{S}^1 = \mathbb{R}$, so first some preliminaries about these. For a general map $f : X \rightarrow Y$, with \tilde{X} the universal cover of X and \tilde{Y} any cover of Y , consider the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array} \quad (338)$$

We want to first show the existence of the lift \tilde{f} that maps between the covering spaces. Consider the map $f \circ \pi_X : \tilde{X} \rightarrow Y$. This will lift to the desired map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ if the lifting criterion is satisfied, namely if (sorry for the profusion of π 's)

$$(f \circ \pi_X)(\pi_1(\tilde{X})) \subset \pi_Y(\pi_1(Y)). \quad (339)$$

This is true for us since \tilde{X} is the universal cover of X , and hence has trivial fundamental group. Thus \tilde{f} exists. Note that we are *not* assuming the existence of a lift of f to a map $g : X \rightarrow \tilde{Y}$, since this imposes too strong a constraint on $\pi_1(X)$.

Now we can start proving the claim. Throughout, we will be cavalier about keeping explicit track of basepoints and related issues.

Suppose that the map f were not nullhomotopic. Then we must have $f(X) = [k] \in \pi_1(S^1) = \mathbb{Z}$ such that $k \neq 0$, where $[k]$ stands for a representative of the element in $\pi_1(S^1)$

which wraps the circle k times. Now consider pulling back $[k]$ into X . The inverse image of a given point on S^1 will of course not usually be unique, but we can choose from among the points in the inverse image so that as we move around continuously in S^1 , we move around continuously in X . The point of this procedure is just to pullback $[k]$ to a curve in X . The pullback of $[k]$ cannot be an open curve in X , since then $f(X)$ would be contractible and $f(X)$ would be nullhomotopic. An example of this is the map

$$g : S^2 \rightarrow S^1, \quad (\theta, \phi) \mapsto 2\theta. \quad (340)$$

This covers the S^1 , but it does so in a contractible way, and the curves in the inverse image of $[1]$ are all contractible. Similarly, the pullback of $[k]$ cannot be contractible, since then $f(X)$ would have been contractible. So, we conclude that the pullback of $[k]$ must be non-contractible in X , and hence if $f(X)$ is not nullhomotopic, then it must map some nontrivial element in $\pi_1(X)$ to $[k]$.

We will now show that this is a contradiction. Consider the (left) action of deck transformations

$$\text{Deck} : \pi(X, x_0) \rightarrow \text{aut}(\pi_X^{-1}(x_0)), \quad \gamma \mapsto (\gamma : x_0 \mapsto [\gamma] \cdot x_0), \quad (341)$$

where $\gamma \cdot x_0$ is the endpoint in \tilde{X} of the curve γ based at x_0 , when that curve is lifted up to \tilde{X} . The lifted map \tilde{f} is such that it allows us to either lift to \tilde{X} and then map, or map and then lift to \tilde{Y} — the results are the same. This means that the action of deck transformations commutes with the mapping between the two spaces:

$$[f(\gamma)] \cdot f(x_0) = \tilde{f}([\gamma] \cdot x_0). \quad (342)$$

Now since we are assuming $f(X)$ has degree k around the S^1 , we can always choose γ to be such that $[f(\gamma)] = [k]$ in $\pi_1(S^1) = \mathbb{Z}$. We know how deck transformations on the covering space of the circle work: they shift us up by one coil on the helix over S^1 . Thus if the S^1 has radius 1, the deck transformation must act as (after choosing an appropriate lift of $\theta = f(x_0)$ into \mathbb{R})

$$[f(\gamma)] \cdot \theta = \theta + 2\pi k. \quad (343)$$

In particular, we see that

$$\tilde{f}([\gamma^n] \cdot x_0) = [f(\gamma)^n] \cdot \theta = \theta + 2\pi nk, \quad (344)$$

for any $n \in \mathbb{Z}$.

Now we get a contradiction: we have assumed that $\pi_1(X)$ is finite, and so in particular, all elements in $\pi_1(X)$ must have finite order. This is in contradiction to the action of the deck transformations we derived above, which implies that all of the elements in $\pi_1(X)$ have infinite order.

As a quick application of this result, we can derive that $\pi_n(S^1) = 0$ for all $n > 1$, which follows from the fact that $\pi_1(S^n) = 0$ for $n > 1$.

28 May 11 — Mixed t Hooft anomaly for the compact boson / Luttinger liquid

Consider the compact boson in two dimensions at radius R :

$$S = \frac{R^2}{4\pi} \int \partial_\mu \phi \partial^\mu \phi. \quad (345)$$

Work on the spacetime $S^1 \times \mathbb{R}$, and write down the charge operators for the momentum and winding number global symmetries. Carry out the analysis using Abelian duality to define a field σ such that $d\phi \propto \star d\sigma$, and find the commutation relations between ϕ and σ .

Next, consider a field with non-zero charge under both momentum and winding number symmetries, and find the algebra of the symmetry charges. By considering subregion charge operators which act on submanifolds of the spatial circle, show that the symmetry is realized projectively. Demonstrate the nontrivial third $U(1)$ cohomology class which parametrizes the mixed t Hooft anomaly. See [6] for inspiration and to see how this works when the symmetry is \mathbb{Z}_2 .

Solution:

First let us write down the dual theory. This is done in the standard way by promoting $d\phi \mapsto D_B \phi$ for B a dynamical one-form, adding the term $\frac{i}{2\pi} \int B \wedge d\sigma$ for σ a zero-form, choosing unitary gauge to set $\phi \rightarrow 0$, and finally doing the Gaussian integral over B . This produces the dual action

$$S = \frac{1}{4\pi R^2} \int \partial_\mu \sigma \partial^\mu \sigma. \quad (346)$$

We can find the relation between the field strengths by looking at where $d\phi$ goes under the mapping. We take $d\phi \rightarrow D_B \phi$ and then kill ϕ to get B . After we do the shift on B to eliminate the $B \wedge d\sigma$ coupling and then do the integral over B , we are left with $R^{-2} \star d\sigma$. Thus under the duality,

$$d\phi \leftrightarrow \frac{1}{R^2} \star d\sigma. \quad (347)$$

From the original action, we see that the canonical momentum for ϕ is

$$\pi_\phi = \frac{R^2}{2\pi} \partial_t \phi = \frac{R^2}{2\pi} \frac{1}{R^2} \epsilon^{tx} \partial_x \sigma = \frac{1}{2\pi} \partial_x \sigma, \quad (348)$$

where x is the direction along the spatial S^1 . Likewise, we could also do canonical quantization on the dual field:

$$\pi_\sigma = -\frac{1}{2\pi} \partial_x \phi. \quad (349)$$

The symmetry here is part of the reason why we chose the factors of 2π in the action as we did.

From the commutation relation

$$[\phi(x), \partial_y \sigma(y)] = 2\pi i \delta(x - y), \quad (350)$$

we will write the commutator of the dual fields as

$$[\phi(x), \sigma(y)] = -2\pi i \Theta(x - y). \quad (351)$$

We could also have chosen the RHS to be $-\pi i \operatorname{sgn}(x - y)$, but the Θ function will be more convenient later on.

The commutation relations above make the non-locality of the duality mapping manifest — the commutator of ϕ and σ is incredibly non-local. This is essentially because ϕ and σ are “fourier transforms” of one another, and so we run into problems if we try to specify both ϕ and σ as well-defined functions simultaneously (the same thing happens in E&M). Thus working directly with σ and ϕ simultaneously is kind of dicey. On a related note, if we try to impose the commutation relations for both π_σ and π_ϕ at the same time, we run into inconsistencies when we compute the $[\phi(x), \sigma(y)]$ commutator. Finally, one may be annoyed with the equation for $[\phi(x), \sigma(y)]$ since we are working on a circle, where this doesn’t make sense. We could go through the trouble of keeping track of basepoints or adding in gauge fields, but this isn’t really needed since we’re only interested in the representation of the fields on a contractible subsection of the S^1 (note to self, come back and think about these points more).

Using our expressions for the momenta of the fields, we see that the charges for the winding number and momentum symmetries are represented on charge w, n fields as

$$Q^w = \exp\left(\frac{iw}{2\pi} \int \partial_x \sigma\right), \quad Q^n = \exp\left(-\frac{in}{2\pi} \int \partial_x \phi\right), \quad (352)$$

where the integrals are over the whole S^1 . Note that the charge operators involve exponentials of operators which don’t commute, which will be important later.

To see if there are any anomalies, we should ask whether or not the symmetry is “splittable”, i.e. whether or not the charge operators can be well-defined as operators acting on submanifolds of the spatial S^1 rather than on all of space. To this end, consider the symmetry generator for an element $(\alpha, \beta) \in U(1)_{\text{momentum}} \times U(1)_{\text{winding}}$ on a given interval $I = [a, b]$:

$$U_I(\alpha, \beta) = \exp\left(-\frac{in[\alpha]}{2\pi} \int_a^b \partial_x \phi\right) \exp\left(\frac{iw[\beta]}{2\pi} \int_a^b \partial_x \sigma\right), \quad (353)$$

where we have used the notation

$$[x] \equiv x \bmod 2\pi. \quad (354)$$

We now want to know whether or not these subsystem symmetry generators realize a linear representation of the $U(1) \times U(1)$ symmetry group. Noting that the two exponentials in the definition of U_I commute, one sees that

$$U_I(\alpha, \beta)U_I(\alpha', \beta') = \Omega_1^2((\alpha, \beta), (\alpha', \beta'); I)U_I(\alpha + \beta, \alpha' + \beta'), \quad (355)$$

where Ω_1^2 is a function of two group elements integrated over a 1-submanifold:

$$\Omega_1^2((\alpha, \beta), (\alpha', \beta'); I) = \exp\left(-\frac{in\Delta(\alpha, \alpha')}{2\pi} \int_b^a \partial_x \phi\right) \exp\left(\frac{iw\Delta(\beta, \beta')}{2\pi} \int_b^a \partial_x \sigma\right), \quad (356)$$

where we have defined the coboundary operator

$$\Delta(x, y) = [x] + [y] - [x + y], \quad (357)$$

which is nontrivial for e.g. $x = y = \pi$. Thus nontriviality of Ω_1^2 indicates that “symmetry fractionalization” occurs. Note that $\Omega_1^2((\alpha, \beta), (\alpha', \beta'); I)$ only actually operates on ∂I , since the fact that $\phi(a)$ commutes with $\sigma(b)$ as $a < b$ means that we can write

$$\Omega_1^2((\alpha, \beta), (\alpha', \beta'); I) = \Omega_0^2((\alpha, \beta), (\alpha', \beta'); b)[\Omega_0^2((\alpha, \beta), (\alpha', \beta'); a)]^*, \quad (358)$$

where

$$\Omega_0^2((\alpha, \beta), (\alpha', \beta'); x) = \exp\left(-\frac{in\Delta(\alpha, \alpha')}{2\pi}\phi(x)\right)\exp\left(\frac{iw\Delta(\beta, \beta')}{2\pi}\sigma(x)\right). \quad (359)$$

For example, if we choose $(\alpha, \alpha') = (\beta, \beta') = (\pi, \pi)$, then

$$\Omega_0^2((\pi, \pi), (\pi, \pi)) = e^{-in\phi(x)}e^{iw\sigma(x)}. \quad (360)$$

The fact that the commutation relations for the operators appearing in Ω_0^2 are very non-local implies that the symmetry is not realized in an “onsite” way on the boundary.

Also, as we will see this form for Ω_0^2 is in keeping with the descent equations for anomalies, which we will elaborate on later. Also, note that Ω_0^2 is only defined up to a coboundary of a group 1-cocycle, since we are allowed to re-define the U_I operators by phases via $U_I((\alpha, \beta)) \mapsto e^{ig((\alpha, \beta))}U_I((\alpha, \beta))$.

Now we need to figure out what the anomaly is, by going one level up in group cohomology (Ω_2^1 is a group 2-cochain, and the anomaly will be captured by a 3-cochain). To do this, we examine associativity of the symmetry operators, since the failure of associativity is captured by 3-cochains. The associativity of the U_I operators requires the cocycle condition on Ω_1^2 , namely

$$(\delta\Omega_1^2)((\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta'')) = 0, \quad (361)$$

where δ is the coboundary operator. One can check that this will hold provided that

$$\Delta(\alpha, \alpha') + \Delta(\alpha + \alpha', \alpha'') - \Delta(\alpha', \alpha'') - \Delta(\alpha, \alpha' + \alpha'') = 0, \quad (362)$$

which is indeed true. Note that we should really be writing e.g. $\Delta([\alpha + \alpha'], \alpha'')$, but we aren’t since $\Delta([\alpha + \alpha'], \alpha'') = \Delta(\alpha + \alpha', \alpha'')$. We stress that if this associativity condition did not hold our theory would not make sense — it is merely a consistency check.

Now we ask if the action is realized associatively on individual points, i.e. whether or not the Ω_0^2 operators are annihilated by δ . This may or may not be true, and if not, it signals an anomaly. We define the cochain Ω^3 (which is just a phase factor — no ϕ or σ operators included) to measure the lack of associativity through

$$\Omega^3((\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta'')) \equiv (\delta\Omega_0^2)((\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta'')). \quad (363)$$

This formula requires some explanation. Writing group elements momentarily as g, h, k , associativity tells us that

$$\begin{aligned} [U_I(g)U_I(h)]U_I(k) &= U_I(g)[U_I(h)U_I(k)] \\ \Omega_1^2(g, h)U_I(gh)U_I(k) &= U_I(g)\Omega_1^2(h, k)U_I(hk) \\ \Omega_1^2(g, h)\Omega_1^2(gh, k)U_I(ghk) &= {}^{U_I(g)}\Omega_1^2(h, k)\Omega_1^2(g, hk)U_I(ghk), \end{aligned} \quad (364)$$

where $U_I(g)$ acts on Ω_1^2 in the way needed to allow us to pull it through to meet $U_I(hk)$. Thus as checked above, we require that $\delta\Omega_1^2 = 0$, where the coboundary operator includes the action of U_I (we didn't mention the action earlier since Ω_1^2 commutes with the U_I 's). However, the individual operators Ω_0^2 which are localized at ∂_I do not need to be co-closed, and so in general we have

$$\Omega_0^2(g, h)\Omega_0^2(gh, k) = \Omega^3(g, h, k) \left({}^{U_I(g)}\Omega_0^2(h, k)\Omega_0^2(g, hk) \right). \quad (365)$$

We can then use the commutation rules to derive the action

$${}^{U_I(\alpha, \beta)}\Omega_0^2((\alpha', \beta'), (\alpha'', \beta'')) = \exp \left(-\frac{inw}{4\pi} [\beta\Delta(\alpha', \alpha'') + \alpha\Delta(\beta', \beta'')] \right) \Omega_0^2((\alpha', \beta'), (\alpha'', \beta'')). \quad (366)$$

Then using $\Theta(0) = 1/2$ in the commutation relations for ϕ and σ operators at the same point, some algebra gives

$$\Omega^3((\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta'')) = \exp \left(\frac{inw}{4\pi} f((\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta'')) \right), \quad (367)$$

where

$$f((\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta'')) = \Delta(\beta, \beta')\Delta(\alpha + \alpha', \alpha'') - \Delta(\beta', \beta'')\Delta(\alpha, \alpha' + \alpha'') + \beta\Delta(\alpha', \alpha'') + \alpha\Delta(\beta', \beta''). \quad (368)$$

As with Ω_1^2 , Ω^3 is only defined up to a 3-coboundary, since we can re-define the Ω_0^2 operators by group 2-cocycles without changing the value of Ω_1^2 .

As an example, consider the case where we consider only the $\mathbb{Z}_2 \subset U(1)$ symmetry of translations of both ϕ and σ by π . Setting all of the group elements in the above to π , we find

$$\Omega^3((\pi, \pi), (\pi, \pi), (\pi, \pi)) = (-1)^{nw}. \quad (369)$$

One can check that we would have obtained the form for Ω^3 had we used different conventions for the commutation relations and symmetry generators, e.g. if we had taken the ϕ, σ commutator to go like $-\pi i \operatorname{sgn}(x - y)$ and taken the $\partial_x \phi$ integral in the symmetry generator to be from $a - \epsilon$ to $b + \epsilon$, where $\epsilon \rightarrow 0$ is used to ensure that the terms in U_I commute with one another.

Ω^3 represents the anomaly, and lives in $H^3(U(1) \times U(1); U(1))$ as expected. First, note that $\Omega^3 = 0$ if either $n = 0$ or $w = 0$. This signals the fact that the anomaly is a mixed anomaly between the two symmetries: either one by itself is non-anomalous, but together they become anomalous (essentially because their respective subsystem symmetry generators do not commute). Also, note that the way it was defined makes Ω^3 look like a coboundary and hence a trivial cohomology class, but this is not so since although $\Omega^3 = \delta\Omega_0^2$, Ω_0^2 is not an element of $C^2(U(1) \times U(1); U(1))$ (group 2-cochains), since it contains ϕ, σ operators.

As a sanity check, we should make sure that $H^3(U(1) \times U(1); U(1))$ is non-zero using other methods. The Künneth formula for group cohomology with $U(1)$ coefficients is a little dicey, so we instead calculate $H^4(U(1)^2; \mathbb{Z})$. We use that $H^p(U(1); \mathbb{Z}) = \mathbb{Z}$ if p is even and $H^p(U(1); \mathbb{Z}) = 0$ else, which is true because the group cohomology of $U(1)$ (with

\mathbb{Z} coefficients) is the simplicial cohomology of the classifying space of $U(1)$, namely \mathbb{CP}^∞ , which has a single generator in each even degree. Thus we get

$$H^4(U(1)^2; \mathbb{Z}) \cong \bigoplus_{i=0}^4 H^i(U(1); \mathbb{Z}) \otimes_{\mathbb{Z}} H^{4-i}(U(1); \mathbb{Z}) \cong \mathbb{Z}^3. \quad (370)$$

In our example, we are only accessing the \mathbb{Z} factor coming from $H^2 \otimes H^2$ (note to self: come back and pay more attention to group cohomology with a topological group — should probably be using the discrete topology.) Since group cohomology with topological groups is kinda weird, suppose our symmetry group was instead $\mathbb{Z}_n \times \mathbb{Z}_n$. Then the Künneth formula gives (recall that $H^p(\mathbb{Z}_n; \mathbb{Z})$ is \mathbb{Z} if $p = 0$, \mathbb{Z}_n if $p > 0$ is even, and 0 if p is odd)

$$H^3(\mathbb{Z}_n^2; U(1)) \cong \mathbb{Z}_n^3, \quad (371)$$

which agrees with our $U(1)$ result in the limit $n \rightarrow \infty$.

Finally, we point out that even if we restrict the symmetry group to $\mathbb{Z}_2^2 \subset U(1)^2$ with $\phi \mapsto \phi + \pi, \sigma \mapsto \sigma + \pi$, we still have an anomaly, as Ω^3 becomes a nontrivial class in

$$H^3(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1)) \cong \mathbb{Z}_2^3. \quad (372)$$

On a related note, even if only the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is preserved, the theory can still not be gapped without breaking the symmetry: for example, mass terms like $\cos \phi$ are not allowed since they are not symmetric, and while $\cos 2\phi$ is allowed, it leads to SSB since it has two inequivalent minima. This is in keeping with the rule that anomalous theories cannot be trivial in the IR unless the symmetry is explicitly broken.

29 May 12 — More on mixed t Hooft anomalies for free fields

In the last problem, we saw that the $U(1) \times U(1)$ symmetry of the compact boson was realized projectively, signaling the existence of a mixed t Hooft anomaly. Understand this in a different light by showing that the $U(1) \times U(1)$ symmetry cannot be gauged, and write down the anomaly polynomial (which is a 4-form). More generally, show this for the $U(1) \times U(1)$ ($D/2 - 1$)-form symmetry possessed by ($D/2 - 1$)-form $U(1)$ gauge theory in D dimensions, where D is even.

Solution:

We will write the action in D dimensions as

$$S = \frac{1}{2g^2} \int_M F \wedge \star F, \quad (373)$$

where F is some $D/2$ -form field which is locally $F = dA$ (depending on conventions there should potentially be some combinatorial prefactors dependent on D up front, but we will

ignore them). Since the field strength F is a $D/2$ form, this theory is self-dual, as $\star F$ is also a $D/2$ form. The dual action is

$$S = \frac{g^2}{2(2\pi)^2} \int \mathcal{F} \wedge \star \mathcal{F}, \quad (374)$$

where the dual field \mathcal{A} (locally $\mathcal{F} = d\mathcal{A}$) is mapped to the original through $\mathcal{F} \leftrightarrow g^2 \star F / 2\pi$. In writing both of these actions, we are *not* assuming that A and \mathcal{A} are globally well-defined. In fact, in the partition function we will need to sum over all possible line bundles for A and \mathcal{A} .

Suppose now that we try to gauge the two symmetries in the theory (the symmetries of shifting A and its dual by flat $(D/2 - 1)$ -forms). In order to gauge the shift symmetries, we introduce a gauge field B which lets us locally transform $A \mapsto A + \alpha$, where α is not necessarily in the kernel of d . There are many ways to see that doing so implies that the \mathcal{A} shift symmetry cannot also be simultaneously gauged, signaling an anomaly. This is essentially because A and \mathcal{A} are related to each other by a “Fourier transform” and so the way in which a gauge field acts on them cannot be “simultaneously diagonalized” on both of the fields.

In order for the shift symmetry of A to be gauged, we must minimally couple to a background field B as

$$S[B] = \frac{1}{2g^2} \int (F - B) \wedge \star(F - B). \quad (375)$$

In writing this expression, we are tacitly assuming that B is a connection on a trivial $(D/2)$ -bundle, so that B is globally well-defined. Relaxing this assumption can be done but leads to keeping track of more details which aren’t super illuminating. For example, if $D = 4$, then we have regular Maxwell theory. Under gauge transformations, both F and B shift by F' , where F' is allowed to be the curvature of any connection on any line bundle (not necessarily trivial, since we are summing over all line bundles for the original gauge field A), and B is a 2-connection on a trivial 2-bundle ($U(1)$ gerbe).

Schematically, we can argue that this leads to a breaking of the shift symmetry for \mathcal{A} as follows. The upgraded gauge-invariant current for the shift symmetry of A is $(B - F)/g^2$, which we get by computing $\delta S[B]/\delta B$. This is conserved provided that the sources of the A gauge field match up with $d^\dagger B$, i.e. provided that $d^\dagger F = d^\dagger B$. This holds as the equation of motion for A . We expect that the current for the dual shift symmetry on \mathcal{A} is then

$$\mathcal{J} \sim \star(F - B). \quad (376)$$

However, the conservation of \mathcal{J} is broken by the curvature of B , since

$$d^\dagger \mathcal{J} \sim d^\dagger \star(F - B) \propto \star dB \neq 0. \quad (377)$$

An analogous problem would have occurred had we started with the dual action, and gauged the shift symmetry on \mathcal{A} by taking

$$\mathcal{F} \mapsto (\mathcal{F} - \mathcal{B}). \quad (378)$$

Doing this would lead to a violation of the conservation of the J current of the form

$$d^\dagger J \propto \star d\mathcal{B}. \quad (379)$$

The derivation of these results depended on the use of the Bianchi identity, i.e. they depended on us working strictly in either the formulation of the action in terms of A or the formulation in terms of \mathcal{A} . Essentially, the point is that no matter which formulation we choose, one of the currents must not be conserved after we introduce a gauge field (choosing a mixed formulation is not allowed since we want a local action). This argument is kind of sloppy since we have used things like $F \sim \star \mathcal{F}$, but $\star F$ and $g^2 \mathcal{F}/2\pi$ aren't really directly proportional — they are mapped to one another under duality, but writing an expression like $\star F = g^2 \mathcal{F}/2\pi$ is kind of delicate: F lives on one side of the duality and \mathcal{F} lives on the other¹⁰.

Now for a more careful argument. To figure out what happens in the dual formulation, we just run Abelian duality on this in the standard way. If the coupling to the background gauge fields is not anomalous, we should get an action like $S[\mathcal{B}] \sim g^2 \int (\mathcal{F} - \mathcal{B})^2$. The usual duality recipe tells us to first add a gauge field, and then integrate out the original field variables. So, we add a $D/2$ -form gauge field a to the action, and then add *another* ($D/2 - 1$)-form field \mathcal{A} with curvature \mathcal{F} whose job is to kill off a to reproduce the original non-gauged theory. Summing over all line bundles L for A and \mathcal{L} for \mathcal{A} , we have the partition function

$$Z = \sum_{L,\mathcal{L}} \int DADaD\mathcal{A} \exp \left(\frac{1}{2g^2} \int (F - a - B) \wedge \star(F - a - B) + \frac{i}{2\pi} \int \mathcal{F} \wedge a \right). \quad (384)$$

Here we are being sloppy and not writing factors of $1/\text{vol } G$ for the various groups of gauge transformations.

In the present partition function A and \mathcal{A} are independent fields. Thus the coupling $\mathcal{F} \wedge a$ is gauge-invariant since \mathcal{F} does not a priori transform under shifts in A (this can be corroborated by checking that under duality, both F and $F + da$ map to $g^2 \mathcal{F}/2\pi$, instead of e.g. $F + d\alpha$ mapping to $\mathcal{F} + g^2 \star d\alpha/2\pi$). Also note that the $\mathcal{F} \wedge a$ coupling is gauge invariant under the $\mathcal{A} \mapsto \mathcal{A} + \alpha$ shift symmetry: since the integral over \mathcal{A} sets a to be exact, after

¹⁰We have been setting $\theta = 0$ so far for simplicity. With a θ term, it's easiest to work in terms of the self-dual and anti-self-dual field strengths. We write the action as

$$\frac{i}{4\pi} \int (\tau F_+ \wedge F_+ + \bar{\tau} F_- \wedge F_-) = \frac{i}{4\pi} (\tau \langle F_+, F_+ \rangle - \bar{\tau} \langle F_-, F_- \rangle), \quad (380)$$

where the modular parameter is $\tau = \frac{\theta}{2\pi} - \frac{2\pi i}{g^2}$. Running Abelian duality tells us that the appropriate dual fields should be identified as

$$\mathcal{F}_+ = i\tau F_+, \quad \mathcal{F}_- = i\bar{\tau} F_-. \quad (381)$$

So, without the θ term the duality between F and \mathcal{F} is just Hodge duality plus an inverting of the coupling constant, while the θ term mixes F and its Hodge dual together.

Now we insert a background field for F , making the replacements $F_\pm \mapsto F_\pm - B_\pm$ in the action. We then run duality on this, which won't be written out explicitly for the sake of expediency. The appropriate dualized background field is

$$\mathcal{B} = \text{Re}(\tau)B + \text{Im}(\tau)\star B, \quad (382)$$

and we find that running the duality produces the action we'd expect, up to a contact term for the dual background fields:

$$\begin{aligned} S_{dual} &= -\frac{i}{4\pi} \int \left(\frac{1}{\tau} (\mathcal{F}_+ - \mathcal{B}_+) \wedge \star(\mathcal{F}_+ - \mathcal{B}_+) - \frac{1}{\bar{\tau}} (\mathcal{F}_- - \mathcal{B}_-) \wedge \star(\mathcal{F}_- - \mathcal{B}_-) \right) - S_{\mathcal{B}}, \\ S_{\mathcal{B}} &= -\frac{i}{4\pi} \int \left(\frac{1}{\tau} \mathcal{B}_+ \wedge \star \mathcal{B}_+ - \frac{1}{\bar{\tau}} \mathcal{B}_- \wedge \star \mathcal{B}_- \right). \end{aligned} \quad (383)$$

As explained in the main text, this extra contact term signals the mixed anomaly.

integrating out \mathcal{A} we are left with $\int_M d\alpha \wedge db$ for some b with $a = db$, and so since $\alpha|_{\partial M} = 0$ (as α is a gauge transformation), then $\int_M d\alpha \wedge db = 0$.

To see that we haven't done anything by writing the partition function in this way, we see that the integral over the globally well-defined part of \mathcal{A} sets $da = 0$, and the sum over \mathcal{L} gives a delta function setting

$$\int_N a \in 2\pi\mathbb{Z} \rightarrow [a] \in H_{2\pi\mathbb{Z}}^{D/2}(M; \mathbb{R}), \quad (385)$$

where $N \subset M$ is any closed $D/2$ manifold, and the subscript on the cohomology group indicates that a must integrate over any $D/2$ manifold to something in $2\pi\mathbb{Z}$. This quantization condition on a means that it is just the same as F : a closed $D/2$ form which is subject to Dirac quantization. Thus since we are summing over all line bundles for A , we can simply do a change of integration variables for A to absorb a into F , recovering the original action without a or \mathcal{A} .

Now we proceed by using the gauge freedom of a to kill off F , which is allowed since F is exact (this amounts to choosing "unitary gauge"). Then

$$Z = \sum_{\mathcal{L}} \int \mathcal{D}a \mathcal{D}\mathcal{A} \exp \left(\frac{1}{2g^2} \int (a + B) \wedge \star(a + B) + \frac{i}{2\pi} \int \mathcal{F} \wedge a \right). \quad (386)$$

Now we kill off the $\mathcal{F} \wedge a$ term by shifting the integration as

$$a \mapsto a - B - \frac{g^2 i}{2\pi} \star \mathcal{F}. \quad (387)$$

Quietly absorbing the Gaussian integral into the integration measure, we get the dual action

$$S_{dual} = \frac{1}{2g_{dual}^2} \int (\mathcal{F} \wedge \star \mathcal{F} + \frac{i}{g} (B \wedge \mathcal{F} + \star \mathcal{F} \wedge \star B)), \quad (388)$$

where the dual coupling constant is $g_{dual} = 2\pi/g$. To be a bit more suggestive, we define

$$\mathcal{B} = \frac{2\pi i}{g^2} \star B. \quad (389)$$

With this we get

$$S_{dual} = \frac{1}{2g_{dual}^2} \int ((\mathcal{F} - \mathcal{B}) \wedge \star(\mathcal{F} - \mathcal{B}) - \mathcal{B} \wedge \star \mathcal{B}) \quad (390)$$

Note that this action is what we would expect to get if we had started by gauging the shift symmetry on \mathcal{A} , except for the anomalous $\mathcal{B} \wedge \star \mathcal{B}$ term, which looks like a mass for the background field. We see that this presentation is not gauge invariant no matter how we choose B to transform under the \mathcal{A} shift symmetry, since the action transforms under gauge transformations as

$$\delta S \sim \int_M \alpha \wedge d \star \mathcal{B} \sim \int_M \alpha \wedge dB, \quad (391)$$

and so the gauge invariance of the dual action is broken by the curvature of the background field. Inserting background fields thus breaks electromagnetic duality, indicating the presence of the mixed anomaly.

Before we move on, we should point out that we have been using one higher $U(1)$ gauge field B to attempt to gauge both symmetries. In general, one might have thought that we should be allowed to use two higher gauge fields, since the full symmetry is a $U(1) \times U(1)$ 1-form symmetry. This turns out to not work, essentially because the two putative gauge fields have to be related to one another in an inconsistent way by duality. Let the two fields be denoted as B_e and $\star B_m$, with B_m neutral under the shift symmetry of A and B_e neutral under that of \mathcal{A} . In order to get a dual action which is gauge-invariant, the B_m fields need to be included in the original action, and so we can write

$$S = \frac{1}{2g^2} \int (dA - B_e - \star B_m) \wedge \star (dA - B_e - \star B_m). \quad (392)$$

In order for this to be a legitimate minimal coupling, we need the terms with B_m to integrate to zero. But this is possible only if

$$\int dA \wedge B_m = 0, \quad (393)$$

i.e. only if $dB_m = 0$, and as such we cannot gauge the \mathcal{A} symmetry with a genuine background field (one which is allowed to be non-flat). Any way we look at it, there's a mixed t Hooft anomaly.

30 May 13 — The simplest t Hooft anomaly

This was inspired by wanting to work through and elaborate on appendix D of “Theta, TR, and Temperature”. Consider the quantum mechanics of a free fermion with periodic boundary conditions in time, with Lagrangian $\mathcal{L} = i\psi^\dagger \partial_t \psi$. Show that this system has a mixed t Hooft anomaly between charge conjugation and $U(1)$. What is the corresponding bulk term that needs to be added to allow the system to be symmetric? Finally, briefly explain why this is equivalent to the model considered in Theta TR and Temperature, viz. the particle on a ring, which is in turn equivalent to QED₂ at $\theta = \pi$. Consider this model with a cosine potential, and find out whether or not the GSD at $\theta = \pi$ is lifted.

Solution:

Projective symmetry action:

Formally from the cohomology classification of anomalies, we know that in quantum mechanics, anomalies in a symmetry group will be classified by the group cohomology $H^2(G; U(1))$, which classifies central extensions of G by $U(1)$ (aka projective representations).

Let's first see that in quantum mechanics, the action of the classical symmetry group (in our case $O(2)$) must either be enlarged to a linear action of a central extension of the

classical symmetry group, or be represented projectively on the Hilbert space of the theory (depending on the way we choose to look at things).

Let us write the group elements of $U(1)$ as α , the generator of \mathbb{Z}_2^C as C , and their representations on the Hilbert space as R_α, R_C , which we will assume to be linear representations. They act on the ψ operators as

$$\alpha\psi\alpha^{-1} = e^{i\alpha}\psi, \quad C\psi C^{-1} = \psi^\dagger. \quad (394)$$

Thus from $R_\alpha\psi^\dagger R_\alpha^{-1} = e^{-i\alpha}\psi^\dagger$ (since the representation is assumed to be unitary), we have

$$e^{-i\alpha}\psi^\dagger = R_\alpha R_C \psi R_C^{-1} R_\alpha^{-1}, \quad (395)$$

and so evidently

$$R_\alpha R_C = R_C R_\alpha^{-1}, \quad (396)$$

and so the symmetry group we expect to get is $U(1) \rtimes \mathbb{Z}_2^C = O(2)$.

Is this symmetry realized linearly on the Hilbert space? The Hilbert space is $\mathcal{H} = \{|0\rangle, |1\rangle = \psi^\dagger|0\rangle\}$. Then

$$R_C|1\rangle = R_C\psi^\dagger R_C^{-1} R_C|0\rangle = \psi R_C|0\rangle. \quad (397)$$

If $R_C|0\rangle \propto |0\rangle$ then $R_C|1\rangle = 0$, and so R_C is not a unitary representation. Thus we can take $R_C|1\rangle = |0\rangle$ (inserting a phase factor in this definition will not change the cohomological classification of the anomaly discussed below), and consequently $R_C|0\rangle = |1\rangle$.

Suppose $|0\rangle$ has charge q_0 under $U(1)$. Then

$$R_\alpha|1\rangle = R_\alpha\psi^\dagger R_\alpha^{-1} R_\alpha|0\rangle = e^{-i\alpha} e^{iq_0\alpha}|1\rangle. \quad (398)$$

Thus, if q_1 is the charge of $|1\rangle$, we have

$$q_1 = q_0 - 1. \quad (399)$$

Now consider evaluating this using C :

$$R_\alpha|1\rangle = R_\alpha R_C|0\rangle = R_\alpha R_C R_\alpha^{-1} R_\alpha|0\rangle = R_C R_\alpha^{-1}|0\rangle = e^{-iq_0\alpha}|1\rangle, \quad (400)$$

and hence we also require

$$q_1 = -q_0. \quad (401)$$

Evidently we must have $q_0 = 1/2, q_1 = -1/2$. But this is a contradiction to our assumption about how the symmetry is realized on the Hilbert space, since now

$$R_{\alpha=2\pi}|0\rangle = -|0\rangle, \quad (402)$$

and so $R_{2\pi}$ is not represented trivially on the Hilbert space! In fact, it is represented as $R_{2\pi} = -1$. Thus the symmetry group actually acts projectively, with relations between representations of generators holding only modulo elements of \mathbb{Z}_2 ¹¹.

¹¹Re-defining $C|0\rangle = e^{i\beta}|1\rangle$ for some phase $e^{i\beta}$ would not have changed the fact that $R_{2\pi} = -1$.

Group extensions:

Projective representations are classified by central extensions of the symmetry group. In quantum mechanics, we are allowed to extend the symmetry group by $U(1)$ — that is, we allow the representation of the symmetry to not be linear, as long as the non-linearity manifests itself purely as $U(1)$ phases. Thus in our case the full quantum symmetry group should fit into the exact sequence

$$1 \rightarrow U(1) \rightarrow G \rightarrow O(2) \rightarrow 1. \quad (403)$$

Central extensions of the above form are classified by the cohomology group

$$H^2(O(2); U(1)) = \mathbb{Z}_2, \quad (404)$$

which can be calculated with spectral sequences starting from the exact sequence $1 \rightarrow U(1) \rightarrow O(2) \rightarrow \mathbb{Z}_2 \rightarrow 1$. The nontrivial element of $H^2(O(2); U(1))$ is the central extension realized by our fermions, where rotations by 2π act as -1 . We can write the extension realized by the fermions as

$$1 \rightarrow U(1) \rightarrow \text{pin}_+(2) \rightarrow O(2) \rightarrow 1, \quad (405)$$

where $\text{pin}_+(2)$ is the double-cover of $O(2)$ for which $C^2 = 1$. The extension is central since the image of \mathbb{Z}_2 in $\text{pin}_+(2)$, namely the identity and the rotation by 2π , is central: $R_C R_{2\pi} = R_{-2\pi} R_C = R_{2\pi} R_C$. Note that since

$$H^2(U(1); U(1)) = H^2(\mathbb{Z}_2^C; U(1)) = 0, \quad (406)$$

we need both the $U(1)$ and the \mathbb{Z}_2^C symmetries to get the anomaly: this is why it is referred to as a mixed t Hooft anomaly.

Let us briefly digress by talking about the difference between $U(1)$ coefficients and \mathbb{Z}_2 coefficients. If we instead performed an extension by \mathbb{Z}_2 , then our allowed quantum symmetry groups would be given by $H^2(O(2); \mathbb{Z}_2)$. We can calculate this given the facts that

$$H^2(O(2); U(1)) \cong \mathbb{Z}_2, \quad H^3(O(2); U(1)) \cong \mathbb{Z} \times \mathbb{Z}_2. \quad (407)$$

We do this by applying the Künneth formula for group cohomology to the cohomology $H^2(1 \times O(2); \mathbb{Z}_2 \otimes_{\mathbb{Z}} U(1))$. Since $\mathbb{Z}_2 \otimes_{\mathbb{Z}} U(1) = \mathbb{Z}_2$ as $0 = (1+1) \otimes \alpha = 1 \otimes (2\alpha)$ implies that taking the \otimes with \mathbb{Z}_2 kills all of the elements in $U(1)$, this cohomology group is actually $H^2(1 \times O(2); \mathbb{Z}_2 \otimes_{\mathbb{Z}} U(1))$. Since $H^k(1; \mathbb{Z}_2)$ is \mathbb{Z}_2 if $k=0$ and is 0 else, we get the following exact sequence:

$$1 \rightarrow \mathbb{Z}_2 \otimes H^2(O(2); U(1)) \rightarrow H^2(O(2); \mathbb{Z}_2) \rightarrow \text{Tor}[\mathbb{Z}_2, H^3(O(2); U(1))] \rightarrow 1. \quad (408)$$

Plugging in the cohomology groups and using $\text{Tor}[\mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}_2] = 0 \oplus \mathbb{Z}_2$, we see that

$$H^2(O(2); \mathbb{Z}_2) \cong \mathbb{Z}_2^2. \quad (409)$$

Thus, restricting the coefficients to \mathbb{Z}_2 leads to a doubling of the number of extensions. This is because with \mathbb{Z}_2 coefficients we can fractionalize the relation $C^2 = 1$ to $C^2 = -1$, as well as fractionalizing the 2π rotation. Thus we get $\text{pin}_{\pm}(2)$, $O(2) \times \mathbb{Z}_2$, and an extension where a 2π acts as the identity but $C^2 = -1$. When we use $U(1)$ coefficients instead this classification

collapses to just two extensions, since the relation $C^2 = -1$ can be turned into $C^2 = +1$ by redefining $C \mapsto iC$. Thus since in quantum mechanics we need to choose $U(1)$ for the coefficient group, we can take $C^2 = +1$ and set the symmetry group realized by our fermions to be $\text{pin}_+(2)$ without loss of generality.

Diagnosing the anomaly by gauging: Now we will diagnose the anomaly by attempting to gauge the $U(1)$ symmetry which sends ψ to $e^{i\alpha}\psi$. Adding a “background field” A (think of it as a chemical potential if you don’t like having gauge fields in one dimension), the new Lagrangian is

$$\mathcal{L} = i\psi^\dagger(\partial_t - iqA)\psi. \quad (410)$$

Charge conjugation sends $\psi \mapsto \psi^\dagger$, and $q \mapsto -q$. The Lagrangian is invariant under charge conjugation because the integration by parts and fermion anticommutation give cancelling minus signs. It is also invariant under local $U(1)$ transformations by construction.

The Lagrangian is gauge invariant, but is the theory invariant under the full $O(2) = U(1) \rtimes \mathbb{Z}_2^C$ symmetry when quantized? To answer this, we look at the partition function. The Hamiltonian with $A = 0$ is zero, while the A term adds a chemical potential to the Hamiltonian so that $H = -q\psi^\dagger A_0 \psi$. The partition function is thus

$$\begin{aligned} Z[A] &= \text{Tr}_{\mathcal{H}} e^{-iH} = \langle 0 | e^{iq \int \psi^\dagger A_0 \psi} | 0 \rangle + \langle 1 | e^{iq \int \psi^\dagger A_0 \psi} | 1 \rangle \\ &= 1 + e^{iq \int A_0}. \end{aligned} \quad (411)$$

Under charge conjugation, this maps to

$$C : Z[A] \mapsto 1 + e^{-iq \int A_0} = e^{-iq \int A_0} Z[A], \quad (412)$$

which is not equal to $Z[A]$ since in general the holonomy of A will not be in $(2\pi/q)\mathbb{Z}$. Thus, we conclude that when we try to couple the theory to a background $U(1)$ field, \mathbb{Z}_2^C is broken—this indicates the presence of a t Hooft anomaly.

How might we cancel this anomaly? Observe that if we added the term $-\frac{q}{2} \int A$ to the Lagrangian, then the partition function would be equal to

$$\begin{aligned} \tilde{Z}[A] &= \text{Tr}_{\mathcal{H}} \exp \left(iq \int \psi^\dagger A \psi - \frac{iq}{2} \int A \right) \\ &= (1 + e^{iq \int A_0}) e^{-iq/2 \int A_0}. \end{aligned} \quad (413)$$

Under charge conjugation,

$$C : \tilde{Z}[A] = 2 \cos \left(\frac{q}{2} \int A_0 \right) \mapsto 2 \cos \left(-\frac{q}{2} \int A_0 \right) = \tilde{Z}[A], \quad (414)$$

and so the partition function is invariant under \mathbb{Z}_2^C . The problem of course is that we have added an incorrectly quantized Chern-Simons term, which breaks the invariance of the Lagrangian under large gauge transformations which change the holonomy of A by $(2\pi/q)\mathbb{Z}$ (and is also not well-defined because of Čech-y reasons; see a later diary entry on the CS term). So this solution doesn’t let us have a symmetric theory either, provided our background gauge field is a $U(1)$ gauge field and not an \mathbb{R} gauge field.

Anomaly inflow then happens by simply allowing the time circle to bound a disk. This allows us to write the $-\frac{q}{2} \int A$ term in a gauge-invariant way as $-\frac{q}{2} \int_{D^2} F$. Simply letting the

circle bound the disk dispenses with the large gauge transformation issue and lets the full theory, namely

$$S = \int_{\partial D^2} i\psi^\dagger(\partial_t - iqA)\psi - \frac{q}{2} \int_{D^2} F, \quad (415)$$

which is now invariant under the full $U(1) \rtimes \mathbb{Z}_2^C$ symmetry.

Bosonic version

cos potential and not lifted b/c of instanton cancelation and so on

31 May 14 — TIs and fermions in three dimensions

Today's problem is quick and easy. Explain topological insulators in terms of anomalies by doing the same thing that we did yesterday but in three dimensions instead of one, and using time-reversal instead of charge conjugation.

Solution:

We consider a massless two-component Dirac fermion, which possesses time reversal symmetry and a $U(1)$ symmetry. We first need to write down conventions for how it transforms under e.g. time reversal. We will use a mostly negative metric signature, with gamma matrices

$$\gamma^0 = X, \quad \gamma^1 = iY, \quad \gamma^2 = iZ. \quad (416)$$

We choose the action of time reversal to square to fermion parity and flip the spin, so we want to take the representation of time reversal to be given by

$$T : \psi(t, x, y) \mapsto K iY \psi(-t, x, y), \quad (417)$$

where K is complex conjugation. Thus we also have

$$T : \bar{\psi} \mapsto -\psi^\dagger iY X K = i\bar{\psi} Y K. \quad (418)$$

One can check that with these conventions, $\bar{\psi} i\cancel{D}_A \psi$ is invariant, as it should be. However, the mass term is not invariant:

$$T : m\bar{\psi}\psi \mapsto m i\bar{\psi} Y iY \psi = -m\bar{\psi}\psi. \quad (419)$$

So, massive Dirac fermions break time reversal (and similarly reflection).

Let us try to gauge the $U(1)$ symmetry of a massless Dirac fermion while preserving T , so that our Lagrangian is

$$\mathcal{L} = i\bar{\psi} \cancel{D}_A \psi. \quad (420)$$

This looks like it's invariant under time reversal, but let's take a closer look. The partition function is $\det i\cancel{D}_A$, which requires regularization. The simplest way to do the regularization

is by doing Pauli-Villars. One might naively think that this could be done without breaking T , since although the PV mass term breaks T , we are taking $M \rightarrow \infty$, and so we might expect the T symmetry to come back after we take the limit. However, we encountered a similar thing earlier in our study of the chiral anomaly: there we added a PV regulator with a mass that broke the chiral symmetry, but saw that even in the $M \rightarrow \infty$ limit, a chiral symmetry breaking effect remained. This will also happen in our time reversal example.

Going over to Euclidean signature and adding the PV field, the partition function is

$$Z_\psi[A] = \frac{\det(\not{D}_A)}{\det(\not{D}_A + M)}. \quad (421)$$

Let the eigenvalues of the Hermitian operator $i\not{D}_A$ be λ_k . Since both of the factors have the same number of eigenvalues, we write the partition function as

$$Z_\psi[A] = \prod_k \frac{i\lambda_k}{i\lambda_k + M}. \quad (422)$$

The absolute value of this part is complicated, but the phase part just comes from the $-iM$ part. We'll roughly get one factor of i for each positive eigenvalue and a factor of $-i$ for each negative eigenvalue, and so we can guess that

$$Z_\psi[A] = |Z_\psi[A]| \exp\left(i\frac{\pi}{2} \sum_k \text{sgn}(\lambda_k)\right) = |Z_\psi[A]| e^{i\pi\eta/2}, \quad (423)$$

where we've defined the η invariant in the last line¹² Since this only depends on $\text{sgn}(M)$, the effects of the regulator field are felt even when its mass is taken to infinity, and in particular, time reversal symmetry is broken (just like the PV field broke chiral symmetry in the ABJ anomaly example).

We can then use the APS index theorem (see e.g. [13]) to write (taking $M > 0$ for definiteness and not writing the gravitational term)

$$\frac{Z_\psi[A]}{T \cdot Z_\psi[A]} = e^{i\pi\eta} = \exp\left(-\frac{i}{4\pi} \int_M A \wedge dA\right). \quad (425)$$

Thus we could try to save time reversal symmetry by adding a CS counterterm for A :

$$\mathcal{L} \mapsto \mathcal{L} + \frac{1}{8\pi} A \wedge dA, \quad (426)$$

but sadly this is not well-defined in strictly three dimensions since the CS level is fractional, provided that the $U(1)$ symmetry is realized linearly and not projectively. This is exactly the same idea as in the previous problem, where our one-dimensional Chern-Simons term $\frac{1}{2} \int A$ was not well-defined.

¹²Technically this needs to be regularized, since as λ_k gets big (bigger than M) these contributions to the phase of $Z_\psi[A]$ vanish. A convenient regulator is a heat kernel type of regulator, where we define

$$\eta = \sum_k \text{sgn}(\lambda_k) e^{-0^+ \lambda_k^2}. \quad (424)$$

Figure 1: A 10 site chain for the localization problem.

Figure 2: A 60 site chain for the localization problem.

If we insist on keeping T symmetry, we can get the same effect as the fractional CS term by writing it as a bulk θ term at $\theta = \pi$ over some 4-manifold X with $\partial X = M$, so that the full action is (assuming for simplicity that A is well-defined throughout X)

$$S = \int_M i\bar{\psi} \not{D}_A \psi + \frac{1}{8\pi} \int_X F \wedge F. \quad (427)$$

This is time reversal symmetric on the boundary (since we've effectively added the fractional CS term) and is also time reversal symmetric in the bulk, since time reversal sends $\theta \mapsto -\theta$ and we've chosen $\theta = \pi$. This whole setup is exactly what a TI is.

32 May 15 — Simple example of Anderson localization

This was a problem in Levitov's spring 2018 Theory of Solids class. Consider a tight-binding model on a chain with N sites, with a random on-site potential drawn from a uniform distribution over the interval $[-W, W]$. Suppose a flux of ϕ threads the ring, so that the holonomy of the gauge field (the product of the phases on the hopping matrix elements) is $e^{i\phi}$. Plot the energies of the 10 levels closest to zero energy as a function of the flux for a few values of N at a fixed realization of the random potential, and see how their distribution changes as N changes.

Solution:

This problem is mostly just writing a little program to make the plots. The Hamiltonian is (sorry for the profusion of i 's)

$$H = - \sum_i \left(t_{i,i+1} e^{i \int_i^{i+1} A} + t_{i,i-1} e^{i \int_i^{i-1} A} \right) - \sum_i V_i, \quad (428)$$

where V is the random potential, t_{ij} is the amplitude to hop from i to j , and the Wilson loops on the hopping matrix elements keep track of the flux through the ring. If the total flux is ϕ , it is convenient to choose a gauge in which the holonomy is distributed uniformly over the chain:

$$H = - \sum_i \left(t_{i,i+1} e^{i\phi/N} + t_{i,i-1} e^{-i\phi/N} \right) - \sum_i V_i. \quad (429)$$

Now we just numerically diagonalize this guy for some fixed values of N . The results are plotted in the various figures. As we see, as we increase N , the levels get more “rigid” and change more slowly as the flux through the ring is increased to one flux quantum. This is because the eigenstates are localized: when N is small enough that the localization scale of the wavefunctions is comparable to the length of the chain, we get interference effects which

show up in the spectrum. But when the chain is much larger than the localization scale the effects of periodic boundary conditions become negligible, and we might as well be working on \mathbb{R} , where adding “flux” like this doesn’t do anything. Also note that the energy levels are symmetric in the flux with respect to $\phi \leftrightarrow -\phi \bmod 2\pi$.

33 May 16 — Information theory things and modular flow in mega-simple qubit systems

Today’s diary entry is a bit different, and is more of a rambling look at a few concepts in quantum information theory and algebraic qft applied to some very very simple examples. Basically, I was reading through Witten’s notes on information theory and wanted to work out a few of the details myself.

We’ll start by looking at the illustrative but very simple example of qubits. This begins with essentially the simplest possible example: a system of two qubits.

Two qubits: entanglement entropy

Consider a two-site system consisting of two qubits, with the Ising Hamiltonian

$$H = -Z_L Z_R, \quad (430)$$

where $Z_{L/R}$ are the third Pauli matrices acting on the left qubit and the right qubit, respectively. We have a global symmetry generated by $X_L X_R$, and so we expect that the ground state will be in the trivial representation of this \mathbb{Z}_2 symmetry group¹³. Thus

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \quad (431)$$

The density matrix ρ has four non-zero entries, with a $1/2$ in each corner. The reduced density matrices are $\rho_L = \rho_R = \mathbf{1}/2$, and so entanglement entropy of a single spin is easily checked to be $S_L = S_R = \ln 2$.

This is a little boring, since the Hamiltonian is classical and has no dynamics. To fix this we can add a momentum term, which takes the form of a transverse field. We write

$$H = -h(X_L + X_R) - J Z_L Z_R. \quad (432)$$

Since this is such a simple system we can get the ground state wavefunction exactly, which due to the presence of the momentum term is now unique:

$$|\psi\rangle \propto (1, \gamma, \gamma, 1)^T, \quad \gamma \equiv \frac{-J + \sqrt{4h^2 + J^2}}{2J}, \quad (433)$$

where I haven’t bothered to normalize it. Note that this is a linear combination of ZZ eigenstates and X_L, X_R eigenstates, and is symmetric under the global symmetry $X_L \otimes X_R$.

Figure 3: A hastily done plot showing the entanglement entropy S_L for the left spin as a function of the ratio of couplings h/J . As $h \rightarrow 0$ we get the superposition $(|00\rangle + |11\rangle)/\sqrt{2}$ which has (the maximal possible) entanglement of a Bell pair, namely $\ln 2$. As $h \rightarrow \infty$ we get the product state $|++\rangle$, and so $S_L \rightarrow 0$.

As $h \rightarrow \infty$ we get a product state $|++\rangle$, and so we expect $S_L = S_R = 0$ in this limit. This is corroborated by a numerical calculation, shown in the first figure.

Now for a few more general comments that will be useful later. First, the entanglement entropy S_A for $A \in \{L, R\}$ is invariant under unitary transformations on A . Indeed,

$$S_A = -\text{Tr}(\rho_A \ln \rho_A) \mapsto -\text{Tr}(U^\dagger \rho_A U \ln(U^\dagger \rho_A U)). \quad (434)$$

The matrix logarithm can be written as a series

$$\ln A = (A - \mathbf{1}) - \frac{1}{2}(A - \mathbf{1})^2 + \frac{1}{3}(A - \mathbf{1})^3 - \dots, \quad (435)$$

which converges provided that $\|A - \mathbf{1}\| < 1$. This holds for us since $\text{Tr} \rho_A \leq 1$ and since ρ_A is positive semidefinite (the case where $\|\rho_A - \mathbf{1}\| = 1$ happens when ρ_A is degenerate, but since $0 \cdot \ln 0 = 0$ this case does not cause problems). From this series, we see that

$$-\text{Tr}(U^\dagger \rho_A U \ln(U^\dagger \rho_A U)) = -\text{Tr}(U^\dagger \rho_A \ln(\rho_A) U) = S_A, \quad (436)$$

and so the entanglement entropy is invariant under a unitary transformation supported on the spin A . Symmetrically, the entanglement entropy is also invariant under a transformation which has support only on the other spin \bar{A} .

However, S_A is not invariant under a general unitary transformation which has support on both A and B . Indeed, for the present two-site example, consider acting with the unitary

$$U = \mathbf{1} \oplus X. \quad (437)$$

This factorizes the wavefunction, leaving the L spin in an X eigenstate and the R spin in a Z eigenstate:

$$U|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle. \quad (438)$$

Then one can check that

$$U^\dagger \rho U = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \rho_L \otimes \rho_R. \quad (439)$$

Thus after applying the unitary, ρ becomes a product state; conjugating by U disentangles the two spins. Thus after applying U , we have $S_L = S_R = 0$.

To summarize, the entanglement entropy is preserved by unitaries of the form $U = U_L \otimes U_R$, but if U does not admit such a \otimes decomposition, the entanglement entropy is not invariant.

¹³Really, we have four different degenerate states, namely $|00\rangle, |11\rangle$, and the two linear combinations. However, after turning on a small transverse field, the symmetric combination gets singled out.

Modular operators and algebraic properties

Now we'll look at how some algebraic QFT things are realized in this simple setting. First, a word on notation: we'll use \mathcal{A} to refer to an algebra of operators, with a subscript specifying additional information. Thus e.g. \mathcal{A}_A will refer to the algebra of operators supported within the region A . These won't be the only algebras of interest though, e.g. one may consider the algebra \mathcal{A}_{Z_i, Z_j} of operators generated by the Pauli matrices Z_i, Z_j at two different sites i, j .

Since we are working with finite-dimensional Hilbert spaces, we can match dimensionalities of different spaces. A precondition for the state $|\psi\rangle$ to be cyclic for the algebra $\mathcal{A}_A = \text{End } \mathcal{H}_A$ (here we assume that \mathcal{A}_A is a factor so that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, although this isn't really needed — in general, \mathcal{A} splits as a direct sum of $\text{End } \mathcal{H}_\alpha$'s; more on this later) is that $\dim \mathcal{A} = \dim \mathcal{H}$, that is we need

$$\dim(\text{End } \mathcal{H}_A) = \dim(\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}). \quad (440)$$

But since

$$\dim(\text{End } \mathcal{H}_A) = \dim(\mathcal{H}_A \otimes \mathcal{H}_A^*) = \dim^2 \mathcal{H}_A, \quad (441)$$

we conclude that $|\psi\rangle$ has a chance to be cyclic for \mathcal{A}_A only if

$$\dim \mathcal{H}_A = \dim \mathcal{H}_{\bar{A}}. \quad (442)$$

Thus in order for the a state to be cyclic for a given algebra \mathcal{A}_A , we must have $|A| = |\bar{A}|$, i.e. the subsystem A must split the global system into two equally-sized chunks. This is in stark contrast to formal continuum QFT where we could usually choose A to be arbitrarily small (and this makes the surprising-ness of the RL theorem in regular QFT seem kinda pathological, since it as soon as you regulate the QFT with a lattice, the algebra you need to generate \mathcal{H} changes from being the operators in an arbitrarily small region to the operators in an entire half of spacetime). Of course, $\dim \mathcal{H}_A = \dim \mathcal{H}_{\bar{A}}$ is just a necessary condition on the algebra: $\mathcal{A}_A|\psi\rangle$ might fail to generate \mathcal{H} if $|\psi\rangle$ is chosen to be some rather unentangled state.

Assuming $\dim \mathcal{H}_A = \dim \mathcal{H}_{\bar{A}}$ so that $|\psi\rangle$ has a chance to be cyclic for \mathcal{A}_A , we use Schmidt decomposition to write

$$|\psi\rangle = \sum_i \lambda_i |i\rangle \otimes |i'\rangle, \quad (443)$$

where the kets on the RHS are bases for \mathcal{H}_A and $\mathcal{H}_{\bar{A}}$. Now if $|\psi\rangle$ is cyclic for \mathcal{A}_A then it must be separating for $\mathcal{A}_{\bar{A}}$, and vice versa (since \mathcal{A}_A and $\mathcal{A}_{\bar{A}}$ commute by the assumption that \mathcal{A}_A is a factor). It will fail to be separating if the Schmidt decomposition above has a nontrivial kernel, that is if the reduced density matrix

$$\rho_A = \sum_i |\lambda_i|^2 |i\rangle \langle i| \quad (444)$$

is not invertible (and likewise for $\rho_{\bar{A}}$). Since ρ_A and $\rho_{\bar{A}}$ share the same eigenvalues (since they came from a pure parent state $\rho = |\psi\rangle \langle \psi|$), if ρ_A is invertible then so is $\rho_{\bar{A}}$, and if ρ_A is degenerate, so is $\rho_{\bar{A}}$.

We have showed that if ρ_A is not invertible then $|\psi\rangle$ is not separating for $\mathcal{A}_{\bar{A}}$, which means that it is not cyclic for \mathcal{A}_A (since being cyclic for \mathcal{A}_A implies being separating for $\mathcal{A}_{\bar{A}}$). Thus

if $|\psi\rangle$ is cyclic for \mathcal{A}_A , then $\det \rho_A \neq 0$. In fact, the converse is also true. This is essentially because if $\det \rho_A \neq 0$, $\det \rho_{\bar{A}} \neq 0$, then every substate in the \bar{A} subsystem is entangled with the A subsystem, and so by acting only on the A we can still create states with arbitrary behavior in \bar{A} . More precisely, suppose we want to create the state

$$|\phi\rangle = \sum_{lm} \gamma_{lm} |l\rangle \otimes |m\rangle. \quad (445)$$

We can assume wolog that the above decomposition is taken with respect to the basis in which $|\psi\rangle$ is Schmidt-decomposed. When we act on $|\psi\rangle$ with some operator $\mathcal{O} = \mathcal{O}_A \otimes \mathbf{1}_{\bar{A}}$, we get

$$\mathcal{O}|\psi\rangle = \sum_{ab} [\mathcal{O}_A]_{ab} \lambda_b |a\rangle \otimes |b\rangle. \quad (446)$$

Thus in order to reproduce the state $|\phi\rangle$, we need to choose

$$[\mathcal{O}_A]_{ab} = \gamma_{ab} / \lambda_b, \quad (447)$$

which we can do if ρ_A is invertible, since then all the λ_b are non-zero. Recapitulating, we have shown that $|\psi\rangle$ is cyclic for \mathcal{A}_A iff $\det \rho_A \neq 0$, $\det \rho_{\bar{A}} \neq 0$.

One may ask whether the ground states of the simple two-qubit Ising Hamiltonians are cyclic / separating with respect to one of the qubits. We expect “generic” states to be cyclic and separating for “generic” algebras, so we expect the answer to be yes. Since $|\psi\rangle$ is cyclic for the qubit L if and only if $\det \rho_L \neq 0$ (and hence $\det \rho_R \neq 0$), this guess is easy to test. Putting the 2-qubit TFIM into Mathematica shows that indeed, for all finite values of h , $\det \rho_L \neq 0$. When $h \rightarrow \infty$ we get a product state, and in that case of course $\det \rho_L = 0$. Essentially, $\det \rho_L$ decreases towards zero monotonically with h in same fashion that S_L does.

Now lets look at the modular operators. We will take $|\psi\rangle$ to be cyclic for \mathcal{A} . In QFT this places comparatively few restrictions on \mathcal{A} , but as stated earlier in the present context we need \mathcal{A} to have the same (finite) dimension as \mathcal{H} . For now, we will assume that \mathcal{A} is a factor, with $\mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. Note that the cyclicity requirement on $|\psi\rangle$ means that $\rho \neq \rho_A \otimes \rho_{\bar{A}}$.

We define the antilinear operator S_ψ (which we will call the “modular generator”) to be the operator which acts as \dagger for the algebra \mathcal{A} :

$$S_\psi \mathcal{O} |\psi\rangle = \mathcal{O}^\dagger |\psi\rangle, \quad (448)$$

for all \mathcal{O} which act as an element in $\mathcal{A} = L(\mathcal{H}_A) \otimes \mathbf{1}_{\bar{A}}$. Such an involution on \mathcal{A} exists because of the inner product on \mathcal{A} defined by using the trace and the density matrix $\rho = |\psi\rangle\langle\psi|$. Since $|\psi\rangle$ is assumed to be cyclic for \mathcal{A} , specifying the action of the modular generator on elements of \mathcal{A} as above is sufficient to determine its action on all of \mathcal{H} .

To figure out what S_ψ is, we just need to know its matrix elements with respect to a basis in the matrix algebra \mathcal{A} . The canonical basis for the matrix algebra is of course $a_{ij} = |i\rangle\langle j|$ (which is also a basis for $\mathcal{A} \cong \mathcal{H}_A \otimes \mathcal{H}_A^*$ since $|\psi\rangle$ is cyclic with respect to \mathcal{A}). When acting on $a_{ij}|\psi\rangle$, we get

$$S_\psi a_{ij} |\psi\rangle = a_{ij}^\dagger \sum_k \lambda_k |kk\rangle = \lambda_i |ji\rangle, \quad (449)$$

where we have concise-ified the notation by denoting $|k\rangle_A \otimes |k\rangle_{\bar{A}} = |kk\rangle$. But we also have

$$S_\psi a_{ij} |\psi\rangle = S_\psi \lambda_j |ij\rangle = \lambda_j^* S_\psi |ij\rangle, \quad (450)$$

and so comparing these two, the nonzero matrix elements of S_ψ are

$$\langle ji|S_\psi|ij\rangle = \frac{\lambda_i}{\lambda_j^*}. \quad (451)$$

Now for S_ψ^\dagger . Since S_ψ is anti-unitary by virtue of it implementing \dagger , we have

$$\langle \alpha|S_\psi^\dagger|\beta\rangle = \langle \beta|S_\psi|\alpha\rangle, \quad (452)$$

where S_ψ^\dagger is understood to act on the right. Note that this is different from the definition of an adjoint for a unitary operator by an application of complex conjugation, since we want antiunitary operators to still satisfy $U^\dagger = U^{-1}$. So then the non-zero matrix elements of S_ψ^\dagger are

$$\langle ji|S_\psi^\dagger|ij\rangle = \frac{\lambda_j}{\lambda_i^*}. \quad (453)$$

Now define the modular operator (related to the modular Hamiltonian) as the “Laplacian”

$$\Delta_\psi = S_\psi^\dagger S_\psi. \quad (454)$$

Its matrix elements are calculated by using

$$\Delta_\psi a_{ij}|\psi\rangle = S_\psi^\dagger \lambda_i |ji\rangle = \lambda_i^* S_\psi^\dagger |ji\rangle. \quad (455)$$

But on the other hand, we also have

$$\Delta_\psi a_{ij}|\psi\rangle = \lambda_j \Delta_\psi |ij\rangle. \quad (456)$$

Then since we know the non-zero matrix elements of S_ψ^\dagger , we see that Δ_ψ is diagonal, with matrix elements

$$\langle ij|\Delta_\psi|ij\rangle = \frac{|\lambda_i|^2}{|\lambda_j|^2}. \quad (457)$$

Or, taking advantage of the particularly simple form of $|\psi\rangle$'s density matrices, we may write

$$\Delta_\psi = \rho_A \otimes \rho_{\bar{A}}^{-1}. \quad (458)$$

Sometimes we may want to work in a basis other than the one in which $|\psi\rangle$ is Schmidt-decomposed. Writing $|\psi\rangle = \sum_{kl} c_{kl}|kl\rangle$, one checks that

$$\langle jk|\Delta_\psi|lm\rangle = \sum_{rs} [c^{-1}]_{mr} [c^{-1}]_{kr}^* c_{ls}^* c_{js}, \quad (459)$$

which also reads $\Delta_\psi = \rho_A \otimes \rho_{\bar{A}}^{-1}$. So, the form (458) is valid in any basis.

Modular flow is defined by evolving with the modular operator. For an operator \mathcal{O} , we write

$$\mathcal{O}_z = \Delta^{iz} \mathcal{O} \Delta^{-iz}. \quad (460)$$

Modular flow preserves expectation values of operators in \mathcal{A} and \mathcal{A}' : if $\mathcal{O} = \mathcal{O}_A \otimes \mathbf{1}_{\bar{A}}$, then

$$\langle \mathcal{O}_z \rangle = \text{Tr}(\rho(\rho_A^{iz} \mathcal{O}_A \rho_A^{-iz} \otimes \mathbf{1}_{\bar{A}})) = \text{Tr}_A(\rho_A^{iz+1} \mathcal{O}_A \rho_A^{-iz}) = \langle \mathcal{O}_0 \rangle, \quad (461)$$

and likewise for $\mathcal{O} \in \mathcal{A}'$. From the expression (458), we see that

$$\Delta^{iz}\mathcal{A}\Delta^{-iz} = \mathcal{A}, \quad \Delta^{iz}\mathcal{A}'\Delta^{-iz} = \mathcal{A}', \quad (462)$$

where as before $\mathcal{A} = L(\mathcal{H}_A) \otimes \mathbf{1}_{\bar{A}}$. For any two operators $\mathcal{O}, \mathcal{O}'$, an interesting function we will look at is a modular-flowed two-point function:

$$f_{\mathcal{O}\mathcal{O}'}(z) \equiv \text{Tr}_{\mathcal{H}} \left(\rho_{\mathcal{O}} \Delta_{\psi}^{iz} \mathcal{O}' \Delta_{\psi}^{-iz} \right). \quad (463)$$

Now let's return to the starting example of two qubits. Define the state

$$\phi_{\theta} = \cos \theta |00\rangle + \sin \theta |11\rangle. \quad (464)$$

The modular operator computed for the algebra \mathcal{A}_L of the left qubit is

$$\Delta_{\phi_{\theta}} = \begin{pmatrix} 1 & & \\ & \cot^2 \theta & \\ & \tan^2 \theta & 1 \end{pmatrix}. \quad (465)$$

Of course, we would get the same modular operator if we used the algebra \mathcal{A}_R . Let us now do some modular flows. For any two operators $\mathcal{O}, \mathcal{O}' \in \text{End}(\mathcal{H}_L)$, we write the function $f_{\mathcal{O}\mathcal{O}'}(z)$ as

$$f_{\mathcal{O}\mathcal{O}'}(z) = \text{Tr}_{\mathcal{H}} \left(\rho_{\phi_{\theta}} (\mathcal{O} \otimes \mathbf{1}_R) \Delta_{\phi_{\theta}}^{iz} (\mathcal{O}' \otimes \mathbf{1}_R) \Delta_{\phi_{\theta}}^{-iz} \right) = \langle \phi_{\theta} | (\mathcal{O} \otimes \mathbf{1}_R) \Delta_{\phi_{\theta}}^{iz} (\mathcal{O}' \otimes \mathbf{1}_R) | \phi_{\theta} \rangle, \quad (466)$$

where we have used

$$\Delta_{\phi_{\theta}}^{iz} |\phi_{\theta}\rangle = |\phi_{\theta}\rangle, \quad (467)$$

which one can check either by the definition of the modular generator S and the definition $\Delta = S^\dagger S$, or by using (458) and Schmidt-decomposing $|\phi_{\theta}\rangle$. The expression for f simplifies to

$$f_{\mathcal{O}\mathcal{O}'}(z) = \text{Tr}_{\mathcal{H}_L} \left(\rho_{\phi_{\theta},L} \mathcal{O} \rho_{\phi_{\theta},L}^{iz} \mathcal{O}' \rho_{\phi_{\theta},L}^{-iz} \right). \quad (468)$$

We see right away that

$$f_{\mathbf{1}\mathcal{O}}(z) = f_{\mathcal{O}\mathbf{1}}(z) = \langle \mathcal{O} \rangle. \quad (469)$$

We also compute

$$f_{ZZ}(z) = 1, \quad f_{XZ}(z) = f_{ZX}(z) = 0, \quad (470)$$

which are a consequence of Z being invariant under the modular flow:

$$\Delta_{\phi_{\theta}}^{iz} (Z_L \otimes \mathbf{1}_R) \Delta_{\phi_{\theta}}^{-iz} = Z_L \otimes \mathbf{1}_R. \quad (471)$$

Thus the only possible f that has a chance to actually evolve under the modular flow is $f_{XX}(z)$, which is

$$f_{XX}(z) = \cos^2 \theta (\tan^2 \theta)^{iz} + \sin^2 (\cot^2 \theta)^{iz}. \quad (472)$$

At the symmetric point $\theta = \pi/4$, $f_{XX}(z) = 1$. As we go further away from the symmetric point, $f_{XX}(z)$ for $z \in \mathbb{R}$ starts to spiral around the origin in the complex plane. The spirals become faster and faster, with a period that goes to zero as θ approaches π or 0 . The figures

Figure 4: Here we take $\theta = 0.3$.

Figure 5: Here we take $\theta = \pi/4 - 0.3$.

Figure 6: $f_{XX}(z)$ as a function of complex z , with $\theta = \pi/8$. Note the singularities at $\text{Im}(z) = -1/2$.

show the trajectory for $z \in \mathbb{R}$ for two different θ values. We can also examine the analytic structure of $f_{XX}(z)$ when $z \in \mathbb{C}$, an example of this is shown in the next figure.

Suppose we instead used the Bell state $\varphi_\theta = \cos \theta |01\rangle + \sin \theta |10\rangle$. Working in the $0, 1$ basis (n.b. not Schmidt-decomposing φ_θ), we get

$$\Delta_{\varphi_\theta} = \begin{pmatrix} \cot^2 \theta & & \\ & 1 & \\ & & 1 \\ & & \tan^2 \theta \end{pmatrix}. \quad (473)$$

The reduced density matrices are

$$\rho_L = \begin{pmatrix} \cos^2 \theta & \\ & \sin^2 \theta \end{pmatrix}, \quad \rho_R = X \rho_L X. \quad (474)$$

Thus if $\mathcal{O}, \mathcal{O}' \in \mathcal{A}_L$, the modular flow function $f_{\mathcal{O}\mathcal{O}'}(z)$ is the same as for the case with the $\cos \theta |00\rangle + \sin \theta |11\rangle$ state. If $\mathcal{O}, \mathcal{O}' \in \mathcal{A}_R$ then we take $\theta \mapsto \theta + \pi/2$, but the form of $f_{\mathcal{O}\mathcal{O}'}(z)$ is the same.

We can also look at relative modular operators. Let ψ be a cyclic vector for the algebra \mathcal{A}_L , and let ϕ be arbitrary. Let ρ and σ be the associated density matrices for these two states. For any $\mathcal{O} \in \mathcal{A}_L$, we define the relative modular generator as

$$S_{\psi|\phi} \mathcal{O} |\psi\rangle = \mathcal{O}^\dagger |\phi\rangle. \quad (475)$$

The relative modular operator is then defined in the expected way as

$$\Delta_{\psi|\phi} = S_{\psi|\phi}^\dagger S_{\psi|\phi}. \quad (476)$$

The matrix elements of these operators can be computed by Schmidt-decomposing both ψ and ϕ in their respective Schmidt bases, which in general are not orthogonal (despite this, it is better to do this than to Schmidt-decompose ψ and non-diagonally decompose ϕ in the same basis). Following the same procedure that we did in the non-relative case, one gets

$$\Delta_{\psi|\phi} = \sigma_L \otimes \rho_R^{-1}. \quad (477)$$

Note that we don't require the reduced density matrices of ϕ to be invertible, only those of ρ (that is, we only require ψ to be cyclic, not ϕ).

Algebraic entropies

One advantage to formulating things algebraically is that we get a natural notion of entropy for algebras, rather than necessarily for spatial regions. For example, we can compute

Figure 7: Various entanglement-related quantities for a two-site TFIM. The entanglement entropy for a single site S_{P_i} and the algebraic entropy for the $X_L X_R$ -algebra are the same, as are the $Z_L Z_R$ and $Y_L Y_R$ two-point functions. The algebraic entropy of the $Z_L Z_R$ algebra and the $Y_L Y_R$ algebra are also identical. By symmetry, $\langle Z_i \rangle = \langle Y_i \rangle = 0$.

the algebra of the X -spin operators, or the algebra of Z -spin operators, or the tensor product of a pair of Pauli algebras located at a pair of non-adjacent sites (thus computing an ‘‘entanglement correlation function’’), and so on.

Given an algebra \mathcal{A} , we choose to work in a basis where the operators $\mathcal{O} \in \mathcal{A}$ are orthogonal with respect to the Hilbert-Schmidt inner product

$$\langle \mathcal{O}_a | \mathcal{O}_b \rangle = \frac{1}{\text{Tr} \mathbf{1}_{\mathcal{A}}} \text{Tr}(\mathcal{O}_a^\dagger \mathcal{O}_b) = \delta_{ab}. \quad (478)$$

We will usually just drop the \dagger since observables will be Hermitian. Now in general we write the density matrix of the algebra as

$$\rho_{\mathcal{A}} = \frac{1}{\text{Tr} \mathbf{1}_{\mathcal{A}}} \sum_{\mathcal{O} \in \mathcal{A}} c_{\mathcal{O}} \mathcal{O}. \quad (479)$$

In the basis which is orthonormal under the above inner product, the coefficients are just the expectation values, and so

$$\rho_{\mathcal{A}} = \frac{1}{\text{Tr} \mathbf{1}_{\mathcal{A}}} \sum_{\mathcal{O} \in \mathcal{A}} \langle \mathcal{O} \rangle \mathcal{O}. \quad (480)$$

Tensor products of Pauli algebras are especially convenient algebras to work with since this property is already built-in, so no changing of bases is required.

Note the decomposition of the density matrix in this way is actually kind of cool, since it tells us that in order to figure out the reduced density matrix of e.g. a given spatial region A , we just need to know the correlation functions of operators entirely contained within A . So, to figure out how entangled A is with the rest of the system, we actually don’t need to know anything about the fields in the other part of the system at all!¹⁴ We only need to know about local stuff going on in A . This is essentially because the entanglement present in generic ground states allows us to build up information in \bar{A} just from information in A .

In previous sections, we made use of the decomposition $\mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. While if we have in mind some collection of qubits then such a decomposition is always possible if A is associated to a spatial region, for algebras not tied to a particular region, this is not a convenient decomposition. In general, if $Z(\mathcal{A}) = 0$ then \mathcal{A} induces a decomposition $\mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ (A is not necessarily associated to a spatial region), on which \mathcal{A} acts as

$$L(\mathcal{H}_A) \otimes \mathbf{1}_{\bar{A}}. \quad (481)$$

Thus if $Z(\mathcal{A}) = 0$ we can write $|\psi\rangle = \sum_{ij} c_{ij} |ij\rangle$, and use the strategy developed previously. If $Z(\mathcal{A}) \neq 0$ however, we get the weaker

$$\mathcal{H} \cong \bigoplus_{\alpha} \mathcal{H}_{\alpha} \otimes \mathcal{H}_{\bar{\alpha}}, \quad (482)$$

¹⁴Apart from the fact that we knew it was a pure state, which makes the result less surprising.

Figure 8: Just for fun, the chaos unleashed when we do a repetition of the previous plot but with a longitudinal field of strength J added (so the title of the plot is wrong, and a $-J \sum_i Z_i$ should be added).

where the sum runs over a complete set of orthogonal projectors $\Pi_\alpha \in Z(\mathcal{A})$, and on which \mathcal{A} acts as

$$\bigoplus_{\alpha} L(\mathcal{H}_\alpha) \otimes \mathbf{1}_{\bar{\alpha}}. \quad (483)$$

This just comes from simultaneously diagonalizing the elements of $Z(\mathcal{A})$ and finding the minimal projectors in the center.

As an example, consider the algebra generated by $X_L = X \otimes \mathbf{1}$ and $X_R = \mathbf{1} \otimes X$. In this case $Z(\mathcal{A}) = \mathcal{A}$, and so since $\mathcal{A} \cong \mathbb{Z}_2^2$ has four irreps we get four projectors

$$\Pi_{\pm_L \pm_R} = \mathbf{1} \pm_L X_L \pm_R X_R + (\pm_L \cdot \pm_R) X_L X_R, \quad (484)$$

where $\pm_L, \pm_R \in \{1, -1\}$. Choosing the $+$ ($-$) sign for L/R projects onto the eigenspace of $X_{L/R}$ with positive (negative) eigenvalue.

Continuing with this example, we can write a generic element in \mathcal{H} as

$$|\psi\rangle = a|++\rangle + b|+-\rangle + c|-+\rangle + d|--\rangle. \quad (485)$$

Suppose $|\psi\rangle$ is an eigenstate of one of the operators in \mathcal{A} . Then by acting on $|\psi\rangle$ with this operator, we see that two of the coefficients in the above decomposition must vanish. But as soon as at least one of the coefficients in the expansion vanishes, the associated density matrix $|\psi\rangle\langle\psi|$ will not be invertible since it annihilates the ket whose coefficient in the decomposition in $|\psi\rangle$ is zero. Thus, more generally we see that if $|\psi\rangle$ is an eigenstate of any operator (other than $\mathbf{1}$) in \mathcal{A} then ψ is not cyclic for \mathcal{A} . Equivalently, we can say that ψ will fail to be cyclic for \mathcal{A} if $\langle \mathcal{O} \rangle = \pm 1$ for some $\mathcal{O} \in \mathcal{A}$ (assuming we are still in the basis in which $\text{Tr}(\mathcal{O}_a \mathcal{O}_b) \propto \delta_{ab}$).

Suppose we fix our state ψ to be the ground state of H . For simplicity, let's again work with the 2-qubit example. If the ground state is a tensor product $\psi = \psi_L \otimes \psi_R$, then it is not cyclic for \mathcal{A}_L . This is because the projector onto $|\psi_L\rangle^\perp$ is in \mathcal{A}_L , and so \mathcal{A}_L is not separating for ψ , meaning that $\mathcal{A}_R = \mathcal{A}'_L$ is not cyclic. But then \mathcal{A}_L is not cyclic either.

What about the algebras $\mathcal{A}_X \ni \mathbf{1}, X_L, X_R, X_L X_R$ and $\mathcal{A}_Z \ni \mathbf{1}, Z_L, Z_R, Z_L Z_R$? Let us work in a basis where all the elements of the algebra are diagonal. First, write the associated density matrix for the algebra as $\rho \propto \sum_{\mathcal{O} \in \mathcal{A}} \langle \mathcal{O} \rangle \mathcal{O}$. As discussed earlier, $\mathcal{A}|\psi\rangle$ will be dense in \mathcal{H} iff $\rho_{\mathcal{A}}$ is invertible. As we just saw, this fails if ψ is an eigenstate of any of the $\mathcal{A} \in \mathcal{A}$ (other than $\mathbf{1}$). This is in keeping with the general symmetry \iff degeneracy idea. Another way to see this result is to note that if ψ is an eigenstate of some $\mathcal{O} \in \mathcal{A}$, then \mathcal{O} acts as the identity on ψ , and thus doesn't contribute to the part of the Hilbert space generated by $\mathcal{A}|\psi\rangle$. Since there are only $\dim \mathcal{H}$ operators in \mathcal{A} , this means that $\mathcal{A}|\psi\rangle$ cannot generate all of \mathcal{H} , and so ψ is not cyclic for \mathcal{A} .

For example, for the TFIM model for 2 sites, the algebra \mathcal{A}_X is never cyclic for the ground state, since the symmetry generated by $X_L X_R$ implies $|\psi\rangle$ is a $X_L X_R$ eigenstate. At $h = 0$ we have the ground state $(|00\rangle + |11\rangle)/\sqrt{2}$, although if we instead had $|00\rangle$ then the ground state would be cyclic for \mathcal{A}_X , since $\langle \mathcal{O} \rangle_{|00\rangle} = 0 \forall \mathcal{O} \in \mathcal{A}_X$ implies $\rho_{\mathcal{A}_X} \propto \mathbf{1}$ is invertible. By contrast, ψ is always cyclic for the \mathcal{A}_Z algebra, unless $h = 0$ in which case $|\psi\rangle \propto |00\rangle + |11\rangle$

Figure 9: A 12-site TFIM chain. Note the evidence of the phase transition at $h/J = 1$.

is a $Z_L Z_R$ eigenstate. When $h \rightarrow \infty$ then $|\psi\rangle = |++\rangle$ is not cyclic for \mathcal{A}_X since all the operators in \mathcal{A}_X have expectation value 1, but it *is* cyclic for \mathcal{A}_Z , for the same reason that \mathcal{A}_X is cyclic for $|00\rangle$.

The other natural algebras to look at are those of the form \mathcal{A}_i , the algebra of Pauli operators at a given site (or collection of sites) i . ψ will fail to be cyclic for these algebras precisely when $\psi = \psi_i \otimes \psi_{\bar{i}}$, since then ρ_i annihilates all vectors orthogonal to ψ_i in \mathcal{H}_i . Recapitulating, we see that if the wavefunction is a product state over different lattice sites this precludes it from being cyclic for algebras built from tensor products of Pauli algebras at different sites, it can still be cyclic for different types of algebras like \mathcal{A}_X and \mathcal{A}_Z .

Finally, we should admit that a 2-qubit system is rather simple. Marginally less trivial is an N -qubit chain (with periodic boundary conditions). My laptop can handle up to $N \sim 12$ exactly. Figure 9 shows a length-12 chain. We won't go into any more detail about the $N > 2$ case for the sake of expediency.

34 May 17 — Energies of dislocations and disclinations in two dimensional solids **unfinished**

This is a problem in Chaikin and Lubensky. The Hamiltonian for dislocations and disclinations for a solid in two dimension is

$$H = \frac{Y_2}{2} \int_{x,x'} U(x, x')(s(x) - \epsilon^{ij} \nabla_j b_i(x))(s(x') - \epsilon^{kl} \nabla_l b_k) + \sum_{\alpha} (E_c b_{\alpha}^2 + E_s s_{\alpha}^2), \quad (486)$$

where the interaction potential is

$$U(x, x') = \int_q \frac{1}{q^4} e^{iq \cdot (x-x')}, \quad (487)$$

$s(x)$ are the disclination densities, and $b_i(x)$ are the dislocation densities (delta functions at the locations of the dislocations equal to the Burgers vectors of the dislocations). The terms with E_c and E_s are self-energies of dislocations and disclinations, respectively. Do several things:

- Find the energy of a single disclination in the middle of a disk of radius R
- Find the Hamiltonian between a collection of dislocations, which should look similar to the Hamiltonian for vortices in the XY model but with a term depending on the relative alignments of the dislocations
- Now consider being above the melting temperature. Find the equilibrium dislocation density in a non-zero disclination field. Use this to show that the effective interaction between disclination gets screened down to a $\ln(R)$ interaction.

Solution:

For a single disclination, we set $b_i = 0$ and $s(x) = \delta(x)$. Thus the energy is

$$E = \frac{Y_2}{2} \int \frac{d^2 q}{4\pi^2} \frac{1}{q^4}, \quad (488)$$

and so imposing a low-momentum cutoff at $q_{\min} \sim 1/R$ where R is the radius of the disk, we see that $E \sim R^2$. Single disclinations are very costly!

Now set $s(x) = 0$ to consider a “gas” of dislocations. The interaction energy is

$$\begin{aligned} E &= \frac{Y_2}{2} \int_{x,x',q} \frac{1}{q^4} e^{iq \cdot (x-x')} \epsilon^{ij} \epsilon^{kl} \partial_j b_i(x) \partial_l b_k(x') \\ &= \frac{Y_2}{2} \int_q \frac{1}{q^4} e^{iq \cdot (x-x')} \epsilon^{ij} \epsilon^{kl} q_j q_l b_i b_k \\ &= \frac{Y_2}{2} \int_q \frac{q^2 \delta_{ik} - q_i q_k}{q^4} b_i(x) b_k(x') e^{iq \cdot (x-x')} \end{aligned} \quad (489)$$

Now we use

$$\int_q \frac{1}{q^2} e^{iqr} = -\ln(r/a) + C, \quad \int_q \frac{q_i q_k}{q^4} e^{iqr} = -\frac{1}{8\pi} \left(\delta_{ik} (2 \ln(r/a) - 1) + 2 \frac{r_i r_k}{|r|^2} + C' \right), \quad (490)$$

and drop the (system-size dependent!) constants C, C' . Here a is the radius of the “vortex cores”. Plugging these in gives

$$E = \sum_{r,r'} \frac{Y_2}{8\pi} b_i(r) b_j(r') \left(-\ln(|r-r'|/a) \delta_{ij} + \frac{(r-r')_i (r-r')_j}{|r-r'|^2} \right) + \sum_r E_c |b_r|^2, \quad (491)$$

where r, r' are locations of the dislocations. The logarithmic part is the same as the interactions between vortices in the XY model, while the other interaction term is special to dislocations and depends on the relative orientation of the dislocations. The physical interpretation of the additional term relates to the presence of disclinations. The disclination density is essentially

$$s(x) \approx \sum_{x'} \frac{b_i(x)(x-x')_i}{|x-x'|^2}, \quad (492)$$

which I think comes from the fact that disclinations are bound states of dislocations. This means that the extra term is something like

$$\sum_{r,r'} s(r) s(r') V(|r-r'|), \quad V(|r-r'|) = |r-r'|^2. \quad (493)$$

This shows that disclinations are super strongly confined in this phase, with a string tension that grows quadratically with their separation.

Now we will do the screening calculation.

35 May 18 — Cohomology of finite groups with $U(1)$ and \mathbb{Z} coefficients

Today is a quick recollection of some algebraic topology basics. Remind yourself why for finite groups G , we have the isomorphism in group cohomology

$$H^k(G; \mathbb{R}/2\pi\mathbb{Z}) \cong H^{k+1}(G; \mathbb{Z}) \quad (494)$$

for $k > 0$.

Solution:

This is basically just a covariant analogue of the contravariant statement about exact sequences of chain complexes inducing long exact sequences in cohomology. First, we have the exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 1. \quad (495)$$

Let C be a free chain complex. $\text{Hom}(C, \cdot)$ is a *covariant* functor (as opposed to $\text{Hom}(\cdot, M)$ which is contravariant), and so the above short exact sequence induces the short exact sequence

$$1 \rightarrow \text{Hom}(C, \mathbb{Z}) \rightarrow \text{Hom}(C, \mathbb{R}) \rightarrow \text{Hom}(C, U(1)) \rightarrow 1. \quad (496)$$

Thus by writing down the three chain complexes side-by-side with the rows of the resulting diagram being exact, we get the associated long exact sequence in cohomology ($H^*(G; \cdot)$ is also a covariant functor)

$$\dots \rightarrow H^k(G; \mathbb{Z}) \rightarrow H^k(G; \mathbb{R}) \rightarrow H^k(G; U(1)) \rightarrow H^{k+1}(G; \mathbb{Z}) \rightarrow \dots \quad (497)$$

The maps from guys with coefficients in \mathbb{Z} to those with coefficients in \mathbb{R} and the subsequent maps to guys with coefficients in $U(1)$ are the maps induced by the maps in $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1)$. The maps between degrees are induced by the coboundary operator,

$$\delta : H^k(G; U(1)) \rightarrow H^{k+1}(G; \mathbb{Z}). \quad (498)$$

Hitting an element in $H^k(G; U(1))$ with δ produces something nontrivial here since the coefficient group is changing: if $\delta\omega^k = 0$ for some $U(1)$ -valued cochain ω^k , then we really mean that $\delta\omega^k = 2\pi\omega^{k+1}$, where ω^{k+1} is \mathbb{Z} -valued.

Now we use the fact that for finite groups G , the group cohomology with coefficients in \mathbb{R} vanishes

$$H^k(G; \mathbb{R}) = 0 \quad (499)$$

for $k > 0$. This can be seen in several ways; one is by writing the UCT for group cohomology for $k > 0$ as

$$H^k(G; M) \cong \text{Hom}(H_k(G; \mathbb{Z}), M) \oplus \text{Ext}(H_{k-1}(G; \mathbb{Z}), M). \quad (500)$$

When G is a finite group the homology $H_k(G; \mathbb{Z})$ will be finite when $k \neq 0$, and so $\text{Hom}(H_k(G; \mathbb{Z}), \mathbb{R}) = 0$ when $k > 0$. If $k = 1$ then we have $H_0(G; \mathbb{Z}) = \mathbb{Z}$, but $\text{Ext}(\mathbb{Z}, \mathbb{R}) = 0$. When $k > 1$ $H_{k-1}(G; \mathbb{Z})$ will be finite, and then $\text{Ext}(H_{k-1}(G; \mathbb{Z}), \mathbb{R})$ will vanish (e.g. $\text{Ext}(\mathbb{Z}_n; M) =$

M/nM). One might also say that \mathbb{R} is an injective $\mathbb{Z}G$ module, so that group cohomology with \mathbb{R} coefficients vanishes (see e.g. Brown's book on group cohomology). Anyway, since the terms with \mathbb{R} coefficients in the long exact sequence die, we get

$$H^k(G; \mathbb{R}/2\pi\mathbb{Z}) \cong H^{k+1}(G; \mathbb{Z}). \quad (501)$$

This is only true for group cohomology with finite groups! However, we can also apply it in the case of Čech cohomology where the coefficients are valued in some sheaf, usually that of the smooth functions on a manifold X valued in a group G (G needn't be Abelian, although if it isn't then the corresponding Čech cohomologies are just sets, and not groups). In this case an n -cochain is a smooth function from intersections $U_1 \cap \dots \cap U_{n+1}$ into G . From the same SES, we get

$$\dots \rightarrow H^k(X; \mathcal{C}^\infty(\mathbb{Z})) \rightarrow H^k(X; \mathcal{C}^\infty(\mathbb{R})) \rightarrow H^k(X; \mathcal{C}^\infty(\mathbb{C}^\times)) \rightarrow H^{k+1}(X; \mathcal{C}^\infty(\mathbb{Z})) \rightarrow \dots \quad (502)$$

Now of course when the group is \mathbb{Z} , we can just replace the Čech cohomology above with regular cohomology valued in \mathbb{Z} . The Čech cohomology groups that involve $\mathcal{C}^\infty(\mathbb{R})$ are all zero (in degree greater than zero)¹⁵. We can prove this as follows. Consider the case $k = 1$. Let u_α be a partition of unity, where each u_α has compact support in U_α and where $\sum_\alpha u_\alpha = \mathbf{1}$. Given a 1-cochain $\lambda_{\alpha\beta}$, consider the 0-cochain

$$\omega_\alpha = \sum_\beta \lambda_{\alpha\beta} u_\beta. \quad (503)$$

Then

$$(\delta\omega)_{\alpha\beta} = \sum_\gamma u_\gamma (\lambda_{\alpha\gamma} - \lambda_{\beta\gamma}) = \sum_\gamma u_\gamma \lambda_{\alpha\beta} = \lambda_{\alpha\beta}, \quad (504)$$

where we used the cocycle condition for λ and the fact that $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$. Therefore $\lambda = \delta\omega$ is exact! A similar argument goes through for all higher cohomology groups. This argument fails in the case of e.g. $\mathcal{C}^\infty(U(1))$ coefficients though, since then we cannot conclude that $\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha} = 0$, only that it is equal to $2\pi n_{\alpha\beta\gamma}$, where $n_{\alpha\beta\gamma} \in \mathbb{Z}$ (here we have lifted a $U(1)$ cochain to an \mathbb{R} cochain $\lambda_{\alpha\beta}$).

Returning to the SES and using the above result, we get the isomorphism

$$H^k(X; \mathcal{C}^\infty(U(1))) \cong H^{k+1}(X; \mathbb{Z}). \quad (505)$$

Note in particular the case when $k = 1$. The RHS gives the Chern class of a $U(1)$ bundle over X , and so this tells us that

$$\begin{aligned} H^1(X; \mathcal{C}^\infty(U(1))) &\cong \{\text{Isomorphism classes of principal } U(1) \text{ bundles over } X\} \\ &\cong H^2(X; \mathbb{Z}), \end{aligned} \quad (506)$$

where the first line comes from the fact that the cohomology group on the left classifies all possible transition functions for $U(1)$ bundles over X , up to isomorphism. This is kind of

¹⁵This is what allows one to prove the equivalence between Čech cohomology with coefficients in the constant sheaf \mathbb{R} and de Rham cohomology: construct the Čech de-Rham bicomplex and use the triviality of all the cohomology groups in the “interior” of the complex to snake from the first column of the complex to the last row.

cool since the bundles are all characterized by a second cohomology group, independent of $\dim X$. This is also essentially the same as saying that for $U(1)$ bundles, only the first Chern class is non-vanishing. The first statement about bundles is true more generally:

$$H^1(X; \mathcal{C}^\infty(G)) \cong \{\text{Isomorphism classes of principal } G \text{ bundles over } X\}. \quad (507)$$

Here we think of these bundles as having a distinguished “identity” element $X \times G \rightarrow X$ given by the trivial bundle. For example, taking G to be some k -dimensional subgroup of $GL(\dim X; \mathbb{R})$ gives us the set of all k -plane bundles over X . Note that if G is not Abelian, the LHS will not be a group in general; it will just be a set with a distinguished “identity” element. In the case where it is a group and G is abelian, like when $G = U(1)$, the group operation corresponds to taking \otimes s of bundles. The analogue of $H^2(X; \mathbb{Z})$ for G -bundles is $H^2(X; \mathcal{C}^\infty(\pi_1(G))) \cong H^2(X; \pi_1(G))$, with the isomorphism holding for compact Lie groups G , since then $\pi_1(G)$ is discrete and abelian. That $\pi_1(G)$ is relevant here comes from the SES $1 \rightarrow \pi_1(G) \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ for \tilde{G} the universal cover, and from then applying the same construction described above to this SES. In particular, taking $G = SO(n)$ with $n > 2$ gives

$$H^1(X; \mathcal{C}^\infty(SO(n))) \cong H^2(X; \mathbb{Z}_2), \quad (508)$$

since $\pi_1(SO(n)) = \mathbb{Z}_2$ for all $n > 2$. The classes on the RHS are the second Stiefel-Whitney classes.

36 May 19 — Functional RG equations

Consider a scalar field theory with arbitrary interactions given by $\mathcal{L}_I[\phi]$. Impose a soft UV cutoff by modifying the action as (Euclidean signature)

$$Z[J] = \int \mathcal{D}\phi e^{-S + \int J\phi}, \quad S = S_0 + S_I, \quad (509)$$

with

$$S_0 = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \phi_p (p^2 + m^2) e^{p^2/\Lambda^2} \phi_{-p}, \quad (510)$$

and

$$S_I = \int \frac{d^4 p}{(2\pi)^4} \mathcal{L}_I[\phi], \quad (511)$$

where \mathcal{L}_I contains all possible interactions for ϕ_p . Here, the current J only contains momentum modes up to some scale $\mu < \Lambda$ (the energy scale below which we are interested in calculating correlation functions), and the purpose of the e^{p^2/Λ^2} term is so that the propagator gets smoothly cutoff at very high momentum. The action S_I explicitly depends on Λ through the dimensionality of the coupling constants appearing in S_I .

By requiring that the change in the generating functional of correlation functions under a change in the cutoff is independent of the current (so that correlation functions at energy scales below Λ are insensitive to changes in Λ)

$$\frac{d}{d \ln \Lambda} Z[J] = \text{independent of } J, \quad (512)$$

derive an expression for $d_{\ln \Lambda} S_I[\phi]$. Note: this is in Schwartz, but I think there are some typos in the problem, so don't worry about trying to derive what he tells you to derive. Polchinski's original paper may be a better reference.

Solution:

Let's first see if we can guess the answer. When we change the cutoff, what needs to happen to $\mathcal{L}_I[\phi]$ so that the partition function is unchanged? Lowering the cutoff means that certain high energy modes get integrated out. In diagrams, this means that lowering the cutoff corresponds to "collapsing" certain high energy propagator lines. There are two types of lines we can collapse: one which is an open line connecting two different vertices, and one which joins back on itself, with both ends at the same vertex. We can select out lines of the first kind by computing $\delta_{\phi_p} \mathcal{L}_I \delta_{\phi_{-p}} \mathcal{L}_I$, and we can select out the latter types of lines by doing $\delta_{\phi_p} \delta_{\phi_{-p}} \mathcal{L}_I$. After we have selected out these lines, we need to integrate over the high energy modes, which we can do with the propagator. Thus we expect something like

$$\frac{d}{d \ln \Lambda} \mathcal{L}_I \sim \int_p \frac{e^{-p^2/\Lambda^2} p^2/\Lambda^2}{p^2 + m^2} \left(\frac{\delta \mathcal{L}_I}{\delta \phi_p} \frac{\delta \mathcal{L}_I}{\delta \phi_{-p}} + \frac{\delta^2 \mathcal{L}_I}{\delta \phi_p \delta \phi_{-p}} \right). \quad (513)$$

The $e^{-p^2/\Lambda^2} p^2/\Lambda^2$ factor ensures that \mathcal{L}_I only changes near the cutoff momentum, which will be corroborated soon.

To figure out what conditions \mathcal{L}_I needs to satisfy, just differentiate Z with respect to Λ :

$$\frac{d}{d \ln \Lambda} Z = \int \mathcal{D}\phi \int_p \left(\frac{p^2}{\Lambda^2} e^{p^2/\Lambda^2} \phi_p (p^2 + m^2) \phi_{-p} - \frac{d}{d \ln \Lambda} \mathcal{L}_I \right) e^{-S}. \quad (514)$$

This must be independent of J , which lets us figure out how \mathcal{L}_I changes. Note that $\frac{d}{d \ln \Lambda} \mathcal{L}_I$ should only have support near $p^2 = \Lambda^2$.

To see what we should write for $\frac{d}{d \ln \Lambda} \mathcal{L}_I$, let's calculate the functional derivatives of e^{-S} . We get

$$\frac{\delta}{\delta \phi_p} e^{-S} = -\frac{1}{(2\pi)^4} \left((p^2 + m^2) \phi_{-p} e^{p^2/\Lambda^2} + \frac{\delta \mathcal{L}_I}{\delta \phi_p} \right) e^{-S}, \quad (515)$$

and

$$\begin{aligned} \frac{\delta^2}{\delta \phi_p \delta \phi_{-p}} e^{-S} &= -\frac{1}{(2\pi)^4} \left((p^2 + m^2) e^{p^2/\Lambda^2} + \frac{\delta^2 \mathcal{L}_I}{\delta \phi_{-p} \delta \phi_p} \right) e^{-S} \\ &\quad + \frac{1}{(2\pi)^8} \left((p^2 + m^2) \phi_{-p} e^{p^2/\Lambda^2} + \frac{\delta \mathcal{L}_I}{\delta \phi_p} \right) \left((p^2 + m^2) \phi_p e^{p^2/\Lambda^2} + \frac{\delta \mathcal{L}_I}{\delta \phi_{-p}} \right) e^{-S}. \end{aligned} \quad (516)$$

We can now make a better educated guess about the factors in $\frac{d}{d \ln \Lambda} \mathcal{L}_I$. We choose

$$\frac{d}{d \ln \Lambda} S_I = (2\pi)^4 \int d^4 p \frac{e^{-p^2/\Lambda^2} p^2/\Lambda^2}{p^2 + m^2} \left(\frac{\delta S_I}{\delta \phi_p} \frac{\delta S_I}{\delta \phi_{-p}} - \frac{\delta^2 S_I}{\delta \phi_p \delta \phi_{-p}} \right). \quad (517)$$

The reason why this ends up working is the following: first, we have, for any momentum p ,

$$0 = \int \mathcal{D}\phi \frac{\delta}{\delta \phi_p} \left[\left(\phi_p e^{p^2/\Lambda^2} + \frac{(2\pi)^4}{2} \frac{1}{p^2 + m^2} \frac{\delta}{\delta \phi_{-p}} \right) e^{-S + \int J \phi} \right], \quad (518)$$

just because at the extremes of the functional integration, ϕ is infinite everywhere in space-time, and so we get a factor of $e^{-\infty}$ from e^{-S} . Now we apply the above relation with $p > \mu$, where again μ is the scale above which the current vanishes (therefore the functional derivative is taken with respect to a high-energy mode). After some algebra that I won't write out, we evaluate the functional derivatives and get

$$\frac{e^{p^2/\Lambda^2} Z[J]}{2} = \int \mathcal{D}\phi \left(\phi_p \frac{p^2 + m^2}{2} e^{2p^2/\Lambda^2} \phi_{-p} + \frac{1}{2(p^2 + m^2)} \left(\frac{\delta^2 S_I}{\delta \phi_p \delta \phi_{-p}} - \frac{\delta S_I}{\delta \phi_p} \frac{\delta S_I}{\delta \phi_{-p}} \right) \right) e^{-S + \int J \phi}. \quad (519)$$

Next, we calculate $d_{\ln \Lambda} Z[J]$:

$$d_{\ln \Lambda} Z[J] = - \int \mathcal{D}\phi \left(\int \frac{d^4 p}{(2\pi)^4} \frac{p^2 + m^2}{2} \phi_p \phi_{-p} \left(-\frac{2p^2}{\Lambda^2} e^{p^2/\Lambda^2} \right) + d_{\ln \Lambda} S_I \right). \quad (520)$$

Now we plug in our ansatz for $d_{\ln \Lambda} S_I$ into this equation, and use (519) to get

$$d_{\ln \Lambda} Z[J] = \int d^4 p e^{-p^2/\Lambda^2} \frac{p^2}{\Lambda^2} (e^{p^2/\Lambda^2} Z[J]), \quad (521)$$

which tells us that the change of the generating function under changes in the cutoff is

$$\frac{dW[J]}{d \ln \Lambda} = \int d^4 p \frac{p^2}{\Lambda^2}, \quad (522)$$

which as required is independent of J , and so if S_I changes according to the functional differential equation provided in our ansatz, correlation functions at energy scales below μ are all independent of Λ .

Note that all of this has been non-perturbative and exact! The equation for $d_{\ln \Lambda} S_I$ gives a *linear* differential equation for how the coupling constants in S_I change as the cutoff varies, and so the effective coupling constants at any lower scale can be found in principle only using our knowledge of the coupling constants at a fixed UV scale. Note that if S_I is only quadratic in ϕ (e.g. $S_I = \sum_k g_k \int \phi_p p^{2k} \phi_{-p}$), then S_I remains quadratic in ϕ , but if it contains any interactions, then after integrating the equation for $d_{\ln \Lambda} S_I$ down from a cutoff of Λ to a cutoff of μ , we generically will generate all possible interactions allowed by symmetry.

37 May 20 — Practice with transfer matrix

Consider a “spin chain” in one dimension with nearest neighbor interactions of the form

$$H = \sum_i |q_i - q_{i+1}|, \quad (523)$$

where $q_i \in [-1, 1]$ are the “spins”. Find the spectrum of the transfer matrix and the correlation length in the limit of $T \rightarrow 0$.

Solution:

The partition function is the trace of the n th power of the transfer matrix \mathbb{T} , so that $\mathbb{T}(q, q')$ is basically the part of the action that connects q and q' . It acts on vectors by integration against the kernel provided by the interaction, and so its eigenvectors ψ with eigenvalue λ will satisfy

$$[\mathbb{T}\psi](q) = \int_{-1}^1 dq' \mathbb{T}(q, q')\psi(q') = \int_{-1}^1 dq' e^{-|q-q'|/T} \psi(q') = \lambda\psi(q). \quad (524)$$

Differentiating this once gives

$$\lambda\partial_q\psi(q) = -\frac{1}{T} \int_{-1}^1 dq' \operatorname{sgn}(q - q') e^{-|q-q'|/T} \psi(q'). \quad (525)$$

Differentiating again and using $\partial_q \operatorname{sgn}(q - q') = 2\delta(q - q')$ gives

$$\lambda\partial_q^2\psi(q) = -\frac{2}{T}\psi(q) + \frac{1}{T^2} \int_{-1}^1 dq' e^{-|q-q'|/T} \psi(q'), \quad (526)$$

and so we have the DE

$$T^2\lambda\psi''(q) = -2T\psi(q) + \lambda\psi(q). \quad (527)$$

Let

$$\omega^2 = \frac{2T - \lambda}{T^2\lambda}, \quad (528)$$

so that the solutions are

$$\psi(q) = A \cos(\omega q) + B \sin(\omega q). \quad (529)$$

We can get boundary conditions relating ψ and ψ' by using the expression for ψ' . This gives

$$T\lambda\partial_q\psi(1) = -\lambda\psi(1) \quad (530)$$

and so, applying this to $\psi(q)$ and equating terms in A and B ,

$$\cos(\omega) = T\omega \sin(\omega), \quad \sin(\omega) = -T\omega \cos(\omega), \quad (531)$$

so that

$$\tan(\omega) = \frac{1}{T\omega}, \quad \text{or} \quad \tan(\omega) = -T\omega. \quad (532)$$

Let us choose e.g. the sin solution. Writing $\omega = n\pi + xT$ for $n \in \mathbb{Z}$, we have

$$xT \approx -Tn\pi - xT^2 \implies \omega \approx n\pi(1 - T). \quad (533)$$

From the definition of ω then, we get

$$\lambda_n = \frac{2T}{1 + n^2\pi^2T^2}. \quad (534)$$

Now we want to get the correlation length. For a generic two point function $\langle q_0 q_l \rangle$, we expect $|\langle q_0 q_l \rangle_c| \sim \exp(-|l|/\xi)$, and so we define the correlation length by

$$\xi = -\lim_{l \rightarrow \infty} \frac{|l|}{\ln(|\langle q_0 q_l \rangle_c|)}. \quad (535)$$

We now want to write this in terms of the spectrum of the transfer matrix. In the thermodynamic limit the expression $Z = \text{Tr} \mathbb{T}^n$ collapses onto the largest eigenvalue λ_0 , so that for n sites it goes to λ_0^n . Let Q be the position operator, so that $Q(q, q') = q\delta(q - q')$. Then the two point function is (projecting onto the ground state wavefunction ψ_0 and assuming PBC on a chain of total length L)

$$\begin{aligned}\langle q_0 q_l \rangle &= \frac{1}{\lambda_0^L} \int dq_0 dq_l \psi_0(q_0) \text{Tr} (\mathbb{T}^{L-l} Q \mathbb{T}^l Q) \psi_0(q_l) \\ &= \frac{1}{\lambda_0^L} \int dq_0 dq_l \psi_0(q_0) q_0 [\mathbb{T}^l](q_0, q_l) q_l \psi_0(q_l) \\ &= \frac{1}{\lambda_0^L} \int dq_0 dq_l \sum_n \psi_0(q_0) q_0 \lambda_n^l \psi_n(q_0) \psi_n(q_l) q_l \psi_0(q_l) \\ &\approx \frac{1}{\lambda_0^L} \int dq_0 dq_l q_0 q_l \left(\lambda_0^l \psi_0^2(q_0) \psi_0^2(q_l) + \lambda_1^l \psi_0(q_0) \psi_1(q_0) \psi_0(q_l) \psi_1(q_l) \right),\end{aligned}\tag{536}$$

where we've kept only the two leading terms in the spectral decomposition of \mathbb{T} . Subtracting off $\langle q^2 \rangle$ to get the connected piece, we have

$$\langle q_0 q_l \rangle_c = \left(\frac{\lambda_1}{\lambda_0} \right)^l \Gamma^2, \quad \Gamma = \int dq q \psi_0(q) \psi_1(q).\tag{537}$$

Taking $l \rightarrow \infty$ and taking the log, we get

$$\xi = \frac{1}{\ln(\lambda_0/\lambda_1)}.\tag{538}$$

For us, this means that the correlation length at $T \rightarrow 0$ is

$$\xi \approx \frac{1}{\pi^2 T^2}.\tag{539}$$

Since this diverges, the continuum limit can be taken safely.

38 May 21 — Basic setup of elasticity theory

Consider some sort of quantum solid in three spacetime dimensions. Write down the Lagrangian in terms of displacement fields, assuming a rotationally-invariant Hamiltonian (I was thinking about dislocation-mediated melting when writing this, and wanted to ignore disclinations, the presence of which restore the rotational symmetry).

Assuming an isotropic situation, write down the propagators for the transverse and longitudinal modes in terms of the compression and shear elastic moduli κ and μ . Identify the stress tensor and then write down the theory in a dual form where the stress tensor is treated as the fundamental variable. Finally, solve the conservation equation on the stress energy tensor and write the Lagrangian in terms of “elastic gauge fields”. See [2] for inspiration if desired.

Solution:

Let R_{ij} and R_{ij}^0 be the (vectorial) positions and equilibrium displacements between the atoms at sites i and j . Let u_i^a denote the a -direction displacement of the i th atom from its equilibrium position. We will expand the potential $V(\{R_{ij}\})$ about a state in which all separations are R_{ij}^0 (ignoring the binding energy of the solid):

$$\begin{aligned} V &= \frac{1}{2} \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} (R_{ij}^a - R_{ij}^{0a})(R_{kl}^b - R_{kl}^{0b}) + \dots \\ &= \frac{1}{2} \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} (u_i^a - u_j^a)(u_k^b - u_l^b), \end{aligned} \quad (540)$$

where there is no linear term since we are expanding about the equilibrium positions and where we used $R_{ij}^a = R_{ij}^{0a} + u_i^a - u_j^a$. Here all the indices are spatial, but it is very helpful to regard the a, b, \dots indices as “flavor” indices, with the u^a representing different goldstone fields.

We assume that V contains only interactions between neighboring sites, so that we can expand u_i^a about u_j^a , writing

$$u_i^a = u_j^a + R_{ij}^{0m} \partial_m u^a + \frac{1}{2} R_{ij}^{0m} R_{ij}^{0n} \partial_n \partial_m u^a + \dots \quad (541)$$

The potential energy thus goes to

$$V = \frac{1}{2} C_{abmn} \partial_m u^a \partial_n u^b + \frac{1}{2} C_{abmnrs} \partial_m \partial_r u^a \partial_n \partial_s u^b, \quad (542)$$

where

$$C_{abmn} = \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} R_{ij}^{0m} R_{kl}^{0n}, \quad C_{abmnrs} = \frac{1}{4} \frac{\partial^2 V}{\partial R_{ij}^a \partial R_{kl}^b} R_{ij}^{0m} R_{ij}^{0n} R_{kl}^{0r} R_{kl}^{0s}. \quad (543)$$

We will ignore the four-derivative part in V for now. Also note that we’ve dropped the cross terms which are cubic in derivatives: this is because they are antisymmetric under $R_{ij} \rightarrow -R_{ij}^0 = R_{ji}^0$.

It’ll be good to have a more specific form for C_{abmn} in terms of various more specific elastic moduli. We can a few different types of things that can happen to the lattice which lead to an increase in energy: compressions, shears, and rotational things (I don’t know the right words to use). Compressions are scalars, in that compressions are radially-symmetric things that transform trivially under rotations (i.e. are in the trivial rep of $O(2) \times O(2)$ —one $O(2)$ for the ab indices and one for the mn indices). We are going to be assuming that the energy V is invariant under rotations, and so we won’t have to worry about rotational things (spin 1 representations) appearing in the decomposition of C_{abmn} . However, we do have to worry about shear forces, where one direction is compressed and the other direction is elongated. Because shear configurations are symmetric under π rotations, they come in the spin 2 representation (think of gravity waves and gravitons). We thus decompose C_{abmn} in terms of minimal projectors as

$$C_{abmn} = 2 (\kappa \Pi_{abmn}^0 + \mu \Pi_{abmn}^2), \quad (544)$$

where κ and μ are the compression and shear elastic moduli, and the projectors are

$$\Pi_{abmn}^0 = \frac{1}{2}\delta_{am}\delta_{bn}, \quad \Pi_{abmn}^2 = \frac{1}{2}(\delta_{mn}\delta_{ab} - \delta_{ma}\delta_{nb} + \delta_{mb}\delta_{na}), \quad (545)$$

which project onto the trivial representation and the 2 representation, respectively.

Now we can write down the Lagrangian (in Euclidean signature). Here it is:

$$\mathcal{L} = \frac{\rho}{2}(\partial_t u)^2 + \frac{1}{2}\partial_m u^a (\kappa\delta_{ma}\delta_{nb} + \mu[\delta_{mn}\delta_{ab} - \delta_{ma}\delta_{nb} + \delta_{mb}\delta_{na}]) \partial_n u^b. \quad (546)$$

Now the δ structure of the various terms becomes clearer: for example, the compression term with κ looks like $(\nabla \cdot u)^2$, which makes sense since compressions should be captured by a divergence in the displacements.

We can write \mathcal{L} more concisely by extending C_{abmn} to $C_{ab\mu\nu}$, which incorporates the time index, through

$$C_{ab\mu\nu} = C_{abmn} + \rho\delta_{ab}\delta_{\mu 0}\delta_{\nu 0}. \quad (547)$$

Hopefully the notation is clear: the first part is the purely spatial part and the second is relevant if both μ and ν are temporal. Thus

$$\mathcal{L} = \frac{1}{2}C_{\mu\nu}^{ab}\partial^\mu u^a \partial^\nu u^b. \quad (548)$$

The stress tensor is thus

$$T_\mu^a = C_{\mu\nu}^{ab}\partial^\nu u^b. \quad (549)$$

Since u^t doesn't make sense, it is good to segregate the indices on T in this way. This form for the stress tensor makes sense because $\partial^\nu u_b$ is, well, a stress. When we source the displacement field by adding a $\int J_a u^a$ term to the action, we get the expected relation

$$\partial^\mu T_\mu^a = J^a, \quad (550)$$

saying that external forces lead to divergences in the stress tensor. Also note that T_μ can be thought of as a 1-form field itself, with a acting as a flavor index.

Let's find the propagators by going to momentum space:

$$\mathcal{L} = \frac{\rho}{2}\omega^2 u^a p^2 u^a + \frac{\kappa}{2}u^a p^a p^b u^a u^b + \frac{\mu}{2}u^a p^2 u^a. \quad (551)$$

Now we invert the kernel with transverse and longitudinal projectors in the usual way. We write

$$\mathcal{L} = \frac{1}{2} [\Pi_{ab}^T(\rho\omega^2 + p^2\mu) + \Pi_{ab}^L(\rho\omega^2 + (\mu + \kappa)p^2)], \quad (552)$$

where as usual $\Pi_{ab}^T = \delta_{ab} - p^a p^b / p^2$ and $\Pi^L = p^a p^b / p^2$. So, the full propagator is

$$G_{ab} = \rho^{-1} \left(\frac{\Pi_{ab}^L}{\omega^2 + \frac{\mu+\kappa}{\rho}p^2} + \frac{\Pi_{ab}^T}{\omega^2 + \frac{\mu}{\rho}p^2} \right). \quad (553)$$

Strictly speaking, this expression for $\langle u^a u^b \rangle$ will only make sense when u^a is single-valued, i.e. when there are no dislocations. If there are dislocations around, we will need instead to compute $\langle \partial_\mu u^a \partial_\nu u^b \rangle$, which is well-defined.

Anyway, from the expression for G_{ab} we identify the speed of the modes as

$$c_T = \sqrt{\mu/\rho}, \quad c_L = \sqrt{(\mu + \kappa)/\rho}. \quad (554)$$

Thus the longitudinal compression mode is always faster than the transverse one. We also see that κ only appears in the longitudinal mode: this is because the longitudinal mode is a compressional wave, while the transverse modes are not. The transverse mode is better to focus on because it is more discriminatory when it comes to dealing with different phases: both liquids and solids have compressional longitudinal modes, but the liquid doesn't have transverse modes, since it is dissipative.

Now we will write the theory in a dual presentation in terms of the stress tensor, rather than in terms of the displacement variables (the strains). This can be done easily since T_μ^a is essentially the momentum conjugate to u^a . To invert $T_\mu^a = C_{\mu\nu}^{ab}\partial^\nu u^b$, we need to invert C . This is strictly speaking not possible, since the coefficient of the Π^1 projector onto the spin 1 representation in the decomposition of C is zero, and so C will annihilate spin 1 vectors. This is okay for us though, since the spin 1 representation is antisymmetric (think $r_n\partial_m - r_m\partial_n$), while in the absence of external torques the stress tensor will be symmetric, and so at least when acting on T it is possible to invert C . So we can write

$$[C^{-1}]_{ab\mu\nu} = \frac{1}{\rho}\delta_{ab}\delta_{\mu 0}\delta_{\nu 0} + \frac{1}{2}(\kappa^{-1}\Pi_{abmn}^0 + \mu^{-1}\Pi_{abmn}^2). \quad (555)$$

The Lagrangian in terms of the stresses is then

$$\mathcal{L} = \frac{1}{2}T_\mu^a[C^{-1}]_{\mu\nu}^{ab}T_\nu^b + iT_\mu^a\partial_\mu u^a, \quad (556)$$

which comes from either doing the canonical procedure for fields and their momenta or from doing a HS transformation on the $\partial u C \partial u$ term.

Let us perform a Hodge decomposition on the 1-form $\partial_\mu u^a$ by writing $u = u_s + u_d$, where u_s is smooth and u_d is a discontinuous part due to the presence of dislocations. This means that $\partial_\mu u^a = du_s^a + du_d^a$, where du_d^a is not generically either closed or exact. Note that the path integral over u_s enforces conservation of energy-momentum. This means that

$$d^\dagger T^a = 0, \quad (557)$$

which we can enforce by writing $T^a = d^\dagger \alpha^a$ for some 2-form α^a . To do this, we have made use of the decomposition

$$\ker(d_p^\dagger) = \text{im}(d_{p+1}^\dagger) \oplus H^p(X, \partial X), \quad (558)$$

where the cohomology group contains all cohomology classes which vanish on the boundary of X . We assume that this cohomology group is trivial, meaning that $d^\dagger T = 0$ implies that T is coexact.

Actually it will be more convenient to write things in terms of the 1-form $A^a \equiv \star \alpha^a$, since this will give us something more gauge-theory-like. Thus the integration of u_s sets

$$T^a = \star dA, \quad (559)$$

where A is a gauge field since $A \mapsto A + d\lambda$ doesn't affect the physical variable T^a . If the above cohomology group were nontrivial, then we would have $T^a = \star(dA + c)$, where $c \in H^2(X; \mathbb{Z})$ is a 2-form parametrizing the nontriviality of the A gauge bundle.

Thus we can write the Lagrangian as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\star dA^a)_\mu [C^{-1}]_{\mu\nu}^{ab} (\star dA^b)_\nu + i(\star dA^a)_\mu \partial_\mu u_d^a \\ &= \frac{1}{2}(\star F^a)_\mu [C^{-1}]_{\mu\nu}^{ab} (\star F^b)_\nu + iA^a \wedge d^2 u^a \\ &= \frac{1}{2}(\star F^a)_\mu [C^{-1}]_{\mu\nu}^{ab} (\star F^b)_\nu + iA^a \wedge \star J_d^a,\end{aligned}\tag{560}$$

where we have defined the dislocation current as

$$J_d^a \equiv \star d^2 u^a.\tag{561}$$

This is the main reason why working in the gauge field formulation is preferred: the dislocation current couples directly to the dual gauge field, rather than to the field-strength variable T_μ^a . We see that dislocations act as electric charges, and interact amongst themselves by exchanging photons. This is very similar to the “spin waves + vortices” picture in the XY model, except with an extra flavor index a and additional elastic moduli added in to the kinetic term in the form of C^{-1} .

39 May 22 — Classical $O(n)$ model in one dimension

Another easy problem today — more practice with the transfer matrix. Consider the $O(n)$ model in one dimension (spin chain of n -component spins on S^{n-1}). The Hamiltonian is

$$H = -J \sum_i S_i \cdot S_{i+1}.\tag{562}$$

Find the correlation length to first order in $(\beta J)^{-1}$. Do so by finding the first two eigenvalues of the transfer matrix.

Solution:

Recall from two days ago that we can get the correlation length by computing

$$\xi = -\lim_{l \rightarrow \infty} \frac{|l|}{\ln |\langle S_0 S_l \rangle_c|}.\tag{563}$$

Also recall that because of the $l \rightarrow \infty$ limit, we could do a spectral decomposition on the transfer matrix and keep only the two leading terms. This let us write

$$\xi^{-1} = \ln(\lambda_0/\lambda_1),\tag{564}$$

where λ_0, λ_1 are the largest and second-largest eigenvalues of \mathbb{T} , respectively.

For a function $\psi(S)$ of the vector S , we have

$$\mathbb{T}\psi(S) = \int dS' \delta(S'^2 - 1) e^{\beta JS \cdot S'} \psi(S'). \quad (565)$$

Note that \mathbb{T} has $O(n)$ symmetry, since it only depends on the dot product. Thus its eigenvectors will transform in various irreps of $O(n)$. The lowest energy irrep is the trivial irrep, the second lowest is the fundamental S irrep, and so on. Thus the largest eigenvalue of \mathbb{T} will come from an eigenvector $\psi_0(S)$ in the trivial representation—that is, a constant. So we just need to compute

$$\lambda_0 \psi_0(S) = \int dS' \delta(S'^2 - 1) e^{\beta JS \cdot S'} \psi_0(S'), \quad (566)$$

where $\psi_0(S') = 1$. We do this by choosing $S = (1, 0, 0, \dots, 0)$. A convenient parametrization of S' is then

$$S' = (\cos \phi, \mathbf{n} \sin \phi), \quad \mathbf{n} \in S^{n-2}. \quad (567)$$

We then write

$$dS' \delta(S'^2 - 1) = d\phi d\mathbf{n} \sin^{n-2} \phi \delta(\mathbf{n}^2 - 1), \quad (568)$$

where n powers of $\sin \phi$ came from the Jacobian in the change of variables and -2 powers came from the δ function.

Doing the integral over \mathbf{n} gives the surface area of the sphere, which we will just write as Γ . So

$$\lambda_0 = \Gamma \int d\phi e^{\beta J \cos \phi} \sin^{n-2} \phi. \quad (569)$$

We are taking $\beta J \gg 1$, so we can do a saddle point approximation. Since $\beta J \gg 1$, the integrand only cares about small ϕ . Thus we can expand in ϕ and take the integral from 0 to ∞ rather than 0 to π . We write

$$\begin{aligned} e^{\beta J \cos \phi} \sin^{n-2} \phi &\approx e^{\beta J} e^{-\beta J \phi^2/2} (1 + \beta J \phi^4/4!) (\phi^{n-2} - (n-2)\phi^n/3!) \\ &\approx e^{\beta J} e^{-\beta J \phi^2/2} \phi^{n-2} (1 - (n-2)\phi^2/3! + \beta J \phi^4/4!). \end{aligned} \quad (570)$$

The integral can then be done since the integrand is Gaussian and we're integrating up to ∞ . We get

$$\lambda_0 = \frac{C}{(\beta J)^{(n-1)/2}} e^{\beta J} \left(1 - \frac{1}{8\beta J} (n^2 - 4n + 3) + O([\beta J]^{-2}) \right), \quad (571)$$

where C is a βJ -independent but n -dependent constant with 2s and π s and stuff (it won't matter since we're calculating λ_0/λ_1 and as we will see the same such constant appears in λ_1).

Now we get λ_1 . We expect that the next-to-lowest energy state will be in the fundamental representation, i.e. the spin 1 representation, of $O(n)$. Thus the appropriate eigenfunction of \mathbb{T} will be $\psi(S) = S$ (remember that S is a vector in S^{n-1}). The eigenvalue equation reads

$$\lambda_1 S = \int dS' \delta(S'^2 - 1) e^{\beta JS \cdot S'} S'. \quad (572)$$

We can again without loss of generality set $S = (1, 0, \dots, 0)$. Working again in the parametrization $S' = (\cos \phi, \mathbf{n} \sin \phi)$, we get

$$\lambda_1(1, 0, \dots, 0) = \int d\mathbf{n} d\phi \delta(\mathbf{n}^2 - 1) \sin^{n-2} \phi e^{\beta J \cos \phi} (\cos \phi, \mathbf{n} \sin \phi). \quad (573)$$

The integration over the S^{n-2} kills all but the first component on the RHS, demonstrating that indeed, S is an eigenvector of \mathbb{T} . So then we have

$$\lambda_1 = \Gamma \int d\phi \sin^{n-2} \phi e^{\beta J \cos \phi} \cos \phi = \partial_{\beta J} \lambda_0. \quad (574)$$

So then

$$\lambda_1 = \frac{1-n}{2\beta J} \lambda_0 + \lambda_0 + O([\beta J]^{-2}), \quad (575)$$

telling us that

$$\xi^{-1} = -\ln \left(1 - \frac{n-1}{2\beta J} \right) \implies \xi \approx \frac{2\beta J}{n-1}. \quad (576)$$

Note in particular that it gets smaller at large n .

40 May 23 — Path integrals for Majorana fermions in one dimension and spin structures

Consider $2n$ real Majorana fermions η_i in one dimension, with action

$$S = \int dt i \sum_j \eta_j \frac{d}{dt} \eta_j. \quad (577)$$

Suppose time is an S^1 . Compute the partition function for both spin structures. Also compute $\text{Tr}_{\mathcal{H}}(g)$ and $s\text{Tr}_{\mathcal{H}}(g)$, where $g \in \text{Spin}(n)$. Here \mathcal{H} is the Hilbert space of the $2n$ Majoranas, with dimension 2^n . When $g = 1$, identify the two traces with the two partition functions.

Solution:

Computing the Pfaffians

First we evaluate the partition functions by taking Pfaffians. There are two spin structures on S^1 , which are permuted by $H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$. This means there are two spin bundles on S^1 , which we will write as S_+ and S_- . S_+ is the trivial \mathbb{Z}_2 bundle and S_- is the mobius strip. We will label the trivial bundle (the double-cover) “non-bounding” NB and the non-trivial bundle “bounding” B . This is because the nontrivial bundle bounds a disk, while the trivial

bundle does not. Why? Well, when we embed the S^1 into a 2-manifold equipped with a spin structure, we need homologically trivial closed loops along which the fermion framing is parallel-transported to act as -1 in $\text{Spin}(2) = U(1)$. Thus the S_- bundle is the one which can be embedded in two dimensions, while the S_+ bundle can only be embedded in two dimensions if it's embedded along a non-contractible loop. Anyway, each of the majoranas η_j , being real fields, should be viewed as sections of either the associated spinor bundle $S_+ \otimes \mathbb{R}$ or the associated spinor bundle $S_- \otimes \mathbb{R}$, depending on the spin structure.

Will we also want to compute the partition function in the presence of a background field for the $SO(2n)$ global symmetry. Writing the twisted Dirac operator as \not{D}_R for $R \in SO(2n)$, we need to compute $\text{Pf}(\not{D}_R) \in \mathbb{R}$ as a function of R . Unfortunately the sign is ambiguous, since we compute $\text{Pf}(\not{D}_R)$ by taking a square root. Now $\text{Pf}(\not{D}_R)$ can be thought of as a section of a real line bundle, the Pfaffian line bundle, over $SO(2n)$. The structure group is $O(1) = \mathbb{Z}_2$, corresponding to a choice of sign in taking the square root. Unfortunately since $\pi_1(SO(2n)) = \mathbb{Z}_2$, this line bundle can be nontrivial, which means that the sign of the partition function may not be well defined.

However, suppose we instead twist the fields by an element in $\text{Spin}(2n)$ instead. There are no nontrivial line bundles over $\text{Spin}(n)$,¹⁶ and so the Pfaffian line bundle must be trivial, giving us a well-defined partition function. Anyway, we will let $Z_{\pm}(R)$ denote the partition function with \pm spin structure, twisted by an additional possible element $R \in \text{Spin}(2n)$.

Now since the twisting by $R \in \text{Spin}(2n)$ can be done by inserting R into either $\text{Tr}_{\mathcal{H}}[R]$ or $\text{Tr}_{\mathcal{H}}[(-1)^F R]$ (depending on the spin structure) and since $(-1)^F$ is $-\mathbf{1}$ in $\text{Spin}(2n)$, which is central,¹⁷ we have

$$Z_{\pm}(R) = Z_{\pm}(V^{-1}RV), \quad \forall V \in \text{Spin}(2n) \quad (579)$$

so that $Z_{\pm}(R)$ is a class function on $\text{Spin}(2n)$. This means we can, by conjugating, work in a basis where R is the exponential of an element in the maximal torus of $\mathfrak{spin}(2n)$. Letting the generators of the (complex, I think) Clifford algebra $\text{Cliff}(2n)$ be γ_i , a generic element of the maximal torus can be written as

$$MT(\mathfrak{spin}(2n)) \ni -i \ln R = \sum_{j=1}^n r_j \gamma_{2j-1} \gamma_{2j}. \quad (580)$$

As required, all of the $\gamma_{2j-1} \gamma_{2j}$ commute with each other, and they all square to $-\mathbf{1}$. Thus we can choose a basis in which each $\gamma_{2j-1} \gamma_{2j}$ is represented by a matrix with n 2×2 blocks on the diagonal, with the matrix $J = -iY$ in the j th block and 0s everywhere else. Thus R is the exponential (no factor of $i!$):

$$R = \exp \left(\bigoplus_{j=1}^n r_j J \right), \quad J = -iY. \quad (581)$$

¹⁶This is because $\pi_1(\text{Spin}(2n)) = 0$. At a mathematical level, we would demonstrate this as

$$\text{Line}_{\mathbb{R}}(M) = [M, BO(1)] = [M, B\mathbb{Z}_2] = H^1(M; \mathbb{Z}_2) = \text{Hom}(\pi_1(M), \mathbb{Z}_2), \quad (578)$$

where M is any manifold and $\text{Line}_{\mathbb{R}}(M)$ are the different real line bundles over M . Thus if $\pi_1(M) = 0$, there are no nontrivial real line bundles over M .

¹⁷ $(-1)^F$ is the nontrivial element in the $\mathbb{Z}_2 \subset Z(\text{Spin}(2n))$ that gets projected out when taking the quotient to $SO(2n)$. $Z(\text{Spin}(2n))$ is either \mathbb{Z}_4 or \mathbb{Z}_2^2 , depending on whether n is odd or even, respectively. The extra $\mathbb{Z}_2 \subset Z(\text{Spin}(2n))$ that remains after taking the quotient to $SO(2n)$ constitutes the center of $SO(2n)$, via $Z(SO(2n)) = \mathbb{Z}_2$.

To put this into the Lagrangian, we just need to construct the spin connection ω appearing in $\partial_\mu - \omega_\mu^{ij} \Sigma_{ij}$, with $\Sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$. From the above construction, normalizing so that the radius of the temporal S^1 to be 1, we see that we may take

$$\omega_t^{2j-1, 2j} = r_j / 2\pi. \quad (582)$$

Indeed, with this choice we have $e^{i \oint dt \omega_t^{ij} \Sigma_{ij}} = e^{iR}$.

We now need to compute the Pfaffian of \not{D}_R . Since the spin connection breaks up into a direct sum, we can consider each 2-by-2 block individually. First, consider the non-bounding partition function $Z_+(R)$. The partition function $Z_+(r_j)$ for a single block (containing a single pair of Majoranas) is¹⁸

$$Z_+(r_j) = \text{Pf} \left(\frac{d}{d\tau} + 2\omega_t^{12} \Sigma_{12} \right) = \text{Pf} \left(\frac{d}{d\tau} + \frac{r_j}{2\pi} J \right). \quad (583)$$

The relevant determinant is then (since we have chosen the S^1 to have radius 1, the eigenfunctions of $d/d\tau$ are $e^{i\omega\tau}$, $\omega \in \mathbb{Z}$)

$$\text{Det} \left(\frac{d}{d\tau} + \frac{r_j}{2\pi} J \right) = \prod_{n \in \mathbb{Z}} (-n^2 + r_j^2/(2\pi)^2). \quad (584)$$

We regularize this rather cheekily with the ζ function:

$$\begin{aligned} \prod_{n \in \mathbb{Z}} (-n^2 + r_j^2/4\pi^2) &= \frac{r_j^2}{4\pi^2} \prod_{n \neq 0} n^2 \prod_{n \neq 0} \left(1 - \frac{r_j^2}{4\pi^2 n^2} \right) \\ &= \frac{1}{\pi^2} \left(\prod_{n=1}^{\infty} n^2 \right)^2 \left(\frac{r_j}{2} \prod_{n=1}^{\infty} \left(1 - \frac{r_j^2}{4\pi^2 n^2} \right) \right)^2 \\ &= 4 \sin^2(r_j/2), \end{aligned} \quad (585)$$

where we used

$$\sin(x) = x \prod_{j=1}^{\infty} (1 - x^2/(\pi^2 j^2)) \quad (586)$$

to deal with the second product. To deal with the first product, we used

$$\partial_s \zeta(s) = \sum_n \partial_s e^{-s \ln n} \implies [\partial_s \zeta](0) = - \sum_n \ln n \quad (587)$$

to write

$$e^{-2[\partial_s \zeta](0)} = e^{\sum_{n=1}^{\infty} \ln(n^2)} = \prod_{n=1}^{\infty} n^2. \quad (588)$$

¹⁸We are in Euclidean signature. In real time, the Dirac operator is $i\not{D}_R = i\partial_t + \omega_t^{ij} \Sigma_{ij}$ (factor of i in front of spin connection to get Hermiticity?), since $\omega_t^{ij} = -\omega_t^{ji}$ and since Σ_{ij} is anti-Hermitian (again, just to avoid confusion: ω is the spin connection for the global $\text{Spin}(2n)$ symmetry, not the $\text{Spin}(1)$ symmetry of the temporal circle). When we go to Euclidean time, the Dirac operator instead becomes $\partial_\tau + \omega_t^{ij} \Sigma_{ij}$.

Using the result $[\partial_s \zeta](0) = -\frac{1}{2} \ln 2\pi$ then tells us that

$$\prod_{n=1}^{\infty} n^2 = 2\pi, \quad (589)$$

which allows us to take care of the first product. This means that¹⁹

$$Z_+(R) = \text{Det}^{1/2} \left(\bigoplus_j^n \left[\frac{d}{d\tau} + \frac{r_j}{2\pi} J \right] \right) = 2^n \prod_j |\sin(r_j/2)|. \quad (591)$$

Here the absolute value bars are in place because we are choosing the positive sign when taking the square root (so that $r_j \sim r_j + 2\pi$). As mentioned before, we can make a consistent choice of this sign for all r_j since $\pi_1(\text{Spin}(2n)) = 0$.

Anyway, we see that if we don't twist by any gauge fields the partition function vanishes; $Z_+(\mathbf{1}) = 0$. This is to be expected, since choosing the $S_+ \otimes \mathbb{R}$ bundle gives us a zero mode of the (untwisted) Dirac operator, causing the partition function to vanish. If we twist all the fermions by a 2π rotation, which amounts to choosing $r_j = \pi$ for all j (since this is what gives $R = -\mathbf{1}$ when exponentiated), then we effectively switch to AP boundary conditions and get $Z_+(R) = 2^n$, the dimension of the Hilbert space (each of the $2n$ Majoranas gives a contribution to the dimension of $\sqrt{2}$).

We can similarly compute partition functions for the bounding bundle $S_- \otimes \mathbb{R}$ twisted by some element $R \in \text{Spin}(2n)$. We just take our result for the $S_+ \otimes \mathbb{R}$ bundle, but shift $r_j \mapsto r_j + \pi$, since this adds an extra 2π twist and hence shifts the boundary conditions, as mentioned above. Thus we get

$$Z_-(R) = 2^n \prod_{j=1}^n |\cos(r_j/2)| \quad (592)$$

for the bounding spin bundle. As expected, this counts the dimension of the Hilbert space (and in particular is non-zero) when there is no extra $\text{Spin}(2n)$ twist ($R = 0$).

We could also obtain this result by using the fact that the boundary conditions are AP to compute, for a single block,

$$\begin{aligned} [Z_-(r_j)]^2 &= \prod_{n \in \mathbb{Z}-1/2} (-n^2 + r_j^2/(2\pi)^2) \\ &= \prod_{n \in \mathbb{Z}-1/2} n^2 \left[\prod_{n=1}^{\infty} \left(1 - \frac{r_j^2}{(n-1/2)^2 (2\pi)^2} \right) \right]^2 \\ &= \left(\prod_{n \in \mathbb{Z}-1/2} n^2 \right) \cos^2(r_j/2) \end{aligned} \quad (593)$$

¹⁹If we had wanted the result in \mathbb{R} time, we'd have obtained (note to self: are you sure there is no factor of i in the spin connection?)

$$Z_-^2 = \det \left(i \frac{d}{dt} + \frac{1}{2} J \right) = \prod_{n \in \mathbb{Z}} (n^2 + 1/4) = 2^{2n} \sinh^2(\pi/2). \quad (590)$$

where we used the product formula

$$\cos(x) = \prod_{j=1}^{\infty} (1 - x^2/(\pi^2(n-1/2)^2)). \quad (594)$$

Now we need to deal with the remaining product. We write

$$\prod_{n \in \mathbb{Z}-1/2} n^2 = \prod_{n \in \mathbb{Z}} \frac{1}{4} (2n-1)^2 = \prod_{n \in 2\mathbb{Z}+1} n^2. \quad (595)$$

Here we have used the hilarious fact that the ζ -regularized number of integers is zero:

$$\prod_{n \in \mathbb{Z}} \alpha = e^{\sum_{n \in \mathbb{Z}} \ln \alpha} = 1. \quad (596)$$

This means that when doing ζ -function regularization, $\prod_{n \in \mathbb{Z}} \alpha f(n) = \prod_{n \in \mathbb{Z}} f(n)$ for any non-zero constant α . We can now write the product as

$$\prod_{n \in 2\mathbb{Z}+1} n^2 = \frac{\prod_{n \in \mathbb{Z} \setminus 0} n^2}{\prod_{n \in \mathbb{Z} \setminus 0} (2n)^2} = 4. \quad (597)$$

Here we have used that the ζ -regularized number of *non-zero* integers is -1 , so that

$$\frac{1}{\prod_{n \in \mathbb{Z} \setminus 0} 4} = e^{-\sum_{n \in \mathbb{Z} \setminus 0} \ln 4} = e^{\ln 4} = 4. \quad (598)$$

Putting this into our expression for $Z_-(r_j)$ and taking the product over all blocks j , we recover (592).

Let's briefly comment on why the two approaches (taking the momentum to be quantized in either \mathbb{Z} or $\mathbb{Z} + 1/2$ and not changing the $\text{Spin}(2n)$ gauge field, or always taking the momentum to be in \mathbb{Z} but modifying the $\text{Spin}(2n)$ gauge field in the case of bounding spin structure) agree. Basically, they agree since the \mathbb{Z}_2 of $\text{Spin}(1)$ is equal to the $(-1)^F$ of $\text{Spin}(2n)$, so that we can absorb the spin structure into the $\text{Spin}(2n)$ gauge field. At the formal level, this is just coming from the fact that we can permute spin structures by tensoring the spin bundle with the nontrivial line bundle ϵ (the mobius band) over the circle: $S_{\pm} \otimes \epsilon \cong S_{\mp}$. Since the Dirac operator is computed with a connection on the bundle $S_{\pm} \otimes E_{\text{Spin}(2n)}$, we can write $(S_{\pm} \otimes \epsilon) \otimes E_{\text{Spin}(2n)} = S_{\pm} \otimes (E_{\text{Spin}(2n)} \otimes \epsilon)$, and transfer the spin structure onto the gauge connection. Anyway, the point of this

$$Z_{\pm}(R) = Z_{\mp}(R_{2\pi} \cdot R), \quad (599)$$

where $R_{2\pi}$ is a 2π rotation (alias $(-1)^F$) in $\text{Spin}(2n)$. As we saw though, the fact that these expressions agree is not immediately obvious from the calculations.

Directly taking the trace

Let's see if we can reproduce these results by computing the trace directly. With no additional twisting going on, we know we need to get 0 for the non-bounding spin structure $S_+ \otimes \mathbb{R}$ and 2^n for the bounding spin structure $S_- \otimes \mathbb{R}$. This means that

$$Z_-(\mathbf{1}) = \text{Tr}_{\mathcal{H}}(\mathbf{1}) = 2^n, \quad Z_+(\mathbf{1}) = s\text{Tr}_{\mathcal{H}}(\mathbf{1}) = \text{Tr}_{\mathcal{H}}[(-1)^F] = 0. \quad (600)$$

Here the supertrace $s\text{Tr}$ is a trace weighted by the matrix $(-1)^F \in \text{Spin}(2n)$.

Note that the *nontrivial bundle* corresponds to the *untwisted* (regular) trace, while the *trivial* bundle corresponds to the *twisted* trace (supertrace). Again, this is because fermions “naturally” have a minus sign associated to them when traveling around closed loops (they naturally have anti-periodic boundary conditions), so that taking the regular trace with the $S_- \otimes \mathbb{R}$ bundle just leads to a sum over states (the two minus signs cancel out). However, with the $S_+ \otimes \mathbb{R}$ bundle, we have given the fermions an extra twist, and the partition function vanishes since the even and odd parts cancel.

Because $Z_{\pm}(R)$ is a class function on $\text{Spin}(2n)$, it can be written as a linear combination of the characters of $\text{Spin}(n)$. In particular, $Z_{\pm}(R)$ will be a sum of characters of $\text{Spin}(2n)$ weighted by $(-1)^{(1\mp 1)j}$, where j is a half-odd-integer or an integer according to whether the representation j of $\text{Spin}(2n)$ restricts to a representation of $SO(2n)$ or not. This doesn’t really help (me) compute the partition function, though.

Let’s start with the bounding $S_- \otimes \mathbb{R}$ bundle. We need to know how to represent R in the basis that we’re taking the trace in. For some reason, I found this a bit tricky, and I think the way I do it below could be improved.

First, we know that $\dim \mathcal{H} = 2^n$, since each pair of Majoranas gives us an $SU(2)$ algebra. As we discussed earlier, R can be chosen to lie in exponential of the maximal torus of $\mathfrak{spin}(2n)$. We can then represent R on \mathcal{H} as a product of factors $\otimes_{j=1}^n R_j$, where each R_j factor is a 2×2 matrix acting on the Majoranas η_{2j-1}, η_{2j} . Then we have

$$Z_-(R) = \prod_{j=1}^n \text{Tr} R_j. \quad (601)$$

Therefore we just need to know how to represent the action of the spin group on a single 2×2 block. Now a single pair of Majoranas can be represented by the matrices X, Z , so that the product $XZ = J$ generates $\text{Spin}(2) = U(1)$. Elements of $\text{Spin}(2)$ are written as $e^{rJ/2}$, with $r = \pi$ giving $e^{\pi J/2} = J$, which anti-commutes with the fermion-odd matrices X and Z : therefore we identify it with $(-1)^F$ (note to self: the factor of $1/2$ is needed to get right results but the derivation is suspect).

With our \otimes decomposition of R , which is basically a reduction $\text{Spin}(2n) \rightarrow [\text{Spin}(2)]^n$ (made possible by the fact that $Z_{\pm}(R)$ is a class function, as mentioned above), we then have

$$\begin{aligned} Z_-(R) &= \text{Tr}_{\mathcal{H}}(R) = \text{Tr}_{\mathcal{H}} \left[\bigotimes_{j=1}^n \exp \left(\frac{1}{2} r_j J \right) \right] \\ &= \prod_{j=1}^n \text{Tr} \begin{pmatrix} \cos(r_j/2) & -\sin(r_j/2) \\ \sin(r_j/2) & \cos(r_j/2) \end{pmatrix} \\ &= 2^n \prod_{j=1}^n \cos(r_j/2), \end{aligned} \quad (602)$$

which matches our earlier result.

Now for the non-bounding bundle $S_+ \otimes \mathbb{R}$. We have

$$Z_+(R) = s\text{Tr}_{\mathcal{H}}(R) = \text{Tr}_{\mathcal{H}}((-1)^F R). \quad (603)$$

We saw above how to represent $(-1)^F$ on \mathcal{H} : just set $r_j = \pi$ for all j . Actually we will take $r_j = -\pi$ to match the sign conventions of the previous section; once we have fixed a choice there is no ambiguity since the Pfaffian bundle over $\text{Spin}(2n)$ is trivial. With this choice,

$$(-1)^F = \bigotimes_{j=1}^n e^{-\pi J/2} = (-J)^{\otimes n}. \quad (604)$$

Now we can calculate

$$\begin{aligned} Z_+(R) &= \text{Tr}_{\mathcal{H}} \left[\bigotimes_{j=1}^n (-J) \exp \left(\frac{1}{2} r_j J \right) \right] \\ &= \text{Tr}_{\mathcal{H}} \left[\bigotimes_{j=1}^n \exp \left(\frac{1}{2} (r_j - \pi) J \right) \right] \\ &= \text{Tr}_{\mathcal{H}} \left[\bigotimes_{j=1}^n \begin{pmatrix} \sin(r_j/2) & \cos(r_j/2) \\ -\cos(r_j/2) & \sin(r_j/2) \end{pmatrix} \right] \\ &= 2^n \prod_{j=1}^n \sin(r_j/2), \end{aligned} \quad (605)$$

which reproduces our calculation using the Pfaffians. (note to self: come back and figure out this stupid factor of $1/2$)

41 May 24 — Anomalous dimension in ϕ^4 theory

When are we going to be sick of doing one-loop calculations with simple qfts? Who knows. Anyway, today is short and simple: remind yourself how to calculate the anomalous dimension of the operator ϕ^2 in ϕ^4 theory. Remember that the anomalous dimension of an operator \mathcal{O} is defined by

$$d_{\ln \mu} \mathcal{O} = \gamma_{\mathcal{O}} \mathcal{O}. \quad (606)$$

Calculate γ_{ϕ^2} to first order in λ .

Solution:

To find the anomalous dimension, we can insert the term $\int J \phi^2$ into the Lagrangian. To one loop order, J gets corrected by a ϕ^4 vertex, with the relevant graph looking like a fish. If we take J at momentum k , then the two (incoming) parts of the tail of the fish will have momenta p and $-k-p$, with the two segments of the body of the fish having momenta q and $k-q$ (we take the k momentum of the current J to be “incoming” with respect to the head of the fish). There is a symmetry factor of 2 associated with the number of ways to order the lines on the tail of the fish, and so the fish diagram is

$$\text{fish} = \frac{i^2 \lambda}{2} \int_q \frac{1}{(q^2 - m^2)((k-q)^2 - m^2)}. \quad (607)$$

We could do Feynman parameters and dim reg, but we care only about the coefficient of the large logarithm, so we can also just directly do the integration: the surface area of an S^3 is $2\pi^2 R^4$, so

$$\text{fish} \sim -2\lambda \frac{\pi^2}{32\pi^4} \ln(\Lambda/\Delta), \quad (608)$$

where Δ is an invariant built out of the external momenta and the mass.

We thus renormalize the current by writing

$$J_b = J_R Z_J = J_R(1 + \delta_J), \quad (609)$$

where

$$\delta_J = \frac{\lambda}{16\pi^2} \ln(\Lambda/\mu), \quad (610)$$

where μ is the RG scale at which we want the one-loop correction to vanish. Since J_b is independent of μ ,

$$\begin{aligned} 0 &= \frac{d}{d \ln \mu} (J_R(1 + \delta_J)) \\ &= \frac{d J_R}{d \ln \mu} (1 + \delta_J) + J_R \frac{d}{d \ln \mu} \left(\frac{\lambda}{16\pi^2} \ln(\Lambda/\mu) \right). \end{aligned} \quad (611)$$

Which means that

$$0 = \gamma_J \left(1 + \frac{\lambda}{16\pi^2} \ln(\Lambda/\mu) \right) + \frac{\beta_\lambda}{16\pi^2} \ln(\Lambda/\mu) - \frac{\lambda}{16\pi^2}. \quad (612)$$

Now β_λ is order λ^2 , since it is computed by looking at the corrections to the ϕ^4 vertex. γ_J is also first order in λ , and so to first order we have $\gamma_J = \frac{\lambda}{16\pi^2}$. The anomalous dimension of ϕ^2 is the negative of this, so

$$\gamma_{\phi^2} = -\frac{\lambda}{16\pi^2}. \quad (613)$$

Alternately, we could have obtained this by calculating $G_2(\phi(x), \phi(x))$ in an RPA-like way, by summing diagrams that look like chains with all possible number of beads on them (each bead is a ϕ^4 bubble attached to the chain at a single point). At the renormalization scale μ , we would sum the series and get

$$G_2 \sim G_{2,0} \frac{1}{1 - \frac{\lambda}{16\pi^2} \ln(\Lambda/\mu)}. \quad (614)$$

Taking the derivative wrt $\ln \mu$ and working to first order in λ reproduces the correct result.

42 May 25 — BF theory basics

Carefully go through sections 1,2, and 3 of “Coupling QFTs to TQFTs” and explain how the various presentations of BF theory are constructed, and explain what the global symmetries are.

Solution:

First, a note on notation: I will largely only write down dA in an integrand when A is actually a well-defined form, so that dA is exact and d actually acts as the exterior derivative. For example, given a connection A on a nontrivial $U(1)$ -bundle, I will try to only write integrals over F_A , the curvature of A , and will try to avoid writing dA . Also, as a warning, I will generally be sloppy about keeping track of various minus signs coming from the super-commutativity of the exterior derivative and from interchanging the order of wedge products. Another thing to keep in mind vis-a-vis factors of 2π : if $B \in H^p(X; \mathbb{Z})$ and $A \in H^q(X; \mathbb{Z})$, then $B \wedge A \in H^{p+q}(X; \mathbb{Z})$, i.e. the wedge product is a product operation on cohomology, meaning that the wedge product of two forms with integer periods is also a form with integer periods. We will thus always divide by 2π -s in such a way that we are only wedging together forms in $H^*(X; \mathbb{Z})$, so as to avoid writing cohomology groups like $H^*(X; 2\pi\mathbb{Z})$.²⁰

First formulation: The first way of writing BF theory is

$$\frac{i}{2\pi} \int_X H \wedge (F_a + nA), \quad (615)$$

where A is a q -form gauge field and a is a $q-1$ form gauge field. Both are defined up to large gauge transformations²¹. For example, if $q=1$ then a is a scalar, and $a \sim a + 2\pi\mathbb{Z}$. If $q=2$, then a is a 1-form, with $a \sim a + \alpha$ for all α such that $\int_{M_1} \alpha \in 2\pi\mathbb{Z}$ for all closed 1-manifolds M_1 . In the above action, H is a $(D-1)$ -form Lagrange multiplier field.²² If $\partial X \neq \emptyset$, we take $H|_{\partial X} = 0$. This type of action comes from a generalized $U(1) \rightarrow \mathbb{Z}_n$ Higgs transition. Indeed, start with the Lagrangian

$$\mathcal{L}_{Higgs} = \frac{\rho^2}{2} |F_a + nA|^2, \quad \rho \rightarrow \infty \quad (617)$$

which describes the deep IR theory for a charge n $(q-1)$ -form field which has been Higgsed. Because we are interested in the $\rho \rightarrow \infty$ limit, we can freely add a term $\frac{\rho^*}{2} |H|^2$ to \mathcal{L} , where $\rho^* = \frac{1}{2\pi\rho}$. Now make the shift $\delta H = (\rho^*)^{-1} i \star (F_a + nA)$ (notice that we couldn't make such a shift if we were imposing a quantization condition on H 's periods). This gives

$$\mathcal{L} = \frac{\rho^*}{2} |H|^2 + \frac{i}{2\pi} H \wedge (F_a + nA), \quad (618)$$

²⁰This is a no-no since $2\pi\mathbb{Z}$ is not a ring with the usual multiplication (e.g. $4\pi^2 \notin 2\pi\mathbb{Z}$), and so the cup product / wedge product operation no longer works.

²¹Large gauge transformations on a q -form gauge field are transformations $A \mapsto A + \alpha$, where α integrates to an element of $2\pi\mathbb{Z}$ on every closed q -manifold (and as such is a closed form). Note that α is a globally well-defined q form, although it may not be exact. Large gauge transformations of course never change the field strength and do not change the global well-definedness of A , despite some statements in the literature along the lines of “large gauge transformations take you between different magnetic flux sectors”.

²²In Anton and Seiberg's paper, H is taken to be quantized as

$$\int_{M_{D-q}} H \in 2\pi\mathbb{Z}, \quad (616)$$

for any closed $D-q$ submanifold M_{D-q} . Thus in this case, we can think of H as the field strength for some $U(1)$ $(D-q-1)$ -form gauge field. I think we actually don't want to put this restriction on H though—if we do then the a field literally serves no purpose in the action, and H cannot come from a field arising from manipulating a Higgs theory (more on this in a sec).

which goes over to the action we wrote above for $\rho \rightarrow \infty$.

Anyway, using this action, we can first do the integral over a : this tells us that H must be closed, and it sets the periods of H to be in $2\pi\mathbb{Z}$. Since the field strength of H vanishes (and we are assuming the spacetime manifold is torsion-free), we can globally Hodge-decompose it as

$$H = d\alpha_H + \omega_H, \quad (619)$$

where ω_H is harmonic and where the absence of a co-exact part comes from the fact that H must be closed. Up to constants relating $\mathcal{D}H$ to $\mathcal{D}\alpha_H$ and prefactors depending on $\dim H^{D-q}(X; \mathbb{Z})$, the path integral is

$$Z = \int \mathcal{D}A \mathcal{D}\alpha_H \sum_{\omega_H \in 2\pi H^{D-q}(X; \mathbb{Z})} \exp \left(\frac{i}{2\pi} \int_X (d\alpha_H + \omega_H) \wedge nA \right). \quad (620)$$

If $\partial X \neq 0$, the cohomology group becomes the relative cohomology $H^{D-q}(X, \partial X; \mathbb{Z})$.

The integral over α_H eliminates the local degrees of freedom (it sets the curvature of A to zero), while because the periods of ω_H over any closed manifold are in $2\pi\mathbb{Z}$, the sum over cohomology classes acts as a δ function setting

$$\int_{M_q} A \in \frac{2\pi}{n} \mathbb{Z} \quad (621)$$

for all closed M_q ²³.

Thus the Lagrangian of the theory is zero after integrating out H and a , and A is set to be a flat \mathbb{Z}_n connection.

We can also get to this conclusion by integrating over H first: this tells us that n copies of A need to be exact; this is another way of saying that A is a \mathbb{Z}_n gauge field.

Symmetries: What are the gauge transformations and the symmetries of the action? We obviously have the gauge transformation $a \mapsto a + d\lambda_{q-2}$. We also need to have gauge transformations on A as well, but in order for these to leave the action invariant, we need F_a to shift as well. So, the fields are tied together in the way they transform, and gauge transformations act as

$$a \mapsto a + d\lambda_{q-2} - n\lambda_{q-1}, \quad A \mapsto A + d\lambda_{q-1}. \quad (624)$$

²³To see this more carefully, by Poincare duality we can write any $\omega_H \in 2\pi H^{D-q}(X; \mathbb{Z})$ as

$$\omega_H = \sum_{c \in H_q(X; \mathbb{Z})} 2\pi m_c \hat{c}, \quad (622)$$

where $m_c \in \mathbb{Z}$ and the hat indicates the Poincare dual. When we put this in the path integral, we get something like (assuming the homology of X is torsion-free for simplicity)

$$\sum_{\{m_c\} \in \mathbb{Z}^{\dim H_q(X; \mathbb{Z})}} \prod_c \exp \left(i \int_c nA \right), \quad (623)$$

where the product is over all homology classes in $H_q(X; \mathbb{Z})$. This acts as a bunch of δ functions which set $n \int_c A \in 2\pi\mathbb{Z}$ for all closed q -manifolds c .

This is a local symmetry of the action, and so it really is a gauge symmetry. But notice that it shifts a by something which is not an exact form! This has consequences for what the global symmetries are.

Now for the global symmetries. First, we see that we can shift A by

$$A \mapsto A + \frac{1}{n}\epsilon_q, \quad \epsilon_q \in 2\pi H^q(X; \mathbb{Z}), \quad (625)$$

so that the integral of ϵ_q over any q -manifold is valued in $2\pi\mathbb{Z}$ (and hence ϵ_q is closed). Due to the n in the action, such a shift leaves the action invariant modulo $2\pi\mathbb{Z}$ provided we also shift the cohomology class of F_a appropriately, which is allowed since we are summing over all $[F_a]$ in the path integral.²⁴ This means that we have a global \mathbb{Z}_n q -form symmetry.

It also looks like we have a global $(q-1)$ -form symmetry, since if we shift $a \mapsto a + \epsilon_{q-1}$, where ϵ_{q-1} is any flat form (with periods equal to arbitrary elements of $\mathbb{R}/2\pi\mathbb{Z}$), then S is left invariant, since it only depends on F_a . What would be the charged objects under this symmetry? Of course, they would be the Wilson loops for a , namely $W_a(M_{q-1}) = \exp(i \int_{M_{q-1}} a)$. Normally, Wilson loop operators are gauge invariant, since they contain integrals over closed surfaces and since gauge transformations shift $U(1)$ gauge fields by exact forms. This is no longer true however, since we have a gauge transformation $a \mapsto a + n\lambda_{q-1}$, where λ_{q-1} is not exact and has no quantization conditions on its periods. Thus the $W_a(M_{q-1})$ Wilson loops are actually not gauge invariant, and the $U(1)$ higher symmetry actually does not exist.

A brief aside: one might be tempted to make $W_a(M_{q-1})$ gauge invariant by attaching a q -manifold M_q to it (with $\partial M_q = M_{q-1}$), and adding an integral of A over M_q to the Wilson operator. There are two problems with this: first, this is only possible if M_{q-1} is homologically trivial, in which case $W_a(M_{q-1})$ can not be charged under a higher symmetry in the first place. Second, the upgraded Wilson operator would be a trivial operator since it would be constructed using an integral of $F_a + nA$ over M_q , but $F_a + nA$ is trivial (which one sees by integrating out H).

So far we have only identified a single \mathbb{Z}_n q -form symmetry, but it turns out that there is another hidden higher \mathbb{Z}_n symmetry. It's easiest to see in the second formulation, so we'll come back to it after we've discussed the second formulation.

Second formulation: To get the second presentation of the action, we can “dualize” a from a $q-1$ form gauge field to a $D-(q-1)-2 = D-q-1$ form gauge field B .

Stupid way: This way is longer, but is what usually goes down during dualization. We'll go through it here just to make sure that the second formulation is actually obtained by dualization.

We do the dualization by adding in a new q form gauge field G and a $D-q-1$ form gauge field B , so that we get the hard-to-look-at expression

$$Z = \int \mathcal{D}A \mathcal{D}H \mathcal{D}G \mathcal{D}B \mathcal{D}a \exp \left(\frac{i}{2\pi} \int_X (H \wedge (F_a - G + nA) + F_B \wedge G) \right), \quad (626)$$

where in the path integral we are summing over all possible bundles for all the fields²⁵ except for G , which is a globally well-defined form (i.e. dG really is trivial in cohomology).

²⁴Note that we wouldn't also have to shift F_a if we had originally taken the periods of H to be quantized in $2\pi\mathbb{Z}$.

²⁵e.g. all $U(1)$ -bundles for A if A is a 1-form, all $U(1)$ gerbes for A if A is a 2-form, etc.

Adding all these fields hasn't actually done anything, which we can see by integrating out B : the globally well-defined part of B gives a δ function setting $dG = 0$, while the sum over cohomology classes of $F_B \in 2\pi H^{D-q}(X; \mathbb{Z})$ sets $\int_{M_q} G \in 2\pi\mathbb{Z}$ for all M_q . This means that G is the exterior derivative of some $q-1$ form gauge field, and so we can send $F_a - G \mapsto F_a$ by a field redefinition on a , recovering the original action.

Now we Hodge decompose G as

$$G = d\alpha_G + d^\dagger \beta_G + \omega_G. \quad (627)$$

We gauge-fix G by setting the exact component of the Hodge decomposition to be the exact part of F_a . Since we are summing over all cohomology classes for G in the path integral, the cohomologically nontrivial part of F_a can be absorbed by shifting ω_G .²⁶ This eliminates a from the theory entirely.

We then do the path integral over H , which sets $G = nA$. So finally we get the BF action in its usual form, namely

$$S = \frac{in}{2\pi} \int_X B \wedge F_A. \quad (628)$$

Again, in this presentation, both F_A and F_B have periods in $2\pi\mathbb{Z}$. Technically, to write it like this we have integrated by parts, trading the integral over $\frac{i}{2\pi} F_B \wedge A$ for one over $\frac{i}{2\pi} B \wedge F_A$. This can be done since although the two integrals are not equal, they differ by an element of $2\pi\mathbb{Z}$ (this is best thought about with DB cocycles — more on this later).

Fast way: Starting from the first formulation (615), just integrate out a directly! As we have explained, the integral over the globally well-defined part of a sets $dH = 0$, while the sum over cohomology classes for F_a enforces the quantization of the periods of H . Thus after integrating out a we can write H as the field strength of a $U(1)$ gauge field, $H = F_B$, which after integrating by parts gives (628).

Symmetries: Let's now check the symmetries in this formulation. The gauge symmetries are just shifts in B and A by exact forms, and we have no gauge symmetries that act on both fields as we had in the first formulation.

As for the global symmetries, we still have the \mathbb{Z}_n q -form symmetry coming from shifting A , as we must. Note that naively looking at the action, we might conclude that we in fact have a $U(1)$ q -form symmetry, since the action only contains F_A which doesn't change under shifting A by a flat q -form, no matter what the holonomy of the flat q -form is. This isn't true though, and the actual symmetry is discrete: one of the ways to see this is to integrate by parts and write the integrand as $F_B \wedge A$, in which the \mathbb{Z}_n character of the shift symmetry on A is manifest. But really, one should formulate the integral using DB cohomology (more on this to come). The advantage of this presentation is that the other global symmetry manifests itself as a $(D-q-1)$ -form \mathbb{Z}_n symmetry coming from shifting B as

$$B \mapsto B + \frac{1}{n} \epsilon_{D-q-1}, \quad \epsilon_{D-q-1} \in 2\pi H^{D-q-1}(X; 2\mathbb{Z}). \quad (629)$$

This is a symmetry because of the quantization on F_A .

²⁶Such a shift is fine since although the shift changes the $F_B \wedge G$ term, it changes it by something in $2\pi\mathbb{Z}$ due to the quantization condition on F_B .

Third formulation: The final presentation of the action is a “magnetic” one, in which we treat F_B as its own variable, and enforce the fact that is actually the curvature of B with a Lagrange multiplier \tilde{B} . So we write the partition function as

$$Z = \int \mathcal{D}B \mathcal{D}F \mathcal{D}\tilde{A} \exp\left(\frac{i}{2\pi} \int_X F \wedge (F_{\tilde{A}} + nB)\right). \quad (630)$$

Hopefully the notation here is clear: $F_{\tilde{A}}$ is the field strength of \tilde{A} , which is locally $d\tilde{A}$, while F is its own independent field (not necessarily the field strength of any q -connection). As we have seen several times already, the integral over the globally-defined part of \tilde{A} sets $dF = 0$, while the summation over the cohomology classes for $F_{\tilde{A}}$ enforce Dirac quantization on F . This means that integrating out \tilde{A} sets F to be the curvature of a q -form gauge field, and so indeed this presentation is equivalent to the usual BF action (note that this is similar in form to, but not exactly the same as, the first presentation).

Symmetries: Let’s now check the symmetries in this formulation. The gauge symmetries, like in the first formulation, act on two fields simultaneously. First, we can shift \tilde{A} by an exact form. Second, we can shift B by $d\lambda_{D-q-2}$ while also shifting \tilde{A} by $-n\lambda_{D-q-2}$.

Similarly to our analysis of the first formulation, we see that B has a $(D - q - 1)$ -form \mathbb{Z}_n global symmetry. One might think that we have a higher symmetry corresponding to constant shifts in \tilde{A} , but since $\exp(i \int \tilde{A})$ is not gauge-invariant, this is not so. We also have the \mathbb{Z}_n q -form symmetry identified earlier, but it is “hidden” in this presentation.

43 May 26 — Writhe and linking numbers

Write down an integral formula for the writhe $w(L)$ of a link in S^3 . If the link has components K_i , the appropriate formula to reproduce is

$$w(L) = \sum_{i,j} \mathcal{L}(K_i, K_j) + \sum_i w(K_i), \quad (631)$$

where \mathcal{L} is the linking number and $w(K_i)$ is the writhe of the K_i link component.

Solution:

The linking number $\mathcal{L}(K_i, K_j)$ can be conveniently defined as the signed intersection number of K_i with the Seifert surface of K_j (or vice versa). Recall that the Seifert surface of a knot is a (possibly non-unique) *orientable* manifold $M_j \subset S^3$ that bounds K_j , so that $\partial M_j = K_j$. Since the intersection product of manifolds is Poincare dual to the wedge product, we guess that the correct formula will come from taking wedge products between the Poincare duals of the knots K_i and the Seifert surfaces.

To this end, let S_i be the Seifert surface of K_i . Its Poincare dual in S^3 is a 1-form $A_i = \hat{S}_i$. To deal with the write of each link component properly, we need to flatten the link components into solid tori of radius r_i (we do this so that after flattening, none of the flattened

K_i intersect one another). Let $B_i = \widehat{K}_i$ be the Poincare dual of the solid tori K_i . In the usual way (Thom classes and stuff) we can choose B_i to be a bump function oriented orthogonally to K_i , with total integral 1 in any plane intersecting K_i transversely. If $\eta_i : S^3 \rightarrow K_i$ is the restriction, then

$$K_i = \eta_i^*(f_i(r)rdr \wedge d\theta), \quad (632)$$

where r, θ are the coordinates on a given cross section of the solid torus, and where $\int_0^{r_i} f_i(r)rdr \wedge d\theta = 1$.

Now, we claim that the answer for the writhe of the link L is the Chern-Simons-like result²⁷

$$w(L) = \int_{S^3} \sum_{i,j} A_i \wedge B_j. \quad (636)$$

To see this, let us first look at the $i \neq j$ terms. These give

$$\int_{S^3} A_i \wedge B_j = \int_{S^3} A_i \wedge \eta_j^*(f_j(r)rdr \wedge d\theta). \quad (637)$$

The integrand will be non-zero when the Seifert surface S_i meets the support of $\eta_j^*(f_j(r)rdr \wedge d\theta)$. If z is the coordinate along the non-bounding cycle of the solid torus K_j , then we can write this as

$$\int_{K_j} A_{i,z} f_j(r)rdr \wedge d\theta \wedge dz = \sum_{p \in S_i \cap K_j} \int \pm f_j(r)rdr \wedge d\theta = \sum_{p \in S_i \cap K_j} \text{sgn}(p). \quad (638)$$

Hopefully the notation is clear: the \pm signs are determined by whether the orientation of A_i agrees or disagrees with the normal vector determined by $\star \eta^*(dr \wedge d\theta)$ in S^3 . Thus for $i \neq j$ we have

$$\int_{S^3} A_i \wedge B_j = \mathcal{L}(K_i, K_j). \quad (639)$$

Now for the $i = j$ case. This is slightly trickier, since $S_i \cap K_i$ isn't well-defined. To handle this, we split apart a given solid torus K into a bunch of smaller solid tori K_λ , so that K looks like a coaxial cable or one of those cables that are used in e.g. suspension bridges. The writhe of K is then given by the linking number of any two distinct K_λ, K_γ (the decomposition is such that as the framing of K twists around in S^3 , all of the small component tori twist

²⁷That Chern-Simons theory is counting the linking number is not surprising. Partly because it's topological so what else could it be doing, but also because the 2-point functions for the gauge field A reproduce the Gauss formula for the linking number. To get the propagator we need to invert d , i.e. we need to solve

$$\epsilon^{\mu\nu\lambda} \partial_\lambda \Delta_{\mu\nu}(x - y) = \delta(x - y), \quad (633)$$

where the derivative acts on x . We can do this with

$$\Delta_{\mu\nu}(x - y) = \epsilon_{\mu\nu\sigma} \frac{(x - y)^\sigma}{4\pi|x - y|^3}, \quad (634)$$

which one can check using $\nabla \cdot \frac{r}{|r|^3} = 4\pi\delta(r)$. Thus we have

$$\left\langle \int_C A_\mu dx^\mu \int_{C'} A_\nu dy^\nu \right\rangle = \frac{1}{4\pi} \int_C \int_{C'} dx^\mu \wedge dy^\nu \epsilon_{\mu\nu\lambda} \frac{(x - y)^\lambda}{|x - y|^3} = \mathcal{L}(C, C') \quad (635)$$

where we've used Gauss' formula for the linking number.

around rigidly, maintaining a constant cross-section). Each K_λ has a Poincare dual 2-form that we write similarly as

$$K_\lambda = \eta_\lambda^*(f_\lambda(r)rdr \wedge d\theta), \quad (640)$$

where now $f_\lambda(r)$ does not integrate to $1/2\pi$, but rather $\sum_\lambda \int f_\lambda(r)rdr \wedge d\theta = 1$, so that the f_λ are a bump-function resolution of the identity for f . Now let A_λ be the Poincare dual to the Seifert surface of K_λ , except modified so that it is a delta function supported on S_λ with weight $\int f_\lambda(r)rdr \wedge d\theta \neq 1$. Then we have, for $\lambda \neq \gamma$,

$$\begin{aligned} \int_{S^3} A_\lambda \wedge B_\gamma &= \sum_{p \in S_\lambda \cap K_\gamma} \pm \left(\int f_\lambda(r)rdr \wedge d\theta \right) \left(\int f_\gamma(r)rdr \wedge d\theta \right) \\ &= w(K) \left(\int f_\lambda(r)rdr \wedge d\theta \right) \left(\int f_\gamma(r)rdr \wedge d\theta \right), \end{aligned} \quad (641)$$

where we used the fact that the linking number of any two distinct smaller component tori is equal to the write of the link K that we are focusing on.

The $\lambda = \gamma$ terms are still problematic, but if we decompose K into a larger and larger number N of constituent smaller solid tori, then since the individual values of $\int f_\lambda(r)rdr \wedge d\theta$ go to zero, and since there are N^2 $\lambda \neq \gamma$ terms but only N $\lambda = \gamma$ terms, in the $N \rightarrow \infty$ limit, we can ignore the $\gamma = \lambda$ terms. Thus

$$\sum_{\gamma, \lambda} \int A_\gamma \wedge B_\lambda = w(K) \left(\sum_\gamma \int f_\gamma(r)rdr \wedge d\theta \right) \left(\sum_\lambda \int f_\lambda(r)rdr \wedge d\theta \right) = w(K). \quad (642)$$

Since the $\{f_\lambda\}$ are a partition of unity for f , and $\int f(r)rdr \wedge d\theta = 1$.

So the $i \neq j$ terms give us the linking numbers of the various link components, while the $i = j$ terms give us the write of each component, and we get

$$w(L) = \sum_{i,j} \mathcal{L}(K_i, K_j) + \sum_i w(K_i), \quad (643)$$

which is what we wanted to show.

44 May 27 — σ model in two dimensions

Consider a σ model in two dimensions, where the target space is the sphere S^N . The action, after introducing the Lagrangian multiplier in the usual way, is

$$S = \frac{1}{2\lambda} \int d^2x (\partial_\mu \phi \partial^\mu \phi - i\lambda \sigma(\phi^2 - 1)). \quad (644)$$

Working in the large N limit, find the two point function for σ , namely $\langle \sigma(-q)\sigma(q) \rangle$. It will help to expand about the equilibrium constant value for σ . The expression should simplify in the large q limit.

Solution:

The equation of motion for ϕ is

$$\phi\Delta\phi = i\lambda\sigma, \quad (645)$$

where we have used $\phi^2 = 1$. Here, $\Delta = -\partial_\mu\partial^\mu$ is the Hodge Laplacian. From this, we expect on dimensional grounds that

$$\langle\sigma(x)\sigma(y)\rangle \sim \frac{1}{|x-y|^4}. \quad (646)$$

Fourier-transforming, we expect

$$\langle\sigma(-q)\sigma(q)\rangle \sim |q|^2. \quad (647)$$

Let's get the exact propagator. Integrating out ϕ , we have

$$Z = \int \mathcal{D}\sigma \exp\left(-\frac{N}{2} \text{Tr} \ln\left(\frac{\Delta}{\lambda} - i\sigma\right) - \frac{i}{2} \int \sigma\right). \quad (648)$$

Again, Δ is the Hodge Laplacian. Since $N \rightarrow \infty$ we can minimize with respect to $\sigma = \text{const}$ to find the equilibrium value of σ . Doing this tells us that

$$\frac{i}{2} = \frac{iN}{2} \int_p \frac{1}{p^2/\lambda - i\sigma}, \quad (649)$$

i.e. that

$$1 = \frac{N}{4\pi} \ln\left(-\Lambda^2/i\lambda\sigma\right), \quad (650)$$

so that the equilibrium value for the mass of ϕ is determined by

$$\sigma = i \frac{M^2}{\lambda}, \quad M^2 = \Lambda^2 e^{-4\pi/N\lambda}. \quad (651)$$

To get the inverse propagator for σ , we just need to find the term in the effective action quadratic in σ . We vary σ as $\sigma = iM^2/\lambda + \delta\sigma$, and isolate the piece in the effective action quadratic in $\delta\sigma$. This comes from the usual polarization bubble type of Feynman diagram. Let Γ be the effective action for σ that we wrote down above. Then

$$\begin{aligned} \frac{\delta^2\Gamma}{\delta\sigma(-q)\delta\sigma(q)} \Big|_{\sigma=iM^2/\lambda} &= -\frac{\delta^2}{\delta\sigma(-q)\delta\sigma(q)} \frac{N}{2} \text{Tr} \left[\frac{1}{\Delta^2 + M^2} i\lambda\delta\sigma \frac{1}{\Delta^2 + M^2} i\lambda\delta\sigma \right] \\ &= \frac{\delta^2}{\delta\sigma(-q)\delta\sigma(q)} \frac{N\lambda^2}{2} \int_{x,y} G(x,y)\delta\sigma(x)G(y,x)\delta\sigma(y), \end{aligned} \quad (652)$$

where G is the Greens function for the massive scalar. This gives

$$\begin{aligned} \frac{\delta^2\Gamma}{\delta\sigma(-q)\delta\sigma(q)} \Big|_{\sigma=iM^2/\lambda} &= N\lambda^2 \int_p \frac{1}{(p^2 + M^2)((p-q)^2 + M^2)} \\ &= N\lambda^2 \int_p \int_0^1 dx \frac{1}{[p^2 + M^2 + x(1-x)q^2]^2} \\ &= \frac{N\lambda^2}{4\pi} \int_0^1 dx \frac{1}{M^2 + x(1-x)q^2} \\ &= \frac{N\lambda^2}{4\pi} \frac{\ln\left(1 + \frac{q(q+\sqrt{4M^2+q^2})}{2M^2}\right)}{q\sqrt{4M^2+q^2}}. \end{aligned} \quad (653)$$

If we take the $q^2 \gg M^2$ limit, things simplify, and the RHS goes to $N\lambda^2 \ln(q^2/M^2)/(4\pi q^2)$. This is the inverse propagator, so the propagator in this limit is thus

$$\langle \sigma(-q)\sigma(q) \rangle = \frac{4\pi}{N\lambda^2} \frac{q^2}{\ln(q^2/M^2)}. \quad (654)$$

45 May 28 — Fun with two dimensional bosonization, the Schwinger model, and θ angles

This is an elaboration on one of the problems in Quantum Fields & Strings, a Course for Mathematicians, Vol II. Consider two flavors of massive fermions in two dimensions coupled to a $U(1)$ gauge field with a θ term, in Euclidean signature:

$$S = \frac{1}{2\pi} \int \left(\sum_i \bar{\psi}_i \not{D}_A \psi_i + \bar{\psi}_i m_i \psi_i \right) + \frac{1}{2e^2} \int F \wedge \star F + \frac{i\theta}{2\pi} \int F. \quad (655)$$

The factor of $1/2\pi$ in front of the fermions is there so that the fermion correlators will have no annoying prefactors. Here the mass term is complex, so that in the representation where $\gamma^0 = X$ and $\psi_i = (\psi_{i,+}, \psi_{i,-})^T$,

$$\bar{\psi}_i m_i \psi_i = m \psi_{i,+}^\dagger \psi_{i,-} + m^* \psi_{i,-}^\dagger \psi_{i,+}. \quad (656)$$

Show that the θ term can be eliminated when either of the fermions are massless. What is the effect of taking one of the masses to ∞ but leaving the other finite? Describe what happens in the limit when both m_i are small, and when both are large.

Solution:

Lets first do a review of how the bosonization goes down for a single flavor of fermions, not coupled to a gauge field. We work in Euclidean signature, with gamma matrices $\gamma^0 = X, \gamma^1 = Y$. Thus the Dirac operator is

$$\not{D} = \begin{pmatrix} 0 & \partial_+ \\ \partial_- & 0 \end{pmatrix}, \quad \partial_\pm = \partial_0 \mp i\partial_1. \quad (657)$$

Note the “reversed” signs! This is so that ∂_+ is a “left-moving” derivative and ∂_- is a “right-moving” derivative. The classical eom are $\partial_+ \psi_- = \partial_- \psi_+ = 0$, so that classically ψ_- is right-moving and ψ_+ is left-moving, since ∂_- kills things that are only a function of $t + ix$ (which we regard as left-moving), while ∂_+ kills things that are only a function of $t - ix$ (which are right-moving). This ensures that the chirality operator Z counts left-movers minus right-movers, so the helpful thing to remember is that $+$ things have positive chirality and hence move left (counterclockwise), while $-$ things have negative chirality and thus move right.

The correlator of the fermions is

$$D_{\alpha\beta}(x - y) = \delta_{\alpha\beta} \frac{x - y}{|x - y|^2}, \quad (658)$$

where x and y are complex coordinates in the plane. One can verify this by computing the derivative explicitly, or by noting that it is derivative of the free boson propagator (since if G is the free boson propagator, then schematically $\partial^2 G = \delta \implies D = \partial G$ satisfies $\partial D = \delta$). The factor of $1/2\pi$ in the Lagrangian was chosen so that no factors of 2π appear in the fermion propagator.

Thus from Wick's theorem and the anticommuting properties of fermions, the correlator of a bunch of α fermions is

$$\langle \psi_\alpha(x_1) \dots \psi_\alpha(x_n) \psi_\alpha^\dagger(y_1) \dots \psi_\alpha^\dagger(y_n) \rangle = \det \left(\frac{1}{x_i - y_j} \right). \quad (659)$$

Now let us consider the compact boson at radius R , with action

$$S = \frac{R^2}{4\pi} \int \partial_\mu \phi \partial^\mu \phi. \quad (660)$$

It will be helpful to decompose the boson into holomorphic and antiholomorphic parts as $\phi = \phi_+ + \phi_-$, where $\partial_+ \phi_- = \partial_- \phi_+ = 0$ (this is possible classically since these are the equations of motion—as far as I can tell we just stipulate that it can be done in general). Note that $\partial_+ \phi_+$ is still holomorphic and $\partial_- \phi_-$ is still antiholomorphic. The action is then

$$S = \frac{R^2}{4\pi} \int \partial_+ \phi_+ \partial_- \phi_-. \quad (661)$$

The operators that we have access to in general are polynomials in ϕ and vertex operators. We will assume for now that the operators $e^{i\phi_\pm}$ also make sense. Consider the correlator of e.g. a bunch of homolomorphic vertex operators. Since the two-point function of the compact boson is logarithmic, we get (recalling that the propagator for e.g. the holomorphic part is $-\frac{1}{2R^2} \ln(z - w)$)

$$\langle e^{i\phi_+(x_1)} \dots e^{i\phi_+(x_n)} e^{-i\phi_+(y_1)} \dots e^{-i\phi_+(y_n)} \rangle = \frac{\prod_{i < j < n} (x_i - x_j)^{1/2R^2} \prod_{i < j < n} (y_i - y_j)^{1/2R^2}}{\prod_{i < j < n} (x_i - y_j)^{1/2R^2}}. \quad (662)$$

This is because the correlators for xy terms have a positive coefficient in the exponential after moving the expectation value to the exponential, while the xx and yy correlators have a negative coefficient. The starting point of bosonization is that this vertex operator correlator is exactly equal to the correlator for fermions when $R = 1/\sqrt{2}$. One can show this e.g. by looking at the zeros and the poles: both functions have poles when some x_i equals some y_j , and both have zeros when two x 's or two y 's are coincident (since the matrix in the determinant becomes degenerate). Thus we have the mapping

$$\psi_\pm \leftrightarrow e^{\mp i\phi_\pm}. \quad (663)$$

The \mp sign in the exponential is a convention, but one which is crucial to keep track of (I hope I have done a correct job of this). This convention means that the fermion Dirac mass

will map to a $\cos(\phi)$ term, so that ϕ describes the fermion density. The other way of doing it is to take $\psi_{\pm} \leftrightarrow e^{-i\phi_{\pm}}$, for which the Dirac mass becomes $\cos(\theta)$, where $\theta = \phi_+ - \phi_-$ is the dual field. Note also that we are forgetting (on purpose) to write a factor of $a^{-1/2}$ on the RHS (where a is a UV cutoff) to make things dimensionally correct. Forgetting to do this is fine as long as we simultaneously forget to make the arguments of the logarithms in the boson propagators dimensionally incorrect.

Let's see where different fermion bilinears go (at $R = 1/\sqrt{2}$). We have (x and y are holomorphic coordinates—sorry for the notation)

$$\begin{aligned} : \psi_+^\dagger(x) \psi_+(x) : &= \lim_{x \rightarrow y} (\psi_+^\dagger(x) \psi_+(y) - D_{++}(x-y)) \\ &\rightarrow \lim_{x \rightarrow y} (e^{-i\phi_+(x)} e^{i\phi_+(y)} - D_{++}(x-y)) \\ &= \lim_{x \rightarrow y} \left(: e^{-i\phi_+(x)} e^{i\phi_+(y)} : \frac{1}{x-y} - \frac{1}{x-y} \right) \\ &= \lim_{x \rightarrow y} \left([1 + (i\partial_+ \phi_+)(y-x)] \frac{1}{x-y} - \frac{1}{x-y} \right) \\ &= -i\partial_+ \phi_+(x), \end{aligned} \tag{664}$$

where $D_{++}(x-y) = \frac{1}{x-y}$ is the fermion propagator, which has no annoying numerical coefficient because of how we've written the free fermion action. We've also used the normal-ordering identity

$$e^X e^Y =: e^X e^Y : e^{\langle XY + \frac{1}{2}(X^2 + Y^2) \rangle}, \tag{665}$$

which can be proved by writing down the series expansions and doing a bit of algebra (remember that the normal-ordering gets rid of *all* contractions between the two operators; for the vertex operators there are infinitely many such contractions to take into account). This calculation illustrates the general theme: we want to bosonize the normal-ordered product of some fermion operators, but our bosonization map only works on the full (un-normal-ordered) operators. So, we un-normal-order, use the bosonization map, and then re-write things in terms of normal-ordered products so that we can do the point-splitting appropriately (remember that Taylor expansions can only be performed *inside* the normal-ordering symbol). Similarly, one finds

$$: \psi_-^\dagger(x) \psi_-(x) := i\partial_- \phi_-(x). \tag{666}$$

Note that this means that quadratic fermion interactions get mapped to the kinetic term on the boson side, meaning that a fermion theory that is naively interacting is actually free: schematically,

$$\psi_+^\dagger \psi_-^\dagger \psi_+ \psi_- \leftrightarrow \partial_\mu \phi \partial^\mu \phi. \tag{667}$$

The off-diagonal bilinears are easy, since $D_{\pm\mp}(x-y) = 0$ means that the OPE is simple. We get

$$: \psi_-(x) \psi_+^\dagger(x) := e^{-i\phi}, \quad : \psi_+(x) \psi_-^\dagger(x) := e^{i\phi}. \tag{668}$$

Finally for the Dirac kinetic term, which is a bit trickier. We will just look at $\psi_+^\dagger \partial_x \psi_+$ for simplicity, which is a bit easier than staying in complex coordinates the whole time. We first need to know that at equal times (I think the sign is right),

$$\langle \psi_+^\dagger(x) (\partial_y \psi_+(y)) \rangle = \partial_y \frac{1}{ix - iy} = -i \frac{1}{(x-y)^2}. \tag{669}$$

Following the familiar strategy,

$$\begin{aligned} :\psi_+^\dagger \partial_x \psi_+ : &= \lim_{\epsilon \rightarrow 0} \left(\psi_+^\dagger(x + \epsilon) \partial_x \psi_+(x) + \frac{i}{\epsilon^2} \right) \\ &\rightarrow \lim_{\epsilon \rightarrow 0} \left(-ie^{i\phi_+(x+\epsilon)} \partial_x \phi_+(x) e^{-i\phi_+(x)} + \frac{i}{\epsilon^2} \right) \end{aligned} \quad (670)$$

By writing out the series expansions for the exponentials, the operator product inside the limit is, using the identity (665) and writing $V(x) = e^{i\phi_+(x)}$ for the vertex operator and remembering that $\langle \phi_+(z)\phi_+(w) \rangle = -\ln(z-w)$ at our choice of boson radius,

$$\begin{aligned} e^{i\phi_+(x+\epsilon)} \partial_x \phi_+(x) e^{-i\phi_+(x)} &= \partial_x \phi(x) : V(x + \epsilon) V^*(x) : \frac{-i}{\epsilon} \\ &\quad + \text{things where } \partial_x \phi_+ \text{ is contracted with } V(x + \epsilon) \\ &= \partial_x \phi(x) : V(x + \epsilon) V^*(x) : \frac{-i}{\epsilon} + \langle \phi_+(x + \epsilon) \partial_x \phi_+(x) \rangle \\ &\quad \times \left(iV^*(x) - \phi_+(x + \epsilon) V^*(x) - \frac{i}{2} \phi_+^2(x + \epsilon) V^*(x) + \dots \right) \\ &= \partial_x \phi(x) : V(x + \epsilon) V^*(x) : \frac{-i}{\epsilon} + i \frac{-i}{\epsilon} V(x + \epsilon) V^*(x) \\ &= \partial_x \phi(x) : V(x + \epsilon) V^*(x) : \frac{-i}{\epsilon} + \frac{1}{\epsilon^2} : V(x + \epsilon) V^*(x) : \\ &\approx \frac{-i}{\epsilon} \partial_x \phi_+ (1 + i\epsilon \partial_x \phi_+(x)) + \frac{1}{\epsilon^2} (1 + i\epsilon \partial_x \phi_+(x)) \\ &= (\partial_x \phi_+(x))^2 + \frac{1}{\epsilon^2}. \end{aligned} \quad (671)$$

Here we have taken $\epsilon \rightarrow 0$ and dropped the $\epsilon^2 \partial_x^2 \phi_+$ term in the expansion, on the grounds that when multiplied by the $1/\epsilon$ it is non-singular and also a total derivative. Putting this back into the earlier formula we see that,

$$:\psi_+^\dagger i \partial_x \psi_+ : \rightarrow (\partial_x \phi_+)^2. \quad (672)$$

This gets us part of the kinetic term. The rest of the kinetic term comes from the other derivative of ψ_+ and the derivatives of ψ_- in the similar way.

This mapping also tells us that the Hamiltonian is

$$H \propto \int dx \left[(\partial_x \phi_+)^2 + (\partial_x \phi_-)^2 \right]. \quad (673)$$

This suggests to us that $\partial_x \phi_\pm$ should be identified with canonical momenta. It turns out that these are the canonical momenta for ϕ_\pm . This fact is hard to see if we write the Lagrangian as $\partial_+ \phi \partial_- \phi$ —more on this in a later diary entry.

Note that the chiral symmetry maps to the shift symmetry in the boson model, and that the currents match (since the momentum $\partial_0 \phi$ generates $\phi \mapsto \phi + \alpha$ translations):

$$\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \leftrightarrow i \partial_0 \phi. \quad (674)$$

Very importantly for the present problem, the mappings above mean that we have the mapping of the mass term to vertex operators. There are a few ways of writing the mass

term. If we take $m_{\mathbb{C}} \in \mathbb{C}$, then we can write the mass term as $\bar{\psi}m_{\mathbb{C}}\psi$; what we really mean by this is

$$\bar{\psi}m_{\mathbb{C}}\psi = m_{\mathbb{C}}\psi_+^\dagger\psi_- + m_{\mathbb{C}}^*\psi_-^\dagger\psi_+. \quad (675)$$

We can also equivalently take two real parameters $m, m_5 \in \mathbb{R}$, and write

$$\bar{\psi}m_{\mathbb{C}}\psi = m\bar{\psi}\psi + m_5\bar{\psi}\gamma\psi = m(\psi_+^\dagger\psi_- + \psi_-^\dagger\psi_+) + m_5(\psi_+^\dagger\psi_- - \psi_-^\dagger\psi_+). \quad (676)$$

Either way, using the bosonization mapping we see that

$$\bar{\psi}m_{\mathbb{C}}\psi \mapsto m_{\mathbb{C}}e^{i\phi} + m_{\mathbb{C}}^*e^{-i\phi} = \Re[m_{\mathbb{C}}]\cos\phi + \Im[m_{\mathbb{C}}]\sin\phi. \quad (677)$$

In particular, when m is real, we get a sine-Gordon $\cos\phi$ interaction (even though the fermions are free!). Note that $[m] = 1$ while $[\bar{\psi}m\psi] = 2$, so the dimensions in the above formula look funny since one would be forgiven for thinking that $\cos\phi, \sin\phi$ are dimensionless. This is reconciled by the fact that at this value of the free scalar radius, the vertex operator $e^{i\phi}$ has anomalous dimension 1. One can see this just by calculating the two point function of the vertex operators, which as we have seen goes as $1/|x - y|^{1/2R^2}$, and so when $R = 1/\sqrt{2}$ the scaling dimension of the vertex operator is 1.

When we add a gauge field, not a lot changes. The Dirac operator gets upgraded to

$$\not{D}_A = \begin{pmatrix} 0 & \partial_+ + iA_+ \\ \partial_- + iA_- & 0 \end{pmatrix}, \quad \partial_\pm = \partial_0 \mp i\partial_1, \quad A_\pm = A_0 \mp iA_1. \quad (678)$$

This means that the coupling between A and the fermion current is

$$S \supset \frac{1}{2\pi} \int (A_+J_- + A_-J_+) dz \wedge d\bar{z}, \quad (679)$$

where $J_\pm = \psi_\pm^\dagger\psi_\pm$. This is the same as $A \wedge \star J$, since the metric in z, \bar{z} coordinates is off-diagonal.

When we do the mapping now, the kinetic term goes to

$$\frac{1}{2\pi} \int \bar{\psi} \not{D}_A \psi \mapsto \frac{1}{2\pi} \int \left(\frac{1}{4} \partial_+ \phi \partial_- \phi + A_+ \partial_- \phi - A_- \partial_+ \phi \right). \quad (680)$$

We can integrate the last two terms by parts, which yields $\frac{i}{2\pi} \phi F_A$, so that ϕ contributes to the θ angle. This expression makes sense even though ϕ is not a zero-form since the quantization condition on F_A means that such ambiguities only affect the action by something in $2\pi\mathbb{Z}$.

One quick comment on this: the term $\frac{1}{2\pi} \phi F_A$ is obviously not invariant under shifting ϕ by a constant, but naively its counterpart $A_+ \partial_- \phi - A_- \partial_+ \phi$ is invariant under the shift, since ϕ only appears under a derivative sign. Actually this conclusion is wrong, and the latter expression *does* change under shifting ϕ by a constant. The proper way to understand this is by using differential cohomology, and interpreting the term as

$$A_+ \star \phi_- - A_- \star \phi_+, \quad (681)$$

where \star is the product operation in Deligne-Bellissimo cohomology. Since the ϕF term is not invariant under the shift symmetry of ϕ and since the shift symmetry of ϕ is the chiral

symmetry on the fermion side, this non-invariance brings out the chiral anomaly from the integration measure and makes it more explicit though the non-invariance of the action.

Now consider making the fermions massive with a mass term $m\bar{\psi}\psi + m_5\bar{\psi}\gamma_5\psi$. We will shortly be looking at T symmetry, so for clarity we will momentarily switch to \mathbb{R} time, with $\gamma^0 = -iY \equiv J, \gamma^1 = X$. We then get

$$\frac{1}{2\pi} \int \bar{\psi} iD_A \psi \mapsto \frac{1}{2\pi} \int \left(\frac{1}{4} \partial_+ \phi \partial_- \phi + m \cos \phi + m_5 \sin \phi + i\phi F_A \right). \quad (682)$$

Let's now look at how T acts. On the fermions,

We can now generalize to the multi-species case given in the introduction to this diary entry. The bosonic theory is (setting the radius of both bosons equal to $1/\sqrt{2}$ since this is required by the mapping)

$$S = \frac{1}{8\pi} \int \sum_i (1 + g_i) \partial_+ \phi_i \partial_- \phi_i + \sum_i \int (m_i e^{i\phi_i} + m_i^* e^{-i\phi_i}) \\ + \frac{1}{2e^2} \int F_A \wedge \star F_A + \frac{i(\theta + \phi_1 + \phi_2)}{2\pi} \int F_A. \quad (683)$$

Here we have added interactions for both of the fermions, in the form

$$S_{int} = \frac{1}{2\pi} \sum_i g_i \int \psi_{+,i}^\dagger \psi_{-,i}^\dagger \psi_{+,i} \psi_{-,i}, \quad (684)$$

which as we saw bosonizes to the free kinetic term.

If at least one of the $m_i = 0$, one of the boson fields has no $e^{i\phi_i}$ interaction, and so we can perform the shift $\phi_i \mapsto \phi_i - \theta$ to eliminate the θ dependence from the action. Hence there is a θ dependence only when *both* fermions are massive.

Consider when both fermion masses are small, $m_i \rightarrow 0$. Also for simplicity, let $g_1 = g_2 = g$. Since when $m = 0$ the theory is quadratic, we can just look at the equations of motion and because of the masslessness, we can ignore the θ dependence. Now define

$$\varphi_\pm = \frac{1}{2}(\phi_1 \pm \phi_2), \quad (685)$$

so that the action is

$$S = \frac{1}{8\pi} \int \sum_{a=\pm} (1 + g_a) \partial_+ \varphi_a \partial_- \varphi_a + \frac{1}{2e^2} \int F_A \wedge \star F_A + \frac{i\varphi_+}{2\pi} \int F_A. \quad (686)$$

The eom for φ_- gives a regular massless wave equation. The equations of motion for A and for φ_+ are

$$-\frac{1}{e^2} \partial_\mu F^{\mu\nu} - \frac{i}{2\pi} \partial_\mu \varphi_+ \epsilon^{\mu\nu} = 0, \quad (687)$$

and

$$-\frac{1+g}{4\pi} \partial_\mu \partial^\mu \varphi_+ + \frac{i}{2\pi} F^{01} = 0. \quad (688)$$

Thus we get a massive wave equation for φ_+ :

$$(\square - m_\varphi^2) \varphi_+ = 0, \quad m_\varphi = \frac{e}{\sqrt{\pi(1+g)}}. \quad (689)$$

So we get one massive and one massless scalar. If we only were bosonizing a single fermion, then we would only have one boson, and the whole theory would be gapped. This is the bosonization way of seeing the main point of the Schwinger model: gauge invariance is not enough to guarantee massless states in two dimensions, since you can get confining gauge fields to do the job.

What happens when one of the fermion masses, say m_1 , goes to ∞ ? In perturbation theory, I don't think we can see that anything happens, other than e.g. the Maxwell term getting corrected by factor that goes to zero as $m_1 \rightarrow \infty$. However, on the bosonic side, we can argue as follows: first, perform a chiral rotation so that $m_1 \in \mathbb{R}^{<0}$. This shifts the θ term by $\pi - \arg(m)$. The mass is now real, and we get a $m_1 \cos(\phi_1)$ potential for the bosons. When $m_i \rightarrow -\infty$ we can take $\phi_1 \rightarrow 0$, and ϕ_1 disappears from the theory. Thus the effect of the heavy fermions is to shift

$$\theta \mapsto \theta + \pi - \arg(m). \quad (690)$$

46 May 29 — Careful explanation of CS level quantization

Explain why the Chern-Simons level is quantized in $U(1)$ gauge theory. Your argument should be valid on any manifold (e.g. even open ones), and should work even when $\int_{M_2} F = 0$ for all 2-submanifolds M_2 of spacetime.

Solution:

The usual explanations which everybody always repeats for why the CS coefficient is quantized are either “place the theory on $S^2 \times S^1$ with a unit of flux through the S^2 and look at large gauge transformations” or “realize it by integrating over a bulk 4-manifold and require independence of the extension”. These are both somewhat unsatisfying to me since I want to know why even on an open manifold (like e.g. all the ones we are interested in the real world; at least time will be \mathbb{R}), the CS term is quantized. That is, what is the reason that CS theory on say \mathbb{R}^3 only makes sense when the level is quantized?

We can answer this by looking more carefully at what $\int_X A \wedge dA$ really means. It is often stated that the integrand is only well-defined up to a total derivative, but in fact the ambiguity in the integrand is much more serious than that.

The correct way to think about things is by using DB cohomology (differential characters), which is essentially a way of defining gauge fields within the framework of the Čech-de-Rham bicomplex. Recall that a $U(1)$ gauge field actually consists of three pieces of data: for a given decomposition of X into patches U_α , this data includes A_α (1-forms on each patch), $\Lambda_{\alpha\beta}$ (\mathbb{R} -valued 0-forms on each double overlap), and $n_{\alpha\beta\gamma}$ (\mathbb{Z} -valued 0-forms on each triple overlap). They relate with one another by

$$\delta_0 A_{\alpha\beta} = A_\alpha - A_\beta = d_0 \Lambda_{\alpha\beta}, \quad \delta_1 \Lambda_{\alpha\beta} = 2\pi d_{-1} n_{\alpha\beta\gamma}, \quad (691)$$

where d_{-1} is just a suggestive way of writing the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and the δ 's are the Čech differentials. The transition functions in the bundle are $g_{\alpha\beta} = \exp(i\Lambda_{\alpha\beta})$, so that sending

$\Lambda_{\alpha\beta} \mapsto \Lambda_{\alpha\beta} + 2\pi m_{\alpha\beta}$ for $m_{\alpha\beta}$ valued in \mathbb{Z} does nothing (one can also check that this changes $n_{\alpha\beta\gamma}$ by a coboundary, and so doesn't affect the cohomology class of $n \in H_C^2(X; \mathbb{Z})$ (the C is for Čech cohomology). We can sum this up by writing A as the triple

$$A = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma}), \quad (692)$$

with

$$\delta_{-1}A = A_\alpha, \quad \delta_0A = d\Lambda_{\alpha\beta}, \quad \delta_1\Lambda_{\alpha\beta} = 2\pi d_{-1}n_{\alpha\beta\gamma}, \quad (693)$$

where $(\delta_{-1}A)_\alpha = A|_{U_\alpha} = A_\alpha$ is the restriction.

Morally speaking, this is kind of like doing a Hodge decomposition. The A_α part (a $(1, 0)$ form, i.e. de Rham degree 1 and Čech degree 0) keeps track of the local curvature (the field strength), while the $\Lambda_{\alpha\beta}$ part (a $(0, 1)$ form) keeps track of the holonomy of the gauge field around non-contractible loops. We can see this because the holonomy is captured by a flat 1-form, i.e. an element $\lambda \in H^1(X; \mathbb{R})$. Since λ is globally well-defined, we can write it simply as

$$\lambda = (\delta_{-1}\lambda, 0, 0). \quad (694)$$

Alternately, since we are only interested in its holonomy, we can just as well write it as

$$\lambda = (0, f_{\alpha\beta}, 0), \quad (695)$$

where the $f_{\alpha\beta}$ are real-valued functions. The holonomy of λ around a given loop can be computed by summing up the $f_{\alpha\beta}$'s along 2-fold intersections of patches along the loop, as we will see later. Note that there is no $n_{\alpha\beta\gamma}$ term here because $\delta_1(f_{\alpha\beta}) = \delta_1(\delta_0\lambda)_{\alpha\beta} = 0$.

The $n_{\alpha\beta\gamma}$ part (a $(-1, 2)$ form) in the decomposition of A keeps track of the topology of the bundle (i.e. the Chern class). This is because, as we will see, the integral of F_A (the curvature of A) over a closed 2-manifold is given by a sum of the $n_{\alpha\beta\gamma}$'s.

Why is all this data needed in order to be able to do integrals? The philosophy is basically “we want the integrals we write down to be independent of the way in which we choose to decompose X into coordinate patches”. With that in mind, consider integrating the gauge field along a 1-cycle that starts at a point $a \in U_\alpha \setminus (U_\alpha \cap U_\beta)$ and ends at $b \in U_\beta \setminus (U_\alpha \cap U_\beta)$, with $U_\alpha \cap U_\beta$ non-empty. To define the integral, we need to integrate part of the way with A_α , and then the rest of the way with A_β . Suppose the transition point between these two is at $p \in U_\alpha \cap U_\beta$. Then tentatively our integral is

$$I(a, b; p) = \int_a^p A_\alpha + \int_p^b A_\beta. \quad (696)$$

The problem is that I is not independent of p ! Indeed, one can check that, for $q \in U_\alpha \cap U_\beta$, we have

$$I(a, b; p) - I(a, b; q) = \Lambda_{\alpha\beta}(p) - \Lambda_{\alpha\beta}(q). \quad (697)$$

The fix is to just add this transition function term into the integral. Thus, the following integral is independent of p :

$$I(a, b) = \int_a^p A_\alpha - \Lambda_{\alpha\beta}(p) + \int_p^b A_\beta. \quad (698)$$

However, recall that we need the shift $\Lambda_{\alpha\beta} \mapsto \Lambda_{\alpha\beta} + 2\pi m_{\alpha\beta}$ to not do anything. But, this shift changes the value of $I(a, b)$ by something in $2\pi\mathbb{Z}$! The only way to fix this is to ensure that the only time we write $I(a, b)$ is in exponentials as $\exp(iqI(a, b))$, where $q \in \mathbb{Z}$ (really we should be taking the integration to be over a closed cycle, but of course the same $2\pi\mathbb{Z}$ ambiguity still occurs). This is just another way of saying that the Wilson loop operators must be taken with integer charge. We know that if the Wilson loop wraps a nontrivial cycle then $q \in \mathbb{Z}$ is required by invariance under large gauge transformations, but here we are saying that *even for topologically trivial cycles*, the charge in the Wilson loop must be taken to be in \mathbb{Z} .

On a related note, this formulation lets us see why flat connections can be either specified as collections $(\lambda_\alpha, 0, 0)$, or entirely in terms of transition functions $(0, f_{\alpha\beta}, 0)$. Indeed, for the first formulation, we write the integral $\int_C \lambda$ for some cycle C as

$$\int_C \lambda = \sum_\alpha \int_{C_\alpha} \lambda_\alpha, \quad (699)$$

since the transition functions vanish. Hopefully the notation is clear: the C_α are the segments of C that lie in the patch U_α . On the other hand, since λ is closed and each U_α is contractible, we can write $\lambda_\alpha = d\omega_\alpha$ for some 0-forms ω_α , and so

$$\int_C \lambda = \sum_{p \in U_\alpha \cap U_\beta} (\omega_\beta(p) - \omega_\alpha(p)). \quad (700)$$

Thus if we define the transition functions $f_{\alpha\beta} = \omega_\alpha - \omega_\beta$, we see that if we only care about the holonomy of λ , we can just as well replace it with the collection $(0, f_{\alpha\beta}, 0)$ — the transition functions entirely determine the holonomy of closed forms.

The next step up in complexity comes from integrating F_A over a surface S . We need to do so in a way that doesn't depend on what sort of way we choose to cover the spacetime manifold with patches. Consider at first the case where S is closed. F_A is closed, and so on each patch it is exact (we can and will always take our patches, as well as their n -fold intersections, to be topologically trivial). So then we have

$$\int_S F_A = \sum_\alpha \int_{C_\alpha} dA_\alpha = \sum_\alpha \int_{\partial C_\alpha} A_\alpha. \quad (701)$$

Here, the $C_\alpha \subset U_\alpha$ are non-overlapping 2-chains contained in each of the patches, such that $\cup_\alpha C_\alpha = S$. Note that there are many ways of choosing the C_α , but different choices do not affect the integral, since they differ by integrals of the form $\int d(A_\alpha - A_\beta) = \int d^2 \Lambda_{\alpha\beta} = 0$. Additionally, we can always choose the C_α so that at most C_α meet at any given point (we can always choose the boundaries of the C_α to be a triangulation of S).

Returning to the integral of F_A over S , and assuming that S is orientable, we see that

$$\int_S F_A = \sum_{\alpha\beta} \int_{\partial C_\alpha \cap \partial C_\beta} (A_\alpha - A_\beta) = \sum_{\alpha\beta} \int_{\partial C_\alpha \cap \partial C_\beta} d\Lambda_{\alpha\beta}. \quad (702)$$

Each of the integrals in the above sum is over a line segment, and so each integral contributes a term like $\Lambda_{\alpha\beta}(b) - \Lambda_{\alpha\beta}(a)$. When we sum over all such line segments, we get three $\Lambda_{\alpha\beta}$

terms at each vertex (where three C_α 's meet), and they appear in the form $\delta_1 \Lambda_{\alpha\beta\gamma} = 2\pi n_{\alpha\beta\gamma}$. Thus we have

$$\int_S F_A = \sum_{\alpha\beta\gamma} 2\pi n_{\alpha\beta\gamma}. \quad (703)$$

This is why we said that the $n_{\alpha\beta\gamma}$ determine the topology of the bundle (if $\partial S \neq \emptyset$ then the only thing that changes is that we get an additional integral of A over the boundary of S). Note that in order to get a non-zero Chern class, the transition functions $\Lambda_{\alpha\beta}$ could not all be constant. Thus in order to create bundles which are twisted, it is not enough to just twist the transition functions by constants: we have to have “twisting” inside of double overlaps as well²⁸.

Let us finally now turn to Chern-Simons theory. Our naive guess of what the relevant integral is would be

$$\sum_\alpha \int_{C_\alpha} A_\alpha \wedge dA_\alpha, \quad (704)$$

where each C_α is now a 3-cycle, the C_α are all non-overlapping, and their union is the full spacetime X . This is not invariant under moving around the boundaries of the C_α though, which is a problem. When we wiggle one of the C_α boundaries, the difference in the integral as written above is an integral like

$$\int_{\delta C_{\alpha\beta}} (A_\alpha \wedge dA_\alpha - A_\beta \wedge dA_\beta) = \int_{\partial \delta C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_\beta, \quad (705)$$

where $\delta C_{\alpha\beta}$ is the volume enclosed by the two different choices of the boundary between C_α and C_β (I may add pictures at some point to make this clearer). Thus, we can take care of this ambiguity by adding in this term to the definition of the Chern-Simons integral, like like how we added $\Lambda_{\alpha\beta}(p)$ into the definition of the Wilson line integral. So, our improved integral for the CS action now looks like

$$\sum_\alpha \int_{C_\alpha} A_\alpha \wedge dA_\alpha - \sum_{\alpha\beta} \int_{C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_\beta, \quad (706)$$

where the $C_{\alpha\beta}$ are the 2-cells where the C_α 3-cells meet. Notice that to find the correction term to the naive $A_\alpha \wedge dA_\alpha$ term, we computed $\delta(A_\alpha \wedge dA_\alpha) = d\Lambda_{\alpha\beta} \wedge dA_\beta$, which we found to be a total derivative (we also used that $dA_\alpha = dA_\beta$). Thus when we took the Čech differential, we got something that was exact in de Rham cohomology. This is in keeping with the general Čech-de-Rham bicomplex structure of this whole construction.

Sadly, even the improved integral is not invariant under re-arranging the patches. Now we have to consider what happens when we wiggle one of the 1-cells $C_{\alpha\beta\gamma}$, which is a common boundary of three of the C_α 's. Drawing some pictures, one can convince oneself that for two choices of $C_{\alpha\beta\gamma}$ that differ by the surface $\delta C_{\alpha\beta\gamma}$, the term that we added to the naive CS integral changes by the term

$$-\int_{\delta C_{\alpha\beta\gamma}} (\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha}) \wedge dA_\gamma = -2\pi \int_{\partial \delta C_{\alpha\beta\gamma}} n_{\alpha\beta\gamma} \wedge A_\gamma, \quad (707)$$

²⁸Another way to say this is that if the transition functions are constants, we can choose a gauge in which the connection is flat: $A_\alpha = 0$ is a connection which obeys $A_\alpha = g_{\alpha\beta}^{-1}(A_\beta - d)g_{\alpha\beta}$. Flat connections can't have non-zero Chern class, and so we conclude that the transition functions need to not be constant if we are to get $\int F_A \neq 0$.

where we used that dA is the same on all three of the patches. Again, we see that the change in the integral is computed by taking a Čech differential, and that taking the differential gives us something exact in de Rham cohomology. Thus to cancel out *this* variation, we modify the CS action to

$$\sum_{\alpha} \int_{C_{\alpha}} A_{\alpha} \wedge dA_{\alpha} - \sum_{\alpha\beta} \int_{C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_{\beta} + 2\pi \sum_{\alpha\beta\gamma} \int_{C_{\alpha\beta\gamma}} n_{\alpha\beta\gamma} \wedge A_{\gamma}. \quad (708)$$

Even this isn't good enough, since we haven't looked at what happens when we wiggle around the 0-cells where four 3-cells meet. At this point, the pattern about how to fix the ambiguity should be clear: we take the Čech differential of the last term we added to the integral, find that we get an exact form, and then add that term back to the integral, but with opposite sign. Doing so gives the final form of the Lagrangian, and so the correct CS action is

$$S = \frac{k}{4\pi} I_{CS}; \\ I_{CS} = \sum_{\alpha} \int_{C_{\alpha}} A_{\alpha} \wedge dA_{\alpha} - \sum_{\alpha\beta} \int_{C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_{\beta} + 2\pi \sum_{\alpha\beta\gamma} \int n_{\alpha\beta\gamma} \wedge A_{\gamma} - 2\pi \sum_{\alpha\beta\gamma\sigma} \int n_{\alpha\beta\gamma} \wedge \Lambda_{\gamma\sigma}. \quad (709)$$

The integrals are evaluated on 3, 2, 1, and 0 cells, in turn.

Now we can note several things about this expression that aren't clear if we were to think of it as $A \wedge dA$. First, notice that I_{CS} is ambiguous up to elements of $(2\pi)^2 \mathbb{Z}$, even if the spacetime manifold is completely trivial and F_A is globally exact. This is because we have an equivalence $\Lambda_{\alpha\beta} \sim \Lambda_{\alpha\beta} + 2\pi m_{\alpha\beta}$ where $m_{\alpha\beta}$ takes values in \mathbb{Z} , which e.g. shifts the $2\pi \int n \wedge \Lambda$ term by something in $(2\pi)^2 \mathbb{Z}$ (the $\int n \wedge \Lambda$ term is really a sum over discrete points in the manifold). If we want S to be invariant modulo $2\pi \mathbb{Z}$, this forces the quantization of k even in topologically trivial scenarios. In fact, it forces $k \in 2\mathbb{Z}$ to be an *even* integer. We know that even levels describe bosonic systems, and this construction is only able to directly handle this case. For the fermionic case where we have a genuine spin TQFT and k is odd, we need a little bit more data: the spin structure needs to be introduced explicitly into the procedure described above, with minus signs coming from the spin structure cancelling out the extra minus signs that come from the above presentation of the action not being completely invariant under re-arrangements of the patches when k is odd. This is essentially the framing anomaly: the theory looks topological, but it actually retains a hidden dependence on the spin structure.

Also, note that changing A by a flat connection (a flat connection can be captured purely by transition functions, i.e. it can be written in the form $(0, f_{\alpha\beta}, 0)$) does *not* leave I_{CS} invariant (even if the spacetime is closed), contrary to what we would expect from writing the action as $A \wedge dA$. Indeed, we see that shifting A by a flat connection only leaves the action invariant if that flat connection is in $\Omega^1_{2\pi\mathbb{Z}/k}(X)$, i.e. only if the holonomy of the action is quantized in units of $2\pi/k$ for all 1-cycles of the spacetime. This is why CS theory has a \mathbb{Z}_k 1-form symmetry, and not a $U(1)$ 1-form symmetry. One again we stress that this argument works on topologically trivial spacetimes with globally exact field strengths.

The same argument shows that e.g. in the three-dimensional BF theory with action

$$\frac{i n}{2\pi} \int B \wedge F_A, \quad (710)$$

the action is *not* invariant under shifting A by a flat form λ , because of the presence of the $\int n_{\alpha\beta\gamma} \wedge \Lambda_{\gamma\sigma}$ correction term. If λ has holonomy $\exp(i \int_C \lambda) = \exp(2\pi i k/n)$ for all 1-cycles C and for $k \in \mathbb{Z}$ then the action changes only by an element in $2\pi i \mathbb{Z}$, and so we have a global \mathbb{Z}_n 1-form symmetry, but not a global $U(1)$ 1-form symmetry.

47 May 30 — RG and the nonlinear σ model

This is from Altland and Simons. Consider the $n\sigma m$ in two dimensions,

$$S = \frac{1}{\lambda} \int d^2x \text{Tr}[\partial_\mu g \partial^\mu g^{-1}]. \quad (711)$$

We will consider the case when an $SU(2)$ symmetry is reduced to a $U(1)$ subgroup (actually we are really interested in $O(3)$ being broken to $O(2)$, but since we will not be interested in global issues the $SU(2) \rightarrow U(1)$ problem is equivalent).

We will find the RG flow for λ by using the background field method. By breaking up g as $g = g_s g_f$ into “slow” and “fast” degrees of freedom and writing $g = \exp(iW)$, expand S to 2nd order in W . It will help to fix a “gauge” in which $g_s dg_s^{-1}$ anticommutes with Z (the third Pauli matrix). Integrate out the fast degrees of freedom, and find the beta function for λ .

Solution:

First let us do a bit of massaging to the action. We can essentially guess the form of the action based on the requirement of global $SU(2)$ invariance and local $U(1) \subset SU(2)$ invariance: we should get something that looks like a covariant derivative. The slow fields are being treated as background fields, so they should enter as the gauge field in the covariant derivative.

Let us write $g \in SU(2)$ by rotating away from the generator Z of the $U(1)$ unbroken subgroup as $g = g_s g_f Z g_f^{-1} g_s^{-1}$. We do this to make the fact that the action describes modes living in the coset space $SU(2)/U(1) = S^2$ manifest: Z is the generator of the unbroken $U(1)$, and $hZh^{-1} \subset U(1)$ for any h in the $U(1)$ subgroup. Let us define

$$\Gamma = g_f Z g_f^{-1}, \quad \Phi_\mu = g_s^{-1} \partial_\mu g_s. \quad (712)$$

Then

$$g_s^{-1} (\partial_\mu g) g_s = \Phi_\mu \Gamma + \partial_\mu \Gamma - \Gamma \Phi_\mu. \quad (713)$$

Putting this into the trace and using the cyclic property of the trace, we indeed get a covariant-derivative-looking action:

$$S = \frac{1}{\lambda} \int \text{Tr} (d\Gamma + [\Phi, \Gamma])^2. \quad (714)$$

It will be helpful to choose a gauge in which Φ is fixed to a particular form, which we can do using invariance of Γ under local $U(1)$ transformations. Let us take $U(1)$ to act on the left, with $g_f \mapsto hg_f$ for $h \in U(1)$. Since Z is the generator of the $U(1)$, doing this transformation

doesn't affect the $[\Phi, \Gamma]$ term (essentially by construction — the form of Γ was chosen so that it lives in the coset space $SU(2)/U(1)$). However, the derivative term changes:

$$d\Gamma \mapsto dh\Gamma - h\Gamma h^{-1}dhh^{-1}. \quad (715)$$

Putting this into the action and using the cyclic property of the trace to conjugate by h , we see that the action shifts to

$$S \mapsto \frac{1}{\lambda} \int \text{Tr} (d\Gamma + [h^{-1}(\Phi + d)h, \Gamma])^2, \quad (716)$$

which is exactly what we'd expect from a gauge transformation. We can then use this freedom to fix a gauge (at least locally) in which Φ is sort of in “unitary gauge”, in that it has no Z component. That is, we can fix Φ so that

$$\{\Phi_\mu, Z\} = 0. \quad (717)$$

This is accomplished I believe with the choice $h = \exp(i\psi(x)Z)$, where

$$\psi(x) = -i \int_{\bullet} dx^\mu \text{Tr}[Z\Phi_\mu] + \psi(\bullet), \quad (718)$$

for some basepoint \bullet .

Now let us finally break apart the action into a slow component, a fast component, and a mixed term that couples slow and fast degrees of freedom. We will parametrize g_f by

$$g_f = \exp(iW), \quad W = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \quad z \in \mathbb{C}. \quad (719)$$

Note that since W is built out of an X and a Y , this form of g_f is preserved by the action of the $U(1)$ subgroup by $h(x)$. One then sees that to second order in W ,

$$g_f = Z(1 - 2iW - 2W^2). \quad (720)$$

Since the form of W means that $ZWZ = -W$, the $(d\Gamma)^2$ term in the action becomes

$$S_f = \frac{(2i)^2}{\lambda} \int \text{Tr}[\partial_\mu WZ\partial^\mu WZ] = \frac{4}{\lambda} \int \text{Tr}[\partial_\mu W\partial^\mu W] = \frac{8}{\lambda} \int \partial_\mu z\partial^\mu \bar{z}. \quad (721)$$

Now the commutator appearing in the action is, to second order in W ,

$$[\Phi_\mu, \Gamma] = 2\Phi_\mu Z + 2iZ\{\Phi_\mu, W\} + 2Z\{\Phi_\mu, W^2\}, \quad (722)$$

where we used $Z\Phi_\mu Z = -\Phi_\mu$ by virtue of our gauge choice. Squaring this and working to quadratic order in W , we see that the slow mode part is

$$S_s = \frac{4}{\lambda} \int \text{Tr}[\Phi^2], \quad (723)$$

while the slow-fast coupling part is

$$S_{sf} = -\frac{4}{\lambda} \int \text{Tr}[Z(\Phi_\mu W + W\Phi_\mu)Z(\Phi_\mu W + W\Phi_\mu)] + \frac{8}{\lambda} \int \text{Tr}[\Phi_\mu\{\Phi_\mu, W^2\}]. \quad (724)$$

We can massage this into

$$S_{sf} = -\frac{8}{\lambda} \int \text{Tr} [\Phi_\mu W Z \Phi^\mu W Z] + \frac{8}{\lambda} \int \text{Tr} [W^2 \Phi^2], \quad (725)$$

or equivalently,

$$S_{sf} = -\frac{8}{\lambda} \int \text{Tr} [\Phi_\mu W \Phi^\mu W - W^2 \Phi^2]. \quad (726)$$

The mixed $d\Gamma$ and commutator cross-terms won't contribute since they are odd after we go to momentum space.

Now we take advantage of the specific form of W and Φ . Since Φ anticommutes with Z , we can write it as

$$\Phi_\mu = \begin{pmatrix} 0 & \phi_\mu \\ \bar{\phi}_\mu & 0 \end{pmatrix}, \quad (727)$$

Thus we get

$$S_{sf} = -\frac{8}{\lambda} \int (\phi^2 z^2 + \bar{\phi}^2 \bar{z}^2 - 2|\phi|^2 z \bar{z}). \quad (728)$$

To find the correction to the coupling constant induced by integrating out the fast modes, we will take $e^{-S_{sf}}$ and expand it as $1 - S_{sf} + \dots$, keeping terms of order W^2 . Thus we will be computing the expectation value of the above integral with respect to the free $\partial_\mu z \partial^\mu \bar{z}$ action. Since only the $\bar{z}_p z_p$ two-point function is non-zero, the first two terms in the above expression for S_{sf} will only have a non-zero expectation value at order $z^2 \bar{z}^2$, which we are dropping. Thus, after we expand the exponential, we may ignore the first two terms. Thus the relevant expectation value to compute is

$$\int \mathcal{D}z \mathcal{D}\bar{z} \left(\frac{8}{\lambda} \int_p \text{Tr}[\Phi^2] \bar{z}_p z_p \right) \exp \left(-\frac{8}{\lambda} \int_p p^2 z_p \bar{z}_p \right). \quad (729)$$

We find this to be

$$\begin{aligned} \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d \ln p}{2\pi} \text{Tr}[\Phi^2] &= -\frac{1}{2\pi} \ln \left(1 + \frac{d\Lambda}{\Lambda} \right) \text{Tr}[\Phi^2] \\ &= -\frac{\text{Tr}[\Phi^2]}{2\pi} d \ln \Lambda, \end{aligned} \quad (730)$$

where $\Lambda - \delta\Lambda$ is the boundary between the fast modes and the slow modes, and where $d\Lambda = -\delta\Lambda$ is negative, since the cutoff is being decreased.

Now we re-exponentiate to find that the new slow action is

$$S_s = \frac{4}{\lambda + d\lambda} \int \text{Tr}[\Phi^2], \quad (731)$$

where

$$\frac{4}{\lambda + d\lambda} = \frac{4}{\lambda} + \frac{d \ln \Lambda}{2\pi}. \quad (732)$$

We re-write this as

$$\lambda + d\lambda = \frac{\lambda}{1 + \frac{1}{8\pi} d \ln \Lambda} = \lambda - \frac{1}{8\pi} d \ln \Lambda, \quad (733)$$

and so the β function is

$$\frac{d\lambda}{d \ln \Lambda} = -\frac{\lambda^2}{8\pi}. \quad (734)$$

Thus we have asymptotic freedom, with weak coupling in the UV (as expected for a NLSM into a manifold with positive curvature, namely S^2) and with a strong-coupling disordered phase in the IR. In terms of the time flow along the RG trajectory, we can write

$$\frac{d\lambda}{dt} = \frac{\lambda^2}{8\pi}. \quad (735)$$

How special is the choice of dimension 2? To get some insight into this, we can consider doing the same calculation in dimension $d = 2 + \epsilon$. The relevant integral is then schematically

$$\int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^2 p}{4\pi^2} p^{-2} = \frac{1}{2\pi\epsilon} (\Lambda^{-\epsilon} - \Lambda^{-\epsilon}(1 - \epsilon d \ln \Lambda)). \quad (736)$$

The affect of the extra $\Lambda^{-\epsilon}$ is to just add a $\epsilon d \ln \Lambda$ contribution to the integral, which is opposite in sign to the $d = 2$ part. Thus the β function is upgraded to

$$\frac{d\lambda}{d \ln \Lambda} = -\frac{\lambda^2}{8\pi} + \epsilon. \quad (737)$$

This tells us that while in $d = 2$ there is no phase transition and the theory just flows to a disordered state (as it must because of the CMW theorem), in $d > 2$ there is a phase transition between the ordered and disordered phases at some finite value of λ .

48 May 31 — RG for the one dimensional Ising model

We have a chill problem today since I spent most of the day climbing. Consider the classical Ising model in a longitudinal field:

$$H = -K \sum_i s_i s_{i+1} - h \sum_i s_i, \quad (738)$$

where $s_i \in \mathbb{Z}_2$. By “integrating out” all of the even spins, find a renormalization group equation for the couplings. Parametrize the couplings by $x = e^{-2K}, y = e^h$.

Solution:

We write the partition function as

$$Z = \sum_{\{s_i | i \in 2\mathbb{Z}\}} \prod_{i \in 2\mathbb{Z}} \sum_{\{s_i | i \in (2\mathbb{Z}+1)\}} \exp \left(\frac{h}{2} (s_i + s_{i+2}) + h s_{i+1} + K (s_i s_{i+1} + s_{i+1} s_{i+2}) \right), \quad (739)$$

where the $1/2$ is needed to prevent double-counting. So then

$$Z[h, K] = \sum_{\{s_i | i \in 2\mathbb{Z}\}} \prod_{i \in 2\mathbb{Z}} e^{\frac{h}{2}(s_i + s_{i+2})} \left(e^h e^{K(s_i + s_{i+2})} + e^{-h} e^{-K(s_i + s_{i+2})} \right). \quad (740)$$

Our ansatz is that this will give us a partition function in the same form as the original, except with different values of h and K and perhaps a shift in the vacuum energy. So we write

$$Z[h, K] = \sum_{\{s_i | i \in \mathbb{Z}\}} \prod_{i \in \mathbb{Z}} \exp \left(\frac{h'}{2}(s_i + s_{i+1}) + K' s_i s_{i+1} - V_0 \right), \quad (741)$$

where V_0 is a vacuum energy density.

In order for this to work out, we need

$$\begin{aligned} e^h (e^{h+2K} + e^{-h} e^{-2K}) &= C e^{h'} e^{K'} \\ e^{-h} (e^{h-2K} + e^{-h} e^{2K}) &= C e^{-h'} e^{K'} \\ 2 \cosh(h) &= C e^{-K'}, \end{aligned} \quad (742)$$

where $V_0 = -\ln C$. Solving this, we get

$$\begin{aligned} e^b &= \frac{C}{2 \cosh(h)}, \\ e^{h'} &= \frac{C^2 e^h}{4 \cosh(h-2K) \cosh(h)}, \\ C^2 &= 4 \cosh(h) \cosh(h-2K) \cosh(h+2K). \end{aligned} \quad (743)$$

It will be helpful to define $x = e^{-2K}$, $y = e^h$, and likewise for their primed counterparts. Then after a bit of algebra, we get

$$x' = (y + y^{-1})^2 C^2, \quad y' = \frac{C^2 y}{(y + y^{-1})(yx + y^{-1}x^{-1})}. \quad (744)$$

This then becomes

$$\begin{aligned} x' &= \frac{y + y^{-1}}{\sqrt{(yx + y^{-1}x^{-1})(yx^{-1} + y^{-1}x)}} \\ y' &= y \sqrt{\frac{yx^{-1} + y^{-1}x}{yx + y^{-1}x^{-1}}}, \end{aligned} \quad (745)$$

which are the sought-after RG equations.

Let's do a sanity check: when $h = 0$, the new Hamiltonian shouldn't have a magnetic field term (i.e. $y' = 1$), and we should get something cosh-like for x' . Indeed, one can check (I won't write it out) that in this limit, the equations reduce to

$$x' = \frac{2}{x + x^{-1}} \implies K' = \frac{1}{2} \ln \cosh(2K), \quad (746)$$

which is the expected result.

49 June 1 — $SU(N)$ WZW basics

Consider the $SU(N)$ WZW model on a two-dimensional spacetime. The action is

$$S = S_{kin} + S_{wzw} = \frac{1}{8\pi} \int \text{Tr}[\partial_\mu g \partial^\mu g^{-1}] + i2\pi \int_{B^3} f^*(\alpha), \quad (747)$$

where g is a map from the spacetime X s (which since we will take the fields to be constant at infinity is topologically an S^2) into $SU(N)$, $f : B^3 \rightarrow SU(N)$ where $\partial B^3 = X$ is a three-ball which bounds spacetime, and where α is some nontrivial form in $H^\bullet(SU(N); \mathbb{R})$.

If $\omega = g^{-1}dg$ is the Maurer-Cartan form on $SU(N)$ pulled back to X , then the forms

$$\lambda_j = \text{Tr}(\omega \wedge j), \quad j = 3, 5, \dots, 2N - 1 \quad (748)$$

are classes in $H^\bullet(SU(N); \mathbb{R})$ pulled back to functions on X . We will mainly be interested in $SU(2)$, so we only have the $\text{Tr}(\omega \wedge \omega \wedge \omega)$ form. So, take

$$f^*(\alpha) = C \text{Tr}(\omega \wedge \omega \wedge \omega), \quad (749)$$

where C is a normalization constant to be determined, which will ensure that the periods of $f^*(\alpha)$ lie in \mathbb{Z} (so that S is independent of the choice of the bounding manifold B^3). We will focus on $SU(2)$ because we are interested in a model which lives in two dimensions, so that the WZW term is defined through the pullback of a cohomologically nontrivial 3-form from the group manifold to S^2 . For $SU(N)$ the 3-form will always be $\text{Tr}(\omega \wedge \omega \wedge \omega)$, and so there isn't much difference between the different N 's. The case of $SU(2)$ is easiest since α is then proportional to the volume form on S^3 , which makes things simple.

First, show that α indeed defines a nontrivial cohomology class. Then calculate the equations of motion from varying g , and interpret them as current conservation equations. Finally, calculate the value of C .

Solution:

First let us check that α is closed. By using the cyclic invariance of the trace, we have

$$d\lambda_3 = \text{Tr}[d\omega \wedge \omega \wedge \omega]. \quad (750)$$

Now

$$d\omega = -g^{-1}dg \wedge g^{-1}dg = -\omega \wedge \omega, \quad (751)$$

so

$$d\lambda_3 = -\text{Tr}[\omega \wedge \omega \wedge \omega \wedge \omega]. \quad (752)$$

When we take the trace, we can apply the normal supercommutativity of the wedge product since

$$\text{Tr}(A \wedge B) = \sum_{ij} A_{ij} \wedge B_{ji} = (-1)^{|A||B|} B_{ji} \wedge A_{ij} = (-1)^{|A||B|} \text{Tr}(B \wedge A). \quad (753)$$

Applying this to the above with $A = \omega$ and $B = \omega \wedge \omega \wedge \omega$, we conclude that $d\lambda_3 = 0$. We will confirm that λ_3 is not exact in a little bit, which will then show that α is indeed a nontrivial cohomology class.

Now for the equations of motion. First, let's do the kinetic term. We have

$$\begin{aligned}
S_{kin}[e^{iW}g] - S_{kin}[g] &= \frac{1}{8\pi} \int \text{Tr} [\partial_\mu(g + iWg)\partial^\mu(g^{-1} - ig^{-1}W)] - S_{kin}[g] \\
&= \frac{i}{8\pi} \int \text{Tr} [\partial_\mu W g \partial^\mu g^{-1} + W \partial_\mu g \partial^\mu g^{-1} + \partial_\mu g (-g^{-1} \partial^\mu W - \partial^\mu g^{-1} W)] \\
&= \frac{i}{8\pi} \int \text{Tr} [W((\partial^2 g)g^{-1} - g\partial^2 g^{-1})],
\end{aligned} \tag{754}$$

where we integrated by parts in the last step.

Now for the wzw term. We first need the variation of the Maurer-Cartan form:

$$\begin{aligned}
\delta\omega &= g^{-1}(\mathbf{1} - iW)d[(\mathbf{1} + iW)g] - \omega \\
&= -ig^{-1}Wdg + ig^{-1}(dWg + Wdg),
\end{aligned} \tag{755}$$

and so the variation in the WZW term is

$$\begin{aligned}
S_{wzw}[e^{iW}g] - S[g] &= 6\pi i C \int_{B^3} \text{Tr} [(-ig^{-1}Wdg + ig^{-1}(dWg + Wdg)) \wedge \omega \wedge \omega] \\
&= -6\pi C \int_{B^3} \text{Tr}[W \wedge dg \wedge dg^{-1}] \\
&= -6\pi C \int_X d^2x W \epsilon^{\mu\nu} \partial_\mu g \partial_\nu g^{-1},
\end{aligned} \tag{756}$$

where we used the cyclicity of the trace.

In a little bit we will show that the normalization constant C needs to be

$$C = -\frac{1}{24\pi^2}. \tag{757}$$

Putting this in, we arrive at the equations of motion:

$$2i\epsilon^{\mu\nu} \partial_\mu g \partial_\nu g^{-1} + (\partial^2 g)g^{-1} - g\partial^2 g^{-1} = 0. \tag{758}$$

We can also re-write this as

$$i\epsilon^{\mu\nu} \partial_\mu g \partial_\nu g^{-1} + (\partial^2 g)g^{-1} - (\partial_\mu g)g^{-1}(\partial^\mu g)g^{-1} = 0, \tag{759}$$

or finally, as

$$i\epsilon^{\mu\nu} \partial_\mu g \partial_\nu g^{-1} + (\partial^2 g)g^{-1} + \partial_\mu g \partial^\mu g^{-1} = 0. \tag{760}$$

Now define the antiholomorphic current \bar{J}_μ by

$$\bar{J} = -(\partial_{\bar{z}} g)g^{-1}, \tag{761}$$

where $z = x^0 + ix^1$ (depending on your preferences there may be a different constant out in front). Then we can compute

$$\partial_z \bar{J} = -(\partial^2 g)g^{-1} - \partial_{\bar{z}} \partial_z g^{-1} = -(\partial^2 g)g^{-1} - \partial_\mu g \partial^\mu g^{-1} - i\epsilon^{\mu\nu} \partial_\mu g \partial_\nu g^{-1} = 0, \tag{762}$$

which vanishes by the equations of motion. Similarly, we also define a holomorphic current J , with

$$J = g^{-1} \partial_z g, \quad \partial_{\bar{z}} J = 0. \quad (763)$$

A few words to help us understand these currents: in keeping with the two currents being “conjugates” of one another, the action of time reversal (sending $g \mapsto g^{-1}$ and $z \mapsto \bar{z}$) exchanges the two currents, since

$$T : J \mapsto g \partial_{\bar{z}} g^{-1} = -(\partial_{\bar{z}} g) g^{-1} = \bar{J}. \quad (764)$$

The currents also couple chirally to the symmetries in the product $SU(2)_L \times SU(2)_R$ (which act on g on the left and right respectively), since under $SU(2)_L$,

$$J \mapsto g^{-1} h^{-1} \partial_z h g = J, \quad \bar{J} \mapsto -h^{-1} (\partial_{\bar{z}} g) g^{-1} h^{-1} = h \bar{J} h^{-1}, \quad (765)$$

while similarly under $SU(2)_R$,

$$J \mapsto h^{-1} J h, \quad \bar{J} \mapsto \bar{J}. \quad (766)$$

Finally, note that the wzw term here is essential for getting the “right” equations of motion (at least, the right ones if we have bosonization in mind as a physical context): without it, the equations of motion would be a conservation of a different kind of current, and would read

$$d^\dagger \omega = 0. \quad (767)$$

Now we check that C is indeed given by $-1/24\pi^2$. We do this by requiring that the action be independent of the exact choice of B^3 , modulo elements of $2\pi i \mathbb{Z}$. This will be the case if

$$2\pi i C \int_{M_3} \text{Tr}[\omega \wedge \omega \wedge \omega] \in \mathbb{Z} \quad (768)$$

for all closed 3-manifolds M_3 . We need only check this for a particular manifold, like S^3 , since as we saw earlier, the integrand is closed. The fact that the integrand is closed also tells us that the above integral will be quantized: we just have to figure out what the correct normalization is. One can also check this by computing the variation of the integrand under $g \mapsto e^{iW} g$: as we saw earlier, the variation is exact, and so the above integral is stationary under any infinitesimal variation of g , meaning that it must be quantized.

Anyway, let’s find the normalization coefficient. We parametrize the S^3 by

$$g = e^{ix^3 n_a \sigma^a} = \cos(x^3) + i n_a \sigma^a \sin(x^3), \quad (769)$$

where x^3 is an angular coordinate running from 0 to π and $n : X \rightarrow S^2$ is a field which maps spacetime to S^2 (this is the usual way of building a three-sphere out of an S^2 and an S^1). Then we have

$$\omega_\mu dx^\mu = g^{-1} \partial_\mu g dx^\mu = i n_a \sigma^a dx^3 + i \partial_j n_a \sigma^a dx^j, \quad (770)$$

where i runs through the spacetime coordinates and we’ve used $(n_a \sigma^a)^2 = 1$.

Putting this in, the integral is

$$I = 6\pi i C \int_{M_3} d^3 x \epsilon^{3jk} \sin^2(x^3) \text{Tr}[n_a \sigma^a \partial_j n_b \sigma^b \partial_k n_c \sigma^c]. \quad (771)$$

Doing the x^3 integral and taking the trace,

$$I = 6\pi^2 C \int_X d^2x \epsilon^{jk} \epsilon^{abc} n_a \partial_j n_b \partial_k n_c. \quad (772)$$

Because of the double-antisymmetrization, this is

$$I = 12\pi^2 C \int_X d^2x \epsilon^{abc} n_a \partial_1 n_b \partial_2 n_c. \quad (773)$$

We recognize the $n \cdot \partial_i n \times \partial_j n \, dx^i \wedge dx^j$ as the pullback of the volume form on S^2 to X through the map n (it is just the Jacobian for the map: $\partial_i n \times \partial_j n$ lies perpendicular to the S^2 , parallel to n , and has magnitude equal to the local magnification of the mapping $n : X \rightarrow S^2$). Thus we have

$$I = 12\pi^2 C \int_{n(X)} \text{vol}_{S^2}. \quad (774)$$

Since this can be non-zero, $\text{Tr}[\omega^3]$ must not be exact and hence it must indeed represent a nontrivial cohomology class.

Since the volume form is closed, the integral is a topological invariant which is of course the winding number. So we have

$$I = 48\pi^3 C w, \quad w \in \mathbb{Z}. \quad (775)$$

Since we have to consider maps with winding number $w = 1$, we see that we need (the minus sign is just convention)

$$C = -\frac{1}{24\pi^2} \implies 2\pi i C \int_{M_3} \text{Tr}[\omega^3] \in \mathbb{Z}. \quad (776)$$

We also see a little bit of the relation between WZW terms and topological terms: after we integrated over the x^3 coordinate, the WZW term became a topological θ term on the compactified spacetime $X \sim S^2$.

50 June 2 — RG in the $SU(N)$ WZW model

Like in yesterday's diary entry, consider the $SU(N)$ WZW model on a two-dimensional spacetime. The action is

$$S = S_{kin} + S_{wzw} = \frac{1}{\lambda} \int \text{Tr}[\partial_\mu g \partial^\mu g^{-1}] - \frac{i}{12\pi} \int_{B^3} \text{Tr}[\omega \wedge \omega \wedge \omega], \quad (777)$$

where g is a map from the spacetime X (which since we will take the fields to be constant at infinity is topologically an S^2) into $SU(N)$, B^3 is a three-ball which bounds spacetime, and where ω is the Maurer-Cartan form on $SU(N)$ pulled back to B^3 .

Using the background field method with an explicit momentum-shell cutoff, find the beta function for λ . Show that $\lambda = 8\pi$ is a fixed point, the existence of which is made possible by the wzw term.

Solution:

To do the RG, we split up $g = g_s g_f$ into low- and high-momentum parts. We find it helpful to parametrize g_f as $g_f = e^W$ for $W \in \mathfrak{su}(N)$. Since we are only interested in finding β_λ to one-loop order, we only need to keep terms quadratic in W (to quadratic order, the measure $\mathcal{D}g_f$ is the same as $\mathcal{D}W$). Thus we can write

$$\partial_\mu g_f \approx \partial_\mu W + \frac{1}{2}\{\partial_\mu W, W\}. \quad (778)$$

Let us focus on S_{kin} first. After some straightforward algebra, we find

$$S_{kin}[g] = S_{kin,s}[g_s] + S_{kin,f}[W] + S_{kin,sf}[g_s, W], \quad (779)$$

where

$$S_{kin,s}[g_s] = \frac{1}{\lambda} \int \text{Tr}[\partial_\mu g_s \partial^\mu g_s^{-1}], \quad S_{kin,f} = -\frac{1}{\lambda} \text{Tr}[\partial_\mu W \partial^\mu W], \quad (780)$$

and

$$\begin{aligned} S_{kin,sf}[g_s, W] &= \frac{1}{2\lambda} \int \text{Tr} [g_s^{-1} \partial_\mu g_s [\partial_\mu W, W] + (\partial_\mu g_s^{-1}) g_s [W, \partial_\mu W]] \\ &= \frac{1}{\lambda} \int \text{Tr} [\star \omega_s \wedge [dW, W]], \end{aligned} \quad (781)$$

where $\omega_s = g_s^{-1} \partial_\mu g_s dx^\mu$.

Now for the wzw term. The well-definedness of this term forces the coefficient to be an integer (times $i/12\pi$) which means the factor in the wzw term can't flow under RG, and so one might naively think that the wzw term would not contribute to the beta function. This is not so, however.

Let us first expand the Maurer-Cartan form. Again, straightforward algebra gives

$$\omega \approx \omega_s(1 + W + W^2/2) - W\omega_s - W\omega_s W + \frac{1}{2}W^2\omega_s + \frac{1}{2}[\partial_\mu W, W] + \partial_\mu W. \quad (782)$$

Now we take the wedge product of three copies of the above expression. We only want to keep terms that have at most two derivatives in the slow fields (since they are slowly varying), and we only need to keep up to quadratic order in W . The very last $\partial_\mu W$ parts contribute a factor of $3\text{Tr}[dW \wedge dW \wedge \omega_s]$. The only other term that is not cubic in ω_s comes from the commutator, and so

$$S_{top}[g_s, W] = -\frac{i}{4\pi} \int_{B^3} \text{Tr} \left[\left(\frac{1}{2} [dW, W] \wedge \omega_s + dW \wedge dW \right) \wedge \omega_s \right]. \quad (783)$$

The integrand is a total derivative:

$$d \left(-\frac{i}{8\pi} \text{Tr} [[dW, W] \wedge \omega_s] \right) = -\frac{i}{8\pi} \text{Tr} [-2dW \wedge dW \wedge \omega_s + [dW, W] \wedge \omega_s \wedge \omega_s], \quad (784)$$

since $d\omega_s = -\omega_s \wedge \omega_s$ (we've also used the sign rules for the supercommutativity of the wedge product). So, we have

$$S_{top}[g_s, W] = -\frac{i}{8\pi} \int_X \text{Tr} [[dW, W] \wedge \omega_s]. \quad (785)$$

To write the full slow-fast action succinctly, it is helpful to introduce

$$\tilde{\omega} = \omega_s - \frac{i\lambda}{8\pi} \star \omega_s, \quad (786)$$

so that

$$S = S_f[W] + S_s[g_s] + S_{sf}[g_s, W], \quad (787)$$

where

$$S_{sf}[g_s, W] = \frac{1}{\lambda} \int \text{Tr}[\star \tilde{\omega} \wedge [dW, W]]. \quad (788)$$

Thus, while the wzw term can't get renormalized through a change of its coefficient (the level), it contributes towards the renormalization of the kinetic term.

Now we expand the exponential $\exp(-S_{sf})$ in the path integral. The linear term vanishes since it contains a single fast momentum from the dW piece, and so the first relevant contribution comes from the S_{sf}^2 term. Thus the effective action for the slow fields is

$$S_{eff,s}[g_s] = S_s[g_s] - \ln \left(1 + \frac{1}{2} \langle S_{sf}^2[g_s, W] \rangle \right), \quad (789)$$

where the expectation value is taken with respect to the $\lambda^{-1} \int \text{Tr}(\partial W)^2$ action. The expectation value is

$$\frac{1}{2} \langle S_{sf}^2[g_s, W] \rangle = -\frac{4}{2\lambda^2} \int_{p,q,p',q'} p_\mu p'_\nu \langle \text{Tr}[\tilde{\omega}_q^\mu W_p W_{-q-p}] \text{Tr}[\tilde{\omega}_{q'}^\nu W_{p'} W_{-q'-p'}] \rangle, \quad (790)$$

where we've gone to momentum space and we've used $2p+q \approx 2p$ since p is a fast momentum and q is a slow momentum.

We take the expectation value by contracting the various W 's. Contracting two W 's in the same trace yields zero: this is because the propagator for the W fields is diagonal in the $\mathfrak{su}(N)$ generators (expand the W 's in terms of T^a 's in the kinetic term for W and use the orthogonality of the T^a 's under the trace inner product), and so contracting two W 's in the same trace produces something like $\text{Tr}[\tilde{\omega} C_2]$, where $C_2 \sim \sum_a T^a T^a$ is the quadratic Casimir, which is central in the Lie algebra. Expanding $\tilde{\omega}$ (which lives in $\mathfrak{su}(N)$ since it is built from the Maurer-Cartan form) in terms of the traceless T^a , one sees that the trace vanishes.

So, we just need to consider the two contractions between the W 's in different traces. This gives (see Altland and Simons chapter 8 for some useful identities)

$$\frac{1}{2} \langle S_{sf}^2[g_s, W] \rangle = -\frac{N}{2\lambda^2} \int_{p,q} G_p G_{p+q} p_\mu p_\nu \text{Tr}[\tilde{\omega}_q^\mu \omega_{-q}^\nu], \quad (791)$$

with the factor of N coming from the sum over the internal loop in the polarization diagram (use the double line notation to see). If we again approximate $p+q \approx p$ then we can do the integral over the fast momentum p easily:

$$\frac{1}{2} \langle S_{sf}^2[g_s, W] \rangle = -\frac{N}{8\pi} \int_q \ln \left(\frac{\Lambda}{\Lambda - \delta\Lambda} \right) \text{Tr}[\tilde{\omega}_q^\mu \tilde{\omega}_{\mu,-q}]. \quad (792)$$

Taking $d\Lambda = -\delta\Lambda$ and expanding,

$$\frac{1}{2} \langle S_{sf}^2[g_s, W] \rangle = -d \ln \Lambda \frac{N}{8\pi} \int \text{Tr}[\tilde{\omega} \wedge \star \tilde{\omega}], \quad (793)$$

where the integral is now in \mathbb{R} space. From the definition of $\tilde{\omega}$, one sees that this integral is

$$\frac{1}{2} \langle S_{fs}^2[g_s, W] \rangle = d \ln \Lambda \frac{N}{8\pi} \left(1 - \left(\frac{\lambda}{8\pi} \right)^2 \right) \int \text{Tr}[\omega_s \wedge \star \omega_s]. \quad (794)$$

Adding this in with the $\omega_s \wedge \omega_s$ term, we see that the effective coupling $\lambda + d\lambda$ is

$$\frac{1}{\lambda + d\lambda} = \frac{1}{\lambda} + d \ln \Lambda \frac{N}{8\pi} \left(1 - \left(\frac{\lambda}{8\pi} \right)^2 \right), \quad (795)$$

which gives the β function

$$\frac{d\lambda}{d \ln \Lambda} = - \frac{N\lambda^2}{8\pi} \left(1 - \left(\frac{\lambda}{8\pi} \right)^2 \right). \quad (796)$$

There are several things to note about this. First, we see that it is asymptotically free for small λ , which is what we expect based on our experience with σ models into spaces with positive curvature. However, we also see that $\lambda = 8\pi$ constitutes a fixed point of the RG flow, which is made possible only by the presence of the wzw term. Such a fixed point had to occur at this value of λ , since it is precisely at this value of λ that the theory admits a bosonization duality to a theory of N flavors of free fermions which, being free, do not flow under RG. Thus the presence of the wzw term is essential for making the bosonization work.

51 June 3 — Functional bosonization

This is from Altland and Simons (but there are some typos in the problem so don't worry too much about reproducing their results — also, our notation will deviate a bit from theirs). Consider fermions in two dimensions, with action

$$S = - \int \bar{\psi} \not{\partial} \psi + \frac{1}{2} \int \rho^T \mathcal{G} \rho - \int (\psi_\sigma^\dagger J_\sigma + J_\sigma^\dagger \psi_\sigma), \quad (797)$$

where \mathcal{G} is an interaction matrix, $\rho = (\rho_+, \rho_-)^T$ are the densities, and $\sigma \in \{\pm\}$ are the left- and right-moving indices. Our plan is to get to the usual bosonization result in a slightly different way.

First, decouple the interactions using a bosonic doublet. It will turn out to be convenient to do a Hodge decomposition on this doublet, since the longitudinal and transverse parts of the decomposition will match with the vector and chiral currents and so this decomposition will behave nicely with respect to the holomorphic / antiholomorphic decomposition of the boson field.

Then, integrate out the fermions, and find the effective action for the components of the Hodge decomposition of the Hubbard-Stratonovich field. Represent the Greens function in the current-current coupling using spin / charge bosonic variables. Finally, integrate out the remaining Gaussian fields and obtain the interacting Luttinger liquid action.

Solution:

First we decouple the interaction, using a bosonic field $\phi = (\phi_+, \phi_-)$. We let ϕ appear in the action as $\frac{1}{2}\phi^T\mathcal{G}^{-1}\phi$, and then shift $\phi \mapsto \phi + i\mathcal{G}\rho$. This leaves us with a $i\rho^T\phi$ coupling, which is just like that of a gauge field. To write the coupling in a covariant form, let

$$\phi_0 = \frac{1}{2}(\phi_+ + \phi_-), \quad \phi_1 = \frac{1}{2i}(\phi_+ - \phi_-). \quad (798)$$

In a representation where the gamma matrices are $\gamma^0 = X, \gamma^1 = Y$, we then have

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi \exp \left(-\frac{1}{2} \int \phi^T \mathcal{G}^{-1} \phi + \int \bar{\psi} \not{D}_\phi \psi - S_{src}[J, \psi] \right), \quad (799)$$

where

$$\not{D}_\phi = \gamma^\mu (\partial_\mu - i\phi_\mu). \quad (800)$$

We now do a Hodge decomposition on ϕ . We will write it as

$$\phi = d\xi + id^\dagger \star \eta. \quad (801)$$

The i is just for convenience, and we have written $\star\eta$ since we'd rather work with zero-forms than two-forms. The Hodge decomposition plays nicely with the chiral nature of the fermions, with ξ relating to the vector current and η to the chiral current. We see this by considering

$$\not{\partial}(e^{i\xi+i\eta Z}\psi) = e^{i\xi+i\eta Z}\not{\partial}\psi + (-\partial_0\xi - i\partial_1\eta X - \partial_1\xi Y + i\partial_0\eta Y)e^{i\xi+i\eta Z}\psi = \gamma^\mu (\partial_\mu + i\phi_\mu)e^{i\xi+i\eta Z}\psi. \quad (802)$$

Thus (if we ignore what happens to $\mathcal{D}\psi$), we can eliminate the coupling to the background field ϕ through a phase rotation by ξ and a chiral rotation by η .

Now we can integrate out the fermions. We expand the Tr ln to second order in ϕ , producing the usual polarization bubble. The effective action then has a term (remembering the -1 from the fermion loop)

$$-\frac{1}{2} \int_{q,p} \phi_q^T \frac{1}{\not{p} - \not{q}} \phi_{-q}. \quad (803)$$

Because ϕ couples to the fermions as a gauge field would, the integration kernel will be diagonal in the spin indices. The propagator for the ψ 's is

$$D_\psi(p)_{\sigma\sigma'} = \delta_{\sigma\sigma'} \frac{i}{\nu + i\sigma p}, \quad (804)$$

which comes from inverting ∂_σ . Note that I am being lazy and not distinguishing between two-momenta and their spatial components: that is, I am writing $q = (\nu, q)$. Sorry not sorry.

We now write the σ part of the above integral as

$$\frac{1}{2} \int_{q,p} \phi_{q,\sigma}^T \frac{1}{\nu + i\sigma p} \frac{1}{\nu + i\sigma p + \omega + i\sigma q} \phi_{-q,\sigma}, \quad (805)$$

where $q = (\omega, q)$. It is helpful to recast this as

$$\frac{1}{2} \int_q \frac{1}{\omega + i\sigma q} \int_p \phi_{q,\sigma}^T \left(\frac{1}{\nu + i\sigma p} - \frac{1}{\nu + i\sigma p + \omega + i\sigma q} \right) \phi_{-q,\sigma}. \quad (806)$$

This looks like it might be zero after doing a contour integral and closing it in either the upper half plane or the lower one (depending on σ), but on the other hand, it's $\sim \int_p d^2 p p^{-2}$ which is divergent (I think there are some subtle things going on as the pole at $-i\sigma p$ gets pushed to ∞). As suggested in the book, we do a somewhat suspect thing and close the integrals in the plane where they give a non-zero answer by the residue theorem. If $\sigma p > 1$ then the pole lies in the lower half plane and we get a clockwise integral, giving $-2\pi i$, while if $\sigma p < 1$ then we get a counterclockwise integral, giving $2\pi i$. So then

$$-\frac{i}{2} \int_q \int_p \frac{dp}{2\pi} \phi_{q,\sigma}^T (\operatorname{sgn}(\sigma p) - \operatorname{sgn}(\sigma(p+q))) \phi_{-q,\sigma}. \quad (807)$$

Now we introduce an explicit cutoff for the spatial momentum integration. Luckily as long as we take $\Lambda \rightarrow \infty$, the answer doesn't depend on the exact value for Λ . So we get

$$-\frac{1}{4\pi} \int_q \phi_{q,\sigma}^T \frac{-i\sigma q}{\omega + i\sigma q} \phi_{-q,\sigma}. \quad (808)$$

Recapitulating, the effective action for the boson fields is

$$S_{eff}[\phi] = \frac{1}{2} \int_q \phi_q^T (\mathcal{G}^{-1} + G_\phi(q)) \phi_{-q}, \quad (809)$$

where

$$G_\phi(q)_{\sigma\sigma'} = \frac{\delta_{\sigma\sigma'}}{2\pi} \frac{i\sigma q}{\omega + i\sigma q}. \quad (810)$$

Now we look at the source term. Before we integrated out the fermions but after we introduced the ϕ fields, we can perform a shift on ψ to eliminate the linear coupling between the fermions and the sources (we have to do this after adding in the ϕ fields since we don't want to mess with the density-density ψ interactions). The source term is then

$$S_{src} = \int_{x,x'} J^\dagger(x) G_{\psi,\phi}(x, x') J(x'), \quad (811)$$

where $G_{\psi,\phi}(x, x')$ is the propagator for the fermions in the background ϕ field.

Using the Hodge decomposition of the ϕ field,

$$S_{src} = \int_{x,x'} J^\dagger(x) e^{-i(\xi+\eta Z)(x)} G_\psi(x, x') e^{i(\xi+\eta Z)(x')} J(x'). \quad (812)$$

Now we can play a trick by representing the fermion propagator with a bosonic doublet of fields. First of all, the actual expression for $G_\psi(x, 0)$ is (just invert ∂_\pm)

$$[G_\psi(x, 0)]_{\sigma\sigma'} = \delta_{\sigma\sigma'} \int_{q,\omega} \frac{e^{-i(xq+\omega\tau)}}{i\sigma q + \omega}. \quad (813)$$

Taking $x > 0$ wolog, the integrand is analytic in the lower half-plane, and so we close the contour for $q \rightarrow -i\infty$. Thus we get zero if $\sigma\omega > 0$ and get a residue of $-2\pi\sigma \exp(\omega(\sigma x - i\tau))$ otherwise, so that

$$[G_\psi(x, 0)]_{\sigma\sigma'} = -\sigma\delta_{\sigma\sigma'} \int_\omega \Theta(-\sigma\omega) e^{\omega(\sigma x - i\tau)}. \quad (814)$$

Doing the integral,

$$[G_\psi(x, 0)]_{\sigma\sigma'} = \frac{\delta_{\sigma\sigma'}}{2\pi} \frac{1}{i\sigma x + \tau}. \quad (815)$$

Now we want to reproduce this with bosons. Consider a doublet of bosons which possesses the following free action:

$$S_\varphi = \frac{1}{2} \int_q \varphi_q^T K_\varphi(q) \varphi_{-q}, \quad (816)$$

where $\varphi = (\varphi_+, \varphi_-)^T$ and

$$K_\varphi(q) = \begin{pmatrix} q^2 + iq\omega & 0 \\ 0 & q^2 - iq\omega \end{pmatrix}. \quad (817)$$

One can think of the components of φ as holomorphic and antiholomorphic modes. The Greens function for these guys is

$$[G_\varphi(x, 0)]_{\sigma\sigma'} = \delta_{\sigma\sigma'} \int_{q,\omega} \frac{e^{-i(xq+\omega\tau)}}{q^2 + i\sigma q\omega}. \quad (818)$$

The poles of the momentum integral are at $q = -i\sigma\omega$, and the integrand is analytic for $q \rightarrow -i\infty$. The residue at the pole is $2\pi/\sigma\omega$ (since the contour is closed clockwise), so we get

$$[G_\varphi(x, 0)]_{\sigma\sigma'} = \sigma \delta_{\sigma\sigma'} \int_\omega \Theta(\sigma\omega) \frac{e^{-\omega(\sigma x + i\tau)}}{\omega}. \quad (819)$$

To do the integral, we impose a small frequency cutoff at a^{-1} (or at $-a^{-1}$, depending on σ), where a is the lattice spacing. The integral can then be expanded in small a . We will get a bunch of constants, which will be normal-ordered away. The surviving piece gives a log, and so

$$[G_\varphi(x, 0)]_{\sigma\sigma'} = -\ln \left(\frac{i\sigma x + \tau}{a} \right). \quad (820)$$

The whole point of going through this is that the correlators of the vertex operators for the φ fields reproduce the form of the fermion correlators:

$$\langle e^{i\phi_\sigma(x, \tau)} e^{-i\phi_{\sigma'}(0, 0)} \rangle = \delta_{\sigma\sigma'} \frac{a}{i\sigma x + \tau}, \quad (821)$$

where the expectation value is over the free φ action. This is exactly equal to the fermion correlator but for a factor of $a/2\pi$. Thus if we absorb this factor into the sources J , we can write the source term as

$$S_{src} = \int_{x,x'} J^\dagger(x) e^{-i(\xi + \eta Z)(x)} \langle e^{i\phi(x)} e^{-i\phi(x')} \rangle e^{i(\xi + \eta Z)(x')} J(x'). \quad (822)$$

Now we play a cute trick: the vertex operators have a Gaussian distribution (since they map to ψ , which has just a free action), and so we can pull the expectation value out of the exponential in $e^{-S_{src}}$, and realize the expectation value by integrating over the φ fields. Thus we just need to take the square root of the above integral, exponentiate it, and path integrate over φ . The partition function is then

$$Z = \int \mathcal{D}\xi \mathcal{D}\eta \int \mathcal{D}\varphi e^{-S_{eff}[\xi, \eta] - S_\varphi[\varphi]} \exp \left(- \int (J^\dagger e^{-i(\xi + \eta Z)} e^{i\varphi} + e^{-i\varphi} e^{i(\xi + \eta Z)} J) \right), \quad (823)$$

with $S_\varphi[\varphi]$ the free action for the φ fields and $S_{eff}[\xi, \eta]$ is the effective action for ϕ that we derived earlier.

This representation of the source term tells us that it would be nice if we had a decomposition of φ to a form like $\xi + \eta Z$. This is easily done by writing

$$\varphi_\pm = \Phi \pm \Theta. \quad (824)$$

One can check that the action S_φ becomes, in this representation,

$$S_\varphi = \frac{1}{2} \int_q (\Phi_q, \Theta_q) \tilde{K}_q \begin{pmatrix} \Phi_{-q} \\ \Theta_{-q} \end{pmatrix}, \quad \tilde{K}_q = \begin{pmatrix} q^2 & iq\omega \\ iq\omega & q^2 \end{pmatrix}. \quad (825)$$

With this representation, we can eliminate the ξ, η fields from the source term by shifting the Φ and Θ fields. The sum of the free actions then becomes, after the shift (letting Ψ denote the (Φ, Θ) doublet and letting Ξ denote the (ξ, η) doublet)

$$S_{eff}[\xi, \eta] + \frac{1}{2} \int_q \left(\Psi_q^T \tilde{K}_q \Psi_{-q} + \Xi_q^T \tilde{K}_q \Psi_{-q} + \Psi_q^T \tilde{K}_q \Xi_{-q} + \Xi_q^T \tilde{K}_q \Xi_{-q} \right). \quad (826)$$

Here a miracle occurs. We change basis from the ϕ field to its Hodge representation Ξ by way of the matrix (I think this is listed incorrectly in the book?)

$$\phi_q = U_q \Xi_q, \quad U_q = \begin{pmatrix} q - i\omega & q - i\omega \\ -q - i\omega & q + i\omega \end{pmatrix}. \quad (827)$$

The miracle is that

$$\tilde{K}_q = -U_q^T G_\phi(q) U_{-q}. \quad (828)$$

This means that the $\Xi^T \tilde{K} \Xi$ term in the last integral we wrote actually cancels with one of the terms in $S_{eff}[\xi, \eta]$ after we complete the switch from the ϕ representation to the (ξ, η) representation.

Recapitulating, the action (without the source term) is

$$S = \frac{1}{2} \int_q \left(\Psi_q^T \tilde{K}_q \Psi_{-q} + \Xi_q^T \tilde{K}_q \Psi_{-q} + \Psi_q^T \tilde{K}_q \Xi_{-q} + \Xi_q^T U_q^\dagger G_q^{-1} U_{-q} \Xi_q \right). \quad (829)$$

Since now only Ψ appears in the source term, we want to integrate out Ξ , which we can now do happily. We integrate it out to get

$$S = \frac{1}{2} \int_q \Psi_q^T \left(\tilde{K}_q - \tilde{K}_q [U_q^\dagger G_q^{-1} U_{-q}]^{-1} \tilde{K}_q \right) \Psi_{-q}. \quad (830)$$

To write this out explicitly, let us write \mathcal{G} as (following the notation in the book now)

$$\mathcal{G} = g_4 \mathbf{1} + g_2 X. \quad (831)$$

We know the explicit form for all the matrices in the above action, and so we can just multiply them out and see what we get. Our final bosonized form for the complete partition function is then

$$Z = \int \mathcal{D}\Theta \mathcal{D}\Phi \exp(-S_0 - S_{src}), \quad (832)$$

where

$$S_0 = \frac{1}{2\pi} \int_q (\Phi_q, \Theta_q) \begin{pmatrix} q^2(1 + 2\pi(g_4 - g_2)) & iq\omega \\ iq\omega & q^2(1 + 2\pi(g_4 + g_2)) \end{pmatrix} \begin{pmatrix} \Phi_{-q} \\ \Theta_{-q} \end{pmatrix}, \quad (833)$$

and

$$S_{src} = \int \left(J_\sigma^\dagger e^{i(\Phi+\sigma\Theta)} + e^{-i(\Phi+\sigma\Theta)} J_\sigma \right). \quad (834)$$

All done!

52 June 4 — The Hopf map

This is from Nakahara. Consider a map $f : S^{2n-1} \rightarrow S^n$. Do a few things:

- Write down the generator v for $H^n(S^n; \mathbb{R})$. Show that $f^*(v)$ is closed and exact, with

$$f^*(v) = d\omega. \quad (835)$$

- Define the Hopf invariant by the self-linking number of the worldlines defined by the Poincare dual of $d\omega$:

$$H(f) = \int_{S^{2n-1}} \omega \wedge d\omega. \quad (836)$$

Show that while ω is not uniquely determined, this ambiguity does not affect $H(f)$. Also show that if $g : S^{2n-1} \rightarrow S^n$ is homotopic to f , then $H(f) = H(g)$.

- Compute $H(f)$ in the case $n = 2$, where f is the Hopf map which relates coordinates on $S^2 \subset \mathbb{R}^3 = \langle x, y, z \rangle$ and $S^3 \subset \mathbb{R}^4 = \langle a, b, c, d \rangle$ by

$$x = 2(ad + cb), \quad y = 2(bc - ad), \quad z = a^2 + b^2 - c^2 - d^2. \quad (837)$$

Solution:

There are a few ways to write down the volume form. One which is kind of cool is

$$v = \frac{1}{r} \sum_i (-1)^{i+1} x_i dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^{n+1}, \quad (838)$$

where the hat means that we omit that term. To see that this is the volume form, we wedge with the 1-form $dr = x_i dx^i$:

$$dr \wedge v = dx^1 \wedge \cdots \wedge dx^{n+1}, \quad (839)$$

which is the volume form on \mathbb{R}^{n+1} . Since we know that this splits as dr times the sphere part, v must indeed be a generator for the top cohomology of S^n .

Obviously v is closed, and so $f^*(v)$ is closed by virtue of the fact that exterior differential commutes with pullbacks. This can be proved by writing down the definitions. For example, if $h : X \rightarrow Y$ is a smooth map (so that locally we can invert the Jacobian) and γ is a 1-form on Y (just for simplicity), then one can check that both $df^*(v)$ and $f^*(dv)$ are given by

$$df^*(v) = f^*(dv) = \frac{\partial}{\partial y^\alpha} \alpha_\mu(h(y)) \frac{\partial x^\mu}{\partial y^\beta} dy^\alpha \wedge dy^\beta. \quad (840)$$

Now $f^*(v)$ is a closed n -form on S^{2n-1} . Since S^{2n-1} has no nontrivial n -cycles to integrate this over, it must be exact. Thus we can write

$$f^*(v) = d\omega, \quad (841)$$

for some $n - 1$ form ω .

Now for the second bullet point. ω is only defined up to an exact form. This doesn't affect $H(f)$ though, since shifting ω by an exact form changes the integrand by a total derivative.

Now, suppose $f \sim g$. We claim that since f and g are homotopic, their Hopf invariants must be equal. Indeed this is the case: to see this, set up the mapping cylinder $Cy = S^{2n-1} \times I$, where on $S^{2n-1} \times \{0\}$ we use the map f and on $S^{2n-1} \times \{1\}$ we use g . Then

$$H(g) - H(f) = \int_{S^{2n-1} \times \{1\}} \omega_g \wedge g^*(v) - \int_{S^{2n-1} \times \{0\}} \omega \wedge f^*(v). \quad (842)$$

Defining f_t in the natural way,

$$H(g) - H(f) = \int_{Cy} d\omega_t \wedge d\omega_t, \quad (843)$$

where $f_t^*(v) = d\omega_t$. Since pullbacks distribute over wedge products, we have

$$\begin{aligned} H(g) - H(f) &= \int_{Cy} f_t^*(v) \wedge f_t^*(v) \\ &= \int_{Cy} f_t^*(v \wedge v) = 0, \end{aligned} \quad (844)$$

since $v \wedge v = 0$ as the wedge product is being carried out in S^n and v is an n -form.

Finally for the third bullet point. This is kind of heinous and we won't write everything out. We will first deploy the form of v given above, so that

$$v = \frac{1}{4\pi r} (x \, dy \wedge dz - y \, dx \wedge dz + z \, dx \wedge dy). \quad (845)$$

Here we've normalized v by dividing by 4π so that it has integral periods.

Life can be made slightly easier by noting that $d(r^2) = x \, dx + y \, dy + z \, dz = 0$, so that the above becomes

$$v = -\frac{1}{4\pi} \frac{dx \wedge dy}{z}. \quad (846)$$

We know x and y in terms of the 4-sphere coordinates a, b, c, d , and so we can just substitute them in the above equation and simplify. This is done with the help of $a \, da + b \, db + c \, dc + d \, dd =$

0, but we won't write out all the algebra. We end up getting (sorry for the awful notation dd by the way)

$$f^*(v) = \frac{1}{\pi}(da \wedge db + dc \wedge dd). \quad (847)$$

This is a total derivative, and so we can identify ω with

$$\omega = \frac{1}{\pi}(a \wedge db + c \wedge dd). \quad (848)$$

Thus the integral is

$$H(f) = \frac{1}{\pi^2} \int_{S^3} (a \wedge db \wedge dc \wedge dd + c \wedge dd \wedge da \wedge db). \quad (849)$$

The two contributions are equal since the a, b, c, d are all on equal footing, and so we only need to worry about the first one.

We can do the integral by remembering that the $n+1$ sphere is a suspension of the n sphere. This means that we can build a coordinate system on the 3 sphere as an iterative process by attaching 1-sphere coordinates to $n < 3$ sphere coordinates that are already in place. Concretely, we start with the 1-sphere $(\cos \phi, \sin \phi)$. Then we add on a sphere to one of the components, so that we get the 2-sphere with coordinates $(\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi)$. Finally we add on another sphere to get the coordinates for the 3-sphere, namely

$$(a, b, c, d) = (\cos \phi \cos \theta \cos \psi, \cos \phi \cos \theta \sin \psi, \cos \phi \sin \theta, \sin \phi). \quad (850)$$

We then have

$$db \wedge dc \wedge dd = \cos^3 \phi \cos \theta^2 \cos \psi \, d\psi d\theta d\phi, \quad (851)$$

so that the integral is

$$H(f) = \frac{2}{\pi^2} \int \cos^4 \phi \cos^3 \theta \cos^2 \psi \, d\psi d\theta d\phi, \quad (852)$$

where the integral over ϕ runs from 0 to 2π and the others go from $-\pi/2$ to $\pi/2$. The integral is $\pi^2/2$, and so

$$H(f) = 1. \quad (853)$$

53 June 5 — Reminder about Stiefel-Whitney classes

Remind yourself of what the first and second Stiefel-Whitney classes are, and what they obstruct. In dimensions $d \leq 4$, show that the second Stiefel-Whitney class, when capped with the fundamental class of an orientable 2-submanifold S , computes the mod 2 self-intersection number of S . It will help to use the Whitney sum formula:

$$w(E \oplus F) = w(E) \cup w(F), \quad (854)$$

where $w = 1 + w_1 + w_2 + \dots$ is the total Stiefel-Whitney class. Finally, convince yourself that all orientable 3-manifolds are spin (do so without doing computations with Wu classes). It may behoove you to look at “The Wild World of Four Manifolds” for inspiration.

Solution:

Suppose we’re given a manifold M of dimension m . Consider the tangent bundle of M . This is a vector bundle with structure group $O(m)$ (reduced from $GL(m; \mathbb{R})$ with the help of a metric), where $O(m)$ acts on the orthonormal frames e_α^i on each patch U_α . Let $t_{\alpha\beta} \in O(m)$ be the transition function between frames on different patches, so that $e_\alpha^i = [t_{\alpha\beta}]_j^i e_\beta^j$. We will throughout assume we are working with a cover which is refined sufficiently so that all the patches and their intersections are simply connected. We can get such a cover by starting from a triangulation and taking each U_α to be the maximal star-shaped region containing only the 0-handle labeled α .

First Stiefel-Whitney class: Define w_1 as the cohomology class of the determinant Cech 1-cochain

$$w_1 = [\det t_{\alpha\beta}] \in H^1(M; \mathbb{Z}_2), \quad (855)$$

where the cohomology is Cech cohomology. From the cocycle condition on the transition functions and multiplicative property of the determinant, we see that $\delta w_1 = 0$. Suppose we perform a rotation on the framing at each patch by $e_\alpha^i \mapsto [h_\alpha]_j^i e_\alpha^j$. We see that this acts on $g_{\alpha\beta}$ with the adjoint action, and so

$$w_1 \mapsto [\det h_\alpha \det^{-1} h_\beta \det t_{\alpha\beta}] = [\det t_{\alpha\beta}], \quad (856)$$

since $\det h_\alpha \det^{-1} h_\beta$ is a \mathbb{Z}_2 coboundary. Thus w_1 is independent of the exact choice of local frame. We should think of this as a \mathbb{Z}_2 gauge field: with the 0-skeleton of M living at the center of patches, $\det t_{\alpha\beta}$ naturally lives on the 1-skeleton, and so w_1 behaves like a \mathbb{Z}_2 gauge field, with local frame rotations playing the role of gauge transformations.

Now we mention what this has to do with orientability. If M is orientable then we can further reduce the structure group to $SO(m)$, thereby trivializing the \mathbb{Z}_2 bundle over the 1-skeleton and setting $w_1 = 0$. Conversely, if w_1 is a trivial cohomology class, then $\det g_{\alpha\beta} = \det h_\alpha \det^{-1} h_\beta$ for some local rotations h_α, h_β , and so M is seen to be orientable after performing a rotation of the framing. Thus, the first Stiefel-Whitney class w_1 measures an obstruction to orientability. This in turn represents an obstruction to finding m linearly independent sections of TM . The intuitive picture for this is that as we go around a path over which w_1 has nontrivial holonomy, two of the basis vectors e^i in any orthonormal frame must change places, so that the orientation flips. This can only be done if two of the e^i are linearly dependent at some point along the path. Equivalently, we can think of w_1 as measuring the obstruction to extending a trivialization of the frame bundle from the 0-skeleton (which can always be trivialized) to a trivialization of the 1-skeleton. If $\det g_{\alpha\beta} = -1$ then the frames at U_α and U_β cannot be smoothly connected, and so the trivialization cannot be extended across the 1-handle which links U_α and U_β .

Since w_1 is a 1-cocycle, it is Poincare dual to a codimension-1 submanifold \widehat{w}_1 of M . According the definition of w_1 , the orientation flips when passing through \widehat{w}_1 . Now if w_1 is nontrivial in cohomology, then \widehat{w}_1 is not a boundary. This means we can travel along a

curve in M that intersects the \tilde{w}_1 surface only once, and thus defining a global orientation is impossible. In this way, the parts of the 1-skeleton of M that intersect \hat{w}_1 transversely are the parts of the 1-skeleton across which a trivialization on the 0-skeleton cannot be extended. In terms of the basis vectors for the framing, they must become degenerate when passing through \hat{w}_1 , so that \hat{w}_1 determines the location where a collection of m basis vectors for the framing must fail to be linearly independent. Said another way, if N is any closed one-dimensional submanifold of M and $[N]$ is its fundamental class, then

$$w_1 \smile [N] \in \mathbb{Z}_2 \quad (857)$$

is equal to 1 if the orientation is preserved around N and -1 if the orientation is reversed.

Since w_1 , a first cohomology class, measures the obstruction to having an orientation, we expect that different orientations are classified by a zeroth cohomology class. Of course this is true — one picks an orientation for each connected component of M — but let us see why this happens in a more systematic way. We start from the exact sequence

$$1 \rightarrow SO(m) \rightarrow O(m) \rightarrow \mathbb{Z}_2 \rightarrow 1, \quad (858)$$

where the second to last map is the determinant map. This gives rise to a long exact sequence in cohomology. Since the groups involved are not Abelian, we have to clarify exactly what cohomology we are dealing with. We will be thinking in a Čech frame of mind, and considering cochains valued in smooth functions (and not constant functions like in regular Čech cohomology, where we assign n -fold intersections of patches with constant group elements) which take values in some group (I'll try to not use sheafy language). Thus the relevant part of the long exact sequence we get is

$$\dots \rightarrow H^0(M; \mathbb{Z}_2) \rightarrow H^1(M; \mathcal{C}^\infty(SO(m))) \rightarrow H^1(M; \mathcal{C}^\infty(O(m))) \rightarrow H^1(M; \mathbb{Z}_2) \rightarrow \dots, \quad (859)$$

since cohomology with coefficients in \mathcal{C}^∞ is the same thing as regular cohomology. Note that the Čech cohomology $H^1(M; \mathcal{C}^\infty(G))$ is only a set, and not a group, if G is not Abelian. This set comes with a distinguished “identity” element, and the exactness of the sequence means that the image of one map is the preimage of this distinguished element under the subsequent map.

We see that if the image of the map into $H^1(M; \mathbb{Z}_2)$ is non-zero, then by exactness the map $H^1(M; \mathcal{C}^\infty(SO(m))) \rightarrow H^1(M; \mathcal{C}^\infty(O(m)))$ cannot be surjective. This means that we have frame bundles that can be trivialized when using $O(m)$ as the structure group, but not $SO(m)$, meaning that the image of $H^1(M; \mathcal{C}^\infty(O(m)))$ determines the non-orientability of the manifold. Furthermore, suppose that M is orientable. Then the map $H^1(M; \mathcal{C}^\infty(SO(m))) \rightarrow H^1(M; \mathcal{C}^\infty(O(m)))$ is surjective, and so we can always reduce the structure group to $SO(m)$, since every $O(m)$ bundle comes from an $SO(m)$ bundle. Furthermore, from the exact sequence we see that different orientations are classified by $H^0(M; \mathbb{Z}_2)$, as promised.

Second Stiefel-Whitney class and beyond: Now for w_2 . Assume now that M is orientable. As we will see, w_2 measures the obstruction to having a spin structure over M . Let \tilde{t} be a lift of the t transition functions (which since $w_1 = 0$ we can take to be in $SO(m)$) to $\text{Spin}(m)$. Now the t 's must satisfy the cocycle condition, but in general the \tilde{t} 's do not: all that we require is that

$$(\delta\tilde{t})_{\alpha\beta\gamma} = f_{\alpha\beta\gamma} \mathbf{1}, \quad f_{\alpha\beta\gamma} \in \mathbb{Z}_2, \quad (860)$$

since the projection $\phi : \text{Spin}(m) \rightarrow SO(m)$ has kernel $\ker \phi = \mathbb{Z}_2$, so if $f_{\alpha\beta\gamma} = -1$ then we still project down to a well-defined oriented frame bundle. There is an ambiguity in the choice of \tilde{t} , since we can lift t to either \tilde{t} or $-\tilde{t}$. Let us specify this choice of lift by

$$t_{\alpha\beta} \mapsto h_{\alpha\beta}\tilde{t}_{\alpha\beta}, \quad (861)$$

where h is a Čech 1-cochain valued in \mathbb{Z}_2 . We see that doing this changes $f_{\alpha\beta\gamma}$ by a 2-coboundary. The cohomology class of the \mathbb{Z}_2 -valued 2-cochain $f_{\alpha\beta\gamma}$ is precisely w_2 , and by construction it measures the obstruction to lifting an $SO(m)$ frame bundle to a Spin bundle: thus w_2 is the obstruction to having a spin structure over M .

How is this the natural degree-2 version of w_1 ? It is the natural analogue since while w_1 measured the inability to extend a trivialization of the frame bundle over the 1-skeleton of M , w_2 measures the obstruction to extend the trivialization over the 2-skeleton. We think about this as follows: since we have assumed $w_1 = 0$, we can trivialize the bundle over the 1-skeleton. Consider a 2-cell c , and look at what the framing does as we travel around ∂c . Since $\pi_1(SO(m)) = \mathbb{Z}_2$, we have two options. If the trivialization on ∂c traces out a trivial path, then we can extend the framing into c . However if it traces out the nontrivial loop in $SO(m)$, then we cannot extend the framing into c (the induced spin structure on ∂c is non-bounding). That this inability to extend the framing is captured by w_2 is seen by realizing that -1 in $\text{Spin}(m)$ corresponds to the nontrivial element of $\pi_1(SO(m))$ (think about Deck transformations), and so if $(\delta\tilde{t})_{\alpha\beta\gamma} = -1$ then the framing on the boundary of the section of the 2-skeleton determined by $U_\alpha \cap U_\beta \cap U_\gamma$ twists by 2π , and hence $w_2 = (\delta\tilde{t})$ indeed measures the obstruction to extending the framing over the 2-skeleton. Also, analogously to the w_1 case, one can show that w_2 defines an obstruction to having $m-1$ linearly independent sections.

By Poincare duality, w_2 defines a codimension-2 submanifold \hat{w}_2 . This submanifold should be thought of as the region in M on which fermions cannot be defined. By the above arguments, we know that the framing must rotate by 2π along curves which link \hat{w}_2 , and so we can think about \hat{w}_2 as representing the location of vortices around which the fermions pick up an extra minus sign: they essentially behave like 2π vortices in a type II superconductor (and w_2 would be nontrivial if we had an odd number of such vortices). More formally, we can say that \hat{w}_2 is such that the 2-handles pierced by it are precisely those 2-handles over which the trivialization of the 1-skeleton of M does not extend.

We can see this things from a more algebraic perspective by looking at the exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(m) \rightarrow SO(m) \rightarrow 1, \quad (862)$$

where the second to last map is the projection. This gives rise to a long exact sequence in cohomology, the relevant part of which is

$$\dots \rightarrow H^2(M; \mathbb{Z}_2) \rightarrow H^1(M; \mathcal{C}^\infty(\text{Spin}(m))) \rightarrow H^1(M; \mathcal{C}^\infty(SO(m))) \rightarrow H^2(M; \mathbb{Z}_2) \rightarrow \dots, \quad (863)$$

again since cohomology with coefficients in $\mathcal{C}^\infty(\mathbb{Z})$ is the same thing as regular cohomology with coefficients in \mathbb{Z} .

We see that if the image of the map into $H^2(M; \mathbb{Z}_2)$ is non-zero, then by exactness the map $H^1(M; \mathcal{C}^\infty(\text{Spin}(m))) \rightarrow H^1(M; \mathcal{C}^\infty(SO(m)))$ cannot be surjective. This means that we have $SO(m)$ frame bundles that do not come from projecting a Spin bundle, meaning that

the image of $H^1(M; \mathcal{C}^\infty(SO(m)))$ in $H^2(M; \mathbb{Z}_2)$ determines the inability of the manifold to possess a spin structure. Thus the image of $H^1(M; \mathcal{C}^\infty(SO(m)))$ in $H^2(M; \mathbb{Z}_2)$ is w_2 . Like in the w_1 case, if we assume that M admits a spin structure, then by the above exact sequence we see that such spin structures are classified (non-canonically) by $H^1(M; \mathbb{Z}_2)$.

Now we want to mention another more geometric way of thinking about w_2 , which works particularly well in four dimensions. Let $S \subset M$ be an oriented and closed 2-submanifold. Consider capping w_2 with the fundamental class of S . We get

$$w_2 \frown [S] = w_2(TM|_S), \quad (864)$$

where on the RHS we are indicating the Stiefel-Whitney class of the restriction of the tangent space to S . Now we use the fact that $TM|_S = TS \oplus TN$ where N is the part of the tangent space normal to the surface S . Now we use the Whitney product formula to conclude that

$$w_2 \frown [S] = w_2(TS) + w_2(TN) + w_1(S) \cup w_1(N), \quad (865)$$

since the terms on the RHS are the only possible things in the product of degree 2. Since S was assumed orientable, the last term vanishes. Since S is 2-dimensional, $w_2(TS)$ is actually the mod 2 reduction of the Euler class $e(S)$ (to be discussed in a little bit), and so will be zero for any compact orientable surface,²⁹ and so $w_2(TS) = 0$. Thus

$$\widehat{w}_2 \cap S = \widehat{w}_2(TN). \quad (866)$$

Now TN is also 2-dimensional, so that $w_2(TN)$ is the mod 2 reduction of $e(TN)$, which is a Poincare dual to a collection of points. In fact, $\widehat{w}_2(TN)$ precisely measures the self-intersection number of S ! One can see this by taking another copy of S and displacing it slightly along directions determined locally by the normal bundle TN , so that this copy of S is a section of TN . Each time that this copy intersects the original surface will correspond to a zero of the section, which is measured by $e(TN)$. This is precisely what we do to define the self-intersection number.³⁰ Recapitulating, we have

$$\widehat{w}_2 \cap S = S \cap S \mod 2, \quad (867)$$

where on both sides we really mean the signed intersection number. This is essentially where the splitting of w_2 into Wu classes comes from.

We now want to convince ourselves that all orientable 3-manifolds have $w_2 = 0$, and consequently all admit a spin structure. The half-hazard way of doing this is the following: consider as before an arbitrary closed oriented submanifold S of M . Again using the product formula, we have

$$w_2 \frown [S] = w_2(TS) + w_1(S) \cup w_1(N), \quad (868)$$

but again by the fact that $w_2(TS)$ is the mod 2 reduction of $e(S)$ and S was assumed orientable, this vanishes since $e(S) \in 2\mathbb{Z}$. We conclude that the integral of w_2 over any closed, oriented surface is zero (I think this implies that w_2 is zero, but perhaps it is feasible for w_2 to have a nontrivial cap product with an unorientable N ? Should come back to this).

²⁹Just because $\chi(M) = 2 - 2g \in 2\mathbb{Z}$ vanishes mod 2.

³⁰Recall that we do this by thinking of fermions as ribbons and defining the linking number as the intersection number of one edge of the ribbon with the Seifert surface determined by the other edge.

The third Stiefel-Whitney class w_3 is usually pretty boring, because it is zero if both w_1 and w_2 are zero. This is because

$$\pi_2(SO(m)) = 0 \quad (869)$$

for all m . This implies that, assuming we have been able to extend the trivialization onto the 2-skeleton, $w_3 = 0$. Indeed, there will be an obstruction to extending the trivialization into a given 3-cell c if the framing on ∂c winds around in a noncontractible way. But this never happens, since $\pi_2(SO(m)) = 0 \forall m$. Thus if we have found a trivialization over the 0,1, and 2 cells, then we can always extend this to a trivialization over the 3-cells, and we don't need to worry about w_3 being nonzero.

As a corollary, the Euler characteristic of any orientable three-manifold is even.³¹ Now in the case of a 3-manifold M , since all the obstructions $\pi_i(SO(m))$ to extending the trivialization are \mathbb{Z}_2 -valued, the vanishing of the SW class w_3 actually means that $e(M) = 0$, not just $e(M) \in 2\mathbb{Z}$, but in other cases where the relevant $\pi_i(SO(m))$ are not just \mathbb{Z}_2 s, the vanishing of the SW classes only provide necessary, but not sufficient, conditions for extending the trivialization. As another way of re-stating the same thing, any orientable spin 3-manifold is parallelizable.

In general, the nonvanishing of the n th Stiefel-Whitney class indicates an obstruction to having $m - n + 1$ linearly independent nonvanishing sections of TM , or equivalently, it measures the obstruction to trivializing the frame bundle up to the n -skeleton of M (note that the nontriviality of w_n is a sufficient, but not necessary, condition for there not to exist $m - n + 1$ linearly independent non-zero sections). As we have mentioned, the top class w_m is the mod 2 Euler characteristic, by virtue of the fact that it is the \mathbb{Z}_2 reduction of the Euler class (representing the obstruction to fully trivializing the frame bundle throughout all of M). That is,

$$w_m \frown [M] = \chi(M) \mod 2. \quad (870)$$

Geometrically, the dual \widehat{w}_m is a collection of points at which there is an obstruction to defining a single vector field, i.e. \widehat{w}_m tells us where vector fields must vanish (so that e.g. \widehat{w}_m is the location of the cowlicks on a hairy ball). Since w_m is a mod 2 index, it doesn't distinguish between positive and negative index critical points. In particular since $\chi(M) \in 2\mathbb{Z}$ for any closed, orientable 2-manifold M , the associated w_2 vanishes, and so fermions can be defined on any of these manifolds (naively one might think that a framing would be hard to define on S^2 since S^2 is not parallelizable, but this is not the case since $\chi(S^2) = 2$ [and of course fermions can exist on a 2-sphere]).

The statement made at the beginning of the last paragraph can be proved by using the Whitney sum formula. Indeed, suppose we have a bundle E which admits n linearly independent nonzero sections. Then we can decompose E as

$$E = E_{\perp} \oplus \mathcal{O}_n, \quad (871)$$

where \mathcal{O}_n is a trivial rank- n bundle. Using the Whitney sum formula and the fact that $w_k(F) = 0$ if $k > \text{Rank}(F)$ for any bundle F , we have

$$w(E) = w(E_{\perp} \oplus \mathcal{O}_n) = w(E_{\perp}) = 1 + w_1(E_{\perp}) + \cdots + w_{m-n}(E_{\perp}). \quad (872)$$

³¹Of course, Poincare duality does better and tells us that all odd-dimensional manifolds actually have zero Euler characteristic.

Therefore we conclude that if E has n linearly independent nonzero sections, $w_k(E) = 0$ for $k > m - n$. In particular, the non-vanishing of $w_{m-n+1}(E)$ implies that there cannot exist n such sections. Changing the names of the variables, we can equivalently say that $w_n(E) \neq 0$ implies that E does not admit $m - n + 1$ linearly independent nonzero sections.

54 June 6 & 7 — Even more on the $O(N)$ and nonlinear σ models in two dimensions

This problem is a way to try to understand some of the content in Polyakov's book — the goal is to understand what's written there and to fill in the details that are omitted.

Consider the $O(N)$ vector model which maps spacetime into the $N - 1$ sphere, so that the symmetry group is $O(N)$. As we have seen earlier, this model admits a saddle point solution in $N \rightarrow \infty$ limit: using the saddle point, one can see how dimensional transmutation occurs and defines a mass scale.

Using the saddle point method, what is the propagator for the massive excitations? What kind of divergences arise and what kind of renormalization must be done? After looking at this (you should compute the field strength renormalization), go back and find the exact propagator and beta function to order g^2 (without using large N) by employing a background field method with a Wilsonian-picture momentum integration. Show how to do the renormalization and compare to the saddle point results. Show that as $N \rightarrow \infty$, the model describes a free theory.

Now consider the case of the $SU(N)$ nl σ m, still in 2 dimensions. Find the propagator and the beta function for g^2 . In the large N limit, is the theory described by the saddle point, like the $O(N)$ model is? Why not?

Solution:

The $O(N)$ vector model: The action, with the Lagrange multiplier λ to enforce that the vector n lives on S^{N-1} , is

$$S = \frac{1}{2g^2} \int (\partial_\mu n_a \partial^\mu n^a + \lambda(n^2 - 1)). \quad (873)$$

Assuming that a massive solution exists and integrating out n ,

$$S = -\frac{1}{2g^2} \int \lambda + \frac{N}{2} \ln \det(-\partial^2 + \lambda). \quad (874)$$

We have already seen how to do the saddle point multiple times: it gives the mass

$$\lambda = m^2 = \Lambda^2 \exp\left(-\frac{4\pi}{Ng^2}\right). \quad (875)$$

Let's find the 2-point function for the saddle point solution, which we will use for comparison later. We will primarily be interested in short-distance behavior (or small mass) in the

following so for $m \ll r^{-1}$ we find

$$\langle n(r) \cdot n(0) \rangle = \frac{Ng^2}{2\pi} \int dp \frac{e^{ipr}}{p^2 + m^2} \approx \frac{Ng^2}{2\pi} \int_0^{1/mr} d\alpha \frac{\alpha e^{i\alpha mr}}{1 + \alpha^2} \quad (876)$$

where $\alpha = p/m$ and the N comes from the N components of the n vector. Thus in the small mr limit we get the log which indicates that as $N \rightarrow \infty$ the n field is free:

$$\langle n(r) \cdot n(0) \rangle \approx -\frac{Ng^2}{2\pi} \ln(mr) = 1 - \frac{Ng^2}{2\pi} \ln(r/a), \quad (877)$$

where we have used the saddle point expression for m and where $\Lambda = a^{-1}$. Now as stated, this result cannot be correct for small N , since for $N = 2$ it predicts that the propagator for the n field has nontrivial field strength renormalization, which cannot be true since in that case the n field is a map into S^1 and therefore has no interactions and is free. Later on we will see that the correct thing to do is to replace N by $N - 2$.

Now we want to consider fluctuations about the saddle point. To this end, define the field ϕ through

$$\lambda = m^2 + \sqrt{\frac{2}{N}}\phi, \quad (878)$$

where the $q = 0$ momentum component of ϕ vanishes. Putting this in the $\text{Tr} \ln$ and expanding to third order gives

$$S_{eff} = S_{eff,\phi=0} - \frac{N}{2} \left(\frac{1}{N} \int_q \phi_q \phi_{-q} \Pi_2(q^2) - \frac{2}{N} \sqrt{\frac{2}{N}} \int_{q,r} \phi_q \phi_r \phi_{-q-r} \Pi_3(q, r) + \dots \right), \quad (879)$$

where the first term is the saddle point, $\Pi_2(q^2)$ is the polarization bubble with two external ϕ legs, $\Pi_3(q^2)$ is the analogous diagram with three external ϕ legs, and so on. In these diagrams, the internal propagators are $G(p) = 1/(p^2 + m^2)$. Note that there is no bubble with a single leg, since we are expanding around a saddle point. The $1/\sqrt{N}$ factor in front of the 3-legged bubble tells us that in the large N limit we can only worry about the polarization bubble as far as renormalizability goes. From this expression, we see that the propagator for the ϕ field is

$$G_\phi(q^2) = \frac{1}{\Pi_2(q^2)}. \quad (880)$$

Thus the λ field has gained some dynamics from the n field. Let us look at the large q limit of this expression to see how renormalization should work. We have

$$\Pi_2(q^2) = \int_p \frac{1}{(p^2 + m^2)((p - q)^2 + m^2)} = \int_p \int_x \frac{1}{(p^2 + q^2(x^2 + x) + m^2)^2}. \quad (881)$$

We can estimate this in the large q limit by noting that only small x will be important to the integral, so that defining $dy = q^2 dx$,

$$\Pi_2(q^2) \approx \int_p \int_y \frac{q^{-2}}{(p^2 + y + m^2)^2} \approx \int_p \frac{q^{-2}}{p^2 + m^2}, \quad (882)$$

where the integral over y was done from 0 to q^2 . So then

$$\Pi_2(q^2) \approx \frac{1}{2\pi} \int dp \frac{pq^{-2}}{p^2 + m^2}. \quad (883)$$

In the $q^2 \gg m^2$ limit then,

$$\Pi_2(q^2) \approx \frac{1}{2\pi q^2} \ln(q^2/m^2), \quad (884)$$

and so the propagator for the ϕ field is

$$G_\phi(q^2) = \frac{2\pi q^2}{\ln(q^2/m^2)}, \quad (885)$$

which diverges as $q \rightarrow \infty$, which is bad. This divergence will also appear in the propagator for the n field, which one can see by considering the first 1-loop correction to the n field two-point function.

The reason for this divergence is essentially due to the fact that we haven't renormalized the mass m^2 of the elementary excitations. As Polyakov points out in the book, the divergence arises because we are confining the particles to the sphere with the λ field, which because of the uncertainty relation leads to the above divergences. So we can fix this by adding a mass term for λ :

$$\mathcal{L} \mapsto \frac{1}{2g^2} [(\partial n)^2 + \lambda(n^2 - 1)] - \frac{\lambda^2}{4\beta}. \quad (886)$$

Solving the equation of motion for λ gives us a Lagrangian with a term $\beta(n^2 - 1)/g^4$, and so in the large β limit (small mass for λ) we recover the σ model. When we do the large N expansion with this term added, we just have to add the new mass term into the ϕ propagator:

$$G_\phi(q^2) \mapsto \frac{1}{\Pi_2(q^2) + \frac{1}{\beta N}}, \quad (887)$$

which resolves the power law divergences at large q .

So, we want to do mass renormalization. The way to do it is to ensure that the inverse propagator has no power divergences like the one appearing above for the naive ϕ propagator. To this end we subtract off the zero momentum part of the self energy to define the mass renormalization, so that the n field Greens function is

$$G_n(q^2) = \frac{1}{q^2 + m^2 + \Sigma(q^2) - \Sigma(0)}. \quad (888)$$

Let us now check that subtracting off $\Sigma(0)$ cancels the power divergences. The first term in $\Sigma(q^2)$ is a diagram with a straight n field line of momentum q that has a ϕ field arc attached to it at momentum p . There are two factors of $\sqrt{2/N}$ coming from the two vertices, and so we can write (we are just interested in the structure of the divergences, so we will take $q^2 \gg m^2$ in what follows)

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N} \int_p \frac{1}{\Pi_2(p^2)} \left(\frac{1}{(p-q)^2} - \frac{1}{p^2} \right). \quad (889)$$

As we can see, if we didn't have the second $\Sigma(0)$ term, the power law divergence of the ϕ field propagator would infect the self energy of the n field, and we would not get something that was logarithmically divergent. So indeed, mass renormalization is the correct procedure for dealing with the power divergence.

Now combine the two fractions in the last integrand into one and then multiply the numerator and denominator by $(p+q)^2$:

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N} \int_p \frac{1}{\Pi_2(p^2)} \left(\frac{(2q \cdot p - q^2)(q^2 + p^2 + 2q \cdot p)}{p^2(p^4 + q^4 - 4(q \cdot p)^2)} \right). \quad (890)$$

Since the denominator is even under $p \mapsto -p$ we can simplify this to

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N} \int_{q < p < \Lambda} dp \frac{p}{2\pi\Pi_2(p^2)} \frac{4(q \cdot p)^2 - p^2 q^2}{p^6(1 + q^4/p^4 - 4(p \cdot q)^2/p^4)} + \text{finite}. \quad (891)$$

We can drop the terms other than the p^6 in the denominator and absorb them into the +finite part. Then using $\int p_\mu p_\nu \rightarrow \frac{1}{2}\delta_{\mu\nu} \int p^2$ since we are in two dimensions, we get

$$\Sigma(q^2) - \Sigma(0) = \frac{1}{N} \int_{q < p < \Lambda} dp \frac{2}{p \ln(p^2/m^2)} + \text{finite}. \quad (892)$$

Now $d \ln(\ln(p^2/m^2)) = 2/[p \ln(p^2/m^2)]$, so

$$\Sigma(q^2) - \Sigma(0) = \frac{1}{N} \ln \left(\frac{\ln(\Lambda^2/m^2)}{\ln(q^2/m^2)} \right) + \text{finite}. \quad (893)$$

As promised, we see that the new propagator has no power law divergences, but it does have a weird nested log structure. Thus mass renormalization has led to logarithmic divergences which can subsequently be cleaned up with field strength renormalization for the n field.

Now let us calculate the propagator directly (i.e. not assuming large N but assuming small $g^2(q)$), without using the saddle point approximation, and see to what extent the saddle point results are reproduced. We do this essentially by the background field method. We split up the field into slow and fast components by letting the slow component be some vector $n_0(x)$ and letting the fast components be fluctuations about that vector. Thus the relevant decomposition is

$$n = \sqrt{1 - \psi^2} n_0 + \psi_a e^a, \quad (894)$$

where the $\{e^a\}$ are a collection of vectors orthogonal to $n_0(x)$ on the sphere³². Note that $n_0(x)$ has unit length and is orthogonal to all of the frames e^a . Now since n lies on the sphere, we know that $\partial_\mu n_0(x)$ will lie in the tangent space at x and hence be orthogonal to $n_0(x)$ and expressible in terms of the e^a 's. Thus for some 1-form ω^a and some $\mathfrak{o}(N-1)$ -valued 1-form A (since $O(N-1)$ is the group which acts on the basis vectors normal to n_0), we may follow Polyakov and write

$$\partial^\mu n_0 = \omega_\mu^a e^a, \quad \partial_\mu e^a = [A_\mu]^{ab} e_b - \omega_\mu^a n_0. \quad (896)$$

³²This is an expansion which is essentially tantamount to going over to Riemann normal coordinates. The action is $\partial_\mu n^a \partial^\mu n_a$, which for our present decomposition becomes schematically

$$\partial_\mu (n_0 + \psi)^a \partial^\nu (n_0 + \psi)^b (\delta_{ab} - \frac{1}{3} R_{acbd} \psi^c \psi^d), \quad (895)$$

where R_{acbd} is the Riemann curvature tensor for the target space S^{N-1} .

The minus sign in the second term is needed since n_0 being \perp to all the e^a implies that the change of n_0 in the e^a direction is equal to the change of e^a in the $-n_0$ direction (draw a picture to check, or notice that the fact that the whole ensemble of frames rotates rigidly implies $n_0 \cdot \partial_\mu e^a + e^a \cdot \partial_\mu n_0 = 0$). Now we put this into the action for the n field (the $n^2 = 1$ constraint is explicitly built into our parametrization, so no need for Lagrange multipliers). The quadratic parts in the fields are

$$S_{\psi^2} = \frac{1}{2g^2} \int \left[(\delta_{ab} \partial_\mu \psi_b - A_\mu^{ab} \psi_b)^2 + B_\mu^a B^{b\mu} (\psi_a \psi_b - \psi_c \psi^c \delta_{ab}) + B_\mu^a B^{a\mu} \right], \quad (897)$$

where we have used the skew symmetry of A by virtue of the fact that it lives in $\mathfrak{o}(N-1)$. The $B_\mu^a B_a^\mu$ term is the slow $(\partial_\mu n_0)^2$ part.

Let us now compute the β function, the field strength renormalization, and the propagator. Because of the decomposition of the n field we have chosen we will work in the Wilsonian point of view where we change the high energy cutoff by a small amount. Note that A_μ has dimension 1 and appears in the action only in the covariant derivative. A_μ really is a gauge field, since it is a connection on a frame bundle which just tells us how to relate one arbitrary choice for the e^a frame at one point to the arbitrary choice made at another. Thus the only way that A can appear in what follows is in the Maxwell term $F_A \wedge \star F_A$, but since we are in two dimensions this is irrelevant. Thus we can ignore the A field in matters concerning renormalization.

Now let us compute the correlation function

$$C_\psi^{ab} = \langle \psi^a \psi^b - \psi_c \psi^c \delta^{ab} \rangle. \quad (898)$$

As mentioned above, we only need to care about the ω fields. Since the ω fields couple quadratically to the ψ 's we have a single tetravalent vertex, and so to one loop order our only diagrams are a ψ loop and a ψ loop with an ω loop glued on. Since the free term for the ψ 's is diagonal in a, b , we get

$$C_\psi^{ab} = (\delta^{ab} - (N-1)\delta^{ab}) \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{dp}{2\pi} \frac{g^2}{p}, \quad (899)$$

since there are $N-1$ fields in the $\psi_c \psi^c$ contraction. So then taking $d\Lambda = -\delta\Lambda$ to be infinitesimal,

$$C_\psi^{ab} = \delta^{ab} (N-2) \frac{g^2}{2\pi} d \ln \Lambda. \quad (900)$$

When we expand the exponential of the action in small ψ , do the integral over the fast fields, and then re-exponentiate, we thus get a term that goes as

$$\frac{1}{2g^2} \int B_\mu^a B^{a\mu} (N-2) \frac{g^2}{2\pi} d \ln \Lambda. \quad (901)$$

Therefore the effective charge is

$$g_{eff}^2 = \frac{g^2}{1 + g^2 \frac{N-2}{2\pi} d \ln \Lambda}. \quad (902)$$

Writing $g_{eff}^2 = g^2 + dg^2$ we obtain the beta function

$$\beta(g^2) = \frac{2-N}{2\pi}g^4, \quad (903)$$

which as we have seen several times so far is asymptotically free when the symmetry group $SO(N)$ is non-Abelian (or when the n field maps into a space with positive curvature). We will need the expression for $g^2(p)$, which we can get by integrating the β function from a reference scale μ to p :

$$g^2(p) = \frac{g^2(\mu)}{1 - \frac{2-N}{2\pi}g^2(\mu)\ln(p/\mu)}. \quad (904)$$

Now we want to get the 2-point function for the n fields and compare it to what we got with the saddle point approximation. We have

$$\begin{aligned} \langle n(r)n(0) \rangle &= \langle n_0(r)n_0(0) \sqrt{(1-\psi^2(r))(1-\psi^2(0))} + \psi_a(r)\psi^a(0) \rangle \\ &\approx \langle n_0(r)n_0(0)(1-\psi^2(0)) \rangle \\ &= \langle n_0(r)n_0(0) \rangle_{p<\Lambda-\delta\Lambda}(1 - \langle \psi_c(0)\psi^c(0) \rangle_{\Lambda-\delta\Lambda < p < \Lambda}), \end{aligned} \quad (905)$$

where we dropped the $\psi(r)\psi(0)$ term since the ψ 's are rapidly fluctuating, and expanded to quadratic order in ψ . We know the ψ^2 expectation value, since $\langle \psi_c\psi^c \rangle = -C_\psi^{aa} + \langle \psi_a\psi_a \rangle$ (no sum over a). So then

$$\langle n(r)n(0) \rangle = \langle n_0(r)n_0(0) \rangle_{p<\Lambda-\delta\Lambda} \left(1 + \frac{N-1}{2\pi} g^2(\Lambda) d\ln\Lambda \right), \quad (906)$$

since the two-point function for a single ψ component goes as $-d\ln\Lambda$.

From here we can read off the field strength renormalization: in order to ensure that the log divergences in the 2 point functions get canceled to one loop order, we need to have

$$\gamma(g^2) = \frac{N-1}{2\pi}g^2 \quad (907)$$

(the positive sign comes from the fact that the free propagator goes to minus of the log term).

Knowing this, we can finally get the expression for the n -field propagator which is not limited to the large N saddle point. Let the propagator be $G_n(p^2)$. Then we have

$$\frac{\partial(G_np^2)}{\partial\ln(p/\mu)} = \gamma(g^2(p))(G_np^2), \quad (908)$$

with μ the RG scale (momentarily going to a continuum RG picture). This equation is most easily proven graphically: the field strength renormalization counter terms appear in the full expression for the propagator in a geometric series of the form $\frac{1}{p^2}\gamma(g^2)\ln(p/\mu)\frac{1}{p^2} + \dots$, which gives rise to the above equation. We can also write this as

$$\frac{\partial\ln(p^2G_n)}{\partial\ln(p/\mu)} = \gamma(g^2(p)). \quad (909)$$

Thus to order g^2 we can use this and our knowledge of γ to get

$$\begin{aligned} G_n(p^2) &= \frac{1}{p^2} \exp \left(\frac{N-1}{2\pi} g^2(\mu) \int \frac{d \ln(p/\mu)}{1 + \frac{N-2}{2\pi} g^2(\mu) \ln(p/\mu)} \right) \\ &= \frac{1}{p^2} \left(1 + \frac{N-2}{2\pi} g^2(\mu) \ln(p/\mu) \right)^{\frac{N-1}{N-2}}. \end{aligned} \quad (910)$$

This would look like what we would expect from a free field provided that the term in the big parenthesis was not raised to a power. Thus as $N \rightarrow \infty$ we have

$$G_n(p^2; N \rightarrow \infty) = \frac{1}{p^2} \left(1 + \frac{N-2}{2\pi} g^2(\mu) \ln(p/\mu) \right), \quad (911)$$

which indeed indicates that the n field becomes free in the $N \rightarrow \infty$ limit.

Now we can compare this to the saddle point propagator we derived earlier. We see from the correlation function (877) that the difference between the saddle point answer and the actual result is just a replacement of N with the correct coefficient $N-2$ (and so in particular we get the correct behavior in the Abelian $N=2$ case).

$SU(N)$ nl σ m: Now we want to look at the case of the $SU(N)$ nonlinear σ model (here the fields are sections of $SU(N)$ bundles [though we will actually use $U(N)$], unlike the case of the $O(N)$ model where $O(N)$ was the symmetry group, not the target space. Sorry for the bad but standard terminology), which is very similar to the $O(N)$ model technically, but which must give us different answers since as we will see the $SU(N)$ version is *not* described by the saddle point in the large N limit.

To get the propagator we need the β function (so that we can get the effective coupling constant at an arbitrary momentum scale) and the anomalous dimension. We already found the β function back in (734). There we had $\lambda = 2g^2$, and we had taken $N=2$. Getting the β function for general N isn't hard, though: we just have N terms in the trace rather than 2, and so we can translate our old result to

$$\beta(g^2) = -\frac{Ng^4}{4\pi}, \quad (912)$$

which holds to order g^4 . Just like the $O(N)$ model we have asymptotic freedom, with it becoming "more" asymptotically free at large N . Integrating this gives the charge at a given momentum scale in terms of the RG scale μ :

$$g^2(p) = \frac{g^2(\mu)}{1 + \frac{N}{4\pi} g^2(\mu) \ln(p/\mu)}. \quad (913)$$

Now we need to know the anomalous dimension. This is also easy to get by looking back at the previous problem we did. We first write $T^a T^a = C_2 \mathbf{1}$ where the T^a are the $SU(N)$ generators. The quadratic casimir here is $C_2 = (N^2 - 1)/2N$ (check with $C_2 = 3/4$ for $SU(2)$, which works since the generators are $X/2, Y/2, Z/2$). By looking back at the previous problem, we find the anomalous dimension

$$\gamma = \frac{N^2 - 1}{2\pi N} g^2. \quad (914)$$

Then we can use the earlier formula for $\partial \ln(Gp^2)$ to get the propagator for the $SU(N)$ field:

$$\begin{aligned} G_U(p^2) &= \frac{1}{p^2} \exp \left(\int d \ln(p/\mu) \frac{C_2}{\pi} \frac{g^2(\mu)}{1 + \frac{N}{4\pi} g^2(\mu) \ln(p/\mu)} \right) \\ &= \frac{1}{p^2} \left(1 + \frac{N}{4\pi} g^2 \ln(p/\mu) \right)^{4C_2/N}. \end{aligned} \quad (915)$$

When we integrate this over p to get the correlator in real space, the exponent gets another power since we are integrating the above expression in parenthesis against $d \ln p$. In the $N \rightarrow \infty$ limit $4C_2/N + 1 \rightarrow 3$, so we get

$$G_U(r, N \rightarrow \infty) \approx \left(1 - \frac{Ng^2}{4\pi} \ln(r/a) \right)^3. \quad (916)$$

This does *not* go to just a single \ln in the $N \rightarrow \infty$ limit, and so the $SU(N)$ theory does not look like a free theory as $N \rightarrow \infty$ (it actually *does* go to a free theory as $N \rightarrow \infty$, but the propagators are still not free and it only becomes free in a way that is hard to see with the present variables).

Given that the propagator is not free, the saddle-point approximation should not be good as $N \rightarrow \infty$. Why is this? Proceeding along the lines of the previous analysis, it would seem like we could take the $\text{Tr} \ln$ in the effective action for the Lagrange multiplier and do a saddle point analysis on it, since the coefficient of the $\text{Tr} \ln$ will contain an N .

This is not the case though, essentially because the matrix nature of the principal fields mean that the number of Lagrange multiplier fields grows with N , which destroys the saddle point. To see this, let us write the action as

$$S = \frac{1}{g^2} \int \text{Tr}[\partial_\mu U \partial^\mu U^\dagger], \quad (917)$$

where $U(x)$ is a $U(N)$ -valued field. There is really no difference between this and the $SU(N)$ case since the determinant part (the Abelian $U(1)$) decouples and doesn't affect issues relating to renormalizability.

As with the $O(N)$ model, we enforce that $UU^\dagger = \mathbf{1}$ using a Lagrange multiplier (note the differing factor of $1/2$ since we now are working with complex fields):

$$S = \frac{1}{g^2} \int \left(\text{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \text{Tr}[\lambda(\mathbf{1} - UU^\dagger)] \right). \quad (918)$$

Integrating over the U 's, which is possible since the addition of λ loosened the unitary constraint on the U 's and turned the integral into a Gaussian, gives

$$S_{eff}[\lambda] = -\frac{1}{g^2} \int \text{Tr} \lambda + N \text{Tr} \ln(-\partial^2 \mathbf{1} + \lambda). \quad (919)$$

Here we get the N from e.g. diagonalizing and seeing that after taking the trace we are left with N complex scalars.

The effective action for λ looks essentially the same as it did for the $O(N)$ vector model, so it is natural to expect that λ gets an expectation value given by some saddle point condition:

$$\langle \lambda \rangle = m^2 \mathbf{1}. \quad (920)$$

Indeed, doing the saddle point analysis by varying with respect to λ_{aa} gives the same answer for $\langle \lambda_{aa} \rangle$ as it did for λ in the $O(N)$ vector model, just with slightly different numerical coefficients accounting for the complex nature of the field.

Just as we defined the ϕ fields in the $O(N)$ vector model, we define Φ fields as

$$\lambda = m^2 \mathbf{1} + \frac{1}{\sqrt{N}} \Phi, \quad (921)$$

where the trace of Φ has no $q = 0$ component, i.e. $\int \text{Tr}\Phi = 0$, so that Φ doesn't affect the expectation value. Now put this into the effective action:

$$S_{\text{eff}} = - \int \text{Tr} \left[\mathbf{1} \left(\frac{m^2}{g^2} - p^2 - m^2 \right) \right] - \frac{1}{2} \int_q \Phi_{ab} [\Pi_2]_{ab,cd}(q^2) \Phi_{cd} \\ + \frac{1}{3\sqrt{N}} \int_{q,r} [\Pi_3]_{ab,cd,ef}(q, r) \Phi_{ab,q} \Phi_{cd,r} \Phi_{ef,-q-r} + \dots, \quad (922)$$

where e.g.

$$[\Pi_2]_{ab,cd}(q^2) = \frac{\pi_2(q^2)}{2} [\delta_{ac} \delta_{bd} + \delta_{bc} \delta_{ad}], \quad [\Pi_3]_{ab,cd,ef}(q^2) = \frac{\pi_3(q, r)}{6} [\delta_{ae} \delta_{fd} \delta_{cb} + \delta_{af} \delta_{ed} \delta_{bc} + \dots], \quad (923)$$

where the \dots indicates all ways to pair the indices in Kronecker delta functions so that a does not appear in the same δ as b and likewise for the pairs cd and ef . Here π_2 is the regular polarization bubble (with a fixed color structure) with internal propagators computed using the mass m^2 and π_3 is the three-legged version.

In the $O(N)$ model the saddle point was stable and we got a free theory since the nonlinear part with the Π_3 kernel made an insignificant contribution as N went to infinity, allowing the corrections to the free Φ propagator to vanish. This is not so in the $SU(N)$ model, though, and the saddle-point does not work. We see this by writing the Π_3 term in double-line notation as a sum of three-tentacled amoeba, where the sum is over the various ways to assign matrix indices to the six ends of the lines in the amoeba. The first order correction to the free Φ propagator caused by the Π_3 term thus looks like $N^{-1} \times \mathbb{P}$, where \mathbb{P} is a double-line version of the regular polarization bubble. Importantly, it has a completely internal circular line that when summed over produces a factor of N , cancelling the N^{-1} that came from the two factors of $1/\sqrt{N}$. Thus this contribution can not be ignored as $N \rightarrow \infty$, and so the nonlinearities are important even at large N , corroborating our finding that the $SU(N)$ model was not free at large N . Proceeding in the standard way and using the Gauss-Bonnet theorem, one can show that all planar diagrams do not vanish as $N \rightarrow \infty$, while diagrams that glue up on higher-genus surfaces become increasingly small.

55 June 8 — Landau parameters and mass renormalization with magnetization

This is one of the problems in Piers Coleman's book (there are a few typos in the chapter and one in the problem though, so watch out). Derive Landau's mass renormalization formulae

is the case where the Fermi liquid has a nonzero magnetization. The starting point is the Hamiltonian in the presence of an external field for both the current and the z component of the spin current:

$$H[\mathbf{v}, \mathbf{w}] = \int \frac{1}{2m} \Psi^\dagger [-i\nabla - \mathbf{v} - Z\mathbf{w}]^2 \Psi + V, \quad (924)$$

where $\Psi = (\psi_\uparrow, \psi_\downarrow)^T$, Z is the third Pauli matrix, and V denotes interactions (which we assume conserve the spin current). \mathbf{v} is an external velocity that couples to the particle number current, and \mathbf{w} is an external field which couples to the spin current.

Find the renormalized mass parameters associated with the background fields in terms of the Landau parameters F_l^a, F_l^s , where F_l^s is the l th multipole moment of the density-density spin-independent part of the interaction, and F_l^a is the l th multipole part of the spin-dependent part (that depends on the dot product of the two spins involved). Assume that there is no net magnetization in the system. Next, find the renormalized masses of the two spin species in the case of a net magnetization, when the density of states N_σ is not independent of σ . See Coleman's book for background on these parameters.

Solution:

We start by writing the energy of the Fermi liquid as

$$E = E_0 + \sum_{p\sigma} \epsilon_{p\sigma}^{(0)} n_{p\sigma} + \frac{1}{2} \sum_{p\sigma, p'\sigma'} f_{p\sigma, p'\sigma'} n_{p\sigma} n_{p'\sigma'}, \quad (925)$$

where $\epsilon_{p\sigma}^{(0)}$ is measured relative to the Fermi surface and is the energy of a single isolated quasiparticle. The first order variation of E is defined as $\epsilon_{p\sigma}$:

$$\frac{\delta E}{\delta n_{p\sigma}} = \epsilon_{p\sigma} = \epsilon_{p\sigma}^{(0)} + \sum_{p'\sigma'} f_{p\sigma, p'\sigma'} n_{p'\sigma'}, \quad (926)$$

which captures the energy of a quasiparticle when interactions with others are included. This is essentially the quasiparticle energy, renormalized by the effect of the interactions. Varying this equation gives us an expression for $\delta\epsilon_{p\sigma}$ in terms of $\delta n_{p\sigma}$. The latter is, for small departures of ϵ from the equilibrium value $\epsilon^{(0)}$,

$$n_{p\sigma} = f(\epsilon_{p\sigma}^{(0)}) + \frac{\partial f}{\partial \epsilon_{p\sigma}}(\epsilon_{p\sigma}^{(0)}) \delta\epsilon_{p\sigma}. \quad (927)$$

f here is the Fermi distribution, so at zero temperature we get

$$n_{p\sigma} = n_{p\sigma}^{(0)} - \delta_D(\epsilon_{p\sigma}^{(0)}) \delta\epsilon_{p\sigma}, \quad (928)$$

where δ_D is the Dirac delta. So putting this expression for $\delta n_{p\sigma}^{(0)}$ into the formula for $\delta\epsilon_{p\sigma}$, we get

$$\delta\epsilon_{p\sigma} = \delta\epsilon_{p\sigma}^{(0)} - \sum_{p'\sigma'} f_{p\sigma, p'\sigma'} \delta_D(\epsilon_{p\sigma}^{(0)}) \delta\epsilon_{p'\sigma'}. \quad (929)$$

The variation we will consider is that imposed by external fields which couple to the particle number current and spin current, which we will assume are conserved. For small applied fields, we see from the problem statement that

$$\delta\epsilon_{p\sigma}^{(0)} = -\frac{1}{m}\mathbf{p} \cdot (\mathbf{v} + \sigma\mathbf{w}). \quad (930)$$

Wolog, we can fix \mathbf{v} and \mathbf{w} to point in e.g. the \hat{z} direction, so that $\delta\epsilon_{p\sigma}^{(0)} \propto \cos\theta_p$.

Now we decompose the interaction into spherical harmonics / Legendre polynomials:

$$f_{p\sigma,p'\sigma'} = \sum_{\alpha=s,a} \sum_{l=0}^{\infty} \frac{1}{2N_{\sigma}(0)} (2l+1) F_l^{\alpha} P_l(\cos\theta), \quad (931)$$

where θ is the angle between \mathbf{p} and \mathbf{p}' . Here the sum over α is over the two representations that the interactions can be in as far as the spin degrees of freedom are concerned. $\alpha = s$ is the trivial representation, which is independent of spin, while $\alpha = a$ is dependent on the dot product of the two spins involved (this is the only other possibility by rotational invariance), and so we can rotate to a basis where $F_l^a \propto \sigma\sigma'$. We have also divided by the density of states so that the F_l^{α} are dimensionless. The choice of $2N_{\sigma}(0)$ will be explained later.

Now we insert this into the expression for $\delta\epsilon_{p\sigma}$. The Dirac delta makes us integrate over the Fermi surface, and we do so with

$$\sum_p \delta_D(\epsilon_{p\sigma}^{(0)}) = N_{\sigma}(0) \int \frac{d\Omega_p}{4\pi}, \quad (932)$$

where $N_{\sigma}(0)$ is the appropriate density of states. Now since we are integrating over the sphere against $\delta\epsilon_{p\sigma}$ which is proportional to $\cos\theta$, we select out the $l = 1$ component of the decomposition of the interaction (again by spin rotational invariance and actual rotational invariance, we can choose these two θ 's to be the same). That the $l = 1$ component gets selected is no surprise, since a uniform velocity field will displace the FS linearly in some direction, which is a dipolar perturbation. The normalization factors ensure that all the 4π 's and stuff cancel. So then

$$\delta\epsilon_{p\sigma} = \delta\epsilon_{p\sigma}^{(0)} - \sum_{\sigma'} \sum_{\alpha} \frac{1}{2N_{\sigma}(0)} N_{\sigma'}(0) F_1^{\alpha} \delta\epsilon_{p\sigma'}. \quad (933)$$

First let us assume that there is no net magnetization, so that $N_{\uparrow}(0) = N_{\downarrow}(0)$. We plug in the form for $\delta\epsilon_{p\sigma}^{(0)}$, and define the renormalized masses by

$$\delta\epsilon_{p\sigma} = -\mathbf{p} \left(\frac{\mathbf{v}}{m^*} + \sigma \frac{\mathbf{w}}{m_s^*} \right), \quad (934)$$

where we have a separate renormalization for the spin part. Putting this in and equating coefficients of the background fields on both sides of the equation, one sees that the coefficient of \mathbf{v} contains only F_1^s (since the F_1^a part dies by antisymmetry under the spin sum on σ'), while the coefficient of \mathbf{w} contains only F_1^a (for the same reason). After a little algebra, we get

$$m^* = (1 + F_1^s)m, \quad m_s^* = (1 + F_1^a)m. \quad (935)$$

Thus, the phrase “mass renormalization” makes sense here, since the F_1^α ’s appear just like counterterms that enter when renormalizing a QFT.

Now we consider the case where there is a net magnetization, with the density of states $N_\uparrow \neq N_\downarrow$. We want to find the renormalized masses m_σ^* , so we just turn on an external velocity field \mathbf{v} . The renormalized masses are defined by

$$\delta\epsilon_{p\sigma} = -\mathbf{p} \cdot \frac{\mathbf{v}}{m_\sigma^*}. \quad (936)$$

We again use (933), and set the coefficient of \mathbf{v} to vanish. Because the density of states are different, the antisymmetric F_1^a piece survives and makes a finite contribution. We thus end up having to solve

$$0 = -\frac{1}{m} + \frac{1}{m_\downarrow^*} + \frac{1}{2N_\downarrow} \left(F_1^s \left[\frac{N_\uparrow}{m_\uparrow^*} + \frac{N_\downarrow}{m_\downarrow^*} \right] + F_1^a \left[-\frac{N_\uparrow}{m_\uparrow^*} + \frac{N_\downarrow}{m_\downarrow^*} \right] \right), \quad (937)$$

together with the companion equation where the second term is $(m_\downarrow^*)^{-1}$. Putting these equations in mathematica and massing them a bit gives

$$\frac{1}{m_\sigma^*} = \frac{1}{2mn_\sigma} \left[\frac{1}{1+F_1^s} + \sigma \frac{M}{1+F_1^a} \right], \quad (938)$$

where $M = n_\downarrow - n_\uparrow$ and $n_\sigma = N_\sigma/(N_\uparrow + N_\downarrow)$. One sees that when $M = 0$ we recover the expected answer.

Note that in the book, he tells you to derive something different. The answer he tells you to get is not right since it doesn’t have the factor of σ in the second term on the LHS, but it also doesn’t have the factor of $1/n_\sigma$ in front. I think this may be related to the somewhat ad-hoc choice of making F_1^α dimensionless by dividing by $2N_\sigma$, which reproduces the correct result in the $M = 0$ case but might not be the most universal choice. After playing around in Mathematica for far too long I couldn’t come up with any way to make the interaction dimensionless that gave the answer listed in the book, so I’ll just go with this. Plus, this factor kind of makes sense, since we expect a larger density of states to give a larger effective mass (there’s more stuff to “get in the way” of quasiparticles moving around the FS).

56 June 9 — Algebraic entanglement entropy reminder

Remind yourself of how entanglement entropy is defined in gauge theories, focusing on lattice Abelian gauge theories to make things easy. Recapitulate why the natural generalization of entanglement entropy is best called algebraic entropy, and explain how to define the entropy of an algebra of operators \mathcal{A} , especially when the center $Z(\mathcal{A})$ is nontrivial. For help see e.g. the appendix in Daniel Harlow’s paper on the RT formula and error correction.

Solution:

A word on notation: \mathcal{A}_A will denote the algebra of operators supported within a spatial region A . When we don't have a particular spatial region in mind and are just referring to an algebra, we will just write \mathcal{A} . Is this bad notation? Yes. We will also infuriatingly use the index a to label direct sum decompositions.

Generalities on algebraic entropy and what to do about nontrivial centers: We first start with addressing how to view entanglement entropy from an algebraic point of view. The idea is to focus on the entanglement entropy of an algebra of operators more abstractly, rather than tying ourselves down by thinking of these operators are corresponding to a certain physical region of space (so really, we should be calling it algebraic entropy). This viewpoint is required if we are interested in gauge theories, but more generally we might want to deal with algebras other than the algebra of all operators supported within a certain region.

Although our focus is on the algebra rather than the Hilbert space, throughout we will assume a finite-dimensional Hilbert space (more importantly, a finite-dimensional representation of the algebra of interest), since we have lattice gauge theory in mind. Seeing to what extent the statements below hold in the infinite-dimensional case requires a bit more work.

Suppose we are given an algebra \mathcal{A} . If \mathcal{A} has trivial center (i.e. if $Z_{\mathcal{A}} = \{\lambda \mathbf{1} \mid \lambda \in \mathbb{C}\}$), then it is a factor, meaning that it induces a tensor product decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, on which it acts as $L(\mathcal{H}_A) \otimes \mathbf{1}_{\bar{A}}$, where $L(\mathcal{H}_A)$ are the linear operators on \mathcal{H}_A .

In such a case, the algebraic entropy of the subalgebra \mathcal{A}_A (which we haven't defined yet) is the same as the entanglement entropy of the region A , namely

$$S(\mathcal{A}_A) = -\text{Tr}(\rho_A \ln \rho_A). \quad (939)$$

If $Z_{\mathcal{A}}$ is nontrivial, things change. In this case, we no longer have a simple $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ factorization. Schematically, this is because the tensor product $U \otimes V$ of two vector spaces is defined to be "the freest possible bilinear product" between U and V , meaning that we impose $U \otimes V \ni (au, v) = (u, av)$ for $a \in \mathbb{C}$, but no other relations. Thinking of $\mathcal{A} = \mathcal{A}_A$ as the collection of operators supported within some spatial region A , then if $Z_{\mathcal{A}}$ is nontrivial we would have relations like $(zu, v) = (u, zv)$ for z a central element and (u, v) an element of a putative tensor product. This only jives with the definition of \otimes if z is a scalar, i.e. if the center is trivial. If $Z_{\mathcal{A}}$ is nontrivial one could imagine doing something like factoring \mathcal{H} as $\mathcal{H} = \mathcal{H}_A \otimes_{Z_{\mathcal{A}}} \mathcal{H}_{\bar{A}}$ where $Z_{\mathcal{A}}$ is treated as the tensor unit, but this is vague and the wrong direction to go in.

Central elements typically come from non-local constraints, like gauge constraints. When the algebras in question are associated to the operators within a spatial region A , the central elements will usually correspond to degrees of freedom living on ∂A . We will see that elements in $Z_{\mathcal{A}}$ contribute a Shannon term to the entanglement entropy, since they commute with everything and as such are "classical". The Shannon term will usually manifest itself as the entropy associated to the boundary conditions on A .

Since stuff in $Z_{\mathcal{A}}$ commutes with everything in \mathcal{A} , we can work in a representation in which all the elements in $Z_{\mathcal{A}}$ are diagonalized. Thus we can write an element $x \in Z_{\mathcal{A}}$ as

$$x = \bigoplus_a \lambda_a \mathbf{1}_{|a| \times |a|}, \quad (940)$$

where $\lambda_a \in \mathbb{C}$ and $|a|$ is the size of the a th block in the diagonalized representation of the center. The sum is an orthogonal direct sum since each generator of $Z_{\mathcal{A}}$ corresponds to a

single block, and if the different blocks overlapped, the generators would not commute with each other (and hence wouldn't be generators of $Z_{\mathcal{A}}$).

Let the projector Π_a be defined to project onto the a th block in the decomposition. They satisfy $\Pi_a \Pi_b = \delta_{ab} \Pi_a$. Since $\dim \mathcal{H} < \infty$ we can and will always take the direct sum decomposition above to be complete, so that the Π_a s are minimal projectors (meaning that there are no projectors that map onto a subspace of an image of one of the Π_a s).

Since $\mathcal{A}_a \equiv \Pi_a \mathcal{A} \Pi_a$ is the component of \mathcal{A} contained within a single block of the decomposition, all the elements in the center act on it either as 0 or as $\lambda_a \mathbf{1}_{|a| \times |a|}$. One can then check that \mathcal{A}_a is a von Neumann algebra with trivial center. Indeed, if $Z_{\mathcal{A}_a}$ were non-trivial, we would diagonalize it as above and it would contain a projector $\Pi_{\tilde{a}}$ which was represented as something other than $\mathbf{1}$ on $\Pi_a \mathcal{H}$. Then we could form the projector $\Pi_{\tilde{a}} \otimes 0_{(\mathbf{1} - \Pi_{\tilde{a}})\mathcal{H}}$, which is a central element of \mathcal{A} and a projector, contradicting the assumed completeness of the Π_a s.

Thus, while \mathcal{A} doesn't induce a tensor factorization of \mathcal{H} , the upshot is that we can use the \mathcal{A}_a s to induce a tensor factorization on subspaces of \mathcal{H} . They won't split the full Hilbert space, but they will split the subspaces $\mathcal{H}_a \equiv \Pi_a \mathcal{H}$ on which they act. Since they are type I factors, we know that they induce a splitting

$$\mathcal{H}_a = \mathcal{H}_{A_a} \otimes \mathcal{H}_{\bar{A}_a}, \quad (941)$$

where A_a is some region associated to the factor \mathcal{A}_a . From their realizations as type I factors, they act on each \mathcal{H}_a as

$$\mathcal{A}_a = L(\mathcal{H}_{A_a}) \otimes \mathbf{1}_{\bar{A}_a}. \quad (942)$$

Since each \mathcal{A}_a induces a tensor product decomposition on each subspace \mathcal{H}_a , the full Hilbert space splits up into a sum of products according to the elements of the center:

$$\mathcal{H} = \bigoplus_a (\mathcal{H}_{A_a} \otimes \mathcal{H}_{\bar{A}_a}). \quad (943)$$

Given a global state ρ we want to construct a density matrix $\rho_{\mathcal{A}}$ for \mathcal{A} , one which will faithfully reproduce expectation values of operators in \mathcal{A} but which will carry no information about what's going on in the commutant \mathcal{A}' . Since we have used the Π_a projectors to decompose \mathcal{A} into different von Neumann algebras, to get $\rho_{\mathcal{A}}$ we need only to retain information about the diagonal components in the block decomposition. This is because any operator $\mathcal{O} \in \mathcal{A}$ must have $\mathcal{O}_{\alpha\beta} \propto \delta_{\alpha\beta}$ since \mathcal{O} commutes with everything in the center, and so as far as computing $\langle \mathcal{O} \rangle$ goes, we only need to worry about the ρ_{aa} components. So we can write (this notation really is hilariously bad)

$$\rho_{\mathcal{A}} = \bigoplus_a p_a \rho_{A_a}, \quad (944)$$

where each component of the density matrix is taken to have unit trace on its parent tensor sub-factor:

$$\text{Tr}_{\mathcal{H}_a} \rho_{A_a} = 1. \quad (945)$$

Since the component density matrices have unit trace, the p_a s are

$$p_a = \text{Tr}_{\mathcal{H}_a} \rho_{\mathcal{A}}. \quad (946)$$

Thus $\sum_a p_a = 1$. There are some different ways to choose normalization factors and stuff for the different components of the density matrix, but I don't think the actual choice is too important.

So, the classical probabilities measure the traces of each part of the full density matrix, while each block of the density matrix has unit trace, allowing it to be defined as a bona fide density matrix on each superselection sector a (we call them superselection sectors since there are no operators with off-diagonal blocks which connect different \mathcal{H}_a subspaces, by virtue of the fact that all operators must commute with everything in $Z_{\mathcal{A}}$). The idea here is that the Hilbert space splits into a decomposition in terms of “boundary conditions”, which are given by the central elements in $Z_{\mathcal{A}}$. Of course, the individual probabilities p_a will have to be calculated on a case-by-case basis; they correspond to the probability of “getting” the superselection sector a .

We can then define an algebraic entropy for \mathcal{A} by taking the von Neumann entropy of $\rho_{\mathcal{A}}$:

$$S(\mathcal{A}) = - \sum_a p_a \ln p_a + \sum_a p_a S(\rho_a). \quad (947)$$

The first part is classical since it comes from elements of the center, which by virtue of their commutativity with everything, are effectively classical. Thus the probability distribution of the superselection sectors are responsible for the classical Shannon term. This is in-line with the idea that you can have states which are mixtures of different superselection sectors (you can have multiple $p_a \neq 0$), but you cannot have superpositions of different sectors (by definition), and so their contribution to $S(\mathcal{A})$ must be in the form of a Shannon term.

Note that the presence of a nontrivial center *reduces* the total entanglement entropy. This is not a very precise statement as it stands: roughly, we mean that if we imagine one algebra \mathcal{A} with nontrivial $Z_{\mathcal{A}}$ and some $\tilde{\mathcal{A}} = \mathcal{A} \cup \{\mathcal{O}_c\}$ with a few operators \mathcal{O}_c that have very small weights in the density matrix and are such that their inclusion forces $Z_{\mathcal{A}}$ to become trivial (think of very massive electric charges for Abelian gauge theory), then $S(\tilde{\mathcal{A}}) > S(\mathcal{A})$. This might seem not to be the case due to the presence of the positive classical Shannon entropy term, but the von Neumann $S(\rho_a)$ contribution increases to more than account for the lack of the Shannon term. That the presence of a nontrivial center reduces the total entropy is of course physical: gauge constraints reduce the size of the effective Hilbert space, which after all is what the entanglement entropy is designed to capture.

Lattice gauge theory examples: The basic variables for us are the holonomies $\exp(i \int_l A)$ (the integral of A along a link l), and the electric fields E_l^r , which measure the flux passing along l in a certain representation r . In the abelian case, $E = i \frac{\delta}{\delta A}$. When we are working with discrete gauge theories, the Hilbert space and algebra of operators are

$$\mathcal{H} = \bigotimes_l \mathbb{C}_l^N, \quad \mathcal{A} = \bigotimes_l \mathrm{Gl}(N, \mathbb{C})_l. \quad (948)$$

Operators on different links thus commute. We will usually work in the basis where E_l^r is diagonalized on each link. The physical Hilbert space is spanned by the states created by loops of constant electric flux. A wilson loop of charge 1 around a plaquette is a creation operator for these states.

Using Gauss' law, we can write the electric field E_l on a boundary link as the product of E_k 's with k links all entirely outside of A . The electric flux variables on the boundary

of the region are thus effectively classical because their conjugate variables $e^{iA_{l \in \partial A}}$ aren't in \mathcal{A}_A . Also, we see that the operators in A , namely \mathcal{A}_A , actually generate *more* than just themselves: they also generate electric field operators lying outside of A . This is in-line with a general result which says that the algebra of operators generated by \mathcal{A}_A is in fact the double commutant $(\mathcal{A}_A)''$. If \mathcal{A}_A is a von Neumann algebra then the operator algebra it generates is itself, but for gauge theory applications this is not the case.

Since $Z_{\mathcal{A}_A}$ is non-trivial for generic choices of subregions in gauge theories, we need to follow the procedure of the previous section to be able to compute an algebraic entropy. What are the projectors Π_a in this case? They are simply the projectors onto electric flux configurations on ∂A . They are complete ($\bigoplus_a \Pi_a = \mathbf{1}$) and they are orthogonal, since no gauge-invariant operator in \mathcal{A}_A can change the electric flux boundary conditions (the only operators that can are Wilson lines that end on the boundary). The gauge constraint means that physically, while we can form mixtures of states with different electric flux configurations, we are not allowed to take superpositions. Thus the density matrix must be diagonalized in the electric boundary flux basis, and we have seen that it indeed is. Also, of course not all electric flux configurations are allowed, since we require charge neutrality.

Now, a somewhat incongruous comment on how we choose the decomposition of the lattice into A and \bar{A} . There are basically two choices: we can choose to bipartition the links between the two subregions, or we can bipartition the plaquettes. The first choice is called an “electric” cut since it divvies up the links on which E lives, and the latter called a “magnetic” cut since it divvies up the plaquettes on which B lives. This is slightly confusing, since the two different choices of cuts produce different results for the entanglement entropy. This is succinctly demonstrated by \mathbb{Z}_N gauge theories, where the boundary contribution vanishes for magnetic cuts and is nonzero with electric cuts. This is of course expected since in the plaquette basis the ground state wavefunction is a product state $|\Psi\rangle \propto \bigotimes_{\text{plaquettes}} (|\square\rangle + |0\rangle)$, where \square represents a \mathbb{Z}_N string fused into the plaquette \square and $|0\rangle$ represents a state with no loops. However, this isn't really so much of an issue: since we are working with electric variables (E and A) rather than magnetic variables (\tilde{A} and B , with $\star F = d\tilde{A}$), we need to make use of electric cuts. Electric cuts are also generic, in that if one creates a bipartition of the lattice by drawing a random closed smooth curve on top of the lattice, one always gets an electric cut. Basically, it is impossible to work with both electric and magnetic variables simultaneously since they are related by a non-local duality, and so our choice of variables is determined by where we put the canonical momentum variables (links or plaquettes). (*note added:* actually, these statements are too strong. You can reproduce the same entropy for both choice of boundary conditions, I think. For a magnetic cut, you still have the appropriate central operators, which are the operators consisting of a product of E operators on the interior part of the star of a boundary vertex)

57 June 10 — Entanglement for a harmonic oscillator

Find the entanglement entropy between two coupled harmonic oscillators as a function of their masses and the coupling strength. Do this by explicitly tracing out one of the oscillators and finding the entropy of the reduced density matrix. This calculation has been done several

times in the literature (see e.g. Cassini's paper or Srednicki's one on black holes) — the goal here is just to understand it for myself and fill in the details that aren't in the papers.

Solution:

If our oscillators are at positions ϕ_1, ϕ_2 , have masses W_{11}, W_{22} , and interact with a coupling W_{12} , then the ground-state wavefunction is (here $\phi^T = (\phi_1, \phi_2)$)

$$\psi = \left(\frac{\det W}{\pi^2} \right)^{1/4} \exp \left(-\frac{1}{2} \phi^T W \phi \right). \quad (949)$$

Thus the full density matrix is

$$\langle \phi | \rho | \phi' \rangle = \sqrt{\frac{\det W}{\pi^2}} \exp \left(-\frac{1}{2} \phi^T W \phi - \frac{1}{2} \phi'^T W \phi' \right). \quad (950)$$

We get the reduced density matrix for the first oscillator by doing the trace over ϕ_2 , setting $\phi_2 = \phi'_2$ and integrating out ϕ_2 . This gives

$$\langle \phi | \rho_1 | \phi' \rangle = \sqrt{\frac{\det W}{\pi W_{22}}} \exp \left(-\frac{1}{2} \left(W_{11} - \frac{W_{12}^2}{2W_{22}} \right) (\phi^2 + \phi'^2) + \frac{W_{12}^2}{2W_{22}} \phi \phi' \right). \quad (951)$$

We see that if $W_{12} = 0$ then we have $\rho_1 = |\psi_G\rangle\langle\psi_G|$, where $|\psi_G\rangle$ is a Gaussian. Since the reduced density matrix is a pure state the entanglement entropy is zero, which one can check by going into an occupation number representation where $|\psi_G\rangle = |0\rangle$. Despite the simple form of ρ_1 , we aren't really in a position to calculate the entanglement entropy: taking the trace is easy, but taking the logarithm of ρ_1 is very difficult, since ρ_1 is not diagonal.

In order to compute the entropy in the W_{12} case, we can either use the replica trick or go into a basis where ρ_1 is diagonalized. We will go for the latter strategy. We need to figure out how to write the reduced density matrix in a simpler way. Experience teaches us that taking the modular Hamiltonian to look like a real Hamiltonian is a good choice (provided that the oscillators are coupled so that the reduced density matrix is likely to look thermal), and so we guess

$$\rho_1 = \frac{1}{Z_1} \exp(-\varepsilon a^\dagger a), \quad (952)$$

for some appropriate a^\dagger, a . Since we only have one oscillator to deal with, there is only one species of creation / annihilation operators. ε is some energy that we'll compute later. The a operators are presumably in the form

$$\pi_1 = \lambda a + \lambda^* a^\dagger, \quad \phi_1 = i(\gamma a - \gamma^* a^\dagger) \quad (953)$$

for some coefficients λ, γ . By imposing $\langle \phi_1 \pi_1 \rangle = i/2$, we get the constraint

$$\gamma = \frac{1}{2\lambda^*}. \quad (954)$$

We then calculate the expectation values of ϕ_1^2 and π_1^2 in this basis, which is easy to do since $e^{-\varepsilon a^\dagger a}$ is diagonal:

$$\langle \phi_1^2 \rangle = |\gamma|^2 (2\langle n \rangle + 1), \quad \langle \pi_1^2 \rangle = |\lambda|^2 (2\langle n \rangle + 1). \quad (955)$$

Figure 10: The entanglement entropy after tracing out one of the oscillators in a coupled pair of oscillators.

This means that if we know the 2-point functions we know λ and γ :

$$|\lambda|^2 = \frac{1}{2} \sqrt{\frac{\langle \phi_1^2 \rangle}{\langle \pi_1^2 \rangle}}, \quad |\lambda|^2 = \frac{1}{2} \sqrt{\frac{\langle \pi_1^2 \rangle}{\langle \phi_1^2 \rangle}}. \quad (956)$$

We would also know $\langle n \rangle$, since

$$2\langle n \rangle + 1 = 2\sqrt{\langle \pi_1^2 \rangle \langle \phi_1^2 \rangle}. \quad (957)$$

We can calculate $\langle n \rangle$ explicitly since the density matrix is thermal, and we get

$$\tanh(\varepsilon) = \frac{1}{2\sqrt{\langle \pi_1^2 \rangle \langle \phi_1^2 \rangle}}. \quad (958)$$

This means that if we can get the 2-point functions, we can get the entanglement entropy. Indeed, since we're in the occupation number basis the trace is now easy to carry out:

$$\begin{aligned} S &= -\text{Tr} \left((1 - e^{-\varepsilon}) e^{-\varepsilon a^\dagger a} [\ln(1 - e^{-\varepsilon}) - \varepsilon a^\dagger a] \right) \\ &= \frac{\varepsilon}{1 - e^{-\varepsilon}} e^{-\varepsilon} - \ln(1 - e^{-\varepsilon}). \end{aligned} \quad (959)$$

Rephrasing this slightly,

$$S = \varepsilon \langle n \rangle + \ln Z_1, \quad Z_1 = \frac{1}{1 - e^{-\varepsilon}}. \quad (960)$$

Plugging in for ε by inverting (958), we get an expression for S in terms of the 2-point functions.

All that remains is to get expressions for $\langle \phi_1^2 \rangle$ and $\langle \pi_1 \rangle$. They are both easy to calculate. The first one is

$$\langle \phi_1^2 \rangle = \frac{W_{22}}{2 \det W}. \quad (961)$$

The second one is

$$\begin{aligned} \langle \pi_1^2 \rangle &= -\sqrt{\frac{\det W}{\pi W_{22}}} \int d\phi_1 \left(\frac{W_{12}^2 - 2W_{11}W_{22}}{2W_{22}} + 2\phi_1^2 \frac{\det^2 W}{W_{22}^2} \right) \exp(-\phi_1^2 [W_{11} - W_{12}^2/W_{22}]) \\ &= \frac{W_{11}}{2}. \end{aligned} \quad (962)$$

We get this simple result, the one for a pure Gaussian, since the coupling in the Hamiltonian takes place only in real space. Note that in order for this to work, we need to have $\det W > 0$.

So, we are finished! One sees that for $W_{12} = 0$ we get the pure Gaussian result $\langle \phi_1^2 \rangle = \langle \pi_1^2 \rangle / W_{11}^2 = 1/(2W_{11})$. This leads to a vanishing entanglement entropy by way of (960), since for these 2-point functions we have the minimally-uncertain $\langle \pi_1^2 \rangle \langle \phi_1^2 \rangle = 1/4$, giving $\varepsilon \rightarrow \infty$ (projecting onto the ground state), resulting in $\langle n \rangle = 0, Z_1 = 1$, and hence $S = 0$.

In Figure 10 we plot the entanglement entropy as a function of W_{12}/W_{11} for a few different values of W_{11}/W_{22} . All the values we show have $W_{11}/W_{22} < 1$, but this is done wolog since the entanglement entropy is symmetric under swapping W_{11} and W_{22} . The divergence in S_1 as $W_{12} \rightarrow W_{11}$ for the $W_{11} = W_{22}$ case is because at these values of the parameters W is singular.

58 June 11 — Intro to thermofield dynamics

Go read about thermofield dynamics somewhere and recapitulate some of the important parts. Focus on harmonic oscillator-like systems for simplicity. Explain the information theory meaning behind the thermofield double states and how Tomita-Takesaki theory makes an appearance. A good reference for this problem is the book by Blasone, Jizba, and Vitiello.

Solution:

The basic problem with temperature is that it forces one to work with states that are not pure. This means that expectation values cannot be calculated by simply taking matrix elements of operators in the usual Fock basis. It also makes doing path integrals hard: there is no real concept of nontrivial asymptotic states for e.g. particle scattering processes, and it is no longer to use path integrals to calculate amplitudes for one thermal state $|\Psi_\infty\rangle$ to evolve to another state $|\Psi_{out}\rangle$, since thermal states cannot be written as ket vectors. Indeed, suppose we had some pure state $|s\rangle = \sum_n s_n |n\rangle$ expressible by transforming the usual Fock basis in some way. Then if \mathcal{O} is an operator with any nonzero off-diagonal components,

$$\langle s|\mathcal{O}|s\rangle = \sum_{m,n} s_n s_m^* \mathcal{O}_{mn} \neq \frac{1}{\text{Tr}[e^{-\beta H}]} \sum_n e^{-\beta E_n} \mathcal{O}_{nn}, \quad (963)$$

no matter what coefficients we choose. While the imaginary time formalism can be of help here, it is only really usable for studying equilibrium processes where we can do away with time, and doing the analytic continuation at the end can be a pain.

To fix these difficulties, we need to insist on having some “thermal pure state” that we can use to compute expectation values and compute transition amplitudes and stuff. This means we need to find some way to make the overall state pure. The way to do this is essentially to couple the system to a heat bath, so that the combined system+heat bath is in a pure state. Since the original thermal state’s density matrix is invertible — thermal states have “a lot” of entanglement and are far from being product states — the minimal way we can purify the thermal state is to just make a copy of it, so that together with its copy it becomes pure. Following convention, tildes will indicate things belonging to the copy, so that e.g. the full Hilbert space is $\mathcal{H} \otimes \tilde{\mathcal{H}}$ and if we are interested in the operator algebra A acting on the original \mathcal{H} , then we also have an associated algebra \tilde{A} acting on $\tilde{\mathcal{H}}$. Recall that if our mixed

density matrix is $\rho = \text{diag}(p_1, \dots, p_n)$ in a basis $\{|n\rangle\}$ for \mathcal{H} , then the purification is

$$\rho_p = \sum_{n,m} \sqrt{p_n p_m} |n \otimes \tilde{n}\rangle \langle m \otimes \tilde{m}| = |N\rangle \langle N|, \quad |N\rangle = \sum_n \sqrt{p_n} |n \otimes \tilde{n}\rangle. \quad (964)$$

When this is applied to our thermal density matrix ρ , we call the resulting purification the thermofield double:

$$|TFD\rangle = \frac{1}{\sqrt{\text{Tr}[e^{-\beta H}]}} \sum_n e^{-\beta E_n/2} |n \otimes \tilde{n}\rangle. \quad (965)$$

By construction, tracing out one of the tensor factors of the pure density matrix $|TFD\rangle \langle TFD|$ gives a thermal state:

$$\text{Tr}_{\tilde{\mathcal{H}}}(|TFD\rangle \langle TFD|) = \frac{1}{\text{Tr}[e^{-\beta H}]} \sum_n e^{-\beta E_n} |n\rangle \langle n|. \quad (966)$$

We see that (again by construction) if $\mathcal{O} = \mathcal{O}_A \otimes \mathbf{1}$ then $\langle TFD | \mathcal{O} | TFD \rangle$ reproduces the correct thermal expectation value of \mathcal{O} (and likewise if $\mathcal{O} = \mathbf{1} \otimes \mathcal{O}_{\tilde{A}}$). The $|\tilde{n}\rangle$ states are the “heat bath” and represent coupling to a large “classical” system, since their purpose in life is to select out the diagonal elements of the density matrix when computing expectation values of operators.

Since the vectors in the tilde Fock basis are just a copy of the ones in the original Fock basis, we also see that $|TFD\rangle$ is a ground state of the following Hamiltonian:

$$\mathcal{H}|TFD\rangle \equiv (H \otimes \mathbf{1} - \mathbf{1} \otimes H)|TFD\rangle = 0. \quad (967)$$

This “doubled system with the copy moving backwards in time” should remind you of either Rindler space or modular flow or both. More on this in a sec.

Let’s look briefly at the obligatory example of the harmonic oscillator in a heat bath. We can write

$$\begin{aligned} |TFD\rangle &= \frac{1}{\sqrt{\text{Tr}[e^{-\beta H}]}} \sum_n \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n!}} e^{-\beta\omega/4} e^{-\beta\omega n/2} (a^\dagger)^n (\tilde{a}^\dagger)^n |0 \otimes \tilde{0}\rangle \\ &= \frac{e^{-\beta\omega/4}}{\sqrt{\text{Tr}[e^{-\beta H}]}} \sum_n \exp\left(e^{-\beta\omega/2} a^\dagger \tilde{a}^\dagger\right) |0 \otimes \tilde{0}\rangle. \end{aligned} \quad (968)$$

Now a little algebra lets us write

$$\frac{1}{\sqrt{\text{Tr}[\exp(-\beta H)]}} = e^{\beta\omega/4} \sqrt{1 - e^{-\beta\omega}}, \quad (969)$$

so that the ground state is

$$|TFD\rangle = \sqrt{1 - e^{-\beta\omega}} \exp\left(e^{-\omega\beta/2} a^\dagger \tilde{a}^\dagger\right) |0 \otimes \tilde{0}\rangle. \quad (970)$$

We then want to make a suggestive (in terms of Rinder space and stuff) definition

$$\cosh \theta = \frac{1}{\sqrt{1 - e^{-\beta\omega}}}, \quad \sinh \theta = \frac{1}{\sqrt{e^{\beta\omega} - 1}}, \quad (971)$$

so that

$$\tanh \theta = e^{-\omega \beta / 2}, \quad (972)$$

allowing us to write

$$|TFD\rangle = \frac{1}{\cosh \theta} \exp \left(\tanh \theta a^\dagger \tilde{a}^\dagger \right) |0 \otimes \tilde{0}\rangle. \quad (973)$$

Now some algebra lets us re-write this as

$$|TFD\rangle = \exp \left(\theta (a^\dagger \tilde{a}^\dagger - a \tilde{a}) \right) |0 \otimes \tilde{0}\rangle, \quad (974)$$

from which we see that the TFD state is a bath of particles and antiparticles ³³. Note that in contrast to the imaginary time formalism, we can easily generalize to the non-equilibrium case simply by letting θ depend on time.

Thus let us define the operator U_θ by

$$U_\theta = e^{\theta(a^\dagger \tilde{a}^\dagger - a \tilde{a})}, \quad (976)$$

so that U_θ lets us map between the doubled Fock basis and the thermofield double. We see that the operators

$$a_\theta = U_\theta a U_\theta^{-1}, \quad \tilde{a}_\theta = U_\theta \tilde{a} U_\theta^{-1} \quad (977)$$

both annihilate $|TFD\rangle$. In terms of the original operators, some annoying algebra gives

$$a_\theta = a \cosh(\theta) - \tilde{a}^\dagger \sinh(\theta), \quad (978)$$

and likewise for \tilde{a}_θ .

One of the points of this whole thing is that in the thermodynamic limit (or in a finite system with infinitely many degrees of freedom allowed by e.g. free boundary conditions), different values of θ (i.e. different temperatures) give rise to different operator algebras (different representations of the CCR) which are unitarily *inequivalent*, so that in some sense temperature is really fundamentally just a parameter that describes the way in which we represent the CCR. One can see this by taking the inner product of TFD states at different temperatures: generalizing our earlier results to incorporate all the different momentum modes,

$$\begin{aligned} \langle TFD_\theta | TFD_{\theta'} \rangle &= \prod_{k,k'} \langle 0 \otimes \tilde{0} | e^{\tanh \theta_k \tilde{a}_k a_k} \frac{1}{\cosh \theta_k \cosh \theta_{k'}} e^{\tanh \theta_{k'} a_{k'}^\dagger \tilde{a}_{k'}^\dagger} | 0 \otimes \tilde{0} \rangle \\ &\propto \exp \left(-V \int_k \ln \cosh \theta_k - V \int_{k'} \ln \cosh \theta_{k'} \right), \end{aligned} \quad (979)$$

which vanishes unless both $\theta = \theta' = 0$ for all k , i.e. if both TFD states are at zero temperature.

³³Showing this last step actually turned out to be kind of a pain in the ass! One first takes the logarithm of the prefactor and then exponentiates it, adds in the $a \tilde{a}$ term with the aid of the BCH formula, and then does some algebra on the various numbers in the exponential. At the end one uses the identity

$$\cosh^{-1}(x) = \ln \left(x + \sqrt{(x+1)(x-1)} \right), \quad (975)$$

applied to $x = (1 - e^{-\beta \omega})^{-1/2}$ (and so the LHS is equal to θ).

We know that at finite temperature, Lorentz symmetry gets broken down to rotations + translations by the introduction of an energy scale. The boost symmetries are responsible for an $SU(1, 1)$ symmetry, since space and time have a relative minus sign in the metric. So we should expect to see the breaking of an $SU(1, 1)$ symmetry somewhere in this formalism.

This becomes most apparent if we define the thermal doublet (suppressing the momentum dependence from now)

$$\Psi_\theta = (a_\theta, \tilde{a}_\theta^\dagger)^T. \quad (980)$$

We define the doublet in this way since it is the object that transforms under changes in temperature by the Bogoliubov transformation identified above:

$$\Psi_\theta = B_\theta \Psi_0 \equiv \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \Psi_0. \quad (981)$$

One can quickly check that $B_\theta \in SU(1, 1)$, i.e. B_θ preserves the matrix Z . Note that when we put back in the momentum dependence, we actually have $B_{\{\theta_k\}} \in \bigoplus_k SU(1, 1)_k$, with one copy of $SU(1, 1)$ for each mode. Thus the unitarity of this transformation in the thermodynamic limit fails, since as we saw earlier the different $|TFD\rangle_\theta$ are not unitarily equivalent in the thermodynamic limit (this is the same as the statement that the charge operator $Q = \int \star J$ in theories with a spontaneously broken continuous symmetry is not unitary since it has infinite norm).

Let us now rewrite the Hamiltonian \mathcal{H} in a more suggestive form. Since $\mathcal{H} = H \otimes \mathbf{1} - \mathbf{1} \otimes H$, we have, in terms of the original $\theta = 0$ operators (which act on the original Fock space)

$$\mathcal{H} = \omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}), \quad (982)$$

which we can write as

$$\mathcal{H} = \omega(\Psi_0^\dagger Z \Psi_0 + 1), \quad (983)$$

where Z as before is the third Pauli matrix. Since $B_\theta \in SU(1, 1)$, \mathcal{H} is actually independent of θ :

$$\mathcal{H} = \omega(\Psi_\theta^\dagger Z \Psi_\theta + 1). \quad (984)$$

This means that the Hamiltonian possesses the full $SU(1, 1)$ boost symmetry (\mathcal{H} annihilates $|TFD\rangle_\theta$ for all θ). However, as we have just seen, the ground states of \mathcal{H} (the $|TFD_\theta\rangle$ states) are *not* invariant under $SU(1, 1)$: they are labelled by a particular θ and permuted by the $SU(1, 1)$ action. Thus we have scenario like SSB, where the Hamiltonian is symmetric but the actual states that are chosen are not. Also like in SSB, in the thermodynamic limit, distinct ground states are unitarily inequivalent.

Now for some more algebraic / informationy comments. As mentioned above, the operators in A commute with those in \tilde{A} , and vice versa. In fact, if A is a factor (i.e. has trivial center, which we can assume wolog after possibly doing a direct sum decomposition of an algebra with non-zero center by projectors onto central elements), we have

$$A' = \tilde{A}. \quad (985)$$

We know that any von Neumann algebra is equal to its double-commutant, which we see here is reflected by the involutive nature of the tilde operation (which is true but which we haven't mentioned):

$$A'' = A \implies \tilde{\tilde{A}} = A. \quad (986)$$

One may think that since the Hamiltonian does not couple the \mathcal{H} and $\tilde{\mathcal{H}}$ degrees of freedom, the two systems would be trivially decoupled. Of course this is not true since $|TFD\rangle$ has a huge amount of entanglement between the A and \tilde{A} subsystems, and so even though the two systems do not talk to one another in the action we generically have correlations between the two subsystems:

$$\langle TFD | \mathcal{O} \otimes \tilde{\mathcal{O}} | TFD \rangle \neq 0. \quad (987)$$

In fact, the thermofield double is cyclic and separating for the algebras A and \tilde{A} (since they are each other's commutants, the fact that $|TFD\rangle$ is cyclic and separating for one implies that it is also cyclic and separating for the other³⁴. The cyclicity means that any state vector in $\mathcal{H} \otimes \tilde{\mathcal{H}}$ can be constructed by acting on $|TFD\rangle$ with an operator of either the form $\mathcal{O} \otimes \mathbf{1}$ or of the form $\mathbf{1} \otimes \tilde{\mathcal{O}}$, which is made possible precisely because of the huge amount of entanglement between the A and \tilde{A} subsystems. The fact that states and operators can be in bijection like this in a finite system is allowed since

$$\dim \text{End}(\mathcal{H}) = \dim(\mathcal{H})^2 = \dim(\mathcal{H} \otimes \tilde{\mathcal{H}}), \quad (988)$$

as $\dim(\tilde{\mathcal{H}}) = \dim(\mathcal{H})$ by construction. Note that when we say “any state vector in $\mathcal{H} \otimes \tilde{\mathcal{H}}$ can be created by acting on $|TFD\rangle$ with an operator in either $\text{End}(\mathcal{H})$ or $\text{End}(\tilde{\mathcal{H}})$, we are working with a *fixed* representation of the CCRs. As we saw earlier, we do have operators that take us between different CCRs (which change the temperature of the thermal state), but the image of this operator acting on $|TFD\rangle$ is not included in $\mathcal{H} \otimes \tilde{\mathcal{H}}$, since the Hilbert space $\mathcal{H} \otimes \tilde{\mathcal{H}}$ is the Hilbert space for a fixed representation of the CCRs. This is like considering the Hilbert space associated to a particular broken-symmetry sector in the case of SSB. The full Hilbert space, which includes all different symmetry-broken sectors, would in our case be something like $\bigoplus_\theta (\mathcal{H} \otimes \tilde{\mathcal{H}})_\theta$.

Note that the full time evolution operator for the system can be written suggestively as

$$\Delta_{TFD}^{is} = \exp(i(H \otimes \mathbf{1} - \mathbf{1} \otimes H)s), \quad (989)$$

where s is some real parameter. Connecting to Tomita-Takesaki theory, we see that the role of modular conjugation is played by the tilde involution which exchanges the A and \tilde{A} algebras:

$$\text{End}(\tilde{\mathcal{H}}) \ni J^{-1} \mathcal{O} J = \tilde{\mathcal{O}} \quad (990)$$

while the modular operator is

$$\Delta_{TFD} = e^{\mathcal{H}}. \quad (991)$$

This matches with the fact that for a Hilbert space split as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the modular operator Δ_Ψ for a state Ψ which is cyclic and separating for $\text{End}(\mathcal{H}_1)$ (and thus $\text{End}(\mathcal{H}_2)$) is

$$\Delta_\Psi = \rho_1 \otimes \rho_2^{-1}, \quad (992)$$

where ρ_i are the reduced density matrices.

³⁴Proof: suppose $\tilde{\mathcal{O}} \in \text{End}(\tilde{\mathcal{H}})$. Then $\tilde{\mathcal{O}}$ commutes with all $\mathcal{O} \in \text{End}(\mathcal{H})$ since operators acting only on \mathcal{H} commute with those acting only on $\tilde{\mathcal{H}}$. Suppose $\tilde{\mathcal{O}}|TFD\rangle = 0$. Then $\tilde{\mathcal{O}}\mathcal{O}|TFD\rangle = 0$ for any $\mathcal{O} \in \text{End}(\mathcal{H})$. But $|TFD\rangle$ is cyclic for $\text{End}(\mathcal{H})$ and so this means that $\tilde{\mathcal{O}}$ annihilates all state vectors. Thus $\tilde{\mathcal{O}} = 0$ and $|TFD\rangle$ is separating for $\text{End}(\tilde{\mathcal{H}})$.

59 June 12 — RPA basics

Today's problem is a quick refresher about the RPA approximation. We consider a system of fermions of spin S that interact with an interaction $V(\mathbf{q}) = \frac{1}{N}U(\mathbf{q})$, where $N = 2S + 1$.

For large N , describe the $1/N$ expansion of the ground-state energy diagrammatically. You should assume that the fermions are living in a uniform positive background so that the total charge in the system is zero. Then find the effective interaction $V_{eff}(\mathbf{q}, \omega)$ in the large- N limit, considering in particular a Coulomb potential $U(\mathbf{q}) = \alpha/r$.

Solution:

First we need to understand the ground state energy from a diagrammatic point of view. This is very standard stuff, but it's good to remind ourselves how it works. Let Z_0 be the vacuum-to-vacuum transition amplitude for the theory with the interactions turned on at some very early time $-T/2$ and the interactions turned off at some very late time $T/2$. We construct Z_0 by glueing together the final state $U^\dagger(-T/2)|0_\infty\rangle$ with the initial state $\langle 0_{-\infty}|U(T/2)$, where $U(T/2)$ is the time evolution operator for time $T/2$. Inserting a resolution of the identity in the space at $T = 0$ where these two states are glued together, and taking T very large so that $U(T/2)$ evolves $|0_\infty\rangle$ to the ground state $|\psi_g\rangle$ of the full theory with the interactions turned on. The amplitude is

$$S_T = \langle 0_{-\infty}|U(T/2)|\psi_g\rangle\langle\psi_g|U^\dagger(-T/2)|0_\infty\rangle. \quad (993)$$

Now in the interaction picture, we write $U(T/2)|\psi_g\rangle = \exp(iH_0T/2)\exp(-iHT/2)|\psi_g\rangle$. So we can write

$$\begin{aligned} \partial_T S_T &= \langle 0_{-\infty}|(iH_0/2 \times e^{iH_0T/2}e^{-iHT/2} - e^{iH_0T/2}e^{-iHT/2} \times iH/2)|\psi_g\rangle\langle\psi_g|U^\dagger(-T/2)|0_\infty\rangle \\ &\quad + \langle 0_{-\infty}|U(T/2)|\psi_g\rangle\langle\psi_g|(iH_0/2 \times e^{iH_0T/2}e^{-iHT/2} - e^{iH_0T/2}e^{-iHT/2} \times iH/2)|0_\infty\rangle \\ &= i(E_0 - E)TS_T, \end{aligned} \quad (994)$$

where E_0 is the eigenvalue of H_0 on $|0_{\pm\infty}\rangle$ and E is the eigenvalue of H on $|\psi_g\rangle$. Thus

$$S_T \propto e^{-i\delta ET}, \quad (995)$$

where δE is the shift in the ground state energy (this shouldn't be surprising).

Now we use the linked-cluster theorem to write $\ln(S_T)/(VT)$ (free energy density) as the sum of linked cluster diagrams (recall that taking the log as done with the replica trick kills off all the disconnected diagrams). So we have

$$\delta E = iV \sum (\text{linked diagrams}). \quad (996)$$

Now we consider what type of linked diagrams go where in the $1/N$ expansion. Each interaction between two fermion propagators gets a $1/N$, while each fermion loop gets an N from the spin summation. Thus we need to look at connected diagrams and assign their power of $1/N$ by $\#(\text{interaction lines}) - \#(\text{fermion loops})$.

The first type of diagrams, which go as $O(N)$, are built from taking two fermion bubbles connected by an interaction line (two tadpoles joined at the tail) and concatenating various

numbers of tadpoles onto them. These will all vanish though, since tadpoles are canceled by the counterterms we put in to ensure overall charge neutrality (the interaction tadpole leg must have zero momentum transfer, and so the loop of the tadpole computes the charge density. Since only fluctuations $\delta\rho$ are nonzero after adding in the positive background, the tadpoles are cancelled).

The next class is $O(1)$, which is exactly like the above tadpole diagrams except the tadpole chain is closed into a closed “pearl necklace” or sorts. These diagrams have the same number of interaction lines as fermion bubbles. This will be the important class of diagrams, with the other diagrams going as $1/N$ or worse. If $\chi(q)$ is the susceptibility, then a diagram like this with n bubbles goes like

$$n \text{ bubble} \sim \frac{1}{P_n} \text{Tr}[(-U\chi)^n], \quad (997)$$

where the powers of N have cancelled, P_n is a symmetry factor, and we have put in the minus sign for the fermion bubbles.

What is P_n ? To find P_n , we need to count the number of ways to wiggle around the components of the diagram to get a copy of the diagram that has the same connectivity and topology, but with the locations of the vertices permuted. For the $n = 2$ contribution, $P_2 = 4$. This is because there are two \mathbb{Z}_2 operations we can perform on the diagram: we can rotate it by an angle of π , or we can exchange the positions of the bubbles, which is done by taking the disks of the fermion bubbles and *rigidly* moving them so that they exchange places. We can also do things familiar from low dimensional topology and put the diagram on an S^2 : the second operation then corresponds to moving one of the bubbles around the sphere, passing “behind” the other (to do all of this carefully we should introduce framings and blah blah blah). These two operations generalize to the n -bubble contribution: there is a \mathbb{Z}_n action (rotating the ring of bubbles by $2\pi/n$) and a \mathbb{Z}_2 action (wrapping one of the bubbles “around” the S^2). Together these generate the dihedral group, which has order $2n$. Thus $P_n = 2n$, and after summing all the bubbles we see that the shift in ground state energy density due to the interaction is

$$\delta E/V = \frac{i}{2} \int_q \ln(1 - U(\mathbf{q})\chi(q)). \quad (998)$$

Now we look at the effective interaction. To find it diagrammatically we do the same thing as we did for δE , except we require that the diagrams be anchored with interaction lines at two “walls” which represent the Greens functions of the electrons participating in the interaction. The $O(N)$ tadpoles are canceled as before, and so again we just get the RPA chain-of-bubbles diagrams. Here since the ends of the chain are fixed in place by the external electron propagators the symmetry factor is just 1, and so we get

$$V_{eff}(\mathbf{q}, \omega) = \frac{U(\mathbf{q})}{1 + U(\mathbf{q})\chi(q)}. \quad (999)$$

Note that V_{eff} carries frequency dependence even though $U(\mathbf{q})$ does not.

Consider e.g. the case when $U(r) = \alpha/r$. Then $U(\mathbf{q})$ goes as $|\mathbf{q}|^{-2}$ in three dimensions, and so

$$V_{eff}(\mathbf{q}, \omega) \sim \frac{1}{|\mathbf{q}|^2 + \chi(q)}. \quad (1000)$$

Thus we see the standard picture how χ leads to screening and so on.

60 June 13 — Photon-atom interactions and stimulated emission

Another simple problem today, since I wasn't feeling very well. This is from Piers's many body theory book. Consider a fermionic atom that can be in two states, with energies $E_+ > E_-$. The Hamiltonian is

$$H = H_{ph} + H_a + H_I, \quad (1001)$$

with

$$\begin{aligned} H_{ph} &= \sum_{\mathbf{q}} \omega_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \\ H_a &= E_+ c_+^\dagger c_+ + E_- c_-^\dagger c_- \\ H_I &= \sum_{\mathbf{q}} g_{\mathbf{q}} (c_+^\dagger c_- + c_-^\dagger c_+) (a_{-\mathbf{q}} + a_{\mathbf{q}}^\dagger), \end{aligned} \quad (1002)$$

where the photon dispersion is $\omega_{\mathbf{q}} = c|\mathbf{q}|$.

By computing the self energies to second order in $g_{\mathbf{q}}$ and taking their imaginary parts, find the lifetimes of the + and - states. You can assume a non-degenerate scenario and can take $f(E_{\pm}) \approx 0$ (where f is the Fermi distribution). Your answer should be in terms of the Bose distribution at frequency $\omega_c = E_+ - E_-$.

Solution:

We need to compute the self energies of the two states, which will come from the usual polarization bubble type of diagram, whereby the atom emits a photon and hops to the other state, and then re-absorbs the photon and returns to the original state. Since the photon coupling in the relevant vertex is $a_{-\mathbf{q}} + a_{\mathbf{q}}^\dagger$, it helps to get the propagator for the combination $b_{\mathbf{q}} = a_{-\mathbf{q}} + a_{\mathbf{q}}^\dagger$ instead of the propagator for $a_{\mathbf{q}}$. We have

$$\begin{aligned} D(\mathbf{q}, t) &= -i\langle 0 | T\{b_{\mathbf{q}}(t)b_{\mathbf{q}}^\dagger(0)\} | 0 \rangle \\ &= -i\langle 0 | T\{a_{-\mathbf{q}}(t)a_{-\mathbf{q}}^\dagger(0) + a_{\mathbf{q}}^\dagger(t)a_{\mathbf{q}}(0)\} | 0 \rangle \\ &= -i\Theta(t)e^{-i\omega_{\mathbf{q}}t} - i\Theta(-t)e^{i\omega_{\mathbf{q}}t}. \end{aligned} \quad (1003)$$

Fourier transforming and not writing the convergence factor,

$$D(\mathbf{q}, \omega) = \frac{1}{\omega - \omega_{\mathbf{q}}} - \frac{1}{\omega + \omega_{\mathbf{q}}} = \frac{2\omega_{\mathbf{q}}}{\omega^2 - \omega_{\mathbf{q}}^2}. \quad (1004)$$

The self energy for the - state is then

$$\Sigma_-(\nu) = -T \sum_{\omega} \int_{\mathbf{q}} g_{\mathbf{q}}^2 \frac{2\omega_{\mathbf{q}}}{(i\omega)^2 - \omega_{\mathbf{q}}^2} \frac{1}{i(\nu - \omega) - E_+}. \quad (1005)$$

Breaking this up into positive and negative frequency parts and then rewriting the product of the propagators in a more useful form,

$$\begin{aligned}\Sigma_- &= -T \sum_{\omega} \int_{\mathbf{q}} g_{\mathbf{q}}^2 \left(\frac{1}{i\omega - \omega_{\mathbf{q}}} - \frac{1}{i\omega + \omega_{\mathbf{q}}} \right) \frac{1}{i(\nu - \omega) - E_+} \\ &= T \sum_{\omega} \int_{\mathbf{q}} \left(g_{\mathbf{q}}^2 \frac{1}{i\nu - (\omega_{\mathbf{q}} + E_+)} \left[\frac{1}{i\omega - \omega_{\mathbf{q}}} - \frac{1}{i\omega - (i\nu - E_+)} \right] - \dots \right),\end{aligned}\quad (1006)$$

where \dots indicates the same term but with $\omega_{\mathbf{q}} \rightarrow -\omega_{\mathbf{q}}$. To do the sum, we do the usual contour integral trick, by integrating against the Bose function $n(z)$. We do a counterclockwise integral around the imaginary axis, which since the sum of all the residues is zero, becomes the negative of the counterclockwise integral around all the other poles, namely the poles of the function in the integrand above. Let's just look at the positive $\omega_{\mathbf{q}}$ part. The poles in $z = i\omega$ are at $\omega_{\mathbf{q}}$ for the first term, and at $i\nu - E_+$ for the second term. So the positive $\omega_{\mathbf{q}}$ part gives

$$-\int_{\mathbf{q}} g_{\mathbf{q}}^2 \frac{n(\omega_{\mathbf{q}}) - n(i\nu - E_+)}{i\nu - \omega_{\mathbf{q}} - E_+}, \quad (1007)$$

where there is no T since T is included in the poles of the integrand. Since ν is a fermionic Matsubara frequency,

$$n(i\nu - E_+) = \frac{1}{-e^{-\beta E_+} - 1} = 1 - f(E_+). \quad (1008)$$

So the above term is

$$-\int_{\mathbf{q}} g_{\mathbf{q}}^2 \frac{1 + n(\omega_{\mathbf{q}}) - f(E_+)}{i\nu - \omega_{\mathbf{q}} - E_+}. \quad (1009)$$

For the negative $\omega_{\mathbf{q}}$ part, the only thing that changes is that the pole shifts from being at $\omega_{\mathbf{q}}$ to being at $-\omega_{\mathbf{q}}$. Then we use

$$n(-\omega_{\mathbf{q}}) = \frac{e^{\beta \omega_{\mathbf{q}}}}{1 - e^{\beta \omega_{\mathbf{q}}}} = -n(\omega_{\mathbf{q}}) - 1. \quad (1010)$$

Putting this together, the full self energy is

$$\Sigma_-(\nu) = -\int_{\mathbf{q}} g_{\mathbf{q}}^2 \left(\frac{1 + n(\omega_{\mathbf{q}}) - f(E_+)}{i\nu - \omega_{\mathbf{q}} - E_+} + \frac{n(\omega_{\mathbf{q}}) + f(E_+)}{i\nu + \omega_{\mathbf{q}} - E_+} \right). \quad (1011)$$

Now let's find the lifetime. We can get the lifetime with

$$\tau_{\sigma}^{-1} = 2\text{Im} [\Sigma_{\sigma}(E_{\sigma} - i\eta)]. \quad (1012)$$

Putting this in, and taking $f(E_+) \approx 0$,

$$\begin{aligned}\tau_{-}^{-1} &= 2\pi \int \frac{d^3 q}{(2\pi)^3} g_{\mathbf{q}}^2 ((1 + n(\omega_{\mathbf{q}}))\delta(-\omega_c - \omega_{\mathbf{q}}) + n(\omega_{\mathbf{q}})\delta(\omega_c - \omega_{\mathbf{q}})) \\ &= \frac{1}{\pi c^3} g(\omega_c)^2 n(\omega_c) \omega_c^2.\end{aligned}\quad (1013)$$

To get the lifetime for the other state, we just have to swap plus signs and minus signs in the above. This leads to the same expression except with ω_c swapped with $-\omega_c$ in the first line of the above equation. Thus

$$\tau_+^{-1} = \frac{1}{\pi c^3} g(\omega_c)^2 [1 + n(\omega_c)] \omega_c^2. \quad (1014)$$

As a sanity check, note that the lifetime in the lower level is longer by a factor

$$\frac{\tau_-}{\tau_+} = \frac{1 + n(\omega_c)}{n(\omega_c)} = e^{\beta(E_+ - E_-)}, \quad (1015)$$

which makes sense as it's equal to the ratio of the Boltzmann weights for the two states.

61 June 14 — Practice with the Kubo formula and electrical conductivity

This is also from Piers' many body theory book, chapter 10. Find the electrical conductivity in a metal with Feynman diagrams by restricting your attention to the polarization bubble diagram. Your starting point should be

$$\sigma^{ab}(i\nu) = e^2 \frac{T}{\nu} \sum_{\mathbf{q}, \omega} v_{\mathbf{q}}^a v_{\mathbf{q}}^b (G(\mathbf{q}, i\omega + i\nu) G(\mathbf{q}, i\omega) - G(\mathbf{q}, i\omega)^2), \quad (1016)$$

which represents the polarization bubble with the zero-frequency part subtracted off. In this formula, the electron propagators G are

$$G(\mathbf{q}, i\omega) = \frac{1}{i\omega - \epsilon_{\mathbf{q}} - \Sigma(i\omega)}, \quad (1017)$$

for some momentum-independent self energy, the $1/\nu$ comes from $-\partial_t A^i = E^i$, the $v_{\mathbf{q}}^a$'s are velocities, and the second term in parenthesis is the diamagnetic part.

Solution:

Our first task is to switch out the momentum integral for an energy integral. Since the Greens functions are rotationally invariant and we are in three dimensions, we can swap out $v^a v^b$ for $v^2/3$. Furthermore, since we are subtracting off the zero frequency part which picks up the finite density of the fermi surface, at frequencies low compared to the Fermi energy we can potentially get away with crudely approximating the density of states as a constant (equal to its value at the FS), and in doing so we can integrate the energy (relative to the FS) from $-\infty$ to $+\infty$. So we can do (not keeping track of spatial volumes)

$$\sum_{\mathbf{q}} v^a v^b \rightarrow \delta^{ab} \int_{\mathbb{R}} d\epsilon N(0) \frac{k_F^2}{3m^2} = \delta^{ab} \int_{\mathbb{R}} d\epsilon \frac{k_F^3}{2\pi^2 m}, \quad (1018)$$

since the density of states at the FS is $N(0) = mk_F/(4\pi^2)$. The number density is

$$n = \int d\epsilon N(\epsilon) = \frac{2m}{3 \cdot 2\pi^2} \sqrt{2m\epsilon_F} \epsilon_F = \frac{k_F^3}{3 \cdot 2\pi^2} = \frac{2}{3} N(0) \frac{k_F^2}{m}, \quad (1019)$$

where we integrated up to the FS. Thus the diagonal part of the conductivity is

$$\sigma^{aa}(i\nu) = \frac{ne^2 T}{m\nu} \int_{\mathbb{R}} d\epsilon \sum_{\omega} (G(\epsilon, i\omega + i\nu) G(\epsilon, i\omega) - G(\epsilon, i\omega)^2). \quad (1020)$$

Now we do the Matsubara sum by integrating against the Fermi distribution. Continuing $i\omega$ to z ,

$$\sigma^{aa}(i\nu) = \frac{ne^2}{2\pi i m \nu} \int_{\mathbb{R}} d\epsilon \int dz f(z) (G(\epsilon, z + i\nu) G(\epsilon, z) - G(\epsilon, z)^2). \quad (1021)$$

The singularities of the summand are branch cuts which run horizontally across the complex plane. For $\alpha \in \mathbb{R}$ and δ infinitesimal, we have branch cuts at $z = \alpha \pm i\delta$ and at $z = \alpha - i\nu \pm i\delta$. The first of these gives

$$\begin{aligned} & \frac{1}{\alpha + i\nu - \epsilon - \Sigma(\alpha + i\nu)} \left(\frac{1}{\alpha - i\delta - \epsilon - \Sigma(\alpha - i\delta)} - \frac{1}{\alpha + i\delta - \epsilon - \Sigma(\alpha - i\delta)} \right) \\ & \rightarrow \frac{2\pi i\delta(\epsilon + \Sigma(\alpha - i\delta) - \alpha)}{\alpha + i\nu - \epsilon - \Sigma(\alpha + i\nu)}, \end{aligned} \quad (1022)$$

where we've subtracted in the given order since the counterclockwise path of the contour means that the $\alpha - i\delta$ part is oriented from $-\infty$ to $+\infty$. The branch cut at $z = \alpha - i\nu \pm i\delta$ gives us a similar expression, namely

$$\frac{2\pi i\delta(\epsilon + \Sigma(\alpha + i\delta) - \alpha)}{\alpha - i\nu - \epsilon - \Sigma(\alpha - i\nu)}. \quad (1023)$$

Putting these in and doing the energy integral, we reduce to an integral of the parameter α along the \mathbb{R} line:

$$\sigma^{aa}(i\nu) = \frac{ne^2}{m\nu} \int_{\mathbb{R}} d\alpha f(\alpha) \left(\frac{1}{i\nu - \Sigma(\alpha + i\nu) + \Sigma(\alpha - i\delta)} + \frac{1}{-i\nu + \Sigma(\alpha + i\delta) - \Sigma(\alpha - i\nu)} \right), \quad (1024)$$

where we have dropped $i\delta$'s when they appear in a sum with other imaginary things.

Now we analytically continue by taking $i\nu \rightarrow \nu + i\delta$. This gives

$$\sigma^{aa}(\nu + i\delta) = i \frac{ne^2}{m} \int_{\mathbb{R}} d\alpha \frac{f(\alpha)}{\nu} \left(\frac{1}{\nu - \Sigma(\alpha + \nu + i\delta) + \Sigma(\alpha - i\delta)} - \frac{1}{\nu - \Sigma(\alpha + i\delta) + \Sigma(\alpha - \nu - i\delta)} \right). \quad (1025)$$

Now shift α by $-\nu/2$ in the first term and by $+\nu/2$ in the second term:

$$\sigma^{aa}(\nu + i\delta) = i \frac{ne^2}{m} \int_{\mathbb{R}} d\alpha \frac{f(\alpha - \nu/2) - f(\alpha + \nu/2)}{\nu} \frac{1}{\nu - \Sigma(\alpha + \nu/2 + i\delta) + \Sigma(\alpha - \nu/2 - i\delta)}. \quad (1026)$$

Now define the average scattering rate by

$$\bar{\tau}^{-1} = \text{Im} [\Sigma(\alpha - \nu/2 - i\delta) + \Sigma(\alpha + \nu/2 - i\delta)] \quad (1027)$$

and the wavefunction renormalization by

$$Z^{-1} = 1 - \frac{1}{\nu} \text{Re} [\Sigma(\alpha - \nu/2) - \Sigma(\alpha + \nu/2)]. \quad (1028)$$

Then we have

$$\sigma^{aa}(\nu + i\delta) = \frac{ne^2}{m} \int_{\mathbb{R}} d\alpha \frac{f(\alpha - \nu/2) - f(\alpha + \nu/2)}{\nu} \frac{1}{\bar{\tau}^{-1} - i\nu Z^{-1}}. \quad (1029)$$

This just comes from decomposing the denominator into real and imaginary parts: the sign on one of the Σ 's in τ is flipped since we flipped the sign of the $i\delta$ in its argument, and we've dropped the $i\delta$'s in the self-energy since we're taking the \mathbb{R} part.

Finally, if we can ignore the α dependence of τ and Z , then we can do the frequency integral at low temperatures and get

$$\sigma^{aa}(\nu + i\delta) = \frac{ne^2}{m} \frac{1}{\bar{\tau}^{-1} - i\nu Z^{-1}}, \quad (1030)$$

since the integral of the difference of the Fermi functions just gives ν . Note that if we set $\nu = 0$ then we get the expected (still at low temperatures)

$$\sigma^{aa}(0 + i\delta) = \frac{ne^2 \tau}{m}, \quad (1031)$$

where $\tau = 2\text{Im}[\Sigma(0 - i\delta)]$. Yay!

62 June 15 — AFM spin waves and itinerant magnetism

This is yet again from Piers' many body theory book, chapter 13. Consider a three-dimensional tight-binding model on a cubic lattice, and develop the mean field theory for the AFM commensurate order parameter

$$M_i^\mu = M^\mu e^{iQ \cdot r_i}, \quad (1032)$$

where μ is a vector index and i is a site on the lattice, and Q is the AFM vector $Q = (\pi, \pi, \pi)$.

Find the mean-field free energy and the gap equation for the magnitude M . Argue that at half-filling, we will always have a transition to a spin-density wave state even for arbitrarily small interaction strength (think of BCS theory and the “dimensional reduction” that the FS does).

Solution:

First we need to get the MF Hamiltonian. We start from (being sloppy about the index notation)

$$H = -t \sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + \mu c_j^\dagger c_j) + U \sum_j n_{\uparrow j} n_{\downarrow j}, \quad (1033)$$

and then fiddle with the interaction in the usual way, writing it more democratically in spin space as (using $n_{\sigma j}^2 = n_{\sigma j}$)

$$U n_{\uparrow j} n_{\downarrow j} = -\frac{U}{6} (c_{\alpha j}^\dagger \sigma_{\alpha\beta}^\mu c_{\beta j})^2 + \frac{U}{2} (n_{\uparrow j} + n_{\downarrow j}) \rightarrow -\frac{I}{2} (c_{\alpha j}^\dagger \sigma_{\alpha\beta}^\mu c_{\beta j})^2, \quad (1034)$$

where $I = U/3$ and we've absorbed the $n_{\uparrow} + n_{\downarrow}$ term into the chemical potential. We then decouple the interaction with the mean field M . Because M^μ is at momentum Q , in momentum space we end up with a coupling like

$$M^\mu (c_{\alpha\mathbf{k}+\mathbf{Q}}^\dagger \sigma_{\alpha\beta}^\mu c_{\beta\mathbf{k}}). \quad (1035)$$

It thus helps to split up the electron operators in the BZ according to \mathbf{Q} . Consider the mini-BZ that stretches from 0 to $\pm\pi/2$ for all coordinates. Any point in the BZ can be reached by adding a \mathbb{Z}_2 -multiple of Q to a point in the mini-BZ, and furthermore almost all (in the mathematical sense) of the points outside the mini-BZ can be associated with a unique point in the mini-BZ after taking off \mathbf{Q} . So then we can just sum over the mini-BZ instead, and keep track of the operators located outside the mini-BZ by stacking on the \mathbf{Q} vector. So then define the four-component spinor

$$\Psi_{\mathbf{k}} = (c_{\uparrow\mathbf{k}}, c_{\downarrow\mathbf{k}}, c_{\uparrow\mathbf{k}+\mathbf{Q}}, c_{\downarrow\mathbf{k}+\mathbf{Q}})^T. \quad (1036)$$

The Hamiltonian has the free c^\dagger part, the $M - c$ coupling part (which connects stuff in the mini-BZ to stuff outside of it, so it will be off-diagonal when acting on $\Psi_{\mathbf{k}}$), and the $M^2/2I$ quadratic part. Neglecting to write the volume of the lattice, we have

$$H = \sum_{\mathbf{k} \in \text{mini-BZ}} \Psi_{\mathbf{k}}^\dagger ((\epsilon_{\mathbf{k}} - \mu) \mathbf{1} \oplus (\epsilon_{\mathbf{k}+\mathbf{Q}} - \mu) \mathbf{1} + X \otimes M_\mu \sigma^\mu) \Psi_{\mathbf{k}} + \frac{1}{2} M^2. \quad (1037)$$

The dispersion is, just read off from the tight-binding model,

$$\epsilon_{\mathbf{k}} = -2t \sum_{\mu} \cos k_{\mu}, \quad (1038)$$

and so we see that we have the important nesting property, namely that $\epsilon_{\mathbf{k}+\mathbf{Q}} = -\epsilon_{\mathbf{k}}$. Note also that the eigenvalues of the $\Psi^\dagger \Psi$ term are

$$E_{\mathbf{k}s} = s \sqrt{\epsilon_{\mathbf{k}}^2 + M^2} - \mu, \quad s = \pm 1. \quad (1039)$$

Now let's get the free energy. We get the usual log det, and then have to evaluate

$$\sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \sum_{\omega} \ln(i\omega + E_{\mathbf{k}s}). \quad (1040)$$

We do the Matsubara sum by integrating against the Fermi distribution. Besides the Matsubara frequencies, we have a branch cut for the log, starting at $E_{\mathbf{k}s}$ and e.g. going out to ∞ along the \mathbb{R} axis. The change in the branches across the \mathbb{R} axis is $2\pi i$, and so we get

$$- \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \int_{E_{\mathbf{k}s}}^{\infty} d\alpha f(\alpha). \quad (1041)$$

Now

$$-\int_a^\infty dx \frac{1}{e^{\beta x} + 1} = a - T \ln(1 + e^{\beta a}), \quad (1042)$$

which you can get by differentiating with respect to a . Using this and putting in $E_{\mathbf{k}s}$ for a , we see that the first term above gives us -4μ , since we have four eigenvalues for each k . The sum over s kills the $s\sqrt{\epsilon^2 + M^2}$ part, so we can borrow two of the μ 's in the -4μ to write the term coming from a on the RHS of the last equation as

$$\sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \left(-2\mu + \frac{\beta E_{\mathbf{k}s}}{2} \right). \quad (1043)$$

Why did we do this dumb manipulation, adding in terms that vanish after summing over s anyway? Because it simplifies the $\ln(1 + e^{\beta a})$ part of the integral. We take the $\frac{\beta E_{\mathbf{k}s}}{2}$ term above and write

$$\sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \left[\frac{\beta E_{\mathbf{k}s}}{2} + \ln(1 + e^{-\beta E_{\mathbf{k}s}}) \right] = \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \ln[2 \cosh(\beta E_{\mathbf{k}s}/2)]. \quad (1044)$$

So, combining this with the remaining -2μ part and the M^2 term, we finally get

$$F = -\frac{1}{\beta} \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \ln \left[2 \cosh \left(\frac{E_{\mathbf{k}s}\beta}{2} \right) \right] + \frac{M^2}{2I} - 2\mu. \quad (1045)$$

Now we can differentiate with respect to M to find the gap equation. This is straightforward, and we get

$$\frac{1}{I} = \frac{1}{2} \sum_{\mathbf{k} \in \text{mini-BZ}} \sum_s \tanh \left(\frac{\beta E_{\mathbf{k}s}}{2} \right) \frac{1}{\sqrt{\epsilon_{\mathbf{k}}^2 + M^2}}. \quad (1046)$$

Now we see why the nesting property is important. Consider some small value of the interaction strength I . Since $\epsilon_{\mathbf{k}+Q} = -\epsilon_{\mathbf{k}}$, $\epsilon_{\mathbf{k}}$ must vanish on a codimension-1 surface in the (full) BZ. Thus at $M = 0$ the RHS diverges, since we are integrating something which has a $1/r$ -type singularity on a codimension-1 submanifold of the integration domain (note to self: this is true for this example and other simple ones — is it true in general? Should be simple to figure out). This means that by making M small, we can make the RHS of the above equation arbitrarily large (provided we also choose T to be small enough so that the tanh piece doesn't cancel the divergence). In particular, at least at $\mu = 0$, we can always find a non-zero solution for M of the above equation, no matter how small I is (think of superconductivity), and so we will always have a transition to a spin-density wave phase even at arbitrarily small I .

63 June 16 — Kondo problem intro

Consider the Anderson impurity Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \left(\epsilon_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + V_{\mathbf{k}} f_\sigma^\dagger c_{\mathbf{k}\sigma} + V_{\mathbf{k}}^* c_{\mathbf{k}\sigma}^\dagger f_\sigma \right) + \sum_\sigma \epsilon_f n_{f\sigma} + U n_{f\uparrow} n_{f\downarrow}. \quad (1047)$$

Here the f (f for f -orbital) operators add / remove electrons from the impurities, and ϵ_f is assumed to lie below the Fermi energy while $2\epsilon_f + U$ is assumed to lie above. Both ϵ_f and U are to be taken to be large parameters.

By splitting the total wavefunction into $\psi_0 + \psi_1 + \psi_2$ where the subscripts indicate the occupancy of the impurity, use perturbation theory to find the effective Hamiltonian for the subspace in which the impurity is singly occupied.

Solution:

Decomposing the wavefunction as suggested in the Schrodinger equation and identifying the terms that go with each component, we have

$$(H_{00} + H_{10})\psi_0 = E\psi_0, \quad (H_{11} + H_{21} + H_{01})\psi_1 = E\psi_1, \quad (H_{22} + H_{12})\psi_2 = E\psi_2, \quad (1048)$$

so that we get the familiar form

$$\left(H_{12} \frac{1}{E - H_{22}} H_{21} + H_{11} + H_{10} \frac{1}{E - H_{00}} H_{01} \right) \psi_1 = E\psi_1. \quad (1049)$$

Here H_{ij} connects the i occupancy subspace to the j occupancy subspace, and so we can read off

$$H_{00} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma}, \quad H_{11} = H_{00} + \epsilon_f, \quad H_{22} = H_{00} + 2\epsilon_f + U \quad (1050)$$

for the diagonal terms and

$$H_{01} = \sum_{\mathbf{k}\sigma} V_{\mathbf{k}}^* c_{\mathbf{k}\sigma}^\dagger f_\sigma, \quad H_{21} = \sum_{\mathbf{k}\sigma} V_{\mathbf{k}} f_\sigma^\dagger c_{\mathbf{k}\sigma}, \quad (1051)$$

where $-\sigma$ is the opposite spin to σ . We also have $H_{10} = H_{01}^\dagger$ and $H_{12} = H_{21}^\dagger$ so that e.g.

$$H_{12} = \sum_{\mathbf{k}\sigma} V_{\mathbf{k}}^* c_{\mathbf{k}\sigma}^\dagger f_\sigma. \quad (1052)$$

Now we need to simplify the effective Hamiltonian. Consider first the leftmost term. We write

$$\frac{1}{E - H_{22}} = \frac{1}{E - U - 2\epsilon_f - H_{00}} = \left[(-U - \epsilon_f) \left(1 - \frac{E - \epsilon_f - H_{00}}{U + \epsilon_f} \right) \right]^{-1}. \quad (1053)$$

Since the Fermi level was chosen to be above ϵ_f but below $2\epsilon_f + U$, the $E - \epsilon_f - H_{00}$ term will be small when acting on ψ_1 and dividing by $U + \epsilon_f$ will make it smaller. Thus we can write

$$\frac{1}{E - H_{22}} \approx -\frac{1}{U + \epsilon_f} \left(1 + \frac{E - \epsilon_f - H_{00}}{U + \epsilon_f} \right). \quad (1054)$$

We then put this in the Hamiltonian and move the fraction through H_{21} so that it acts on ψ_1 . When we move H_{00} through H_{21} we pick up the extra term

$$[H_{00}, H_{21}] = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} V_{\mathbf{k}} c_{\mathbf{k}\sigma} f_\sigma^\dagger, \quad (1055)$$

and so the first term in the effective Hamiltonian is

$$-H_{12} \frac{1}{U + \epsilon_f} \left(H_{21} + H_{21} \frac{E - \epsilon_f - H_{00}}{U + \epsilon_f} - \frac{1}{U + \epsilon_f} \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} V_{\mathbf{k}} c_{\mathbf{k}\sigma} f_{\sigma}^{\dagger} \right). \quad (1056)$$

The second term will be droppable after acting on ψ_1 . We can then also take the $\epsilon_{\mathbf{k}}$ part and put it back in the denominator. So then we can write

$$-H_{12} \frac{1}{U + \epsilon_f - \epsilon_{\mathbf{k}}} H_{21}, \quad (1057)$$

and we get the expected energy transfer in the denominator (the \mathbf{k} in the denominator makes sense after putting in the sums for H_{12} and H_{21} —it is the momentum of the latter).

We do the exact same thing for the other term in the effective Hamiltonian (which takes us through the unoccupied intermediate state). The algebra is the same as before, and we get

$$-H_{10} \frac{1}{\epsilon_{\mathbf{k}} - \epsilon_f} H_{01}. \quad (1058)$$

Putting this together, the effective Hamiltonian is (ignoring constants)

$$H_{\text{eff}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} V_{\mathbf{k}} V_{\mathbf{k}'}^* \left(\frac{f_{\sigma}^{\dagger} f_{\sigma'} c_{\mathbf{k}\sigma} c_{\mathbf{k}'\sigma'}^{\dagger}}{\epsilon_{\mathbf{k}'} - \epsilon_f} + \frac{f_{\sigma'}^{\dagger} f_{\sigma} c_{\mathbf{k}'\sigma'}^{\dagger} c_{\mathbf{k}\sigma}}{U + \epsilon_f - \epsilon_{\mathbf{k}}} \right). \quad (1059)$$

To write this in a more transparent form, we define

$$S_f^{\mu} = \frac{1}{2} \sum_{ab} f_a^{\dagger} [\sigma^{\mu}]_{ab} f_b, \quad s_{\mathbf{k}\mathbf{k}'}^{\mu} = \frac{1}{2} \sum_{ab} c_{\mathbf{k}a}^{\dagger} [\sigma^{\mu}]_{ab} c_{\mathbf{k}'b}. \quad (1060)$$

Using

$$\sigma_{ab}^{\mu} \sigma_{cd}^{\mu} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}, \quad (1061)$$

we have

$$\sum_{\sigma\sigma'} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}'\sigma'} f_{\sigma'}^{\dagger} f_{\sigma} = 2s_{\mathbf{k}\mathbf{k}'}^{\mu} S_f^{\mu} + \frac{1}{2} \sum_{\sigma\sigma'} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}'\sigma} f_{\sigma'}^{\dagger} f_{\sigma'}, \quad (1062)$$

since the LHS has the $\delta_{ad}\delta_{bc}$ structure and the second term on the RHS is put in to cancel the $\delta_{ab}\delta_{cd}$ part. So, now we would like to get the terms in H_{eff} into a form like the form of the LHS above. This just involves commuting a few operators past each other: when we commute the c 's in the middle term of H_{eff} we get a number operator in f 's, which when acting on the low energy space is equal to the identity, and so we just get an extra constant. When commuting the f 's in the third term, we get an extra term with the operators $c_{\mathbf{k}'\sigma}^{\dagger} c_{\mathbf{k}\sigma}/(U + \epsilon_f - \epsilon_{\mathbf{k}})$. This comes in with a minus sign relative to the terms of the form $c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}'\sigma'} f_{\sigma'}^{\dagger} f_{\sigma}$, and so we get

$$H_{\text{eff}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} \left(2J_{\mathbf{k}\mathbf{k}'} S_f^{\mu} s_{\mathbf{k}\mathbf{k}'}^{\mu} + T_{\mathbf{k}\mathbf{k}'} \sum_{\sigma} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}'\sigma} \right), \quad (1063)$$

where the coefficients are

$$J_{\mathbf{k}\mathbf{k}'} = V_{\mathbf{k}} V_{\mathbf{k}'}^* \left(\frac{1}{U + \epsilon_f - \epsilon_{\mathbf{k}}} + \frac{1}{\epsilon_{\mathbf{k}'} - \epsilon_f} \right) \quad (1064)$$

and

$$T_{\mathbf{k}\mathbf{k}'} = -\frac{1}{2} V_{\mathbf{k}} V_{\mathbf{k}'}^* \left(\frac{1}{U + \epsilon_f - \epsilon_{\mathbf{k}}} - \frac{1}{\epsilon_{\mathbf{k}'} - \epsilon_f} \right), \quad (1065)$$

with the relative minus sign in the last expression coming from the commuting we did to the c 's. The form of J is of course expected, since it's just the strength of the tunneling between the impurity and the conduction band times the sum of the energy denominators for the two relevant virtual processes. As confirmed by our analysis, since we're doing second order perturbation theory (implying we get a negative correction to the energy), and since the interactions between the impurity and the conduction band are happening in the singlet channel, we get an *antiferromagnetic* exchange coupling between the impurity and the spin density of the conduction electrons.

64 June 17 — More about the Kondo problem and doing RG with the SW transformation

This is from Altland and Simons, but we will use different notation / a slightly different approach (plus there are some typos in the book). Today's problem uses the tools of yesterday's problem to implement an RG analysis on the Anderson impurity Hamiltonian, generalized to allow for anisotropic exchange couplings. We start with

$$H = H_0 + \sum_{\mathbf{kq}} \left(J_z S_f^z \left[c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{q}\uparrow} - c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{q}\downarrow} \right] + J_+ S_f^+ c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{q}\uparrow} + J_- S_f^- c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{q}\downarrow} \right), \quad (1066)$$

where

$$H_0 = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} \quad (1067)$$

and where the S_f^a operators are the spin operators for the (spin 1/2) impurity.

Let the conduction electron band extend from $-\Lambda$ to Λ . The high energy degrees of freedom that need to be integrated out during the RG step are the electrons at the top of the band and holes at the bottom of the band. The RG will proceed in the Wilsonian perspective, by shaving off small regions around the band edges at $\pm\Lambda$.

Do a SW transformation on the Hamiltonian by decomposing $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, where \mathcal{H}_0 contains state vectors with at least one hole within some small energy window $\delta\Lambda$ of the lower band edge, \mathcal{H}_2 contains vectors with at least one electron within a small window $\delta\Lambda$ of the upper band edge, and \mathcal{H}_1 is the low-energy submanifold. Proceeding in the usual way, find the effective Hamiltonian and obtain the beta functions for J_z and J_\pm . Show that in the antiferromagnetic case of the Kondo problem, the theory is asymptotically free and find the Kondo temperature (the dimensionally transmuted scale in the problem—think Λ_{QCD}).

Solution:

As we saw yesterday, the Schrodinger equation reads

$$\left(H_{12} \frac{1}{E - H_{22}} H_{21} + H_{11} + H_{10} \frac{1}{E - H_{00}} H_{01} \right) \psi_1 = E \psi_1. \quad (1068)$$

Figure 11: The two kinds of processes we need to worry about. The dashed lines are the impurity spins and the solid lines are conduction band electrons.

From the definition of the 0, 1, 2 decomposition, the term with $1/(E - H_{00})$ is essentially the particle-hole conjugate of the term with $1/(E - H_{22})$. Let's look at the later first. We have a bunch of terms, that go as $J_z^2, J_+J_-, J_+J_z, J_-J_z$, and flip-flops thereof.

Let's just start by looking at the $S_f^- S_f^+$ term. It is

$$J_- J_+ \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} S_f^- c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{q}\downarrow} \frac{1}{E - H_{22}} S_f^+ c_{\mathbf{q}\downarrow}^\dagger c_{\mathbf{k}\uparrow}, \quad (1069)$$

where \mathbf{k}, \mathbf{k}' are slow momenta and \mathbf{q} is a fast momentum. We were able to write the momentum structure in the way we did above since H_{22} is zero when acting on states with no fast electrons and since H_{22} conserves momentum.

Now we do the integral over \mathbf{q} : letting $\rho(\Lambda)$ be the density of states at the band edge, we get (using the fact that $E - H_{22}$ becomes $E - \Lambda + \epsilon_{\mathbf{k}} - H_0$ when acting on a state with an electron taken up to the band edge from a state with energy $\epsilon_{\mathbf{k}}$)

$$J_- J_+ \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda S_f^- S_f^+ c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{k}\uparrow} \frac{1}{E - \Lambda + \epsilon_{\mathbf{k}} - H_0}, \quad (1070)$$

where $\delta\Lambda$ is the small energy window we've integrated out, over which the density of states is assumed constant. When we take the energy relative to the ground state and keep in mind that the cutoff Λ is much larger than any other scales we're interested in, we can write (using $S_f^+ S_f^- = 0 \oplus 1 = 1/2 - S_f^z$)

$$J_- J_+ \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} - S_f^z \right) c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{k}\uparrow} \frac{1}{E - \Lambda}. \quad (1071)$$

We see that the S_f^z piece of this term will go towards renormalizing J_z , since it's diagonal in the spin index of the conduction band electrons. If we were to look at the $S_f^+ S_f^-$ term, we would get (following the same argument)

$$J_+ J_- \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} + S_f^z \right) c_{\mathbf{k}'\downarrow}^\dagger c_{\mathbf{k}\downarrow} \frac{1}{E - \Lambda}. \quad (1072)$$

Now we look at the particle-hole conjugated versions of these two contributions. To find them, we do the same thing as in the above but with $E - H_{00}$ in the denominator. The process mediated by this term involves a creation of a hole at the lower band edge together with a slow-momentum electron, and then a subsequent annihilation of the hole with a different slow-momentum electron. This is shown in part b of Figure 11 (time is read left-to-right), with the diagram leading to the previous contributions shown in part a. As we see from the figure, the particle-hole conjugate version has an extra overall minus sign (the crossed fermion lines) and involves the spin operators changing places, since the slow electron which escapes

now participates in the interaction at an earlier time. This means that the ph conjugates contribute

$$-J_- J_+ \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} + S_f^z \right) c_{\mathbf{k}'\uparrow}^\dagger c_{\mathbf{k}\uparrow} \frac{1}{E - \Lambda} \quad (1073)$$

and

$$-J_+ J_- \sum_{\mathbf{k}\mathbf{k}'} \rho(\Lambda) \delta\Lambda \left(\frac{1}{2} \mathbf{1} - S_f^z \right) c_{\mathbf{k}'\downarrow}^\dagger c_{\mathbf{k}\downarrow} \frac{1}{E - \Lambda}. \quad (1074)$$

The actual denominators are of these terms are slightly different than their ph conjugate partners, but since we're sending $E - \Lambda \pm \epsilon_{\mathbf{k}} \rightarrow E - \Lambda$ for slow \mathbf{k} , the differences don't show up.

Adding up all of the terms collected so far, we see that the $1/2$ terms cancel while the $\pm S_f^z$ terms add, so that the effective J_z coupling becomes

$$J_z \mapsto J_z - 2J_+ J_- \rho(\Lambda) \delta\Lambda \frac{1}{E - \Lambda}. \quad (1075)$$

Now we need to find the renormalization of the J_{\pm} couplings, which will come from spin-flipping terms that go as either $J_z J_+$ or $J_z J_-$. Again we first look at the terms involving $1/(E - H_{22})$. Choosing e.g. the $J_- J_z$ term, we get a contribution

$$-J_- J_z \sum_{\mathbf{k}\mathbf{k}'} \frac{S_f^-}{2} \rho(\Lambda) \delta\Lambda c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}'\downarrow} \frac{1}{E - \Lambda}, \quad (1076)$$

where we used $S_f^z S_f^- = -S_f^-/2$. The ph-conjugate has switched spin operators and an extra minus sign which cancel, so it adds with the above term. The $J_+ J_z$ term is obtained in the same way. Thus the J_{\pm} couplings change as

$$J_{\pm} \mapsto J_{\pm} - 2J_z J_{\pm} \rho(\Lambda) \delta\Lambda \frac{1}{E - \Lambda}. \quad (1077)$$

Now we can get the β functions, since we know the effective couplings: they are³⁵

$$\beta_{J_s} = \frac{dJ_s}{d\ln\Lambda} = -2J_s J_z \rho \quad s = \pm 1, \quad \beta_{J_z} = \frac{dJ_z}{d\ln\Lambda} = -2J_+ J_- \rho, \quad (1078)$$

where we replaced $E - \Lambda$ with $-\Lambda$ since if the Wilsonian RG we're doing is going to work we need the “average energy scale” associated with the slow momentum modes (the eigenvalue of ψ_1 under H) to be super small compared to Λ . Taking the ratio of the β functions, we see that

$$\frac{dJ_z}{dJ_{\pm}} = \frac{J_{\mp}}{J_z} \implies J_z^2 - J_+ J_- = \text{const.} \quad (1079)$$

Consider the case where J_+, J_- , and J_z all have the same sign. Note that then J_z is monotonically increasing along the RG trajectory (flowing from small to large distances). If we start with a ferromagnetic coupling ($J_z < 0$) and start with $J_z^2 > J_+ J_-$, then the flow is

³⁵We've ignored the terms quadratic in J_z since they are independent of the impurity spins and just lead to a renormalization of the kinetic energy term H_0 .

towards trivial couplings. If $J_z^2 < J_+J_-$ then it seems like J_z is allowed to cross through zero with the J_\pm afterwards becoming increasingly negative, which is weird.

However, if we have an antiferromagnetic coupling (which as we saw above, we do) then the β function for J_z is asymptotically free and J_z increases without bound (meaning it increases until our perturbative calculation breaks down). This is of course the cause of the resistivity minimum.

Finally, let us consider for simplicity the case where $J = J_\pm = J_z$ are all antiferromagnetic. Introducing the dimensionless variable (at a particular energy scale we suppress)

$$g \equiv J\rho, \quad (1080)$$

we see from the β function for g that at a given energy scale T , (here Λ is an arbitrary reference point for the RG)

$$g(T) = \frac{g(\Lambda)}{1 + 2g(\Lambda) \ln(T/\Lambda)}. \quad (1081)$$

The dimensionally transmuted scale at which we lose perturbation theory (the Kondo temperature) is the root of the numerator, and so T_K is

$$T_K = \Lambda \exp\left(-\frac{1}{2g(\Lambda)}\right). \quad (1082)$$

65 June 18 — Beta function in the Kondo problem to 2-loop order

One last problem on the Kondo problem. Today we'll look at the beta function in more detail.

The Hamiltonian is

$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} JS_f^a \left(\frac{1}{2} c_{\mathbf{k}\alpha}^\dagger [\sigma^a]_{\alpha\beta} c_{\mathbf{k}'\beta} \right) \quad (1083)$$

Defining the dimensionless constant

$$g = J\rho, \quad (1084)$$

where ρ is the density of states at the band edge (for a given cutoff), show that the beta function for g is

$$\beta_g = -2g^2 + 2g^3 + O(g^4). \quad (1085)$$

Work at $T = 0$ for simplicity. Note that unlike yesterday, we are assuming isotropic couplings.

Solution:

To handle the impurity spins in Feynman diagrams, we write the impurity spin as $S_f^a = \frac{1}{2} f_\alpha^\dagger \sigma_{\alpha\beta}^a f_\beta$. We need to then project onto the singly-occupied f states, which we do by adding

the chemical potential term $\lambda_f(f_\alpha^\dagger f_\alpha - 1) = [i\pi/(2\beta)](f_\alpha^\dagger f_\alpha - 1)$ to the Hamiltonian. This works since the normal Hamiltonian is zero for unoccupied or doubly-occupied f fermion states, and it ensures that the unphysical occupations cancel out in the partition function (only “unphysical” since we are restricting ourselves to the singly-occupied space—physically, the impurity spin could be unoccupied). However since we are taking $T \rightarrow 0$ we won’t usually have to worry about adding this chemical potential term into the impurity spin propagator.

First let’s get the self energy for the impurity spin. One of the diagrams is taken care of by normal ordering, and so the diagram we need to compute looks like

(1086)

where the dashed lines represent the impurity spin and the solid lines are the electrons.

First we will find the polarization bubble $\Pi_2(i\omega)$, which has one impurity line and one electron line. We will assume that all the lines in the bubble are pointing to the right, so that the diagram is (note: we will not keep careful track of the spin operators σ and S_f for a bit—they only produce some factors of 2 when summed over which can fairly easily be put back in later)

$$\Pi_2(\omega) = \frac{1}{\beta} \sum_{\omega} \int_{\epsilon} \rho(\epsilon) \frac{1}{i\omega - \epsilon} \frac{1}{i\nu - i\omega - \lambda_f}. \quad (1087)$$

We’ve written it in this way since even though we’re taking $\beta \rightarrow \infty$, it actually turns out to be slightly easier to do the Matsubara sum compared to the integrals. The Matsubara sum is standard, and after doing it we do integral over ϵ from the lower band edge $-\Lambda$ up to Λ , and then set $\lambda_f = 0$. This gives (here $\omega > 0$)

$$\Pi_2(\omega) = \rho \ln \left(\frac{\Lambda}{|\omega|} \right), \quad (1088)$$

where we took a flat density of states.

Now we can calculate the self energy using the melon we drew above. The top two lines of the melon are just Π_2 , while the bottom line just contributes an extra electron propagator. There are three different melons (three places to put the dashed line) and each has a symmetry factor of 2, so the self energy is (here we take the top two lines in the melon to point right and the bottom line to point left)

$$\Sigma(\omega) = -\frac{3J^2}{2} \sum_{\mathbf{k}} \int \frac{d\nu}{2\pi} \Pi_2(\omega + \nu) \frac{1}{i\nu - \epsilon_{\mathbf{k}}}. \quad (1089)$$

When we do the integral over energy we get a contribution from the half-residue of $-i\pi \operatorname{sgn}(\nu)\rho$, so

$$\Sigma(\omega) = \frac{3i}{4} g^2 \int_{-\Lambda}^{\Lambda} d\nu \ln \left(\frac{\Lambda}{|\omega + \nu|} \right) \operatorname{sgn}(\nu), \quad (1090)$$

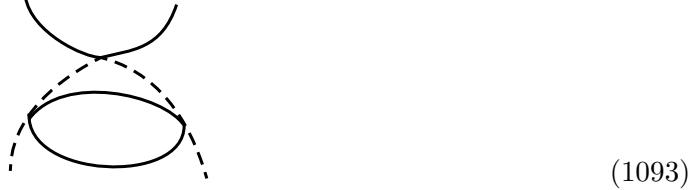
which gives us (for $\omega > 0$)

$$\Sigma(\omega) = -\frac{3i}{2} g^2 \omega \ln(\Lambda/\omega). \quad (1091)$$

The self energy determines the wavefunction renormalization for the conduction electrons. We get it from the residue of $[i\omega - \Sigma(\omega)]^{-1}$, and so at an energy scale μ where we're doing the renormalization, to $O(g^2)$, we have

$$Z(\mu) = 1 - \frac{3}{2}g^2 \ln(\Lambda/\mu). \quad (1092)$$

The last two-loop diagram we need to know is the 2-loop vertex correction. This contribution is illustrated by the worst looking figure I've ever made:



To do RG, we need to evaluate this diagram at a single energy scale. The RG condition we will choose is to have all the legs at the same energy μ (all the legs except for the top one in the electron bubble are assumed to be pointing to the right). With this choice, the two internal f impurity spin legs have the same frequency. Thus we need to compute something like

$$\frac{1}{2} J^2 (-JS_f^a \sigma^a) \int_{\mathbf{k}, \mathbf{k}'} \int_{\nu, \gamma} \frac{1}{[i(\mu + \gamma - \nu)]^2} \frac{1}{i\gamma - \epsilon_{\mathbf{k}}} \frac{1}{i\nu - \epsilon_{\mathbf{k}'}} , \quad (1094)$$

where due to our choice of arrows in the diagrams there are no minus signs from fermion bubbles, the $1/2$ is a symmetry factor, and the σ^a is the spin operator for the conduction electrons (there are also spin operators for the other vertices which we are omitting). Note that except for the $-JS_f \sigma$ interaction piece, the above integral is just the derivative with respect to $i\mu$ of the diagram we computed for $\Sigma(i\omega)$, except without the factor of 3. So then the diagram of (1093), evaluated at the scale μ , is

$$-\frac{1}{2} (JS_f^a \sigma^a) g^2 \ln(\Lambda/\mu). \quad (1095)$$

Now only the easy part is left: getting the 1-loop corrections to the vertex. We actually essentially already did this in the last problem. They can just be computed using our knowledge of the polarization bubble. The contributions from the particle (s-channel) and hole (t-channel) processes have opposite signs, but the ordering of the σ^a operators is also different. So the 1-loop diagrams give the counterterms

$$-J^2 \rho \ln(\Lambda/\mu) (\sigma^a \sigma^b - \sigma^b \sigma^a) S_f^a S_f^b. \quad (1096)$$

We can antisymmetrize the $S_f^a S_f^b$ by adding a factor of $1/2$. The product of the two commutators is $-\epsilon^{abc} \epsilon^{abd} \sigma^c S_f^d = -2\sigma^a S_f^a$, and so the end result of the 1-loop terms is to produce the counterterm

$$2J^2 \rho \ln(\Lambda/\mu) \sigma^a S_f^a. \quad (1097)$$

Collecting all of the counterterms and noting that $1/2 + 3/2 = 2$, the counterterms for the $J\sigma^a S_f^a$ interaction are

$$\text{counterterms} = 2 \ln\left(\frac{\Lambda}{\mu}\right) (J^2 \rho - J^3 \rho^2). \quad (1098)$$

Multiplying everything by ρ gives us the β function for g by differentiating the RHS of the above equation with respect to $\ln \mu$:

$$\beta_g = -2(g^2 - g^3) + O(g^4). \quad (1099)$$

Alternately, from a more Wilsonian point of view, we could say that the effective coupling constant for the $\sigma^a S_f^a$ term with the RG conditions imposed at the energy scale μ is

$$g(\mu) = g - 2g^2 \ln(\Lambda/\mu) + 2g^3 \ln(\Lambda/\mu), \quad (1100)$$

and so taking $\Lambda \mapsto \Lambda + d\Lambda$, $g(\mu) \mapsto g(\mu) + dg$, and $\ln(1 + d\Lambda/\Lambda) = d \ln \Lambda$, we have

$$dg(\mu) = d \ln \Lambda (-2g^2 + 2g^3), \quad (1101)$$

which of course gives us the same β function.

66 June 19 — Level repulsion for kindergarteners

Today's problem is super basic but was new to me so it was still worth a diary entry. Consider a 2-by-2 Hamiltonian

$$H = \begin{pmatrix} H_1 & \Delta \\ \Delta^* & H_2 \end{pmatrix}. \quad (1102)$$

Assume that the probability distribution for the components of H is a Gaussian:

$$P(H_1, H_2, \Delta) \propto \exp\left(-\frac{1}{2}\text{Tr}H^2\right). \quad (1103)$$

Let E_+, E_- be the two eigenvalues of H (with $E_+ > E_-$), and define the variable $\xi = E_+ - E_-$. Find the probability distribution for ξ , first for real Δ and then for complex Δ .

Solution:

First let $\Delta \in \mathbb{R}$. It will be helpful to change to variables

$$H_{\pm} = \frac{H_1 \pm H_2}{2}, \quad (1104)$$

in terms of which the eigenvalues are

$$E_{\pm} = H_+ \pm \sqrt{\Delta^2 + H_-^2}. \quad (1105)$$

First we need the probability distribution for getting two values E_+, E_- . This is

$$P(E_+, E_-) \propto \int dH_+ dH_- d\Delta \delta(E_+ - H_+ - \sqrt{\Delta^2 + H_-^2}) \delta(E_- - H_- - \sqrt{\Delta^2 + H_-^2}) e^{-(E_+^2 + E_-^2)/2}. \quad (1106)$$

Now do the integral over H_+ :

$$P(E_+, E_-) \propto \int dH_- d\Delta \delta(E_+ - E_- - 2\sqrt{\Delta^2 + H_-^2}) e^{-(E_+^2 + E_-^2)/2}. \quad (1107)$$

This is just an integral over \mathbb{R}^2 , with $r = \sqrt{\Delta^2 + H_-^2}$ being the radial coordinate. Thus we get a factor of r in the integration measure and

$$P(E_+, E_-) \propto (E_+ - E_-) e^{-(E_+^2 + E_-^2)/2}. \quad (1108)$$

Thus we have

$$P(\xi) \propto \int dE_+ dE_- \delta(E_+ - E_- - \xi) (E_+ - E_-) e^{-(E_+^2 + E_-^2)/2}, \quad (1109)$$

so that

$$P(\xi) \propto \int dE_+ \xi e^{-(E_+ - \xi/2)^2 - \xi^2/4} \implies P(\xi) \propto \xi e^{-\xi^2/4}. \quad (1110)$$

Note that this goes to zero as $\xi \rightarrow 0$, which is level repulsion.

For the more general case of $\Delta \in \mathbb{C}$, the only thing that changes is we get a probability distribution like

$$P(E_+, E_-) \propto \int dH_- d\Delta_1 d\Delta_2 \delta(E_+ - E_- - 2\sqrt{\Delta_1^2 + \Delta_2^2 + H_-^2}), \quad (1111)$$

and so now we're doing an integral over \mathbb{R}^3 instead and get an r^2 in the measure. Thus for $\Delta \in \mathbb{C}$ the level repulsion is even stronger, going as ξ^2 :

$$P(\xi) \propto \xi^2 e^{-\xi^2/4}. \quad (1112)$$

67 June 20 — Slave bosons for the Hubbard model unfinished

Consider the Hubbard model at infinite U :

$$H = -t \sum_{\langle i,j \rangle, \sigma} (|i;0\rangle\langle i;\sigma| \otimes |j;\sigma\rangle\langle j;0| + h.c.) - \mu \sum_{j\sigma} |j;\sigma\rangle\langle j;\sigma|. \quad (1113)$$

The notation here is such that $|i;\sigma\rangle$ represents a state where the i th site is occupied by a fermion with spin projection σ along the z axis ($|i;0\rangle$ denotes an unoccupied site). We have $n_f \leq 1$ at every site. We will look at this problem with slave bosons, rather than by doing a HS decoupling of the $Un_f n_f$ interaction.

Rewrite the model in terms of slave bosons that keep track of valence fluctuations (unoccupied sites). Show how to choose a gauge in which the slave-boson amplitude is real everywhere. Find the mean field free energy, and show that the effective mass of the fermions diverges as the slave-boson field amplitude goes to zero.

Solution:

The slave bosons b will occupy sites vacated by the fermions. This means that the sum of the slave boson number and the fermion number must be a constant integer:

$$b^\dagger b + n_f = Q \in \mathbb{Z}. \quad (1114)$$

We can choose the normalization of the charge so that we have the constraint $Q = 1$. Both the slave bosons and the fermions then carry unit charge under a fictitious gauge field.

In terms of the slave bosons, the various terms in H are

$$|i;0\rangle\langle i;\sigma| \otimes |j;\sigma\rangle\langle j;0| + h.c. = b_i^\dagger f_{\sigma i} f_{\sigma j}^\dagger b_j + b_j^\dagger f_{\sigma j} f_{\sigma i}^\dagger b_i, \quad |i\sigma\rangle\langle i;\sigma| = f_{\sigma i}^\dagger f_{\sigma i}. \quad (1115)$$

To implement the gauge constraint, we add a Lagrange multiplier field A , and couple it to the combination $A_i(n_{fi} + b_i^\dagger b_i - Q)$. Then going over to coherent states to write the path integral and adding in the kinetic terms, the action is

$$S = \sum_i \int d\tau \left(\bar{f}_i(\partial_\tau + A_i - \mu) f_i + \bar{b}_i(\partial_\tau + A_i) b_i - t \sum_{\langle ij \rangle \sigma} (b_i^\dagger f_{\sigma i} f_{\sigma j}^\dagger b_j + b_j^\dagger f_{\sigma j} f_{\sigma i}^\dagger b_i) - A_i Q \right). \quad (1116)$$

Let us decompose the slave bosons as $b_i = r_i e^{i\phi_i}$. Then ϕ appears in two places in the action: in the b kinetic term and in the interaction term, where it enters as $e^{i\phi_i - i\phi_j}$. To get rid of its appearance in the later term, perform the change of variables $f_i \mapsto f_i e^{i\phi_i}$ (this is not a gauge transformation). This adds in a $\partial_\tau \phi$ term to the f kinetic term. Thus we have

$$S = \sum_i \int d\tau \left(\bar{f}_i(\partial_\tau + A_i - \mu + i\partial_\tau \phi_i) f_i + r_i(\partial_\tau + A_i + i\partial_\tau \phi_i) r_i - t \sum_{\langle ij \rangle \sigma} (r_i f_{\sigma i} f_{\sigma j}^\dagger r_j + r_j f_{\sigma j} f_{\sigma i}^\dagger r_i) - A_i Q \right). \quad (1117)$$

Now we go to unitary gauge to kill the $\partial_\tau \phi$ parts. We Hodge decompose A as $A = d\alpha + d^\dagger \beta + \omega$, where $d\alpha$ is the gauge degree of freedom. We gauge-fix by choosing α to cancel the $i\partial_\tau \phi$ terms. Then re-naming $d^\dagger \beta + \omega$ as A , we have

$$S = \sum_i \int d\tau \left(\bar{f}_i(\partial_\tau + A_i - \mu) f_i + r_i(\partial_\tau + A_i) r_i - t \sum_{\langle ij \rangle \sigma} (r_i f_{\sigma i} f_{\sigma j}^\dagger r_j + r_j f_{\sigma j} f_{\sigma i}^\dagger r_i) - (A_i - i\partial_\tau \phi) Q \right). \quad (1118)$$

The $r_i \partial_\tau r_i$ term is a total derivative and since $Q \in \mathbb{Z}$ and ϕ has 2π periods, the $\partial_\tau \phi Q$ term can be dropped. Thus

$$S = \sum_i \int d\tau \left(\bar{f}_i(\partial_\tau - \mu) f_i - t \sum_{\langle ij \rangle \sigma} r_i r_j (f_{\sigma i} f_{\sigma j}^\dagger + f_{\sigma j} f_{\sigma i}^\dagger) - A_i(Q - r_i^2 - n_{f,i}) \right). \quad (1119)$$

Now we can do mean field.

68 June 21 — WZW Miscellanea

Today's problem is a smattering of little things about the WZW term. The format will be a series of mini-questions. I found these questions listed as exercises in Abanov's lecture notes on topological terms in QFT.

Preliminaries: For a vector n that lives in S^2 , define

$$W = \frac{1}{16\pi i} \int_D \text{Tr}[\hat{n} \wedge d\hat{n} \wedge d\hat{n}], \quad (1120)$$

where D is a two-disk and $\hat{n} = n^a \sigma^a$. Show that the variation of W only depends on the values that the n field takes on ∂D . Also show that

$$S_{WZW} = 4\pi SW[n^a] \quad (1121)$$

is well-defined as an action on the manifold ∂D provided that $S \in \frac{1}{2}\mathbb{Z}$.

First we rewrite W as

$$W = \frac{1}{16\pi i} \int_D i\epsilon^{abc} \text{Tr}[n^a dn^b \wedge dn^d \sigma^c \sigma^d] = \frac{1}{8\pi} \int_D \epsilon^{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n), \quad (1122)$$

where we've written out the derivative explicitly since $dn \wedge \times dn$ is bad notation. Now we vary this, sending $n \mapsto n + \delta n$. Since $n^1 = 1$, $\delta n \cdot n = 0$. But since $\partial_\mu n$ is also orthogonal to n , $\partial_\mu n \times \partial_\nu n$ is parallel to n , and so $(\delta n) \cdot (\partial_\mu n \times \partial_\nu n) = 0$. Thus the variation is

$$\delta W = \frac{1}{8\pi} \int_D \epsilon^{abc} n^a (\delta n^b \wedge dn^c + dn^b \wedge \delta n^c). \quad (1123)$$

Integrating both terms by parts and again using that $(\delta n) \cdot (\partial_\mu n \times \partial_\nu n) = 0$ as well as the supercommutativity of \wedge , we get

$$\delta W = \frac{1}{4\pi} \int_{\partial D} dx^\mu \epsilon^{abc} \delta n^a \partial_\mu n^b n^c, \quad (1124)$$

and so indeed, the variation only cares about the fields on the boundary. This is because W is measuring the area on the target S^2 swept out by the n field. This area only depends on the trajectory of the n field as one moves around the circle ∂D . Indeed, the boundary values of n on ∂D trace out some region on the target S^2 , and the values of the field on the interior of D "fill in" the area enclosed by this trajectory in a smooth way. By thinking of this visually, it's clear that changing the way that D "fills in" this region cannot change the total (signed) area swept out by the n field.

Now for the well-definedness of S_{WZW} . The difference in two extensions to the 2-manifold D from the given ∂D is the integral of S_{WZW} over a closed 2-manifold M . Since δW only depended on the values of n on the boundary, since $\partial M = 0$ we know that the values of W evaluated over M must be quantized. All we need to do is check the coefficient. We've got

$$S_{WZW} = S \int_M \epsilon^{abc} \frac{1}{2} n^a (dn^b \wedge dn^c). \quad (1125)$$

But the integrand on the RHS is precisely the pullback of one half of the volume form on S^2 by the map $n : M \rightarrow S^2$ (the $1/2$ because of the antisymmetrization and since the volume form is $\epsilon^{abc} n^a \partial_x n^b \partial_y n^c$ if the coordinates are x, y). So then

$$S_{WZW} = S \int_{n(M) \subset S^2} \text{vol} = 4\pi Sw, \quad (1126)$$

where $w \in \mathbb{Z}$ is the winding number. Thus we get something well-defined provided that $S \in \frac{1}{2}\mathbb{Z}$. The most suggestive way to write S_{WZW} is probably

$$S_{WZW} = 2S \frac{2\pi}{\Omega_2} \frac{1}{2} \int_D \epsilon^{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n), \quad (1127)$$

where $\Omega = 4\pi$. If we let D be a sphere, then the integral gives us an integer multiple of $2\Omega_2$ (where the 2 comes from the antisymmetrization of the derivatives in spacetime—if we used a \wedge for the spacetime derivatives we wouldn't need the 2).

Spin precession: Now add a magnetic field h , so that

$$S[h] = S_{WZW}[n] - S \int dt h^a n^a. \quad (1128)$$

Assume h is independent of time for simplicity. Derive spin precession in an $SU(2)$ invariant way.

We just compute the equations of motion. To do this, add a Lagrange multiplier to enforce $n^2 = 1$ so that we can vary n freely:

$$S[h, \lambda] = S_{WZW}[n] - S \int dt h^a n^a + \frac{1}{2} \lambda(n^2 - 1). \quad (1129)$$

We can use our knowledge from the first part to write

$$\delta S = \int dt \left(S \epsilon^{abc} \partial_t n^b n^c + \lambda n^a - Sh^a \right) \delta n^a. \quad (1130)$$

Taking the term in the parenthesis and dotting it with n ,

$$\lambda = Sh^a n^a - S \epsilon^{abc} n^a \partial_t n^b n^c = Sh^a n^a. \quad (1131)$$

Thus the equation of motion gives

$$\epsilon^{abc} \partial_t n^b n^c + n^a (h^b n^b) - h^a = 0, \quad (1132)$$

or as a vector equation,

$$\partial_t n \times n = h - (h \cdot n)n. \quad (1133)$$

Note that as expected, if we set n parallel to h then n doesn't want to move, while if we set n normal to h then n wants to precess about the h axis.

Quantization: Derive the commutation relations for the spin operator from S_{WZW} . Use this to check that the equations of motion you derived earlier correspond to the Heisenberg equations of motion.

To derive the spin commutation relations we should figure out what the symplectic form is. Since we only have the ϕ and θ variables, they will label the phase space. Since the phase space is a sphere, we expect a closed but not exact symplectic form. We want to start from the term

$$S_{WZW} = \frac{S}{2} \int_D \epsilon^{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n). \quad (1134)$$

We can proceed in kind of a dumb way by varying this to find the symplectic current and then taking a second variation to find what the symplectic form is, after which the Poisson brackets can be read off. Luckily we already computed the variation in terms of the n field. Plugging in the coordinate representation for n , doing a fair bit of algebra (use Mathematica as it's fairly heinous), and using the equations of motion gives the symplectic current. Taking another variation gives the symplectic form, which if I did the algebra correctly is

$$j = -\delta(S \cos \theta) \wedge \phi \implies \Omega = -\delta(S \cos \theta) \wedge \delta\phi, \quad (1135)$$

where the expression is evaluated at some given initial time (symplectic currents are integrated over codimension 1 Cauchy slices, which in our case are just points). Thus from the symplectic form we conclude that the symplectic partner for ϕ is $S \cos \theta$, and so

$$\{\phi, S \cos \theta\} = 1. \quad (1136)$$

Thus when we take the Poisson bracket of two quantities we do it as follows:

$$\{X, Y\} = -\frac{1}{S \sin \theta} (\partial_\phi X \partial_\theta Y - \partial_\theta Y \partial_\phi X), \quad (1137)$$

which indeed gives $\{\phi, S \cos \theta\} = 1$. As expected, these expressions only make sense within a coordinate patch, and the symplectic form is not globally exact.

Anyway, now we can use this to compute the Poisson bracket of the different n vectors. The relevant ones are $S^x = S \cos \phi \sin \theta$, $S^y = S \sin \phi \sin \theta$, $S^z = S \cos \theta$. For example,

$$\{S^y, S^z\} = -\frac{S}{\sin \theta} (-\cos \phi \sin^2 \theta + 0) = S^x. \quad (1138)$$

Of course, the general rule is $\{S^a, S^b\} = \epsilon^{abc} S^c$. When we pass to quantum mechanics we change to commutators and add in the i , and as a result derive the spin commutation relations. Note that we did not start from anything involving Pauli matrices to do this.

Now we check the Heisenberg equations of motion. The WZW term doesn't enter into the Hamiltonian since it is linear in time derivatives (but of course it still affects the equations of motion, unlike e.g. a theta term). Then we can compute

$$[H, S^a] = [S^a, S^b] h^b = i \epsilon^{abc} S^c h^b \implies \partial_t S^a = \epsilon^{abc} S^b h^c \implies \partial_t n^a = \epsilon^{abc} n^b h^c. \quad (1139)$$

Is this equivalent to the equations of motion we derived earlier? Yes: putting this into the equations of motion implies

$$h^a - (h^b n^b) n^a = \epsilon^{abc} \epsilon^{bde} n^d h^e n^c = (\delta_{cd} \delta_{ae} - \delta_{ad} \delta_{ce}) n^d h^e n^c, \quad (1140)$$

which is a trivial equality.

Relation to the θ term: Now suppose that the value of n on the 1-manifold is constrained to lie on a circle of constant latitude, with θ a constant. Find S_{WZW} in this case (note that we do not impose this restriction on the extension of n into the disk [and it will in general not be possible to impose such a condition]). Show that making this restriction on θ turns the WZW term into a θ term.

Also, in an AFM chain, show how the WZW term gives rise to a θ term that is nontrivial when $2S$ is odd. Sort of conversely, motivate why an $S = 1$ chain with a θ term placed on an open line has two spin $1/2$'s at the boundary.

First let's compute δW with this restriction. In spherical coordinates,

$$\delta n = (-\delta\phi \sin\phi \sin\theta, \delta\phi \cos\phi \sin\theta, 0)^T. \quad (1141)$$

Similarly, $\partial_\mu n$ also has no z component. Thus the integrand of δW goes like $n^z(\delta n \times \partial_t n)^z$. But from the form of δn , the cross product vanishes, and so $\delta W = 0$. This means that with this restriction on θ , W is quantized. Thus in order to compute W , we are free to choose a convenient extension of n into the 2-manifold as well as a convenient shape for the 2-manifold, since W cannot change if we change either of these choices. So we will choose the bounding 2-manifold to be the unit disk, with coordinates ϕ, r . We are even allowed to choose the ϕ coordinate of the disk to be an integer multiple of the ϕ coordinate on the S^2 that n maps into, where the integer is the winding number of the spacetime (really just time) S^1 into the $S^1 \subset S^2$ defined by the constant value of θ . If this winding number is w , then with this restriction we can choose n to be

$$n = (-\sin(w\phi) \sin(r\theta), \cos(w\phi) \sin(r\theta), \cos(r\theta))^T. \quad (1142)$$

The first two components reduce to w times the volume form on S^1 at $r = 1$ where the field is constrained to lie on the constant θ circle. This field extension we've chosen looks like a cowlick on the head of a particularly curly-haired person, with the hair all standing straight up at the center of their head.

Putting this into (no Jacobian changing from Cartesian coordinates since our integrand is a differential form)

$$S_{WZW} = S \int_{D^2} dr d\phi \epsilon^{abc} n^a (\partial_r n^b \partial_\phi n^c) \quad (1143)$$

and doing some algebra gives

$$S_{WZW} = 2\pi i Sw(\cos\theta - 1), \quad (1144)$$

which is just measuring the area on the sphere defined by the line of constant latitude at the given value of θ . So, we get a θ term (where the action is proportional to the volume form on the temporal S^1) with coefficient iSw . As it must, when $\theta = \pi$ (the polar angle of the n vector, not the θ of the θ term—sorry!) this gives us something in $2\pi i\mathbb{Z}$. Also note that when the vector is restricted to lie on the equator, we get a theta term at theta-angle $\theta = \pi$ (relevant for the AFM spin chain).

For the AFM spin chain, the action has a term which is just $\sum_i S_{WZW}[n_i]$, where n_i is the n -field at the site i . Since we expect a staggered configuration to be a good “mean field”, we can as a first pass examine $\sum_i S_{WZW}[(-1)^i n_i]$, where now n_i is slowly varying. Since $S_{WZW}[n]$ measures the area on the target S^2 enclosed by the trajectory of the n field,

$S_{WZW}[n_j] + S_{WZW}[-n_{j+1}] = S_{WZW}[n_j] - S_{WZW}[n_{j+1}]$ measures a difference in the areas swept out by the two curves along their trajectories. Drawing a picture shows that this area difference is proportional to $\int dt \epsilon^{abc} n^a \partial_x n^b \partial_t n^c$, where x is the coordinate along the chain. Integrating this over the length of the chain, we get

$$\frac{1}{2} S \int \epsilon^{abc} n^a dn^b \wedge dn^c. \quad (1145)$$

Note that this term now maps the spacetime S^2 into the target S^2 , and is defined without reference to a bounding 3-manifold. Since it integrates to $2\pi S w$ where w is the winding number, it is a θ term which only contributes if $2S$ is odd. Thus the WZW terms sum up and interfere with each other to produce a θ term.

Now suppose we are given the $n\sigma m$ description of an open AFM chain with the θ term, for $2S$ even. Suppose we already know that the chain is gapped for this choice of S , so that we can focus on the θ term. Now the θ term is the integral of the volume form over the chain. The volume form is closed and is also exact since spacetime has trivial H^2 , so the θ term must reduce to something localized on the ends of the chain. Indeed, plugging in the form for n means the θ term is, after some algebra,

$$\frac{S}{2} \int dx dt \sin \theta (\partial_t \theta \partial_x \phi - \partial_x \theta \partial_t \phi). \quad (1146)$$

Let's assume that we're working with periodic time. Then we can integrate by parts and write the term as

$$-\frac{S}{2} \int_{\partial C} \cos \theta \partial_t \phi, \quad (1147)$$

where the integral is over the ends of the chain. This means that we get WZW terms on the chain ends. In particular if $S = 1$ then we get WZW terms with spin 1/2, suggesting that the critical $S = 1$ AFM spin chain hosts spin 1/2's at its edges.

69 June 22 — WZW action from fermions

This problem is one of the problems I found listed as an exercise in Abanov's lecture notes on topological terms in QFT.

Consider a fermion in 0+1 dimensions coupled to a vector:

$$S = \int d\tau \bar{\psi} (\partial_\tau + M n^a \sigma^a) \psi, \quad (1148)$$

where M is a mass and n maps into S^2 . Integrate out the fermion and find the leading two terms in a $1/M$ expansion. To do this, it is helpful to first compute δS and then un-do the variation at the very end.

Integrating over the fermions,

$$\delta S = -\delta \text{Tr} \ln(\partial_\tau + M \hat{n}), \quad (1149)$$

where we have used the notation $\hat{n} = n^a \sigma^a$. Let $D = \partial_\tau + M\hat{n}$ be the “covariant derivative”. Then

$$\delta S = -\text{Tr}(\delta D D^{-1}) = -\text{Tr}(\delta D D^\dagger (D D^\dagger)^{-1}), \quad (1150)$$

which we have written in this way since expanding $D D^\dagger$ in $1/M$ is easier. Now since $\hat{n}^2 = n^2 \mathbf{1} = \mathbf{1}$,

$$D D^\dagger = (\partial_\tau + M\hat{n})(-\partial_\tau + M\hat{n}) = -\partial_\tau^2 + M^2 + M\partial_\tau\hat{n}. \quad (1151)$$

We then expand the inverse of this in the following way, writing $G_f = (-\partial_\tau^2 + M^2)^{-1}$ for the free propagator:

$$(D D^\dagger)^{-1} = \frac{G_f}{\mathbf{1} + G_f M \partial_\tau \hat{n}} = \frac{G_f (\mathbf{1} - G_f M \partial_\tau \hat{n})}{1 - (G_f M \partial_\tau \hat{n})^2}, \quad (1152)$$

where the denominator on the RHS is now just a number. To the leading orders in M we can actually just replace the denominator by 1, and so

$$\delta S = -\text{Tr}[(M\delta\hat{n})(-\partial_\tau + M\hat{n})G_f(\mathbf{1} - G_f M \partial_\tau \hat{n})]. \quad (1153)$$

We do the trace by

$$\text{Tr}[\mathcal{O}] = \int d\tau \int \frac{d\omega}{2\pi} e^{i\omega t} \text{Tr}_\sigma[\mathcal{O}] e^{-i\omega t}, \quad (1154)$$

where Tr_σ indicates a trace over the spin degrees of freedom. We wrote the trace like this (by inserting a resolution of the identity in frequency space) since the operator \mathcal{O} in question isn't local in time (it involves a bunch of G_f 's), but is local in frequency space. Writing this out,

$$\delta S = -\frac{M}{2\pi} \int d\tau d\omega \text{Tr}_\sigma \left[\delta\hat{n}(i\omega + M\hat{n}) \frac{1}{\omega^2 + M^2} \left(\mathbf{1} - \frac{1}{\omega^2 + M^2} M \partial_\tau \hat{n} \right) + \delta\hat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau^2 \hat{n} \right]. \quad (1155)$$

Dropping things that will die after taking the spin trace and things that are odd in frequency,

$$\delta S = \frac{M}{2\pi} \int d\tau d\omega \text{Tr}_\sigma \left[\delta\hat{n} M \hat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau \hat{n} - \delta\hat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau^2 \hat{n} \right]. \quad (1156)$$

The relevant integral is

$$\int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + M^2)^2} = \frac{1}{4M^3}, \quad (1157)$$

and so

$$\delta S = \int d\tau \text{Tr}_\sigma \left[\frac{1}{4} \delta\hat{n} \hat{n} \partial_\tau \hat{n} - \frac{1}{4M} \delta\hat{n} \partial_\tau^2 \hat{n} \right]. \quad (1158)$$

Taking the trace and integrating the second term by parts,

$$\delta S = \int d\tau \left(-\frac{i}{2} \epsilon^{abc} \delta n^a \partial_\tau n^b n^c + \frac{1}{2M} \delta(\partial_\tau n)^a \partial_\tau n^a \right). \quad (1159)$$

Looking at yesterday's problem, we see that the first term is precisely the variation of the WZW term for a single spin 1/2 in 0+1 dimensions (spin 1/2 because of the prefactor), while the second term is the variation of a kinetic term for the vector. So the effective action is

$$S_{eff} = S_{kin} + S_{WZW} = \frac{1}{4M} \int d\tau (\partial_\tau n^a)^2 - \frac{1}{8\pi i} \int_D \text{Tr}[\hat{n} \wedge d\hat{n} \wedge d\hat{n}], \quad (1160)$$

where D is any two-disk that bounds the temporal circle.

70 June 23 — Fractional charge of fermion mass solitons in two dimensions

Today we return to a somewhat familiar problem involving the crazy things that fermion masses can do. We work in two dimensions, with the action

$$S = \int_{M_2} \left(\bar{\psi} \not{D}_A \psi + \bar{\psi} M e^{i\phi\bar{\gamma}} \psi \right), \quad (1161)$$

where ϕ is a scalar field and $\bar{\gamma} = Z$ is the chirality operator (we are working in Euclidean signature). Assume M is large, and consider a time-dependent ϕ which asymptotes to the constants ϕ_{\pm} at $x = \pm\infty$. What is the electric charge carried by this kink?

Solution:

To find the charge induced by the kink, we need to integrate out the fermions. Doing so is slightly tricky, though: we don't want to use $(\not{\psi} + M e^{i\phi\bar{\gamma}})^{-1}$ as a free propagator since inverting the exponential is a pain. However, we can't do the usual expansion of the Tr ln while treating the mass term as a vertex, since we are interested in large M and so we won't be able to truncate the expansion. Furthermore, treating the mass term as a vertex is bad since it will turn out that the induced charge doesn't depend on M . Thus we need to find a way to include M in a propagator in a simple way such that we can do a $1/M$ expansion.

The way to do this turns out to make use of the trick we did yesterday: first compute the variation of the effective action and then integrate over the variation at the end. So in the usual way we get

$$\delta S = -\text{Tr} \left(\delta D D^\dagger (D D^\dagger)^{-1} \right), \quad (1162)$$

where this time (note that in our notation $\not{D}_A = \not{\partial} + i\not{A}$ so that $\not{D}_A^\dagger = -\not{D}_A$)

$$D = \not{D}_A + M e^{i\phi\bar{\gamma}}, \quad D^\dagger = -\not{D}_A + M e^{-i\phi\bar{\gamma}}. \quad (1163)$$

The reason for doing this is that the product $D D^\dagger$ has within it the term $G_f^{-1} = -\partial_\mu \partial^\mu + M^2$ which we can use to do an expansion in large M . If I did the algebra right, then

$$D D^\dagger = G_f^{-1} - \frac{M}{2i} \sin(\phi\bar{\gamma}) (\not{\partial} + i\not{A}) + A_\mu A^\mu - i(\not{\partial}\not{A}) - 2iA^\mu \partial_\mu - iM(\not{\partial}\phi)\bar{\gamma}e^{-i\phi\bar{\gamma}}, \quad (1164)$$

where we used $-i\not{\partial}\not{A} = (\not{\partial}\not{A}) - 2iA^\mu \partial_\mu + i\not{A}\not{\partial}$. This is a bit of a mess, but when we invert it the large M limit can save the day, since the largest factor of M comes in G_f^{-1} which is nice and simple. So we invert $D D^\dagger$ essentially in the same way that we did yesterday. After doing this and dropping terms that are small in $1/M$, we get

$$\begin{aligned} \delta S = & +\text{Tr} \left[(i\delta\not{A} + i\bar{\gamma}M\delta\phi e^{i\phi\bar{\gamma}})(\not{\partial} + i\not{A} - M e^{-i\phi\bar{\gamma}})G_f \left(\mathbf{1} + G_f \left(\frac{M}{2i} \sin(\phi\bar{\gamma})(\not{\partial} + i\not{A}) \right. \right. \right. \\ & \left. \left. \left. - A^2 + i\not{\partial}\not{A} - 2iA^\mu \partial_\mu + iM\not{\partial}\phi\bar{\gamma}e^{-i\phi\bar{\gamma}} \right) \right) \right]. \end{aligned} \quad (1165)$$

Just so we can find our way through this mess slightly easier, let us note that

$$\int \frac{d^2 k}{(2\pi)^2} G_f^2 = \int \frac{dk}{2\pi} \frac{k}{(k^2 + M^2)^2} = \frac{1}{4M^2\pi}, \quad (1166)$$

so that when we do the integrals we are going to be picking up factors of $1/M^2$ (the terms involving just a single G_f will die for spin trace / momentum oddness reasons).

The expression for δS above looks like a disaster, but actually a lot of things drop out under the trace or under the implicit momentum integration (by oddness). In fact, we are just interested in the charge of the kink, so we can restrict our attention to terms linear in A (since we want to find the current). If we take a critical look at δS , we see that the key surviving term is the guy coming from the $i\partial A$ piece. This part gives

$$\delta S = \text{Tr} [\bar{\gamma} M^2 \delta\phi G_f^2 \partial A] + \dots \quad (1167)$$

Doing the trace is easy since the derivative is just acting on A , and so

$$\delta S = \int d^2 x \int \frac{d^2 k}{(2\pi)^2} \text{Tr}_\sigma [\bar{\gamma} M^2 \delta\phi \partial A] \frac{1}{k^2 + M^2} + \dots, \quad (1168)$$

where the trace is just the spin trace. The trace against $\bar{\gamma}$ selects out an antisymmetric structure for ∂ and A which gives us the field strength, and since $\text{Tr}_\sigma[\mathbf{1}] = 2$ we have

$$\delta S = \frac{1}{2\pi} \int d^2 x \delta\phi F + \dots, \quad (1169)$$

which is a θ term controlled by the phase of the fermion mass, as we would have expected. Thus integrating over the variation, we see that the current is

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi, \quad (1170)$$

and so the charge induced by the kink is (assuming space is a line)

$$Q = \int_{\mathbb{R}} j^0 = \int_{\mathbb{R}} \partial_x \phi \frac{1}{2\pi} = \frac{1}{2\pi} (\phi(\infty) - \phi(-\infty)). \quad (1171)$$

Thus the charge is set by degree to which the phase of the fermion mass winds along the kink. The most common scenario, since we have two degenerate states at $\phi = \pi, 0$, is to have $\phi(\infty) - \phi(-\infty) = \pi$, so that the mass term goes from being $M \cdot \mathbf{1}$ at $-\infty$ to $M \cdot \mathbf{1}$ at ∞ . In this case, the charge of the kink is $Q = 1/2$.

The kink also localizes a fermion zero mode. A low-tech way to see this is to just explicitly solve $H\psi = 0$ for some choice of $\phi(x)$. For example, suppose that $\phi(x) = \pi\Theta(x)$ so that the mass term is $M \cdot \mathbf{1}$ at $-\infty$ and $M \cdot \mathbf{1}$ at ∞ . Then for example at $x > 0$ the equation $H\psi = 0$ reads

$$(i\partial_x - A_x)\psi_L = M\psi_R, \quad -(i\partial_x - A_x)\psi_R = M\psi_L, \quad (1172)$$

so that $-(i\partial_x - A_x)^2\psi_L = M^2\psi_L$, which we can solve handily as

$$\psi_L(x) \propto \exp \left(i \int_0^x A - M|x| \right), \quad (1173)$$

which is localized around the kink and which could have been guessed from the fact that the phase of zero modes is given by the parallel transport formula (since $D_A\psi = 0$ is the requirement that ψ be parallel-transported) and their magnitude should be localized to the region where the mass changes.

71 June 24 — Fermion nonconservation and the ABJ anomaly

Just realized that there was still a problem in P&S that I wanted to do—it's from chapter 19.

Examine the ABJ relation

$$\Delta N_L - \Delta N_R = -\frac{e^2}{16\pi^2} \int \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} \quad (1174)$$

in four dimensions, where $N_{L/R}$ are the number of left- and right-handed fermions $\psi_{L/R}$ and the Δ measures differences between $t = \infty$ and $t = -\infty$. Take the background field to be

$$A^\mu = (0, 0, Bx, A), \quad (1175)$$

with B constant and A constant in space but possibly varying adiabatically in time. First, find the Hamiltonian. Then, solve the Schrodinger equation for the two fields $\psi_{L/R}(x)$. You should encounter a harmonic oscillator at some point during the calculation.

Consider putting the fermions in a box with sides of length L , with periodic boundary conditions. Find the degeneracy of the energy levels. Then change A adiabatically. What happens to the number of left- and right-handed fermions? Show that this checks out with the ABJ equation.

Solution:

Now $\epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} = 4\epsilon^{0ijk} F_{0i} F_{jk} = 8E_i B^i$, so that the ABJ formula is

$$N_L - N_R = -\frac{e^2}{2\pi^2} \int E_i B^i. \quad (1176)$$

The goal of this problem is to check this relation in a rather direct way.

First let's get the Hamiltonian. We have

$$\mathcal{H} = \pi_\psi D_0 \psi - \mathcal{L}. \quad (1177)$$

Here π_ψ is the regular canonical momentum for ψ , namely $\pi_\psi = \bar{\psi} i\gamma^0 = i\psi^\dagger$. Note that the $p\partial_t q$ term is modified by replacing ∂_t with the covariant version D_0 (we are treating A^μ as a background field, not a dynamical one), which completely cancels the D_0 part of the Lagrangian.

We will work in mostly negative signature, so that the gamma matrices are $\gamma^i = iY \otimes \sigma^i$. When combined with the $\gamma^0 = X \otimes \mathbf{1}$ from $\bar{\psi}$, we get the matrix $-Z \otimes \sigma^i$. Thus in the basis $\psi = (\psi_L, \psi_R)^T$, we have

$$\mathcal{H} = -i\psi_R^\dagger \sigma^i (\partial_i - ieA_i) \psi_R + i\psi_L^\dagger \sigma^i (\partial_i - ieA_i) \psi_L. \quad (1178)$$

Now the only spatial coordinate that appears in the vector potential is x , so that we have translation symmetry for y and z . Thus if we have $\int \mathcal{H} = E$ for e.g. the right-handed fermions then we are led to consider the Eigenvalue problem

$$-i\sigma^i(\partial_i - ieA_i)\psi_R = E\psi_R, \quad \psi_R = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} e^{ik_y y + ik_z z}. \quad (1179)$$

Writing out the eigenvalue equation, we get the coupled equations

$$\begin{aligned} (E - k_z + eA)\phi_1 + (i\partial_x + ik_y - ieBx)\phi_2 &= 0 \\ (E + k_z - eA)\phi_2 + (i\partial_x - ik_y + ieBx)\phi_1 &= 0. \end{aligned} \quad (1180)$$

So then after some algebra,

$$(\partial_x^2 + E^2 + B)\phi_1 = [(k_z - A)^2 + (k_y - Bx)^2]\phi_1, \quad (1181)$$

where we are temporarily letting $e = 1$. More suggestively,

$$(-\partial_x^2 + (k_y - Bx)^2 + (k_z - A)^2 - B)\phi_1 = E^2\phi_1, \quad (1182)$$

which is the Harmonic oscillator we were told we were going to find. Notice that k_y just sets the location of the center of the oscillator potential, but does not actually appear in the expression for the energy levels E (which we can get explicitly if we want, but we won't need the exact expressions).

What would have happened if we had done this with ψ_L instead? The only difference for ψ_L is that the eigenvalue equation is

$$+i\sigma^i(\partial_i - ieA_i)\psi_L = E\psi_L, \quad \psi_L = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} e^{ik_y y + ik_z z}, \quad (1183)$$

and so the only change is to replace E with $-E$, which won't affect conclusions about the degeneracy.

Now we put the Fermions in a box with side length L . The momenta are then $k_i \in (2\pi)/L\mathbb{Z}$. From the oscillator equation we saw that k_y sets the center of the oscillator potential through $x_c = k_y/B$ and doesn't affect the energy, while k_z sets the energy. So for a given energy (determined by k_z), different choices of k_y will lead to degenerate levels. However, we only have access to a certain number of k_y due to the finiteness of the box: taking the harmonic oscillator wavefunctions to be localized on a scale in the y direction much smaller than L , we require that $x_c < L$, or

$$k_y < BL \quad (1184)$$

since of course we need the wavefunction to be within the box. Since each value of k_y is spaced $\delta k_y = 2\pi/L$ apart, the degeneracy of each k_z level is

$$n = \frac{BL^2}{2\pi}, \quad (1185)$$

independent of k_z .

Now we consider an adiabatic change in A . Since this gives us a nonzero $E_z = \partial_t A$ and since the magnetic field is $B\hat{z}$, we have a nonzero $E_i B^i$ during this process and expect

$\Delta N_L - \Delta N_R \neq 0$ from the ABJ formula. We will consider a change in A so that the initial and final configurations are related by a gauge transformation with winding 2π around the x direction. So, we can set the initial A to be $A_i = 0$ and the final A to be $A_f = 2\pi/L$, which is a large gauge transformation since the holonomy of the gauge field around the x direction in the final state is in $2\pi\mathbb{Z}$. This gives the minimal value we can have for $\int F \wedge F$, since in this context $\int F \wedge F$ measures the number of large gauge transformations (each a winding by 2π) that occur.

Since the initial and final states are related by a large gauge transformation, the solving of the eigenvalue problem proceeds in the same way at t_i and t_f , and we get the same harmonic oscillator spectrum. However, let's look at where the levels go during the evolution: solving for E in the harmonic oscillator Hamiltonian for ϕ_1 , we see that E depends on A through $E \sim k_z - A$. Thus increasing A by $A \mapsto A + 2\pi/L$ is tantamount to *decreasing* k_z by $2\pi/L$, and so during the adiabatic evolution one level for the ψ_R fermions sinks below the Fermi level (wherever that may be). Conversely, we saw that when solving for the left-handed analogue φ_1 , we just had to replace $E \mapsto -E$, and so for the left-handed fermions we have $E \sim A - k_z$, and so the change is equivalent to *increasing* k_z by $2\pi/L$, which pulls up a level's worth of ψ_L fermions out of the Fermi sea. Thus during the change in A , the vacuum loses R fermions but gains L fermions.

Since there are $BL^2/(2\pi)$ states in each level and since doing a large gauge transformation moves the ψ_R fermions down a level and the ψ_L fermions up a level, we have (now restoring e)

$$\Delta N_R - \Delta N_L = -\frac{BL^2}{2\pi} - \frac{BL^2}{2\pi} = -\frac{Be^2 L^2}{\pi}. \quad (1186)$$

Does this agree with the anomaly formula? Let's check: during the time when A is changing (which is the only time during which $E \neq 0$), the non-zero part of E is $E_z = \frac{2\pi}{LT}$, where $T = t_f - t_i$ is the time of the adiabatic evolution. Thus according to the ABJ formula,

$$\Delta N_R - \Delta N_L = -\frac{e^2}{2\pi^2} \int d^4x E_i B^i = -\frac{e^2}{2\pi^2} \int d^3x B \frac{2\pi}{L} = -\frac{Be^2 L^2}{\pi}, \quad (1187)$$

which is exactly what we got with the more direct approach. Yay!

72 June 25 — QED_3 as a σ model

The goal of this problem is to work out the details of a description of QED_3 as a σ model that I read about in [10].

The starting point is QED_3 with $N = 2$ flavors of Dirac fermions and the square of an $SU(2)$ vector that we want to use as a potential order parameter field. The action is

$$S = \int (\bar{\psi} i \not{D}_A \psi + g(\bar{\psi} \sigma^a \psi)^2), \quad (1188)$$

where σ^a is a Pauli matrix in flavor space (not spin space!) and we've omitted the $F \wedge \star F$

term. Your mission is to show that at long distances,³⁶ this behaves like an $O(4)$ model with a theta term at $\theta = \pi$.

To show this, first decouple the $g(\bar{\psi}\sigma^a\psi)^2$ term with a 3-component vector N^a . After this is done, treat the vector as having a fixed length, with $N^a = Mn^a$ for n a unit vector in S^2 . Working with M large, integrate out the fermions and find the current. There is also a Hopf term hiding in the resulting effective action, which you should find (it will give spin to the skyrmions).

Switching to a \mathbb{CP}^1 representation for the vector, show that you get a mixed CS term between the electromagnetic gauge field and the \mathbb{CP}^1 gauge field. Show that this in turn reduces to the $O(4)$ model with a θ term at $\theta = \pi$.

Solution:

First we do the decoupling. We add the term $N^2/2g$ to the action where N is a 3-component vector, and then shift

$$N^a \mapsto N^a + ig\bar{\psi}\sigma^a\psi. \quad (1189)$$

We then take $N = Mn$ for n a unit vector (this restriction is supposed to not affect the phase diagram), and so

$$S = \int \bar{\psi} (iD_A + iMn^a\sigma^a) \psi. \quad (1190)$$

Let's first find the current after integrating out the fermions, which we will see gives electric charge to the solitons. Using the trick of two days ago, we have

$$J^\mu = i \frac{\delta}{\delta A_\mu} \text{Tr} \ln (iD_A + iMn^a\sigma^a) = i \text{Tr} (\gamma^\mu (D^\dagger D)^{-1} D^\dagger), \quad (1191)$$

with

$$D = i\partial + iMn^a\sigma^a. \quad (1192)$$

Here γ^μ are the Pauli matrices acting on spin space (we are in Euclidean signature), and we have set $A = 0$ in D since to find the current we can take the functional derivative with respect to J and then set the gauge field to zero. We then expand

$$(D^\dagger D)^{-1} = \frac{1}{-\partial^2 + M - M(\partial n^a)\sigma^a} = \frac{G_f}{\mathbf{1} - G_f M(\partial n^a)\sigma^a}, \quad (1193)$$

where as usual $G_f = (-\partial^2 + M^2)^{-1}$. To write it in this form, we have to remember that γ^μ and σ^a are both Pauli matrices, but commute since they act on different tensor factors. We expand this as

$$(D^\dagger D)^{-1} \approx G_f (\mathbf{1} + G_f M(\partial n^a)\sigma^a)(\mathbf{1} + G_f M(\partial n^b)\sigma^b). \quad (1194)$$

After putting this into our expression for J^μ and dropping things which get obviously killed by the momentum integration or the spin trace,

$$J^\mu = i \text{Tr} \left(\gamma^\mu \left[2G_f^2 M^2 (\partial n^a)\sigma^a + G_f^3 M(\partial n^a)\sigma^a(\partial n^b)\sigma^b \right] iMn^c\sigma^c \right). \quad (1195)$$

³⁶And long distances are really all we can talk about: from the kinetic term, we see that the dimension of ψ is 1, so that $[g] = -1$ and the model is (naively) non-renormalizable. We will be relating this to the $O(4)$ nlsm in three dimensions, which is also non-renormalizable, and we will only be able to make statements about IR physics in the following.

The first term goes like $(\partial_\mu n^a)n^a$ (with the index structure required if it wants to survive the flavor trace) which dies since n being a unit vector means that it's orthogonal to ∂n . In order to survive the traces over the spin and flavor indices, we need a $\epsilon^{\mu\nu\lambda}\epsilon^{abc}$ index structure. Thus

$$J^\mu = M^3 \text{Tr}(\mathbf{1}_{4 \times 4}) \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + M^2)^3} \epsilon^{\mu\nu\lambda} \epsilon^{abc} n^a \partial_\nu n^b \partial_\lambda n^c, \quad (1196)$$

which is starting to look a lot like a skyrmiony thing. Indeed, the integral gives $1/(32\pi|M|^3)$ and so

$$J^\mu = \frac{\text{sgn}(M)}{8\pi} \int \epsilon^{abc} n^a [\star(dn^b \wedge dn^c)]^\mu. \quad (1197)$$

Thus, we have shown that skyrmions carry electric charge! The current that couples to electromagnetism is precisely the topological charge density.

There are a few more parts of the effective action that we need to get. One is the kinetic term for n . We get the kinetic term by taking a variation with respect to n^a . Writing the variation of the trace as $\text{Tr}[\delta D D^\dagger (D D^\dagger)^{-1}]$ and expanding, to leading order in the large M expansion the relevant term comes when the $i\partial$ in D^\dagger hits the term in the expansion for $(D D^\dagger)^{-1}$ that is linear in M . We get

$$\begin{aligned} \delta S &= -\text{Tr}[M^2 \delta n^a \sigma^a G_f^2 \partial^2 n^b \sigma^b] + \dots \\ &= -4M^2 \int d^3x \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + M^2)^2} \delta n^a \partial_\mu \partial^\mu n^a + \dots \end{aligned} \quad (1198)$$

The integral is $1/8\pi M$, and so after integrating by parts over spacetime and integrating over the variation (we have $(\delta \partial_\mu n^a)\partial^\mu n^a$ which is a total variational derivative) we get

$$\delta S = \frac{M}{4\pi} \int d^3x (\partial_\mu n^a)^2. \quad (1199)$$

The last thing in the effective action we need is a topological term, which bestows spin upon the skyrmions. Since it's non-perturbative we have to find a clever way of getting to it. Such a way can be found in e.g. [1] (or in the classic paper by Wilczek and Zee), which we now go through. Basically, the idea is to consider a particular field history for a skyrmion and find what the topological term is in that case, and then write down the general topological term by writing down a covariant version of the specific topological term.

We consider a field configuration on e.g. $S^2 \times S^1$ with a skyrmion that rotates by 2π along the S^1 factor. This trajectory measures the spin angular momentum of the skyrmion, since the phase acquired during the trajectory is $e^{iS} = e^{2\pi i J}$. If the skyrmion is to be rotated around z axis as we pass around the S^1 , then we can consider the configuration where

$$n^a \sigma^a = e^{i\sigma^z \alpha(t)/2} n_0^a \sigma^a e^{-i\sigma^z \alpha(t)/2}, \quad (1200)$$

where t is the S^1 coordinate, $\alpha(2\pi) = \alpha(0) + 2\pi$, and n_0 is a static reference vector.

To get at the angular momentum of the skyrmion in a perturbative way, we can perturb in $\partial_t \alpha$ since the evolution is adiabatic. To bring a time derivative of α into the functional determinant, we consider performing the transformation $\psi \mapsto e^{i\alpha(t)\sigma^z/2} \psi$. This is not single-valued though, and it changes the boundary conditions of the fermions around the S^1 (from periodic to antiperiodic or vice-versa). We can fix the boundary conditions by doing a

further transformation which only affects the phase by $\psi \mapsto e^{i\alpha(t)/2}\psi$, which cancels out the change in boundary conditions. Doing this transformation kills the exponentials in the $e^{i\sigma^z\alpha(t)/2}n_0^a\sigma^a e^{-i\sigma^z\alpha(t)/2}$ term, and picks up a time derivative term from the Dirac operator. Thus we have to evaluate

$$\text{Tr} \ln \left(iD_A + \frac{i\gamma^0}{2}(\mathbf{1} + \sigma^z)\partial_t\alpha + iMn_0^a\sigma^a \right). \quad (1201)$$

The angular momentum of the skyrmion is obtained by differentiating the effective action with respect to the angular frequency with which the skyrmion rotates (think $E = m\omega^2 r^2/2 \implies \partial_\omega E = mvr = J$), and so to get the spin of the skyrmion we need to functionally differentiate the above with respect to $\partial_t\alpha$. This is easy though, since we can just look back at how we computed the variation of the effective action with respect to the gauge field— $\partial_t\alpha$ appears in the functional determinant in the exact same way as A_0 , except with an extra $(\mathbf{1} + \sigma^z)/2$ tacked on. Computing the variation in the same way as before and then integrating over the variation gives the topological term (a Hopf term, as we will see shortly) on this manifold:

$$S_H = \frac{i}{2} \int_{S^2 \times S^1} \partial_t\alpha J^0, \quad (1202)$$

where J^0 is the time component of the topological current given earlier (selected out because of the γ^0 in the trace). Note that since the topological charge is conserved, the Hopf term vanishes if α is topologically trivial, i.e. if it does not wind by some element of $2\pi\mathbb{Z}$ (since if it does not wind we can integrate by parts and get zero).

So, we've found a topological term that is constructed exactly in the same way as the coupling of A_μ to the topological current, except that A_μ is replaced by parts of the n field (in our example it is replaced by $\partial_t\alpha$). Because of the normalization of the topological current and the factor of $1/2$ in front, we see that the above term gives the skyrmions spin $1/2$.

To find the general presentation of the Hopf term, we just have to “covariantize” the particular form of S_H found above. One way to do this is to write it in terms of the $SU(2)$ matrix U which rotates between σ^z and $n^a\sigma^a$: with some algebra one can check that we get

$$S_H = -i\pi \frac{1}{24\pi^2} \int \text{Tr}[U^\dagger dU \wedge U^\dagger dU \wedge U^\dagger dU]. \quad (1203)$$

Actually, a slightly cooler presentation of this action is as a linking number a la the usual interpretation of the Hopf invariant. To write it down in this way we need to recast stuff in the \mathbb{CP}^1 language.

Let's then switch over to \mathbb{CP}^1 variables. The kinetic term for the n vector becomes the $|D_a z|^2$ term (here $a_\mu = -iz^\dagger \partial_\mu z$), where $n^a\sigma^a = 2zz^\dagger - \mathbf{1}$. What about the $A^\mu J_\mu$ term coupling the skyrmions to the electromagnetic field? The topological current J_μ maps to $i \star da/2\pi = i \star (dz^\dagger \wedge dz)/2\pi$ in the \mathbb{CP}^1 variables, which one can see either with a fair bit of algebra or with a “what else could it be” argument (the coefficient is fixed by the integrality of the topological charge).³⁷

³⁷More carefully, working in a parametrization where

$$z = \begin{pmatrix} \cos(\theta/2)e^{i(\gamma+\phi)/2} \\ \sin(\theta/2)e^{i(\gamma-\phi)/2} \end{pmatrix}. \quad (1204)$$

Finally, the Hopf term is just like the $A_\mu J^\mu$ coupling, except it has no A and only involves the n field. This means that in the \mathbb{CP}^1 language, it becomes a CS term for the \mathbb{CP}^1 gauge field, $a \wedge da$ (this is another way to see that it computes the Hopf invariant). Thus our new action is

$$S = \frac{M}{4\pi} \int |(\partial_\mu - ia_\mu)z|^2 + \frac{i}{2\pi} \int A \wedge da + \frac{1}{2e^2} \int F_A \wedge \star F_A + S_H, \quad (1207)$$

where S_H is now the CS term for a .

Since the \mathbb{CP}^1 variables were introduced to deal with a theory involving a vector living in S^2 , we do not yet have the $O(4)$ model we were promised. To get an $O(4)$ model, we have to access the full $O(4)$ symmetry of the z variables, instead of the $O(3)$ symmetry we get when acting on the quotient $S^3/S^1 = S^2$. Thus to get an $O(4)$ symmetry, we need to “eliminate the \mathbb{CP}^1 gauge field a ”, since it is responsible for quotienting out by S^1 .

We can see how this might happen by doing the integral over A and checking what the resulting action for the z fields is. The parts in the action involving A are

$$S_A = \frac{1}{2e^2} \int F_A \wedge \star F_A + \frac{i}{2\pi} \int a \wedge F_A. \quad (1208)$$

To integrate over A carefully, we first add an extra field ϕ in the path integral that enforces the exactness of F , and then treat F as an unconstrained field that we path-integrate over independently. The new action for the gauge fields reads

$$S_A = \frac{1}{2e^2} \int F \wedge \star F + \frac{i}{2\pi} \int a \wedge F_A + \frac{i}{2\pi} \int \phi \wedge dF. \quad (1209)$$

We can now perform the shift

$$F \mapsto F + \alpha \star a, \quad (1210)$$

where α is some constant to be determined. A little algebra shows that choosing $\alpha = -ie^2/2\pi$ cancels out the CS coupling of F to a . After doing this shift, we have

$$S_A = \frac{1}{2e^2} \int F \wedge \star F + \frac{i}{2\pi} \int \phi \wedge dF + \frac{e^2}{4\pi^2} \int \phi \wedge d \star a + \frac{e^2}{8\pi} \int a \wedge \star a. \quad (1211)$$

If we work in transverse gauge $d^\dagger a = 0$ things become simpler since then $d \star a = 0$ and the extra $\phi \wedge d \star a$ term dies. Then we can do the integral over ϕ to set $F = dA$, and so

$$S_A = \frac{1}{2e^2} \int F_A \wedge F_A + \frac{e^2}{8\pi} \int a \wedge \star a. \quad (1212)$$

Thus we have traded the CS coupling between the gauge fields for a mass term for a . The Maxwell field is now totally decoupled, and just gives a constant in the path integral which

Then (factors of 2 and minus signs?!)

$$da = -idz^\dagger \wedge dz = -\sin(\theta/2) \cos(\theta/2) d\theta \wedge d\phi = \sin(\theta/2) d\theta \wedge d\phi, \quad (1205)$$

which is the the pull-back of the volume form on S^2 , and hence $\star da$ is indeed the topological current. This can also be written as

$$[F_a]_{\mu\nu} = (D_{[\mu} z)^{\dagger} D_{\nu]} z, \quad (1206)$$

which can be checked with a little bit of algebra.

we can drop. Furthermore, the massiveness of a induced by the electromagnetic field means that we can integrate it out and get

$$S = \frac{M}{4\pi} \int |\partial_\mu z|^2 + S_H + \dots, \quad (1213)$$

where \dots include terms that are not $O(4)$ symmetric (note that S_H is $O(4)$ symmetric). In this way of writing it, the $O(4)$ symmetry comes from the fact that we can rotate among the four components of the complex spinor z (recall $|z|^2 = 1$). All the terms in \dots are seemingly less relevant at long distances than the $|\partial_\mu z|^2$ term. Whether the \dots terms can really be ignored or not is a tricky dynamical question that apparently no one knows the answer to. However, it is reasonable to hypothesize (based on the present understanding of the “duality web; more on this is a future diary entry) that these terms are indeed irrelevant, which frees up the full $O(4)$ symmetry of the model.

So accepting this somewhat shaky conclusion, we have what we wanted to find: an $O(4)$ model with a theta term at $\theta = \pi$ (it's a θ term since we started out with a Hopf term for the $n \in S^2$ vector, but after switching to z variables and killing off the gauge fields it becomes a θ term for an $O(4)$ field since we are now mapping $S^3 \rightarrow S^3$ rather than $S^3 \rightarrow S^3/S^1$). Oh yeah, one final thing: do we have an $O(4)$ symmetry in the formulation in terms of the U 's? Yes, it acts as

$$O(4) \cong (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2, \quad (1214)$$

with the two $SU(2)$ factors acting on the U matrices on the left and on the right, respectively (this is a symmetry since in S_θ all the U 's are sandwiched by U^\dagger 's, and vice versa—the quotient by \mathbb{Z}_2 avoids double-counting the center $Z(SU(2)) = \mathbb{Z}_2$).

73 June 26 — Number of zero modes allowed by magnetic fluxes in two dimensions

Consider massless Dirac fermions on a spatial \mathbb{R}^2 (2+1 dimensions) with a background field A . The Hamiltonian is

$$H = \gamma^0 \gamma^j (i\partial_j + A_j), \quad (1215)$$

where we will take $\gamma^0 = Z, \gamma^x = X, \gamma^y = Y$. Show that if the total magnetic flux through the plane is $\int F = \Phi > 0$, then H generically supports

$$\left\lceil \frac{\Phi}{2\pi} \right\rceil - 1 \quad (1216)$$

zero modes. If the magnetic flux is oriented positively (negatively) with respect to the plane, you should find that all the zero modes are generically right (left) handed.

Solution:

The proof is actually pretty quick once one figures out the trick. Letting the candidate zero mode be $\psi = (\psi_L, \psi_R)^T$, $H\psi = 0$ reads

$$(\partial - iA_-)\psi_L = 0, \quad (\bar{\partial} - iA_+)\psi_R = 0, \quad (1217)$$

with $\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, and $A_{\pm} = \frac{1}{2}(A_x \pm iA_y)$. These parallel transport equations can be solved in the usual way by attaching Wilson lines to the fermions, but this gives us yucky non-local expressions that are not well-suited for our present needs. Better is to choose the gauge $d^\dagger A = 0$, allowing us to write $A = d^\dagger \phi$ since we're on \mathbb{R}^2 (this is really Coulomb gauge $\partial_j A^j = 0$ not $\partial_\mu A^\mu = 0$ since we are only working in space). From now on, all d 's and \star 's and so on will take place in \mathbb{R}^2). Since we don't want to work with 2-forms, we will actually write $A = -\star d\lambda$, where $\lambda = -\star \phi$ is a zero-form (the minus sign is just for convenience). This means that

$$A_- = -\partial_y \lambda - i\partial_x \lambda, \quad A_+ = -\partial_y \lambda + i\partial_x \lambda, \quad (1218)$$

so that we need to solve

$$[\partial_x - \sigma(\partial_x \lambda) - i\partial_y + i\sigma(\partial_y \lambda)] \psi_\sigma = 0, \quad (1219)$$

where $\sigma = +1$ for the L fermions and $\sigma = -1$ for the R fermions. So evidently we need

$$\psi_L = f(\bar{z}) e^\lambda, \quad \psi_R = f(z) e^{-\lambda}, \quad (1220)$$

where $f(\bar{z})$ and $f(z)$ are polynomial functions in \bar{z} and z , respectively. We've included these functions since they get killed by ∂ and $\bar{\partial}$ in turn and so don't affect the equations for the ψ_σ 's (they need to be polynomials since we don't want any singularities and since we need the fields to be single-valued).

Now we need to make sure that these candidate solutions are normalizable and have appropriate fall-off behavior at infinity. To examine this, we need to know what happens to λ at $|z| = \infty$. Now $dA = F$ reads $-d\star d\lambda = F$, which is $\partial_j \partial^j \lambda = F$ (hence the weird sign choice). Thus

$$\int d^2 r' G(r, r') F(r') = \lambda(r), \quad (1221)$$

where

$$G(r, r') = -\frac{1}{2\pi} \ln \left(\frac{|r - r'|}{a} \right) \quad (1222)$$

is the propagator for free scalars in two dimensions (a is some short-distance cutoff). Let us now suppose that $F(r') \rightarrow 0$ as $r' \rightarrow \infty$, and take $r \rightarrow \infty$ in the above expression. Then we write

$$G(r, r') = -\frac{1}{2\pi} \left(\ln(r/a) + \frac{1}{2} \ln(1 + r'^2/r^2 - 2(r \cdot r')/r^2) \right). \quad (1223)$$

The second term can be dropped in the integral: it only contributes at very large r' , and we have assumed that $F(r')$ dies off at large r' . Thus as $r \rightarrow \infty$,

$$\lambda(r) \rightarrow -\frac{1}{2\pi} \int d^2 r' \ln(r/a) F(r') = \frac{\Phi}{2\pi} \ln(a/r), \quad (1224)$$

where $\Phi = \int F$ is the total flux through the plane. This means that our candidate fermion fields behave at $|z| \rightarrow \infty$ like

$$\psi_L(z, \bar{z}) = f(\bar{z}) \left(\frac{a}{|z|} \right)^{\Phi/2\pi}, \quad \psi_R(z, \bar{z}) = f(z) \left(\frac{a}{|z|} \right)^{-\Phi/2\pi}. \quad (1225)$$

Thus for the fields to be finite at infinity, we need $\Phi > 0$ for the L fermions and $\Phi < 0$ for the R fermions: the handedness of the magnetic flux means that only fermions whose handedness matches that of the flux can be zero modes (or whose handedness is opposite to that of the flux; maybe our labelling of L/R was unfortunate and should be swapped).

Let us take $\Phi > 0$ wolog, and suppose that $f(\bar{z})$ is a polynomial of degree $l \in \mathbb{Z}^{>0}$ in \bar{z} . Then in order for $\int |\psi_L|^2$ to be finite, we count powers of $|z|$ and see that the degree of the polynomial has to satisfy

$$l < \frac{\Phi}{2\pi} - 1. \quad (1226)$$

Since a degree l polynomial gives us $l + 1$ linearly independent functions to choose from, the number of degenerate zero modes is evidently

$$N = \left\lceil \frac{\Phi}{2\pi} \right\rceil - 1, \quad (1227)$$

as promised. If the flux Φ is negative, we get the same number of zero modes, and in that case

$$N = \left\lfloor \frac{\Phi}{2\pi} \right\rfloor + 1. \quad (1228)$$

74 June 27 — Lasagna models and how to make SPTs

Today's problem (and title) is inspired by an exercise suggested in John McGreevy's notes on SPTs that I found online. The theme is coupled wire constructions in various contexts.

First, consider a bunch of 1+1D free fermion wires ($2N$ of them) indexed by j and coupled together in the following fashion:

$$S = \int dx d\tau \sum_{j=0}^{2N-2} \left(\psi_j^\dagger (\partial_\tau + (-1)^j v \partial_x) \psi_j - (-1)^j w_j (\psi_{j+1}^\dagger \psi_j + \psi_j^\dagger \psi_{j+1}) \right), \quad (1229)$$

where v is some velocity (note the $(-1)^j$ factor in the hopping, which is crucial and which I believe is missing from McGreevy's notes + the original paper by Senthil and Vishwanath). Suppose w_j takes values w_e or w_o depending on $j \bmod 2$. Describe the different phases we can have and find the continuum limit when $t_e = t_o$.

Describe a similar thing that occurs upon coupling chains of $SU(2)_1$ theories with alternating signs for their WZW terms. You should find a theory with a theta term at $\theta = \pi$.

Finally, do the same thing for coupled critical majorana chains. You should produce a $p \pm ip$ superconductor.

Solution:

First for the fermion lasagna. We are basically doing the Kitaev wire construction, just in one higher dimension and with complex fermions. Since we are starting at $j = 0$, the left-most edge of the lasagna is left-moving, while since we are ending the sum at an even value

of j , the rightmost edge of the lasagna is right-moving. Thus if we consider the case when $w_e \gg w_o$, the left-moving edge is strongly coupled to the $j = 1$ chain and the right-moving edge is strongly coupled to the $j = 2N - 1$ chain, and similarly for the interior chains. This hybridizes all of the fermions and leaves us with something that is gapped and boring (the w_e terms become a mass term for a Dirac fermion built out of the even and odd wire fermions). However, in the opposite case where $w_0 \gg w_e$, while the interior chains hybridize, the $j = 0$ and the $j = 2N - 1$ boundary chains are uncoupled, and give us a lasagna with $\sigma_{xy} = e^2/h$ coming from the single Dirac fermion on the edge, which is not connected to any mass term.

When $w_e = w_o$, we recover translation symmetry by $2a$, where a is the inter-wire spacing. Taking $N \rightarrow \infty$ with Na held fixed gives us the continuum limit. We can write things in terms of a Dirac fermion by defining, for j even,

$$\Psi_j = (\psi_{L,j}, \psi_{R,j})^T = (\psi_j, \psi_{j+1})^T. \quad (1230)$$

We then expand e.g. $\psi_{j+2} = \psi_j + 2a\partial_y\psi_j$, where y is the direction normal to the chains. Putting this in and integrating by parts, we get

$$S = \int d^3x \left(\psi_L^\dagger(\partial_\tau + v\partial_x)\psi_L + \psi_R^\dagger(\partial_\tau - v\partial_x)\psi_R - w\psi_L^\dagger\partial_y\psi_R + w\psi_R^\dagger\partial_y\psi_L \right). \quad (1231)$$

Note that to end up with something that is massless, the $(-1)^j$ factor in the action (which I don't see in the original sources) is crucial—without it, we would get a mass that goes as $w_e + w_o$. Letting the gamma matrices be X, Y, Z , we get

$$S = \int d^3x \bar{\Psi}(\gamma^0\partial_x + v\gamma^1\partial_x + w\gamma^2\partial_y)\Psi. \quad (1232)$$

In particular, at $w = v$ we get a massless Dirac cone.

Now we do a different variant related to bosonic IQH transitions, where the layers of the lasagna are $SU(2)_1$ WZW models. Each WZW layer (indexed by j) is

$$S_j = \frac{1}{\lambda} \int_L \text{Tr}[dg^\dagger \wedge \star dg] - \frac{i}{6 \cdot 2\pi} \int_{B_3} \text{Tr}[\omega \wedge \omega \wedge \omega], \quad \omega = g^\dagger dg, \quad (1233)$$

where B_3 is a 3-ball bounding the lasagna layer L and g is the $SU(2)$ variable (or really, $U(2)$ variable) being integrated over. The numbers are chosen since

$$\frac{1}{12 \cdot \Omega_3} \int_{S^3} \text{Tr}[\omega^3] \in \mathbb{Z}, \quad \Omega_3 = 2\pi^2. \quad (1234)$$

Here $1/12$ is really $1/6 \cdot 1/2$, where (I believe) the $1/2$ comes from cancelling $\text{Tr}[\mathbf{1}] = 2$ and the $1/6 = 1/3!$ is from the antisymmetrization on the derivatives.

Each layer is the edge theory of an SPT, and we will reconstruct that SPT by making a lasagna out of the edge theories. This will illustrate the general strategy this problem is trying to make clear (see McGreevy's notes):

- Start with a bunch of SPT- or SPT-like states, and take them to criticality. That is, tune them so that they are almost the trivial phase. For example, for the Kitaev wire, tune to the critical point where all the couplings in the chain are equal.

- These critical states will form the edge theory of a new SPT. Stack them together in some “alternating” way and couple them together by adding some new term to the Hamiltonian.
- The result will be a higher-dimensional SPT. One can then repeat the process by taking this SPT to the critical point separating it from the trivial phase, and then perform the stacking again (note to self: what does this have to do with the dimensional-reduction approach to classifying anomalies with splittable symmetries?)

Anyway, back to the current example: the setup is to consider a lasagna of $SU(2)_1$'s with alternating chiralities, like with the fermions. Here the chirality is set by the sign of the WZW term (they're not really chiral in the true sense, but they become chiral in the quantum hall context when only one of the bosons that constitute g is coupled to the $U(1)$ gauge theory). We also need to add a coupling between the chains, which we do with a “ferromagnetic” coupling $t\text{Tr}[g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j]$. If we write g as (see the paper by Senthil and Vishwanath)

$$g = \begin{pmatrix} a_1 & -a_2^\dagger \\ a_2 & a_1^\dagger \end{pmatrix} \quad (1235)$$

where the a_i are boson operators, then after multiplying the matrices we see that the coupling term is a boson hopping term like $t(a_{I,j}^\dagger a_{I,j+1} + h.c.)$, for $I = 1, 2$. Anyway, the lasagna is (allowing for different even/odd hoppings as before)

$$S = \sum_{j=0}^{2N-2} \left(\frac{1}{\lambda} \int_{L_j} \text{Tr}[dg^\dagger \wedge \star dg] + (-1)^j \frac{i}{12\pi} \int_{B_{3,j}} \text{Tr}[\omega_j^3] - t_j \int_{L_j} \text{Tr}[g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j] \right). \quad (1236)$$

Note that unlike with the fermions, we don't need the $(-1)^j$ factor in front of the interchain hopping term.

As before, if the even t_j 's are all strong compared to the odd ones, then we “hybridize” the whole system and are left with nothing interesting, while in the opposite limit we are left with surface chains of free $SU(2)_1$ theories at the edges (a bosonic IQH state). The most interesting case is when we have uniform coupling. Let's first look at the coupling term. The ferromagnetic coupling means that we want $g_{j+1} \sim g_j$ for all j , and so we can do

$$g_j^\dagger g_{j+1} + g_{j+1}^\dagger g_j \rightarrow 2 \cdot \mathbf{1} + g_j^\dagger \partial_z g_j + g_j \partial_z g_j^\dagger + \frac{1}{2}(g_j^\dagger \partial_z^2 g_j + g_j \partial_z^2 g_j^\dagger) \rightarrow -\partial_z g_j^\dagger \partial_z g_j, \quad (1237)$$

where z is the direction normal to the layers and in the last step we dropped a constant, integrated by parts, and used $g_j^\dagger \partial_z g_j + g_j \partial_z g_j^\dagger = \partial_z(g_j^\dagger g_j) = 0$.

As for the WZW terms, we've already done a similar computation (a few days ago) with an alternating sum of WZW models in the AFM spin chain, where we saw that they combined to give a θ term (the extra dimension used to define the WZW actions became the dimension along the chain). Here we have the same thing, but one dimension up. Now S_{WZW} is the volume in S^3 enclosed (or “exclosed”, our choice of coefficient means that it doesn't matter) by the image of the S^3 vector field n used to define g . We can write a given WZW term as

$$S_{WZW}[g] = \frac{2\pi i}{\Omega^3} \int_{B_3} d^2x du \epsilon^{abcd} n^a \partial_t n^b \partial_x n^c \partial_u n^d, \quad (1238)$$

where u is the extra “fake” coordinate of the 3-ball. The factor of $2\pi i/\Omega_3$ ensures that if the image of n for a given Lasagna layer bounds all of S^3 (which is the same as bounding none of it), we get something in $2\pi i\mathbb{Z}$.

Therefore, defining an n -field n_j at each site, to first order in $|n_{2j} - n_{2j+1}|$, we can compute the difference between two successive WZW terms as

$$S_{WZW}[g_{2j}] - S_{WZW}[g_{2j-1}] \approx \frac{2\pi i}{\Omega_3} \int_{S^3} d^3x \epsilon^{abcd} (n_{2j}^a - n_{2j-1}^a) \partial_t n_{2j}^b \partial_x n_{2j}^c n_{2j}^d. \quad (1239)$$

We arrive at this expression the following reasoning: the difference in the volume bounded by the two spheres defined by the trajectories of n_{2j} and n_{2j-1} (i.e. the volumes bounded by the images of each two-dimensional lasagna layer $L_j \approx S^2$) is equal to $n_{2j}^a - n_{2j-1}^a$ (the distance between the two spheres embedded in S^3) wedged with the pullback of the volume element in S^2 by n (the volume element of the image of a patch of two-dimensional spacetime mapped into S^3 by n). Drawing a picture may help—it’s just the analogue of the drawable S^1 into S^2 example relevant for the AFM chain.

Replacing the difference in n vectors by a derivative and summing over all j , we get

$$\sum_j (-1)^j S_{WZW}[n_j] = \frac{2\pi i}{2 \cdot 6\Omega_3} \int_M \epsilon^{abcd} n^a \wedge dn^b \wedge dn^c \wedge dn^d, \quad (1240)$$

where M is the full spacetime manifold. The factor of $1/6$ is the price we pay for antisymmetrizing when taking the wedge product, while the all-important prefactor of $1/2$ comes when we write

$$n_{2j}^a \approx n_{2j-1}^a + \frac{\delta z}{2} \partial_z n_{2j-1}^a, \quad (1241)$$

where z is the direction normal to the chains and $\delta z/2$ is the spacing between chains. The actual lattice constant is δz , namely twice the spacing between chains, since the fact that the chains come in with alternating signs means that we only have translation symmetry by a distance of twice the chain separation (you can also think that since we are summing up terms of the form $S_{WZW}[n_{2j}] - S_{WZW}[n_{2j-1}]$, we are only summing over half of the chains, namely the even ones). Without this factor of $1/2$, the topological term we get would disappear, so it is very important.

Written in terms of the g variables then, we can write³⁸

$$\sum_j (-1)^j \frac{i}{12\pi} \int_{B_{3,j}} \text{Tr}[\omega_j^3] = \frac{\pi i}{2 \cdot 6\Omega_3} \int_M \text{Tr}[\omega^3] = \frac{1}{2} \frac{i}{12\pi} \int_M \text{Tr}[\omega^3]. \quad (1242)$$

Since our boundary conditions on g will be such that $M \approx S^3$, the integral is a θ term at $\theta = \pi$ and as such is valued in ± 1 (since if we were to multiply the RHS by 2, we would get a WZW term integrated over a closed manifold, which we know has to give something in $2\pi i\mathbb{Z}$). This is clearest from the expression of the integral in terms of the n field: after the $1/6$ is removed by the antisymmetrization in the wedge product, we see that we get the term

$$S_{\theta=\pi} = i\pi \int_M n^*(\text{vol}_{S^3}) \in \pi i\mathbb{Z}. \quad (1243)$$

³⁸Note that this presentation of the result in terms of the g variables has an “extra” factor of $1/2$ relative to the prefactor for the presentation in terms of the n fields, due to the fact that $\text{Tr}[1] = 2$.

Now for majorana lasagna. The simplest lasagna to make is the Kitaev wire. We start with a collection of majoranas in 0+1 dimensions, and couple them together in the same way as before, except with factors of i to make the whole thing Hermitian, and without the $(-1)^j$ factor we needed for the Dirac fermions:

$$H = i \sum_j (t_e \chi_j^L \chi_j^R + t_o \chi_j^R \chi_{j+1}^L). \quad (1244)$$

Here L, R denote the left and right spots in a given site j , which contains places for two majoranas (each j is a site for a single \mathbb{C} fermion). When $t_e \gg t_o$ all of the majoranas are part of a $i\chi_j^L \chi_R$ mass term (which becomes $\sim c_j^\dagger c_j$), and so we get a boring gapped phase. When $t_o \gg t_e$ then we define $d_j = (\chi_j^R + i\chi_{j+1}^L)/2$ and get a Hamiltonian which is gapped in the bulk ($H \sim \sum d_j^\dagger d_j$), but which contains dangling edge modes. When $t_e = t_o$ we have a critical chain. In the continuum limit, we define $t_e/2 = t_o/2 = v$ and get

$$H = iv \int dx (\chi_L \partial_x \chi_R + \chi_R \partial_x \chi_L). \quad (1245)$$

The next step up from this is to use critical majorana chains as the layers of a 2+1-dimensional lasagna, with the chains alternating in the sign of v . The coupled Hamiltonian is

$$H = i \sum_j \int dx (v \chi_j^L \partial_x \chi_j^L - v \chi_j^R \partial_x \chi_j^R + w_e \chi_j^L \chi_j^R + w_o \chi_j^R \chi_{j+1}^L). \quad (1246)$$

Here as before, each j label contains two majorana chains, one that is left-moving (χ_j^L) and one that is right-moving (χ_j^R). As usual, when $w_e \gg w_o$ then all of the χ_j 's are involved in a mass term, and we get something which is gapped. When $w_o \gg w_e$, the bulk is gapped but we have surviving Majorana modes on the edge: this is a $p \pm ip$ topological superconductor (with the sign in $p \pm ip$ set by the sign of v).

The critical case is when $w_e = w_o$ which describes the phase transition. We can take the continuum limit of the coupling terms in the same way that we did for constructing the critical Kitaev chain, and we get

$$H = i \int dxdy [v(\chi_L \partial_x \chi_L - \chi_R \partial_x \chi_R) + w(\chi_L \partial_y \chi_R + \chi_R \partial_y \chi_L)], \quad (1247)$$

with y the direction normal to the chains.

A final word on the Lagrangian and matrix choices: in Minkowski signature, the time derivative part enters as $i(\chi_L \partial_t \chi_L + \chi_R \partial_t \chi_R)$, which is Hermitian. If we go to Euclidean signature, we drop the i and still get something sensible since ∂_τ now gets two minus signs under Hermitian conjugation. The general Lagrangian is then

$$\mathcal{L} = \bar{\chi}(\gamma^\mu \partial_\mu + m)\chi, \quad (1248)$$

where $\bar{\chi}$ is formed with the charge conjugation matrix and the γ^μ are chosen as follows:

$$\bar{\chi} = \chi^T Y, \quad \gamma^0 = iY, \quad \gamma^1 = X, \quad \gamma^2 = Z. \quad (1249)$$

This ensures that e.g. the mass term is the correct bilinear form with the intertwiner for $\text{Spin}(3)$, namely $i\chi_L \chi_R - i\chi_R \chi_L$. In our example with differing w_e, w_o , one can check that $m \propto w_e - w_o$ while the ∂_y term is proportional to $w_e + w_o$.

75 June 29 — More WZW things

The goal of today's problem is to try to get more familiar with the WZW term, to see how it is responsible for generating anomalies, to become acquainted with its currents, and so on. Most of what we want to know is in the big yellow book—try to avoid looking until you're done.

We will be working with WZW for $SU(2)_k$ ³⁹:

$$S = \frac{k}{8\pi} \int_M \text{Tr}(dg^\dagger \wedge \star dg) + \frac{ik}{12\pi} \int_B \text{Tr}[\omega \wedge \omega \wedge \omega], \quad \omega = g^\dagger dg. \quad (1250)$$

As usual, M is some 2-manifold and B is a 3-ball bounded by M . First, look at the WZW part of the action and show that it transforms projectively under the global symmetry $g \mapsto gh$. Then define holomorphic and antiholomorphic currents, and show how they transform under $SU(2)_L \times SU(2)_R$.

The consider the conformal Ward identity (you are allowed to look at previous diary entries to get the variation of the action). Using the currents as the operators inserted into the Ward identity, find the OPE of the currents with themselves. After splitting up the currents with a mode expansion, find the commutator algebra of the modes in the usual way (in radial quantization, using contour integrals).

Solution:

We can see why the WZW term comes from cohomology by seeing where it goes under a global symmetry transformation of $SU(2)_R$, with $g \mapsto gh$. Under this

$$\omega \mapsto h^\dagger \omega h + \eta, \quad \eta \equiv h^\dagger dh. \quad (1251)$$

Putting this into S and expanding,

$$S_{WZW}[gh] = S_{WZW}[g] + S_{WZW}[h] + \frac{ik}{4\pi} \int \text{Tr}[\eta \wedge \eta \wedge {}^h\omega + \eta \wedge {}^h\omega \wedge {}^h\omega], \quad {}^h\omega \equiv h^\dagger \omega h. \quad (1252)$$

Now $\eta \wedge \eta = -d\eta$, while

$$d^h\omega = -\eta^h\omega + h^\dagger d\omega h - {}^h\omega \wedge \eta = -\eta \wedge {}^h\omega - {}^h\omega \wedge \eta - {}^h\omega \wedge {}^h\omega. \quad (1253)$$

Solving for ${}^h\omega \wedge {}^h\omega$ and putting this into our expression for $S_{WZW}[gh]$, we get

$$S_{WZW}[gh] = S_{WZW}[g] + S_{WZW}[h] + \frac{ik}{4\pi} \int d\text{Tr}[\eta \wedge {}^h\omega]. \quad (1254)$$

So, the symmetry acts linearly on the WZW action up to a boundary term: this is emblematic of SPTs / anomalies: when we consider the symmetry action on some open manifold (or submanifold), we get a linear representation of the symmetry up to a term supported on the boundary of that manifold (see earlier diary entries).

³⁹Note that in the big yellow book, the coefficients in front of the integrals are different. This is because they use a trace with a different normalization (the trace we are using is not normalized by the Dynkin index; it is the straight-up trace).

When we consider the kinetic S_k term as well, a similar computation shows

$$S_k[gh] = S_k[g] + S_k[h] - \frac{k}{4\pi} \int \text{Tr}(\eta \wedge \star^h \omega). \quad (1255)$$

When we add these two terms, the form that couples to η in the “cocycleness” δS of the action, namely

$$\delta S(g, h) \equiv S[g] + S[h] - S[gh], \quad (1256)$$

is exactly the same type of form defined in our RG analysis of the WZW model back on June 2. To get something useful out of this, we need to switch to using ∂_z and $\partial_{\bar{z}}$. When we do this, $\delta S_k(g, h)$ becomes after a bit of algebra

$$\delta S_k(g, h) = \frac{k}{2\pi} \int d^2x \text{Tr}[\eta_z \omega_{\bar{z}} + \eta_{\bar{z}} \omega_z]. \quad (1257)$$

On the other hand, the WZW part is

$$\delta S_{WZW}(g, h) = \frac{k}{2\pi} \int d^2x \text{Tr}[\eta_z \omega_{\bar{z}} - \eta_{\bar{z}} \omega_z]. \quad (1258)$$

Note that $\delta S_{WZW}(g, h)$ is real since the i in the original prefactor cancels with the i generated by going to ∂_z and $\partial_{\bar{z}}$ since $\delta S_{WZW}(g, h)$ only contains terms with one derivative in x and one in y (where $z = x + iy$). Putting these together, the total “coboundary” of the action is (switching from an integral over $d^2x = dx \wedge dy$ to one over $dz \wedge d\bar{z}$ at the cost of a factor of $i/2$),

$$\delta S(g, h) = \frac{ik}{2\pi} \int dz \wedge d\bar{z} \text{Tr}[h^\dagger(\partial_z h)g^\dagger\partial_{\bar{z}} g]. \quad (1259)$$

Note in particular that if h is holomorphic, then $\delta S(g, h) = 0$.

This was done for the action of $SU(2)_R$. If we instead consider the $SU(2)_L$ action $g \mapsto fg$, then we instead have

$$\omega \mapsto g^\dagger \lambda g + \omega, \quad \lambda \equiv f^\dagger df. \quad (1260)$$

Thus for the action of $SU(2)_L$, it is the Cartan form for the element doing the symmetry action (namely f) that gets conjugated, instead of ω . This ends up meaning that we end up getting the same thing for $\delta S_{WZW}(f, g)$, except that η and ${}^h\omega$ change places. Since they are both 1-forms, this gives us a minus sign, and so essentially the only thing that changes is that $\delta S_{WZW}(f, g)$ term gets a minus sign relative to $\delta S_k(f, g)$, which doesn’t get a minus sign since it doesn’t have wedge products. Thus we find that for the $SU(2)_L$ action,

$$\delta S(f, g) = \frac{k}{\pi} \int d^2x \text{Tr}[f^\dagger(\partial_{\bar{z}} f)g^\dagger\partial_z g]. \quad (1261)$$

Note that this vanishes if f is anti-holomorphic.

So, we reach the following conclusion: the $SU(2)_L$ symmetry is implemented anomalously (i.e. the action is only invariant up to a boundary term), unless the action is done by a holomorphic function $f(z)$. Likewise, the $SU(2)_R$ symmetry is implemented anomalously unless the action is done by an anti-holomorphic function $h(\bar{z})$. This tells us that conservation of the holomorphic current (the expression for which will be recalled shortly) comes from the

$SU(2)_L$ symmetry, while conservation of the antiholomorphic current comes from the $SU(2)_R$ symmetry.

Now we turn to look at the currents. We will defer to a previous diary entry (the one for June 1) where we calculated these and found δS . We found that the holomorphic (left-moving, since it doesn't depend on \bar{z}) and the anti-holomorphic (right-moving, since it's independent of z) currents are⁴⁰

$$J = -k(\partial_z g)g^\dagger, \quad \bar{J} = kg^\dagger\partial_{\bar{z}}g. \quad (1262)$$

Under infinitesimal $SU(2)_L$ transformations $g \mapsto g + \gamma g$, and infinitesimal $SU(2)_R$ transformations $g \mapsto g - g\bar{\gamma}$ (we will see why the minus sign is natural in a second), a quick calculation shows that the holomorphic current varies as

$$SU(2)_L : \delta J = [J, \gamma] - k\partial_z\gamma, \quad SU(2)_R : \delta J = kg\partial_z\bar{\gamma}g^\dagger, \quad (1263)$$

so that when the variation is holomorphic J is invariant under $SU(2)_R$. Similarly, the anti-holomorphic current transforms as

$$SU(2)_R : \delta\bar{J} = -[\bar{J}, \bar{\gamma}] - k\partial_{\bar{z}}\bar{\gamma}, \quad SU(2)_L : \delta\bar{J} = kg^\dagger(\partial_{\bar{z}}\gamma)g. \quad (1264)$$

As anticipated in our calculation of the “coboundary” δS , the holomorphic current is identified with the $SU(2)_L$ symmetry and the antiholomorphic one with the $SU(2)_R$ symmetry. The conservation of these two currents (which comes from the equations of motion as we saw in a previous diary entry) implies that we have the symmetry

$$g(z, \bar{z}) \mapsto \Gamma(z)g(z, \bar{z})\bar{\Gamma}^{-1}(\bar{z}) \quad (1265)$$

for any homomorphic (antiholomorphic) Γ ($\bar{\Gamma}$). We've chosen the right action to involve an inverse since it makes various formulae nicer later on and is the more natural choice. The infinitesimal version of this is, writing $\Gamma(z) = \mathbf{1} + \gamma(z)$,

$$g \mapsto g + \gamma g - g\bar{\gamma}. \quad (1266)$$

Consider now the ward identity for some operator \mathcal{O} . To compute $\langle \delta \mathcal{O} \rangle$, we need to know the variation of the action. Fortunately this was worked out in a previous diary entry (June 1), whose results we will steal. After going through the annoying step of switching to $\partial_z, \partial_{\bar{z}}$ we find for the variation (1266)

$$\delta S = -\frac{k}{2\pi} \int d^2x \text{Tr}[\gamma\partial_{\bar{z}}J + \bar{\gamma}\partial_z\bar{J}]. \quad (1267)$$

We can pull the derivatives out of the trace for free, since γ and $\bar{\gamma}$ are killed by $\partial_{\bar{z}}$ and ∂_z , respectively. Then since $dx \wedge dy = (i/2)dz \wedge d\bar{z}$, we can go over to an integration over $dz \wedge d\bar{z}$ and then integrate the total derivatives to get

$$\delta S = -\frac{i}{4\pi} \oint dz \text{Tr}[\gamma J] + \frac{i}{4\pi} \oint d\bar{z} \text{Tr}[\bar{\gamma}\bar{J}], \quad (1268)$$

⁴⁰We're actually changing conventions slightly for the currents compared to the last diary entry—there we were following Altland and Simon's conventions, and here we follow more standard ones (which are conjugated by g and since we are no longer at $k = 1$ have a k in front)

where the relative minus sign comes from the fact that we're taking both of the \oint 's to be oriented right-handedly.

Following the usual procedure of performing a shift in integration variables in the path integral which computes $\langle \mathcal{O} \rangle$, we have, for $\gamma, \bar{\gamma}$ chosen to have compact support on some ball centered on w (in radial quantization),

$$\langle \delta \mathcal{O}(w) \rangle = -\frac{i}{4\pi} \oint dz \langle \text{Tr}[\gamma J] \mathcal{O}(w) \rangle + \frac{i}{4\pi} \oint d\bar{z} \langle \text{Tr}[\bar{\gamma} \bar{J}] \mathcal{O}(w) \rangle, \quad (1269)$$

where the contours are taken on paths enclosing the point w .

We can get the OPEs for the currents by choosing $\mathcal{O} = J^a$, where $J = J^a t^a$ with t^a the Pauli matrices. Since $\bar{\gamma}$ is antiholomorphic J is invariant under $SU(2)_R$, and so we see from our earlier result that

$$\delta J^a = i f^{abc} J^b \gamma^c - k \partial_z \gamma^a, \quad (1270)$$

where for us $f^{abc} = \epsilon^{abc}$. Note that when we put this in, the LHS of (1269) will only contain γ , and so the OPE between J and \bar{J} must only contain non-singular terms (since the $\bar{\gamma}$ term on the RHS needs to die). Thus (not writing the expectation value brackets and taking the trace)

$$i f^{abc} J^b(w) \gamma^c(w) - k \partial_w \gamma^a(w) = -\frac{i}{2\pi} \oint dz \gamma^b(z) J^b(z) J^a(w). \quad (1271)$$

From this, we can read off the OPE for the holomorphic currents. the f^{abc} term on the LHS has no derivatives, so we need to pick it up with a $1/(z-w)$ pole. Since the $k \partial_w \gamma^a(w)$ term has one derivative, we need to match it with a $1/(z-w)^2 = -\partial_w(z-w)^{-1}$ term. So then taking into account the $2\pi i$'s from the residues, we deduce that the singular parts of the OPE are

$$J^a(z) J^b(w) \sim \frac{k \delta_{ab}}{(z-w)^2} + i f^{abc} \frac{J^c(z)}{z-w}. \quad (1272)$$

This is the current algebra we've been looking for.

Now we can define the modes of the current by their “angular momentum”, i.e. what we get when we integrate the current against z^n . Since $\partial_{\bar{z}} J^a = 0$ we can expand the current as

$$J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J_n^a. \quad (1273)$$

Here we picture J_n^a as an operator acting at the origin in radial quantization that sets up an associated state. The choice of power ensures that J_n^a can be found by integrating $J^a(z)$ against z^n . The commutator of two modes is

$$[J_n^a, J_m^b] = \oint_0 dw \oint_w dz z^n w^m J^a(z) J^b(w). \quad (1274)$$

Here the w integral is taken on a contour centered on the origin, while for a given w the z integral is taken on a contour that encloses w and the $z^n w^m$ selects out the desired components of the mode expansion. This is the usual thing you get when writing the (radial) commutator: for each w you end up doing two contours surrounding the origin along circles or radii slightly larger / smaller than $|w|$, which you then deform into a small contour enclosing w .

Anyway, now we just insert the OPE into the integral on the LHS. Remembering that we get the residue of the second-order pole with $d_z[(z-w)^2(z^n w^m/(z-w)^2)]|_{z=w}$, we do the integrals and get

$$[J_n^a, J_m^b] = if^{abc} J_{n+m}^c + kn\delta_{a,b}\delta_{n,-m}. \quad (1275)$$

The $\delta_{n,-m}$ comes since after doing the z integral we have an integral over w^{n-1-m} which sets $n = m$, with the prefactor of n coming from taking the derivative when finding the residue. Finally, that the mode J_{n+m}^c is selected out can be seen by plugging in the mode expansion for J^c and noting that the integrand $z^n w^{m-l-1}/(z-w)$ is only non-zero if $l = m+n$. This is the algebra we've been looking for. We also get a similar algebra for the antiholomorphic currents by putting \bar{J} into the conformal Ward identity. Also note that since the $J\bar{J}$ OPE has to have no singular terms, the two algebras are decoupled:

$$[J_n^a, \bar{J}_m^b] = 0. \quad (1276)$$

76 June 30 — Quantization for Chern-Simons

The goal of today's problem is to try to understand the connection between CS theory and the WZW action through the quantization of the former (motivated by reading the paper by Jackiw and others back in 1989 on the quantization of Chern-Simons theory). We will also try to work out the details of some of the results in [5].

We will be working with CS theory / WZW actions for $SU(2)_k$ for concreteness:

$$S_{CS} = \frac{k}{4\pi} \int \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A). \quad (1277)$$

Quantize the theory using holomorphic quantization, by first quantizing the fields and then imposing the constraint of gauge-invariance. Check that the generators of gauge transformations form a linear representation of the gauge group, and find the wavefunctional for the quantized fields. Your answer should involve the WZW action.

Solution:

When dealing with a system with constraints (for us, Gauss' law), we can either solve the constraint and then quantize, or the other way around. In what follows we will adopt the latter strategy: this means we will quantize the fields in the normal way, and solve the constraint afterwards by requiring that the generator of the constraint act trivially on the wavefunction.

From the $A \wedge \partial_t A$ term in the CS action, we can write $A = t^a A^a$ and take the trace and directly read off the commutator⁴¹

$$[A_i^a(z), A_j^b(w)] = \epsilon_{ij} \frac{4\pi i}{k} \delta_{ab} \delta(z-w). \quad (1278)$$

⁴¹In the commutator, the δ function is normalized to have unit integral with the measure $dx \wedge dy$, not with $dz \wedge d\bar{z}$ (sorry)

To get the right coefficient, we need to remember that $t^a = \sigma^a/2$, so that $\text{Tr}(t^a t^b) = \delta^{ab}/2$, turning the part of the action that survives in the $A_0 = 0$ gauge into $(k/8\pi) \int \epsilon^{ij} (\partial_t A_i^a) A_j^a$.

Slightly more carefully, we can find the symplectic form by varying the action. We get, for a spacetime X ,

$$\delta S = \frac{k}{2\pi} \int \text{Tr}[\delta A \wedge (dA + A \wedge A)] - \frac{k}{4\pi} \int_{\partial X} \text{Tr}[A \wedge \delta A] \quad (1279)$$

where the sign of the last term comes from the signs in the product rule for d (we take the two differentials d and δ to commute). The boundary term gives the symplectic current, and so varying it again gives

$$\Omega = -\frac{k}{4\pi} \int_{\partial X} \text{Tr}[\delta A \wedge \delta A]. \quad (1280)$$

The fact that the symplectic form is a wedge product of δA with itself means e.g. that Wilson loops in A don't commute with themselves and is responsible for all the stuff we know and love about CS theory. Also note that $\delta A \wedge \delta A \neq 0$ even in the $U(1)$ case—this may look at first glance like the wedge product of two one-forms, but it is actually two $(1, 1)$ forms (one degree in de Rham and one variational degree), and is non-vanishing (the degree relevant for the graded commutation rules is the total degree in the $d\delta$ bicomplex).

Anyway, we see from the symplectic form that we can choose e.g. either A_x or A_y as the canonical momentum. Actually, in keeping with yesterday's problem, it will be better to work with a holomorphic polarization, choosing A_z as the coordinate and $\bar{A} \equiv A_{\bar{z}}$ as the momentum. So then we have

$$[A^a(z), \bar{A}^b(w)] = \frac{4\pi i}{k} \delta_{ab} \delta(z - w). \quad (1281)$$

As in Maxwell theory, A_0 has no momentum and imposes a Gauss' law constraint. Varying the action with respect to A_0^a , the first $A \wedge dA$ part gives us two copies of $\partial_{[i} A_{j]}^a/2$ (the $1/2$ from the trace), while the second A^3 part gives us three copies of $\epsilon^{ij} A_i^b A_j^c f^{abc}/4$, where various factors of 2 from the matrix structure and trace have cancelled. Thus after a bit more algebra, we see that the constraint from A_0 means that for any physical state $|\Psi\rangle$,

$$(\partial_z \bar{A}^a - \partial_{\bar{z}} A^a + i f^{abc} A^b \bar{A}^c) |\Psi\rangle = 0. \quad (1282)$$

Thus the field strength will be the generator of gauge transformations on a given Cauchy slice (for us, the z, \bar{z} plane). This is in contrast with Maxwell theory, where the generator of gauge transformations is $\star F$, and not F .

To check that the field strength generates gauge transformations when it acts on the gauge field, define the operator

$$U(\lambda) = \exp \left(-i \frac{k}{4\pi} \int \lambda^a \wedge F^a \right). \quad (1283)$$

Using the commutation relations we have

$$\bar{A}^a = -\frac{4\pi i}{k} \frac{\delta}{\delta A^a}, \quad (1284)$$

and so we can write $U(\lambda^a)$ as (after integrating by parts)

$$U(\lambda) = \exp \left(\int \left[\partial_z \lambda^a \frac{\delta}{\delta A^a} - i \frac{k}{4\pi} \partial_{\bar{z}} \lambda^a A^a - i f^{abc} \lambda^a A^b \frac{\delta}{\delta A^c} \right] \right). \quad (1285)$$

Then expanding the exponentials and a little bit of algebra shows that

$$U(\lambda) A^a [U(\lambda)]^\dagger = A^a + \partial_z \lambda^a + i f^{abc} A^b \lambda^c, \quad (1286)$$

which is exactly what we want.

When deriving the action of $U(\lambda)$ on A^a , the part proportional to $\partial_{\bar{z}} \lambda^a$ canceled out. It is still essential to keep though, since it generates gauge transformations for the antiholomorphic part. Indeed, a little bit of algebra gives

$$U(\lambda) \frac{\delta}{\delta A^a} [U(\lambda)]^\dagger = \frac{\delta}{\delta A^a} + i \frac{k}{4\pi} \partial_{\bar{z}} \lambda^a + i f^{abc} \lambda^c \frac{\delta}{\delta A^b}, \quad (1287)$$

so that

$$U(\lambda) \bar{A}^a [U(\lambda)]^\dagger = \bar{A}^a + \partial_{\bar{z}} \lambda^a + i f^{abc} \bar{A}^b \lambda^c, \quad (1288)$$

as required.

Now let's find the wavefunctionals, which will be holomorphic functionals of A . In what follows, it will be convenient to be able to work with group elements $g = e^\lambda, h = e^\gamma$, as well as Lie algebra elements. We find the wave functionals by requiring that they are invariant under the action of $U(g)$. This means we have to know how the exponentiation of $\int \text{Tr}[\lambda \wedge F]$ acts on arbitrary functionals of A . This is slightly tricky, since the exponentiation of F is difficult—it contains operators that don't commute among themselves.

Let us break up $U(g)$ into two parts: the part which implements the gauge transformation (the part with the $\delta/\delta A$'s), and the part containing $\text{Tr}[\partial_{\bar{z}} \lambda A]$. So define

$$-\frac{ik}{2\pi} \int \text{Tr}[\lambda \wedge F] = \mathcal{G}(g) + \frac{ik}{2\pi} \int \text{Tr}[g^\dagger \partial_{\bar{z}} g A], \quad (1289)$$

where

$$\mathcal{G}(g) \equiv \int \left[\partial_z \lambda^a \frac{\delta}{\delta A^a} - i f^{abc} \lambda^a A^b \frac{\delta}{\delta A^c} \right] \quad (1290)$$

is the logarithm of the part of $U(g)$ which does the gauge transformation.

Let $|\Psi[A]\rangle$ be a candidate wavefunctional, and write the action of $U(g)$ on it craftily as

$$U(g) |\Psi[A]\rangle = U(g) e^{-\mathcal{G}(g)} |\Psi[g A]\rangle. \quad (1291)$$

If $U(g)$ only performed gauge transformations, then we would have $U(g) e^{-\mathcal{G}(g)} = \mathbf{1}$. Because of the extra part in $U(g)$ though, this operator is nontrivial.

Now we need to find out what $U(g) e^{-\mathcal{G}(g)}$ is. Since manipulating stuff in the exponentials is difficult, let us bring down the stuff in the exponentials using the “fake one-parameter evolution” trick. Namely, introduce a homotopy parameter $\phi \in [0, 1]$ and consider $U(g^\phi) e^{-\mathcal{G}(g^\phi)}$. At $\phi = 0$ the g field is just the identity on all of spacetime, while it becomes equal to the value of interest at $\phi = 1$. Geometrically, what we are doing is extending the spatial manifold to be realized as the boundary of a three-manifold, where the added direction is the ϕ direction.

Since $g^\phi = \mathbf{1}$ at $\phi = 0$ we can compactify space to a point at $\phi = 0$, and so this three-manifold looks like a cone. Anyway, we use $g^\phi = e^{\phi\lambda}$ to compute the derivative

$$\partial_\phi(U(g^\phi)e^{-\mathcal{G}(g^\phi)}) = U(g^\phi) \left(\frac{ik}{2\pi} \int \text{Tr}[g^\dagger \partial_{\bar{z}} g A] + \mathcal{G}(g) \right) e^{-\mathcal{G}(g^\phi)} - U(g^\phi)e^{-\mathcal{G}(g^\phi)} \mathcal{G}(g), \quad (1292)$$

since the exponents are just linear in ϕ . When we bring the integral through the $e^{-\mathcal{H}(g^\phi)}$ it gets gauge-transformed, and so since the $\mathcal{G}(g)$'s cancel,

$$\partial_\phi(U(g^\phi)e^{-\mathcal{G}(g^\phi)}) = U(g^\phi)e^{-\mathcal{G}(g^\phi)} \frac{ik}{2\pi} \int \text{Tr}[g^\dagger \partial_{\bar{z}} g (g_\phi^\dagger A g_\phi + g_\phi^\dagger \partial_z g_\phi)], \quad (1293)$$

where g_ϕ is the same as g^ϕ but is used since $(g^\phi)^\dagger$ looks uglier.

Now to simplify this mess. The first term on the RHS is a total derivative since

$$\frac{d}{d\phi} \text{Tr}[(\partial_{\bar{z}} g_\phi) g_\phi^\dagger A] = \text{Tr}[\partial_{\bar{z}}(g_\phi \lambda) g_\phi^\dagger A - (\partial_{\bar{z}} g_\phi) \lambda g_\phi^{-1}] = \text{Tr}[(\partial_{\bar{z}} \lambda) g_\phi^\dagger A g_\phi], \quad (1294)$$

as $\partial_\phi g_\phi = g_\phi \lambda$. We break the second term up as

$$\text{Tr}[g^\dagger \partial_{\bar{z}} g (g_\phi^\dagger \partial_z g_\phi)] dz \wedge d\bar{z} = \frac{1}{2} \text{Tr}[g_\phi^\dagger \partial_z g_\phi \partial_{\bar{z}} \lambda + g_\phi^\dagger \partial_{\bar{z}} g_\phi \partial_z \lambda] dz \wedge d\bar{z} - \frac{1}{2} \text{Tr}[g_\phi^\dagger dg_\phi \wedge g^\dagger dg], \quad (1295)$$

where the wedge product (taken only on the spatial slice; not involving the time coordinate) treats ∂_z as coming first and $\partial_{\bar{z}}$ as coming second. The second term on the RHS is

$$\frac{1}{2} \text{Tr}[g_\phi^\dagger \partial_z g_\phi \partial_{\bar{z}} \lambda + g_\phi^\dagger \partial_{\bar{z}} g_\phi \partial_z \lambda] = \frac{d}{d\phi} \text{Tr}[g_\phi^\dagger \partial_z g_\phi g_\phi^\dagger \partial_{\bar{z}} g_\phi] \quad (1296)$$

since two of the terms after taking the derivative on the RHS cancel. Finally, in the last term with the wedge products, we can use $\lambda = g_\phi^\dagger \partial_\phi g_\phi$ to plug in for λ . Then we can antisymmetrize the three derivatives and divide by a factor of 3 to get

$$\frac{1}{2} \int \text{Tr}[g_\phi^\dagger dg_\phi \wedge g^\dagger dg] = \frac{1}{2 \cdot 3} \int d^2 z \epsilon^{\alpha\beta\gamma} \text{Tr}[g_\phi^\dagger \partial_\alpha g_\phi g_\phi^\dagger \partial_\beta g_\phi g_\phi^\dagger \partial_\gamma g_\phi], \quad (1297)$$

where on the RHS α, β, γ run over z, \bar{z} , and ϕ . Look at how WZW-like this is! Since the integral on the RHS is only over space, the RHS is also a total ϕ derivative, and it equals

$$\frac{d}{d\phi} \frac{1}{6} \int_{B_{3,\phi}} \text{Tr}[g_{\phi'}^\dagger dg_{\phi'} \wedge g_{\phi'}^\dagger dg_{\phi'} \wedge g_{\phi'}^\dagger dg_{\phi'}], \quad (1298)$$

where $B_{3,\phi}$ is a bounding 3-ball extending from $\phi' = 0$ to $\phi' = \phi$.

Recapitulating, the whole integral on the RHS of (??) is a total derivative. This means that $U(g^\phi)e^{-\mathcal{G}(g^\phi)}$ is actually the exponential of the argument of the total derivative, which means that after setting $\phi = 1$, we have found $U(g)e^{-\mathcal{G}(g)}$. Keeping track of the various factors of $k/2\pi$, we get

$$U(g)e^{-\mathcal{G}(g)} = \frac{ik}{4\pi} \int \text{Tr}[g^\dagger(\partial_{\bar{z}} g)A + g^\dagger \partial_z g g^\dagger \partial_{\bar{z}} g] + \frac{ik}{24\pi} \int_{B^3} \text{Tr}[g^\dagger dg \wedge g^\dagger dg \wedge g^\dagger dg]. \quad (1299)$$

Thus when acting on wavefunctionals, $U(g)$ both implements gauge transformations and multiplies the wavefunctionals by this exponential factor. We write it as

$$U(g)|\Psi[A]\rangle = e^{i\Omega[g,A]}|\Psi[^g A]\rangle \equiv \exp\left(\frac{ik}{4\pi} \int \text{Tr}[A\bar{J}_g]\right) e^{iS[g]}|\Psi[^g A]\rangle, \quad (1300)$$

where we have suggestively written the current as $\bar{J}_g = g^\dagger \partial_{\bar{z}} g$ and where $S[g]$ is the WZW action (both kinetic and topological terms).

Since $U(g)$ is a representation of the gauge group, we need it to satisfy

$$U(h)U(g) = U(gh). \quad (1301)$$

Note the perverse ordering of the group elements on the RHS. Such a perversion is needed since gauge transformations act as

$$A \mapsto g^{-1}Ag + g^{-1}\partial_z g, \quad (1302)$$

so that for the product gh , it is g which acts first, and h which acts second.

Do the $U(g)$ form a linear representation? Consider the product $U(h)U(g)$ acting on $|\Psi[A]\rangle$. If this is equal to the action of $U(gh)$, then we need

$$\Omega[g, A] + \Omega[h, ^g A] = \Omega[gh, A] \quad \text{mod } 2\pi\mathbb{Z}. \quad (1303)$$

That is, we need Ω to have a coboundary which vanishes mod $2\pi\mathbb{Z}$. After some algebra, this condition means that we need

$$\frac{ik}{4\pi} \int \text{Tr}[g^\dagger \partial_z g h^\dagger \partial_{\bar{z}} h] + S[g] + S[h] = S[gh]. \quad (1304)$$

Looking back at our previous diary entry (1259), we see that this is actually true! The coboundary of the WZW action is precisely equal to the term needed to ensure that the $U(g)$ form a linear representation of the gauge group.

Now that we know how the $U(g)$'s act on candidate wavefunctionals, we need to actually find a particular solution for $|\Psi[A]\rangle$. But this is easy: we want it to be invariant under the action of the gauge group, and so we can project onto the trivial representation of the gauge group by integrating over all gauge transformations, that is, by acting with $\int \mathcal{D}g U(g)$ on any candidate wavefunctional. For example, we may just take

$$\Psi[A] = \int \mathcal{D}g \exp\left(iS[g] + \frac{ik}{4\pi} \int \text{Tr}[A\bar{J}_g]\right). \quad (1305)$$

We see that the wavefunctional for the CS gauge fields is obtained just by plugging them in as sources for the WZW theory.

Finally we briefly touch on some stuff we have brushed over. What happens if the spatial manifold X has a boundary? Recall that

$$\delta S = \frac{k}{2\pi} \int \text{Tr}[\delta A \wedge F_A] - \frac{k}{4\pi} \int_{\partial X} \text{Tr}[A \wedge \delta A], \quad (1306)$$

We will fix our boundary conditions by fixing $A|_{\partial X}$ to be some specific (not for sure zero) function. Then the boundary term in δS is an integral over $\text{Tr}[A\delta\bar{A}]$. In order to receive no boundary corrections to the equations of motion, we have to add the counterterm

$$S_\partial = \frac{1}{4\pi} \int_{\partial X} \text{Tr}[A\bar{A}]. \quad (1307)$$

This counterterm then also shows up in the action of $U(g)$ that we use to find the wavefunctional, in order to ensure that gauge invariance is maintained.

Secondly, what if we are on a spatial manifold with nontrivial 1-cycles? In this case, since the constraint from A_0 merely fixes the gauge field to be flat, we can have fields with nontrivial holonomy. I think the best way to deal with the problem in this case is to adopt the “solve the constraint and then quantize” approach, whereby we first decompose $A = U^{-1}\alpha U + U^{-1}dU$, and then quantize. Here U is single-valued and α keeps track of the holonomy.

77 July 1 — Chern-Simons Propagator

Today is a quickie: a simple calculation I realized I'd never done before.

Find the propagator for Abelian CS + Maxwell theory:

$$S = \frac{k}{4\pi} \int A \wedge F_A + \frac{1}{2e^2} \int F_A \wedge \star F_A. \quad (1308)$$

Show that the theory describes massive excitations and find the mass. If the propagator is massive, how does the CS term bestow long-ranged statistical interactions between particles?

Solution:

Let's find the answer for the mass in an easy way first so we can check our work. First we find the equations of motion, which we write as

$$\partial_\mu F^{\mu\nu} - \frac{\kappa}{2} \epsilon^{\nu\mu\lambda} F_{\mu\lambda} = 0, \quad \kappa \equiv \frac{e^2 k}{2\pi}. \quad (1309)$$

This is equivalently written as⁴²

$$(d^\dagger \star -\kappa) \star F = 0. \quad (1310)$$

Now we act on both sides with the operator $\star d + \kappa$, so that

$$(\star d + \kappa)(\star d - \kappa) \star F = (d^\dagger d - \kappa^2) \star F = 0. \quad (1311)$$

Now since $F = dA$, $d^\dagger \star F = 0$ identically. Thus

$$(d^\dagger d + dd^\dagger - \kappa^2) \star F = 0 \implies (\partial_\mu \partial^\mu - \kappa^2)(\star F)^\nu = 0, \quad (1312)$$

⁴² $d^\dagger = -\star d\star$ in Euclidean signature in three dimensions when acting on 1-forms while $d^\dagger = +\star d\star$ when acting on 2-forms. The hodge star satisfies $\star^2 = 1$ when acting on any degree form.

indicating that $\star F$ is a massive vector field with mass

$$m = \kappa = \frac{e^2 k}{2\pi}. \quad (1313)$$

When $e^2 \rightarrow \infty$ we have $m \rightarrow \infty$, which means that at strong coupling (in the deep IR), we have an infinite mass (we are “projecting onto the LLL”) and we can focus only on the Chern-Simons term.

Now let’s find the propagator and check this. We will choose to use Feynman gauge, by adding the gauge-fixing term

$$S_{gf} = \frac{1}{2e^2} \int d^\dagger A \wedge \star d^\dagger A \quad (1314)$$

to the action. This turns the Maxwell term into $\int A \wedge \star (d^\dagger d + dd^\dagger) A$. Since the thing in the parenthesis is the Hodge Laplacian, we just get a k^2 term. So then in momentum space, we need to find

$$G_{\mu\nu} = \left(iC\epsilon^{\mu\lambda\nu}k_\lambda + Dg^{\mu\nu}k^2 \right)^{-1}, \quad C \equiv \frac{k}{4\pi}, \quad D \equiv \frac{1}{2e^2}. \quad (1315)$$

The strategy for inverting this guy is to try to break it up into projectors, as usual. We let

$$\Pi_L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}, \quad \Pi_T = \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu}. \quad (1316)$$

The extra piece $iC\epsilon^{\mu\lambda\nu}k_\lambda$ is in the image of Π_T (since it’s orthogonal to Π_L , but it is not itself a projector. We look for a solution of the form

$$G^{\mu\nu} = X\Pi_T^{\mu\nu} + Y\Pi_L^{\mu\nu} + Z\epsilon^{\mu\lambda\nu}k_\lambda. \quad (1317)$$

We find the constants X, Y, Z just by multiplying this ansatz by $iC\epsilon^{\mu\lambda\nu}k_\lambda + Dg^{\mu\nu}k^2$ and setting the result equal to $g^{\mu\nu}$. This gives three equations (the coefficients of the three types of terms in G) in three unknowns, which we solve for in terms of C, D . The algebra is kind of boring and gives

$$G^{\mu\nu} = \frac{1}{k^2(k^2 - C^2/D^2)} \left(g^{\mu\nu} \frac{k^2}{D} - \frac{1}{D} k^\mu k^\nu - \frac{iC}{D^2} \epsilon^{\mu\lambda\nu} k_\lambda \right). \quad (1318)$$

This tells us that we have something with mass

$$m = \frac{C}{D} = \frac{ke^2}{2\pi}, \quad (1319)$$

which agrees with our previous result. As a sanity check, we note that we get the right answer when $C = 0$. Also, note that D^{-1} has mass dimension 1, so that all the dimensions of the various terms agree. We can also resolve the question posed at the beginning: if the theory is massive, how can it lead to nontrivial statistical interactions between particles which are infinitely long ranged? We see that the answer to this is that although we have a pole at $k^2 = C^2/D^2$, we also have a pole at $k^2 = 0$, since the last iC/D term has just a single power of momenta. This zero-momentum pole is what allows us to transmit the information needed to transmute the statistics of the particles.

78 July 2 — Magic at the $SU(2) \times SU(2)$ radius

Today we are going to learn more about the compact scalar in two dimensions and its magical properties at special values of the compactification radius.

The action is

$$S = \frac{R^2}{4\pi} \int d^2z \partial X \bar{\partial} X. \quad (1320)$$

From the equation of motion $\bar{\partial} \partial X = 0$ we can separate X as $X = X_L(z) + X_R(\bar{z})$. Define the operators

$$J^3 = i\partial X, \quad J^\pm =: e^{\pm i2X_L} : \quad (1321)$$

where we are tacitly assuming that J^\pm is well-defined. Using (note: some people have different factors of 2 but I think this is correct; recall $d^2z = 2d^2x$)

$$\langle X(z)X(0) \rangle = -\frac{1}{R^2} \ln |z|, \quad (1322)$$

Find the value of R for which the vertex operators J^\pm are well-defined and have scaling dimension 1. Letting

$$J^1 = \frac{1}{2}(J^+ + J^-), \quad J^2 = \frac{1}{2i}(J^+ - J^-), \quad (1323)$$

find the OPEs of the J^a 's, and show that the algebra obeyed by the charge operators is the $SU(2)$ algebra. Note to self: there are some factors of 2 and $\sqrt{2}$ that I never really convinced myself were correct. Oh well.

Solution:

First we find the scaling dimension of J^3 and J^\pm . For J^3 , we just differentiate the $\ln |z|$ propagator twice and see that

$$\Delta_{J^3} = 1. \quad (1324)$$

I think we've also found the scaling dimension for J^\pm in an earlier diary entry, so I'll be somewhat brief. From

$$\langle X(z, \bar{z})X(0) \rangle = -\frac{1}{2R^2}(\ln z + \ln \bar{z}), \quad (1325)$$

we get

$$\langle X_L(z)X_L(0) \rangle = -\frac{1}{2R^2} \ln z, \quad \langle X_R(\bar{z})X_R(0) \rangle = -\frac{1}{2R^2} \ln \bar{z}, \quad \langle X_L(z)X_R(0) \rangle = 0. \quad (1326)$$

Now, the correlation function $\langle J^\pm(z)(J^\mp)^\dagger(0) \rangle = 0$ since we violate “charge neutrality”. That is, if we have a current (which for us is $j = \sum_i \sigma_i \delta(z_i)$ for $\sigma_i \in \mathbb{Z}$) and expand $X_L = \sum_i \alpha_i X_i$ with $\int X_i X_j = \delta_{ij}$, then if we have a coupling term $\int j X_L$ in the action, the path integral over X_0 enforces $\int j = 0$, i.e. charge neutrality. Since we stick in such a $\int j X_L$ term when we are computing the scaling dimension of the vertex operators J^\pm , we see that the

two-point function of the vertex operators is only nonzero when their charge is neutral. Then we can write

$$\langle e^{\pm i2X_L(z)} e^{\mp i2X_L(0)} \rangle = \exp\left(-\frac{1}{2} \int d^2w d^2w' j(w) G(w, w') j(w')\right), \quad (1327)$$

where $j = 2(\pm\delta(z) \mp \delta(0))$. When we normal-order the vertex operators to compute the 2-point functions of the J^\pm we kill the points in the integral with support at $w = w'$, and so using the expression for the propagator we found,

$$\langle J^\pm(z) (J^\pm)^\dagger(0) \rangle = \frac{1}{z^{2/R^2}}, \quad (1328)$$

so that $\Delta_{J^\pm} = 1/R^2$. This means that the J^\pm operators have scaling dimension 1 when $R = 1$. Note that in some places (several string theory books) it's $R = \sqrt{2}$ —I think these differences come from different conventions for the $R^2/2\pi$ in the action. Note that at this radius, the scaling dimensions of J^\pm and J^3 are the same: as we will see, this permits the existence of a symmetry which rotates them into one another. A hint of this extra symmetry comes from looking at the mass

$$m^2 = \frac{n^2}{R^2} + w^2 R^2 + \dots, \quad (1329)$$

where \dots is an oscillator contribution that may or may not be there (it's there e.g. when this theory comes from compactification in the string theory context). n/R is the momentum (which comes in units of $1/R$ because of $x \sim x + 2\pi R$), while wR with w the winding number is the energy coming from the string tension (I believe with these conventions the string tension is $T = 1/2\pi$ so that $w2\pi RT = wR$). Sending $R \leftrightarrow 1/R$ and $n \leftrightarrow w$ is a symmetry of the spectrum, which acts as a self-duality when $R = 1$. When $R \neq 1$ we just have a $U(1) \times U(1)$ symmetry for the momentum and winding separately, but when $R = 1$ we will see that we get an $SU(2) \times SU(2)$ symmetry that rotates winding and momentum into one another.

Oh yeah, also note that at $R = 1$ the 2-point function for charge-1 vertex operators would be

$$\langle :e^{\pm iX_L(z)}: :e^{\mp iX_L(0)}: \rangle = \frac{1}{z^{1/2}}, \quad (1330)$$

which is nonsense since it is not single-valued. So for $R = 1$, only the charge 2 vertex operators are defined.

Now let's compute OPEs between the various J 's, specializing to the choice of $R = 1$ where both currents have the same scaling dimensions. The first one between two J^3 's is easy since we just have to do one contraction:

$$J^3(z) J^3(w) \sim \frac{1}{2(z-w)^2}. \quad (1331)$$

When we do the OPE for J^3 and J^\pm , we just have to contract the ∂X from J^3 with one of the X 's from the expansion of the vertex operator: the contractions among the X 's in the vertex operator are removed by the normal ordering. So since $\partial X(z, \bar{z}) = \partial(X_L(z) + X_R(\bar{z})) = \partial X_L(z)$, some algebra gives the OPE

$$J^3(z) J^\pm(w) = i\partial X_L(z) :e^{\pm i2X_L(w)}: \sim \pm\partial_z \ln(z-w) J^\pm(w) \sim \pm \frac{1}{z-w} J^\pm(w), \quad (1332)$$

where as usual \sim means equality up to non-singular terms (in this case, just the fully normal ordered term). This allows us to compute the OPE

$$J^3(z)J^1(w) \sim \frac{1}{z-w} \frac{J^+ - J^-}{2} \sim \frac{iJ^2(w)}{z-w}. \quad (1333)$$

Likewise,

$$J^3(z)J^2(w) \sim -\frac{iJ^1(w)}{z-w}. \quad (1334)$$

Now for the vertex operator OPEs. We find the OPE with the general prescription used to convert normal-ordering things to time-ordered things:

$$\mathcal{O}_1(z)\mathcal{O}_2(w) = \exp\left(-\frac{1}{2}\int dz'dw' \ln(z-w) \frac{\delta}{\delta X_L(z',1)} \frac{\delta}{\delta X_L(w',2)}\right) : \mathcal{O}_1(z)\mathcal{O}_2(w) :, \quad (1335)$$

where we are still at $R = 1$ and where the \mathcal{O}_i are functionals of X_L and the notation $X_L(z, i)$ means that the functional derivative acts only on \mathcal{O}_i . For example, we have

$$J^+(z)J^-(w) \sim \sum_{k=1}^{\infty} \frac{\ln^k(z-w)}{2^k k!} (-1)^k 2^{2k} : e^{2iX(z)} e^{-2iX(w)} : \sim \frac{1}{(z-w)^2} : e^{2iX(z)} e^{-2iX(w)} : \quad (1336)$$

Now we can expand the $X(z)$ exponential about $X(w)$ since it's inside the normal ordering, and so

$$J^+(z)J^-(w) \sim \frac{1}{(z-w)^2} + \frac{2i\partial X(w)}{z-w}. \quad (1337)$$

When we compute J^-J^+ the only difference is a minus sign when expanding the vertex operators inside the normal ordering, and so

$$J^-(z)J^+(w) \sim \frac{1}{(z-w)^2} - \frac{2i\partial X(w)}{z-w}. \quad (1338)$$

When we compute the $J^\pm J^\pm$ OPEs, we get an extra $(-1)^k$ which cancels the one appearing in the J^+J^- OPE, which renders all of the terms non-singular, so that

$$J^\pm(z)J^\pm(w) \sim 0, \quad (1339)$$

which we expect from the fact that the LHS is not charge-neutral. We then get

$$J^1(z)J^1(w) \sim J^2(z)J^2(w) \sim \frac{J^+(z)J^-(w) + J^-(z)J^+(w)}{4} \sim \frac{1}{2(z-w)^2}, \quad (1340)$$

as well as

$$J^1(z)J^2(w) \sim -J^2(z)J^1(w) \sim \frac{J^-J^+ - J^-J^+}{4i} = -\frac{\partial X(w)}{z-w} = iJ^3(w). \quad (1341)$$

Collecting these together, we get

$$J^a(z)J^b(w) = \frac{1}{2(z-w)^2} \delta_{ab} + i\epsilon^{abc} \frac{J^c}{z-w}, \quad (1342)$$

for $a, b, c = 1, 2, 3$. The $SU(2)$ -ness of this of course comes from the $(z - w)^{-1}$ term. Indeed, this term is responsible for making the algebra of the charges the $SU(2)$ algebra. To compute $[Q^a, Q^b]$, we do the usual trick: $Q^a Q^b$ looks like two concentric circles in radial quantization radially separated by a small distance ϵ : we turn the two associated contour integrals to an integral like $\oint_0 dw \oint_w dz$ where the subscripts indicate the center of the contour, and since only the $1/(z - w)$ pole contributes to the integral we get

$$[Q^a, Q^b] = \oint_0 dw \int_w dz [J^a(w), J^b(z)] = i\epsilon^{abc} \oint_0 dw \int_w dz \frac{J^c(z)}{z - w} = i\epsilon^{abc} Q^c, \quad (1343)$$

which is what we wanted.

We've only been dealing with the holomorphic part X_L , but the same story plays out for X_R . Since $X_L X_R \sim 0$, the L story and the R story are completely independent, and together they generate an $SU(2)_L \times SU(2)_R$ symmetry at the self-dual point.

This manifestation of the duality can be written in a perhaps more familiar form by writing down mode expansions for X_L and X_R and identifying the $m = 0$ term in the expansions, which are the momenta p_L, p_R of the modes. From $p \propto \int dz \partial X - \int d\bar{z} \bar{\partial} X$ (think: $\partial + \bar{\partial} \propto -i\partial_x$) and $w \propto \int dx^\mu \partial_\mu X \propto \int dz \partial X + \int d\bar{z} \bar{\partial} X$, one gets (still at $R = 1$)

$$p_L = n + w, \quad p_R = n - w, \quad (1344)$$

where we are at $\alpha' = 1$. Exchanging momentum and winding sends $p_L \rightarrow p_L, p_R \rightarrow -p_R$, so that the symmetry acts oddly on the antiholomorphic component. Therefore define the field $\tilde{X} = X_L - X_R$ as a guess for what the image of the field X is under duality (or better, just take the definition from $\partial \tilde{X} = \partial X, \bar{\partial} \tilde{X} = -\bar{\partial} X$). This passes basic sanity checks since the fact that $X_L X_R \sim 0$ means that X and \tilde{X} have all the same correlation functions, the same stress-energy tensor, and so on. Then by acting on \tilde{X} with $\partial \pm \bar{\partial}$, we see that the fields are related as

$$d\tilde{X} = \star dX, \quad (1345)$$

which is exactly the type of duality we are familiar with from the particle-vortex duality approach (at $R = 1$ it is a self-duality). More on this and lots of other cool things from the string theory side to perhaps appear in future diary entries.

79 July 3 — Verlinde formula for finite groups

This is problem 10.8 in the big yellow book, chapter 10. In what follows, it is helpful to recall the orthogonality relations for characters of a finite group G , namely

$$\begin{aligned} \sum_{j \in \text{Rep}(G)} \chi_j(a) \chi_j^*(b) &= \frac{|G|}{|C_a|} \delta_{a,b} \\ \frac{1}{|G|} \sum_{g \in G} \chi_j(g) \chi_k^*(g) &= \delta_{j,k} \end{aligned} \quad (1346)$$

where a, b label representative elements of conjugacy classes C_a, C_b and $\text{Rep}(G)$ is the irreps of G . The second one is the counterpart of $\int dx e^{ix(k-k')} = \delta(k - k')$ and the first one is the dual.

Define the group S matrix as

$$S_j(a) = \sqrt{\frac{|C_a|}{|G|}} \chi_j(a). \quad (1347)$$

The matrix elements here are $[S]_{ja} = S_j(a)$, with the mixing of conjugacy class and irreps allowed since for finite groups the number of conjugacy classes is the same as the number of irreps.

Use the orthogonality relations for the character to prove the Verlinde formula for the group G , relating the fusion coefficients of the group to the S matrix elements. Also prove a dual formula giving the structure constants of the class algebra in terms of the S matrix elements. Getting to these results requires several intermediate steps, which are sub-problems that are listed in the big yellow book—see there for details (although I'm not sure if the formula in part h is correct?).

Solution:

Let's first see if we can derive the orthogonality relations, just for fun. We'll get the second one first. We need Schur's lemma. Let $\psi : V_i \rightarrow V_j$ be a map between the two vector spaces that the irreps ρ_i, ρ_j act on, and let it be a map which commutes with the action of the representation, in that

$$\psi \rho_i(g) = \rho_j(g) \psi, \quad \forall g \in G. \quad (1348)$$

Then since elements in $\ker \psi$ are thus killed by $\psi \rho_i(g)$, $\ker \psi$ is $\rho_i(g)$ -invariant. Since ρ_i is an irrep, we must either have $\ker \psi = 0$ or $\ker \psi = V_i$. If it's the latter, then $\psi = 0$, so wolog we can take it to be the former. Similarly, since $\text{im } \psi$ is $\rho_j(g)$ -invariant, then if $\text{im } \psi \neq 0$, $\text{im } \psi = V_j$. This means that either ψ is 0, or it is an isomorphism $\psi : V_i \cong V_j$. From $\psi \rho_i(g) = \rho_j(g) \psi$, we see that it is invariant under conjugation by $\rho_i(g)$ for all $g \in G$, and thus must be central in $GL(V)$. Thus it must be of the form $\psi = \gamma \mathbf{1}$ for some constant γ .

We can make maps that satisfy the condition above by averaging over the group. For any map ϕ and representations ρ_i, ρ_j , define the average by

$$\tilde{\phi} = \frac{1}{|G|} \sum_{g \in G} \rho_i^{-1}(g) \phi \rho_j(g). \quad (1349)$$

By using the linearity of the representations and shifting the sum, one checks that $\tilde{\phi} \rho_i(g) = \rho_j(g) \tilde{\phi}$. Thus $\tilde{\phi} = 0$ unless $i = j$. Assume this is the case, so that $\tilde{\phi} = \gamma \mathbf{1}$, and take the trace. Then using the cyclicity of the trace,

$$\gamma \dim_i = \frac{1}{|G|} \sum_{g \in G} \text{Tr}[\phi] \implies \tilde{\phi} = \frac{\text{Tr}[\phi]}{\dim_i} \mathbf{1}, \quad (1350)$$

where \dim_i is the dimension of ρ_i .

To get the character orthogonality relation, pick some representation ρ_i let $\phi = E_{\alpha\beta}$ be one of the basis elements of $GL(\dim_i)$. Then $\tilde{\phi} = \mathbf{1}\delta_{\alpha\beta}/\dim_i$. So then since the RHS is zero if the two representation were different, we can write

$$\delta_{ij} \frac{\delta_{\gamma\sigma}\delta_{\alpha\beta}}{\dim_i} = \frac{1}{|G|} \sum_{g \in G} [\rho_i(g^{-1})]_{\gamma\alpha} [\rho_j(g)]_{\beta\sigma}. \quad (1351)$$

Now take $\gamma = \alpha$ and $\beta = \sigma$, and sum over α, β . On the RHS, this gives a sum over $g \in G$ of the product $\chi_i(g^{-1})\chi_j(g)$. On the LHS, this gives a sum over α, β of $\delta_{\alpha\beta}/\dim_i = 1$. So then

$$|G|\delta_{ij} = \sum_{g \in G} \chi_i(g^{-1})\chi_j(g). \quad (1352)$$

Now we need the relation

$$\chi_j(g^{-1}) = \chi_j^*(g). \quad (1353)$$

Proof: since G is finite, $g^n = 1$ for some $n \in \mathbb{Z}$, and thus $\chi_j(g^{-1}) = \chi_j(g^{n-1})$. Working in a basis where g is diagonal, since $g^{n-1}g = 1$, the matrix elements on the diagonals of g , g^{n-1} must be roots of unity, and since the product of each pair of diagonal entries is 1, they must be conjugates. Thus $\rho_j(g^{-1}) = \rho_j^*(g)$ and so $\chi_j(g^{-1}) = \chi_j^*(g)$.

Finally, we use that $\chi_i(g)$ is a class function to change the sum over $g \in G$ to a sum over conjugacy classes. Doing so gives

$$\sum_a |C_a| \chi_j(a) \chi_k^*(a) = |G| \delta_{j,k} \quad (1354)$$

which is the orthogonality relation we wanted.

The other orthogonality relation is in some sense the dual of the first, where duality exchanges the conjugacy class index a with the representation index i . Conjugacy classes and irreps are dual to one another by way of the characters:

$$\chi : \text{Rep}(G) \otimes \text{Conj}(G) \rightarrow \mathbb{C}, \quad (1355)$$

so that the intuition is $\text{Rep}(G) \cong (\text{Conj}(G))^*$. Thus we can think of the i index in $\chi_i(a)$ as a contravariant index and a as a covariant index. When G is Abelian $\text{Conj}(G) = G$, and so we get the familiar $\text{Rep}(G) \cong G^* \cong G$ for finite G , with the duality between group elements and representations being the Fourier transform.

To get the other orthogonality relation, define the character table matrix by $X_{ij} = \chi_i(g_j)$, where i is an irrep and g_j is some (representative of) a conjugacy class. Again, we're allowed to write it like this since there are just as many conjugacy classes as irreps⁴³. Also defining the matrix $D_{ij} = \delta_{ij}|C_{g_j}|/|G|$, the orthogonality relation we already derived reads

$$\mathbf{1} = XDX^\dagger. \quad (1357)$$

⁴³Proof: the conjugacy classes $C_a = \sum_{g \in C_a} g$ form a basis of the center $Z(\mathbb{C}[G])$, so that the number of conjugacy classes is

$$N_C = \dim(Z(\mathbb{C}[G])). \quad (1356)$$

On the other hand, $\mathbb{C}[G]$ is isomorphic to a product $\prod_i \text{Mat}_{\dim_i}(\mathbb{C})$, where the product runs over irreps. Since the centers of the matrix algebras are all one-dimensional regardless of their size, the dimension of the center of $\mathbb{C}[G]$ is equal to the number of terms in the product, proving the claim.

Now we multiply by X^{-1} on the left and X on the right to get $\mathbf{1} = DX^\dagger X$. This means that

$$\delta_{ij} = \sum_k \frac{|C_{g_i}|}{|G|} \chi_k^*(g_i) \chi_k(g_j). \quad (1358)$$

Switching back to denoting the conjugacy classes by a and rearranging gives

$$\sum_{j \in \text{Rep}(G)} \chi_j(a) \chi_j^*(b) = \frac{|G|}{|C_a|} \delta_{a,b}, \quad (1359)$$

which is what we wanted.

Now for the Verlinde formula. Consider the tensor product

$$\rho_i \otimes \rho_j = \bigoplus_k N_{ij}^k \rho_k. \quad (1360)$$

Taking the trace of both sides, writing $\text{Tr}[\rho_i \otimes \rho_j] = \text{Tr}[\rho_i] \text{Tr}[\rho_j]$, and then evaluating the representations on a particular (representative of) a conjugacy class C_a , we have

$$\chi_i(a) \chi_j(a) = \sum_k N_{ij}^k \chi_k(a). \quad (1361)$$

Of course, the χ_j are class functions, and so the exact choice of representative of the conjugacy class is not important. Now multiply both sides by $|C_a| \chi_l^*(a)$ and sum over all conjugacy classes a , so that orthogonality lets us write (re-labeling $l \rightarrow k$)

$$N_{ij}^k = \frac{1}{|G|} \sum_a |C_a| \chi_i(a) \chi_j(a) \chi_k^*(a). \quad (1362)$$

In terms of the S matrix,

$$N_{ij}^k = \sum_a \sqrt{\frac{|G|}{|C_a|}} S_i(a) S_j(a) S_k^*(a). \quad (1363)$$

Since $\chi_0(a) = 1$ for all a , $S_0(a) = (|C_a|/|G|)^{1/2}$, and so we have the Verlinde formula

$$N_{ij}^k = \sum_a \frac{S_i(a) S_j(a) S_k^*(a)}{S_0(a)}. \quad (1364)$$

A few comments on the S matrix. First, it is unitary. Indeed,

$$[SS^\dagger]_{ja} = \sum_b S_j(b) S_b^\dagger(a) = \sum_b S_j(b) S_a^*(b) = \sum_b \frac{|C_b|}{|G|} \chi_j(b) \chi_a^*(b) = \delta_{j,a}. \quad (1365)$$

The S matrix is symmetric for Abelian groups, because for finite Abelian groups Rep and Vec are the same. If G is Abelian then every irrep is one dimensional, and it acts by multiplication by some $g \in G$, putting the irreps and the group elements (which since G is Abelian are the same thing as conjugacy classes) in bijection. In this case $S_j(a) = \chi_j(a)/\sqrt{|G|}$, and $\chi_j(a) = \chi_a(j)$ since the representations just act by group multiplication and G is Abelian.

However, S is not symmetric as constructed for non-Abelian groups, since the conjugacy classes can have different orders which mess things up. For example in the group S_3 ,

$$S_{(12)}((123)) = \sqrt{\frac{2}{6}}\chi_{(12)}((123)) = -1/\sqrt{3}, \quad (1366)$$

while

$$S_{(123)}((12)) = \sqrt{\frac{3}{6}}\chi_{(123)}((12)) = -1/\sqrt{2}. \quad (1367)$$

Now we will get a dual version of the Verlinde formula, obtained with the $\text{Rep}(G) \cong (\text{Conj}(G))^*$ duality. Instead of summing over conjugacy classes to get the fusion coefficients, we will sum over irreps. To this end we introduce the class algebra. In what follows, when we write a conjugacy class C_a , we mean the sum $\mathbb{Z}[G] \ni C_a = \sum_{g \in C_a} g$. We can form an algebra with the classes through the following product operation:

$$\star : \mathbb{Z}[G] \times \mathbb{Z}[G] \rightarrow \mathbb{Z}[G], \quad C_a \star C_b = \sum_c \mathcal{N}_{ab}^c C_c. \quad (1368)$$

Just to be explicit, this is nothing more than

$$C_a \star C_b = \sum_{g \in C_a} \sum_{h \in C_b} gh. \quad (1369)$$

Here the coefficients \mathcal{N}_{ab}^c are integers and we can restrict to $\mathbb{Z}[G]$ from $\mathbb{C}[G]$. This is because if $C_c \ni f = gh$ appears in the product $C_a \star C_b$ for $g \in C_a, h \in C_b$ then so does $g'fg'^{-1} = (g'gg'^{-1})(g'hg'^{-1})$, so if one element in a conjugacy class appears on the RHS, all elements appear. This product is the conjugacy class dual of the \otimes of irreps. As an example, for S_3 we have

$$C_{(12)} \star C_{(12)} = 3C_{\mathbf{1}} + 3C_{(123)}, \quad C_{(123)} \star C_{(123)} = C_{(123)} + 2C_{\mathbf{1}}, \quad C_{(123)} \star C_{(12)} = 2C_{(12)}. \quad (1370)$$

Since we associate the sign representation of S_3 with the class of (123) and the two-dimensional representation with the class of $C_{(12)}$, there is no (is there? maybe we can normalize by the sizes of conjugacy classes or something?) obvious relation between the \mathcal{N} coefficients and the N coefficients.

Now take both sides of the \star product decomposition formula and act on them with some representation ρ_j , and then take the trace. This gives

$$\chi_j(C_a \star C_b) = \sum_c \mathcal{N}_{ab}^c \chi_j(C_c). \quad (1371)$$

Now multiply by $\chi_j^*(C_d)$ and sum over all irreps $j \in \text{Rep}(G)$. We can simplify the resulting expression by using the orthogonality of characters in their “covariant” representation index, which for the classes reads

$$\sum_{j \in \text{Rep}(G)} \chi_j(C_c) \chi_j^*(C_d) = |C_c| |C_d| \sum_{j \in \text{Rep}(G)} \chi_j(c) \chi_j^*(d) = \delta_{c,d} |C_c| |C_d| \frac{|G|}{|C_c|}. \quad (1372)$$

since the characters are class functions. Using this and re-naming $d \rightarrow c$,

$$\mathcal{N}_{ab}^c = \frac{1}{|G||C_c|} \sum_j \chi_j(C_a \star C_b) \chi_j^*(C_c). \quad (1373)$$

Now since $gC_ag^{-1} = C_a \quad \forall g \in G$, we have

$$C_a \in Z(\mathbb{C}[G]) \implies \chi_j(C_a) = \lambda_a \dim_j, \quad (1374)$$

for some constant λ . Thus we have

$$\chi_j(C_a \star C_b) = \lambda_a \lambda_b \dim_j = \frac{1}{\dim_j} \chi_j(C_a) \chi_j(C_b) = \frac{|C_a||C_b|}{\dim_j} \chi_j(a) \chi_j(b). \quad (1375)$$

This means that

$$\mathcal{N}_{ab}^c = \frac{|C_a||C_b|}{|G|} \sum_{j \in \text{Rep}(G)} \frac{1}{\dim_j} \chi_j(a) \chi_j(b) \chi_j^*(c). \quad (1376)$$

Since $\chi_j(1) = \dim_j$, we have

$$\mathcal{N}_{ab}^c = \frac{|C_a||C_b|}{|G|} \sum_{j \in \text{Rep}(G)} \frac{\chi_j(a) \chi_j(b) \chi_j^*(c)}{\chi_j(1)}. \quad (1377)$$

Finally, we just need to write this in terms of the S matrix. This gives the dual Verlinde formula

$$\mathcal{N}_{ab}^c = \sqrt{\frac{|C_a||C_b|}{|C_c|}} \sum_{j \in \text{Rep}(G)} \frac{S_j(a) S_j(b) S_j^*(c)}{S_j(1)}. \quad (1378)$$

Note that because of these square roots, unless the group G is Abelian, the coefficients

$$\mathcal{M}_{ab}^c = \sum_{j \in \text{Rep}(G)} \frac{S_j(a) S_j(b) S_j^*(c)}{S_j(1)} \quad (1379)$$

will *not* generally be integers. From our expression for N_{ij}^k we saw that the “dual version” of the \mathcal{M}_{ab}^c ’s (where we replace conjugacy classes with irreps) are integers, so unless G is Abelian there is an asymmetry between the conjugacy class approach and the irrep approach.

80 July 4 — T is not a conformal primary when $c \neq 0$, and the Hamiltonian on the cylinder

Today was a mostly non-physics day so I’m taking it easy and doing a calculation that I’ve seen in many places but never worked out for myself.

By using your knowledge of the TT OPE, use the conformal Ward identity to show that under the conformal transformation ξ_μ , the stress tensor changes by

$$\delta_\xi \langle T \rangle = (\xi \partial + 2\partial\xi)T + \frac{c}{12}\partial^3\xi. \quad (1380)$$

Now let $z = e^w$ be the mapping from cylindrical coordinates $w = \sigma^0 + i\sigma^1$ (with $\sigma^1 \sim \sigma^1 + 2\pi$ the spatial coordinate—what's the best notation here?) to the plane where time increases radially. Show that

$$T(w) = z^2 T(z) - \frac{c}{24}. \quad (1381)$$

Then from

$$H = \int \frac{d\sigma^1}{2\pi} T_{00}, \quad (1382)$$

show that the Hamiltonian on the cylinder is

$$H = L_0 + \bar{L}_0 - (c + \bar{c})/24. \quad (1383)$$

Solution:

The conformal Ward identity for an operator X says that

$$\delta_\xi \langle X \rangle = \int d^2\sigma \partial_\mu \langle T^{\mu\nu} \xi_\nu X \rangle = \oint d\sigma^\mu \langle \epsilon_{\mu\nu} T^{\nu\lambda} \xi_\lambda X \rangle. \quad (1384)$$

When we go to complex coordinates we get a $-i/2$ out front from the change in the ϵ tensor, since it becomes $\epsilon = -Y/2$. We also get a factor of 4 when we lower the indices on $T^{\bar{z}\bar{z}}$ to T_{zz} , and a factor of 1/2 when raising the index on $\xi_{\bar{z}}$ to $\xi \equiv \xi^z$. Finally, we get a $-1/2\pi$ from changing T_{zz} to $T = -2\pi T_{zz}$. So the conformal Ward identity in complex coordinates is

$$\delta_\xi \langle X(w) \rangle = \frac{1}{2\pi i} \oint dz \langle T\xi X(w) \rangle - \frac{1}{2\pi i} \oint d\bar{z} \langle \bar{T}\bar{\xi} X(w) \rangle. \quad (1385)$$

Here the contour is taken to be a small circle enclosing $\text{Supp}(X(w))$ (or the support of a suitably smeared version of X). Now we take $X = T$ and plug in the general form of the TT OPE (and use $T\bar{T} \sim 0$ to separate the holomorphic and antiholomorphic parts). We then expand $\xi(z)$ about $z = w$. Only the $\partial^3 \xi(z-w)^3$ term is able to integrate to something nonzero with the $(c/2)/(z-w)^4$ term, only the $\partial\xi(z-w)$ term is able to make the $2T/(z-w)^2$ term nonzero, and only the zeroth order term can make the $\partial T/(z-w)$ part nonzero. So after doing the integral we get

$$\delta_\xi \langle T \rangle = (\xi\partial + 2\partial\xi)T + \frac{c}{12} \partial^3 \xi \quad (1386)$$

as required.

Finding the finite version of this is a pain, but we can look it up in Polchinski:

$$T(w) = (\partial_w z)^2 T(z) + \frac{c}{12} \{z; w\}, \quad (1387)$$

where $\{z; w\}$ is the Schwartzian derivative:

$$\{z; w\} = \frac{\partial_w^3 z}{\partial_w z} - \frac{3}{2} \left(\frac{\partial_w^2 z}{\partial_w z} \right)^2. \quad (1388)$$

Thus the stress tensor isn't a conformal primary unless $c = 0$. Checking that this works is straightforward and unilluminating so I won't write it out.

We need to do the coordinate transformation $z = e^w$ between the plane (z) and the cylinder (w). The Schwartzian derivative in this case is easy:

$$\{z; w\} = -1/2. \quad (1389)$$

So putting this in, the cylinder stress tensor is

$$T(w) = z^2 T(z) - \frac{c}{24}, \quad (1390)$$

which is what we wanted.

Now for the Hamiltonian. We write it as

$$H = -\frac{1}{2\pi i} \int (dw T + d\bar{w} \bar{T}) = \frac{1}{2\pi i} \int (dz z^{-1} T(w) + d\bar{z} \bar{z}^{-1} \bar{T}(\bar{w})), \quad (1391)$$

since $dz = zdw$. Using the transformation law for T , this is

$$H = \frac{1}{2\pi i} \int (dz z^{-1} [z^2 T(z) - c/24] - d\bar{z} \bar{z}^{-1} [\bar{z}^2 \bar{T}(\bar{z}) - \bar{c}/24]). \quad (1392)$$

The integral selects out the $n = 0$ components of the Laurent expansions of T, \bar{T} due to the conventions on shifting the powers by 2 in the expansion. The central charge pieces are integrated against $1/z, 1/\bar{z}$ and so they survive, and then since $\int dz/z = -\int \bar{z}/\bar{z}$, we get

$$H = L_0 + \bar{L}_0 - (c + \bar{c})/24, \quad (1393)$$

as expected. That the Hamiltonian has the $L_0 + \bar{L}_0$ is no surprise, since this is the operator that generates dilations: since $L_0 = \frac{1}{2\pi i} \oint dz z T$, for a primary X we have

$$L_0 X = \frac{1}{2\pi i} \oint dz z \frac{hX(w)}{(z-w)^2} = hX(w), \quad (1394)$$

so that indeed, L_0 performs the dilations, and thus belongs in H since dilations in the plane in radial quantization are the same as time evolution.

81 July 5 — Linear dilaton CFT

This is an exercise John McGreevy assigned to his QFT class. Consider the linear dilaton CFT, which is a free scalar plus a coupling to gravity:

$$S = \frac{1}{2\pi\alpha'} \int d^2x \partial X \bar{\partial} X + \frac{1}{2\pi} \int d^2x Q X R, \quad (1395)$$

where Q is a constant, which may be either real or imaginary (if we are thinking about strings we should be writing out the spacetime index on the X like $Q_\mu X^\mu$, but for this problem we will just think of X as a scalar) and R is the two-dimensional Ricci curvature scalar.

Show that in flat space, while the coupling to gravity doesn't affect the equations of motion, it does change the stress tensor, which is

$$T = -\frac{1}{\alpha'} : \partial X \partial X : + Q \partial^2 X. \quad (1396)$$

Here Q is found by computing the TT OPE, and check that the TT OPE has the right form for stress tensors in CFTs. Then compute the scaling dimension of the vertex operator $:e^{ikX}:$.

Solution:

In flat space the extra term in the action vanishes, so we can use the usual $-\alpha' \ln |z-w|^2/2$ propagator in what follows. However the stress tensor will change since it comes from varying the metric away from flat space. Since there is a \sqrt{g} in the measure, we need to compute the variation

$$\delta(\sqrt{g}R) = \delta(\sqrt{g}g^{\nu\mu}R_{\mu\nu}) = (\delta\sqrt{g})R + \sqrt{g}R_{\mu\nu}\delta g^{\nu\mu} + \sqrt{g}g^{\nu\mu}\delta R_{\mu\nu}, \quad (1397)$$

with $R_{\mu\nu}$ the Ricci tensor. The variation of \sqrt{g} is

$$\delta\sqrt{g} = \frac{1}{2\sqrt{g}}\delta e^{\text{Tr ln } g} = \frac{1}{2}\sqrt{g}\text{Tr}[g^{-1}\delta g] = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\nu\mu}, \quad (1398)$$

where in the last step we used $\delta(g^{\mu\nu}g_{\nu\lambda}) = \delta(\delta^\mu_\lambda) = 0$. The variation of the Ricci scalar is more heinous to derive, but luckily it can be found in Wald, in section 7.5 (the one on perturbations). It turns out that $g^{\nu\mu}\delta R_{\mu\nu}$ is a total derivative:

$$g^{\nu\mu}\delta R_{\mu\nu} = \nabla^\mu v_\mu \equiv \nabla^\mu[\nabla^\nu(\delta g_{\mu\nu}) - g^{\lambda\sigma}\nabla_\mu(\delta g_{\lambda\sigma})]. \quad (1399)$$

Thus the variation of the dilaton term with respect to the metric is

$$\begin{aligned} \delta S_Q &= \frac{1}{2\pi} \int d^2x Q X \sqrt{g} \left(\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right] \delta g^{\mu\nu} + \nabla^\mu v_\mu \right) \\ &= \frac{1}{2\pi} \int d^2x Q X \sqrt{g} (G_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu v_\mu), \end{aligned} \quad (1400)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \quad (1401)$$

is the Einstein tensor.

Here's where being in two dimensions helps. Because of the symmetries of the Riemann curvature tensor, it has only one independent component in two dimensions: since $R_{\mu\nu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$ and $R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$, the only independent non-zero component is R_{0101} . Then the Ricci tensor is

$$R_{\mu\nu} = R_{0101} \begin{pmatrix} g^{11} & -g^{01} \\ -g^{10} & g^{00} \end{pmatrix} = \frac{R_{0101}}{\det g} g_{\mu\nu}, \quad (1402)$$

since the matrix is $(\det g^{-1})(g^{\mu\nu})^{-1}$. On the other hand, we can contract the Ricci tensor explicitly and see that the curvature scalar is

$$R = 2 \frac{R_{0101}}{\det g}, \quad (1403)$$

so that

$$R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}, \quad (1404)$$

which implies the vanishing of the Einstein tensor in two dimensions, $G_{\mu\nu} = 0$. Thus from (1400) we see that if we were to ignore the scalar X so that we just had gravity, we can conclude that in two dimensions the variation of the Einstein-Hilbert action is actually a total divergence, and so vanishes on closed spacetimes. This is because we should think of the Einstein-Hilbert action in this case as being $\int F$ and measuring the topology of the spacetime (more precisely, $\sqrt{g}R$ is the Euler density, so that the Einstein-Hilbert action computes the Euler characteristic).

Now we integrate the remaining $X\nabla^\mu v_\mu$ term by parts. We get, now in flat space,

$$\delta S_Q = \frac{1}{2\pi} \int d^2x Q(\partial^\mu \partial^\nu X - \partial^\lambda \partial_\lambda X g^{\mu\nu}) \delta g_{\mu\nu}, \quad (1405)$$

and so with the identification $\delta S = -\frac{1}{2} \int T^{\mu\nu} \delta g_{\mu\nu}$, we see that S_Q gives a contribution to the stress tensor of

$$T_Q^{\mu\nu} = \frac{1}{\pi} Q(g^{\mu\nu} \partial_\lambda \partial^\lambda - \partial^\mu \partial^\nu) X. \quad (1406)$$

Now we switch over to complex coordinates and use the definition $T = -2\pi T_{zz} = -(\pi/2)T^{\bar{z}\bar{z}}$ to get

$$T_{zz} = \frac{-Q}{2\pi} (\partial_0^2 - \partial_1^2 - 2i\partial_0\partial_1) X, \quad (1407)$$

so that the dilaton contribution to the holomorphic part of the stress tensor is

$$T_Q = Q\partial^2 X. \quad (1408)$$

Now we want to find the central charge. We thus need to look at the $1/(z-w)^4$ term in the $T(z)T(w)$ OPE. This part only comes from terms that have been fully contracted, since we need four derivatives acting on propagators to give a $1/(z-w)^4$ term. The usual piece is

$$\begin{aligned} \frac{1}{\alpha'^2} 2(\langle \partial X(z) \partial X(w) \rangle)^2 &= \frac{1}{2} (\partial_z \partial_w \ln |z-w|^2)^2 \\ &= \frac{1/2}{(z-w)^4}, \end{aligned} \quad (1409)$$

which gives the usual contribution of 1 to the central charge. The other piece is

$$Q^2 \langle \partial_z^2 X(z) \partial_w^2 X(w) \rangle = \frac{\alpha'}{2} \frac{6}{(z-w)^4}, \quad (1410)$$

so that

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + O(1/z^2), \quad c = 1 + 6\alpha' Q^2. \quad (1411)$$

Note that the central charge gets *reduced* by the introduction of the dilaton coupling if Q is imaginary.

Now we should check to make sure that the $O(1/z^2)$ in the above equation really is $O(1/z^2)$, and not $O(1/z^3)$ since otherwise we're in trouble. Recall that the form of the TT OPE needs to be

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{z^2}T(w) + \frac{1}{z}\partial T(w). \quad (1412)$$

One potentially troublesome term in the linear dilaton TT OPE is

$$-\frac{1}{\alpha'} Q \langle \partial_z X(z) \partial_w^2 X(w) \rangle \partial_z X(z) = \frac{2Q}{\alpha'} \frac{\alpha'}{2} \partial_z \partial_w^2 \ln |z-w|^2 \partial_z X(z) = \frac{2Q \partial_z X(z)}{(z-w)^3}. \quad (1413)$$

What saves us from this term is the other contraction with $z \leftrightarrow w$, since the denominator is odd under this shift. The two cross-contractions between the Q term and the regular term thus give

$$\frac{2Q(\partial_z X(z) - \partial_w X(w))}{(z-w)^3} \rightarrow \frac{2Q \partial_z^2 X(z)}{(z-w)^2}, \quad (1414)$$

which is exactly the term we need for the $2T(w)/z^2$ piece of the TT OPE. It is straightforward to check that the remainder of the $(z-w)^{-2}T(w)$ term and the $(z-w)^{-1}\partial T(w)$ term are produced as well.

Let's now find the conformal dimension of the vertex operator by taking the OPE with T . We need to only look at the leading piece of the OPE which goes as $1/(z-w)^2$. The part from the $: \partial X \partial X :$ term can be found by noting that $1/(z-w)^2$ terms only occur when both ∂X operators are contracted with the vertex operator. So after figuring out the combinatorial factors,

$$-\frac{1}{\alpha'} : \partial X \partial X :: e^{ikX} := \frac{1}{\alpha'} k^2 \left(\frac{\alpha'}{2} \partial_z \ln |z-w|^2 \right)^2 \sum_{j=0}^{\infty} \frac{1}{(j+2)!} \binom{j+2}{2} 2(ik)^j : X(w)^j : + O(1/z), \quad (1415)$$

so that the relevant term is

$$-\frac{1}{\alpha'} : \partial X \partial X :: e^{ikX} := \frac{k^2 \alpha' / 4}{(z-w)^2} : e^{ikX} : + O(1/z), \quad (1416)$$

which gives a contribution of $k^2 \alpha' / 4$ to the conformal dimension. The contribution from the dilaton term is

$$ikQ \sum_{j=0}^{\infty} \frac{1}{j!} \langle \partial^2 X(z) X(w) \rangle : (iX(w))^j : = \frac{ik\alpha' Q / 2}{(z-w)^2} : e^{ikX} :, \quad (1417)$$

which gives a contribution of $ik\alpha' Q / 2$ to the conformal dimension. Thus the conformal dimension of the vertex operator is

$$h = \frac{k^2 \alpha'}{4} + \frac{ik\alpha' Q}{2}. \quad (1418)$$

This gives a real conformal dimension if Q is imaginary. If Q is real (does this make sense physically?) then the scaling dimension Δ is unaffected by the coupling to the dilaton.

82 July 6 — Bosonization and torus partition functions

I got this problem from a pset that John McGreevy assigned to his QFT class and posted online. Update: actually, looks like a large part of this is in the big yellow book (of course).

Find the partition function for a free compact boson on a torus with sides given by the complex numbers ω_1, ω_2 . Show that this is the same as the partition function for a Dirac fermion on the torus, with all spin structures taken into account.

Solution:

Let the two sides of the torus be given by $\omega_1 \in \mathbb{R}, \omega_2 \in \mathbb{C}$, with modular parameter $\tau = \omega_2/\omega_1$. The time evolution operator along a direction parallel to ω_2 for a “time” s is

$$U(s) = \exp\left(-\frac{s}{|\omega_2|} [\text{Im}(\omega_2)H - i\text{Re}(\omega_2)P]\right). \quad (1419)$$

The signs are the way they are since H acts as $+i\partial_t$ (so that we translate in time with $e^{-iHt} \rightarrow e^{-\tau H}$) while P acts as $-i\partial_x$ (so that we translate with e^{+iP}). We can use the expression we found for H two days ago, namely $H = L_0 + \bar{L}_0 - (c + \bar{c})/24$. This was derived for a cylinder of radius 1, and so if the radius is instead ω_1 , then the Hamiltonian is (by dimensional analysis)

$$H = \frac{2\pi}{\omega_1} \left(L_0 + \bar{L}_0 - \frac{c + \bar{c}}{24} \right). \quad (1420)$$

Since we are specializing to a free compact boson, we will just write the last part as $(c + \bar{c})/24 = 1/12$ in what follows.

On a cylinder of radius 1, we get the momentum operator from

$$P = \int d\sigma T_{\tau\sigma} = i \frac{1}{2\pi i} \int (dw T(w) + d\bar{w} \bar{T}(w)). \quad (1421)$$

Recalling from two days ago how to map $T(w)$ onto the plane, we get

$$P = i \frac{1}{2\pi i} \oint \left(z^{-1} dz z^2 T(z) + \bar{z}^{-1} d\bar{z} \bar{z}^2 \bar{T}(z) + (z^{-1} dz + \bar{z}^{-1} d\bar{z}) \frac{c}{24} \right). \quad (1422)$$

The central charge term dies since the contour with dz is minus the one with $d\bar{z}$, while the other integrals select out the $n = 0$ component of the Laurent expansions for the stress tensors, which enter with opposite sides. When on a cylinder of radius ω_1 then,

$$P = \frac{2\pi i}{\omega_1} (L_0 - \bar{L}_0). \quad (1423)$$

The partition function is then

$$Z = \text{Tr} \left[e^{2\pi i(L_0 + \bar{L}_0)\frac{\tau - \bar{\tau}}{2}} e^{2\pi i(L_0 - \bar{L}_0)\frac{\tau + \bar{\tau}}{2}} e^{-2\pi i c \frac{\tau - \bar{\tau}}{24}} \right]. \quad (1424)$$

This can be cleaned up by defining

$$q \equiv e^{2\pi i\tau}. \quad (1425)$$

Then we have

$$Z = \text{Tr} \left[q^{L_0-1/24} \bar{q}^{\bar{L}_0-1/24} \right]. \quad (1426)$$

To do the trace, we need the spectrum of L_0 . The action is

$$S = \frac{1}{8\pi} \int dx dt [(\partial_t X)^2 - (\partial_x X)^2]. \quad (1427)$$

Now we expand, momentarily ignoring winding number issues,

$$X = \sum_n X_n e^{inx/r}, \quad (1428)$$

where r is the radius of the spatial circle—for us, $\omega_1 = 2\pi r$. In what follows, all sums over roman letter variables will be sums over \mathbb{Z} . The momentum is $\pi_n = \partial_t X_{-n} r/2$, and the Hamiltonian is

$$H = \frac{1}{r} \sum_n \left(\pi_n \pi_{-n} + \frac{n^2}{4} X_n X_{-n} \right). \quad (1429)$$

This is a bunch of harmonic oscillators—from the momentum term we see that $m = 1/2$, so that the frequency of each oscillator is just $\omega_n = |n|$.

We will use this result in a bit, but first we will need to remember topological issues. Let us treat the zero mode X_0 separately from the other modes in the sum. From the commutator $[X_0, H] = 2i\pi_0/r$, we can get the time evolution of the zero mode. This further simplifies, since the compact nature of the boson forces $\pi_0 = k/R$ for $k \in \mathbb{Z}$. We also need to add in a term that keeps track of the winding number of X : it is Rmx/r (with $X \sim X + 2\pi R$ defining the boson radius), which shifts as $Rmx/r \mapsto Rmx/r + 2\pi Rm$ around the spatial circle. Thus the decomposition for X

$$X(x, t) = X_0(0) + \frac{2k}{r}\tau + \frac{Rm}{r}x + \sum_{n \neq 0} X_n(t)X_{-n}(t). \quad (1430)$$

Now we need to go to complex coordinates. There are many options, but it seems like the best choice are

$$z = e^{(\tau+ix)/r}, \quad \bar{z} = e^{(\tau-ix)/r}. \quad (1431)$$

This is kind of unpleasant since $\tau = it$ is imaginary time, but oh well—it ends up giving the answer in the form written in the problem statement. The $1/r$ in the exponents is so that r disappears from the final expression for X . With this choice of coordinates

$$x = -\frac{ir}{2}(\ln z - \ln \bar{z}), \quad t = -\frac{ir}{2}(\ln z + \ln \bar{z}). \quad (1432)$$

Putting this into $X(t)$:

$$X(z, \bar{z}) = X_0 - i(\ln z)p_L - i(\ln \bar{z})p_R + i \sum_{n \neq 0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}), \quad (1433)$$

where the momenta are

$$p_L = \left(\frac{k}{R} + \frac{mR}{2} \right), \quad p_R = \left(\frac{k}{R} - \frac{mR}{2} \right). \quad (1434)$$

We can now get expressions for L_0, \bar{L}_0 by using $T = -\frac{1}{2} : \partial X \partial X :$. So taking the derivative, we have

$$L_0 = \frac{1}{4\pi i} \oint dz z \left(p_L^2 z^{-2} + \sum_{i,j \neq 0} a_i a_j z^{-i-j-2} + 2p_L \sum_{j \neq 0} a_j z^{-j-2} \right). \quad (1435)$$

The last term dies while in the second term i gets set to $-j$, so

$$L_0 = \frac{1}{2} p_L^2 + \sum_{j>0} a_{-j} a_j. \quad (1436)$$

Similarly,

$$\bar{L}_0 = \frac{1}{2} p_R^2 + \sum_{j>0} \bar{a}_{-j} \bar{a}_j. \quad (1437)$$

Here the sum over $j > 0$ means a sum from $\mathbb{Z} \ni j = 1$ to $j = \infty$.

We can now finally get the partition function. For each oscillator mode j , the sum over occupation numbers gives $1/(1 - q^j)$. This is because as we saw earlier, the frequency of the j th mode is simply $|j|$. So the oscillator contribution to q^{L_0} is $\prod_{j>0} (1 - q^j)^{-1}$. The zero modes are accounted for just by summing over all momenta k and winding numbers m , and so since the antiholomorphic oscillator contribution is the conjugate of the holomorphic contribution,

$$Z(q) = \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}, \quad (1438)$$

where

$$\eta(q) \equiv q^{1/24} \prod_{j>0} (1 - q^j). \quad (1439)$$

Note that the theory is self-dual at the radius $R = \sqrt{2}$ (not $1/\sqrt{2}$ like earlier because of how we defined the coupling constant for the action). In the following we will consider the radius $R = 1$, which from earlier diary entries we know to be a value for which a fermion description works. Explicitly, at this radius we have

$$Z(q; R = 1) = \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m/2)^2} \bar{q}^{\frac{1}{2}(k-m/2)^2}. \quad (1440)$$

Our goal now is to relate this to fermions. First we break up the sum into m even and m odd:

$$Z(q; R = 1) = \frac{1}{|\eta(q)|^2} \sum_{k,m} \left(q^{\frac{1}{2}(k+m)^2} \bar{q}^{\frac{1}{2}(k-m)^2} + q^{\frac{1}{2}(k+m+1/2)^2} \bar{q}^{\frac{1}{2}(k-m-1/2)^2} \right). \quad (1441)$$

The first term is actually

$$\begin{aligned} \frac{1}{2|\eta(q)|^2} \left(\left| \sum_k q^{k^2/2} \right|^2 + \left| \sum_k (-1)^k q^{k^2/2} \right|^2 \right) &= \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}k^2} \bar{q}^{\frac{1}{2}m^2} \frac{1 + (-1)^{k+m}}{2} \\ &= \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m)^2} \bar{q}^{\frac{1}{2}(k-m)^2}, \end{aligned} \quad (1442)$$

since the term $\frac{1+(-1)^{k+m}}{2}$ projects onto configurations where $k = m \bmod 2$, which is exactly fulfilled by the pair $k+m, k-m$ (we've re-labeled the summation variables—the point is that for any integers k, m , the combination $k+m, k-m$ survives the projection by $(1+(-1)^{k+m})/2$).

We can use a similar trick for the second term in $Z(q; R = 1)$: since the members of the combination $k+m, k-m$ always have the same parity, we can change the sum to run over all integers $x = k+m, y = k-m$ such that $x = y \bmod 2$. We can then instead sum over *all* x, y , provided that we insert the projector $(1 + (-1)^{x+y})/2$. Doing this, and then relabeling $x \rightarrow k, y \rightarrow m$ for consistency of notation, we have

$$\frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m+1/2)^2} \bar{q}^{\frac{1}{2}(k-m-1/2)^2} = \frac{1}{2|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+1/2)^2} \bar{q}^{\frac{1}{2}(m+1/2)^2} (1 + (-1)^{k+m}), \quad (1443)$$

Consider the second term, proportional to $(-1)^{k+m}$. The exponential part is symmetric under the shift $k \mapsto -k - 1$, but the $(-1)^{k+m}$ part picks up a minus sign, and so the second term is zero. Thus we can only keep the first term of this bit. It too factors between holomorphic and antiholomorphic contributions, and so the full partition function is

$$Z(q; R = 1) = \frac{1}{2|\eta(q)|^2} \left(\left| \sum_k q^{k^2/2} \right|^2 + \left| \sum_k (-1)^k q^{k^2/2} \right|^2 + \left| \sum_k q^{\frac{1}{2}(k+1/2)^2} \right|^2 \right). \quad (1444)$$

This is starting to look more fermiony! The three terms are theta functions, and so we can write Z more compactly as (preserving the order of the terms as in the last equation)

$$Z(q; R = 1) = \frac{1}{2|\eta(q)|^2} (|\theta_3(\tau)|^2 + |\theta_4(\tau)|^2 + |\theta_2(\tau)|^2). \quad (1445)$$

The theta functions can be written as infinite products—see e.g. Polchinski. The product form for the theta functions all contain a factor that cancels the product in the $1/|\eta(q)|^2$ in the denominator, and leaves us with

$$Z(q; R = 1) = \frac{1}{2|q^{1/24}|^2} \left(\prod_{j>0} |(1 + q^{j-1/2})^2|^2 + \prod_{j>0} |(1 - q^{j-1/2})^2|^2 + |q^{1/8}|^2 \prod_{j \geq 0} |(1 + q^j)^2|^2 \right). \quad (1446)$$

Our goal is in sight, since we are seeing the different boundary conditions for the fermions appearing.

To keep going, we will need the fermion partition functions. The Hamiltonian for the fermions is derived in the same way as the boson Hamiltonian, which we already did above. It will give us an oscillator contribution coming from the L_0, \bar{L}_0 operators, as well as a central charge piece that appears when we switch from the cylinder to the plane. Let's focus on a single real fermion with antiperiodic boundary conditions around the spatial circle: the central charge is $1/2$, while the oscillator expansion gives

$$L_0 = \sum_{k>0} (k - 1/2) \lambda_k \lambda_{-k}, \quad (1447)$$

where the λ_k 's are Majorana operators and the sum is offset by $1/2$ to get the boundary conditions right (note: our convention is such that the action for a single real fermion is

$\frac{1}{4\pi} \int \psi \gamma^0 \bar{\psi} \psi$). So, using the same logic that we used for the compact boson, the partition function for antiperiodic boundary conditions around both cycles is (NSNS / BB spin structure; B for “bounding”)

$$Z_{NSNS}(q) = \text{Tr}_A[q^{L_0-1/48} \bar{q}^{\bar{L}_0-1/48}] = \frac{1}{|q^{1/48}|^2} \prod_{k>0} |1 + q^{k-1/2}|^2. \quad (1448)$$

When we work with periodic boundary conditions around the temporal cycle, we need to insert $(-1)^F$ to implement the supertrace. This sends $q \rightarrow -q$ in the above and so

$$Z_{NSR}(q) = \text{Tr}_A[(-1)^F q^{L_0-1/48} \bar{q}^{\bar{L}_0-1/48}] = \frac{1}{|q^{1/48}|^2} \prod_{k>0} |1 - q^{k-1/2}|^2. \quad (1449)$$

When we have periodic boundary conditions in space, the form for L_0 changes in two ways: first, the momenta live in \mathbb{Z} rather than $\mathbb{Z} + \frac{1}{2}$, and second, they change by a constant since the vacuum energy on the cylinder depends on the boundary conditions (see e.g. chapter 6 of the big yellow book). In particular,

$$L_0 = \sum_{k \geq 0} \lambda_{-k} \lambda_k + \frac{1}{16}. \quad (1450)$$

Thus for the RNS (periodic in space, antiperiodic in time) spin structure, the partition function is

$$Z_{RNS}(q) = \text{Tr}_P[q^{L_0-1/48} \bar{q}^{\bar{L}_0-1/48}] = \frac{1}{|q^{1/48-1/16}|^2} \prod_{k \geq 0} |1 + q^k|^2 = |q^{1/24}|^2 \prod_{k \geq 0} |1 + q^k|^2. \quad (1451)$$

Finally, when we do the RR torus, we get zero because of the zero mode: when we change the thing in the product to $1 - q^k$, the $k = 0$ term kills the partition function. So $Z_{RR}(q) = 0$.

A massless Dirac fermion for us is the same as two real fermions which are completely independent expect for the requirement that their boundary conditions match. Thus to get $Z_{\text{Dirac}}(q)$ (with all spin structures counted), we just need to sum the squares of the real fermion partition functions over spin structures. This gives

$$\begin{aligned} Z_{\text{Dirac}}(q) &= \sum_{r,s \in \{NS,R\}} Z_{rs}(q)^2 \\ &= \frac{1}{2} \left(\frac{1}{|q^{2/48}|^2} \prod_{k>0} |(1 + q^{k-1/2})^2|^2 + \frac{1}{|q^{2/48}|^2} \prod_{k>0} |1 - q^{k-1/2}|^2 + |q^{2/24}|^2 \prod_{k \geq 0} |1 + q^k|^2 + 0^2 \right) \\ &= \frac{1}{2|q^{1/24}|^2} \left(\prod_{j>0} |(1 + q^{j-1/2})^2|^2 + \prod_{j>0} |(1 - q^{j-1/2})^2|^2 + |q^{1/8}|^2 \prod_{j \geq 0} |(1 + q^j)^2|^2 \right) \\ &= Z(q; R = 1). \end{aligned} \quad (1452)$$

Thus we've shown that the compact boson on the torus (at $R = 1$) is the same as a Dirac fermion, which in turn is a pair of Ising models coupled together in a certain way.

Note that for this correspondence to work, we have to sum over all spin structures for the fermion (of course, in order to have modular invariance we needed to sum over all spin structures since they [except RR] are permuted into one another by modular transformations. Checking the modular invariance of the final expression can be done by looking up the transformation properties of the theta functions, see e.g. Polchinski). Note that we can see here why people talk about modular invariance giving you information about how the holomorphic and anti-holomorphic sectors talk to each other, information which isn't available on the plane. Indeed, if the two sectors didn't talk to each other then we would have $Z_{\text{Dirac}}(q) = Z_{\text{Dirac}/2}(q)Z_{\text{Dirac}/2}(\bar{q})$ for some function $Z_{\text{Dirac}/2}(q)$. This factorization property is true for a particular spin structure, but not for the whole partition function, and so the boundary conditions introduced by the torus and the requirement of modular invariance let us see the constraints on the ways that the two sectors can talk to each other.

A more suggestive way to write the sum over spin structures is to write the fermion partition function as

$$Z_{\text{Dirac}}(q) = \text{Tr}_{NS} \left[\frac{\mathbf{1} + (-1)^F}{2} q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24} \right] + \text{Tr}_R \left[\frac{\mathbf{1} + (-1)^F}{2} q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24} \right], \quad (1453)$$

where the subscript on the trace indicates the spatial boundary conditions⁴⁴. The point of writing it this way is that it takes the form of fermions coupled to a \mathbb{Z}_2 gauge field, with the projectors $(\mathbf{1} + (-1)^F)/2$ enforcing that all states in the Hilbert space be a singlet under the \mathbb{Z}_2 symmetry sending $\psi \mapsto -\psi$. That is, we can view the temporal part of the spin structure as coming from a \mathbb{Z}_2 gauge field (modular invariance then forces us to sum over both spatial boundary conditions and trace over the whole $\mathcal{H}_{NS} \oplus \mathcal{H}_R$ Hilbert space). The point of this remark is that if we are writing down a mapping between a bosonic theory and a fermionic one, since we never have local fermion operators in the bosonic Hilbert space—only operators which are pairs of fermions. Because of this, the sign of any fermion operators we write down must be ambiguous up to a sign, and so we expect there to be a \mathbb{Z}_2 gauge redundancy in any putative fermionic dual model we write down. In this case, we see that this thinking is correct.

83 July 7 — Vertex correlators

Today we have an exercise from the big yellow book, chapter 9. We will be considering a (non-compact) boson with action

$$S = \frac{1}{4\pi} \int dz d\bar{z} \partial\phi \bar{\partial}\phi. \quad (1454)$$

Our goal will be to derive the correlation functions of vertex operators in a careful way.

To define the vertex operator, do a mode expansion and separate out the zero mode as follows:

$$\mathcal{V}_\alpha(z, \bar{z}) = :e^{i\alpha\Phi}: V'_\alpha(z) \bar{V}'_\alpha(\bar{z}), \quad (1455)$$

⁴⁴We could instead use $\mathbf{1} - (-1)^F$ in the second trace as well. This \pm ambiguity is due to the fact that the term it appears in is the *RR* spin structure term, which is zero (the ambiguity relates to the two degenerate states on the *RR* torus which differ by whether or not the zero mode is filled)

and show that the zero mode is

$$\Phi(z, \bar{z}) = \phi_0 - ia_0 \ln(z\bar{z}) \quad (1456)$$

while the $V'_\alpha(z)$'s are

$$V'_\alpha(z) = :e^{i\alpha\phi'(z)} := \exp\left(-\alpha \sum_{n>0} \frac{1}{n} a_{-n} z^n\right) \exp\left(\alpha \sum_{n>0} \frac{1}{n} a_n z^{-n}\right), \quad (1457)$$

and likewise for the $V'_\alpha(\bar{z})$'s. Here, $\phi'(z)$ denotes the holomorphic part of ϕ with the zero mode removed.

Find the $\langle \phi'(z)\phi'(w) \rangle$ correlator, and find the n -pt correlation function of the $V'_\alpha(z)$'s. Find the n -point correlation function of the $:e^{i\alpha\phi'(z,\bar{z})}:$ zero mode vertex operators, and use these results to find the correlators for the full vertex operators $\mathcal{V}_\alpha(z, \bar{z})$.

Solution:

First lets review the mode expansion. On a cylinder with circumference L , we do $\phi = \sum_n \phi_n e^{2\pi i x/L}$. Finding the Hamiltonian is straightforward:

$$H = \frac{2\pi}{L} \sum_{n \in \mathbb{Z}} (\pi_n \pi_{-n} + \frac{n^2}{4} \phi_n \phi_{-n}), \quad \pi_n = \frac{L}{4\pi} \partial_t \phi_{-n}. \quad (1458)$$

We can solve the Hamiltonian by introducing oscillators a_n . We will work with the conventions in the big yellow book, so that for $n \neq 0$,

$$\phi_n = \frac{i}{n} (a_n - \bar{a}_{-n}), \quad [a_n, a_m] = n\delta_{n+m}. \quad (1459)$$

Thus ϕ is decomposed as

$$\phi(x) = \phi_0 + i \sum_{n \neq 0} \frac{1}{n} (a_n - \bar{a}_n) e^{2\pi i n x / L}. \quad (1460)$$

We commutate this with the Hamiltonian to get the time dependence, which is straightforward. In particular, the time dependence of the zero mode is $\phi_0 + \frac{4\pi}{L} \pi_0 t$. As we have done in the last couple of days, we let $z = e^{2\pi(\tau - ix)/L}$, $\bar{z} = e^{2\pi(\tau + ix)/L}$ where $\tau = it$. With these conventions,

$$t = -i \frac{L}{4\pi} \ln(z\bar{z}). \quad (1461)$$

Putting this in and writing a_0 for π_0 , the decomposition for ϕ is

$$\phi(z, \bar{z}) = \Phi(z, \bar{z}) + i \sum_{n \neq 0} \frac{1}{n} (a_n z^{-n} + \bar{a}_n \bar{z}^{-n}), \quad (1462)$$

where the zero mode part is

$$\Phi(z, \bar{z}) = \phi_0 - ia_0 \ln(z\bar{z}). \quad (1463)$$

Now define the field

$$\phi'(z) = \sum_{n \neq 0} \frac{1}{n} a_n z^{-n} \quad (1464)$$

to be the holomorphic part of $\phi(z, \bar{z})$ with the zero mode removed. We write its two point function as

$$\langle \phi'(z) \phi'(w) \rangle = \left\langle \sum_{m < 0, n > 0} \frac{1}{nm} a_n a_m z^{-n} w^{-m} \right\rangle, \quad (1465)$$

since a_n with $n > 0$ annihilates the vacuum (remember that a_n with $n < 0$ act as the usual creation operators). Using the commutator, we can set $m = -n$ and get

$$\langle \phi'(z) \phi'(w) \rangle = \sum_{n > 0} \frac{1}{n} \frac{w^n}{z^n} = -\ln \left(1 - \frac{w}{z} \right). \quad (1466)$$

Now we can get the correlator of a product of $V'_\alpha(z) =: e^{i\alpha\phi'(z)} :$ operators. To do the normal ordering we just need to move the creation operators (a_n with $n < 0$) to the left of the annihilation operators, and so

$$V'_\alpha(z) = : e^{i\alpha\phi'(z)} : = \exp \left(-\alpha \sum_{n > 0} \frac{1}{n} a_{-n} z^n \right) \exp \left(\alpha \sum_{n > 0} \frac{1}{n} a_n z^{-n} \right), \quad (1467)$$

To get the correlators involving a product of $V'_\alpha(z) =: e^{i\alpha\phi'(z)} :$ operators, we need to know how to normal-order the product. To do this, we use

$$e^A e^B = e^B e^A e^{[A,B]}, \quad \text{if } [A, B] \in \mathbb{C}. \quad (1468)$$

The commutator of the terms appearing in the exponentials in the definition of the $V'_\alpha(z)$ is a c-number, so we can use this formula. Applying this to move all the $a_{-n}, n > 0$ operators to the left in the product lets us figure out how to do the normal ordering. A bit of algebra gives (see e.g. the big yellow book, appendix 6.A)

$$\left\langle \prod_j V'_{\alpha_j}(z_j) \right\rangle = \left\langle : \prod_j V'_{\alpha_j}(z_j) : \right\rangle \exp \left(- \prod_{i < k} \alpha_i \alpha_k \langle \phi'(z_i) \phi'(z_k) \rangle \right). \quad (1469)$$

Using the correlator that we just derived and the fact that the fully normal-ordered term is equal to 1, we get

$$\left\langle \prod_j V'_{\alpha_j}(z_j) \right\rangle = \prod_{j < k} \left(1 - \frac{z_k}{z_j} \right)^{\alpha_j \alpha_k}. \quad (1470)$$

Putting this together with the antiholomorphic piece, we have

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \left\langle \prod_j : e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle \prod_{j < k} |z_j - z_k|^{2\alpha_j \alpha_j} |z_j|^{-2\alpha_j \alpha_k}. \quad (1471)$$

So, we just need the zero mode part. Remembering that $\Phi(z, \bar{z})$ is formed from a linear combination of position and momenta operators for the zero mode, we can use the same

approach we used above to do the normal-ordering. We regard a_0 as the annihilation operator and ϕ_0 as the creation operator, so that to normal-order we need to shuffle the a_0 's to the left. This gives

$$\begin{aligned} \left\langle \prod_j : e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle &= \left\langle : \prod_j e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle \exp \left(\sum_{i < k} [-ia_0, \phi_0] \alpha_i \alpha_k \ln(z_i \bar{z}_i) \right) \\ &= \left\langle : \prod_j e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle \prod_{i < k} |z_i|^{2\alpha_i \alpha_k}. \end{aligned} \quad (1472)$$

Unlike with the non-zero-mode operators, the fully normal-ordered part actually does something. The vacua are parametrized by the value of the zero mode. So we work with the coherent vacua $|\beta\rangle$ such that $a_0|\beta\rangle = \beta|\beta\rangle$. Then we see that

$$a_0 e^{-i\alpha\phi_0} |\beta\rangle = i \frac{\delta}{\delta\phi_0} e^{-i\alpha\phi_0} |\beta\rangle = e^{-i\alpha\phi_0} (\alpha + a_0) |\beta\rangle = (\alpha + \beta) e^{-i\alpha\phi_0} |\beta\rangle, \quad (1473)$$

so that $e^{-i\alpha\phi_0} |\beta\rangle = |\beta + \alpha\rangle$ shifts us between different vacua (think of electric flux operators). In the fully normal-ordered piece above, all the ϕ_0 operators stand to the left of the a_0 operators, and so we can let them act directly on the left vacuum bra. If our vacuum state is $|\beta\rangle$, then

$$\left\langle : \prod_j e^{i\alpha_j \Phi(z_j, \bar{z}_j)} : \right\rangle = e^{i \sum \alpha_i \beta} \langle \beta | \gamma \rangle = \delta_{\sum_i \alpha_i, 0}, \quad \gamma = \beta + \sum_j \alpha_j, \quad (1474)$$

and so the zero mode implements the charge-neutrality condition for us.

Putting everything together, we have

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \delta_{\sum_i \alpha_i, 0} \prod_{j < k} |z_j - z_k|^{2\alpha_j \alpha_k}. \quad (1475)$$

We can use this result to figure out the OPE of two vertex operators. We have

$$: e^{i\alpha\phi(z)} :: e^{i\beta\phi(w)} :=: e^{i(\alpha\phi(z) + \beta\phi(w))} : |z - w|^{2\alpha\beta}. \quad (1476)$$

Note that the RHS is non-zero even when $\alpha + \beta \neq 0$. It needs to be non-zero since we need to be able to take OPEs of vertex operators that don't satisfy charge neutrality and produce a non-zero result, since e.g. the product $\mathcal{V}_1 \mathcal{V}_1 \mathcal{V}_{-2}$ has a non-zero vev but can be evaluated by first performing the OPE on the first two factors. Charge neutrality enters when we take the vev of the above equation, as the vev of the fully-normal-ordered part on the RHS actually vanishes when charge neutrality is not satisfied:

$$\langle : e^{i(\alpha+\beta)\phi} : \rangle = \langle e^{i\Phi(\alpha+\beta)} \rangle \langle : e^{i(\alpha+\beta) \sum_{n \neq 0} \phi_n e^{2\pi i n x/L}} : \rangle = \langle e^{i\Phi(\alpha+\beta)} \rangle = \delta_{\alpha+\beta, 0}, \quad (1477)$$

where we have used that the normal-ordered exponential of the non-zero modes of ϕ is equal to 1, and the fact that Φ and $\phi_{n \neq 0}$ commute.

84 July 7 — Vertex correlators II

Today we have another quick and easy exercise from the big yellow book, chapter 9, which is the functional way of getting to the result of yesterday's diary entry.

Consider the real-space propagator

$$K(x, y) = -\ln(m^2[(x-y)^2 + a^2]), \quad (1478)$$

which is the free boson correlator regulated by a mass m (long-distance cutoff) and a lattice spacing a (short-distance cutoff).

Compute the vertex correlation function with functional methods and show that you get the result we obtained yesterday at the conformal point where $m, a \rightarrow 0$.

Solution:

We use

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = Z[J] = Z[0] \exp \left(-\frac{1}{2} \int_{x,y} J(x) K(x, y) J^\dagger(y) \right), \quad (1479)$$

where the relevant current for us is

$$J(x) = i \sum_j \alpha_j \delta(x - x_j). \quad (1480)$$

Thus

$$Z[J] = \exp \left(\frac{1}{2} \sum_{j,k} \alpha_j \alpha_k \left(\ln(ma)^2 + \ln \left[\frac{|z_j - z_k|^2}{a^2} + 1 \right] \right) \right). \quad (1481)$$

Since we are interested in sending $a^2 \rightarrow 0$, we will take $|z_j - z_k|^2 \gg a^2$ for all $j \neq k$, allowing us to get rid of the $+1$ in the \ln unless $j = k$, in which case the \ln vanishes. So then

$$\begin{aligned} Z[J] &= (ma)^{\sum_j \alpha_j)^2} \prod_{i < k} \left(\frac{|z_i - z_k|^2}{a^2} \right)^{\alpha_i \alpha_k} \\ &= m^{(\sum_j \alpha_j)^2} a^{\sum_l \alpha_l^2} \prod_{i < k} |z_i - z_k|^{2\alpha_i \alpha_k}. \end{aligned} \quad (1482)$$

We see that when we take $m \rightarrow 0$, $Z[J]$ vanishes unless $\sum_j \alpha_j = 0$. Even if we have charge neutrality, we still have the prefactor of $a^{\sum_l \alpha_l^2}$. This isn't a problem though, since it factors as $\prod_l a^{\alpha_l^2}$, so that if we renormalize the vertex operators with the short-distance cutoff by

$$\tilde{\mathcal{V}}_\alpha(z, \bar{z}) \equiv a^{-\alpha^2} \mathcal{V}_\alpha(z, \bar{z}), \quad (1483)$$

then we have

$$\left\langle \prod_j \tilde{\mathcal{V}}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \delta_{\sum_i \alpha_i, 0} \prod_{j < k} |z_j - z_k|^{2\alpha_j \alpha_k}, \quad (1484)$$

which is the same correlator that we found yesterday. In fact, our need to renormalize the vertex operators in the above way is not surprising, and is equivalent to the statement that the vertex operators have anomalous dimension α^2 . From the 2-point function $\langle \tilde{\mathcal{V}}_\alpha(z_1, \bar{z}_1) \tilde{\mathcal{V}}_\alpha^\dagger(z_2, \bar{z}_2) \rangle \sim |z_1 - z_2|^{-2\Delta_\alpha}$ we see that the scaling dimension of the vertex operator $\tilde{\mathcal{V}}_\alpha$ is $\Delta_\alpha = \alpha^2$. Thus the renormalization defined above lets us go between operators with dimensionless two-point functions (\mathcal{V} 's) to those with two point functions whose dimensions give the scaling dimension.

85 July 8 — Vacuum energy and boundary conditions

Today is another quickie: deriving a statement made in Ginzburg's lectures on CFT about vacuum energies.

Consider a free Dirac fermion, with boundary conditions on the cylinder such that

$$\psi(\sigma) = e^{2\pi i \gamma} \psi(\sigma + 2\pi), \quad \gamma \in [0, 1], \quad (1485)$$

where σ is the spatial coordinate of the (radius 1) cylinder. Show that the vacuum energy is given by

$$E_0 = \frac{1}{12} - \frac{1}{2}\gamma(1 - \gamma). \quad (1486)$$

Solution:

There are a few ways to do this problem. The first way is to use ζ function regularization to normal-order the L_0 operators on the cylinder. First, we note that since our complex fermion has the OPEs

$$\psi^\dagger(z)\psi^\dagger(w) \sim \psi(z)\psi(w) \sim 0, \quad \psi^\dagger(z)\psi(w) \sim \frac{1}{z-w}, \quad (1487)$$

if we write $\psi(z) = \frac{1}{\sqrt{2}}(\lambda(z) + i\eta(z))$ then we must have

$$\lambda(z)\lambda(w) \sim \eta(z)\eta(w) \sim \frac{1}{z-w}, \quad \lambda(z)\eta(w) \sim 0, \quad (1488)$$

i.e. the OPE forces the two Majoranas making up the Dirac fermion to decouple. Thus the energy momentum tensor and importantly for us the vacuum energy contribution to the dilatation generator can be obtained just by taking the answer for a single Majorana fermion and multiplying by 2. Thus on the cylinder we have

$$L_0 = \frac{1}{2} \sum_n n(:\lambda_{-n}\lambda_n:+:\eta_{-n}\eta_n:) = \sum_{n>0} (\lambda_{-n}\lambda_n + \eta_{-n}\eta_n) - \sum_{n>0} n, \quad n \in \mathbb{Z} + \gamma. \quad (1489)$$

The last constant part is what shifts the vacuum energy density. We evaluate it with ζ function regularization. Letting

$$\zeta(q, r) = \sum_{n=0}^{\infty} (n+r)^q, \quad (1490)$$

we see that we need to evaluate $\zeta(-1, \gamma)$. Luckily this is easily looked up:

$$\zeta(-1, \gamma) = -\frac{1}{2}(\gamma^2 - \gamma + 1/6), \quad (1491)$$

and so the vacuum energy is evidently

$$E_0 = -\zeta(-1, \gamma) = \frac{1}{12} - \frac{1}{2}\gamma(1 - \gamma), \quad (1492)$$

as predicted.

The second, more rigorous way that doesn't use the ζ function is to find $\langle T \rangle$ directly using the mode expansion on the Dirac fermion. We decompose the Dirac fermion in a mode expansion as

$$\psi(w) = \sum_{k \in \mathbb{Z}^{>0} + \gamma} (\alpha_k e^{-kw} + \beta_k^\dagger e^{kw}), \quad (1493)$$

where $w = \tau - ix$. This mode expansion comes from doing the expansion for the fermion on the circle and then getting the time dependence by commuting with the Dirac Hamiltonian. Notice that we are only summing over *positive* k in the above: here $\alpha_k = c(-k_F - k)$ destroys a left-moving particle while $\beta_k^\dagger = c(-k_F + k)$ creates a left-moving hole. The corresponding anitholomoprhic guy is the same thing but with right-moving momenta and coordinates:

$$\bar{\psi}(w) = \sum_{k \in \mathbb{Z}^{<0} + \gamma} (\alpha_k e^{k\bar{w}} + \beta_k^\dagger e^{-k\bar{w}}). \quad (1494)$$

The signs in the exponents come from requiring the time dependence $e^{-\tau|k|}$, so that when $k < 0$ we need to invert the sign of $\bar{w} = \tau + ix$ in the exponent.

The modes satisfy the algebra $\{\alpha_k^\dagger, \alpha_l\} = \delta_{k,l}$; same for the β modes. Note that unlike the real fermions, there is no $\psi_0^2 = 1/2$ mode to worry about since α and α^\dagger are distinct. When we map these guys into the plane, we take $z = e^w$ and multiply by a factor of $(dz/dw)^{-h} = z^{-1/2}$ since the fermions have conformal dimension $h = 1/2$ to get

$$\psi(z) = \sum_{k \in \mathbb{Z}^{>0} + \gamma} (\alpha_k z^{-k-1/2} + \beta_k^\dagger z^{k-1/2}). \quad (1495)$$

We can now calculate the expectation value $\langle \psi^\dagger(z)\psi(w) \rangle$:

$$\begin{aligned} \langle \psi^\dagger(z)\psi(w) \rangle &= \sum_{k,l \in \mathbb{Z}^{>0} + \gamma} z^{-k-1/2} w^{l-1/2} \langle \beta_k \beta_l^\dagger \rangle = \sum_{k \in \mathbb{Z}^{>0} + \gamma} z^{-k-1/2} w^{k-1/2} \\ &= \frac{1}{\sqrt{wz}} w^\gamma z^{-\gamma} \frac{z}{z-w}, \end{aligned} \quad (1496)$$

since when we complex conjugate z^x we get z^{-x} for $x \in \mathbb{R}$ since z is actually the exponential of a purely imaginary number (and so $z^* \neq \bar{z}$ with these conventions unfortunately, only when we pretend that τ is real). Also, here w is now a coordinate on the plane, and is *not* the earlier w , which was the cylinder coordinate. Sorry! Also note that regardless of γ , unlike with \mathbb{R} fermions we don't have to worry about treating a zero mode separately.

Anyway, we have

$$\partial_z \langle \psi^\dagger(z) \psi(w) \rangle = (-\gamma + 1/2) \frac{w^{\gamma-1/2} z^{-\gamma-1/2}}{z-w} - \frac{w^{\gamma-1/2} z^{-\gamma+1/2}}{(z-w)^2}, \quad (1497)$$

and similarly for the derivative wrt w . Now the holomorphic stress tensor has the expectation value

$$T(w) = \frac{1}{2} \lim_{z \rightarrow w} \left\langle \partial_z \psi^\dagger(z) \psi(w) - \psi^\dagger(z) \partial_w \psi(w) \right\rangle + \frac{1}{\epsilon^2}, \quad (1498)$$

where we've subtracted $2 \cdot \frac{1}{2} \partial_z \frac{1}{z-w}$ evaluated at $z = w + \epsilon$ in line with the usual normal ordering prescription. Putting in our expressions for the derivatives,

$$\langle T(w) \rangle = \frac{1}{2} \lim_{z \rightarrow w} \left(\frac{1/2 - \gamma}{z-w} (z^{-1/2-\gamma} w^{\gamma-1/2} + z^{1/2-\gamma} w^{\gamma-3/2}) - 2 \frac{z^{1/2-\gamma} w^{\gamma-1/2}}{(z-w)^2} \right) + \frac{1}{\epsilon^2}. \quad (1499)$$

Evaluating the term in the parenthesis for $z = w + \epsilon$ we find that it is equal to

$$\frac{1}{2} \lim_{z \rightarrow w} (\dots) = -\frac{1}{\epsilon^2} + \frac{1 - 4\gamma - 4\gamma^2}{8w^2} + O(\epsilon). \quad (1500)$$

The singular part cancels the $1/\epsilon^2$ introduced by the normal ordering, and since we are dealing with free fields there are no more singular parts leftover to be cancelled, leaving only the $1/w^2$ piece. Thus

$$\langle T(z) \rangle = \frac{1 - 4\gamma - 4\gamma^2}{8z^2}. \quad (1501)$$

Sanity check: when $\gamma = 1/2$ so that we have the normal anti-periodic boundary conditions for the fermions on the cylinder, we have $\langle T(z) \rangle = 0 \checkmark$.

Now let's go over to the cylinder. The holomorphic (left-moving) part of the Hamiltonian is found by (now w is the cylinder coordinate again—jeez this is awful notation)

$$H_L = \frac{1}{2\pi i} \int dw T(w) = \frac{1}{2\pi i} \int dz z^{-1} T(w) = \frac{1}{2\pi i} \int dz z^{-1} (z^2 T(z) - 1/24), \quad (1502)$$

where we used the transformation rule for T we derived earlier (with coordinates $z = e^w$) and put in the central charge $c = 1$. We see that the $n = 0$ mode of the Laurent expansion for T is selected out, which picks up the extra piece contributing to $\langle T(z) \rangle$ that we found above. So

$$E_0 = \langle L_0 \rangle + \frac{1 - 4\gamma - 4\gamma^2}{8} - \frac{1}{24} = \frac{1}{12} - \frac{1}{2}\gamma(\gamma - 1), \quad (1503)$$

since $\langle L_0 \rangle = 0$ as L_0 is a sum of normal-ordered oscillators. We see that this approach gives exactly the same vacuum energy that we derived using ζ function regularization! Cool.

86 July 9 — Fermion partition functions on the torus: the functional approach

Today is a check on our understanding of fermion path integrals on the torus: the goal is to confirm some statements in Ginzburg's CFT lectures.

For a free Majorana fermion on a torus with a given spin structure, find $Z(q)$, where $q = e^{2\pi i \tau}$ and τ is the modular parameter. Do so using functional methods and ζ function regularization methods (the preferred regulator since it respects modular invariance) rather than using the operator approach. You only need to reproduce the q dependence; don't worry about constants and stuff.

Solution:

We start from

$$Z_{XY}(q) = \text{Pf}_{XY}(\partial) \text{Pf}_{XY}(\bar{\partial}) = \sqrt{\det_{XY}(\partial\bar{\partial})}, \quad (1504)$$

where $X, Y \in \{NS, R\}$ are spin structure labels. The (un-normalized) eigenfunctions of $\partial\bar{\partial}$ are

$$\psi_{n,m}(\alpha, \beta) = \exp\left(\frac{2\pi i}{2i\text{Im}(\tau)} [(n + \alpha)(z - \bar{z}) + (m + \beta)(\tau\bar{z} - \bar{\tau}z)]\right), \quad \alpha, \beta \in \{0, 1/2\}, \quad (1505)$$

where we have assumed that the spacelike edge of the torus stretches from 0 to 1 in the complex plane. One checks that

$$\begin{aligned} (z \mapsto z + 1) : \psi_{n,m}(\alpha, \beta) &\mapsto (-1)^{2\beta} \psi_{n,m}(\alpha, \beta) \\ (z \mapsto z + \tau) : \psi_{n,m}(\alpha, \beta) &\mapsto (-1)^{2\alpha} \psi_{n,m}(\alpha, \beta), \end{aligned} \quad (1506)$$

so that α sets the timelike boundary conditions and β sets the spacelike boundary conditions.

From the eigenvalues we get that

$$\det_{XY}(\partial\bar{\partial}) = \prod_{n,m} \frac{\pi^2}{\text{Im}(\tau)^2} |n + \alpha - (m + \beta)\tau|^2. \quad (1507)$$

We will ignore the $\pi^2/\text{Im}(\tau)^2$ part, which by using ζ function regularization ends up contributing something with τ dependence $\sqrt{\text{Im}(\tau)}$ to the partition function. This is important for maintaining modular invariance—we won't worry about it now, but will just remember that we need to re-instate this bit if we want to get something modular invariant.

First consider the case where $X, Y = R, R$, i.e. $\alpha = \beta = 0$. Then from the zero mode in the partition function, we get $Z_{RR}(q) = 0$ as expected. What about $X, Y = NS, R$, i.e. $\alpha, \beta = 1/2, 0$? This is periodic in space and anti-periodic in time. We have

$$\det_{NS,R}(\partial\bar{\partial}) \propto \prod_{n \in \mathbb{Z}} (n + 1/2)^2 \prod_{n,m \in \mathbb{Z}, m \neq 0} |n + 1/2 - m\tau|^2. \quad (1508)$$

Now we need

$$\prod_{n \in \mathbb{Z}} (n + x) = e^{i\pi x} - e^{-i\pi x}, \quad (1509)$$

so that the first product is just a constant, and so after some algebra (combining the products for $m > 0$ and $m < 0$),

$$\det_{NS,R}(\partial\bar{\partial}) \propto \prod_{m \in \mathbb{Z}^{>0}} |q^{-m}(1 + q^m)^2|^2. \quad (1510)$$

Now we use

$$\prod_{m>0} q^{-m} = e^{-\zeta(-1) \ln q} = q^{1/12}, \quad (1511)$$

so that, taking the square root to get the partition function,

$$Z_{NS,R}(q) \propto (q\bar{q})^{1/24} \prod_{m \in \mathbb{Z} \geq 0} |1 + q^m|^2. \quad (1512)$$

Now for $X, Y = R, NS$ i.e $\alpha, \beta = 0, 1/2$. Then

$$\det_{R,NS}(\partial\bar{\partial}) \propto \prod_{n \in \mathbb{Z}} |n - \tau/2|^2 \prod_{n,m \in \mathbb{Z}, m \neq 0} |n - (m + 1/2)\tau|^2. \quad (1513)$$

The first product gives us

$$\prod_{n \in \mathbb{Z}} |n - \tau/2|^2 = |q^{-1/4}(1 - q^{1/2})|^2. \quad (1514)$$

The second product is

$$\prod_{n,m \in \mathbb{Z}, m \neq 0} |n - (m + 1/2)\tau|^2 = \prod_{m \in \mathbb{Z} > 0} |q^{-(m+1/2)/2} q^{-(m-1/2)/2} (1 - q^{m+1/2})(1 - q^{m-1/2})|^2. \quad (1515)$$

We can change the $(1 - q^{m-1/2})$ to a $(1 - q^{m+1/2})$ by also including a factor of $(1 - q^{1/2})$, which combines with the factor in the previous product to produce an $m = 0$ term in the full product. Taking everything together and taking the square root, we get

$$Z_{R,NS}(\partial\bar{\partial}) \propto (q\bar{q})^{-1/48} \prod_{m \in \mathbb{Z} \geq 0} |1 - q^{m+1/2}|^2, \quad (1516)$$

which is exactly what we expect: the periodic boundary conditions in time give us a supertrace by sending $q \rightarrow -q$, while we have the half-odd-integer momenta needed for antiperiodic boundary conditions in space.

Finally for $\alpha = \beta = 1/2$, the $NSNS$ spin structure. We split up the product into $m = 0$ and $m \neq 0$ parts as before. The $m = 0$ part is

$$\prod_{n \in \mathbb{Z}} |n + 1/2 - \tau/2|^2 = |q^{-1/4}(1 + q^{1/2})|^2. \quad (1517)$$

The $m \neq 0$ part is dealt with as before: the fact that $\alpha = 1/2$ means that we get a trace instead of a supertrace, and the same sort of algebra leads to

$$Z_{NS,NS}(\partial\bar{\partial}) \propto (q\bar{q})^{-1/48} \prod_{m \in \mathbb{Z} \geq 0} |1 + q^{m+1/2}|^2. \quad (1518)$$

87 July 10 — Orbifolding basics

Today is an elaboration on part of a problem in the big yellow book, chapter 10. Show that on the torus,

$$Z_{\text{Ising}}^2 = Z_{\text{orb}}(R=1), \quad (1519)$$

where Z_{Ising} is the partition function of a single majorana fermion and $Z_{\text{orb}}(R=1)$ is the free boson at compactification radius $R=1$ orbifolded under the $\mathbb{Z}_2 X \mapsto -X$ symmetry.

Solution:

Let's first write down Z_{Ising} , which is easy using our knowledge from the previous few diary entries. In addition to the trace of the $q^{L_0}\bar{q}^{\bar{L}_0}$ part, we need to know the vacuum energy contribution to partition function (i.e. the part that is $(q\bar{q})^{-c/24}$ if boundary conditions are not an issue). We find this contribution with the regularization

$$-\frac{1}{2} \sum_{n \in \mathbb{Z} + \alpha} = \frac{1}{24} - \frac{\alpha}{4}(1 - \alpha), \quad (1520)$$

which we essentially derived a few days ago (it comes from doing the normal-ordering in the oscillator sum in the Hamiltonian. This is done *on the torus*, not on the plane, and so the modding is not shifted by $1/2$). If the boundary conditions in space are anitperiodic then the fermion is modded in $\mathbb{Z} + 1/2$ on the torus, and hence we can take $\alpha = 1/2$ for NS spatial boundary conditions, giving a contribution of $-1/48$. If the boundary conditions are periodic, then we just set $\alpha = 0$ so that we get a $+1/24$ contribution. So then since each spin structure factors as holomorphic and antiholomorphic parts, we can just write down (remember that the the RR spin structure gives zero)

$$Z_{\text{Ising}} = \frac{1}{2} \left(\left| \frac{\theta_2}{\eta(q)} \right| + \left| \frac{\theta_3}{\eta(q)} \right| + \left| \frac{\theta_4}{\eta(q)} \right| \right), \quad (1521)$$

where

$$\begin{aligned} \frac{\theta_2}{\eta(q)} &= \frac{1}{\sqrt{2}} q^{1/24} \prod_{n \in \mathbb{Z} \geq 0} (1 + q^n), \\ \frac{\theta_3}{\eta(q)} &= q^{-1/48} \prod_{n \in \mathbb{Z} \geq 0} (1 + q^{n+1/2}), \\ \frac{\theta_4}{\eta(q)} &= q^{-1/48} \prod_{n \in \mathbb{Z} \geq 0} (1 - q^{n+1/2}), \end{aligned} \quad (1522)$$

and the η function is

$$\eta(q) = q^{1/24} \prod_{n \in \mathbb{Z} > 0} (1 - q^n). \quad (1523)$$

The $1/\sqrt{2}$ factor is the quantum dimension of the σ primary field and is inserted to have the usual expression for the modular S matrix (note to self: haven't actually tracked this $1/\sqrt{2}$ around to see if this is correct).

Now for the orbifolded boson under the \mathbb{Z}_2 symmetry taking $X \rightarrow -X$. Let's recall how we arrive at the partition function. We want to "gauge" the \mathbb{Z}_2 symmetry (better: we want to do a Fourier transformation) by projecting the Hilbert space onto states which are singlets under the \mathbb{Z}_2 , which we do with the operator $(\mathbf{1} + (-1)^X)/2$, where $(-1)^X$ is our dumb way of writing the operator which does the symmetry action on X (it inserts a \mathbb{Z}_2 symmetry defect that wraps around the spatial cycle of the torus and changes the temporal boundary conditions). Inserting this operator (the tube algebra Hamiltonian) ruins modular invariance since it's asymmetrical under S , and so we have also have to sum over spatial boundary conditions by inserting defects that wrap the temporal cycle. We end up summing over all (four) topologically distinct way to place \mathbb{Z}_2 symmetry defect lines on the torus (for us, they are added with no relative phases between them).

The torus with no symmetry defects just gives a contribution of $Z(R)/2$, the regular unorbifolded version. The torus with a symmetry defect inserted which wraps the spatial cycle gives the holomorphic contribution

$$\frac{q^{-1/24}}{2} \text{Tr}_P [(-1)^X q^{L_0}]. \quad (1524)$$

Here the trace is done with periodic boundary conditions in space, and so we have to take into account zero modes. The basis states for \mathcal{H}_0 (the zero-mode part of the Hilbert space) that diagonalize the action of $(-1)^X$ are $|k, w\rangle \pm | -k, -w\rangle$ (k is momentum, w is winding) for $k, w \in \mathbb{Z}$, since $(-1)^X$ acts as -1 on the momenta and winding numbers. However, recall that the k, w dependence of the spectrum of L_0 comes from the term $\frac{1}{2}p_L^2$, where $p_L = (k/R + wR/2)$ (for \bar{L}_0 , send $w \mapsto -w$). Thus the spectrum of L_0 is unchanged under the action of $(-1)^X$, and so all the terms in the trace above with the exception of $|0, 0\rangle$ die in pairs. Thus we don't even have to worry about the zero modes, and so taking the trace over the nonzero oscillator modes in the usual way, we get the holomorphic part

$$\frac{q^{-1/24}}{2} \prod_{n \in \mathbb{Z}_{>0}} \frac{1}{1 + q^n}. \quad (1525)$$

Note that the $+$ sign in the denominator comes from the action of $(-1)^X$ on the left bra in the trace.

The torus with a symmetry defect inserted along the temporal cycle just changes the modding of the boson modes to be in $\mathbb{Z} + 1/2$, since it changes the spatial boundary conditions to be antiperiodic. Thus there are no zero modes to worry about, and the only subtlety is the change in the vacuum energy contribution due to the altered boundary conditions. The relevant normal-ordering result for bosons is

$$\frac{1}{2} \sum_{n \in \mathbb{Z} + \alpha} n = \frac{1}{2} \left(-\frac{1}{12} + \frac{\alpha}{2}(1 - \alpha) \right), \quad (1526)$$

which gives us the $-1/24$ for periodic boundary conditions that we used before, and which gives $+1/48$ for antiperiodic boundary conditions (just like for the fermions, but with opposite signs). So then we get the holomorphic term

$$\frac{2q^{1/48}}{2} \prod_{n \in \mathbb{Z}_{>0}} \frac{1}{1 - q^n}. \quad (1527)$$

Here, the factor of 2 in the numerator comes from the fact that we have two vacua which have identical L_0 eigenvalues (± 1 eigenvalues under $(-1)^X$). The 2 in the denominator comes from the $1/2$ in the projector onto \mathbb{Z}_2 -invariant states.

Finally, the torus with defects wrapped around both cycles: this is just like the above, except with $+q^n$ in the denominator, with the sign coming from the $(-1)^X$ acting on the oscillator modes in the trace. So we get the term

$$\frac{2q^{1/48}}{2} \prod_{n \in \mathbb{Z}^{>0}} \frac{1}{1+q^n}. \quad (1528)$$

Adding everything together, remembering to square everything because of the antiholomorphic parts, and using the definitions of the θ functions and the η function, we get (note: this is in TBYB, but I think the formula there has some incorrect factors of 2?)

$$Z_{orb}(R) = \frac{1}{2} \left(Z(R) + 2 \left| \frac{\eta(q)}{\theta_2} \right| + 2 \left| \frac{\eta(q)}{\theta_3} \right| + 2 \left| \frac{\eta(q)}{\theta_4} \right| \right). \quad (1529)$$

Note that only the first of the four terms actually depends on the compactification radius R , since it is the only term in which the presence of the zero modes contributed something nonzero to the partition function.

Specializing to $R = 1$, and recalling our result from a few days ago about the duality between Dirac fermions and the $R = 1$ boson, we have

$$Z(R = 1) = Z_{\text{Dirac}} = \frac{1}{2} \left(\left| \frac{\theta_2}{\eta} \right|^2 + \left| \frac{\theta_3}{\eta} \right|^2 + \left| \frac{\theta_4}{\eta} \right|^2 \right). \quad (1530)$$

Now we need to make use of

$$\eta(q)^3 = \frac{1}{2} \theta_2 \theta_3 \theta_4, \quad (1531)$$

which we use to substitute $\eta = \theta_2 \theta_3 \theta_4 / (2\eta^2)$ in for the η 's in $Z_{orb}(R = 1)$. We get

$$Z_{orb}(R = 1) = \frac{1}{4} \left(\left| \frac{\theta_2}{\eta} \right| + \left| \frac{\theta_3}{\eta} \right| + \left| \frac{\theta_4}{\eta} \right| \right)^2 = Z_{\text{Ising}}^2, \quad (1532)$$

which is what we wanted to show. Thus orbifolding lets us “decouple” the two majorana fermions making up a Dirac fermion from one another and allows them to have separate spin structures.

One comment on orbifolds of orbifolds (for G finite and Abelian). The orbifold of an orbifold is the original theory, essentially because the Fourier transform is an involution. To orbifold an orbifold, we sum over all ways of twisting the orbifold, where an orbifold twisted by some function $\beta : G \rightarrow U(1)$ means

$$Z_{orb}^\beta = \frac{1}{|G|} \sum_{g,h} \beta(h) Z_g^h, \quad (1533)$$

where Z_g^h denotes the torus with a g twist in the spatial direction and an h twist in the temporal direction. To get something modular invariant, we then need to write

$$Z_{orb}^\omega = \frac{1}{|G|} \sum_{g,h} \omega(g, h) Z_g^h, \quad (1534)$$

where $\omega(g, h) = \beta(g)\beta(h)$. To orbifold the orbifold we sum over all such functions $\omega(g, h)$, which is the dual version (in the sense of $\text{Rep} \leftrightarrow \text{Vec}$ duality) of projecting onto a G -singlet state. Thus the procedure looks like

$$Z \xrightarrow{\text{orbifold}} \frac{1}{|G|} \sum_{g,h} Z_g^h \xrightarrow{\text{orbifold the orbifold}} \frac{1}{|G|^2} \sum_{\omega \in \text{Rep}^2(G)} \sum_{g,h} \omega(g, h) Z_g^h = Z, \quad (1535)$$

so that orbifolding squares to the identity (remember that we are assuming G is finite and Abelian so that Rep is in bijection with G). Recall that since we are summing over genuine representations here, the 2-cocycle $\omega(g, h) = \beta(g)\beta(h)$ for $\beta \in \text{Rep}(G)$ is exact. That this works is easy to explicitly verify e.g. for the \mathbb{Z}_2 case we've been considering, with the last equality in the above chain holding because the Fourier transform is involutive. If $G = \mathbb{R}$ then we can telegraphically illustrate this as

$$Z(x=0) \xrightarrow{\text{orbifold}} \int dx Z(x) \xrightarrow{\text{orbifold the orbifold}} \int dx \int dk e^{ikx} Z(x) = Z(x=0), \quad (1536)$$

showing that orbifolding is an involution. Here the orbifold is $Z(k=0)$ (which is a singlet under translations).

88 July 11 — Cardy formula

This is a basic result that I'd seen a few times but had never derived. Show that at high energies, modular invariance implies that

$$\ln \rho(E) = \sqrt{\frac{2\pi E c}{3}}, \quad (1537)$$

where c is the central charge. To get here, work on a torus with modular parameter large and purely imaginary. **Solution:**

We will work on a torus with modular parameter $\tau = i\beta \in i\mathbb{R}$, with the real leg of the torus stretching from 0 to 1 as usual. We will take the “low temperature” limit of $\beta \gg 1$ so that the torus is very stretched out.

Since $\beta \gg 1$ and $\bar{\tau} = -\tau$, we have $q = \bar{q} = e^{2\pi i\tau} = q^{-2\pi\beta} \rightarrow 0$. The partition function is (assuming $c = \bar{c}$)

$$Z(q) = q^{-c/12} \text{Tr}[q^{L_0 + \bar{L}_0}]. \quad (1538)$$

Since $q \rightarrow 0$, we can approximate the partition function by only the terms where the modes are unoccupied, so (assuming $c > 0$)

$$Z(q) \approx q^{-c/12}. \quad (1539)$$

Now we use modular invariance with $S(q) = e^{-2\pi i/\tau} = e^{-2\pi/\beta} \rightarrow 1$, and so the modular-transformed version of the partition function is actually dominated by *high* energy modes:

the super stretched nature of the torus means that the high and low energy parts of the partition function get linked through the modular S transformation and we can write

$$S[Z(q)] \approx \int dE \rho(E) e^{-E/\beta}, \quad (1540)$$

since $H = 2\pi(L_0 + \bar{L}_0 - c/12)$.

Now we make the rather suspect assumption that we can use a saddle-point approximation (note to self: can you back this up?), with the condition on $E = E(\beta)$ being

$$\frac{\partial \ln \rho(E)}{\partial E} \Big|_{E=E_*} = \beta^{-1}. \quad (1541)$$

So then by modular invariance,

$$\ln Z(q) = \ln(S[Z(q)]) \implies \frac{\pi c \beta}{6} = \ln \rho(E_*) - \beta^{-1} E_*. \quad (1542)$$

Hitting both sides of the RHS with ∂_β so that we can use our saddle point condition, we get

$$\frac{\pi c}{6} = \partial_\beta E_* \partial_E \ln \rho(E)|_{E=E_*} + \frac{E_*}{\beta^2} - \beta^{-1} \partial_\beta E_* = \frac{E_*}{\beta^2} \implies \beta = \sqrt{\frac{6E_*}{\pi c}}. \quad (1543)$$

We can then use the relation that we took the derivative of to get

$$\ln \rho(E_*) = \frac{\pi c \beta}{6} + \frac{E_*}{\beta} = \sqrt{\frac{2\pi E_* c}{3}}. \quad (1544)$$

Since the saddle point energy E_* is large, we see that at high energy, the (logarithm of the) density of states is controlled by the central charge.

89 July 12 — Alternate route to WZW central charge

This is a slightly elaborated version of yet another problem in the big yellow book, chapter 15.

Consider the WZW theory at level k with horizontal lie algebra obtained from the group G . The current algebra is captured by the OPE

$$J^a(z) J^b(w) \sim \frac{k \delta_{ab}}{(z-w)^2} + \frac{i f^{abc} J^c}{z-w}. \quad (1545)$$

Show that the central charge is

$$c = \frac{k \dim G}{k+g}, \quad (1546)$$

where g is the dual coxeter number (one half of the quadratic casimir of the adjoint representation), satisfying

$$g \delta_{cd} = \frac{1}{2} \sum_{a,b} f_{abc} f_{abd} = \frac{1}{2} C_2(G) \delta_{cd}. \quad (1547)$$

In the expression for c , $\dim G$ is the dimension of the Lie algebra itself, not the dimension of the particular representation that the currents act on the fields with.

Show that this is the central charge *without* taking the TT OPE. You will need to find how the Virasoro algebra fits into the current algebra and will need to consider the action of various currents and Virasoro generators on a WZW primary field. **Solution:**

First we need to figure out the form of the energy momentum tensor (we are just trying to motivate the Sugawara construction). The action is (at the conformal fixed point)

$$S[g] = \frac{k}{8\pi} \int d^2x \text{Tr}[\partial_\mu g^{-1} \partial^\mu g] - \frac{ik}{12\pi} \int_{M_3} \text{Tr}[\omega \wedge \omega \wedge \omega], \quad (1548)$$

where as usual $\omega = g^{-1}dg$. When we write this in complex coordinates the second term doesn't change, but the first term changes to $\frac{k}{4\pi} \int \text{Tr}[\partial g^{-1} \bar{\partial} g]$ since $\partial = (\partial_0 - i\partial_1)/2$ while $|d^2x| = \frac{1}{2}|d^2z|$. Now for a scalar field we have

$$T^{\mu\nu} = -g_{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\sigma \phi g^{\sigma\nu}. \quad (1549)$$

The only thing that changes for us is that we need to take the trace on the RHS. Now in our conventions $T = -2\pi T_{zz} = -\frac{\pi}{2} T^{\bar{z}\bar{z}}$ since in complex coordinates $g^{\mu\nu} = 2X$. So then for us,

$$T = -\pi \text{Tr} \left[\frac{\partial \mathcal{L}}{\partial \bar{\partial} g \phi} \partial g \right]. \quad (1550)$$

Now for the WZW Lagrangian, only the first kinetic term will contribute to T since the second term doesn't contain the metric (it's built out of wedge products, and so we use ϵ to contract indices rather than the metric). Alternatively, since it is linear in time derivatives it doesn't contribute to the Hamiltonian, and then by symmetry we can argue that it doesn't contribute to T . Finally, we can argue that since T is local but the topological term doesn't depend on the choice of M_3 , it cannot contribute to T . Anyway, this means we can write the classical stress tensor as

$$T_c = -\frac{k}{2} \text{Tr}[\partial g^{-1} \partial g] = \frac{1}{2k} \text{Tr}[k(\partial g)g^{-1}k(\partial g)g^{-1}] = \frac{1}{2k} \sum_a J_a J_a, \quad (1551)$$

where the current is

$$J = J^a t^a = -k(\partial g)g^{-1}. \quad (1552)$$

So, this is the stress tensor we expect classically. Quantumly we try an ansatz with the same Sugawara form, except with a different coefficient, so that the ansatz for the actual stress tensor is

$$T = \frac{1}{\gamma} \sum_a (J^a J^a), \quad (1553)$$

where (...) denotes normal-ordering, which is the full taking-off-all-singular-parts normal ordering. Since the theory is interacting, the normal-ordering is more complicated than for free fields, and means that unlike for free theories, the coefficient γ will receive quantum corrections (double contractions from interactions) to the classical value of $\gamma_c = 2k$.

Now we will figure out the actual value of γ . First, some preliminary work. Since the J^a 's are $(1, 0)$ currents, we define the mode expansion as

$$J^a(z) = \sum_n z^{-n-1} J_n^a. \quad (1554)$$

We get the commutator of the current modes by doing the usual double contour integral with the help of the $J^a J^b$ OPE provided by the current algebra:

$$\begin{aligned} [J_n^a, J_m^b] &= \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz z^n w^m \left(\frac{k\delta_{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w} \sum_l w^{-l-1} J_l^c \right) \\ &= nk\delta_{n+m}\delta_{ab} + \sum_l \frac{1}{2\pi i} if^{abc} \oint dw w^{m+n-l-1} J_l^c \\ &= nk\delta_{n+m}\delta_{ab} + if^{abc} J_{m+n}^c. \end{aligned} \quad (1555)$$

Sticking the expressions for the current modes into the stress tensor we find that the Virasoro generators (defined as usual with $T = \sum_n L_n z^{-n-2}$ so that L_n has conformal dimension n) can be written in terms of the current as

$$L_n = \frac{1}{\gamma} \sum_m : J_m^a J_{n-m}^a : \quad (1556)$$

Here the normal-ordering means just what it does as if the J modes were oscillator modes of a free field: the operator with the larger mode index gets put to the right.

We also need the commutation relation between the Virasoro generators and the current modes:

$$[L_n, J_m^a] = -m J_{n+m}^a, \quad (1557)$$

which just comes from the fact that J^a , a $(1, 0)$ current, is a primary with conformal dimension $h = 1$ (again, this is easy to check from the double-contour method of computing the commutator). Note that the J_0^a modes generate a symmetry corresponding to the Lie algebra G , as they commute with the Hamiltonian.

In the following, we will let $|\phi\rangle$ be a WZW primary state. This means the same thing as it does for primary states of the Virasoro algebra, namely

$$J_0^a |\phi\rangle = -t^a |\phi\rangle, \quad J_{n>0}^a |\phi\rangle = 0. \quad (1558)$$

Here, t^a is the representation matrix assigned to the generator a (it lives in the Lie algebra of G). We won't need to specify the exact irrep in what follows. That J_n^a annihilates $|\phi\rangle$ if $n > 0$ just means that $|\phi\rangle$ is a highest weight (or maybe better, “lowest weight” state for the current algebra. Also, note that $|\phi\rangle$ is also automatically a Virasoro primary (this isn't always the case for current algebras), since the normal-ordering in the expression of L_n in terms of the currents means that if $n > 0$, in the action of L_n on $|\phi\rangle$, the right-most current operator always has a mode number which is greater than zero.

Now on one hand, we have

$$[J_1^a, L_{-1}] |\phi\rangle = J_0^a |\phi\rangle = -t^a |\phi\rangle. \quad (1559)$$

On the other hand, we have

$$[J_1^a, L_{-1}]|\phi\rangle = J_1^a L_{-1}|\phi\rangle = \frac{1}{\gamma} J_1^a (2J_{-1}^b J_0^b) |\phi\rangle = -\frac{2}{\gamma} [J_1^a, J_{-1}^b] t^b |\phi\rangle, \quad (1560)$$

since only the $m = 0, m = -1$ terms in L_{-1} act nontrivially on $|\phi\rangle$. From the commutations we derived earlier, this turns into

$$-\frac{2}{\gamma} \left(k\delta_{ab}t^b + i\frac{1}{2}f_{abc}[t^c, t^b] \right) |\phi\rangle. \quad (1561)$$

Using the definition of the dual coxeter number g , this becomes

$$[J_1^a, L_{-1}]|\phi\rangle = -\frac{2}{\gamma}(k+g)t^b |\phi\rangle. \quad (1562)$$

Reconciling these two ways of writing the action of the commutator lets us conclude that

$$\gamma = 2(k+g), \quad (1563)$$

so the correct quantum stress tensor is

$$T = \frac{1}{2(k+g)} \sum_a (J^a J^a). \quad (1564)$$

Now we can get the central charge. As one does when computing unitarity constraints, we will compute the norm of the state $L_{-2}|\phi\rangle$. We will do this in two different ways: using the Virasoro algebra commutation relations and using the expression of L_{-2} in terms of the currents. We choose L_{-2} since it is the smallest weight Virasoro generator that lets us access the central charge: the central charge appears in the Virasoro algebra relations together with $\delta_{n+m}(n^3 - n)$, which gives us zero for $n = 0, 1$ but not $n = 2$.

Using the Virasoro commutation relations, we have

$$||L_{-2}|\phi\rangle||^2 = \langle\phi|[L_2, L_{-2}]|\phi\rangle = 4\langle\phi|L_0|\phi\rangle + \frac{c}{2}, \quad (1565)$$

so that

$$\frac{c}{2} = -\frac{4}{\gamma} \langle\phi|t^a t^a|\phi\rangle + \langle\phi|[L_2, L_{-2}]|\phi\rangle. \quad (1566)$$

Now we need to evaluate $\langle\phi|[L_2, L_{-2}]|\phi\rangle$ using the current modes, which is actually kind of gross. Keeping the terms in the mode expansion that survive, (namely $m = -2, -1, 0$) we have (repeated indices are summed)

$$\langle\phi|[L_2, L_{-2}]|\phi\rangle = \frac{1}{\gamma^2} \langle\phi|(-2t^a J_2^a + J_1^a J_1^a)(-2J_{-2}^b t^b + J_{-1}^b J_{-1}^b)|\phi\rangle. \quad (1567)$$

We now must painstakingly commute the terms in the left group through until they act on the ket on the right. After doing the first series of commutations, we have

$$\begin{aligned} \langle\phi|[L_2, L_{-2}]|\phi\rangle &= \gamma^{-2} \langle\phi| \left[4t^a (if^{abc} J_0^c + 2k\delta_{ab}) t^b - 2t^a if^{abc} J_1^c J_{-1}^b - 2J_1^a if^{abc} J_{-1}^c t^b \right. \\ &\quad \left. + if^{abc} if^{acd} J_1^d J_{-1}^b - it^c f^{abc} J_1^a J_{-1}^b + k\delta_{ab} J_1^a J_{-1}^b + (k\delta_{ab} + if^{abc} J_0^c)(k\delta_{ab} + if^{abd} J_0^d) \right] |\phi\rangle. \end{aligned} \quad (1568)$$

Now we let the J_0^a 's act on the primary fields and replace the remaining $J_1^a J_{-1}^b$'s with $i f^{abc} J_0^c + k \delta_{ab}$. A few of the resulting terms die by the antisymmetry of the structure constants. A few contain three t^a generators, but they always appear with some f^{abc} 's so we can antisymmetrize them into two t^a 's plus an extra f^{abc} . Doing this and again using the definition of the dual coexeter number, we find

$$\langle \phi | [L_2, L_{-2}] | \phi \rangle = \frac{1}{\gamma^2} [\langle \phi | t^a t^a | \phi \rangle (8k + 8g) + (2kg + 2k^2) \dim G], \quad (1569)$$

with $\dim G = \delta_{aa}$. Now the first bit on the RHS is $\gamma^{-2} \langle \phi | t^a t^a | \phi \rangle (8k + 8g) = 4\gamma^{-1} \langle \phi | t^a t^a | \phi \rangle$, which cancels against the $-4\gamma^{-1} \langle \phi | t^a t^a | \phi \rangle$ term in (1566). Thus we have found that the central charge is given by

$$\frac{c}{2} = \frac{2kg + 2k^2}{\gamma^2} \dim G \implies c = \frac{k \dim G}{k + g}. \quad (1570)$$

As examples, for $\mathfrak{su}(n)$ (for $n \geq 2$), the dual coxeter number is $g = n$, so that

$$c_{\widehat{\mathfrak{su}}(n)_k} = \frac{k(n^2 - 1)}{k + n}. \quad (1571)$$

For the familiar case of $n = 2, k = 1$ we have $c = 1$, which agrees with the calculation we did earlier showing how the compact boson at the self-dual radius could be mapped onto the $\widehat{\mathfrak{su}}(2)_1$ CFT. Another interesting case is $n = k = 2$, for which

$$c_{\widehat{\mathfrak{su}}(2)_2} = \frac{3}{2}, \quad (1572)$$

which hints at representations in terms of either three Majoranas or in terms of a free Majorana and a free boson (both are possible). Likewise, for $\widehat{\mathfrak{u}}(n)_k$ we have

$$c_{\widehat{\mathfrak{u}}(n)_k} = 1 + c_{\widehat{\mathfrak{su}}(n)_k}, \quad (1573)$$

where the $+1$ is from the decoupled Abelian $U(1)$ factor. For example, when $k = 1$ we have

$$c_{\widehat{\mathfrak{u}}(n)_k} = n, \quad (1574)$$

which is compatible with a free-fermion realization in terms of n Dirac fermions.

For $\widehat{\mathfrak{so}}(n)$ we have $g = n - 2$ so that

$$c_{\widehat{\mathfrak{so}}(n)_k} = \frac{\frac{k}{2}(n^2 - n)}{(n - 2) + k}. \quad (1575)$$

In particular, for $k = 1$ we have

$$c_{\widehat{\mathfrak{so}}(n)_1} = \frac{n}{2}, \quad (1576)$$

which is compatible with $\widehat{\mathfrak{so}}(n)_1$ being realized by n free Majorana fermions. For $k = g = n - 2$, we have

$$c_{\widehat{\mathfrak{so}}(n)_1} = \frac{n^2 - n}{4}, \quad (1577)$$

compatible with a realization in by $(n^2 - n)/2$ free Majoranas.

90 July 13 — Free fermion representation of $\widehat{\mathfrak{so}}(N)_g$ current algebra

This is a problem from the big yellow book, chapter 15. Consider Majorana fermions transforming in the adjoint representation of $SO(N)$. Show how to build currents with these fermions that satisfy the $\widehat{\mathfrak{so}}(N)_g$ current algebra, where g is the dual coxeter number (see yesterday's diary entry). Compute the central charge. Answer:

$$c = \frac{1}{4}N(N - 1), \quad (1578)$$

which is what one would expect from $\dim[\mathfrak{so}(N)]$ flavors of real fermions.

Solution:

The fermions have dimension $1/2$ while WZW currents have dimension $(1, 0)$, and so the J^a 's we write down will need to be bilinear in the fermions. The natural choice, since we are working in the adjoint representation, is

$$J^a(z) = \alpha i f^{abc} \psi_b(z) \psi_c(z) \quad (1579)$$

where α is some yet-to-be-determined constant. We can fix it by computing the OPE for the current. Using $\psi_a(z)\psi_b(w) \sim \delta_{ab}(z-w)^{-1}$, this is straightforward: remembering the minus signs when moving fermions around, we have

$$\begin{aligned} J^a(z)J^d(w) &\sim -\alpha^2 f^{abc} f^{def} \left(\frac{\delta_{ec}\delta_{bf} - \delta_{cf}\delta_{be}}{(z-w)^2} + \frac{1}{z-w} [\psi_b\psi_f\delta_{ce} - \psi_b\psi_e\delta_{cf} - \psi_c\psi_f\delta_{be} + \psi_c\psi_e\delta_{bf}] \right) \\ &\sim -\alpha^2 \left(-\frac{4g\delta_{ad}}{(z-w)^2} + \frac{2}{z-w} f^{abc} (\psi_b\psi_f f^{dcf} + \psi_c\psi_e f^{deb}) \right) \\ &\sim \frac{4g\alpha^2}{(z-w)^2} \delta_{ad} + i \left(\frac{4\alpha^2 i}{z-w} f^{abc} f^{dcf} \psi_b\psi_f \right). \end{aligned} \quad (1580)$$

Now the anticommuting property of the fermions lets us use the identity

$$f^{abc} f^{dcf} \psi_b \psi_f = \frac{1}{2} f^{ade} f^{efg} \psi_f \psi_g. \quad (1581)$$

To derive this, we use the Bianchi identity in the form

$$f^{ade} f^{efg} \psi_f \psi_g = -(f^{aeg} f^{dfg} + f^{deg} f^{fae}) \psi_f \psi_g. \quad (1582)$$

Plugging this in, using the antisymmetry of the ψ_a 's and relabelling a bunch of variables gives the sought-for identity. Putting (1581) in to the $J^a J^d$ OPE, we see that we get the OPE for an affine $\widehat{\mathfrak{so}}(N)$ algebra provided that $\beta = 1/2$ (because this choice of β fixes the $i f^{ade} J^e / (z-w)$ term appearing in the OPE to have a coefficient of unity). So then recapitulating, the properly normalized currents are

$$J^a = \frac{1}{2} i f^{abc} \psi_b(z) \psi_c(w), \quad (1583)$$

and they have the OPE

$$J^a(z)J^b(w) \sim \frac{g\delta_{ab}}{(z-w)^2} + if^{abc}\frac{J^c(w)}{z-w}. \quad (1584)$$

Thus the current algebra is the $\widehat{\mathfrak{so}}(N)_g$ algebra.

The stress tensor is constructed using the Sugawara strategy described in yesterday's diary entry, and we can use the formula derived yesterday to compute the central charge. Since we are at level $k = g$, we have

$$c = \frac{k \dim \mathfrak{so}(N)}{k+g} = \frac{1}{2} \dim \mathfrak{so}(N). \quad (1585)$$

Now for $\mathfrak{so}(N)$ we have $\dim \mathfrak{so}(N) = \frac{1}{2}N(N-1)$,⁴⁵ meaning that

$$c = \frac{1}{4}N(N-1), \quad (1587)$$

which is equal to $1/2$ (central charge of each Majorana) times the number of fermion fields. This hints that even though from the Sugawara construction the theory doesn't look free, it actually is. This can be verified by carefully checking the equivalence of the seemingly interacting Sugawara ($J^a J^a$) stress tensor and $N(N-1)/2$ copies of $T_\psi = -\frac{1}{2}\psi\partial\psi$. Showing this is straightforward; see the big yellow book chapter 15 for hints.

91 July 14 — More free fermion representations of current algebras (unfinished)

This is a problem built from some problems in the big yellow book, chapter 15. **Solution:**

This is actually pretty simple. We let

$$J^a(z) = \frac{i}{2}\epsilon^{abc}\psi_b(z)\psi_c(z), \quad (1588)$$

which gives the correct OPEs. For example

$$J^1(z)J^2(w) = - : \psi_2(z)\psi_3(z) :: \psi_3(w)\psi_1(w) : \sim -\frac{1}{z-w} : \psi_2(z)\psi_1(w) : \sim \frac{1}{z-w}iJ^3(w). \quad (1589)$$

⁴⁵Why? Consider the symmetrizer map $S : M \mapsto MM^T - \mathbf{1}$. Now $O(N) = \ker(S)$, while $\dim \text{im}(S) = \sum_{i=1}^N i = \frac{1}{2}N(N+1)$ is the dimension of all symmetric matrices. So then

$$\dim O(N) = \dim GL(N) - \dim \text{im}(S) = N - \frac{1}{2}N(N+1) = \frac{1}{2}N(N-1). \quad (1586)$$

Then since $\dim \mathfrak{so}(N) = \dim \mathfrak{o}(N) = \dim O(N)$, $\dim \mathfrak{so}(N) = \frac{1}{2}N(N-1)$.

On the other hand, when we take the OPE of e.g. J^1 with itself, we get

$$J^1(z)J^1(w) = - : \psi_2(z)\psi_3(z) : : \psi_2(w)\psi_3(w) : \sim \frac{1}{(z-w)^2}, \quad (1590)$$

since the terms with one contraction go like $\frac{1}{z-w}\partial\psi_a(w)(z-w)$, which is regular. Thus we have the OPE

$$J^a(z)J^b(w) \sim \frac{\delta_{ab}}{(z-w)^2} + if^{abc}J^c(w). \quad (1591)$$

blah blah Next up is the free fermion + free boson representation.

92 July 15 — Two more interpretations of the central charge

Today we're looking at two standard things relating to the central charge that I'd heard about but never taken a close look at.

The first thing: show that the partition function of any 2d CFT on a sphere of radius a obeys the relation

$$\frac{d \ln Z_{S^2}}{d \ln a} = \frac{c}{3}. \quad (1592)$$

To derive this, you will need to think about the trace anomaly and the central charge.

The second thing: show that the entanglement entropy for an interval $[a, b]$ is

$$S \sim c \ln |a - b|. \quad (1593)$$

To do this, you will need to use the replica trick and think about twist operators.

Solution:

For the first thing, we will need to find how Z varies as we change a . The metric on the sphere, in coordinates projected onto the plane, is $g_{\mu\nu} = \mathbf{1}_{\mu\nu} 4a^2 / (1 + |x|^2)^2$ so that the line element is $ds^2 = \frac{4a^2}{(1+|r|^2)^2} (dx^2 + dy^2)$. This means that when we vary the radius of the sphere, the variation of the metric is proportional to the metric itself: for $a \mapsto a + \delta a$ we use $g_{\mu\nu}(x) \propto a^2$ to write

$$g_{\mu\nu} \mapsto g_{\mu\nu} + 2(\delta a)a^{-1}g_{\mu\nu}. \quad (1594)$$

So then applying the Ward identity (with no operator insertions), we have

$$\delta \ln Z = -\frac{1}{2} \int d^2x \langle T^{\mu\nu}(x) \delta g_{\mu\nu}(x) \rangle = - \int d^2x \sqrt{g} \langle T^{\mu\nu}(x) \rangle \delta a a^{-1} g_{\mu\nu}(x), \quad (1595)$$

and so

$$\frac{d \ln Z}{d \ln a} = - \int d^2x \sqrt{g} \langle \text{Tr } T_{\mu\nu} \rangle. \quad (1596)$$

Because of the trace anomaly (which is relevant since we're on a sphere), this will be non-zero.

We will calculate $\langle \text{Tr } T_{\mu\nu} \rangle$ by working infinitesimally and finding $\delta \langle T \rangle$, where the variation is a Weyl transformation of the metric. We can get away with doing this because we know

what the form of the answer will be. Indeed, since $\langle \text{Tr}T_{\mu\nu} \rangle = 0$ classically and since we are working with a CFT, the only way in which $T_{\mu\nu}$ could fail to be traceless is for there to be some of local anomaly coming from a contact term (the kind of thing which ruins the tracelessness of T in e.g. QED in four dimensions comes from scales generated during the RG flow: since we are working with a genuine CFT this sort of thing cannot happen). Since this involves UV physics and can only depend on the spacetime metric, $\langle \text{Tr}T_{\mu\nu} \rangle$ must be proportional to R (the Ricci scalar), since this is the only metric-dependent, local, mass-dimension-2 scalar function that could fit the bill. We then just have to determine the coefficient β in $\langle \text{Tr}T_{\mu\nu} \rangle = \beta R$, and so finding the variation of the trace of $T_{\mu\nu}$ under a Weyl transformation is good enough for our present goal.

We will choose coordinates where the metric takes the form $g_{\mu\nu} = e^{\phi(x)}\eta_{\mu\nu}$, so that an infinitesimal variation of the metric away from flat space is $\delta g_{\mu\nu}(x) = \delta\phi(x)\eta_{\mu\nu}$. The Ricci scalar can be found after some pretty heinous algebra to be

$$R = -e^{-\phi}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi. \quad (1597)$$

This implies a fact that we will need later, namely that

$$\sqrt{g}R(x) = -\square\phi(x) = -4\partial\bar{\partial}\phi(x), \quad (1598)$$

where the factor of 4 comes from our conventions where e.g. $\partial = \frac{1}{2}(\partial_0 - i\partial_1)$.

Now we return to finding the trace of T . Upon varying $g_{\mu\nu}$ away from flat space, we have (going to be using σ, σ' instead of x, y for Cartesian coordinates from now on)

$$\delta\langle T_\mu{}^\mu(\sigma) \rangle = -\frac{1}{2} \int d^2\sigma' \langle T_\mu{}^\mu(\sigma) T_\nu{}^\nu(\sigma') \rangle \delta\phi(\sigma'), \quad (1599)$$

since the variation of the metric is proportional to $\eta_{\mu\nu}$. Now we need the OPE between the two traces of the stress tensor on the RHS of this equation.

Getting this in a precise way is a bit tricky: we will hybridize an argument in some notes by Komargodski and an argument in Tong's CFT notes. Going back to Cartesian coordinates, we examine the two point function $\langle T_{\mu\nu}(q)T_{\alpha\beta}(p) \rangle$, where q, p are momenta. By energy conservation, this two-point function needs to be killed by $\partial_\mu, \partial_\nu, \partial_\alpha$, and ∂_β . It also needs to contain a δ function enforcing momentum conservation, and so it needs to have the form

$$\langle T_{\mu\nu}(q)T_{\alpha\beta}(p) \rangle = \delta(p+q) \left[\frac{f(q^2)}{2} (\Pi_{\mu\alpha}^T(q)\Pi_{\nu\beta}^T(q) + \Pi_{\mu\beta}^T(q)\Pi_{\nu\alpha}^T(q)) + g(q^2)\Pi_{\mu\nu}^T(q)\Pi_{\alpha\beta}^T(q) \right], \quad (1600)$$

where of course the transverse projectors are $\Pi_{\alpha\beta}^T(q) = q_\alpha q_\beta - q^2\eta_{\alpha\beta}$. We can fix the form of the functions f, g by requiring scale invariance: thus they must be algebraic in q^2 , and they have to go as $1/q^2$ because of requiring the two-point function to be invariant under $q \mapsto \lambda q$ for $\lambda \in \mathbb{R}$ (remember that the delta function also transforms!). So, we can write $f(q^2) = a/q^2, g(q^2) = b/q^2$. Anyway, by taking $\mu = \nu$ and summing, we get

$$\langle T_\mu{}^\mu(p)T_{\alpha\beta}(q) \rangle = \delta(p+q) \left[\frac{a}{q^2}\Pi_{\mu\alpha}^T(q)\Pi_{\beta}^{T\mu}(q) + \frac{b}{q^2}\Pi_{\mu}^{T\mu}(q)\Pi_{\alpha\beta}^T(q) \right] = -\delta(p+q)(a+b)\Pi_{\alpha\beta}^T(q). \quad (1601)$$

Contracting one more time and Fourier transforming, we have

$$\langle T_\mu^\mu(\sigma) T_\nu^\nu(\sigma') \rangle \propto \square \delta(\sigma - \sigma'), \quad (1602)$$

where the proportionality constant $(a + b)$ is something that we can't determine with this method. The point of these couple of steps is to show that the two-point function of the traces has to be proportional to a double-derivative of a δ function. To find the proportionality constant, we need another argument.

In complex coordinates, conservation of energy is $\partial T^{z\bar{z}} + \bar{\partial} T^{\bar{z}\bar{z}} = 0$, or $\partial T_{z\bar{z}} - \bar{\partial} T/(2\pi) = 0$. Now with our conventions $T_\mu^\mu(\sigma) T_\nu^\nu(\sigma') = -16T_{z\bar{z}}(z)T_{\bar{w}\bar{w}}(w)$. Using conservation of energy then, we have

$$\langle \partial_z T_{z\bar{z}}(z) \partial_w T_{\bar{w}\bar{w}}(w) \rangle = -\frac{1}{4\pi^2} \bar{\partial}_z \bar{\partial}_w \langle T(z) T(w) \rangle. \quad (1603)$$

We know what the RHS is, since we know the TT OPE. We have

$$\langle \bar{\partial}_z T_{z\bar{z}}(z) \bar{\partial}_w T_{\bar{w}\bar{w}}(w) \rangle \sim \frac{1}{4\pi^2} \bar{\partial}_z \bar{\partial}_w \frac{c/2}{(z-w)^4} + \dots = \frac{c}{48\pi^2} \partial_z^2 \partial_w \bar{\partial}_z \bar{\partial}_w \frac{1}{z-w} + \dots, \quad (1604)$$

where \dots represents things that contain $\bar{\partial}_z T(z)$ and derivatives thereof. Now we need to use

$$\bar{\partial}_z \frac{1}{z-w} = 2\pi \delta(z-w, \bar{z}-\bar{w}), \quad (1605)$$

which one can prove by using $\int_R d^2 z \bar{\partial} f(z, \bar{z}) = -i \oint_{\partial R} f(z, \bar{z})$ with the function $f = 1/(z-w)$ and taking R to be a region containing the point w . Thus the OPE we've been looking at is

$$\langle \bar{\partial}_z T_{z\bar{z}}(z) \bar{\partial}_w T_{\bar{w}\bar{w}}(w) \rangle \sim -\partial_z \partial_w \left(\frac{c}{24\pi} \partial_z \bar{\partial}_z \delta(z-w, \bar{z}-\bar{w}) \right) + \dots, \quad (1606)$$

so then using the earlier relation we found between the various stress energy tensor two-point functions, we have

$$\frac{1}{16} \partial_z \partial_w \langle T_\mu^\mu(\sigma) T_\nu^\nu(\sigma') \rangle \sim -\partial_z \partial_w \left(\frac{c}{24\pi} \partial_z \bar{\partial}_z \delta(z-w, \bar{z}-\bar{w}) \right) + \dots \quad (1607)$$

This is where the earlier analysis we did of the two-point function of T will come in handy: we know that the two point function of the traces has to be proportional to $\square \delta(\sigma - \sigma')$. This lets us do two things: first, it lets us drop the \dots (terms that go as $\bar{\partial}_z T$; of course classically this is zero anyway) on the RHS, since we know that no operators (only c-numbers) can appear on the RHS (actually, I guess the extra terms also die because they have nonzero spin, and we have rotational invariance). It also lets us strip away the derivatives and conclude that

$$\langle T_\mu^\mu(\sigma) T_\nu^\nu(\sigma') \rangle \sim -\frac{2c}{3\pi} \partial_z \bar{\partial}_z \delta(z-w, \bar{z}-\bar{w}). \quad (1608)$$

Naively this conclusion could only be reached modulo singular terms in the kernel of $\partial_z \partial_w$, but since we know that the LHS has to be proportional to $\square \delta(x-y)$, such terms will not appear. Now we can change the δ function and the derivatives over to σ, σ' coordinates at the cost of a factor of $1/8$ (2 from each derivative and 2 from the δ function since $dz d\bar{z} \rightarrow 2dx dy$), so that

$$\langle T_\mu^\mu(\sigma) T_\nu^\nu(\sigma') \rangle \sim -\frac{c}{12\pi} \square \delta(\sigma - \sigma'). \quad (1609)$$

Finally, putting this into our expression for $\delta\langle T_\mu^\mu \rangle$, we have

$$\delta\langle T_\mu^\mu(\sigma) \rangle = \frac{c}{24\pi} \int d^2\sigma' \square \delta(\sigma - \sigma') \delta\phi(\sigma') = \frac{c}{24\pi} \square \delta\phi(\sigma). \quad (1610)$$

Using our earlier result for the variation of the Ricci scalar, we get

$$\delta\langle T_\mu^\mu(\sigma) \rangle = -\frac{c}{24\pi} \delta R(\sigma). \quad (1611)$$

Therefore, integrating over the variation, we can conclude that

$$\langle T_\mu^\mu(\sigma) \rangle = -\frac{c}{24\pi} R(\sigma), \quad (1612)$$

which holds in all geometries, not just those infinitesimally close to flat space (as we discussed earlier, the form $\langle T_\mu^\mu(\sigma) \rangle \propto R$ is required, and the coefficient of proportionality of course is geometry-independent).

Thus using our expression for $\delta \ln Z$ under a change in a (the radius of the sphere), we finally have

$$\frac{d \ln Z}{d \ln a} = -\frac{c}{24\pi} \int d^2\sigma \sqrt{g} R(\sigma). \quad (1613)$$

On a sphere the Ricci scalar is $2/a^{246}$, so since $\int d^2\sigma \sqrt{g} = 4\pi a^2$,

$$\frac{d \ln Z}{d \ln a} = \frac{c}{3}, \quad (1615)$$

which is what we wanted to show.

One final comment is that this whole derivation relied on determining the contact term of $\text{Tr } T_{\mu\nu}$ with itself. It's often the case in QFT that contact terms like this are non-universal and depend on our regularization scheme. This isn't the case here, since the contact term in question is determined by the TT OPE, i.e. by correlation functions of the stress tensor at *separated* points. This information is universal, and ensures that the contact term is universal as well.

Now for the entanglement entropy part of the diary entry. We use the replica trick in the usual way, writing for $A = [a, b]$,

$$S_A = \lim_{n \rightarrow 1} \frac{\text{Tr}[\rho_A^n] - 1}{1 - n} = -\lim_{n \rightarrow 1} \partial_n \text{Tr}[\rho^n]. \quad (1616)$$

As usual, we calculate the n th power of the reduced density matrix by calculating the partition function on the n -sheeted replica manifold. If we let $\Sigma(a)$ denote an n -fold twist operator (which inserts the endpoint of a $z^{1/n}$ branch cut at a), then traveling around a with $\Sigma(a)$

⁴⁶A handy formula for the Ricci scalar is $R = 2\theta/A$, where θ is the angle between a vector and the image of itself parallelly-transported around the boundary of a small geodesic ball (a ball bounded by geodesics), and A is the area of the loop (I think the factor of 2 is correct; some places seem to not have it). For the sphere, parallel transporting a vector halfway around the equator and then back along the prime meridian rotates the vector by π , so that the Ricci scalar is

$$R = 2 \frac{\pi}{4\pi a^2/4} = 2/a^2. \quad (1614)$$

inserted is equivalent to moving between sheets of the Riemann surface. Since each of the n sheets (each a copy of \mathbb{C}) has the twist operators inserted, the trace in the expression for S_A is the same as the n th power of the two-point function of the twist operators:

$$\text{Tr}[\rho_A^n] = \langle \Sigma(a)\Sigma(b) \rangle \sim \frac{1}{|a-b|^{2n\Delta_\Sigma}}. \quad (1617)$$

We can get the scaling dimension Δ_Σ of the twist operator by the following argument. Let w be the coordinate on the n -sheeted Riemann surface used in calculating ρ^n , and let z be a coordinate on \mathbb{C} . Now zoom in to one of a, b , and consider a map which locally takes the Riemann surface to \mathbb{C} in a region around this point. We can map the Riemann surface to the plane by taking $z = w^{1/n}$, which “unwinds” the Riemann surface and which is single valued when acting on w (we can e.g. think about w living in a space where $e^{2\pi i n} = 1$ but $e^{2\pi i} \neq 1$). Then we can use the transformation law for the stress tensor to write

$$\langle T(w) \rangle = \left\langle \partial_w z T(z) + \frac{c}{24} \{z; w\} \right\rangle = \frac{c}{24} \{w^{1/n}; w\}, \quad (1618)$$

since on the plane the vev of the stress tensor vanishes (here we are thinking about the $T(w)$ insertion as occurring only on a single sheet of the replica manifold). The Schwartzian derivative is easily computed:

$$\{w^{1/n}; w\} = (n^{-1} - 1)(n^{-1} - 2)w^{-2} - \frac{3}{2}(n^{-1} - 1)^2 w^{-2} = \frac{1}{2}(1 - n^{-2}), \quad (1619)$$

so that

$$\langle T(w) \rangle = \frac{1}{w^2} \frac{c}{24} (1 - n^{-2}). \quad (1620)$$

Thus we read off the conformal dimension of the twist operators as⁴⁷

$$h_\Sigma = \frac{c}{24} (1 - 1/n^2), \quad (1622)$$

so that the scaling dimension is $\Delta_\Sigma = c(1 - n^{-2})/12$. Some sanity checks: when $n \rightarrow 1$ the twist operator becomes trivial, and $\Delta_\Sigma \rightarrow 0$ as required. Secondly, as $n \rightarrow \infty$ we have $\Delta_\Sigma \rightarrow c/12$. This makes sense because $n \rightarrow \infty$ gives a replica manifold that becomes a cylinder: the two ends of the branch cut become the two ends of the cylinder, with $n \rightarrow \infty$ meaning that wrapping around the twist operators never returns one to one’s starting point (the cohomology of the cylinder is \mathbb{Z} , which has no elements of finite order—this is the limit $\mathbb{Z}_n \rightarrow \mathbb{Z}$ for $n \rightarrow \infty$). This agrees with $\Delta_\Sigma \rightarrow c/12$ since we know that on the cylinder, $\langle T_{cyl}(z) \rangle = cz^{-1}/24$ (see previous diary entry), giving $h_\Sigma = c/24$ and $\Delta_\Sigma = c/12$.

⁴⁷Note that we have cheated somewhat here by working locally around one of the branch cut points, essentially assuming that the branch cut runs from $z = 0$ to $z = \infty$, when in reality it runs from a to b . We can correct for this by mapping the branch cut to one stretching from 0 to ∞ by using the conformal mapping $z = (w-a)^{1/n}/(w-b)^{1/n}$ (instead of $z = w^{1/n}$). Now the resulting OPE with the stress tensor needs to have singularities at each of the termination points of the branch cuts and needs to vanish when $a = b$, and so the general form of the vev of $T(w)$ is in fact

$$\langle T(w) \rangle = \frac{c}{24} (1 - n^{-2}) \frac{(a-b)^2}{(w-a)^2(w-b)^2}. \quad (1621)$$

This of course gives us the same conformal dimension as derived in the non-footnoted text.

Anyway, using the scaling dimension we get

$$\text{Tr}[\rho_A^n] = \alpha |a - b|^{-\frac{c}{6}(n-n^{-1})}, \quad (1623)$$

where α is some constant that we won't be able to determine. This gives

$$S_A = - \lim_{n \rightarrow 1} \partial_n \text{Tr}[\rho^n] = - \lim_{n \rightarrow 1} \partial_n e^{-\frac{\alpha c}{6}(n-n^{-1}) \ln |a-b|} = \alpha \frac{c}{3} \ln |a-b|, \quad (1624)$$

as required (the argument of the log will be made dimensionless with some short-distance cutoff that we haven't been writing).

93 July 16 — β function for general two-dimensional nonlinear σ models and Ricci flow

Consider a nl σ m in two dimensions, with action⁴⁸

$$S = \frac{1}{4\pi\alpha} \int_{\Sigma} d^2\sigma \gamma^{\mu\nu} g_{ij}(X) \partial_{\nu} X^i \partial_{\mu} X^j. \quad (1625)$$

Here $X : \Sigma \rightarrow M$ for some Riemannian manifold with metric g_{ij} and some Riemann surface Σ with metric $\gamma_{\mu\nu}$, and where $g_{ij}(X) = X^* g_{ij}$ is the pullback of the metric on M by X .

Show that the beta function(al) for g_{ij} is, to one loop order, given by the Ricci flow equation

$$\beta_{ij} = \frac{dg_{ij}}{d\ln \mu} = R_{ij} = \frac{1}{2} R g_{ij}, \quad (1626)$$

where R_{ij} is the Ricci tensor, R is the Ricci scalar, and we have used

$$R_{ij} = \frac{1}{2} g_{ij} R, \quad (1627)$$

which holds in two dimensions (see the previous diary entry on the linear dilaton CFT). This is a weak-coupling result, which you should derive by assuming that the geometry of M varies slowly compared to $\sqrt{\alpha}$. Note that the coupling for the theory is roughly $\sim 1/r$ for r the radius of curvature, so that if $R_{ij} > 0 \implies \beta_{ij} > 0$, then β_{ij} increases as we flow to the UV, we see that $R_{ij} > 0$ implies the theory is asymptotically free.

Note that unlike a free boson in two dimensions, here we are letting X be *dimensionful*, with mass dimension -1 , and hence $[\alpha] = -2$ so that $\sqrt{\alpha}$ is a length which can be compared with a radius of curvature, which means that weak coupling is when M is “big” compared to the “string scale”. This is the more string theory oriented way of writing things down. Alternatively we could let X be dimensionless and write the action as $l^2 \int \partial X^i \partial X^j g_{ij}$, where now the invariant distance is $ds^2 = l^2 g_{ij} dX^i dX^j$, with ds^2 and l both dimensionless. The small parameter in this case is then l^{-1} .

⁴⁸In string theory, it is more natural to let X have Greek indices since the target space is spacetime, and to let γ have Roman indices since γ lives on the worldsheet. Here we are thinking of Σ as spacetime, and so the conventions are reversed.

Solution:

First let's assess the sensibility of such a result. We see that for target manifolds with positive curvature, the theory is asymptotically free, since large radii of curvature translate to weak coupling (we've also derived this earlier for the case where $M = S^n$). This makes perfect sense, since the strength of the coupling at a given point in M is given by the inverse radius of curvature at that point: going to the UV by "zooming in" on a positive curvature region means increasing the radius of curvature locally, which leads to a smaller coupling constant, agreeing with asymptotic freedom. Geometrically, this is Ricci flow: positive curvature regions get smaller and negative curvature regions get larger. Also, we could have got the answer by using dimensional analysis to write, to leading order,

$$\beta_{ij} = aR_{ij} + bg_{ij} + cRg_{ij}. \quad (1628)$$

However, $b = 0$ since we know that when the target space is flat, the theory is totally conformal and $\beta_{ij} = 0$. Moreover, we know that $R_{ij} = \frac{1}{2}g_{ij}R$ in two dimensions, so that $\beta_{ij} \propto R_{ij}$. Then using remarks similar to those above, we could fix the sign of the coefficient by physical reasoning: we see that these general arguments get us nearly all the way to the answer!

To do things more carefully, we will adopt the background field approach. We will let the "slow" degree of freedom (the background field) be denoted by ϕ , so that the full field is $X^i = \phi^i + \gamma^i$, where γ^i is the "fast" degree of freedom representing fluctuations away from the background field. We want to integrate out the γ^i and see how this changes the coupling constant (the metric) for the slow degrees of freedom. The problem with this is that γ^i is defined as the difference between X^i and ϕ^i , which for a given spacetime coordinate map to different points on the target manifold M . Thus γ^i is not actually a vector, since it transforms non-covariantly under coordinate transformations on M .

Instead of the object γ^i , we can work with vectors by using the following prescription. Let $\lambda^i(s)$ be the geodesic (we assume there is only one) passing between ϕ^i and X^i , with $\lambda^i(0) = \phi^i$ and $\lambda^i(1) = X^i$. Furthermore, define

$$\zeta^i \equiv \left. \frac{d\lambda^i(s)}{ds} \right|_{s=0} \quad (1629)$$

as the tangent to the geodesic at ϕ^i . ζ^i is of course a vector, and we will use it as the integration variable instead of γ^i . We can expand X^i about the slow field ϕ^i in terms of ζ^i by flowing along the geodesic from ϕ^i with the help of $e^{s\zeta}$:

$$\lambda(s) = e^{s\zeta}\phi = \phi + s\nabla_\zeta X|_\phi + \frac{1}{2}s^2\nabla_\zeta^2 X|_\phi + \dots \quad (1630)$$

with X^i obtained by evaluating this at $s = 1$.

We now want to find an explicit expansion for $(e^{s\zeta}\phi)^i$ in terms of ζ^i (i.e. in terms of s). We write

$$(e^{s\zeta}\phi)^i = \phi^i + s\zeta^i + \frac{1}{2}s^2C_2^i + \frac{1}{3!}s^3C_3^i + \dots \quad (1631)$$

for some as-yet-undetermined coefficients C_n^i . We also need to expand the Christoffel symbols about ϕ :

$$\Gamma_{ij}^k(e^{s\zeta}\phi) = \Gamma_{ij}^k(\phi) + s\partial_l\Gamma_{ij}^k(\phi) \left(s\zeta^i + \frac{1}{2}s^2C_2^i + \frac{1}{3!}s^3C_3^i + \dots \right)^l + \dots \quad (1632)$$

Now recall the geodesic equation, which we get by requiring that the covariant derivative of the tangent vector along the geodesic (namely ζ^i) vanish:

$$\frac{d^2\zeta^i}{ds^2} + \Gamma_{jk}^i \frac{d\zeta^j}{ds} \frac{d\zeta^k}{ds} = 0. \quad (1633)$$

We plug the above power series into the geodesic equation (after plugging in the expansion of $e^{s\zeta}\phi$ into the series for the Christoffel symbols), and equate powers in s . The s^0 term tells us that

$$C_2^i = -\Gamma_{jk}^i \zeta^j \zeta^k. \quad (1634)$$

The next order term is

$$C_3^i = \left(2\Gamma_{km}^i \Gamma_{lj}^m - \frac{1}{3} \partial_{(j} \Gamma_{kl)}^i \right) \zeta^k \zeta^l \zeta^j, \quad (1635)$$

and the higher order terms won't be important to us.

So, putting this in for C_2^i , we can write the expansion of $e^{\zeta}\phi$ about ϕ as

$$X^i = (e^{\zeta}\phi)^i = \phi^i + \zeta^i - \frac{1}{2} \Gamma_{jk}^i \zeta^j \zeta^k + O(\zeta^3). \quad (1636)$$

Now suppose we had chosen coordinates about ϕ^i in which the geodesics were straight lines passing through ϕ . These are Riemann normal coordinates, in which geodesics passing through ϕ are used to construct a coordinate system which is locally $\mathbb{R}^{\dim M}$. In Riemann normal coordinates then, the linear order approximation $\phi^i + s\zeta^i$ is actually exact, since the geodesic is a straight line. So then looking at the series expansion for X^i , we see that in Riemann normal coordinates, all of the C_n^i must vanish. This means in particular that

$$\Gamma_{ij}^k(\phi) = \partial_{(i} \Gamma_{kl)}^j(\phi) = 0 \quad (\text{in Riemann normal coordinates}). \quad (1637)$$

This is consistent with the fact that the normal coordinate system is locally $\mathbb{R}^{\dim M}$. Note that this statement is made at the origin of the coordinate system, and does not hold at other points (Christoffel symbols without an argument will always be assumed to be evaluated at ϕ). One upside is that the Riemann tensor in normal coordinates is (just by definition)

$$R^i_{jkl} = \partial_k \Gamma_{lj}^i - \partial_l \Gamma_{kj}^i \quad (\text{in Riemann normal coordinates}), \quad (1638)$$

since the terms quadratic in Γ die. By writing $\frac{1}{3}(R^i_{jkl} + R^i_{lkj})$ in terms of the Christoffel symbols in this way, and then adding and subtracting $(1/3)\partial_k \Gamma_{lj}^i$ and using that $\partial_{(i} \Gamma_{kl)}^j = 0$, the derivative of the Christoffel symbol at the origin is related to the curvature tensor as (still in Riemann normal coordinates)

$$\frac{1}{3}(R^i_{jkl}(\phi) + R^i_{lkj}(\phi)) = \frac{1}{3}(3(\partial_k \Gamma_{lj}^i)(\phi) - \partial_{(l} \Gamma_{jk)}^i(\phi)) = (\partial_k \Gamma_{lj}^i)(\phi). \quad (1639)$$

Finally we need to know how to expand more general tensors about the origin ϕ in terms of the ζ 's. In normal coordinates, this is just the normal Taylor expansion since we locally are in $\mathbb{R}^{\dim M}$. However, if we write it in terms of covariant things, the expansion will work for any coordinate system. So, we will presently work in normal coordinates and re-write the Taylor expansion in terms of covariant derivatives.

Consider a $(2,0)$ tensor T_{ij} (two covariant indices). Since the Christoffel symbols at the origin vanish in normal coordinates, the first order term in the expansion is

$$\partial_k T_{ij}(\phi) \zeta^k = \nabla_k T_{ij}(\phi) \zeta^k. \quad (1640)$$

The second order term is trickier though. We have

$$\frac{1}{2} \nabla_k \nabla_l T_{ij} \zeta^k \zeta^l = \frac{1}{2} \nabla_k (\partial_l T_{ij} - \Gamma_{il}^m T_{mj} - \Gamma_{jl}^m T_{im}) \zeta^k \zeta^l. \quad (1641)$$

After we take the ∇_k and evaluate at ϕ (the origin), all the un-differentiated Christoffel symbols will die. Thus, using our expression for the derivative of the Christoffel symbols in terms of the Riemann curvature tensor,

$$\frac{1}{2} (\nabla_k \nabla_l T_{ij})(\phi) \zeta^k \zeta^l = \frac{1}{2} \left(\partial_k \partial_l T_{ij}(\phi) - \frac{1}{3} (R^m{}_{lki} + R^m{}_{ikl}) T_{mj}(\phi) - \frac{1}{3} (R^m{}_{lkj} + R^m{}_{jkl}) T_{im}(\phi) \right) \zeta^k \zeta^l. \quad (1642)$$

Now we use the Bianchi identity in the form

$$R^m{}_{lki} + R^m{}_{ikl} = -R^m{}_{kil} \quad (1643)$$

on the curvature tensors in the above expansion. After solving for $\frac{1}{2} \partial_k \partial_l T_{ij}(\phi)$ in terms of the other covariant stuff, we can finally rewrite the Taylor expansion for T as

$$T_{ij}(e^\zeta \phi) = T_{ij}(\phi) + \nabla_k T_{ij}(\phi) \zeta^k + \frac{1}{2} \zeta^k \zeta^l \left(\nabla_k \nabla_l T_{ij}(\phi) - \frac{1}{3} R^m{}_{kil} T_{mj}(\phi) - \frac{1}{3} R^m{}_{kjil} T_{im}(\phi) \right) + O(\zeta^3). \quad (1644)$$

Behold, everything is covariant! Thus, this expansion holds in any coordinate system, not just in Riemann normal coordinates. Also note that if T is taken to be the metric, then $\nabla_k g_{ij} = 0$ means

$$g_{ij}(e^\zeta \phi) = g_{ij}(\phi) - \frac{1}{3} R_{ikjl} \zeta^k \zeta^l + O(\zeta^3). \quad (1645)$$

Again, this holds in any coordinate system.

With this preparatory work out of the way, we can start massaging the action. We need to expand both the metric and the derivatives. We know how to do the former; the latter is, in Riemann normal coordinates centered on ϕ^i , (note to self: convince yourself that the derivatives on ϕ don't need to be covariant ones)

$$\begin{aligned} \partial_\mu \left(\phi^i + \zeta^i - \frac{1}{2} \Gamma_{jk}^i \zeta^j \zeta^k \right) &\approx \partial_\mu \phi^i + \partial_\mu \zeta^i - \frac{1}{6} (\partial_\mu \phi^m) (R^i_{lmk} + R^i_{kml}) \zeta^k \zeta^l, \\ &= \partial_\mu \phi^i + \nabla_\mu \zeta^i - \frac{1}{3} \partial_\mu \phi^m R^i_{lmk} \zeta^k \zeta^l, \end{aligned} \quad (1646)$$

where the covariant derivative acting on ζ is

$$\nabla_\mu \zeta^i = \partial_\mu \zeta^i + \partial_\mu \phi^m \Gamma_{mj}^i \zeta^j. \quad (1647)$$

The derivatives on ϕ just convert the derivative into the i, j, k, \dots index space⁴⁹. Again, we derived this in normal coordinates, but the answer holds in a generic coordinate system since it's gauge-invariant under gauge transformations (coordinate transformations) on ζ^i . This is good news for us, since we can use the above expression in the action, the fields of which generically cannot be written in normal coordinates for more than one point in the target manifold (unless of course if $R_{ijkl} = 0$).

Now we put the above into the action, along with the expansion of the metric, and keep everything below third order in ζ . This produces

$$S = \frac{1}{4\pi\alpha} \int \sqrt{\gamma} \gamma^{\mu\nu} \left[g_{ij}(\phi) (\partial_\mu \phi^i \partial_\nu \phi^j + \nabla_\mu \zeta^i \nabla_\nu \zeta^j + 2\nabla_\mu \zeta^i \partial_\nu \phi^j) + R_{ijkl}(\phi) \partial_\mu \phi_i \partial_\nu \phi^k \zeta^j \zeta^l \right] \quad (1649)$$

Unfortunately, we cannot do Feynman diagrams with this action. The reason is that the kinetic term for ζ is dependent on ϕ , since the kinetic term involves the full metric $g_{ij}(\phi)$. This makes doing calculations with the ζ propagator essentially impossible.

To fix this, we need to change the metric in the ζ kinetic term to a flat one. We do this by switching over to vielbeins, with $\zeta^i = e_a^i \zeta^a$. The e_a^i are orthonormal frames and $g_{ij} = e_i^a e_j^b \eta_{ab}$, so that the kinetic term is

$$\gamma^{\mu\nu} g_{ij} \partial_\mu \zeta^i \partial_\nu \zeta^j = \eta_{ab} \gamma^{\mu\nu} \partial_\mu \zeta^a \partial_\nu \zeta^b, \quad (1650)$$

so that the ζ^a propagator is the usual propagator which can now be used in Feynman diagrams. From now on, we will assume a flat spacetime for simplicity, so that $\gamma^{\mu\nu} = \eta^{\mu\nu}$.

To translate the rest of the action into the vierbein formulation, we just need to note how the covariant derivatives change. Let us denote the spin connection (a matrix-valued 1-form on Σ) as $\omega_\mu{}^a{}_b$. The spin connection is built out of the regular affine connection with Γ , plus a term which keeps track of how the basis frames rotate as we travel around Σ , which gives it the properties of an $SO(\dim M)$ gauge field over Σ (in Euclidean signature). This extra term is the analogue of the Maurer-Cartan form $g^{-1}dg$ in gauge theory, and is written as $e_b^i \partial_\mu e_i^a$. Using the vielbeins to deal with the mixed indices in the Christoffel symbols, the spin connection is

$$\omega_\nu{}^a{}_b = e_i^a e_b^k \Gamma_{\nu k}^i - e_i^a \partial_\nu e_b^i = \partial_\nu \phi^j e_i^a e_b^k \Gamma_{jk}^i - e_i^a \partial_\nu e_b^i. \quad (1651)$$

The minus sign in front of the $e_i^a \partial_\nu e_b^i$ term is chosen so that the vielbein frames are covariantly constant: taking the covariant derivative of the framing e_a^i and using the Christoffel symbols to deal with the lower index and the spin connection to deal with the upper index, we have (recall that covariant indices get a minus sign in the covariant derivative and contravariant indices get a plus sign)

$$D_\mu e_j^a = \partial_\mu e_j^a - \Gamma_{ij}^k e_k^a \partial_\mu \phi^i + \omega_\mu{}^a{}_b e_j^b = 0, \quad (1652)$$

⁴⁹This is made possible because the Christoffel symbols transform as a $(2, 1)$ tensor (two co-vector indices and one vector index) up to an extra term

$$\Gamma_{ij}^k = \partial_a x^k \left[\Gamma_{bc}^a \partial_i x^b \partial_j x^c + \partial_i \partial_j x^a \right], \quad (1648)$$

where the transformation is from x^i coordinates to x^a coordinates. This can be checked using algebra and $\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. This means that if we want to only transform one of the bottom indices of the Christoffel symbol, then we can treat Γ as a tensor, essentially since the non-tensorial piece is $\partial_i \partial_b x^a = \partial_i \delta_b{}^a = 0$; this allows us to write $\nabla_\mu \zeta^i$ as above.

which follows from the definition of the spin connection and the definition of the inverse framing $e_b^i e_k^b = \delta_k^i$. In what follows, we will use D_μ to denote covariant derivatives in the vielbein (gauge) formalism, and ∇_μ for covariant derivatives in the regular coordinate basis.

We can use these properties to convert the action into the vielbein formalism. After swapping out the covariant derivatives ∇_μ for the gauge covariant derivatives D_μ , we get

$$S = \frac{1}{4\pi\alpha} \int \eta^{\mu\nu} \left[g_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j + \eta_{ab} \left(D_\mu \zeta^a D_\nu \zeta^b + 2D_\mu \zeta^a e_j^b \partial_\nu \phi^j \right) + e_a^j e_b^l R_{ijkl}(\phi) \partial_\mu \phi_i \partial_\nu \phi^k \zeta^a \zeta^b \right]. \quad (1653)$$

Now the kinetic term for ζ is nice and ready for doing Feynman diagrams. The relevant vertices are a $\omega\omega\zeta\zeta$ vertex, a $k^\mu\omega\zeta\zeta$ vertex, and a $\zeta\zeta\partial\phi\partial\phi R$ vertex. The gauge field ω won't actually contribute to any β functions, since gauge theory in two dimensions has no divergences⁵⁰.

All this means that getting the β function for the metric is incredibly simple. We can do it either by figuring out what counterterms need to be added or by letting the fast fields ζ^i be defined over an energy range from Λ to $\Lambda + d\Lambda$. Adopting this approach, there is only one diagram that contributes to the renormalization of $g_{ij}(\phi)$, namely the one with two outgoing $\partial_\mu \phi^i$ legs joined at a bubble formed from a single ζ^a propagator. This diagram gives

$$(\text{diagram})^{ik} = \int_k e_a^j e_b^l (\zeta^a \zeta^b)(k^2) R_{ijkl} = \ln \left(\frac{\Lambda + d\Lambda}{\Lambda} \right) \eta^{ab} e_a^j e_b^l R_{ijkl} = d \ln \Lambda R_{ik}. \quad (1655)$$

In the last step, we have used (basically, the trace is the same in all coordinates)

$$\eta^{ab} e_a^j e_b^l R_{ijkl} = R_{ilk}^m g_{mj} e_a^j e_n^l \eta^{ab} = R_{ilk}^m g_{mj} g^{jl} = R_{imk}^m = R_{ik}. \quad (1656)$$

So finally, we differentiate the effective g_{ij} (the one which includes the above radiative correction) with respect to $\ln \Lambda$ and find that

$$\beta_{ij} = R_{ij} \quad (1657)$$

as required.

94 July 17 — Another look at currents and operator splitting applied to bosonization

Today is a fast one—verifying a statement I've seen a bunch but never worked out. Consider a free Dirac fermion in two dimensions. Identify the currents (vector and chiral) and compute

⁵⁰Divergences for gauge theories are two degrees lower than naive power counting suggests, I think roughly since we loose one degree of freedom to the non-dynamical A_0 and another to gauge invariance. One can check e.g. that the two diagrams contributing to the renormalization of the ω propagator give a contribution

$$(\text{two diagrams})^{\mu\nu}(q) \sim \int_k \left(\frac{\delta^{\mu\nu}}{k^2} - \frac{2k^\mu(k-q)^\nu}{k^2(k-q)^2} \right) \sim \int_{k,x} \frac{\delta^{\mu\nu} q^2}{(k^2 - \Delta_q)^2}, \quad (1654)$$

which indeed has an integrand going as $1/k^3$ rather than the naive $1/k$. Here the μ, ν are the spacetime indices for the external ω legs.

their commutators, being careful to do the point splitting of the operators that constitute the currents. You should find e.g. for the vector current that

$$[j_\mu(x), j_\nu(y)] = C\partial_x \delta(x - y), \quad (1658)$$

where C is some constant that depends on how one normalizes the currents. Show how this is matched up with the currents of a free boson in line with the usual bosonization formulae.

Solution:

The regular fermion vector currents are $j^\mu = \bar{\psi}\gamma^\mu\psi$, so that in Euclidean signature with $\gamma^0 = X, \gamma^1 = Y$ we have for $\psi = (\psi_L, \psi_R)^T$,

$$j_R \equiv \frac{1}{2}(j_0 + ij_1) = \psi_R^\dagger \psi_R, \quad j_L \equiv \frac{1}{2}(j_0 - ij_1) = \psi_L^\dagger \psi_L. \quad (1659)$$

We will choose the coefficient in front of the action to be such that

$$\langle \psi_L^\dagger(z, \bar{z}) \psi_L(w, \bar{w}) \rangle = \frac{1}{z - w}, \quad \langle \psi_R^\dagger(z, \bar{z}) \psi_R(w, \bar{w}) \rangle = \frac{1}{\bar{z} - \bar{w}}. \quad (1660)$$

Since we get this from $\frac{1}{\pi}\bar{\partial}z^{-1} = \delta(z, \bar{z})$, we want the coefficient in front of the action to be $g = 1/2\pi$.

Anyway, now we can compute the (equal-time) commutator of the currents. Of course, j_R and j_L commute. We then have (as usual, the following is to be understood in the OPE sense of having an implicit expectation value)

$$\begin{aligned} [j_R(x), j_R(y)] &= [:(\psi_R^\dagger \psi_R)(x) : : (\psi_R^\dagger \psi_R)(y) :] \\ &= \lim_{\epsilon, \eta \rightarrow 0} [: \psi_R^\dagger(x + \epsilon) \psi_R(x - \epsilon) : : \psi_R^\dagger(y + \eta) \psi_R(y - \eta) :] \\ &\sim i \lim_{\epsilon, \eta \rightarrow 0} \left(\frac{1}{x + \epsilon - y + \eta} \delta(x - y - \epsilon - \eta) - \frac{1}{x + \epsilon - y + \eta} \psi_R^\dagger(y + \eta) \psi_R(x - \epsilon) \right. \\ &\quad \left. - \frac{1}{y + \eta - x + \epsilon} \psi_R^\dagger(x + \epsilon) \psi_R(y - \eta) - \frac{1}{(y + \eta - x + \epsilon)(x + \epsilon - y + \eta)} \right) \\ &\quad - (x \leftrightarrow y, \epsilon \leftrightarrow \eta), \end{aligned} \quad (1661)$$

where the factor of i came from the i in $1/(\bar{z} - \bar{w})$ evaluated at $\bar{z} = -ix, \bar{w} = -iy$. There may be other factors of 2 somewhere but I'm not going to worry about them too much. The most singular term is symmetric under the interchange $x \leftrightarrow y, \epsilon \leftrightarrow \eta$ and so it dies, while the two middle terms are also symmetric under the interchange. So

$$\begin{aligned} [j_R(x), j_R(y)] &\sim i \lim_{\epsilon, \eta \rightarrow 0} \left(\frac{1}{x + \epsilon - y + \eta} \delta(x - y - \epsilon - \eta) - \frac{1}{y + \eta - x + \epsilon} \delta(y - x - \eta - \epsilon) \right) \\ &= i \lim_{\epsilon, \eta \rightarrow 0} \left(\frac{\delta(x - y - [\epsilon + \eta]) - \delta(x - y + [\epsilon + \eta])}{\eta + \epsilon} \right) = -i\partial_x \delta(x - y). \end{aligned} \quad (1662)$$

When we compute the commutator for j_L the only thing changes is that we have a $-i$ up front instead of a $+i$ by virtue of the ψ_L 2-point function being $1/(z - w)$, so

$$[j_R(x), j_R(y)] \sim i\partial_x \delta(x - y). \quad (1663)$$

Now we can go and rewrite this in terms of the spacetime components of the current. Since $j_0 = j_R + j_L$, $j_1 = -i(j_R - j_L)$ and the $j_{R,L}$ commutators are opposite in sign, we have

$$[j_0(x), j_0(y)] = [j_1(x), j_1(y)] = 0, \quad [j_0(x), j_1(y)] \sim -2\partial_x \delta(x - y). \quad (1664)$$

We could probably have chosen a smarter normalization for the currents so that this dumb factor of 2 wasn't there, but too late. Actually from now on I think it'll be good to go over into real time. Doing this means we need to multiply j_1 or j_0 by i , depending on the signature we want (since we need to change either of the γ matrices to i times itself in order to get the right Clifford algebra relations with the new metric—for definiteness we will let the real-time γ matrices be $\gamma^0 = X$, $\gamma^1 = iY$), and so in real time we have

$$[j_0(x), j_1(y)] \sim -2i\partial_x \delta(x - y). \quad (1665)$$

Now consider a free boson ϕ , and define the currents

$$\mathcal{J}_0 = \sqrt{2}\partial_x \phi, \quad \mathcal{J}_1 = -\sqrt{2}\partial_t \phi, \quad (1666)$$

so that $\mathcal{J}_\mu = \sqrt{2}\epsilon_{\mu\nu}\partial^\nu\phi$ is equal to (the dumb factor of $\sqrt{2}$ times) the topological current, which if ϕ is smooth is trivially conserved. Since the conservation of the fermion current will fail at the locations of certain operator insertions, this tells us that these operator insertions create a topological singularity in the dual ϕ field (so that ϕ is not integrable). Anyway, since $\partial_t\phi$ is the momentum we have

$$[\mathcal{J}_\mu(x), \mathcal{J}_\nu(y)] = -2i\epsilon_{\mu\nu}\partial_x \delta(x - y) = [j_\mu(x), j_\nu(y)]. \quad (1667)$$

Thus we have found a way to represent the fermion current as the topological current of a boson. Note in particular that under bosonization,

$$-j_\mu j^\mu \mapsto -2\epsilon_{\mu\nu}\partial^\nu\phi\epsilon^{\mu\sigma}\partial_\sigma\phi = 2\partial_\mu\phi\partial^\mu\phi, \quad (1668)$$

so that the current bilinear for the fermions becomes the free kinetic term for the boson. This is actually not surprising if we think about the Sugawara construction for the stress tensor: since in models of current algebras we have $T \sim \sum_a : J^a J^a :$, it's not that crazy to think that the current bilinear will bosonize to $\partial\phi\bar{\partial}\phi$, since this is a similar sort of object to the stress tensor.

Anyway, what about the chiral current for the fermions? In two dimensions for our choice of γ matrices we have

$$\gamma^0\gamma^5 = XZ = -iY = \epsilon^{01}iYg_{11}, \quad (1669)$$

since for us, $g_{11} = -1$ and $\epsilon^{P(\mu\nu)} = \text{sgn}(P)$, $\epsilon_{\mu\nu} = -\epsilon^{\mu\nu}$. We also have

$$\gamma^1\gamma^5 = iYZ = \epsilon^{10}Xg_{00}. \quad (1670)$$

Putting these together means that

$$\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu. \quad (1671)$$

So this means that the chiral current is related to the vector current via

$$j^{\mu 5} = \epsilon^{\mu\nu}j_\nu, \quad (1672)$$

which means that the bosonic avatar of the chiral current is

$$\mathcal{J}_\mu^5 = \sqrt{2}\epsilon^{\mu\nu}\epsilon_{\nu\sigma}\partial^\sigma\phi = \sqrt{2}\partial_\mu\phi, \quad (1673)$$

which is conserved by virtue of the free boson's equation of motion.

95 July 18 — Bosonizing the spin 1/2 chain and the sine-Gordon model

Bosonize the $SU(2)$ -symmetric spin 1/2 AFM spin chain (Heisenberg model) using *Abelian* bosonization. You are allowed to use the results of yesterday's diary entry. This is partly just a test of your ability to remember how bosonization works. Try not to look at books!

Solution :

The spin chain is described by (setting the prefactor $J = 1$ for simplicity)

$$H = \sum_{j,a} S_j^a S_{j+1}^a. \quad (1674)$$

The grand strategy is to write things in terms of spinless fermions by way of a Jordan-Wigner transformation, and then do bosonization on these fermions. Hold on you may say, to do bosonization we will need something with central charge $c = 1$, i.e. a two-component fermion. So don't we need spinful fermions? We do not, since the two “spin” components will come from the left- and right-moving fermion excitations around the two Fermi points.

Anyway, we will work in a basis where S^z is diagonalized. The appropriate strings are built out of $\prod_{i < j} (1 - 2c_i^\dagger c_i) = \prod_{i < j} (-1)_i^F$, and the raising / lowering operators are

$$S_j^+ = (-1)^j \prod_{i < j} (-1)_i^F c_j, \quad S_j^- = (-1)^j \prod_{i < j} (-1)_i^F c_j^\dagger. \quad (1675)$$

The raising and lowering operators have been staggered by a factor of $(-1)^j$ since we anticipate expanding round a staggered spin configuration (this of course doesn't affect the operator algebra). One then uses $[S_j^+, S_j^-] = Z$ to get

$$S_j^z = -\frac{1}{2}(-1)_j^F. \quad (1676)$$

Checking that all these operators commute with one another as they should is straightforward.

Now we need to plug these into the Hamiltonian to get H as a function of the c operators. This is straightforward and produces (rescaling J by a factor of 4)

$$H = J \sum_j \left[(-1)_j^F (-1)_{j+1}^F - 2(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) \right]. \quad (1677)$$

Note that the presence of the $S^z S^z$ term is responsible for giving us the interactions: if the Hamiltonian only had $U(1)$ symmetry (instead of $SU(2)$), we would have free fermions. Also note that if we had a very strong anisotropy for the $S^z S^z$ term we would need $(-1)_j^F = -(-1)_{j+1}^F$, so that the fermion occupancy on the chain would be staggered. This is the CDW state.

Now we want to go over to continuum fermions. We build a Dirac fermion out of the the fermionic excitations around each of the Fermi points at $k_F = \pm\pi/2$ (since the hopping

term gives a $\cos k$ dispersion for $-\pi/2 \leq k \leq \pi/2$). Let the lattice spacing be a , and define continuum fields L, R via

$$c_j = \sqrt{a}(R_j e^{ik_F j} + L_j e^{-ik_F j}) = \sqrt{a}(i^j R_j + i^{-j} L_j). \quad (1678)$$

The \sqrt{a} here is to get the dimensions right: we want the continuum fermions to have mass dimension $[L] = [R] = 1/2$, while the c_j fermions are dimensionless by virtue of $\{c_i, c_j^\dagger\} = \delta_{ij}$. One of the hopping terms is then

$$c_j^\dagger c_{j+1} \approx a(i^{-j} R_j^\dagger + i^j L_j^\dagger)(i^{j+1} R_j + i^{j+1} a \partial_x R_j + i^{-j-1} L_j + i^{-j-1} a \partial_x L_j). \quad (1679)$$

Higher derivative terms get suppressed by higher powers of a . When we sum over j , terms that oscillate with j will die—they represent high energy events that hop fermions between the two Fermi points, with operators like $L_j^\dagger \partial R_j$, and so we can get rid of them. The expression above has $iR^\dagger R$ and $iL^\dagger L$ terms, but when we add the Hermitian conjugate these terms die, and so

$$a^{-1} \sum_j (c_j^\dagger c_{j+1} + h.c.) = -2i \int dx (R^\dagger \partial_x R - L^\dagger \partial_x L), \quad (1680)$$

which gives us the free Dirac fermion.

Now for the interaction term. Since $k_F = \pi/2$, we are at half filling for the spinless fermions. Thus $\langle c_j^\dagger c_j \rangle = 1/2$, and so

$$-(-1)_j^F = 2c_j^\dagger c_j - 1 = 2 : c_j^\dagger c_j := 2a : (i^{-j} R_j^\dagger + i^j L_j^\dagger)(i^j R_j + i^{-j} L_j) : \quad (1681)$$

The interaction term then becomes, in the continuum variables,

$$\begin{aligned} a^{-1} \sum_j (-1)_j^F (-1)_{j+1}^F &\approx \int dx : \left(R^\dagger R + L^\dagger L + (-1)^j (L^\dagger R + R^\dagger L) \right) : \\ &\quad \times : \left(R^\dagger R + L^\dagger L - (-1)^j (L^\dagger R + R^\dagger L) \right) : \end{aligned} \quad (1682)$$

Here we have dropped all derivatives, since they all contain an extra factor of a that make them comparatively small (remember that we are assuming the L, R vary slowly over the lattice scale). We can drop the terms that go as $(-1)^j$, but we still get an Umklapp term from the $(-1)^{2j} = 1$ term. From yesterday's problem, we recall the currents $j_0 = R^\dagger R + L^\dagger L$, $j_1 = L^\dagger L - R^\dagger R$ (the sign of j_1 is dictated by our choice of $\gamma^1 = -iY$, which we did so that $\gamma^5 = Z$ and not $-Z$). We then have the current bilinear $(j_\mu)^2 = 2[(R^\dagger R)^2 + (L^\dagger L)^2]$. Note that I have stopped indicating the normal ordering, for notation's sake. However, it is important to remember that it is there, so that e.g. $(R^\dagger R)^2$ really means $(: R^\dagger R :)^2$. When we do bosonization, we will need to be careful to only bosonize things that have been normal-ordered.

After some algebra, we can then write the interaction term as (there are some factors of 2 that we've absorbed into a , all we care about is the relative factor between the different terms)

$$H_I = \int dx \left(j_\mu j^\mu - 2 \left[(L^\dagger R)^2 + (R^\dagger L)^2 \right] \right). \quad (1683)$$

This means that the full action in terms of the fermions becomes ($\Psi = (L, R)^T$)

$$S = \frac{1}{2\pi} \int dx dt \left(i\bar{\Psi}\not{\partial}\Psi - j_\mu j^\mu + 2 \left[(L^\dagger R)^2 + (R^\dagger L)^2 \right] \right). \quad (1684)$$

I think the $1/2\pi$ factor in front will be the most convenient for avoiding gross $\sqrt{4\pi}$'s and stuff, but I'm not sure. We'll see how it goes.

Now let us bosonize. We will roughly follow the normalization conventions in Witten's lectures in Quantum Fields and Strings part II. As we saw from yesterday's entry, the current-current interaction $-j_\mu j^\mu$ miraculously bosonizes into a free $\partial_\mu \phi \partial^\mu \phi$ term! The free Dirac term also maps to $\partial_\mu \phi \partial^\mu \phi$. With our convention for the action, I believe the correct bosonic image of the free Dirac term is

$$\mathcal{B}(S_D) = \frac{1}{8\pi} \int \partial_\mu \phi \partial^\mu \phi, \quad (1685)$$

which is a compact boson at radius $R = 1/\sqrt{2}$. In terms of the holomorphic / anti-holomorphic components of the boson, the mapping is

$$\mathcal{B}(L) = e^{i\phi_+}, \quad \mathcal{B}(R) = e^{-i\phi_-}, \quad (1686)$$

which one can check reproduces the correct scaling dimensions (the vertex operators $e^{\pm i\phi_\pm}$ have a two-point function that goes like $1/(x-y)^{1/2R^2}$, which is what we want since $R = 1/\sqrt{2}$). If we were to be a bit more careful, we should probably write this as $\mathcal{B}(L) = \frac{1}{\sqrt{2\pi a}} e^{i\phi_+}$ where a is the short-distance cutoff. This ensures that the bosonized fermion has the same dimension of the fermion, and still produces the right correlators since when we are remembering to include the cutoff the propagator for the boson goes like $\ln |r/a|$ instead of just $\ln |r|$. Anyway, we'll suppress the cutoff dependence in what follows.

We can then conclude that

$$\mathcal{B}([LR^\dagger]^2) = e^{2i\phi}, \quad (1687)$$

which means that the extra interaction term in the fermionic version of S maps to a sum of vertex operators. The freeness of the bosonized theory is thus ruined by the extra interacting part in S . Anyway, we can now map the full action over to bosons:

$$\mathcal{B}(S) = \int dx dt \left[\frac{(1+1)}{8\pi} \partial_\mu \phi \partial^\mu \phi + \frac{1}{\pi} \cos 2\phi \right]. \quad (1688)$$

The only part of this that is questionable is second number 1 in $1+1$; this came from our conventions in the previous diary entry which I think jive with the ones today but which I haven't checked closely. This 1 can (but won't) be checked by checking the relevance of the $\cos 2\phi$ interaction, by rescaling the fields so that the factor in front of the kinetic term is 1/2: we let $\varphi = \phi/\sqrt{2\pi}$, and the interaction cosine becomes $\cos 2\phi = \cos(\beta\varphi)$, with $\beta = \sqrt{8\pi}$. I may come back and see if this is indeed correct later.

Finally, a side comment on adding interactions to the fermion Hamiltonian. The interactions that are both tractable and interesting are the current-current terms, which can be written in terms of densities as products like $\rho_L \rho_R$ and which bosonize to the free term. Consider on the other hand an interaction I like

$$H \ni I = \int d^2x d^2y \rho(x) V(x-y) \rho(y), \quad (1689)$$

where $V(x - y)$ is taken to be a contact interaction $\delta(x - y)$. Using

$$\rho = \rho_L + \rho_R + (\psi_R^\dagger \psi_L e^{-2ik_F x} + h.c.), \quad (1690)$$

we see that (up to an Umklapp term), the density-density contact term becomes precisely $\rho_L^2 + \rho_R^2$. Since we are thinking in terms of Hamiltonians we do the point-splitting in space, and so this bosonizes to $(\partial_x \phi_L)^2 + (\partial_x \phi_R)^2$, which just renormalizes the speed of light for the bosons, as we expect from a contact term. So the $\rho_{R/L}^2$ terms are less interesting than the $\rho_L \rho_R$ ones.

96 July 19 — The spin 1/2 chain and the LSM theorem

Demonstrate the existence of the anomaly between translation and spin symmetry in the spin 1/2 AFM chain by using Abelian bosonization via Jordan-Wigner as in yesterday's diary entry, and by looking at the full $U(2)_1$ WZW model. Read the first bit of section two of [?] for some inspiration regarding anomalies.

Solution:

The bosonization of the S^z operator is

$$\begin{aligned} \mathcal{B}[S_j^z] &= -\frac{1}{2} \mathcal{B}[(-1)_j^F] = \mathcal{B}[: c_j^\dagger c_j :] = a \mathcal{B} \left[: R^\dagger R + L^\dagger L + (-1)^j (R^\dagger L + L^\dagger R) : \right] \\ &= a \mathcal{B} \left[: j_0 : + (-1)^j \bar{\Psi} \Psi \right]. \end{aligned} \quad (1691)$$

Here we've used $c_j^\dagger c_j - 1/2 \rightarrow : c_j^\dagger c_j :$. It's important to do this since we are only defining the bosonization map \mathcal{B} to act on normal-ordered operators.

As we saw last time, the time component of the fermion current bosonizes to the space current of the boson current, times some prefactor. The Dirac mass term becomes a $\cos(\phi)$ interaction. Rather than following the conventions adopted in the previous diary entry, we will follow more standard conventions with more factors of $\sqrt{\pi}$ (see e.g. Fradkin's book). With these conventions, the j_0 fermion current bosonizes to

$$\mathcal{B}[: j_0 :] = \frac{1}{\sqrt{\pi}} \partial_x \phi. \quad (1692)$$

In these conventions, the current commutators for the fermions have a $1/\pi$ in front the delta function—hence the $1/\sqrt{\pi}$ here. The Dirac mass term becomes a cosine interaction with parameter $\beta = \sqrt{4\pi}$ since each fermion is now bosonized as

$$\mathcal{B}[R] = \frac{1}{\sqrt{2\pi a}} : e^{i\sqrt{4\pi}\phi_+} :, \quad \mathcal{B}[L] = \frac{1}{\sqrt{2\pi a}} : e^{-i\sqrt{4\pi}\phi_-} :. \quad (1693)$$

The a 's are inserted to give the images of the fermions the right dimensions, and we have omitted the Klein factors needed to ensure that the images of the fermions anticommute

(recall that ϕ_+ commutes with ϕ_-). Note the opposite signs in the exponentials! Also note that with these conventions, the period of the chiral components and of the field ϕ is $\sqrt{\pi}$ since

$$\phi_{\pm} \sim \phi_{\pm} + \sqrt{\pi}\mathbb{Z}. \quad (1694)$$

Our bosonizations of R, L mean that

$$\mathcal{B}[\bar{\Psi}\Psi] = \frac{1}{\pi a} : \cos(\sqrt{4\pi}\phi) :, \quad (1695)$$

so that

$$\mathcal{B}[S_j^z] = \frac{a}{\sqrt{\pi}} \partial_x \phi + (-1)^j \frac{1}{\pi} : \cos(\sqrt{4\pi}\phi) : \quad (1696)$$

Note also that the staggered magnetization (the Ising order parameter) is the fermion mass term:

$$M_j^z \equiv \frac{S_j^z - S_{j+1}^z}{2} = 2a(R^\dagger L + L^\dagger R) = 2a\bar{\Psi}\Psi, \quad \mathcal{B}[M_j^z] = \frac{2}{\pi} : \cos(\sqrt{4\pi}\phi) :. \quad (1697)$$

Now for the images of the S_j^{\pm} operators under \mathcal{B} . For this, it is helpful to introduce the dual field $\theta = -\phi_+(t-x) + \phi_-(t+x)$. Letting $\partial \equiv \frac{1}{2}(\partial_t + \partial_x)$, $\bar{\partial} \equiv \frac{1}{2}(\partial_t - \partial_x)$ so that the right-mover ϕ_+ is “antiholomorphic” with $\partial\phi_+ = 0$ and the left-mover is “holomorphic” with $\bar{\partial}\phi_- = 0$, this decomposition means that

$$\partial\theta = \partial\phi, \quad \bar{\partial}\theta = -\bar{\partial}\phi, \quad (1698)$$

i.e. that

$$\partial_t\theta = \partial_x\phi, \quad \partial_x\theta = \partial_t\phi. \quad (1699)$$

Note that neither of these equations has a minus sign! This is because we are working in real time with metric $\eta_{\mu\nu} = Z_{\mu\nu}$. We can write this more succinctly as

$$\partial_\mu\theta = \epsilon_{\mu\nu}\partial^\nu\phi, \quad (1700)$$

which explicitly is $\partial_t\theta = \epsilon_{tx}\partial^x\phi = -\epsilon^{tx}(-\partial_x)\phi = +\partial_x\phi$, $\partial_x\theta = \epsilon_{xt}\partial^t\phi = -\epsilon^{xt}\partial_t\phi = \partial_t\phi$, where $\epsilon^{ab} = \text{sgn}(ab) = -\epsilon_{ab}$. From the identification $\partial_x\theta = \partial_t\phi$, we can identify the canonical momentum for ϕ with $\partial_x\theta$, so that we have the equal-time relations

$$[\phi(x), \partial_y\theta(y)] = i\delta(x-y). \quad (1701)$$

We will take a rather backdoor approach to finding $\mathcal{B}[S_j^{\pm}]$ by using $[S_j^{\pm}, S_i^z] = \mp\delta_{ij}S_j^{\pm}$ and by noting that since the JW images of the S_j^{\pm} have strings that give them their commutation relations with the S_j^z , the result for $\mathcal{B}[S_j^{\pm}]$ will not be local in the ϕ fields. Since we know it needs to have a commutator with $\partial_x\phi$ proportional to itself, we guess $\mathcal{B}[S_j^{\pm}] \sim (-1)^j e^{\pm i\sqrt{4\pi}\theta}$. This is the right form for the answer since it's single-valued under e.g. $\phi_+ \mapsto \phi_+ + \sqrt{\pi}\mathbb{Z}$ and since

$$[\partial_x\phi(x), e^{i\sqrt{4\pi}\theta(y)}] = \sqrt{4\pi}\delta(x-y). \quad (1702)$$

This can also be verified on the fermion side, where the relevant commutator is $[R^\dagger R + L^\dagger L, L^\dagger R^\dagger] = 2L^\dagger R^\dagger$ ($e^{i\sqrt{4\pi}\theta} \rightarrow L^\dagger R^\dagger$). In order for this ansatz to work, we need to see how it

commutes with the $\cos(\sqrt{4\pi}\phi)$ part of S^z . But indeed, since the cosine goes to $R^\dagger L + L^\dagger R$ and we have $[R^\dagger L + L^\dagger R, R^\dagger L^\dagger] = 0$, everything works. So to conclude (actually, not totally sure about S_j^\pm). This is what Max and Ryan wrote down in their paper but I think it might be a little more subtle),

$$\mathcal{B}[S_j^z] \sim \frac{a}{\sqrt{\pi}} \partial_x \phi + (-1)^j \frac{1}{\pi} : \cos(\sqrt{4\pi}\phi) :, \quad \mathcal{B}[S_j^\pm] \sim (-1)^j : e^{i\sqrt{4\pi}\theta} : \quad (1703)$$

Now let's figure out how the symmetries act on the bosonized fields. For translation, when $j \mapsto j+1$ we see that the fermions map as

$$T : R \mapsto iR, \quad L \mapsto -iL, \quad (1704)$$

which comes from $e^{\pm ik_F(j+1)} = \pm e^{ik_F j}$. This then implies

$$T : \phi_\pm \mapsto \phi_\pm + \sqrt{\pi}/4, \quad \phi \mapsto \phi + \sqrt{\pi}/2, \quad \theta \mapsto \theta. \quad (1705)$$

Note that the Dirac mass $\bar{\Psi}\Psi$ is odd under translation, and that our identification of the period of the ϕ_\pm fields with $\sqrt{\pi}$ means that translation actually acts as a \mathbb{Z}_2 internal symmetry on the boson fields. When we write the Bosonized version of the Heisenberg chain action we get a $\cos(4\sqrt{\pi}\phi)$ term, which thankfully is invariant under translation. Also note that Max and Ryan say that $\theta \mapsto \theta + \sqrt{\pi}/2$ in their paper, which doesn't make sense to me yet. This additional shift of θ would help make the pure translation anomaly more clear; note to self: come back to this eventually.

The other symmetry is the $SO(3)$ symmetry, although it is easier and more standard to look at the $U(1)$ subgroup of rotations about \hat{z} . Rotating by an angle α means $(S^x, S^y)^T \mapsto R_\alpha(S^x, S^y)^T$, which one checks sends $S^\pm \mapsto e^{\pm i\alpha} S^\pm$, while of course S^z is invariant. Thus we can identify the action on the boson variables as

$$U(1)_z : \phi \mapsto \phi, \quad \theta \mapsto \theta + \frac{\alpha}{\sqrt{4\pi}}, \quad (1706)$$

with the action chosen so that rotating by $\alpha \in 2\pi\mathbb{Z}$ does nothing. That this $U(1)$ acts trivially on ϕ is good since we have a $\cos\phi$ term in the bosonized action, which would cause problems if ϕ wasn't invariant.

We can now figure out the charge operators and diagnose the anomaly that is responsible for the LSM theorem. From the commutation rules, they are

$$Q_T(M) = \frac{\sqrt{\pi}}{2} \int_M \partial_x \theta, \quad Q_\alpha(M) = \frac{\alpha}{\sqrt{4\pi}} \int_M \partial_x \phi, \quad (1707)$$

so that doing e.g. $e^{iQ_T(M)} \phi(x) e^{-iQ_T(M)}$ translates ϕ by $\sqrt{\pi}/2$ if $x \in M$ as it should.

The anomaly can now be seen from the fact that the charge operators don't commute with each other, meaning that if we try to gauge one of the symmetries, we will break the other. As another way of saying this, the charge operators are not splittable: they must be integrals over all of space, otherwise we run into problems since the boundary points of their integration domains carry quantum numbers of the opposite symmetry. For example, consider acting with $e^{iQ_T(I)}$ for $I = [a, b]$ an interval. This basically inserts two lattice points

at each of a, b . Then from our bosonized expression for S^z , we see that the endpoints of this interval carry spin 1/2:

$$\left[\int_{a-\delta}^{a+\delta} dx \frac{1}{\sqrt{\pi}} \partial_x \phi, e^{iQ_T(I)} \right] = \frac{1}{\sqrt{\pi}} [\partial_x \phi(a), e^{i\frac{\sqrt{\pi}}{2}[\theta(a)-\theta(b)]}] = \frac{1}{2}, \quad (1708)$$

for any $0 < \delta < |a - b|$. This is just coming from the fact that e.g. if we start with an AKLT-like picture where the microscopic spin 1/2's pair up into singlets, then inserting a lattice site leaves a “dangling” spin 1/2. This means that if we were to try to gauge the translation symmetry we would not have states in a representation of $SO(3)$ symmetry, but rather a in a representation of the \mathbb{Z}_2 extension $SU(2)$ (since T acts as a \mathbb{Z}_2 symmetry). A similar thing happens when we insert a $U(1)$ flux by acting with $Q_\alpha(I)$. Also note that while we have been discussing $U(1) \subset SO(3)$ spin symmetry, we actually could get away with a lot less. In fact, the \mathbb{Z}_2 subgroup of the spin symmetry is still enough to enable the presence of an anomaly.

From the relation of the critical spin 1/2 chain and the bulk 2+1 dimensional SPT, we expect that the theory can be gapped out if we add perturbations which don't respect the symmetry. Let's do an example to check that this is the case.

Suppose we break translation symmetry by adding the perturbation

$$\delta H = J_t \sum_i (-1)^i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+). \quad (1709)$$

In the continuum, this goes over to (look back to a few diary entries ago to see how we pass to the continuum limit)

$$\delta H \rightarrow -iJ_t a \int (L^\dagger R - R^\dagger L), \quad (1710)$$

where we are omitting irrelevant terms which disappear in the IR (terms which have a factor of $(-1)^i$ in the sum over lattice sites). When we bosonize, this becomes the interaction

$$\delta H \leftrightarrow \frac{J_t}{\pi} \int \sin(\sqrt{4\pi}\phi). \quad (1711)$$

This is a relevant interaction: indeed, the scaling dimension of the interaction $\sin(\beta\phi)$ is

$$\Delta = \frac{\beta^2}{2R^2}, \quad (1712)$$

so that the dividing point between relevance and irrelevance is at $\beta = 2R$, with $\beta > 2R$ being irrelevant (R is determined by the coefficient of the kinetic term being $R^2/4\pi$). For our present conventions $R = \sqrt{2\pi}$ so that the marginal point is $\beta = \sqrt{8\pi}$. Thus since $\sqrt{4\pi} < \sqrt{8\pi}$, adding δH indeed produces a massive spectrum (in the bosonization of the unperturbed chain, translation symmetry ensured that we could only have $\beta = 2\sqrt{4\pi}$ interactions, which are irrelevant and leave the spectrum massless).

97 August 1 — WZ consistency conditions and descent equations

After a bit of a break traveling, it's time to return to the diary.

In the BRST formalism, show that the BRST operator is exact when acting on both the gauge field and on the ghosts. Use these facts to derive the descent equations linking the different types of anomalies in different dimensions together. This is pretty standard stuff, but I got rather confused about signs / commutativity issues reading the presentation in e.g. Nakahara, and decided that for posterity's sake it'd be good to write out the details.

Solution:

We will let S be the BRST operator which implements gauge transformations along the fiber of the bundle \mathcal{A}/\mathcal{G} , where \mathcal{A} is the space of all gauge field configurations and \mathcal{G} is the group of local gauge transformations. The anomaly has a chance to be nontrivial when the cohomology of S is nontrivial, i.e. when the topology of \mathcal{A}/\mathcal{G} is nontrivial.

As usual, we will work with a bicomplex of forms that have a degree under d (the degree of the differential form) and a degree under S (the ghost number), with d and S anticommuting:

$$dS + Sd = 0. \quad (1713)$$

Both d and S have the following graded product rule:

$$d(a \wedge b) = da \wedge b + (-1)^{|a|} a \wedge db, \quad S(a \wedge b) = Sa \wedge b + (-1)^{|a|} a \wedge Sb, \quad (1714)$$

where $|a|$ is the total degree (differential form degree + ghost number) of a (it needs to be the total degree since d and S anticommute), and the wedge product is meant to take place in both complexes.

We need to figure out how S acts on the ghosts and the gauge fields. To do this, it is helpful to define a basepoint A_\bullet from which we can reach other points in \mathcal{A} . We can reach a point A by (using math conventions for is and stuff)

$$A = g^{-1}(A_\bullet + d)g, \quad (1715)$$

with $F = gF_\bullet g^{-1}$. Note that since A, A_\bullet are differential 1-forms, they both anticommute with S . The ghost ω is defined by varying g along the fiber, and then transporting the variation back to the origin with g^{-1} :

$$\omega = g^{-1}Sg = g^{-1}\delta g|_{\text{fiber}}. \quad (1716)$$

The action of S on the ghost is easy:

$$S\omega = -g^{-1}Sgg^{-1}Sg = -\omega^2, \quad (1717)$$

so that

$$S^2\omega = -S\omega^2 = -(S\omega)\omega + \omega(S\omega) = \omega^3 - \omega^3 = 0, \quad (1718)$$

since ω has total degree 1.

The action of S on the gauge field is (using $Sd = -dS$ and $A_\bullet = g(A - d)g^{-1}$)

$$\begin{aligned} SA &= S[g^{-1}(A_\bullet + d)g] = -g^{-1}SgA - g^{-1}A_\bullet Sg - g^{-1}d(Sg) \\ &= -\omega A - (A - d)g^{-1}Sg - g^{-1}d(gg^{-1}Sg) \\ &= -\omega A - A\omega - dw \\ &= -D_A\omega. \end{aligned} \tag{1719}$$

S is also nilpotent on A , since

$$\begin{aligned} SSA &= -SD_A\omega \\ &= \omega^2 A + \omega(SA) - (SA)\omega - A\omega^2 - d(\omega^2) \\ &= \omega^2 A - \omega^2 A - \omega A\omega - \omega d\omega + \omega A\omega + A\omega^2 + (d\omega)\omega - A\omega^2 - (d\omega)\omega + \omega(d\omega) \\ &= 0. \end{aligned} \tag{1720}$$

Let α_n be the anomaly, which is a top-dimensional form closed under S (so that $S\alpha_n = d\alpha_n = 0$). Here the subscript and superscript indicate that it is a differential form of degree n , with ghost number 0. Now $dS\alpha_n = 0$, so locally we have

$$\alpha_n = d\alpha_{n-1}^0. \tag{1721}$$

Then we also have

$$dS\alpha_{n-1}^0 = -S\alpha_n = 0, \tag{1722}$$

so that locally

$$S\alpha_{n-1}^0 = d\alpha_{n-2}^1. \tag{1723}$$

The pattern continues, and we derive the descent equations

$$S\alpha_{n-k}^{k-1} = d\alpha_{n-k-1}^k, \tag{1724}$$

for $0 \leq k < n$.

The usual example of an anomaly which satisfies the descent equations is the $(n/2)$ th Chern character

$$\alpha_n = \text{ch}_{n/2}(F) = \frac{1}{(n/2)!} \text{Tr} \left(\frac{iF}{2\pi} \right)^{n/2}. \tag{1725}$$

This is the total derivative of a Chern-Simons term, whose gauge variation is the total derivative of ω wedged with the $(n/2 - 1)$ th Chern character, and so on. For a $U(1)$ gauge field, the basic relations are

$$\alpha_4 = \frac{1}{8\pi^2} F \wedge F = \frac{1}{8\pi^2} d(A \wedge F) = d\alpha_3^0, \tag{1726}$$

and

$$S\alpha_3 = \frac{1}{8\pi^2} d\lambda \wedge F = \frac{1}{8\pi^2} d(\lambda \wedge F) = d\alpha_2^1, \tag{1727}$$

where we've written $d\lambda$ for the gauge transformation instead of ω for clarity.

98 August 2 —Asymptotic symmetry in an RG flow

This is from P&S, chapter 12. Consider two massless copies of ϕ^4 theory interacting quadratically:

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 + (\partial\theta)^2) - \frac{\lambda}{4!}(\phi^2 + \theta^2) - \frac{2\rho}{4!}\phi^2\theta^2. \quad (1728)$$

Compute the β functions for λ , ρ , and ρ/λ . Show that $\beta_{\rho/\lambda}$ has fixed points at 0, 1, 3, and that the one at $\rho/\lambda = 1$ is stable, so that if e.g. we start with $\rho/\lambda < 3$ then we flow to a phase where $\rho = \lambda$, which is characterized by the model obtaining an $O(2)$ global symmetry in the IR.

Solution:

We just want the β functions to leading order in the couplings. Let us find the counterterm for ρ to one-loop order by examining the one-loop corrections to a vertex with two external ϕ lines and two external θ lines. There are six diagrams, counting the original ρ term and the δ_ρ counterterm. The four other diagrams are s and t channel $2\phi \rightarrow 2\theta$ diagrams, and two u channel diagrams (warning, I may have gotten mixed up with s and u), which differ by the type of field (θ or ϕ) which propagate in the loop. The basic integral is

$$\text{some typical diagram} = -C \int_q \int_x \frac{1}{(q^2 - \Delta)^2}, \quad (1729)$$

where x is a Feynman parameter, C is a constant that depends on the type of diagram, and Δ is some function of the square of the external momenta (e.g. the s, t, u channels $(p_1 + p_2)^2, (p_1 + p_4)^2$, etc.) and x . In dim reg, this gives

$$\text{some typical diagram} = -\frac{Ci}{4\pi^2} \int_x \left(\frac{2}{\epsilon} - \ln(\Delta/\Lambda) + \dots \right), \quad (1730)$$

with Λ the UV cutoff. Basically all we then need to do is compute the different C s for the different diagrams.

The s channel diagram has two ρ vertices. Each vertex contributes a factor of $-i(2\rho/4!) \cdot 2 = -i\rho/3$, where the extra factors of 2 are the symmetry factor. Thus the 1-loop corrections for the s and t channels are (just writing the \ln terms and putting p_1, p_2 on the “bottom” of the diagrams and p_3, p_4 on the “top”)

$$-(\rho/3)^2 \frac{i}{(4\pi)^2} \left(\ln \frac{-(p_1 + p_2)^2}{\Lambda^2} + \ln \frac{-(p_1 + p_4)^2}{\Lambda^2} \right). \quad (1731)$$

The u channel diagrams have one $-i(\lambda/4!) \cdot 4!$ vertex and one $-i\rho/3$ vertex, plus a symmetry factor of 1/2 since the two internal lines are identical. These thus add and produce

$$-(\rho/3)\lambda \frac{i}{(4\pi)^2} \ln \frac{-(p_2 + p_4)^2}{\Lambda^2}. \quad (1732)$$

The ρ counterterm enters into the diagrams as $-i\delta_\rho/3$, since the original ρ interaction appeared with a $1/3$ prefactor. We will fix our renormalization conditions so that the effective interaction between two θ s and two ϕ s is $\rho/3$ for the choice of momenta where $s = t = u = -M^2$ (the RG conditions are imposed at spacelike momenta as usual). Thus the renormalization-scale dependent part of the counterterm needs to be

$$\delta_\rho \sim \frac{1}{16\pi^2} (\lambda\rho + 2\rho^2/3) \ln(M^2/\Lambda^2). \quad (1733)$$

This gives us the β function for ρ :

$$\beta_\rho = \frac{d\rho}{d\ln M} = \frac{1}{8\pi^2} (\lambda\rho + 2\rho^2/3) \ln(M^2/\Lambda^2). \quad (1734)$$

Now for β_λ , which we evaluate by focusing on the correction to graphs with four external ϕ lines. In addition to the bare λ term and the $-i\delta_\lambda$ counterterm, there are six one-loop graphs. There are s, t, u channel graphs for graphs with an internal ϕ loop that go as λ^2 , and likewise there are s, t, u channel graphs where the internal loop is a θ . The former three have a factor of $\lambda^2/2$ where the $1/2$ is the symmetry factor of the internal loop, and the latter three similarly have a factor of $(\rho/3)^2/2$. Thus the M -dependent part of the counterterm needed to reduce the full term to $-i\lambda$ at our RG scale is

$$\delta_\lambda \sim \frac{1}{16\pi^2} (3\lambda^2/2 + \rho^2/6) \ln(M^2/\Lambda^2), \quad (1735)$$

where we've remembered to multiply by 3 since each s, t, u channel result is the same. Thus β_λ is

$$\beta_\lambda = \frac{1}{8\pi^2} (3\lambda^2 + \rho^2/6). \quad (1736)$$

The β function for the ratio is

$$\begin{aligned} \beta_{\rho/\lambda} &= \frac{1}{\lambda} \left(\beta_\rho - \frac{\rho}{\lambda} \beta_\lambda \right) \\ &= \frac{1}{8\pi^2\lambda} (\lambda\rho + 2\rho^2/3 - 3\rho\lambda/2 - \rho^3\lambda/6). \end{aligned} \quad (1737)$$

Note that this has fixed points at $\rho = 0$, $\rho = \lambda$, and $\rho = 3\lambda$. The $\rho = \lambda$ fixed point has an “emergent” global $O(2)$ symmetry, so we would like to know whether this fixed point is attractive or not. Indeed it is: writing $\rho = 3\lambda(1 - \eta)$ for small η , one gets

$$\beta_{\rho/\lambda} \approx \frac{3\lambda\eta}{8\pi^2}, \quad (1738)$$

so that ρ/λ gets smaller at long distances, approaching the $\rho/\lambda = 1$ fixed point. Likewise if $\rho = \eta\lambda$ then $\beta_{\rho/\lambda} \approx -\eta\lambda/(16\pi^2)$, so that ρ/λ increases as we flow to larger distances, again approaching the $\rho/\lambda = 1$ symmetric fixed point.

99 August 3 — unfinished Thirring model beta function

Consider the massless Thirring model, describing fermions in two dimensions with a current-current interaction:

$$S = \frac{1}{2\pi} \int \bar{\psi} i\partial^\mu \psi + \lambda \int j_V^\mu j_V^\mu, \quad j_V^\mu = : \bar{\psi} \gamma^\mu \psi :. \quad (1739)$$

From Abelian bosonization, we know that the current-current interaction maps to the free kinetic term for the bosons. Thus we know that not only is this model secretly free, it is actually just the compact boson CFT! If this is the case, we had better have that $\beta_\lambda = 0$. Show that this is the case, at least to order λ^2 .

Solution:

First let us point out an amusing feature of the model: we have a \mathbb{Z}_2 symmetry which sends $C_- : \lambda \mapsto -\lambda$. One can see that this is implemented by the field transformation $C_- : \psi_+ \mapsto \psi_+, \psi_- \mapsto \psi_-^\dagger$ which charge conjugates the right-handed (negative chirality) component (with the notation $\psi = (\psi_+, \psi_-)^T$). The kinetic term is invariant, while it's easy to check that the transformation exchanges the vector and axial currents:

$$C_- : j_V^\mu \mapsto j_A^\mu, \quad j_A^\mu = : \bar{\psi} \gamma^\mu \bar{\gamma} \psi :. \quad (1740)$$

Since the axial and vector currents are related in two dimensions by Hodge duality $j_A^\mu = \epsilon^{\mu\nu} j_V^\nu$, we see that the current-current interaction maps as

$$C_- : j_V^\mu j_{V\mu} \mapsto j_A^\mu j_{A\mu} = -j_V^\mu j_{V\mu}, \quad (1741)$$

so that indeed, $C_- : \lambda \mapsto -\lambda$ (this can also be checked explicitly since the current-current term is $\psi_+^\dagger \psi_+ \psi_-^\dagger \psi_-$). The \mathbb{Z}_2 symmetry $\lambda \sim -\lambda$ means that there can be no contribution to β_λ which is odd in λ .

Note that if we were to do bosonization, the current-current interaction would become a kinetic term, and so the value of the boson radius R would get changed by an amount controlled by λ . The symmetry C_- then becomes a \mathbb{Z}_2 duality for the free compact boson.

We can also check the vanishing of the β function directly by calculating Feynman diagrams. The one-loop $O(\lambda^2)$ correction to the fermion propagator is zero since it contains $\int_k \frac{k}{k^2} = 0$. The one-loop correction to the λ vertex is formed by gluing two legs on each of two $j_V^\mu j_V^\mu$ vertices together to form a closed loop. This diagram naively contributes to the β function since it has a logarithmic divergence: $\int_k \frac{k(k+p)}{k^2(k+q)^2} \sim \ln \Lambda$. However, the trace over the virtual fermion indices actually means that the diagram vanishes. Indeed, inserting the γ^μ matrices from the currents means that the diagram, we see that the spin trace kills the diagram:

$$\text{1-loop } j_V^\mu j_V^\mu \text{ correction} \sim \lambda^2 \int_q \frac{\text{Tr}[\not{q} \gamma^\mu (\not{p} + \not{k}) \gamma^\nu g_{\mu\nu}]}{q^2(p+q-k)^2} = 0, \quad (1742)$$

where k, p are external momenta. That the spin trace kills the diagram can be seen by using the Clifford algebra relations for the γ^μ matrices and by using rotational invariance. So, the

$O(\lambda^2)$ correction to the vertex vanishes. The $O(\lambda^2)$ correction to the fermion propagator (which is a melon diagram) also vanishes, since it is $\sim \lambda^2 \int_{k,q} \frac{\text{Tr}[kq(k+p-q)]}{k^2 q^2 (k+q)}$, which vanishes since we are taking the trace over an odd number of γ^μ matrices.

Thus we have no contribution to β_λ at $O(\lambda^2)$. There is also no $O(\lambda^3)$ contribution by the $\mathbb{Z}_2 C_-$ symmetry argument given earlier. Furthermore, any graph containing as a subgraph one of the two graphs discussed above will vanish, which precludes a huge class of diagrams from contributing to β_λ . While it takes a little more work to refine these arguments to a proof that β_λ is actually zero, we at least see that it vanishes to the first few orders in perturbation theory.

100 August 4 — Canonical momenta and commutation relations in bosonization

For the free boson in two dimensions, what is the canonical momentum of ϕ_\pm ? Use bosonization to derive / motivate the answer.

Solution:

The correct commutation relations for the ϕ_\pm are a bit inobvious from the Lagrangian $\partial\phi\bar{\partial}\phi$, so we'll approach things via bosonization. First we need to know where the R density goes (we are working in \mathbb{R} time for this problem!)

$$\begin{aligned} \mathcal{B}[:n_R(x):] &= \mathcal{B}[\psi_R^\dagger(x+\epsilon)\psi_R(x) - i\epsilon^{-1}] \\ &= e^{i\phi_R(x+\epsilon)}e^{-i\phi_R(\epsilon)} - \frac{i}{\epsilon} \\ &=: e^{i\phi_R(x+\epsilon)}e^{-i\phi_R(\epsilon)} : \frac{i}{\epsilon} - \frac{i}{\epsilon} \\ &= -\partial_x\phi_R(x). \end{aligned} \tag{1743}$$

The overall sign here at the end isn't really important and may be incorrect anyway. Like everything in bosonization, the prefactors and minus signs are an unmitigated mess. In the above, we have used the fact that the (equal-time) correlator for the R (L) fermions is $1/(-ix)$ ($1/ix$). In our conventions we will have $\mathcal{B}[\psi_L(x)] = e^{-i\phi_L(x)}$ (this sign is different than in past diary entries, and is chosen so that $\int \partial_x\theta$ is the thing measuring the fermion number), and thus $\mathcal{B}[:n_L(x):] = \partial_x\phi_L(x)$. Thus $\mathcal{B}[:n_R(q):] = q\phi_R(q)$, and the same for $n_L(q)$ but with a minus sign.

Now the L and R fermion densities commute, so the ϕ_L and ϕ_R fields must also commute (thus the identification of the single fermions with the vertex operators is not strictly correct—we need the Klein factors). This might seem slightly surprising if we had written the Lagrangian as $\mathcal{L} \sim \partial\phi\bar{\partial}\phi = \partial\phi_L\bar{\partial}\phi_R$.

We can now determine the appropriate commutation relations for the $\phi_{L/R}$ using the

commutation relations of the density operators. We have⁵¹

$$\rho_R(q) = \int_p \psi_p^\dagger \psi_{p-q} = \int_p \left(: \psi_p^\dagger \psi_{p-q} : + \langle \psi_p^\dagger \psi_{p-q} \rangle \right) = \int_p \left(: \psi_p^\dagger \psi_{p-q} : + \delta_{q,0} \theta(-p) \right). \quad (1744)$$

In the last step we used that the right-movers are only occupied in the ground state if their momentum is negative (relative to k_F). Now (momentarily dropping R subscripts on ψ s)

$$\begin{aligned} [\rho_R(q), \rho_R(p)] &= \int_{l,k} \left(\delta_{l-q,k} \psi_l^\dagger \psi_{k-p} - \delta_{l,k-p} \psi_k^\dagger \psi_{l-q} \right) \\ &= \int_l \left(\psi_l^\dagger \psi_{l-q-p} - \psi_{l+p}^\dagger \psi_{l-q} \right) \\ &= \int_l \left(: \psi_l^\dagger \psi_{l-q-p} : - : \psi_{l+p}^\dagger \psi_{l-q} : + \delta_{q,-p} \theta(-l) - \delta_{p,-q} \theta(-l-p) \right) \quad (1745) \\ &= \delta_{p,-q} \int_l (\theta(-l) - \theta(-l-p)) \\ &= \delta_{p,-q} p. \end{aligned}$$

Here the units are right since ρ_q is dimensionless, and we take $\int_q \delta_{q,0} = 1$ so that the dimension of the δ and p cancel. Thus in \mathbb{R} space we get

$$[\rho_R(x), \rho_R(y)] = \int_p p e^{ip(x-y)} = -i \partial_x \delta(x-y). \quad (1746)$$

Bosonizing, this means that

$$[\partial_x \phi_R(x), \partial_y \phi_R(y)] = i \partial_x \delta(x-y) \implies [\phi_R(x), \partial_y \phi_R(y)] = i \delta(x-y). \quad (1747)$$

Thus the canonical momentum conjugate to $\phi_R(x)$ is actually just its derivative, $\partial_x \phi_R(x)$. We can also write this as

$$[\phi_R(x), \phi_R(y)] = \frac{i}{2} \operatorname{sgn}(x-y). \quad (1748)$$

This commutation relation is what allows the vertex operators $e^{i\phi_R(x)}$ to be fermionic.

When we repeat this procedure for the ψ_L 's, the only thing that is different is the expectation value $\langle \psi_{L,p}^\dagger \psi_{L,p-q} \rangle = \delta_{q,0} \theta(p)$, which gives a minus sign so that the momentum conjugate to $\phi_L(x)$ is $-\partial_x \phi_L(x)$, and we have

$$[\phi_L(x), \phi_L(y)] = -\frac{i}{2} \operatorname{sgn}(x-y). \quad (1749)$$

Now let's see if we recover the Lagrangian. We know from an earlier diary entry that the Hamiltonian is (ignoring constant prefactors)

$$H = \int dx \left[(\partial_x \phi_L)^2 + (\partial_x \phi_R)^2 \right]. \quad (1750)$$

⁵¹Recall that normal-ordering is done so that all the operators which annihilate the ground state are placed to the right. This is *not* the same as placing all annihilation operators to the right: for a system with a finite density of fermions, an annihilation operator with $k > k_F$ will kill the ground state, while a creation operator with $k < k_F$ will also kill the ground state. Thus normal ordering acts nontrivially on things like $\psi^\dagger(x)\psi(x)$, since each of the operators involved involves a sum of many different operators, some of which need to be moved to the right, and some of which do not.

Thus the action should be

$$S = \int dx dt (\partial_x \phi_R (\partial_t - \partial_x) \phi_R - \partial_x \phi_L (\partial_t + \partial_x) \phi_L). \quad (1751)$$

This looks rather mysterious since it does not have any terms quadratic in time derivatives, even though we know that the action is $\partial\phi\bar{\partial}\phi$, which is quadratic in time derivatives. Furthermore in this presentation the ϕ_L and ϕ_R are decoupled, whereas in the $\partial\phi\bar{\partial}\phi$ presentation they are not. We can resolve these conundrums by defining $\phi = \phi_L + \phi_R$, $\theta = \phi_L - \phi_R$. Then after some algebra and integrating by parts, we get (again not writing constant prefactors)

$$S = \int dx dt (-(\partial_x \phi)^2 - (\partial_x \theta)^2 + 2\partial_x \phi \partial_t \theta). \quad (1752)$$

Now the equation of motion for θ says that $\partial_x^2 \theta = \partial_x \partial_t$, so that assuming the derivatives of the fields vanish at $\pm\infty$, we have $\partial_x \theta = \partial_t \phi$ (this is just the usual $d\theta = \star d\phi$ thing), and so the action goes to (again after integrating by parts)

$$S = \int dx dt [(\partial_t \phi)^2 - (\partial_x \phi)^2], \quad (1753)$$

which is finally what we expect from $\int \partial\phi\bar{\partial}\phi$ (in our signature $\partial = -\partial_x + \partial_t$, $\bar{\partial} = \partial_x + \partial_t$).

Finally, we note that since $\partial_x \phi \leftrightarrow n_R - n_L$ and $\partial_x \theta \leftrightarrow n_R + n_L$, we have that $\theta(x)$ counts the total fermion number (relative to $-\infty$), while $\phi(x)$ counts the net chirality (these statements are dependent on the sign conventions made for the bosonization mapping). Inserting $e^{i\theta}$ creates a vortex in ϕ , around which $\oint \partial\phi = 2\pi$. This means that inserting $e^{i\theta}$ at a given time changes the chiral charge, since $\oint \partial_x \phi$ takes on different values before and after the insertion. This jives with the fact that the vertex operator for θ bosonizes to

$$e^{i\theta(x)} \leftrightarrow \psi_L^\dagger(x) \psi_R(x), \quad (1754)$$

which is indeed a scattering operator that indeed changes the net value of $n_R - n_L$. Likewise, a vertex operator for ϕ creates a vortex for θ , which means that it must change the fermion number $n_L + n_R$. This in turn jives with the fact that it bosonizes to $e^{i\phi} \leftrightarrow \psi_L^\dagger \psi_R^\dagger$. Relatedly, we can kind of motivate why in this formulation spatial translations map to shifts in θ by constants. The vertex operator $e^{i\theta}$ shifts a right-moving fermion to a left-moving one, and thus shifts the total momentum. Since it shifts the momentum, it should not commute with the momentum $\int dx T_{01}$, which if this paragraph is to be believed is really the same as $\int dx \partial_x \phi$. In terms of the holomorphic and antiholomorphic fields, we then use the commutation relations of the $\phi_{L/R}$ to conclude that spatial translations do $\phi_L \mapsto \phi_L + c$, $\phi_R \mapsto \phi_R - c$. Again, this is expected from the fermion side, if we write $\psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x)$.

101 August 5 — Chiral almost-symmetry-breaking in two dimensions

Today we're looking at an awesome paper by Witten [?]. The goal is to explore the exact relation between QLRO and masslessness in two dimensions. We will make an attempt to

work with simplified normalization conventions so as to avoid a lot of the tedious numerical factors present in Witten's paper.

The model is

$$S = \frac{1}{8\pi} \int d\sigma \wedge \star d\sigma + \frac{1}{2\pi} \int i\bar{\psi}\partial\psi + \frac{g}{2} \int \bar{\psi}(\cos\sigma + i\bar{\gamma}\sin\sigma)\psi, \quad (1755)$$

where $\bar{\gamma} = Z$ is the chirality operator.

Some comments on conventions that are not related to the problem statement but are useful to have as a reference. The radius of σ (defined by the coefficient of the action being $R^2/4\pi$) has been set to $1/\sqrt{2}$ to make the algebra simpler later on (hopefully). Here by "radius" of the boson, we just mean the coefficient in front of the kinetic term: we are still identifying $\sigma \sim \sigma + 2\pi$. Since the propagator is $-R^{-2} \ln |x - y|$, for us we have $-\frac{1}{2} \ln |x - y|$, while if the coefficient of the kinetic term were $1/2$ so that $R = \sqrt{2\pi}$, we would have the more familiar $-\frac{1}{2\pi} \ln |x - y|$. Finally note that the propagators for the holomorphic and antiholomorphic components of σ are $-\frac{1}{2R^2} \ln(z - w)$ and $-\frac{1}{2R^2} \ln(\bar{z} - \bar{w})$, respectively. This means that in order for $e^{i\alpha\sigma_+}$ to be identified as a fermion, we need $\alpha^2/(2R^2) = 1$. Thus the (positive-chirality) fermion is identified with the vertex operator in the following way:

$$\psi_+ \leftrightarrow e^{i\sqrt{2}R\sigma_+}. \quad (1756)$$

When the coefficient in front of the kinetic term is $1/2$ then the fermion is $e^{i\sqrt{4\pi}\sigma_+}$, while when it is $1/8\pi$ the fermion is the simpler $e^{i\sigma_+}$. This is all just for posterity's sake.

Anyway, the chiral symmetry of interest is the $U(1)$ symmetry

$$U(1)_A : \psi \mapsto e^{i\alpha\bar{\gamma}/2}\psi, \quad \sigma \mapsto \sigma + \alpha. \quad (1757)$$

A naive guess would be that for large coupling, we get spontaneous symmetry breaking of $U(1)_A$ by virtue of a fermion mass, so that the spectrum contains a Goldstone boson and massive fermions. But of course we are in two dimensions, so this is impossible.

Instead, we will show that the spectrum of the model is a massless boson and a massive fermion, but that the symmetry is unbroken. We will show the latter by demonstrating that all the symmetry-breaking Greens functions vanish.

Solution:

We first bosonize $\psi = (\psi_+, \psi_-)^T$, with

$$\mathcal{B}[\psi_\pm] = e^{\pm i\phi_\pm}. \quad (1758)$$

Then the interaction goes to

$$\mathcal{B}[\bar{\psi}\psi] = -\mathcal{B}[\psi_- \psi_+^\dagger + \psi_+ \psi_-^\dagger] = -(e^{-i\phi} + e^{i\phi}) = -2\cos\phi. \quad (1759)$$

Here the minus sign arises since we only use the bosonization map on normal-ordered things, and the normal ordering requires moving the ψ_\pm^\dagger past the ψ_\mp for a minus sign. Likewise,

$$\mathcal{B}[\bar{\psi}\bar{\gamma}\psi] = \mathcal{B}[\psi_- \psi_+^\dagger - \psi_+ \psi_-^\dagger] = -2i\sin\phi. \quad (1760)$$

Thus the action becomes, using $\cos \sigma \cos \phi - \sin \sigma \sin \phi = \cos(\phi + \sigma)$,

$$S = \frac{1}{8\pi} \int (d\sigma \wedge \star d\sigma + d\phi \wedge \star d\phi) - g \int \cos(\phi + \sigma). \quad (1761)$$

Note that the interacting part only involves the combination $\phi + \sigma$. This means that we should define new variables

$$\lambda \equiv \phi + \sigma, \quad \gamma \equiv \phi - \sigma. \quad (1762)$$

The action is then written suggestively as

$$S = \frac{1}{8\pi} \int \left(\frac{1}{2} d\gamma \wedge \star d\gamma + \left[1 - \frac{1}{2} \right] d\lambda \wedge \star d\lambda \right) - g \int \cos(\lambda). \quad (1763)$$

Thus we have produced a theory consisting of a free boson and an interacting boson. Note that the cosine interaction is well-defined since the periodicity of λ_{\pm} is the same as that of σ, ϕ , namely $\lambda \sim \lambda + 2\pi$. The scaling dimension of the cosine operator is the scaling dimension of $e^{i\lambda_{\pm}}$, which for a boson of radius R is $\Delta = 1/(2R^2)$. Since λ in our example has $R = 1/2$, the cosine is marginal. Thus we expect (do we? The interaction is only marginal) that the spectrum should contain a free γ boson, and a massive fermion (the kink / antikink soliton for λ).

To write the theory in terms of the physical variables then, we need to fermionize λ with a fermion η such that

$$\eta_{\pm} \leftrightarrow e^{\pm i\lambda_{\pm}}. \quad (1764)$$

Since the coefficient of the λ kinetic term is not $1/8\pi$, we will not get simply the free Dirac kinetic term — the extra $-1/16\pi$ in the kinetic term for λ will lead to a current-current interaction. Now the current-current interaction maps as

$$\mathcal{B}[j_{\mu} j^{\mu}] = \mathcal{B}[: \eta_+^\dagger \eta_+ :: \eta_-^\dagger \eta_- :] = 4\partial\lambda \bar{\partial}\lambda = d\lambda \wedge \star d\lambda, \quad (1765)$$

so that in terms of the physical variables we have

$$S = \frac{1}{2\pi} \int \left(\bar{\eta} i \not{\partial} \eta - \frac{1}{8} j_{\mu} j^{\mu} \right) + \frac{g}{2} \int \bar{\eta} \eta + \frac{1}{16\pi} \int d\gamma \wedge \star d\gamma. \quad (1766)$$

So we have finally arrived at what we wanted: a theory with a free decoupled boson and a massive fermion. However, there is *no* SSB (as there must not be, in accordance with the CMW theorem), even though the fermion in the spectrum is massive. We can check this by computing the chirality-nonconserving Greens functions that would be nonzero in the case of SSB, like $\langle \psi_{\pm} \psi_{\mp}^\dagger \rangle$. To do this we need

$$\psi_{\pm} \leftrightarrow e^{\pm i\phi_{\pm}} = e^{\pm i(\frac{\lambda}{2} + \frac{\gamma}{2})_{\pm}}. \quad (1767)$$

This means that e.g.

$$\langle \psi_{\pm}(x) \psi_{\mp}^\dagger(0) \rangle \leftrightarrow \langle e^{\pm i(\lambda/2)_{\pm}(x)} e^{\pm i(\lambda/2)_{\mp}(0)} \rangle \langle e^{\pm i(\gamma/2)_{\pm}(x)} e^{\pm i(\gamma/2)_{\mp}(0)} \rangle. \quad (1768)$$

In order to find out what this is, we need to know the correlation functions of the vertex operators for the rescaled field $\lambda/2$. This is slightly nontrivial since multiplicative rescaling

of the fields does not preserve the holomorphic / antiholomorphic decomposition of the fields, i.e. we have

$$(a\phi')_{\pm} \neq a \cdot \phi_{\pm}, \quad a \in \mathbb{C}, \quad (1769)$$

essentially because ϕ_{\pm} do not commute with each other and since momenta and position get scaled oppositely in order to preserve the CCR. A little digression on this since it's interesting: the holomorphic and antiholomorphic parts of the field are defined by (this is a non-local definition as it must be)

$$\phi_{\pm}(x) = \frac{1}{2} \left(\phi(x) \pm \int_{-\infty}^x dx' \Pi(x') \right), \quad (1770)$$

where $\Pi = \partial_x \theta$ is the momentum (which leads to the standard $\theta = \phi_+ - \phi_-$). The nonlocal nature of this definition is needed to ensure that the chiral vertex operators anticommute with one another and thus have a chance to become fermions⁵² (passing one $e^{i\phi_{\pm}}$ around another encircles the latter in a $\Pi(x')$ string, which after being wiggled straight must pass over the former vertex operator, which gives the interaction needed for fermionic statistics). Rescaling this, we get the non-homogenous transformation

$$(a\phi)_{\pm}(x) = \frac{1}{2} \left(a\phi(x) \pm a^{-1} \int_{-\infty}^x dx' \Pi(x') \right), \quad a \in \mathbb{C}. \quad (1771)$$

Solving for $\phi(x)$ and the momentum integral in terms of the original (unscaled) field, we see that under rescaling the chiral components of the fields get mixed by a “boost”

$$\begin{pmatrix} (a\phi)_+ \\ (a\phi)_- \end{pmatrix} = \begin{pmatrix} \cosh(\ln a) & \sinh(\ln a) \\ \sinh(\ln a) & \cosh(\ln a) \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \quad (1772)$$

This Bogoliubov transformation is consistent with the map $\phi \mapsto a\phi, \theta \mapsto a^{-1}\theta$, required in order to preserve the CCR (note that in accordance with yesterday's diary entry, the CCR here are $[\phi_{\pm}(x), \partial_x \phi_{\pm}(y)] \propto i\delta(x-y)$: if we just imposed the CCR on θ, ϕ then we could also get away with having $\phi_{\pm}, \partial_x \phi_{\pm}$ commute but $\phi_{\pm}, \partial_x \phi_{\mp}$ not commute).

So, this provides us with another example where multiplying the action by a constant is very nontrivial in quantum field theory! Just rescaling the field by a constant is thus not as innocuous as it seems: it actually does a “Lorentz boost” / Bogoliubov transformation that mixes the chiral components.

Now one may wonder whether it is $\tilde{\phi}_{\pm} \equiv (a\phi)_{\pm}$ which is left / right moving, or whether it is ϕ_{\pm} . Thinking about this is perhaps easier in the Hamiltonian formulation. Here since $:\psi_L^\dagger \partial_x \psi_L:$ gets mapped to $(\partial_x \phi_+)^2$ while the same for ψ_R gets mapped to $(\partial_x \phi_-)^2$, the free fermion Hamiltonian is bosonized to

$$H \propto \int dx [(\partial_x \phi_+)^2 + (\partial_x \phi_-)^2] \quad (1773)$$

⁵²Here by “become fermions” we mean in the weak sense, where the vertex operators behave like fermions when inserted into correlation functions. They aren't *really* fermions in the constructive sense since they do not change the fermion number. To have the vertex operators be fermions in the strong sense of them being equal to fermion fields as operators, we need Klein factors. The Klein factors basically just change the parity of the fermion vacuum by subtracting a fermion from the Dirac sea and then re-arranging all the existing fermions to create the new vacuum state. In any case, for now we will only need bosonization in the weak sense of matching correlation functions, so these subtleties won't come up.

Now the re-scaling of the ϕ field is accomplished by adding interactions for the fermions of the form $j_\mu j^\mu$. In terms of fermions this is proportional to $\rho_L \rho_R$, which bosonizes to $(\partial_x \phi_+)(\partial_x \phi_-)$. Thus adding this interaction term to the Hamiltonian means that the bosonized Hamiltonian now has a cross term between $\partial_x \phi_+$ and $\partial_x \phi_-$. We can re-write the Hamiltonian without the cross term, but this requires making a field re-definition. Since only the Hamiltonian with the new re-defined fields separates into a sum of left- and right-moving parts, only the new re-defined fields are holomorphic / anti-holomorphic. This makes sense since adding the $j_\mu j^\mu$ interaction mixes the left- and right-moving fermions, and so the bosonized images of the original fermions should not be purely left- or right-moving.

Anyway, applying this to the problem at hand, we find

$$\langle \psi_+(x) \psi_-^\dagger(0) \rangle \sim \langle e^{i(5\lambda_+(x)/4 - 3\lambda_-(x)/4)} e^{i(5\lambda_-/4(0) - 3\lambda_+(0)/4)} \rangle \langle e^{i(5\gamma_+(x)/4 - 3\gamma_-(x)/4)} e^{i(5\gamma_-/4(0) - 3\gamma_+(0)/4)} \rangle. \quad (1774)$$

Now we don't need to know what the correlator of the λ vertex operators are (this is tricky since λ is not free), but we do know what the γ vertex operator correlator is, since γ is free. In a previous diary entry we saw that in order for the correlator to be nonzero, we had to have “charge neutrality” for both of chiral fields γ_\pm in the vertex operators if the correlator was to be non-zero, otherwise the correlator vanishes because of infrared effects. Since in this case the correlators are not charge-neutral ($5/4 - 3/4 \neq 0$), we get

$$\langle \psi_+(x) \psi_-^\dagger(0) \rangle = 0. \quad (1775)$$

By checking the other relevant Greens functions, we see that indeed, the symmetry is unbroken. That the cause of the correlators being zero is coming from infrared effects (the condition of charge neutrality) is comforting, since we know infrared effects are behind the CMW theorem which is what prevents the symmetry breaking from happening.

One final note: in Witten's paper he writes down the mapping $\psi \leftrightarrow \eta e^{i\gamma}$, although strictly speaking this doesn't make sense since we actually need to split γ into holomorphic and antiholomorphic pieces. Witten says this is a technicality, which confuses me since he's done some rescaling of the field, and as we have seen the field rescaling scrambles the holomorphic and antiholomorphic components of the field together.

102 August 6 — β functions from the OPE and perturbations of CFTs

Today we will look at how we can use the algebraic structure of a CFT to describe the RG flow in the vicinity of its fixed point (I read about this problem in a review on bosonization somewhere; sadly I forget which reference. This is pretty standard stuff, though). Anyway, consider perturbing a CFT by adding to the action the term

$$S = S_{CFT} + \int d^d x g_\alpha \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha}, \quad (1776)$$

where the sum on α is implied and g_α are *dimensionless* couplings, so that the $a^{-d+\Delta_\alpha}$ is needed to make the integrand dimensionally correct. Of course, for studying perturbations

around a fixed point, the most interesting operators to choose will be marginal ones, for which $\Delta_\alpha = d$.

Working perturbatively in the couplings g_α , find the $O(g_\alpha g_\beta)$ β functions for the g_α . You will need to use the OPE.

Solution:

There are two ways of getting the result, which are actually rather similar: one uses dimensional analysis and block spin RG, while the other does the RG in a more symmetric way. Both methods take place in \mathbb{R} space, due to the fact that we will need to use the OPE coming from the CFT.

Both methods start by writing the partition function as (here ϕ is some stand-in for an arbitrary collection of fields)

$$Z = \int \mathcal{D}\phi e^{-S_0[\phi]} \left(1 - g_\alpha \int d^d x \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} + \frac{1}{2} g_\alpha g_\beta \int d^d x d^d y \mathcal{O}^\alpha(x) \mathcal{O}^\alpha(y) a^{-2d+\Delta_\alpha+\Delta_\beta} - \dots \right). \quad (1777)$$

The plan will be to do some sort of real space renormalization group step, and see how the effective coupling constants in front of each term in the expansion change.

First for the block spin method. This one is rather heuristic, so forgive the handwaving in what follows. Let's first look at the linear term, which we won't need any CFT to deal with. On one hand, we can write it as (this is not the best way of getting the first-order beta function, but it is instructive for what will follow⁵³)

$$g_\alpha \int d^d x \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = \sum_I \sum_{i \in I} g_\alpha \tilde{\mathcal{O}}^\alpha(x_i). \quad (1779)$$

Here I denotes a block spin site consisting of lattice sites x_i , and $\tilde{\mathcal{O}}^\alpha(x_i)$ is the dimensionless operator defined by $\tilde{\mathcal{O}}^\alpha = a^{\Delta_\alpha} \mathcal{O}^\alpha$. Now let the linear size of each block be L . Then just by dimensional analysis, we have

$$g_\alpha \int d^d x \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = \sum_I a^{-d+\Delta_\alpha} \int_{\text{block}} d^d x g_\alpha \mathcal{O}^\alpha = \sum_I g_\alpha (L/a)^{d-\Delta_\alpha} \tilde{\mathcal{O}}^\alpha(x_I). \quad (1780)$$

We could write the result of the integration as being proportional to $L^{d-\Delta_\alpha}$ since because we are perturbing around a CFT, there is no dimensionful parameter from the theory (like a correlation length) to use in place of L to obtain the required dimensionality (other than I guess a , but this can't appear since we need the beta functions to not be explicitly dependent

⁵³Recall a simpler way: we rescale $a \mapsto a + da$, $g \mapsto g + dg$ and ask what dg needs to be such that the partition function is preserved to first order in the couplings. We have

$$g_\alpha \int d^d x \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = (g_\alpha + dg_\alpha) \int d^d x \mathcal{O}(x) a^{-d+\Delta_\alpha} (1 + (-d + \Delta_\alpha) d \ln a) \implies dg_\alpha = d \ln a (d - \Delta_\alpha) g_\alpha, \quad (1778)$$

which gives in our (high energy) convention $\beta_\alpha = (\Delta_\alpha - d)g_\alpha$ to first order in the coupling.

on a). Taking the block size L to be infinitesimally larger than the lattice spacing, $L = a + da$, we get

$$g_\alpha \int d^d x \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} \rightarrow \sum_I g_\alpha (1 + (d - \Delta_\alpha) d \ln a) \tilde{\mathcal{O}}^\alpha(x_I). \quad (1781)$$

Thus we see that we get the same thing as the original term (which was $\sum_i g_\alpha \tilde{\mathcal{O}}^\alpha(x_i)$), just with a different coupling constant $g_\alpha(L)$. Then defining the β function as $\beta_\alpha = -d \ln a g_\alpha$ (note the minus sign! This is because we have already done a tone of field theory problems using the field theory definition of the β function as $dg/d \ln \Lambda$ and want to keep the notation consistent. This means that irrelevant operators have positive β functions), we see that to first order in the couplings, we get the expected answer:

$$\beta_\alpha = (\Delta_\alpha - d) g_\alpha + O(g_\alpha^2). \quad (1782)$$

There are several things to object to about this, some of which may be fixable. First, what do you mean “take L to be infinitesimally larger than a ? You’re on a lattice!” Indeed, taking $L = a + da$ is very formal. Secondly, the dimensional analysis part was kind of lame, since we actually had no way of computing any sort of geometry-dependent factors that are associated with the block spin procedure are. The kind of wonky combination of mixing integrating in the continuum with the block spin approach is also ugly. Worst of all, when we evaluated the integral with dimensional analysis, we should have obtained a sum $\sum_\beta \tilde{\mathcal{O}}^\beta$ of different dimensionless operators that could be produced during the RG step, but we only kept the original $\beta = \alpha$ operator.

Now we look at the second order piece. We write it as

$$\begin{aligned} \frac{1}{2} g_\alpha g_\beta \int d^d x d^d y \mathcal{O}^\alpha(x) \mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} &= \frac{1}{2} \sum_{I \neq J} g_\alpha g_\beta (1 + (2d - \Delta_\alpha - \Delta_\beta)) \\ &\times (1 - d \ln a) \tilde{\mathcal{O}}^\alpha(x_I) \tilde{\mathcal{O}}^\beta(x_J) + \frac{1}{2} g^\alpha g^\beta \sum_I \int_{\text{block}} d^d x d^d y \mathcal{O}^\alpha(x_I + x) \mathcal{O}^\beta(x_I + y) a^{-2d+\Delta_\alpha+\Delta_\beta}. \end{aligned} \quad (1783)$$

The first term becomes the quadratic part of the expansion of the exponential of the action after performing the RG step, and so will not be important in what follows (plus for the case we are most interested in, where both $\mathcal{O}^\alpha, \mathcal{O}^\beta$ are marginal, this term has no affect on the β function).

Now since the distances between different sites in a single block are “below the resolution” of the theory after doing the blocking, we will take the distance $|x - y|$ to be small enough for us to profitably use the OPE. Of course this is a bit hand wavy (what about all those neighboring lattice sites whose connecting links cut a block boundary?), but we will stick with it. Using this and then doing the integral over the intra-block coordinates using dimensional analysis (again, the only dimensionful scale we have by virtue of perturbing around a CFT

is L), the second term in the previous equation is,

$$\begin{aligned} \frac{1}{2}g_\alpha g_\beta \sum_I \int_{\text{block}} d^d x d^d y C_\gamma^{\alpha\beta} \frac{1}{|x-y|^{\Delta_\alpha + \Delta_\beta - \Delta_\gamma}} a^{-2d+\Delta_\alpha+\Delta_\beta} \mathcal{O}^\gamma(x_I + x) \\ = \frac{1}{2}g_\alpha g_\beta \sum_I C_\gamma^{\alpha\beta} \tilde{\mathcal{O}}^\gamma(x_I) (L/a)^{2d-\Delta_\alpha-\Delta_\beta} \\ \rightarrow \frac{1}{2}g_\alpha g_\beta \sum_I C_\gamma^{\alpha\beta} \tilde{\mathcal{O}}^\gamma(x_I) (1 + (2d - \Delta_\alpha - \Delta_\beta)d \ln a), \end{aligned} \quad (1784)$$

where we have taken $L \rightarrow a + da$ in the last step and assumed that $2d - \Delta_\alpha - \Delta_\beta \neq 0$. Actually, the more relevant case (or in light of the subject, maybe I should say the more interesting case) is when all the perturbing operators are marginal. In this case dimensional analysis produces a $\ln(L/a)$, and the second term instead becomes

$$\frac{1}{2}g_\alpha g_\beta \sum_I C_\gamma^{\alpha\beta} \tilde{\mathcal{O}}^\gamma(x_I) d \ln a. \quad (1785)$$

We see that this term gives a contribution to the g_γ beta function, since it has the effect of just changing the $\tilde{\mathcal{O}}^\gamma$ coupling constant in the block spin theory. Remembering the sign difference between this and the linear term in the expansion (1777), we see that this gives

$$\beta_\gamma = +\frac{1}{2}g_\alpha g_\beta C_\gamma^{\alpha\beta}, \quad (1786)$$

where we have assumed that the associated operator \mathcal{O}_γ is marginal so that no linear part appears (again, this is the most interesting case for perturbing about a CFT fixed point).

Of course, all the gripes about the non-rigorous nature of this method that we raised when deriving the linear part of the β function can be raised here. In order to feel better about our result, we briefly discuss a way to make it a bit more precise by using a more symmetric approach for dealing with the second order term which I just learned about from Cardy's book (Renormalization and Scaling in Statistical Physics).

The basic idea is that we can remain in the continuum and treat the cutoff not as a lattice spacing per se, but rather just as the closest distance that operators are allowed to get from one another. So, we picture the operators as hard spheres, with the radii of the sphere being set by the cutoff. This makes dealing with the second order term super easy:⁵⁴

$$\begin{aligned} \frac{1}{2}g_\alpha g_\beta \int_{|x-y|\geq a} d^d x d^d y \mathcal{O}^\alpha(x) \mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} &= \frac{1}{2}g_\alpha g_\beta \left(\int_{|x-y|\geq L} d^d x d^d y \mathcal{O}^\alpha(x) \mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} \right. \\ &\quad \left. + \int_{a \leq |x-y| < L} d^d x d^d y \mathcal{O}^\alpha(x) \mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} \right). \end{aligned} \quad (1787)$$

Doing the renormalization step means changing the effective cutoff to L by doing the second integral on the RHS and absorbing the result into a rescaling of the coupling constants. We

⁵⁴This kind of "hard sphere gas of operators" picture isn't really helpful for thinking about how changing the cutoff affects the first (linear) term in the expansion of Z , i.e. the one which produces the regular $(\Delta_\alpha - d)g_\alpha$ part—but of course this piece can be obtained from more general reason anyway.

do this by using the OPE: taking $L = a + da$ and using translation invariance we get

$$\begin{aligned} \frac{1}{2}g_\alpha g_\beta \int_{a \leq |x-y| < L} d^d x d^d y \mathcal{O}^\alpha(x) \mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} \\ = \frac{1}{2}g_\alpha g_\beta C_\gamma^{\alpha\beta} \int d^d x \frac{1}{a^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}} da A(S^{d-1}) a^{-d-1+\Delta_\alpha+\Delta_\beta} \mathcal{O}^\gamma(x) \\ = \frac{1}{2}g_\alpha g_\beta C_\gamma^{\alpha\beta} A(S^{d-1}) d \ln a \int d^d x \mathcal{O}^\gamma(x) a^{-d+\Delta_\gamma}, \end{aligned} \quad (1788)$$

where $A(S^{d-1})$ is the area of the unit S^{d-1} . This gives essentially the same contribution to the β function as we got with the rather hand-waving block spin method, up to a factor of $A(S^{d-1})$. Indeed, assuming again that \mathcal{O}^γ is marginal, we get (again, remembering that $\beta_\gamma = -dg_\gamma(L)/d \ln a$)

$$\beta_\gamma = \frac{1}{2}A(S^{d-1})g_\alpha g_\beta C_{\alpha\beta}^\gamma. \quad (1789)$$

The unasthetic factor of the sphere area can be gotten rid of by absorbing into the coupling constants.

As an example, consider e.g. $\mathfrak{su}(2)_k$. The current operators J, \bar{J} for the $SU(2)_L$ and $SU(2)_R$ symmetries, respectively) have dimension 1, and so current-current interactions are marginal (they also must be dimension 2 since the stress tensor is built out of $J^a J^a$ terms via the Sugawara construction). Consider deforming a WZW CFT with an anisotropic “Thirring model” type current-current interaction:

$$\mathcal{L}_{int} = \sum_a \lambda_a J^a \bar{J}^a. \quad (1790)$$

Since the $J^a \bar{J}^a$ terms are all marginal, the beta functions $\beta_a = -d \ln a \lambda_a(L)$ (sorry for the bad notation) is determined to lowest order by the quadratic term in our expression for the beta function. Now recall that the OPE is

$$J^a J^b \sim i \epsilon^{abc} \frac{J^c(z)}{z-w} + \frac{k \delta^{ab}}{(z-w)^2}. \quad (1791)$$

Thus we have (no implicit summation)

$$(J^a \bar{J}^a)(z) \cdot (J^b \bar{J}^b)(w) \supset \sum_c \frac{1}{|z-w|^2} |\epsilon^{abc}| (J^c \bar{J}^c)(w). \quad (1792)$$

We can then conclude that the $O(\lambda^2)$ β functions for the various interactions are (the numerical prefactor isn't important)

$$\beta_a = \frac{\pi}{2} \sum_{b,c} |\epsilon^{abc}| \lambda_b \lambda_c. \quad (1793)$$

As another application of this, we can do an easy check of the one-loop β function in the ϕ^4 model. We can only check up to the one-loop result since we have only kept terms quadratic in the \mathcal{O}^α in the expansion of the partition function. We will fix notation by

$$S = \int \left(\frac{1}{2} (\partial\phi)^2 + r\phi^2 + u\phi^4 \right). \quad (1794)$$

In e.g. dimension $d = 4 - \epsilon$, the dimension of r is 2 while the dimension of u is ϵ (really 2ϵ , but we are taking $\epsilon \rightarrow 0$). This gives us the first-order terms in the β functions. Then we need the OPEs (schematic notation and writing a lot of numbers to make the combinatorics transparent)

$$\phi^2 \cdot \phi^2 \sim \frac{2 \cdot 2}{x^2} \phi^2, \quad \phi^2 \cdot \phi^4 \sim \frac{4 \cdot 2}{x^2} \phi^4 + \frac{\frac{4!}{2 \cdot 2} \cdot 2}{r^4} \phi^2, \quad (1795)$$

and

$$\phi^4 \cdot \phi^4 \sim \frac{\left(\frac{4!}{3!}\right)^2 \cdot 3!}{r^6} \phi^2 + \frac{\left(\frac{4!}{2!^2}\right)^2 \cdot 2}{r^4} \phi^4, \quad (1796)$$

where we have ignored the most singular parts where all of the legs have been contracted and ingored the ϕ^6 part in the last term. Then we can read off the OPE coefficients needed for calculating the β functions from the above formula. We will assume the coupling constants have been rescaled to get rid of the annoying factor of $A(S^{d-1})/2$ in our expression for the second-order contribution to the β functions. We get (still in the high-energy convention where we get β by differentititing wrt $\ln \Lambda$ and not $\ln a$)

$$\beta_r = -2r + 96u^2 + 24ur, \quad \beta_u = -\epsilon u + 72u^2 + 16ru. \quad (1797)$$

From here one can compare to e.g. Peskin and Schroeder after traking down how the conventions differ. One can also use these to solve the for the WF fixed point, etc etc.

103 August 6 — Orbifolding orbifolds and gauging higher form discrete symmetries

Today we are revisting an old diary entry which contained stuff on how taking the orbifold of an orbifold does nothing. Here we look at the same sort of result from a different perspective. The problem statement: read the bit of the introduction in [?] that talks about gauging 1-form \mathbb{Z}_2 symmetries, and understand it. Write down what you understand and what it has to do with orbifolding.

Solution:

Our starting point will be a theory in 2+1 dimensions with a discrete global 0-form symmetry G , which we will take to be \mathbb{Z}_N for simplicity (nothing prevents us from going to more general Abelian G other than a desire to make the notation simple).

Let us couple this theory to a background 1-form \mathbb{Z}_N gauge field α . Since the symmetry is discrete, the only gauge-invaraint data in the gauge field comes from its cohomology class. Letting the partition function on a 3-manifold X in the presence of the background field α be $Z[X; \alpha]$, the gauged partition function is obtained by path-integrating over α , i.e. by summing over all cohomology classes in $H^1(X; \mathbb{Z}_N)$ (coboundaries are pure gauge). The gauged partiton function Z_g is

$$Z_g[X] = \frac{1}{\dim H^0(X; \mathbb{Z}_N)} \sum_{\alpha \in H^1(X; \mathbb{Z}_N)} Z[X; \alpha]. \quad (1798)$$

Here, the normalization out front comes from the “volume of the zero mode”, i.e. it is the “volume” of the group of global symmetries, which are \mathbb{Z}_N -valued functions that are constant on each path component of X (this is the “gobal part of the group of gauge transformations” which we regard as remaining genuine global symmetries during the gauging process).

The construction of $Z_g[X]$ is an orbifold because orbifolds are made by summing over all possible twisted boundary conditions for the fields (where the twists are elements of \mathbb{Z}_N represented on $U(1)$), and this is exactly what the gauging — summing over all values for the gauge field holonomy — is doing (the gauge field holonomy is the same as the value of the twist on the boundary condition for the field). Of course we said at the start that we were interested in 2+1 dimensions (for TQFT reasons), but the gauging here and the usual two-dimensional notion of an orbifold really are the same thing.

Intuitively, the gauging procedure was accomplished by doing the following. First, we equip X with a cell decomposition (dual to a triangulation). We will be thinking in the wavefunction picture, and will take $X = \Sigma \times \mathbb{R}$. Then, we define the split charge operators $e^{iQ_f(g)}$, where f is a face in some given spatial slice and $g \in \mathbb{Z}_N$. $e^{iQ_f(g)}$ is defined to create an oriented domain wall on ∂f accross which charged operators are acted on by R_g (a representation of \mathbb{Z}_N)⁵⁵. If we had a continuous symmetry, we would have e.g. $Q_f(g) = g \int_f \star j$ for j some 1-form conserved ($d^\dagger j = 0$) current. To gauge the symmetry, i.e. to project onto \mathbb{Z}_N -singlet states, we cut up Σ into a bunch of pieces, and sow them back together with each piece sowed to the others with added g twists. That is, to gauge the symmetry we act with all possible split charge operators, by applying the operator

$$\Pi = \prod_f \sum_g e^{iQ_f(g)} \tag{1799}$$

on some reference vacuum state $|0\rangle$. A particular element in the resulting sum for Π looks like a “ g soap bubble foam”. This should be familiar from e.g. the construction of the toric code wavefunction, which is created by applying the operator $\prod_{\square} (\mathbf{1} + ZZZZ)_{\square}$ to the vacuum, where $ZZZZ$ denotes the product of Z operators around $\partial\square$. Note that any operator which is gauge variant has zero expectation value in the state $\Pi|0\rangle$, because of the phase interference accumulated during the sum over all possible soap bubble films.

The holonomy of the gauge field is determined by the net group element accumulated as one passes through a closed loop in the foam. Now the operator $\prod_f \sum_g e^{iQ_f(g)}$ cannot change the holonomy of the gauge field since all the f ’s are homologically trivial. We can pick up different holonomies by explicitly summing over different twists in the domain wall configurations in the partition function. Doing this completes the gauging procedure.

Now we will do another orbifold. We will see that this is equivalent to “ungauging”. Indeed, suppose we want to set the holomony of α around a given cycle C to be trivial, i.e. we suppose we want to poject onto states with no “magnetic flux” threading C . Killing off the holonomy of α is tantamount to setting $\alpha = 0$, since the gauge group is discrete. Then we see that if we let

$$W_C \equiv e^{\frac{2\pi i}{N} \int_C \alpha} \tag{1800}$$

be the Wilson loop around some cycle C , inserting a sum of W_C ’s raised to all possible powers

⁵⁵If our symmetry was anomalous, this step would be problematic — the charge operators would not be “splittable” in this way.

in the partition function will do the trick⁵⁶:

$$Z_{C\text{trivial}} = \frac{1}{\dim H^0(X; \mathbb{Z}_N)} \sum_{\alpha \in H^1(X; \mathbb{Z}_N)} (1 + W_C + W_C^2 + \cdots + W_C^{N-1}) Z[X; \alpha]. \quad (1801)$$

Since W_C is an N -th root of unity, the insertion of $\sum_{n \in \mathbb{Z}_N} W_C^n$ acts as a δ function sending $W \rightarrow 1$. This fixes the holonomy of the gauge field to be trivial around C . The un-gauged theory is thus recovered by taking the above idea and using a product over all homology classes for C :

$$Z = \frac{1}{\dim H^0(X; \mathbb{Z}_N)} \sum_{\alpha \in H^1(X; \mathbb{Z}_N)} \prod_{C \in H_1(X)} (1 + W_C + W_C^2 + \cdots + W_C^{N-1}) Z[X; \alpha], \quad (1802)$$

which un-gauges (the holonomy, which is the entire physical content of the gauge field, has disappeared) and returns us to our starting point.

Why is this equivalent to orbifolding the orbifold? It is equivalent because inserting Wilson loops in the partition function twists the sum in the partition function by the holonomy of the Wilson loop, and twisting all possible cycles by all possible twists is what an orbifold does. Recall that in the CFT example of a free boson with a \mathbb{Z}_2 orbifold, the orbifold of the orbifold on a torus was constructed as

$$Z \xrightarrow{\text{orb}} \frac{1}{4} \sum_{\omega \in \text{Rep}^2(\mathbb{Z}_2)} \sum_{g, h \in \mathbb{Z}_2} \omega(g, h) Z_g^h = Z, \quad (1803)$$

where $\omega(g, h) = \beta(g)\beta(h)$ with $\beta(g)$ the analogue of an insertion of the Wilson loop (the identity) if $g = 1$ (if $g = 0$), and where Z_g^h is the partition function on the torus with boundary conditions twisted or untwisted according to h, g . In the present example we have just generalized this by working on a 3-manifold with general $H^1(X; \mathbb{Z})$, but the idea is the same.

Now we will address the 1-form symmetry / anyon condensation point of view. While adding the α gauge field made the original 0-form \mathbb{Z}_N symmetry local, its existence also added another global symmetry, namely a global \mathbb{Z}_N 1-form symmetry generated by the Wilson lines (the *generators* are the Wilson lines, while the charged objects are the t' Hooft lines, essentially). We will denote the charged objects as T_C , where $C \subset \Sigma$ is a 1-dimensional submanifold. When we are thinking of a cell decomposition of Σ , we usually will want the C in T_C to be defined on the dual lattice. The 1-form symmetry does not act on α in a simple way). Anyway, note that this is a 1-form symmetry since the charge operators are codimension $p + 1$, so since the Wilson lines are 1-dimensional, we have $3 - p - 1 = 1 \implies p = 1$. If this symmetry is not spontaneously broken, then the states are eigenstates of W_C for all C , so that W_C just acts on states by multiplying them by the Wilson line holonomy. If the symmetry is spontaneously broken, then acting with W_C takes us from one ground state to another. Since W_C inserts electric flux along C , the ground states are labelled by the values of the electric flux around each non-contractible cycle $C \subset \Sigma$.

Now we will explain how inserting the superposition of Wilson loops used above to ungauging the 0-form symmetry of the theory is equivalent to gauging the 1-form symmetry. First let's

⁵⁶Recall how the string-net Hamiltonian, which is exactly the projector onto plaquettes with zero flux, is a sum of “Wilson loop” insertions.

write the orbifold of the orbifold in a more suggestive way. The product over homology classes of the sum of wilson loops is equivalent to the sum

$$\prod_{C \in H_1(X; \mathbb{Z}_N)} \sum_{n \in \mathbb{Z}_N} W_C^n = \sum_{\{n_i\} \in \mathbb{Z}_N^{\dim H_1(X)}} W_{C_1}^{n_1} W_{C_2}^{n_2} \cdots W_{C_{\dim H_1(X)}}^{n_{\dim H_1(X)}}, \quad (1804)$$

where the C_i enumerate a set of generators for $H_1(X)$. This can be beautified by using Poincare duality, where we use a 2-cochain $\beta \in H^2(X; \mathbb{Z}_N)$ to keep track of the different C_j and their associated exponents n_i . To do this, we recall that $\int \beta \cup \alpha$ is dual to the intersection product $\hat{\beta} \cap \hat{\alpha}$. Thus $\int \beta \cup \alpha = \int_{\hat{\beta}} \alpha$, which selects out the holonomy of α along the 1-dimensional submanifold determined by the dual $\hat{\beta}$. To sum over all values for the holonomy for a given cycle C , we just need to sum $\exp\left(\frac{2\pi i}{N} \int \beta \cup \alpha\right)$ over the different $\beta \in H^2(X; \mathbb{Z}_N)$ whose Poincare duals have a nontrivial homology class only due to the cylce C . Taking the product over all cycles can then be done simply by summing over *all* cohomologically distinct β . Thus an alternate way to write the orbifold of the orbifold is as

$$Z_{orb^2} = \frac{1}{\dim H^1(X; \mathbb{Z}_N)} \sum_{\beta \in H^2(X; \mathbb{Z}_N)} \sum_{\alpha \in H^1(X; \mathbb{Z}_N)} \exp\left(\frac{2\pi i}{N} \int \beta \cup \alpha\right) Z[X; \alpha] = Z, \quad (1805)$$

where the last equality follows from the argument above: projecting onto $\alpha = 0$ (as a cohomology class) returns us to the original (ungauged) partition function.

To make the connection with gauging the 1-form symmetry, note that we can re-write the orbifold² partition function as

$$Z_{orb^2} = \frac{\dim H^0(X; \mathbb{Z}_N)}{\dim H^1(X; \mathbb{Z}_N)} \sum_{\beta \in H^2(X; \mathbb{Z}_N)} Z[X; \beta], \quad (1806)$$

with

$$Z[X; \beta] = \frac{1}{\dim H^0(X; \mathbb{Z}_N)} \sum_{\alpha \in H^1(X; \mathbb{Z}_N)} \exp\left(\frac{2\pi i}{N} \int \beta \cup \alpha\right) Z[X; \alpha]. \quad (1807)$$

$Z[X; \beta]$ is to be interpreted as the guaged theory coupled to a background field β for the 1-form global symmetry, and the sum over background fields β in Z_{orb^2} means that in Z_{orb^2} , this symmetry has been gauged. The normalization factor in front of the sum in Z_{orb^2} comes from the fact that now $\dim H^1(X; \mathbb{Z}_N)$ is the “volume” of the gauge group, and the factor of $\dim H^0(X; \mathbb{Z}_N)$ is needed to cancel the one appearing in $Z[X; \beta]$. This works out because after accounting for the fact that the sum over β projects onto $\alpha = 0$, we have

$$\begin{aligned} Z_{orb^2} &= \frac{\dim H^0(X; \mathbb{Z}_N)}{\dim H^1(X; \mathbb{Z}_N)} \sum_{\alpha \in H^1(X; \mathbb{Z}_N)} \prod_{C \in H_1(X)} \delta\left(\int_C \alpha\right) \frac{1}{\dim H^0(X; \mathbb{Z}_N)} Z[X; \alpha] \\ &= \frac{1}{\dim H^1(X; \mathbb{Z}_N)} \sum_{\alpha \in H^1(X; \mathbb{Z}_N)} Z = Z, \end{aligned} \quad (1808)$$

where the δ function is only nonzero if $\int_C \alpha = 0 \pmod{N}$.

What of the global \mathbb{Z}_2 symmetry? If it is to have survived, what is it generated by? It needs to be generated by some topological codimension 1 surface operator. The natural

candidate for this is $Q(M_2) = \int_{M_2} \beta$, which one can check does the job. We will do a whole diary entry on why Wilson operators in discrete gauge theories are always the generators of a global symmetry in the future, so we defer a more detailed discussion until then.

Now for the intuitive picture of what we're doing. In the 0-form symmetry case, we did the gauging by defining split symmetry operators that acted on open submanifolds of the codimension 1 manifold that full charge operators are defined on. Acting with all possible split charge operators gave us a g soap bubble foam which did the gauging for us. To gauge the 1-form symmetry, we do the same thing, but this time with the operators that are dual to the domain walls, which are the Wilson loops. The W_C 's are also the full charge operators for the 1-form symmetry, so that the split charge operators are W_L 's, with L some open line. Acting on a reference state with all possible W_L 's creates a “ g -foam” of codimension two objects, i.e. it covers Σ in a bunch of “ g hair”. Having a superposition of all open Wilson lines is equivalent to condensing the e particle, which is the anyon we take to be defined as the thing that is created (along with the associated anti-thing) at the ends of an open Wilson line⁵⁷. Thus anyon condensation is closely related to gauging a 1-form symmetry. Now doing the condensation confines particles that braid nontrivially with e . In particular, this means that the T_C loops have vanishing (well, area-law decaying) expectation values due to their statistical interference with the e condensate. Of course, this is exactly what gauging is supposed to do: give all gauge-variant operators zero expectation value. So another way to look at gauging of the 1-form symmetry is to think of it as confinement of the particles created when an operator charged under the symmetry is cut open.

Let's do a recap for \mathbb{Z}_2 gauge theory. The soap-bubble projector operator that does the gauging in our formulation is $\Pi = \prod_v (1 + A_v)$, where $A_v = \prod_{l \in \partial v} X_l$. Here the split charge operator acting on a little patch of the manifold centered on the vertex v is just A_v . In this formulation, the 1-form symmetry of interest is generated by the Wilson loops $\prod_{l \in \gamma} Z_l$, where γ is some cycle. The charged operators are the loops $T_\gamma = \prod_{l \in \gamma^*} X_l$, where γ^* is a path in the dual lattice. If the symmetry is spontaneously broken, the different ground states are distinguished by the eigenvalue of T_{γ^*} for each homologically distinct γ^* . Since $W_\gamma = \prod_{l \in \gamma} Z_l$ changes the eigenvalue of T_{λ^*} if $\lambda^* \cap \gamma \neq 0 \pmod{2}$, the W_γ move us between different ground states, as they should. To gauge the symmetry, we need to set $\langle T_{\gamma^*} \rangle = 0$ for all γ^* (really, just for all homologically nontrivial γ^*). We do this by condensing e particles (excitations of the $\sum_v A_v$ term of the Hamiltonian) by letting open-ended W_γ 's condense in the system. This gives the T_{γ^*} vanishing (exactly vanishing if $[\gamma^*] \neq 0$) vevs, and the symmetry is gauged.

⁵⁷Note that in the above, we actually never worked with open Wilson lines. While splitting the Wilson lines works, it's slight overkill. Instead, since the charged objects we need to give zero vev to all wrap nontrivial cycles, it is enough to sum over all homologically nontrivial Wilson loop insertions. It's really the intersection number of the inserted Wilson loops with the charged operators that count: if we can get intersection numbers that are non-zero mod N between the charged objects and Wilson loops, and if we then sum over all Wilson loop insertions, the charged objects will have zero vev.

104 August 7 — Cooler proof of Goldstone’s theorem for p -form symmetries

Today we looking at a proof of the generalized Goldstone’s theorem which was presented in [?]. The proof is complimentary to the one I did, but in my opinion is more slick — wish I’d thought of it! Anyway, look through their arguments, figure it out, and fill in the missing details.

Solution:

The proof of Goldstone’s theorem will start from the Ward identity. In what follows, we will ignore factors of i and q (charge), as well as signs (these are all irrelevant for our purposes).

Let \mathcal{O}_C be an operator charged under a p -form symmetry, with $\text{Supp}(\mathcal{O}_C) = C$ a p -dimensional submanifold of spacetime X (or an appropriately smeared bump-function version of a p -dimensional submanifold if we are being pedantic). We will assume that SSB occurs⁵⁸. If $p \geq 1$, the vacuum expectation value of the un-renormalized operator \mathcal{O} is then allowed to vanish up to as fast as a “perimeter law”, meaning that it may vanish as fast as $e^{-g^2 L^p/a}$, where g is a coupling constant and a is a UV cutoff. If this is the case, we will always renormalize \mathcal{O} by subtracting off the UV divergence; this can be done with a simple multiplicative renormalization and ensures that the renormalized \mathcal{O} has a finite, cutoff-independent vev. We will assume that such renormalization has been done in what follows.

Under an infinitesimal transformation, we let \mathcal{O} transform as

$$\mathcal{O} \mapsto \mathcal{O} + \mathcal{O} \int_C \lambda, \quad (1809)$$

for some p -form λ . Since we have a p -form symmetry when $d\lambda = 0$, the action must vary as

$$e^{-S} \mapsto e^{-S} \left(1 + \int \star J \wedge d\lambda \right), \quad (1810)$$

where J is a $(p+1)$ -form current which is classically conserved, $d^\dagger J = 0$. This conservation law ensures that the charge operator $Q(M_{D-p-1}) \sim \int_{M_{D-p-1}} \star J$ is topological. So then after integrating by parts (we assume that λ is compactly supported) the Ward identity reads

$$\langle \mathcal{O}_C \int_X \lambda \wedge \hat{C} \rangle = \langle \mathcal{O}_C \int_X d \star J \wedge \lambda \rangle. \quad (1811)$$

We then conclude that

$$\langle \mathcal{O}_C \rangle \hat{C}(x) = \langle \mathcal{O}_C (d \star J)(x) \rangle. \quad (1812)$$

Note that because we have assumed SSB, we can choose an \mathcal{O}_C which is both charged under the symmetry and is such that the LHS is nonzero.

⁵⁸We are taking this to mean that there is some charged operator \mathcal{O}_C with $\langle \mathcal{O}_C \rangle \neq 0$. I guess technically this is too strong of an assumption — we just need for there to be some nonzero Greens function which is charged under the symmetry. Since Greens functions with non-local objects are annoying and complicate the notation, we will not address in this subtlety in what follows,

Now we pick an open $D - p$ manifold M_{D-p} which intersects C transversely at a point, and then integrate the ward identity over this manifold. We get

$$\langle \mathcal{O}_C \rangle \int_X \widehat{M}_{D-p} \wedge \widehat{C} = \int_{\partial M_{D-p}} \langle \mathcal{O} \star J \rangle. \quad (1813)$$

Now by our choice of M_{D-p} , this simplifies to

$$\langle \mathcal{O}_C \rangle = \int_{\partial M_{D-p}} \langle \mathcal{O}_C \star J \rangle. \quad (1814)$$

Note that the LHS is *independent* of the choice of M_{D-p} ! In fact, it is just a constant. Thus we can make ∂M_{D-p} have support arbitrarily far away from the support of \mathcal{O}_C , and the RHS must remain a constant. This implies that we have the correlator

$$\langle \mathcal{O}_C \star J(r) \rangle \sim \frac{1}{r^{D-p-1}}, \quad (1815)$$

where r is some typical distance away from C . For example, for $p = 0$ C is just a point, and we can take M_D to be a D -ball of radius r centered on C . For e.g. $p = 1$ and $D = 3$, we might take C to be the z axis and M_2 to be a solid disk in the xy plane centered at the origin and with radius r .

Anyway, the point is that this power law correlation function implies that we must have massless particles in the spectrum: if we had no massless particles, such a long-ranged correlation function would not be possible. Now we usually expect that for SSB the current will be realized as $J = dA$ for some p -form A ⁵⁹. The action for the Goldstones is then the usual

$$S = \frac{1}{2g^2} \int dA \wedge \star dA, \quad (1816)$$

where g is the “superfluid stiffness”. Does this jive with the Ward identity? Let us test it for the case where $\mathcal{O}_C = \exp(i \int_C A)$. The AA correlator, going as $1/k^2$, goes as $\langle A(r)A(0) \rangle \sim \int d^D k k^{-2} e^{ikr} \sim r^{2-D}$, so that

$$\langle e^{i \int_C A} \star J(r) \rangle \sim \partial_r \langle \int_C A(0)A(r) \rangle \sim \partial_r r^{2-D+p} \sim \frac{1}{r^{D-p-1}}, \quad (1817)$$

⁵⁹What about in electromagnetism with electric charges; $d^\dagger F = \rho dF = 0$? We expect that in the Coulomb phase the $U(1)_m$ is still spontaneously broken, otherwise given the explicit breaking of $U(1)_e$, the photon would not be around in the Coulomb phase (which it is because you are reading this). On the other hand, the $U(1)_m$ current is $\star F$, and if this were realized as $\star F = d\tilde{A}$, then we would have $d^\dagger F = \star d^2 \tilde{A} = 0$, a contradiction. Alternatively, we could write $\star F = d\tilde{A}$, but \tilde{A} would be a singular field, for which $d^2 \tilde{A} \neq 0$. This is just the electromagnetic dual of the statement that the vector potential is singular when magnetic monopoles are around. This isn’t just a global issue since we are treating the matter as dynamical (i.e. we are not defining the electric charges by excising little bits that change the topology of spacetime). Thus, we have found a counterexample to the claim that the current is realized as $J = dA$ whenever SSB occurs. Note that in the generalized global symmetries + holography paper, the authors say that $\star F \neq d\tilde{A}$ means that $U(1)_m$ is unbroken, but again this can’t be the case since we know the ‘t Hooft operators have a perimeter law in the Coulomb phase (the Wilson lines also go as a perimeter law — this does not come into conflict with the fact that they have nontrivial linking with the ‘t Hooft loops since in the Coulomb phase we have the massless photon, which can “communicate” between two linked loops and keep track of their commutation relations (or we might say that the ‘t Hooft operator needs a topological surface attached to it in order to be well-defined in this phase)). Also note that the fact that the Wilson lines can now split apart is of no concern, since the ends of the Wilson lines have a holonomy around the ‘t Hooft lines which reproduces the statistical interaction between closed Wilson and ‘t Hooft loops).

which is indeed what the Ward identity requires.

105 August 8 — Some stuff about gauging higher symmetries

Today is a fast one. We will review some things scattered in various papers by Kapustin and Seiberg. Some of this has appeared in earlier diary entries but today we will have a slightly different take on things.

Solution:

Suppose we have a theory with a $D - q - 2$ form $U(1)$ symmetry, which we will refer to as the magnetic symmetry. Suppose further that we want to gauge the $\mathbb{Z}_N \subset U(1)_m$ subgroup, which contains the symmetries which shift the magnetic potential \tilde{A} as

$$\mathbb{Z}_N : \tilde{A} \mapsto \tilde{A} + \frac{2\pi}{N} \lambda, \quad \lambda \in H^{D-q-2}(X; \mathbb{Z}). \quad (1818)$$

The operators which implement this symmetry are \mathbb{Z}_N powers of the magnetic flux operators

$$U(M_{q+1}) = \exp \left(\frac{2\pi i}{N} \int_{M_{q+1}} F \right). \quad (1819)$$

As a result, upon doing the gauging, these operators need to act as the identity — they need to become transparent. In particular, they need to act as the identity *even when* $\partial M_{q+1} = 0$ (the symmetry needs to be splittable if it is to be gauged). Since they are transparent their N th powers are transparent, which means that in the gauged theory we have, taking $\partial M_{q+1} \neq \emptyset$,

$$e^{i \int_{M_q} A} \rightarrow \mathbf{1}, \quad (1820)$$

for all closed M_q . Note that while we have used notation that is suggestive of $U(1)$ gauge theory, this does not have to be the case, i.e. F is some arbitrary $(q+1)$ -form field, that may or may not satisfy $\star F = d\tilde{A}$ (we might even have started with a discrete symmetry instead of $U(1)$, but this would require some notational changes).

Anyway, this means that once we gauge the \mathbb{Z}_N subgroup, the Wilson “lines” (or surface, or points, etc.) must all become transparent. Note that we could not conclude this without first taking the N th power, since the operators $\exp(\frac{2\pi i k}{N} \int A)$ are not well-defined for $k \neq 0 \pmod{N}$ (the Wilson lines are only genuine line operators when their charge is integral). This is because of the whole invariance-under-changing-patches thing; see the earlier diary entry.

What does the transparency of the Wilson operator mean? A few things. First, suppose e.g. that $q = 1$. Then the Wilson lines are really Wilson lines, and gauging the \mathbb{Z}_N subgroup of $U(1)_m$ means that we are in a string-net condensate, since the strings act as c-numbers on the physical states. Another nice physical picture occurs when the original magnetic $U(1)_m$ symmetry is a 1-form symmetry, so that $q = D - 3$. For example, take an Abelian gauge theory in three dimensions — we will let it be a finite gauge theory, so that the symmetry was a \mathbb{Z}_N

higher symmetry all along, and during the gauging we gauged the whole thing. The charge operators for this symmetry are one-dimensional, and we will refer to them as Wilson lines (this is slightly annoying, since before we were referring to the boundary operators of split charge operators as Wilson lines—oh well). Gauging the symmetry turns them transparent, and means that the operator $e^{i \int_L A}$ must act as the identity for any open path L (we labelled A as F earlier; sigh). But this open Wilson line operator exactly corresponds to the insertion of two e anyons at ∂L ! Thus gauging the magnetic symmetry is equivalent to condensing e anyons. The take-home message is that gauging symmetries is equivalent to condensing stuff, and the physical interpretation of the condensed stuff depends on the type of symmetry and the ambient dimension.

Now we return to writing the charge operators for $U(1)_m$ as $\exp(\frac{2\pi i}{N} \int F)$, with $F = dA$. From our discussion above, we see that if there is an electric symmetry that the Wilson operators are charged under, the gauging of the \mathbb{Z}_N subgroup must break this symmetry (technically $\exp(i \int_C A)$ is only be charged if C is non-contractible, but from our construction C is actually a boundary since the Wilson loops which get a vev are those which bound the split symmetry operators. This isn't really a problem, since we can pierce the manifold which C bounds with an operator charged under $U(1)_m$, which effectively renders C non-contractible and allows $\exp(i \int_C A)$ to be charged under the electric symmetry). This is the t Hooft anomaly, which we can phrase as follows: gauging a symmetry means that split symmetry operators must act as c-numbers on the vacuum. If these operators are themselves charged under another symmetry (i.e. if $e^{iQ_A} e^{iQ_B} e^{-iQ_A} \neq e^{iQ_B}$, which means that the charge operators for the two symmetries don't commute), then since they are forced to have nonzero vevs, that other symmetry must be spontaneously broken.

Now let's see what happens with a slightly more explicit example: q -form gauge theory. To gauge the \mathbb{Z}_N subgroup of $U(1)_m$, we need to couple the conserved current, namely $\star F$, to a background \mathbb{Z}_N $(D - q - 1)$ -form gauge field B_n . Thus the action is

$$S = \frac{1}{2g^2} \int F \wedge \star F + \frac{i}{2\pi} \int F \wedge B_n. \quad (1821)$$

Here the $i/2\pi$ is just for convenience. The discrete gauge field B_n is kind of unpleasant to work with directly, so we follow the usual pattern of introducing an extra Lagrange multiplier field \tilde{A} that will enforce the constraint. \tilde{A} is a regular $U(1)$ gauge field, and $F_{\tilde{A}}$ has periods quantized in the usual manner. Thus, the action is

$$S = \frac{1}{2g^2} \int F \wedge \star F + \frac{i}{2\pi} \int F \wedge B + \frac{Ni}{2\pi} \int F_{\tilde{A}} \wedge B, \quad (1822)$$

where \tilde{A} is dynamical, and now B is a regular $U(1)$ gauge field. This works since the integral over the exact part of $F_{\tilde{A}}$ sets B to be flat and the sum over cohomology classes of $F_{\tilde{A}}$ force the holonomy of B to be quantized in units of $2\pi/n$ — this turns B into a \mathbb{Z}_N $(D - q - 1)$ -form gauge field as we wanted.

Now we integrate out B . This sets $F = nF_{\tilde{A}}$. Since $F_{\tilde{A}}$ has periods in $2\pi\mathbb{Z}$, we see that the coupling to the gauge field sets the periods of F to lie in $2\pi N\mathbb{Z}$. This means that the effect of gauging the \mathbb{Z}_N subgroup of $U(1)_m$ is to restrict the sum over cohomology classes for F in the path integral to only those cohomology classes that have periods in $N\mathbb{Z}$. Intuitively, we might explain this as follows: gauging the $\mathbb{Z}_N \subset U(1)_m$ subgroup means that “magnetic

monopoles” (a stand-in term for whatever the objects charged under $U(1)_m$ are) now have a gauge symmetry which shifts them by $e^{2\pi i/N}$. Thus only conglomerates of N monopoles are physical (gauge-invariant), and thus the only bundles that we can include in the path integral are those where the “monopole number” (generalized Chern number) is a multiple of N .

106 August 9 — Maxwell in two dimensions

Today is a little brain-warmer that I realized I'd never done. It'll be used in a future diary on QED_2 that I plan to write. It was inspired by the appendix of “Theta, time-reversal, and temperature”.

Anyway, consider pure $U(1)$ gauge theory in two dimensions, with theta angle θ . Put in on $\mathbb{R} \times S^1$, where the S^1 has circumference L and the \mathbb{R} is time. Find the spectrum as a function of θ .

Solution:

First let us fix a gauge. We will choose a gauge in which A_1 is constant. This means we need to find an α such that

$$\begin{aligned} \partial_x^2 \alpha(x, t) = \partial_x A_1(x, t) &\implies \partial_x \alpha(x, t) = A_1(x, t) - A_1(0, t) + c \\ &\implies \alpha(x, t) = \int_0^x dx' (A_1(x', t) - A_1(0, t)) + cx + d. \end{aligned} \quad (1823)$$

This ensures that $A' = A - d\alpha$ will satisfy $\partial_x A'_1 = 0$ for all t . Note that α is time-dependent! We can get the constants by requiring that α be well-defined, at least modulo 2π . This doesn't allow us to fix d (as d parametrizes the global $U(1)$ symmetry, it cannot be fixed), but it allows us to fix c so that (setting $d = 0$)

$$\alpha(x, t) = \int_0^x dx' (A_1(x', t) - A_1(0, t)) + x \frac{\int_0^L dx' A_1(x', t)}{L}. \quad (1824)$$

One sees that at both $x = 0$ and $x = L$ we have $\alpha = 0$. This means that in the following, we can work with the variable

$$\phi(t) \equiv \int_0^L dx A_1(x, t), \quad (1825)$$

so that the gauge-fixed $A_1(t)$ is $\phi(t)/L$.

Now we integrate out A_0 to enforce

$$\partial_x \star F = 0, \quad (1826)$$

so that the electric field is a constant. We are left with the action (we have done the dx integral since everything is independent of x)

$$S = \frac{1}{2g^2 L} \int dt (\partial_t \phi)^2 + \frac{\theta}{2\pi} \int dt \partial_t \phi. \quad (1827)$$

The canonical momentum is

$$p = \frac{1}{g^2 L} \partial_t \phi + \frac{\theta}{2\pi}. \quad (1828)$$

Dimensionality check: $[g] = 1$, so p is dimensionless as it should be. After some algebra we then find the Hamiltonian

$$H = \frac{g^2 L}{2} \left(p - \frac{\theta}{2\pi} \right)^2. \quad (1829)$$

Now we are in $U(1)$ gauge theory, not \mathbb{R} gauge theory (this is slightly artificial since there are no charges), so that $\phi \sim \phi + 2\pi$ as a result of large gauge transformations being gauged (that they are gauged is what we mean by a $U(1)$ gauge theory). So since ϕ is periodic, the eigenfunctions p are just $e^{in\phi}$ for $n \in \mathbb{Z}$, and thus the spectrum is

$$E_n = \frac{g^2 L}{2} \left(n - \frac{\theta}{2\pi} \right)^2, \quad n \in \mathbb{Z}. \quad (1830)$$

This has the expected twofold ground state degeneracy at $\theta = \pi$, exhibits the periodicity $\theta \sim \theta + 2\pi$, and so on. Note that the energy levels are linearly proportional to the circumference of the circle, so that all the energy in the different states comes from the energy density of the vacuum. That is, there are no particles, which of course we know must be the case for gauge theory in two dimensions. Also note that the spectrum is dependent on θ , even though the added term $\theta \int F$ is topological, and hence it is independent of the metric and doesn't contribute to $T_{\mu\nu}$. In particular, it doesn't contribute to T_{00} , the Hamiltonian. But through its modification of the canonical momentum, it still has an effect on the spectrum.

What do the different n levels represent? They essentially represent the different values that the quantized electric flux $\star F$ can take on. We see that the electric flux is determined via

$$\star F = g^2(n - \theta/2\pi). \quad (1831)$$

The θ term contributes to the electric flux in the usual way, with $\theta \mapsto \theta + 2\pi$ equivalent to changing the electric flux by one unit (this is one of the reasons why θ acts as a background electric field). Thus the different n levels are distinguished by the value of the electric flux which is threaded around the circle. We can go between different n by applying the Wilson line operator, since

$$[e^{i \int dx A_1(t)}, (\star F)(t)] = -g^2 e^{i \int dx A_1(t)}, \quad (1832)$$

which means that the operator $e^{i\star F/g^2}$ generates the 1-form symmetry which shifts the holonomy. Since $\star F$ is constant, the charge operator $e^{i\star F/g^2}$ is independent of position, which is the statement of “current conservation \implies topological charge operator” for a point charge operator. Since of course this 1-form symmetry cannot be broken in two dimensions, we know we will be able to label the states by their 1-form charges, which is just the obvious statement that we can label states by their electric fluxes. Anyway, if $W = e^{i\phi}$ is the Wilson loop, at equal times we have

$$W(\star F/g^2 + 1) = \star F W. \quad (1833)$$

Thus we get the obvious statement that acting with the Wilson line increases the electric flux by g^2 . This takes $n \rightarrow n + 1$, and so the Wilson loop takes us between the different E_n . We

can also phrase this in terms of the easily-checked similarity transform (here $H(p; \theta)$ is the Hamiltonian)

$$WH(p; \theta)W^\dagger = H(p; \theta + 2\pi), \quad (1834)$$

which again demonstrates $\theta \sim \theta + 2\pi$. However, since there are no charges in the theory, the Wilson loop is kind of a pathological operator, since there is no way to apply it “gradually”. In order to actually have it at our disposal, we would need to have charges (very massive ones would be fine) that we could pair-create and use to make W . Since this option is not available to us (pure gauge theory in two dimensions has no particles, as we have said), the different E_n levels are actually disconnected from one another, regardless of L . Also note that if we were to put our theory on T^2 instead of $\mathbb{R} \times S^1$, we would have to be more careful with the quantization procedure, since the operator $e^{i \int A_1}$ would not make sense: the electric flux needs to jump by 1 when crossing the Wilson line, but the Wilson line does not divide T^2 into disjoint pieces, so we get a contradiction (for a similar reason, $e^{i \int A_0}$ would not make sense). This is a trivial example of a higher-symmetry-enforced selection rule.

107 August 10 — Anomalies and current OPEs

Today we try to get a better understanding of why central-extension-y terms which appear in current OPEs encode anomalies. We will be in two dimensions throughout.

First let us consider a theory with holomorphic and antiholomorphic currents, with OPE

$$J(z)J(w) \sim \frac{k}{(z-w)^2} + \dots, \quad \bar{J}(\bar{z})\bar{J}(\bar{w}) \sim \frac{\bar{k}}{(\bar{z}-\bar{w})^2} + \dots, \quad J\bar{J} \sim 0, \quad (1835)$$

where the \dots could include further terms linear in the current, like e.g. for WZW models / current algebras and stuff (the \dots does *not* stand for nonsingular stuff). We assume the classical eom for the currents are either, as in WZW models, $\bar{\partial}J = \partial\bar{J} = 0$, or $\bar{\partial}J + \partial\bar{J} = 0$ as e.g. for the vector / axial currents in QED. By coupling the theory to a background gauge field, find the anomaly in terms of the k 's. What happens if $k \neq \bar{k}$, or if the gauge field only couples to one of the two currents (think quantum spin Hall / the standard model)?

Next, take a slightly more general approach and deduce a more general form of the current-current OPEs. You should include a non-singular contact term for the $J\bar{J}$ OPE, which will be important to deal with. Demonstrate the mixed anomaly between the current and the appropriately-defined axial current.

Solution:

First let us remember what contact terms in the current-current correlators mean. We will be fast and schematic. Consider a two-point function with a contact term

$$\langle J(x)J(y) \rangle = f(x-y) + g\delta(x-y) + \dots \quad (1836)$$

Since we get this by taking $\delta_{A(x)}\delta_{A(y)}Z[A]$ for a background field which couples to J , the contact term must come from a counterterm like $\int A^2$ which has been added to the action.

Similarly we could have a counterterm like $\int A\partial A$ which would give us a contact term like $\partial\delta(x - y)$, and so on; higher derivative counterterms give more singular contact terms, and counterterms higher order in A give contact terms to higher point functions of J . Often these contact terms are non-universal and can be modified at will without affecting the physics. Sometimes this is not the case though, e.g. when the contact terms are determined by an OPE in a CFT (since then they are determined by correlation functions at *separated* points), or when they are required by gauge invariance (like the $A_\mu A^\mu$ contact term required by gauge invariance in scalar QED). Since the contact terms tell us about counterterms involving gauge fields, if we know the contact term structure we can learn about whether or not the theory has a gauge-invariant action when background gauge fields are added.

Anyway, now we turn to the problem stated above. To diagnose the anomaly, we should try to gauge the symmetry generated by J . To couple to the gauge fields in a gauge-invariant way, we need to couple \bar{A} with J and A with \bar{J} (the gauge field A is not holomorphic, the notation just means that it is the z -component of the 1-form $A_\mu dx^\mu$). Now take the path integral and expand in the background fields:

$$Z[A, \bar{A}] \approx \left\langle 1 - \int (\bar{J}A + J\bar{A}) + \frac{1}{2} \int_{z,w} (\bar{J}A + J\bar{A})(z) \cdot (\bar{J}A + J\bar{A})(w) \right\rangle. \quad (1837)$$

Let us now look at the gauge invariance of this expression. The linear term is gauge invariant since the equation of motion holds on J with no other operators inserted. Any anomalous variation will come from the quadratic term. Using the OPEs (allowed since there are no other operator insertions), we get

$$\delta Z[A, \bar{A}] = \delta \frac{1}{2} \int_{z,w} \left(\frac{k}{(z-w)^2} \bar{A}(z, \bar{z}) \bar{A}(w, \bar{w}) + \frac{\bar{k}}{(\bar{z}-\bar{w})^2} A(z, \bar{z}) A(w, \bar{w}) \right). \quad (1838)$$

Let's just look at the first term. We have, for $A \mapsto A + d\gamma$, (and not being too careful about factors of π and 2, and using shitty notation where A and its z component are the same so that $A \mapsto A + d\gamma$ really means $A \mapsto A + \partial\gamma$, $\bar{A} \mapsto \bar{A} + \bar{\partial}\gamma$)

$$\begin{aligned} \delta \frac{1}{2} \int_{z,w} \partial_w \frac{k}{z-w} \bar{A}(w, \bar{w}) \bar{A}(z, \bar{z}) &= \int \partial_w \frac{k}{z-w} \bar{A}(w, \bar{w}) \partial_{\bar{z}} \gamma(z, \bar{z}) \\ &= \int_{z,w} \partial_{\bar{z}} \frac{k}{z-w} \partial_w \bar{A}(w, \bar{w}) \gamma(z, \bar{z}) \\ &= \int_{z,w} \delta^2(z-w, \bar{z}-\bar{w}) k \partial_w \bar{A}(w, \bar{w}) \gamma(z, \bar{z}) \\ &= k \int_z \gamma \partial \bar{A}. \end{aligned} \quad (1839)$$

Thus the variation of the partition function is

$$\delta Z[A, \bar{A}] = \int \gamma (k \partial \bar{A} + \bar{k} \bar{\partial} A). \quad (1840)$$

Now in the special case that $k = \bar{k}$, this can be canceled by adding the local counter-term

$$S_{ct} = (k + \bar{k}) \int_z A \bar{A}, \quad (1841)$$

or since the variation is the integral of $(k + \bar{k})\gamma F$, it can also be canceled by the variation of a Chern-Simons theory in three dimensions (note that I have not been keeping track of factors of 2π and stuff so the CS term won't have the proper normalization if we use the literal expression above—the needed numerical factors will enter e.g. from the step where we replaced $\partial_z \frac{1}{z-w}$ with the delta function). Anyway, note that if $k \neq \bar{k}$, then no such local counterterm will do, and no anomaly cancellation is possible. Note that we would also derive an un-cancellable anomaly if one of the couplings $J\bar{A}$ or $\bar{J}A$ wasn't present in the gauged partition function. This further illustrates the general phenomenon of anomalies being tied to chirality: if the currents are intrinsically chiral, or if the gauging is done in a chiral way, there is an anomaly. This fits with our experience of gauge anomalies in hep-ph scenarios coming from fermions which are chirally coupled to gauge fields.

A super dumb example of a scenario in which the $\int A\bar{A}$ term needs to be employed is in scalar electrodynamics. For example, consider a compact scalar, with just a free action. The current associated with the shift symmetry is $J = \partial\phi$, $\bar{J} = \bar{\partial}\phi$. Suppose we want to gauge this symmetry via minimal coupling of a gauge field (A, \bar{A}) to the current. The OPEs are of course $JJ \supset 1/(z-w)^2$, $\bar{J}\bar{J} \supset 1/(\bar{z}-\bar{w})^2$, so that $k = \bar{k}$ and the anomaly can be canceled by a term $A\bar{A} = A_\mu A^\mu$. Of course such a term is present, since we know that the full gauged action has the kinetic term $(d\phi - A)^2$, which contains the A^2 term.

Now we back up a little bit and consider the current OPEs from a bit more general point of view. In momentum space, we have⁶⁰

$$JJ = \frac{q_+^2}{q^2} k, \quad J\bar{J} = -K, \quad \bar{J}\bar{J} = \frac{q_-^2}{q^2} \bar{k}, \quad (1842)$$

where K is some arbitrary constant that we can choose by hand (as part of how we define the regularization procedure—it comes from $\delta_{A(z)}\delta_{\bar{A}(z)}$, which picks up the $\int A\bar{A}$ counterterm). These counterterms only affect correlation functions at coincident points). Since the K term is momentum-independent, it only contributes a δ function contact term in real space and so doesn't affect any correlation functions at separated points—this is why we needn't worry about its arbitrariness. Here q_+ Fourier transforms to $\partial = (\partial_0 - i\partial_1)/2$ and q_- goes to $\bar{\partial} = (\partial_0 + i\partial_1)/2$. The \pm sign indicates their chirality, so that ∂ kills right-movers (negative chirality) and $\bar{\partial}$ kills left-movers (positive chirality). In real space this works since schematically we have e.g. $q_+^2/q^2 = q_+/q_- \rightarrow \partial_{\bar{z}}^1 = \frac{1}{z^2}$.

Anyway, now let's look at current conservation. In our notation $\partial_\mu J^\mu$ is written as $\partial\bar{J} + \bar{\partial}J$. Using the OPEs, we find

$$\langle(\partial\bar{J} + \bar{\partial}J)\bar{J}\rangle = q_-(\bar{k} - K), \quad \langle(\partial\bar{J} + \bar{\partial}J)J\rangle = q_+(k - K). \quad (1843)$$

Thus we can preserve current conservation only if $k = \bar{k}$ (as we saw before, we get a gauge anomaly if $k \neq \bar{k}$, which is compatible with this condition on current conservation), and if we choose $K = k$.

Now let us define an axial current, which we will write as $\mathcal{J}, \bar{\mathcal{J}}$. The currents are related as

$$\mathcal{J} = J, \quad \bar{\mathcal{J}} = -\bar{J}. \quad (1844)$$

⁶⁰Here we are focusing just on the anomalous parts. More generally we can have terms like $JJ = \frac{q_+^2}{q^2} f(q^2)$ for $f(q^2)$ some dimensionless function. This won't appear in the examples we're interested in, and in any case we can absorb $f(\infty)$ into the k 's so that in the UV the form for the OPEs below suffices.

One sees that the word axial is appropriate by e.g. by looking at the case of Dirac fermions: there we have $J = j_0 - ij_1 = 2L^\dagger L$, $\bar{J} = j_0 + ij_1 = 2R^\dagger R$, for $j^\mu = \bar{\Psi} \not{D}_A \Psi$ and $\Psi = (L, R)^T$. The chiral current is $\mathcal{J}_0 = L^\dagger L - R^\dagger R$, $\mathcal{J}_1 = i(L^\dagger L + R^\dagger R)$ for gamma matrices equal to the Pauli matrices, and so indeed $\mathcal{J} = 2L^\dagger L$, $\bar{\mathcal{J}} = -2R^\dagger R$. Thus current conservation for the axial current is

$$d^\dagger \mathcal{J} = \partial \bar{\mathcal{J}} + \bar{\partial} \mathcal{J} = -\partial \bar{J} + \bar{\partial} J = -idJ, \quad (1845)$$

which is basically the usual Hodge duality formula for relating symmetries with mixed anomalies. Thus if both currents are conserved, then the regular (vector) current is both closed and co-closed. This in turn means that if both currents are conserved, we have

$$\partial \bar{J} = \partial \bar{\mathcal{J}} = 0, \quad \bar{\partial} J = \bar{\partial} \mathcal{J} = 0, \quad (1846)$$

so that if both currents are conserved the un-barred currents really are holomorphic and the barred currents really are antiholomorphic.

Using the OPE, we can check that

$$\langle d^\dagger \mathcal{J} J \rangle = q_+(k + K), \quad \langle d^\dagger \mathcal{J} \bar{J} \rangle = -q_-(\bar{k} + K). \quad (1847)$$

Thus the axial current is only conserved if we have both $k = \bar{k}$ (as usual), and if we choose $k = -K$. Now we see the mixed anomaly between J and \mathcal{J} —it's impossible to choose our regularization conventions (alias K) in such a way that both $d^\dagger J = 0$ and $d^\dagger \mathcal{J} = 0$. Now the anomaly means that if we gauge the symmetry generated by J (as we usually do), \mathcal{J} conservation will get broken, which we confirm by setting $K = -k = -\bar{k}$ and computing

$$\langle d^\dagger \mathcal{J} (\bar{J} A + J \bar{A}) \rangle = k(q_+ \bar{A} - q_- A) = k \star F. \quad (1848)$$

This tells us that $\langle d^\dagger \mathcal{J} \rangle = k \star F$, at least to lowest order. This comes from putting $d^\dagger \mathcal{J}$ in the path integral and expanding the $e^{-\int J_\mu A^\mu}$ term to first order, to create the usual bubble diagram with one photon leg. We see this in a slightly different way by writing

$$\begin{aligned} \partial_\mu \langle \mathcal{J}^\mu(z) \rangle_{A, \bar{A}} &= -\partial \frac{\delta Z[A, \bar{A}]}{\delta A(z)} + \bar{\partial} \frac{\delta Z[A, \bar{A}]}{\delta \bar{A}(z)} \\ &= \left(-\partial \frac{\delta}{\delta A(z)} + \bar{\partial} \frac{\delta}{\delta \bar{A}(z)} \right) \left\langle 1 - \int_u (J \bar{A} + \bar{J} A)(u) \right. \\ &\quad \left. + \frac{1}{2} \int_{u,v} (J \bar{A} + \bar{J} A)(u) (J \bar{A} + \bar{J} A)(v) \right\rangle_{0,0} \\ &= \frac{1}{2} \left(-\partial \frac{\delta}{\delta A(z)} + \bar{\partial} \frac{\delta}{\delta \bar{A}(z)} \right) \int_{u,v} \left(\frac{k}{(u-v)^2} \bar{A}(u) \bar{A}(v) \right. \\ &\quad \left. + \frac{\bar{k}}{(\bar{u}-\bar{v})^2} A(u) A(v) - \delta(u-v) K(\bar{A}(u) A(v) + A(u) \bar{A}(v)) \right) \\ &= \partial_z \int_u \left(\bar{\partial}_u \frac{1}{\bar{u}-\bar{v}} \bar{k} A(u) + K \delta(z-u) \bar{A}(u) \right) \\ &\quad - \bar{\partial}_z \int_u \left(\partial_u \frac{1}{u-z} k \bar{A}(u) + K \delta(z-u) A(u) \right) \\ &= -\bar{k} \bar{\partial} A(z) + K \partial \bar{A}(z) + k \partial \bar{A}(z) - K \bar{\partial} A(z), \end{aligned} \quad (1849)$$

where we used that $\langle d^\dagger \mathcal{J}(z) \rangle_{0,0} = 0$. If we were to do this for $d^\dagger J$ instead of $d^\dagger \mathcal{J}$ we would have gotten

$$\langle d^\dagger J(z) \rangle_{A,\bar{A}} = \bar{k}\bar{\partial}A(z) - K\partial\bar{A}(z) + k\partial\bar{A}(z) - K\bar{\partial}A(z), \quad (1850)$$

meaning that as we saw before, we need $K = k = \bar{k}$ for conservation of the vector current. So making this choice, we get (remember we are not being careful with factors of 2 and stuff)

$$\langle d^\dagger \mathcal{J}(z) \rangle_{A,\bar{A}} = k(\partial\bar{A}(z) - \bar{\partial}A(z)) = k \star F(z). \quad (1851)$$

Thus we have completed our jillionth derivation of the chiral anomaly.

We have only been working up to the one-loop level. However, we know that this result is one-loop exact for the following reason: by gauge invariance the only possibility for $d^\dagger \mathcal{J}$ is $f(e^2) \star F$, where e is the gauge coupling. But $\int d^\dagger \mathcal{J} \in \mathbb{Z}$ and $\frac{1}{2\pi} \int F \in \mathbb{Z}$, and so $f(e^2)$ cannot continuously depend on e . Thus it must be independent of e , barring super pathological counterexamples. Since the answer for $d^\dagger \mathcal{J}$ is thus independent of the gauge coupling, it is one-loop exact (the gauge coupling appears where \hbar appears in the action, and diagrams with l loops go as \hbar^{-1+l} by Euler characteristic reasons). .

108 August 11 — Subsystem symmetries and their Elitzur's theorem

Today is a short one. Look at the paper [?] and understand their proof of a generalized Elitzur's theorem for subsystem symmetries. They show that e.g. in a gapped system, SSB of a discrete symmetry is impossible if the charge operators are one-dimensional, and SSB of a continuous symmetry is impossible if they are two-dimensional. Of course this cannot be true, since we know that e.g. a 1-form discrete symmetries can be broken in three dimensions. Show what goes wrong with their argument.

Solution:

The argument goes as follows. Consider a q form symmetry, so that the charge operators act on $D - q - 1$ manifolds. Consider a theory where the fields are schematically denoted by ϕ . Furthermore let $S[\phi]$ be some local action. Consider a charge operator $Q(M)$, where M is a closed $D - q - 1$ submanifold of spacetime X . Let us break up the fields as $\eta(x) = \phi(x)$ for $x \in M$ and $\bar{\eta}(x) = \phi(x)$ for $x \in X \setminus M$. Finally let $\mathcal{O}[\eta, \bar{\eta}]$ be some (not necessarily local) operator charged under $Q(M)$. Cheekily rewrite the vev of \mathcal{O} as

$$\langle \mathcal{O}[\eta, \bar{\eta}] \rangle = \int \mathcal{D}\bar{\eta} \left(\frac{\int \mathcal{D}\eta \mathcal{O} e^{-S[\eta, \bar{\eta}]} }{\int \mathcal{D}\eta e^{-S[\eta, \bar{\eta}]} } \right) \frac{\int \mathcal{D}\eta e^{-S[\eta, \bar{\eta}]} }{Z_0}, \quad (1852)$$

where Z_0 is the partition function with no operator insertions. As usual, to get a vev for \mathcal{O} we need to either add a symmetry-breaking field or fix boundary conditions appropriately. We will assume the latter approach, so that X must technically not be compact.

Now we can just do

$$|\langle \mathcal{O}[\eta, \bar{\eta}] \rangle| \leq \int \mathcal{D}\bar{\eta} \left| \frac{\int \mathcal{D}\eta \mathcal{O} e^{-S[\eta, \bar{\eta}_m]} }{\int \mathcal{D}\eta e^{-S[\eta, \bar{\eta}_m]} } \right| \frac{\int \mathcal{D}\eta e^{-S[\eta, \bar{\eta}]} }{Z}, \quad (1853)$$

where $\bar{\eta}_m$ is the value of the field configuration on $X \setminus M$ such that the absolute value of the expression in parenthesis in (1852) is maximized (assuming that it's bounded). Then we see that we just have a factor of $Z_0/Z_0 = 1$ in addition to the term evaluated at $\bar{\eta} = \bar{\eta}_m$, so

$$|\langle \mathcal{O}[\eta, \bar{\eta}] \rangle| \leq |\langle \mathcal{O}[\eta, \bar{\eta}_m] \rangle_{S[\eta, \bar{\eta}_m]}|. \quad (1854)$$

This means that we can bound the expectation value of \mathcal{O} by the expectation value it takes on in the presence of a fixed field configuration for the fields living on $X \setminus M$. Now if the original action was local and well-behaved, then $S[\eta, \bar{\eta}_m]$ should be a local and well-behaved action for the η field.

Now comes the catch: since $Q(M)$ acts as a global symmetry on M , the RHS of the above equation is essentially the expectation value that an operator in a $D - q - 1$ dimensional theory has in the presence of a global 0-form symmetry generated by $Q(M)$. Thus we can apply the regular CMW theorem for 0-form symmetries to conclude that $|\langle \mathcal{O}[\eta, \bar{\eta}] \rangle| = 0$ if M is $d \leq 2$ dimensional and it generates a continuous symmetry, or if M is $d \leq 1$ dimensional and it generates a discrete symmetry⁶¹.

Now we know examples of higher symmetries that are spontaneously broken, and yet have charge operators whose dimension comes into conflict with the above result. So what gives? The point is that these counter examples all occur (to my knowledge) when the q -form symmetry in question arises from a gauge theory. This means that the decomposition $\int D\phi \rightarrow \int D\eta D\bar{\eta}$ is impossible, since the Hilbert space does not factorize as $\mathcal{H}_M \otimes \mathcal{H}_{X \setminus M}$. This is not just an issue of M not being “smooth” in X : we could thicken it up into a D -dimensional submanifold, or we could try to smoothly interpolate between η and $\bar{\eta}$: nothing we could do would let us do the field decomposition in this way. The dimensional reduction approach that this method uses doesn't work for gauge theories, since their nonlocal-ness means that the degrees of freedom in different directions are all inter-related and can't get separated in the way they would need to be to make this argument work.

109 August 12 — Alternate approach to Wilson line expectation values in Chern-Simons

This is kind of a cop-out since I didn't figure out anything cooler to do. Consider some simple Abelian Chern-Simons theories, like $U(1)_k$ or some simple X -type K matrix theory. Compute $\langle W^q(C) \rangle$ and $\langle W^q(C)W^p(C') \rangle$, where $W(C) = \exp(i \int_C A)$. The usual way to do this is to solve the classical equations of motion in the presence of a source of charge q so that e.g. $F = (2\pi q/k) \star j$, where j is the source worldline. Find a more direct way to do it using manipulations in the path integral. Assume that both C and C' are homologically trivial.

Solution:

⁶¹Here we are assuming the generic case where $\mathcal{O}[\eta, \bar{\eta}]$ is local when restricted to M (this is usually the case since charged operators transversely intersect charge operators). But in principle, the support of $\mathcal{O}[\eta, \bar{\eta}]|_M$ could be anywhere up to $D - q - 1$ dimensional.

The strategy is the same thing using shifts of integration variables and Poincare duality that we know and love so well by now. For the insertion of two Wilson loops, for $U(1)_k$ we have

$$\langle W^q(C)W^p(C') \rangle = \frac{1}{Z} \int \mathcal{D}A \exp \left(i \frac{k}{4\pi} \int A \wedge dA + i \int A \wedge (q\hat{C} + p\hat{C}') \right). \quad (1855)$$

To get rid of the Wilson loop insertion, we perform the shift

$$A \mapsto A - \frac{2\pi}{k} d^{-1}(q\hat{C} + p\hat{C}'). \quad (1856)$$

Note that \hat{C} and \hat{C}' are both 2 forms, but taking the d^{-1} turns them into 1 forms. One can check using some integrations by parts that this shift kills the term in the exponent that is linear in A . This produces

$$\begin{aligned} \langle W^q(C)W^p(C') \rangle &= \frac{1}{Z} \int \mathcal{D}A \exp \left(i \frac{k}{4\pi} \int A \wedge dA - i \frac{2\pi}{k} pq \int \hat{C} \wedge \frac{1}{d}\hat{C}' - i \frac{\pi}{k} q^2 \int \hat{C} \wedge \frac{1}{d}\hat{C} \right. \\ &\quad \left. - i \frac{\pi}{k} p^2 \hat{C}' \wedge \frac{1}{d}\hat{C}' \right). \end{aligned} \quad (1857)$$

Now writing $C = \partial D$ and $C' = \partial D'$ so that $\hat{C} = dD$ and $\hat{C}' = dD'$, we have

$$\langle W^q(C)W^p(C') \rangle = \exp \left(-i \frac{2\pi}{k} pq \int \hat{D} \wedge d\hat{D}' - i \frac{\pi}{k} q^2 \int \hat{D} \wedge d\hat{D} - i \frac{\pi}{k} p^2 D' \wedge d\hat{D}' \right). \quad (1858)$$

The terms with the self-CS interaction of \hat{D} and \hat{D}' are the expectation values of single Wilson loops, which we see by setting e.g. $p = 0$:

$$\langle W^q(C) \rangle = \exp \left(-i \frac{q^2 \pi}{k} \int \hat{D} \wedge d\hat{D} \right). \quad (1859)$$

This is ill-defined since the integral computes the intersection of ∂D with D , which doesn't make sense. We can regulate it by using a framing of the curve C . Given such a framing, we replace $d\hat{D}$ in the above integral with $d\tilde{D}'$, where $\partial D' = C'$ is a copy of C displaced infinitesimally along the vector field defined by the framing. The integral $\int \hat{D} \wedge d\hat{D}'$ then becomes the linking number of C and C' , which depends only on the topological class of the framing (how many times the framing winds as it travels around C). Thus this framing-assisted regularization is just a choice of how to do point-splitting regularization for the Wilson operator. In \mathbb{R}^3 or S^3 we can always choose the framing so that the linking number of C and C' is zero, but for more general manifolds this may not be possible. In what follows we will actually choose a framing that winds by 2π along C if C is homologically trivial in the ambient spacetime (which we will assume to be the case). The reason for doing this will become clear in a second. Note that we are not loosing much by doing this, since we have a controlled way of determining how the answer changes upon changing the framing (just like how we may not have a canonical RG scale to choose from when renormalizing a QFT, but we know how the coupling constants at different scales are related, so there is no problem with the physics).

Anyway, doing the framing regularization so that the self-intersection number is equal to 1, we obtain

$$\langle W^q(C) \rangle = (-1)^{q^2/k}. \quad (1860)$$

This means that with this convention, a lone Wilson loop computes the topological spin $s = q^2/(2k) \bmod 1$ of the relevant anyon⁶². This comes from the fact that with our convention, the ribbon formed by C and its deformed copy has a 2π twist in it, so that unlinked loops compute the topological spin. In another convention where the framing is topologically trivial, unlinked loops would simply have expectation value 1. Now we have

$$\langle W^q(C)W^p(C') \rangle = \langle W^q(C) \rangle \langle W^p(C') \rangle \exp\left(-2\pi i \frac{pq}{k} \mathcal{L}(C, C')\right), \quad (1861)$$

where $\mathcal{L}(C, C') = \int \hat{D} \wedge d\hat{D}'$ is the linking number of C and C' . In particular, note that a line with charge k is transparent with respect to all other lines. If k is odd this transparent line has spin $(-1)^k = -1$, and so odd k theories contain a transparent fermion—this is why they are spin TQFTs.

If we were to repeat this exercise with e.g. the Abelian CS theory with K matrix kX , then we would start with

$$\langle W^q(C)W^p(C') \rangle = \frac{1}{Z} \int \mathcal{D}A \exp\left(i \frac{k}{4\pi} \int (A \wedge dB + B \wedge dA) + i \int (qA \wedge C + pB \wedge C')\right), \quad (1862)$$

where we have assumed that C is an A line and C' is a B line. If they were both A lines or both B lines, then we see we could perform a shift on just one of the fields so that $\langle W^q(C)W^p(C') \rangle = 1$. This is a check that the A and B fields are bosons (there is no self-interaction to change their statistics). Also note here that the lack of a self-interaction in the action means that the single loop expectation values $\langle W^p(C) \rangle$ don't need to be renormalized: they are equal to 1 identically. Anyway, we can perform the shifts

$$A \mapsto A - \frac{2\pi}{k} q \frac{1}{d} \hat{C}', \quad B \mapsto B - \frac{2\pi}{k} p \frac{1}{d} \hat{C}', \quad (1863)$$

which produces the familiar formula

$$\langle W^q(C)W^p(C') \rangle = \exp\left(-2\pi i \frac{pq}{k} \mathcal{L}(C, C')\right). \quad (1864)$$

110 August 13 — Comments on discrete gauge theories, dualities, and global symmetries

For the first part of the problem, show how to gauge the different \mathbb{Z}_N symmetries in \mathbb{Z}_N BF theory. You should work in the continuum.

⁶²The topological spin is defined only modulo 1 since a Maxwell term (which we always imagine to be sitting around; it's just less relevant than the CS term in the IR) leads to massive spin-1 particles (photons) that don't have any braiding phase with the sources. Hence by computing Wilson line vevs we can't distinguish a given anyon from the same anyon with a massive photon attached to it, and so the spin of the anyons is only well-defined modulo 1.

The second part of this problem concerns itself with trying to understand / elaborate on some of the content in [8] regarding discrete gauge theories.

Ising models (and \mathbb{Z}_N generalizations thereof) are often said to be dual to gauge theories. An example we see mentioned quite often is the mapping between the Ising model in two dimensions and a \mathbb{Z}_2 gauge theory. First, understand this duality mapping in a \mathbb{Z}_N Ising-like spin model⁶³ (this means deriving it, figuring out where the operators go, understanding the various phases. etc) and pay careful attention to the global symmetries on both sides of the duality map. In particular, the spin model has a \mathbb{Z}_N 0-form global symmetry: how is this reproduced on the gauge theory side?

Next, work out the analogous duality mapping for an Ising-like \mathbb{Z}_N spin system in 3 dimensions, again keeping track of the global symmetries. Make some comments about analogous duality mappings for q -form \mathbb{Z}_N gauge theories.

Solution:

First let's remind ourselves of what symmetries a \mathbb{Z}_N q -form gauge theory possesses. One way to write the action is as

$$S = \frac{i}{2\pi} \int_X G \wedge (d\phi - nA), \quad (1865)$$

where A is a $U(1)$ q -form gauge field (i.e. $\int_{M_{q+1}} F_A \in 2\pi\mathbb{Z} \forall$ closed $M_{q+1} \subset X$), G is a $D-q$ form with 2π -quantized periods, and ϕ is a $q-1$ form, again with $d\phi$ having 2π -quantized periods. Integrating out G sets $A = \frac{1}{n}d\phi$, meaning that A is constrained to be flat and that n copies of A are a large gauge transformation.

An alternate way to write this that makes the symmetries manifest is to integrate out ϕ , which sets $G = F_B$ for some $U(1)$ $D-q-1$ form $U(1)$ gauge field B . We usually integrate by parts (for a careful discussion of why this works even though the integrations involve bare gauge fields, see the earlier diary entry on DB cohomology) and write the result as

$$S = \frac{in}{2\pi} \int B \wedge F_A. \quad (1866)$$

We have *two* global symmetries in this theory. One is the q -form global symmetry on A . This is a \mathbb{Z}_N symmetry (*not* a $U(1)$ symmetry), which is most clearly seen when we re-write the integrand as $A \wedge F_B$ (again, see earlier diary entry). We similarly have a $D-q-1$ form \mathbb{Z}_N symmetry (we will use the notation $\mathbb{Z}_N^{(D-q-1)}$) that shifts B . From the action, we see that the Wilson operators for A (B) generate the $\mathbb{Z}_N^{(D-q-1)}$ symmetry (the $\mathbb{Z}_N^{(q)}$ symmetry). Note how the “magnetic” symmetry on B is shifted by a degree from the magnetic symmetry in $U(1)$ gauge theory, which is a $U(1)_m^{(D-q-2)}$ symmetry.

When the \mathbb{Z}_N gauge theory arises in the context of coupling to something with a $\mathbb{Z}_N^{(q-1)}$ global symmetry, the $\mathbb{Z}_N^{(q)}$ symmetry will be explicitly broken, since the Wilson operators for A will be allowed to terminate on the $(q-1)$ -dimensional charged objects. However, the $\mathbb{Z}_N^{(D-q-1)}$ “hidden” magnetic symmetry will remain. Thus whenever we couple a theory to

⁶³I.e. \mathbb{Z}_N -valued spin variables, with a Hamiltonian that is some function of $S_i S_{i+l}^\dagger$.

a \mathbb{Z}_N gauge field, we turn a global q -form symmetry into a local one, but at the same time we add a $\mathbb{Z}_N^{(D-q-1)}$ global symmetry to the theory. This extra symmetry is “hidden” in the natural variables that we usually write down the theory in, and is generated by the Wilson operators of the q -form gauge field.

Finally, let us look at what the global symmetries are when a q -form \mathbb{Z}_N field is coupled to a $(q+1)$ -form \mathbb{Z}_N field. This is the kind of coupling we need to turn on to gauge the \mathbb{Z}_N q -form symmetry of the \mathbb{Z}_N BF theory. Now if the original field whose symmetry is to be gauged is A and the gauge field is \mathcal{A} , we have

$$S \supset \frac{i}{2\pi} \int (NF_A \wedge B - \mathcal{A} \wedge B + N\mathcal{A} \wedge F_B), \quad (1867)$$

for the appropriate degree fields \mathcal{A}, \mathcal{B} . The last term sets \mathcal{A} to be a \mathbb{Z}_N $(q+1)$ -form field (F_B is a properly quantized $U(1)$ field strength). The second term is what allows us to gauge the $\mathbb{Z}_N^{(q)}$ global symmetry. Under a “local” action of this symmetry (i.e. a change in higher transition functions such that the cocycle condition on $(q+2)$ -fold overlaps of patches fails by an N th root of unity), $\frac{N}{2\pi}F_A$ changes by an element in $H^{q+1}(X; \mathbb{Z})$. We can write this as $F_A \mapsto F_A + \frac{1}{N}F_\alpha$, where F_α is a properly quantized $U(1)$ field strength. This change is canceled out by the corresponding shift $\mathcal{A} \mapsto \mathcal{A} + F_\alpha$, and so the whole action is gauge-invariant.

There are four symmetries to look at: first, the “electric” symmetry that shifts A is gauged. Second, the “magnetic” symmetry which shifts B is explicitly broken by the coupling to \mathcal{A} : shifting $B \mapsto B + \lambda$ for a λ a $(D-q-1)$ form with periods quantized in $\frac{2\pi}{N}$ shifts the action by something in $\frac{2\pi i}{N}\mathbb{Z}$. Likewise, the electric shift symmetry of \mathcal{A} is broken by the presence of B (gauge transformations on \mathcal{A} still act trivially, though). The only remaining global symmetry is the magnetic symmetry of \mathcal{A} , which acts by shifting \mathcal{B} . Thus the global symmetry of this system is $\mathbb{Z}_N^{(D-q-2)}$. In general if we have a chain of a q form field coupled to a $q+1$ form coupled to a \dots coupled to a $q+n$ form, only the magnetic symmetry of the $q+n$ form survives as a genuine global symmetry.

Now we can take a look at the two-dimensional \mathbb{Z}_N spin model. From the above, we know that if it indeed maps to a \mathbb{Z}_N gauge theory, then we will have a $\mathbb{Z}_N^{(D-1-1)} = \mathbb{Z}_N^{(0)}$ global symmetry generated by the Wilson line for the gauge field. This is exactly the right symmetry we need to match with the symmetry of the spin model.

The partition function is (hopefully the notation won’t get too horrible)

$$Z = \sum_{s_i} \prod_l E(\zeta^{s_i - s_{i+l}}). \quad (1868)$$

Here the spin variables on each site are $S_i = \zeta^{s_i}$, where $\zeta = e^{2\pi i/N}$. E is some energy functional that gives the Boltzmann weight for a particular configuration of two neighboring spins. In the product, i and $i+l$ are determined as the sites at ∂l . Fourier transforming,

$$Z = \sum_{s_i} \sum_{a_l} \prod_l \tilde{E}(\zeta^{a_l}) \zeta^{a_l(s_i - s_{i+l})}. \quad (1869)$$

The coupling between a and s is schematically $\int a \wedge \star ds = \int s \wedge \star d^\dagger a$, and the sum over all s_i configurations thus implements the delta function $\delta(d^\dagger a)$. The constraint $d^\dagger a = 0$ becomes

a flatness constraint on the dual lattice, which we can then deal with in the usual way. On the direct lattice though, a_l is not a gauge field: there is no local symmetry that shifts a_l by something exact. The local symmetry will only exist on the dual lattice.

Now we will show how this is compatible with a gauge theory presentation. The gauge theory we start with is

$$Z_g = \sum_{s_P, \tilde{a}_L, \tilde{s}_I} \prod_L \zeta^{s_P(\tilde{d}\tilde{a})_P} \tilde{E}(\zeta^{\tilde{a}_L + (\tilde{d}\tilde{s})_L}). \quad (1870)$$

For notation, we have used the following: fields with a tilde over them are the ones which naturally live on the dual lattice. The links, sites, and plaquettes. of the dual lattice are denoted in capital letters: L, I, P . Here $\tilde{d}\tilde{a}$ is the (signed) sum of \tilde{a}_L 's around a plquette on the dual lattice. We can equivalently write $(\tilde{d}\tilde{a})_P$ as $(\tilde{d}\tilde{a})_i$, where i is at the center of P . In the continuum, the coupling of s and $\tilde{d}\tilde{a}$ goes to the integral $\int s \wedge \star d\tilde{a} = \int s \wedge \star d^\dagger a$, where $d\tilde{a} = d^\dagger a$. The basic rule for mapping to the dual lattice is e.g. $\tilde{a} = \star a$, with d acting on the dual lattice becoming $\star d$ on the original lattice. Anyway, in this partition function we see that the $s_i = s_P$ variables are Lagrange multipliers which enforce the flatness of the \tilde{a} gauge field. One should think of s_i (or really, the two-cochains $s_P \in C^2(X^*; \mathbb{Z}_N)$) as being the B in BF theory.

To see that this Z is the same as the spin model Z , we work in unitary gauge: we Hodge decompose \tilde{a}_L , and fix the exact part in the decomposition equal to $-\tilde{d}\tilde{s}$. This kills off the \tilde{s}_I variables. Then we can re-write the $s\tilde{d}\tilde{a}$ term using $\int s \wedge \star d\tilde{a} = \int s \wedge \star d^\dagger a = \int a \wedge \star ds$ to get

$$Z_g = \sum_{s_i, \tilde{a}_L} \prod_L \zeta^{\tilde{a}_L(s_i - s_{i+\star L})} \tilde{E}(\zeta^{\tilde{a}_L}) = Z, \quad (1871)$$

so that Z_g is the same as the \mathbb{Z}_N spin theory. The dual spins \tilde{s}_I basically play the role of soaking up the gauge redundancy of the \tilde{a} field, leaving a field (\tilde{a} after unitary gauge fixing) that is not invariant under the shift of an exact form, which can then map onto the “momentum” field a_l of the direct lattice.

Now although in the Z_g formulation the sum over s_P sets the \tilde{a} field to be flat, in general topology prevents us from fixing $\tilde{a} = d\tilde{\lambda}$ to be trivial. If $H^1(X) = 0$ however we can choose a gauge in which \tilde{a} is gauge equivalent to zero. This gives

$$Z_g|_{H^1(X)=0} = \sum_{\tilde{s}_I} \prod_L \tilde{E}(\zeta^{\tilde{s}_I - \tilde{s}_{I+L}}), \quad (1872)$$

which is the Fourier-transformed spin system on the dual lattice. However, in general we need to keep s_P and \tilde{a}_L , and so the true dual of the original \mathbb{Z}_N spin system is a Fourier-transformed dual spin system, living on the dual lattice and coupled to a \mathbb{Z}_N gauge field. We write this duality as

$$\mathbb{Z}_N^{(0)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(0)} \wedge \widetilde{\mathbb{Z}_N}^{(1)}, \quad (1873)$$

where the RHS denotes the dual gauge theory. As discussed before, the global symmetries on both sides match: the global $\mathbb{Z}_N^{(0)}$ on the LHS is generated by the Wilson line on the RHS, and acts on the “monopole operators” s_P . Note that if we hadn’t kept the original s_P variables in the Z_g formulation, the origin of the $\mathbb{Z}_N^{(0)}$ symmetry on the RHS would be somewhat hidden.

Let us briefly discuss the operator content of both sides of the duality. As we just mentioned, the s_i operators (or the “vertex operators” ζ^{s_i}) become the “monopoles for the gauge theory. Inserting ζ^{s_x} into the partition function and performing the Fourier transform leads to a coupling like $\sum_i s_i(\delta_{i,x} + (d^\dagger a)_i)$, which sets the divergence of the “momentum” a_l to be a delta function concentrated at x (this is just Gauss’ law). When we go to the dual lattice $d^\dagger a$ goes to $d\tilde{a}$, and so the constraint from s_i leads to $(d\tilde{a})_P = \delta_{P,\tilde{x}}$, where \tilde{x} is the plaquette in the dual lattice associated with x . Thus inserting the original spins, which are the “magnetic” lagrange multipliers in the dual formulation, leads to places where the topologicalness of the dual gauge field is violated.

The dual spins are not gauge invariant by themselves, so we can only insert operators of the form

$$W_\gamma = \zeta^{k\tilde{s}_I} \prod_{L \in \gamma} \zeta_L^{k\tilde{a}_L} \zeta^{k\tilde{s}_J}, \quad (1874)$$

where γ is a path on the dual lattice from I to J and $k \in \mathbb{Z}_N$. When we insert this operator in the gauge theory and perform the gauge fixing by setting the exact part of \tilde{a} to kill the dual spins, we get

$$\langle W_\gamma \rangle = \sum_{\tilde{a}_L^f, s_P} \prod_L \tilde{E}(\zeta^{\tilde{a}_L^f}) \zeta^{\tilde{a}_L^f((ds)_L + k\delta_{L \in \gamma})}. \quad (1875)$$

Here the superscript on \tilde{a}_L^f denotes the gauge-fixing. Now un-doing the Fourier transform ($a_l = \star \tilde{a}_L^f$ is the momentum variable), we get

$$\langle W_\gamma \rangle = \sum_{s_i} \prod_l E(\zeta^{(ds)_l + k(\star\gamma)_l}). \quad (1876)$$

Here $\star\gamma$ is γ on the direct lattice, and is obtained by rotating all the segments of γ by $\pi/2$. Thus we see that computing the partition function with the insertion of W_γ is equivalent to computing the partition function with a modified Hamiltonian for the spin model: the Hamiltonian gets modified on all links in $\star\gamma$, i.e. for all links that intersect γ transversely. For example in the case where $N = 2$ we might have an Ising model nearest-neighbor interaction, and the insertion of W_γ would flip the sign of the interaction between ferromagnetic and anti-ferromagnetic on all the links in $\star\gamma$ (of course, provided $k = 1$).

In the symmetry-breaking phase $\langle \zeta^{s_i} \rangle \neq 0$ of the spin model, the Wilson operators $\langle W_\gamma \rangle$ have an exponential decay that goes as $|\min(\gamma)|$, where $\min(\gamma)$ is the shortest path connecting I to J : we have linear confinement. This is because the insertion of W_γ changes the sign of the coupling along γ (it is a disorder operator), so that in the symmetry-breaking phase the value of $\langle \zeta^{s_i} \rangle$ jumps by an amount determined by k upon crossing γ . In order for $\langle \zeta^{s_i} \rangle$ to be well-defined there must be some other line across which $\langle \zeta^{s_i} \rangle$ jumps by the opposite amount. Since the Hamiltonian is not modified on this line, this line will have a tension determined by the strength of the coupling in the Hamiltonian. Note that the line defined by γ itself is not tensionful, since the modification of E along γ means that jumps in $\langle \zeta^{s_i} \rangle$ are not energetically costly there. Thus the minimal energy configuration will be one where $\langle \zeta^{s_i} \rangle$ flips along γ , and then again along the shortest line connecting I and J (again, only the latter line is energetically costly). Thus we get a linearly confining phase. On the other hand, if $\langle \zeta^{s_i} \rangle = 0$ then the W_γ operators have nonzero vevs. This is the Higgs phase of the gauge theory.

How does this work in higher dimensions? Basically, everything goes through in the same way. We first Fourier-transform the partition function by introducing “momentum” variables a_l on the links of the original lattice. These are then dual to gauge fields $\tilde{a} = \star a$, which are cochains in $C^{D-1}(X^*)$. To write the gauge theory version Z_g of the partition function, we add dual “matter fields” \tilde{s} , which are cochains in $C^{D-2}(X^*; \mathbb{Z}_N)$ (i.e. $(D-2)$ -form gauge fields on the dual lattice). As before, the spin system partition function is recovered upon fixing the unitary gauge. Everything else goes through in the same way—this is why we’ve been using differential forms. Thus we get that in D dimensions, the \mathbb{Z}_N spin system is dual to a theory on the dual lattice with a $(D-2)$ -form \mathbb{Z}_N gauge field coupled to a $(D-1)$ -form \mathbb{Z}_N gauge field. We write this as

$$\mathbb{Z}_N^{(0)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(D-2)} \wedge \widetilde{\mathbb{Z}_N}^{(D-1)}. \quad (1877)$$

Again, the global symmetries match. The LHS has a $\mathbb{Z}_N^{(0)}$ symmetry, while the only global symmetry on the RHS⁶⁴ is the one generated by the Wilson operators of the $(D-1)$ -form gauge field: since the charge operators are $(D-1)$ -dimensional, this is a regular zero-form symmetry, which matches with the LHS. Note that in keeping with duality, the \mathbb{Z}_N symmetry on the LHS is an electric symmetry (the symmetry “acts on the fundamental fields”), while on the RHS it is a magnetic symmetry (the symmetry is “generated by the fundamental fields”).

Dualizing a \mathbb{Z}_N gauge theory (a non-topological one) is also done in a similar way. For example, in three dimensions the partition function is

$$Z = \sum_{a_l} \prod_p E(\zeta^{da_p}) = \sum_{a_l, \tilde{b}_L} \prod_p \tilde{E}(\zeta^{\tilde{b}_L}) \zeta^{(da_L)\tilde{b}_L} = \sum_{a_l, \tilde{b}_L} \prod_p \tilde{E}(\zeta^{\tilde{b}_L}) \zeta^{(d^\dagger \tilde{a})_L \tilde{b}_L}, \quad (1878)$$

where $\tilde{a}_P = \star a_l$. Following the usual procedure, this is equivalent to (using $\int d^\dagger \tilde{a} \wedge \star \tilde{b} = \int \star \tilde{a} \wedge d\tilde{b}$)

$$Z = \sum_{\tilde{b}_L, \tilde{s}_I, a_P} \prod_L \tilde{E}(\zeta^{\tilde{b}_L + (d\tilde{s})_L}) \zeta^{\tilde{a}_P (\tilde{d}\tilde{b})_P}, \quad (1879)$$

which we see upon fixing unitary gauge to kill \tilde{s}_I . This is a spin model coupled to a topological \mathbb{Z}_N gauge field, so we have

$$\mathbb{Z}_N^{(1,g)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(0)} \wedge \widetilde{\mathbb{Z}_N}^{(1)}. \quad (1880)$$

Here the $\mathbb{Z}_N^{(1,g)}$ means a non-topological \mathbb{Z}_N 1-form gauge theory. Note that the LHS doesn’t have the same magnetic symmetry coming from the lagrange multiplier enforcing flatness that the topological gauge theories have. It does have the electric 1-form symmetry though, which is matched with the magnetic 1-form symmetry of the $\widetilde{\mathbb{Z}_N}^{(1)}$ factor on the RHS.

What happens if we were to gauge the 1-form symmetry on the LHS? This is done by coupling to a topological 2-form \mathbb{Z}_N gauge theory. The only symmetry left is the magnetic

⁶⁴At the risk of repeating myself: the “electric” symmetry of the $\mathbb{Z}_N^{(D-2)}$ factor is gauged, and the “magnetic” symmetry of the $(D-2)$ -form field along with the electric symmetry of the $(D-1)$ -form field are both explicitly broken by the coupling between the two fields. The only remaining global symmetry is the magnetic symmetry of the $(D-1)$ -form field.

symmetry for the 2-form, which is generated by the Wilson surface and hence is a $\mathbb{Z}_N^{(0)}$ symmetry. Thus we expect that the result is just a \mathbb{Z}_N spin model. Indeed, this is what we get: the partition function is

$$Z = \sum_{a_l, B_p, \lambda_c} \prod_{p,c} E(\zeta^{(da)_p + B_p}) \zeta^{\lambda_c (dB)_c}, \quad (1881)$$

where λ_c is a Lagrange multiplier enforcing flatness for the background field B_p . So then Fourier transforming,

$$\begin{aligned} Z &= \sum_{a_l, B_p, \lambda_c, b_p} \prod_{p,c} \tilde{E}(\zeta^{b_p}) \zeta^{\lambda_c (dB)_c + b_p((da)_p + B_p)} \\ &= \sum_{\tilde{s}_I, \tilde{a}_P, \tilde{B}_L, \tilde{\lambda}_I, \tilde{b}_L} \prod_{I,L} \tilde{E}(\zeta^{\tilde{b}_L + (d\tilde{s})_L}) \zeta^{\tilde{\lambda}_I (d\tilde{B})_I + \tilde{b}_L((d^\dagger \tilde{a})_L + \tilde{B}_L)}. \end{aligned} \quad (1882)$$

We can now integrate out the \tilde{B}_L field. It appears as $\int \star \tilde{B} \wedge (\tilde{b} + d\tilde{\lambda})$, which sets \tilde{b} to be exact. Thus \tilde{b} is pure gauge, and the term $\int \tilde{b} \wedge \star d^\dagger \tilde{a} \rightarrow \int \tilde{\lambda} \wedge d(\star d^\dagger \tilde{a})$ vanishes. Thus since all the fields except \tilde{s} disappear and as predicted, we get a $\mathbb{Z}_N^{(0)}$ spin model:

$$Z = \sum_{\tilde{s}_I} \prod_L \tilde{E}(\zeta^{(d\tilde{s})_L}). \quad (1883)$$

This is written as

$$\mathbb{Z}_N^{(1,g)} \wedge \mathbb{Z}_N^{(2)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(0)}. \quad (1884)$$

The story is similar if we start with a higher-form \mathbb{Z}_N gauge theory, or if we work in more general dimensions. Going through the arguments above, we find

$$\mathbb{Z}_N^{(q,g)} \leftrightarrow \mathbb{Z}_N^{(D-q-2,g)} \wedge \widetilde{\mathbb{Z}_N}^{(D-q-1)}. \quad (1885)$$

The q -form electric symmetry on the LHS matches with the $(D - (D - q - 1) - 1) = q$ form magnetic symmetry on the RHS. As was the case for $q = 1, D = 3$, gauging the electric symmetry on the LHS by coupling to a topological \mathbb{Z}_N $(q+1)$ -form field removes the topological gauge field from the RHS of the duality, and we get

$$\mathbb{Z}_N^{(q,g)} \wedge \mathbb{Z}_N^{(q+1)} \leftrightarrow \widetilde{\mathbb{Z}_N}^{(D-q-2,g)}. \quad (1886)$$

This time the magnetic $D - (q+1) - 1$ form symmetry on the LHS matches with the electric $D - q - 2$ form symmetry on the RHS.

111 August 14 — The Witten effect

I never really understood Witten's original paper when I read it, so I wanted to see if there was a different way of understanding it. So, the task for today is to explain the Witten effect,

both in pure $U(1)$ gauge theory and in a situation where some larger non-Abelian gauge group is Higgsed down to $U(1)$.

Solution:

We'll first do the easy part of looking at $U(1)$ gauge theory in four dimensions. We write the action as (in Minkowski signature)

$$S = \frac{1}{2e^2} \int F \wedge \star F - \frac{\theta}{8\pi^2} \int F \wedge F. \quad (1887)$$

Here $\frac{1}{8\pi^2} \int F \wedge F = \frac{1}{2} \int (F/2\pi) \wedge (F/2\pi)$, which is in \mathbb{Z} if the spacetime X is spin, so that we have the correct normalization of the θ term, with $\theta \sim \theta + 2\pi$. This is checked by remembering that the “instanton number” for $U(1)$ gauge theory is the second Chern character, which is

$$\text{ch}_2 = \frac{1}{2}(c_1 \wedge c_1 - 2c_2), \quad (1888)$$

where the c_i 's are the Chern classes. Since $c_2 = 0$ for Abelian theories and $c_1 = F/(2\pi i)$, we see that the θ term is $\theta \int \text{ch}_2$, which is the proper normalization.

Anyway, back to the problem. Of course, one way to motivate the Witten effect is to look at the equation of motion near a domain wall where θ jumps by some amount $\Delta\theta$. Not paying attention to getting the numbers right, this gives

$$d^\dagger F \propto d^\dagger \star (F \wedge \theta) = \star(dF \wedge \theta) + \star(F \wedge d\theta), \quad (1889)$$

which means that the effective magnetic and electric currents look like

$$j_e^\mu \sim j_m^\mu \theta + \Delta\theta \epsilon^{\mu\nu\lambda z} F_{\nu\lambda} \delta(z), \quad (1890)$$

where we have taken the domain wall to lie in the xy plane. In particular, the $\mu = 0$ component says that

$$\rho_e \sim \theta \rho_m + \Delta\theta B^z \delta(z). \quad (1891)$$

From the first term on the LHS, we see that monopoles in a $\theta \neq 0$ medium get electric charge attached to them. Alternatively, we can consider a spherical shell of material at θ , surrounded by a vacuum at $\theta = 0$. If there is magnetic flux B^r leaving the surface of the shell (this could be due to a magnetic monopole inside the shell or could come from a nontrivial 1st Chern class), then although $\rho_m = 0$ at the interface of the outer part of the shell with the vacuum, the second term on the RHS means that after integration we have $Q_e = \Delta\theta Q_m$ (where we have used Gauss' law for the magnetic field). This again shows how sources of magnetic field (be they monopoles or Chern classes) pick up electric charge when θ is turned on.

Now for a more precise justification. Consider the 1-form symmetry

$$A \mapsto A + \lambda \epsilon, \quad \epsilon \in H^1(X; \mathbb{Z}), \quad (1892)$$

where λ is a constant. Here large gauge transformations mean we identify $\lambda \sim \lambda + 2\pi$. The charge operator which generates the symmetry in the Hamiltonian formalism is

$$Q_{[\epsilon]}^{(1)} = \int_{\Sigma} \epsilon \wedge \frac{\delta}{\delta A}, \quad (1893)$$

where Σ is space. Since $e^{2\pi Q_{[\epsilon]}^{(1)}}$ acts as the identity, the charges of this symmetry are quantized in \mathbb{Z} . To get the canonical momentum of A , we just differentiate the action and get

$$\frac{\delta}{\delta A} = i \left(\frac{\star F}{e^2} - \frac{\theta}{2\pi} F \right). \quad (1894)$$

Now the electric charge operator is

$$Q_e(M) = \frac{1}{e^2} \int_M \star F \quad (1895)$$

while the magnetic one is

$$Q_m(M) = \frac{1}{2\pi} \int_M F, \quad (1896)$$

where the $1/2\pi$ normalization comes from normalizing $Q_m \in \mathbb{Z}$. So we get

$$\begin{aligned} Q_{[\hat{\epsilon}]}^{(1)} &= \int_{\Sigma} \epsilon \wedge \left(\frac{\star F}{e^2} - \frac{\theta}{2\pi} F \right) = \int_{\hat{\epsilon} \subset \Sigma} \left(\frac{\star F}{e^2} - \frac{\theta}{2\pi} F \right) \\ &= Q_e([\hat{\epsilon}]) - \frac{\theta}{2\pi} Q_m([\hat{\epsilon}]). \end{aligned} \quad (1897)$$

In particular, we see that the electric charge of a dyon is

$$Q_e([\hat{\epsilon}]) = Q_{[\hat{\epsilon}]}^{(1)} + \frac{\theta}{2\pi} Q_m([\hat{\epsilon}]). \quad (1898)$$

Now $Q_{[\hat{\epsilon}]}$ and $Q_m([\hat{\epsilon}])$ are both quantized in \mathbb{Z} , so for $\theta \notin 2\pi\mathbb{Z}$ the electric charge is not integral. We also see that $T : \theta \mapsto \theta + 2\pi$ acts on the charge lattice (q, m) by $T : (q, m) \mapsto (q + m, m)$. Of course, in order to actually have nonzero charges we need $H_2(X; \mathbb{Z})$ to be nontrivial so that we can have nontrivial choices for $\hat{\epsilon}$ ($H^2(X; \mathbb{Z})$ can of course be made nontrivial by excising small balls from spacetime and placing magnetic monopoles inside of them).

Now for the non-Abelian version. We will do the usual example where we have $SU(2)$ broken down to $U(1)$ by giving a scalar ϕ in the fundamental of $SO(3)$ a vev (if we gave a vev to a scalar in the fundamental of $SU(2)$, the gauge group would be broken down completely). The symmetry breaking allows us to get dyons with charge assignments as in the $U(1)$ case but on spacetimes that have $H_2(X; \mathbb{Z}) = 0$, e.g. \mathbb{R}^4 .

We will find it helpful to work with ϕ in the adjoint of $SU(2)$, rather than the fundamental of $SO(3)$ (of course they are the same, only the notation is different). This means we will write ϕ as a matrix in $\mathfrak{su}(2)$ as $\phi = \sigma^a \phi^a$. This is slightly more convenient compared to writing ϕ as a three-vector and having it transform under three-dimensional matrices.

The action is (Minkowski signature)

$$S = \int \left(-\frac{1}{2g^2} \text{Tr}[F \wedge \star F] - \frac{\theta}{8\pi^2} \int \text{Tr}[F \wedge F] + \text{Tr}[d_A \phi \wedge \star d_A \phi] + \lambda \left(\text{Tr}[\phi^2] - \frac{v^2}{2} \right)^2 \right), \quad (1899)$$

where the covariant derivative is $d_A \phi = d\phi - i[A, \phi]$. Locally then the potential makes ϕ want to go like e.g. $v\sigma^3/2$, provided $\lambda \neq 0$. We will look for a monopole solution where at infinity ϕ goes to $\phi = \frac{r^a \sigma^a v}{2r}$ at infinity. If $\lambda = 0$ then we can have $\phi = \frac{r^a \sigma^a v}{2r} + O(1/r)$, but

if $\lambda \neq 0$ then this leads to an infinite potential energy. We also need to choose asymptotic falloff conditions on the gauge field so that the kinetic term $|d_A\phi|^2$ is finite when integrated over space: i.e., we need $d_A\phi \sim O(1/r^2)$ as $r \rightarrow \infty$. We can ensure that this is the case provided that we choose A as follows:

$$A = 2\frac{i}{v^2}[\phi, d\phi] + \frac{2}{v}\mathcal{A}\phi, \quad (1900)$$

where \mathcal{A} is a $U(1)$ gauge field. This works since

$$\begin{aligned} d_A\phi &= d\phi + \frac{2}{v^2}[[\phi, d\phi], \phi] = d\phi - \frac{2}{v^2}\epsilon^{abc}\epsilon^{cde}\phi^a(d\phi)^b\phi^d\sigma^e \\ &= d\phi - \frac{2}{2v^2}(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})\phi^a(d\phi)^b\phi^d\sigma^e = d\phi - \frac{2}{v^2}(\text{Tr}[\phi^2]d\phi - \phi d\text{Tr}[\phi^2]) \\ &\sim \frac{1}{v^2r^2}(\phi/r - d\phi), \end{aligned} \quad (1901)$$

where in the last step we have kept the leading terms as $r \rightarrow \infty$. Since this goes as $1/r^2$, the kinetic term $|d_A\phi|^2$ has a finite integral over \mathbb{R}^3 . Note that we don't actually need to know the functional form of ϕ for this to work (we can only solve for ϕ analytically when $\lambda = 0$).

The important part here is that the abelian gauge field \mathcal{A} that we tacked on doesn't contribute to $d_A\phi$. It represents the gauge freedom in the unbroken $U(1)$ subgroup at infinity (rotations about the radial axis). When we multiply A by ϕ and take the trace the $[\phi, d\phi]$ part gets killed, and so \mathcal{A} is proportional to the projection of A onto the radial direction:

$$\mathcal{A} = \frac{1}{v}\text{Tr}[\phi A](1 + \dots), \quad (1902)$$

where if $\lambda = 0$ we can have $\dots \sim O(r^{-1})$; otherwise since we need $\text{Tr}[\phi^2] \rightarrow v^2/2 + O(r^{-2})$ we have $\dots \sim O(r^{-2})$. The identification of the $U(1)$ gauge field makes sense since projecting onto the \hat{r} direction in $SU(2)$ by tracing with ϕ selects out the generator of rotations about the radial direction, which leave $\langle\phi\rangle$ invariant (the structure group is reduced to $U(1)$ at infinity). Similarly, the $U(1)$ field strength is defined asymptotically as

$$\mathcal{F} = \frac{1}{v}\text{Tr}[\phi F]. \quad (1903)$$

Since $F \sim O(1/r^2)$ at infinity in order for the gauge field kinetic term to have finite energy, the parts of ϕ which go as negative powers of r can be ignored (thus we have not written any $(1 + O(r^{-1}))$ factor on the RHS of the above equation), and the $U(1)$ field strength becomes exactly the projection of F onto the radial direction (locally we can write $\phi \rightarrow v\sigma^3$ and then $\mathcal{F} \rightarrow F^3$).

Now we need to examine the residual “zero mode” gauge transformations that act on the $U(1)$ gauge field \mathcal{A} . We need to look for “gauge transformations” which act nontrivially on the fields at infinity (and hence are not gauged). We want to leave the scalar field configuration invariant, and since it transforms in the adjoint, our gauge transformation parameter should be something built out of ϕ so that its action on ϕ is trivial. We also need the gauge transformation parameter to be purely radial, so that it only affects \mathcal{A} . It also needs to not mess with the falloff conditions we've imposed on e.g. $d_A\phi$ so that the energetics are

unchanged. A transformation which fits the bill is given by $U_\alpha = \exp(i\alpha\phi/v)$, where $\alpha \in \mathbb{R}$. Actually, the fact that this is the generator is kind of obvious: it performs rotations about the \hat{r} direction, which is exactly what the unbroken $U(1)$ does. This maps $A \mapsto A + \alpha d_A \phi/v$ (this has to be nontrivial since we can't have $d_A \phi = 0$ identically: if we did, we would have a global $U(1)$ symmetry (global symmetries are parametrized by covariantly constant things). But we know this can't happen, since the structure group does not globally reduce to $U(1)$). Under the transformation, \mathcal{A} changes by

$$\delta\mathcal{A} = \frac{1}{v} \text{Tr}[\phi d_A \phi] = \frac{1}{2v} d\text{Tr}[\phi^2]. \quad (1904)$$

If $\lambda = 0$ this can have an $O(r^{-1})$ contribution, and we get the familiar $\mathcal{A} \sim r^{-1}$ falloff behaviour of a gauge field in the Coulomb phase. Now since when $\alpha \in 2\pi\mathbb{Z}$ we have $U_\alpha \rightarrow \mathbf{1}$, the operators $U_{2\pi k}, k \in \mathbb{Z}$ act as gauged gauge transformations. Therefore the “physical gauge transformations” are parametrized by $\alpha \in [0, 2\pi)$ and thus give a $U(1)$ symmetry as expected.

Since the symmetry is $U(1)$, the charges associated to the asymptotic $U(1)$ symmetry will be integral. The charge operator for the symmetry is

$$\mathcal{U}(\alpha) = \exp\left(-i\frac{\alpha}{v} \int_{\mathbb{R}^3} \text{Tr}[d_A \phi \wedge \delta_A]\right). \quad (1905)$$

From the Lagrangian we read off

$$\delta_A = \frac{1}{g^2} \star F - \frac{\theta}{4\pi^2} F, \quad (1906)$$

so that $\mathcal{U}(\alpha) = e^{-i\alpha Q_{\mathcal{A}}}$, where

$$\begin{aligned} Q_{\mathcal{A}} &= \frac{1}{v} \int_{\mathbb{R}^3} \text{Tr} \left[d_A \phi \wedge \left(\frac{1}{g^2} \star F - \frac{\theta}{4\pi^2} F \right) \right] \\ &= \frac{1}{v} \int_{S_\infty^2} \left(\frac{1}{g^2} \text{Tr}[\phi \wedge \star F] - \frac{\theta}{4\pi^2} \text{Tr}[\phi \wedge F] \right) - \frac{1}{v} \int_{\mathbb{R}^3} \text{Tr} \left[\phi \wedge \left(\frac{1}{g^2} d_A \star F - \frac{\theta}{4\pi^2} d_A F \right) \right], \end{aligned} \quad (1907)$$

where we've used the fact that ϕ being in the adjoint means that e.g. $\text{Tr}[\phi \wedge F]$ is $SU(2)$ -neutral, so that we may write $\text{Tr}[d_A \phi \wedge F] = d\text{Tr}[\phi \wedge F] - \text{Tr}[\phi \wedge d_A F]$. Also in the above, the hodge star is taken with respect to the full spacetime. The Bianchi identity means $d_A F = 0$, while we have

$$\text{Tr}[\phi \wedge d_A \star F] \rightarrow \text{Tr}[\phi(d_A)_i F^{0i}] \propto \text{Tr}[\phi \sigma^a \epsilon_{abc} \phi^b (d_A \phi)_0^c] = 0, \quad (1908)$$

where we have assumed our monopole solution is such that ϕ is covariantly constant in time (of course, we see from this formula that a moving monopole will produce an electric field, just like a moving electric charge produces a magnetic field). Thus only the surface integrals at infinity contribute, and we have

$$\begin{aligned} Q_{\mathcal{A}} &= \frac{1}{v} \int_{S_\infty^2} \left(\frac{1}{g^2} \text{Tr}[\phi \wedge \star F] - \frac{\theta}{4\pi^2} \text{Tr}[\phi \wedge F] \right) \\ &= Q_e - \frac{\theta}{2\pi} Q_m. \end{aligned} \quad (1909)$$

Now Q_A is quantized in \mathbb{Z} and so is Q_m (it is the first Chern class of the $U(1)$ bundle at infinity, and in fact is valued in $2\mathbb{Z}$, because of the factor of 2 from the trace or alternatively because the thing getting the vev was in the adjoint of $SU(2)$ i.e. the fundamental of $SO(3)$, instead of the fundamental of $SU(2)$). This is in fact the minimal possible magnetic charge since in these conventions, if we introduced a field charged in the fundamental of $SU(2)$ to do a Dirac string experiment, the field would have electric charge $1/2$). This implies, as in the Abelian case, that the electric charge of a monopole is non-integral, and dependent on the value of θ .

112 August 15 — More on the Schwinger model, its phases, and the theta term

Today's problem came from wanting to understand a statement made by Zohar Komargodski in his lecture on 1-form symmetries at a Stony Brook workshop.

Consider the massive Schwinger model on the spacetime $S^1 \times \mathbb{R}$, where the fermions carry charge $q \in \mathbb{Z}$ and have a real mass $m \in \mathbb{R}$ (for simplicity—axial rotations allow us to interchange the phase of the fermion mass with the θ angle, so this is done without loss of generality):

$$S = \int_{S^1 \times \mathbb{R}} \left(\frac{1}{2\pi} \bar{\psi} (\not{p} - iq\not{A}) \psi + \frac{m}{2\pi} \bar{\psi} \psi + \frac{1}{2e^2} F \wedge \star F - i \frac{\theta}{2\pi} F \right). \quad (1910)$$

Because the fermion has charge q , this theory has a \mathbb{Z}_q 1-form symmetry.

Explain what happens in the limits $m \ll e$ and $m \gg e$. What are the differences between $q = 1$ and e.g. $q = 2$? As usual, most of the action happens at $\theta = \pi$.

Solution:

$m \gg e$. First we do the case of large mass. It is then reasonable to throw the fermions away, and look at the pure gauge theory. We already analyzed QED₂ with a θ term in a previous diary entry, where we showed that the spectrum was labeled by the different quantized values of the electric flux:

$$E_n = \frac{e^2}{2} \left(n - \frac{\theta}{2\pi} \right)^2 = \frac{e^2}{2} F_{01}^2, \quad (1911)$$

where we have set the circumference of the circle to 1 for simplicity. When $\theta = \pi$ we have a degeneracy corresponding to the choice of $F_{01} = \pm 1/2$. This degeneracy can get lifted by a mixing between the two ground states of order e^{-mL} if $q = 1$, but is exact if $q > 1$.

We can also look at this from the boson side. The action bosonizes to (using the quantum fields and strings part II conventions and working in Euclidean time)

$$S = \int \left(\frac{1}{8\pi} d\phi \wedge \star d\phi - \frac{m}{\pi} \cos \phi + \frac{1}{2e^2} F \wedge \star F - i \frac{\theta + q\phi}{2\pi} F \right). \quad (1912)$$

Note that changing $m \mapsto -m$ is the same as changing $\phi \mapsto \phi + \pi$, which from the fact that ϕ appears in the θ term reminds us of why TIs occur inside regions of spacetime where the fermion mass has a sign opposite to the sign it has in vacuum. Anyway, sending $m \rightarrow \infty$ freezes out the boson and we get pure QED at $\theta = \pi$, which as we have said has two degenerate ground states (strictly speaking, only for $q > 1$).

$m \ll e$. Now we look at small mass. In fact, we start with $m = 0$. Here the θ dependence can be completely removed by a shift in ϕ : this is the chiral anomaly, since a shift in ϕ is generated by the vector current for ϕ , which is dual to the axial current for the fermions. The action for $m = 0$ is quadratic and thus easy to solve. We can integrate out the gauge field and use the results of previous diary entries to write the effective action as

$$S = \int \left(\frac{1}{8\pi} d\phi \wedge \star d\phi + \frac{e^2}{2} \min_{k \in \mathbb{Z}} (k - q\phi/2\pi)^2 \right), \quad (1913)$$

which holds as long as ϕ is slowly varying (if ϕ were non-compact, we would just have a quadratic mass term). Thus we see that we end up with a massive scalar, and so the massless Schwinger model is in fact actually massive. Since ϕ is valued in $[0, 2\pi)$, there is only one minimum of the potential if $q = 1$, but if $q > 1$ then we have q distinct minima.

Now we turn on a small mass. After shifting ϕ to kill the θF coupling, the potential for ϕ is

$$V(\phi) = -\frac{m}{\pi} \cos(\phi - \theta/q) + \frac{e^2}{2} \min_{k \in \mathbb{Z}} (k - q\phi/2\pi)^2. \quad (1914)$$

Now the effect of θ is more important. Let's take $\theta = \pi$. If $q = 1$ then θ has the effect of shifting the minimum of the oscillating part of the potential to π . Superimposing the cosine on top of the quadratic potential has the effect of creating two distinct minima if the mass is large enough. We thus get the attractive picture of an Ising-type phase transition where two minima merge into one as m is varied, although where exactly this happens is hard to say, since it relies on us trusting this form of the effective potential for ϕ beyond the regime of parameters for which it was derived. Since the Ising transition is described by free fermions, we see that at some value of $m \sim e$ the confinement disappears. The picture here is a line of alternating ± 1 charges, which since they change the flux by $\Delta F_{01} = 1$ are the domain walls for the Ising order parameter. When the domain walls start to proliferate confinement goes away at the massless Ising point since the electric fields of the domain walls cancel the background electric field coming from the θ term.

If $q = 2$ then the quadratic part of $V(\phi)$ has two minima. Upon adding the mass term, there are still two minima. They are not equally spaced in ϕ because the cosine part of the potential is not symmetric about $\phi = 0$ while the quadratic part is. A similar picture holds for $q > 2$: the q distinct minima have their positions shifted, but for small m there remain q different minima.

The different minima are distinguished by the value of the electric flux around the ring, which comes spaced in integer units. Shifting the flux by q units is the same as shifting ϕ by 2π , and since we identify ϕ with $\phi + 2\pi$, different vacua that differ in their electric fluxes by q are connected. This is of course due to the fact that we have charge q particles which can propagate around the circle and change the flux by q . However, there is no process which can change the electric flux by $p < q$ units. Thus, even though we are on a circle, different vacua related by $\Delta F < q$ do not mix—they do not even mix in a way which is exponentially small

in the particle mass / size of the circle. They only mix through a Wilson operator $e^{ip\oint A}$, which is nonlocal. Thus we genuinely have q distinct superselection sectors, even when space is compact.

So, at $\theta = \pi, q = 1$ we have the following picture: for $m \gg e$ we have two degenerate ground states distinguished by the electric flux, with an exponentially small mixing between them. Then at $m \ll e$ we only have one ground state, and so at some finite $m \sim e$ we have an Ising transition. If we e.g. change to $q = 2$, then the $m \gg e$ story is the same (although the two ground states do not mix), but for $m \ll e$ we still have two ground ground states (again with no mixing). Thus it is natural to guess that for $q = 2$ the Ising transition is eliminated. If $q > 2$, then it seems like as m is increased, there will be phase transitions where pairs of distinct minima merge into each other, eventually pairing down to leave behind the two minima of the $m \gg e$ limit (note to self: come back and think about this).

113 August 16 — Duality in the Ising model

Today's problem came from wanting to understand a statement in [8] about duality in the Ising model. Read the mini-section on page 38, and explain / elaborate on everything.

Solution:

Let X be a Riemann surface equipped with a choice of spin structure η . Let Z_+ denote a partition function for a spin structure chosen so that $\text{Arf}(\eta) = 0$ (the spin structure can be extended to a bounding three-manifold), and Z_- a partition function for a spin structure with $\text{Arf}(\eta) = 1$ (the spin structure is non-bounding). We want to examine what happens when the theory is pushed away from the self-dual conformal point by adding in a perturbation given by the energy operator $m \int \epsilon^{65}$. Since $\epsilon \sim \psi \bar{\psi}$ is a fermion mass, this is indeed the right perturbation for tuning the theory away from the critical point.

Write the perturbed partition function as

$$Z_f[\eta] = \left\langle 1 - m \int_z \epsilon(z, \bar{z}) + \frac{m^2}{2} \int_{z,w} \epsilon(z, \bar{z}) \epsilon(w, \bar{w}) - \dots \right\rangle_\eta, \quad (1915)$$

where the expectation value is computed in the CFT (i.e. at the critical point). We claim that Z_+ is even under $m \mapsto -m$, while Z_- is odd, that is, we claim that $\langle \epsilon^k \rangle$ for k even is only nonzero when $\text{Arf}(\eta) = 0$, while for k odd it is only nonzero for $\text{Arf}(\eta) = 1$.

We now take a look at why this is true. Many thanks to Wenjie Ji for patiently explaining to me why this is true. We will work on the torus for simplicity.

First, we note that $\langle \epsilon \rangle_\eta = 0$ if the spin structure η has antiperiodic boundary conditions in the spatial direction. This is because we are computing $\langle 0 | \psi \bar{\psi} | 0 \rangle$ or $\langle 0 | (-1)^F \psi \bar{\psi} | 0 \rangle$, both of which vanish since the ψ s are primary fields which create states orthogonal to $|0\rangle$ when

⁶⁵ ϵ is an energy operator since $\epsilon \sim \psi \bar{\psi}$ where $\psi \sim X \prod Z, \bar{\psi} \sim Y \prod Z$ means that $\epsilon_j \sim \psi_j \bar{\psi}_j + \psi_j \bar{\psi}_{j+1} + h.c. \sim X_j X_{j+1} + Z_j$ contains the terms that appear in the Hamiltonian $H \sim \sum (XX + Z)$.

acting on $|0\rangle$. Since $\psi\bar{\psi} \sim 1$, inserting an even number of ϵs results in something non-zero, while for the above reason inserting an odd number of ϵs gives zero.

Now for the spin structures with periodic boundary conditions in space. These boundary conditions are created by computing the expectation value in the states $|\sigma\rangle$ and $|\mu\rangle$, which differ by the occupation number of the fermion zero modes, and hence differ in their fermion parity (here the zero mode is a zero mode of the Hamiltonian, not a zero of the action [which doesn't exist if the time direction is antiperiodic]). More precisely, the two states differ in their $(-1)^F = (-1)^{F_L+F_R}$ eigenvalue, where $(-1)^{F_L}$ counts whether the holomorphic zero mode is filled, and $(-1)^{F_R}$ counts whether the antiholomorphic zero mode is filled. Since we are assuming the ∂ conditions for both ψ and $\bar{\psi}$ are the same, the ground states only carry a representation of the Clifford algebra associated to the total (non-chiral) zero mode algebra.

So, to do the trace, we need to sum over the two different ground states $|\sigma\rangle$ and $|\mu\rangle$, which differ in the occupation number of the zero modes. To compute e.g. $\langle\sigma|\epsilon|\sigma\rangle$, we use the operator-state correspondence to write $|\sigma\rangle = \sigma(0)|0\rangle$. Thus to get the expectation value of ϵ , we need to know the OPEs between $\psi, \bar{\psi}$, and σ, μ . Looking this up in the Big Yellow Book, we see that

$$\epsilon(z, \bar{z})\sigma(0) = i\psi(z)\bar{\psi}(\bar{z})\sigma(0) = i\psi(z)\frac{e^{-i\pi/4}}{\sqrt{2\bar{z}}}\mu(0) = \frac{1}{2|z|}\sigma(0). \quad (1916)$$

Similarly,

$$\epsilon(z, \bar{z})\mu(0) = i\psi(z)\bar{\psi}(\bar{z})\mu(0) = i\psi(z)\frac{e^{i\pi/4}}{\sqrt{2\bar{z}}}\mu(0) = -\frac{1}{2|z|}\mu(0). \quad (1917)$$

Note that the phase factors, which aren't always written, are very important here! Now we see that

$$\langle\epsilon(z, \bar{z})\rangle_{RN} = \langle\sigma|\epsilon(z, \bar{z})|\sigma\rangle + \langle\mu|\epsilon(z, \bar{z})|\mu\rangle = 0. \quad (1918)$$

More generally, we see that $\langle\epsilon^k\rangle_{RN} = 0$ for odd k , while it is non-zero for even k . Of course, this could also have been concluded by doing an S transformation on $\langle\epsilon\rangle_{NR} = 0$, which we already knew was true.

On the other hand, for the RR spin structure, we have

$$\langle\epsilon(z, \bar{z})\rangle_{RR} = \langle\sigma|(-1)^F\epsilon(z, \bar{z})|\sigma\rangle + \langle\mu|(-1)^F\epsilon(z, \bar{z})|\mu\rangle = \langle\sigma|\epsilon(z, \bar{z})|\sigma\rangle - \langle\mu|\epsilon(z, \bar{z})|\mu\rangle = \frac{1}{|z|}\langle 0|\mathbf{1}|0\rangle \neq 0, \quad (1919)$$

where we took $|\mu\rangle$ to have an odd number of total zero modes and in the last step used $\sigma \otimes \sigma \sim \mu \otimes \mu \sim \mathbf{1} + \epsilon$ and $\langle\epsilon\rangle = 0$ (here $\langle\rangle$ without any subscript denotes the standard NN boundary conditions expectation value). So we conclude that on the RR torus, odd powers of ϵ are the ones that give nonzero expectation values⁶⁶.

Summing up, if $\text{Arf}(\eta) = 0$ the series only includes even powers of m , while if $\text{Arf}(\eta) = 1$ the series only includes odd powers. Now when we sum over spin structures, the full partition function for the bosons is

$$Z[m] = Z_+[m] \pm Z_-[m], \quad (1920)$$

⁶⁶From a Lagrangian point of view, we could also argue that since $\int \mathcal{D}\psi_0 \mathbf{1} = 0$ (Grassmann integration is the same as differentiation), if there exists some ψ_0 such that $S[\psi_0] = 0$, then $Z = \int \prod_k \mathcal{D}\psi_k e^{-S} = 0$ (such a zero-action mode only exists on the RR torus). However, if we insert an ϵ into the partition function then we can do a mode expansion on it, with the term $\langle\epsilon\rangle_{RR} \supset \int \mathcal{D}\prod_k \mathcal{D}\psi_k \psi_0 \bar{\psi}_0 e^{-S}$ surviving and giving a non-zero expectation value.

where the \pm sign can be chosen freely (see e.g. the Big Yellow Book, chapter 11). This \pm sign corresponds to projecting onto different (total) fermion parity sectors for the torus with periodic spatial ∂ conditions, which projects onto either the $|\sigma\rangle$ states or the $|\mu\rangle$ states. Since μ and σ are order-disorder duals of one another, we expect that this \pm sign is switched under duality. Indeed, based on our comments above, we see that doing duality by taking $m \leftrightarrow -m$ is equivalent to flipping the \pm sign, due to the evenness / oddness of the two partition functions.

We claim that doing duality, i.e. flipping the sign in the linear combination of Z_+ and Z_- is equivalent to tensoring with a Kitaev chain. Indeed, we will see that the partition function of the Kitaev chain in the topological phase is just a sign which depends on the spin structure in the right way to change the sign of Z_- .

Now let's explain this, starting with some more general comments. On a manifold with nontrivial topology, a bosonic theory can only be dual to a fermionic one if the duality relates a bosonic partition function to a sum over spin structures of fermionic partition functions, so that the bosonic theory has no spin structure dependence. A particular spin structure η , or a particular gauge field α , can be selected out by putting the analogue of e^{ikx} in the sum, in accordance with the relation between the two theories being a Fourier transform. In the following we will find it convenient to think about gauge fields as elements of $H_1(X; \mathbb{Z}_2)$, so that the gauge field is defined by having a \mathbb{Z}_2 twist across α (this changes the boundary conditions for cycles β such that $\alpha \cap \beta = 1$). So,

$$Z_f[\eta] = \frac{1}{2} \sum_{\alpha} (-1)^{\eta(\alpha)} Z_b[\alpha], \quad (1921)$$

and⁶⁷

$$Z_b[\alpha] = \frac{1}{2} \sum_{\eta} (-1)^{\eta(\alpha)} Z_f[\eta]. \quad (1923)$$

Here the action of η on $\alpha \in H_1(X; \mathbb{Z}_2)$ is given by $\eta(\alpha) = 0$ if η assigns anti-periodic (N) boundary conditions to α , and $\eta(\alpha) = 1$ if it assigns periodic (R) boundary conditions. This jives with the fact that antiperiodic boundary conditions are the “natural” ones, so that they are the identity in the \mathbb{Z}_2 group law. For example, if we label the nontrivial cycles on the torus as a_1, a_2 , and $a_1 a_2$, then e.g.

$$\eta_{NN}(0) = \eta_{NN}(a_1) = \eta_{NN}(a_2) = 0, \quad \eta_{NN}(a_1 a_2) = 1, \quad (1924)$$

while

$$\eta_{NR}(0) = \eta_{NR}(a_1) = \eta_{NR}(a_1 a_2) = 0, \quad \eta_{NR}(a_2) = 1, \quad (1925)$$

and

$$\eta_{RR}(0) = 0, \quad \eta_{RR}(a_1) = \eta_{RR}(a_2) = \eta_{RR}(a_1 a_2) = 1. \quad (1926)$$

⁶⁷The normalisation factors here are specific to a torus. If we are not on the torus, we have to write the more cumbersome normalization factor

$$Z_f[\eta] = \frac{1}{\sqrt{2}^{\dim H_1(X; \mathbb{Z}_2)}} \sum_{\alpha} (-1)^{\eta(\alpha)} Z_b[\alpha]. \quad (1922)$$

Here the first label in the spin structure is the spatial boundary conditions and the second is the temporal ones. For us, spin structures will be identified with elements of $H^1(X; \mathbb{Z}_2)$. Here the 1-cocycleness comes from the failure of η to be a homomorphism on 1-chains:

$$\eta(a) + \eta(b) - \eta(a + b) = [a] \cap [b]. \quad (1927)$$

One can check the consistency of this with the assignments of $\eta_{XY}(a)$ above.

Another way of writing $\eta(\alpha)$ for a gauge field α is by computing the integral $\eta(\alpha) = \int_{\widehat{\eta}} \alpha$. Here, the submanifold $\eta \in H_1(X; \mathbb{Z}_2)$ is determined by the location of the branch cuts needed to determine the spin structure (each branch cut twists the boundary conditions of the fermions, with the “default” boundary condition [no branch cuts passed] being anti-periodic). Thus on T^2 , the spin structure NR has a Poincare dual \widehat{NR} given by the cycle which wraps the spatial cycle of the torus (so that the temporal boundary conditions are periodic).

As a sanity check, we check that the Fourier transform is involutive:

$$\begin{aligned} Z_b[\alpha] &= \frac{1}{2} \sum_{\eta} (-1)^{\eta(\alpha)} Z_f[\eta] = \frac{1}{4} \sum_{\eta} \sum_{\alpha'} (-1)^{\eta(\alpha) + \eta(\alpha')} Z_b[\alpha'] \\ &= \frac{1}{4} \sum_{\alpha'} \sum_{\eta} (-1)^{\eta(\alpha + \alpha')} \int \widehat{\alpha} \cup \widehat{\alpha}' Z_b[\alpha'] \\ &= \frac{1}{4} \sum_{\alpha'} \sum_{\eta} (-1)^{\eta(\alpha')} \int \widehat{\alpha} \cup \widehat{\alpha}' Z_b[\alpha' + \alpha] \\ &= \frac{1}{2} \sum_{\alpha'} \delta(\alpha') (-1)^{\int \widehat{\alpha} \cup \widehat{\alpha}'} Z_b[\alpha' + \alpha] \\ &= Z_b[\alpha] \end{aligned} \quad (1928)$$

where we have used $\int \widehat{\alpha} \cup \widehat{\alpha} = 0$. We have also used

$$\frac{1}{2} \sum_{\eta} (-1)^{\eta(\alpha)} = \delta(\alpha). \quad (1929)$$

This is easy to check explicitly: if $\alpha = 0$, then $(-1)^{\eta(\alpha)} = 1$ for all spin structures. If $\alpha \neq 0$, then two spin structures have $(-1)^{\eta(\alpha)} = -1$, while the other two have $(-1)^{\eta(\alpha)} = +1$, and so the sum gives zero.

We can also check that this Fourier transform formula is consistent with what we know from CFT. Recall that the Virasoro characters for the Ising CFT are given by

$$\chi_1 = \frac{1}{2}(Z_f[NN] + Z_f[NR]), \quad \chi_\epsilon = \frac{1}{2}(Z_f[NN] - Z_f[NR]) \quad (1930)$$

and

$$\chi_{\sigma/\mu} = \frac{1}{2}(Z_f[RN] \mp Z_f[RR]), \quad (1931)$$

where in the spin structure XY , X labels the spatial boundary conditions and Y the temporal ones. The choice of \mp sign depends on what phase of the Ising model we are in (more on this

later). The partition function for the Ising model with periodic boundary conditions for the spins, i.e. for a trivial gauge field $\alpha = 0$, is

$$Z_b[0] = |\chi_1|^2 + |\chi_\epsilon|^2 + |\chi_\sigma|^2. \quad (1932)$$

Now consider putting antiperiodic boundary conditions around the temporal cycle for the spin in the Ising model: this means working with a gauge field $\alpha = a_t$ which has nontrivial holonomy around the temporal cycle. The energy operator ϵ and the identity are not affected, but σ gets mapped to minus itself, and so we might guess that the partition function in the presence of a_t is

$$Z_b[a_t] = |\chi_1|^2 + |\chi_\epsilon|^2 - |\chi_\sigma|^2. \quad (1933)$$

In terms of the different spin structures, this is

$$Z_b[a_t] = \frac{1}{2}(Z_f[NN] + Z_f[NR]) - \frac{1}{2}(Z_f[RN] \mp Z_f[RR]). \quad (1934)$$

Since $(NN)(a_t) = (NR)(a_t) = 0$ (the Poincare dual of the former spin structure is trivial while the dual of the second one is the spatial cycle, along which a_t has trivial holonomy) while $(RN)(a_t) = (RR)(a_t) = 1$, from our Fourier transform formula we see that we should assign minus signs to the RN and RR sectors. Indeed, this is exactly what happens from flipping the sign of $|\chi_\sigma|^2$ in the expression for Z_b . One can then use modular invariance of the Ising spin partition function to confirm the Fourier transform formula for the other choices of Ising spin boundary conditions.

Now let's see what the partition functions are in the different phases. In the symmetry-broken phase, nontrivial gauge field twists are not allowed: they create domain walls in Z , and since the symmetry-broken phase ground state is an eigenstate of Z , a domain wall which wraps a nontrivial cycle creates an inconsistency in the spin configuration, and so $Z_b[\alpha] = \delta_{\alpha,0}$. This means that the fermionic partition function is just

$$Z_f[\eta] = 1. \quad (1935)$$

So, the symmetry-broken phase for the bosons maps to the trivial phase for the fermions.

For the bosons, we expect that in the paramagnetic (symmetric) phase, the partition function will be insensitive to the presence of twists caused by the gauge field: in an eigenstate of X , the operator which creates a domain wall in Z acts trivially. So, we have $Z_b[\alpha] = 1$ in the symmetric phase, and thus

$$Z_f[\eta] = \frac{1}{2} \sum_{\alpha} (-1)^{\eta(\alpha)}. \quad (1936)$$

Now if $\eta \neq RR$, $\eta(\alpha)$ is equal to 1 for only one cycle, and the sum produces $(3 - 1)/2 = 1$. On the other hand, if $\eta = RR$ then three of the cycles are assigned 1, and the sum produces $(1 - 3)/2 = -1$. So then in the symmetric phase, the fermion partition function is the Arf invariant:

$$Z_f[\eta] = (-1)^{\text{Arf}(\eta)}. \quad (1937)$$

This is precisely the partition function for the topological phase of the Kitaev chain. As a check that these work, we can use these fermionic partition functions to reproduce the

bosonic ones. Taking $Z_f[\eta] = 1$ gives

$$Z_b[\alpha] = \frac{1}{2} \sum_{\eta} (-1)^{\eta(\alpha)} = \delta_{\alpha,0}, \quad (1938)$$

which is the partition function for the symmetry-breaking phase⁶⁸. Likewise, for the Kitaev chain in the topological phase,

$$Z_b[\alpha] = \frac{1}{2} \sum_{\eta} (-1)^{\eta(\alpha) + \text{Arf}(\eta)} = 1, \quad (1939)$$

which is uniform in α and thus also matches the symmetric-phase partition function for the bosons (just check the last equality explicitly: for all spin structures, there is always one -1 term in the sum).

This confirms that tensoring with a Kitaev chain implements the duality in the Ising model, since on the bosonic side interchanges the partition functions of the symmetric and symmetry-broken phases.

One more thing: we've seen that in the symmetric phase for the Ising spins (the topological phase for the fermions), there is a $1 - (-1)^F$ projection in the path integral in the R sector, so that with periodic boundary conditions the fermion parity is odd (while it is even in the symmetry-breaking phase). Let's do a sanity check for this result. From the JW transformation, the Majorana fermions ψ satisfy $\psi_i = \prod_{j=i}^{i+L} Z_j \psi_{i+L}$ for a chain of length L (here the Hamiltonian is $H \sim -J \sum XX - h \sum Z$). The product $\prod_j Z_j = G$ is the generator of the global \mathbb{Z}_2 symmetry. So $\psi_i|0\rangle = G\psi_{i+L}|0\rangle = -\psi_{i+L}G|0\rangle = -\lambda_G\psi_{i+L}|0\rangle$, where λ_G is the \mathbb{Z}_2 charge of the ground state and we have used that the Majorana fermions anticommute with the G operator since it is a string of Z 's. If the ground state has trivial \mathbb{Z}_2 charge then we are in the N sector and if it has -1 \mathbb{Z}_2 charge we are in the R sector. But G is also equal to the fermion parity operator, and so we learn that in the symmetric phase the N sector has even parity while the R sector has odd parity.

This can also be understood by actually computing H in momentum space. It is (for complex fermion operators c_k ; the derivation is straightforward)

$$H = \sum_k (2h - 2J \cos(k)) c_k^\dagger c_k - iJ \sin k (c_k c_{-k} + c_k^\dagger c_{-k}^\dagger). \quad (1940)$$

In the N sector the fermion modes always come in pairs symmetric about $k = 0$ and so $(-1)^F = 1$. In the R sector, we have a mode at $k = 0$ and a mode at $k = \pi$. The latter is always un-filled since it is always at high energy. The former is filled, and hence the ground state has odd parity, provided that $J > h$, i.e. provided we are in the paramagnetic phase. This provides another sanity check. A similar computation can be done for Majorana fermions γ_i , where in momentum space we have two unpaired modes, $\gamma_0^\dagger = \gamma^0$ and $\gamma_\pi^\dagger = \gamma_\pi$. Only the former is filled for the R spin structure and gives us the required odd parity. Also note that here duality does $\gamma_0 \mapsto \gamma_0, \gamma_\pi \mapsto -\gamma_\pi$, since duality in the Majorana language is

⁶⁸Note how summing over spin structures produces a δ function for the gauge field α (since if $\alpha \neq 0$ then two spin structures give a -1 phase and two give a $+1$), but that summing over gauge fields does not produce a δ function for the spin structure: $\frac{1}{2} \sum_{\alpha} (-1)^{\eta(\alpha)} \neq \delta(\eta)$ (we would have to decide what exactly we mean by $\delta(\eta)$ also). Instead, this sum gives the Arf invariant.

equivalent to translation through half a unit cell ($i \mapsto i + 1$ for the Majorana index). At the self-dual point this symmetry prevents us from hybridizing the unpaired modes with a term like $\gamma_0\gamma_\pi$. Also note that this \mathbb{Z}_2 duality symmetry is anomalous: it acts as a \mathbb{Z}_2 symmetry on the Majoranas but it actually squares to a translation, so that it cannot be implemented in an on-site way in terms of the original Ising (spin) variables. Indeed, there is no operator that we can write down in terms of the spin variables that is the charge operator for this symmetry.

Finally, we expand on why duality acts as translation by one site, and explain why this is equivalent to changing the sign of the fermion mass m . Recall that at the critical point, the Majorana chain is

$$H = iJ \sum_j^{2N} \eta_j \eta_{j+1}, \quad (1941)$$

where the chain has N physical sites. Let us define the Majorana fields ξ, γ by

$$\eta_j = \frac{1}{\sqrt{2}}(\xi_j + (-1)^j \gamma_j). \quad (1942)$$

The factor of $1/\sqrt{2}$ ensures that they have the usual $\{\xi_i, \xi_j\} = \{\gamma_i, \gamma_j\} = 2\delta_{ij}$ Clifford algebra relation. The $(-1)^j$ is needed since ξ, γ are the slowly varying fields which represent linearizations about the two points where the dispersion touches zero (the chemical potential for the Majoranas vanishes). Writing H in momentum space gives $H \sim \sum_{k>0} \eta_k^\dagger \eta_k \sin(k)$, so that the dispersion has zeros at $k = 0$ (the ξ mode) and at $k = \pi$ (the γ mode). Hence γ_j comes with a factor of $e^{\pi i j}$.

Anyway, putting this expansion into the Hamiltonian, we see that the $(-1)^j$ factors kill the off-diagonal terms, and so in the continuum we get a massless Majorana as expected:

$$H = \frac{iJ}{2} \sum_j (\xi_j \xi_{j+1} + \gamma_{j+1} \gamma_j) \implies S = \frac{iJ}{2} \int \bar{\Xi} \not{\partial} \Xi, \quad (1943)$$

where $\Xi = (\xi, \gamma)^T$. Here we are taking $\gamma^0 = X, \gamma^1 = Y$.

Now let's add the term

$$\delta H = im \sum_j (-1)^j \eta_j \eta_{j+1}. \quad (1944)$$

Since the coupling is alternating on each bond, the coupling strength changes within physical sites and between physical sites. Thus we expect that this term should be the one which drives us away from the critical point, where all the hopping strengths are equal. Indeed, if we put in our expansion for η , we get

$$\delta H = im \sum_j ((-1)^j [\xi_{j+1} \xi_j - \gamma_{j+1} \gamma_j] + \xi_{j+1} \gamma_j - \gamma_{j+1} \xi_j). \quad (1945)$$

The terms that vary as $(-1)^j$ die when we go to the continuum since they oscillate fast and cancel out, and so we have

$$\delta H = im \int dx (-\gamma - \partial_x \gamma) \xi + (\xi + \partial_x \xi) \gamma \implies \delta S = 2 \int im \xi \gamma = - \int \bar{\Xi} im Z \Xi, \quad (1946)$$

where the derivative terms have canceled. So δH is indeed a Majorana mass term. Now we know from above that $m \mapsto -m$ should be equivalent to doing duality. And indeed, from the definition of δH we see that sending $m \mapsto -m$ is equivalent to $j \mapsto j + 1$, i.e. it is equivalent to translation by one site (half of a physical lattice constant). This is another check that duality is realized by translation through half a physical lattice site. In the continuum, it can be implemented via the transformation $\Xi \mapsto Z\Xi$, which leaves the kinetic term invariant but changes the mass term as $m \mapsto -m$. That Z is the right matrix to use here makes sense if we look at (1942), where we see that duality (translation by a lattice site) should take $\xi \mapsto \xi$ and $\gamma \mapsto -\gamma$, which is exactly what Z does.

114 August 17 and 18 — Let's learn about Lie algebras and weights

Today's problem is a little different — instead of solving a particular problem we will just learn about some math facts. I'm just going to go learn about weights and roots and all that, and write down what I learn. Hopefully the compendium of facts will be useful later on.

Solution:

Let \mathfrak{g} be a semisimple Lie algebra coming from a group G . Semisimple means \mathfrak{g} is a \oplus of simple Lie algebras, where a simple Lie algebra is one with no nontrivial ideals other than \mathfrak{g} itself, i.e. no nontrivial subalgebras \mathfrak{i} with $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ other than $\mathfrak{i} = \mathfrak{g}$. An example of a simple Lie algebra is $\mathfrak{su}(2)$, and an example of a semi-simple one is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the latter has the nontrivial ideals $(\mathfrak{g}_1, 0)$ and $(0, \mathfrak{g}_2)$. Note that since \mathbb{R} is Abelian, it has many nontrivial ideals (e.g. $\mathfrak{i} = \{0, x\}$ for $x \in \mathbb{R}$), and so \mathbb{R} is not simple (all simple Lie algebras must be non-Abelian). Equivalently, one can say that a semisimple Lie algebra is one with no nontrivial Abelian ideals. Basically, semisimplicity is ruined by $U(1)$ factors in the Lie group. For example, $\mathfrak{u}(n)$ is not simple, since the diagonal elements (the scalar trace part) constitute a nontrivial ideal.

Let A be an element of \mathfrak{g} such that the zero eigenvalue of the adjoint action Ad_A has greatest multiplicity. I.e. let A be such that $[A, H] = 0$ is satisfied for the greatest number of independent $H \in \mathfrak{g}$. Let H_i be a basis for the space $\ker(\text{Ad}_A)$. The dimension of $\ker(\text{Ad}_A)$ is called the rank of \mathfrak{g} , which we will write as r . We will see momentarily that the H_i generate the Cartan subalgebra of \mathfrak{g} (i.e. they generate a maximal torus in \mathfrak{g}).

Roots:

Now let R_α denote the eigenvectors of the Ad_A action with eigenvalues $\alpha \neq 0$, so that $[A, R_\alpha] = \alpha R_\alpha$. In fact, all the non-zero eigenvalues α of the adjoint are non-degenerate, so that the eigenspace for α is one-dimensional (we won't prove this). This means we can write $\mathfrak{g} = \ker \oplus \text{im}$ by doing (not bothering to distinguish the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ from \mathfrak{g} as is the physics tradition)

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha} V_{\alpha}, \quad (1947)$$

where V_α are the one-dimensional vector spaces generated by the R_α , and \mathfrak{t} is the subalgebra generated by the H_i (the zero eigenspace of the adjoint action). Accordingly, we can decompose A as

$$A = c^i H_i + c^\alpha R_\alpha. \quad (1948)$$

Now consider the action of Ad_A on $[H_i, R_\alpha]$. By expanding the commutator one gets

$$[A, [H_i, R_\alpha]] = \alpha[H_i, R_\alpha], \quad (1949)$$

so that $\text{Ad}_A([H_i, R_\alpha]) \in V_\alpha$. But $\dim V_\alpha = 1$, so we must have

$$[H_i, R_\alpha] = \alpha_i R_\alpha \equiv \alpha(H_i) R_\alpha \quad (1950)$$

for some constant α_i which is the evaluation of α on the generator H_i . Now since $H_i \in \ker(\text{Ad}_A)$, we have

$$0 = [A, H_i] = c^j [H_j, H_i] + c^\alpha \alpha_i R_\alpha. \quad (1951)$$

Thus by linear independence of the R_α , we need $c^\alpha = 0$ for all α , and so $A = c^i H_i \in \mathfrak{t}$. Also, since we picked H_i arbitrarily from the generators of the kernel of Ad_A , we also need $[H_i, H_j] = 0$ for all i, j . Thus \mathfrak{t} is the Cartan subalgebra of \mathfrak{g} . That is, the H_i constitute a maximal set of simultaneously diagonalizable generators of \mathfrak{g} . The A such that Ad_A has the biggest kernel is then a linear combination of these diagonalized generators. The exact choice of linear combination is essentially just a choice of basis.

The α (or sometimes the α_i , or sometimes the R_α) are called the roots of the Lie algebra. Each α can be viewed as a vector in \mathbb{R}^r , with components α_i . The best way to think about things is to define α to live in the dual Lie algebra \mathfrak{t}^* via

$$\alpha : \mathfrak{t} \rightarrow \mathbb{C}, \quad H \mapsto (\text{eigenvalue of } R_\alpha \text{ under } \text{Ad}_H). \quad (1952)$$

A math fact is that all of the H_i can be chosen to be Hermitian (wrt the Killing form, which is built out of the structure constants as $g_{ij} = f_{ik}^l f_{lj}^k$), and thus the α can be chosen to be real. This means that if α is a root then so too is $-\alpha$, with the associated eigenvector being $R_{-\alpha} = R_\alpha^\dagger$. This just follows from applying \dagger to $[H_i, R_\alpha] = \alpha_i R_\alpha$. Another math fact is that if α is a root, then $n\alpha$ is only a root if $n = -1, 0, 1$. Finally, note from the \oplus decomposition of \mathfrak{g} that the number of roots for a given representation is $d - r$, where d is the dimension of the representaiton.

R_α naturally has the interpretation as a raising operator, while $R_{-\alpha}$ is naturally interpreted as a lowering operator. Indeed, if $|W_\mu\rangle$ is an eigenstate of H_i with eigenvalue μ_i (here $\mu \in \mathbb{R}^r$ — recall that we can make the H_i simultaneously diagonal so i can be chosen freely from \mathbb{Z}_r), then

$$H_i R_{\pm\alpha} |W_\mu\rangle = ([H_i, R_{\pm\alpha}] + R_{\pm\alpha} \mu_i) |W_\mu\rangle = (\mu_i \pm \alpha_i) |W_\mu\rangle. \quad (1953)$$

Also, by using the Jacobi identity for commutators with $A, R_\alpha, R_{-\alpha}$, one finds that $[A, [R_\alpha, R_{-\alpha}]] = 0$, so that $[R_\alpha, R_{-\alpha}]$ belongs to \mathfrak{t} . In fact, the precise linear combination is $[R_\alpha, R_{-\alpha}] = \alpha^i H_i$. Thus the commutator of the raising and lowering operators is consistent with being some sort of σ^z thing.

Given R_α and $R_{-\alpha}$, we can form an $\mathfrak{su}(2)$ Lie algebra by defining

$$H_\alpha = 2 \frac{\alpha^i}{\alpha^2} H_i \equiv (\alpha^\vee)^i H_i. \quad (1954)$$

This definition makes use of a metric (the Killing form $g_{ij} = f_{il}^k f_{jk}^l$), which is slightly annoying since metrics can be re-scaled. A perhaps better way to define the H_α is just by equating them to the commutator $[R_\alpha, R_{-\alpha}]$. Anyway, the generators of this $\mathfrak{su}(2)$ are then⁶⁹

$$\mathfrak{su}(2)_\alpha = \langle R_\alpha + R_{-\alpha}, -i(R_\alpha - R_{-\alpha}), H_\alpha \rangle. \quad (1956)$$

Think of these as X, Y, Z . Some people would undoubetdly prefer all of these generators to be divided by 2, but too late⁷⁰. Checking that the $\mathfrak{su}(2)$ commutation relations (really, the $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$) are satisfied is straightforward. The H_α are known as the co-roots of the Lie algebra, and appropriately lie in $(\mathfrak{t}^*)^* = \mathfrak{t}$ through the map $H_\beta : \alpha \mapsto \alpha(H_\beta)$ (we will also equivalently refer to the coefficients $\alpha_i^\vee = 2\alpha_i/\alpha^2$ as the co-roots). The co-roots span a lattice in \mathfrak{t} called $\Lambda_r^\vee(G)$.

The co-roots obey a quantization condition that will be very helpful to have when we think about monopoles. Namely, we have the following quantization condition:

$$\alpha_i^\vee \beta^i \in \mathbb{Z}, \quad \forall \beta \in \Lambda_r(G), \alpha_i^\vee \in \Lambda_r^\vee(G), \quad (1958)$$

which we can also write as $\beta(H_\alpha) \in \mathbb{Z}$ for all $\beta, \alpha \in \Lambda_r(G)$. The integers that occur in this quantization condition are known as the Cartan integers. In particular, from $[H_\alpha, R_\alpha] = 2R_\alpha$, we have $\alpha(H_\alpha) = 2$. These facts just follow from angular momentum quantization in $\mathfrak{su}(2)_\alpha$, and are just saying that the eigenvalues of the diagonal generator of $\mathfrak{su}(2)$ when acting adjointly are integers. Thus we can think of $H_\alpha \in \Lambda_r^*(G)$, since it is a function

$$H_\alpha : \Lambda_r(G) \rightarrow \mathbb{Z}, \quad \beta \mapsto \beta(H_\alpha). \quad (1959)$$

Thus we have the inclusion $\Lambda_r^\vee(G) \subseteq \Lambda_r^*(G)$. Since the co-roots always have integral eigenvalues in any representation, we can identify $\Lambda_r^\vee(G)$ with the kernal of the exponential map from \mathfrak{g} to \tilde{G} .

One important fact is that for all the simply-laced Lie algebras (like $\mathfrak{su}(N)$), which are the ones for which all the roots are the same length, we can normalize the metric so that $\alpha^2 = 2 \forall \alpha \in \Lambda_r(G)$. This means that the co-roots and the roots are actually the same for the simply-laced case.

Weights:

Weights are the eigenvalues of the H_i in general representations. Note that this means the roots are just the weights when the representation is taken to be the adjoint (but really

⁶⁹Technically this is $\mathfrak{sl}(2, \mathbb{C})$ rather than $\mathfrak{su}(2)$. This is because we are really working with the complexification

$$\mathfrak{su}(2) \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C}). \quad (1955)$$

Remember that the \otimes here just means we are allowed to take \mathbb{C} -linear combinations of the generators of $\mathfrak{su}(2)$, rather than \mathbb{R} -linear combinations (it has nothing to do with the field we use for entries in the matrices of $\mathfrak{su}(2)$). When we allow for linear combinations like $aX + ibZ$, we no longer can satisfy Hermiticity, but tracelessness is preserved. Thus the complexification becomes $\mathfrak{sl}(2, \mathbb{C})$.

⁷⁰An alternate normalization scheme along these lines would be e.g.

$$\mathfrak{su}(2)_\alpha = \left\langle \frac{R_\alpha + R_{-\alpha}}{\sqrt{2\alpha^2}}, \frac{R_\alpha - R_{-\alpha}}{i\sqrt{2\alpha^2}}, H_\alpha \right\rangle, \quad (1957)$$

with $H_\alpha = \frac{1}{2}[R_\alpha, R_{-\alpha}] = \alpha^i H_i / \alpha^2$.

one should think of them as living in \mathfrak{t}^*). Usually when people just say “weight”, they mean a weight taken in a fundamental representation, but it seems like they often also mean a weight taken in any representation. Such a weight μ is determined by

$$AW_\mu = \mu W_\mu, \quad H_i W_\mu = \mu_i W_\mu, \quad (1960)$$

where W_μ is a vector transforming in some representation. The number of weights for a given representation is of course the dimension of that representation. We will write $\mu \in \Lambda_w(G)$ to denote a weight in the lattice formed by the weights of \mathfrak{g} , in all representations of G (unless specified otherwise). Note the weights really do depend on G , and not just on \mathfrak{g} . If we have a representation of \mathfrak{g} that does not lift to one of G , then it is not included in $\Lambda_w(G)$. Since the roots are the weights in the adjoint, and since the adjoint representation of \mathfrak{g} always lifts to one on G regardless of the choice of G , we have

$$\Lambda_r(G) \subseteq \Lambda_w(G). \quad (1961)$$

Later we will identify the quotient $\Lambda_w(G)/\Lambda_r(G)$ with something involving a magnetic group.

One potential source of confusion is that several of the physics papers I’ve been reading do not distinguish between the fundamental weights and the weights. The fundamental weights w_i are defined by the orthogonality condition

$$w^i \alpha_j^\vee = \delta_j^i, \quad \forall \alpha_i^\vee \in \Lambda_r^\vee(G). \quad (1962)$$

Thus the fundamental weights form a lattice which we can identify with $(\Lambda_r^\vee(G))^*$, since \mathbb{Z} -valued linear combinations of the w_i allow us to generate all \mathbb{Z} -valued functions on $\Lambda_r^\vee(\tilde{G})$. The fundamental weights are the weights for the covering group \tilde{G} , so that we can write the lattice they generate as

$$\Lambda_w(\tilde{G}) = (\Lambda_r^\vee(\tilde{G}))^*. \quad (1963)$$

This means that the lattice formed by the fundamental weights encompasses the lattices formed by the weights for any other choice of Lie group G with Lie algebra \mathfrak{g} :

$$\Lambda_w(G) \subseteq \Lambda_w(\tilde{G}), \quad (1964)$$

since \tilde{G} is the “biggest” Lie group with Lie algebra \mathfrak{g} (again, since the roots only care about the Lie algebra, $\Lambda_r^\vee(G) = \Lambda_r^\vee(\tilde{G})$).

Since $\alpha(H_\beta) \in \mathbb{Z}$ for all roots α, β , we have $\Lambda_r(\tilde{G}) \subseteq (\Lambda_r^\vee(\tilde{G}))^* = \Lambda_w(\tilde{G})$. We will show later on that

$$\Lambda_w(\tilde{G})/\Lambda_r(\tilde{G}) \cong Z(\tilde{G}). \quad (1965)$$

Again, note that we have been writing the various lattices to be a function of G or \tilde{G} , rather than \mathfrak{g} . This is a little misleading for the root lattice, since it only depends on the Lie algebra. Indeed, since the roots come from the weights in the adjoint representation which does not see global things like $Z(\tilde{G})$, it is sensitive only to \mathfrak{g} and we may write $\Lambda_r(\mathfrak{g})$ or $\Lambda_r(\tilde{G})$ instead of $\Lambda_r(G)$. However, the weights do know about the global structure of the group through their normalization. The fundamental weights, which can be constructed of the dual of the co-roots, only depend on the Lie algebra. Thus, given a Lie algebra, we can always construct the lattice of fundamental weights. Depending on our choice of Lie group,

the representations of the Lie group will be found by selecting out appropriate sublattices of $\Lambda_w(\tilde{G})$.

To elaborate on this, one way to distinguish regular weights vs fundamental weights is by the following method. Let $H_I(G) \subset \mathfrak{t}$ be the set of all elements of the Cartan algebra that exponentiate to the identity, i.e.

$$\exp(H) = \mathbf{1} \in G \quad \forall H \in H_I(G), \quad (1966)$$

where we have used the particular choice of exponential map $\exp : \mathfrak{g} \rightarrow G$ appropriate for the chosen Lie group. For example, if $G = \tilde{G}$ is simply connected, then the lattice of such $H_I(G) \subset \mathfrak{t}$ is the co-root lattice. If $\mu \in \mathfrak{t}^*$ is such that $\mu(H) \in \mathbb{Z}^{71}$ for all $H \in H_I(G)$, then μ is a weight in $\Lambda_w(G)^{72}$.

For example, consider the exponentiation of Z in $\mathfrak{su}(2)$ (the only generator of the Cartan subalgebra). Suppose in our conventions for the exponential map that $\exp(Z)$ generates a 4π rotation about \hat{z} . Then $Z \in H_I(SU(2))$, since a 4π rotation is trivial in $SU(2)$. The associated weight $\mu_{\pm}(Z) = \pm 1$ is thus a weight vector in $\Lambda_w(SU(2))$, and so $\Lambda_w(SU(2)) = \mathbb{Z}$. However, if $G = SO(3)$ then $(Z/2)$ also exponentiates trivially, so that $Z/2 \in H_I(SO(3))$. But μ_{\pm} no longer assigns integers to all the elements of $H_I(G)$, since $\mu_{\pm}(Z/2) = \pm 1/2$. Thus $\mu_{\pm} \notin \Lambda_w(SO(3))$, and the weight lattice of $SO(3)$ is smaller. It is in fact generated by the weight $\mu'(Z) = \pm 2$, which assigns Z to twice one of its eigenvalues. Then $\mu'(H) \in \mathbb{Z}$ for all $H \in H_I(SO(3))$, and so the weight lattice of $SO(3)$ is $\Lambda_w(SO(3)) = 2\mathbb{Z}$.

Finally, just to build intuition: for a defining representation of dimension d , the fundamental weights are a collection of d vectors in \mathbb{R}^r . The vectors connecting these vectors are the roots. This is because the roots are the raising and lowering operators for the $\mathfrak{su}(2)_\alpha$'s, which connect states of different H_α eigenvalues (which are precisely the weights). The fact that the roots always come in pairs just comes from the fact that if a root vector connects two weights, then its negative also connects them, just in the opposite direction.

Weyl group:

The Weyl group is the group that acts by rotating about the X (or Y) axes of the $\mathfrak{su}(2)_\alpha$'s. Stated another way, it is the symmetry group of the maximal torus \mathfrak{t} (the Weyl groups for different choices of the maximal torus are of course isomorphic via conjugation in \mathfrak{g}). Letting

$$\mathcal{W}_\alpha = \exp \left(i\pi \frac{R_\alpha + R_{-\alpha}}{2} \right), \quad (1967)$$

the Weyl group acts on a general matrix $B \in \mathfrak{g}$ as

$$\text{Weyl} : B \mapsto \mathcal{W}_\alpha B \mathcal{W}_\alpha^\dagger. \quad (1968)$$

The image of this is in \mathfrak{g} since we translate “out” of the Lie algebra with the first exponential, but then translate back to the tangent space of the identity with the second exponential. From the definition of \mathcal{W}_α , we see that it performs a π rotation about an axis perpendicular to the “quantization axis” of the $\mathfrak{su}(2)_\alpha$ (the dumb $1/2$ factor is there because of the normalization

⁷¹Recall that μ acts on H by assigning H to one of its eigenvalues.

⁷²We are tacitly including a factor of $2\pi i$ in the exponential map if we are thinking about matrix algebras, so that $\exp(H) = e^{2\pi i H}$.

conventions on the $\mathfrak{su}(2)_\alpha$ generators we chose). Equivalently, it reflects vectors through the hyperplane normal to the quantization axis of $\mathfrak{su}(2)_\alpha$ (namely, the vector $\alpha \in \mathbb{R}^r$). So this just subtracts off twice the projection of a weight μ onto the unit vector α from μ . That is,

$$\text{Weyl} : \mu^i \mapsto \mu^i - 2 \frac{(\mu_j \alpha^j) \alpha^i}{\alpha^2} = \mu^i - \alpha^i \mu(H_\alpha). \quad (1969)$$

From the angular momentum quantization in $\mathfrak{su}(2)_\alpha$, we see that μ is shifted by $n\alpha$, for some $n \in \mathbb{Z}$.

As an example, the Weyl groups of $SU(n)$ and that of $U(n)$ are both S_n , the symmetric group on n objects, which acts by permuting the entries of the (diagonal) matrices in \mathfrak{t} ($SU(2)$ and $U(1)$ have the same roots since their Lie algebras only differ by a $U(1)$ factor which doesn't show up in the roots since the roots are defined by the adjoint action, which is blind to Abelian subalgebras). A math fact is that the action of the Weyl group maps roots to roots. In particular, if we are interested in the adjoint representation we can take $\beta = \alpha$ to identify β with $-\beta$.

Let's go through all this for the simplest possible example: $\mathfrak{g} = \mathfrak{su}(2)$ (again, this is really $\mathfrak{g} = \mathfrak{su}(2) \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$). Part of this we have already done above.

The Cartan subalgebra is one dimensional, with \mathfrak{t} in our normalization being generated just by Z . The weights are the eigenvalues of Z , namely ± 1 . The roots are the eigenvalues of the adjoint action of Z . The eigenvectors are σ^\pm , since $[Z, \sigma^\pm] = \pm 2\sigma^\pm$, and so the roots are ± 2 . Thus the fundamental weight lattice and the root lattice are

$$\Lambda_w(\tilde{G}) = \mathbb{Z}, \quad \Lambda_r(G) = 2\mathbb{Z}. \quad (1970)$$

The roots are ± 2 , and are precisely the vectors which connect the fundamental weights ± 1 to each other. If we were to take $G = SO(3)$, the root lattice would remain unchanged but the weight lattice would be changed, since the fundamental weights ± 1 now no longer lift to representations of the Lie group (since the map $\exp : \mathfrak{g} \rightarrow G$ is different for the two choices of G). The weight lattice is instead

$$\Lambda_w(SO(3)) = 2\mathbb{Z} = \Lambda_r(\mathfrak{su}(2)). \quad (1971)$$

Thus in this case the weight and root lattices are equal. This is because all the representations of $SO(3)$ are obtained from the adjoint representation of $SU(2)$. Note that we again have that $\Lambda_w(G)/\Lambda_r(G) = 1$ is the center of $G = SO(3)$.

The Weyl group acts by sending $\mu \mapsto -\mu$, and so identifies the weights $+1$ and -1 . Thus the weight lattice modulo the action of the Weyl group is

$$\Lambda_w(G)/\text{Weyl} = \mathbb{Z}_2. \quad (1972)$$

This is the center of $SU(2)$, which is no accident. Similarly, we have

$$\Lambda_w(G)/\Lambda_r(G) = \mathbb{Z}_2, \quad (1973)$$

which also is no accident. The global information about the group is tied up in various lattices in interesting ways, which we turn to now.

Global aspects and covering spaces:

Let \tilde{G} be the simply connected universal cover of G , and let $\tilde{T} \subset \tilde{G}$ be a maximal torus of \tilde{G} that is the image of \mathfrak{t} under the exponential map, which we define to act on \mathfrak{t} by $E : H \mapsto \exp(2\pi i H)$ (note that E is a homomorphism acting on \mathfrak{t} since we can freely combine exponentials). Now in general, given a Lie algebra \mathfrak{g} , there may be multiple exponential maps that one can define, each of which “delinearizes” \mathfrak{g} into a different Lie group. However, a math fact is that there is always a unique simply connected Lie group \tilde{G} which can be obtained from exponentiating \mathfrak{g} , such that all other Lie groups G obtained from \mathfrak{g} are quotients

$$G = \tilde{G}/\Gamma, \quad \Gamma \subseteq Z(\tilde{G}). \quad (1974)$$

The exponential map which takes us to \tilde{G} is the only one of the various exp maps that is injective, and until further notice, this map is the one we mean when we talk about the exponential map.

We write

$$\tilde{T} \cong \mathfrak{t}/\Lambda_I(\tilde{G}), \quad (1975)$$

where the kernel of E being quotiented out by is the integer lattice, which consists of those elements $H \in \mathfrak{t}$ such that $\mu(H) \in \mathbb{Z}$ for all weights $\mu \in \Lambda_w(\tilde{G})$. We write this as

$$\Lambda_I(\tilde{G}) = \Lambda_w^*(\tilde{G}), \quad (1976)$$

where the duality involves the group \mathbb{Z} , so that $f \in \Lambda_w^*(\tilde{G})$ means that f is a \mathbb{Z} -valued function on the weight lattice of \tilde{G} . Finally, define the central lattice by $\Lambda_Z(\tilde{G})$, which consists of all those elements in \mathfrak{t} that get exponentiated to something in $Z(\tilde{G})$. From the definition,

$$Z(\tilde{G}) = \Lambda_Z(\tilde{G})/\Lambda_I(\tilde{G}). \quad (1977)$$

Now we need the following: for any lattices Λ_a, Λ_b , we have

$$\Lambda_a \subseteq \Lambda_b \implies \Lambda_b^* \subseteq \Lambda_a^*. \quad (1978)$$

Proof: let Λ_b be generated by the vectors e_i^b , so that any $l \in \Lambda_b$ can be written as $l = m^i e_i^b$, $m^i \in \mathbb{Z}$. Then any $f \in \Lambda_b$ becomes a function on Λ_a through linearity, since we can write any $l' \in \Lambda_b$ as $l' = n^i e_i^b = n^i (k_i e_i^a)$ for some $n^i, k_i \in \mathbb{Z}$. So indeed, $\Lambda_b^* \subseteq \Lambda_a^*$. We will also need

$$\Lambda_b/\Lambda_a \cong \Lambda_a^*/\Lambda_b^*. \quad (1979)$$

Proof: by linearity, we can focus on where functions send a single generator of the lattices, so it is enough to consider the case where the lattices are one-dimensional, generated by the numbers e^a, e^b . Since $\Lambda_a \subseteq \Lambda_b$, we can take $e^a = k, e^b = 1$ for some $k \in \mathbb{Z}$ wolog, so that the quotient on the LHS is \mathbb{Z}_k . Now on the RHS, a function in Λ_b^* can assign the point k anything in $k\mathbb{Z}$ by linearity, while a function in Λ_a^* can assign the point k anything in \mathbb{Z} . Thus there are a \mathbb{Z}_k 's worth of functions on Λ_a that do not lift to functions on Λ_b , and so the quotient on the RHS is also \mathbb{Z}_k . Applying this argument for each generator of the lattice gives us the result.

One more fact we will need is that

$$\Lambda_Z(\tilde{G}) \cong \Lambda_r^*(\tilde{G}). \quad (1980)$$

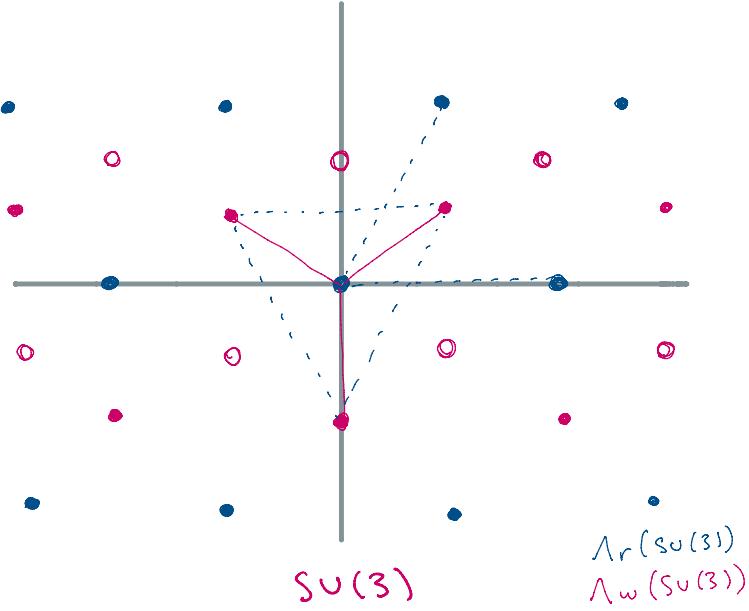


Figure 12: The fundamental weight lattice for $SU(3)$. The blue points indicate the sublattice formed by the root vectors.

Note that $\Lambda_r^*(\tilde{G}) = \Lambda_r^*(G)$ since the roots are defined wrt the adjoint action, which only cares about the structure of \mathfrak{g} .

Proof: let $H \in \mathfrak{t}$. Then $H \in \Lambda_Z(\tilde{G})$ iff $\exp(2\pi i H)$ commutes with $\exp(iR_\alpha)$ for all $\alpha \in \Lambda_r(\tilde{G})$. This is because the R_α generate the part of \mathfrak{g} orthogonal to the maximal torus \mathfrak{t} , and so their exponentials generate all the stuff in the Lie group which does not commute with the exponential of H . Thus $H \in \Lambda_Z(\tilde{G})$ iff we have $e^{-iR_\alpha} e^{2\pi i H} e^{iR_\alpha} = e^{2\pi i H}$. The BCH formula is simplified since $[H, R_\alpha] = \alpha(H)R_\alpha$:

$$e^X e^Y e^{-X} = e^{e^s Y}, \quad [X, Y] = sY. \quad (1981)$$

Making use of this gives $s = 2\pi i \alpha(H)$, and so we see that H is in the central lattice iff $\alpha(H) \in \mathbb{Z}$ for all roots α , which proves the claim.

Now we can use the results of the last few paragraphs to show that

$$Z(\tilde{G}) \cong \Lambda_Z(\tilde{G}) / \Lambda_I(\tilde{G}) \cong \Lambda_r^*(\tilde{G}) / \Lambda_w^*(\tilde{G}) \cong \Lambda_w(\tilde{G}) / \Lambda_r(\tilde{G}). \quad (1982)$$

Phew! As a trivial check of this, for $G = SO(3)$ we get $Z(SU(2)) = \mathbb{Z}/2\mathbb{Z}$, as required.

As a less trivial check, we can look at $SU(3)$, which also lets us illustrate some other properties of the various lattices we've been talking about. $SU(3)$ has rank 2, and has three generating vectors for the fundamental weight lattice $\Lambda_w(SU(3))$. The roots are computed by forming the six different differences of pairs of the generating vectors for the fundamental weight lattice. This is shown in figure 12. The red solid dots indicate the basis vectors for the fundamental weight lattice, while the blue solid dots indicate the root lattice. The red open dots indicate the sublattice formed by adding any two of the fundamental weight lattice generators together, while adding any three fundamental weight lattice generators together

gives something in the root lattice. Thus we have a three-fold periodicity: starting on a given sublattice and adding any three fundamental weight lattice generators will always give one a point in the original sublattice. Any point in a given sublattice can be connected to any other point in the same sublattice through a root vector, and so upon taking the quotient $\Lambda_w(SU(3))/\Lambda_r(SU(3))$, the whole lattice collapses to just three points. This is consistent with $Z(SU(3)) = \Lambda_w(SU(3))/\Lambda_r(SU(3)) = \mathbb{Z}_3$. Also note that the same sort of quotient can be taken by modding out by the action of the Weyl group, which performs reflections in the planes orthogonal to the roots. By taking a look at the figure, one can see that these reflections take the various sublattices to themselves, and that modding out by all three types of reflections collapses each lattice to a single point.

Now we see what can be said when the group in question is not the simply-connected \tilde{G} , but something smaller. We will see that G is always the quotient of \tilde{G} by some finite subgroup. Now every representation of G lifts to a representation of \tilde{G} (one in which the deck transformations are represented trivially), and so $\Lambda_w(G) \subseteq \Lambda_w(\tilde{G})$ (the weights form a sublattice of the lattice generated by the fundamental weights, which are not weights of G unless $G = \tilde{G}$). Also, the root lattice $\Lambda_r(G) = \Lambda_r(\tilde{G})$ is contained in the weight lattice, as we saw earlier (just since roots are particular choices of weights). Thus

$$\Lambda_r(G) \subseteq \Lambda_w(G) \subseteq \Lambda_w(\tilde{G}). \quad (1983)$$

Then we can use what we've learned in the last little section to take the dual (maps into \mathbb{Z}) and produce

$$\Lambda_w^*(\tilde{G}) \subseteq \Lambda_w^*(G) \subseteq \Lambda_r^*(G). \quad (1984)$$

Now as we just saw the rightmost lattice is $\Lambda_Z(\tilde{G})$, while the rightmost lattice is $\Lambda_I(\tilde{G})$ since if an element in \mathfrak{t} has integral eigenvalues under the fundamental representation of \tilde{G} its exponentiation will be trivial (but elements in $\Lambda_w^*(G) \subset \mathfrak{t}$ are not necessarily trivial when exponentiated!). So we get

$$\Lambda_I(\tilde{G}) \subseteq \Lambda_w^*(G) \subseteq \Lambda_Z(\tilde{G}). \quad (1985)$$

Now we exponentiate these sublattices. The rightmost one of course gives us $Z(\tilde{G})$. Define

$$\exp(2\pi i \Lambda_w^*(G)) \equiv \Gamma_G. \quad (1986)$$

Then we have

$$\Gamma_G \cong \Lambda_w^*(G)/\Lambda_I(\tilde{G}), \quad (1987)$$

since \exp is a homomorphism when acting on \mathfrak{t} . So then

$$\Gamma_G \cong \Lambda_w^*(G)/\Lambda_w^*(\tilde{G}) \cong \Lambda_w(\tilde{G})/\Lambda_w(G) \quad (1988)$$

measures the difference between the lattice generated by the fundamental weights and the actual weights of G . $\Gamma_G \subset Z(\tilde{G})$ is the subgroup that determines how we obtain G from \tilde{G} :

$$G = \tilde{G}/\Gamma_G. \quad (1989)$$

This in turn implies⁷³

$$\pi_1(G) = \Gamma_G. \quad (1993)$$

So to summarize, if we are given a Lie group G , it can always be obtained from its universal cover by quotienting by some subgroup Γ_G of $Z(\tilde{G})$. This is one way to prove that if G is a topological group, $\pi_1(G)$ is always Abelian (since it is a subgroup of $Z(\tilde{G})$).

Conversely, given any subgroup Γ_G of $Z(\tilde{G})$ for \tilde{G} simply connected, one can always construct an associated Lie algebra G satisfying $\tilde{G}/\Gamma = G$ (by building it up from its weight lattice).

Dual lattices and dual groups

The last topic in our mini survey of Lie algebra stuff are GNO-dual or Langlands-dual Lie algebras, which are important when discussing magnetic charge quantization in gauge theories.

Given a Lie algebra \mathfrak{g} and Lie group G , we can consider the root system \mathfrak{g}^\vee formed by the coroots of \mathfrak{g} . This is a simple root system, and so we can always exponentiate it to form a simply-connected Lie group \tilde{G}^\vee . The dual Lie group we are interested in will be a quotient of \tilde{G}^\vee by some subgroup of $Z(\tilde{G}^\vee)$. The subgroup is selected by the requirement that the weights of G and G^\vee be dual:

$$\Lambda_w(G^\vee) \equiv (\Lambda_w(G))^*. \quad (1994)$$

The fundamental weights for the dual Lie algebra are by definition the dual of the co-root lattice of \mathfrak{g}^\vee . But the roots of \mathfrak{g}^\vee are the co-roots of \mathfrak{g} , and so the co-roots of \mathfrak{g}^\vee are the roots of \mathfrak{g} , since for any root α , we have $(\alpha^\vee)^\vee = \alpha$. Thus the fundamental weights for \mathfrak{g}^\vee are the dual of the root lattice of \mathfrak{g} :

$$\Lambda_w(\tilde{G}^\vee) = \Lambda_r(\tilde{G})^*. \quad (1995)$$

Using the formula for the center of \tilde{G} that we proved earlier on \tilde{G}^\vee , we see that the universal covers of the group and its dual have the same center:

$$Z(\tilde{G}^\vee) = \Lambda_w(\tilde{G}^\vee)/\Lambda_r(\tilde{G}^\vee) = \Lambda_r^*(\tilde{G})/\Lambda_r^*(\tilde{G}) = \Lambda_r^*(\tilde{G})/\Lambda_w^*(\tilde{G}) = \Lambda_w(\tilde{G})/\Lambda_r(\tilde{G}) = Z(\tilde{G}), \quad (1996)$$

where we used that the roots of \tilde{G} are the co-roots of \tilde{G}^\vee and vice versa.

Now as before, let Γ_G be such that $\tilde{G}/\Gamma_G = G$. Recall that this group was computed by taking the quotient of the two weight lattices: $\Gamma_G = \Lambda_w(\tilde{G})/\Lambda_w(G)$. This same formula holds

⁷³Proof: intuitively, \tilde{G} is the universal cover of G , with fiber Γ_G . Thus upon taking the quotient, a path in \tilde{G} that moves between different sheets of the cover becomes a closed path in the quotient, and contributes to the fundamental group. More formally, consider the fibration

$$\Gamma_G \hookrightarrow \tilde{G} \rightarrow \tilde{G}/\Gamma_G. \quad (1990)$$

This induces a long exact sequence in homotopy groups. Since $\pi_1(\tilde{G}) = 0$ by assumption, the relevant part of the sequence is

$$0 \rightarrow \pi_1(\tilde{G}/\Gamma_G) \rightarrow \pi_0(\Gamma_G) \rightarrow \pi_0(\tilde{G}) \rightarrow \pi_0(\tilde{G}/\Gamma_G) \rightarrow 0. \quad (1991)$$

Now since Γ_G is discrete we have $\pi_0(\Gamma_G) = \Gamma_G$. Since this is finite, the next homomorphism from this into \mathbb{Z} must be the zero map, since all elements in \mathbb{Z} have infinite order. Thus using exactness of the sequence we obtain

$$\pi_1(\tilde{G}/\Gamma_G) \cong \Gamma_G, \quad (1992)$$

as claimed.

for the dual group Γ_G^\vee , and so

$$\Gamma_G^\vee = \Lambda_w(\tilde{G}^\vee)/\Lambda_w(G^\vee) = \Lambda_w^*(G^\vee)/\Lambda_w^*(\tilde{G}^\vee) = \Lambda_w(G)/\Lambda_r(\tilde{G}). \quad (1997)$$

Now let's see how the two subgroups Γ_G, Γ_G^\vee are related. We compute

$$\Gamma_G = \frac{\Lambda_w(\tilde{G})/\Lambda_r(\tilde{G})}{\Lambda_w(G)/\Lambda_r(\tilde{G})} = Z(\tilde{G})/\Gamma_G^\vee. \quad (1998)$$

Thus the two subgroups which we use to obtain G and G^\vee are complementary to one another inside of the center $Z(\tilde{G})$. For example, suppose we are working with the group $\tilde{G} = SU(ab)$, $a, b \in \mathbb{Z}$, and suppose that we obtain G by quotienting by $\Gamma_G = \mathbb{Z}_a$. Then from the above we see that $\Gamma_G^\vee = \mathbb{Z}_b$, and so

$$G = SU(ab)/\mathbb{Z}_a, \quad G^\vee = SU(ab)/\mathbb{Z}_b. \quad (1999)$$

115 August 19 — Allowed spectrum of charges in Abelian and non-Abelian gauge theory

Today the goal is to figure out what types of electric and magnetic charges are allowed to be possessed by line operators in non-Abelian gauge theory.

For two dyons of charges $(q, m), (q', m')$, show that for $U(1)$ gauge theory the quantization condition is

$$qg' - q'g \in \mathbb{Z}, \quad (2000)$$

and determine the non-Abelian analogue of this formula.

Describe the allowed line operators in a non-Abelian gauge theory based on a general compact Lie group G . By “line operators”, we mean operators that are literally supported on a line, and do not come with any surface operator (topological or otherwise) attached.

Solution:

Let's first do $U(1)$ gauge theory, which is the easiest to understand. The charge quantization condition we want to derive is

$$qg' - q'g \in \mathbb{Z}. \quad (2001)$$

The minus sign is the tricky part. Basically, the minus sign is the minus sign in $\star\star = -1$, which holds on 2-forms in four-dimensional spacetime in Lorentzian signature ($\star^2 = (-1)^{p(D-p)+1}$ on a Lorentzian-signature D manifold when acting on p -forms, while for Euclidean signature the exponent is shifted by one). From the Lorentz force law and EM duality $E \mapsto B, B \mapsto -E$, the force on a dyon (q, g) moving at velocity v in the field of another (motionless) dyon (q', g') is (ignoring factors of $1/4\pi$)

$$m \frac{dv^i}{dt} = (qq' + gg') \frac{r^i}{r^3} + (qg' - q'g) \epsilon^{ijk} v_j \frac{r_k}{r^3}, \quad (2002)$$

where the all-important minus sign comes from doing $B \mapsto -E$ in the Lorentz force term. We want to get an angular momentum out of this so that we can find something which is quantized, and so we cross both sides with r : $r \times d_t(v) = d_t(r \times v)$ since $v \times v = 0$, so

$$m \frac{d(r \times v)}{dt} = (qg' - q'g) \frac{r \times (r \times v)}{r^3}. \quad (2003)$$

Using the identity for two epsilon symbols with one index contracted between them, the cross product goes to $[r \times (r \times v)]^i = r^2 v^i - (r \cdot v)r^i$ (maybe the sign is wrong). When we divide by r^3 we get $v^i/r - (r \cdot v)r^i/r$, which is exactly the time derivative of r^i/r^2 . So then we have total time derivatives on both sides, and we conclude that

$$(r \times p)^i = (qg' - q'g)\hat{r}^i. \quad (2004)$$

Thus from quantization of angular momentum, we see that $qg' - q'g$, with the minus sign, is the correct thing to put a quantization condition on.

Now we go to looking at the spectrum of line operators in a general non-Abelian gauge theory. We want to examine the quantization condition on the magnetic charge carried by a given dyonic line operator \mathcal{O} . Consider a (small) S^2 linking \mathcal{O} . We can take the magnetic field on this S^2 to be uniform, with the gauge field on the S^2 being (some people prefer not to have the factor of $1/2$ to be tacked on)

$$A_{\pm} = \frac{B}{2}(\pm 1 - \cos \theta)d\phi, \quad (2005)$$

where B is a covariantly constant matrix determined by the magnetic field⁷⁴. A proof of why the field is covariantly constant is in the following footnote⁷⁵. Here the coordinates A_+

⁷⁴More precisely, it is a covariantly constant section of the adjoint bundle on the S^2 . Recall that sections of the adjoint bundle $\text{Ad}P$ are gauge-invariant things (like field strengths). The adjoint bundle is given by taking the product of a principal G bundle P over the relevant spacetime (sub)manifold with \mathfrak{g} , and then quotienting by the adjoint action Ad , so $\text{Ad}P = (P \times \mathfrak{g})/\sim_{\text{Ad}}$. The identification here is $(g \cdot \phi, F) \sim (\phi, \text{Ad}_{g^{-1}}F)$, or $(g \cdot \phi, \text{Ad}_g F) \sim (\phi, F)$, which is telling us to mod out by gauge transformations.

⁷⁵An S^2 linking the line operator in question sees a magnetic field

$$F_{ij} = \epsilon_{ijk} \frac{x^k}{4\pi|x|^3} B(x) = \text{vol}_{S^2} \wedge B(x), \quad (2006)$$

since in this configuration there is a monopole at the center of the S^2 . Note that B has x dependence since setting B to be a constant would not be a gauge-invariant thing to do in the non-Abelian case. However, B is covariantly constant.

To show this, consider the equations of motion on the S^2 , namely $D_i F^{ij} = 0$. This reads

$$0 = (d^\dagger \text{vol}_{S^2})^j B(x) + (\text{vol}_{S^2})^{ij} D_i B(x). \quad (2007)$$

Since the volume form on S^2 is co-closed (it is harmonic), we get that

$$\epsilon_{ijk} x^j D_k B(x) = 0. \quad (2008)$$

Now consider the Bianchi identity, $D_{(i} F_{jk)} = 0$. This reads

$$0 = d\text{vol}_{S^2} \wedge B(x) + \text{vol}_{S^2} \wedge DB(x). \quad (2009)$$

Again the first term vanishes, and this implies that

$$x^k D_k B(x) = 0, \quad (2010)$$

are used for the northern hemisphere, and A_- is used for the southern hemisphere. The two expressions differ at the equator by $d(B\phi)$, which needs to be $-ig^{-1}dg$ for some well-defined g since the gauge fields on different patches are glued together with exterior derivatives of transition functions. We see that $g = e^{Bi\phi}$ on the equator, and so in order for g to be well-defined we need

$$e^{2\pi i B} = \mathbf{1} \quad (2011)$$

to hold as a matrix equation.

Another essentially equivalent way to get the quantization is as follows. Gauge transformations act on B adjointly by conjugation. We now need a math fact: for any $B \in \mathfrak{g}$, we can always find a $B' = S^\dagger BS$ such that $B' \in \mathfrak{t}$. Basically, we can always make a rotation in the gauge group to diagonalize the uniform field B (all elements in \mathfrak{g} lie in some maximal torus). Since the components of the gauge field now commute with each other, we can use Stokes' theorem and write the Wilson loop as an integral over either the southern or northern hemispheres of the S^2 . Demanding that these two integrals give consistent Wilson loops gives us the same quantization condition as before.

Now since $B \in \mathfrak{t}$, we can write $B = \beta^i H_i$ for some coefficient vector $\beta \in \mathbb{R}^r$. When we take the Wilson line to be in some representation, since the H_i can be simultaneously diagonalized we can replace them with their eigenvalues μ_i , which are the weights of the given representation. Thus the quantization condition is that

$$\beta_i \mu^i \in \mathbb{Z}, \quad \mu \in \Lambda_w(G). \quad (2012)$$

Looking back at yesterday's entry, we see that we can satisfy this condition by taking

$$\beta^i = 2 \frac{\alpha_i}{\alpha^2} = \alpha_i^\vee, \quad \alpha \in \Lambda_r(G), \alpha^\vee \in \Lambda_r^\vee(G) \quad (2013)$$

so that the quantization condition is satisfied by virtue of quantization of angular momentum in $\mathfrak{su}(2)_\alpha$. This is why we often think of the allowed t'Hooft lines as coming from representations that are created with \otimes s of the adjoint representation, in contrast to Wilson lines which can be in any representation: if the β^i can always be written in terms of a root in $\Lambda_r(G)$, then since $\Lambda_r(G)$ is the root lattice for the adjoint representation, all allowed magnetic charges must come from \otimes s of adjoint representations (the \otimes operation corresponding to the fusion of t'Hooft lines).

This is not strictly true though, since while $\beta \in \Lambda_r^\vee(G)$ is sufficient for satisfying the quantization condition, it is not always necessary. The allowed values for β actually come from a sublattice of the co-root lattice. The most general choice of β would be to take $\beta \in (\Lambda_w(G))^*$, where the dual indicates functions into \mathbb{Z} . Looking back at the previous problem, we see that this is precisely what it means for β to be a weight of the dual group G^\vee . Thus $\Lambda_w(G^\vee)$ parametrizes the allowed values of magnetic charge.

The diagonalization we made to rotate the magnetic field so that $B \in \mathfrak{t}$ was made with a gauge transformation that was constant on the S^2 which we were using to study the quantization condition. This doesn't completely fix B though, since there are still rotations

since the antisymmetrizations from the wedge product and the definition of the volume form cancel out.

So together with the previous equation derived from the eom, we see that the vector $D_k B(x)$ is orthogonal to both the radial direction and to the directions tangent to the S^2 . Thus it vanishes identically, and so $D_k B(x) = 0$ as claimed.

we can do within the Cartan subalgebra which represent residual gauge redundancies. These redundancies are precisely given by the action of the Weyl group for the dual Lie algebra \mathfrak{g}^\vee (remember that the Weyl group is given by reflections about the roots, so that it only depends on the Lie algebra, and not on the choice of Lie group). A math fact is that if $WBW^\dagger \in \mathfrak{t}$ for $B \in \mathfrak{t}$, then W implements a Weyl transformation. So, the Weyl group contains all the residual gauge transformations not fixed by our choice of magnetic field $B \in \mathfrak{t}$. Now the roots for the dual group are the co-roots of the original group, which means that the Weyl group acts in the same way on both the lattice of G and the lattices of G^\vee , since

$$\text{Weyl} : \mu \mapsto \mu - \alpha(\alpha_i^\vee \mu^i) = \mu - \alpha^\vee(\alpha_i \mu^i), \quad (2014)$$

so that Weyl and Weyl^\vee act in the same way. Thus our tentative classification scheme for magnetic charges is to label them by elements of the quotient $\Lambda_w(G^\vee)/\text{Weyl}$.

A slightly better way to classify the charges is to realize that no matter what the exact Lie group and dual Lie group are (given a particular \mathfrak{g}), magnetic lines in the dual root lattice $\Lambda_r(\mathfrak{g}^\vee) = \Lambda_r^\vee(\mathfrak{g})$ will always be allowed. Indeed, for $\mu \in \Lambda_w(G)$ and $H \in \Lambda_r^\vee(\mathfrak{g})$ we have $\mu(H) \in \mathbb{Z}$ regardless of the exact choice of $G = \tilde{G}/\Gamma_G$, so that magnetic charges in $\Lambda_r(\mathfrak{g}^\vee)$ are always allowed. Thus to obtain a classification which distinguishes between the line operators that are allowed for different choices of Γ_G , we can quotient the lattice of all possible magnetic charges (viz. $\Lambda_w(G^\vee)$) by the lattice of those that will be there no matter what (viz. $\Lambda_r(\mathfrak{g}^\vee)$). Thus we propose to classify magnetic charges by the quotient

$$\Lambda_w(G^\vee)/\Lambda_r(\mathfrak{g}^\vee) = \Lambda_w^*(G)/\Lambda_r^\vee(G) = \Lambda_w(\tilde{G})/\Lambda_w(G), \quad (2015)$$

where we have used various manipulations derived in the last diary entry. Using the last section of that diary entry, we see that

$$\Lambda_w(G^\vee)/\Lambda_r(\mathfrak{g}^\vee) = \Gamma_G = \pi_1(G), \quad (2016)$$

where again $\tilde{G}/\Gamma_G = G$. That we get such a classification for the magnetic charges of ‘t Hooft operators in terms of $\pi_1(G)$ makes perfect sense, since the ‘t Hooft operators can be defined by the holonomy (valued in $\pi_1(G)$) they induce in gauge field configurations. So the full class of allowed magnetic charges is $\Lambda_w(G^\vee)$ (up to Weyl invariance, more on this in a sec), and once we mod out by the lines which always appear regardless of the Lie group, we see that they are classified by $\Gamma_G = \pi_1(G)$.

A similar statement can be made for the electric operators. Since the allowed electric operators are determined by the allowed representations that we can take the trace in, they are classified by $\Lambda_w(G)$ (up to Weyl invariance). But the lines in the adjoint representation, corresponding to points in the $\Lambda_r(\mathfrak{g})$ lattice, always appear (they are the “worldlines of the gauge fields”, since the gauge fields transform adjointly under constant gauge transformations) regardless of the choice of Lie group ($\Lambda_r(\mathfrak{g}) \subset \Lambda_w(G)$ for all G with Lie algebra \mathfrak{g}). Thus we propose to classify electric operators by the quotient

$$\Lambda_w(G)/\Lambda_r(G) = \Lambda_w^*(G^\vee)/\Lambda_w^*(\tilde{G}) = \Lambda_w(\tilde{G}^\vee)/\Lambda_w(G^\vee) = \Gamma_G^\vee = \pi_1(G^\vee), \quad (2017)$$

where we used that e.g. $\Lambda_r^\vee(G) = \Lambda_r^\vee(G^\vee)$ and $\Lambda_w(\tilde{G}) = (\Lambda_r^\vee(\tilde{G}))^*$. Note the nice symmetry of this quotient with the quotient for the magnetic charges!

So far we have been considering lines that were either purely electric or purely magnetic. What happens if we have dyonic lines? Some of the details for this are in [?], so we will be brief. Basically, you can only have a consistent dynonic line if the electric and magnetic fields commute with one another (so that they can be fused together on a line in an unambiguous way). This is already ensured if we take the charges to be given by the classification scheme above, since everything done above involved only the Cartan subalgebras of \mathfrak{g} and \mathfrak{g}^\vee . In more detail, one first fixes a magnetic field B , and then chooses an electric field in the centralizer of B in G . The centralizer G_B gives rise to a Lie algebra \mathfrak{g}_B , whose Cartan algebra is still the Cartan algebra of G since the elements in the Cartan algebra commute with one another. The Weyl group acts on B , and we also have a redundancy coming from Weyl_B acting on the electric sector, where Weyl_B is the subgroup which fixes B . But the combined action of Weyl_B and $\text{Weyl}/\text{Weyl}_B$ on the electric sector is just the same as having the full Weyl act, and so we just get a single (diagonal) action of Weyl on the electric and magnetic sectors. Thus the whole analysis goes through unchanged for dyonic operators.

If we propose to classify line operators by $\Gamma_G^\vee \times \Gamma_G$, what about Weyl invariance? It turns out that this is already accounted for, since the Weyl group acts trivially on $Z(\tilde{G})$, and hence on the above assignments of both magnetic and electric charges ($\Gamma_G, \Gamma_G^\vee \subset Z(\tilde{G})$). Proof: recall that a weight vector μ under the action of the Weyl group changes by $\delta\mu = \alpha(\alpha_i^\vee \mu^i)$, where α^\vee is some co-root. The inner product here is just $\mu(H_\alpha)$, which since H_α is an $\mathfrak{sl}(2, \mathbb{C})$ generator, is an integer. Thus $\delta\mu \in \Lambda_r(G)$, and so the Weyl group acts on vectors by adding integer multiples of roots to them (see the $SU(3)$ figure from the last diary entry for a nontrivial example of how this plays out). In the quotient $Z(\tilde{G}) = \Lambda_w(\tilde{G})/\Lambda_r(\tilde{G})$ this action is trivial, and so Weyl acts trivially on $Z(\tilde{G})$.

Summarizing, we can classify the lattice L of line operators by

$$L = (\Lambda_w(G)/\Lambda_r(G)) \times (\Lambda_w(G^\vee)/\Lambda_r(G^\vee)) = \Gamma_G^\vee \times \Gamma_G = \pi_1(G^\vee) \times \pi_1(G) \subset Z(\tilde{G})^2. \quad (2018)$$

Since $\Gamma_G^\vee \times \Gamma_G$ is an Abelian group, multiplying equivalence classes of line operators is done easily by using addition in the group. The simplest cases are when one of Γ_G, Γ_G^\vee is the center of \tilde{G} . If $\Gamma_G = Z(\tilde{G})$ then $\Lambda_w(G) = \Lambda_r(G)$ and $\Gamma_G^\vee = \mathbb{Z}_1$, so that the spectrum of line operators modulo gauge field world lines only includes magnetic line operators. Likewise, if the gauge group is simply connected so that $\Gamma_G = \mathbb{Z}_1$ then we have no 't Hooft lines (since $\pi_1(G) = 0$), and the spectrum has only electric line operators. Also note that this implies that no matter what the gauge group is, the number of line operators is always equal to the order of the center:

$$|L| = |\Gamma_G||\Gamma_G^\vee| = |Z(\tilde{G})|. \quad (2019)$$

Now we take another slightly approach to identifying L by looking at the analogue of the quantization condition on $qm' - q'm$ for the non-Abelian case. The same angular momentum argument goes through unmodified (I think—the argument for the $U(1)$ unfortunately took S -duality for granted, though), but the charges involved are now matrices and need to get turned into numbers with the help of an inner product. Since the electric (magnetic) charges are in \mathfrak{t}^* (\mathfrak{t}), the inner product is just the evaluation map $\langle , \rangle : \mathfrak{t}^* \otimes \mathfrak{t} \rightarrow \mathbb{C}$. For two dyons with electric / magnetic field strengths (Q, B) and (Q', B') , we thus require

$$\langle Q, B' \rangle - \langle Q', B \rangle \in \mathbb{Z}. \quad (2020)$$

Let us pretend we didn't know about the Γ_G, Γ_G^\vee groups calculated previously, and just wanted to go about solving the quantization condition directly. By the physical arguments given earlier, we know that the lattice of line operators has to sit inside $Z(\tilde{G})^2$:

$$L \subset (\Lambda_w(\tilde{G})/\Lambda_r(\tilde{G})) \times \Lambda_w(\tilde{G}^\vee)/\Lambda_r(\tilde{G}^\vee) = Z(\tilde{G})^2, \quad (2021)$$

where the group \tilde{G}^\vee is defined so that its roots are the co-roots of \tilde{G} .

Solving the quantization condition by calculating the inner product is simple in our reduced classification scheme in terms of $\Gamma_G^\vee \times \Gamma_G \subset Z(\tilde{G})^* \times Z(\tilde{G})$ (we have been ignoring the difference between $Z(\tilde{G})$ and $Z(\tilde{G})^*$ since $Z(\tilde{G})$ is Abelian), since we can use the group law in $Z(\tilde{G})$. For simplicity, let us first assume that $Z(\tilde{G}) = \mathbb{Z}_N$. We can thus write a given electric line in $\Lambda_w(\tilde{G})/\Lambda_r(\mathfrak{g})$ as $\frac{q}{N}R$, where $R \in \Lambda_r(\mathfrak{g})$ is some root that we fix and $q \in \mathbb{Z}_N$. Similarly, we can choose a given magnetic line to be mH , where $H \in \Lambda_w(\tilde{G}^\vee) = \Lambda_r^*(\tilde{G})$ is a fundamental magnetic weight that has inner product 1 with R (this is possible since the fundamental weights are the dual of the roots), and $m \in \mathbb{Z}_N$. Thus the quantization condition for two dyons $((q/N)R, mH), ((q'/N)R, m'H)$ is

$$\frac{1}{N}(qm' - q'm)\langle R, H \rangle = \frac{qm' - q'm}{N} \in \mathbb{Z}. \quad (2022)$$

More generally, if $Z(\tilde{G})$ is an Abelian group with $n \mathbb{Z}_k$ factors, then we can proceed as above but assign electric lines as $q_1R_1 + \dots + q_nR_n$, and likewise for magnetic lines. One then ends up with the same quantization condition as above in each \mathbb{Z}_k factor. Thus if we specify two dyons by $(q, m), (q', m') \in Z(\tilde{G})^2$, the quantization condition is

$$qm' - q'm = 0 \text{ in } Z(\tilde{G}). \quad (2023)$$

Let's now again specialize to the case where $Z(\tilde{G}) = \mathbb{Z}_N$, which is basically the only case of interest (all simple Lie groups have $Z(\tilde{G})$ cyclic except $\text{Spin}(4n)$, where it is \mathbb{Z}_2^2). How do the solutions to the quantization condition (2023) relate to the subgroups Γ_G, Γ_G^\vee that we identified earlier? Hopefully, both the solutions to (2023) and the choices of Γ subgroups enumerate the same list of sublattices of the “full” charge lattice \mathbb{Z}_N^2 . Let's now see why this is indeed the case.

Without loss of generality we can write

$$\Gamma_G = \mathbb{Z}_a, \quad \Gamma_G^\vee = \mathbb{Z}_b, \quad (2024)$$

for some $a, b \in \mathbb{Z}$ such that $ab = N$. The most obvious way of embedding these groups into the full lattice \mathbb{Z}_N^2 is to take $\Gamma_G = (0, \frac{N}{a}k) \subset \mathbb{Z}_N^2$ for $k \in \mathbb{Z}_a$, and likewise to take $\Gamma_G^\vee = (\frac{N}{b}l, 0) \subset \mathbb{Z}_N^2$ for $l \in \mathbb{Z}_b$ (here the first factors are electric charges and the second factors are magnetic charges). Then the lattice of allowed operators is

$$L = \{(N/b)l, (N/a)k \mid (l, k) \in \mathbb{Z}_b \times \mathbb{Z}_a\} \quad (2025)$$

This of course satisfies the quantization condition (2023) for all l, k . However, depending on the choice of groups involved, this lattice will not be the only lattice allowed, as there may be multiple ways of embedding the Γ_G, Γ_G^\vee groups into the full lattice \mathbb{Z}_N^2 . This is related to the Witten effect.

In our conventions, the embedding of the group Γ_G^\vee into \mathbb{Z}_N^2 , which determines the allowed electric lines, will always be uniquely defined as

$$\Gamma_G^\vee = \mathbb{Z}_b = \{((N/b)k, 0) \mid k \in \mathbb{Z}_b\} \subset \mathbb{Z}_N^2. \quad (2026)$$

That is, the group Γ_G^\vee determines the purely electric operators in the charge lattice. This is because $\Gamma_G^\vee = \Lambda_w(G)/\Lambda_r(G)$ classifies the allowed representations of the gauge group (modulo those constructed from \otimes s of the adjoint). Each representation R always defines a purely electric line operator via $W_C = \text{Tr}_R[\exp(\int_C A)]$.

By contrast, we have some freedom when it comes to the magnetic operators and the embedding of Γ_G . This freedom essentially comes from our ability to make a re-definition of what we mean by a magnetic charge. For example, when we were deriving the constraints on allowed magnetic charges for line operators, we probed a magnetic line operator \mathcal{O} with a purely electric line, and then constrained the allowed magnetic lines as a function of the different representations the electric line could be taken in. But since electric lines don't have statistical phases with other electric lines, the quantization conditions on the magnetic charge of \mathcal{O} would be unchanged if we replaced \mathcal{O} with $\mathcal{O} \otimes W$, where W is a purely electric line. Thus we could define our magnetic operators to come attached with electric lines, and the whole story would go through unchanged. Theories with different types of W 's attached to the magnetic operators \mathcal{O} are related by the Witten effect (i.e. usually by a shift of some θ angle), and correspond to different ways of embedding Γ_G in the full \mathbb{Z}_N^2 charge lattice. For example, if $\Gamma_G = \mathbb{Z}_N$, we may choose to embed it in \mathbb{Z}_N as any of the N distinct subgroups $\{(nl, l)\}$, with $n, l \in \mathbb{Z}_N$.

In general, we can say that the lattice L of line operators fits into an exact sequence

$$0 \rightarrow \Gamma_G^\vee \rightarrow L \rightarrow \Gamma_G \rightarrow 0. \quad (2027)$$

The injection is unique and the choice of Γ_G^\vee completely determines the purely electric operators we have access to. There are usually multiple ways of projecting onto the magnetic group Γ_G though, each of which gives us a distinct solution to the quantization condition. Note that this sequence will not always be split, which means that there may be no subset of lines in the full charge lattice whose magnetic charges fuse in a Γ_G subalgebra (if there were, there would exist a splitting homomorphism $\Gamma_G \rightarrow L$ with that subalgebra as its image).

Again returning to $Z(\tilde{G}) = \mathbb{Z}_N$, we see that if $\Gamma_G = \mathbb{Z}_1$, there is only one theory, whose charge lattice is the purely electric $\mathbb{Z}_N \times 0$ sublattice of \mathbb{Z}_N^2 . As already mentioned, if $\Gamma_G = \mathbb{Z}_N$, there are N distinct theories, which differ in the charge assigned to the fundamental magnetic line and which are permuted by the modular T operation. Also note that regardless of the choice of Γ_G , if there is a single line (q, m) with $m = 1$, then there are no purely electric lines in the \mathbb{Z}_N^2 charge lattice (this follows from $|L| = N$). Likewise, if there is a line $(1, 0)$ then there are no magnetically charged lines (and as previously mentioned we must have $\Gamma_G = \mathbb{Z}_1$).

For $Z(\tilde{G}) = \mathbb{Z}_N$, we can take $\Gamma_G^\vee = \mathbb{Z}_b$, $\Gamma_G = \mathbb{Z}_a$ for $N = ab$. The different equivalence classes of extensions are given by

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_a, \mathbb{Z}_b) = \mathbb{Z}_a \otimes_{\mathbb{Z}} \mathbb{Z}_b = \mathbb{Z}_{\gcd(a,b)}. \quad (2028)$$

In this classification, the split extensions are trivial. The number of different split extensions is classified by different ways of taking semidirect products, i.e. different maps $\mathbb{Z}_b \rightarrow \text{Aut}(\mathbb{Z}_a) =$

\mathbb{Z}_a^* , where \mathbb{Z}_a^* is the multiplicative group (of order $|\mathbb{Z}_a^*| = \phi(a)$). Thus the number of split extensions is $|\text{Hom}(\mathbb{Z}_b \rightarrow \mathbb{Z}_a^*)|$ (note to self: where was I going with this paragraph?)

Two last things worth re-iterating before wrapping up. First, we have classified the allowed line operators in the theory given \tilde{G} and Γ_G by picking out a certain subset of the charge lattice. Operators with charges running over all values of the charge lattice exist no matter what Γ_G is, but unless they are part of the given subset, they must come with surfaces (which may be topological depending on the phase) attached to them, and cannot be defined on homologically non-trivial cycles. Also, it is good to remember the distinction between magnetic charge (i.e. “GNO magnetic charge”) and t’ Hooft flux. The former is basically a lattice point in $\Lambda_w(G^\vee)$, while the latter is an element in $\pi_1(G)$. In particular, we have no operators with t’ Hooft flux in theories where the gauge group is simply connected, while we still have operators with nonzero GNO magnetic charge.

116 August 20, 21 — Higher symmetries in non-Abelian gauge theories

Consider non-Abelian gauge theory for some gauge group G , obtained from a simply-connected covering space \tilde{G} by $G = \tilde{G}/\Gamma_G$. What are the higher symmetries in this theory? How are theories with different choices of Γ_G related from a higher-symmetry point of view? You should try to understand the comments in [8, 7] and explain what these papers are doing in detail.

Solution:

The difference between gauge theories based on different quotients $G = \tilde{G}/\Gamma_G$ for finite Γ_G comes in the line operators that are allowed in the theory, as we saw in yesterday’s diary entry. The difference between the different G s is the types of transition functions that they allow: a given set of transition functions may satisfy the cocycle condition (closed under the Čech differential) for one choice of Γ_G , but not another.

How does this affect the allowed line operators? Naively the Wilson operators do not care about the transition functions and the different choices of Γ_G , since they are integrals of a Lie-algebra-valued quantity. But this is too hasty. To see why, we write the Wilson line by splitting it up into a bunch of patches. We take the path C to lie in the union of a collection of patches $\{U_\alpha\}$, with the segment of the Wilson line W_C lying in U_α denoted by W_α . The naive formula for $W(C)$ is then

$$W(C) = \text{Tr} \left[\prod_\alpha W_\alpha \right], \quad (2029)$$

but this is not quite correct. Indeed, it is not invariant under changing the local trivializations on each U_α . Under a change in trivialization g_α which is constant on each patch, we have $W_\alpha \mapsto g_\alpha^\dagger W_\alpha g_\alpha$, and so if we e.g. change the trivialization on a single patch, our formula for $W(C)$ is not invariant. The correct thing to do is to glue each W_α together with transition functions $t_{\alpha\beta}$:

$$W(C) = \text{Tr} [W_\alpha t_{\alpha\beta} W_\beta t_{\beta\gamma} W_\gamma \dots]. \quad (2030)$$

Under a change in transition functions of $\{g_\alpha\}$, we have $t_{\alpha\beta} \mapsto g_\alpha^\dagger t_{\alpha\beta} g_\beta$, and $W(C)$ is left invariant. This construction is related to the DB cohomology approach for integrating gauge fields, which is easier to spell out in more detail in the Abelian case.

Anyway, we now see how the choice of transition functions affects the Wilson loop. For example take $SU(N)$, and consider twisting the transition functions such that a single transition function appearing in $W(C)$ gets twisted by $t_{\alpha\beta} \mapsto e^{2\pi i/N} t_{\alpha\beta}$ (we are not changing any of the trivializations, just a transition function—this can be done without creating extra field strength only if C is homologically nontrivial). Then under this change we see that $W(C) \mapsto e^{2\pi i/N} W(C)$, despite the fact that we haven't actually changed the coordinate-patch realizations of the gauge field $\{A_\alpha\}$ at all. Thus it is good to keep in mind that although $W(C)$ only involves the gauge field, the gauge field is really a principal bundle, which carries more information (viz. information about the transition functions) than just the 1-forms A_α .

The example we will be focusing on primarily is $\tilde{G} = SU(N)$. The fundamental Wilson lines depend on the transition functions, and may not be good line operators when Γ_G is taken to be nontrivial. However, Wilson lines in the adjoint are always good line operators, regardless of Γ_G . We know this from the previous diary entries since the root lattice always gives legit line operators, but now we can see it in a different way.

For N the fundamental of $SU(N)$, we have

$$N \otimes \bar{N} = 1 \oplus A, \quad (2031)$$

where A is the adjoint. Thus for all $SU(N)$ we can write a matrix in the adjoint as follows:

$$[U_A]_{kl}^{ij} = [U_N]_j^i [U_N^\dagger]_l^k - \frac{1}{N} \delta_l^i \delta_k^j. \quad (2032)$$

In this expression, i, j are fundamental indices and k, l are antifundamental indices. Here the second factor projects out the $\mathbf{1}$ in the direct sum, and the $1/N$ is so that when we put in $U_N = \mathbf{1}$ then we get $\text{Tr}[\mathbf{1}_A] = N^2 - 1 = \dim(A)$. The index structure is fixed to be $\delta_l^i \delta_k^j$, since this is the \otimes of invariant symbols (δ functions) for the two pairs of N, \bar{N} indices.

This means that the Wilson loop in the adjoint can be computed from the Wilson loop in the fundamental by

$$W_A(C) = \text{Tr}_A e^{i \oint_C A} = \text{Tr} \left\{ \left[e^{i \oint_C A} \right]_k^i \left[e^{-i \oint_C A} \right]_l^j - \frac{1}{N^2} \delta_k^i \delta_l^j \right\}. \quad (2033)$$

The trace sets $i = k$ and $j = l$, so

$$W_A(C) = \left| \text{Tr}_f e^{i \oint_C A} \right|^2 - 1. \quad (2034)$$

Thus we see that since the center symmetry changes the fundamental Wilson line by a phase, it leaves $W_A(C)$ invariant. Thus no matter what choice of $\Gamma_G \subset \mathbb{Z}_N = Z(SU(N))$ we make, the adjoint Wilson lines will always be well-defined well defined operators, blind to the allowed t' Hooft line operators.

With these introductory comments out of the way, let's now see how this works in a more detailed way. We'll focus on the simple example of the relation between $SU(N)$ and $PSU(N)$ gauge theory, although quotienting $SU(N)$ by subgroups of \mathbb{Z}_N can be done analogously.

Of course, the real difference between $SU(N)$ and $PSU(N)$ gauge theories is in the transition functions, and passing from $SU(N)$ to $PSU(N)$ means changing the transition functions. This isn't something that's easy to do in a transparent way as far as the variables naturally appearing in a QFT are concerned, so we will try to encode the changed transition functions into a field that appears in the action.

We want to “gauge” the \mathbb{Z}_N part of the transition functions to obtain $PSU(N)$ gauge theory, which we will do by coupling the $SU(N)$ theory to a \mathbb{Z}_N gauge field. Recall that discrete gauge fields are basically just \mathbb{Z}_N transition functions: they have no local degrees of freedom on patches; all of their physical content is in transition functions between patches. Coupling $SU(N)$ to a \mathbb{Z}_N gauge field will then allow us to identify two states that differ by twisting the transition functions by something in \mathbb{Z}_N , and we will have obtained a $PSU(N)$ theory. Since we are in the continuum, we will write down the \mathbb{Z}_N gauge field using the BF theory approach, where we only deal with $U(1)$ fields. Since the \mathbb{Z}_N symmetry we want to gauge is a 1-form symmetry, the gauge field will be a \mathbb{Z}_N 2-form field B . We use the presentation of the BF action where NB gets set to be the field strength of a $U(1)$ gauge field through a Lagrange multiplier coupling:

$$S \supset \frac{i}{2\pi} \int H \wedge (F_{\mathcal{A}} - NB), \quad (2035)$$

where $F_{\mathcal{A}}$ is the field strength for a $U(1)$ gauge field \mathcal{A} (we are avoiding writing it as $d\mathcal{A}$ since \mathcal{A} may not be a 1-form). Here H is quantized to have $2\pi\mathbb{Z}$ periods around all closed 2-manifolds.

If F_A is the $SU(N)$ field strength, the naive thing to do would be to take $F_A \mapsto F_A - B\mathbf{1}$ in the action. This is what we would do if we were trying to gauge a $U(1)$ 1-form symmetry, since gauging the 1-form symmetry that acts on A means that F_A changes by total derivatives under gauge transformations. But this isn't right: locally we can replace B with $\frac{1}{N}F_{\mathcal{A}}$, which means that the action $\|F_A - \frac{1}{N}F_{\mathcal{A}}\|^2$ now has more local degrees of freedom than it did when we started, which is definitely not what we want. The correct thing to do is to cancel out these extra local degrees of freedom with the \mathcal{A} field. We will do this by taking a $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$ gauge theory, and killing off the $U(1)$ factor to produce an $SU(N)/\mathbb{Z}_N = PSU(N)$ theory.

To this end, consider the $U(N)$ field

$$\mathcal{A} = A + \frac{1}{N}\mathcal{A}\mathbf{1}. \quad (2036)$$

Here A is traceless and \mathcal{A} is the $U(1)$ part of the $U(N)$, with $\text{Tr}\mathcal{A} = \mathcal{A}$. By saying that this is a $U(N)$ field, we really just mean that the transition functions of A and those of \mathcal{A} are correlated in a way such that their combination gives legit $U(N)$ transition functions. In particular, we can let the transition functions for the $SU(N)$ part fail to be closed in $\check{C}^1(M; \mathbb{Z})$, provided that the transition functions for \mathcal{A}/N compensate this (\mathcal{A} can do this while still remaining a legit $U(1)$ gauge field because of the $1/N$ in front of it in the definition for \mathcal{A} : \mathcal{A} is a well-defined $U(1)$ gauge field, but \mathcal{A}/N is not). This identification is the quotient by \mathbb{Z}_N in $[SU(N) \times U(1)]/\mathbb{Z}_N$.

Since we now can shift the transition functions of the A bundle by N th roots of unity through a gauge transformation, we almost have a $PSU(N)$ gauge field. The only problem is that it has an extra $U(1)$ local degree of freedom that the $PSU(N)$ theory doesn't have.

This extra local degree of freedom is eliminated by adding the background field B to the action though, via

$$S \supset \frac{1}{2g^2} \int \text{Tr} [(F_{\mathcal{A}} - B\mathbf{1}) \wedge \star(F_{\mathcal{A}} - B\mathbf{1})]. \quad (2037)$$

This is the proper way to couple the $SU(N)$ theory to a \mathbb{Z}_N gauge field: neither B nor \mathcal{A} by themselves is the \mathbb{Z}_N gauge field; the \mathbb{Z}_N gauge field involves both of them. This is the price we pay for wanting to work in the continuum. Note that A is not really the $PSU(N)$ field: while the bundle for A is allowed to have transition functions which fail the cocycle condition by something in \mathbb{Z}_N , we need the gauge fields \mathcal{A}, B to make sure that the places where the cocycle condition fails can be removed by gauge transformations, so that such places do not contribute to a physical field strength. So it's really the whole package that constitutes the $PSU(N)$ field.

Now it may seem like we've actually just gotten back to our starting point by adding B , since it appears in the combination $\frac{1}{N}F_{\mathcal{A}}\mathbf{1} - B\mathbf{1}$, which seems to vanish upon integrating out H . But integrating out H only says that $F_{\mathcal{A}} = NB$ and does *not* imply that $\frac{1}{N}F_{\mathcal{A}} = B$. Locally it does, and so we have indeed added no new local degrees of freedom. But globally, knowing NB does not let you know B . Knowing NB means that you know B locally, and means that you know $(e^{i\int_{\Sigma} B})^N$ for all closed 2-manifolds Σ . In fact since $NB = F_{\mathcal{A}}$, we have $(e^{i\int_{\Sigma} B})^N = 1$ for all Σ , and so knowing NB doesn't give you any information about the holonomies of B around closed 2-manifolds (which are always N th roots of unity). It may help to again explain exactly what happens when we integrate out H . Doing a Hodge decomposition $F_{\mathcal{A}} = d\alpha + \omega_{\mathcal{A}}$, $B = d\lambda + d^\dagger\epsilon + \omega_B$, $H = d\gamma + \omega_H$, $\omega_{\mathcal{A}} \in 2\pi H^2(M; \mathbb{Z})$, $\omega_H \in H^2(M; \mathbb{R}/2\pi\mathbb{Z})$, $\omega_B \in 2\pi H^2(M; \mathbb{Z})$ (recall that $\int_{\Sigma} H \in 2\pi\mathbb{Z}$ for all closed 2-manifolds Σ), we get a δ function setting $\epsilon = 0$, so that the non-cohomological degrees of freedom are all pure gauge and thus disappear. Upon summing over $\omega_H \in 2\pi H^2(M; \mathbb{Z})$ we get the term

$$\sum_{\omega_H \in 2\pi H^2(M; \mathbb{Z})} \exp\left(\frac{i}{2\pi} \int \omega_H \wedge (\omega_{\mathcal{A}} - N\omega_B)\right) = \sum_{\widehat{\omega}_H \in H_2(M; \mathbb{Z})} \exp\left(i \int_{\widehat{\omega}_H} (\omega_{\mathcal{A}} - N\omega_B)\right). \quad (2038)$$

Given the quantization of $\omega_{\mathcal{A}}$, this means that we can constrain

$$\omega_B \in \frac{2\pi}{N} H^2(M; \mathbb{Z}), \quad (2039)$$

but ω_H is free to take on any cohomology class in this cohomology group (classes in $H^2(M; \mathbb{Z})$ are gauged under the 1-form gauge symmetry though, so only the classes in $(2\pi)/NH^2(M; \mathbb{Z}_N)$ are physically distinct). In particular, integrating out H does not actually set $B = \frac{1}{N}F_{\mathcal{A}}$, since the cohomology class of B is not fixed after integrating out H . This class is the sole degree of freedom carried by the \mathbb{Z}_N gauge field.

Thus, the combination $\frac{1}{N}F_{\mathcal{A}} - B$ carries no local degrees of freedom, but carries global \mathbb{Z}_N degrees of freedom: it is the \mathbb{Z}_N 2-form gauge field that want to couple to the $SU(N)$ fields. As a sanity check, if we have a situation where $\int_{\Sigma} B \in 2\pi\mathbb{Z}$ for all Σ , then B is pure gauge and we can make a gauge transformation to eliminate \mathcal{A} and B from the theory entirely. This gives us back the pure $SU(N)$ theory as required.

We can now look at the operators in the theory. The Wilson line in the fundamental of $SU(N)$ now is not gauge invariant under 1-form gauge transformations $\mathcal{A} \mapsto \mathcal{A} + N\lambda$, and

so we must write it as a surface operator by attaching a B surface:

$$W_f(C) \mapsto \text{Tr} \left[\exp \left(i \oint_C \mathcal{A} \right) e^{-i \int_{\Sigma} B \mathbf{1}} \right] = \text{Tr} \left[\exp \left(i \oint_C A \right) \right] e^{\frac{i}{N} \oint_C \mathcal{A}} e^{-i \int_{\Sigma} B}, \quad (2040)$$

where $\partial\Sigma = C$ and we used $e^{x\mathbf{1}} = \mathbf{1}e^x$ for $x \in \mathbb{C}$. Because of the attached surface operator the fundamental Wilson lines are no longer part of the lines operators in the theory. The adjoint lines are the new “smallest charge” electric line operators: as we saw earlier they depend on the fundamental Wilson lines through the product $|W_f(C)|^2$. Since the Abelian part relating to the holonomy of \mathcal{A} cancels when the square is taken, the adjoint Wilson loops don’t see the \mathcal{A} field and are consequently gauge-invariant well-defined line operators. Note that similar things like this are discussed in Tong’s gauge theory notes, but unlike in his notes we are saying that the line operators in the $PSU(N)$ theory are the adjoint representation Wilson lines, not N -fold powers of the fundamental line. N -fold powers don’t get you anything since you still need a surface operator attached to the line to ensure gauge invariance⁷⁶. Also e.g. for $SU(3)$ we have

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \quad (2041)$$

Since 10 cannot be built from \otimes s of the adjoint 8, the RHS is not invariant under the \mathbb{Z}_N 1-form symmetry and thus taking three fundamental Wilson lines is not quite the right way to get something invariant under the 1-form symmetry.

Now, are the global symmetries in the $PSU(N)$ theory right? As we’ve seen before, a pure \mathbb{Z}_N 2-form gauge field comes with a \mathbb{Z}_N 2-form electric symmetry and a $\mathbb{Z}_N(D-2-1)$ -form magnetic symmetry. For us $D=4$ so the magnetic symmetry is a 1-form symmetry. The 2-form electric symmetry sends $B \mapsto B + \gamma$ for a flat 2-form γ with periods in $\frac{2\pi}{N}\mathbb{Z}$. This symmetry is broken by the coupling to the $U(N)$ field \mathcal{A} , which is good since the $PSU(N)$ theory shouldn’t have any 2-form symmetries. However, in keeping with our discussion the other day about \tilde{G} and \tilde{G}/Γ_G gauge theories, we know that the $PSU(N)$ theory should have a \mathbb{Z}_N ’s worth of t’ Hooft operators, which are the charged objects for a \mathbb{Z}_N 1-form symmetry. This is precisely the 1-form magnetic symmetry of the B field. The t’ Hooft loop is constructed with \tilde{A} , where $F_{\tilde{A}} = H$ (writing H like this is possible since we took H to be a closed 2-form with periods in $2\pi\mathbb{Z}$). So the t’ Hooft operator is

$$T(C) = \exp \left(i \oint_C \tilde{A} \right). \quad (2042)$$

The last type of operator we have is

$$\mathcal{W}(\Sigma) = \exp \left(i \int_{\Sigma} B \right), \quad (2043)$$

where Σ is closed. From the commutation relation between B and \tilde{A} (roughly $[\tilde{A}, B] = i/N$), we see that $\mathcal{W}(\Sigma)$ is the charge operator for the magnetic 1-form symmetry, and that it has the correct commutation relations with the fundamental Wilson line $W_f(C)$ (the nontrivial

⁷⁶The surface operator is now $e^{-iN \int_{\Sigma} B}$. If Σ is closed then this is always equal to 1, and so the attached surface operator is topological (independent of the exact choice of Σ). But the surface operator still needs to be there, and if C does not bound a surface then the N -fold power of $W_f(C)$ can’t be defined in a gauge-invariant way.

commutation relation is due to the B surface operator attached to $W_f(C)$). It thus computes the integral of ω_2 over Σ , where $\omega_2 \in H^2(M; \mathbb{Z}_N)$ is the 2nd Stiefel-Whitney class for the $PSU(N)$ bundle restricted to Σ .

Now given this, how would we get back to the $SU(N)$ theory? We would need to gauge the \mathbb{Z}_N 1-form magnetic symmetry with a \mathbb{Z}_N 2-form gauge field, whose magnetic 1-form symmetry would become the electric symmetry of the $SU(N)$ theory. This new gauge field needs to have the effect of forcing $F_{\mathcal{A}}$ to be quantized in periods of $2\pi N\mathbb{Z}$, since as we saw this turns the $PSU(N)$ transition functions into $SU(N)$ transition functions. So we can add some fields with the action

$$\frac{i}{2\pi} \int H' \wedge (F_{\mathcal{A}} - NF_{\mathcal{A}'}) , \quad (2044)$$

where H' is yet another Lagrange multiplier 2-form gauge field and \mathcal{A}' is a $U(1)$ gauge field. The periods of H are not quantized, so that integrating out H sets the cohomology classes of $F_{\mathcal{A}}$ and $NF_{\mathcal{A}'}$ to be equal. This makes $F_{\mathcal{A}}$ pure gauge under the original 1-form \mathbb{Z}_N gauge symmetry and so \mathcal{A} and B can be eliminated from the action, leaving us with an $SU(N)$ field. The electric symmetry of $SU(N)$ in this presentation is the 1-form symmetry that shifts \mathcal{A}' by a closed form that has periods in $\frac{2\pi}{N}\mathbb{Z}$ (again, the fact that this is the symmetry can be most easily seen by integrating by parts and writing the relevant term in the action as $\frac{iN}{2\pi} \int F_{H'} \wedge \mathcal{A}'$, and recalling that H' is a $U(1)$ 2-form gauge field so that $F_{H'}$ has periods in $2\pi\mathbb{Z}$). The electric 2-form symmetry of the added \mathbb{Z}_N gauge field that we used to get back to $SU(N)$ is broken by the coupling to $F_{\mathcal{A}}$, and so all the global symmetries are properly accounted for.

So, summarizing, we've seen that $SU(N)$ has an electric $\mathbb{Z}_N^{(1)}$ symmetry, while $PSU(N)$ as a magnetic $\mathbb{Z}_N^{(1)}$ symmetry, and that these two symmetries can be related to one another through a gauging procedure. We know that the electric symmetry in $SU(N)$ is only \mathbb{Z}_N since we always have gluons in the adjoint representation, so Wilson lines in the adjoint can end and hence can't carry a 1-form charge. How do we know that we haven't missed e.g. a magnetic symmetry in $SU(N)$? One way to argue is to say that we can couple the theory to a Higgs field in the adjoint: this leads to dynamical t'Hooft Polyakov monopoles as we have seen in an earlier entry, which carry magnetic charge N in the magnetic lattice. Since adding the Higgs field doesn't break any symmetries, the symmetries of the Higgsed theory should be the same as the un-Higgsed one, and so the un-Higgsed one can't have any magnetic 1-form symmetries. Actually, apparently the pure glue theory actually does have dynamical charge N monopoles, created at the intersection of N Wilson lines (should find the reference for this).

We now briefly cover θ angles. After gauging to $PSU(N)$, the θ term is (remember that the $1/8\pi^2$ comes from expanding $\exp(F/2\pi)$)

$$S \supset \frac{i\theta}{8\pi^2} \int \text{Tr} [(F_{\mathcal{A}} - B\mathbf{1}) \wedge (F_{\mathcal{A}} - B\mathbf{1})] . \quad (2045)$$

Since $F_{\mathcal{A}}$ is traceless, this is

$$S \supset \frac{i\theta}{8\pi^2} \int \text{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] + \frac{i\theta N}{8\pi^2} \int B \wedge B - \frac{i\theta}{4\pi^2} \int F_{\mathcal{A}} \wedge B . \quad (2046)$$

Since $\text{Tr}[F_{\mathcal{A}}] = F_{\mathcal{A}}$, we can write this as

$$S \supset i\theta c_2 + \frac{i\theta N}{8\pi^2} \int B \wedge B - \frac{i\theta}{4\pi^2} \int F_{\mathcal{A}} \wedge B + \frac{i\theta}{8\pi^2} \int F_{\mathcal{A}} \wedge F_{\mathcal{A}}. \quad (2047)$$

Here the second Chern class is

$$c_2 = \frac{1}{2(2\pi)^2} (\text{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] - \text{Tr}[F_{\mathcal{A}}] \wedge \text{Tr}[F_{\mathcal{A}}]), \quad (2048)$$

and is always integral. Recall that we get this from the second-order contribution to $\det(\mathbf{1} + F_{\mathcal{A}}/2\pi)$, which we compute as $\exp(\text{Tr} \ln(\mathbf{1} + F_{\mathcal{A}}/2\pi))$ by using the Taylor series for the log. Using the constraint from integrating out H , we can eliminate \mathcal{A} and write everything in terms of B (we do this by writing $F_{\mathcal{A}} = NB$. This is not the same as writing $B = N^{-1}F_{\mathcal{A}}$, which is not a replacement that we are making):

$$S_\theta = i\theta \int c_2 - (N^2 - N) \frac{i\theta}{8\pi^2} \int B \wedge B, \quad (2049)$$

where now B is constrained to have periods in $2\pi/N$ around all 2-submanifolds. We see that shifting θ by 2π is now nontrivial: where we dropped $2\pi i \int c_2 \in 2\pi\mathbb{Z}$. The N term is certainly non-trivial because of the quantization on B , while the +1 part is in $2\pi\mathbb{Z}$ if we are working on a spin manifold. If we are not on a spin manifold, this factor can contribute a \pm sign to the path integral⁷⁷.

The $\int B \wedge B$ term here prompts us to go back to our original action and add a counterterm of this form. We have several options for counterterms to add, given by an integer p .

⁷⁷The claim is that on a spin manifold M ,

$$\int_M \frac{F}{2\pi} \wedge \frac{F}{2\pi} \in 2\mathbb{Z}, \quad (2050)$$

for any 2-form $F \in 2\pi H^2(M; \mathbb{Z})$ while on a non-spin manifold, the RHS is replaced by \mathbb{Z} . Why does the (non)admitance of a spin structure determine how gauge fields integrate? This is because if a four-manifold M is spin, the intersection form

$$H^2(M; \mathbb{Z}) \times H^2(M; \mathbb{Z}) \rightarrow \mathbb{Z} \quad (2051)$$

is even. This is because mod 2 we have, for any $A \in H^2(M; \mathbb{Z})$,

$$A \frown A = \omega_2 \frown A \mod 2, \quad (2052)$$

where ω_2 is the second Stiefel Whitney class. Thus if M is spin then ω_2 lifts to an even class in $H^2(M; \mathbb{Z})$, and so since $(F/2\pi) \in H^2(M; \mathbb{Z})$, the integral above must indeed be even if M is spin.

To prove the last equation, one notes that $\omega_2 \frown A = \omega_2(TM|_A)$, i.e. $\omega_2 \frown A$ measures the Stiefel-Whitney class of A embedded in M (using Poincare duality to think of A as a 2-manifold). Now $TM|_A = TA \oplus TN$, where TN is the component of the tangent bundle of M normal to A . Then since A is an orientable manifold, the Whitney product formula reads

$$\omega_2(TM|_A) = \omega_2(TA) + \omega_2(TN). \quad (2053)$$

The first term on the RHS is the mod 2 Euler class of A , which is trivial for A a 2-manifold, since $\chi(A) = 2 - 2g \in 2\mathbb{Z}$. The second term is the mod 2 Euler class of TN , which is precisely the self-intersection number of A mod 2. This is because $\omega_2(TN)$ measures the zeros of vector fields in TN mod 2, and the zeros in TN precisely come from self-intersection points of A (the intersection at each self-intersection point can be made transverse, so at these points the tangent space of A generates the full tangent space of M ; hence any vector field in TN must vanish at these points).

The counter-term we add is

$$S_{ct} = \frac{ipN}{4\pi} \int B \wedge B. \quad (2054)$$

Under a gauge transformation that shifts B by F_λ (the curvature of a $U(1)$ gauge field), this counter-term changes by

$$\delta S_{ct} = ipN\pi \int \frac{F_\lambda}{2\pi} \wedge \frac{F_\lambda}{2\pi} + \frac{ipN}{2\pi} \int B \wedge F_\lambda. \quad (2055)$$

The first term is in $2\pi\mathbb{Z}$ if we are on a spin manifold, but is non-trivial in general. There is no other term in the action that will give us a quadratic term in F_λ , so in order for S_{ct} to make sense we must have $pN \in 2\mathbb{Z}$ (if we are on a spin manifold, just $pN \in \mathbb{Z}$ is okay).

The second $B \wedge F_\lambda$ term is trivial if we use the constraint that B has periods in $\frac{2\pi}{N}\mathbb{Z}$ coming from integrating out H , since then the term is valued in $2\pi p\mathbb{Z}$. Integrating out H gets rid of the operator responsible for the t' Hooft lines though (recall $H = F_{\tilde{A}}$), so it would be better if we could keep H around (we want to talk about what happens to the t' Hooft lines when θ is shifted, which is tricky to do if we have to integrate it out to insure gauge invariance. I admit this argument is a little shaky). Basically, we want to ensure that the action is gauge invariant without needing to integrate out any of the Lagrange multipliers. We see that we can cancel the second term directly if we take

$$H = F_{\tilde{A}} \mapsto F_{\tilde{A}} - pF_\lambda \quad (2056)$$

under the 1-form gauge transformation. This adds an extra term $ipN \int F_A \wedge F_\lambda / 2\pi$, but this is trivial since both F_A and F_λ are quantized in $2\pi\mathbb{Z}$ (by definition of A and λ , not because of Lagrange multipliers). Finally, note that if we take $p \mapsto p + 2N$ then the action changes by $\frac{iN^2}{2\pi} \int B \wedge B$, which is trivial (after integrating out H). So we have the periodicity $p \sim p + 2N$ (for generic parity N). On a spin manifold, or if N is even, we can do better and write $p \sim p + N$. This is the same as the fact that in Abelian CS theory with odd level, there are $2k$ particles with the k th one a fermion, while for k even the periodicity is smaller and there are only k particles. Adding this counter-term means that doing $\theta \mapsto \theta + 2\pi$ is the same as doing $p \mapsto p + 1 - N$ (if spacetime is spin then it is just $p \mapsto p + 1$).

Let's now check that the line operators we predicted to exist in the last diary entry are indeed the ones that are realized. As expected, for $\theta \neq 0$, a pure charge-1 t' Hooft line doesn't exist, since $e^{i\oint_C \tilde{A}}$ is not gauge invariant (recall that we made $F_{\tilde{A}} \mapsto F_{\tilde{A}} - pF_\lambda$ under the 1-form gauge symmetry). However, we see that its gauge variance is exactly canceled by the gauge variance of p copies of the antifundamental Wilson line (since $\delta F_A = NF_\lambda$ under the 1-form gauge transformation), so that the t' Hooft line becomes

$$T(C; p) = e^{\frac{i}{N} \oint_C \tilde{A}} \text{Tr} \left[\exp \left(-ip \oint_C \mathcal{A} \right) \right], \quad (2057)$$

demonstrating that shifting the θ angle by 2π attaches electric charge 1 (really, -1) to the magnetic lines. By changing θ and thus changing p , we can realize the full range of $PSU(N)$ charge sublattices that we found in the previous diary entry, generated by the lines $(k, 1)$ for $k \in \mathbb{Z}_N$ the electric charge of the minimal magnetically charged line.

Before we end, we briefly mention that some quotient theories have charge lattices that are not related by shifting θ . To find examples, we need Γ_G to be a proper subgroup of

the center. The simplest example where we can quotient by a subgroup of the center is for $G = SU(4)/\mathbb{Z}_2$. Recall that the charge lattice is given by the exact sequence

$$1 \rightarrow \Gamma_G^\vee \rightarrow L \rightarrow \Gamma_G \rightarrow 1. \quad (2058)$$

For this example we have $\Gamma_G = \mathbb{Z}_2$ (and hence $\Gamma_G^\vee = \mathbb{Z}_2$), and so there are two possible choices for L : the split extension $L = \mathbb{Z}_2 \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2^2$, or the non-trivial extension in $H^2(\mathbb{Z}_2; \mathbb{Z}_2)$, namely $L = \mathbb{Z}_4$. In the former case we have two generators for the charge lattice, namely $(0, 2)$ and $(2, 0)$. In the later case, we have a single generator with magnetic charge 2, namely $(1, 2)$. Changing θ can only relate theories that differ in the way in which they define the electric charge of the generator(s) of the charge lattice; it cannot relate two theories with cohomologically distinct extensions L . Indeed, we see that the higher symmetries in the problem are not given by two 1-form symmetries $\Gamma_G^\vee \times \Gamma_G$, but rather are determined by L . In the case of $L = \mathbb{Z}_4$, there is only a single mixed electromagnetic 1-form symmetry, while for $L = \mathbb{Z}_2^2$ we have both electric and magnetic 1-form \mathbb{Z}_2 symmetries. This gives us an understanding of why the two $SU(4)/\mathbb{Z}_2$ theories are not related by a shift in a θ angle: they have different global symmetries.

117 August 25 — Compendium of various things relating to Chern classes / characters and (fractional) instanton numbers

Today's entry is just a collection of some basic things about instantons and their normalizations in different gauge groups that I thought would be handy to have around as a reference.

Solution:

The first half of this diary entry will be a compendium of math facts. For us, the instanton number I for a gauge bundle E over a four-dimensional spacetime X will be defined as

$$I = \int_X \text{ch}_2(E), \quad (2059)$$

where $\text{ch}(E)$ is the second chern character. Recall that the Chern characters are obtained from $\text{Tr}[e^{F/2\pi}]$ as

$$\text{ch}_k(E) = \frac{1}{k!} \text{Tr} \left[(F/2\pi)^{\wedge k} \right]. \quad (2060)$$

We define I as the integral of $\text{ch}_2(E)$ and not of $c_2(E)$ (the second chern class), since we want I to be nonzero when we choose the gauge group to be $U(1)$ (and also, since it is the chern character, not the chern class, that appears in the index formula). For gauge groups like $SU(N)$ with traceless generators, $c_2(E)$ and $\text{ch}_2(E)$ are *almost* the same. The difference comes from torsion phenomena: the Chern classes can have torsion contributions, so that even when $\text{Tr}[F] = 0$, we can have e.g. $c_1 \neq 0$, provided that c_1 is pure torsion. This can happen when the gauge group is the quotient of some simply connected group, like in the

case of $SU(N)/\mathbb{Z}_N$. These torsionful contributions are ignored by Chern-Weil theory, but are important to keep track of.

In contrast, the Chern characters are defined as classes in $H^*(X; \mathbb{Q})^{78}$, and as such *never* have any torsionful elements. They are calculated purely from the local curvature, and are only sensitive to data about the gauge group's Lie algebra (whereas the Chern classes care about the full Lie group).

The chern characters satisfy

$$\text{ch}(E \otimes F) = \text{ch}(E) \wedge \text{ch}(F), \quad \text{ch}(E \oplus F) = \text{ch}(E) + \text{ch}(F). \quad (2061)$$

The former can be seen by plugging in $F_{E \otimes F} = \mathbf{1}_G \otimes F_H + F_G \otimes \mathbf{1}_H$ into the formulae for the chern characters, while the latter is straightforward to see since the chern characters involve only a single trace. On the other hand, the Chern class of the direct sum is the wedge of the Chern classes, instead of the sum:

$$c(E \oplus F) = c(E) \wedge c(F). \quad (2062)$$

This is the Whitney sum formula and can be seen from the definition of the Chern classes in terms of the expansion of $\det(\mathbf{1} + F_A/2\pi)$, and the fact that $\det(A \oplus B) = \det(A)\det(B)$. I'm unaware of any simple formula for $c(E \otimes F)$, unless $E \cong \bigoplus_i \mathcal{L}_i, F \cong \bigoplus_j \mathcal{L}'_j$ for line bundles $\mathcal{L}_i, \mathcal{L}'_j$. In that case, we have

$$c(E \otimes F) = c\left(\bigoplus_{i,j} \mathcal{L}_i \otimes \mathcal{L}'_j\right) = \bigwedge_{i,j} (1 + c_1(\mathcal{L}_i \otimes \mathcal{L}'_j)) = \bigwedge_{i,j} (1 + c_1(\mathcal{L}_i) + c_1(\mathcal{L}'_j)), \quad (2063)$$

where we used the Whitney sum formula and the fact that $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$. This can be seen by recalling that the first Chern class can be defined by the Euler class of the underlying real bundle. Since the expression for the Euler class involves the log of the transition functions, and since the transition functions of $\mathcal{L} \otimes \mathcal{L}'$ are the product of the transition functions for \mathcal{L} and \mathcal{L}' , the Euler class of $\mathcal{L} \otimes \mathcal{L}'$ splits as a sum of the Euler classes of each line bundle—hence $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$.

Something else we sometimes need to do is to determine characteristic classes / instanton numbers for product bundles $E \otimes F$, where E is a principal G -bundle and F is a principal H -bundle. The answer for the instanton number is what you would expect: for theories not involving a $U(1)$ factor so that their Lie algebra generators are traceless, we have

$$I_{E \otimes F} = \text{ch}_2(E \otimes F) = I_G \dim H + I_H \dim G, \quad (2064)$$

where the dimension means the dimension of the defining representaiton of the associated Lie algebras. This follows from $\text{ch}(E \otimes F) = \text{ch}(E) \wedge \text{ch}(F)$: taking the second order terms, we have

$$\text{ch}_2(E \otimes F) = \text{Tr}_G[\mathbf{1}_G] \frac{1}{8\pi^2} \text{Tr}_H[F_H \wedge F_H] + \text{Tr}_H[\mathbf{1}_H] \frac{1}{8\pi^2} \text{Tr}_G[F_G \wedge F_G], \quad (2065)$$

⁷⁸The rational coefficients here are simply because the mod 1 contributions to the Chern characters come from the coefficients in the expansion of the exponential. Since the cohomology groups are isomorphic, we may equivalently just write $H^*(X; \mathbb{R})$.

which gives us what we want.

Now we will look at the topic which originally inspired this diary entry: instantons in $PSU(n)$ gauge theories. Normal instantons come from transitions between pure gauge field configurations in different homotopy classes of $\pi_3(G)$, where the 3 in $\pi_3(G)$ is a spatial slice (or region thereof) where the gauge fields asymptote to a constant (the elements in $\pi_3(G)$ are the glueing data for nontrivial bundles on S^4). These instantons can live in any \mathbb{R}^4 -like region of a given 4-manifold, regardless of its topology. Furthermore they will exist for all choices of (non-Abelian) gauge groups, since $\pi_3(G) = \mathbb{Z}$ for all simple compact non-Abelian Lie groups G . These instantons are common to all gauge groups G that descend from some simply connected group \tilde{G} by quotienting by some finite Γ_G (which may be \mathbb{Z}_1). To show this, one uses the long exact sequence coming from $1 \rightarrow \Gamma_G \rightarrow \tilde{G} \rightarrow G \rightarrow 1$. This sequence contains

$$\dots \rightarrow \pi_4(\tilde{G}/G) \rightarrow \pi_3(G) \rightarrow \pi_3(\tilde{G}) \rightarrow \pi_3(\tilde{G}/G) \rightarrow \dots . \quad (2066)$$

Now $\pi_{k>0}(\tilde{G}/G) = \pi_{k>0}(\Gamma_G) = 0$ since Γ_G is discrete and the homotopy groups are basepoint preserving (the basepoint is fixed to be a given element of the target space for the definition of the homotopy group, so we don't get a $|\Gamma_G|$'s worth of constant maps, we just get a \mathbb{Z}_1 's worth). Thus we have an isomorphism $\pi_3(G) \cong \pi_3(\tilde{G})$, and so the “small” instantons associated with $\pi_3(G)$ have the same instanton number no matter what Γ_G is.

As in [3], we will normalize the instanton number so that the minimal “small” instanton has instanton number $I = 1$. This minimal small instanton can always be taken to be a minimal $SU(2)$ instanton, on an S^3 around which $A \sim U^\dagger dU, U \sim e^{ix^a T^a}$, for an appropriately chosen trio of generators $T^z, T^+, T^- \in \{T^a\}$, with T^z, T^+, T^- generating an $\mathfrak{su}(2)$ Lie algebra. Recall that this embedding of $\mathfrak{su}(2)$ can always be done: we pick a pair of roots T^+, T^- that are eigenvalues under the action of Ad_A where A is such that Ad_A has maximal kernel, and then from these generators we can construct a T^z in the Cartan subalgebra of \mathfrak{g} that together with the T^\pm generates an $\mathfrak{su}(2)$. Thus for all choices of (compact, simple) Lie group G , we can always embed an $SU(2)$ instanton through a choice of $\mathfrak{su}(2) \rightarrow \mathfrak{g}$. This induces a map $SU(2) \rightarrow G$, and the normalization of the instanton number depends on the index of this map.

One foolproof way to find the normalization for the instanton number is to compute the instanton number by requiring that for a minimal small instanton field configuration F , we have

$$1 = \frac{1}{N_g} \int \text{Tr}_{Ad_g} \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right) . \quad (2067)$$

Here Tr_{Ad_g} is taken in the adjoint representation of \tilde{G} , which is always a representation for all $G = \tilde{G}/\Gamma_G$, and N_g is a normalization constant that fixes the equality. For example, consider $SU(2)$. We know the bundle E with a minimal $SU(2)$ instanton is such that⁷⁹

$$1 = \int c_1(E) = \frac{1}{2} \int \text{Tr}_f \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right) . \quad (2068)$$

⁷⁹Sort of random comment: this is an integer since it is the integral of the second Chern class, which is always integral. In particular, it is integral on any manifold, spin or otherwise. In contrast, the instanton number for a $U(1)$ field, defined through the second Chern character, is not generically integral on a non-spin manifold, since the intersection form on a non-spin manifold is not for sure even. Thus we should remember that the Chern classes are good \mathbb{Z} characteristic classes, while the Chern characters are not.

In the fundamental, we have the normalization

$$\mathrm{Tr}(T_f^a T_f^b) = \frac{\delta^{ab}}{2}. \quad (2069)$$

On the other hand, in the adjoint we have

$$\mathrm{Tr}(T_{Ad}^a T_{Ad}^b) = N\delta^{ab}, \quad (2070)$$

so that if we take the minimal $SU(2)$ instanton F^a but change the representation to the adjoint, the answer changes by a factor of $2N$. Thus we have

$$N_{\mathfrak{su}(N)} = 4N. \quad (2071)$$

When $\Gamma_G \neq \mathbb{Z}_1$ we can have “large” instantons that contribute to the instanton number I but which make rational contributions to I instead of integral contributions. This is because if the topology of spacetime is nontrivial, we can have G bundles which are not \tilde{G} bundles. This is not as contrived a scenario as it seems, since nontrivial spacetime topologies can be created by inserting t’ Hooft operators with nontrivial t’ Hooft flux, which exist if $\pi_1(G) = \Gamma_G$ is nontrivial. To visualize the types of processes that give nonzero “large” instanton number, we can think about the $U(1)$ case, where our intuition is aided by the fact that the integral in $\int \mathrm{ch}_2(E)$ can be interpreted as a self-intersection number of the Poincare dual of F (such an interpretation is only possible in the non-Abelian case if $E = \bigoplus_i \mathcal{L}_i$ is a direct sum of line bundles so that F is diagonal). For example, if we consider a process in which two initially separated magnetic flux loops pass through each other to form a Hopf link and then later re-separate, then the self-intersection number of $\hat{F}/2\pi$ is 2, and we get $I = 1$.

$SU(N)$ and $PSU(N)$

We will now specialize to the case where $G = PSU(N)$ for concreteness. The degree to which a given $PSU(N)$ bundle E does not lift to an $SU(N)$ bundle is determined by a class

$$\omega_2(E) \in H^2(X; \mathbb{Z}_n), \quad (2072)$$

where X is spacetime. We can construct E by taking an $SU(2)$ bundle \tilde{E} and relaxing the cocycle condition on the transition functions to only hold modulo an N th root of unity: $[g_{ij}g_{jk}g_{ki}]_{ab} = \delta_{ab}e^{2\pi i f_{ijk}/N}$, where the f_{ijk} are integers. The choice of f_{ijk} determines the $\omega_2(E)$ class, which when integrated over a given closed 2-submanifold tells us the fractional flux passing through that manifold.

As may be intuitively clear, the instanton number for $PSU(N)$ bundles can be (in our normalization) an element of $\frac{1}{N}\mathbb{Z}$. This is because if E is a $PSU(N)$ bundle, then $E^{\otimes N}$ is an $SU(N)$ bundle, since the transition functions of $E^{\otimes N}$ are N -fold \otimes s of the transition functions for E , which ensures that the cocycle condition holds exactly in $E^{\otimes N}$ (i.e., not just up to an N th root of unity). The instanton number for $E^{\otimes N}$ is found from

$$\mathrm{ch}(E^{\otimes N}) = \mathrm{ch}(E)^{\wedge N} = 1 + N\mathrm{ch}_2(E) + \dots, \quad (2073)$$

where we used $\mathrm{ch}_1(E) = 0$ on account of the tracelessness of the $SU(N)$ generators. Thus the instanton number is $\mathrm{ch}_2(E) = \frac{1}{N}\mathrm{ch}_2(E^{\otimes N})$. We know that $\int \mathrm{ch}_2(E^{\otimes N}) \in \mathbb{Z}$ since $E^{\otimes N}$ is an $SU(N)$ bundle, and so the minimal instanton number for a $PSU(N)$ bundle is $\frac{1}{N}$.

Now we need to show that such a fractional instanton number can actually be realized. We will work on a spin manifold in what follows. First, let us fix a class ω_2 . We will elaborate / work out in detail a computation described in [12]. The goal is to explicitly construct a $PSU(N)$ bundle that will get us the minimal possible I of $1/N$. First, let \mathcal{L} be the line bundle over X with first Chern class reducing to $\omega_2 \bmod N$:

$$\omega_2 = c_1(\mathcal{L}) \bmod N. \quad (2074)$$

Here the LHS is viewed as an element in $H^2(X; \mathbb{Z})$, but we will usually use the correspondence between elements of $H_{dR}^*(X; \mathbb{R})$ with quantized periods and those in $H^*(X; \mathbb{Z})$ to think of it as an actual 2-form in the de Rham sense. From \mathcal{L} we can form the bundle $\mathcal{L}^{-1/N}$, defined to have transition functions which are $1/N$ th roots of the transition functions of \mathcal{L} . In particular, the cocycle conditions in $\mathcal{L}^{-1/N}$ are only satisfied up to N th roots of unity. We can then construct a $PSU(N)$ bundle E as follows:

$$E = \mathcal{L}^{-1/N} \otimes \left(\mathcal{L} \oplus \bigoplus_{i=1}^{N-1} T_i \right), \quad (2075)$$

where T_i is a trivial line bundle. The $\mathcal{L}^{-1/N}$ means that E is not an $SU(N)$ bundle. However, the $\mathcal{L}^{-1/N}$ factor does not turn the thing in parenthesis from an $SU(N)$ bundle into a $PSU(N)$ bundle, since the thing in parenthesis is not an $SU(N)$ bundle: indeed, it has nonzero first chern class, which precludes it from being an $SU(N)$ bundle. Indeed, (vector bundles associated to) $SU(N)$ principal bundles always have zero first Chern class: this is seemingly obvious because $\text{Tr}(F) = 0$, but the tracelessness of F doesn't mean that the first Chern class can't have a part which is pure torsion. In the case of $SU(N)$, such a torsionful part does not exist, however, and so the tracelessness argument works.

Now if E is to be a $PSU(N)$ bundle then since at the Lie algebra level $PSU(N)$ and $SU(N)$ are identical, E must also have a first Chern character which vanishes⁸⁰. This indeed is true: we first use

$$\text{ch}(E) = \text{ch}(\mathcal{L}^{-1/N}) \wedge \left(\text{ch}(\mathcal{L}) + \sum_{i=1}^{N-1} \text{ch}(T_i) \right). \quad (2076)$$

Taking the first degree component gives

$$\text{ch}_1(E) = -\frac{1}{N} c_1(\mathcal{L}) \cdot N + 1 \cdot c_1(\mathcal{L}) = 0 \quad (2077)$$

as required.

Anyway, the construction of building E from “fractional” line bundles makes it clear that it is a $PSU(N)$ bundle. If $\lambda_{ij} = e^{i2\pi g_{ij}}$ are the transition functions for \mathcal{L} , then the transition functions for E are the matrices

$$\Lambda_{ij} = \text{diag}(e^{i2\pi g_{ij}(1-\frac{1}{N})}, e^{-i2\pi g_{ij}/N}, \dots, e^{-i2\pi g_{ij}/N}). \quad (2078)$$

Note that while we still have $\det(\Lambda_{ij}) = 1$, $\delta\Lambda$ is no longer trivial:

$$(\delta\Lambda)_{ijk} = e^{-2\pi i f_{ijk}/N} \mathbf{1}, \quad (2079)$$

⁸⁰The first Chern class of E must only vanish modulo possible torsionful contributions.

where the $f_{ijk} \in \mathbb{Z}$ are as before determined by the class ω_2 . This means that the Λ_{ij} are transition functions for a $PSU(N)$ bundle, but not for an $SU(N)$ bundle⁸¹.

Now we will compute the instanton number of E , working modulo integral classes (i.e., just focusing on the fractional part). This is done just by computing the second Chern class of E . We use the Whitney sum formula to write

$$c(E) = (1 + c_1(\mathcal{L}^{-1/N}) + c_1(\mathcal{L})) \wedge \bigwedge_{i=1}^{N-1} (1 + c_1(\mathcal{L}^{-1/N})). \quad (2080)$$

Taking the degree-2 part, we have

$$c_2(E) = \frac{N^2 - N}{2} c_1(\mathcal{L}^{-1/N}) \wedge c_1(\mathcal{L}^{-1/N}) + (N-1) c_1(\mathcal{L}) \wedge c_1(\mathcal{L}^{-1/N}). \quad (2081)$$

Now the wedge product of the chern classes is

$$c_1(\mathcal{L}^{-1/N}) \wedge c_1(\mathcal{L}^{-1/N}) = \frac{1}{N^2} \omega_2 \cup \omega_2 = \frac{1}{N^2} 2 \text{Pf}(\omega_2). \quad (2082)$$

Here we have followed Witten's notation and denoted $\text{Pf}(\omega_2)$ as the "square root" of the intersection form, $\text{Pf}(\omega_2) = (\omega_2 \cup \omega_2)/2$. The dividing by 2 here is legitimate since we assumed that we are working on a spin manifold, so that the intersection form is always even. Then we get

$$c_2(E) = \text{Pf}(\omega_2) \left(1 - \frac{1}{N} - 2 + 2\frac{1}{N} \right) = - \left(1 - \frac{1}{N} \right) \text{Pf}(\omega_2). \quad (2083)$$

Since the pfaffian is in $H^4(X; \mathbb{Z})$, we get

$$I = \frac{1}{N} \text{Pf}(\omega_2) + \text{integer}, \quad (2084)$$

which confirms that the instanton number is valued in $\frac{1}{N}\mathbb{Z}$ on a spin manifold⁸², while on a non-spin manifold $I \in \frac{1}{2N}\mathbb{Z}$. Thus, if we let the instanton number l be defined so that it gives 1 for a small $SU(N)$ instanton (the same as a small $SU(2)$ instanton), we have

$$l_{PSU(N)} = \frac{1}{N} c_1(PSU(N)). \quad (2086)$$

We can also do the computation by computing $\text{ch}_2(E)$, which should agree with $c_2(E)$ since $c_1(E) = 0$. The calculation goes as follows:

$$\text{ch}(E) = \text{ch}(\mathcal{L}^{-1/N}) \wedge (\text{ch}(\mathcal{L}) + N - 1). \quad (2087)$$

⁸¹Here it is very important that $\delta\Lambda$ is a constant N th root of unity times $\mathbf{1}$: having different N th roots of unity along each entry of the diagonal would be no good, since only the diagonal \mathbb{Z}_N is moded out by when passing to $PSU(N)$.

⁸²It's kind of interesting that we can get the minimal instanton number by working with a direct sum of line bundles, since for $SU(N)$ bundles this isn't the case. Indeed, take e.g. an $SU(2)$ bundle $E = \mathcal{L} \oplus \mathcal{L}^*$. Then it's easy to check that

$$\text{ch}_2(E) = 2\text{ch}_2(\mathcal{L}), \quad (2085)$$

which on a spin manifold is in $2\mathbb{Z}$ (since on a spin manifold $\text{ch}_2(\mathcal{L})$ is an integral class), twice the minimum allowed value. This tells us that the minimal instantons for $SU(N)$ are the ones that involve twisting around more than just two axes (and hence cannot be composed into a \oplus of line bundles).

Now unlike chern classes, the chern characters ch_k for line bundles do not vanish after $k = 1$. For example, $\text{ch}_2(\mathcal{L}) = \frac{1}{2}\text{c}_1(\mathcal{L}) \wedge \text{c}_1(\mathcal{L})$. So taking the degree 2 component of the above, and working modulo terms that are integral classes,

$$\begin{aligned} \text{ch}_2(E) &\sim N\text{ch}_2(\mathcal{L}^{-1/N}) + \text{ch}_2(\mathcal{L}) + \text{ch}_1(\mathcal{L}^{-1/N}) \wedge \text{ch}_1(\mathcal{L}) \\ &= 2\text{Pf}(\omega_2) \left(\frac{N}{2N^2} + \frac{1}{2} - \frac{1}{N} \right) \\ &= \left(1 - \frac{1}{N} \right) \text{Pf}(\omega_2), \end{aligned} \tag{2088}$$

which agrees with the Chern class result, and where \sim denotes equality up to integral classes (actually this is only up to a half an integral class if N is odd and we are on a non-spin manifold).

To be completely explicit, we can do the case for $SU(2)$. Let A_m be the $U(1)$ gauge field for a 2π monopole whose worldline wraps some nontrivial cycle in spacetime, and let

$$A_{SO(3)} = \begin{pmatrix} A_m/2 & 0 \\ 0 & -A_m/2 \end{pmatrix}. \tag{2089}$$

Note that this has zero first Chern character as required,⁸³ and that it is constructed as a \oplus of “fractional” line bundles: the total $SO(3)$ bundle is $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$, where \mathcal{L} is a legit $U(1)$ line bundle. The instanton number for this field configuration is

$$I = \frac{1}{16\pi^2} \int F_{A_m} \wedge F_{A_m} \in \frac{1}{4}\mathbb{Z} \tag{2090}$$

on general manifolds, while it is in $\frac{1}{2}\mathbb{Z}$ on spin manifolds. This is in keeping with us being able to get strength $1/2N$ instantons for $PSU(N)$ gauge theories on a generic manifold, and strength $1/N$ instantons on spin manifolds. Note how this instanton picks out a particular direction in $SU(2)$ space, namely the σ^z direction, unlike the small instantons which wrap about the entire internal $SU(2)$ space.

One comment: the fractional part is due entirely to ω_2 , as noted in [12]. That is, we have small instantons, which give integral contributions to the instanton number, and large instantons caused by ω_2 which can be fractional, and no other types of instantons. We see this as follows: to examine different ways to construct a $G = \tilde{G}/\Gamma_G$ bundle, we start from a trivialization over the 1-skeleton of X . This is always possible for orientable X and connected G . Then, we try to extend this bundle over the 2-skeleton. If $\pi_1(G) = \Gamma_G$ is nontrivial, this may not be possible: if the trivialization on the 1-skeleton winds by an element of $\pi_1(G)$ along the boundary of a given 2-cell, the trivialization is not extendable into that 2-cell (this is a global obstruction if the product of all such holonomies in $\pi_1(G)$ across all 2-cells is nontrivial). 2-cells where there is an obstruction to extending the trivialization are determined by ω_2 . Now we try to extend the trivialization over the 3-cells. This may be obstructed by an element in $\pi_2(G)$. However, since G is a topological group, we have the

⁸³However, its first Chern class is a nontrivial element in \mathbb{Z}_2 cohomology—the second SW class of the $SO(3)$ bundle, which is torsionful (indeed, the second SW class for a complex vector bundle is the mod-2 reduction of that bundle’s first Chern class).

magical fact that $\pi_2(G) = \mathbb{Z}_1$, and so there is no obstruction at this level. Finally we try to extend into the 4-skeleton: this is obstructed by $\pi_3(G)$. But as we have seen $\pi_3(G) = \pi_3(\tilde{G})$ parametrizes the “small” instantons, which are the same for both G and its universal cover, and so the contribution of $\pi_3(G)$ elements to I is always integral in our normalization. Thus the only possible contribution to the fractional part of I is ω_2 .

A final way to see all of this is to write the $PSU(N)$ field in terms of a $U(N)$ field and a \mathbb{Z}_N 2-form gauge field, as was done in the previous diary entry. Using the same notation as in that diary entry, we have

$$I = \frac{1}{8\pi^2} \int \text{Tr}[(F_{\mathcal{A}} - B\mathbf{1}) \wedge (F_{\mathcal{A}} - B\mathbf{1})], \quad (2091)$$

where $\mathcal{A} = A_{SU(N)} + \mathcal{A}\mathbf{1}/N$ is a $U(N)$ gauge field, and \mathcal{A} is a properly quantized $U(1)$ gauge field, with $F_{\mathcal{A}} = NB$ enforced through a Lagrange multiplier constraint. Then

$$I = c_2(E_{U(N)}) + \frac{i}{8\pi^2} \int (F_{\mathcal{A}} \wedge F_{\mathcal{A}} + NB \wedge B - 2F \wedge B) \rightarrow c_2(E_{U(N)}) + \frac{i}{8\pi^2}(N^2 - N) \int B \wedge B, \quad (2092)$$

where we used the Lagrange multiplier constraint. The first term, the second Chern class of the $U(N)$ bundle, is in \mathbb{Z} . However, the second term is in $\frac{1}{N}\mathbb{Z}$ on a spin manifold, since $B/2\pi$ has periods in $1/N$. Thus the instanton number for $PSU(N)$ theories is valued in $\frac{1}{N}\mathbb{Z}$. B here is the 2-form that measures ω_2 of the bundle, and so as above, we see that the fractional part of the instanton number comes from the “large” instantons (the small ones are determined by the $c_2(E_{U(N)})$ factor).

$Sp(N)$ and $PSp(N)$

First, let’s disambiguate the notation: here, by $Sp(N)$, we mean the *compact* group

$$Sp(N) \equiv U(2N) \cap Sp(2N; \mathbb{C}), \quad (2093)$$

where $Sp(2N; \mathbb{C})$ is the *non-compact* group of complex $2N$ - $2N$ matrices that preserve $J \otimes \mathbf{1}_{N \times N}$, in the sense that

$$U \in Sp(N) \implies U^\dagger U = 1, \quad U^T J U = J \otimes \mathbf{1}_{N \times N}, \quad (2094)$$

where $J \equiv (-iY) \otimes \mathbf{1}_{N \times N}$, $J^2 = -\mathbf{1}_{2N \times 2N}$ is our choice of symplectic form. The Lie algebra for the compact symplectic group⁸⁴ can be obtained by writing a general Lie algebra element T as a linear combination

$$\mathfrak{sp}(N) \ni T = i\mathbf{1} \otimes A + X \otimes B_1 + Y \otimes B_2 + Z \otimes B_3, \quad (2095)$$

where A is traceless and antisymmetric, and the B_i ’s are symmetric. Both A and the B_i ’s are real (they have to be (anti-)Hermitian in order for $e^{i\alpha T}$ to be unitary, and they have to be (anti-)symmetric in order for $e^{i\alpha T}$ to preserve the symplectic form). In this presentation we

⁸⁴Why’s it called symplectic? Since it preserves iY , which is the antisymmetric form used in the commutation relations for the symplectic form on phase space: if $v = (x, p)^T$, then $v^T J v = i$ is the CCR, and we can send $v \mapsto Rv$ for any $R \in Sp(N)$ preserving the CCR.

see clearly how $\mathfrak{su}(2)$ is embedded in $\mathfrak{sp}(N)$. Additionally, we see that $Sp(1) = SU(2)$. The center of $Sp(N) = \mathbb{Z}_2$, as can be easily checked by looking for diagonal things that preserve $J \otimes \mathbf{1}$.

To get the normalization for the instanton number straight, we need to look at how $SU(2)$ embeds into $Sp(N)$. First, note that there can only be a single full $SU(2)$ factor in $Sp(N)$, since $Z(Sp(N)) = \mathbb{Z}_2 = Z(SU(2))$ means that we can't have multiple copies without having a quotient by their centers as well. We can also find such a full $SU(2)$ just by looking at matrices of the form $U \otimes \mathbf{1}_{N \times N}$, where $U \in SU(2)$. These are obviously unitary, and a quick check shows that they are also in $Sp(2N; \mathbb{C})$. Furthermore setting $U = -\mathbf{1}_{2 \times 2}$ gives the center of $Sp(N)$, so we know that $Sp(N)$ really does have a full $SU(2)$ inside of it (i.e., the $SU(2)$ doesn't appear in a form where it's quotiented by \mathbb{Z}_2 in some way). Thus minimal $SU(2)$ instantons have instanton number 1 in $Sp(N)$, and so

$$l_{Sp(N)} = p_1(Sp(N)). \quad (2096)$$

Explicitly, we can write the gauge field $A^{Sp(N)}$ for a minimal instanton in terms of the $SU(2)$ minimal instanton gauge field $A_{SU(2)}$ as e.g.

$$A_\mu^{Sp(N)} = A_\mu^{SU(2)} \otimes E_{11}, \quad (2097)$$

where E_{11} is the matrix with a 1 in the upper leftmost entry, and zeros everywhere else. Since E_{11} is symmetric, $A_\mu^{Sp(N)}$ is indeed in the Lie algebra $\mathfrak{sp}(N)$. Writing it like this, it's clear that $l_{Sp(N)} = p_1(Sp(N))$.

Now for the quotient groups $PSp(N) = Sp(N)/\mathbb{Z}_2$. How might we obtain a $PSp(N)$ bundle that's not an $Sp(N)$ bundle? We consider the bundle $E_{SO(3)} = \mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$, which is an $SO(3)$ bundle that does not lift to an $SU(2)$ bundle. Similarly to as in our discussion of $SU(N)$, \mathcal{L} is the line bundle whose first Chern class reduces mod 2 to some class $w_2 \in H^2(X; \mathbb{Z}_2)$. Since roughly the transition functions fail the cocycle condition on triple overlaps by the value of w_2 on the triple overlap, the cocycle conditions of $\mathcal{L}^{1/2}$ fail by an amount controlled by $w_2/2$. We then use the diagonal embedding $SU(2) \rightarrow Sp(N)$ to use $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$ to create a $PSp(N)$ bundle that doesn't lift to an $Sp(N)$ bundle. Since the diagonal $SU(2) \rightarrow Sp(N)$ embedding sends

$$SU(2) \ni U \mapsto U \otimes \mathbf{1}_{N \times N} \in Sp(N), \quad (2098)$$

the $PSp(N)$ bundle we get is a direct sum of N copies of $E_{SO(3)}$ ⁸⁵:

$$E_{PSP(N)} = E_{SO(3)}^{\oplus N} = (\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2})^{\oplus N}. \quad (2099)$$

Using manipulations like the ones used for looking at $PSU(N)$ bundles, we see that the instanton number mod 1 (i.e. the part of the instanton number that doesn't come from small instantons) is

$$[\text{ch}_2(E_{PSP(N)})]_1 = \frac{N}{2} [\text{ch}_2(\mathcal{L})]_2 = \frac{N}{2} \frac{P(w_2)}{2}, \quad (2100)$$

⁸⁵This is more obvious if we write the embedding as $\mathbf{1}_{N \times N} \otimes U$, and change our definition so that the elements in $Sp(N)$ preserve $\mathbf{1}_{N \times N} \otimes J$.

where $P(w_2)$ is the Pontryagin square (a \mathbb{Z}_4 class), which I believe is the slightly more correct way to write the $w_2 \cup w_2$ appearing in what we earlier denoted $\text{Pf}(w_2)$, and where $[]_k$ denotes the mod k reduction. Since $P(w_2)/2$ is an integer class on a spin manifold by the even-ness of the intersection form, on spin manifolds we can have fractional instantons for $PSp(N)$ if N is odd, but not if N is even.

Before moving on, let's just clarify why we needed to choose the diagonal embedding of $SU(2)$ into $Sp(N)$, instead of e.g. the embedding $U \mapsto E_{11} \otimes U$ used to compute the normalization of $l_{Sp(N)}$ (I'm writing the tensor product in the opposite order since I find it slightly easier to visualize). Indeed, suppose we chose this embedding for the $SO(3)$ bundle. Then we would end up with a bundle whose transition functions could fail the cocycle condition by the matrix $-\mathbf{1}_{2 \times 2} \oplus \mathbf{1}_{2N-2 \times 2N-2}$. In a $PSp(N)$ bundle, the transition functions are only allowed to fail the cocycle condition by the matrix $-\mathbf{1}_{2N \times 2N}$, since this is the thing that gets quotiented out by upon passing to $PSp(N)$. In contrast, if we choose the diagonal embedding $U \mapsto \mathbf{1} \otimes U$, then we get a bundle whose transition functions fail the cocycle condition by $-\mathbf{1}_{2N \times 2N}$, which is what we want. Thus, we must choose the diagonal embedding.

$SO(N)$

Let's look at the normalization of the instanton number for $SO(N)$. First for $SO(3)$, which we've already mentioned above. To find the normalization, we compute the value that a minimal $SU(2)$ instanton has when lifted to the adjoint representation. This is easy: we can take the same $U^\dagger dU$ with $U \sim e^{ir^a T^a}$ type of instanton, we just have to change the T^a 's. Now for $SU(N)$ we have $\text{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$, while for $SO(3)$ we have $\text{Tr}[T^a T^b] = 2\delta^{ab}$. So $\int p_1(E_{SO(3)})$ for an $SO(3)$ bundle with a minimal $SU(2)$ instanton is $4 \int p_1(E_{SU(2)})$. Thus for $SO(3)$,

$$l_{SO(3)} = \frac{1}{4} p_1(SO(3)). \quad (2101)$$

The notation $p_1(SO(3))$ has the hopefully obvious meaning “ $p_1(E)$ for some $SO(3)$ bundle E ”. Note that this conclusion was reached for an arbitrary manifold, spin or not spin. If we restrict ourselves to spin manifolds, the Pontryagin class is even, so that $\int l_{SO(3)} = \frac{1}{2}\mathbb{Z}$ on spin manifolds. This can be proved decomposing the $SO(3)$ bundle as $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$ and realizing that the second chern character depends on the even-ness of the intersection form, or by using the relation

$$p_1(E) \mod 2 = P(w_2), \quad (2102)$$

where P is the Pointryagin square: on spin manifolds $P(w_2)$ is a class in $2\mathbb{Z}$, and so $\int p_1(E)$ is indeed even (w_2 is the second SW class for the *gauge bundle*, not the tangent bundle). It's kind of interesting that the quantization of instanton number for simply connected Lie groups doesn't depend on whether the base manifold is spin (since there the instanton number is also the Chern class, which is integral on any manifold), but that for quotients of simply connected Lie groups, the quantiation of the instanton number does depend on whether the base manifold is spin.

Now for $SO(N \geq 4)$. We use

$$SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2, \quad (2103)$$

where the quotient is the diagonal \mathbb{Z}_2 (some people write this with \otimes instead of \times , which I don't like: the tensor unit is \mathbb{C} , which means that we would already be making the $/\mathbb{Z}_2$ identification!). Note that since $Z(SU(2) \times SU(2)) = \mathbb{Z}_2^2$, taking the quotient leaves behind a factor of \mathbb{Z}_2 in the center, which is just right to match with $Z(SO(4)) = \mathbb{Z}_2$.

Regular (non-fractional) instantons are created in $SO(N > 3)$ through embedding a minimal $SU(2)$ instanton into one of the $SU(2)$ factors in the decomposition for the subgroup $SO(4) \subset SO(N)$. Now, we can form fractional instantons in $SO(N)$ by embedding an $SU(3)$ instanton inside of $SO(N)$. The way this embedding works is also through the $SO(4)$ subgroup (note to self: can we show there are no other ways to do the embedding?), but it is the embedding into the diagonal subgroup of $[SU(2) \times SU(2)]/\mathbb{Z}_2$. The reason that the embedding must be done though the diagonal subgroup is because $SU(3)$ has trivial center, and so we need to embed $SU(3)$ in the diagonal subgroup so that the quotient by \mathbb{Z}_2 gives us something without a -1 central element. Anyway, the point of this is that the minimal fractional instanton number in $SO(N)$ will be *twice* that in $SU(3)$, since both $SU(2)$ factors contribute. So

$$l_{SO(N)} = \frac{1}{2}p_1(SO(N)), \quad N \geq 4. \quad (2104)$$

Again, this holds over arbitrary manifolds, be they spin or not spin. If the manifold is spin, we can conclude that $\int l_{SO(N)} \in \mathbb{Z}$ since in that case $p_1(SO(N))$ is an even class, as discussed earlier.

118 August 26 — Dyon spin, statistics, and statistical transmutation from θ angles

Consider $U(1)$ gauge theory in four dimensions. Explain why, if the 2π monopoles are bosonic at $\theta = 0$, they become fermionic when $\theta = 2\pi$. What if the theory is coupled to fermionic matter, so that the $q = 1$ electric charges are fermions? Derive the spin and statistics for the dyons in the charge lattice for all values of θ , and explain why the spectrum depends on the value of θ , even though $\theta \int F \wedge F$ is a topological term which doesn't contain the metric and hence doesn't contribute to $T_{\mu\nu}$. Why does the induced electric charge created by the θ term not contribute to the statistics of the monopoles? Also, what is the periodicity of θ on (non)-spin manifolds?

Now consider the discrete analogue, namely \mathbb{Z}_N BF theory. How can you turn the monopoles (magnetic lines) into fermions? How can you turn the electric lines into fermions?

Solution:

We start with yet another derivation of the Witten effect, prompted by reading an old paper by Wilczek that made no sense (he did what we are going to do here but with a pure gauge transformation and somehow got a non-zero electric field which is bizarre). Consider a single monopole in spatial \mathbb{R}^3 of unit magnetic charge, with the magnetic field being set

up by a vector potential A_0 . Consider a change of the gauge field A which takes place over a time Δt :

$$A(t) = A_0 + \left[\frac{t - t_0}{\Delta t} \Theta(t - t_0) \Theta(t_0 + \Delta t - t) + \Theta(t - (t_0 + \Delta t)) \right] (0, \alpha f'(r), 0, 0), \quad (2105)$$

where we are in (t, r, θ, ϕ) coordinates. Here α is any real number and $f(r)$ is a smoothed step function with $f(\infty) = 1, f(0) = 0$. Since $A(t) - A_0 = 0$ for r near the origin, adding this change to the gauge field can be thought of as a change only of the gauge field on the coordinate patch that does not encompass the origin—thus it is well-defined even though the full $A(t)$ must be constructed by gluing patches. This $A(t)$ leads to a radial electric field

$$E_r(t) = \frac{\alpha f'(r)}{\Delta t} \Theta(t - t_0) \Theta(t_0 + \Delta t - t). \quad (2106)$$

This electric field gives zero contribution to the energy if we take $\Delta t \rightarrow \infty$, since $\int d^3r dt E_r(t)^2 \propto \frac{1}{\Delta t} \rightarrow 0$. However, it does contribute to the θ term:

$$\frac{\theta}{8\pi^2} \int F \wedge F = \frac{\theta}{8\pi^2} \int \frac{1}{4} \cdot 4 \cdot 2B_i E^i = \frac{\theta\alpha}{4\pi^2} 4\pi \int_0^\infty dr r^2 B_r f'(r) = \frac{\alpha\theta\Phi_B}{4\pi^2}, \quad (2107)$$

where in the last step we integrated by parts and used $f(0) = 0$ to kill the $\nabla \cdot B$ term. Here the magnetic flux Φ_B is measured in units where a unit monopole has 2π flux. Thus the θ term contributes a phase to the path integral.

What is the physical interpretation of this phase? The final and initial gauge configurations differ by

$$A(t > t_0 + \Delta t) - A(t < t_0) = \alpha \partial_r f. \quad (2108)$$

This is a function that goes to the constant α at spatial infinity. Thus it is a “large” gauge transformation, better called an asymptotic symmetry, which rotates the boundary conditions of the sections of the $U(1)$ bundle in question by a phase $e^{i\alpha}$ (if $\alpha = 2\pi$ then this is a legit gauge transformation). The electric charge of a system is defined as the representation that the system transforms under when acted on by asymptotic symmetries like this, and so we identify the relative phase between these two configurations (the phase picked up by the θ term in the path integral) with $e^{i\alpha q}$, where q is the electric charge of the monopole. Thus since this holds for all α , we have

$$e^{i\theta\Phi_B\alpha/2\pi} = e^{i\alpha q}, \implies q = \frac{\theta\Phi_B}{4\pi^2}. \quad (2109)$$

In particular, a unit monopole comes attached with an electric charge of $\theta/2\pi$.

Since for any current j such that $\langle e^{i \int A \wedge \star j} \rangle \neq 0$ we have that $\star j \in d\Omega^1(X; \mathbb{R})$ ⁸⁶, we can invert the d and write $d^{-1} \star j = D$ for some $D \in \Omega^2(X; \mathbb{R})$. In the case where $\partial X \neq 0$, D

⁸⁶This is just because there does not exist a solution to the classical eom otherwise: $d \star F = \star j$ tells us that $\star j$ is exact, since $\star F$ is always a globally well-defined 2-form. Define the Poincare dual $\hat{\mathcal{J}}$ of $\mathcal{J} = \star j$. Then

$$\star j \in d\Omega^3(X; \mathbb{R}) \implies \hat{\mathcal{J}} \in B_1(X, \partial X; \mathbb{Z}). \quad (2110)$$

Here $B_1(X, \partial X; \mathbb{Z})$ are the 1-chains which are relative boundaries, i.e. for each element $C \in B_1(X, \partial X; \mathbb{Z})$ there is a 2-manifold M such that ∂M consists of C and 1-submanifolds of ∂X . This follows from Poincare duality applied to relative cohomology groups (we are being a bit cavalier about switching between \mathbb{R} and \mathbb{Z})

may have support on ∂X . D is basically just the worldsheet swept out by the electric fluxes that link the two points created when we take the intersection of the Poincare dual of $\star j$ with any constant time slice.

Let's remind ourselves of why dyons can have fermionic statistics, thinking classically without a θ term. The angular momentum for a configuration with electromagnetic fields E, B is (I derived this by finding $T_{\mu\nu} \sim g_{\mu\nu}F \wedge \star F - F_{\mu\sigma}F^{\sigma}_{\nu}$, and taking T_{0i} to get the momentum, but surely there's a better way)

$$L_i = \int d^3r \epsilon_{ijk} r^j \epsilon^{klm} E^l B^m. \quad (2111)$$

Now let B^i be a monopole of strength g at the origin and E^i be sourced by an electric point charge q at position r_0^i . Using the triple product $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, we get

$$L_i = \frac{1}{4\pi} \int d^3r \frac{r_{0,i} r^2 - r_i(r_j r_0^j)}{r^3 |r - r_0|^3}. \quad (2112)$$

Now we get crafty and use

$$\partial_i \hat{r}^j = \frac{\delta_{ij}}{|r|} - \frac{r_j r_i}{|r|^3}. \quad (2113)$$

Using this we can re-write the integrand above as $E^j \partial_j \hat{r}_i$, and then integrate by parts. The boundary term is like $\int d^2\Omega_{S^2_\infty} \hat{r}^j \rightarrow 0$ since we are integrating over all angles. This gives an integral over $(\nabla \cdot E) \hat{r}^j$. Then using $\nabla \cdot E = 4\pi\delta(r - r_0)$, we have

$$L_i = -q g \hat{r}_i. \quad (2114)$$

Note that L_i is pointed along the vector from the charge to the monopole, and that $|L|$ is *independent* of the distance between the charge and the monopole.

This electromagnetic angular momentum needs to be added on to the usual angular momentum of charged particles in order to get a conserved quantity. Using $\partial_t \hat{r} = |r|^{-3}(\dot{r} -$

coefficients: we usually want to think of j as being in $C^1(X; \mathbb{Z})$, but we usually want to think of F as being in \mathbb{R} -valued dR cohomology, so we are sloppily mixing the two).

What sorts of current loops \mathcal{J} are allowed by this condition? All contractible current loops are of course allowed. Non-contractible loops are only allowed on non-compact manifolds: this is because on non-compact manifolds we can have non-contractible loops that are in $B_1(X, \partial X; \mathbb{Z})$ (think of the loop on a cylinder). This is just another way of saying that we can't have a single charge on a compact manifold (think of a current line wrapping a temporal circle), since the flux can't be well-defined everywhere. On a non-compact manifold, we can have a single charge since flux can end on the boundary: thus if we are on a non-compact manifold we can choose \mathcal{J} to be a non-contractible element of $B_1(X, \partial X; \mathbb{Z})$, and this implies that $d \star F$ must be trivial in $H^3(X; \mathbb{R})$ and must be non-trivial in $H^3(X, \partial X; \mathbb{R})$ (since in this case $(d \star F)|_{\partial X} = 0$ but $(\star F)|_{\partial X} \neq 0$; flux lines are ending on the boundary). A corollary of this / another way of saying the same thing is that $\langle e^{i \oint_C A} \rangle = 0$ if the curve C is nontrivial in $H_1(X, \partial X; \mathbb{Z})$: the only Wilson loops that can have vevs are ones integrated around curves which are relative boundaries (otherwise there is a trivial shift in integration variables [one which doesn't affect the boundary conditions] which the Wilson line transforms nontrivially under). A quick comment on a common current confusion: if we add background matter through $A \wedge \star j$, it seems like the 1-form symmetry on A is unbroken, since $\star j = dD$ means $\delta(A \wedge \star j) = \lambda \wedge dD$ for a flat 1-form λ , which seems to vanish upon integration by parts. The key is that for any allowed $\star j = dD$, we have $D|_{\partial X} \neq 0$, and since λ is flat we also have $\lambda|_{\partial X} \neq 0$ (provided $\lambda \notin H^1(X, \partial X; \mathbb{R})$ in which case no Wilson lines can have vevs under the symmetry coming from shifting A by λ). Thus an integration by parts actually gives $\int_{\partial X} \lambda \wedge D \neq 0$, and so the coupling to the current indeed breaks the symmetry.

$r(r \cdot \dot{r})$), one can check that the conserved angular momentum for a particle of charge q moving in a monopole field of strength g is now ($m = 1$ units)

$$L_i = \epsilon_{ijk} r^j \dot{r}^k - e g \hat{r}_i. \quad (2115)$$

If these angular momentum generators are correct, they need to satisfy the correct commutation relations. When we quantize, we write the angular momentum generators as

$$L_i = \epsilon_{ijk} r^j \pi^k - e g \hat{r}_i, \quad (2116)$$

where $\pi^k = p^k - eA^k$ is the kinetic momentum and $p^k \leftrightarrow -i\partial^k$ is the canonical momentum. Since π^k parallel transports in the presence of A , its commutator picks up the field strength:

$$[\pi_i, \pi_j] = ieF_{ij}. \quad (2117)$$

One can then check, with the help of the identity $-\epsilon^{ilm}\epsilon^{ilm} + \epsilon^{ilm}\epsilon^{jnl} = \epsilon^{ijk}\epsilon^{knm}$, that

$$[\epsilon_{ijk} r^j \pi^k, \epsilon_{jol} r^l \pi^o] = ie \epsilon_{ijl} \epsilon_{lmn} r^m \pi^n + ie F_{ij}. \quad (2118)$$

The first term on the RHS is what we want if we want the usual $SU(2)$ relations for angular momentum, while the (uniform by assumption) F_{ij} term is some vaguely central-extensiony thing that screws up the commutation relations. So if our angular momentum generators are correct, the commutators with the $-e\hat{r}$ term need to cancel the field strength term above. This is indeed what happens: we compute

$$[\epsilon^{ilm} r_l \pi_m, -e\hat{r}^j] = ir_l e \epsilon^{ilm} \frac{1}{|r|^3} (\delta_{jm} r^2 - r_j r_m). \quad (2119)$$

When we subtract the $i \leftrightarrow j$ counterpart to get the full expression in the commutator, we find that these commutators contribute a total of $-2ie^{ijk}e\hat{r}^k$. One of these goes into the definition of L_k , and so

$$[L_i, L_j] = i\epsilon_{ijk} L^k + ie F_{ij} - ie g \epsilon_{ijk} \hat{r}^k. \quad (2120)$$

We've defined g so that $\int F = 4\pi g$ on a small sphere surrounding the monopole, and so F is given by $F_{ij} = \epsilon_{ijk} g \hat{r}^k$. Thus the two extra terms precisely cancel each other, and we recover the correct $SU(2)$ commutation relations.

Anyway, let's go back and look at the extra $-e\hat{r}$ term added on to L_i . In these conventions the quantization condition on the magnetic charge is $4\pi g e \in 2\pi\mathbb{Z}$, where e is the minimal electric charge. Thus $g \in \frac{1}{2}\mathbb{Z}$. Taking $g = 1/2$ for the minimal monopole and $q = 1$, we see that the minimal dyon has $L_i = -\frac{1}{2}\hat{r}_i$. So, if we do a rotation about the \hat{r} axis through an angle of 2π , we get a phase of $e^{2\pi i \hat{r}_i L^i} = -1$. Thus the minimal dyon is a fermion (via spin-statistics; more on this in a moment), as we would expect.

This same argument also works for deriving the quantization condition for a pair of dipoles $(q_1, g_1), (q_2, g_2)$: we put the electric and magnetic fields generated by the two dyons into the integral for angular momentum, and use the same trick described above. This gives $L_i = -(q_1 g_2 - q_2 g_1) \hat{r}_i$, with r_i the separation vector between the monopoles. Note the relative minus sign between the two terms! This is kind of counter-intuitive, since the two dyons are being braided around one another with the same handedness.

Let's now check that the spin-statistics argument is correct by computing the self-statistics of a dyon. We do this by moving one (e, g) dyon in a π semicircle around the other (the two-dyon system has translation invariance so we can scoot the rotated system back to the original one after the π rotation for free). The electric charge of the moving dyon picks up a phase $e^{ie \int_C A(r)}$, where A is sourced by the magnetic monopole at the origin and C is the semicircular contour. By electromagnetic duality $(e, g) \mapsto (-e, g)$ the moving magnetic monopole picks up a phase $e^{-ig \int_C \tilde{A}(r)}$. We find $\tilde{A}(r)$ by noting that the Lagrangian for a single dyon contains the couplings $er_e^i A_i(r)$ and $-gr_g^i \tilde{A}_i(r)$. Since the total dyon system is translationally invariant, the total canonical momentum $p_e + p_g$ is conserved, $p_e + p_g = 0$. From varying the action, we have

$$p_e + p_g = m_e \dot{r}_e + m_g \dot{r}_m + eA(r_e - r_g) - g\tilde{A}(r_g - r_m) = 0. \quad (2121)$$

Since this has to hold when e.g. the electric and magnetic charges are at rest, we require

$$eA(r) = g\tilde{A}(-r). \quad (2122)$$

Now for a monopole we can use the solution

$$A_i(r) = \frac{g \epsilon_{ijk} r^j \hat{n}^k}{r r - r_l \hat{n}^l}, \quad (2123)$$

where \hat{n}^j is some unit vector that we may choose freely. This solution is valid on a large enough patch on S^2 for our purposes, and we will take $\hat{n} = \hat{z}$ to be the unit vector in the plane normal to the movement of the dyon, so that $r^i \hat{n}_i = 0$. For motion in the xy plane then, we have

$$A(r) = -\frac{g}{r} d\phi, \quad \tilde{A}(r) = \frac{e}{r} d\phi. \quad (2124)$$

Thus the total phase accumulated during the exchange is

$$\exp \left(-ieg \int_C r d\phi \frac{1}{r} + ieg \int_C r d\phi \frac{-1}{r} \right) = e^{2\pi i eg} = (-1)^{2eg}, \quad (2125)$$

which indeed gives us fermionic statistics if $eg \in \frac{1}{2}(2\mathbb{Z} + 1)$, which agrees with spin-statistics.

Another way to see this is to make a “gauge transformation” to get rid of the potentials, at the expense of making the wavefunction not single-valued. I really don’t like this way of doing things since wavefunctions should always be single valued and one should never do singular gauge transformations, but because a lot of other people seem to do similar things it’s good to understand what the argument is. If we let r denote the vector pointing from dyon 1 to dyon 2, the Hamiltonian is (setting the mass of the dyons to 1/2 for simplicity)

$$H = (-i\partial_1 - eA(r) + g\tilde{A}(r))^2 + (-i\partial_2 - eA(-r) + g\tilde{A}(-r))^2 + (e^2 + g^2) \frac{1}{|r|}. \quad (2126)$$

Now note that $eA(r) - g\tilde{A}(r) = 2egd\phi/|r|$. Thus we can eliminate the gauge fields in H if we make a “gauge transformation”

$$\psi(|r|, \phi) \mapsto \tilde{\psi}(|r|, \phi) \equiv \exp(-2ige\phi) \psi(|r|, \phi), \quad (2127)$$

where ϕ is the angular coordinate in the plane of the dyon's motion, with one of the dyons fixed at the origin. So the Schrodinger equation is now

$$\left(-\partial_1^2 - \partial_2^2 + \frac{e^2 + g^2}{|r|}\right) \tilde{\psi}(|r|, \phi) = E \tilde{\psi}(|r|, \phi), \quad (2128)$$

where $\tilde{\psi}$ has a non-single-valued part $e^{-2ige\phi}$. Changing $\phi \mapsto \phi + \pi$ exchanges the dyons, and does

$$\tilde{\psi} \mapsto (-1)^{2ge} \tilde{\psi}. \quad (2129)$$

Thus we again find that the dyons are fermions if $2ge \in 2\mathbb{Z} + 1$.

Recapitulating, we have seen that bound states of a unit charge and a unit monopole are fermions, both in terms of their spin and in terms of their statistics. We have also seen that turning on a θ term changes the electric charge of the monopoles. This raises the question: are the statistics of the monopoles affected by θ ? This seems reasonable because of the charge attachment, but definitely can't be true since we are in three dimensions and can only have bosons and fermions, but can tune θ continuously, implying that if the statistics did depend on θ then the statistics would vary continuously, which is a contradiction. Indeed, we will see that the electric field induced on the monopole is a "polarization effect" and makes no contribution to either the spin or the statistics of the monopole. This means that only the "microscopic" charge and monopole number are relevant for determining dyon spins and statistics. For example, suppose at $\theta = 0$ the pure minimal monopole is a boson. When we increase θ , the charge of the monopole increases, but it remains a boson. At $\theta = 2\pi$, it becomes a bosonic $(1, 1)$ dyon. This means at $\theta = 2\pi$ the new charge-neutral monopole is really a $q = -1$ "microscopic" charge bound to the $(1, 1)$ bosonic dyon. Since only the microscopic charge is relevant for determining the statistics, the new charge-neutral monopole is a fermion. More on this to follow.

Let's now verify the claim that the induced electric charge doesn't contribute to spin or statistics. For the spin, there is a very simple argument: applying Gauss' law $d(\star F/e^2 + \theta F/4\pi^2) = 0$ around a monopole of flux $\Phi = 2\pi m$ tells us that the induced electric field is

$$E_{ind}^i = \frac{m\theta}{2\pi} \frac{r^i}{r^3}. \quad (2130)$$

In particular, the induced electric field is purely radial, and parallel to the magnetic field (since the θ term is an $E \cdot B$ term). Now as we recalled earlier the angular momentum goes like $L \sim \int r \times (E \times B)$, and so the contribution to the total angular momentum from the induced field vanishes. Thus the spin of the monopoles is independent of θ .

Now we look at the statistics. We could do a wavefunction approach like we did previously, but here we will do something more field-theory-centric. Write the action in the presence of sources for monopoles and electric charges as (we're in \mathbb{R} time since it seemed to be easiest to keep track of signs that way—can't promise that all the signs are correct though)

$$S = \int \left[-\frac{1}{2e^2} F \wedge \star F + \frac{\theta}{8\pi^2} F \wedge F + q A \wedge \star j \right], \quad (2131)$$

where $\star j$ is dual to the electric worldlines. We will keep track of the monopoles by doing the decomposition

$$F = dA + 2\pi\beta + \omega. \quad (2132)$$

Here β is the magnetic monopole part such that $\star d\beta$ is the monopole current, normalized so that its Poincare dual has \mathbb{Z} periods (we are *not* assuming β is coexact; this is not exactly a Hodge decomposition). The reason why is so that we can more easily deal with monopoles). Here ω is a harmonic component that will only be activated when we have flux threading 2-cycles of spacetime (the monopoles are treated as locations where $dF \neq 0$, rather than excised balls in spacetime). The harmonic component decouples from the rest of the action and gives $S_\omega \sim \int \omega \wedge \star \omega + \theta \int \omega \wedge \omega$. We will avoid talking about it any further, and will thus only be dealing with non-harmonic forms in what follows. Thus the Hodge Laplacian will always be invertible on the forms we'll be working with.

The action is then re-written as

$$S = \int \left[-\frac{1}{2e^2} A \wedge \left((\star d^\dagger d + \star dd^\dagger) A + 4\pi d \star \beta - 2e^2 \frac{\theta}{2\pi} d\beta - 2e^2 q \star j \right) - \frac{2\pi^2}{e^2} \beta \wedge \star \beta + \frac{\theta}{2} \beta \wedge \beta \right], \quad (2133)$$

where we have taken boundary conditions so that $A|_{\partial X} = 0$ and inserted a gauge-fixing term in Feynman gauge. Now we make the shift

$$A \mapsto A - \frac{1}{2} \square^{-1} \star \left(4\pi d \star \beta - 2e^2 \frac{\theta}{2\pi} d\beta - \frac{q}{2e^2} \star j \right). \quad (2134)$$

Here the $-$ sign is needed since $\star^2 = +1$ on 1-forms in Lorentzian signature. If the worldlines of the monopoles and electric sources meet ∂X transversely, which we will assume, then this shift preserves the boundary conditions on A . This renders the A part of the action to just be $\int F \wedge \star F$, which we absorb into the normalization of the measure. This leaves us with

$$S = \int \left[\frac{1}{8e^2} (4\pi d \star \beta - 2e^2 \star \mathcal{J}_\theta) \wedge \square^{-1} \star (4\pi d \star \beta - 2e^2 \star \mathcal{J}_\theta) - \frac{2\pi^2}{e^2} \beta \wedge \star \beta + \frac{\theta}{2} \beta \wedge \beta \right], \quad (2135)$$

where we've defined the current

$$\mathcal{J}_\theta \equiv qj + \frac{\theta}{2\pi} \star d\beta = qj + \frac{\theta}{2\pi} m, \quad (2136)$$

which is a linear combination of the charge and monopole currents j and m , in accordance with the induced electric charges stuck onto the monopoles because of the θ term. The terms involving β but not dependent on θ are

$$S \supset -\frac{2\pi^2}{e^2} \int (-d^\dagger \beta \wedge \star \square^{-1} d^\dagger \beta + \beta \wedge \star \beta) = -\frac{1}{2\tilde{e}^2} \int d\beta \wedge \star \square^{-1} d\beta, \quad (2137)$$

where $\tilde{e} = e/2\pi$ is the dual charge. Here we've used $\square = dd^\dagger + d^\dagger d$ to write

$$\begin{aligned} \int (-d^\dagger \beta \wedge \star \square^{-1} d^\dagger \beta + \beta \wedge \star \beta) &= \int (-\beta \wedge \star \square^{-1} dd^\dagger \beta + \beta \wedge \star \square^{-1} (dd^\dagger + d^\dagger d)\beta) \\ &= \int \beta \wedge \star d^\dagger \square^{-1} d\beta = \int d\beta \wedge \star \square^{-1} d\beta, \end{aligned} \quad (2138)$$

since d, d^\dagger commute with \square and hence with \square^{-1} .

We can also see this by going to momentum space: ignoring constants coming from combinatorial factors, we take $\beta \wedge \star \beta \rightarrow \frac{1}{q^2} \beta_{\mu\nu} q^2 \beta^{\mu\nu}$, and then use

$$\beta \wedge \star \beta = \beta \wedge \star \square \square^{-1} \beta = -\partial^\sigma \beta_{\sigma\lambda} \partial^\gamma \frac{1}{-\partial^2} \beta_{\gamma\lambda} - \partial_\sigma \beta^{\rho\omega} \epsilon^{\sigma\lambda\omega\rho} \partial_\lambda \frac{1}{-\partial^2} \beta^{\lambda\alpha} \rightarrow \frac{1}{q^2} \left(q^\sigma q^\gamma \beta_{\sigma\lambda} \beta_{\gamma\lambda} + q_\sigma q_\lambda \epsilon^{\sigma\lambda\omega\rho} \beta^{\rho\omega} \beta^{\lambda\alpha} \right). \quad (2139)$$

The first term cancels with

$$-d^\dagger \beta \wedge \star \square^{-1} d^\dagger \beta \rightarrow -\frac{1}{q^2} q^\sigma q^\gamma \beta_{\sigma\lambda} \beta_{\gamma\lambda}, \quad (2140)$$

which indeed leaves only the second term.

Since d, d^\dagger commute with the Hodge Laplacian $dd^\dagger + d^\dagger d$, we can write the term containing β but not θ as

$$S \supset -\frac{1}{2\tilde{e}^2} \int d\beta \wedge \star \square^{-1} d\beta = \frac{1}{2\tilde{e}^2} \int m \wedge \square^{-1} \star m, \quad (2141)$$

where as before $m = \star d\beta$ is the monopole current. This is the electromagnetic dual of the electric current-current Coulomb interaction for the monopoles. The fact that such an interaction was induced could also have been argued in the following way: due to the fact that $dA + 2\pi\beta$, the effective action for β needs to be invariant under the shift $\delta\beta = d\lambda$ for λ a 1-form, since it can be compensated by a shift in A . Thus the effective action for β should involve the projector onto the coexact forms. Indeed, we have

$$\int d\beta \wedge \star \square^{-1} d\beta = \int \beta \wedge \star \frac{d^\dagger d}{\square} \beta, \quad (2142)$$

with $d^\dagger d/\square$ the projector onto the coexact forms: $\square^{-1} d^\dagger d d\lambda = 0$, while

$$\frac{d^\dagger d}{\square} d^\dagger \omega = d^\dagger \frac{\square - d^\dagger d}{\square} \omega = d^\dagger \omega, \quad (2143)$$

so that it acts as **1** on coexact forms (it is not defined on the harmonic forms since they are in $\ker \square$).

The θ dependence of the action is

$$S \supset \int \left[\frac{e^2}{2} \mathcal{J}_\theta \wedge \square^{-1} \star \mathcal{J}_\theta - 2\pi d^\dagger \beta \wedge \square^{-1} \star \mathcal{J}_\theta + \frac{\theta}{2} \beta \wedge \beta \right], \quad (2144)$$

where we've used $\star^2 = 1$ on 1-forms in real time in four dimensions. The first term is the usual current-current interaction for electrically charged sources: here the current is upgraded to include a term proportional to θ and the monopole current, reflecting the fact that the electric fields of the monopoles contribute to the usual electric interaction between Wilson lines. The second two terms are

$$\int \left[-2\pi d^\dagger \beta \wedge \square^{-1} \star \mathcal{J}_\theta + \frac{\theta}{2} \beta \wedge \beta \right] = \int \left[-2\pi q d^\dagger \beta \wedge \square^{-1} \star j - \theta d^\dagger \beta \wedge \square^{-1} d\beta + \frac{\theta}{2} \beta \wedge \beta \right] \quad (2145)$$

The last two terms actually cancel since (I think the signs are correct) in momentum space

$$0 = A[q_\alpha \beta_{\mu\nu} \beta_{\lambda\sigma}] q^\alpha \epsilon^{\mu\nu\lambda\sigma} \implies \beta_{\mu\nu} \beta_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} = \frac{1}{q^2} 4q^\alpha \beta_{\alpha\mu} q_\nu \beta_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma}. \quad (2146)$$

Here the first equality simply follows from being in four dimensions (A represents antisymmetrization on the indices) and the factor of 4 comes from our ability to contract the q outside the antisymmetrizer with any four β indices. In the final equality, the left term corresponds to the $\beta \wedge \beta$ piece, while the right term corresponds to the $d^\dagger \beta \wedge \square^{-1} d\beta$ piece. Another way to see this is (okay, I'm really just having fun at this point) to Hodge-decompose β as $\beta = d\alpha + d^\dagger \gamma$ (recall β had no harmonic component), and then write

$$\int (d\alpha + d^\dagger \gamma) \wedge (d\alpha + d^\dagger \gamma) = 2 \int d\alpha \wedge d^\dagger \gamma, \quad (2147)$$

and

$$-\int d^\dagger (d\alpha + d^\dagger \gamma) \wedge \square^{-1} d(d\alpha + d^\dagger \beta) = \int d\star d\alpha \wedge \square^{-1} \star d d^\dagger \gamma = \int d\alpha \wedge \square^{-1} (d^\dagger d + dd^\dagger) d^\dagger \gamma = \int d\alpha \wedge d^\dagger \gamma. \quad (2148)$$

The θ -dependent β terms are the last expression minus half the second-to-last expression, and so they indeed cancel.

Recapitulating, the full action is

$$S = \int \left[\frac{e^2}{2} \mathcal{J}_\theta \wedge \square^{-1} \star \mathcal{J}_\theta + \frac{1}{2e^2} m \wedge \square^{-1} \star m - 2\pi q \beta \wedge \square^{-1} \star dj \right]. \quad (2149)$$

The first two terms are the electric and magnetic current-current interactions, respectively, while the last bit is the AB term. Why is it an AB term? We can write it as

$$S_{AB} = -2\pi q \int \square^{-1} dj \wedge \star \beta = -2\pi q \oint dx^\mu \int d^4y \epsilon_{\mu\nu\lambda\sigma} \frac{x^\nu - y^\nu}{|x-y|^4} (\star \beta)^{\lambda\sigma}(y), \quad (2150)$$

where the \oint is over the Poincare dual of $\star j$. This is a linking number between the current loop and the surface dual to $\star \beta$, which turns out to be the AB phase. This is seen a bit more explicitly by writing $\star j = dD$ (if j is not of this form, $\langle e^{i \oint A \wedge \star j} \rangle = 0$), and using current conservation to write

$$\square j = d^\dagger dj \implies \square^{-1} dj = d(d^\dagger d)^{-1} \star dD = dd^{-1} \star d^{-1} dD = \star D, \quad (2151)$$

where we haven't bothered to keep track of potential minus signs. Thus S_{AB} is

$$S_{AB} = -2\pi q \int D \wedge \beta = -2\pi q \int_{\hat{D}} \beta. \quad (2152)$$

Here the integral is over a disk bound by the current loop. For a geometry where a spatial current loop is drawn in spatial \mathbb{R}^3 with a monopole sitting at the origin, this just becomes

$$S_{AB} = -q \int_{\hat{D}} dA_{mag} = -qm\Omega(\hat{D}), \quad (2153)$$

where m is the monopole strength, A_{mag} the monopole part of the vector potential, and $\Omega(\hat{D})$ the solid angle enclosed by the loop ($d\beta = 0$ away from the origin of \mathbb{R}^3 since the monopole current is $m \propto \delta(\vec{x})dt$, and so the dependence on \hat{D} is topological). This is precisely the AB phase we expect.

The important thing here is that the calculation of the AB phase involved only j , and not \mathcal{J}_θ . This means that the monopoles have statistical interactions only with genuine microscopic electric currents, and they do not have any statistical interactions with the electric charge bound to them by the θ term. Thus the statistics of a dyon is calculated solely through its microscopic charge assignments, and its statistics are unchanged as θ is varied.

Now we will discuss the periodicity of θ , which is a bit subtle. We can see the difference in the periodicity of θ for fermionic / bosonic theories even without talking about spin structures and the evenness of the intersection form. For bosonic theories (both $(1, 0)$ and $(0, 1)$ are bosons, with notation (e, m)), the $(1, 1)$ dyon is a fermion, as explained above. More generally, (e, m) is a fermion whenever e and m are both odd. Thus the $m = 0$ row of the charge lattice is totally bosonic, the $m = 1$ has alternating bosons / fermions, the $m = 2$ row has all bosons, etc. Now when $\theta \mapsto \theta + 2\pi$, the $m = 0$ row is invariant, the $m = 1$ row moves to the right by one unit, the $m = 2$ row moves to the right by two units, and so on (increasing θ changes the charge, but not the statistics, of a given dyon, since the induced charge doesn't enter in to the statistical calculation). Thus the $m \in (2\mathbb{Z} + 1)e$ rows are not invariant under the shift in θ , but are invariant under a 4π shift. So when e, m are bosons, $\theta \sim \theta + 4\pi$.

This is equivalent to the statement that on a non-spin manifold, the quantization of $\int(F/2\pi) \wedge (F/2\pi)$ is in \mathbb{Z} , not in $2\mathbb{Z}$. On a spin manifold, we have transparent fermions that we can bind to any of the particles in the charge lattice, changing their statistics. Thus on a spin manifold, the statement that the statistics of the $m = 1$ row gets changed when shifting θ by 2π is not meaningful. On a non-spin manifold, (on which the theory for which both e, m are bosons can be defined) we don't have these transparent fermions, and so the shift of the $m = 1$ row is meaningful.

Hold on, one might say, if the $(1, 1)$ dyon is a fermion, how can we define the theory on a non-spin manifold? We can't use a spinc connection since the charge $(1, 0)$ object is a boson, and therefore we have no spin-statistics relation. So what's going on? The answer is⁸⁷ that even though $(1, 1)$ is a fermion, it doesn't need a spin structure on the spacetime manifold to have a well-defined framing. The point is that there is no choice of fundamental fields for which $(1, 1)$ is a local operator: it is always a non-local object. We can give it a framing (a spin structure on its worldline) by using e.g. the vector that points from the electric charge to the magnetic charge. If it were a local operator we couldn't use its internal structure to give it a framing, so we'd have to give it a framing by using the one induced by the framing of the tangent space of the ambient manifold. If the ambient manifold is non-spin then this would be impossible. So non-spin manifolds preclude defining neutral *local* fermions, but not nonlocal (emergent) ones.

By contrast if e is a fermion and m is a boson, things change. Now the $m \in 2\mathbb{Z}$ rows of the lattice are alternating fermion-boson, while the $m \in (2\mathbb{Z} + 1)$ rows are all bosonic. Shifting θ by 2π only shifts the $m \in (2\mathbb{Z} + 1)$ rows of the lattice, which is trivial in this case since the statistics of all the dyons in the odd rows is bosonic. Thus $\theta \sim \theta + 2\pi$ when e is a fermion (regardless of whether we are on a spin manifold, or a non-spin manifold with a

⁸⁷Thanks to Ryan Thorngren to helping me understand this! :D

spinc connection). The case where both e and m are fermions is anomalous and probably warrants its own diary entry, to be written sometime in the future.

119 August 30 — unfinished More on discrete θ angles

Today's diary entry is the discrete version of yesterday's. We will look at the consequences of the discrete $B \wedge B$ theta term in \mathbb{Z}_N BF-type gauge theory in four dimensions.

Solution:

The action with sources included is

$$S = \int \left(\frac{n}{2\pi} F_A \wedge B + \frac{kn}{4\pi} B \wedge B + j \wedge \star A + \Sigma \wedge \star B \right). \quad (2154)$$

Now we normally have a gauge transformation on B where $\delta B = F_\lambda$ for F_λ the field strength of a $U(1)$ gauge field. We want the action to be gauge invariant without having to integrate any fields out to impose quantization conditions (i.e. we want the action to be invariant without using our knowledge that the periods of B will be quantized in $2\pi\mathbb{Z}/n$), and so we require that $\delta A = -k\lambda$ under this shift. Imposing this invariance in the presence of the currents means

$$k\lambda \wedge \star j - \lambda \wedge d \star \Sigma = 0 \implies j = \frac{1}{k} d^\dagger \Sigma. \quad (2155)$$

This forces $\Sigma = \Sigma_j + \Sigma_c$, where $\partial \widehat{\star \Sigma_j} = \widehat{\star j}$ and $\partial \widehat{\star \Sigma_c} = 0$. Here $\star \Sigma_j$ is dual to the surfaces attached to the Wilson lines to render them gauge invariant, while $\star \Sigma_c$ is dual to the worldsheets of closed strings that are charged under B .

Integrating out A says that

$$\star j = -\frac{n}{2\pi} F_B. \quad (2156)$$

Since the current is conserved, working in a gauge where $d^\dagger B = 0$ we can take d^\dagger of both sides and write (we are in Euclidean time in four dimensions, so $d^\dagger = -\star d \star$, and $\star^2 = (-1)^{p^2}$ on p -forms)

$$\star dj = -\frac{n}{2\pi} \square B \implies B = -\frac{2\pi}{n} \square^{-1} \star dj. \quad (2157)$$

This is only a solution for B up to elements of $H^2(X) = \ker \square$. We will assume for simplicity that this cohomology group vanishes, since what we really care about are the correlation functions of the various operators, which we can study on e.g. \mathbb{R}^4 . Relaxing this assumption is no big deal, since the harmonic parts of B decouple from most other things in the action. Also, since we always take $\star j = dD$ for \widehat{D} some disk (or more precisely, any 2-manifold with $\partial D \setminus \partial X = \widehat{\star j}$, where X is spacetime—recall that if j is not of this form, the path integral vanishes [also, in our case, the associated Wilson line would not be gauge invariant]), one may also write $dD = -\frac{n}{2\pi} F_B \implies B = -\frac{2\pi}{n} D$. This solution for B is only defined up to elements in $\ker(d)$. If we impose $d^\dagger B = 0$ then the exact part drops out, and we get a solution for B up to elements of $H^2(X)$. As before we assume this vanishes, and so we get the same

result. In BF theory, we know that the periods of B are quantized. Is this reproduced by our solution? Let M_2 be any closed 2-chain in $H_2(X; \mathbb{Z})$. Then

$$\frac{2\pi}{n} \int_{M_2} \square^{-1} \star dj = \frac{2\pi}{n} \int \star \widehat{M}_2 \wedge \square^{-1} dj = \frac{2\pi}{n} \int M_2 \wedge D \in \frac{2\pi}{n} \mathbb{Z}, \quad (2158)$$

since we take $\star j$, and hence D , to be in the image of $H^*(X; \mathbb{Z})$ under the inclusion into de Rham cohomology. So indeed, B has the periods we expect.

Now we put this solution back into the action:

$$S = \frac{1}{2\pi} \int \left(\frac{4\pi^2 k}{2n} \square^{-1} dj \wedge \square^{-1} dj - \Sigma \wedge \frac{2\pi}{n} \square^{-1} dj \right). \quad (2159)$$

120 September 2 — Basics of large N Yang-Mills

Consider $SU(N)$ gauge theory, for N large. Explain why the t' Hooft limit is essential for getting analytic control on the theory, rather than just the $N \rightarrow \infty$ limit. Then derive the matrix structure of the propagator and the Feynman rules. Finally, relate the Wilson line in the adjoint to the Wilson line in the fundamental. How does this relationship simplify in the large N limit?

Solution:

$SU(N)$ YM theory does not have a sensible $N \rightarrow \infty$ limit without also taking $g \rightarrow 0$. To see why, recall that the $SU(N)$ beta function is

$$\frac{dg^2}{d \ln \mu} = -\frac{22g^4}{48\pi^2} C(SU(N)) = -\alpha g^4 N, \quad (2160)$$

where α is a positive number and we've used the fact that the quadratic casimir of the adjoint representation (aka twice the dual coxeter number) is $f_{abc}f_{dbc} = \delta_{ad}C(SU(N)) \implies C(SU(N)) = N$. If we integrate the β function, then we get

$$g^2(\mu) = \frac{Ng_0^2}{1 - \alpha Ng_0^2 \ln(\Lambda_0/\Lambda_\mu)}, \quad (2161)$$

where the 0 subscripts are at some reference scale (like the UV). Λ_{QCD} is the scale at which perturbation theory breaks down, found by setting the denominator to zero. This gives

$$\Lambda_{QCD} = \Lambda_0 e^{-(\alpha Ng_0^2)^{-1}}. \quad (2162)$$

Thus we see that if we take $N \rightarrow \infty$ without also taking $g \rightarrow 0$ so that Ng^2 is fixed, we get $\Lambda_{QCD} \rightarrow \Lambda_0$, and so the strongly-coupled scale becomes equal to the UV cutoff scale, leaving us unable to say anything useful about anything.

So with this in mind, we define the finite coupling constant of interest as $\lambda = g^2 N$. Thus the action is

$$S = \frac{N}{\lambda} \int \text{Tr}[F \wedge \star F]. \quad (2163)$$

Despite the N in front, we cannot just take the classical saddle point. This is essentially because while a non-classical field configuration will be suppressed by the N out front, it will be amplified by the fact that as N gets large there are many many more non-classical field configurations to have the fields in. So the theory is still very quantum.

To determine the Feynman rules, we will deal with the matrices A directly, rather than their components A^a in some vector representation. Now A transforms adjointly under the (global part of the) gauge group:

$$A_b^a \mapsto [U^\dagger]^{ad} A_e^d U_{eb} = [U^*]^{da} U_{eb} A_e^d. \quad (2164)$$

Thus the upper index of A transforms in \bar{N} while the lower index transforms in N , reflecting the fact that $N \otimes \bar{N} = Ad \oplus \mathbf{1}$.

Now just as how we can write something transforming with spin 1/2 under $SU(2)$ as $a|\uparrow\rangle + b|\downarrow\rangle$, we can write

$$A_b^a = \sum_{\mathcal{A}=1 \dots N^2} A^{\mathcal{A}} [T^{\mathcal{A}}]_b^a. \quad (2165)$$

Here we are thinking of A as a vector transforming under the adjoint of $SU(N)$: the tuple (a, b) is a composite vector index. The $T^{\mathcal{A}}$'s are basis vectors, and the $[T^{\mathcal{A}}]_b^a$'s are their components. Maybe the notation $v_{a,b}^{\mathcal{A}}$ would be slightly better to emphasize that we are thinking of the $T^{\mathcal{A}}$ as vectors, rather than generator matrices. Indeed, \mathcal{A} runs over N^2 different values, since the vectors being represented live in an N^2 dimensional space (the space of $N \times N$ unitary matrices).

Now since the adjoint representation has dimension $N^2 - 1$, we need to remove one of the $T^{\mathcal{A}}$'s from the basis, in order to get a vector space of the right dimensionality, as $\dim \mathfrak{su}(N) = N^2 - 1$. We want to remove the generator corresponding to the trivial representation, which is proportional to the identity matrix. Let us work in the normalization

$$\langle T^{\mathcal{A}} | T^{\mathcal{B}} \rangle = \text{Tr}[T^{\mathcal{A}} T^{\mathcal{B}}] = \delta^{\mathcal{AB}}. \quad (2166)$$

Then the generator proportional to $\mathbf{1}$ that we need to remove is $[T^{N^2}]_b^a = \frac{1}{\sqrt{N}} \delta_b^a$. With this generator taken out, the completeness relation is now

$$\sum_{\mathcal{A}=1}^{N^2-1} [|T^{\mathcal{A}}\rangle \langle T^{\mathcal{A}}|]_{db}^{ac} = \sum_{\mathcal{A}=1}^{N^2-1} [T^{\mathcal{A}}]_b^a [T^{\mathcal{A}}]_d^c = \left(\delta_d^a \delta_b^c - \frac{1}{N} \delta_b^a \delta_d^c \right). \quad (2167)$$

If we were working with $U(N)$ so that we could have generators with nonzero trace, we would have the full N^2 generators and we would have $\mathbf{1} = \delta_d^a \delta_b^c$ on the RHS. Taking out the generator for the $SU(N)$ case means that $\text{Tr}[\sum_{\mathcal{A}} |T^{\mathcal{A}}\rangle \langle T^{\mathcal{A}}|] = \frac{1}{2}(N^2 - 1)$.

Anyway, the point of writing A like this is that it allows us to figure out what the index structure of the propagator is. The kinetic term in the action looks like

$$\text{Tr}[dA \wedge \star dA] = \sum_{\mathcal{A}, \mathcal{B}} \langle T^{\mathcal{A}} | T^{\mathcal{B}} \rangle dA^{\mathcal{A}} \wedge \star dA^{\mathcal{B}} = \sum_{\mathcal{A}} dA^{\mathcal{A}} \wedge \star dA^{\mathcal{A}}, \quad (2168)$$

so that the propagator is only non-zero when it connects two A 's with the same generator T^A . Thus

$$\begin{aligned} \langle A_{\mu b}^a(x) A_{\nu d}^c \rangle &= \sum_{\mathcal{A}, \mathcal{B}} [T^{\mathcal{A}}]_b^a [T^{\mathcal{B}}]_d^c \langle A_{\mu}^{\mathcal{A}}(x) A_{\nu}^{\mathcal{B}}(y) \rangle = \sum_{\mathcal{A}} [|T^{\mathcal{A}}\rangle \langle T^{\mathcal{A}}|]_{db}^{ac} \langle A_{\mu}^{\mathcal{A}}(x) A_{\nu}^{\mathcal{A}}(y) \rangle \\ &= D_{\mu\nu}(x - y) \left(\delta_d^a \delta_b^c - \frac{1}{N} \delta_b^a \delta_d^c \right). \end{aligned} \quad (2169)$$

Here $D_{\mu\nu}(x - y)$ is the regular vector propagator in whichever gauge fixing condition we feel like adopting. Together with this propagator, the Feynman rules are easy to write down in double-line notation.

Now we can look at general correlation functions. All operators of interest will be gauge invariant and hence will involve traces (they will have no free indices). We can focus on operators with a single trace, since operators with more traces can be built from single-trace ones. We find connected correlation functions for single-trace operators \mathcal{O}_i by adding $\sum_i N \int J_i \mathcal{O}_i$ to the action, and then finding $\prod_j (N^{-1} \delta_{J_j}) W[J]$, where $Z[J] = e^{-W[J]}$ as usual. The factors of N here are just so that $W[J]$ is the same order in N as the vacuum partition function. This is order $O(N^2)$, which can be seen from evaluating the simplest planar vacuum-to-vacuum graphs. In general, we see from the action that the amplitude of a given graph is determined by

$$\mathcal{A} \sim \left(\frac{N}{\lambda} \right)^{v-e} N^{f+s} = \lambda^{e-v} N^{\chi+s}, \quad (2170)$$

where v, e, f, s are the vertices, edges, faces, and sources of the Feynman diagram. We can then use $\chi = 2 - 2g - s$, where g is the genus and s is the number of holes (sources), to write

$$\mathcal{A} \sim \lambda^{e-v} N^{2-2g}. \quad (2171)$$

Thus no matter how many (pure glue) sources we insert, the leading-order in N diagrams that contribute to $W[J]$ will be planar and go as N^2 (drawing some pictures to experimentally test this is fun).

Now from the way we are computing connected correlation functions, we see that every functional differentiation with respect to a source that we need to perform multiplies the correlation function by $1/N$. Thus $\langle \mathbf{1} \rangle$ goes as $O(N^2)$, $\langle \mathcal{O}_1 \rangle$ goes as N^1 , and in general an n -point connected correlation function goes as $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_c \sim N^{2-n}$. This means that as $N \rightarrow \infty$ all the 2-point functions of single-trace operators factorize: $\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle_c + \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \rightarrow (\langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \sim O(N^2)) + O(1)$. In particular, $\langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle / \langle \mathcal{O} \rangle^2 = \langle \mathcal{O} \mathcal{O} \rangle_c / \langle \mathcal{O} \rangle^2 \sim N^{-2}$, so that as $N \rightarrow \infty$ the fluctuations become small.

Now we turn to the computation of the Wilson loop. We are interested in computing the Wilson loop in the adjoint representation. In general, we have

$$W_{R_1 \otimes R_2}(C) = P \exp \left(i \oint_C dx^\mu A_\mu^\alpha (T_{R_1}^\alpha \otimes \mathbf{1} + \mathbf{1} \otimes T_{R_2}^\alpha) \right) = W_{R_1}(C) \otimes W_{R_2}(C). \quad (2172)$$

Here we have used that $T_{R_1 \otimes R_2}^\alpha = T_{R_1}^\alpha \otimes \mathbf{1} + \mathbf{1} \otimes T_{R_2}^\alpha$ (it's an easy check to see that the Lie bracket holds, and that $e^{i\theta^\alpha T_{R_1 \otimes R_2}^\alpha} = e^{i\theta^\alpha T_{R_1}^\alpha} \otimes e^{i\theta^\alpha T_{R_2}^\alpha}$), and then used that things which commute can be separately path-ordered.

We are most interested in the case of $SU(N)$, for which $N \otimes \bar{N} = Ad \oplus \mathbf{1}$. We need to project out the trivial representation from $W_{N \otimes \bar{N}}(C)$. We do this by (using the notation $a\bar{a} \in \mathbb{Z}_N^2$ for an adjoint index)

$$[W_{Ad}(C)]_{b\bar{b}}^{a\bar{a}} = [W_N(C)]_b^a [W_{\bar{N}}(C)]_{\bar{b}}^{\bar{a}} - \frac{1}{N} \delta_b^a \delta_{\bar{b}}^{\bar{a}}. \quad (2173)$$

The index structure on the last term is such that it only activates for indices that are diagonal in both the N and \bar{N} factors, and is determined by taking the \otimes of the intertwiners (the δ functions) for the indices in the N and \bar{N} representations (delta functions like δ_a^b that connect two indices transforming in the same representation are not invariant symbols unless $N \cong \bar{N}$ which only happens if $N = 2$, so we need to have δ functions with an index structure that connects N and \bar{N} indices). The $1/N$ normalization is to ensure that when we take the gauge field to vanish, we get $\text{Tr}[W_{Ad}(C)|_{A=0}] = \dim(Ad) = N^2 - 1$.

Taking the trace and using the Hermiticity of the generators in the fundamental representation to write

$$W_{\bar{N}}(C) = P \exp \left(i \oint_C A^\alpha (-T_N^\alpha)^* \right) = P \exp \left(i \oint_C A^\alpha T_N^\alpha \right)^* = W_N(C)^*, \quad (2174)$$

we see that

$$\langle \text{Tr } W_{Ad}(C) \rangle = \langle |\text{Tr } W_N(C)|^2 \rangle - 1. \quad (2175)$$

Since $W_N(C)$ is a single-trace operator, in the $N \rightarrow \infty$ limit the two point function is dominated by the disconnected part⁸⁸, Thus at large N , we can move the square outside of the expectation value:

$$\langle \text{Tr } W_{Ad}(C) \rangle \approx_{N \rightarrow \infty} |\langle \text{Tr } W_N(C) \rangle|^2. \quad (2176)$$

Thus the coefficient of the adjoint area law is twice that of the fundamental line area law. This is actually kind of crazy, since we know that for small N the adjoint Wilson line always has perimeter law, since adjoint strings can break and end on gluons (adjoint sources can be screened by gluons). Looking through the old QCD literature, apparently the adjoint line goes as (schematically)

$$\langle \text{Tr } W_{Ad}(C) \rangle \approx N^2 e^{-\sigma A} + e^{-\sigma P}, \quad (2177)$$

⁸⁸Since the Wilson lines are nonlocal, this might be a little bit subtle to see. Consider first the 1-point function for the fundamental Wilson line. To zeroth order in the t' Hooft coupling, it just looks like a single fundamental line, drawn in the shape of C . This is $O(N)$, since there is one trace. To next order, we have to integrate over all ways for a propagator to connect two points on the fundamental line together (the Abelian Wilson line is the exponential of this double integral). The propagator is a double line, and so we are integrating over diagrams that look like two loops, which are parallel along the propagator line. This diagram thus has N dependence of λN : two gluon-quark-quark vertices that go as 1 in our choice of coupling, one propagator that goes as λ/N , and two sums over N for the two loops. Higher order terms have more propagators connecting the loop to itself, but adding a propagator in a planar way increases the number of propagators by one and the number of faces by one, resulting in an extra power of λ but the same $O(N)$ N -dependence (as usual non-planar diagrams are suppressed). Thus the 1-point function for the Wilson line is $O(N)$.

The connected correlation for two fundamental Wilson lines is $O(1)$ however. Indeed, consider the $O(\lambda)$ contribution to the connected part: it looks like two single (fundamental) lines, with a single double-line propagator connecting them. This diagram has one propagator and one loop, so it goes as $\lambda^1 N^0$. Adding further propagators cannot increase the N -dependence, and only increases the λ dependence. So $|\langle W_N(C) \rangle|^2$ is larger than $\langle |W_N(C)|^2 \rangle_c$ by a factor of N^2 , in line with what we expect for two-point functions of single-trace operators.

where σ is a string tension, A is the area and P is the perimeter. Since normally $A \gg P$, the perimeter law piece dominates. However if we make $N \rightarrow \infty$, the area-law-scaling piece can actually win out—and it does if the fundamental lines are confined, as we saw above. Note however that for large enough Wilson lines, there is always a cross over to perimeter law for any finite N .

121 September 2 — unfinished More on gauging higher symmetries

Solution:

We write the BF theory as

$$S = \frac{in}{2\pi} \int F \wedge B + \frac{i}{2\pi} \int F \wedge F_{\tilde{A}}. \quad (2178)$$

Here F is unconstrained and can have whatever periods it likes. We have a gauge symmetry $B \sim B + F_\lambda$, where λ is any $U(1)$ connection. To see the meaning of this gauge symmetry, integrate out F to get $F_{\tilde{A}} = nB$. Shifting B by F_λ means that $\delta F_{\tilde{A}} = nF_\lambda$. Since $\int F_{\tilde{A}}$ measures electric flux, this is saying that inserting n units of electric flux acts trivially—just what we want for a \mathbb{Z}_n Higgs phase. The operator which generates the gauge transformation is $\exp(i \int_{\hat{\lambda}} F)$, where $\hat{\lambda}$ is Poincare dual to λ in space, so that $\hat{\lambda} \in C_1(\Sigma; \mathbb{R})$. Since this measures the magnetic flux passing through $\hat{\lambda}$, the fact that this acts as **1** means that the total magnetic flux passing through any surface must be a multiple of $2\pi/n$.

122 September 5 — Inequivalent gauge fields with same field strength in non-Abelian gauge theory (unfinished)

Consider a gauge theory on \mathbb{R}^d . If the gauge group is $U(1)$, the only gauge invariant observables are the Wilson lines. Putting aside boundary condition issues, all such Wilson lines are determined by the field strength, so that F contains all gauge-invariant information about the gauge field. Show that this is not true if the gauge group is non-Abelian.

Solution:

Since $SU(2)$ is the simplest non-Abelian gauge theory, we'll look to $SU(2)$ for an example.

123 September 30 — Acceleration along Killing vectors

Today is a short one, and kind of a cheat since it was part of a homework assignment in AdS/CFT class. Consider an observer moving along a Killing vector K^μ , with four-velocity U^μ such that

$$K^\mu = \alpha(x)U^\mu. \quad (2179)$$

Show that the proper acceleration is

$$a^\mu = \nabla^\mu \ln \alpha. \quad (2180)$$

Use this to compute the proper acceleration for $\rho = \text{const}$ observers in Rindler spacetime and $r = \text{const}$ observers in Schwarzschild.

Solution:

This is straightforward; we just test the given formula and make sure it works:

$$\begin{aligned} a^\mu &= -\frac{1}{2(-K_\nu K^\nu)} \nabla^\mu (K_\lambda K^\lambda) = \frac{1}{K_\nu K^\nu} K_\lambda \nabla^\mu K^\lambda \\ &= -\frac{1}{\sqrt{-K_\nu K^\nu}} U_\lambda \nabla^\mu K^\lambda. \end{aligned} \quad (2181)$$

Since $\nabla^{(\mu} K^{\nu)} = 0$ because K^μ generates an isometry,

$$a^\mu = \frac{1}{\sqrt{-K_\nu K^\nu}} U_\lambda \nabla^\lambda K^\mu = U_\lambda \nabla^\lambda U^\mu, \quad (2182)$$

which is indeed the correct formula for proper acceleration. In the last step we have used

$$U_\lambda (\nabla^\lambda [-K_\nu K^\nu]^{-1/2}) U^\mu \propto U_\lambda (\nabla^\lambda K^\nu) K_\nu K^\mu \propto U_\lambda U_\nu \nabla^{(\lambda} K^{\nu)} K^\mu = 0. \quad (2183)$$

Consider then Rindler spacetime $ds^2 = -\rho^2 d\eta^2 + d\rho^2$, and consider an observer at constant ρ which moves along the Killing vector $K^\mu = (\partial_\eta)^\mu = (1, 0)$. Then

$$\sqrt{-K^\mu K_\mu} = \sqrt{-g_{00}} = \rho, \quad (2184)$$

so that

$$a^\mu = (0, \rho^{-1}), \quad (2185)$$

since $g^{\rho\rho} = 1$. Thus the magnitude of the proper acceleration is

$$a = \sqrt{a^\mu a_\mu} = \frac{1}{\rho}. \quad (2186)$$

Likewise for Schwarzschild with an observer at constant r , the Killing vector is again the timelike vector $K^\mu = (1, 0, 0, 0)$. It has magnitude $\sqrt{f(r)}$, where $f(r) = 1 - r_S/r$. Then the acceleration is purely radial:

$$a_\mu = \nabla_\mu \sqrt{f} = \delta_{\mu,r} \frac{f'(r)}{2f} dr, \quad (2187)$$

so that the magnitude of the acceleration is (using $g^{rr} = f$)

$$a = \frac{f'(r)}{2\sqrt{f(r)}}. \quad (2188)$$

If we want to convert this to the acceleration seen by an observer at infinity, we just need to re-scale by redshift factor $\sqrt{-K_\nu K^\nu} = \sqrt{f}$, so that the acceleration seen at infinity is

$$a_\infty = \frac{f'(r)}{2}. \quad (2189)$$

In particular, the surface gravity of the black hole is found by evaluating this at $r = r_S$, for which we get

$$\kappa = a_\infty(r_S) = \frac{1}{4GM}. \quad (2190)$$

124 October 15 — Kerr-Newmann metric

A general black hole in asymptotically flat space of mass M angular momentum J and charge Q has the metric

$$ds^2 = -\frac{\rho^2 \Delta}{\Sigma} dt^2 + \frac{\Sigma}{\rho^2} \sin^2 \theta (d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \quad (2191)$$

where the constants are

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr, \quad a = J/M, \quad (2192)$$

and

$$\Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \omega = \frac{a}{\Sigma} (r^2 + a^2 - \Delta). \quad (2193)$$

- a) Find T_H , the horizon radius, the horizon area, and the angular velocity at the black hole horizon.
- b) what happens when the black hole is extremal, i.e. when $M^2 = a^2 + Q^2$?
- c) Show that for $J = 0$, the region near the horizon of an extremal black hole is $\text{AdS}_2 \times S^2$.

Solution:

- a) recalling the Schwarzschild metric, the horizon radius r_+ should occur where the dr^2 part of the metric changes sign. Solving the quadratic equation $\Delta = 0$ for r and taking the larger value for r , we get

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}. \quad (2194)$$

Note that this gives the correct $r_+ = 2M$ when $J = Q = 0$ (we're in units where $G_N = 1$).

To get the horizon area, we need to integrate the area element over the sphere $dr = dt = 0$ at $r = r_+$:

$$A = \int \sqrt{\det g|_{r=r_+, t=0}} d\phi d\theta = \int \sqrt{\Sigma} \sin \theta d\phi d\theta = 2\pi \int_0^\pi (r_+^2 + a^2) \sin \theta d\theta = 4\pi(r_+^2 + a^2). \quad (2195)$$

To get the Hawking temperature, we consider an observer moving radially along $d\phi = d\theta = 0$ at some value of θ, ϕ (we can take $\phi = 0$ wolog since ϕ doesn't appear in the metric). Then the metric reduces to

$$ds^2 \rightarrow - \left(\frac{\rho^2 \Delta}{\Sigma} - \frac{\Sigma \sin^2 \theta \omega^2}{\rho^2} \right) dt^2 + \frac{\rho^2}{\Delta} dr^2. \quad (2196)$$

Let's expand this near the horizon $r = r_+$, to first order in $r - r_+$. We will define

$$dx = \frac{1}{\sqrt{\Delta}} dr, \quad (2197)$$

so as to simplify the dr^2 term in the metric. Since $\Delta = 0$ at the horizon, only terms with a $\Delta'(r_+) = 2(r_+ - M)$ will survive. Furthermore since we are interested in T_H / the surface gravity, we can set θ to be any angle we like, since we know in equilibrium the temperature will be the same over the entire horizon. We will pick $\theta = 0$, so that the metric above is approximately

$$ds^2 \approx - \frac{(\partial_r \Delta)(r_+)(r - r_+)}{r_+^2 + a^2} dt^2 + \rho^2 dx^2 = - \frac{2(r_+ - M)(r - r_+)}{r_+^2 + a^2} dt^2 + \rho^2 dx^2. \quad (2198)$$

Integrating the definition for dx gives

$$x = \frac{2}{\sqrt{2(r_+ - M)}} \sqrt{r - r_+}, \quad (2199)$$

so that

$$r - r_+ = \frac{r_+ - M}{2} x^2. \quad (2200)$$

Putting this into the metric and defining $\rho dx = dy$, we get, after analytically continuing to $\tau = it$,

$$ds^2 \approx \frac{(r_+ - M)^2}{(r_+^2 + a^2)^2} y^2 d\tau^2 + dy^2. \quad (2201)$$

Thus we are prompted to write

$$\theta \equiv \tau \frac{r_+ - M}{r_+^2 + a^2}, \quad (2202)$$

and identify $\theta \sim \theta + 2\pi$ in order to get a smooth geometry. On τ this identification is

$$\tau \sim \tau + 2\pi \frac{r_+^2 + a^2}{r_+ - M} = \tau + \frac{A}{2(r_+ - M)}. \quad (2203)$$

Since $\tau \sim \tau + \frac{1}{T}$ in field theory, we find that T_H is

$$T_H = \frac{2(r_+ - M)}{A}. \quad (2204)$$

The angular velocity of the horizon is found by evaluating ω at $r = r_+$. There we have

$$\omega|_{r_+} = a(r_+^2 + a^2)/\Sigma = \frac{a}{r_+^2 + a^2} = \frac{4\pi a}{A}. \quad (2205)$$

b) The black hole is extremal if $M^2 = a^2 + Q^2$. This means that $r_+ = M$, giving $T_H = 0$. The entropy (alias area) is finite, though:

$$S = \frac{A}{4} = \pi(M^2 + a^2). \quad (2206)$$

c) For $J = 0$ we have $a = 0$, and setting $M = Q$ gives $\Delta = (r - M)^2$. Since $\Sigma \rightarrow r^4$ and $\rho \rightarrow r^2$, the metric then becomes

$$ds^2 = -(1 - M/r)^2 dt^2 + r^2 \sin^2 \theta d\phi^2 + \frac{1}{(1 - M/r)^2} dr^2 + r^2 d\theta^2. \quad (2207)$$

The proper distance to the horizon from any $r > r_+$ is infinite, since the integral over dr of $1/(1 - M/r)^2$ diverges as the integration limit is taken to $r \rightarrow r_+ = M$.

Near the horizon, we write $r \approx M + x$ and expand e.g. $1 - M/r \approx x/M$. Dropping the terms that go as x, x^2 in front of the ϕ, θ parts of the metric, we get

$$ds^2 \approx -\frac{x^2}{M^2} dt^2 + \frac{M^2}{x^2} dx^2 + M^2(\sin^2 \theta d\phi^2 + d\theta^2), \quad (2208)$$

which is the metric for the space $\text{AdS}_2 \times S^2$, where the S^2 has radius M .

125 October 16 — Basic black hole thermodynamics

This is another homework problem, this time from Hong's AdS/CFT class. It's a continuation of yesterday's entry.

a) verify the first law of black hole thermodynamics for the Kerr-Newmann metric. b) Take $Q = 0$. Suppose the black hole loses all of its angular momentum through some classical process. How much of its initial mass can be radiated away? c) Take $J = 0$ and $M > Q$. Suppose we want to reach extremality by throwing charged particles into the black hole. Show that if the particles being thrown in are not themselves black holes, an infinite number of particles need to be thrown in before the black hole becomes extremal.

Solution:

a) The first law of BH thermodynamics is

$$dM = TdS + \Phi dQ + \Omega dJ, \quad (2209)$$

where $S = A/4$. Now for us, the electric potential at the horizon is $\Phi = Er_+ = 4\pi Qr_+/A$. Putting this into the first law, we get

$$dM = \frac{r_+ - M}{2A} dA + \frac{4\pi J}{MA} dJ + \frac{4\pi Qr_+}{A} dQ. \quad (2210)$$

Re-arranging,

$$dA = \frac{8\pi}{r_+ - M} \left((r_+^2 + J^2/M^2) dM - \frac{J}{M} dJ - Qr_+ dQ \right). \quad (2211)$$

This had better agree with our formula for the area. Using the area formula, we have

$$dA = 8\pi(r_+dr_+ + ada) = 8\pi \left[r_+ \left(dM + \frac{MdM - ada - QdQ}{r_+ - M} \right) + ada \right]. \quad (2212)$$

Now

$$ada = \frac{J}{M} \left(\frac{dJ}{M} - \frac{JdM}{M^2} \right), \quad (2213)$$

so that

$$dA = 8\pi \left(dM \left(r_+ + \frac{Mr_+}{r_+ - M} - \frac{J^2}{M^3} + \frac{J^2}{M^3} \frac{r_+}{r_+ - M} \right) + dJ \left(-\frac{r_+}{r_+ - M} \frac{J}{M^2} + \frac{J}{M^2} \right) - QdQ \frac{r_+}{r_+ - M} \right). \quad (2214)$$

After pulling out the common factor of $r_+ - M$, this agrees with the above expression for dA derived from the first law. So, the first law checks out.

b) The second law of BH thermodynamics says that classically, A is monotonically increasing: $A_f \geq A_i$ for some process $i \rightarrow f$. If we want to radiate away as much mass as possible, we should take our initial black hole to be spinning as fast as possible. For $Q = 0$, the extremal solution thus has $a = M$, or $J = M^2$. The initial area is then

$$A_i = 4\pi(M_i^2 + J^2/M_i^2) = 8\pi M_i^2. \quad (2215)$$

On the other hand, when the black hole radiates to the point where $J = 0$, we have the Schwarzschild relation

$$A_f = 4\pi r_+^2 = 16\pi M_f^2. \quad (2216)$$

Setting $A_f = A_i$ to get the biggest possible amount of radiated mass, we get

$$M_f^2 = M_i^2/2, \quad (2217)$$

or $M_f/M_i = 1/\sqrt{2} \approx 0.71$. Thus at most 29 percent of the black hole's initial mass can be radiated away.

c) If the particles with charge, mass q, m we are throwing into the black hole are pushing the black hole closer to extremality, we need $q/m > 1$. But in order for them to fall in we need $qQ < Mm$, so that $1 < q/m < M/Q$. Define

$$\chi \equiv Q/M. \quad (2218)$$

Now

$$\dot{\chi} = \chi(\partial_t \ln Q - \partial_t \ln M) = \frac{q}{M} - \frac{mQ}{M^2}, \quad (2219)$$

where $\dot{Q} = q$ and $\dot{M} = m$ (each timestep is signified by throwing one particle q, m into the BH). Now we re-write this as

$$\dot{\chi} = \chi \frac{m}{M} \left(\frac{q}{m} \chi^{-1} - 1 \right) < \chi \left(\frac{q}{m} \chi^{-1} - 1 \right). \quad (2220)$$

Here we used $m/M < 1$: this is true since if $m \geq M$, the fact that $q > m$ would mean that the "particle" being added to the black hole was a super-extremal black hole itself. Now since $q/m < M/Q$,

$$\dot{\chi} < \chi (\chi^{-2} - 1) = \chi^{-1} - \chi. \quad (2221)$$

This means that the time it takes to bring the black hole up to $Q = M$ is bounded below by

$$\Delta t \geq \int_{\chi_0}^1 d\chi \frac{1}{\chi^{-1} - \chi} = -\frac{1}{2} \int_{\chi_0}^1 \frac{-2\chi d\chi}{1 - \chi^2} = -\frac{1}{2} \ln(1 - \chi^2) \Big|_{\chi_0}^1, \quad (2222)$$

which is divergent for all $\chi_0 < 1$. Thus it takes an infinite amount of time to build up to the point $Q/M = 1$.

126 October 17 — The n -state Potts model

Today is another homework problem, this time from Ashvin's condensed matter class. Consider the n -state Potts model. The Hamiltonian is

$$H_P = -J \left[\sum_j \sum_{l=0}^{n-1} Z_j^{-1} Z_{j+1} + g \sum_j \sum_{l=0}^{n-1} X_j \right]. \quad (2223)$$

Here Z is the diagonal matrix $Z = \text{diag}(1, \omega, \omega^2, \dots, \omega^{n-1})$ for $\omega = e^{2\pi i/n}$, and X is the shift matrix (generator of the alternating part of S_n), which is a “backwards” permutation taking the l th eigenstate of Z to the $l-1$ eigenstate. The commutation relation is

$$ZX = \omega^{-1} XZ. \quad (2224)$$

- a) What is the global symmetry? What are the eigenvalues / eigenstates of X ? What are the ground state(s) in the limit b) $g \rightarrow \infty$ and c) $g \rightarrow 0$? d) What is the critical value g_c for the phase transition? e) Do a mean-field analysis and plot the mean-field energy for a few different g s. f) Minimize the mean field energy as a function of g and locate the phase transition. What order is it?

Solution:

- a) If we think of X as measuring “electric flux”, then eigenstates of X are those with definite electric flux. This means that they are uniform superpositions of the eigenstates of the conjugate variable, Z . We thus compute the eigenstates by Fourier transforming those of Z : they are

$$|Q\rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} e^{iQl} |l\rangle, \quad (2225)$$

where $|l\rangle$ are the eigenstates of Z . Acting with X does $X : |l\rangle \mapsto |l-1\rangle$, so that

$$X|Q\rangle = e^{iQ}|Q\rangle. \quad (2226)$$

The model has an internal \mathbb{Z}_3 symmetry rotating the “spins” on every site by $2\pi/n$, which is performed by the operator $\prod_j X_k$.

b) When $g \rightarrow \infty$, we must work in a state with zero electric flux. Thus the system is in a unique ground state

$$|GS\rangle_{g \rightarrow \infty} = |Q = 0\rangle = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} |l\rangle. \quad (2227)$$

c) When $g \rightarrow 0$, we need to satisfy the interaction term. Since there is no X in the Hamiltonian we can work in the Z eigenbasis. If site j is in the eigenstate $|l_j\rangle$ and site $j+1$ is in $|l_{j+1}\rangle$, then the interaction term between them is

$$\sum_m \omega^{m(l_{j+1}-l_j)} = \delta_{l_j, l_{j+1}}. \quad (2228)$$

Since this term appears with negative coefficient in the Hamiltonian, we want the δ constraint to be satisfied. Thus this forces neighboring sites to be in identical Z eigenstates, and we have n degenerate ground states

$$|GS_l\rangle_{g \rightarrow 0} = \bigotimes_j |l_j\rangle \quad (2229)$$

(here $l_j \in \mathbb{Z}_n$ is the same for all j).

d) Let $j+$ denote the site $j + 1/2$ on the dual lattice and let j_- denote the dual lattice site $j - 1/2$. We define dual operators by

$$\tilde{X}_{j+} = Z_j^{-1} Z_{j+1}, \quad \tilde{Z}_{j-}^{-1} \tilde{Z}_{j+} = X_j. \quad (2230)$$

The latter equation is satisfied if we write $\tilde{Z}_{j\pm}$ as the string

$$\tilde{Z}_{j\pm} = \prod_{j < j_\pm} X_j. \quad (2231)$$

Now we want to compute the commutator between \tilde{X}_{j+} and \tilde{Z}_{k+} . Suppose first that the two dual sites are equal, $j+ = k+$. To move \tilde{X}_{j+} to the right of \tilde{Z}_{j+} , we need to pass the two Z 's in the definition of \tilde{X} through the chain of X operators created by \tilde{Z} . Since we put the string in \tilde{Z} “to the left”, the Z_{j+1} operator goes through for free, while the Z_j^{-1} operator picks up an ω . Thus we have

$$\tilde{X}_{j+} \tilde{Z}_{k+} = \omega \tilde{Z}_{j+} \tilde{X}_{j+}. \quad (2232)$$

Now suppose that the two dual operators are not at the same site. If the \tilde{X} is at a site to the left of the \tilde{Z} , it commutes though for free since the operators involved act on different \otimes factors. If it is to the left of the \tilde{Z} , then the Z it contains picks up a factor of ω^{-1} , while the Z^{-1} picks up a factor of ω which cancels the ω^{-1} , and so the two operators commute. Thus the commutation relation is

$$\tilde{X}_{j+} \tilde{Z}_{k+} = \omega^{\delta_{j,k}} \tilde{Z}_{k+} \tilde{X}_{j+}, \quad (2233)$$

which is the same relation as the one the X, Z operators satisfy.

We know that there's a phase transition at some g_c because the ground state degeneracy must change. Note that in terms of the dual variables, the Hamiltonian is

$$H = -J \sum_j \left[g \sum_{l=0}^{n-1} \tilde{Z}_j^{-l} \tilde{Z}_{j+1}^l + \sum_{l=0}^{n-1} \tilde{X}_j \right], \quad (2234)$$

which is the same as the original Hamiltonian when $g = 1$. Therefore we identify $g_c = 1$ as the self-dual point where the phase transition is likely to happen.

e) We take the mean field ansatz

$$|\Psi(x_1, \dots, x_n)\rangle = \bigotimes_j |\hat{n}(\{x_i\})\rangle_j, \quad |\hat{n}(\{x_i\})\rangle = \sum_{i=1}^n x_i |0\rangle, \quad \vec{x} \in S^{n-1}. \quad (2235)$$

Here the fact that $\vec{x} \in S^{n-1}$ ensures that $|\Psi\rangle$ is normalized correctly.

What is the variational energy? Let's first look at the X term. Because of the sum over l , this is a projector onto the eigenstate of X with eigenvalue 1. In the basis where Z is diagonal, this is a projector onto the uniform sum $\sum_j |j\rangle$. It is represented by a matrix with 1 in every entry. Thus

$$\langle \Psi(\vec{x}) | X | \Psi(\vec{x}) \rangle = \left(\sum_j x_j \right)^2. \quad (2236)$$

Now for the nearest neighbor term. Indexing the entries of $Z_j^{-l} \otimes Z_{j+1}^l$ as (a, b) , we see from $\sum_j \omega^{jk} \propto \delta k, 0$ that the matrix $Z_j^{-l} \otimes Z_{j+1}^l$ is diagonal with entries $(a, b) = \omega^{l(a-b)}$. Summing over l gives a non-zero result only when $a = b$, and so $\sum_l Z_j^{-l} \otimes Z_{j+1}^l$ has entries $(a, b) = \delta_{a,b} n$. Thus

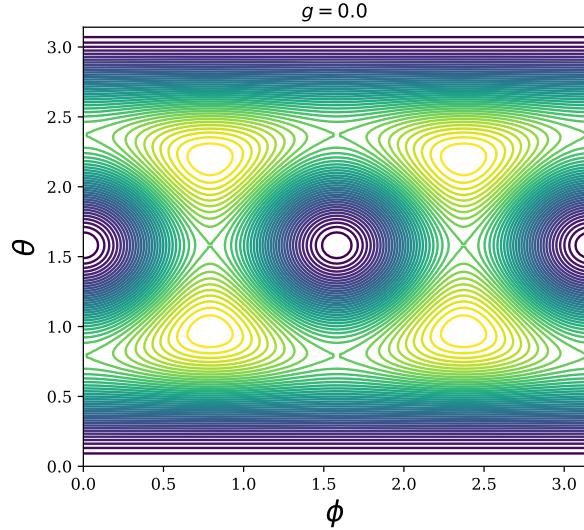
$$\langle \Psi(\vec{x}) |_j \otimes \langle \Psi(\vec{x}) |_{j+1} \sum_l Z_j^{-l} \otimes Z_{j+1}^l | \Psi(\vec{x}) \rangle_j \otimes | \Psi(\vec{x}) \rangle_{j+1} = n \sum_i x_i^4. \quad (2237)$$

Putting this together, on a length L chain we just get the contribution above for every j , and so the variational energy is

$$E(\vec{x}) = -JN \left[n \sum_i x_i^4 + g \left(\sum_j x_j \right)^2 \right]. \quad (2238)$$

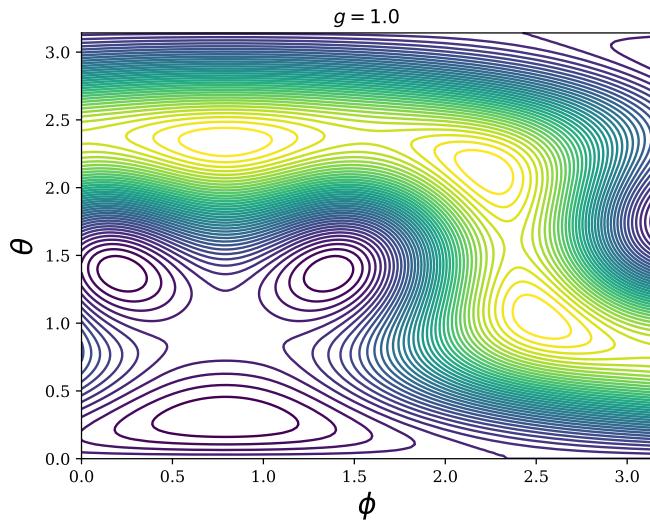
Now we specialize to the case of the 3-state Potts model, parametrizing the coordinates on S^2 by θ, ϕ . Only half of the sphere gives a physically distinct wavefunction, since $|\Psi\rangle$ and $-|\Psi\rangle$ are equivalent minima. We will choose the hemisphere $(\theta, \phi) \in [0, \pi]^2$. Setting $JN = 1$,

when $g = 0$ $E(\theta, \phi)$ looks like

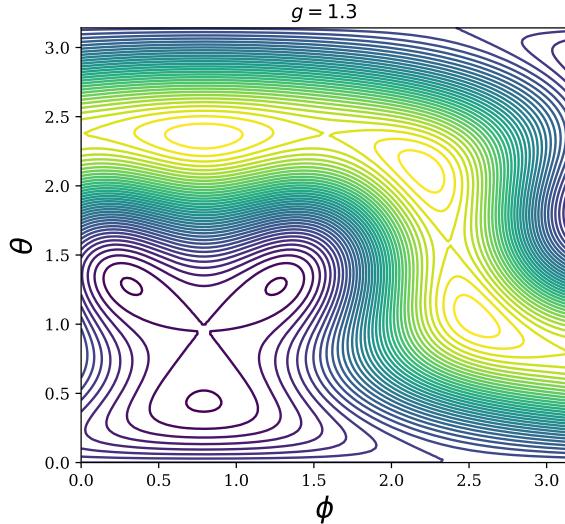


We see that there are three distinct minima, in agreement with what we predicted earlier (there is a single minimum at the top of the figure (the point $(0, 0, 1)$), which is identified with the minimum at the bottom of the figure. Likewise, the two minima at $\phi = 0, \pi$ are identified). They are the eigenstates of the Z operators, and consequently appear at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

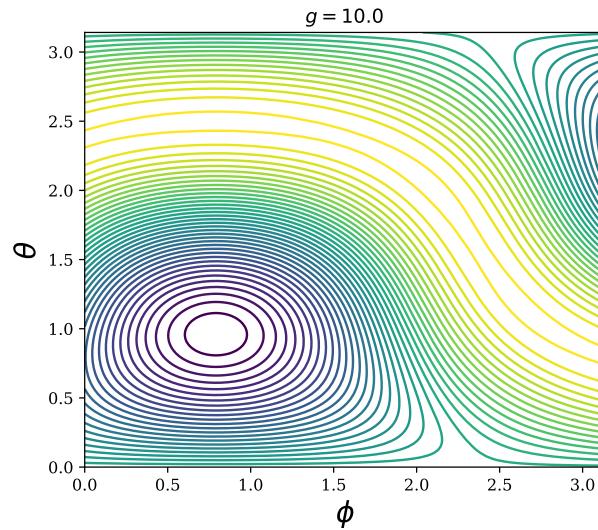
When we set $g = 1$ the plot looks like



The three minima start to merge as g is increased:

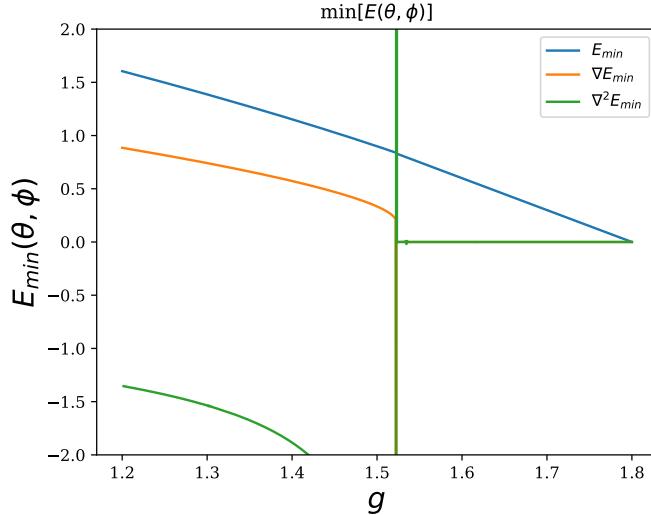


When g is large we need to get a \otimes state from satisfying the $\sum_l X^l$ term. Since this state will be one in the eigenvalue 1 eigenstate of X , it will have coordinates on the sphere $x = y = z$. Thus we expect a single minimum at $\theta = \pi/4, \phi = \pi/4$. Indeed, this is what happens at large g :



f) We now plot $\min[E(\theta, \phi)]$, again with $JN = 1$. The interesting behavior happens

around $g = 1.5$:



Here we have subtracted constants from E_{\min} and ∇E_{\min} so that they both hit 0 at $g = 1.8$. The behavior for $g < 1.2$ and $g > 1.8$ is what you would expect from the figure. We see that we have a second order phase transition at $g \approx 1.5225$, since $\nabla^2 E_{\min}$ is singular there. We know the self-dual point is at $g = 1$, so we see that mean-field theory over-estimates the critical value of g . This is what we expect from a mean-field treatment: the mean-field ansatz neglects fluctuations and assumes a \otimes state, so that it is biased towards ordered states (for us, small g).

127 October 18 — Symmetries of the Hubbard model

Consider the Hubbard model in arbitrary dimensions on a bipartite lattice. What are its global symmetries? Pay special attention to half filling.

Solution:

We will work in one dimension for simplicity, although the generalization is straightforward as long as the lattice is bipartite. The Hubbard model Hamiltonian is

$$H = -t \sum_{i\sigma} (c_{i\sigma}^\dagger c_{i+1,\sigma} + h.c.) + U \sum_i n_{i\uparrow} n_{i\downarrow} - \sum_{i\sigma} \mu n_{i\sigma}. \quad (2239)$$

Up to a constant, we can also write it as

$$H = -t \sum_{i\sigma} (c_{i\sigma}^\dagger c_{i+1,\sigma} + h.c.) + U \sum_i (n_{i\uparrow} - u)(n_{i\downarrow} - u), \quad u \equiv \mu/U. \quad (2240)$$

Now define Majoranas by

$$c_{i\sigma} = \frac{\chi_{i\sigma} + i\eta_{i\sigma}}{2}. \quad (2241)$$

Here the $1/2$ means that the Majoranas satisfy the Clifford algebra relation with the 2 on the RHS, in order for $\{c, c^\dagger\}$ to work out properly. Then the number operator is

$$n_{i\sigma} = \frac{1}{2}(1 + i\chi_{i\sigma}\eta_{i\sigma}) = \frac{1}{4}(2 + i\lambda_{i\sigma}^T iY \lambda_{i\sigma}), \quad (2242)$$

where $\lambda = (\chi, \eta)^T$. This means the potential term is

$$U \sum_i (n_{i\uparrow} - u)(n_{i\downarrow} - u) = U \sum_i \left(\alpha + \frac{i}{4} \lambda_{i\uparrow}^T iY \lambda_{i\uparrow} \right) \left(\alpha + \frac{i}{4} \lambda_{i\downarrow}^T iY \lambda_{i\downarrow} \right), \quad (2243)$$

where we have defined

$$\alpha \equiv \frac{1}{2} - \frac{\mu}{U}. \quad (2244)$$

We also need the Majorana form for the kinetic term, which is

$$-t \sum_{i\sigma} (c_{i\sigma}^\dagger c_{i+1,\sigma} + h.c.) = -\frac{ti}{2} \sum_i \Lambda_i^T (iY \oplus iY) \Lambda_{i+1}, \quad \Lambda_i \equiv (\lambda_{i\uparrow}, \lambda_{i\downarrow})^T. \quad (2245)$$

Now, what are the global symmetries of H ? Regardless of what μ is, H conserves spin and particle number, so we have at least a $G = (SU(2) \times U(1))/\mathbb{Z}_2$ symmetry (this is a global symmetry and not a local one because of the kinetic term, and the $1/\mathbb{Z}_2$ is needed to avoid over-counting the common \mathbb{Z}_2 subgroup of $SU(2)$ and $U(1)$). These are (as far as I can tell) the only symmetries of the model for generic μ .

Consider now the special case when $\mu = U/2$ so that $\alpha = 0$. Now

$$(\lambda_{i\sigma}^T iY \lambda_{i\sigma})^2 = 4(\chi_{i\sigma}\eta_{i\sigma})^2 = -4. \quad (2246)$$

This means that when $\alpha = 0$, the potential term is, up to a constant,

$$-\frac{U}{16} \sum_i (\lambda_{i\uparrow}^T iY \lambda_{i\uparrow} + \lambda_{i\downarrow}^T iY \lambda_{i\downarrow})^2 = -\frac{U}{16} \sum_i (\Lambda_i^T (iY \oplus iY) \Lambda_i)^2. \quad (2247)$$

Define $A \equiv iY \oplus iY$, so that at $\alpha = 0$ the Hamiltonian is

$$H = -\frac{it}{2} \sum_i \Lambda_i^T A \Lambda_{i+1} - \frac{U}{16} \sum_i (\Lambda_i^T A \Lambda_i)^2. \quad (2248)$$

Consider $\Lambda_i \mapsto \mathcal{O}_i \Lambda_i$. In order to preserve the commutation relations for the Majoranas, we need $\mathcal{O} \in O(4)$. Consider first the case when $\mathcal{O}_i = \mathcal{O}$ is the same on every site. Then we also need $\mathcal{O}^T A \mathcal{O} = A$, i.e. we need

$$\mathcal{O} \in O(4) \cap Sp(4; \mathbb{R}) \cong U(2) \cong (SU(2) \times U(1))/\mathbb{Z}_2. \quad (2249)$$

This is the spin rotation + charge conservation symmetry we found before.

However, there is another sneaky symmetry. Because the potential term has the square of the bilinear form, we can also let $\mathcal{O}_i^T A \mathcal{O}_i = -A$, provided that we choose \mathcal{O}_{i+1} such that $\mathcal{O}_i^T A \mathcal{O}_{i+1} = A$. This can be done by staggering the operators by taking $\mathcal{O}_i = -\mathcal{O}_{i+1}$. Now, let $\tilde{\mathcal{O}} \in O(4) \cap Sp(4; \mathbb{R})$, and define

$$\mathcal{O}_i = (-1)^i \mathcal{Z} \tilde{\mathcal{O}}, \quad \mathcal{Z} \equiv Z \oplus Z. \quad (2250)$$

Multiplying $\tilde{\mathcal{O}}$ by \mathcal{Z} implements a particle-hole conjugation in addition to the rotation performed by $\tilde{\mathcal{O}}$. Since a particle-hole conjugation doesn't leave the kinetic term invariant (it changes it by a minus sign), we add the compensating factor of $(-1)^i$, which ensures that $\mathcal{O}_i^T A \mathcal{O}_{i+1} = A$. Since the definition of \mathcal{O}_i can be done for each $\tilde{\mathcal{O}} \in U(2)$, this gives us another $U(2)$ "pseudospin" symmetry. Note that this only works when $\mu = U/2$ (this choice favors $\langle n_{i\sigma} \rangle = 1/2$ for both σ , which corresponds to an average occupation of one particle per site: this is half-filling). Thus at half-filling we have two $U(2)$'s, while away from half-filling we just have the $SU(2) \times U(1)$.

We can also look at this symmetry from a bosonization point of view. Recall how we would bosonize the $U = 0$ theory: working at long distances we can write $c_j = e^{ik_F j} \psi_{R,j} + e^{-ik_F j} \psi_{L,j}$ and keep the single-derivative terms, giving us an action of two Dirac fermions: $S \propto \int \sum_\sigma \bar{\Psi}_\sigma \not{\partial} \Psi_\sigma$, where $\Psi = (\psi_L, \psi_R)^T$. This gets bosonized to the $k = 1$ WZW model (which keeps track of the $SU(2)$ spin sector), plus a decoupled free boson that represents the $U(1)$ charge sector. A check that $k = 1$ is the right level is that the central charge of the $SU(2)_k$ WZW model is $c = 3k/(k+2)$, which gives the correct $c = 1$ to combine with the free $U(1)$ sector to produce a total of $c = 2$ when $k = 1$, matching the two Dirac fermions. From the way the WZW model is constructed, we have a manifest $(SU(2) \times SU(2))/\mathbb{Z}_2$ symmetry by $g \mapsto V^\dagger g U$, plus a $U(1)$ symmetry for the compact boson.

What happens when we turn on U , for an arbitrary filling (chemical potential)? We need to know what the U term maps to under bosonization. We can get the form of the answer by symmetry: the U term preserves $U(1)$ particle number conservation, as well as the diagonal $SU(2)$ symmetry which rotates the spins (diagonal in that it acts identically on ψ_L and ψ_R). First, define the currents

$$j_s = \psi_{s,\alpha}^\dagger \psi_{s,\alpha}, \quad J_s^a = \frac{1}{2} \psi_{s,\alpha}^\dagger \sigma_{\alpha\beta}^a \psi_{s,\beta}, \quad s \in \{L, R\}. \quad (2251)$$

Recall from some diary entry way back that the $U(1)$ currents bosonize to the charge sector boson ϕ as

$$j_L \mapsto -i\bar{\partial}\phi, \quad j_R \mapsto i\partial\phi, \quad (2252)$$

while the $SU(2)$ currents go to (the numerical factors are almost for sure wrong, but this won't matter now)

$$J_L^a \mapsto -i\text{Tr}[(\bar{\partial}g)g^\dagger \sigma^a], \quad J_R^a \mapsto i\text{Tr}[g\partial g^\dagger \sigma^a]. \quad (2253)$$

So, what kind of interactions can appear while preserving the diagonal $U(1)$ and $SU(2)$? We can have $j_s j_s$ terms, as well as $J_s^a J_s^a$ terms. Since the stress tensor is $T \sim v : \partial\phi\partial\phi :$ (and likewise for \bar{T} , and likewise for the WZW sector), adding these terms just changes the velocity v and doesn't affect the WZW theory. Up to four fermion terms, we can add the current interactions

$$S_{current} = \int (\gamma_1 j_L j_R + \gamma_2 J_L^a J_R^a). \quad (2254)$$

The γ_1 term gets bosonized to the free action for ϕ and so just contributes to a change in the radius of the $U(1)$ charge sector boson. The γ_2 term (which is invariant under the diagonal $SU(2)$ since the $SU(2)$ rotation preserves the dot product $J_L^a J_R^a$) is more serious and we will examine it separately. In addition to these, we can also get something invariant under the

$SU(2)$ by using the intertwiner (invariant symbol) coming from $2 \otimes 2 \ni \mathbf{1}$. The invariant symbol is $\epsilon^{\alpha\beta}$ since

$$\epsilon^{\alpha\beta} \mapsto U_{\alpha\gamma} U_{\beta\delta} \epsilon^{\gamma\delta} = \epsilon^{\alpha\beta} \det U = \epsilon^{\alpha\beta}, \quad (2255)$$

where the second-to-last step just uses the definition of the determinant (recall how $\det U$ is a product of entries of U where no two elements in any given row or column are multiplied together). Thus $\psi_{s,\alpha} \epsilon^{\alpha\beta} \psi_{s,\beta}$ is invariant under the $SU(2)$ spin symmetry (one of the ψ 's can't be daggered since otherwise the bilinear wouldn't be invariant). To make this invariant under the diagonal $U(1)$ we need to add a corresponding term with $\psi_{s',\alpha}^\dagger$ s. If we choose $s' = s$ then we again get a redefinition of the velocity v , and so the most general interesting term along these lines we can add is

$$S_\epsilon = \int \gamma_3 \left(\epsilon^{\alpha\beta} \psi_{L,\alpha}^\dagger \psi_{L,\beta}^\dagger \cdot \epsilon^{\gamma\lambda} \psi_{R,\gamma} \psi_{R,\lambda} + h.c. \right). \quad (2256)$$

Under bosonization, we use

$$\psi_{L,\alpha}^\dagger \psi_{R,\beta} \mapsto g_{\alpha\beta} e^{i\phi}, \quad (2257)$$

where the $g_{\alpha\beta}$ keeps track of the spins and the $e^{i\phi}$ is because the ϕ parts are already normal-ordered since ψ_L^\dagger, ψ_R contract to give zero. Thus the γ_3 term maps as

$$\gamma_3 \left(\epsilon^{\alpha\beta} \psi_{L,\alpha}^\dagger \psi_{L,\beta}^\dagger \cdot \epsilon^{\gamma\lambda} \psi_{R,\gamma} \psi_{R,\lambda} + h.c. \right) \mapsto \gamma_3 g_{\alpha\gamma} g_{\beta\lambda} \epsilon^{\alpha\beta} \epsilon^{\gamma\lambda} \cos \phi = \gamma_3 \cos \phi, \quad (2258)$$

since $\det g = 1$. Thus the γ_3 term is actually g -independent and it induces a mass for ϕ (it is relevant, which can be seen by re-scaling the kinetic term for ϕ , which was changed from the canonical normalization by the γ_1 term), which causes the $U(1)$ CDW part to be Mott-insulated away and disappear from the IR physics.

A final possible term that could appear which is allowed by the diagonal $U(1) \times SU(2)$ is the relevant $\text{Tr}[g]$, but this is actually not allowed as long as we preserve translation symmetry, which since it maps $\psi_L \mapsto i\psi_L$ and $\psi_R \mapsto -i\psi_R$ sends $g \mapsto -g$, and forbids $\text{Tr}[g]$. Since we have a WZW model at $k = 1$ only degree 1 polynomials in g are independent, and so we don't have to worry about a $\text{Tr}[g^2]$ term (which dies by Fermi statistics).

So we have accounted for all the possible things that the U term can add except for the γ_2 term, which is now the only term remaining that can reduce the spin part of the symmetry down from $SU(2)_L \times SU(2)_R$. Since the γ_2 term breaks the $SU(2)_L \times SU(2)_R$ symmetry of the $U = 0$ free theory down to the diagonal $SU(2)$, the symmetry of the theory at long distances will depend on what γ_2 flows to in the IR. Now since the $J_{L/R}^a$ s are currents, they have dimension 1 and hence the γ_2 term is marginal. We can compute the leading term in the β function for gamma since we know the current-current OPE. Recall from an earlier diary entry that for a marginal perturbation $\sum_i \int \lambda_i \mathcal{O}_i$, we have, in an appropriate normalization,

$$\beta_{\lambda_i} = \frac{d\lambda_i}{d \ln \Lambda} = -C_{ijk} \lambda_j \lambda_k + O(\lambda^3). \quad (2259)$$

Here C_{ijk} is the coefficient of \mathcal{O}_i appearing in the OPE of \mathcal{O}_j with \mathcal{O}_k . Since the $J_{L/R}^a$ s are currents, we can get their OPEs immediately from their commutation relations (write the commutator of the charge operator created by integrating the current with a local operator, deform the commutator to a contour integral enclosing the local operator, and use the OPE).

Indeed, suppose under J_s^a we have $[Q_s^a, \mathcal{O}_i] = R_s^a \mathcal{O}_i$, where R_s^a is some representation of the symmetry defined by the current and $Q_s^a = \oint dz J_s^a(z)$. Then following the instructions in the above parenthetical, we get the OPE

$$J_s^a(z) \mathcal{O}_i(w) = \frac{1}{2\pi i} \frac{R_s^a \mathcal{O}_i}{z - w} + \dots \quad (2260)$$

In particular, when we take $\mathcal{O}_i(w)$ to be one of the $SU(2)$ currents, we recover the correct OPE (again, we are not really paying attention to numerical factors)

$$J_s^a(z) J_s^b(w) \supset \epsilon^{abc} \frac{J_s^c(z)}{z - w}. \quad (2261)$$

This means that the β function for γ_2 is

$$\beta_{\gamma_2} \propto \gamma_2^2. \quad (2262)$$

Since we are using the “high-energy” definition of the β function, this means that provided γ_2 is positive, γ_2 decreases as we flow to the IR, and so adding the γ_2 term actually does nothing in the IR: we just flow back to the $\gamma_2 = 0$ WZW fixed point, provided that there are no other intermediate fixed points along the way. This means that the $SU(2)_L \times SU(2)_R$ symmetry is actually likely to survive in the IR for the Hubbard model, even at finite U !

The last thing remaining to do is to check that γ_2 is indeed positive for the Hubbard model. This can be checked to indeed be the case (actually, I think that we have the simple $\gamma_2 = U$) by expanding out the definition of J_s^a and using the identity $\sigma_{\alpha\beta}^a \sigma_{\gamma\lambda}^a = 2\delta_{\alpha\gamma}\delta_{\beta\lambda} - \delta_{\alpha\beta}\delta_{\gamma\lambda}$.

128 October 19 — Precession of Mercury’s orbit from EFT

This is a fun one from Schwartz. Write the Lagrangian for GR as

$$\mathcal{L} = M_P^2 \left(-\frac{1}{2} h \square h + h^2 \square h \right) - h T, \quad (2263)$$

where $h = h_{00}$, with $h_{\mu\nu}$ the first-order perturbation to $g_{\mu\nu}$ away from $\eta_{\mu\nu}$, and $T = T_{00}$ is (we are in $\hbar = c = 1$ units)

$$T = M \delta^3(x), \quad (2264)$$

with M the mass of the sun. Looking only at T_{00} and h_{00} is valid here since we know the Schwarzschild solution only differs from $\eta_{\mu\nu}$ in the dt^2 and dr^2 parts, and we can focus only on the dt^2 part since we are interested in computing GR-induced changes to the gravitational potential set up by the sun.

Anyway, solve for h to $O(\lambda^4)$, where $\lambda \equiv 1/M_P = \sqrt{G_N}$ is the inverse planck mass in natural units. From the resulting gravitational potential, find the precession of Mercury’s orbit.

Solution:

First re-scale h by $h \mapsto M_P h = h/\lambda$, so that

$$\mathcal{L} = -\frac{1}{2}h\Box h + \lambda h^2\Box h - \lambda hT. \quad (2265)$$

We will do perturbation theory in λ . The equations of motion are (up to a $(\partial h)^2$ term that Schwartz neglects for some reason?)

$$\Box h = \lambda\Box(h^2) - \lambda T. \quad (2266)$$

We just want to get an idea for how the calculation works — we won't be keeping too careful track of numerical factors. Now write $h = h_0 + \lambda h_1 + \dots$. The lowest order gives us $h_0 = 0$ and (all the action is taking place in space, [we are assuming a static situation] so \Box is really $\partial_i\partial^i$, in mostly-negative signature)

$$h_1(x) = - \int d^3y \left[\frac{1}{\Box} \right]_{x,y} T(y) = \int d^3y \frac{1}{|x-y|} M\delta^3(y) = \frac{M}{x}. \quad (2267)$$

When we restore units, this gives us the regular Newtonian potential. The next order in λ tells us that $h_2 = 0$, and the order after that tells us that

$$\Box h_3 = \Box(h_1^2) \implies h_3 = \left(\frac{1}{\Box} T \right)^2 = \frac{M^2}{r^2}. \quad (2268)$$

Restoring h so that it is dimensionless (in natural units), we get

$$h(r) \approx \lambda^2 \frac{M}{r} + \lambda^4 \frac{M^2}{r^2}. \quad (2269)$$

Restoring units gives the Newtonian potential. Thus the potential energy for a Mercury of mass m is the Newtonian one, plus a $1/r^2$ term:

$$V(r) \approx -\frac{G_N M m}{r} \left(1 + \frac{G_N M}{c^2 r} \right), \quad (2270)$$

where we have restored the c^2 on dimensional grounds. Thus the correction goes as $\approx 3.3\text{km}/r$, which is clearly tiny, even for Mercury.

Now we want to find the correction this induces to the precession of Mercury's perihelion radius, which is a fun walk back through undergraduate physics. The kinetic term in the Lagrangian is $\frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$, where (r, ϕ) are the coordinates in the plane of Mercury's orbit. This of course leads to the conservation of $L = mr^2\dot{\phi}$. The eom for r is

$$md_t^2r = md_t(\dot{\phi}\partial_\phi r) = m(\partial_t^2\phi\partial_\phi r + \dot{\phi}^2\partial_\phi^2 r) = \frac{L^2}{mr^3} - \partial_r V(r). \quad (2271)$$

Since there are a lot of $1/rs$, we define $\rho \equiv 1/r$. We want to take the above and use angular momentum conservation to get an equation for $\partial_\phi^2 u$. We use

$$\partial_t^2\phi\partial_\phi r + \dot{\phi}^2\partial_\phi^2 r = -\frac{\partial_t^2\phi}{u^2}\partial_\phi u - \dot{\phi}^2 \left(\partial_\phi^2 u \frac{1}{u^2} - \frac{\dot{\phi}^2}{u^3}(\partial_\phi u)^2 \right) = -\frac{\dot{\phi}^2}{u^2}\partial_\phi^2 u - (\partial_t L)\partial_\phi u = -\frac{u^2 L^2}{m^2}\partial_\phi^2 u. \quad (2272)$$

This means

$$\partial_\phi^2 u = -u - \frac{m}{L^2} \partial_u V(u), \quad (2273)$$

which for us gives

$$\partial_\phi^2 u = -u + \frac{m^2}{L^2} GM \left(1 + \frac{2GM}{c^2} u \right), \quad (2274)$$

which we write as

$$\partial_\phi^2 u = u(\Gamma - 1) + \alpha, \quad \Gamma = 2m^2 G^2 M^2 / (c^2 L^2), \quad \alpha = m^2 G M / L^2. \quad (2275)$$

Now we can solve this: $\Gamma - 1 < 0$ since $\Gamma \approx (3.3\text{km})^2 c^2 / (r^2 v_m^2) \ll 1$ where r is Mercury's semimajor axis and v_m is Mercury's typical velocity around the sun ($r \sim 6 \times 10^7 \text{km}$), and so

$$u(\phi) = A \cos \left(\phi \sqrt{1 - \Gamma} \right) + \frac{\alpha}{1 - \Gamma}. \quad (2276)$$

We let Mercury be at perihelion when $\phi = 0$. When ϕ goes to 2π , we see that u does not exactly come back to $u(\phi = 0)$, because of the square root in the cosine. This causes the location of Mercury's perihelion to rotate. The second perihelion instead happens when

$$\phi = \frac{2\pi}{\sqrt{1 - \Gamma}} \approx 2\pi(1 + \Gamma/2) \implies \delta\phi \approx \frac{2\pi m^2 G^2 M^2}{c^2 L^2}. \quad (2277)$$

Putting in numbers for Mercury, this gives a precession of $\sim 26''$ per century, which is about half of what the actual answer is ($43''$ per century). Given that we weren't paying attention to constants when getting the eom, this isn't bad! Same order for magnitude, at least.

129 November 3 — SF Mott insulator transition with 2d compact boson

Consider the 1+1 dimensional Bose Hubbard model in the following form:

$$H = \frac{U}{2} \sum_i n_i^2 + \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j). \quad (2278)$$

When can we expand the cosine to quadratic order? When we can, what is the correlator for the ϕ vertex operators? When does the term $\cos(p\phi)$ modify the theory significantly?

Solution:

We will first expand the cosine to quadratic order and calculate the vertex operator correlation function, and then go back to see when doing this makes sense. Note that when we expand the cosine in this way the zero momentum mode of ϕ doesn't appear in the Hamiltonian, so that the zero-momentum part of H is just $U n_{q=0}^2 = \frac{U}{L^2} N^2$, where N is the total number of particles and L^2 is the spacetime volume. This has non-zero compressability

of $\chi = 1/U$, and so we know that it cannot be in the incompressible Mott phase. Thus the harmonic expansion is for sure only valid in the SF phase.

Anyway, turning the cosine into $\frac{1}{2}a^2(\partial\phi)^2$ and then turning $\sum a^2$ into an integral, we see that the continuum Hamiltonian goes to

$$H = \frac{1}{2} \int dx (Un^2 + J(\partial_x\phi)^2). \quad (2279)$$

Define $\chi = U^{-1}$ as the susceptibility. This is the susceptibility since $\chi^{-1} = \partial_n\mu = \partial_n^2 H = U$. Now from $\partial_t\phi = \partial_n H$, we get $\partial_t\phi = nU$. Putting this in to get the Lagrangian and changing the sign of the $(\partial_x\phi)^2$ term upon passing to Euclidean signature, we get

$$S_E = \frac{1}{2} \int (\chi(\partial_t\phi)^2 + J(\partial_x\phi)^2) = \frac{K}{2} \int (c^{-1}(\partial_t\phi)^2 + c(\partial_x\phi)^2), \quad (2280)$$

where $c \equiv \sqrt{J/\chi}$, $K \equiv \sqrt{J\chi}$. The propagator is then

$$G(r) = -\frac{1}{2\pi K} \ln |(r+a)m|, \quad (2281)$$

where a is a short distance cutoff and m is a mass which ensures the argument of the log is properly dimensionless. This propagator works since

$$-K\Box \frac{1}{2\pi K} \int d^2y \ln[(|x-y|+a)m]\phi(x) = \frac{1}{2\pi} \int d^2y \partial_\mu \frac{(x-y)^\mu}{(|x-y|+a)^2} \phi(x) = \phi(y), \quad (2282)$$

after we send $a \rightarrow 0$ (here $\Box = \partial_\mu \partial^\mu$ with c absorbed into the derivative). Thus

$$\langle e^{i\phi(r)} e^{-i\phi(0)} \rangle = \exp \left(\int d^2x d^2y \delta(x-r) G(x-y) \delta(y) \right) = \exp \left(-\frac{1}{2\pi K} \ln[m(r+a)] \right) = \frac{1}{(r/a)^{1/2\pi K}}, \quad (2283)$$

where we have taken $m \rightarrow a^{-1}$. Thus if we define the dimensionful vertex operators $V(r) = e^{i\phi(r)} a^{-1/4\pi K}$, we have

$$\langle V(r) V^*(0) \rangle = \frac{1}{r^\eta}, \quad \eta = \frac{1}{2\pi K}. \quad (2284)$$

The above has been derived under the assumption that ϕ is non-compact. Restoring ϕ to be compact means including vortex configurations in the path integral. If these vortex configurations are relevant, they will lead to an exponentially decaying correlator for the ϕ vertex operators, and the above calculation will not be valid. Now the vortices are created by the dual vertex operators $\tilde{V} = e^{i\theta}$, so accounting for minimal (winding number ± 1) vortices can be done by adding $\cos\theta$ to the action. Note that here we are using conventions where $\theta(x) = 2\pi \int_{-\infty}^x n$, so that $e^{i\theta}$ creates a 2π vortex. This means that $\frac{1}{2\pi}\partial_x\theta$ is conjugate to ϕ , since it is equal to n . Likewise, it means that the conjugate momentum for θ is $\pi_\theta = \partial_x\phi/2\pi$. Thus we can re-write the Hamiltonian as

$$H = \frac{1}{2} \int dx \left(\frac{1}{4\pi^2\chi} (\partial_x\theta)^2 + 4\pi^2 J \pi_\theta^2 \right). \quad (2285)$$

Going to the Euclidean Lagrangian,

$$S_E = \frac{1}{2} \int \left(\frac{(\partial_t\theta)^2}{4\pi^2 J} + \frac{1}{4\pi^2\chi} (\partial_x\theta)^2 \right). \quad (2286)$$

Rewriting this in the same way as before,

$$S_E = \frac{\tilde{K}}{2} \int (c^{-1}(\partial_t \theta)^2 + c(\partial_x \theta)^2), \quad \tilde{K} \equiv \frac{1}{4\pi^2 \sqrt{\chi J}} = \frac{1}{4\pi^2 K}. \quad (2287)$$

To determine whether the power-law result for the ϕ vertex operators is valid, we need to determine whether $\cos \theta$ is relevant. So, we need the scaling dimension of the (dimensionful) $\tilde{V}(r)$ operators, which can be calculated using the above action. We see that

$$\langle \tilde{V}(r) \tilde{V}(0) \rangle = \frac{1}{r^\eta}, \quad \eta = \frac{1}{2\pi \tilde{K}} = 2\pi K. \quad (2288)$$

Thus the scaling dimension of the vortex operator is

$$\Delta_{\cos \theta} = \pi K, \quad (2289)$$

and so our treatment of ϕ as non-compact (and the form for the $V(r)$ correlator) is only valid if $\Delta_{\cos \theta} > 2 \implies K > 2/\pi$.

Now let us add $\Delta H = -\lambda \sum_i \cos(p\phi_i)$ to the action. The universal properties of the transition will remain unaffected if this term is irrelevant. The cosine term consists of vertex operators of the form $e^{ip\phi}$. According to our discussion above, they have the two-point function

$$\langle e^{ip\phi(r)} e^{-ip\phi(0)} \rangle = \frac{1}{(r/a)^{p^2/2\pi K}}. \quad (2290)$$

Their scaling dimension is accordingly

$$\Delta_p = \frac{p^2}{4\pi K}. \quad (2291)$$

Setting $\Delta_p = 2$ gives the condition for the operator to be marginal, and so the $\cos(p\phi)$ term will be irrelevant provided that

$$p > \sqrt{8\pi K}. \quad (2292)$$

130 November 4 — Weakly interacting bosons

Consider bosons interacting through a potential V that is weak enough so that the bosons are mostly condensed. Write the boson annihilation operator as $\psi(r) = \psi + \phi(r)$, where the $\phi(r)$ are fluctuations about the condensate. Working quadratically in ϕ , diagonalize the Hamiltonian and find the dispersion.

Solution:

Let $\psi(r) = \psi + \phi(r)$, where ψ is a constant parametrizing the condensate and $\phi(r) \equiv \delta\psi(r)$ represents fluctuations about the condensate. Expanding the Hamiltonian to quadratic order

in ϕ , we have (after normal-ordering and assuming $V(r) = V(|r|)$)

$$H = \int_r \left[\frac{|\nabla\phi|^2}{2m} - \mu(\psi^*\phi + \psi\phi^\dagger) - \mu|\psi|^2 - \mu|\phi|^2 \right] + \int_{r,r'} V(r-r') \left(\frac{1}{2}|\psi|^4 + |\psi|^2(\psi\phi(r')^\dagger + \psi^*\phi(r')) \right. \\ \left. + |\psi|^2|\phi(r')|^2 + \frac{1}{2} \left(\psi^2\phi(r)^\dagger\phi(r')^\dagger + (\psi^*)^2\phi(r)\phi(r') \right) + |\psi|^2\phi(r)^\dagger\phi(r') \right) + \mathcal{O}(\phi^3) \quad (2293)$$

We want to choose ψ so that $\langle\phi(r)\rangle = 0$. Linear terms in ϕ are currents which source an expectation value for ϕ , and so in order to ensure that this expectation value vanishes we need the Hamiltonian to be independent of terms linear in ϕ, ϕ^\dagger . Taking a look at e.g. the ϕ^\dagger piece, we see that for the terms in $\delta H/\delta\phi^\dagger(r)$ which are c numbers to vanish for all r , we need

$$\mu\psi = \int_{r'} V(r')|\psi|^2\psi \implies \psi = \sqrt{\frac{\mu}{V_0}}e^{i\alpha}, \quad \alpha \in [0, 2\pi), \quad V_0 = \int_{r'} e^{-i0\cdot r'}V(r'). \quad (2294)$$

We have a $U(1)$'s worth of choices for the phase of ψ , and wolog we will take it to be real by setting $\alpha = 0$. Thus the “condensate strength” is determined by the chemical potential and the $q = 0$ component of the potential.

Putting in this value for ψ , the Hamiltonian is (dropping constants)

$$H = \int_r \frac{|\nabla\phi|^2}{2m} + \frac{\mu}{V_0} \int_{r,r'} V(r-r') \left[\phi(r)^\dagger\phi(r') + \frac{1}{2} \left(\phi(r)^\dagger\phi(r')^\dagger + \phi(r)\phi(r') \right) \right], \quad (2295)$$

where the $-\mu|\phi|^2$ part has canceled with one of the potential terms (the fluctuations have no chemical potential). We write this in momentum space by taking $\phi(r) = \int_q e^{iqr}\phi_q$, so that

$$H = \int_q \frac{q^2}{2m} \phi_q^\dagger\phi_q + \frac{\mu}{V_0} \int_{q,q',r,r'} V(r-r') \left(\phi_q^\dagger\phi_{q'} e^{i(-qr+q'r')} + \phi_q\phi_{q'} e^{i(qr+q'r')} + \phi_q^\dagger\phi_{q'}^\dagger e^{-i(qr+q'r')} \right) \\ = \int_q \frac{q^2}{2m} \phi_q^\dagger\phi_q + \frac{\mu}{V_0} \int_q V(q) \left(\phi_q^\dagger\phi_q + \frac{1}{2}[\phi_q\phi_{-q} + \phi_q^\dagger\phi_{-q}^\dagger] \right) \\ = \frac{1}{2} \int_q : \Phi_q^\dagger \mathcal{H}_q \Phi_q :, \quad (2296)$$

where we have defined

$$\Phi_q \equiv (\phi_q, \phi_{-q})^T, \quad \mathcal{H}_q = \begin{pmatrix} \frac{q^2}{2m} + \frac{\mu}{V_0}V(q) & \frac{\mu}{V_0}V(q) \\ \frac{\mu}{V_0}V(q) & \frac{q^2}{2m} + \frac{\mu}{V_0}V(q) \end{pmatrix}. \quad (2297)$$

To diagonalize this, we want a family of matrices M_q such that $M_q^\dagger \mathcal{H}_q M_q$ is diagonal. The diagonalized boson operators are then defined through $\Gamma_q = M\Phi_q$, with $\Gamma_q = (\gamma_q, \gamma_{-q}^\dagger)^T$. Their commutation relations are

$$[\gamma_q, \gamma_{q'}^\dagger] = i\delta_{q+q'} \det M_q, \quad (2298)$$

and so we should use a boost

$$M_q = \begin{pmatrix} \cosh(\theta_q) & \sinh(\theta_q) \\ \sinh(\theta_q) & \cosh(\theta_q) \end{pmatrix}. \quad (2299)$$

Since we are doing a boost and not a rotation (i.e. since M_q is not unitary), the eigenvalues of the diagonalized Hamiltonian $M_q \mathcal{H}_q M_q$ will not be the same as those of \mathcal{H}_q . This means we actually have to find θ_q in order to get the spectrum.

Writing $\mathcal{H}_q = a\mathbf{1} + bX$, we have

$$M_q \mathcal{H}_q M_q = [a \cosh(2\theta_q) + b \sinh(2\theta_q)]\mathbf{1} + [b \cosh(2\theta_q) + a \sinh(2\theta_q)]X. \quad (2300)$$

Requiring that the coefficient of X vanish means that

$$\theta_q = \frac{1}{2}\operatorname{atanh}(-b/a). \quad (2301)$$

Plugging this in gives, after some algebra

$$M_q \mathcal{H}_q M_q = \sqrt{a^2 - b^2}\mathbf{1}. \quad (2302)$$

The diagonalized Hamiltonian is then

$$H = \int_q \gamma_q^\dagger \epsilon(q) \gamma_q, \quad \epsilon(q) = \sqrt{\frac{q^2}{2m} \left(\frac{q^2}{2m} + \frac{2\mu V(q)}{V_0} \right)}. \quad (2303)$$

If we simplify by assuming a contact interaction so that $V(q) \approx V_0$, the dispersion simplifies. In particular, for small q we have

$$\epsilon(q \rightarrow 0) \approx |q| \sqrt{\mu/m} = c|q|, \quad (2304)$$

as expected for phonons (the sound mode in the superfluid). On the other hand, when q is very large, we have the regular free particle $\epsilon(q) \approx q^2/2m$.

To discuss the finite lifetime of the excitations, we need to go beyond the approximation where the fluctuations ϕ are non-interacting. In particular, we should go back to the original Hamiltonian and keep the terms cubic and quartic in ϕ , which produce interactions whereby a phonon decays into other phonons. We should then do the same Bogoliubov transformation on the Hamiltonian, producing a theory with a diagonalized kinetic term, plus cubic and quartic interactions for the γ_q . We then have the tools needed to compute Feynman diagrams, and so we can calculate the γ_q self-energy $\Sigma(\omega)$, which appears in the propagator derived from a coherent state path integral as $G(\omega, q) = (\omega - \epsilon(q) - \Sigma(q))^{-1}$, where $\epsilon(q)$ is the dispersion found earlier. The imaginary part of this self-energy gives the lifetime, since after Fourier transforming to the time domain the propagator becomes

$$G(q, t) \propto \exp[i t \tilde{\epsilon}(q) - t \operatorname{Im}(\Sigma(q))], \quad (2305)$$

where we've assumed Σ is roughly frequency-independent and have defined $\tilde{\epsilon}(q) = \epsilon(q) + \operatorname{Re}(\Sigma(q))$. This means that if we can compute the imaginary part of Σ , we can get the lifetime.

131 November 5 — Tower of states

Explain what the Anderson tower of states is. Go over the mega-simple example of a spin 1/2 AFM on a bipartite lattice by reducing the Hamiltonian to an effective Hamiltonian for the two “spins” made by combining the spins on all the sites of each of the sublattices. This was on a problem set for Ashvin’s condensed matter class.

Solution:

Consider a (spin 1/2, for simplicity) AFM spin system on a bipartite lattice (so that we don’t have to worry about frustration). The Hamiltonian is

$$H = J \sum_{\langle ij \rangle} S_i \cdot S_j = \frac{J}{N} \sum_{\langle ij \rangle} \sum_{p,q} S_p \cdot S_q e^{i(qi+pj)} = 2J \sum_{\mu} \sum_q S_q \cdot S_{-q} \cos(q_{\mu}), \quad (2306)$$

where N is the number of lattice sites.

The two most relevant wave vectors are $\vec{0}$ and (π, π, \dots, π) . The BZ is such that the latter wave vector is $S_{\pi} \propto \sum_j (-1)^{|j|} S_j$, where the minus sign takes on alternating values on the two sublattices. Thus the former q measures the total spin, while the latter measures the extent of the Neel ordering. If we only take these two wave vectors, we get the Hamiltonian

$$H = 2J(S_0 \cdot S_0 - S_{\pi} \cdot S_{\pi}) = \frac{2J}{N}(S_{tot} \cdot S_{tot} - S_{Neel} \cdot S_{Neel}) = \frac{4J}{N}S_A \cdot S_B, \quad (2307)$$

where A, B are the two sublattices and e.g. $S_A = \sum_{j \in A} S_j$. We can write this more suggestively as

$$H = \frac{2J}{N} [(S_A + S_B)^2 - S_A^2 - S_B^2]. \quad (2308)$$

As usual, we are writing $S_A + S_B$ for $S_A \otimes \mathbf{1} + \mathbf{1} \otimes S_A$ ⁸⁹. So, the ground state is evidently one annihilated by $S_A + S_B$ and where the spins on each sublattice add together to maximize

⁸⁹A little details on adding spins, since I remember always getting confused from the lack of explicit tensor products in physics books. The generators for a tensor product of representations R_1, R_2 for a Lie algebra are

$$T_{R_1 \otimes R_2} = \mathbf{1}_1 \otimes T_2 + T_1 \otimes \mathbf{1}_2. \quad (2309)$$

Thus e.g. for two spin 1/2s, A and B , we have e.g. $S_{A \otimes B}^x = \frac{1}{2}(\mathbf{1} \otimes X + X \otimes \mathbf{1})$ and

$$S_{A \otimes B}^2 = \frac{1}{4} \sum_a (\mathbf{1} \otimes \sigma^a + \sigma^a \otimes \mathbf{1})^2 = \frac{3}{2}\mathbf{1} \otimes \mathbf{1} + \sum_a \frac{1}{2}\sigma^a \otimes \sigma^a. \quad (2310)$$

Evaluating this on $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ and $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ shows that the quantum numbers are assigned in the way we know they are. Generalizing to N spin 1/2s tensored together, we get

$$S_{1/2 \otimes N} = \frac{1}{4} \sum_a \sum_{\{i_j\} | \sum i_j = 2} \bigotimes_j (\sigma^a)^{ij}. \quad (2311)$$

The raising / lowering operators for the N spin 1/2s are accordingly $\sigma^{\pm} \otimes \mathbf{1} \otimes \dots + \mathbf{1} \otimes \sigma^{\pm} \otimes \mathbf{1} \otimes \dots$, with highest / lowest weight states the expected $\bigotimes |\uparrow\rangle$ and $\bigotimes |\downarrow\rangle$, respectively. Evaluating the above expression for S^2 on $\otimes |\uparrow\rangle$ shows that the X and Y terms cancel, leaving

$$S_{1/2 \otimes N}^2 |\uparrow \dots \uparrow\rangle = \left(\frac{3N}{4} + 2 \binom{N}{2} \right) \mathbf{1}^{\otimes N} |\uparrow \dots \uparrow\rangle = \frac{N}{2} \left(\frac{N}{2} + 1 \right) |\uparrow \dots \uparrow\rangle, \quad (2312)$$

as expected.

S_A, S_B . Here we are assuming that the number of spins on A and B are the same (namely $N/2$), so that the total number of spin 1/2s is even, allowing us to form a singlet. Also since $S_A + S_B$ commutes with the quadratic terms, we can both maximize S_A^2, S_B^2 and simultaneously ensure that $S_A + S_B$ acts as 0. Anyway, this means that the ground state is a spin singlet, and preserves the spin symmetry: there is no spontaneous symmetry breaking in the ground state, consistent with us working in finite volume.

This means in particular that the naive Neel state

$$|Neel\rangle \propto \bigotimes_{i \in A} |\uparrow\rangle_i \otimes \bigotimes_{j \in B} |\downarrow\rangle_j \quad (2313)$$

is *not* the ground state of H . In fact, it is not even an eigenstate, since it transforms nontrivially under S^2 . Rather, it is a superposition of eigenstates

$$|Neel\rangle = \sum_S C_{(N/4, N/4), (N/4, -N/4)}^{(S, 0)} |S, 0\rangle, \quad (2314)$$

where the C 's are the CG coefficients in an appropriate normalization (e.g. the relevant C 's for the $1/2 \otimes 1/2$ case are both $1/\sqrt{2}$). All the components in the sum have $S^z = 0$ and so $|Neel\rangle$ correctly has $S^z = 0$, but it is not an eigenstate of H since it is built out of a linear combination of all possible values for $S \in 0, \dots, N/2$.

What are the low-energy excitations? First we can consider what happens when we violate the $(S_A + S_B)^2$ term in H but satisfy the $S_A^2 + S_B^2$ term. Now $(S_A + S_B)^2$ can take values in $0, \dots, N(N+1)$ (renaming $N/2$ as N), and so one of these excitations costs

$$\Delta E_{S_A + S_B} = \frac{J}{N}, \quad (2315)$$

and in particular the energy gap goes to zero as $1/L^d$ in d dimensions, since this is how the total spin scales. These are the excitations in the tower of states; in this case the tower is the same as the ladder formed with the raising / lowering operators of $SU(2)$.

The other excitations are ones in which the spins on a given sublattice do not combine to form the maximum possible spin of $N/2 \cdot (N/2 + 1)$ (recall the maximum total spin on each sublattice is $N/2$ in our current notation). The difference in S_A^2 between the highest weight and next-to-highest weight states is N , and so one of these excitations costs energy

$$\Delta E_{S_A^2} = 2J, \quad (2316)$$

which is independent of N and thus independent of system size (normally we expect the energy of the “Goldstone mode” to vanish as $1/L$, which we can’t see in this model since our neglect of all wave vectors except 0, π has left us with pathological non-local couplings between the spins).

Basically the same thing happens in a model where we replace the $S_A^2 + S_B^2$ term with an external field, now thinking of S_A, S_B as fundamental spin $N/2$'s (and not composite):

$$H = \frac{J}{N} S_A \cdot S_B - h(S_A^z + S_B^z). \quad (2317)$$

Since S_A^2, S_B^2 are now fixed as quadratic casimirs (unlike before where they specified the representation we selected out from $(1/2)^{\otimes N}$), we can also write H up to constants as

$$H = \frac{J}{2N}(S_A + S_B)^2 - h(S_A^z + S_B^z). \quad (2318)$$

Here again the ground state is a singlet with both spins pointing in the Z direction. The lowest excitation in the “tower of states” is at $\Delta E = J/2N$, while the excitation created by changing the S_A^z eigenvalue by 1 has a constant energy gap of h .

Let’s compute the susceptibility $\delta\langle S^z \rangle / \delta h$, now generalizing slightly to the case where each sublattice has N sites of spin S each. To do this we need the partition function. Since S_A and S_B are spins of the same magnitude N , we have the decomposition $N \otimes N = 0 \oplus 1 \oplus \dots \oplus 2NS$, where each representation appears with unit multiplicity. Choosing a value for $(S_A + S_B)^2$ selects out a given irrep in the direct sum decomposition, while choosing a value for $(S_A^z + S_B^z)$ selects out a given basis state within that irrep. So, specifying the pair S_{tot}^2, S_{tot}^z uniquely fixes a state in the direct sum decomposition. Thus the partition function is

$$Z = \sum_{j=0}^{2NS} e^{-\frac{\beta J}{2N}j(j+1)} \sum_{l=-j}^j e^{\beta hl}. \quad (2319)$$

The sum over l can be done easily but the remaining sum is hard, even if we take $2NS \rightarrow \infty$. At zero temperature though, we can say something. The ground state will have S_{tot}^z maximized for a given value of S_{tot} , so that $S_{tot}^z = S_{tot}$. Thus the energy in the ground state is $JS_{tot}^2/N - hS_{tot}$, which means that

$$\langle S_{tot}^z \rangle_0 = \frac{hN}{J} \implies \chi = \frac{N}{J}. \quad (2320)$$

Sanity check: when $N/J \rightarrow \infty$ there is no antiferromagnetic term, so that any external field causes the spins to align, giving $\chi \rightarrow \infty$. On the other hand, when $N/J \rightarrow 0$ the system is locked into antiferromagnetic ordering and can’t respond to external fields, so that $\chi \rightarrow 0$.

132 November 6 — Vandermonde determinant and matrix integrals

What is the integration measure on the space of N by N Hermitian matrices? Write it down in a form involving an integration over the eigenvalues.

Solution

Let M be a Hermitian matrix. We can write it as $M = U\Lambda U^\dagger$, where U is unitary and Λ is diagonal. The distance on the space of Hermitian matrices is provided by the trace

$$ds^2 = \text{Tr}(dM dM). \quad (2321)$$

Now

$$dM = U(d\Lambda)U^\dagger + UU^\dagger(dU)\Lambda U^\dagger + U\Lambda(dU^\dagger)UU^\dagger = U(d\Lambda + U^\dagger dU\Lambda - \Lambda U^\dagger dU)U^\dagger, \quad (2322)$$

since $(dU^\dagger)U = -U^\dagger dU$. Now define $\omega \equiv U^\dagger dU$. The trace distance is then

$$\mathrm{Tr}(dM dM) = \mathrm{Tr}[(d\Lambda + [\omega, \Lambda])^2]. \quad (2323)$$

The commutator is

$$[\omega, \Lambda]_{ij} = \omega_{ij}\lambda_j - \omega_{ij}\lambda_i, \quad (2324)$$

where $\Lambda_{ij} = \lambda_i \delta_{ij}$. Thus the trace of the commutator squared gives

$$\mathrm{Tr}([\omega, \Lambda]^2) = \sum_{ij} \omega_{ij}(\lambda_j - \lambda_i)\omega_{ji}(\lambda_i - \lambda_j) = - \sum_{ij} \omega_{ij}\omega_{ji}(\lambda_i - \lambda_j)^2. \quad (2325)$$

Now define $dN \equiv -i\omega$. Now $\omega^\dagger = -\omega$, so that dN is Hermitian. Furthermore, we may choose dN to have zeros on the diagonal. Indeed, notice that the relation $M = U\Lambda U^\dagger$ is preserved under the map

$$U \mapsto UD, \quad D \in U(1)^N. \quad (2326)$$

The notation here means that D is a diagonal unitary matrix. Making this shift changes

$$\omega \mapsto D^\dagger U^\dagger (dU)D + D^\dagger dD. \quad (2327)$$

We claim that we can choose D such that the diagonal part of this is zero. Doing so means that for all i ,

$$D_i^\dagger dD_i = -D_i^\dagger \omega_{ii} D_i. \quad (2328)$$

Now write $D = e^{iH^D}$, where H^D is Hermitian and diagonal. Also write $U = e^{iH^U}$, where H^U is Hermitian. Then $\omega = idH^U$, and the above equation is solved provided that we take

$$H_i^D = -H_{ii}^U. \quad (2329)$$

Thus, taking $U_{ij} \mapsto U_{ij}e^{-i\delta_{ij}H_{ii}^U}$ kills off the diagonal parts of dN . Thus we may write

$$dN_{ij} = \frac{da_{ij} + idb_{ij}}{\sqrt{2}}, \quad dN_{ii} = 0, \quad (2330)$$

where a, b are both real, so that a is symmetric and b is antisymmetric. With this notation we have

$$\mathrm{Tr}([\omega, \Lambda]^2) = \frac{1}{2} \sum_{ij} dN_{ij} dN_{ji} (\lambda_i - \lambda_j)^2 = \sum_{i < j} |dN_{ij}|^2 (\lambda_i - \lambda_j)^2 = \sum_{i < j} (da_{ij}^2 + db_{ij}^2)(\lambda_i - \lambda_j)^2. \quad (2331)$$

The quadratic in $d\Lambda$ part of $\mathrm{Tr}(dM dM)$ is just $\sum_i d\lambda_i^2$, while the remaining term is

$$\mathrm{Tr}[\{d\Lambda, [\omega, \Lambda]\}] = 0, \quad (2332)$$

since Λ and $d\Lambda$ commute. Thus the metric is

$$\mathrm{Tr}(dM dM) = \sum_i d\lambda_i^2 + \sum_{i < j} (da_{ij}^2 + db_{ij}^2)(\lambda_i - \lambda_j)^2. \quad (2333)$$

The integration measure is obtained from $\sqrt{\det g}$. In our case, $\det g$ is just

$$\det g = \prod_{i < j} (\lambda_i - \lambda_j)^4 \implies \sqrt{\det g} = \prod_{i < j} (\lambda_i - \lambda_j)^2 \equiv \Delta^2(M). \quad (2334)$$

The fact that the difference in eigenvalues is raised to the fourth power on the LHS is since we get a factor of $(\lambda_i - \lambda_j)^2$ from both the da^2 and db^2 parts of the metric. Thus the integration measure is

$$\mathcal{D}M = \prod_i d\lambda_i \prod_{j < k} da_{jk} db_{jk} \Delta^2(M). \quad (2335)$$

Thus the integration over a, b is like integrating over angles in spherical coordinates, the integration over λ is like integrating over the radial direction, and the Jacobian $\Delta^2(M)$ is like r or $r^2 \sin \theta$. If the function we're integrating is only a function of the eigenvalues of M , then the angular integration over a, b just results in a constant.

133 November 7 — Baby example of RN metric and D branes

Today we'll look at the spacetime due to “D0 branes” in four dimensions. In $G_N = 1$ units and ignoring the electric potential, the RN metric is

$$ds^2 = -f dt^2 + f^{-1} d\rho^2 + \rho^2 d\Omega_2^2, \quad f = 1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}. \quad (2336)$$

- a) For $M > Q$, find the outer event horizon.
- b) At the extremal point, what does the geometry reduce to?
- c) What happens when $Q > M$?
- d) Re-write the metric as

$$ds^2 = -g(r)dt^2 + h(r)(dr^2 + r^2 d\Omega_2^2). \quad (2337)$$

- e) Find the explicit metric in the case of $Q = M$.

Solution

- a) The horizon(s) occur where the radial and time coordinates switch between being timelike and spacelike. This occurs when $f = 0$, i.e. when

$$0 = \rho^2 - 2M\rho + Q^2 \implies \rho = M \pm \sqrt{M^2 - Q^2}. \quad (2338)$$

The outer horizon is the $+$ sign and is what we will mean by “horizon” from now on.

- b) Now let the black hole be extremal, with $M = Q$. Then the horizon is at $\rho_+ = M$. Near the horizon, we expand $\rho = \rho_+ + Mz = M(1 + z)$

$$f(\rho_+ + Mz) = 1 - \frac{2}{1+z} + \frac{1}{(1+z)^2} = z^2 + O(z^3). \quad (2339)$$

Thus the metric goes to

$$ds^2 = -z^2 dt^2 + \frac{1}{z^2} dz^2 + M^2(1+z)^2 d\Omega_2^2 \approx \frac{1}{u^2} (du^2 + \eta_{\mu\nu} dx^\mu dx^\nu) + M^2 d\Omega_2^2, \quad (2340)$$

where in the last step we have defined $u \equiv 1/z$, dropped the z -dependent pieces in the sphere part of the metric, and written $\eta_{\mu\nu}dx^\mu dx^\nu = -dt^2$ to be suggestive. The first part of the metric is AdS_2 , and the second part is S^2 . The two parts don't talk to each other, and so the spacetime is of the form $AdS_2 \times S^2$. Note that the radius of the sphere is finite as $z \rightarrow 0$.

c) For $Q > M$ there are no real solutions to $f = 0$, and so if the BH is super-extremal then it doesn't have a horizon (naked singularity).

d) We want to write the metric as

$$ds^2 = -g(r)dt^2 + h(r)(dr^2 + r^2d\Omega_2^2). \quad (2341)$$

From the dt^2 term, we know that $f(\rho) = g(r)$. We also know from the S^2 part that

$$\rho(r) = r\sqrt{h(r)}. \quad (2342)$$

Finally we look at the $d\rho^2$ part of the metric, which tells us that

$$\frac{d\rho}{\sqrt{f}} = \sqrt{h(r)}dr. \quad (2343)$$

e) Again specify to $M = Q$, so that $f = 1 - 2M/\rho + M^2/\rho^2 = (1 - M/\rho)^2$. Then we have

$$\frac{d\rho}{1 - M/\rho} = \sqrt{h(r)}dr = \rho d\ln r \implies d\ln(\rho - M) = d\ln r, \quad (2344)$$

so that we can take e.g. $r = \rho - M$. Then $g(r) = (1 - M/\rho)^2 = r^2/(r + M)^2$, while $h(r) = (1 + M/r)^2$. So the metric goes to

$$ds^2 = -\frac{r^2}{(r + M)^2}dt^2 + (1 + M/r)^2(dr^2 + r^2d\Omega_2^2). \quad (2345)$$

When we take the $r \rightarrow 0$ limit, corresponding to approaching the outer horizon $\rho \rightarrow M$, we get

$$ds_{r \rightarrow 0}^2 = -\frac{r^2}{M^2}dt^2 + \frac{M^2}{r^2}dr^2 + M^2d\Omega_2^2. \quad (2346)$$

The first dt^2 and dr^2 coordinates constitute an AdS_2 with the scale M , while the last part constitutes an S^2 of *constant* radius. In particular the radius of the sphere is finite even at the horizon, which in these coordinates is at $r = 0$.

134 November 8 — Bosonization, the XY chain, and different types of Mott insulators as SPTs

Bosonize the XY spin chain and discuss the symmetries the spin chain possesses and their realization on the bosonized side. Then add a ZZ term to the Hamiltonian. What consequences does this term have? When the coefficient of this term is large, the system becomes a Mott insulator. How many types of Mott insulator are there? Talk about this in the language of SPTs.

Solution:

The example we'll be focusing on is the bosonization of the XY chain, with Hamiltonian

$$H = -\frac{J}{2} \sum_i (X_i X_{i+1} + Y_i Y_{i+1}) = -J \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+). \quad (2347)$$

Identifying the fermions as $c_i = S_i^- \prod_{j>i} Z_j = S_i^- \prod_{j< i} (-1)_j^F$, we have

$$H = -J \sum_i (c_i^\dagger (-1)_i^F c_{i+1} + c_i (-1)_i^F c_{i+1}^\dagger) = -J \sum_i (c_i^\dagger c_{i+1} + h.c.), \quad (2348)$$

which is free fermions at half-filling. We go to the continuum via $c_j = e^{-ik_F j} \psi_L + e^{ik_F j} \psi_R$, and get the usual Dirac Lagrangian. Translation acts on the fields as $\psi_L \mapsto -i\psi_L$, $\psi_R \mapsto i\psi_R$; since we assume the ψ_σ fields vary slowly on the scale of a lattice spacing we drop the $a\partial_x \psi_\sigma + \dots$ part that comes after doing translation (so that translation acts as an internal symmetry).

Unlike in previous diary entries, we will actually be careful about Klein factors. There are a jillion different normalization conventions for the various factors that appear everywhere; we will bosonize via

$$\mathcal{B}[\psi_{L/R}] = \gamma_{L/R} e^{-i\phi_{L/R}}. \quad (2349)$$

This is probably closest to the conventions in QFT and strings part II, although here the bosonizations of both ψ_L and ψ_R have the same sign in the exponent. This ensures that $e^{i\phi}$ changes the total fermion number, while $e^{i\theta}$ changes the eigenvalue of the total γ^5 , so that θ measures the total number of fermions while ϕ measures $N_L - N_R$. Here our conventions are

$$\phi = \phi_L + \phi_R, \quad \theta = \phi_L - \phi_R. \quad (2350)$$

Here also the γ_σ 's are Klein factors that are representations of $Cl(2)$:

$$\{\gamma_\sigma, \gamma_\rho\} = 2\delta_{\rho,\sigma}. \quad (2351)$$

In our conventions we have the commutators

$$[\phi_\sigma(x), \phi_\rho(y)] = -i\pi\delta_{\rho,\sigma}(-1)^\sigma \text{sgn}(x-y), \quad (2352)$$

where we have defined $(-1)^R = -1$, $(-1)^L = +1$ ($(-1)^\sigma$ has alias γ_5). Note that with this commutation relation, ϕ_L and ϕ_R commute! Thus we need the Klein factors in order to have normal anticommutation relations between $\mathcal{B}[\psi_L]$ and $\mathcal{B}[\psi_R]$. One also checks that the commutation relation above ensures that the ψ_σ are self-fermionic. In this notation the canonical momentum is thus

$$\pi_\sigma = (-1)^\sigma \frac{\partial_x \phi_\sigma}{2\pi}. \quad (2353)$$

This means that the canonical momentum for the ϕ field has an unaesthetic factor of $1/4\pi$:

$$\pi_\phi = \frac{\partial_x \theta}{4\pi}. \quad (2354)$$

There are two symmetries in the problem that will be of interest to us. The first is translation, which as we have seen leaves $\psi_\sigma^\dagger \psi_\sigma$ invariant but acts as -1 on $\psi_L^\dagger \psi_R, \psi_R^\dagger \psi_L$. In particular, we see that $\phi \mapsto \phi, \theta \mapsto \theta + \pi$ under translation. We also have a type of particle-hole symmetry since we are at half-filling. The particle-hole transformation is

$$C : c_j \mapsto c_j^\dagger \implies \psi_L \mapsto \psi_R^\dagger, \psi_R \mapsto \psi_L^\dagger. \quad (2355)$$

The fact that it exchanges left and right movers is familiar from the usual form of charge conjugation as $\psi \mapsto \gamma_2 \psi^*$. This of course is not a symmetry of the Hamiltonian, since it sends H to $-H$ (we are reflecting the $\cos k$ band structure about the zero-energy line). In order to make this a symmetry, we need to translate the flipped band structure by π . This is done by staggering the PH transformation on each site. We write this symmetry as \tilde{C} :

$$\tilde{C} : c_j \mapsto (-1)^j c_j^\dagger \implies c_j \mapsto e^{3ik_F j} \psi_R^\dagger + e^{ik_F j} \psi_L^\dagger \implies \psi_L \mapsto \psi_L^\dagger, \psi_R \mapsto \psi_R^\dagger, \quad \tilde{C} H \tilde{C} = H. \quad (2356)$$

The fact that L and R are not exchanged is simply due to the fact that \tilde{C} involves a π momentum shift, which takes L fermions to R fermions and vice versa. In the spin language, this symmetry acts as a π rotation about X , so that

$$\tilde{C} : X \mapsto X, Y \mapsto -Y, Z \mapsto -Z. \quad (2357)$$

Since the π rotation sends S^\pm to S^\mp , one might think that it does $c_j \mapsto c_j^\dagger$ for all j . But it also acts nontrivially on the $\prod_{j < i} Z_j$ strings attached to the S_i^\pm operators. It acts trivially on strings with an even number of Z_j operators, and as -1 on those with an odd number; hence the $(-1)^j$ in the action of \tilde{C} .

In bosonized language, translation acts trivially on ϕ and sends $\theta \mapsto \theta + \pi$, while \tilde{C} does

$$\tilde{C} : \phi \mapsto -\phi, \quad \theta \mapsto -\theta. \quad (2358)$$

The combination of both \tilde{C} and translation does

$$\tilde{C}T : \phi \mapsto -\phi, \quad \theta \mapsto -\theta + \pi. \quad (2359)$$

The \tilde{C} and T symmetries prevent mass terms from being added. Indeed, both, the $\bar{\psi}\psi = \psi_L^\dagger \psi_R + h.c.$ and the $i\bar{\psi}\gamma^5\psi$ mass terms get mapped to minus themselves under translation. On the other hand, we see that the $\bar{\psi}\psi$ mass term is mapped to minus itself under \tilde{C} , while the chiral mass is invariant. Conversely, the $\bar{\psi}\psi$ mass term is invariant under $\tilde{C}T$, while the chiral mass term gets multiplied by -1 . This suggests that the two mass terms are

$$\mathcal{B}[\bar{\psi}\psi] \sim \sin \theta, \quad \mathcal{B}[i\bar{\psi}\gamma^5\psi] \sim \cos \theta, \quad (2360)$$

where we have only written \sim because of Klein factor issues. Indeed, doing this carefully we have

$$\mathcal{B}[\bar{\psi}\psi] = \mathcal{B}[\psi_L^\dagger \psi_R + h.c.] = \gamma_L \gamma_R e^{i\phi_L - i\phi_R} + \gamma_R \gamma_L e^{i\phi_R - i\phi_L} = 2i\gamma_L \gamma_R \sin \theta. \quad (2361)$$

Note the crucial Klein factors, without which we would have concluded that the bosonized mass term was a cosine, as is often erroneously done in the literature. Similarly, we have

$$\mathcal{B}[i\bar{\psi}\gamma^5\psi] = i\gamma_L \gamma_R e^{i\phi_L - i\phi_R} - i\gamma_R \gamma_L e^{i\phi_R - i\phi_L} = 2i\gamma_L \gamma_R \cos \theta. \quad (2362)$$

Again, the Klein factors are crucial. Note that this matches with what we guessed from the symmetry analysis above.

We will be interested in what happens when a $\Delta Z_i Z_{i+1}$ term is added to H . Since Z_i goes to a term involving the fermion density, this term will generate interactions for the fermions. In particular, such a term will contain a term

$$\sum_j \Delta Z_j Z_{j+1} \ni \int dx \lambda \cos(2\theta). \quad (2363)$$

This is invariant under both translation symmetry and \tilde{C} , neither of which are broken by the added ZZ term. It is generated by the Umklapp term $(\psi_R^\dagger \psi_L)^2 + h.c.$ which is translation invariant since it carries momentum 2π and which is non-zero since the terms being squared are secretly point-split. Doing the point-splitting procedure leads to some derivative terms and also the $\cos(2\theta)$ term above. The other terms in the fermionization of the ZZ term will a) renormalize the velocity of light and b) change the radius of the boson (larger Δ will make it larger). The degree to which the radius of the boson is changed determines the relevance of the $\cos(2\theta)$ term, and so whether or not the $\cos(2\theta)$ term is relevant depends on how large Δ is (since as the radius gets larger, $\cos(2\theta)$ gets more relevant, as vortices become less energetically costly at large boson radii: small radii project onto zero winding, just as only the zero momentum mode of a field on a circle becomes relevant when the radius of the circle goes to zero, while a continuum of momenta open up when the radius becomes large).

Let us now assume that Δ is large enough so that the added $\cos(2\theta)$ is relevant. The relevance of the cosine drives the theory into a Mott insulating phase. The point though is that there are two types of Mott insulators, since we have two scenarios depending on the sign of λ :

$$\lambda > 0 \implies \langle \theta \rangle \in \{\pm\pi/2\}, \quad \lambda < 0 \implies \langle \theta \rangle \in \{0, \pi\}. \quad (2364)$$

Both scenarios spontaneously break translation symmetry in a \mathbb{Z}_2 fashion, since the degenerate vacua are in both cases mapped into one another under $\theta \mapsto \theta + \pi$. In addition, we see that the $\lambda > 0$ scenario spontaneously breaks \tilde{C} particle-hole symmetry ($\theta \mapsto -\theta$), while the ground states remain symmetric under \tilde{CT} . On the other hand, the $\lambda < 0$ scenario has SSB for the \tilde{CT} symmetry, and preserves \tilde{C} .

Since the $\cos(2\theta)$ term leads to SSB for the translation symmetry, it effectively generates mass terms for the fermions that were previously forbidden by translation invariance. Indeed, let us sneakily write 2θ as $\langle \theta \rangle + \theta$. Then we have

$$\lambda \cos(2\theta) \rightarrow \lambda (\cos\langle \theta \rangle \cos \theta - \sin\langle \theta \rangle \sin \theta). \quad (2365)$$

Thus depending on the sign of λ , which controls $\langle \theta \rangle$, we get different mass terms: we can write the generated mass term as

$$\bar{\psi} e^{i\gamma^5 \Omega} \psi = \cos \Omega \bar{\psi} \psi + i \sin \Omega \bar{\psi} \gamma^5 \psi \mapsto \cos \Omega (i\gamma_L \gamma_R \sin \theta) + \sin \Omega (i\gamma_L \gamma_R \cos \theta), \quad \Omega = \langle \theta \rangle + \pi/2. \quad (2366)$$

This means that the mass term is

$$\lambda > 0 \implies H \ni \pm M \int dx i\gamma_L \gamma_R \sin \theta, \quad \lambda < 0 \implies H \ni \pm M \int dx i\gamma_L \gamma_R \cos \theta, \quad (2367)$$

where the \pm signs are determined by which ground state is chosen by SSB. Note that in what we're doing, we can always fix a “gauge” in which the states we're acting on have eigenvalue 1 under $i\gamma_L\gamma_R$, so that the Klein factors become unimportant at this stage.

Let's now do a sanity check and identify these two Mott insulators. the $\lambda > 0$ gives our fermions a non-chiral $\bar{\psi}\psi$ mass, which as we have seen breaks translation symmetry and \tilde{C} symmetry. Now the $\bar{\psi}\psi$ mass is easily checked to correspond in the c_j language to a modulated chemical potential term, so that

$$\lambda > 0 \implies H \ni \pm M \sum_j (-1)^j c_j^\dagger c_j. \quad (2368)$$

As a sanity check, we see that this term is odd under T (translation) and under \tilde{C} (remember to normal-order the $c_j^\dagger c_j$ to check the latter), while it preserves $\tilde{C}T$. The physical picture of the two ground states are chains where the electrons are localized to either all even sites, or all odd sites. Acting either with T or with \tilde{C} exchanges these two ground states. This is exactly what is expected for a strong anti-ferromagnetic coupling between the spins (since electron occupation \leftrightarrow spin up in Z basis), which makes sense because $\lambda \sim \Delta$, so that $\lambda > 0$ corresponds to an antiferromagnetic coupling.

Similarly, having $\lambda < 0$ gives our fermions a chiral mass, which breaks T and $T\tilde{C}$ but is invariant under charge conjugation \tilde{C} . Indeed, mapping this mass term back to the c_j language gives

$$\lambda < 0 \implies H \ni \pm M \sum_j ((-1)^j c_j^\dagger c_{j+1} + h.c.), \quad (2369)$$

which is a modulated hopping. This term breaks T but is invariant under \tilde{C} , so that it also breaks $T\tilde{C}$, as required. The physical picture here is of a Mott insulator coming from a dimerized phase, with the two ground states being related by translation since they differ in whether the dimers form on the even or odd links.

Summarizing, we have two types of Mott insulators. The first type (type I from here on) is one where the fermion occupation number is staggered, while the second (type II) is a dimerized phase. The type I insulator has SSB for \tilde{C} , but preserves $T\tilde{C}$, while the type II insulator has SSB for $T\tilde{C}$ but preserves \tilde{C} .

For each insulator of a given type (there are two such insulators of each type, related by translation), we can view one insulator as a trivial vacuum and the other as an SPT protected by the relevant unbroken symmetry.

Since SPTs are diagnosed by their edges, in order to elucidate this claim we should study domain walls where M changes sign, corresponding to boundaries between the vacuum and the putative type I / type II SPTs (as usual, the choice of whether M or $-M$ corresponds to the vacuum is arbitrary; only the relative sign is important [since the relative sign is chosen by the way in which T is spontaneously broken]).

Consider a first a domain wall for the type I Mott insulator. The domain wall corresponds to a pair of adjacent lattice sites that are either both occupied or both unoccupied. Evidently this domain wall carries charge $1/2$, since it can be created by inserting a single occupied / unoccupied site into a chain with uniform M , which is at half-filling so that each site carries charge $1/2$. That the domain wall (edge of the SPT) carries fractional charge under \tilde{C} is a

consequence of the fact that the symmetry protecting the SPT is $T\tilde{C}$. More precisely, we see that a domain wall with M changing sign at y is created by the operator

$$U_\pi(y) = \exp \left(i\pi \int_{-\infty}^y dx \frac{\partial_x \phi}{4\pi} \right). \quad (2370)$$

Indeed, using the commutation relations we see that acting on a state with a definite spatial profile for θ , we have (using $\hat{\theta}$ to denote an operator just for now)

$$U_\pi(y)e^{i\hat{\theta}(x)}|\theta\rangle = e^{i\theta(x)}|U_\pi(y)\theta\rangle, \quad U_\pi(y)e^{i\hat{\theta}(x)} = e^{i\pi\Theta(y-x)}e^{i\hat{\theta}(x)}|U_\pi(y)\theta\rangle. \quad (2371)$$

Evidently the θ value of $|U_\pi(y)\theta\rangle$ is the same as that of $|\theta\rangle$ at all $x > y$, while it is increased by θ for all $x < y$. Thus $U_\pi(y)$ does indeed create a domain wall across which θ jumps by π . Also, we see that this domain wall carries charge 1/2. One way to make this precise is to check that the domain wall has particle number 1/2: since $-\partial_x \theta / 2\pi$ is the total fermion number density⁹⁰, we see that, letting $N(y-a, y+a)$ denote the total fermion number in the interval $[y-a, y+a]$,

$$e^{2\pi i \hat{N}(y-a, y+a)} U_\pi(y)|\theta\rangle = e^{-2\pi i \int_{y-a}^{y+a} \partial \theta / 2\pi} U_\pi(y)|\theta\rangle = e^{-i\pi} U_\pi(y) e^{2\pi i \hat{N}(y-a, y+a)} |\theta\rangle. \quad (2378)$$

This means that the particle number in the interval $[y-a, y+a]$ in the state $|U_\pi(y)\theta\rangle$ is different by 1/2 from the particle number in the state $|\theta\rangle$: thus the domain wall carries $U(1)$ charge 1/2.

⁹⁰We might naively have guessed that the total density was instead $\partial_x \phi_L + \partial_x \phi_R = \partial_x \phi$. To identify the density correctly, we can think about determining the phase of the fermion operators ψ_σ through the WKB approximation, where the phase is determined through $e^{i \int_0^x dx' k(x')}$. The right-moving part of the electron creation operator is

$$\psi_R^\dagger(x) e^{ik_F x} \sim e^{i \int_0^x dx' k(x')} = e^{ik_F x + i \int^x dx' \delta k_F(x')}, \quad (2372)$$

where $\delta k_F(x')$ is the variation of the fermi level caused by a variation in electron density (I know mixing \mathbb{R} space and k space like this is weird, but we're working in the WKB approximation where we're assuming that the density varies sufficiently slowly that we can define a local Fermi level in this way). The density of right-moving electrons is $(k_F + \delta k_F)/2\pi$, so that $\delta k_F(x) = 2\pi \delta n_R$. Thus

$$\psi_R^\dagger(x) e^{ik_F x} \sim e^{ik_F x} e^{i2\pi \int^x dx' \delta n_R(x)}. \quad (2373)$$

Since we defined ϕ_R through $\mathcal{B}[\psi_R^\dagger] = e^{i\phi_R}$, we see that

$$\phi_R(x) = 2\pi \int^x dx' \delta n_R(x') \implies \frac{\partial_x \phi_R}{2\pi} = \delta n_R(x). \quad (2374)$$

The left moving electrons are treated in a similar way: since the ψ_L fields live near the point of the FS at $-k_F + \delta k_F$, we have

$$\psi_L^\dagger(x) e^{-ik_F x} \sim e^{-ik_F x + i \int^x dx' \delta k_F(x')}. \quad (2375)$$

Now a positive δk_F corresponds to a smaller density of L fermions in our conventions. Thus $\delta k_F(x') = -2\pi \delta n_L(x')$, and so

$$\psi_L^\dagger(x) \sim e^{-i2\pi \int^x dx' \delta n_L(x')} \implies \frac{\partial_x \phi_L(x)}{2\pi} = -\delta n_L(x), \quad (2376)$$

since we have $\mathcal{B}[\psi_L^\dagger] = e^{i\phi_L}$. Thus the total deviation of the total density away from $2k_F/2\pi$ is

$$\delta n_R + \delta n_L = \frac{1}{2\pi} (-\partial_x \phi_L + \partial_x \phi_R) = -\frac{\partial_x \theta}{2\pi}. \quad (2377)$$

The type II insulator is entirely analogous. Here the domain wall is a location where two dimers share an end on a single site, or where two dimers are missing from consecutive links. The operator creating the domain wall is the same as before (since as for the type I case the domain wall is a π shift of θ), and as above it carries fractional $U(1)$ charge.

135 November 9 — Learning about AdS space

The AdS metric in a presentation which makes the $SO(2, d)$ symmetry manifest is

$$ds^2 = -dX_{-1}^2 - dX_0^2 + \sum_{i=1}^d dX_i^2, \quad (2379)$$

where the coordinates are constrained to cut out a hyperbola:

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^d X_i^2 = R^2. \quad (2380)$$

Do several things. a) find coordinates in which the metric is the simpler

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2} dr^2. \quad (2381)$$

b) find a transformation where the $1/r^2$ factor is pulled out front. c) find a formulation of ds^2 with only one timelike coordinate. d) what is the Penrose diagram for AdS?

Solution:

a) Written as a hyperboloid in $(2, d)$ Lorentzian space, the coordinates of AdS_{d+1} are not all independent: there is one linear relation among them. We want to eliminate this dependency by introducing a new variable r and getting rid of X_{-1} and X_d . Now

$$dX^\mu = \frac{dr}{R} x^\mu + \frac{r}{R} dx^\mu \implies \sum_{i=0}^d dX_i dX^i = dX_d^2 + \frac{dr^2}{R^2} x^\mu x_\mu + \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{2rdr}{R^2} dx^\mu x_\mu. \quad (2382)$$

Here $\mu = 0, \dots, d-1$ and the μ indices are raised / lowered with the mostly-positive Minkowski metric. so that

$$ds^2 = -dX_{-1}^2 + dX_d^2 + \frac{1}{R^2} (dr^2 x^\mu x_\mu + r^2 dx_\mu dx^\mu + r dr d(x_\mu x^\mu)). \quad (2383)$$

Using the constraint, we can use

$$x_\mu x^\mu = \frac{R^2}{r^2} (X_{-1}^2 - X_d^2 - R^2) \quad (2384)$$

to re-write this as

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu - dX_{-1}^2 + dX_d^2 + \frac{dr^2}{r^2} (X_{-1}^2 - X_d^2 - R^2) - 2 \frac{dr^2}{r^2} (X_{-1}^2 - X_d^2 - R^2) + \frac{2dr}{r} (X_{-1} dX_{-1} - X_d dX_d). \quad (2385)$$

Expressing r in terms of X_{-1} and X_d and simplifying by giving all relevant terms a $1/r^2$ to factor out,

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2} dr^2 + \frac{1}{(X_d + X_{-1})^2} \left((dX_{-1} + dX_d)^2 (X_d^2 - X_{-1}^2) + 2(dX_d + dX_{-1})(X_{-1} dX_{-1} - X_d dX_d)(X_d + X_{-1}) + (dX_d^2 - dX_{-1}^2)(X_d + X_{-1})^2 \right). \quad (2386)$$

Somewhat amazingly the terms in the big parentheses all cancel, and we get the simple

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2} dr^2. \quad (2387)$$

b) If we let $z = R^2/r$, then $dr = -\frac{R^2}{z^2} dz$, so that $(R^2/r^2)dr^2 = (z^2/R^2)(R^4 z^{-4} dz^2)$ and the metric goes to

$$ds^2 = \frac{R^2}{z^2} (dz^2 + dx_\mu dx^\mu). \quad (2388)$$

c) Now we take

$$X_0 = R\sqrt{1+r^2} \cos t, \quad X_{-1} = R\sqrt{1+r^2} \sin t, \quad \sum_{i=1}^d X_i^2 = (Rr)^2. \quad (2389)$$

The last relation means that the spatial X_i coordinates form a sphere of radius Rr . It also means that since e.g.

$$dX_0 = -R\sqrt{1+r^2} \sin t dt + R(1+r^2)^{-1/2} \cos t r dr, \quad (2390)$$

we have

$$dX_0^2 + dX_{-1}^2 = R^2(1+r^2)dt^2 + \frac{R^2 r^2}{1+r^2} dr^2. \quad (2391)$$

Thus the metric is now

$$ds^2 = -R^2 \left((1+r^2)dt^2 + \frac{r^2 dr^2}{1+r^2} \right) + R^2 dr^2 + r^2 R^2 d\Omega_{d-1}^2 = R^2 \left(-(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + d\Omega_{d-1}^2 \right). \quad (2392)$$

Note that now the metric only has a single timelike coordinate. Now we let $r = \tan \rho$, so that $1+r^2 = 1/\cos^2 \rho$ and $dr^2 = (1+\tan^2 \rho)^2 d\rho^2$. Then the metric is

$$ds^2 = R^2(1+r^2) \left(-dt^2 + d\rho^2 + \frac{r^2}{1+r^2} d\Omega_{d-1}^2 \right) = \frac{R^2}{\cos^2 \rho} (-dt^2 + d\rho^2 + \sin^2 \rho d\Omega_{d-1}^2). \quad (2393)$$

d) The overall $(R/\cos \rho)^2$ factor in front doesn't affect the causal structure since the latter is determined by right rays where $ds^2 = 0$, a condition which doesn't care about the Weyl rescaling of the metric. Thus the Penrose diagram can be determined just by looking at

the terms inside of the parentheses. dt^2 gives us a copy of \mathbb{R} , and so we get a sort of solid cylinder, where the axis of the cylinder is at $\rho = 0$ ($r = 0$) and the surface of the cylinder is at $\rho = \pi/2$ ($r = \infty$). The surface of this cylinder is a copy of $\mathbb{R} \times S^{d-1}$. Note that this boundary $\mathbb{R} \times S^{d-1}$ lies at finite proper distance away from any point in the interior of the cylinder, despite the fact that the boundary is at $r = \infty$.

136 November 12 — Large N matrix model quantum mechanics, eigenvalue distributions,

This is an elaboration on a problem from a pset assigned in Hong Liu's class on AdS / CFT. We consider a matrix model with partition function

$$Z = \int \mathcal{D}M \exp\left(-\frac{N}{g}\text{Tr}[V(M)]\right), \quad (2394)$$

where $V(M)$ is a polynomial potential (so that the action is a function only of the eigenvalues of M), and the integral over M runs over all $N \times N$ Hermitian matrices. We will eventually specialize to $V(x) = x^2/2 + x^4$. We will also denote the eigenvalue density by

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i), \quad (2395)$$

where $\{\lambda_i\}$ are the eigenvalues. Taking $N \rightarrow \infty$, we will assume $\rho(\lambda)$ approaches a continuous function supported on some interval $I \subset \mathbb{R}$.

Do several things:

a Find an expression for Z to leading order in the $N \rightarrow \infty$ limit.

b Define the complex function

$$F(\xi) = \int_I d\lambda \frac{\rho(\lambda)}{\xi - \lambda}. \quad (2396)$$

Discuss the analytic properties of $F(\xi)$.

c Show that $F(x) \in \mathbb{R}$ if $x \in \mathbb{R} \setminus I$, and find $F(x \rightarrow \infty)$.

d Use the previous results to determine the form of F .

e Find $\rho(\lambda)$ explicitly.

f At what value of the coupling does the free energy have a non-analytic part? Find the leading non-analytic behavior of the free energy near the critical point.

g You should find that the critical coupling is negative, $g_c = -1/48$. Does this make sense? The fact that $g_c < 0$ means that this theory has a perturbative expansion about $g = 0$ that has a finite radius of convergence—unlike what happens in normal QFT, we get more than just an asymptotic expansion. Why is this?

a) First, we use the Vandermonde determinant (see entry in the summer physics diary) to turn the measure $\mathcal{D}M$ into an integral over the eigenvalues, weighted by an exponential involving a term like $\ln|\lambda_i - \lambda_j|$, which contributes to the action. To get the saddle point equation, we vary the action with respect to a particular eigenvalue λ_i and set the result to zero: this gives

$$\frac{N}{g} V'(\lambda_i) = 2 \sum_{j:j < i} \frac{1}{\lambda_i - \lambda_j} - 2 \sum_{j:j > i} \frac{1}{\lambda_j - \lambda_i} = 2 \sum_{j:j \neq i} \frac{1}{\lambda_i - \lambda_j}. \quad (2397)$$

Turning the sum into an integral using the eigenvalue distribution $\rho(\lambda)$, we get

$$\frac{N}{2g} V'(\lambda) = P \int d\mu \frac{\rho(\mu)}{\lambda - \mu}, \quad (2398)$$

where the principal value has been taken since the sum avoids terms with $i = j$ where $\lambda_i - \lambda_j = 0$.

The saddle point value for the partition function is just obtained by evaluating the matrix exponential on the saddle point distribution of eigenvalues. So the free energy is

$$\mathcal{F} = -\ln Z \approx \frac{N}{g} \sum_i V(\lambda_i) - \ln \prod_{i \neq j} |\lambda_i - \lambda_j| \rightarrow \frac{N^2}{g} \int d\lambda \rho(\lambda) V(\lambda) - N^2 P \int d\lambda d\mu \rho(\lambda) \rho(\mu) \ln |\lambda - \mu|, \quad (2399)$$

where λ is a distribution of eigenvalues satisfying the saddle point equation. Here the logarithm comes from putting the vandermonde determinant in the exponential.

b) Now define the complex function

$$F(\xi) = \int_I d\lambda \frac{\rho(\lambda)}{\xi - \lambda}, \quad \xi \in \mathbb{C}. \quad (2400)$$

Here $I = \text{supp}(\rho) \subset \mathbb{R}$ is assumed to be a union of intervals in \mathbb{R} . Using the Dirac identity, we can take $\xi = \mu - i\epsilon$ for μ real and send $\epsilon \rightarrow 0$ to get

$$F(\mu - i\epsilon) = i\pi \int d\lambda \rho(\lambda) \delta(\mu - \lambda) + P \int d\lambda \frac{\rho(\lambda)}{\mu - \lambda}. \quad (2401)$$

Here we have used the equation of motion to replace the principal part of the integral above with the derivative of $V(\mu)$. Thus we see that

$$\text{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu), \quad \text{Re}[F(\mu - i\epsilon)] = \frac{V'(\mu)}{2g}, \quad \mu \in I. \quad (2402)$$

Note that these properties hold only for $\mu \in I$: if $\mu \in \mathbb{R} \setminus I$ then

$$\text{Im}[F(\mu - i\epsilon)] = 0, \quad \text{Re}[F(\mu - i\epsilon)] = \int d\lambda \frac{\rho(\lambda)}{\mu - \lambda}, \quad \mu \in \mathbb{R} \setminus I. \quad (2403)$$

In particular, the real part of $F(\mu - i\epsilon)$ needn't be related to $V'(\mu)$ if $\mu \notin I$, since the saddle-point equation relating $V'(\mu)$ to the principal part of the relevant integral was derived under the assumption that $\mu \in I$.

Note that $F(\xi)$ is analytic everywhere, and on $I \subset \mathbb{R}$ it has a branch cut ($\rho(\lambda)$ is assumed to be well-behaved in the $N \rightarrow \infty$ limit). We see that across the branch cut at $\mu \in I$, $F(\xi)$ changes by $2\pi i\rho(\mu)$.

c) Note that if we take $\xi \in \mathbb{R}$ and send $\xi \rightarrow \infty$, we have

$$F(\xi \rightarrow \infty) = \int_I d\lambda \frac{\rho(\lambda)}{\xi} (1 + \lambda/\xi + \dots) = \frac{1}{\xi} + O(\xi^{-2}), \quad (2404)$$

where we have used the normalization of $\rho(\lambda)$. Note that $\text{Re}[F(\mu - i\epsilon)]$ does not go to $V'(\mu)/2g$ for $\mu \notin I$ (unless V is logarithmic, which we will assume to not be the case).

d,e) We can use this information to find out what $F(\xi)$ is. In the following we will assume for simplicity that I is a single connected interval centered on zero, so that $I = [-a, a]$ for some $a \in \mathbb{R}$. This will be the case if we have a potential $V(\Lambda)$ with a unique minimum at 0, like $V(\Lambda) = \frac{1}{2}\Lambda^2 + \Lambda^4$. We will determine a self-consistently using the constraints we've derived on F .

Since we know that $F(\xi)$ is analytic but has a branch cut at $I = [-a, a]$ on the \mathbb{R} axis, we expect that $\sqrt{\xi^2 - a^2} = \sqrt{\xi - a}\sqrt{\xi + a}$ will show up in $F(\xi)$ in order to give us the right branch cut structure, and in order to make $\text{Im}[F(\mu - i\epsilon)]$ nonzero only when $\mu \in I$. Since $F(\xi)$ is analytic, we expect $F(\xi) = g(\xi) + f(\xi)\sqrt{\xi^2 - a^2}$, where f, g are some polynomials in ξ with positive powers and real coefficients (as $\text{Im}[F(\mu - i\epsilon)] = 0$ if $\mu \notin I$).

The requirement that the real part of $F(\mu - i\epsilon)$ go to $V'(\mu)/2g$ when $\mu \in I$ tells us that $g(\xi) = V'(\xi)/2g$. We can then get $f(\xi)$ by requiring $F(\xi \rightarrow \infty) \rightarrow 1/\xi + O(1/\xi^2)$:

$$F(\xi \rightarrow \infty) \approx \frac{V'(\xi)}{2g} + f(\xi) \left(\xi - \frac{a^2}{2\xi} + O(\xi^{-3}) \right). \quad (2405)$$

Thus

$$f(\xi) = \frac{1}{\xi^2 - a^2/2} + \frac{\xi V'(\xi)}{2g(a^2/2 - \xi^2)}, \quad (2406)$$

with a^2 to be determined by requiring $f(\xi)$ to be a \mathbb{R} polynomial with positive powers. We know that $\deg(f) = \deg(V') - 1$, which again follows from our knowledge of $F(\xi \rightarrow \infty)$.

We will now specialize to the case

$$V(\lambda) = \frac{1}{2}\lambda^2 + \lambda^4. \quad (2407)$$

So then $\xi V'(\xi) = \xi^2 + 4\xi^4$, and

$$f(\xi) = \frac{1}{2g(\xi^2 - a/2)}(2g - \xi^2 - 4\xi^4). \quad (2408)$$

Since $V'(\lambda)$ is third order, we know that $f(\xi)$ will be second order, which allows us to stop at the leading order expansion for the square root for now. Writing $f(\xi) = A + B\xi + C\xi^2$ we see that $B = 0$, $C = -2/g$, and

$$\frac{1}{2g} + A = \frac{ca^2}{2}, \quad (2409)$$

so that

$$f(\xi) = -\frac{1}{2g}(1 + 2a^2 + 4\xi^2). \quad (2410)$$

This then determines the eigenvalue distribution to be, using $\text{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu)$ for $\mu \in [-a, a]$,

$$\rho(\mu) = -\frac{1}{\pi}f(\mu)\sqrt{a^2 - \mu^2} = \frac{1}{2\pi g}(1 + 2a^2 + 4\mu^2)\sqrt{a^2 - \mu^2}. \quad (2411)$$

As an aside, we can recover the Wigner distribution by looking at $V(\lambda) = \frac{1}{2}\lambda^2$. In this case, since we know that the degree of $f(\xi)$ is two less than the degree of $V(\xi)$, $f(\xi)$ must be a constant. Working it out and solving for a^2 in the manner described below gives (I think)

$$\rho(\mu)|_{V(\lambda)=\lambda^2/2} = \frac{1}{2\pi g}\sqrt{4g - \mu^2}, \quad (2412)$$

which is the famous Wigner distribution.

Anyway, now we return to the quartic potential (2407). To get a , we need to get the $1/\xi$ piece of $F(\xi \rightarrow \infty)$, which requires expanding the square root to include the $-a^4/8\xi^4$ term. Setting the coefficient of the $1/\xi$ piece to 1 means that

$$3a^4 + a^2 - 4g = 0 \implies a^2 = \frac{1}{6}\left(-1 + \sqrt{1 + 48g}\right). \quad (2413)$$

Recapitulating, we have shown that

$$F(\xi) = \frac{1}{2g}\left(\xi + 4\xi^3 - \left[4\xi^2 + \frac{2}{3} + \frac{1}{3}\sqrt{1 + 48g}\right]\sqrt{\xi^2 + \frac{1}{6} - \frac{1}{6}\sqrt{1 + 48g}}\right). \quad (2414)$$

e) We can now get an explicit expression for the free energy

$$\mathcal{F}/N^2 \approx \frac{1}{g} \int d\lambda \rho(\lambda)V(\lambda) - P \int d\lambda d\mu \rho(\lambda)\rho(\mu) \ln |\lambda - \mu|. \quad (2415)$$

The second term in the free energy with the \ln is hard to integrate, but we have another option: we can integrate the equations of motion to obtain

$$\frac{V(\lambda) - V(0)}{2g} = P \int d\mu \rho(\mu) [\ln |\lambda - \mu| - \ln |\mu|], \quad (2416)$$

which means that (since $V(0) = 0$)

$$P \int d\lambda d\mu \rho(\lambda)\rho(\mu) \ln |\lambda - \mu| = \frac{1}{2g} \int d\lambda \rho(\lambda)V(\lambda) + P \int d\mu \rho(\mu) \ln |\mu|. \quad (2417)$$

Putting this into the second integral in the expression for the free energy,

$$\mathcal{F}/N^2 \approx \int d\lambda \rho(\lambda) \left(\frac{V(\lambda)}{2g} - \ln |\lambda| \right). \quad (2418)$$

The first term is

$$\frac{1}{2g} \int_{-a}^a d\lambda \rho(\lambda)(\lambda^2/2 + \lambda^4) = -\frac{a^4}{128g^2}(2 + 10a^2 + 9a^4), \quad (2419)$$

while the second term is

$$-2 \int_0^a d\lambda \rho(\lambda) \ln \lambda = \frac{a^2}{16g} (2 + a^2(3 + 6 \ln 4) + \ln 16 - 4(1 + 3a^2) \ln a). \quad (2420)$$

Now we add these two together, and carry out an expansion in small $\epsilon = g - g_c = g + 1/48$. Here the critical point $g_c = -1/48$ is the coupling at which the free energy becomes singular. This point is supposed to mark the phase transition where “complicated” Feynman diagrams dominate and the Feynman diagrams go over to form a “continuum geometry”, or something like that. The picture is that (according to comments I heard from Klebanov at the Jerusalem winter school) the quantity $\Delta = g - g_c$ is a “chemical potential for triangles in the triangulation of spacetime”. In the Louiville model (gravity theory with just the dilaton) that is supposed to describe the other side of the phase transition, Δ appears as a cosmological constant: writing the metric in conformal gauge as $g = e^\phi \eta$, we get an action like

$$\int \left[\frac{1}{2} (\partial\phi)^2 + \Delta e^\phi \right]. \quad (2421)$$

The fact that $g_c < 0$ means that near the transition point, the potential for the eigenvalues is actually unstable, since $V(\pm\infty) = -\infty$. And yet, the theory we’ve been working with is supposed to make sense for a finite range of negative g ! Somehow, taking the $N \rightarrow \infty$ limit first is enough to stabilize what would otherwise be an unstable potential, allowing the $g < 0$ theory to make sense. The transition comes when the eigenvalues start “spilling over” the brim of the potential and running off towards $\pm\infty$.

Hold on, you may say: at $g \rightarrow g_c a^2 < 0$, but doesn’t a^2 always have to be positive, since the eigenvalues we’re integrating over must always be real? So, isn’t $g < 0$ already ruled out? I guess the philosophy here is that the important thing to look at is really the singular behavior of the partition function: we used the WKB approximation to get the partition function, and while within this approximation $a^2 < 0$ strictly speaking doesn’t make sense, after getting our expression for \mathcal{F} we can just work with it directly, forgetting about the approximation where it came from. These last two paragraphs show that I clearly have a lot more to understand about this theory!

Anyway, doing the expansion with Mathematica, we find that the free energy has an (imaginary) constant part, a term proportional to ϵ , one proportional to ϵ^2 , and then one proportional to $\epsilon^{5/2}$, which is the leading singular part. So, the leading non-analytic behavior of \mathcal{F} is a $5/2$ power dependence on the distance from the critical point (n.b. the cancellation of the $\epsilon^{1/2}, \epsilon^{3/2}$ terms is nontrivial!).

137 November 29 — Yet more on the $SU(2)$ point of the $c = 1$ CFT

This is a fast one, and comes from a problem in a pset assigned in Ashvin Vishwanath’s fall 2018 class on quantum matter. Consider the fermionized description of the anisotropic XXZ chain:

$$H = \sum_i \left[\frac{1}{2} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + g S_i^z S_{i+1}^z \right]. \quad (2422)$$

Write down an expression for $S^z(r)$ in terms of the low energy fermion fields, including components both at zero momentum and at momentum $q = \pi$.

What value of g corresponds to the self-dual $SU(2)$ point? Find out by requiring that the $q = \pi$ component of the S^z spin density have the same power law exponent as the 2-point function of the S^+ operator. We will work in conventions where the bosonized Lagrangian is written as $\mathcal{L} = (\partial_\mu \theta / 2\pi)^2 / (2K)$.

Solution:

Since $S_r^\pm = (-1)^r (-1)^{\sum_{r' < r} n_{r'}}$ (the $(-1)^r$ is needed to cancel a minus sign from moving a string operator past a creation operator in $S_i^+ S_{i+1}^-$ in order to produce a correct-sign hopping), we have $S_r^z = S_r^+ S_r^- - 1/2 = n_r - 1/2$. Thus in terms of the low energy fields,

$$\begin{aligned} S_r^z &= c_r^\dagger c_r - 1/2 = (e^{-ik_F r} R^\dagger(r) + e^{ik_F r} L^\dagger(r))(e^{ik_F r} R(r) + e^{-ik_F r} L(r)) - 1/2 \\ &= n_R(r) + n_L(r) + (-1)^r (L^\dagger(r)R(r) + R^\dagger(r)L(r)) - 1/2, \end{aligned} \quad (2423)$$

since $k_F = \pi/2$. While $(-1)^r$ carries momentum π so does $L^\dagger(r)R(r) + h.c.$, so that $\sum_j S_j^z$ has net zero momentum as required.

The part of the S^z 2-point function that goes as $(-1)^r$ is the part involving scattering from one of the Fermi points to the other:

$$\langle S_r^z S_0^z \rangle \ni (-1)^r \langle L^\dagger(r)R(r)R^\dagger(0)L(0) \rangle \rightarrow (-1)^r \langle e^{-i\theta(r)} e^{i\theta(0)} \rangle, \quad (2424)$$

since in our conventions $L^\dagger(x) \rightarrow e^{i\phi_L(x)}$, $R(x) \rightarrow e^{-i\phi_R(x)}$ and $\theta = \phi_R - \phi_L$. From the action we read off that the propagator for the θ field is

$$G_\theta(r) = -2\pi K \ln |r|, \quad (2425)$$

and so

$$\langle S_r^z S_0^z \rangle \rightarrow \frac{1}{|r|^{2\pi K}}. \quad (2426)$$

Now we look at the S^\pm correlator. We can figure out the image of S_r^\pm under bosonization from its commutation relation with Z_r , which we know maps to $n_r - 1/2 = \partial_x \theta / (2\pi) - 1/2$. Since $[Z_r, S_r^\pm] = \pm S_r^\pm$, we guess that $S_r^\pm \rightarrow e^{\pm i\phi(r)}$. This identification is natural since we know $\phi(r)$ gets shifted by the $U(1)$ symmetry of rotations about the z axis, in the same way that S^+ does. Indeed, from the number-phase relation $[\phi(x), \partial_y \theta(y)/2\pi] = i\delta(x - y)$, we check that

$$[Z_r, S_r^\pm] \rightarrow [\partial_r \theta, e^{\pm i\phi(r)}] = \pm e^{\pm i\phi(r)} \rightarrow \pm S_r^\pm, \quad (2427)$$

as required. Actually this doesn't completely fix the bosonization of S_r^\pm . We will actually include an explicit factor of $(-1)^r$ in its bosonization (as we wrote down above), so that

$$S_r^\pm \rightarrow (-1)^r e^{\pm i\phi(r)}. \quad (2428)$$

(the reason for doing this is to get the correct sign for the fermion kinetic term).

Anyway, this means that

$$\langle S_r^+ S_0^- \rangle = (-1)^r \langle e^{i\phi(r)} e^{-i\phi(0)} \rangle = (-1)^r e^{G_\phi(r)}. \quad (2429)$$

The propagator $G_\phi(r)$ for ϕ is determined by T duality: if the coefficient of the free θ action is $R^2/4\pi$, then the coefficient for the free ϕ action is $1/(4\pi R^2)$. For us $R^2 = 1/2\pi K$, and so the coefficient for the ϕ action will be $K/2$. Thus we have

$$G_\phi(r) = -\frac{1}{2\pi K} \ln |r| \implies \langle S_r^+ S_0^- \rangle \rightarrow (-1)^r \frac{1}{|r|^{1/2\pi K}}. \quad (2430)$$

If we require that the power law exponents in the $(-1)^r$ parts of the S^\pm and S^z two point functions match, which is a necessary requirement if the theory is to have $SU(2)$ symmetry, then we require that $2\pi K = 1/(2\pi K)$, so that we predict the $SU(2)$ point to be located at $K = 1/2\pi$. We know from above that the operator $\cos 2\theta$ becomes marginal when $K = 1/2\pi$, so that the $SU(2)$ point is characterized by the radius at which the $\cos 2\theta$ Umklapp term crosses over between relevance and irrelevance.

138 November 30 — Specific heats for fermions and bosons

Today's diary entry is super simple, and also comes from a problem in Ashvin's class. Calculate the specific heat for free Dirac fermions in 1+1D, as well as for free relativistic massless bosons. Now do the same in 2+1D — what does the result mean for bosonization?

Solution:

First we get C for the Dirac fermions in two dimensions. The L and R components of a massless Dirac fermion are decoupled, and so we can just calculate C for a single component and then multiply the result by 2. The energy of excitations with respect to the Dirac sea where all negative energy states are filled is $v_F|k|$, and so we have, for a single component,

$$\begin{aligned} \langle E \rangle &= -\partial_\beta \ln Z = -\partial_\beta \ln \left[\sum_{n_k=0,1} e^{-\beta v_F \sum_k |k| n_k} \right] = -\partial_\beta \int_{\mathbb{R}} \frac{dk}{2\pi} \ln \left[1 + e^{-\beta v_F |k|} \right] = \frac{v_F}{\pi} \int_0^\infty \frac{k}{e^{\beta v_F k} + 1} \\ &= \frac{T^2}{\pi v_F} \int_0^\infty dl \frac{l}{e^l + 1} = \frac{\pi T^2}{12 v_F}. \end{aligned} \quad (2431)$$

Thus the heat capacity for a single component is $\pi T/(6v_F)$, and so we have

$$C_{Dirac}^{1+1} = \frac{\pi T}{3v_F}. \quad (2432)$$

Now for the bosons. The calculation is almost the same:

$$\begin{aligned} \langle E \rangle &= -\partial_\beta \ln \left[\sum_{n_k \in \mathbb{Z}_{\geq 0}} \prod_k e^{-\beta |k| v_F} \right] = -\partial_\beta \int_0^\infty \frac{dk}{\pi} \ln \left[\frac{1}{1 - e^{-\beta k v_F}} \right] = \frac{v_F}{\pi} \int_0^\infty dk \frac{k}{e^{\beta k v_F} - 1} \\ &= \frac{\beta^2}{\pi v_F} \int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi T^2}{6v_F}, \end{aligned} \quad (2433)$$

and so we get

$$C_{boson}^{1+1} = \frac{\pi T}{3v_F} = C_{Dirac}^{1+1}. \quad (2434)$$

Thus bosonization in two dimensions has a chance of relating a relativistic scalar with a Dirac fermion.

Alternatively, we could get this from our knowledge that in a CFT on a cylinder of circumference L , the vev of the holomorphic stress-energy tensor is (using the conventions in the Big Yellow Book and setting the “speed of light” to 1):

$$\langle T \rangle = -\left(\frac{2\pi}{L}\right)^2 \frac{c}{24} \implies \langle T_{00} \rangle = \langle T_{zz} \rangle + \langle T_{\bar{z}\bar{z}} \rangle = -\frac{2}{2\pi} \langle T \rangle = \frac{\pi c}{6L^2}. \quad (2435)$$

Thus since a cylinder of circumference L maps to a one-dimensional quantum system at temperature $T = 1/L$,

$$C = \partial_T \langle T_{00} \rangle = \frac{\pi c T}{3}. \quad (2436)$$

Since the bosons and fermions both have $c = 1$ (the ψ_L, ψ_R are decoupled and each have central charge 1/2), we recover $C = \pi T/3$ for both systems.

Now we go to $2+1$ dimensions. The only things that change are the numbers, basically. For the fermions, since the Eigenvalues of H are $\pm v_F |k|$, we have

$$\langle E \rangle = -\partial_\beta \int_0^\infty \frac{dk}{2\pi} k \ln \left[1 + e^{-\beta v_F k} \right] = \frac{v_F}{2\pi} \int_0^\infty dk \frac{k^2}{e^{\beta v_F k} + 1} = \frac{T^3}{2\pi v_F^2} \int_0^\infty dx \frac{x^2}{e^x + 1} = \frac{3\zeta(3)T^3}{4\pi v_F^2}, \quad (2437)$$

and so

$$C_{Dirac}^{2+1} = \frac{9\zeta(3)T^2}{4\pi v_F^2}. \quad (2438)$$

For the bosons, we have

$$\langle E \rangle = -\partial_\beta \int_0^\infty \frac{dk}{2\pi} k \ln \left[\frac{1}{1 - e^{-\beta k v_F}} \right] = \frac{T^3}{2\pi v_F^2} \int_0^\infty dx \frac{x^2}{e^x - 1} = \frac{\zeta(3)T^3}{\pi v_F^2}, \quad (2439)$$

and so

$$C_{bosons}^{2+1} = \frac{3\zeta(3)T^2}{\pi v_F^2}. \quad (2440)$$

This is not equal to the specific heat of the Dirac fermions, unless we let the fermions and bosons have different v_F . If they have different v_F they definitely can't be related by bosonization since the two theories have different causal structures, and so we can conclude that in three dimensions more ingredients (e.g. gauge fields) are needed to make bosonization work. This isn't surprising since in contrast to the two-dimensional case where particle statistics aren't really meaningful, in three dimensions braiding statistics is well-defined, and bosons at fermions are different statistically.

139 December 1 — Mass terms in bosonization and FQH from coupled wires

Today's diary entry is yet another problem from a pset in Ashvin's class. We consider a set of N 1+1D fermionic chains, each described by the bosonized fields ϕ_i, θ_i , with conventions such that $\partial_x \theta_i / 2\pi$ gives the density fluctuations away from half-filling on the i th wire, and such that $[\phi_j(x), \partial_y \theta_i(y)] = 2\pi i \delta_{ij} \delta(x - y)$. We couple the wires together with the term

$$H \ni \sum_{\alpha} \lambda_{\alpha} \cos \left(n^{\alpha} \cdot \phi + \frac{1}{2} m^{\alpha} \cdot \theta \right), \quad (2441)$$

where the $n^{\alpha}, m^{\alpha} \in \mathbb{Z}^N$, and where α is an index for the mass terms (here $\phi = (\phi_1, \dots, \phi_N)^T$ and likewise for θ).

- a) When can two cosine terms $(n, m), (n', m')$ be simultaneously minimized?
- b) Find the condition on $p, q \in \mathbb{Z}$ such that $\phi_L = p\phi_1/2 + q\phi_1$ is not made massive by the terms above (ϕ_L is an edge operator that does not get coupled to the other fields by the usual inter-wire hopping terms).
- c) Now consider the case where

$$H \ni \sum_j \cos(\phi_j - \phi_{j+1} - m(\theta_j + \theta_{j+1})/2). \quad (2442)$$

Show that these terms commute and find the magnetic field at which translation symmetry along the chain allows them to be added to H , provided that the constituent fermions have unit charge.

- a) We consider adding the terms

$$\lambda_a \cos \left(n^a \cdot \phi + \frac{1}{2} m^a \cdot \theta \right), \quad (2443)$$

where a is an index labeling the different cosine terms and $\phi = (\phi_1, \dots, \phi_N), \theta = (\theta_1, \dots, \theta_N)$ obey $[\phi_j(x), \theta_k(y)] = -\pi i \operatorname{sgn}(x - y)$, with sgn the sign function. In order for a single such term with (n, m) to be minimizable, the argument of the cosine should be expressible as a c-number in a given basis. This means that the two terms in the argument of the cosine should commute with one another. Now

$$[n \cdot \phi(x), m \cdot \theta(y)]/2 = n_j m_k [\phi^j(x), \theta^k(y)/2] = -\pi i \operatorname{sgn}(x - y) n \cdot m. \quad (2444)$$

This is zero (mod 2π) if $n \cdot m \in 4\mathbb{Z}$. Here the modulo 2π condition is relevant since we are typically interested in the commutator of the cosine and the vertex operators like $e^{in \cdot \phi(x)}$, in which the term in the above equation will appear in an exponential.

Now consider two cosine terms, labeled by (n, m) and (n', m') . Then in addition to the above requirement on each of the two pairs, the simultaneous minimization of the two terms requires that they commute with one another, so that

$$[n \cdot \phi(x) + m \cdot \theta(x)/2, n' \cdot \phi(y) + m' \cdot \theta(y)/2] = -\pi i \operatorname{sgn}(x - y) [m \cdot n' + n \cdot m']. \quad (2445)$$

So the two cosine terms will for sure be simultaneously diagonalizable if $n \cdot m = n' \cdot m' = n \cdot m' + m \cdot n' = 0$. More generally, we just need that e.g. the term in the brackets in the last equation be in $2\mathbb{Z}$.

b) In order for $p\theta_1/2 + q\phi_1$ to not be made massive, it must not be expressible as a linear combination of fields that have been made massive. That is, consider the $2N$ -dimensional vector space V with basis vectors $(\phi_1, 0, \dots, 0), (0, \theta_1/2, 0, \dots, 0), (0, 0, \phi_2, 0, \dots, 0)$, and so on. The vectors $n^a \cdot \phi + \frac{1}{2}m^a \cdot \theta$ generate a d -dimensional subspace of V (assuming wolog that they are all linearly independent), and $p\theta_1/2 + q\phi_1$ will be “left behind” provided that it is not contained within this subspace.

c) These terms can be simultaneously minimized since the terms $j, j-1$ that have fields in common have the commutator:

$$[\phi_j - \phi_{j+1} - m(\theta_j + \theta_{j+1})/2, \phi_{j-1} - \phi_j - m(\theta_{j-1} + \theta_j)/2] = -[\phi_j, m\theta_j/2] + [m\theta_j/2, \phi_j] = 0. \quad (2446)$$

The ϕ parts represent hopping $L_j^\dagger R_j^\dagger L_{j+1} R_{j+1}$. Now if the right Fermi point of wire j is at k_F , then the left Fermi point of wire $j+1$ is at $k_F(-1+2/\nu)$, where $\nu = 2k_F\hbar/B$ (we will take the electric charge of the fermions to be $e = 1$). Thus this hopping term has a total momentum transfer of $2k_F + 2k_F + 2(-2 + 2/\nu)k_F = 4k_F/\nu$. Now under translation, θ_j transforms as $T : \theta_j \mapsto \theta_j + 2k_Fx$, for all j . In order for the cosine term to conserve momentum, the momentum of $m(\theta_j + \theta_{j+1})/2$ needs to equal $4k_F/\nu$, and so we need $2mk_F = 4k_F/\nu$, i.e. we need to take $m = 2/\nu$, meaning that $B = mk_F\hbar$.

140 December 2 — MFT for Mott insulators and SFs in two dimensions

Today’s entry is pretty basic but is something that I hadn’t worked out before. Show that in the 1+1 dimensional Fermi-Hubbard model at half filling, mean field theory predicts a CDW state for arbitrarily small repulsive interactions, and an SCing state for arbitrarily small attractive interactions. Of course both of these conclusions are incorrect, with a gapless phase surviving for finite interactions. Explain why mean field theory goes down in this way and why there is actually a gapless phase.

Solution:

First, a note that I’m writing this super late and without paper at hand, so there will likely be numerical mistakes. The Hamiltonian is the usual

$$H = -\frac{1}{2} \sum_j (c_j^\dagger c_{j+1} + h.c.) + U \sum_j (n_j - 1/2)(n_{j+1} - 1/2). \quad (2447)$$

First let’s examine the CDW state, which we expect to occur for some large enough U . We take as our MFT ansatz

$$n_j = \frac{1}{2}(1 + \Delta(-1)^j) + : \delta n_j :, \quad (2448)$$

where δn_j has zero expectation value and is assumed to be small. Putting this into H and dropping the term quadratic in δn_j ,

$$H = -\frac{1}{2} \sum_j (c_j^\dagger c_{j+1} + h.c.) - U \sum_j \left(\frac{\Delta^2}{4} + \Delta(-1)^j : \delta n_j : \right). \quad (2449)$$

Now going to momentum space,

$$H \rightarrow \sum_{0 \leq k < \pi} \Psi_k^\dagger \mathcal{H}_k \Psi_k - \Delta^2 UV/4, \quad \mathcal{H}_k = \begin{pmatrix} \cos k & -\Delta U \\ -\Delta U & -\cos(k) \end{pmatrix}, \quad \Psi_k = (c_k, c_{k+\pi})^T. \quad (2450)$$

The eigenvalues of \mathcal{H}_k are $E_\pm(k) = \pm \sqrt{\cos^2 k + \Delta^2 U^2}$, and so getting the ground state by filling all of the $E_-(k)$ modes means the ground state energy is

$$\mathcal{E}_0 \rightarrow - \int dk \sqrt{\cos^2 k + \Delta^2 U^2} - VU\Delta^2/4. \quad (2451)$$

Minimizing this (and normalizing by $V = 1$ for simplicity),

$$\frac{1}{4U} = \int \frac{dk}{\sqrt{\cos^2 k + U^2 \Delta^2}}, \quad (2452)$$

which has a solution for arbitrarily small positive U since $\cos^2(k)$ vanishes at $k = \pm\pi/2$. We can approximate (ignoring numerical constants so that the important part of the integral is $\int_k dk / \sqrt{k^2 + U^2 \Delta^2} \rightarrow \ln(k^2 + U^2 \Delta^2)|_0^\Lambda \implies \ln(\Lambda^2/U^2 \Delta^2) \approx 1/U$)

$$\Delta \approx \frac{\Lambda}{U} e^{-1/U}. \quad (2453)$$

Does this solution for Δ give a lower energy than the $\Delta = 0$ state? Indeed it does: the energy with Δ turned on is schematically

$$\mathcal{E}_\Delta = \alpha \Delta^2 - \beta \int_{k,\Delta} \frac{U^2 \Delta}{\sqrt{\cos^2 k + U^2 \Delta^2}} \approx \alpha \Delta^2 - \gamma \int_\Delta \Delta \ln(\Lambda^2/U^2 \Delta^2) = \Delta^2 (\lambda + \omega \ln \Delta), \quad (2454)$$

where all the greek letters are *positive* numerical constants that we don't care about. Now when $\Delta = 0$ we get $\mathcal{E}_\Delta \rightarrow 0$, but because of the logarithm, we see that we can always choose Δ to be small enough so that $\lambda + \omega \ln \Delta < 0$, meaning that \mathcal{E}_Δ is most negative when $\Delta \neq 0$. Thus MFT predicts a CDW at any finite $U > 0$.

We can also see the CDW instability by looking at what happens when we add a perturbation H_Δ to H_0 , where H_0 is the hopping term and H_Δ carries momentum π , e.g.

$$H = H_0 + H_\Delta, \quad H_\Delta = \Delta \sum_j (-1)^j c_j^\dagger c_j \rightarrow \Delta \int dx (R^\dagger L + L^\dagger R), \quad (2455)$$

where $c_j = e^{-ik_F j} R + e^{ik_F j} L$ (we are setting the Fermi velocity to unity). We now go to second-order perturbation theory, with the second order correction to the energy being (the

first-order correction vanishes, since in the ground state $|0\rangle$ where all states below $k_F = \pi/2$ are filled the matrix element $\langle 0|H_\Delta|0\rangle$ vanishes)

$$E_2 = \sum_{l \neq 0} \frac{|\langle l|H_\Delta|0\rangle|^2}{\mathcal{E}_0 - \mathcal{E}_l}, \quad (2456)$$

where l run over the non-groundstate eigenstates of the free Hamiltonian and \mathcal{E}_l are the free Hamiltonian energies. Now since the energy denominator will be small only when $|l\rangle$ is a state with excitations near the Fermi surface, we can take $H_0 \approx \int dx (iR^\dagger \partial_x R - iL^\dagger \partial_x L)$ for our purposes, so that the energies of the $|l\rangle$ states are linear in momentum. Now the mass term H_Δ has momentum transfer π , so that it moves a hole just below the left Fermi point to a particle just above the right Fermi point, and vice versa. Thus when the relevant matrix element $\langle l|H_\Delta|0\rangle$ is non-zero, $\mathcal{E}_l \sim l$, where l is a momentum. So rather schematically we see that the second order correction to the energy is

$$E_2 \sim \Delta \int_0^\Lambda dk \frac{1}{k} \rightarrow \infty, \quad (2457)$$

so that E_2 diverges logarithmically. Here Λ is some unimportant upper cutoff (note that we have a divergence here from the *small* momentum modes right by the Fermi surface).

What happens to the CDW instability away from half-filling? Let us more generally write the interaction term as $\sum_j U(n_j - \nu)(n_{j+1} - \nu)$. This adds to the MF Hamiltonian (still assuming a CDW at momentum π since this is where the divergence in the susceptibility is) a term $\sum_j U\delta\nu n_j$, where $\delta\nu = \nu - 1/2$. The negative-energy branch of the spectrum is then $E_- = U\delta\nu - \sqrt{U^2(\delta\nu)^2 + \cos^2 k + \Delta^2 U^2}$, which in turn leads to the gap equation (assuming $\Delta \neq 0$)

$$\frac{1}{4U} = \int \frac{dk}{\sqrt{U^2(\delta\nu)^2 + \cos^2 k + \Delta^2 U^2}}. \quad (2458)$$

Does this lead to an instability for arbitrarily small U ? Because of the $U^2(\delta\nu)^2$ term in the denominator, the integrand can no longer be made arbitrarily large by making Δ appropriately small. If we make the same approximation to the integral as before, and drop unimportant numerical factors, the self-consistent equation ends up reading (I think)

$$\Delta^2 \sim -(\delta\nu)^2 + (\Lambda/U)^2 e^{-1/U}. \quad (2459)$$

Now for a fixed cutoff Λ , a small enough U will lead to $\Delta^2 < 0$, which is a contradiction. Thus away from half-filling the system is not immediately unstable to a CDW, as expected.

The same analysis can be applied for the SCing instability, which also comes about from a diverging susceptibility. Consider adding the term

$$H_\Delta = \Delta \sum_j (c_j c_{j+1} + h.c.) \rightarrow 2\Delta \int dx (iLR - iR^\dagger L^\dagger) + \dots, \quad (2460)$$

where \dots denotes less relevant operators / ones that oscillate rapidly. The matrix element $\langle 0|H_\Delta|l\rangle$ will be non-zero if $|l\rangle$ contains a pair of particles or holes with opposite momenta

$k, -k$. The energy of such a pair is linear in k , and so we again get a (logarithmically) divergent second-order correction to the energy in the same way as before.

Doing MFT for the SCing instability works in essentially the same way as for the CDW instability; we just replace $c_j c_{j+1} = \Delta + (c_j c_{j+1} - \Delta)$, and drop terms quadratic in the deviation about Δ . The symmetry $\cos(k) = \cos(-k)$ is now what allows us to derive the instability by defining the spinor $\Psi_k = (c_k, c_{-k}^\dagger)$, with the associated Hamiltonian

$$\mathcal{H}_k = \begin{pmatrix} \cos k - \mu & e^{ik} \Delta U \\ e^{-ik} \Delta U & -[\cos(k) - \mu] \end{pmatrix}, \quad (2461)$$

where we took $\Delta \in \mathbb{R}$ for simplicity and where μ is a chemical potential determined by the filling. For a filling fraction of ν , we have $\mu = 2\nu U$. Going through the analysis yields the exact same gap equation as in the CDW case, except with $\cos(k) \mapsto \zeta(k) \equiv \cos(k) - \mu$ and $U \mapsto -U$. As long as the chemical potential is not so large that $\zeta(k)$ the gap equation is solved in the same way as in the CDW case, by linearizing $\zeta(k)$ about its vanishing point. Thus as the SCing instability only relies on the $k \mapsto -k$ TRS symmetry of the diagonal part of the Hamiltonian, it is insensitive to the precise value of the filling ν (at the technical level, the difference is that the chemical potential caused by $\nu \neq 1/2$ shows up in \mathcal{H}_k as proportional to $\mathbf{1}$ in the CDW case, but proportional to Z in the SC case).

141 December 3 — Topological robustness of Fermi surfaces

Today's entry is something that somehow escaped my attention until now but is actually pretty cool: explain why the existence of a fermi surface is protected by a topological invariant relating to the winding of a fermion Greens function. I read about this in Volovik's book.

Solution:

For a Fermi “surface” of codimension $p + 1$ in (ω, k) space (we will always write k for both $|\vec{k}|$ and \vec{k} , hopefully the distinction will be clear from context), a topological invariant is obtained by integrating a p -form around an S^p that links the Fermi surface. The p -form is constructed by taking the trace of wedge products of the Maurer-Cartan form

$$\omega_{\mathcal{G}} = \mathcal{G} d\mathcal{G}^{-1}, \quad \mathcal{G}(\omega, k) = \frac{1}{i\omega - \mathcal{H}(k)}, \quad (2462)$$

where \mathcal{H} is the Hamiltonian density. Note that $\mathcal{G}(\omega, k)$ is defined as the full interacting Greens function at *imaginary* frequency. The nontriviality of the topological invariant will come from the properties of the singularities of \mathcal{G} in ω, k space. Since we want to characterize the robustness of the Fermi surface, we want the only singularities in \mathcal{G} to come at the Fermi surface, where $\mathcal{H}(k)$ and ω vanish. If we were at \mathbb{R} frequencies this would not be the case,

since then we'd have singularities in \mathcal{G} whenever we had quasiparticles going on-shell. The topological invariant is constructed via

$$N = \frac{1}{\mathcal{N}} \oint_{S^p} \text{Tr}[\omega_{\mathcal{G}}^p], \quad (2463)$$

with \mathcal{N} an (imaginary) normalization constant⁹¹.

The simplest example of this is a prototypical codimension-2 Fermi surface (i.e. an S^2 in three spatial dimensions) for a single free fermion. In this case we have

$$N = \frac{1}{2\pi i} \oint_C dz^\mu \frac{1}{i\omega - \mathcal{H}(k)} \partial_\mu(i\omega - \mathcal{H}(k)). \quad (2464)$$

Here z^μ is a stand-in for the coordinate along the contour. We will choose the contour to be a small circle of radius R linking the Fermi sphere in the (ω, k_x) plane. Taking R to be small, we can expand $\mathcal{H}(k)$ about the FS as $\mathcal{H}(k) \approx v_F k$, where now k is measured relative to the FS. Wolog we can set $v_F = 1$, so that the invariant is

$$N = -\frac{1}{2\pi i} \oint dz^\mu \frac{i\omega + k}{\omega^2 + k^2} \partial_\mu(i\omega - k). \quad (2465)$$

Parametrizing the contour in the (ω, k_x) plane by $(R \cos \theta, R \sin \theta)$, we have

$$\begin{aligned} N &= -\frac{1}{2\pi i} \oint d\theta (-\sin \theta, \cos \theta)_j (i, -1)^j [i \cos \theta + \sin \theta] \\ &= -\frac{1}{2\pi i} \oint d\theta (-i \sin^2 \theta - i \cos^2 \theta) = 1. \end{aligned} \quad (2466)$$

Thus the winding around the Fermi surface is nontrivial, and the FS is topologically protected (N is independent of the contour since $\omega_{\mathcal{G}}$ is closed: $d\mathcal{G} \wedge d\mathcal{G}^{-1} = 0$).

How do we get $N = -1$? We just have to change the sign of $\mathcal{H}(k)$. Indeed, if we do this then we have

$$N = -\frac{1}{2\pi i} \oint dz^\mu \frac{i\omega - k}{\omega^2 + k^2} \partial_\mu(i\omega + k) = -\frac{1}{2\pi i} \oint d\theta (-\sin \theta, \cos \theta)_j (i, 1)^j [i \cos \theta - \sin \theta] = -1. \quad (2467)$$

Of course, we know of many examples where Fermi surfaces can be destroyed by various types of instabilities, like the CDW or SCing instabilities. Evidently for this to happen, we need to have some way of changing the Greens function holonomy we calculated above so that it becomes trivial. Indeed, imagine starting from a non-interacting problem, and then slowly turning on interactions which potentially lead to an instability of the Fermi surface. Assuming that turning on the interactions adiabatically is not a singular process, since $N \neq 0$ when the interactions are turned off and since $N \in \mathbb{Z}$, it cannot become zero when the interactions

⁹¹In fact, this is only the topological invariant when p is odd, since ω^p defines a nontrivial class in $H^\bullet(X)$ only for odd p : when $p \in 2\mathbb{Z}$ $\text{Tr}(\omega^p) = -d\text{Tr}(\omega^{p-1})$ is exact (there may be an omitted combinatorial prefactor). This may be related to the fact that Fermi surfaces of even codimension (in full ω, \vec{k} space) are stable (like regular Fermi surfaces in spatial dimension $d = 3$), while those of odd dimension are unstable (e.g. Fermi lines in $d = 3$, which are codimension 3).

are slowly switched on (having $N = 0$ is required if we want to get an instability of the FS which leads to a gapped phase, since in a gapped phase \mathcal{G} has no singularities).

Since we have seen that taking $\mathcal{H}(k) \mapsto -\mathcal{H}(k)$ changes the sign of N , we see that one potential way to create an instability is to find some way of pairing up Fermi surfaces of particles ($N = 1$) and Fermi surfaces of holes ($N = -1$) to create a Fermi surface with net zero winding number, which then has no obstruction to being destroyed by an instability that creates a gap.

As simple examples, both the CDW and SCing instabilities in 1+1 dimensions are created in this way. In each case, we write the Hamiltonian (as we would do in e.g. a MFT analysis) as a bilinear in terms of the spinors

$$\Psi_{CDW} = (c_k, c_{k+\pi})^T, \quad \Psi_{SC} = (c_k, c_{-k}^\dagger)^T. \quad (2468)$$

The crucial property of both of these spinors is that the Hamiltonian matrices they are associated with have a diagonal component proportional to Z (the Pauli matrix), e.g. $\cos(k)Z$ (because e.g. in the CDW case we have $\cos(k + \pi) = -\cos(k)$; in the SCing case the minus sign is due to the mixed creation / annihilation nature of Ψ_{SC} and the symmetry $\cos(-k) = \cos(k)$). More generally, suppose we can write the Hamiltonian as a bilinear in spinors such that after diagonalizing $\mathcal{H}(k)$ ⁹², we have $\mathcal{H}(k) = E(k)Z \approx v_F k Z$. Again setting $v_F = 1$, the index is then

$$\begin{aligned} N &= -\frac{1}{2\pi i} \text{Tr} \left[\oint dz^\mu \begin{pmatrix} \frac{1}{i\omega-k} & 0 \\ 0 & \frac{1}{i\omega+k} \end{pmatrix} \partial_\mu \begin{pmatrix} i\omega-k & 0 \\ 0 & i\omega+k \end{pmatrix} \right] \\ &= -\frac{1}{2\pi i} \oint dz^\mu \frac{1}{\omega^2 + k^2} ((i\omega - k)\partial_\mu(i\omega + k) + (i\omega + k)\partial_\mu(i\omega - k)) \\ &= 0. \end{aligned} \quad (2469)$$

So in this case, the index when interactions are switched off is vanishing, which means that introducing interactions adiabatically does have the possibility of creating a gap and leading to an instability of the FS.

Finally, we re-iterate that this quantization is a very robust result, and does not change when e.g. the strength of interactions are slowly changed. For example, as long as our propagator is of the form $Z/(i\omega - v_F k)$, the quantization of N is quite insensitive to the exact values of Z and v_F : in FL theory changing the strength of interactions only changes the residue Z and renormalizes the Fermi velocity, neither of which change N .

142 December 7 — Quantization in AdS

This is a problem taken from a pset assigned in Hong Liu's holography class. The problem is as follows: consider a scalar field in AdS_{d+1} , with action

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2). \quad (2470)$$

⁹²Sending $\mathcal{H} \mapsto U^\dagger \mathcal{H} U$ for unitary U leaves $\text{Tr}(\omega \mathcal{G})$ invariant, which is a relatively quick thing to check.

Here Greek indices run over time, $d - 1$ “space” dimensions, and the radial coordinate z (sorry). Near the boundary at $z = 0$, we have

$$\phi(z \rightarrow 0, x) = A(x)z^{d/2-\nu} + B(x)z^{d/2+\nu}, \quad \nu \equiv \sqrt{d^2/4 + m^2R^2}, \quad (2471)$$

where R is the AdS radius, and A, B are some functions of the “spacetime coordinates”.

a) Define an inner product for wavefunctions and show its time independence. b) What is the condition on ν for a given ϕ to be normalizable? c) Find the stress tensor and show that it is covariantly conserved. d) Define the energy E as the integral of $\sqrt{-g}g^{tt}T_{tt}$ over a given Cauchy slice, and find the explicit form of $\partial_t E$. e) When does the energy flux at the boundary vanish? f) Show that E is finite if the chosen wavefunction is normalizable, and infinite otherwise.

a) We define the inner product as

$$\langle \phi, \psi \rangle_{\Sigma_t} = -i \int_{\Sigma_t} dz d\vec{x} \sqrt{-g} g^{tt} (\phi^* \partial_t \psi - \partial_t \phi^* \psi), \quad (2472)$$

where Σ_t is any Cauchy surface, and the g^{tt} is required to make the integral invariant under rescaling of t . Taking the difference of the inner products at different times, we have

$$\langle \phi, \psi \rangle_{\Sigma_{t'}} - \langle \phi, \psi \rangle_{\Sigma_t} = -i \int_M d^d x_{\perp}^{\mu} (\phi^* \nabla_{\mu} \psi - \nabla_{\mu} \phi^* \psi), \quad (2473)$$

where M is the timelike boundary of the spacetime volume bounded by the two Cauchy slices, located at $z = 0$. Here we have used the fact that the integral over the bounded volume vanishes, on account of

$$\nabla_{\mu} (\phi^* \nabla^{\mu} \psi - \nabla^{\mu} \phi^* \psi) \sqrt{-g} = 0, \quad (2474)$$

by virtue of the equations of motion, viz. $\nabla^2 = m^2$ when acting on ϕ and ψ . The integral on the RHS of (2473) vanishes if ϕ and ψ are properly normalized at infinity, and so the inner product is time-independent.

b) ϕ has the asymptotic expansion

$$\phi(z \rightarrow 0) = A(x)z^{d/2-\nu} + B(x)z^{d/2+\nu}. \quad (2475)$$

Now since $\sqrt{-g} \propto z^{-(d+1)}$ and $g^{tt} \propto z^2$, we have

$$\langle \phi, \phi \rangle_{\Sigma} \sim -i \int_{\Sigma} dz d\vec{x} (AA'z^{1-2\nu} + BB'z^{2\nu+1} + 2AB'z). \quad (2476)$$

To get something finite, we need the total power of z to be greater than -1 . This is always satisfied by the B mode and so the B mode is always normalizable. For the A mode to be normalizable we need $1 - 2\nu > -1$, so the A mode is normalizable only if

$$0 \leq \nu < 1. \quad (2477)$$

c) There are two terms that contribute to the stress tensor: the Lagrangian density and the $\sqrt{-g}$ in the measure. The variation of the former wrt the metric is simple, while the latter is found by using

$$\delta \sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta e^{\text{Tr} \ln g} = -\frac{\sqrt{-g}}{2} \delta \text{Tr} \ln g \implies \frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} = -\frac{\sqrt{-g}}{2} g^{\mu\nu}. \quad (2478)$$

Since the variation of the Lagrangian density is just $\sqrt{-g}\partial^\mu\phi\partial^\nu\phi$, we have

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}} \left(\sqrt{-g}\partial^\mu\phi\partial^\nu\phi - \frac{\sqrt{-g}g^{\mu\nu}}{2}(\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2) \right) \\ &= \nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}(\nabla_\lambda\phi\nabla^\lambda\phi + m^2\phi^2), \end{aligned} \quad (2479)$$

where in the last step we have replaced ordinary derivatives with covariant derivatives since they act in the same way on ϕ , which is a scalar.

Now let's verify that T is covariantly conserved. This is straightforward since the metric is covariantly constant, meaning that we don't have to worry about the differences between raised and lowered indices, and that the covariant derivatives pass straight through the $g^{\mu\nu}$:

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla^2\phi\nabla^\nu\phi + \nabla_\mu\phi\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\nabla_\mu\nabla^\lambda\phi\nabla_\lambda\phi - m^2\phi\nabla^\nu\phi \\ &= \nabla^\nu\phi(\nabla^2\phi - m^2\phi) = 0, \end{aligned} \quad (2480)$$

again by virtue of the equations of motion. Thus the stress tensor is covariantly conserved.

d) Define the energy as

$$E = \int_{\Sigma_t} dz d\vec{x} \sqrt{-g}g^{tt}T_{tt}. \quad (2481)$$

The g^{tt} here is needed so that under time rescalings it cancels the rescaling of T_{tt} , so that the whole action rescales like $\sqrt{-g} \mapsto \lambda\sqrt{-g}$ under $t \mapsto \lambda^{-1}t$, which is appropriate for an energy. Another way to write $\sqrt{-g}g^{tt}$ would be $\sqrt{-g_t}n^t$, where g_t is the induced metric on Σ_t and $n^t = (0, z/R, 0, \dots, 0)$ is the temporal unit vector in coordinates (z, t, \vec{x}) . Under time rescalings $\sqrt{-g}g^{tt}$ transforms as a vector; as does $\sqrt{-g_t}n^t$ since $\sqrt{-g_t}$ is a scalar under time rescalings (as the metric on Σ_t involves no dt^2 piece).

We can write the time derivative of E as

$$\begin{aligned} \partial_t E &= \frac{1}{\delta t} \left(\int_{V_{\delta t}} d^d x dz \sqrt{-g} \nabla^\mu T_{t\mu} - \int_{M_{\delta t}} d^d x \sqrt{-g_{M_{\delta t}}} n^z T_{tz} \right) \\ &= - \int_{\Sigma_t} d\vec{x} \sqrt{g_{\partial\Sigma_t}} n^t n^z T_{tz}, \end{aligned} \quad (2482)$$

where the various manifolds are defined straightforwardly: $V_{\delta t}$ is the spacetime volume sandwiched between Σ_t and $\Sigma_{t+\delta t}$, and $M_{\delta t}$ is the timelike component of its boundary. In the last step, we have used the fact that the time integral in the second term on the first line just produces a factor of δt .

e) First, we need

$$n^t = (0, z/R, \vec{0}), \quad n^z = (z/R, 0, \vec{0}), \quad (2483)$$

which comes from e.g. $n^z n_z = 1$. Now the induced metric on $\partial\Sigma_t$ has determinant $g_{\partial\Sigma_t} = +(R/z)^{2d-2}$, and so

$$F|_{z=0} = \partial_t E \sim \int_{\Sigma_t} d\vec{x} z^{-d+3} T_{tz} = \int_{\Sigma_t} d\vec{x} z^{-d+3} \nabla_t \phi \nabla_z \phi. \quad (2484)$$

Now suppose $\phi \sim z^\omega$. Then

$$F|_{z=0} \sim z^{-d+3+2\omega-1}, \quad (2485)$$

and so if the flux is to be zero we must have

$$F|_{z=0} = 0 \implies \omega > d/2 - 1. \quad (2486)$$

Now for the Bz^Δ mode this is always true, since $\Delta = d/2 + \nu$. For the $Az^{d-\Delta}$ mode, this condition reads $\nu < 1$, which is precisely the condition that the A mode be normalizable. So we see that (non)normalizable modes have (non)zero energy flux at infinity.

f) The part of T_{tt} with the smallest power of z is the $(\nabla_t \phi)^2$ part. Again, suppose $\phi \sim z^\omega$. Then the contribution to E with the smallest power of z is

$$E \sim \int_{\Sigma_t} dz d\vec{x} z^{-d-1} z^2 z^{2\omega} \sim z^{-d+2+2\omega}. \quad (2487)$$

If the energy is to be finite, we need $2(1+\omega) > d$. For the A mode we have $\omega = d/2 + \nu$, and the energy is always finite. For the B mode we have $\omega = d/2 - \nu$, and

$$E < \infty \implies \nu < 1. \quad (2488)$$

Of course this is the same condition on ν for the B mode to be normalizable. So, normalizable modes, in either quantization scheme, are the ones with finite E .

143 December 8 — AdS Propagators

This is another problem from Hong's holography class. a) How is Lorentzian AdS different from Euclidean AdS? We will use the latter spacetime in what follows. b) Let ϕ be a massive scalar field, and find the bulk-to-boundary propagator $K(z, x; x')$. c) Find a relation between K and the bulk-to-bulk propagator G in terms of the limit of G as one of its arguments approaches the boundary. d) Write down a general boundary correlation function in terms of a limit of a bulk correlation function.

a) From the $1/z^2$ dependence of the metric, we see that in Euclidean AdS, the distances in the x coordinates vanish at $z = \infty$, and so $z = \infty$ is just a single point, unlike in Lorentzian AdS. Another way of seeing this is to recognize that Euclidean AdS is the Poincare disk, with $z \rightarrow \infty$ corresponding to the single point at the center of the disk.

b) We want to get the boundary-to-bulk propagator. Using the equations of motion, the propagator K at $z = \infty$ needs to satisfy

$$(\partial_M (\sqrt{-g} g^{MN} \partial_N \phi) - m^2 \sqrt{-g}) K = 0. \quad (2489)$$

Since $z = \infty$ is a single point in the bulk, in the $z \rightarrow \infty$ limit $K(x, z; x')$ can only depend on z (not x since all x are the same at $z = \infty$, and not x' by rotational invariance of the Poincare disk). Thus, putting in the z dependence of the metric, we have

$$\left[\partial_z \left((R/z)^{d+1} (z/R)^2 \partial_z \right) - m^2 (R/z)^{d+1} \right] K(z \rightarrow \infty) = 0. \quad (2490)$$

Assuming a power-law $K(z) \propto z^\alpha$, we have

$$(1-d)\alpha + \alpha(\alpha-1) - m^2R^2 = 0 \implies \alpha = \frac{d}{2} \pm \sqrt{d^2/4 + m^2R^2}. \quad (2491)$$

We will see later that the requirement that K go to a δ function at the $z = 0$ boundary requires us to select out the larger root (which we denote as Δ), and so

$$K(z \rightarrow \infty) = Cz^\Delta, \quad (2492)$$

for some $C \in \mathbb{R}$.

Now we can use the homogeneity of AdS (despite how it looks when drawn as a Poincare disk, no point is special) to get $K(x, z; 0)$: we first perform the transformation

$$z \mapsto \frac{z}{z^2 + x^2}, \quad x^\mu \mapsto \frac{x^\mu}{z^2 + x^2} \quad (2493)$$

on $K(z \rightarrow \infty)$. We then use translation invariance in the x^μ directions (rotational invariance of the Poincare disk) to get the Poisson form

$$K(x, z; x') = C \left(\frac{z}{z^2 + (x - x')^2} \right)^\Delta. \quad (2494)$$

Here we see that the power of Δ gives us a δ function when $x = x'$. This power is also correct since we have

$$K(z \rightarrow 0, x; x') = z^{d-\Delta} \delta^d(x - x'). \quad (2495)$$

In standard quantization, ϕ has z^Δ scaling, so that $\int d^d x' K(z, x; x') \phi_0(x')$ has the correct scaling.

c) We can relate the bulk-to-bulk propagator to K by using the bulk-to-boundary map and one of Greens identities, namely

$$\int_M d^{d+1}x \sqrt{-g} (\phi_1 G^{-1} \phi_2 - \phi_2 G^{-1} \phi_1) = \int_{\partial M} d^d x \sqrt{-g_\partial} (\phi_1 n^\mu \partial_\mu \phi_2 - \phi_2 n^\mu \partial_\mu \phi_1), \quad (2496)$$

where g_∂ is the induced metric on the boundary, G is the bulk propagator, and n^μ is the unit normal on the boundary. The trick is then to employ this identity with $\phi_1 = K(z, x; x')$, $\phi_2 = G(z, x; z'', x'')$. Now since the LHS is over the bulk and since $(\nabla^2 - m^2) = G^{-1}$ annihilates K in the bulk (the only place it doesn't annihilate K is at coincident points on the boundary), the LHS is

$$LHS = \int_M d^{d+1}x \sqrt{-g} K(z, x; x') (\nabla^2 - m^2) G(z, x; z'', x'') = K(x'', z''; x'), \quad (2497)$$

by definition of G . On the other hand, since $\sqrt{-g_\partial} = (R/z)^d$ and $n^\mu = z$, the RHS is

$$RHS = \int_{\partial M} d^d x z^{-d+1} K(z, x; x') \partial_z^{\leftrightarrow} G(z, x; z'', x''), \quad (2498)$$

where $\partial^{\leftrightarrow}$ denotes the antisymmetrized derivative. Here we have dropped the R dependence since it will cancel out in the end.

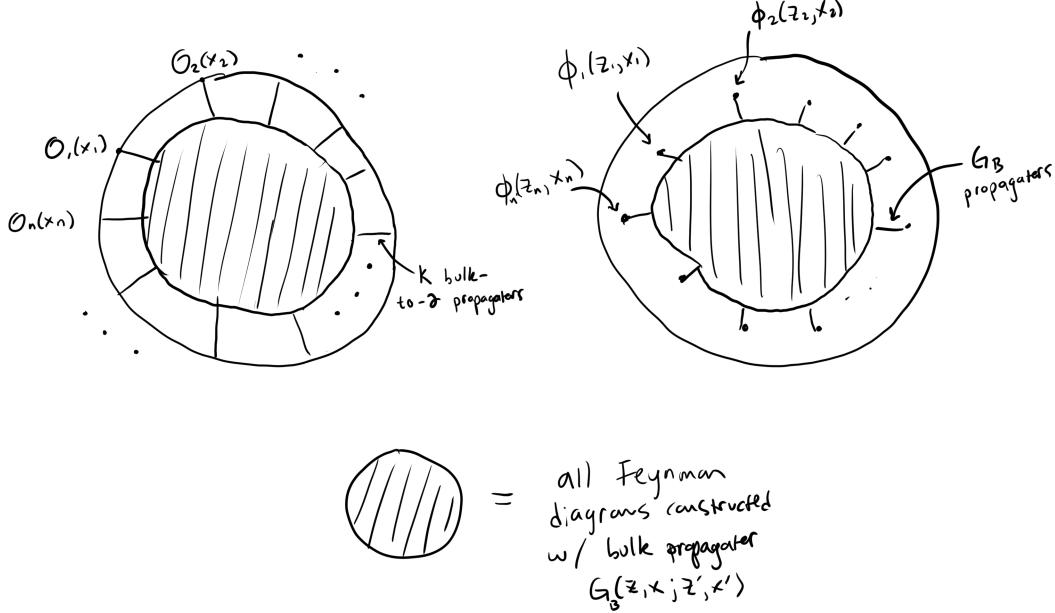


Figure 13: One of the ugliest figures I've ever made. Here, the shaded blob stands for all possible Feynman diagrams constructed from the bulk propagator.

Using the asymptotic $z \rightarrow 0$ form for K as written above, we can explicitly take the derivative with respect to z and get

$$RHS = \int_{\partial M} d^d x z^{-d+1} \delta(x - x') \left(z^{d-\Delta} \partial_z G(z, x; z'', x'') - (d - \Delta) z^{d-\Delta-1} G(z, x; z'', x'') \right). \quad (2499)$$

Now since $G(z, x; z'', x'')$ is normalizable, we know that it has the same $z \rightarrow 0$ scaling as the bulk normalizable mode, namely z^Δ . Thus

$$RHS = z^{-d+1} (z^{d-\Delta} \Delta z^{-1} - (d - \Delta) z^{d-\Delta-1}) G(z, x; z'', x''), \quad (2500)$$

where we are implicitly taking the $z \rightarrow 0$ limit. In the notation we used in class, $\Delta = d/2 + \nu$, and so

$$RHS = \lim_{z \rightarrow 0} (2\Delta - d) z^{-\Delta} G(z, x; z'', x'') = 2\nu z^{-\Delta} G(z, x; z'', x''). \quad (2501)$$

Setting this equal to LHS and moving the $2\nu z^{-\Delta}$ over to the other side and re-labeling some dummy variables, we get

$$\lim_{z \rightarrow 0} G(z, x; z', x') = \frac{z'^\Delta}{2\nu} K(z, x; x'). \quad (2502)$$

d) Let ϕ_i be the bulk scalar dual to a boundary operator \mathcal{O}_i . The correlation function for a product of \mathcal{O}_i 's at various points on the boundary can be determined by computing all

Feynman diagrams in the bulk that have external legs on the boundary. Thus, a correlation function of n \mathcal{O}_i 's involves $n K$ propagators (which connect the boundary \mathcal{O}_i 's to the part of the Feynman diagrams that live in the bulk, plus a bunch of G propagators which constitute the bulk part of the Feynman diagrams. On the other hand, we can consider the same class of Feynman diagrams, but with the external legs all made up of G propagators which terminate at points that have some small value of z . This is a bulk correlation function of ϕ_i fields. Taking the $z \rightarrow 0$ limit then gets us back to the correlation function of the \mathcal{O}_i 's. So, the only difference between the two correlation functions is whether we use K or G for the external legs. If we use G 's, then we need to take the $z \rightarrow 0$ limit for one of G 's arguments—luckily the previous part told us how to do this. So, using our result from c , we have

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{CFT} = \lim_{\{z_i\} \rightarrow 0} \prod_i (2\nu_i z_i^{-\Delta_i}) \langle \phi_1(z_1, x_1) \cdots \phi_n(z_n, x_n) \rangle. \quad (2503)$$

All of this is illustrated in figure 13.

144 December 9 — (Massive) Vectors in AdS

Yep, another problem from Hong's holography class. Consider a massive vector field in AdS:

$$S = - \int d^{d+1}x \sqrt{-g} \left(\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} m^2 A_M A^M \right), \quad (2504)$$

where M, N run over (z, x^μ) . a) When $m^2 = 0$, find the asymptotic behavior of A_μ at the boundary $z \rightarrow 0$. b) What is the scaling dimension of the boundary current J^μ corresponding to the bulk gauge field A ? c) When $m \neq 0$, what is the asymptotic behavior of A_M at the boundary? d) Now what is the scaling dimension of J^μ ? e) What happens to A_z when $m \neq 0$? f) What are the differences between the massive and massless cases?

a) The equation of motion when $m = 0$ is just

$$\partial_M (\sqrt{-g} F^{MN}) = 0 \implies \partial_M (z^{-d-1} g^{ML} g^{NO} \partial_{[L} A_{O]}) = 0, \quad \forall N. \quad (2505)$$

Let us assume the asymptotic behavior $A_\mu(z \rightarrow 0) \sim z^\Delta$, and work in a gauge where $A_z = 0$. The only derivative in the equations of motion we then care about in the $z \rightarrow 0$ limit is the one with $L = z$, and so setting $N = \mu$ we have

$$\partial_z (z^{-d-1+2+2} \partial_z z^\Delta) = 0 \implies (-d+3)\Delta + \Delta(\Delta-1) = 0, \quad (2506)$$

and so we have two options: $\Delta = 0$ or $\Delta = d-2$. Thus we can write

$$A_\mu(z \rightarrow 0) = a_\mu(x) + b_\mu(x) z^{d-2}. \quad (2507)$$

The a_μ is the non-renormalizable piece, while the b_μ part is renormalizable.

b) The non-renormalizable piece a_μ is the part that is relevant for computing the scaling dimension of the current in the CFT dual to A_μ , since a_μ is the part which we interpret as

a change in boundary conditions. To determine the scaling dimension of J^μ , we can look at the boundary integral

$$\int_{\partial AdS} d^d x a_\mu J^\mu. \quad (2508)$$

Now we consider performing the isometry $z \mapsto \lambda z$, $x^\mu \mapsto \lambda x^\mu$. a_μ is independent of z but it carries a covector index, so it transforms with a factor of λ . Writing $J^\mu(x/\lambda) = \lambda^{\Delta_J} J^\mu(x)$, we have

$$\int_{\partial AdS} d^d x a_\mu J^\mu \mapsto \lambda^{-d+1+\Delta_J} \int_{\partial AdS} d^d x a_\mu J^\mu. \quad (2509)$$

Since we need this term to be invariant, we find that $\Delta_J = d - 1$, as expected of a current in a d -dimensional CFT.

Note that A_μ and A^μ have different scaling behaviors since they differ by e.g. $g^{\mu\mu}$, which scales as z^2 . To determine the dimension of J^μ we need to integrate it against something with a covariant index, so it is the scaling of A_μ , not A^μ , which is needed.

c) When the vector field is massive, the equation of motion becomes

$$\partial_M (\sqrt{-g} F^{MN}) - \sqrt{-g} m^2 A^N = 0, \quad \forall N. \quad (2510)$$

Again, let $A_\mu \sim z^\Delta$ near the boundary. Then we have

$$\partial_z (z^{-d-1} \partial_z A_\mu g^{zz} g^{\mu\nu}) - z^{-d-1} m^2 A_\mu g^{\mu\nu} = 0, \quad (2511)$$

so that

$$\Delta \partial_z (z^{-d+3+\Delta-1}) - z^{-d+1+\Delta} R^2 m^2 = 0 \implies \Delta(-d + 2 + \Delta) - R^2 m^2 = 0, \quad (2512)$$

where the R^2 comes from the inverse metric factors. There are thus two possible choices for the scaling behavior of A_μ which are compatible with the equations of motion, and we can write

$$A_\mu = a_\mu z^{\Delta_+} + b_\mu z^{\Delta_-}, \quad \Delta_\pm = 1 - \frac{d}{2} \pm \sqrt{(d-2)^2/4 + m^2 R^2}. \quad (2513)$$

Sorry for the profusion of Δ 's! It's just entrenched as a theme by this point and there's no going back.

d) In standard quantization, the non-renormalizable part will be the Δ_+ piece. Looking at the boundary term $\int_{\partial AdS} b_\mu z^{\Delta_+} J^\mu$ and performing the re-scaling of x and z tells us that $-d + 1 + \Delta_J - \Delta_+ = 0$, and so in this case J^μ has scaling dimension

$$\Delta_J = d - 1 + \Delta_+ = \frac{d}{2} + \sqrt{(d-2)^2/4 + m^2 R^2}. \quad (2514)$$

Sanity check: when $m = 0$ we recover $\Delta_J = d - 1$, as required.

e) When $m \neq 0$, we can no longer use gauge invariance to fix $A_z = 0$. The z component of the equations of motion reads, focusing only on the z -dependence of A_z ,

$$\partial_z (\sqrt{-g} [g^{zz}]^2 \partial_z A_z) - \sqrt{-g} m^2 g^{zz} A_z = 0. \quad (2515)$$

Since all the components of the metric have the same z dependence, the z dependence of A_z is fixed in the same way as that of the A_μ .

f) In the massless case, we have gauge invariance under $A \mapsto A + d\chi$. Accordingly, the boundary J^μ operator must be divergenceless, and so it should be thought of as a conserved current. Since we integrate conserved currents over codimension 1 manifolds to get numbers, we need the dimension of J^μ to be $d - 1$. By contrast, when $m \neq 0$, there is no gauge invariance, and J^μ is not a conserved current; hence its scaling dimension is not fixed at $d - 1$.

145 December 9 — Wilson Loop Vevs in $\mathcal{N} = 4$ SYM using AdS/CFT

Yep, another problem from a pset in Hong's holography class. This time, we're computing Wilson loops in $\mathcal{N} = 4$ SYM with holography. The problem is as follows: by evaluating the saddle-point of the NG action corresponding to a geometry in which two quarks have been inserted a distance L apart in the boundary CFT, find the potential energy $V(L)$ coming from the interaction between the two quarks (the relevant Wilson loop here is a rectangle of sides L, t , where $t \gg L$). How does the result behave in the high temperature and low-temperature limits?

We can compute Wilson loop vevs in $\mathcal{N} = 4$ SYM in the limit $g_s \rightarrow 0$ (no sum over different topologies) and $\alpha' \rightarrow 0$ (when we can use the saddle-point solution to the string path integral). On the CFT side this limit is nontrivial since it corresponds to $N, \lambda \rightarrow \infty$.

We just need to compute the classical string action, since

$$\langle W(C) \rangle = Z_{str}[\partial\Sigma = C] \approx e^{iS_{cl}[\partial\Sigma=C]}, \quad (2516)$$

where Σ is the string worldsheet.

At finite T (here T is temperature, not the temporal length of the Wilson loop, which we will write as t), the appropriate bulk geometry to use is an AdS-Schwarzschild black hole at temperature T . The metric is

$$ds^2 = \frac{R^2}{z^2} \left(-(1 - \bar{z}^d)dt^2 + d\vec{x}^2 + \frac{1}{1 - \bar{z}^d}dz^2 \right), \quad (2517)$$

where we've defined

$$\bar{z} \equiv z/z_0, \quad T = \frac{d}{4\pi z_0}. \quad (2518)$$

z_0 is the location of the horizon, which is closer to the $z = 0$ boundary at larger temperatures.

Let us choose the contour C to run in the $x^1 - t$ plane. We can then parametrize the worldsheet with coordinates $(\tau, \sigma) = (t, x^1)$. We are interested in the energy of two quarks a distance L apart. If the temporal length T of the curve C is much larger than L , then the shape of Σ is determined by a function $z(\sigma) = z(x^1)$, with boundary conditions $z(\pm L/2) = 0$. We will use the NG action (rather than the Polyakov action) to compute S_{cl} , since we aren't ever going to need to quantize anything. The induced metric on the worldsheet is then determined by

$$ds_w^2 = \frac{R^2}{z^2} \left(-dt^2(1 - \bar{z}^d) + d\sigma^2 \left[1 + \frac{z'^2}{1 - \bar{z}^d} \right] \right), \quad (2519)$$

with $z' = \partial_\sigma z$.

The NG action is (again, assuming $t \gg L$ so that the Lagrangian on the classical solution can be treated as time-independent)

$$S_{NG} = -\frac{R^2 t}{\pi \alpha'} \int_0^{L/2} \frac{d\sigma}{z^2} \sqrt{1 - \bar{z}^d + z'^2}, \quad (2520)$$

where we pulled out the R^4/α'^2 from the determinant of the induced metric and used the symmetry $z(\sigma) = z(-\sigma)$ that must be satisfied by the classical solution.

We can eliminate the z' inside the square root by using the equations of motion. Since \mathcal{L} is independent of σ , we have

$$-z' \frac{\partial \mathcal{L}}{\partial z'} + \mathcal{L} = c, \quad (2521)$$

where c is a constant. For us, this is

$$\frac{z'^2}{z^2 \sqrt{1 + z'^2 - \bar{z}^d}} = \frac{\sqrt{1 + z'^2 - \bar{z}^d}}{z^2} + c, \quad (2522)$$

or

$$\frac{1 - \bar{z}^d}{z^2 \sqrt{1 + z'^2 - \bar{z}^d}} = c. \quad (2523)$$

We can get c by noticing that at $\sigma = 0$, $z' = 0$ by symmetry. So, let $z_* \equiv z(0)$. Then

$$c = \frac{\sqrt{1 - \bar{z}_*^d}}{z_*^2}. \quad (2524)$$

The energy of the Wilson line configuration is then computed as

$$E(L) = \frac{\sqrt{\lambda}}{\pi} \int_0^{z_*} \frac{dz}{z^2 z'} \sqrt{1 + z'^2 - \bar{z}^d}, \quad (2525)$$

since $\lambda = R^4/\alpha'^2$. Solving for z' in terms of z and c , we have

$$z' = \sqrt{(1 - \bar{z}^d) \left(\frac{1 - \bar{z}^d}{c^2 z^4} - 1 \right)}. \quad (2526)$$

Putting this into the integral and doing some housekeeping, we get

$$E(L) = \frac{\sqrt{\lambda}}{\pi} \int_0^{z_*} \frac{dz}{z^2 \sqrt{1 - c^2 z^4 / (1 - \bar{z}^d)}}. \quad (2527)$$

Now $E(L)$ has a $z \rightarrow 0$ divergence, but this just corresponds to the diverging mass of the two quarks. Recalling that the quark mass goes as the inverse of their z -coordinates, we expect the quark mass to show up as a $1/\epsilon$ divergence if we cut the integral off below at ϵ .

Now in order to get $E(L)$, we need an expression for z_* in terms of L . We can get an integral equation which gets us part way there by solving for z' and integrating from $\sigma = -L/2$ to $\sigma = 0$:

$$\frac{L}{2} = \int_0^{z_*} dz \left[(1 - \bar{z}^d) \left(\frac{1 - \bar{z}^d}{c^2 z^4} - 1 \right) \right]^{-1/2}. \quad (2528)$$

To see what's happening here more clearly, there are two limits we can take. The first is the $T \rightarrow 0$ limit (or equivalently, the small L limit). In this limit we can send $\bar{z} \rightarrow 0$ and $c^2 \rightarrow z_*^{-2}$, since when $T = 0$ the horizon is pushed to $z_0 = \infty$. In this limit, our integral equation determining z_* is

$$\frac{L}{2} = \int_0^{z_*} dz \frac{1}{\sqrt{z_*^4/z^4 - 1}} = z_* \alpha, \quad (2529)$$

where $\alpha \approx 0.6$ is determined in terms of Elliptic integrals. This means that at $T = 0$, the maximal $z(\sigma)$ value on the worldsheet extends a distance into the bulk which grows linearly with L .

Now we can calculate $V(L)$ in this limit, by using $E(L) = 2M + V(L)$ and subtracting off the divergent mass piece:

$$V(L) = \frac{\sqrt{\lambda}}{\pi z_*} \left(\int_{\epsilon/z_*}^1 \frac{dx}{x^2 \sqrt{1-x^4}} - \frac{1}{\epsilon} \right). \quad (2530)$$

The integral can be evaluated in terms of hypergeometric functions. Doing this, sending $\epsilon \rightarrow 0$, and using our expression for z_* , we get

$$V(L) = -\frac{2\alpha\sqrt{\lambda/\pi}\Gamma(3/4)}{L\Gamma(1/4)}. \quad (2531)$$

The most important features here are the $1/L$ dependence (from scale invariance in the CFT), and the interesting $\sqrt{\lambda}$ coupling dependence.

Now we can look at what happens at higher T , or equivalently, at Wilson lines that have L large enough so that z_* approaches the horizon at z_0 . By looking at (2528), we see that z_* is monotonically increasing with L (this is a bit gross to show, but it ultimately comes down to $\partial_{z_*} c < 0$).

As we keep increasing L , there reaches a point $L_{\text{screening}}$ where (2528) has no solution. Looking back, we see that this must mean that $c = 0$: this is when z_* “disappears” behind the black hole horizon. For $L > L_{\text{screening}}$, we no longer can have a Wilson line connecting the two quarks: the worldsheet ends up splitting apart, and terminating on the black hole. In the way we've been doing things, this corresponds to the trivial solution $V(L) = 0$, and the quarks are fully screened. The crossover between the $1/L$ dependence of the potential and the fully screened potential can be found numerically, but I'll be content with this simple understanding of the two limits.

146 December 12 — RG for fermions a la Shankar

I had to do a mini final project for a class Ashvin Vishwanath taught at Harvard; my topic was on understanding Shankar's RG approach to systems with Fermi surfaces. So, today's diary entry is a bit of cheat: just a copying-and-pasting of the notes I wrote up.

In these notes, we review how to think about systems with a Fermi surface from an RG point of view. The RG program has some interesting technical modifications when applied

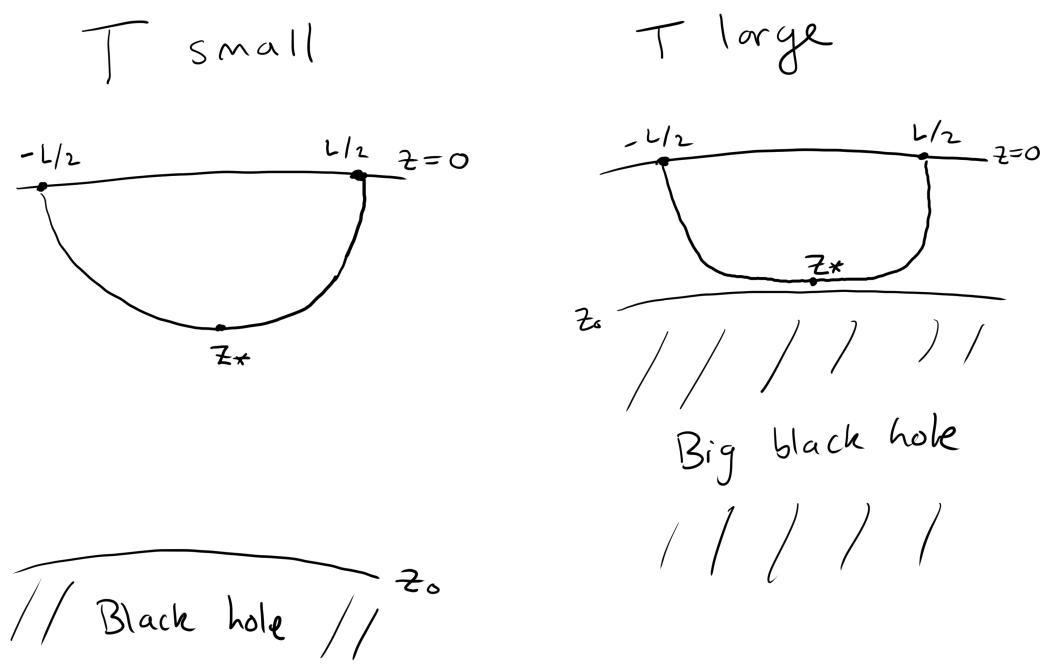


Figure 14: Another one of the ugliest figures I've ever made; explanation is in the text.

to a system with a Fermi surface, since momenta are not scaled towards zero, but rather towards some codimension-1 (in the example we will look at) surface in momentum space. More importantly, the RG approach provides a more modern justification of Fermi liquid theory, as well as a clear perspective on how certain types of instabilities (e.g. the BCS instability) can develop.

The original and classic article on the subject, which we will draw most heavily from, is [11]. Polchinski's notes [9] are also helpful in providing a heuristic effective field theory-based understanding of the relevant material. Finally, the approach taken in [4] provides a lot of useful details that are absent in [11], and proved to be a very helpful reference. While these notes are a bit longer than they should be, many of the calculational details in the middle parts may be skipped over without loosing much.

Illustration of the idea: $d = 1$

We start by illustrating the RG program in the case of a 1+1 dimensional Fermi system. One-dimensional systems are different both qualitatively and quantitatively than $d > 1$ systems, but starting with $d = 1$ will help us build intuition for how calculations in higher dimensions work.

Suppose we are given a system of free Fermions in $d = 1$ at finite density. The low energy theory will then consist of left- and right-moving free gapless fermions $\psi = (\psi_L, \psi_R)$, after linearizing the spectrum about the two Fermi points $\pm k_F$. We are interested in the stability of the Fermi surface with respect to turning on interactions. That is, we are interested in examining the stability of the free fermion fixed point.

To do the RG program, we will write down the most general possible perturbations to the free fermion theory, and examine their relevance. First we need the scaling of ψ under RG, where each RG step corresponds to integrating out modes with $v_F k$ within some narrow energy window of width $d\Lambda$, where k is measured relative to k_F . The window is taken to be $[\Lambda - d\Lambda, \Lambda]$, where $\Lambda \ll K_F$ is the UV cutoff. In imaginary time, the free action is

$$S_0 = \int_{|k|<\Lambda} dk \int_{\mathbb{R}} d\omega \bar{\psi}(i\omega - k)\psi, \quad (2532)$$

where we have set $v_F = 1$ for simplicity (note that k is measured wrt the Fermi surface). After we (trivially) integrate out the modes with $\Lambda s < |k| < \Lambda, \omega \in \mathbb{R}, s < 1$ (note that we integrate over *all* frequency modes), we just have to perform the re-scaling $k', \omega' = k/s, \omega/s$, so that

$$S_0 \mapsto \int_{|k|<\Lambda} dk' \int_{\mathbb{R}} d\omega' s^3 \bar{\psi}(sk', s\omega') (i\omega' - k') \psi(sk', s\omega'). \quad (2533)$$

Since we are interested in the properties of the RG trajectories near the free fermion fixed point, we evidently need $\psi(sk, sw) = s^{-3/2}\psi(k, \omega)$. By dimensional analysis this means that $\psi(sx, st) = s^{-1/2}\psi(x, t)$. We will also need the scaling of the mixed $\psi(t, k)$, which is similarly $\psi(s^{-1}t, sk) = s^{-1/2}\psi(t, k)$. Note that in $d = 1$ the scaling dimension of ψ matches its mass dimension, but that this will not be true for $d > 1$, since under RG only one (of the d) momentum components is rescaled. For example, this means that $\psi(s^{-1}t, sk) = s^{-1/2}\psi(t, k)$ in *any* dimension, not just $d = 1$.

The most relevant departure from the free fermion action we can have is a quadratic term. Under RG,

$$\int_{k,\omega;\Lambda s} \mu(k,\omega) \bar{\psi}(k,\omega) \psi(k,\omega) \mapsto \int_{k,\omega;\Lambda} s^{-1} \left(\sum_{\alpha,\beta} k^\alpha \omega^\beta s^{\alpha+\beta} \mu_{\alpha\beta} \right) \bar{\psi}(k,\omega) \psi(k,\omega), \quad (2534)$$

where the $\mu_{\alpha\beta}$ are the Taylor series coefficients for the chemical potential. We see that the μ_{00} term is relevant, scaling as s^{-1} (recall $s < 1$), the μ_{01} and μ_{10} terms are marginal, while all else are irrelevant (around the free fermion point). The marginal parts just go towards being absorbed in the free action, while μ_{00} contributes to the chemical potential. Although the chemical potential is relevant, we will typically be thinking of working at fixed density, so that μ_{00} is fixed.

In general, an interaction between $2N$ fermion fields looks like

$$\int \prod_{i=1}^{2N-1} dk_i d\omega_i u(\{k_i\}, \{\omega_i\}) \prod_j^N \bar{\psi}(k_j, \omega_j) \psi(k_{j+N}, \omega_{j+N}), \quad (2535)$$

and so if u is a constant, it scales as $s^{2(2N-1)} s^{-3N} = s^{N-2}$. This is relevant if $N = 1$ (as we just saw), marginal if $N = 2$, and irrelevant otherwise. So the only other types of interactions we need to look at are quartic ones, and we can restrict our attention to constant interactions, since k - and ω -dependent interactions are irrelevant⁹³.

Since the 4-Fermi interactions are marginal at tree level, we need to go to at least 1-loop to see whether they affect the stability of the free fixed point. The 1-loop corrections to the effective action come in two classes: there are 1-loop “circle touching a line” diagrams that renormalize the free term (wavefunction renormalization) and the chemical potential, and there are 1-loop diagrams that renormalize the U vertex. In all of these diagrams, the internal lines are taken to lie within the energy shell $[\Lambda - d\Lambda, \Lambda]$.

The 1-loop correction to μ_{00} is easily computed: denoting μ' as the new coupling after taking into account 1-loop corrections, we have

$$\mu'_{00}(\Lambda s) = \mu_{00}(\Lambda) - \frac{U}{4\pi^2} \int_{s\Lambda < |k| < \Lambda} dk \int_{\mathbb{R}} d\omega \frac{e^{i\eta\omega}}{i\omega - k} \quad (2536)$$

where the minus sign comes from the fermion bubble and η is a positive infinitesimal. Since η is positive we need to close the ω contour in the upper half plane. The pole at $\omega = -ik$ will then only get picked up if $k < 0$, and so we have

$$\mu'_{00}(\Lambda s) = \mu_{00}(\Lambda) - \frac{U}{2\pi} \int_{s\Lambda < |k| < \Lambda} dk \theta(-k) = \mu_{00}(\Lambda) - \frac{U}{2\pi} \Lambda(1-s). \quad (2537)$$

Writing $s = 1 - dt$ for t increasing towards the IR, and defining the dimensionless coupling $\bar{\mu}_0 \equiv \Lambda^{-1} \mu_{00}(\Lambda)$, we have

$$\bar{\mu}'_0 = \frac{1}{1-dt} \left[\bar{\mu}_0 - \frac{U}{2\pi} dt \right] \implies \bar{\mu}'_0 - \bar{\mu}_0 = \bar{\mu}_0 dt - \frac{U}{2\pi} dt, \quad (2538)$$

⁹³This is a little bit glib. In reality, the k - and ω -dependence may not be irrelevant because the interactions may be singular functions of ω or k . For a discussion of things along this line, see e.g. Sec 3 of [4].

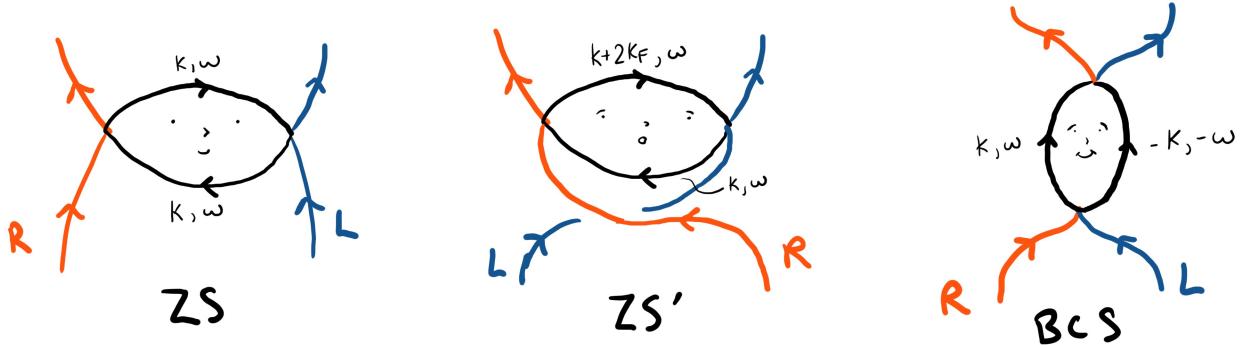


Figure 15: 1-loop diagrams that contribute to the renormalization of the four-fermion vertex to order U^2 .

where we used $\mu'_{00}(\Lambda s) = s\mu'_{00}(\Lambda)$ since μ'_{00} has mass dimension 1. This means that the flow for $\bar{\mu}_0$ is

$$\frac{d\bar{\mu}_0}{dt} = \bar{\mu}_0 - \frac{U}{2\pi}. \quad (2539)$$

The fixed point for the free-fermion chemical potential evidently moves away from zero when we turn on interactions and becomes positive, equal to $\bar{\mu}_0^* = U/2\pi$. This is because the free fermion fixed point is characterized by a fixed density (as we are holding K_F fixed during the RG), and so when interactions are turned on, in order to maintain constant density one has to introduce a chemical potential to counteract the attractive / repulsive effects of the interactions.

Now for the 1-loop renormalization of U . The relevant diagrams are shown in the figure—we are labeling them according to the conventions in [11]. The first ZS diagram enters with coefficient 1, while the middle ZS' diagram comes with a minus sign because of the fermion exchange. The BCS diagram comes with a factor of $-1/2$ due to the symmetry of exchanging the two internal propagators; the minus sign can be checked by expanding the interaction in the exponential to second order and using Wick's theorem. Since the ω - and k -dependent parts of the U interaction are irrelevant, to compute the corrections to U we can take the external legs to be at zero momentum (relative to K_F) and to be at zero frequency. Thus, letting U' denote the renormalized interaction, we have

$$U' - U = \frac{U^2}{4\pi^2} \int_{\Lambda s < |k| < \Lambda} dk \int_{\mathbb{R}} d\omega \left[\sum_{\alpha=\pm 1} \frac{1}{(i\omega - E(\alpha K_F + k))^2} - \frac{1}{(i\omega - E(K_F + k))(i\omega - E(-K_F + k))} \right. \\ \left. - \sum_{\beta=\pm 1} \frac{1}{2} \frac{1}{(i\omega - E(\beta K_F + k))(-i\omega - E(-\beta K_F - k))} \right]. \quad (2540)$$

The first ZS diagram is zero: both of the poles are in the same half-plane, and so closing the ω contour in the opposite half-plane leads to a vanishing result. The second ZS' diagram does contribute: $E(K_F + k)$ and $E(-K_F + k)$ always have opposite signs for

$0 < |k| < 2K_F$, and so the ω integral contains a pole in both half-planes. Taking the residue at $\omega = -iE(K_F + k)$, we get the contribution

$$ZS' = \frac{U^2}{2\pi} \int_{\Lambda s < |k| < \Lambda} dk \frac{1}{E(K_F + k) - E(-K_F + k)}. \quad (2541)$$

Here the positive sign comes from the fact that the contour is closed in the lower half plane, and is thus performed with negative handedness. Since we are treating the dispersion as linear within $\pm\Lambda$ of K_F , we have $E(K_F + k) = -E(-K_F + k)$. Thus this diagram gives

$$ZS' = \frac{U^2}{4\pi} \int_{\Lambda s < |k| < \Lambda} dk \frac{1}{E(K_F + k)}. \quad (2542)$$

Now since the momentum of the lower internal line in the ZS' diagram can only lie within one of the two intervals of thickness $\Lambda(1-s)$ on either side of the right Fermi point, the momentum integration gives us a factor of $2\Lambda(1-s)$, and so since we can take $E(K_F + k) \approx \Lambda$ in the integration region,

$$ZS' = \frac{U^2}{2\pi}(1-s). \quad (2543)$$

The third BCS diagram simplifies since $E(K) = E(-K)$: this is the symmetry which leads to the BCS instability. Then the BCS integral has a pole in both half-planes, giving a non-zero result no matter which way we close the contour. After doing the contour integral around the pole at $\omega = iE(K)$, we get

$$BCS = - \sum_{\beta=\pm 1} \frac{1}{2} \frac{U^2}{4\pi^2} \frac{2\pi i}{-i} \int_{\Lambda s < |k| < \Lambda} dk \frac{1}{-2E(\beta K_F + k)} = -\frac{U^2}{2\pi}(1-s) \quad (2544)$$

Here we have used that there are four momentum regions that contribute, each of width $\Lambda(1-s)$ (one above and one below each Fermi point).

Since the BCS and ZS' graphs give exactly opposite contributions, we have

$$\beta_U = 0, \quad (2545)$$

at least to order U^2 . Thus U remains exactly marginal. Note that within our approximation of linearizing about the Fermi points, this result is independent of the filling, and so does not require us to be at the special value of $\nu = 1/2$ where the system is most susceptible to the onset of CDW order⁹⁴. The vanishing of the β function for U actually extends to all loops as can be seen from certain Ward identities, but due to space constraints we will have to be content with the 1-loop analysis. Summarizing, in $d = 1$, small interactions are exactly marginal, and there are no relevant operators to take us away from the free Fermion fixed point, which is thus stable for a finite range of U .

The basic idea remains the same in higher dimensions, but instead of having two “flavors” of fermions (left- and right-movers), we have an infinite number of “flavors”, one for each direction on the Fermi surface. We will look at $d = 2$ dimensions first before briefly commenting on $d = 3$.

⁹⁴Of course, we know that at half filling $\nu = 1/2$ and strong enough U , a CDW state is formed. Where does the instability come from? It comes from the neglected Umklapp term, $e^{4iK_F} R^\dagger \partial R L^\dagger \partial L + h.c.$, which will survive in the long-distance limit provided we hold the filling fraction at $1/2$. Eventually as the interactions are increased, the scaling dimensions of the ψ fields change sufficiently to overcome the two extra powers of momentum in the Umklapp term and turn it from being irrelevant to being relevant.

$d = 2$

In one dimension the only scattering processes that weren't irrelevant near the free fermion fixed point came from $LR \leftrightarrow LR$ scattering. In two dimensions, there are more options. To start we will assume a circular Fermi line, with θ its angular coordinate. Label a generic scattering process by $12 \leftrightarrow 34$, where the momenta of all four fields are constrained to lie within a distance of Λ from the Fermi circle. The larger Λ is, the more options we have for assigning momenta to the four fields. These options become increasingly constrained as $\Lambda \rightarrow 0$ in the deep IR. Because of this, and because k -dependent corrections to the interaction vertex are irrelevant about the free fixed point, we can determine the interactions that survive in the IR by focusing only on those coupling functions $U(12 \rightarrow 34)$ which are such that all of $1, 2, 3, 4$ lie precisely on the Fermi circle, since the other coupling functions will disappear when we take $\Lambda \rightarrow 0$.

By drawing circles of radius K_F around the origin and around $K_1 + K_2 = K_3 + K_4$, one sees with the usual construction that momentum conservation allows only three possibilities: $K_1 = K_3, K_2 = K_4$, $K_1 = K_4, K_2 = K_3$, and $K_1 = -K_2, K_3 = -K_4$. We will denote the coupling for the first process as $F(\theta_1 - \theta_2)$, the coupling for the second by $-F(\theta_1 - \theta_2)$ (by Fermi statistics), and the coupling for the last process by $V(\theta)$, where θ is the angle of scattering between the two zero-momentum pairs. All are easily seen to be marginal to tree level, so we need to go to 1-loop to examine their fates.

We will first look at the BCS interaction $V(\theta)$. Fix the incoming pair of electrons to be at angles $\theta_1, \theta_1 + \pi$, and the final pair to be at angles $\theta_2, \theta_2 + \pi$ (due to the irrelevance of ω -dependent interactions, we set all external frequencies to zero, as usual). Look at the ZS diagram. By drawing shells of radius $K_F \pm \Lambda$ around the origin and the endpoint of the vector $\theta_1 + \theta$ (here θ is the momentum in the lower internal loop line), we see that as we take $s \rightarrow 1$, there are only two options: either $\theta = \theta_2$, or $\theta = -\theta_1$. More precisely, the available phase space for the internal loop momentum goes as $(1-s)^2 = dt^2 \rightarrow 0$. So, the ZS diagram does not contribute. The exact same reasoning applies to the ZS' diagram, and so kinematic constraints eliminate both of these corrections in the IR limit.

Thus only the BCS diagram can contribute. In this diagram, the internal loop momenta are free to range over the entire Fermi circle, and we get (closing the frequency integral in the upper half-plane, then doing the momentum integral in the usual way)

$$dV(\theta) = -\frac{1}{2} \frac{2\Lambda(1-s)}{2\pi} \int d\theta' V(\theta') V(\theta - \theta') \frac{1}{2\Lambda}. \quad (2546)$$

Here the factor of $2\Lambda(1-s)$ arises since there are two shells (at $K_F \pm \Lambda$) that the internal loop momentum can live in. This is solved by decomposing the interaction into harmonics as $V(\theta) = \sum_n e^{-il\theta} V_n$. Multiplying by $e^{in\theta}$ and integrating over θ , the above equation becomes

$$\frac{dV_n}{dt} = -\frac{V_n^2}{4\pi}, \quad (2547)$$

where $dt = -d\ln \Lambda$ is the parameter along the RG flow. Thus negative (positive) interactions are marginally (ir)relevant: initially negative V_n lead to instabilities as they head off to $-\infty$ under RG, while initially positive $V(\theta)$ are made marginally irrelevant and disappear under RG. So, unlike in one dimension, attractive interactions in the BCS channel are made marginally relevant at one-loop, and the SCing instability is not canceled by a compensating

CDW-type instability: the Fermi surface is unstable with respect to arbitrarily small attractive interactions. A derivation of this β function in the high-energy style is as follows (this is just the worked-out version of a statement that appears in Polchinski's TASI notes).

We want to know the leading order correction to the BCS interaction, which comes from the 1-loop BCS diagram. We will evaluate the diagram with all legs fixed at an energy E . In \mathbb{R} time then, ignoring the momentum-dependence of the interaction and the angular dependence of V for simplicity (i.e. looking at the $n = 1$ mode), we have (letting k denote a momentum on the FS and l denote a momentum orthogonal to the FS)

$$\begin{aligned} \text{BCS diagram} &= V^2 \int \frac{dE' d^{d-1} k dl}{(2\pi)^{d+1}} \frac{1}{[(1+i\epsilon)(E+E') - v_F(k)l][(1+i\epsilon)(E-E') - v_F(k)l]} \\ &= V^2 \int \frac{d^{d-1} k dl}{(2\pi)^d} \frac{1}{(1+i\epsilon)(2E - (1-i\epsilon)v_F(k)l - v_F(k)l)} \\ &= \frac{V^2}{2} \int \frac{d^{d-1} k dl}{(2\pi)^d} \frac{1}{E - v_F(k)l} \\ &\approx -\frac{V^2 N}{4\pi} \ln(\Lambda/E), \end{aligned} \tag{2548}$$

where Λ is the upper cutoff and $N = \int \frac{d^{d-1} k}{(2\pi)^{d-1}} v_F(k)^{-1}$ is the dos at the Fermi level, which we will set to 1 for simplicity. This logarithmic divergence means that we should add to the action the counterterm

$$\delta V = +\frac{V^2}{4\pi} \ln(\Lambda/E). \tag{2549}$$

This counterterm means that at the energy scale E , the effective BCS interaction is just given by V , at least to this order. Then we obtain the β function for V as

$$\frac{dV}{d \ln \Lambda} = \frac{V^2}{4\pi} \implies \frac{dV}{dt} = -\frac{V^2}{4\pi}, \tag{2550}$$

which matches the Wilsonian derivation of the β function. Note that this is independent of the spatial dimension d : all the action comes from the integral over the momentum transverse to the FS, so the fact that the FS is codimension 1 means that the RG equation for the BCS interaction is always what it is in 1 dimension. The other diagrams contributing to the 1-loop renormalization of the coupling in the BCS channel don't contribute: they are independent of the energy E assigned to the incoming / outgoing legs, and the dimensionality of the Λ dependence is instead saturated by K_F , with the diagrams going as Λ/K_F . Since we send $\Lambda/K_F \rightarrow 0$, these diagrams do not contribute. This is the high-energy approach analogue of the reduced phase space which caused such diagrams to not contribute in the Wilsonian approach.

Now we look at the renormalization of the forward scattering interaction $F(\theta)$, where the two left legs on the associated Feynman diagrams are at identical angles, and the two right legs are also at identical angles. Because of this, the momentum in the ZS loop is totally unconstrained, and runs over the whole Fermi circle. However, both poles in the integrand lie in the same half-plane, and so it makes zero contribution to $dF(\theta)$.

The ZS' diagram makes zero contribution to β_F for kinematic reasons: the internal momenta are restricted to lie in a phase space volume that goes as $(1-s)^2$, and so it doesn't

contribute to β_F for the same reason that the ZS' didn't contribute to the renormalization of $V(\theta)$. The same is true for the BCS diagram's contribution to β_F . Thus none of the diagrams contribute to a renormalization of F , and F remains exactly marginal to one-loop.

The picture in $d = 2$ is thus the following: if we start with an attractive interaction in the BCS channel the free fermion point is unstable to a SCing transition, while if we start with a repulsive interaction, the IR theory is determined by the marginal $F(\theta)$ interactions—this is Landau's Fermi-liquid theory.

$d = 3$

The basic idea is the same as in $d = 2$, so we will be brief. When we shrink down to $\Lambda \ll K_F$, the available scattering events are again restricted kinematically: by drawing spheres around the origin and the endpoint of the sum of the incoming momentum, we see that for a process $12 \rightarrow 34$, we need $3, 4$ to lie in the cone defined by rotating the vectors $1, 2$ about the vector $1 + 2$. That is, unless the incoming momenta are in the BCS channel, in which case the outgoing momentum 3 can be chosen freely. Let the coupling constant function for the former process be $F(\theta, \phi)$ (θ is the opening angle between $1, 2$ and ϕ is the angle of rotation between the plane defined by $1, 2$ and the plane defined by $3, 4$), and let the coupling function for the latter process be $V(\Omega_1, \Omega_2)$ (Ω_i are unit vectors on S^2 , and V is sensitive only to their dot product).

First look at the renormalization of $V(\Omega_1, \Omega_2)$. The ZS and ZS' diagrams again do not contribute due to the vanishing phase space available for the internal loop momentum (seen by drawing the three-dimensional version of the pictures described earlier). The BCS diagram does contribute, and we get

$$\frac{dV(\Omega_1, \Omega_2)}{dt} = -\frac{1}{4\pi} \int d^2\Omega V(\Omega_1, \Omega) V(\Omega, \Omega_2). \quad (2551)$$

Now $V(\Omega_1, \Omega_2)$ is only a function of $\Omega_1 \cdot \Omega_2$, so we can decouple this equation with Legendre polynomials: $V(\Omega_1, \Omega_2) = \sum_l V_l P_l(\Omega_1 \cdot \Omega_2)$. Putting this in and using the addition theorem for Legendre polynomials to simplify the RHS gives $dV_l = -\frac{1}{4\pi} V_l^2$, which is the same β function as in two dimensions—the BCS instability occurs in the same fashion as before.

The renormalization of $F(\theta, \phi)$ is also the same as in $d = 2$, namely that the ZS diagram doesn't contribute, while the other two diagrams are suppressed by powers of Λ/K_F for phase space reasons, and so $F(\theta, \phi)$ remains marginal⁹⁵. It turns out that interactions $F(\theta, \phi), F(\theta, \phi')$ with $\phi \neq \phi'$ do not mix (inserting one such interaction as a loop correction to the other interaction leads to a diagram that vanishes as $(d\Lambda)^k$, $k > 1$ because of kinematic constraints), and so as long as we are interested in measuring the system with small-momentum probes, we can focus on the $F(\theta, 0)$. Decomposing these interactions into

⁹⁵In all of our analysis of the β functions for the couplings, we might worry that the wavefunction renormalization would make a correction that we haven't captured. This is not so, though: the wavefunction renormalization comes from $\psi \mapsto \psi/\sqrt{1 + \partial_\omega \Sigma|_{\omega=0}}$ where Σ is the full self-energy, which is designed to ensure that ω -dependent corrections to the propagator don't change the form of the $\bar{\psi}i\omega\psi$ free term. This ends up changing the interaction vertex by $U \mapsto (1 + \partial_\omega \Sigma|_{\omega=0})^2 U$, since the interaction has four ψ legs. However, the first diagram contributing to $\partial_\omega \Sigma$ is the “deathy hallows / sunrise diagram”, which is at $O(U^2)$. Thus corrections to U due to the wavefunction renormalization are $O(U^3)$, and our $O(U^2)$ results for the β functions are unaffected by this subtlety.

Legendre polynomials gives us a collection of marginal parameters, which form the basis for Fermi-liquid theory in $d = 3$.

Before closing, we briefly comment on the extension to higher loops: going beyond 1-loop turns out to be very simple in the limit $\Lambda \rightarrow 0$, because kinematic constraints limit the types of higher-loop diagrams that can contribute. Consider for example corrections to the propagator. First, we see that “cactus diagrams” always contribute, since such diagrams can consist entirely of forward-scattering vertices. With a bit of work one can show that all other diagrams vanish because they are proportional to $(d\Lambda)^k$ with $k > 1$, again because of the kinematic constraints imposed by the Fermi surface. The cactus diagrams can be easily accounted for to arbitrary loop orders by performing geometric sums, and so in the IR the renormalization program is very simple. Since in the IR only forward scattering processes survive, we have an infinite number of conservation laws, namely the conservation of particle number at each point on the Fermi surface.

More things to learn / open questions

I'll close by briefly mentioning some things which went unmentioned / things I do not understand. First, I have ignored the possibility of having a nested Fermi surface (other than in $d = 1$, where all Fermi surfaces are nested). Here, “nesting” means that for all k on the Fermi surface, there is some q such that $E(k + q) = -E(k)$. Having such a symmetry of the spectrum is needed to gap out the Fermi surface (for topological reasons, which one can see by realizing that the Fermi surface is a vortex in (k, ω) space—no room to go into that here). For example, the instability in the BCS case is made possible by the symmetry $E(k) = E(-k)$. The existence of a nesting vector changes the kinematics of scattering processes, and certain diagrams which we threw away on the basis of the small phase space available for the internal lines make non-zero contributions when we have nesting: these new diagrams can lead to CDW-like instabilities for arbitrarily small interactions, which can then gap out the Fermi surface.

One question that I could not find the answer to in the literature is the following: how can nesting play an important role in so many compounds that we see in nature? Nesting depends very sensitively on the exact shape of the Fermi surface, but since the shape of the Fermi surface is controlled by a relevant term like $\mu\bar{\psi}\psi$, having the shape of the Fermi surface be precisely nested in the IR requires extreme fine-tuning in the UV⁹⁶. Are there some sort of symmetries that constrain the shape of the Fermi surface in such compounds? I'm not sure.

There are also a few subtleties regarding the actual RG calculations that I do not yet fully understand. First of all, I've been working at $T = 0$ throughout. What happens when $T > 0$? It turns out that various diagrams we have neglected make a non-zero contribution when $\Lambda < k_B T$, so that in the deep IR our analysis is incomplete at any temperature. I don't yet understand the full implications of this, although some answers in this direction are contained in [4]. Similarly, I have been a bit glib in neglecting the k and ω dependence of the four-fermion interactions, on the grounds of the irrelevance of this dependence near the free fermion fixed point. It turns out that the interactions can actually be *singular* as a

⁹⁶The size of the Fermi surface is also controlled by a relevant operator of the same form, but since the Fermion density is something that we can fine-tune in the lab, having a system with a certain size of Fermi surface is not as worrisome. By contrast, there is no experimental way to control the shape of the FS (as far as I'm aware).

function of k and ω , meaning that treating them as constants is not allowed. Exactly what the ramifications of this are is unclear.

Of course, the big gaping open problem is how to think about non-Fermi liquids (in greater than one dimension, at least) in a systematic way. The RG approach really emphasizes how strange non-Fermi liquids are (at least, ones that are not describable as “Fermi surface + something”, where “something” is e.g. an order parameter field for some broken symmetry): at least for weak coupling and generically shaped Fermi surfaces, it’s hard to imagine how we get anything other than a Fermi liquid / something with an instability to a gapped state. We know that there must be some fixed point other than the free fermion fixed point out there, but I of course have no idea what that fixed point could look like.

Appendix: Phonons and the BCS instability

To get a SCing instability we need a negative *BCS*-channel interaction V . Putting subtleties like the KL mechanism aside, this can only come from attractive electronic interactions induced by interactions between the electrons and other things, like phonons, which somehow must be strong enough to overcome the Coulomb repulsion. Here we follow + elaborate on [9] and look at the effect of including electron-phonon coupling. We will work in greater than 1 spatial dimension, since in the 1-dimensional case the way that the phonon coupling renormalizes is rather different.

The electrons start with a repulsive interaction in the BCS channel, which as we have seen evolves under *RG* as (in the isotropic case) $dV/dt = -\alpha V^2$ for $\alpha > 0$ some constant (strictly speaking, V should be V_n , one of the angular modes of the interaction; in the following we will focus on the s-wave $n = 0$ channel). Taking $dt = -d \ln \Lambda$ and integrating the β function gives

$$V(\Lambda) = \frac{V(\Lambda_0)}{1 + \alpha V(\Lambda_0) \ln(\Lambda_0/\Lambda)}. \quad (2552)$$

Letting $V(\Lambda_0)$ be the bare Coulomb interaction, we see that the Coulomb interaction gets logarithmically screened at low energies. For the problem at hand, we will be interested in $V(\Lambda)$, with Λ the Debeye scale, viz. the scale below which the phonons start to decouple (Λ is the maximum allowed phonon frequency, so above Λ all the phonon modes are saturated). The reason for this is that the phonon interaction will essentially not be renormalized until we run the flow down to Λ , and so the value of V at Λ determines the strength of the repulsive interaction that we will need to compare with the phonon-induced interaction in order to determine if superconductivity occurs. The reason why the phonon-induced interaction doesn’t get renormalized until we get down to Λ is because it is retarded, with phonon interaction lines in diagrams occurring only for frequency transfers less than Λ . In the BCS channel, where the corrections to the phonon interaction contain two phonon lines and two internal electron lines, the phonon lines must carry frequency Λ_t , where Λ_t is the energy of the energy shell being integrated out during the RG procedure (because the external lines all are on the FS and so have ≈ 0 frequency). But since the interaction is turned off for $\Lambda_t > \Lambda$, the phonon interaction is not renormalized until we get down to Λ , as claimed. At least, this is the intuitive idea—below we will give a more careful treatment.

Anyway, since $\Lambda_0 \gg \Lambda$ as Λ_0 is the huge bare electronic energy scale,⁹⁷ this effect is

⁹⁷ $\Lambda \approx 10$ eV for a typical bandwidth of the conduction electrons.

important to keep in mind since it favors the onset of superconductivity: in most cases the logarithm $\ln(\Lambda_0/\Lambda)$ is $O(10)^{98}$ and α is $O(1)$, so that $V(\Lambda)$ is never larger than $\sim 1/10$, no matter how large $V(\Lambda_0)$ is; this favors superconductivity. We will see shortly that $\Lambda_0 \gg \Lambda$, accounting for the large amount of screening quoted above.

We treat the phonons in the following way. The action for just the phonons is

$$S_{ph} = \int dt d^3q d^3k \left(\partial_t \phi^i(q) \partial_t \phi_i(q) + \frac{1}{M} \Delta_{ij}(q) \phi^i(q) \phi^j(-q) \right). \quad (2553)$$

Here M is the mass of the ions in the underlying lattice, and the ϕ fields are defined (following Polchinski's conventions with regards to where the factors of M go) as $\phi^i = \sqrt{M} R^i$, with R^i the displacement field in the x^i direction (i is a spatial index). Δ_{ij} is the Laplacian for the particular lattice under consideration.

The phonon momentum is not scaled under the RG transformations towards the Fermi surface, and so if we determine the scaling dimension of ϕ from the time derivative term, we conclude that $\phi \mapsto s^{-1/2}\phi$ under RG. With this scaling, the second term scales as s^{-2} , and is thus relevant. Note that the two terms, while both kinetic terms for ϕ , scale differently under RG: the scaling of ϕ is not determined by its mass dimension. The scaling of the second term means that the dimensionless magnitude of its coefficient, namely m/M , will become $O(1)$ at a scale Λ_p such that

$$\Lambda_p = \Lambda_0 \sqrt{m/M}. \quad (2554)$$

At this scale, the second term starts being the one which controls the size of the fluctuations of ϕ , and so we should switch to using it as the thing that determines the scaling of ϕ under RG. We thus conclude that for energies $E < \Lambda_p$ we have $\phi \mapsto s^{+1/2}\phi$ under RG (and so the time-derivative term is now irrelevant). We can think of Λ_p as the scale below which the phonons become "massive" and decouple. Of course they are not really massive since they are Goldstones, but they effectively decouple since their interactions with e.g. the fermions go to zero as their momentum goes to zero, in accordance with Goldstones only having derivative interactions (as a consequence of $\phi \mapsto \phi + c$ shift symmetry). Recapitulating, the phonons scale as

$$\phi \mapsto \begin{cases} s^{-1/2}\phi, & E > \Lambda_p, \\ s^{+1/2}\phi, & E < \Lambda_p \end{cases}. \quad (2555)$$

Now we consider the effect of phonon-electron interactions, using the coupling

$$S_{ph-e} = \frac{1}{\sqrt{M}} \int dt d^3q d^3k_1 d^3k_2 \phi^i(q) g_i(q) \psi^\dagger(k_2) \psi(k_1) \delta(k_1 - q - k_2), \quad (2556)$$

with $g_i(0) = 0$ in accordance with the decoupling of the Goldstones at zero momentum. Under our RG step the phonon momentum is invariant (generic phonon momenta are not in the $d\Lambda$ shell), so we see that

$$g_i \mapsto \begin{cases} s^{-1/2}g_i, & E > \Lambda_p \\ s^{+1/2}g_i & E < \Lambda_p \end{cases}, \quad (2557)$$

⁹⁸Typical Debye temperatures are $T_D \sim 200K$ and so $\Lambda \sim 0.01$ eV, giving $\ln(\Lambda_0/\Lambda) \sim \ln(10^3) \sim 10$.

i.e. it scales in the same way that ϕ does. Thus, the interaction is relevant at high energy scales, but then becomes irrelevant after we go below the Debeye scale.

To examine the strength of the interaction, we look at the dimensionless coupling $\sqrt{m/M} \ll 1$. Under RG, before we get to Λ_p , the coupling is modified by $\sqrt{\Lambda_0/\Lambda}$ at the scale Λ . So by the time we reach $\Lambda = \Lambda_p$, the dimensionless coupling is now $\sqrt{m/M}(M/m)^{1/4} = (m/M)^{1/4}$, which is still small.

We get our IR theory from the UV one by the following steps: first, we flow from Λ_0 , the bandwidth of the conduction electrons, down to the scale Λ_p . By this time the Coulomb interaction has been greatly screened down, with a dimensionless coupling constant of order $O(1)$. Now by the time we get to Λ_p , the electron-phonon interaction switches over to irrelevance, and so we can integrate it out. Integrating out the phonons generates a 4-fermion interaction, which is attractive since it comes from second-order perturbation theory. At tree-level the induced interaction, being built from two vertices, is suppressed as $\sqrt{m/M}$, and hence vanishes in the $M \rightarrow \infty$ limit. How are we to get a non-vanishing attractive interaction?

The point is that the scaling of the interaction can change depending on the kinematics, as we have seen earlier. From our earlier analysis, we know that the BCS channel interaction should differ in scaling from a generic interaction by a factor of s^{-1} : in Polchinski's article, he makes a rather sketchy argument about this extra s^{-1} coming from a part of the momentum-conserving δ functions that are now sensitive to the components of the momenta normal to the FS. If we take care of this extra s^{-1} in the BCS channel by giving each phonon vertex a factor of $s^{-1/2}$ then the scaling of each vertex goes as s^{-1} above Λ_p , which is more relevant and means that by the time we get to Λ_p , the dimensionless coupling constant is now $\sqrt{m/M}(\Lambda_0/\Lambda_p) = 1$. Thus the picture that emerges is that by the time we get to Λ_p , an attractive $O(1)$ interaction is induced in the BCS channel, while the attractive interactions in all other channels are suppressed by $\sqrt{m/M}$. Will the attractive $O(1)$ interaction in the BCS channel be strong enough to overcome the Coulomb repulsion? Of course this is a non-universal question; the point is that it would not be too surprising that it is, due to the fact that by the time we get to Λ_p , the Coulomb interaction has already suffered some serious screening.

References

- [1] A. Abanov and P. B. Wiegmann. Theta-terms in nonlinear sigma-models. *Nuclear Physics B*, 570(3):685–698, 2000.
- [2] A. J. Beekman, J. Nissinen, K. Wu, K. Liu, R.-J. Slager, Z. Nussinov, V. Cvetkovic, and J. Zaanen. Dual gauge field theory of quantum liquid crystals in two dimensions. *Physics Reports*, 683:1–110, 2017.
- [3] Z. Bi and T. Senthil. An adventure in topological phase transitions in 3+ 1-d: Non-abelian deconfined quantum criticalities and a possible duality. *arXiv preprint arXiv:1808.07465*, 2018.
- [4] N. Dupuis. Fermi liquid theory: a renormalization group approach. *The European Physical Journal B-Condensed Matter and Complex Systems*, 3(3):315–331, 1998.

- [5] S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg. Remarks on the canonical quantization of the chern-simons-witten theory. *Nuclear Physics B*, 326(1):108–134, 1989.
- [6] D. V. Else and C. Nayak. Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge. *Physical Review B*, 90(23):235137, 2014.
- [7] D. Gaiotto, A. Kapustin, Z. Komargodski, and N. Seiberg. Theta, time reversal and temperature. *Journal of High Energy Physics*, 2017(5):91, 2017.
- [8] A. Kapustin and N. Seiberg. Coupling a qft to a tqft and duality. *Journal of High Energy Physics*, 2014(4):1, 2014.
- [9] J. Polchinski. Effective field theory and the fermi surface. *arXiv preprint hep-th/9210046*, 1992.
- [10] T. Senthil and M. P. Fisher. Competing orders, nonlinear sigma models, and topological terms in quantum magnets. *Physical Review B*, 74(6):064405, 2006.
- [11] R. Shankar. Renormalization-group approach to interacting fermions. *Reviews of Modern Physics*, 66(1):129, 1994.
- [12] E. Witten. Supersymmetric index in four-dimensional gauge theories. *arXiv preprint hep-th/0006010*, 2000.
- [13] E. Witten. Fermion path integrals and topological phases. *Reviews of Modern Physics*, 88(3):035001, 2016.