Critical current

June 1, 2021

- the quasiparticles can contribute to the current. the response to magnetic fields is determined by the ability of the SC to set up screening currents. hence the qps should lead to a lower critical field?
- in nodal SCs, expect leading dependence of $H_{c\perp}$ to be linear in T for small T. Note that linear dependence is also what one gets in LG theory. Ideally would like to know where crossover in the s-wave case is.
- qp poisoning can help explain why $B_{\parallel,c}$ is small even if we hypothesize that TBG has triplet / SVL pairing — no need to talk about shifted fermi surfaces (effects are still small) or Zeeman pair breaking
- disorder probably doesn't matter due to the coherence length being tiny — this can be measured by T-dep of $B_{c,\perp}$ near T_c where LG is reliable, hence doesn't depend on assumption of large gap

1 Setup

We will assume a d-wave gap for simplicity, and assume that the gap has nodes at $\mathcal{T}$-related momenta. For concreteness, consider a d_{xy} order parameter with nodes on the x, y axes. We can decompose the electron annihilation operator as[1]

$$\psi = \psi_1 e^{iKx} + \psi_2 e^{iKy} + \psi_3 e^{-iKx} + \psi_4 e^{-iKy}, \tag{1}$$

where the nodes are at momenta $(\pm K, 0), (0, \pm K)$. The most natural way to proceed is to combine nodes at opposite momenta into Dirac fermions. ψ_1 and ψ_3 can hybridize with the order parameter while conserving momentum; as can ψ_2 and ψ_4. Therefore these Dirac fermions are most naturally written in a Nambu basis as

$$\Psi_+ = \begin{pmatrix} \psi_1 \\ \psi_3^\dagger \end{pmatrix}, \qquad \Psi_- = \begin{pmatrix} \psi_2 \\ \psi_4^\dagger \end{pmatrix}. \tag{2}$$

The $i\mathbb{R}$ time Lagrangian governing the qps is

$$\mathcal{L}_{qp} = \Psi_+^\dagger \left(-i\omega_n + v_F k_x \sigma^z + v_\Delta k_y \sigma^x\right) \Psi_+ + \Psi_-^\dagger \left(-i\omega_n + v_F k_y \sigma^z + v_\Delta k_x \sigma^x\right) \Psi_-. \tag{3}$$

[1] We will be ignoring the physical spin throughout.

If the ψ fermions form a Fermi liquid, we can identify v_F with the Fermi velocity and $v_\Delta \propto \Delta_0$, with Δ_0 the maximum size of the gap. Regardless of the details though, v_Δ will vanish as $\Delta \to 0$. Note also that since we are in 2+1D, interactions are irrelevant.

We claim that the correct way to couple these fields to a background gauge field is as

$$\mathcal{L}_{qp} = \Psi_+^\dagger \left(-i\omega_n + v_F(k_x + A_x\sigma^z)\sigma^z + v_\Delta k_y \sigma^x\right)\Psi_+ + \Psi_-^\dagger \left(-i\omega_n + v_F(k_y + A_y\sigma^z)\sigma^z + v_\Delta k_x \sigma^x\right)\Psi_-. \tag{4}$$

Here the σ^zs multiply the gauge field due to the fact that we're in a Nambu basis.

Due to the fact that there is no gauge field appearing in the terms proportional to v_Δ, this does not look gauge invariant. This is okay however, as v_Δ is related to the (non-gauge-invariant) order parameter, and so there is more to the gauge transformations of these terms than meets the eye. It may be helpful here to go back to real space. The terms proportional to v_Δ are the low-energy representations of $\int_{\mathbf{x},\mathbf{y}} \Psi_\pm^\dagger(\mathbf{x})\Delta(\mathbf{x},\mathbf{y})\sigma^x\Psi_\pm(\mathbf{y})$, which do not need the background field to be gauge invariant. Expanding these terms in momentum space makes it look like we have a problem, but this is just due to the fact that gauge transformations become complicated in momentum space.[2]

2 Stiffness

As a function of a constant vector potential A, the free energy is[3]

$$\mathcal{F}[A] = \rho_{\mu\nu}A^\mu A^\nu + O(A^3) \tag{5}$$

At finite T, $\rho_{\mu\nu}$ receives contributions from both the superconductor and from the nodal quasiparticles. At low T the T-dependence will come from the nodal quasiparticles.

We now compute the contribution of these qps to $\rho_{\mu\nu}$. At first it may seem like the $\Psi_\pm$ cannot affect $\rho_{\mu\nu}$. Indeed, a nonzero contribution means that the qps have a finite paramagnetic / diamagnetic response, which comes from induced currents. Since the $\Psi_\pm$ do not have a well-defined charge, it may seem like this response must vanish. But this is too fast: since the $\Psi_\pm$ involve fields with both opposite charge *and* opposite momentum, they have well-defined currents, allowing them to have a nontrivial paramagnetic / diamagnetic response. The response is proportional to their density, which is made nonzero in the presence of a field, since the field acts as a chemical potential for the qps, as we saw above.

We can calculate the response by finding the coefficient of A_iA_j in the effective action

[2] One can also note that adding the naive terms like $\Psi_+^\dagger v_\Delta(k_y + A_y\sigma^z)\sigma^x\Psi_+$ to $\mathcal{L}_{qp}$ gives a Hamiltonian that is imaginary at the nodal points — not good.

[3] I see no reason for writing m everywhere; hence we are absorbing it into ρ.

obtained by integrating out the $\Psi_\pm$ fields. The Ψ_+ fields give the contribution

$$\begin{aligned}\mathcal{F}[A] &\supset \frac{A_x^2 v_F T}{2v_\Delta}\sum_n \int_{\mathbf{k}} \frac{\mathrm{Tr}[(i\omega_n + k_x\sigma^z + k_y\sigma^x)^2]}{(\omega_n^2+k^2)^2} \\ &= \frac{A_x^2 v_F T}{v_\Delta}\sum_n \int_{\mathbf{k}} \frac{-\omega_n^2 + k^2}{(\omega_n^2+k^2)^2} \\ &= -\frac{A_x^2 v_F}{4v_\Delta T}\int_{\mathbf{k}} \mathrm{sech}^2(k/2T) \\ &= -c\frac{A_x^2 v_F T}{v_\Delta}, \qquad c \equiv \frac{\ln 2}{\pi}.\end{aligned} \tag{6}$$

The Ψ_- fields give the same contribution but with $x \leftrightarrow y$; hence the quasiparticle contribution to ρ is

$$\rho^{ij}_{qp} = -\delta^{ij}\frac{cv_F}{v_\Delta}T. \tag{7}$$

The full stiffness is then obtained just by adding on the $T=0$ diamagnetic part ρ_0 (ignoring the variations of ρ_0 with T, which are subleading for $T/\Delta \ll 1$), giving $\rho^{ij} = \delta^{ij}\rho$, with

$$\rho = \rho_0 - \frac{cv_F}{v_\Delta}T. \tag{8}$$

Note that the stiffness is *isotropic*, even though the gap is not. This comes from the averaging between the two nodal directions, and the response would remain isotropic upon including more nodes (in a symmetric fashion). The factor of $1/v_\Delta$ is just a reflection of the fact that $1/v_\Delta$ controls the qp DOS at the nodes. This expression also matches with that given in Lee+Wen.

3 Critical current

Now we will compute the critical current. We will work in a slightly more general setting with pairs of nodes labeled by n at angles $\theta_n, \theta_n + \pi$. The nodes at $\theta_n, \theta_n+\pi$ combine to form a Dirac fermion Ψ_n, which at node n has velocity $v_{F,n}$ ($v_{\Delta,n}$) normal to (along) the gap direction. As we saw above, a constant vector potential $\mathbf{A}$ simply acts as a chemical potential of magnitude $\mathbf{A}\cdot\hat{\boldsymbol{\theta}}_n$ for each Ψ_n. The current is then obtained as

$$\mathbf{j} = \rho_0\mathbf{A} - \sum_n \frac{1}{v_{\Delta,n}}\hat{\boldsymbol{\theta}}_n \int_{\mathbf{k}} f(k - v_F\mathbf{A}\cdot\hat{\boldsymbol{\theta}}_n), \tag{9}$$

with f the Fermi function (for $T \ll \Delta_0$, which is the regime we are interested in, nonlinear contributions to $\mathbf{j}$ from the superfluid part can be ignored). Note that for a SC with an isotropic gap, the current above would consist of only the first term (up to corrections exponentially small in $e^{-\Delta_0/T}$) until $Av_F = \Delta_0$.

At $T=0$, the integral is just determined by geometry, with

$$\int_{\mathbf{k}} f(k - v_F\mathbf{A}\cdot\hat{\boldsymbol{\theta}}_n) = \frac{1}{4\pi}(v_F\mathbf{A}\cdot\hat{\boldsymbol{\theta}}_n)^2. \tag{10}$$

Therefore

$$\mathbf{j}(T=0) = \rho_0 \mathbf{A} - \sum_n \frac{v_{F,n}^2}{4\pi v_{\Delta,n}} \hat{\boldsymbol{\theta}}_n (\mathbf{A} \cdot \hat{\boldsymbol{\theta}}_n)^2. \tag{11}$$

Consider e.g. a d_{xy} gap. If we take $\mathbf{A}$ to point along $\hat{\mathbf{x}}$ and let $v_{F,n}^2/v_{\Delta,n} \equiv \mathsf{v}$ be the same for all n, then $j_y = 0$ and

$$j_x(T=0) = \rho_0 A - \mathsf{v}\frac{A^2}{4\pi} \implies j_{c,x} = \frac{\pi \rho_0^2}{\mathsf{v}}. \tag{12}$$

On the other hand, consider a $d_{x^2-y^2}$ gap, but keep $\mathbf{A}$ along $\hat{\mathbf{x}}$. Then j_y still vanishes, and we get

$$j_x(T=0) = \rho_0 A - \mathsf{v}\frac{A^2}{4\pi\sqrt{2}} \implies j_{c,x} = \sqrt{2}\frac{\pi \rho_0^2}{\mathsf{v}}, \tag{13}$$

a factor of $\sqrt{2}$ larger than in the previous case, since now the field is a factor of $\sqrt{2}$ less efficient at exciting qps.

For a gap with more nodes, the anisotropy is significantly reduced. For example, for a $d_{xy(x^2-y^2)}$ gap the biggest difference in critical currents at $T=0$ is between those along the $\hat{\mathbf{x}}$ and $R_{\pi/8}\hat{\mathbf{x}}$ directions, which differ by a factor of only

$$\frac{j_{c,0}}{j_{c,\pi/8}} = \frac{1+1/\sqrt{2}}{2(\cos(\pi/8)^3 + \cos(3\pi/8)^3)} \approx 1.01. \tag{14}$$

Expressions for $0 < T \ll v_F v_c$ where v_c is the critical velocity can be obtained from the Sommerfeld expansion of the Fermi function, giving a dependence of j_c on T^2. To order T^2, this gives for a d_{xy} gap

$$j_{c,x} = \frac{\pi \rho_0^2}{\mathsf{v}} - \mathsf{v}\frac{\pi T^2}{12}. \tag{15}$$

The quadratic dependence here is essentially due to the particle-hole symmetry of the Dirac fermions at the nodes: to linear order in T fermions moving parallel and antiparallel to $\mathbf{j}$ are excited in equal numbers, meaning that the leading T dependence is T^2. This crosses over to being linear when $T \sim v_F v_c$. When T is close to T_c (the temperature for which the critical current vanishes), the critical value of A will be small, and we can use the expansion employed above to determine ρ_{qp}^{ij}. This tells us that the critical temperature is

$$T_c = \frac{\pi \rho_0}{\mathsf{v} \ln 2}. \tag{16}$$

Note that $T_c \sim \rho_0 v_\Delta / v_F$. The v_Δ/v_F factor grows with Δ_0, but ρ_0 is proportional to the electron density, meaning that at low electron densities we can have $T_c/\Delta_0 \ll 1$.

The behavior of the critical current near T_c can be obtained by an expansion in $A v_F/T$. We need to expand to the leading nonlinear order in $A v_F/T$ in order to derive an expression for j_c. Taking a d_{xy} gap with $\mathbf{A} \parallel \hat{\mathbf{x}}$ for simplicity, we have

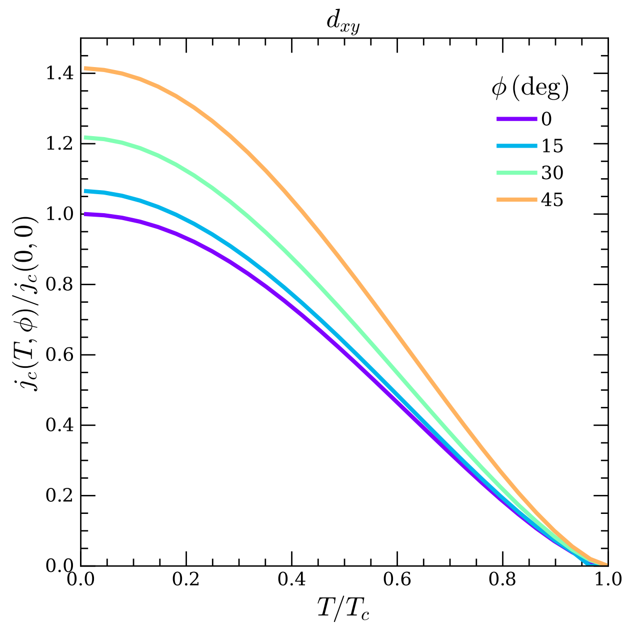
$$j_x \approx \rho(T) A + 2\frac{\mathsf{v} v_F^2 A^3}{3!} \int_{\mathbf{k}} \partial_k^3 f(k) = \rho(T) A - \frac{\mathsf{v} v_F^2 A^3}{24\pi T}, \tag{17}$$

where $\rho(T)$ is given in (8). This gives a critical current of

$$j_c(v_F A/T \ll 1) \approx d\rho(T)^{3/2}T^{1/2}, \qquad d \equiv \frac{2}{3v_F}\sqrt{\frac{8\pi}{\mathsf{v}}}, \tag{18}$$

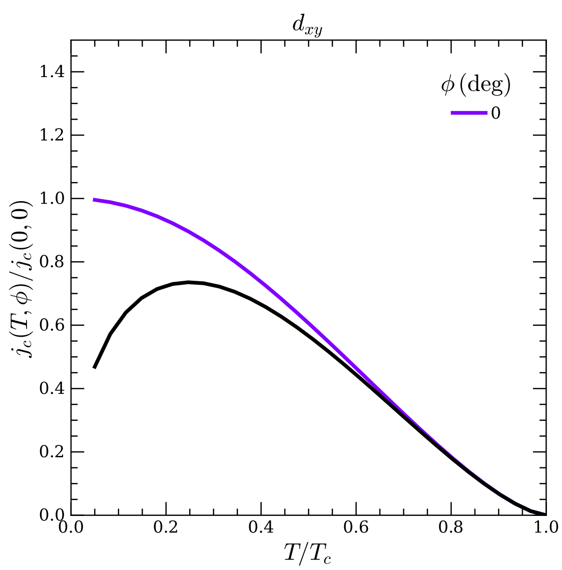
which near T_c goes as $(1-T/T_c)^{3/2}$. This power-law dependence is also what one gets within GL theory, as it must since we are working at $T/T_c \lesssim 1$.

We now illustrate these expectations with some plots.[4] For d_{xy} symmetry, we have e.g.



(19)

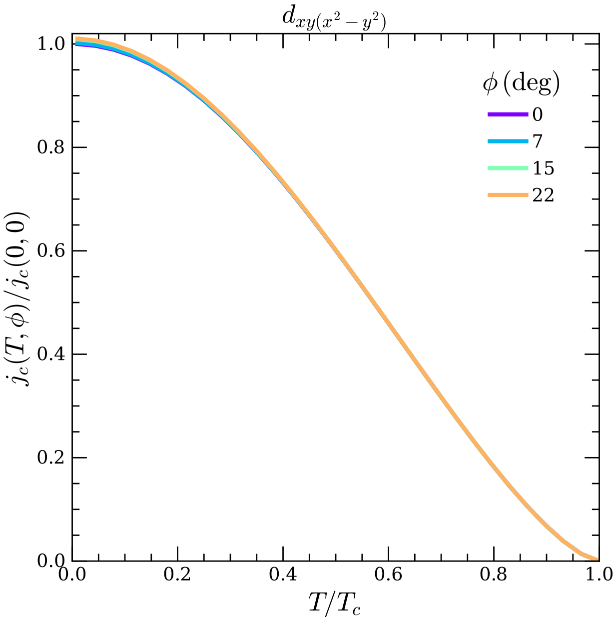
Here ϕ is the angle of the current relative to the x axis (with the plots being periodic in ϕ mod $\pi/4$). The $v_F A/T$ expansion in (18) works quite well: with the fit in black, we have



(20)

[4]One slightly confusing aspect of (9) is that the current is in general not parallel to the applied field. This is because the qps at node n only flow along $\hat{\boldsymbol{\theta}}_n$, and so if the field is not directed along a high-symmetry direction, the current will not be parallel to the field. Therefore when finding the critical current along a given direction, we have to search over different field directions.

As we said above, the angular dependence is much smaller for a $d_{xy(x^2-y^2)}$ gap:



(21)

where now ϕ is periodic mod $\pi/8$.

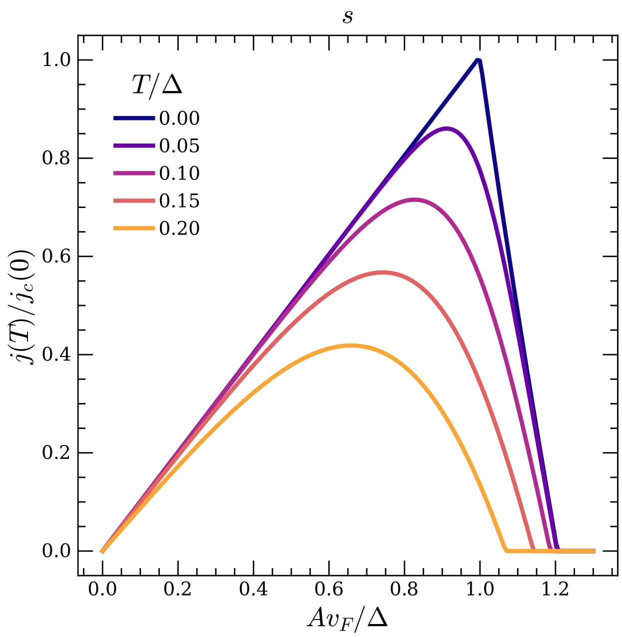
3.1 Comparison with s-wave

Let's compare this with what we'd expect from a fully-gapped SC. Here of course the relationship between j and A will be very different, since at $T = 0$ j is linear in A until $v_F A \sim \Delta_0$. It is however not so obvious that the functional form of the T-dependence of $j_c(T)$ will be different from the nodal case, since in both instances the T-dependence occurs from the thermal activation of qps, which we might expect to take the same T^2 form in both cases. We will see however that in the s-wave case the T-dependence is linear all the way until $T \sim T_c$.

For a uniform s-wave gap and with $\mathbf{A} \parallel \hat{\mathbf{x}}$ wolog, we have

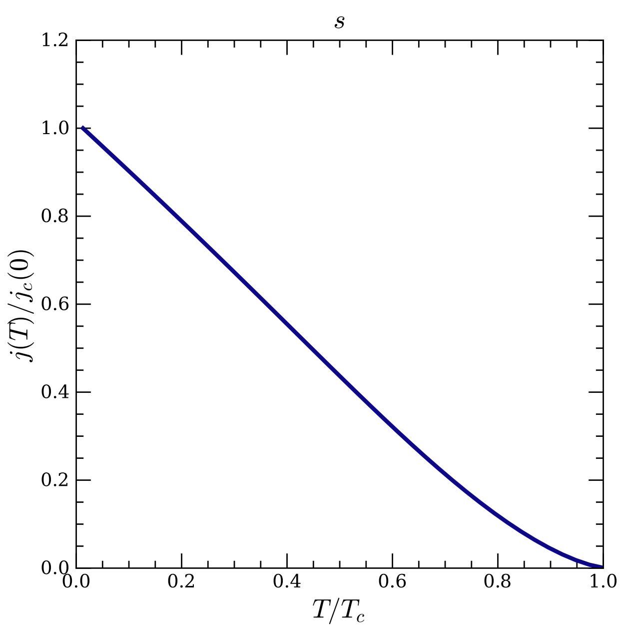
$$j_x = \nu v_F(v_F A) - 2\nu v_F \int d\xi d\theta \, \cos\theta \, f\left(\sqrt{\xi^2 + |\Delta|^2} - v_F A\cos\theta\right), \tag{22}$$

where ν is the DOS at the Fermi level. When $T/\Delta \ll 1$ the qps cannot provide a backflow current until rather large fields $v_F A \sim \Delta$. Until these fields the current is linear in A, and above these fields the current drops very quickly. For example, a typical plot looks like



(23)

We now plot j_c as a function of T:



(24)

The distinguishing feature here is the absence of non-linear behavior at low T/T_c. At $T/T_c \lesssim 1$ however, the nodes are washed out and the nodal and s-wave cases look functionally identical. The nonlinear tail at $T/T_c \approx 1$ arises

Note that we have not taken into account thermal suppression of the gap, i.e. we have neglected the T-dependence of Δ. This is completely justified for the nodal case where we found $\Delta/T_c \gg 1$, and even for the s-wave case where T_c is of order Δ the T-dependence of Δ is subleading.

4 Derivation of Halperin-Nelson formula for nonlinear IV relation

In this section we give a derivation of the nonlinear IV relation of Halperin and Nelson. We do this because it's fun, and because the original paper omitted the calculation.

4.1 Heuristic derivation

In the presence of a SF velocity $\mathbf{v}$, the energy of a vortex-antivortex pair separated by a vector $\mathbf{r}$ is

$$E_p(\mathbf{r}) \sim 2\pi K \ln(r) - \mathbf{r} \times \mathbf{v}. \tag{25}$$

This energy is minimal when $\mathbf{r} \perp \mathbf{v}$; in what follows we will take $\mathbf{r} \parallel \hat{\mathbf{y}}$ and $\mathbf{v} \parallel \hat{\mathbf{x}}$. The location of the energy barrier at which the vortex pair can unbind is then[5]

$$r_* \sim 2\pi K/v. \tag{26}$$

The rate γ at which unbound vortex pairs are produced from the vacuum per unit area is then

$$\nu \sim f e^{-E_p(r_*)/T} \sim f v^{2\pi K/T}, \tag{27}$$

where f is a tunneling attempt frequency per unit area, which is some non-universal constant that is presumably independent of v and K.

To calculate the IV relation, we use the Josephson relation for the voltage difference

$$V \sim \frac{d\Delta\phi}{dt} \sim \frac{dv}{dt}. \tag{28}$$

The decay of the SF velocity is caused by unbound vortices moving across the sample in the direction normal to $\mathbf{v}$. A vortex which traverses the whole sample decreases the SF velocity by $2\pi/L$, where L is the linear size of the sample in the direction transverse to the current. However, individual vortices will not generically make it all the way across the sample; instead they will only propagate for a mean free path λ. In terms of the MFP, we then have

$$\frac{dv}{dt} \sim -\frac{2\pi}{L}(\nu\lambda L) \sim -\nu\lambda. \tag{29}$$

To find λ, we argue in the following way. First, we calculate the area A_v within which we expect on average to find one unbound vortex. On one hand, a given vortex can exist anywhere within an annulus of radius r_* and thickness λ, so that

$$A_v \sim \lambda r_*. \tag{30}$$

On the other hand, we also have $A_v = (\nu\tau)^{-1}$, with τ the vortex lifetime. But the vortex lifetime is $\tau = \lambda/s_v$, where s_v is the mean velocity at which the unbound vortices move. If

[5] For small currents and temperatures not too close to T_c, the unbinding scale r_* will be much larger than the typical size of bound vortex-antivortex pairs. Thus in what follows we will take K to be its fully renormalized value, after we have flowed to the point where the vortex fugacity vanishes.

we take the force on an unbound vortex to be $F \sim v - \eta s_v$ where η is some drag coefficient, then $s_v \sim v$. Therefore

$$\tau = \lambda/v \implies A_v = \frac{v}{\nu\lambda}. \tag{31}$$

Putting these two expressions for A_v together, we get

$$\lambda \sim v/\sqrt{\nu} \implies \frac{dv}{dt} \sim -v\sqrt{\nu}. \tag{32}$$

Using the above expression for ν and $v \sim I$ with I the supercurrent,

$$V \sim I^{1+\pi K/T}. \tag{33}$$

At the BKT transition we have $\pi K/T = 2$, reproducing the $V \sim I^3$ scaling.

4.2 Renormalization of ρ_s by I

The bound vortex pairs existing below R_* contribute to a backflow current which renormalizes the SF density.

This is however a subleading effect, and does not become important until one is very close to T_c, where the separation between the bound vortex pairs becomes large. In Pablo's experiments, there is an extended range of currents where $V \propto I^3$ to a rather good approximation. Given that the effect above leads to a renormalization of ρ_s (and hence of T_c) with changing I^3, strictly speaking it precludes having $V \propto I^3$ over a finite range of currents at fixed T. Therefore we can conclude that this renormalization of ρ_s is unimportant for the present purposes.

5 Critical current in a field

Let us now examine the behavior of the critical current in the presence of a magnetic field. Because the critical current for out-of-plane fields is often determined by non-universal things like vortex pinning, we will focus on an in-plane field in what follows.

At zero current, the critical field $B_{c\parallel}$ seen in experiments $B_{c\parallel} \sim 1T$ is apparently much lower (by a factor of ≈ 40) than the value one would derive within GL theory for a thin-film SC. This is according to Pablo's original TBG paper, but in fact the estimate for what we should "expect" for $B_{c\parallel}$ isn't clear. In GL theory a film of thickness δ has

$$H_{c\parallel} \sim \frac{\sqrt{-rm}}{\delta} = \frac{H_c\lambda}{\delta}, \tag{34}$$

but it is unclear how one is supposed to estimate $H_c\lambda$ in TBG. In any case, note that $H_{c\parallel}$ obtained in this way is independent of the SF density.

One possibility is that $B_{c\parallel}$ is set by Zeeman pair-breaking. This can likely be ruled out because a) it is incompatible with the observed angular dependence of $B_{c\parallel}$ at finite currents, and b) the small value of $B_{c\parallel}$ seems hard to reconcile with a gap of order $\Delta \sim$ 2-4 meV.

We then turn to considering orbital effects. If we take for granted a gap of order $\Delta \sim$ 2-4 meV, direct depairing effects coming from opposite shifts of the K, K' valley Fermi surfaces naively seem to be too small to be in accordance with $B_{c\parallel} \sim 1T$.

Ruling out these possibilities, we then need to ask whether or not a nodal gap can explain things. Since Zeeman effects can only be playing a minor role as we argued above, any suppression of the SC needs to come from orbital effects of $B_\parallel$ on the nodal qps.

The only way for the nodal qps to couple to the orbital field is by generating magnetization current. Let us work with the vector potential

$$\mathbf{A} = \mathbf{z} \times \mathbf{B}. \tag{35}$$

To get a crude idea for how the qps respond to the field, we will write the qp field as $\Psi_\pm = \Psi_{\pm,t} + \Psi_{\pm,b}$, with the two components living on the "top" and "bottom" of the graphene sheet, respectively. The top / bottom fields couple to the gauge field

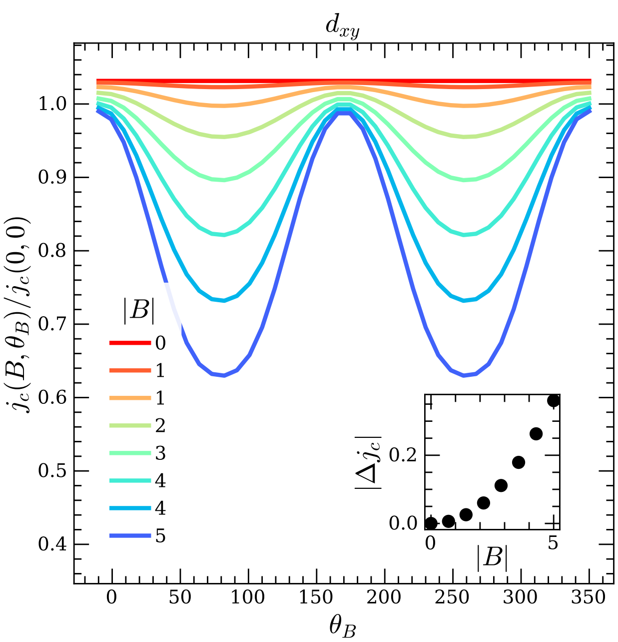
$$\mathbf{A}_{t/b} = \pm\frac{\delta}{2}(-B_y, B_x, 0)^T, \tag{36}$$

and will form oppositely-directed currents in order to generate a magnetization which couples to $\mathbf{B}$ (here δ is the thickness of TBG).

Within this crude model, the current as a function of SF velocity $\mathbf{v}$ is just

$$\mathbf{j} = \rho_0\mathbf{v} - \sum_{n,\alpha=t/b} \frac{1}{v_{\Delta,n}}\hat{\boldsymbol{\theta}}_n \int_{\mathbf{k}} f(k - v_F(\mathbf{v} + \mathbf{A}_\alpha)\cdot\hat{\boldsymbol{\theta}}_n). \tag{37}$$

It is straightforward to check that within this model, the critical current is maximal along $\pm\mathbf{B}$, and minimal along $\pm\wedge\mathbf{B}$. Furthermore, the magnitude of the anisotropy in j_c is seen to be a quadratic function of $B_\parallel$. This produces plots like



(38)

where θ_B is the angle between $\mathbf{B}$ and $\mathbf{j}$, and where we have taken $\mathbf{j}$ to point near a nodal direction.[6]

Note that the field leads to an anisotropic superfluid density. At $T = 0$, one finds

$$\rho_{ij} = \rho_0 \delta_{ij} - \frac{\mathsf{v}\delta}{4\pi} \sum_n \hat{\boldsymbol{\theta}}_n^i \hat{\boldsymbol{\theta}}_n^j |\hat{\boldsymbol{\theta}}_n \times \mathbf{B}|. \tag{39}$$

This means that e.g. for a d_{xy} gap, the stiffness along the field direction is unmodified, while the stiffness normal to the field is decreased. This gives a critical field of

$$B_{c\parallel} = \frac{2\pi\rho_0}{\mathsf{v}\delta} \quad (\text{node}), \qquad B_{c\parallel} = \sqrt{2}\frac{2\pi\rho_0}{\mathsf{v}\delta} \quad (\text{anti}-\text{node}). \tag{40}$$

The variations are smaller in the case of a $d_{xy(x^2-y^2)}$ gap by the same factor as before.

Note that these values for $H_{c\parallel}$ *do* depend on the zero-field SF density, unlike the GL estimate.

[6] If $\mathbf{j}$ points along an antinodal direction there is instead a $\pi/2$-periodicity in the d_{xy} case, while there remains a π-periodicity in the $d_{xy(x^2-y^2)}$ case.