QFT diary

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Preface:

This is a diary containing worked-out QFT problems. Most entries are elaborations on calculations in papers which I wanted to work out in detail, some are problems which arose when doing research, and some are textbook problems. There are doubtless many typos, and I have not been very diligent about adding citations. Moreover, many entries were written near the beginning of grad school, when I didn't understand much about QFT. I take no responsibility for any misguided beliefs that my younger self decided to write down.

Below is an index of problems arranged by subject matter; some diary entries appear under multiple sections as appropriate.

Bosonization and dualities

Conformal field theory

Gauge theory and TQFT

Kinematics, symmetries, and anomalies

Miscellaneous

Classic perturbative calculations

Renormalization and RG

Sigma models

Topological terms

Notation:

- Usual QI notation for Paulis holds
- Unless noted otherwise, $J = -iY$
- $\zeta_d = d$th root of unity
- $\overline{\mathbb{Z}} = 2\pi\mathbb{Z}$
- $d^\dagger = \star d \star$
- $\int_k = \int \frac{d^d k}{(2\pi)^d}$, $\int_x = \int d^d x$, etc.
- wide hats $\widehat{A}$ denote Poincare duals
- In e.g. spectral representations of operators, states that look like $|\psi_i\rangle$ will always denote norm-1 vectors which are not generically linearly independent, while states that look like $|i\rangle$ will always denote orthonormal bases
- normalization factors for wavefunctions / density operators will often be neglected

Setting conventions for the compact boson and bosonizaton

Today we'll be discussing the compact boson in 1+1D and bosonization (both of bosons and fermions). A large part of what follows will be involved with trying to set standards vis-a-vis conventions for the compact boson and the bosonization procedure. The need for doing this stems from the fact that every source in the literature does things differently. First we'll deal just with the boson theory by itself, and later discuss fermions.

Compact boson

We take a 1+1D compact boson at radius R to be described by the following action:

$$S = \frac{R^2}{4\pi} \int d\phi \wedge \star d\phi. \tag{1}$$

Here by "radius" of the boson, we just mean the coefficient in front of the kinetic term: we are always identifying $\phi \sim \phi + 2\pi$, but not restricting ourselves to kinetic terms with coefficients of e.g. 1/2 (we could say that we are fixing the target space to always be the same S^1, but allowing ourselves to consider different metrics on it). The Euclidean-space propagator for the ϕ fields with this normalization is

$$\langle \phi(x,t)\phi(0)\rangle \sim -\frac{1}{R^2}\ln|x-y|. \tag{2}$$

Therefore if the coefficient in front of the integral is 1/2, we would have $-\frac{1}{2\pi}\ln|x-y|$.

Instead of working with the field ϕ, whose equation of motion gives a spectrum containing both positive and negative momentum modes, it is often helpful to work with chiral "fields" $\phi_\pm$, which have the equations of motion

$$\bar\partial\phi_+ \overset{\text{eom}}{=} 0, \qquad \partial\phi_- \overset{\text{eom}}{=} 0, \qquad \partial = \frac{\partial_t - \partial_x}{2}, \qquad \bar\partial = \frac{\partial_t + \partial_x}{2}. \tag{3}$$

This means that the mode ϕ_+ has a spectrum with modes only for positive momentum (right-moving), while ϕ_- is the opposite. The full field ϕ is $\phi = \phi_+ + \phi_-$, which can be classically split up in this way just because solutions to the wave equation in 1+1D are given by $f(x-t)+g(x+t)$. It is very important to stress the fact that $\phi_\pm$ are only (anti)holomorphic *on the equations of motion*, i.e. we have $\bar\partial\phi_+(x) = \partial\phi_-(x) = 0$ only when inserted in correlation functions, and only when there are no other operators inserted at x in the correlation function, i.e. the usual Ward identity is

$$\langle \partial_\mp\phi_\pm(x)\mathcal{O}\rangle \propto \frac{\delta}{\delta\phi_\pm(x)}\langle\mathcal{O}\rangle. \tag{4}$$

Therefore it is of course not permissible to e.g. take the action $\int d\phi\wedge\star d\phi = \int \partial\phi\bar\partial\phi$ and replace it with $\int \partial\phi_+\bar\partial\phi_-$—manipulations like this lead to total nonsense, because the path integral is over all field configurations of $\phi_\pm$, not just (anti-)holomorphic ones.

In terms of the chiral fields, the action is instead (note the $R^2/2\pi$ in front, not $R^2/4\pi$!)

$$S = \frac{R^2}{2\pi}\int \left(-\partial_t\phi_+\partial_x\phi_+ + \partial_t\phi_-\partial_x\phi_- - (\partial_x\phi_+)^2 - (\partial_x\phi_-)^2\right). \tag{5}$$

This ensures that the equations of motion are $(\partial_t \pm \partial_x)\phi_\pm \overset{\text{eom}}{=} 0$, as required. One can also check that this gives the correct log propagators: the momentum space propagator is

$$G_\pm(p,\omega;q,\nu) = \frac{\pi}{R^2}\delta(p-q)\delta(\omega-\nu)\Theta(\pm p)\frac{1}{k(k\mp\omega)}, \tag{6}$$

so that

$$G_\pm(x,t) = \frac{\pi}{R^2}\int_{\mathbb{R}}\frac{dk}{2\pi}\,\Theta(\pm k)\frac{e^{ik(x\mp t)-a|k|}}{k} \sim -\frac{1}{2R^2}\ln\left[\frac{ia}{x\mp t+ia}\right], \tag{7}$$

where in the first step we did the contour integral over ω and in the second step we did the integral by first differentiating wrt x and then re-integrating, choosing the constant term so that $G_\pm(x,\pm t) = 0$, which is a convinient normalization. Of course only derivatives and exponentials of $\phi_\pm$ are actually well-defined fields, but this correlator is still useful for calculating things. The ias are needed for dimensions to work out, with a essentially being a short-distance cutoff used to regulate the theory.[1] The scaling dimensions and conformal spins of the vertex operators $e^{i\alpha\phi_\pm}$ are therefore[2]

$$\Delta_\pm = \pm s_\pm = \frac{\alpha^2}{4R^2}. \tag{9}$$

Recall that these are defined via Now in order for an operator to be local its 2-point function must be single-valued, which means that $s_{\mathcal{O}} \in \frac{1}{2}\mathbb{Z}$ (since $z/\bar{z}$ is charge 2 under rotations). So unless $R^2 = \frac{1}{2n}$ for some $n \in \mathbb{Z}^+$, the $V_\pm =: e^{i\phi_\pm}:$ vertex operators are *not* local.[3]

While we have written $\phi = \phi_+ + \phi_-$ and shown that ϕ has the same propagator as that of the original boson action (1), we will now justify the correspondence between the two ways of writing the action more carefully. For this it is helpful to introduce the field

$$\theta \equiv R^2(\phi_- - \phi_+). \tag{10}$$

The R^2 and the perhaps unexpected minus sign is to make our lives easier in the future. If we re-write the action in terms of these variables, we get

$$S = \frac{1}{4\pi}\int \left(2\partial_x\theta\partial_t\phi - R^2(\partial_x\phi)^2 - R^{-2}(\partial_x\theta)^2\right). \tag{11}$$

[1]Sanity check: the propagator for the field $\phi \equiv \phi_+ + \phi_-$ is then

$$G_\phi(x,t) = -\frac{1}{R^2}\ln\sqrt{x^2-t^2+ia}, \tag{8}$$

which is the same thing we wrote down earlier, just in $\mathbb{R}$ time.

[2]The scaling dimension can be any nonzero number since the unitary bound on scalars is $\Delta \geq (d-2)/2$, which gives no nontrivial constraint for $d=2$.

[3]Note that as long as $R^2 = m^2/(2n)$ for $m,n \in \mathbb{Z}^+$, mth powers of $V_\pm$ are local.

The equation of motion for θ is then $\partial_x\theta \overset{\text{eom}}{=} R^2\partial_t\phi$ (this is the reason for the weird sign in the def of θ), while the eom for ϕ is $R^2\partial_x\phi \overset{\text{eom}}{=} \partial_t\theta$. Therefore

$$d\phi \overset{\text{eom}}{=} \frac{1}{R^2} \star d\theta, \tag{12}$$

where we have to remember to use the mixed-signature $\star$. If we then integrate out θ by shifting $\delta\theta = \partial_x^{-1}\partial_t\phi$, we find an action identical to (1) (in the signature $(+,-)$), establishing the correspondence between the two presentations. Also note that the eom are preserved under the duality

$$T : \phi \mapsto \frac{1}{R^2}\theta, \qquad \theta \mapsto R^2\phi, \tag{13}$$

since $\star^2 = \mathbf{1}$ acting on 1-forms in $\mathbb{R}^{1,1}$. We can check that $R = 1$ is the right self-dual radius because at this point the exponentials of $\theta \pm \phi$ both have scaling dimension 1; hence they can act as chiral conserved currents and generate the $SU(2)$ that we know to be emergent at the self-dual point.

The commutation relations for the various fields involved all follow from the above actions. In this scheme we have $[\phi_\sigma, \phi_{\sigma'}] \propto \delta_{\sigma\sigma'}$, which is not true in some other conventions. The nonzero commutators for the chiral fields are

$$\pi_\pm = \mp\frac{R^2}{\pi}\partial_x\phi_\pm \implies [\phi_\pm(x), \phi_\pm(y)] = \pm i\frac{\pi}{2R^2}\mathrm{sgn}(x-y). \tag{14}$$

There is a factor of 2 that is a little bit subtle here—from the action we might have guessed that instead $\pi_\pm = \mp(R^2/2\pi)\partial_x\phi_\pm$, but this is not correct. One way to see this is by varying the action: we get[4]

$$\delta S = \int \delta\phi_\pm \left(\mp\frac{R^2}{\pi}\partial_t\partial_x\phi_\pm - \frac{\delta H}{\delta\phi_\pm}\right), \tag{16}$$

which implies from Hamilton's equations that $\pi_\pm = \mp\frac{R^2}{\pi}\partial_x\phi_\pm$ as claimed. We can check this by requiring that $\partial_t\phi_\pm = \mp\partial_x\phi_\pm$ hold as a consequence of $\partial_t\phi_\pm = i[H, \phi_\pm]$ (of course this is just another way of doing the same calculation), which gives

$$\mp\partial_x\phi_\pm = \mp i\frac{R^2}{2\pi}[(\partial_x\phi_\pm)^2, \phi_\pm] = \mp i\frac{R^2}{\pi}[\partial_x\phi_\pm, \phi_\pm]\partial_x\phi_\pm, \tag{17}$$

giving us the desired commutator.[5] This factor of 2 is the same factor of 2 as in Chern-Simons theory (where we don't have a Hamiltonian to check) which means in

[4] Pedantic point here: when deriving this we need to integrate by parts. While we can't write $\int \partial_t\phi_\pm\partial_x\phi_\pm = -\int \phi_\pm\partial_t\partial_x\phi_\pm$ since $\phi_\pm$ by itself is not well-defined, we can however write

$$\int \partial_t\delta\phi_\pm\partial_x\phi_\pm = -\int \delta\phi_\pm\partial_t\partial_x\phi_\pm, \tag{15}$$

since $\delta\phi_\pm$, like $\partial\phi_\pm$, is well-defined.

[5] Yet a third way to check is to write the chiral fields as

$$\phi_\pm(x) = \pm\int_0^{\pm\infty} dp\, \frac{1}{2\pi\sqrt{|p|R^2/\pi}}(\phi_p e^{ipx} + \phi_p^\dagger e^{-ipx}), \tag{18}$$

and then to compute the commutator explicitly (by regulating it with a factor of $e^{-\eta p^2}$; the integral in the commutator then becomes $\propto \mathrm{Erf}((x-y)/\sqrt{\eta})$, which has the correct $i\pi\mathrm{sgn}(x-y)$ limit as $\eta \to 0$).

that context that the momentum of the gauge field A is $k \star A/2\pi$ (with the $\star$ taken in space), instead of $k \star A/4\pi$.

The calculated commutators for $\phi_\pm$ tell us that

$$[\phi(x), \theta(y)] = -i\pi \text{sgn}(x-y) \implies \pi_\phi = \frac{1}{2\pi}\partial_x \theta, \tag{19}$$

which agrees with the canonical momentum derived from the action for ϕ and θ we wrote above.

We can use these results to compute commutators of vertex operators. This works by writing, for X, Y Gaussian variables with c-number commutator,[6]

$$e^X \odot e^Y = e^X e^Y e^{-\frac{1}{2}\langle (X+Y)^2\rangle} = e^{X+Y} e^{\frac{1}{2}[X,Y]-\frac{1}{2}\langle (X+Y)^2\rangle} = e^Y e^X e^{[X,Y]-\frac{1}{2}\langle (X+Y)^2\rangle} = e^Y \odot e^X e^{[X,Y]}, \tag{21}$$

where $\odot$ means "operator product" (the colons for normal-ordering look ugly!), so that $e^X \odot e^Y =: e^X :: e^Y :$ and $e^X e^Y =: e^X e^Y :$. Basically, $A \odot B$ is used to denote a product that is not fully normal-ordered, with AB denoting a single operator with normal-ordering $: AB :$.

Spectrum of local operators

All of this is fine, but very formal. It is very formal because the fields we've been manipulating, the $\phi_\pm$s and their linear combinations, aren't really well-defined. Indeed, their two-point functions are nonsensical. The fields that are well-defined are of course exponentials and derivatives of the $\phi_\pm$. In fact even exponentials of $\phi_\pm$ are problematic, since as we mentioned they are non-local for generic values of R.

As we said above, when we say that the field ϕ is compact with $\phi \cong \phi + 2\pi$, what we really mean is that we restrict ourselves to only considering vertex operators for ϕ of the form $V_{n,0} = e^{in\phi}$, with $n \in \mathbb{Z}$ (the notation will become clear in a sec). That is, we take $\phi \in \mathbb{R}$ (which we did when computing correlators), but impose that all physical operators be invariant under shifting ϕ by 2π. Since the conformal spin of $V_{n,0}$ is $s_{n,0} = 0$, $V_{n,0}$ is always non-chiral, and has a well-defined two-point function. Note that this definition of compactness is *not* the same as saying that we restrict ourselves only to combinations of chiral vertex operators of the form $e^{in\phi_\pm}$! We can impose $\phi \cong \phi + 2\pi$, but somewhat confusingly this is not the same as having $\phi_\pm \cong \phi_\pm + 2\pi$, despite $\phi = \phi_+ + \phi_-$.

So, what about vertex operators of θ? The vertex operators $e^{in\theta}$ are also non-chiral and have well-defined correlators. However, they are not generically local with respect to the $V_{n,0}$. We will find the allowed vertex operators for θ by requiring that they create self-consisent field configurations for ϕ.

We can write $e^{i\alpha\theta(x)}$ as $e^{i\alpha \int_C d\theta}$, where C is a path extending from x out to infinity. This operator is only local if correlation functions are independent of the choice of C.

[6] We need

$$e^X \odot e^Y = e^X e^Y e^{\langle XY + \frac{1}{2}(X^2+Y^2)\rangle}, \tag{20}$$

which can be proved by writing down the series expansions and doing a bit of algebra (remember that the normal-ordering gets rid of *all* contractions between the two operators; for the vertex operators there are infinitely many such contractions to take into account).

From the commutation relations, no θ vertex operators can detect C, but ϕ vertex operators can. When $V_{n,0}$ moves through te curve C, it picks up a phase of $e^{2\pi i\alpha n}$. Hence for $e^{i\alpha\theta}$ to be local, we need $\alpha \in \mathbb{Z}$.[7]

Since the spectrum of the theory is generated by exponentials / derivatives of linear combinations of ϕ, θ (or $\phi_\pm$, either way), the claim is that

$$V_{n,w} = e^{in\phi + iw\theta}, \qquad (n,w) \in \mathbb{Z}^2 \tag{22}$$

generate the full spectrum of local vertex operators. As a check, we compute the OPE

$$\begin{aligned} V_{n,w}(z) \odot V_{n',w'}(w) &= \frac{V_{n+n',w+w'}}{(z-w)^{-(n-R^2w)(n'-R^2w')/2R^2}(\bar{z}-\bar{w})^{-(n+R^2w)(n'+R^2w')/2R^2}} \\ &= V_{n+n',w+w'}|z-w|^{-(nn'R^{-2}+ww'R^2)}\left(\frac{z-w}{\bar{z}-\bar{w}}\right)^{\frac{1}{2}(wn'+w'n)}. \end{aligned} \tag{23}$$

This OPE is evidently only well-defined provided that $wn' + w'n \in \mathbb{Z}$, and so indeed by taking $(n,w) \in \mathbb{Z}^2$, the operators $V_{n,w}$ are always well-defined local operators with sensible OPEs (and since we can have $wn' + w'n$ be the minimal value of 1, they generate all such local vertex operators).[8]

From the above OPE, we read off

$$\Delta_{n,w} = \frac{1}{2}(n^2/R^2 + w^2R^2), \qquad s_{n,w} = -nw, \tag{25}$$

with $s \in \mathbb{Z}$ as required. Note that the spin is independent of R, essentially by construction. T-duality acts as

$$T\,:\,V_{n,w} \mapsto V_{w,n},\; R \mapsto R^{-1}, \tag{26}$$

as expected.

As mentioned above, for some values of R the spectrum includes operators that are genuinely chiral, with $\Delta_{n,w} = \pm s_{n,w}$. For this to be the case, we need to have $n/R = wR$ for some n, w. This means that we must have $R^2 = n/w$, and so we only have chiral operators when $R^2 \in$. Thus only for rational values of R^2 do there exist local operators that are exponentials only of either ϕ_+ or ϕ_-.

[7]If one tries to think about the $\phi_\pm$ fields as well-defined entities in their own right, madness sets in quickly: $\phi \cong \phi + 2\pi$ means that if the $\phi_\pm$ are legit fields, we should also have e.g. $\phi_+ \cong \phi_+ + 2\pi$. But this then seems to say $\theta \cong \theta + 2\pi R^2$, which is a contradiction unless we are at the self-dual point $R^2 = 1$ (where the $\phi_\pm$ fields produce legit current operators). Therefore in general its best to only work with vertex operators and derivatives of ϕ, θ unless we're at a special point where there's an important chiral symmetry.

[8]Note that while $V_{n,0} \odot V_{m,0} = V_{n+m,0}$, $V_{n,w} \neq V_{n,0} \odot V_{0,w}$: instead, we have

$$V_{n,0} \odot V_{0,w} \sim V_{n,w}(\varepsilon/\bar{\varepsilon})^{nw/2}, \tag{24}$$

where ε is a point-splitting distance (the $\sim$ is because there's another numerical factor coming from combining the exponentials between $e^{in\phi}e^{iw\theta}$ and $e^{in\phi+iw\theta}$). Since this depends on the choice of ε, $V_{n,w}$ cannot be split-up as a product of operators in a well-defined way.

Bosonizing bosons

In this subsection we use a cond-mat hydrodynamically-flavored line of reasoning to explain the ubiquity of the compact boson theory in one-dimensional problems.

The basic setting is non-relativistic[9] system of complex bosons with translation and $U(1)_N$ particle number symmetries. The basic UV Hamiltonian is

$$H = \frac{1}{2}\int dx\,\left(\frac{1}{m}|\partial_x\psi|^2 + V\rho^2\right), \tag{27}$$

where ρ is the density operator and V is a hard-core repulsion. The basic strategy is to write down an EFT in the IR whose variables keep track of the densities of the two conserved charges (momentum and $U(1)_N$ charge density). We know that neither of the two symmetries will truly be broken since we are in 1+1D; however we know that both symmetries can be very nearly broken, and in what follows we will use fluctuations about both nearly-ordered states to construct a hydrodynamic EFT.

The limit of weak interactions, where the system is close to a superfluid, is easy to deal with. Indeed we know that we have to get the action of a compact boson in the IR, since this is the action describing the "Goldstone modes" of the "broken" symmetry. This action can just be written down on phenomenological grounds, with the form of the parameters in the action fixed using common sense. But we want to do a bit better, viz. we want to provide a slightly more explicit mapping of the boson operators to the operators appearing in the "Goldstone mode" action, and relate the parameters in this action explicitly to V and the SF density. To this end, we write $\psi = \sqrt{\rho}e^{i\phi}$ and drop the fluctuations in ρ; this gives

$$H = \frac{1}{2}\int dx\,\left(K(\partial_x\phi)^2 + V\rho^2\right) \tag{28}$$

where $K = \rho_0/m = \langle\rho\rangle/m$. Now from $[\rho(x), e^{i\phi(y)}] = \delta(x-y)e^{i\phi(x)}$, we can introduce a field θ such that $\rho = \frac{1}{2\pi}\partial_x\theta$ (this is just for suggestive notation as we pass from H to S). Therefore we can write the action as[10]

$$S = \int\left(\frac{1}{2\pi}\partial_x\theta\partial_t\phi - \frac{K}{2}(\partial_x\phi)^2 - \frac{V}{8\pi^2}(\partial_x\theta)^2\right). \tag{29}$$

Now integrating out θ,

$$\begin{aligned} S &= \frac{1}{2}\int\left(\frac{1}{V}(\partial_t\phi)^2 - K(\partial_x\phi)^2\right) = \frac{R^2}{4\pi}\int\left(\frac{1}{v}(\partial_t\phi)^2 - v(\partial_x\phi)^2\right) \\ R &= \sqrt{2\pi}\left(\frac{K}{V}\right)^{1/4}, \qquad v = \sqrt{KV}. \end{aligned} \tag{30}$$

[9]The relativistic case is the familiar XY model, which manifestly has the form of a compact boson in the IR.

[10]The time derivative term here is as usual fixed in accordance with the commutation relations that we have imposed. It can also be derived from the non-relativistic term $i\psi^\dagger\partial_t\psi$ present in the action for the original bose fields.

The near-SF is thus equivalent to a compact boson in the large-radius limit. The extent to which this system fails to be a SF is measured by the correlation function

$$\langle V_{1,0}(x)V_{-1,0}(y)\rangle \sim \frac{1}{|x-y|^{\frac{1}{2\pi}\sqrt{V/K}}}. \tag{31}$$

So of course we do not actually have a SF because of the algebraic falloff, but the power of the decay is arbitrarily small in the weakly interacting $K/V \to \infty$ limit.

In the limit of strong (repulsive) interactions, the natural starting point is the (Wigner) crystal. Again from a phenomenological point of view we know that the IR action has to be that of a compact boson (the "Goldstone" for translation); the only nontrivial part is finding the radius of the compact boson in terms of the interaction strength and the lattice spacing of the crystal, and providing a mapping between the boson operators and those of the IR action.

The phenomenological approach works like this: let us introduce a dimensionless field θ which measures the displacements of the atoms from their equilibrium position, with a shift in θ of 2π corresponding to a translation of all atoms in the crystal by one lattice constant a. That is, we let $x = x_0 + a\theta/2\pi$, where x is the position operator and x_0 a series of delta functions at the lattice sites. From this we see that for small fluctuations of the lattice, $\partial_x\theta$ keeps track of the fluctuations of the equilibrium number density through

$$\rho = \rho_0 + \frac{1}{2\pi}\partial_x\theta. \tag{32}$$

Then the phenomenological Goldstone action is, taking the IR limit of $\sum_i \left(\frac{1}{2}m(\partial_t\delta x)^2 + \frac{1}{2}Va(\delta\rho)^2\right)$,

$$S = \frac{1}{2}\int \left[\frac{ma}{\pi^2}(\partial_t\theta)^2 - \frac{V}{4\pi^2}(\partial_x\theta)^2\right] = \frac{R^2}{4\pi}\int \left(\frac{1}{v}(\partial_t\theta)^2 - v(\partial_x\theta)^2\right), \tag{33}$$

where now $R = \pi^{-1/2}(maV)^{1/4}$ and $v = \sqrt{V/4ma}$ (the latter is $\sqrt{T/\mu}$ since V has dimensions of energy times length). The extent to which this system fails to be a crystal is determined by the correlation functions of the wavevector $k = 2\pi/a$ component of the number density. To be notationally suggestive, we will define the momentum k_F by

$$2k_F \equiv \frac{2\pi}{a} = 2\pi\rho_0. \tag{34}$$

Now we can translate the whole crystal by one lattice spacing by changing θ by 2π. Under this change, the $\mathbb{R}$-space density of the $2k_F$ wavevector part of the density, viz. ρ_{2k_F}, has a phase change of 2π, and hence the relation between the two must be[11]

$$\rho_{2k_F}(x) \propto e^{i(2\pi\rho_0 x + \theta)}. \tag{35}$$

Therefore we have

$$\langle \rho_{2k_F}(x)^\dagger \rho_{2k_F}(y)\rangle \sim \frac{1}{|x-y|^{\pi/\sqrt{maV}}}, \tag{36}$$

[11]Note that the $2k_F$ component of the density is related to a vertex operator of θ, while the zero-momentum component is related to $\partial_x\theta$.

which is algebraically decaying but with a power that is arbitrarily small in the strongly-interacting $maV \to \infty$ crystalline limit (maV has dimensions of $\hbar^2$ and so is properly dimensionless).

Now we will use a more explicit operator mapping to "derive" the compact boson action. Since we are coming from the starting point of a crystal, we want to do the mapping in a subspace where the density operator ρ is a sum of integer-weight delta functions. This will be the case if the combination $2\pi\rho_0 x + \theta$ is constrained to take values only in $2\pi\mathbb{Z}$, since then $\rho = \partial_x(\rho_0 x + \theta/2\pi)$ will be an appropriate sum of delta functions. Now working explicitly with a discontinuous field like this is of course a pain, and in any case we will eventually want to relax this constraint. Therefore we will incorporate the constraint on ρ by adding in an appropriate delta function:

$$\rho = (\rho_0 + \partial_x\theta/2\pi)\sum_{n\in\mathbb{Z}} e^{in(2\pi\rho_0 x+\theta)}. \tag{37}$$

The commutation relations between $\partial_x\theta$ and ϕ are fixed by $[\rho(x), e^{i\phi(y)}] = \delta(x-y)e^{i\phi(x)} \implies [\partial_x\theta(x)/2\pi, \phi(y)] = -i\delta(x-y)$, which gives us the expected momentum for ϕ.

To write the mapping of the boson field ψ, we need to take $\sqrt{\rho}$. But this is actually straightforward, since the square root of a sum of delta functions is just proportional to the same sum of delta functions. Therefore we can write

$$\psi \sim \sqrt{\rho_0 + \partial_x\theta/2\pi}\sum_{n\in\mathbb{Z}} e^{in(2\pi\rho_0 x+\theta)}e^{i\phi}. \tag{38}$$

Therefore we see how the spectrum of the compact boson theory comes out of the original model: all operators built from polynomials in the ψ fields are manifestly given by vertex operators (plus derivatives of θ, ϕ). Furthermore we expect that for IR questions we can soften the constraint on ρ by dropping most of the terms appearing in the sum which enforce the discreteness constraint, since terms with larger n oscillate more quickly in $\mathbb{R}$-space by an amount given by the UV scale $\rho_0 = 1/a$.

Anyway, now we put this relation into the boson Hamiltonian. The first term $|\partial_x\psi|^2$ is rather complicated—it involves the simple $\rho_0(\partial_x\phi)^2 + \dots$ (where $\dots$ are higher in derivatives and hence irrelevant), but also the complicated $(\partial_x\sqrt{\rho})^2$. However, one sees that all the terms in the expansion of $(\partial_x\sqrt{\rho})^2$ are actually all irrelevant, since they are all of the form $(\partial_x^2\theta)^2$ or $(\partial_x\theta)^2 P(\cos\theta, \sin\theta)$, where Ps are polynomials in various cosines and sines of θ. Therefore we can drop all the terms in the $|\partial_x\psi|^2$ term except for the gradient term for ϕ. Similarly the interaction term $V(\delta\rho)^2$ just becomes $\propto V(\partial_x\theta)^2$ after dropping irrelevant terms, and so the Hamiltonian is

$$H = \frac{1}{2}\int dx\, \left(\frac{\rho_0}{m}(\partial_x\phi)^2 + \frac{V}{4\pi^2}(\partial_x\theta)^2\right). \tag{39}$$

If we then integrate out ϕ to get an action just in terms of θ, we get the same compact boson action for θ as above, with the same radius.

In the general case where we are somewhere between a SF and a crystal, we simply map the boson operators with an extrapolation between their images in the two limits:

$$\psi \sim \sqrt{\rho_0 + \partial_x\theta/2\pi}\sum_{n\in\mathbb{Z}} U_n e^{in(2\pi\rho_0 x+\theta)}e^{i\phi}, \tag{40}$$

where the U_n are phenomenological coefficients. When we are close to a SF all the U_n are nearly zero expect for U_0, while when we are close to a crystal the U_n are nearly independent of n.

Bosonizing fermions: formal approach

Now we will discuss a field-theory flavored way of motivating bosonization (of fermions). This has the advantage of being rather clean and easy to work with, but the disadvantage of being slightly subtle once interactions are added and of having the overall appearance of a magic trick (which it is not).

The strategy in the field theory approach is to "rigorously" establish the mapping for the case of free fermions, and then make a rather sketchy argument about the generalization to the interacting case.

The compact boson theory discussed in the last section is, of course, a bosonic theory: the spectrum of operators, viz. $\{V_{n,w}, d\theta, d\phi\}$, are all bosonic (and the derivatives can be obtained from the vertex operators by taking OPEs).[12] To get a fermionic theory, we have to generalize slightly. From the fact that $s_{n,w} = -nw$, we see that all we have to do is to generalize the operator algebra to e.g. include operators either with $n \in \frac{1}{2}\mathbb{Z}$ or $w \in \frac{1}{2}\mathbb{Z}$ (but not both). Taking one of n, w to be fractional effectively attaches a JW string branch cut to the vertex operator, and provides the commutation relations we expect from a fermion. The choices of whether we allow for fractional n or w are related by T-duality, and correspond to whether we want the free fermions to occur at $R = 1/\sqrt{2}$ or $R = \sqrt{2}$. Therefore without loss of generality we can study fermionic theories by looking the theory whose spectrum is generated by the operators $V_{n,\frac{1}{2}w}$, for $n, w \in \mathbb{Z}$.

Unfortunately, it turns out that this notation makes a bunch of formulas that appear later rife with ugly factors of 2. We will therefore introduce the variables

$$\Phi \equiv \phi, \qquad \Theta \equiv \frac{R^2}{2}(\phi_- - \phi_+) = \frac{\theta}{2}, \qquad \pi_\Phi = \frac{1}{\pi}\partial_x\Theta \tag{41}$$

Here Φ is introduced just to make the notation look slightly more visually pleasing. We then define the vertex operators

$$\mathcal{V}_{n,w} = V_{n,w/2} = e^{i(n\Phi + w\Theta)} = e^{i\phi_+(n - R^2w/2) + i\phi_-(n + R^2 w/2)}, \tag{42}$$

which, for the free action (1), have scaling dimensions and spins given by

$$\Delta_{n,w} = \frac{1}{2}\left(\frac{n^2}{R^2} + w^2\frac{R^2}{4}\right), \qquad s_{n,w} = -nw/2. \tag{43}$$

The space of vertex operators is thus still obtained from two compact fields, each still with periodicity 2π,[13] except now the vertex operators can be fermionically nonlocal.

[12] We are working on $\mathbb{R}^2$ throughout, and hence are not caring about global issues like spin structure dependence.

[13] If we like, we could stick with the old ϕ, θ notation and just say that we are increasing the periodicity condition on θ to $\theta \sim \theta + 4\pi\mathbb{Z}$.

From our discussion of the compact boson, we see that at $R = \sqrt{2}$, the chiral fields[14]

$$V_\pm = e^{2i\phi_\pm} = \mathcal{V}_{1,\mp 1} \tag{44}$$

are well-defined in the sense that their two-point functions are single-valued and have the same correlation functions as free fermions (as well as the same self-anti-commutation relations as fermions, so they are only local to the extent that fermions are local). This means that all correlation functions of the $\psi_{L/R}$ fields calculated with the free Dirac action will be identical to those calculated with the vertex operators $e^{i\phi_\pm}$ in a compact boson theory at $R = \sqrt{2}$.

We will thus write the bosonization map *for free fermions* as[15] (note that this mapping *only works at* $R = \sqrt{2}$)

$$\begin{aligned} \mathcal{B}[\psi_R] &= \gamma_R \mathcal{V}_{1,-1} = \frac{\gamma_R}{\sqrt{a}} e^{i(\Phi-\Theta)} = \frac{\gamma_R}{\sqrt{a}} e^{i2\phi_+}, \\ \mathcal{B}[\psi_L] &= \gamma_L \mathcal{V}_{1,1} = \frac{\gamma_L}{\sqrt{a}} e^{i(\Phi+\Theta)} = \frac{\gamma_L}{\sqrt{a}} e^{i2\phi_-} \end{aligned} \tag{45}$$

where a is a UV cutoff needed to get the dimensions correct, which until now we have been hiding in the implicit normal-ordering of the vertex operators, and where γ_σ are Klein factors (Majorana fermions) needed so that $\mathcal{B}[\psi_R]$ anticommutes with $\mathcal{B}[\psi_L]$ in our quantization scheme. From now on, the γ_σs and the $(a)^{-1/2}$s will only be written out when needed.

Note that the translation $U(1)_T$ and particle-number $U(1)_N$ symmetries act on the fermions (in the IR) as

$$\begin{aligned} U(1)_T &: \psi_{L/R} \mapsto e^{\mp i\rho\delta x/2}\psi_{L/R} \\ U(1)_N &: \psi_{L/R} \mapsto e^{i\alpha}\psi_{L/R}, \end{aligned} \tag{46}$$

where $\rho = 2k_F$ is the density (note that $U(1)_T$ acts axially). Hence on Φ, Θ we have (not writing out as)

$$U(1)_T : \Phi \mapsto \Phi, \quad \Theta \mapsto \Theta - \frac{\rho}{2}\delta x, \qquad U(1)_N : \Phi \mapsto \Phi + \alpha, \quad \Theta \mapsto \Theta. \tag{47}$$

The factor of 1/2 in the $U(1)_T$ transformation of Θ means that if we translate by $\delta x = 2\pi/\rho$, which is the distance over which we expect to find one fermion, Θ shifts by π, which is non-trivial. This tells us that $e^{i\Theta}$ counts fermion parity, a conclusion which we will confirm shortly. From the commutation relations above the generators of the two $U(1)$s are

$$Q_T = -\frac{\rho}{2\pi}\int d\Phi, \qquad Q_N = \frac{1}{\pi}\int d\Theta. \tag{48}$$

The mixed anomaly between the two $U(1)$s then can be understood from the commutation relations of the above charge densities; since this is done in another diary entry we won't go in to further detail.

[14]The factor of 2 in the exponent is an unavoidable causality of our notation—this seemed like the least annoying place for factors of 2 to live, so we'll just deal with it.

[15]When the coefficient in front of the kinetic term for the action (1) is normalized to be 1/2, which is another popular choice, the fermions are $e^{\pm i\sqrt{4\pi}\phi_\pm}$.

We will find it convenient to define the fields

$$\varphi_R = \Phi - \Theta, \qquad \varphi_L = \Phi + \Theta, \qquad [\varphi_{R/L}(x), \varphi_{R/L}(y)] = \pm\frac{2\pi i}{R^2}\mathrm{sgn}(x-y). \tag{49}$$

At the free fermion radius $R = \sqrt{2}$ we have $\varphi_{R/L} = 2\phi_\pm$, but this is not true for general R (in particular, the $\varphi_{R/L}$ are *not* chiral at generic radii). In terms of these fields then,[16]

$$\mathcal{B}[\psi_{L/R}] = \frac{1}{\sqrt{a}}e^{i\varphi_{L/R}}, \tag{50}$$

The bosonization map is the statement that the two actions

$$\mathcal{B} : \frac{1}{2\pi}\int \bar{\psi} i\partial\!\!\!/ \psi \leftrightarrow \frac{1}{4\pi}\int \sum_{\sigma=L,R}\left((-1)^\sigma \partial_t\varphi_\sigma\partial_x\varphi_\sigma - v\partial_x\varphi_\sigma\partial_x\varphi_\sigma\right), \tag{51}$$

generate the same correlation functions, where $(-1)^\sigma$ is -1 for $\sigma = R$ and $+1$ for $\sigma = L$, and v is the velocity of the dirac fermions. The RHS is the same as (5) since we are at $R = \sqrt{2}$. In terms of the Φ, Θ fields, we may write

$$\mathcal{B} : \frac{1}{2\pi}\int \bar{\psi} i\partial\!\!\!/ \psi \leftrightarrow \frac{1}{2\pi}\int \left(2\partial_t\Phi\partial_x\Theta - v(\partial_x\Phi)^2 - v(\partial_x\Theta)^2\right). \tag{52}$$

The statement here is again that these two actions generate the same correlation functions provided we identify operators using $\mathcal{B}$. That is, since $\psi_{L/R}$ has the same correlation functions as $e^{i\phi_{L/R}} = e^{i\phi_\pm}$ in the free theory, the claim is that

$$\langle \mathcal{O}[\psi]\rangle_{\frac{1}{2\pi}\bar{\psi}i\partial\!\!\!/\psi} = \langle \mathcal{O}[\mathcal{B}[\psi]]\rangle_{R=\sqrt{2}}, \tag{53}$$

where $\mathcal{O}[\psi]$ is any polynomial of ψ fields at arbitrary positions. The claim is that the spectrum of operators $\mathcal{V}_{n,w}$ (and their derivatives) exhaust all operators in the fermion theory. It's clear that we get all polynomials of the fermions by taking products of the $\mathcal{V}_{1,\pm1}$s—the operators with n and / or w odd have less obvious fermionic counterparts; we will see in a sec that they are related to $(-1)^F$ operators.

Just to make the claim about the matching of correlation functions in the free theory completely explicit, we know that in the fermionic theory we have (looking at e.g. the R fermions wolog)

$$\langle \psi_R(x_1)\dots\psi_R(x_n)\psi_R^\dagger(y_1)\dots\psi_R^\dagger(y_n)\rangle = \det\left(\frac{1}{x_i - y_j}\right). \tag{54}$$

On the other hand, the vertex operators give (with the implicit normal-ordering eliminating the $i = j$ terms)

$$\langle e^{i\varphi_R(x_1)}\dots e^{i\varphi_R(x_n)}e^{-i\varphi_R(y_1)}\dots e^{-i\varphi_R(y_n)}\rangle = \frac{\prod_{i<j<n}(x_i - x_j)\prod_{i<j\le n}(y_i - y_j)}{\prod_{i<j\le n}(x_i - y_j)}. \tag{55}$$

[16] The $\pm$ sign in the exponent is convention: many times it is instead written as $e^{\pm i\phi_\pm}$. Changing these conventions, which amounts to mapping $\psi_L \leftrightarrow \psi_L^\dagger$, simply swaps the physical interpretations of the ϕ and θ via T-duality. In the present conventions, Φ is a phase variable (Φ getting a vev is a "SF"), while Θ is a density variable (Θ getting a vev is a "crystal").

This is indeed exactly equal to the determinant; one can show this e.g. by looking at the zeros and the poles: both functions have poles when some x_i equals some y_j, and both have zeros when two x's or two y's are coincident (since then the matrix in the determinant becomes degenerate).

Interactions are dealt with by expanding the exponential $e^{iS_{int}}$ as a bunch of correlation functions:

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{i(S+S_{int})} = \int \mathcal{D}\phi\,\mathcal{D}\theta\, \exp\left(iS_{R=\sqrt{2}}[\phi,\theta] + i\mathcal{B}[S_{int}]\right). \tag{56}$$

In my opinion what we've done so far is rigorous (for physicists). The formula for $\mathcal{B}[\psi_{L/R}]$ above is useful in that tells us that the parition functions in the two theories above are identical, and furthermore provides us with a may of matching correlation functions in the two free theories.

Bosonization mapping for a few operators

In accordance with the above discussion, we need to figure out how to bosonize products of $\psi_{L/R}$ fields in the free theory. We do this by resolving products of operators by point-splitting in the usual way. We will point-split by displacing the operators in space, since this is most convenient for employing the commutation relations when doing calculations. Shankar's book is an okay reference for some of the following.

We will need to bosonize some operators that take the form of normal-ordered products / derivatives of fermion operators. However, but our bosonization map as written above only works on the constituent fermions themselves, since they are the fields whose correlation functions are matched on the boson side. So in order to map more complicated operators we un-normal-order them and express them in terms of the $\psi_{L/R}$, then bosonize by using the fact that the bosonization map is a homomorphism

$$\mathcal{B}[\mathcal{O}_1 \odot \mathcal{O}_2] = \mathcal{B}[\mathcal{O}_1] \odot \mathcal{B}[\mathcal{O}_2] \tag{57}$$

for $\mathcal{O}_i$ any single-fermion operators, and finally re-write things in terms of normal-ordered products to find the image of the given operator under bosonization (also remember that Taylor expansions can only be performed *inside* the normal-ordering symbol, at the very last step).

For example, let us consider the R fermion density. We first need to remember that

$$G_{L/R}(x,t) = \frac{i}{t \mp x + ia}, \tag{58}$$

where the $+ia$ convergence factor usually won't we written. The fact that the $\pm$ sign appears on the x and not the t is important for some calculations, so we will try to keep track of it correctly.[17] The i here is because the propagator comes from inverting $i^2\not{\partial}$, not $i\not{\partial}$.[18] Also note that the $1/2\pi$ in front of the fermion action means there's no

[17]The easy way to remember this is that in the Dirac action, the derivatives appear as $\partial_t \pm v\partial_x$. It is checked by Fourier transforming with the Feynman propagator (i.e. with $i\varepsilon\mathrm{sgn}(k)$ in the denominator): $\langle\psi_{L/R}(k)\psi_{L/R}(k)^\dagger\rangle \propto \pm\Theta(k)e^{-k0^+}$, which gives the desired result.

[18]There are various conventions for this, but ours is the one in which $G_R(z-w) = \langle\psi_R(z)\psi_R^\dagger(w)\rangle$, with no factor of i.

2π in the above propagator. Anyway, we can now write

$$\begin{aligned}(\psi_R^\dagger\psi_R)(z) &= \lim_{\varepsilon\to 0}\left(\psi_R^\dagger(z+\varepsilon)\odot\psi_R(z) - \frac{i}{\varepsilon}\right)\\ &\to \lim_{\varepsilon\to 0}\left(e^{-i\varphi_R(z_\varepsilon)}\odot e^{i\varphi_R(z)} - \frac{i}{\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0}\left(e^{-i\varepsilon\partial_x\varphi_R(z)+\dots}\frac{i}{\varepsilon} - \frac{i}{\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0}\left([1-\varepsilon(i\partial_x\varphi_R)(z)]\frac{i}{\varepsilon} - \frac{i}{\varepsilon}\right)\\ &= \partial_x\varphi_R(z),\end{aligned} \tag{59}$$

where the i when combining the two vertex operators into a single exponential comes from the BCH-formula phase of $e^{\frac{1}{2}[\varphi_R(z+\varepsilon),\varphi_R(z)]} = e^{i\pi\,\mathrm{sgn}(\varepsilon)/2} = i$, provided that we point-split in the "correct" way (recall we are at $R=\sqrt{2}$). Of course this feels pretty arbitrary (and I'm going to stop paying attention to signs too carefully at this point so as to retain my sanity), and this is one of the reasons why the field theory approach is a bit annoying.

A similar calculation for the L fields gives the opposite sign[19]

$$\mathcal{B}[\psi_{R/L}^\dagger(x)\psi_{R/L}(x)] = \pm\partial_x\varphi_{R/L}(x) = \pm\partial_x\Phi \mp \partial_x\Theta. \tag{60}$$

Therefore the currents map as[20]

$$\mathcal{B}[2\pi j^\mu] = \mathcal{B}[(\psi_L^\dagger\psi_L+\psi_R^\dagger\psi_R, \psi_R^\dagger\psi_R-\psi_L^\dagger\psi_L)^\mu] = 2(\partial_x\Theta,\partial_x\Phi)^\mu. \tag{62}$$

Since $d\Phi \overset{\text{eom}}{=} R^{-2}\star d\theta = 2R^{-2}\star d\Theta$ and we are at $R=\sqrt{2}$, we have (making the replacement on the equations of motion here is exact since both Φ and Θ appear quadtratically in the action — if cosines were added to the action then manipulations like this would not be legit)

$$\mathcal{B}[j] \overset{\text{eom}}{=} \frac{1}{\pi}\star d\Theta \overset{\text{eom}}{=} \frac{1}{\pi}d\Phi. \tag{63}$$

In particular, the density is (this is exact, not just on the eom)

$$\mathcal{B}[j^0] = \frac{1}{\pi}\partial_x\Theta \tag{64}$$

Therefore the operator $e^{i\Theta(x)}$ counts the fermion parity to the left (or right) of x[21]:

$$e^{i\Theta(x)} = e^{i\pi\int^x j^0}. \tag{65}$$

[19] There's a minus sign from the fact that φ_Ls have an opposite sign in the commutator when re-combining the exponentials.

[20] Note that the spatial part of the current here is $n_R - n_L$; in other parts of the diary it's been the other way around. Also note that with these conventions, because of the $1/2\pi$ in the fermion action,

$$n_\sigma = \frac{1}{2\pi}\psi_\sigma^\dagger\psi_\sigma. \tag{61}$$

[21] This nice factor-of-2-free result, which vibes nicely with the conventions in part of the CMT literature is our reason for choosing to work with the $V_{n,w/2}$ vertex operators.

We thus have the physical interpretation of Θ as the field which counts the fermion density, which shifts by $\delta\Theta = \pi$ at the location of a fermion (θ shifts by $\delta\theta = 2\pi$ at each fermion). Therefore in a Euclidean time picture, fermion-odd operators insert π vortices in Θ. Note that the Thirring interaction bosonizes as $\mathcal{B}[j_\mu j^\mu] = -\frac{1}{\pi^2}\partial_x\varphi_R\partial_x\varphi_L$.

The off-diagonal bilinears are easy, since the OPE is trivial:

$$\mathcal{B}[\psi_L(x)\psi_R^\dagger(x)] = e^{i2\Theta(x)}, \qquad \mathcal{B}[\psi_L(x)\psi_R(x)] = e^{i2\Phi(x)}. \tag{66}$$

Finally for the bosonization of kinetic terms for the fermions. Since

$$\langle\psi_R^\dagger(z)(\partial_w\psi_R(w))\rangle = -i\frac{1}{(z-w)^2}, \tag{67}$$

we have, focusing on the ∂_x term for concreteness, (it's better to get rid of the derivative first by point-splitting and then bosonize rather than the other way around)

$$\begin{aligned}\psi_R^\dagger\partial_x\psi_R &= \lim_{\epsilon\to 0}\left(\psi_R^\dagger(z)\frac{\psi_R(z+\varepsilon)-\psi_R(z-\varepsilon)}{2\varepsilon} + \frac{i}{\epsilon^2}\right)\\ &\to \lim_{\epsilon\to 0}\left(e^{-i\varphi_R(z)}\odot\frac{1}{2\varepsilon}\left(e^{i\varphi_R(z+\varepsilon)} - e^{i\varphi_R(z-\varepsilon)}\right) + \frac{i}{\epsilon^2}\right)\end{aligned} \tag{68}$$

The RHS is, remembering the is coming from recombining the exponentials,

$$\frac{i}{2\varepsilon}e^{-i\varphi_R(z)+i\varphi_R(z+\varepsilon)}\frac{1}{-\varepsilon} - \frac{i}{2\varepsilon}e^{-i\varphi_R(z)+i\varphi_R(z-\varepsilon)}\frac{1}{+\varepsilon} + \frac{i}{\varepsilon} \approx \frac{-i}{2\varepsilon^2}\left(2 + i\varepsilon^2\partial_x^2\varphi_R - \varepsilon^2(\partial_x\varphi_R)^2\right) + \frac{i}{\varepsilon^2}, \tag{69}$$

where we have expanded the exponentials to $O(\varepsilon^2)$. Up to total derivatives, this just gives $i\frac{1}{2}(\partial_x\varphi_R)^2$, and therefore

$$\mathcal{B}[\psi_+^\dagger i\partial_x\psi_+] = -\frac{1}{2}(\partial_x\varphi_R)^2. \tag{70}$$

This gets us part of the kinetic term. The rest of the kinetic term comes from the other derivative of ψ_+ and the derivatives of ψ_- in the similar way. For $\psi_+^\dagger i\partial_t\psi_+$ we just get $-\partial_t\varphi_R\partial_x\varphi_R/2$, while for the ψ_- terms we get opposite signs on both the ∂_t and ∂_x terms (from the opposite sign in the commutation relations when combining the exponential). One then checks that the two kinetic terms indeed map into one another, and that even the coefficients are correct!

It's easy to get lost in the formalities here, but note that we're still just doing hydrodynamics. The two conserved quantities are again momentum and $U(1)_N$ charge, and as we have seen above, bosonization gives us a way to represent their currents j in terms of free bosons (we have two symmetries but only one current because of the mixed anomaly between the two symmetries). The resulting action is just an EFT formed from the conserved currents. This is of course true in the non-interacting case, but since it really comes from general EFT principles the general idea holds when interactions are turned on as well.

Adding interactions

As a simple example, consider adding the term

$$S_{int} = -\frac{1}{4\pi}\int U_{\alpha\beta}\rho_\alpha\rho_\beta, \tag{71}$$

where $\rho_\alpha = \psi^\dagger_\alpha\psi_\alpha = 2\pi n_\alpha$. The off-diagonal part $\rho_L\rho_R$ is a $j_\mu j^\mu$ Thirring-type interaction (viz. $\frac{\pi}{2}U_{LR}j_\mu j^\mu$), while the forward scattering terms $U_{\sigma\sigma}$ will be seen to renormalize the velocities. Indeed, the bosonized version of this is

$$\begin{aligned} S_b &= \frac{1}{4\pi}\int\left(\sum_\sigma(-1)^\sigma\partial_t\varphi_\sigma\partial_x\varphi_\sigma - \sum_\sigma(\partial_x\varphi_\sigma)^2(v_\sigma+U_{\sigma\sigma}) + 2U_{LR}\partial_x\varphi_L\partial_x\varphi_R\right) \\ &= \frac{1}{4\pi}\int\left(-\partial_t\varphi^T Z\partial_x\varphi - \partial_x\varphi^T\mathcal{H}\partial_x\varphi\right), \qquad \mathcal{H} = \begin{pmatrix} v'_R & -U_{LR} \\ -U_{LR} & v'_L\end{pmatrix}, \end{aligned} \tag{72}$$

where the renormalized velocities are $v'_\sigma \equiv v_\sigma + U_{\sigma\sigma}$ and $\varphi = (\varphi_R, \varphi_L)^T$. We can calculate the OPE of the vertex operators by diagonalizing the Hamiltonian. We will preserve the commutation relations (first term in the action) if we can diagonalize $\mathcal{H}$ with something in $SO(1,1)$, i.e. a matrix of the form $M = \mathbf{1}\cosh\psi + X\sinh\psi$. This can always be done if $\mathcal{H}$ is positive-definite, which we of course assume on physical grounds. A bit of algebra (in the diary entry on correlators in Luttinger liquids) shows that

$$M^T\mathcal{H}M = \begin{pmatrix} -U_{LR}\sinh(2\psi) + v'_R\cosh^2\psi + v'_L\sinh^2\psi & 0 \\ 0 & -U_{LR}\sinh(2\psi) + v'_L\cosh^2\psi + v'_R\sinh^2\psi\end{pmatrix}, \tag{73}$$

provided that

$$\tanh(2\psi) = \frac{U_{LR}}{(v'_R+v'_L)/2}. \tag{74}$$

This is always possible if

$$|U_{LR}| < \frac{v'_R+v'_L}{2}. \tag{75}$$

Now on the other hand, the condition that $\mathcal{H}$ be positive-definite can be checked to be that $|U_{LR}| < \sqrt{v'_R v'_L}$. Since the geometric mean is always $\le$ the arithmetic mean, $\mathcal{H}$ being positive-definite automatically guarantees that there's a boost M diagonalizing it. This stability condition $|U_{LR}| < \sqrt{v'_L v'_R}$ is less obvious on the fermion side (also note that because the condition is only on $|U_{LR}|$, the fermions are [equally] unstable to both attractive *and* repulsive Thirring-type interactions). Also note that when $v_L = v_R$ so that $v'_R = v'_L \equiv v'$, we get (after using e.g. $\cosh(\tanh^{-1}(x)) = (1-x^2)^{-1/2}$)

$$M^T\mathcal{H}M = \widetilde{v}\mathbf{1}, \qquad \widetilde{v} = v'/\gamma, \qquad M = \gamma\begin{pmatrix} 1 & -U_{LR}/v' \\ -U_{LR}/v' & 1\end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-U_{LR}^2/v'^2}}, \tag{76}$$

which is exactly what we expect from a Lorentz boost.

To deal with the more general situation (just for fun), define

$$\bar{v} \equiv (v'_R+v'_L)/2, \qquad \delta \equiv (v'_R - v'_L)/2. \qquad \gamma \equiv (1-U_{LR}^2/\bar{v}^2)^{-1/2}. \tag{77}$$

Then we have

$$\cosh^2\psi = \frac{\gamma+1}{2}, \qquad \sinh^2\psi = \frac{\gamma-1}{2}, \qquad \sinh(2\psi) = \gamma U_{LR}/\bar{v}, \tag{78}$$

which allows us to write

$$M^T\mathcal{H}M = \bar{v}\begin{pmatrix}\gamma^{-1}+\delta/\bar{v} & \\ & \gamma^{-1}-\delta/\bar{v}\end{pmatrix} \equiv \begin{pmatrix}\widetilde{v}_R & \\ & \widetilde{v}_L\end{pmatrix}. \tag{79}$$

In terms of the Φ, Θ fields, (sanity check: the correct momentum for Φ is recovered) some algebra gives

$$S_b = \frac{1}{2\pi}\int \left(2\partial_t\Phi\partial_x\Theta - (\partial_x\Phi)^2(\bar{v}-U_{LR}) - (\partial_x\Theta)^2(\bar{v}+U_{LR}) + 2\delta\partial_x\Phi\partial_x\Theta\right). \tag{80}$$

From this presentation, the stability bound on $|U_{LR}|$ in the case when $\delta = 0$ is more obvious. We can integrate out Θ since it appears quadratically, using

$$\partial_x\Theta \overset{\text{eom}}{=} \frac{\partial_t\Phi + \delta\partial_x\Phi}{\bar{v}+U_{LR}} \tag{81}$$

which gives, after some algebra

$$S_b = \frac{R^2}{4\pi}\int\left(\frac{1}{\mathsf{v}}(\partial_t\Phi+\delta\partial_x\Phi)^2 + \mathsf{v}(\partial_x\Phi)^2\right), \qquad R^2 \equiv 2\sqrt{\frac{\bar{v}-U_{LR}}{\bar{v}+U_{LR}}}, \qquad \mathsf{v} \equiv \sqrt{\bar{v}^2 - U_{LR}^2}. \tag{82}$$

The rather unfortunate 2 in the definition of R^2 comes from our use of $\Theta = \theta/2$ i.e. our choice that the Φ, Θ fields are 4π, not 2π periodic. There's essentially no perfect notational choice here, so we'll just live with it.

To get a feel for what the interaction does, consider for simplicity the case where $\delta = 0$, and work in units where $\mathsf{v} = 1$. Then to lowest order in U_{LR}, the action is

$$S_{b;v'_L=v'_R} \approx \frac{R_{eff}^2}{4\pi}\int d\Phi\wedge\star d\Phi, \qquad R_{eff}^2 = 2(1-U_{LR}/\bar{v}). \tag{83}$$

Therefore adding a small repulsive interaction between the two chiral fermions has the effect of decreasing the boson radius. Sanity check: this means that repulsive interactions increase the scaling dimension of the SF ordering term $\cos\Phi$, and decrease that of the crystal ordering term $\cos\Theta$, while attractive interactions do the opposite. This is exactly what we expect on physical grounds—attractive interactions favor k-space order (SF), while repulsive ones favor $\mathbb{R}$-space order (crystal).

Another thing to note is how T-duality works in terms of the fermionic parameters. Again since we are using $\Theta = \theta/2$, T-duality is no longer just $R \mapsto R^{-1}$, but rather

$$T: R \mapsto \frac{2}{R}, \tag{84}$$

which is still an involution. We see that this is equivalent to $T: U_{LR} \mapsto -U_{LR}$, so that here T-duality acts to change the sign of the interaction. Thus attractive and repulsive interactions are actually exactly equivalent to one another, as we noted above. On the fermion side, we can implement the sign flip of U_{LR} by doing charge conjugation on only a single chirality, e.g.

$$T : \psi_+ \mapsto \psi_+, \qquad \psi_- \mapsto \psi_-^*, \tag{85}$$

which flips the sign of the interaction. One can also check that this action exchanges axial and vector currents on the fermion side, in keeping with the fact that it exchanges the regular and topological currents on the boson side.

Anyway, let us now find the scaling dimensions of the spectrum of vertex operators. The OPEs are straightforward to calculate. Let us label vertex operators $\mathcal{V}_{n,w}$ by the vector $\mathsf{n} = (n, w)^T$. The R/L components of this vector are $((n-w)/2, (n+w)/2)^T = S\mathsf{n}$, where $S = \frac{1}{2}(\mathbf{1} + J)$. We then find the OPE

$$\mathcal{V}_{\mathsf{n}}(x,t) \odot \mathcal{V}_{\mathsf{m}}(0,0) = \mathcal{V}_{\mathsf{n}+\mathsf{m}}(x,t) \frac{1}{(x+\widetilde{v}_R t)^{-\mathsf{n}^T S^T M|R\rangle\langle R|MS\mathsf{m}}(x-\widetilde{v}_L t)^{-\mathsf{n}^T S^T M|L\rangle\langle L|MS\mathsf{m}}} + \dots, \tag{86}$$

where the $\dots$ are less singular. This means that the conformal dimension is, skipping some algebra,

$$\Delta_{\mathsf{n}} = \frac{1}{2}\mathsf{n}^T S^T M^2 S\mathsf{n} = \frac{\gamma}{4}\mathsf{n}^T \begin{pmatrix} 1+U/\bar{v} & \\ & 1-U/\bar{v} \end{pmatrix} \mathsf{n} = \frac{\gamma}{4}\left(n^2(1+U/\bar{v}) + w^2(1-U/\bar{v})\right). \tag{87}$$

As we saw earlier for the simpler example, repulsive interactions make the Θ (w) vertex operators more relevant, favoring CDW order, while attractive interactions make the Φ (n) operators more relevant, favoring SF order.

The conformal spin is

$$s_{n,w} = \frac{1}{2}\mathsf{n}^T S^T MZMS\mathsf{n} = \frac{1}{2}\mathsf{n}^T S^T ZS\mathsf{n} = -\frac{1}{4}\mathsf{n}^T X\mathsf{n} = -\frac{nw}{2}. \tag{88}$$

Note in particular that the conformal spin is unchanged by interactions or changes in velocity, as it should be (which is a consequence of the fact that $M \in SO(1,1)$ preserves the commutation relations).

Let us also compute the correlation functions of the UV fermions. In real space, the dominant part of this correlation function is

$$\langle \psi(x)\psi^\dagger(0)\rangle \sim \frac{\cos(k_F x)}{|x|^{1/R^2+R^2/4}}. \tag{89}$$

Fourier transforming, we then have

$$\langle \psi_k \psi_k^\dagger \rangle \sim |k-k_F|^\eta + |k+k_F|^\eta, \tag{90}$$

where

$$\eta \equiv \frac{1}{R^2} + \frac{R^2}{4} - 1. \tag{91}$$

Note that $\eta \geq 0$ for all R^2, with the equality saturated only at $R^2 = 2$ (where the distribution is of course instead a step function).

For nontrivial interactions there are no quasiparticles, in the sense that the UV fermions no longer have overlap with the excitations (hydro modes) which diagonalize the Hamiltonian. One can see this either by the fact that the hydro modes have different quantum numbers than the UV fermions, or by the fact that the fermion-fermion correlators have branch cuts, with no well-defined quasiparticle pole. The cool thing is that even though $\psi_{L/R} \sim e^{i\varphi_{L/R}}$ has power-law correlations with a coefficient that is a continuous function of the interaction parameters, the current still has scaling dimension exactly equal to one and is not renormalized by interactions (as it always bosonizes to $\sim (\partial_x\Theta, \partial_x\Phi)$), in keeping with the Ward identity.

It's worth comparing for a second between the approach above and the treatment in a few field theory texts, e.g. Shankar's QFT in CMT book and Witten's lectures on Abelian bosonization, both of which make it seem like bosonization in the presence of interactions can be done exactly while keeping a Lorentz-invariant structure.

From a field theory point of view, given that $\mathcal{B}[j^0] = \frac{1}{2\pi}\partial_x\Phi$, it is natural to make things covariant by writing $\mathcal{B}[j] = \frac{1}{2\pi}d\Phi$. Now as we saw, this is not the correct way to bosonize—when we point-split correctly the j^1 component of the current maps to something with a ∂_x derivative, namely $\partial_x(\varphi_L - \varphi_R)$. This is consistent with the action of spacetime symmetries since under e.g. time reversal that maps $j \mapsto -j$ (as a form). But if we were to take $\mathcal{B}[j] = \frac{1}{2\pi}d\Phi$ at face value, we'd have

$$S_{\text{free}}[\psi] - \frac{g}{\pi}\int j_\mu j^\mu \quad \overset{?}{\leftrightarrow} \quad \frac{1-g}{8\pi}\int d\phi \wedge \star d\phi. \tag{92}$$

When reading Witten and Shankar one gets the feeling that this relation is exact, but this can't be correct: as we discussed earlier, T-duality on the boson side is the same as the "KW Duality" sending $j \mapsto \star j$ on the Fermion side, which acts as $\psi_L \mapsto \psi_L^\dagger$, $\psi_R \mapsto \psi_R$. This sends $g \mapsto -g$, but is also supposed to do $R^2 \mapsto \frac{2}{R^2}$, which is not compatible with the above equation. Indeed we explicitly saw above, the above is only correct to leading order in g, and hence is a perturbative statement (dimensionally correct since the velocity is being suppressed). The full relation between R and g is non-linear, as seen by the formula above with the square roots and such, and is only derived in the QFT framework by doing a self-consistent point-splitting in e.g. space only.

To summarize: while the field theory way of thinking is slicker and nicer for doing calculations, the cond-mat way of doing this is more rigorous and intuitive; one should learn the cond-mat way but usually calculate things the QFT way (just like in RG).

Relevance of symmetry-breaking perturbations

The two symmetries of the system we've been studying so far are the $U(1)_N$ of particle number conservation, which shifts Φ, and the $U(1)_T$ of translation, which shifts Θ.

If we restrict our attention to actions which are symmetric under both symmetries, the previous subsection covers all possibilities, up to the effects of irrelevant derivative interactions. If we allow ourselves to consider perturbations which break the symmetries though, we can add sines and cosines of integer multiples of the Φ and Θ fields (since we usually don't want to add operators with nonzero spin to the theory, we can restrict our attention to just $\cos(n\Phi)$ and $\cos(m\Theta)$, without any mixed Θ-Φ terms).[22] Furthermore note that to preserve $(-1)^F$ we need $n \in 2\mathbb{Z}$, and since we only want to consider perturbations which are local, we also need to take $m \in 2\mathbb{Z}$.

The minimal perturbations are therefore $\cos(2\Theta), \cos(2\Phi)$. At the free fixed point, the dimensions of these two are in fact equal:

$$\Delta_{SC}^{free} = \Delta_{CDW}^{free} = 1. \tag{93}$$

The fact that $\Delta_{CDW}^{free} < 2$ is the statement of Peierls instability: a translation-breaking potential at wavevector $2k_F$ always drives an instability. Note that the filling with the most relevant $U(1)_T$-symmetric cosine is half-filling with $k_F = \pi/2$, which permits $\cos(4\Theta)$ as a $U(1)_T$-preserving perturbation. For free fermions this is comfortably irrelevant, with a scaling dimension of 4. Note that as expected, the $U(1)_T$-breaking perturbations become more relevant as the strength of the interactions is increased (i.e. as the radius of the boson is decreased) since the interactions favor $\mathbb{R}$-space Wigner-crystal ordering; similarly, getting closer to a superfluid by decreasing the interactions makes the $U(1)_N$-breaking perturbations more relevant.

Bosonization: CMT approach

We can use the results in the subsection on bosonizing bosons to get a much more intuitive, but also less rigorous, derivation of the bosonization formulae outlined in the previous section.

Bosonizing fermions is essentially the same as bosonizing bosons, which we have already discussed from an EFT point of view. In order to have notation that's consistent with the previous section we will add an extra minus sign in the expression for the bosonzied density operator, so that

$$\rho = \rho_0 - \partial_x\Theta/\pi. \tag{94}$$

In the Wigner crystal limit, we thus tack on the constraint enforcing the discreteness of the lattice by writing ρ as

$$\rho = (\rho_0 - \partial_x\Theta/\pi)\sum_{n\in 2\mathbb{Z}} e^{in(\pi\rho_0 x - \Theta)}. \tag{95}$$

Extending this approach to fermions is very easy—we just add on JW tails to fermionize the operators appearing in H (we add the tails in the same way in both the SF and crystal limits). The JW strings need to count the fermion number to the left (say) of a given fermion, and so they must be given by $(-1)^{\int^x \rho} = e^{i(\pi\rho_0 x - \Theta(x))}$. Then the fermions are given generically by

$$\psi(x) \sim \sqrt{\rho_0 - \partial_x\Theta/\pi}\sum_{n\in 2\mathbb{Z}} U_n e^{i(n+1)(\pi\rho_0 x - \Theta)} e^{i\Phi(x)} \sim \sqrt{k_F + \partial_x\Theta}\sum_{n\in 2\mathbb{Z}+1} U_n e^{in(k_F x - \Theta(x))} e^{i\Phi}, \tag{96}$$

[22] Note that we might think that adding e.g. $\cos(2\Theta - 2k_F x)$ would be a way to add a symmetry-allowed density modulation, but this in fact vanishes: it comes from a term like $\int dx\, e^{2ik_F x}\psi_L^\dagger\psi_R + h.c.$, but this vanishes because the support of the $\psi_{L/R}$ in momentum space is narrow enough to preclude the $\psi_L^\dagger\psi_R$ term from having the required $2k_F$ momentum transfer to survive integration. The correct density modulation term is instead just $\int dx\, L^\dagger R + h.c.$, which is nonzero but breaks $U(1)_T$.

where U_n are some phenomenological constants. Therefore the effect of the JW tails is to shift the sum of the exponents of the Θ vertex operators from $2\mathbb{Z}$ to $2\mathbb{Z}+1$. Including the $\partial_x\Theta$ term in the square root is done to account for situations in which we imagine k_F varying semiclassically throughout space, with the k_F in the square root representing a spatial average of the Fermi momentum. In a situation where k_F is fixed, then by Luttingers theorem there is no zero-momentum modulation in the density (the Fermi sea only sloshes back and forth, it does not pulse in size). In the non-interacting case only $U_{\pm 1}$ are nonzero; larger U_ns come from processes which transfer momenta $2nk_F$, which are only possible in the presence of interactions. Therefore in the free limit, if we write $k_F = 1/a$ as a UV cutoff, we reproduce exactly the formulae (45) motivated through more formal field-theory methods. One can also check that the symmetry actions work out in the same way as in the bosonic case.

Of course, while this method for mapping the fermions agrees in the IR with the previous field theory approach, it also gives us an idea of what happens when we back away from this limit, and has the conceptual advantage that we didn't have to start with respect to a reference free theory with a certain Fermi surface. Indeed, this approach relied only on basic hydrodynamically-flavored reasoning, and at no point did we bring up complicated ways of counting bosonic and fermionic excitations with respect to a Fermi sea, normal ordering prescriptions, etc. etc.

Two dimensional bosonization, the Schwinger model, and θ angles

This is an elaboration on one of the problems in Quantum Fields & Strings, a Course for Mathematicians, Vol II. Consider two flavors of massive fermions in two dimensions coupled to a $U(1)$ gauge field with a θ term, in Euclidean signature:

$$S = \frac{1}{2\pi}\int\left(\sum_i \bar{\psi}_i \not{D}_A \psi_i + \bar{\psi}_i m_i \psi_i\right) + \frac{1}{2e^2}\int F\wedge\star F + \frac{i\theta}{2\pi}\int F. \tag{97}$$

The factor of $1/2\pi$ in front of the fermions is there so that the fermion correlators will have no annoying prefactors. Here the mass term is complex, so that in the representation where $\gamma^0 = X$ and $\psi_i = (\psi_{i,+}, \psi_{i,-})^T$,

$$\bar{\psi}_i m_i \psi_i = m\psi^\dagger_{i,+}\psi_{i,-} + m^*\psi^\dagger_{i,-}\psi_{i,+}. \tag{98}$$

We will discuss this theory and its bosonization, show that the θ term can be eliminated when either of the fermions are massless, and describe what happens in various limits for the massive case.

Let us first do a brief recapitulation of our chosen bosonization conventions for a single flavor of Dirac fermion in this setting. We will be in $i\mathbb{R}$ time, with gamma matrices $\gamma^0 = X, \gamma^1 = Y$. Thus the Dirac operator is

$$\not{D} = \begin{pmatrix} 0 & \partial_+ \\ \partial_- & 0 \end{pmatrix}, \qquad \partial_\pm = \frac{1}{2}(\partial_0 \mp i\partial_1). \tag{99}$$

Note the "reversed" signs! This is so that ∂_+ is a "left-moving" derivative and ∂_- is a "right-moving" derivative. The classical eom are $\partial_+\psi_- = \partial_-\psi_+ = 0$, so that classically ψ_- is right-moving and ψ_+ is left-moving, since ∂_- kills things that are only a function of $t+ix$ (which we regard as left-moving), while ∂_+ kills things that are only a function of $t - ix$ (which are right-moving). This ensures that the chirality operator Z counts left-movers minus right-movers, so the helpful thing to remember is that $+$ things have positive chirality and hence move left (counterclockwise), while $-$ things have negative chirality and thus move right.

The correlator of the fermions is

$$D_{\alpha\beta}(x-y) = \delta_{\alpha\beta}\frac{x-y}{|x-y|^2}, \tag{100}$$

where x and y are complex coordinates in the plane. One can verify this by computing the derivative explicitly, or by noting that it is derivative of the free boson propagator (since if G is the free boson propagator, then schematically $\partial^2 G = \delta \implies D = \partial G$ satisfies $\partial D = \delta$). The factor of $1/2\pi$ in the Lagrangian was chosen so that no factors of 2π appear in the fermion propagator.

We will use the bosonization rules worked out in the diary entry that does an overview of how bosonization works, viz.

$$\mathcal{B}[\psi_{L/R}] = \frac{1}{\sqrt{a}}e^{i\varphi_\pm}, \tag{101}$$

where the actions for both fields are

$$\mathcal{B}: \frac{1}{2\pi}\int \bar{\psi} i\not{\partial}\psi \leftrightarrow \frac{1}{4\pi}\int \sum_{\sigma=\pm}\left(i\sigma\partial_\tau\varphi_\sigma\partial_x\varphi_\sigma - \partial_x\varphi_\sigma\partial_x\varphi_\sigma\right). \tag{102}$$

In terms of the Φ, Θ fields defined by $\varphi_+ = \Phi + \Theta$, $\varphi_- = \Phi - \Theta$, we may write

$$\mathcal{B}: \frac{1}{2\pi}\int \bar{\psi} i\not{\partial}\psi \leftrightarrow \frac{1}{2\pi}\int \left(2i\partial_\tau\Phi\partial_x\Theta - (\partial_x\Phi)^2 - (\partial_x\Theta)^2\right). \tag{103}$$

The fermion currents map as

$$\mathcal{B}[2\pi j^\mu] = \mathcal{B}[(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-, \psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-)^\mu] = 2(\partial_x\Theta, \partial_x\Phi)^\mu. \tag{104}$$

Very importantly for the present problem, the mappings above mean that we have the mapping of the mass term to vertex operators. There are a few ways of writing the mass term. If we take $m_{\mathbb{C}} \in \mathbb{C}$, then we can write the mass term as $\bar{\psi} m_{\mathbb{C}} \psi$; what we really mean by this is

$$\bar{\psi} m_{\mathbb{C}} \psi = m_{\mathbb{C}} \psi_+^\dagger \psi_- + m_{\mathbb{C}}^* \psi_-^\dagger \psi_+. \tag{105}$$

We can also equivalently take two real parameters $m, m_5 \in \mathbb{R}$, and write

$$\bar{\psi} m_{\mathbb{C}} \psi = m\bar{\psi}\psi + m_5 \bar{\psi}\bar{\gamma}\psi = m(\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+) + m_5(\psi_+^\dagger \psi_- - \psi_-^\dagger \psi_+). \tag{106}$$

Either way, using the bosonization mapping we see that

$$\bar{\psi} m_{\mathbb{C}} \psi \mapsto m_{\mathbb{C}} e^{-i\Theta} + m_{\mathbb{C}}^* e^{i\Theta} = 2\,\mathrm{Re}\,[m_{\mathbb{C}}] \cos\Theta + 2\mathrm{Im}[m_{\mathbb{C}}] \sin\Theta. \tag{107}$$

In particular, when m is real, we get a sine-Gordon $\cos\Theta$ interaction (even though the fermions are free!).

When we add a gauge field, not a lot changes. The Dirac operator gets upgraded to

$$\not{D}_A = \begin{pmatrix} 0 & \partial_+ + iA_+ \\ \partial_- + iA_- & 0 \end{pmatrix}, \quad A_\pm = A_0 \mp iA_1. \tag{108}$$

This means that the coupling between A and the fermion current is

$$S \supset \int (A_+ J_- + A_- J_+) dz \wedge d\bar{z}, \tag{109}$$

where $J_\pm = \psi_\pm^\dagger \psi_\pm / 2\pi$. This is the same as $A \wedge \star J$, since the metric in $z, \bar{z}$ coordinates is off-diagonal.

When we do the mapping now, the kinetic term goes to

$$\mathcal{B} \,:\, \frac{1}{2\pi} \int \bar{\psi} \not{D}_A \psi \mapsto S_0[\Theta, \Phi] + \frac{i}{2\pi} \int dx\, d\tau \,\, \left(A_+ \partial_x(\Phi - \Theta) + A_- \partial_x(\Phi + \Theta) \right). \tag{110}$$

Let us now integrate out Φ; we will be able to do this even in the presence of the mass term since Φ still appears quadratically. We get

$$\mathcal{B} \,:\, \frac{1}{2\pi} \int \bar{\psi} \not{D}_A \psi \mapsto \frac{1}{2\pi} \int \left(d\Theta \wedge \star d\Theta + 2i(A_+ \partial_- \Theta - A_- \partial_+ \Theta) dz \wedge d\bar{z} \right) \tag{111}$$

We can integrate the last two terms by parts, which yields $\frac{i}{\pi}\Theta F_A$, so that 2Θ contributes to the θ angle.[23] This expression makes sense even though Θ is not a legit zero-form since the quantization condition on F_A means that such ambiguities only affect the action by something in $4\pi\mathbb{Z}$.

`<diversion>`

One quick comment on this: the term $\frac{1}{\pi}\Theta F_A$ is obviously not invariant under shifting Θ by a constant, but naively its counterpart $A_+ \partial_- \Theta - A_- \partial_+ \Theta$ is invariant under the shift, since Θ only appears under a derivative sign. Actually this is conclusion

[23] The fact that it is 2Θ and not Θ can be checked from the chiral anomaly: under $\psi \mapsto e^{i\bar{\gamma}\alpha}\psi$ we have $\delta\Theta = \alpha$, while at the same time we know that the action needs to shift by $d^\dagger j_A = \frac{\alpha}{\pi} \int F_A$ — hence the factor of 2 is correct.

is wrong, and the latter expression *does* change under shifting Θ by a constant. The proper way to understand this is by using differential cohomology, and interpreting the term as $A_+ \star \Theta - A_- \star \Theta$ where $\star$ is the product operation in Deligne-Bellison cohomology. Since the ΘF_A term is not invariant under the shift symmetry of Θ and since the shift symmetry of Θ is the chiral symmetry on the fermion side, this non-invariance brings out the chiral anomaly from the integration measure and makes it more explicit though the non-invariance of the action.

`</diversion>`

Now consider making the fermions massive with a mass term $m\bar{\psi}\psi + m_5\bar{\psi}\bar{\gamma}\psi$. The full action is now

$$\mathcal{B} \,:\, \frac{1}{2\pi}\int \bar{\psi} i \not{D}_A \psi \mapsto \frac{1}{2\pi}\int \left(d\Theta \wedge \star d\Theta + 4\pi(m\cos\Theta + m_5 \sin\Theta) + 2\Theta F_A\right). \tag{112}$$

We can now generalize to the multi-species case given in the introduction to this diary entry. The bosonic theory is (adding in a kinetic term for the gauge field, which we assume to come with a θ term)

$$S = \frac{1}{4\pi}\int \sum_i R_i^2 d\Theta \wedge \star d\Theta + \sum_i (m_i e^{i\Theta_i} + m_i^* e^{-i\Theta}) + \frac{1}{2e^2}\int F_A \wedge \star F_A + \frac{\theta/2 + \Theta_1 + \Theta_2}{\pi}\int F_A. \tag{113}$$

Here we have also left open for the two bosons to have different radii, which can be modified away from the free-fermion value of $R = \sqrt{2}$ by Thirring current-current interactions for the fermions.

If at least one of the $m_i = 0$, one of the boson fields has no $e^{i\Theta_i}$ interaction, and so we can perform the shift $\Theta_i \mapsto \Theta_i - \theta/2$ to eliminate the θ dependence from the action. Hence the parameter θ is meaningful only when *both* fermions are massive.

Consider when both fermion masses are small, $m_i \to 0$. Also for simplicity, let $R_1 = R_2 = R$. Since when $m = 0$ the theory is quadratic, we can just look at the equations of motion and because of the masslessness, we can ignore the θ dependence. Now define the bosonic fields

$$\xi_\pm = \Theta_1 \pm \Theta_2, \tag{114}$$

so that the action is (now in $\mathbb{R}$ time)

$$S = \frac{R^2}{8\pi}\int \sum_{a=\pm} d\xi_a \wedge \star d\xi_a + \frac{1}{2e^2}\int F_A \wedge \star F_A + \frac{\xi_+}{\pi}\int F_A. \tag{115}$$

Since ξ_- doesn't see the gauge field, the eom for ξ_- gives a regular massless wave equation. The equations of motion for A and for ξ_+ are

$$\frac{1}{e^2} d^\dagger F = \frac{1}{\pi} \star d\xi_+, \tag{116}$$

and

$$\frac{R^2}{4\pi}\Box \xi_+ = \frac{1}{\pi} \star F \tag{117}$$

We then plug the first equation into the second by solving for F; this gives $\star F = \frac{e^2}{\pi}\xi_+$ and so we get a massive wave equation for ξ_+:

$$(\Box - m_\varphi^2)\xi_+ = 0, \qquad m_\varphi = \frac{2e}{R\sqrt{\pi}}. \tag{118}$$

So we get one massive and one massless scalar[24] (which is rather interesting as Θ_1, Θ_2 entered the original action completely decoupled from one another!). Note that if we only were bosonizing a single fermion, then we would only have one boson, and the whole theory would be gapped. This is the bosonization way of seeing the main point of the Schwinger model: gauge invariance is not enough to guarantee massless states in two dimensions, since you can get confining gauge fields to do the job.

What happens when one of the fermion masses, say m_1, goes to ∞? In perturbation theory, I don't think we can see that anything happens, other than e.g. the Maxwell term getting corrected by factor that goes to zero as $m_1 \to \infty$. However, on the bosonic side, we can argue as follows: first, perform a chiral rotation so that $m_1 \in \mathbb{R}^{<0}$. This shifts the θ term by $\pi - \arg(m)$. The mass is now real, and we get a $m_1 \cos(\Theta_1)$ potential for the bosons. When $m_i \to -\infty$ we can take $\Theta_1 \to 0$, and Θ_1 disappears form the theory. Thus the effect of the heavy fermions is to shift

$$\theta \mapsto \theta + \pi - \arg(m). \tag{120}$$

ethan: *come back and elaborate*

Functional bosonization

This is from Altland and Simons (but there are some typos in the problem so don't worry to much about reproducing their results — also, our notation will deviate a bit form theirs, and will unfortunately also deviate from other diary entries).

Consider fermions in two dimensions, with action

$$S = -\int \bar\psi \partial\!\!\!/ \psi + \frac{1}{2}\int \rho^T \mathcal{G} \rho - \int (\psi_\sigma^\dagger J_\sigma + J_\sigma^\dagger \psi_\sigma), \tag{121}$$

[24] Another way of getting to this result is to take $d^\dagger A = 0$ gauge from the outset, which allows us to write $A^\mu = \varepsilon^{\mu\nu}\partial_\nu\varphi$ for a scalar φ, with $F_A \wedge \star F_A$ becoming $\varphi\Box^2\varphi$. The coupling $\xi_+\Box\varphi$ can then be eliminated by shifting ξ_+, which when done produces a term $\varphi\Box\varphi$. Therefore the φ term has a propagator like $(\partial^2 + \partial^4/e^2)^{-1}$, which leads to a q-space correlation function like (this is just schematic)

$$\frac{1}{q^2 + q^4/e^2} \sim \frac{1}{q^2} - \frac{q^2+e^2}{,} \tag{119}$$

which indeed splits up into massive and massless parts.

where $\mathcal{G}$ is an interaction matrix, $\rho = (\rho_+,\ \rho_-)^T$ are the densities, and $\sigma \in \{\pm\}$ are the left- and right-moving indices (the currents J_σ are just there to generate correlation functions—they are not the fermion currents coming from the $U(1)$ particle number symmetries). Our plan is to get to the usual bosonization result in a slightly different way.

First, we will decouple the interactions using a bosonic doublet. It will turn out to be convenient to do a Hodge decomposition on this doublet, since the longitudinal and transverse parts of the decomposition will match with the vector and chiral currents and so this decomposition will behave nicely with respect to the holomorphic / antiholomorphic decomposition of the boson field.

We will then integrate out the fermions and find the effective action for the components of the Hodge decomposition of the Hubbard-Stratonovich field, eventually obtaining the usual interacting Luttinger liquid action.

This diary entry was written a fair amount of time before the diary entry on the standardization of bosonization conventions, and in retrospect I think the approach outlined below is conceptually rather murky. Nevertheless, I've decided to keep it for posterity's sake.

First we decouple the interaction, using a bosonic field $\boldsymbol{\phi} = (\phi_+,\ \phi_-)$. We let $\boldsymbol{\phi}$ appear in the action as $\frac{1}{2}\boldsymbol{\phi}^T\mathcal{G}^{-1}\boldsymbol{\phi}$, and then shift $\boldsymbol{\phi} \mapsto \boldsymbol{\phi} + i\mathcal{G}\rho$. This leaves us with a $i\rho^T\boldsymbol{\phi}$ coupling, which is just like that of a gauge field. To write the coupling in a covariant form, let ϕ_μ be the "vector" field with components

$$\phi_0 = \frac{1}{2}(\phi_+ + \phi_-), \qquad \phi_1 = \frac{1}{2i}(\phi_+ - \phi_-). \tag{122}$$

In a representation where the gamma matrices are $\gamma^0 = X, \gamma^1 = Y$, we then have

$$Z = \int \mathcal{D}\phi\mathcal{D}\psi \exp\left(-\frac{1}{2}\int \phi^T\mathcal{G}^{-1}\phi + \int \bar{\psi}\not{D}_\phi\psi - S_{src}[J,\psi]\right), \tag{123}$$

where

$$\not{D}_\phi = \gamma^\mu(\partial_\mu - i\phi_\mu). \tag{124}$$

We now do a Hodge decomposition on the 1-form ϕ (without the bold font ϕ means the 1-form $\phi_\mu dx^\mu$; with the bold font it means the 2-component scalar (ϕ_+, ϕ_-)—admittedly not the best notation). We will write it as

$$\phi = d\xi + id^\dagger \star \eta. \tag{125}$$

The i is just for convenience, and we have written $\star\eta$ since we'd rather work with zero-forms than two-forms. The Hodge decomposition plays nicely with the chiral nature of the fermions, with ξ relating to the vector current and η to the chiral current. We see this by considering

$$\not{\partial}(e^{i\xi+i\eta Z}\psi) = e^{i\xi+i\eta Z}\not{\partial}\psi + (-\partial_0\xi - i\partial_1\eta X - \partial_1\xi Y + i\partial_0\eta Y)e^{i\xi+i\eta Z}\psi = \gamma^\mu(\partial_\mu + i\phi_\mu)e^{i\xi+i\eta Z}\psi. \tag{126}$$

Thus (if we ignore what happens to $\mathcal{D}\psi$), we can eliminate the coupling to the background field ϕ through a phase rotation by ξ and a chiral rotation by η.

Now we can integrate out the fermions. We expand the $\mathrm{Tr}\ln$ to second order in ϕ, producing the usual polarization bubble. The effective action then has a term (remembering the -1 from the fermion loop)

$$S_{eff} \supset -\frac{1}{2}\int_{q,p} \boldsymbol{\phi}_q^T \frac{1}{\not{p}(\not{p}-\not{q})}\boldsymbol{\phi}_{-q}. \tag{127}$$

Because ϕ_μ couples to the fermions as a gauge field would, the integration kernel is diagonal in the spin indices. The propagator for the ψ's is

$$D_\psi(p)_{\sigma\sigma'} = \delta_{\sigma\sigma'}\frac{i}{\nu + i\sigma p}, \tag{128}$$

which comes from inverting ∂_σ. Note that I am being lazy and not distinguishing between two-momenta and their spatial components: that is, I am writing $q = (\nu, q)$. Sorry not sorry.

We now write the σ part of the above integral as

$$\frac{1}{2}\int_{q,p} \boldsymbol{\phi}_{q,\sigma}^T \frac{1}{\nu + i\sigma p}\frac{1}{\nu + i\sigma p + \omega + i\sigma q}\boldsymbol{\phi}_{-q,\sigma}, \tag{129}$$

where $q = (\omega, q)$. It is helpful to recast this as

$$\frac{1}{2}\int_q \frac{1}{\omega + i\sigma q}\int_p \boldsymbol{\phi}_{q,\sigma}^T \left(\frac{1}{\nu + i\sigma p} - \frac{1}{\nu + i\sigma p + \omega + i\sigma q}\right)\boldsymbol{\phi}_{-q,\sigma}. \tag{130}$$

This looks like it might be zero after doing a contour integral and closing it in either the upper half plane or the lower one (depending on σ), but on the other hand, it's $\sim \int_p d^2p\, p^{-2}$ which is divergent (I think there are some subtle things going on as the pole at $-i\sigma p$ gets pushed to ∞). As suggested in the book, we do a somewhat suspect thing and close the integrals in the plane where they give a non-zero answer by the residue theorem. If $\sigma p > 1$ then the pole lies in the lower half plane and we we get a clockwise integral, giving $-2\pi i$, while if $\sigma p < 1$ then we get a counterclockwise integral, giving $2\pi i$. So then

$$-\frac{i}{2}\int_q \int_p \frac{dp}{2\pi}\; \boldsymbol{\phi}_{q,\sigma}^T \left(\mathrm{sgn}(\sigma p) - \mathrm{sgn}(\sigma(p+q))\right)\boldsymbol{\phi}_{-q,\sigma}. \tag{131}$$

Now we introduce an explicit cutoff for the spatial momentum integration. Luckily as long as we take $\Lambda \to \infty$, the answer doesn't depend on the exact value for Λ. So we get

$$-\frac{1}{4\pi}\int_q \boldsymbol{\phi}_{q,\sigma}^T \frac{-i\sigma q}{\omega + i\sigma q}\boldsymbol{\phi}_{-q,\sigma}. \tag{132}$$

Recapitulating, the effective action for the boson fields is

$$S_{eff}[\boldsymbol{\phi}] = \frac{1}{2}\int_q \boldsymbol{\phi}_q^T \left(\mathcal{G}^{-1} + G_{\boldsymbol{\phi}}(q)\right)\boldsymbol{\phi}_{-q}, \tag{133}$$

where

$$G_{\phi}(q)_{\sigma\sigma'} = \frac{\delta_{\sigma\sigma'}}{2\pi}\frac{i\sigma q}{\omega + i\sigma q}. \tag{134}$$

Now we look at the source term. Before we integrated out the fermions but after we introduced the ϕ fields, we can perform a shift on ψ to eliminate the linear coupling between the fermions and the sources (we have to do this after adding in the ϕ fields since we don't want to mess with the density-density ψ interactions). The source term is then

$$S_{src} = \int_{x,x'} J^{\dagger}(x) G_{\psi,\phi}(x,x') J(x'), \tag{135}$$

where $G_{\psi,\phi}(x,x')$ is the propagator for the fermions in the background ϕ field.

Using the Hodge decomposition of the ϕ field,

$$S_{src} = \int_{x,x'} J^{\dagger}(x) e^{-i(\xi+\eta Z)(x)} G_{\psi}(x,x') e^{i(\xi+\eta Z)(x')} J(x'). \tag{136}$$

Now we can play a trick by representing the fermion propagator with a bosonic doublet of fields. First of all, the actual expression for $G_{\psi}(x,0)$ is (just invert $\partial_{\pm}$)

$$[G_{\psi}(x,0)]_{\sigma\sigma'} = \delta_{\sigma\sigma'} \int_{q,\omega} \frac{e^{-i(xq+\omega\tau)}}{i\sigma q + \omega}. \tag{137}$$

Taking $x > 0$ wolog, the integrand is analytic in the lower half-plane, and so we close the contour for $q \to -i\infty$. Thus we get zero if $\sigma\omega > 0$ and get a residue of $-2\pi\sigma \exp(\omega(\sigma x - i\tau))$ otherwise, so that

$$[G_{\psi}(x,0)]_{\sigma\sigma'} = -\sigma\delta_{\sigma\sigma'} \int_{\omega} \Theta(-\sigma\omega) e^{\omega(\sigma x - i\tau)}. \tag{138}$$

Doing the integral,

$$[G_{\psi}(x,0)]_{\sigma\sigma'} = \frac{\delta_{\sigma\sigma'}}{2\pi}\frac{1}{i\sigma x + \tau}. \tag{139}$$

Now we want to reproduce this with bosons. Consider a doublet of bosons which possesses the following free action:

$$S_{\varphi} = \frac{1}{2}\int_q \varphi_q^T K_{\varphi}(q) \varphi_{-q}, \tag{140}$$

where $\varphi = (\varphi_+,\ \varphi_-)^T$ and

$$K_{\varphi}(q) = \begin{pmatrix} q^2 + iq\omega & 0 \\ 0 & q^2 - iq\omega \end{pmatrix}. \tag{141}$$

One can think of the components of φ as holomorphic and antiholomorphic modes. The Greens function for these guys is

$$[G_{\varphi}(x,0)]_{\sigma\sigma'} = \delta_{\sigma\sigma'} \int_{q,\omega} \frac{e^{-i(xq+\omega\tau)}}{q^2 + i\sigma q\omega}. \tag{142}$$

The poles of the momentum integral are at $q = -i\sigma\omega$, and the integrand is analytic for $q \to -i\infty$. The residue at the pole is $2\pi/\sigma\omega$ (since the contour is closed clockwise), so we get

$$[G_\varphi(x,0)]_{\sigma\sigma'} = \sigma\delta_{\sigma\sigma'} \int_\omega \Theta(\sigma\omega) \frac{e^{-\omega(\sigma x + i\tau)}}{\omega}. \tag{143}$$

To do the integral, we impose a small frequency cutoff at a^{-1} (or at $-a^{-1}$, depending on σ), where a is the lattice spacing. The integral can then be expanded in small a. We will get a bunch of constants, which will be normal-ordered away. The surviving piece gives a log, and so

$$[G_\varphi(x,0)]_{\sigma\sigma'} = -\ln\left(\frac{i\sigma x + \tau}{a}\right). \tag{144}$$

The whole point of going through this is that the correlators of the vertex operators for the φ fields reproduce the form of the fermion correlators:

$$\langle e^{i\phi_\sigma(x,\tau)} e^{-i\phi_{\sigma'}(0,0)} \rangle = \delta_{\sigma\sigma'} \frac{a}{i\sigma x + \tau}, \tag{145}$$

where the expectation value is over the free φ action. This is exactly equal to the fermion correlator but for a factor of $a/2\pi$. Thus if we absorb this factor into the sources J, we can write the source term as

$$S_{src} = \int_{x,x'} J^\dagger(x) e^{-i(\xi+\eta Z)(x)} \langle e^{i\phi(x)} e^{-i\phi(x')} \rangle e^{i(\xi+\eta Z)(x')} J(x'). \tag{146}$$

Now we play a cute trick: the vertex operators have a Gaussian distribution (since they map to ψ, which has just a free action), and so we can pull the expectation value out of the exponential in $e^{-S_{src}}$, and realize the expectation value by integrating over the φ fields. Thus we just need to take the square root of the above integral, exponentiate it, and path integrate over φ. The parition function is then

$$Z = \int \mathcal{D}\xi\mathcal{D}\eta \int \mathcal{D}\varphi e^{-S_{eff}[\xi,\eta] - S_\varphi[\varphi]} \exp\left(-\int (J^\dagger e^{-i(\xi+\eta Z)} e^{i\varphi} + e^{-i\varphi} e^{i(\xi+\eta Z)} J)\right), \tag{147}$$

with $S_\varphi[\varphi]$ the free action for the φ fields and $S_{eff}[\xi,\eta]$ is the effective action for ϕ that we derived earlier.

This representation of the source term tells us that it would be nice if we had a decomposition of φ to a form like $\xi + \eta Z$. This is easily done by writing

$$\varphi_\pm = \Phi \pm \Theta. \tag{148}$$

One can check that the action S_φ becomes, in this representation,

$$S_\varphi = \frac{1}{2}\int_q (\Phi_q,\ \Theta_q) \widetilde{K}_q \begin{pmatrix} \Phi_{-q} \\ \Theta_{-q} \end{pmatrix}, \qquad \widetilde{K}_q = \begin{pmatrix} q^2 & iq\omega \\ iq\omega & q^2 \end{pmatrix}. \tag{149}$$

With this representation, we can eliminate the ξ, η fields from the source term by shifting the Φ and Θ fields. The sum of the free actions then becomes, after the shift (letting Ψ denote the $(\Phi,\ \Theta)$ doublet and letting Ξ denote the $(\xi,\ \eta)$ doublet)

$$S_{eff}[\xi,\eta] + \frac{1}{2}\int_q \left(\Psi_q^T \widetilde{K}_q \Psi_{-q} + \Xi_q^T \widetilde{K}_q \Psi_{-q} + \Psi_q^T \widetilde{K}_q \Xi_{-q} + \Xi_q^T \widetilde{K}_q \Xi_{-q}\right). \tag{150}$$

Here a miracle occurs. We change basis from the ϕ field to its Hodge representation Ξ by way of the matrix (I think this is listed incorrectly in the book?)

$$\phi_q = U_q \Xi_q, \qquad U_q = \begin{pmatrix} q - i\omega & q - i\omega \\ -q - i\omega & q + i\omega \end{pmatrix}. \tag{151}$$

The miracle is that

$$\widetilde{K}_q = -U_q^T G_\phi(q) U_{-q}. \tag{152}$$

This means that the $\Xi^T \widetilde{K} \Xi$ term in the last integral we wrote actually cancels with one of the terms in $S_{eff}[\xi, \eta]$ after we complete the switch from the ϕ representation to the (ξ, η) representation.

Recapitulating, the action (without the source term) is

$$S = \frac{1}{2} \int_q \left(\Psi_q^T \widetilde{K}_q \Psi_{-q} + \Xi_q^T \widetilde{K}_q \Psi_{-q} + \Psi_q^T \widetilde{K}_q \Xi_{-q} + \Xi_q^T U_q^\dagger \mathcal{G}_q^{-1} U_{-q} \Xi_q \right). \tag{153}$$

Since now only Ψ appears in the source term, we want to integrate out Ξ, which we can now do happily. We integrate it out to get

$$S = \frac{1}{2} \int_q \Psi_q^T \left(\widetilde{K}_q - \widetilde{K}_q [U_q^\dagger \mathcal{G}_q^{-1} U_{-q}]^{-1} \widetilde{K}_q \right) \Psi_{-q}. \tag{154}$$

To write this out explicitly, let us write $\mathcal{G}$ as (following the notation in the book now)

$$\mathcal{G} = g_4 \mathbf{1} + g_2 X. \tag{155}$$

We know the explicit form for all the matrices in the above action, and so we can just multiply them out and see what we get. Our final bosonized form for the complete partition function is then

$$Z = \int \mathcal{D}\Theta \mathcal{D}\Phi \exp\left(-S_0 - S_{src}\right), \tag{156}$$

where

$$S_0 = \frac{1}{2\pi} \int_q (\Phi_q,\ \Theta_q) \begin{pmatrix} q^2(1 + 2\pi(g_4 - g_2)) & iq\omega \\ iq\omega & q^2(1 + 2\pi(g_4 + g_2)) \end{pmatrix} \begin{pmatrix} \Phi_{-q} \\ \Theta_{-q} \end{pmatrix}, \tag{157}$$

and

$$S_{src} = \int \left(J_\sigma^\dagger e^{i(\Phi + \sigma\Theta)} + e^{-i(\Phi + \sigma\Theta)} J_\sigma \right). \tag{158}$$

All done!

Another look at currents and operator splitting applied to bosonization

Today is a fast one. Consider a free Dirac fermion in two dimensions. We will be identifying the currents (vector and chiral) and computing their commutators, being careful to do the point splitting of the operators that constitute the currents. We will find e.g. for the vector current that

$$[j_\mu(x), j_\nu(y)] = C\partial_x \delta(x-y), \tag{159}$$

where C is some constant that depends on how one normalizes the currents.

The regular fermion vector currents are $j^\mu = \bar\psi\gamma^\mu\psi$, so that in Euclidean signature with $\gamma^0 = X, \gamma^1 = Y$ we have for $\psi = (\psi_L, \psi_R)^T$,

$$j_R \equiv \frac{1}{2}(j_0 + ij_1) = \psi_R^\dagger\psi_R, \qquad j_L \equiv \frac{1}{2}(j_0 - ij_1) = \psi_L^\dagger\psi_L. \tag{160}$$

We will choose the coefficient in front of the action to be such that

$$\langle\psi_L^\dagger(z,\bar z)\psi_L(w,\bar w)\rangle = \frac{1}{z-w}, \qquad \langle\psi_R^\dagger(z,\bar z)\psi_R(w,\bar w)\rangle = \frac{1}{\bar z - \bar w}. \tag{161}$$

Since we get this from $\frac{1}{\pi}\bar\partial z^{-1} = \delta(z,\bar z)$, we want the coefficient in front of the action to be $g = 1/2\pi$.

Anyway, now we can compute the (equal-time) commutator of the currents. Of course, j_R and j_L commute. We then have (as usual, the following is to be understood in the OPE sense of having an implicit expectation value)

$$\begin{aligned}
[j_R(x), j_R(y)] &= [:(\psi_R^\dagger\psi_R)(x):, :(\psi_R^\dagger\psi_R)(y):] \\
&= \lim_{\epsilon,\eta\to 0}[:\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon):, :\psi_R^\dagger(y+\eta)\psi_R(y-\eta):]. \\
&\sim i\lim_{\epsilon,\eta\to 0}\Big(\frac{1}{x+\epsilon-y+\eta}\delta(x-y-\epsilon-\eta) - \frac{1}{x+\epsilon-y+\eta}\psi_R^\dagger(y+\eta)\psi_R(x-\epsilon) \\
&\quad - \frac{1}{y+\eta-x+\epsilon}\psi_R^\dagger(x+\epsilon)\psi_R(y-\eta) - \frac{1}{(y+\eta-x+\epsilon)(x+\epsilon-y+\eta)}\Big) \\
&\quad - (x\leftrightarrow y, \epsilon\leftrightarrow\eta),
\end{aligned} \tag{162}$$

where the factor of i came from the i in $1/(\bar z - \bar w)$ evaluated at $\bar z = -ix, \bar w = -iy$. There may be other factors of 2 somewhere but I'm not going to worry about them too much. The most singular term is symmetric under the interchange $x \leftrightarrow y, \epsilon \leftrightarrow \eta$ and so it dies, while the two middle terms are also symmetric under the interchange.

So

$$[j_R(x), j_R(y)] \sim i \lim_{\epsilon,\eta\to 0} \left(\frac{1}{x+\epsilon-y+\eta}\delta(x-y-\epsilon-\eta) - \frac{1}{y+\eta-x+\epsilon}\delta(y-x-\eta-\epsilon) \right)$$
$$= i \lim_{\epsilon,\eta\to 0} \left(\frac{\delta(x-y-[\epsilon+\eta]) - \delta(x-y+[\epsilon+\eta])}{\eta+\epsilon} \right) = -i\partial_x \delta(x-y). \tag{163}$$

When we compute the commutator for j_L the only thing changes is that we have a $-i$ up front instead of a $+i$ by virtue of the ψ_L 2-point function being $1/(z-w)$, so

$$[j_R(x), j_R(y)] \sim i\partial_x \delta(x-y). \tag{164}$$

Now we can go and rewrite this in terms of the spacetime components of the current. Since $j_0 = j_R + j_L, j_1 = -i(j_R - j_L)$ and the $j_{R,L}$ commutators are opposite in sign, we have

$$[j_0(x), j_0(y)] = [j_1(x), j_1(y)] = 0, \qquad [j_0(x), j_1(y)] \sim -2\partial_x\delta(x-y). \tag{165}$$

We could probably have chosen a smarter normalization for the currents so that this dumb factor of 2 wasn't there, but too late. Actually from now on I think it'll be good to go over into real time. Doing this means we need to multiply j_1 or j_0 by i, depending on the signature we want (since we need to change either of the γ matrices to i times itself in order to get the right Clifford algebra relations with the new metric—for definiteness we will let the real-time γ matrices be $\gamma^0 = X, \gamma^1 = iY$), and so in real time we have

$$[j_0(x), j_1(y)] \sim -2i\partial_x\delta(x-y). \tag{166}$$

Now consider a free boson ϕ, and define the currents

$$\mathcal{J}_0 = \sqrt{2}\partial_x\phi, \qquad \mathcal{J}_1 = -\sqrt{2}\partial_t\phi, \tag{167}$$

so that $\mathcal{J}_\mu = \sqrt{2}\epsilon_{\mu\nu}\partial^\nu\phi$ is equal to (the dumb factor of $\sqrt{2}$ times) the topological current, which if ϕ is smooth is trivially conserved. Since the conservation of the fermion current will fail at the locations of certain operator insertions, this tells us that these operator insertions create a topological singularity in the dual ϕ field (so that ϕ is not integrable). Anyway, since $\partial_t\phi$ is the momentum we have

$$[\mathcal{J}_\mu(x), \mathcal{J}_\nu(y)] = -2i\epsilon_{\mu\nu}\partial_x\delta(x-y) = [j_\mu(x), j_\nu(y)]. \tag{168}$$

Thus we have found a way to represent the fermion current as the topological current of a boson. Note in particular that under bosonization,

$$-j_\mu j^\mu \mapsto -2\epsilon_{\mu\nu}\partial^\nu\phi\epsilon^{\mu\sigma}\partial_\sigma\phi = 2\partial_\mu\phi\partial^\mu\phi, \tag{169}$$

so that the current bilinear for the fermions becomes the free kinetic term for the boson. This is actually not surprising if we think about the Sugawara construction for the stress tensor: since in models of current algebras we have $T \sim \sum_a : J^a J^a :$, it's not that crazy to think that the current bilinear will bosonize to $\partial\phi\bar{\partial}\phi$, since this is a similar sort of object to the stress tensor.

Anyway, what about the chiral current for the fermions? In two dimensions for our choice of γ matrices we have

$$\gamma^0\gamma^5 = XZ = -iY = \epsilon^{01} iY g_{11}, \tag{170}$$

since for us, $g_{11} = -1$ and $\epsilon^{P(\mu\nu)} = \mathrm{sgn}(P),\ \epsilon_{\mu\nu} = -\epsilon^{\mu\nu})$. We also have

$$\gamma^1\gamma^5 = iYZ = \epsilon^{10} X g_{00}. \tag{171}$$

Putting these together means that

$$\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu. \tag{172}$$

So this means that the chiral current is related to the vector current via

$$j^{\mu 5} = \epsilon^{\mu\nu} j_\nu, \tag{173}$$

which means that the bosonic avatar of the chiral current is

$$\mathfrak{J}^5_\mu = \sqrt{2}\epsilon^{\mu\nu}\epsilon_{\nu\sigma}\partial^\sigma\phi = \sqrt{2}\partial_\mu\phi, \tag{174}$$

which is conserved by virtue of the free boson's equation of motion.

Bosonizing the spin $1/2$ chain and the sine-Gordon model

Today we will bosonize the $SU(2)$-symmetric spin 1/2 AFM spin chain (Heisenberg model) using *Abelian* bosonization. This is of course in books / the literature, but I wanted to go through it at least once on my own.

The spin chain is described by (setting the prefactor $J = 1$ for simplicity)

$$H = \sum_{j,a} S^a_j S^a_{j+1}. \tag{175}$$

The grand strategy is to write things in terms of spinless fermions by way of a Jordan-Wigner transformation, and then do bosonization on these fermions.[25]

[25]Hold on you may say, to do bosonization we will need something with central charge $c = 1$, i.e. a two-component fermion. So don't we need spinful fermions? We do not, since the two "spin" components will come from the left- and right-moving fermion excitations around the two Fermi points.

Anyway, we will work in a basis where S^z is diagonalized. The appropriate strings are built out of $\prod_{i<j}(1-2c_i^\dagger c_i) = \prod_{i<j}(-1)_i^F$, and the raising / lowering operators are

$$S_j^+ = (-1)^j \prod_{i<j}(-1)_i^F c_j, \qquad S_j^- = (-1)^j \prod_{i<j}(-1)_i^F c_j^\dagger. \tag{176}$$

The raising and lowering operators have been staggered by a factor of $(-1)^j$ since we anticipate expanding round a staggered spin configuration (this of course doesn't affect the operator algebra). One then uses $[S_j^+, S_j^-] = Z$ to get

$$S_j^z = -\frac{1}{2}(-1)_j^F. \tag{177}$$

Checking that all these operators commute with one another as they should is straightforward.

Now we need to plug these into the Hamiltonian to get H as a function of the c operators. This is straightforward and produces (rescaling J by a factor of 4)

$$H = J\sum_j \left[(-1)_j^F(-1)_{j+1}^F - 2(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1})\right]. \tag{178}$$

Note that the presence of the S^zS^z term is responsible for giving us the interactions: if the Hamiltonian only had $U(1)$ symmetry (instead of $SU(2)$), we would have free fermions. Also note that if we had a very strong anisotropy for the S^zS^z term we would need $(-1)_j^F = -(-1)_{j+1}^F$, so that the fermion occupancy on the chain would be staggered. This is the CDW state.

Now we want to go over to continuum fermions. We build a Dirac fermion out of the the fermionic excitations around each of the Fermi points at $k_F = \pm\pi/2$ (since the hopping term gives a $\cos k$ dispersion for $-\pi/2 \le k \le \pi/2$). Let the lattice spacing be a, and define continuum fields L, R via

$$c_j = \sqrt{a}(R_j e^{ik_F j} + L_j e^{-ik_F j}) = \sqrt{a}(i^j R_j + i^{-j} L_j). \tag{179}$$

The $\sqrt{a}$ here is to get the dimensions right: we want the continuum fermions to have mass dimension $[L] = [R] = 1/2$, while the c_j fermions are dimensionless by virtue of $\{c_i, c_j^\dagger\} = \delta_{ij}$. One of the hopping terms is then

$$c_j^\dagger c_{j+1} \approx a(i^{-j}R_j^\dagger + i^j L_j^\dagger)(i^{j+1}R_j + i^{j+1}a\partial_x R_j + i^{-j-1}L_j + i^{-j-1}a\partial_x L_j). \tag{180}$$

Higher derivative terms get suppressed by higher powers of a. When we sum over j, terms that oscillate with j will die—they represent high energy events that hop fermions between the two Fermi points, with operators like $L_j^\dagger \partial R_j$, and so we can get rid of them. The expression above has $iR^\dagger R$ and $iL^\dagger L$ terms, but when we add the Hermitian conjugate these terms die, and so

$$a^{-1}\sum_j(c_j^\dagger c_{j+1} + h.c.) = -2i\int dx\,(R^\dagger\partial_x R - L^\dagger\partial_x L), \tag{181}$$

which gives us the free Dirac fermion.

Now for the interaction term. Since $k_F = \pi/2$, we are at half filling for the spinless fermions. Thus $\langle c_j^\dagger c_j \rangle = 1/2$, and so

$$-(-1)_j^F = 2c_j^\dagger c_j - 1 = 2 : c_j^\dagger c_j := 2a : (i^{-j} R_j^\dagger + i^j L_j^\dagger)(i^j R_j + i^{-j} L_j) : \tag{182}$$

The interaction term then becomes, in the continuum variables,

$$\begin{aligned} a^{-1} \sum_j (-1)_j^F (-1)_{j+1}^F \approx \int dx \ &: \big(R^\dagger R + L^\dagger L + (-1)^j (L^\dagger R + R^\dagger L) \big) : \\ &\times : \big(R^\dagger R + L^\dagger L - (-1)^j (L^\dagger R + R^\dagger L) \big) : \end{aligned} \tag{183}$$

Here we have dropped all derivatives, since they all contain an extra factor of a that make them comparatively small (remember that we are assuming the L, R vary slowly over the lattice scale). We can drop the terms that go as $(-1)^j$, but we still get an Umklapp term from the $(-1)^{2j} = 1$ term. From yesterday's problem, we recall the currents $j_0 = R^\dagger R + L^\dagger L$, $j_1 = L^\dagger L - R^\dagger R$ (the sign of j_1 is dictated by our choice of $\gamma^1 = -iY$, which we did so that $\gamma^5 = Z$ and not $-Z$). We then have the current bilinear $(j_\mu)^2 = 2[(R^\dagger R)^2 + (L^\dagger L)^2]$. Note that I have stopped indicating the normal ordering, for notation's sake. However, it is important to remember that it is there, so that e.g. $(R^\dagger R)^2$ really means $(: R^\dagger R :)^2$. When we do bosonization, we will need to be careful to only bosonize things that have been normal-ordered.

After some algebra, we can then write the interaction term as (there are some factors of 2 that we've absorbed into a, all we care about is the relative factor between the different terms)

$$H_I = \int dx \ \left(j_\mu j^\mu - 2 \left[(L^\dagger R)^2 + (R^\dagger L)^2 \right] \right). \tag{184}$$

This means that the full action in terms of the fermions becomes ($\Psi = (L, R)^T$)

$$S = \frac{1}{2\pi} \int dxdt \ \left(i\bar{\Psi} \partial\!\!\!/ \Psi - j_\mu j^\mu + 2 \left[(L^\dagger R)^2 + (R^\dagger L)^2 \right] \right). \tag{185}$$

I think the $1/2\pi$ factor in front will be the most convenient for avoiding gross $\sqrt{4\pi}$'s and stuff, but I'm not sure. We'll see how it goes.

Now let us bosonize. We will roughly follow the normalization conventions in Witten's lectures in Quantum Fields and Strings part II. With these conventions, the Dirac action is bosonized to

$$\mathcal{B}(S_D) = \frac{1}{8\pi} \int \partial_\mu \phi \partial^\mu \phi, \tag{186}$$

which is a compact boson at radius $R = 1/\sqrt{2}$. In terms of the holomorphic / anti-holomorphic components of the boson, the mapping is

$$\mathcal{B}(L) = e^{i\phi_+}, \qquad \mathcal{B}(R) = e^{-i\phi_-}, \tag{187}$$

which one can check reproduces the correct scaling dimensions (the vertex operators $e^{\pm i\phi_\pm}$ have a two-point function that goes like $1/(x-y)^{1/2R^2}$, which is what we want since $R = 1/\sqrt{2}$). If we were to be a bit more careful, we should probably write this as $\mathcal{B}(L) = \frac{1}{\sqrt{2\pi a}} e^{i\phi_+}$ where a is the short-distance cutoff (and we should probably be writing $\sim$ instead of $=$). This ensures that the bosonized fermion has the same dimension of the fermion, and still produces the right correlators since when we are remembering to include the cutoff the propagator for the boson goes like $\ln|r/a|$ instead of just $\ln|r|$. Anyway, we'll suppress the cutoff dependence in what follows.

We can then conclude that

$$\mathcal{B}([LR^\dagger]^2) = e^{2i\phi}, \tag{188}$$

which means that the extra interaction term in the fermionic version of S maps to a sum of vertex operators. The freeness of the bosonized theory is thus ruined by the extra interacting part in S. Anyway, we can now map the full action over to bosons:

$$\mathcal{B}(S) = \int dx dt \left[\frac{(1+1)}{8\pi} \partial_\mu \phi \partial^\mu \phi + \frac{1}{\pi} \cos 2\phi \right]. \tag{189}$$

The only part of this that is questionable is second number 1 in $1+1$; this comes from the bosonization of the Thirring-type interaction using today's bosonization conventions. This 1 can (but won't, sorry) be checked by checking the relevance of the $\cos 2\phi$ interaction, by rescaling the fields so that the factor in front of the kinetic term is $1/2$: we let $\varphi = \phi/\sqrt{2\pi}$, and the interaction cosine becomes $\cos 2\phi = \cos(\beta\varphi)$, with $\beta = \sqrt{8\pi}$.

Finally, a side comment on adding interactions to the fermion Hamiltonian. The interactions that are both tractable and interesting are the current-current terms, which can be written in terms of densities as products like $\rho_L \rho_R$ and which bosonize to the free term. Consider on the other hand an interaction I like

$$H \ni I = \int d^2x\, d^2y\, \rho(x) V(x-y) \rho(y), \tag{190}$$

where $V(x-y)$ is taken to be a contact interaction $\delta(x-y)$. Using

$$\rho = \rho_L + \rho_R + (\psi_R^\dagger \psi_L e^{-2ik_F x} + h.c.), \tag{191}$$

we see that (up to an Umklapp term)), the density-density contact term becomes precisely $\rho_L^2 + \rho_R^2$. Since we are thinking in terms of Hamiltonians we do the point-splitting in space, and so this bosonizes to $(\partial_x \phi_L)^2 + (\partial_x \phi_R)^2$, which just renormalizes the speed of light for the bosons, as we expect from a contact term. So the $\rho_{R/L}^2$ terms are less interesting than the $\rho_L \rho_R$ ones.

Canonical momenta and commutation relations in bosonization

We will find the CCRs for the free chiral components of the compact boson by using a bosonization approach. This result has already been derived in previous diary entries, but here we will use a different approach, viz. by using the known commutators of fermion density operators.

First we need to know where the R density goes (we are working in $\mathbb{R}$ time for this problem!)

$$\begin{aligned}\mathcal{B}[:n_R(x):] &= \mathcal{B}[\psi_R^\dagger(x+\epsilon)\psi_R(x) - i\epsilon^{-1}] \\ &= e^{i\phi_R(x+\epsilon)}e^{-i\phi_R(\epsilon)} - \frac{i}{\epsilon} \\ &= :e^{i\phi_R(x+\epsilon)}e^{-i\phi_R(\epsilon)}: \frac{i}{\epsilon} - \frac{i}{\epsilon} \\ &= -\partial_x\phi_R(x).\end{aligned} \tag{192}$$

The overall sign here at the end isn't really important and may be incorrect anyway. Like everything in bosonization, the prefactors and minus signs are an unmitigated mess. In the above, we have used the fact that the (equal-time) correlator for the R (L) fermions is $1/(-ix)$ ($1/ix$). In our conventions we will have $\mathcal{B}[\psi_L(x)] = e^{i\phi_L(x)}$ (this sign is different than in some past diary entries, and is chosen so that $\int \partial_x\theta$ is the thing measuring the fermion number), and thus $\mathcal{B}[:n_L(x):] = \partial_x\phi_L(x)$. Thus $\mathcal{B}[:n_R(q):] = q\phi_R(q)$, and the same for $n_L(q)$ but with a minus sign. In fact this bosonization relation for the fermion densities is probably a more natural place to start than the relation for the individual fermion operators themselves—the physical hydrodynamic fields θ, ϕ (or rather, their derivatives) are related to the densities of the fermions, and are $(-1)^F$ even.

Anyway, the L and R fermion densities commute, so the ϕ_L and ϕ_R fields must also commute.[26] We can determine the appropriate commutation relations for the $\phi_{L/R}$ using the commutation relations of the density operators. We have[27]

$$\rho_R(q) = \int_p \psi_p^\dagger\psi_{p-q} = \int_p \left(:\psi_p^\dagger\psi_{p-q}: + \langle\psi_p^\dagger\psi_{p-q}\rangle\right) = \int_p \left(:\psi_p^\dagger\psi_{p-q}: + \delta_{q,0}\theta(-p)\right). \tag{193}$$

[26] Thus the identification of the single fermions with the vertex operators is not strictly correct—we need the Klein factors to get the statistics straight. Some sources (e.g. Shankar) work with different wonky conventions where the mixed commutator of the $\phi_{L/R}$ is nonzero so that the Klein factors can be done away with, but this isn't conceptually ideal—thinking from a CFT perspective, it's always best to have the L and R sectors be completely decoupled.

[27] Recall that normal-ordering is done so that all the operators which annihilate the ground state are placed to the right. This is *not* the same as placing all annihilation operators to the right: for a system with a finite density of fermions, an annihilation operator with $k > k_F$ will kill the ground state, while a creation operator with $k < k_F$ will also kill the ground state. Thus normal ordering acts nontrivially on things like $\psi^\dagger(x)\psi(x)$, since each of the operators involved involves a sum of many different operators, some of which need to be moved to the right, and some of which do not.

In the last step we used that the right-movers are only occupied in the ground state if their momentum is negative (relative to k_F). Now (momentarily dropping R subscripts on ψs)

$$\begin{aligned}
[\rho_R(q), \rho_R(p)] &= \int_{l,k} \left(\delta_{l-q,k}\psi_l^\dagger\psi_{k-p} - \delta_{l,k-p}\psi_k^\dagger\psi_{l-q}\right) \\
&= \int_l \left(\psi_l^\dagger\psi_{l-q-p} - \psi_{l+p}^\dagger\psi_{l-q}\right) \\
&= \int_l \Big(:\psi_l^\dagger\psi_{l-q-p}: - :\psi_{l+p}^\dagger\psi_{l-q}: + \delta_{q,-p}\theta(-l) - \delta_{p,-q}\theta(-l-p)\Big) \\
&= \delta_{p,-q}\int_l (\theta(-l) - \theta(-l-p)) \\
&= \delta_{p,-q}p.
\end{aligned} \tag{194}$$

Here the units are right since ρ_q is dimensionless, and we take $\int_q \delta_{q,0} = 1$ so that the dimension of the δ and p cancel. Thus in $\mathbb{R}$ space we get

$$[\rho_R(x), \rho_R(y)] = \int_p p e^{ip(x-y)} = -i\partial_x\delta(x-y). \tag{195}$$

Bosonizing, this means that

$$[\partial_x\phi_R(x), \partial_y\phi_R(y)] = i\partial_x\delta(x-y) \implies [\phi_R(x), \partial_y\phi_R(y)] = i\delta(x-y). \tag{196}$$

Thus the canonical momentum conjugate to $\phi_R(x)$ is actually just its derivative, $\partial_x\phi_R(x)$. We can also write this as

$$[\phi_R(x), \phi_R(y)] = \frac{i}{2}\mathrm{sgn}(x-y). \tag{197}$$

This commutation relation is what allows the vertex operators $e^{i\phi_R(x)}$ to be fermionic.

When we repeat this procedure for the ψ_L's, the only thing that is different is the expectation value $\langle\psi_{L,p}^\dagger\psi_{L,p-q}\rangle = \delta_{q,0}\theta(p)$, which gives a minus sign so that the momentum conjugate to $\phi_L(x)$ is $-\partial_x\phi_L(x)$, and we have

$$[\phi_L(x), \phi_L(y)] = -\frac{i}{2}\mathrm{sgn}(x-y). \tag{198}$$

Now let's see if we recover the Lagrangian. We know from an earlier diary entry that the Hamiltonian is (ignoring constant prefactors)

$$H = \int dx\, \left[(\partial_x\phi_L)^2 + (\partial_x\phi_R)^2\right]. \tag{199}$$

Thus the action should be

$$S = \int dx\, dt\, \left(\partial_x\phi_R(\partial_t - \partial_x)\phi_R - \partial_x\phi_L(\partial_t + \partial_x)\phi_L\right). \tag{200}$$

This looks rather mysterious since it does not have any terms quadratic in time derivatives, even though we know that the action is $\partial\phi\bar{\partial}\phi$, which is quadratic in time derivatives. Furthermore in this presentation the ϕ_L and ϕ_R are decoupled, whereas in the

$\partial\phi\bar{\partial}\phi$ presentation they are not if we just substitute in $\phi = \phi_L + \phi_R$. Of course the way to make sense of this is to introduce $\phi = \phi_L + \phi_R, \theta = \phi_L - \phi_R$. Then after some algebra and integrating by parts, we get (again not writing constant prefactors)

$$S = \int dx\, dt\, \left(-(\partial_x\phi)^2 - (\partial_x\theta)^2 + 2\partial_x\phi\partial_t\theta\right). \tag{201}$$

Now the equation of motion for θ says that $\partial_x^2\theta = \partial_x\partial_t\phi$, so that $\partial_x\theta = \partial_t\phi$ (this is just the usual $d\theta = \star d\phi$ thing), and so the action goes to (again after integrating by parts)

$$S = \int dx\, dt\, \left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right], \tag{202}$$

which is finally what we expect from $\int \partial\phi\bar{\partial}\phi$ (in our signature $\partial = -\partial_x + \partial_t, \bar{\partial} = \partial_x + \partial_t$).

Finally, we note that since $\partial_x\phi \leftrightarrow n_R - n_L$ and $\partial_x\theta \leftrightarrow n_R + n_L$, we have that $\theta(x)$ counts the total fermion number (relative to $-\infty$), while $\phi(x)$ counts the net chirality (these statements are dependent on the sign conventions made for the bosonization mapping). Inserting $e^{i\theta}$ creates a vortex in ϕ, around which $\oint \partial\phi = 2\pi$. This means that inserting $e^{i\theta}$ at a given time changes the chiral charge, since $\oint \partial_x\phi$ takes on different values before and after the insertion. This jives with the fact that the vertex operator for θ bosonizes to

$$e^{i\theta(x)} \leftrightarrow \psi_L^\dagger(x)\psi_R(x), \tag{203}$$

which is indeed a scattering operator that indeed changes the net value of $n_R - n_L$. Likewise, a vertex operator for ϕ creates a vortex for θ, which means that it must change the fermion number $n_L + n_R$. This in turn jives with the fact that it bosonizes to $e^{i\phi} \leftrightarrow \psi_L^\dagger\psi_R^\dagger$. Relatedly, we can kind of motivate why in this formulation spatial translations map to shifts in θ by constants. The vertex operator $e^{i\theta}$ shifts a right-moving fermion to a left-moving one, and thus shifts the total momentum. Since it shifts the momentum, it should not commute with the momentum $\int dx\, T_{01}$, which is $\sim \int dx\, \partial_x\phi$. In terms of the holomorphic and antiholomorphic fields, we then use the commutation relations of the $\phi_{L/R}$ to conclude tht spatial translations do $\phi_L \mapsto \phi_L + c, \phi_R \mapsto \phi_R - c$. Again, this is expected from the fermion side, by using the usual decomposition $\psi(x) = e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x)$.

Chiral almost-symmetry-breaking in two dimensions via bosonization

Today we're looking at an awesome paper by Witten [?]. The goal is to explore the exact relation between QLRO and masslessness in two dimensions. We will make an attempt to work with simplified normalization conventions so as to avoid a lot of the tedious numerical factors present in Witten's paper.

The model is

$$S = \frac{1}{8\pi}\int d\sigma\wedge\star d\sigma + \frac{1}{2\pi}\int i\bar\psi\partial\!\!\!/\psi + \frac{g}{2}\int\bar\psi(\cos\sigma + i\bar\gamma\sin\sigma)\psi, \tag{204}$$

where $\bar\gamma = Z$ is the chirality operator. Note that we can write the last interaction term as $\bar\psi e^{i\bar\gamma\sigma}\psi$, and that this interaction is marginal since the radius of the σ boson is $R_\sigma = 1/\sqrt{2}$, giving the dimension of $e^{i\bar\gamma\sigma}$ to be $1/(2R_\sigma)^2 = 1$.

The chiral symmetry of interest is the $U(1)$ symmetry

$$U(1)_A : \psi\mapsto e^{i\alpha\bar\gamma/2}\psi, \qquad \sigma\mapsto\sigma+\alpha, \tag{205}$$

which is a symmetry of the action since $\bar\psi\mapsto\bar\psi e^{+i\alpha\bar\gamma}$. A naive guess would be that for large coupling, we get spontaneous symmetry breaking of $U(1)_A$ by virtue of σ acquiring a vev, thereby leading to a fermion mass term. If this happened, the spectrum would contain a Goldstone boson and massive fermions. But of course we are in two dimensions, so this SSB scenario is impossible.

Instead, we will show that the spectrum of the model is a massless boson and a massive fermion as expected from SSB, but that the symmetry is unbroken. We will show the latter by demonstrating that all the symmetry-breaking Greens functions vanish.

We first bosonize $\psi = (\psi_+,\psi_-)^T$, with the conventions

$$\mathcal{B}[\psi_\pm] = e^{\pm i\phi_\pm}. \tag{206}$$

Then the interaction between the fermions and σ is dealt with by using

$$\mathcal{B}[\bar\psi\psi] = -\mathcal{B}[\psi_-\psi_+^\dagger + \psi_+\psi_-^\dagger] = -(e^{-i\phi}+e^{i\phi}) = -2\cos\phi \tag{207}$$

where the minus sign arises due to today's bosonization conventions. Likewise,

$$\mathcal{B}[\bar\psi\bar\gamma\psi] = \mathcal{B}[\psi_-\psi_+^\dagger - \psi_+\psi_-^\dagger] = -2i\sin\phi. \tag{208}$$

Thus the action becomes, using $\cos\sigma\cos\phi - \sin\sigma\sin\phi = \cos(\phi+\sigma)$,

$$S = \frac{1}{8\pi}\int(d\sigma\wedge\star d\sigma + d\phi\wedge\star d\phi) - g\int\cos(\phi+\sigma). \tag{209}$$

Note that the interacting part only involves the combination $\phi+\sigma$[28] (this scenario in which one of the two fields decouples and becomes free is exactly what happens in the Schwinger model, where the spectrum contains a decoupled free boson and an interacting massive one). This means that we should define new variables

$$\lambda\equiv\phi+\sigma, \qquad \gamma\equiv\phi-\sigma. \tag{210}$$

[28]This is again of course still relevant, since it has dimension $2/(2R_\sigma^2) = 2$.

The action is then written suggestively as

$$S = \frac{1}{8\pi}\int\left(\frac{1}{2}d\gamma\wedge\star d\gamma + \frac{1}{2}d\lambda\wedge\star d\lambda\right) - g\int\cos(\lambda). \tag{211}$$

Thus we have produced a theory consisting of a free boson and an interacting boson. Since the interaction is marginal, we expect that the spectrum should contain a free γ boson, and a massive fermion (the kink / antikink soliton for λ). We expect the fermion to be massive here because the marginal cosine will be pushed towards marginal relevance by loop corrections. This happens because the effective action obtained from the momentum-shell scheme will contain a term like $-\frac{g^2}{2}\int d^2x\, d^2y\, \cos(\lambda(x)-\lambda(y))$, which when expanded contains a term proportional to $+g^2(\partial\lambda)^2$. Therefore the $\cos(\lambda)$ leads to a renormalization which increases the radius of λ, and since the dimension of $\cos(\lambda)$ goes as $1/R^2$, an increased radius decreases the dimension of the cosine, making it marginally relevant.[29]

To write the theory in terms of the physical variables then, we will fermionize λ with a fermion η such that

$$\eta_\pm \leftrightarrow e^{\pm i\lambda_\pm}, \tag{213}$$

which will turn the cosine into a fermion mass term. It might feel a bit silly to be going back to fermion variables, but we needed this intermediate purely bosonic step in order to decouple the fields into the massive and massless parts. Since the coefficient of the λ kinetic term is not $1/8\pi$ (the radius of λ is $1/2$ rather than $1/\sqrt{2}$), we will not get simply the free Dirac kinetic term — the extra $-1/16\pi$ in the kinetic term for λ will lead to a current-current interaction in the new action. If we look back at our diary entry on general bosonization conventions, we see that the relation between the boson radius and the Thirring interaction strength U looks like $R^2 = \frac{1}{2}\sqrt{(1-U)/(1+U)}$, hence in our case we get an interaction at $U = 3/5$. Thus in these variables the action becomes

$$S = \frac{1}{2\pi}\int\left(\bar{\eta}i\partial\!\!\!/\eta + 2\pi U j_\mu j^\mu\right) + \frac{g}{2}\int\bar{\eta}\eta + \frac{1}{16\pi}\int d\gamma\wedge\star d\gamma. \tag{214}$$

So we have finally arrived at what we wanted: a theory with a free decoupled boson and a massive (but interacting; the solitons are not free) fermion. However, there is *no* SSB (as there must not be, in accordance with the CMW theorem), even though the fermion in the spectrum is massive. We can check this by computing the chirality-nonconserving Greens functions that would be nonzero in the case of SSB, like $\langle\psi_\pm\psi^\dagger_\mp\rangle$. To do this we need

$$\psi_\pm \leftrightarrow e^{\pm i\phi_\pm} = e^{\pm i\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)_\pm}. \tag{215}$$

[29] In the XY model with an added vortex term the situation is opposite, with the cosine tipped towards irrelevance. That is, consider (still in $i\mathbb{R}$ time)

$$S = \frac{1}{2\pi}\int\left(\partial_t\phi\partial_x\theta + (\partial_x\phi)^2 + (\partial_x\theta)^2 + g\cos\theta\right). \tag{212}$$

Here the radius of ϕ is 2, so that the cosine is marginal. By the same argument, the RG at one-loop generates a term in the action which schematically $+g^2(\partial\theta)^2$, thus decreasing the "standard deviation" of the θ field. When θ is integrated out this then *decreases* the radius of ϕ, which in turn makes $\cos(\theta)$ less relevant, since the dimension of $\cos(\theta)$ goes as R^2. Therefore cosines of the dual field become marginally irrelevant while those of the primary field become marginally relevant.

This means that e.g.

$$\langle \psi_\pm(x)\psi_\mp^\dagger(0)\rangle \leftrightarrow \langle e^{\pm i(\lambda/2)_\pm(x)} e^{\pm i(\lambda/2)_\mp(0)}\rangle\langle e^{\pm i(\gamma/2)_\pm(x)} e^{\pm i(\gamma/2)_\mp(0)}\rangle. \tag{216}$$

In order to find out what this is, we need to know the correlation functions of the chiral vertex operators for the rescaled field $\lambda/2$ (note to self: should come back and just write things in terms of the fields and their duals rather than the chiral components, which is much more confusing). This is slightly nontrivial since multiplicative rescaling of the fields does not preserve the holomorphic / antiholomorphic decomposition of the fields, i.e. we have

$$(a\phi')_\pm \neq a\cdot\phi_\pm, \quad a\in\mathbb{C}, \tag{217}$$

essentially because $\phi_\pm$ do not commute with eachother and since momenta and position get scaled oppositely in order to preserve the CCR. A little digression on this since it's interesting: the holomorphic and antiholomorphic parts of the field are defined by (this is a non-local definition as it must be)

$$\phi_\pm(x) = \frac{1}{2}\left(\phi(x) \pm \int_{-\infty}^x dx'\,\Pi(x')\right), \tag{218}$$

where $\Pi = \partial_x\theta$ is the momentum (which leads to the standard $\theta = \phi_+ - \phi_-$). The nonlocal nature of this definition is needed to ensure that the chiral vertex operators anticommute with one another and thus have a chance to become fermions[30] (passing one $e^{i\phi_\pm}$ around another encircles the latter in a $\Pi(x')$ string, which after being wiggled straight must pass over the former vertex operator, which gives the interaction needed for fermionic statistics). Rescaling this, we get the non-homogenous transformation

$$(a\phi)_\pm(x) = \frac{1}{2}\left(a\phi(x) \pm a^{-1}\int_{-\infty}^x dx'\,\Pi(x')\right), \qquad a\in\mathbb{C}. \tag{219}$$

Solving for $\phi(x)$ and the momentum integral in terms of the original (unscaled) field, we see that under rescaling the chiral components of the fields get mixed by a "boost"

$$\begin{pmatrix}(a\phi)_+\\(a\phi)_-\end{pmatrix} = \begin{pmatrix}\cosh(\ln a) & \sinh(\ln a)\\ \sinh(\ln a) & \cosh(\ln a)\end{pmatrix}\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}. \tag{220}$$

This Bogoliubov transformation is consistent with the map $\phi\mapsto a\phi, \theta\mapsto a^{-1}\theta$, required in order to preserve the CCR (note that in accordance with yesterday's diary entry, the CCR here are $[\phi_\pm(x), \partial_x\phi_\pm(y)] \propto i\delta(x-y)$: if we just imposed the CCR on θ,ϕ then we could also get away with having $\phi_\pm, \partial_x\phi_\pm$ commute but $\phi_\pm, \partial_x\phi_\mp$ not commute).

Now one may wonder whether it is $\widetilde{\phi}_\pm \equiv (a\phi)_\pm$ which is left / right moving, or whether it is $\phi_\pm$. Thinking about this is perhaps easier in the Hamiltonian formulation.

[30]Here by "become fermions" we mean in the weak sense, where the vertex operators behave like fermions when inserted into correlation functions. They aren't *really* fermions in the constructive sense since they do not change the fermion number. To have the vertex operators be fermions in the strong sense of them being equal to fermion fields as operators, we need Klein factors. The Klein factors basically just change the parity of the fermion vacuum by subtracting a fermion from the Dirac sea and then re-arranging all the existing fermions to create the new vacuum state. In any case, for now we will only need bosonization in the weak sense of matching correlation functions, so these subulties won't come up.

Here since $:\psi_L^\dagger \partial_x \psi_L:$ gets mapped to $(\partial_x \phi_+)^2$ while the same for ψ_R gets mapped to $(\partial_x \phi_-)^2$, the free fermion Hamiltonian is bosonized to

$$H \propto \int dx\, \left[(\partial_x \phi_+)^2 + (\partial \phi_-)^2\right] \tag{221}$$

Now the re-scaling of the ϕ field is accomplished by adding in interactions for the fermions of the form $j_\mu j^\mu$. In terms of fermions this is proportional to $\rho_L \rho_R$, which bosonizes to $(\partial_x \phi_+)(\partial_x \phi_-)$. Thus adding this interaction term to the Hamiltonian means that the bosonized Hamiltonian now has a cross term between $\partial_x \phi_+$ and $\partial_x \phi_-$. We can re-write the Hamiltonian without the cross term, but this requires making a field re-definition. Since only the Hamiltonian with the new re-defined fields separates into a sum of left- and right-moving parts, only the new re-defined fields are holomorphic / anti-holomorphic. This makes sense since adding the $j_\mu j^\mu$ interaction mixes the left- and right-moving fermions, and so the bosonized images of the original fermions should not be purely left- or right-moving.

Anyway, applying this to the problem at hand, we find

$$\langle \psi_+(x) \psi_-^\dagger(0) \rangle \sim \langle e^{i(5\lambda_+(x)/4 - 3\lambda_-(x)/4)} e^{i(5\lambda_-/4(0) - 3\lambda_+(0)/4)} \rangle \langle e^{i(5\gamma_+(x)/4 - 3\gamma_-(x)/4)} e^{i(5\gamma_-/4(0) - 3\gamma_+(0)/4)} \rangle. \tag{222}$$

Now we don't need to know what the correlator of the λ vertex operators are (this is tricky since λ is not free), but we do know what the γ vertex operator correlator is, since γ is free. In a previous diary entry we saw that in order for the correlator to be nonzero, we had to have "charge neutrality" for both of chiral fields $\gamma_\pm$ in the vertex operators if the correlator was to be non-zero, otherwise the correlator vanishes because of infrared effects. Since in this case the correlators are not charge-neutral $(5/4 - 3/4 \neq 0)$, we get

$$\langle \psi_+(x) \psi_-^\dagger(0) \rangle = 0. \tag{223}$$

By checking the other relevant Greens functions, we see that indeed, the symmetry is unbroken. That the cause of the correlators being zero is coming from infrared effects (the condition of charge neturality) is comforting, since we know infrared effects are behind the CMW theorem which is what prevents the symmetry breaking from happening.

More on the massless Schwinger model and its spectrum

Applications of bosonization in 1+1D is becoming a bit of a tired topic in this diary, but today we have something slightly new: an alternate way of deriving confinement in the (massless) Schwinger model. We will be somewhat fast and loose with factors of i and signs relating to squares of $\star$ (which squares to -1 on 1-forms in 1+1D in Euclidean time).

In 1+1D, we can fix $d^\dagger A = 0$ gauge and write $A = \star d\phi$ for ϕ a scalar. The kinetic term for the gauge field is then (here $\Box = -\partial_\mu \partial^\mu$ is positive-definite)

$$\int F_A \wedge \star F_A = \int d \star d\phi \wedge d^\dagger d\phi = \int d\phi \wedge \star \Box d\phi = \int \phi \Box^2 \phi. \tag{224}$$

The fermions then couple to the gauge field via the term

$$S \ni \frac{1}{2\pi} \int i\bar{\psi}\gamma^\mu \epsilon^{\mu\nu} \partial_\nu \phi \psi. \tag{225}$$

Now consider performing a chiral rotation by $\psi \mapsto e^{-i\bar{\gamma}\phi}$, where $\bar{\gamma} = -i\gamma_1\gamma_2 = -iXY = Z$. Then the kinetic term changes by

$$\delta(\bar{\psi}\partial\!\!\!/\psi) = -i\bar{\psi}\gamma^\mu \bar{\gamma}\partial_\mu \phi \psi = i\bar{\psi}(\epsilon^{\mu\nu}\partial_\nu \phi)\gamma^\mu \psi, \tag{226}$$

which is just what is needed to kill the coupling of ψ to ϕ. Of course when we do this the chiral anomaly comes into play, and gives us a term $\frac{1}{2\pi}\int (\phi d \star d\phi) = -\frac{1}{2\pi}\int \phi \Box \phi$. Lastly, we bosonize the fermions with a compact field θ, sticking to Witten's conventions in QFTs and strings. Putting all the pieces together, the action is

$$S = \frac{1}{2}\int \left(\frac{1}{4\pi}|d\theta|^2 + \frac{1}{\pi}|d\phi|^2 + \frac{1}{e^2}\phi \Box^2 \phi \right). \tag{227}$$

The ϕ propagator is evidently

$$G^\phi(p) = \frac{1}{p^2/\pi + p^4/e^2} = \frac{\pi}{p^2} - \frac{\pi}{p^2 + e^2/\pi}. \tag{228}$$

Therefore we have two modes: one massive with mass $m^2 \equiv e^2/\pi$, and one massless. The gauge field propagator is then

$$G^A_{\mu\nu}(p) = \epsilon^{\mu\alpha}\epsilon^{\nu\beta} p_\alpha p_\beta \langle \phi(p)\phi(-p)\rangle = \frac{\varepsilon^{\mu\alpha}\varepsilon^{\nu\beta} p_\alpha p_\beta}{p^2} \frac{\pi m^2}{p^2 + m^2} = (\delta^{\mu\nu} - p^\mu p^\nu/p^2)\frac{\pi m^2}{p^2 + m^2}, \tag{229}$$

and so the gauge field is rendered massive by its coupling to the fermions.

It turns out that the massless θ particle and the massless mode of ϕ cancel each other out, leaving behind only a massive mode. To see this, we first take $\theta \mapsto \theta + 2\phi$, which eliminates the $|d\phi|^2$ term and gives us a $\theta \Box \phi/\pi$ term. Then we take $\phi \mapsto \phi - m^2 \Box^{-1}\theta$, which kills the mutual ϕ, θ coupling. This gives an effective action for θ which is

$$S = \frac{1}{8\pi}\int \theta(\Box + m^2)\theta. \tag{230}$$

This tells us that the theory consists of a single massive pseudo-scalar boson (it's "pseudo-scalar" since $\psi^\dagger_\pm \psi_\mp \to e^{\pm i\theta}$ means $P : \theta \mapsto -\theta$, at least in the simple case with a Pin^+ structure, where P acts as $X = \gamma^0$).

One important thing to realize is that the expectation value of the chiral fermion bilinear is nonzero:

$$\langle \psi^\dagger_L \psi_R \rangle = \frac{1}{a} e^{-\langle \theta^2 \rangle/2} = e^{-G^\theta(x=0;m)/2} \approx \frac{1}{a} e^{\ln(ma)} = m, \tag{231}$$

where a is a short-distance cutoff.[31] Many people say that the fact that this is non-zero indicates that we have chiral symmetry breaking. This would not contradict the CMW theorem, since in this case we have long-ranged interactions, provided by the gauge field (the absence of a Goldstone is also okay—the massless part of the gauge field was eaten by the term that we had to add to the action in accordance with the chiral anomaly). But this line of reasoning is not really correct, since the anomaly means that *we never actually had chiral symmetry in the first place, so there is nothing to break.*

Now let us look at correlators of the bilinears $\sigma \equiv \bar\psi\psi$.[32] First, as a check of asymptotic freedom, we can compute the 2-point functions $\langle \sigma_s(x) \sigma_{s'}(0) \rangle$ at $x \to 0$, where $\sigma_\pm \equiv \psi^\dagger_\pm \psi_\mp$ and where s, s' are signs. If $s = s'$, then we find

$$\langle \sigma_\pm(x) \sigma_\pm(0) \rangle_{x\to 0} = \langle \sigma^2_\pm(0) \rangle e^{G^\theta(x\to 0)} \approx m^2 e^{-\ln(x^2/m^2)} = \frac{1}{x^2}. \tag{233}$$

On the other hand if $s = -s'$ then the sign in the exponent switches, and we get

$$\langle \sigma_\pm(x) \sigma_\mp(0) \rangle \approx x^2 m^4 \to 0. \tag{234}$$

Note that these results are exactly in accordance with what we'd get from free field theory; hence the model is asymptotically free. More complicated correlators are those involving the scalar σ. Using the usual manipulations for expectation values of exponentials, we get

$$\langle \sigma(x) \sigma(0) \rangle = \langle \sigma^2(0) \rangle 4 \cosh(G^\theta(x;m)). \tag{235}$$

Now besides the scalar σ, we also have the pseudo-scalar $\widetilde\sigma_\pm \equiv \bar\psi \bar\gamma \psi = i(\psi^\dagger_+ \psi_- - \psi^\dagger_- \psi_+)$. The minus sign turns the cosh into a sinh:

$$\langle \widetilde\sigma(x) \widetilde\sigma(0) \rangle = \langle \widetilde\sigma^2(0) \rangle 4 \sinh(G^\theta(x;m)). \tag{236}$$

[31] With the current conventions,

$$G^\theta(0;m) = 4\pi \int_0^{a^{-1}} \frac{dp}{2\pi} \frac{p}{p^2+m^2} = \ln(a^{-2}/m^2+1) \approx -2\ln(am). \tag{232}$$

[32] Chirally-invariant correlators of two fermions, i.e. $\langle \psi^\dagger_\pm(x) \psi_\pm(0) \rangle$ are hard since the mass term for θ screws up a holomorphic / anti-holomorphic decomposition for θ, and means that the only correlators that are easy to compute are those of $\sigma_\pm = \psi^\dagger_\pm \psi_\mp$ ($\sigma = \sigma_+ + \sigma_-$), since $\sigma_\pm$ bosonizes to $e^{\pm i\theta}$.

To evaluate these expressions, we expand the hyperbolic functions in powers of G^θ and then go to momentum space. Therefore we need to evaluate integrals like

$$\int \prod_i^k \left(\frac{d^2 p_i}{4\pi^2} \frac{1}{p_i^2 + m^2} \right) \frac{1}{\left(q - \sum_i p_i\right)^2 + m^2} \tag{237}$$

for some fixed q^2. Terms with k even will appear in the expansion of the σ correlator, and terms with k odd will appear in the expansion of the $\widetilde{\sigma}$ correlator.

First, note that only the expansion of the $\sinh(G^\theta)$ will give an isolated pole in momentum space (only for $k = 0$ in the above equation, which is the first term in the expansion of sinh, will we get a simple pole at $q^2 = -m^2$; all other singularities are part of branch cuts, and are not isolated). This confirms the result that the boson in the spectrum is a pseudo-scalar, since there is a simple pole only in the correlation function of the pseudo-scalar $\widetilde{\sigma}$ field.

In general, I think it is true that the successive terms in the $\langle \sigma(x)\sigma(0)\rangle$ correlation function contribute branch cut singularities at $q^2 = -(2n)^2 m^2, n \in \mathbb{Z}$, while the successive terms in the $\langle \widetilde{\sigma}(x)\sigma(0)\rangle$ correlation function contribute branch cut singularities at $q^2 = -(2n+1)m^2, n \in \mathbb{Z}$. The first singular contribution to $\langle \sigma(x)\sigma(0)\rangle$ is determined by the integral (ignoring 2πs)

$$\begin{aligned}
I &= \int_p \frac{1}{(p^2+m^2)((p-q)^2+m^2)} \\
&= \int_x \int_p \frac{1}{[x((p-q)^2+m^2)+(1-x)(p^2+m^2)]^2} \\
&= \int_x \int_p \frac{1}{(p^2+m^2+q^2(x-x^2))^2} \\
&= \frac{1}{2}\int_x \frac{1}{m^2+q^2(x-x^2)} \\
&= \frac{1}{q\beta} \ln\left(1+\frac{q}{2m^2}(q+\beta)\right), \qquad \beta \equiv \sqrt{4m^2+q^2}.
\end{aligned} \tag{238}$$

This is singular precisely when $q = 2im$, indicating the contribution of particle production involving a particle with mass m to the Greens function. In fact we can already see the singularity before we do the x integral: taking $q^2 = -\lambda m^2$, the denominator vanishes when

$$x = \frac{1}{2} \pm \sqrt{\lambda^2 - 4\lambda}, \tag{239}$$

which tells us that we have singularities as soon as $\lambda > 4$: this gives us a branch point starting at $q = 2im$, as found above.

Now we turn to the leading term in the expansion for the $\widetilde{\sigma}$ correlator. With two integrals, things are much more heinous:

$$\begin{aligned}
I &= \int_{p,k} \frac{1}{(p^2+m^2)(k^2+m^2)((q-p-k)^2+m^2)} \\
&= 2\int_{x,y}\int_{p,k} \frac{1}{(x(p^2+m^2)+y(k^2+m^2)+z((q-p-k)^2+m^2))^3} \qquad z \equiv 1-x-y \\
&= 2\int_{x,y}\int_{p,k} \frac{1}{(m^2+p^2+k^2+z[-2q\cdot(p+k)+q^2+2p\cdot k])^3}.
\end{aligned} \tag{240}$$

Now we need to eliminate the dot product between q and $p+k$. Consider shifting $\delta p = \delta k = \alpha q$. Then the terms involving q in the denominator become

$$2q\cdot(p+k)[\alpha - z + \alpha z] + q^2(z + 2\alpha^2 z + 2\alpha^2) \implies \alpha = \frac{z}{1+z}. \tag{241}$$

Then we shift $\delta p = -zk$, ending up with

$$\begin{aligned}
I &= 2\int_{x,y}\int_{p,k} \frac{1}{(m^2+p^2+k^2(1-z^2)+\gamma q^2)^3} \qquad \gamma \equiv (1+3z^2)/(1+z) \\
&= \int_{x,y} \frac{1}{2(1-z^2)(m^2+\gamma q^2)}
\end{aligned} \tag{242}$$

The integral can't be done analytically, but if we look at when the denominator vanishes, we can check that it does so at $q^2 = -9m^2$, which is exactly what we'd expect for a contribution coming from a 3-particle intermediate state. (Note to self: come back and work this out more carefully sometime)

More WZW things

The goal of today's problem is to try to get more familiar with the WZW term, e.g. to see how it relates to various anomalies, to become acquainted with its symmetry properties, etc.

We will be working with WZW for $SU(2)_k$[33] :

$$S = \frac{k}{8\pi}\int_M \mathrm{Tr}(dg^\dagger \wedge \star dg) + \frac{ik}{12\pi}\int_B \mathrm{Tr}[\omega\wedge\omega\wedge\omega], \qquad \omega = g^\dagger dg. \tag{243}$$

As usual, M is some 2-manifold and B is a 3-ball bounded by M. First, we will show that the WZW part of the action transforms projectively under the global symmetry

[33] Note that in the big yellow book, the coefficients in front of the integrals are different. This is because they use a trace with a different normalization (the trace we are using is not normalized by the Dynkin index; it is the straight-up trace).

$g \mapsto gh$. We will then similarly define holomorphic and antiholomorphic currents, and show how they transform under $SU(2)_L \times SU(2)_R$.

We will then consider the conformal Ward identity, and find the OPE of the currents with themselves. After splitting up the currents with a mode expansion, we will then get the commutator algebra of the modes in the usual way (in radial quantization, using contour integrals).

We can see why the WZW term comes from cohomological thinking by seeing where it goes under a global symmetry transformation of $SU(2)_R$, with $g \mapsto gh$. Under this action, the MC form for g gets conjugated and spits out a MC form for h:

$$\omega \mapsto h^\dagger \omega h + \eta, \qquad \eta \equiv h^\dagger dh. \tag{244}$$

Putting this into S and expanding,

$$S_{WZW}[gh] = S_{WZW}[g] + S_{WZW}[h] + \frac{ik}{4\pi}\int \mathrm{Tr}[\eta \wedge \eta \wedge^h \omega + \eta \wedge^h \omega \wedge^h \omega], \qquad {}^h\omega \equiv h^\dagger \omega h. \tag{245}$$

Now $\eta \wedge \eta = -d\eta$, while

$$d^h\omega = -\eta^h\omega + h^\dagger d\omega h -^h \omega \wedge \eta = -\eta \wedge^h \omega -^h \omega \wedge \eta -^h \omega \wedge^h \omega. \tag{246}$$

Solving for ${}^h\omega \wedge^h \omega$ and putting this into our expression for $S_{WZW}[gh]$, we get

$$S_{WZW}[gh] = S_{WZW}[g] + S_{WZW}[h] + \frac{ik}{4\pi}\int d\mathrm{Tr}[\eta \wedge^h \omega]. \tag{247}$$

So, the symmetry acts linearly on the WZW action up to a boundary term. This is emblematic of SPTs / anomalies: when we consider the symmetry action on some open manifold (or submanifold), we get a linear representation of the symmetry up to a term supported on the boundary of that manifold (see earlier diary entries).

When we consider the kinetic S_k term as well, a similar computation shows

$$S_k[gh] = S_k[g] + S_k[h] - \frac{k}{4\pi}\int \mathrm{Tr}(\eta \wedge \star^h\omega). \tag{248}$$

When we add these two terms, the form that couples to η in the coboundary δS of the action, namely

$$\delta S(g,h) \equiv S[g] + S[h] - S[gh], \tag{249}$$

is exactly the same type of form defined in our RG analysis of the WZW model back in a previous diary entry.

To get something useful out of this, we need to switch to using ∂_z and $\partial_{\bar z}$. When we do this, $\delta S_k(g,h)$ becomes, after a bit of algebra

$$\delta S_k(g,h) = \frac{k}{2\pi}\int d^2x\, \mathrm{Tr}[\eta_z \omega_{\bar z} + \eta_{\bar z}\omega_z]. \tag{250}$$

On the other hand, the WZW part is

$$\delta S_{WZW}(g,h) = \frac{k}{2\pi}\int d^2x\,\mathrm{Tr}[\eta_z\omega_{\bar z} - \eta_{\bar z}\omega_z]. \tag{251}$$

Note that $\delta S_{WZW}(g,h)$ is real since the i in the original prefactor cancels with the i generated by going to ∂_z and $\partial_{\bar z}$ since $\delta S_{WZW}(g,h)$ only contains terms with one derivative in x and one in y (where $z = x+iy$). Putting these together, the total coboundary of the action is (switching from an integral over $d^2x = dx\wedge dy$ to one over $dz\wedge d\bar z$ at the cost of a factor of $i/2$),

$$\delta S(g,h) = \frac{ik}{2\pi}\int dz\wedge d\bar z\,\mathrm{Tr}[h^\dagger(\partial_z h)g^\dagger\partial_{\bar z}g]. \tag{252}$$

Note in particular that if h is holomorphic, then $\delta S(g,h)=0$.

This was done for the action of $SU(2)_R$. If we instead consider the $SU(2)_L$ action $g\mapsto fg$, then we instead have

$$\omega\mapsto g^\dagger\lambda g+\omega, \qquad \lambda\equiv f^\dagger df. \tag{253}$$

Thus for the action of $SU(2)_L$, it is the Cartan form for the element doing the symmetry action (namely f) that gets conjugated, instead of ω. This ends up meaning that we end up getting the same thing for $\delta S_{WZW}(f,g)$, except that η and ${}^h\omega$ change places. Since they are both 1-forms, this gives us a minus sign, and so essentially the only thing that changes is that $\delta S_{WZW}(f,g)$ term gets a minus sign relative to $\delta S_k(f,g)$, which doesn't get a minus sign since it doesn't have wedge products. Thus we find that for the $SU(2)_L$ action,

$$\delta S(f,g) = \frac{k}{\pi}\int d^2x\,\mathrm{Tr}[f^\dagger(\partial_{\bar z}f)g^\dagger\partial_z g]. \tag{254}$$

Note that this vanishes if f is anti-holomorphic.

So, we reach the following conclusion: the $SU(2)_L$ symmetry is implemented anomalously (i.e. the action is only invariant up to a boundary term), unless the action is done by a holomorphic function $f(z)$. Likewise, the $SU(2)_R$ symmetry is implemented anomalously unless the action is done by an anti-holomorphic function $h(\bar z)$. This tells us that conservation of the holomorphic current (the expression for which will be recalled shortly) comes from the $SU(2)_L$ symmetry, while conservation of the antiholomorphic current comes from the $SU(2)_R$ symmetry.

Now we turn to look at the currents. We will refer to a previous diary entry where we calculated these and found δS. We found that the holomorphic (left-moving, since it doesn't depend on $\bar z$) and the anti-holomorphic (right-moving, since it's independent of z) currents are[34]

$$J = -k(\partial_z g)g^\dagger, \qquad \bar J = kg^\dagger\partial_{\bar z}g. \tag{255}$$

[34] We're actually changing conventions slightly for the currents compared to the last diary entry—there we were following Altland and Simon's conventions, and here we follow more standard ones (which are conjugated by g and since we are no longer at $k=1$ have a k in front)

Under infinitesimal $SU(2)_L$ transformations $g \mapsto g + \gamma g$, and infinitesimal $SU(2)_R$ transformations $g \mapsto g - g\bar{\gamma}$ (we will see why the minus sign is natural in a second), a quick calculation shows that the holomorphic current varies as

$$SU(2)_L: \;\; \delta J = [J, \gamma] - k\partial_z \gamma, \qquad SU(2)_R: \;\; \delta J = kg\partial_z \bar{\gamma} g^\dagger, \tag{256}$$

so that when the variation is holomorphic J is invariant under $SU(2)_R$. Similarly, the anti-holomorphic current transforms as

$$SU(2)_R: \;\; \delta \bar{J} = -[\bar{J}, \bar{\gamma}] - k\partial_{\bar{z}} \bar{\gamma}, \qquad SU(2)_L: \;\; \delta \bar{J} = kg^\dagger (\partial_{\bar{z}} \gamma) g. \tag{257}$$

As anticipated in our calculation of the coboundary δS, the holomorphic current is identified with the $SU(2)_L$ symmetry and the antiholomorphic one with the $SU(2)_R$ symmetry. The conservation of these two currents (which comes from the equations of motion as we saw in a previous diary entry) implies that we have the symmetry

$$g(z, \bar{z}) \mapsto \Gamma(z) g(z, \bar{z}) \bar{\Gamma}^{-1}(\bar{z}) \tag{258}$$

for any homolorphic (antiholomorphic) Γ ($\bar{\Gamma}$). We've chosen the right action to involve an inverse since it makes various formulae nicer later on and is the more natural choice. The infinitesimal version of this is, writing $\Gamma(z) = \mathbf{1} + \gamma(z)$,

$$g \mapsto g + \gamma g - g\bar{\gamma}. \tag{259}$$

Consider now the ward identity for some operator $\mathcal{O}$. To compute $\langle \delta \mathcal{O} \rangle$, we need to know the variation of the action. Fortunately this was also worked out in a previous diary entry, whose results we will steal. After going through the annoying step of switching to $\partial_z, \partial_{\bar{z}}$ we find for the variation (259)

$$\delta S = -\frac{k}{2\pi} \int d^2x \, \mathrm{Tr}[\gamma \partial_{\bar{z}} J + \bar{\gamma} \partial_z \bar{J}]. \tag{260}$$

We can pull the derivatives out of the trace for free, since γ and $\bar{\gamma}$ are killed by $\partial_{\bar{z}}$ and ∂_z, respectively. Then since $dx \wedge dy = (i/2) dz \wedge d\bar{z}$, we can go over to an integration over $dz \wedge d\bar{z}$ and then integrate the total derivatives to get

$$\delta S = -\frac{i}{4\pi} \oint dz \, \mathrm{Tr}[\gamma J] + \frac{i}{4\pi} \oint d\bar{z} \, \mathrm{Tr}[\bar{\gamma} \bar{J}], \tag{261}$$

where the relative minus sign comes from the fact that we're taking both of the $\oint$'s to be oriented right-handedly.

Following the usual procedure of performing a shift in integration variables in the path integral which computes $\langle \mathcal{O} \rangle$, we have, for $\gamma, \bar{\gamma}$ chosen to have compact support on some ball centered on w (in radial quantization),

$$\langle \delta \mathcal{O}(w) \rangle = -\frac{i}{4\pi} \oint dz \, \langle \mathrm{Tr}[\gamma J] \mathcal{O}(w) \rangle + \frac{i}{4\pi} \oint d\bar{z} \, \langle \mathrm{Tr}[\bar{\gamma} \bar{J}] \mathcal{O}(w) \rangle, \tag{262}$$

where the contours are taken on paths enclosing the point w.

We can get the OPEs for the currents by choosing $\mathcal{O} = J^a$, where $J = J^a t^a$ with t^a the Pauli matrices. Since $\bar{\gamma}$ is antiholomorphic J is invariant under $SU(2)_R$, and so we see from our earlier result that

$$\delta J^a = i f^{abc} J^b \gamma^c - k\partial_z \gamma^a, \tag{263}$$

where for us $f^{abc} = \epsilon^{abc}$. Note that when we put this in, the LHS of (262) will only contain γ, and so the OPE between J and $\bar{J}$ must only contain non-singular terms (since the $\bar{\gamma}$ term on the RHS needs to die). Thus (not writing the expectation value brackets and taking the trace)

$$i f^{abc} J^b(w)\gamma^c(w) - k\partial_w \gamma^a(w) = -\frac{i}{2\pi}\oint dz\, \gamma^b(z) J^b(z) J^a(w). \tag{264}$$

From this, we can read off the OPE for the holomorphic currents. the f^{abc} term on the LHS has no derivatives, so we need to pick it up with a $1/(z-w)$ pole. Since the $k\partial_w\gamma^a(w)$ term has one derivative, we need to match it with a $1/(z-w)^2 = -\partial_w(z-w)^{-1}$ term. So then taking into account the $2\pi i$'s from the residues, we deduce that the singular parts of the OPE are

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + i f^{abc}\frac{J^c(z)}{z-w}. \tag{265}$$

This is the current algebra we've been looking for.

Now we can define the modes of the current by their "angular momentum", i.e. what we get when we integrate the current against z^n. Since $\partial_{\bar{z}} J^a = 0$ we can expand the current as

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J^a_n. \tag{266}$$

Here we picture J^a_n as an operator acting at the origin in radial quantization that sets up an associated state. The choice of power ensures that J^a_n can be found by integrating $J^a(z)$ against z^n. The commutator of two modes is

$$[J^a_n, J^b_m] = \oint_0 dw \oint_w dz\, z^n w^m J^a(z) J^b(w). \tag{267}$$

Here the w integral is taken on a contour centered on the origin, while for a given w the z integral is taken on a contour that encloses w and the $z^n w^m$ selects out the desired components of the mode expansion. This is the usual thing one gets when writing the (radial) commutator: for each w we end up doing two contours surrounding the origin along circles or radii slightly larger / smaller than $|w|$, which we then deform into a small contour enclosing w.

Anyway, now we just insert the OPE into the integral on the LHS. Remembering that we get the residue of the second-order pole with $d_z[(z-w)^2(z^n w^m/(z-w)^2)]|_{z=w}$, we do the integrals and get

$$[J^a_n, J^b_m] = i f^{abc} J^c_{n+m} + kn\delta_{a,b}\delta_{n,-m}. \tag{268}$$

The $\delta_{n,-m}$ comes since after doing the z integral we have an integral over w^{n-1-m} which sets $n = m$, with the prefactor of n coming from taking the derivative when finding the residue. Finally, that the mode J^c_{n+m} is selected out can be seen by plugging in the mode expansion for J^c and noting that the integrand $z^n w^{m-l-1}/(z-w)$ is only non-zero if $l = m + n$. This is the algebra we've been looking for. We also get a similar algebra for the antiholomorphic currents by putting $\bar{J}$ into the conformal Ward identity. Also note that since the $J\bar{J}$ OPE has to have no singular terms, the two algebras are decoupled:

$$[J^a_n, \bar{J}^b_m] = 0. \tag{269}$$

Magic at the $SU(2) \times SU(2)$ radius

Today we are going to learn more about the compact scalar in two dimensions and its magical properties at special values of the compactification radius.

The action is

$$S = \frac{R^2}{4\pi} \int d^2z\, \partial X \bar{\partial} X. \tag{270}$$

From the equation of motion $\bar{\partial}\partial X = 0$ we can separate X as $X = X_L(z) + X_R(\bar{z})$ inside correlation functions and away from other operators.

Define the operators

$$J^3 = i\partial X, \qquad J^{\pm} =: e^{\pm i 2 X_L} : \tag{271}$$

where we are tacitly assuming that $J^{\pm}$ is well-defined. Using

$$\langle X(z) X(0) \rangle = -\frac{1}{R^2} \ln |z|, \tag{272}$$

we will find the value of R for which the vertex operators $J^{\pm}$ are well-defined and have scaling dimension 1. Letting

$$J^1 = \frac{1}{2}(J^+ + J^-), \qquad J^2 = \frac{1}{2i}(J^+ - J^-), \tag{273}$$

we will show that the algebra obeyed by the charge operators is the $SU(2)$ algebra.

First we find the scaling dimension of J^3 and $J^{\pm}$. For J^3, we just differentiate the $\ln |z|$ propagator twice and see that

$$\Delta_{J^3} = 1 \tag{274}$$

as is appropriate for a conserved current.

I think we've also found the scaling dimension for $J^\pm$ in an earlier diary entry, so I'll be somewhat brief. From

$$\langle X(z,\bar{z})X(0)\rangle = -\frac{1}{2R^2}(\ln z + \ln \bar{z}), \tag{275}$$

we get

$$\langle X_L(z)X_L(0)\rangle = -\frac{1}{2R^2}\ln z, \qquad \langle X_R(\bar{z})X_R(0)\rangle = -\frac{1}{2R^2}\ln \bar{z}, \qquad \langle X_L(z)X_R(0)\rangle = 0. \tag{276}$$

Now, the correlation function $\langle J^\pm(z)(J^\mp)^\dagger(0)\rangle = 0$ since we violate "charge neutrality". That is, if we have a current (which for us is $j = \sum_i \sigma_i \delta(z_i)$ for $\sigma_i \in \mathbb{Z}$) and expand $X_L = \sum_i \alpha_i X_i$ with $\int X_i X_j = \delta_{ij}$, then if we have a coupling term $\int jX_L$ in the action, the path integral over X_0 enforces $\int j = 0$, i.e. charge neutrality. Since we stick in such a $\int jX_L$ term when we are computing the scaling dimension of the vertex operators $J^\pm$, we see that the two-point function of the vertex operators is only nonzero when their charge is neutral. Then we can write

$$\langle e^{\pm i2X_L(z)}e^{\mp i2X_L(0)}\rangle = \exp\left(-\frac{1}{2}\int d^2w d^2w'\, j(w)G(w,w')j(w')\right), \tag{277}$$

where $j = 2(\pm\delta(z)\mp\delta(0))$. When we normal-order the vertex operators to compute the 2-point functions of the $J^\pm$ we kill the points in the integral with support at $w = w'$, and so using the expression for the propagator we found,

$$\langle J^\pm(z)(J^\pm)^\dagger(0)\rangle = \frac{1}{z^{2/R^2}}, \tag{278}$$

so that $\Delta_{J^\pm} = 1/R^2$. This means that the $J^\pm$ operators have scaling dimension 1 when $R = 1$, giving them a chance to be conserved currents. Note that in some places (several string theory books) it's $R = \sqrt{2}$—these differences come from different conventions for the $R^2/2\pi$ in the action. The fact that the scaling dimensions of $J^\pm$ and J^3 are all the same permits the existence of a symmetry which rotates them into one another. A hint of this extra symmetry comes from looking at the mass

$$m^2 = \frac{n^2}{R^2} + w^2R^2 + \dots, \tag{279}$$

where $\dots$ is an oscillator contribution. n/R is the momentum (which comes in units of $1/R$ because of $x \sim x + 2\pi R$), while wR with w the winding number is the energy coming from the string tension (I believe with these conventions the string tension is $T = 1/2\pi$ so that $w2\pi RT = wR$). Sending $R \leftrightarrow 1/R$ and $n \leftrightarrow w$ is a symmetry of the spectrum, which acts as a self-duality when $R = 1$. When $R \neq 1$ we just have a $U(1) \times U(1)$ symmetry for the momentum and winding separately, but when $R = 1$ we will see that we get an $SU(2) \times SU(2)$ symmetry that rotates winding and momentum into one another.

Also note that in these conventions at $R = 1$ the 2-point function for charge-1 vertex operators is

$$\langle : e^{\pm iX_L(z)} :: e^{\mp iX_L(0)} :\rangle = \frac{1}{z^{1/2}}, \tag{280}$$

which is not single-valued. So for $R = 1$, only the charge 2 vertex operators are legit local operators.

Now let's compute OPEs between the various J's, specializing to the choice of $R = 1$ where both currents have the same scaling dimensions. The first one between two J^3's is easy since we just have to do one contraction:

$$J^3(z)J^3(w) \sim \frac{1}{2(z-w)^2}. \tag{281}$$

When we do the OPE for J^3 and $J^\pm$, we just have to contract the ∂X from J^3 with one of the X's from the expansion of the vertex operator: the contractions among the X's in the vertex operator are removed by the normal ordering. So since $\partial X(z, \bar{z}) = \partial(X_L(z) + X_R(\bar{z})) = \partial X_L(z)$, some algebra gives the OPE

$$J^3(z)J^\pm(w) = i\partial X_L(z) : e^{\pm i2X_L(w)} : \sim \pm\partial_z \ln(z-w) J^\pm(w) \sim \pm\frac{1}{z-w} J^\pm(w), \tag{282}$$

where as usual $\sim$ means equality up to non-singular terms (in this case, just the fully normal ordered term). This allows us to compute the OPE

$$J^3(z)J^1(w) \sim \frac{1}{z-w}\frac{J^+ - J^-}{2} \sim \frac{iJ^2(w)}{z-w}. \tag{283}$$

Likewise,

$$J^3(z)J^2(w) \sim -\frac{iJ^1(w)}{z-w}. \tag{284}$$

Now for the vertex operator OPEs. We find the OPE with the general prescription used to convert normal-ordering things to time-ordered things:

$$\mathcal{O}_1(z)\mathcal{O}_2(w) = \exp\left(-\frac{1}{2}\int dz'dw' \ln(z-w)\frac{\delta}{\delta X_L(z',1)}\frac{\delta}{\delta X_L(w',2)}\right) : \mathcal{O}_1(z)\mathcal{O}_2(w) :, \tag{285}$$

where we are still at $R = 1$ and where the $\mathcal{O}_i$ are functionals of X_L and the notation $X_L(z, i)$ means that the functional derivative acts only on $\mathcal{O}_i$. For example, we have

$$J^+(z)J^-(w) \sim \sum_{k=1}^{\infty} \frac{\ln^k(z-w)}{2^k k!}(-1)^k 2^{2k} : e^{2iX(z)}e^{-2iX(w)} : \sim \frac{1}{(z-w)^2} : e^{2iX(z)}e^{-2iX(w)} : \tag{286}$$

Now we can expand the $X(z)$ exponential about $X(w)$ since it's inside the normal ordering, and so

$$J^+(z)J^-(w) \sim \frac{1}{(z-w)^2} + \frac{2i\partial X(w)}{z-w}. \tag{287}$$

When we compute J^-J^+ the only difference is a minus sign when expanding the vertex operators inside the normal ordering, and so

$$J^-(z)J^+(w) \sim \frac{1}{(z-w)^2} - \frac{2i\partial X(w)}{z-w}. \tag{288}$$

When we compute the $J^\pm J^\pm$ OPEs, we get an extra $(-1)^k$ which cancels the one appearing in the J^+J^- OPE, which renders all of the terms non-singular, so that

$$J^\pm(z)J^\pm(w)\sim 0, \tag{289}$$

which we expect from the fact that the LHS is not charge-neutral. We then get

$$J^1(z)J^1(w)\sim J^2(z)J^2(w)\sim \frac{J^+(z)J^-(w)+J^-(z)J^+(w)}{4}\sim\frac{1}{2(z-w)^2}, \tag{290}$$

as well as

$$J^1(z)J^2(w)\sim -J^2(z)J^1(w)\sim\frac{J^-J^+-J^-J^+}{4i}=-\frac{\partial X(w)}{z-w}=iJ^3(w). \tag{291}$$

Collecting these together, we get

$$J^a(z)J^b(w)=\frac{1}{2(z-w)^2}\delta_{ab}+i\epsilon^{abc}\frac{J^c}{z-w}, \tag{292}$$

for $a,b,c=1,2,3$. The $SU(2)$-ness of this of course comes from the $(z-w)^{-1}$ term. Indeed, this term is responsible for making the algebra of the charges the $SU(2)$ algebra. To compute $[Q^a,Q^b]$, we do the usual trick: Q^aQ^b looks like two concentric circles in radial quantization radially separated by a small distance ϵ: we turn the two associated contour integrals to an integral like $\oint_0 dw\oint_w dz$ where the subscripts indicate the center of the contour, and since only the $1/(z-w)$ pole contributes to the integral we get

$$[Q^a,Q^b]=\oint_0 dw\int_w dz\,[J^a(w),J^b(z)]=i\epsilon^{abc}\oint_0 dw\int_w dz\frac{J^c(z)}{z-w}=i\epsilon^{abc}Q^c, \tag{293}$$

which is what we wanted.

We've only been dealing with the holomorphic part X_L, but the same story plays out for X_R. Since $X_LX_R\sim 0$, the L story and the R story are completely independent, and together they generate an $SU(2)_L\times SU(2)_R$ symmetry at the self-dual point.

This manifestation of the duality can be written in a perhaps more familiar form by writing down mode expansions for X_L and X_R and identifying the $m=0$ term in the expansions, which are the momenta p_L,p_R of the modes. From $p\propto\int dz\partial X-\int d\bar z\bar\partial X$ (think: $\partial+\bar\partial\propto -i\partial_x$) and $w\propto\int dx^\mu\partial_\mu X\propto\int dz\partial X+\int d\bar z\bar\partial X$, one gets (still at $R=1$)

$$p_L=n+w,\qquad p_R=n-w, \tag{294}$$

where we are at $\alpha'=1$. Exchanging momentum and winding sends $p_L\to p_L, p_R\to -p_R$, so that the symmetry acts oddly on the antiholomorphic component. Therefore define the field $\widetilde{X}=X_L-X_R$ as a guess for what the image of the field X is under duality (or better, just take the definition from $\partial\widetilde{X}=\partial X,\bar\partial\widetilde{X}=-\bar\partial X$). This passes basic sanity checks since the fact that $X_LX_R\sim 0$ means that X and $\widetilde{X}$ have all the same correlation functions, the same stress-energy tensor, and so on. Then by acting on $\widetilde{X}$ with $\partial\pm\bar\partial$, we see that the fields are related as

$$d\widetilde{X}=\star dX, \tag{295}$$

which is exactly the type of duality we are familiar with from the particle-vortex duality approach (at $R = 1$ it is a self-duality). More on this and lots of other cool things from the string theory side to perhaps appear in future diary entries.

T is not a conformal primary when $c \neq 0$, and the Hamiltonian on the cylinder

Today is quick and easy—doing a calculation that I've seen in many places but never worked out for myself.

By using our knowledge of the TT OPE, we will use the conformal Ward identity to show that under the conformal transformation ξ_μ, the stress tensor changes by

$$\delta_\xi \langle T \rangle = (\xi\partial + 2\partial\xi)T + \frac{c}{12}\partial^3\xi, \tag{296}$$

which means that T is not a primary unless we are in the trivial case where $c = 0$ (the first two terms in parenthesis are the usual transformation rules for the holomorphic part of a two-index tensor primary).

Now let $z = e^w$ be the mapping from cylindrical coordinates $w = \sigma^0 + i\sigma^1$ (with $\sigma^1 \sim \sigma^1 + 2\pi$ the spatial coordinate—maybe not the best notation) to the plane where time increases radially. We will show that

$$T(w) = z^2 T(z) - \frac{c}{24}. \tag{297}$$

Then using

$$H = \int \frac{d\sigma^1}{2\pi} T_{00}, \tag{298}$$

we will see that the Hamiltonian on the cylinder is

$$H = L_0 + \bar{L}_0 - (c + \bar{c})/24. \tag{299}$$

The conformal Ward identity for an operator X says that

$$\delta_\xi \langle X \rangle = \int d^2\sigma\, \partial_\mu \langle T^{\mu\nu} \xi_\nu X \rangle = \oint d\sigma^\mu \langle \epsilon_{\mu\nu} T^{\nu\lambda} \xi_\lambda X \rangle. \tag{300}$$

When we go to complex coordinates we get a $-i/2$ out front from the change in the ϵ tensor, since it becomes $\epsilon = -Y/2$. We also get a factor of 4 when we lower the indices on $T^{\bar{z}\bar{z}}$ to T_{zz}, and a factor of 1/2 when raising the index on $\xi_{\bar{z}}$ to $\xi \equiv \xi^z$. Finally, we get a $-1/2\pi$ from changing T_{zz} to $T = -2\pi T_{zz}$. So the conformal Ward identity in complex coordinates is

$$\delta_\xi \langle X(w) \rangle = \frac{1}{2\pi i} \oint dz \, \langle T\xi X(w) \rangle - \frac{1}{2\pi i} \oint d\bar{z} \, \langle \bar{T}\bar{\xi} X(w) \rangle. \tag{301}$$

Here the contour is taken to be a small circle enclosing $\mathrm{Supp}(X(w))$ (or the support of a suitably smeared version of X). Now we take $X = T$ and plug in the general form of the TT OPE (and use $T\bar{T} \sim 0$ to separate the holomorphic and antiholomorphic parts). We then expand $\xi(z)$ about $z = w$. Only the $\partial^3\xi(z-w)^3$ term is able to integrate to something nonzero with the $(c/2)/(z-w)^4$ term, while only the $\partial\xi(z-w)$ term is able to make the $2T/(z-w)^2$ term nonzero, and only the zeroth order term can make the $\partial T/(z-w)$ part nonzero. So after doing the integral we get

$$\delta_\xi \langle T \rangle = (\xi\partial + 2\partial\xi)T + \frac{c}{12}\partial^3\xi \tag{302}$$

as required.

Finding the finite version of this is a pain, but we can look it up in Polchinski:

$$T(w) = (\partial_w z)^2 T(z) + \frac{c}{12}\{z; w\}, \tag{303}$$

where $\{z; w\}$ is the Schwartzian derivative:

$$\{z; w\} = \frac{\partial_w^3 z}{\partial_w z} - \frac{3}{2}\left(\frac{\partial_w^2 z}{\partial_w z}\right)^2. \tag{304}$$

Thus the stress tensor isn't a conformal primary unless $c = 0$. Checking that this Schwartzian derivative formula works is straightforward and unilluminating so I won't write it out.

We now need to do the coordinate transformation $z = e^w$ between the plane (z) and the cylinder (w) to get $T(w)$. The Schwartzian derivative in this case is easy:

$$\{z; w\} = -1/2. \tag{305}$$

So putting this in, the cylinder stress tensor is

$$T(w) = z^2 T(z) - \frac{c}{24}, \tag{306}$$

which is what we wanted.

Now for the Hamiltonian. We write it as

$$H = \frac{1}{2\pi i}\int (dw \, T(w) - d\bar{w} \, \bar{T}(\bar{w})) = \frac{1}{2\pi i}\int (dz \, z^{-1} T(w) - d\bar{z} \, \bar{z}^{-1} \bar{T}(\bar{w})), \tag{307}$$

since $dz = z dw$. Using the transformation law for T, this is

$$H = \frac{1}{2\pi i}\int \left(dz \, z^{-1}[z^2 T(z) - c/24] - d\bar{z} \, \bar{z}^{-1}[\bar{z}^2 \bar{T}(\bar{z}) - \bar{c}/24]\right). \tag{308}$$

The integral selects out the $n = 0$ components of the Laurent expansions of $T, \bar{T}$ due to the conventions on shifting the powers by 2 in the expansion. The central charge pieces are integrated against $1/z, 1/\bar{z}$ and so they survive, and then since $\int dz/z = -\int d\bar{z}/\bar{z}$, we get

$$H = L_0 + \bar{L}_0 - (c + \bar{c})/24, \tag{309}$$

as expected. That the Hamiltonian has the $L_0 + \bar{L}_0$ part is no surprise, since this is the operator that generates dilitations: since $L_0 = \frac{1}{2\pi i}\oint dz\, zT$, for a primary X we have

$$L_0 X = \frac{1}{2\pi i}\oint dz\, z\frac{hX(w)}{(z-w)^2} = hX(w), \tag{310}$$

so that indeed, L_0 performs the dilitations, and thus belongs in H since dilitations in the plane in radial quantization are the same as time evolution. Thus on the cylinder H still does dilitations, but it is accompanied by a constant piece that keeps track of a (physically meaningful) vacuum energy.[35]

Linear dilaton CFT

This is an exercise John McGreevy assigned to his QFT class. Consider the linear dilaton CFT, which is a free scalar plus a coupling to gravity:

$$S = \frac{1}{2\pi\alpha'}\int d^2x\,\sqrt{g}\,\partial X\bar{\partial}X + \frac{1}{2\pi}\int d^2x\,\sqrt{g}\,QXR, \tag{311}$$

where Q is a constant, which may be either real or imaginary (if we are thinking about strings we should be writing out the spacetime index on the X like $Q_\mu X^\mu$, but for this problem we will just think of X as a scalar) and R is the two-dimensional Ricci curvature scalar.

We will show that in flat space, while the coupling to gravity doesn't affect the equations of motion, it does change the stress tensor, which is

$$T = -\frac{1}{\alpha'} : \partial X\partial X : + Q\partial^2 X. \tag{312}$$

We will then find the central charge by computing the TT OPE, and check that the TT OPE has the right form for stress tensors in CFTs. Lastly we will compute the scaling dimension of the vertex operator $:e^{ikX}:$.

[35]Normally we don't think of keeping track of a constant term in H as being a meaningful thing to do. Here though, the point of keeping track of it is to compare the differences in vacuum energies with different boundary conditions; more on this later.

In flat space, which we will be working in, the extra term in the action vanishes, so we can use the usual $-\alpha' \ln |z-w|^2/2$ propagator in what follows. However the stress tensor *will* change, since it comes from varying the metric away from flat space. Since there is a $\sqrt{g}$ in the measure, we need to compute the variation

$$\delta(\sqrt{g}R) = \delta(\sqrt{g}g^{\nu\mu}R_{\mu\nu}) = (\delta\sqrt{g})R + \sqrt{g}R_{\mu\nu}\delta g^{\nu\mu} + \sqrt{g}g^{\nu\mu}\delta R_{\mu\nu}, \tag{313}$$

with $R_{\mu\nu}$ the Ricci tensor. The variation of $\sqrt{g}$ is the usual

$$\delta\sqrt{g} = \frac{1}{2\sqrt{g}}\delta e^{\mathrm{Tr}\ln g} = \frac{1}{2}\sqrt{g}\mathrm{Tr}[g^{-1}\delta g] = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\nu\mu}, \tag{314}$$

where in the last step we used $\delta(g^{\mu\nu}g_{\nu\lambda}) = \delta(\delta^{\mu}{}_{\lambda}) = 0$. The variation of the Ricci scalar is more heinous to derive, but luckily it can be found in Wald, in section 7.5 (the one on perturbations). It turns out that $g^{\nu\mu}\delta R_{\mu\nu}$ is a total derivative:

$$g^{\nu\mu}\delta R_{\mu\nu} = \nabla^{\mu}[\nabla^{\nu}(\delta g_{\mu\nu}) - g^{\lambda\sigma}\nabla_{\mu}(\delta g_{\lambda\sigma})] \equiv \nabla^{\mu}v_{\mu}. \tag{315}$$

Thus the variation of the dilaton term with respect to the metric is

$$\begin{aligned}\delta S_Q &= \frac{1}{2\pi}\int d^2x\, QX\sqrt{g}\left(\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} + \nabla^{\mu}v_{\mu}\right)\\ &= \frac{1}{2\pi}\int d^2x\, QX\sqrt{g}(G_{\mu\nu}\delta g^{\mu\nu} + \nabla^{\mu}v_{\mu}),\end{aligned} \tag{316}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{317}$$

is the Einstein tensor.

Here's where being in two dimensions helps: because of the symmetries of the Riemann curvature tensor, it has only one independent component in two dimensions. That is, since $R_{\mu\nu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$ and $R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$, the only independent non-zero component is R_{0101}. Then the Ricci tensor is

$$R_{\mu\nu} = R_{0101}\begin{pmatrix} g^{11} & -g^{01} \\ -g^{10} & g^{00}\end{pmatrix} = \frac{R_{0101}}{\det g}g_{\mu\nu}, \tag{318}$$

since the matrix is $(\det g^{-1})(g^{\mu\nu})^{-1}$. On the other hand, we can contract the Ricci tensor explicitly and see that the curvature scalar is

$$R = 2\frac{R_{0101}}{\det g}, \tag{319}$$

so that

$$R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu}, \tag{320}$$

which implies the vanishing of the Einstein tensor in two dimensions, $G_{\mu\nu} = 0$. Thus from (316) we see that if we were to ignore the scalar X so that we just had gravity, we can conclude that in two dimensions the variation of the Einstein-Hilbert action is actually a total divergence, and so vanishes on closed spacetimes. This is because we should think of the Einstein-Hilbert action in this case as being $\int F$ and measuring the topology of the spacetime (more precisely, $\sqrt{g}R$ is the Euler density, so that the Einstein-Hilbert action computes the Euler characteristic), and hence it is locally a total derivative.

Now we integrate the remaining $X\nabla^\mu v_\mu$ term by parts. We get, still in flat space,

$$\delta S_Q = \frac{1}{2\pi}\int d^2x\, Q(\partial^\mu\partial^\nu X - \partial^\lambda\partial_\lambda X g^{\mu\nu})\delta g_{\mu\nu}, \tag{321}$$

and so with the identification $\delta S = -\frac{1}{2}\int T^{\mu\nu}\delta g_{\nu\mu}$, we see that S_Q gives a contribution to the stress tensor of

$$T_Q^{\mu\nu} = \frac{1}{\pi}Q(g^{\mu\nu}\partial_\lambda\partial^\lambda - \partial^\mu\partial^\nu)X = \frac{1}{\pi}Q\Pi_T^{\mu\nu}X. \tag{322}$$

Now we switch over to complex coordinates and use the definition $T = -2\pi T_{zz} = -(\pi/2)T^{\bar{z}\bar{z}}$ to get

$$T_{zz} = \frac{-Q}{2\pi}(\partial_0^2 - \partial_1^2 - 2i\partial_0\partial_1)X, \tag{323}$$

so that the dilaton contribution to the holomorphic part of the stress tensor is

$$T_Q = Q\partial^2 X. \tag{324}$$

Now we want to find the central charge. We thus need to look at the $1/(z-w)^4$ term in the $T(z)T(w)$ OPE. This part only comes from terms that have been fully contracted, since we need four derivatives acting on propagators to give a $1/(z-w)^4$ term. The usual piece is

$$\begin{aligned}\frac{1}{\alpha'^2}2(\langle\partial X(z)\partial X(w)\rangle)^2 &= \frac{1}{2}\left(\partial_z\partial_w \ln|z-w|^2\right)^2 \\ &= \frac{1/2}{(z-w)^4},\end{aligned} \tag{325}$$

which gives the usual contribution of 1 to the central charge. The other piece is

$$Q^2\langle\partial_z^2 X(z)\partial_w^2 X(w)\rangle = \frac{\alpha'}{2}\frac{6}{(z-w)^4}, \tag{326}$$

so that

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + O(1/z^2), \qquad c = 1 + 6\alpha' Q^2. \tag{327}$$

Note that the central charge gets *reduced* by the introduction of the dilaton coupling if Q is imaginary.

Now we should check to make sure that the $O(1/z^2)$ in the above equation really is $O(1/z^2)$, and not $O(1/z^3)$ since otherwise we're in trouble. Recall that the form of the TT OPE needs to be

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{z^2}T(w) + \frac{1}{z}\partial T(w). \tag{328}$$

One potentially troublesome term in the linear dilaton TT OPE is

$$-\frac{1}{\alpha'}Q\langle\partial_z X(z)\partial_w^2 X(w)\rangle\partial_z X(z) = \frac{2Q}{\alpha'}\frac{\alpha'}{2}\partial_z\partial_w^2 \ln|z-w|^2\partial_z X(z) = \frac{2Q\partial_z X(z)}{(z-w)^3}. \tag{329}$$

What saves us from this term is the other contraction with $z \leftrightarrow w$, since the denominator is odd under this shift. The two cross-contractions between the Q term and the regular term thus give

$$\frac{2Q(\partial_z X(z) - \partial_w X(w))}{(z-w)^3} \to \frac{2Q\partial_z^2 X(z)}{(z-w)^2}, \tag{330}$$

which is exactly the term we need for the $2T(w)/z^2$ piece of the TT OPE. It is straightforward to check that the remainder of the $(z-w)^{-2}T(w)$ term and the $(z-w)^{-1}\partial T(w)$ term are produced as well.

Let's now find the conformal dimension of the vertex operator by taking the OPE with T. We need to only look at the leading piece of the OPE which goes as $1/(z-w)^2$. The part from the $:\partial X \partial X:$ term can be found by noting that $1/(z-w)^2$ terms only occur when both ∂X operators are contracted with the vertex operator. So after figuring out the combinatorial factors,

$$-\frac{1}{\alpha'} :\partial X\partial X::e^{ikX}: = \frac{1}{\alpha'}k^2\left(\frac{\alpha'}{2}\partial_z \ln|z-w|^2\right)^2 \sum_{j=0}^{\infty}\frac{1}{(j+2)!}\binom{j+2}{2}2(ik)^j :X(w)^j: + O(1/z), \tag{331}$$

so that the relevant term is

$$-\frac{1}{\alpha'} :\partial X\partial X::e^{ikX}: = \frac{k^2\alpha'/4}{(z-w)^2}:e^{ikX}: + O(1/z), \tag{332}$$

which gives a contribution of $k^2\alpha'/4$ to the conformal dimension. The contribution from the dilaton term is

$$ikQ\sum_{j=0}^{\infty}\frac{1}{j!}\langle\partial^2 X(z)X(w)\rangle :(iX(w))^j: = \frac{ik\alpha' Q/2}{(z-w)^2}:e^{ikX}:, \tag{333}$$

which gives a contribution of $ik\alpha' Q/2$ to the conformal dimension. Thus the conformal dimension of the vertex operator is

$$h = \frac{k^2\alpha'}{4} + \frac{ik\alpha' Q}{2}. \tag{334}$$

This gives a real conformal dimension if Q is imaginary. If Q is real the scaling dimension Δ is unaffected by the coupling to the dilaton, while the conformal dimensions and spin are rendered imaginary.

Bosonization and torus partition functions

I got this problem from a pset that John McGreevy assigned to his QFT class and posted online. Update: actually, looks like a large part of this is in the big yellow book (of course).

We fill find the partition function for a free compact boson on a torus with sides given by the complex numbers ω_1, ω_2. We will then show that this is the same as the partition function for a Dirac fermion on the torus, with all spin structures taken into account.

Let the two sides of the torus be given by $\omega_1 \in \mathbb{R}$, $\omega_2 \in \mathbb{C}$, with modular parameter $\tau = \omega_2/\omega_1$. The time evolution operator along a direction parallel to ω_2 for a "time" s is

$$U(s) = \exp\left(-\frac{s}{|\omega_2|}\left[\mathrm{Im}(\omega_2)H - i\mathrm{Re}(\omega_2)P\right]\right). \tag{335}$$

The signs are the way they are since Schrodinger decided that H acts as $+i\partial_t$ (so that we translate in time with $e^{-iHt} \to e^{-\tau H}$) while P acts as $-i\partial_x$ (so that we translate with e^{+iP}). We can use the expression we found for H two days ago, namely $H = L_0 + \bar{L}_0 - (c+\bar{c})/24$. This was derived for a cylinder of radius 1, and so if the radius is instead ω_1, then the Hamiltonian is (by dimensional analysis)

$$H = \frac{2\pi}{\omega_1}\left(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}\right). \tag{336}$$

Since we are specializing to a free compact boson, we will just write the last part as $(c+\bar{c})/24 = 1/12$ in what follows.

On a cylinder of radius 1, we get the momentum operator from

$$P = \int d\sigma\, T_{\tau\sigma} = i\frac{1}{2\pi i}\int (dw\, T(w) + d\bar{w}\, \bar{T}(w)). \tag{337}$$

Recalling from two days ago how to map $T(w)$ onto the plane, we get

$$P = i\frac{1}{2\pi i}\oint \left(z^{-1}dz\, z^2 T(z) + \bar{z}^{-1}d\bar{z}\, \bar{z}^2\bar{T}(z) + (z^{-1}dz + \bar{z}^{-1}d\bar{z})\frac{c}{24}\right). \tag{338}$$

The central charge term dies since the contour with dz is minus the one with $d\bar{z}$, while the other integrals select out the $n = 0$ component of the Laurent expansions for the stress tensors, which enter with opposite sides. When on a cylinder of radius ω_1 then,

$$P = \frac{2\pi i}{\omega_1}(L_0 - \bar{L}_0). \tag{339}$$

The partition function is then

$$Z = \mathrm{Tr}\left[e^{2\pi i(L_0+\bar{L}_0)\frac{\tau-\bar{\tau}}{2}}e^{2\pi i(L_0-\bar{L}_0)\frac{\tau+\bar{\tau}}{2}}e^{-2\pi ic\frac{\tau-\bar{\tau}}{24}}\right]. \tag{340}$$

This can be cleaned up by defining

$$q \equiv e^{2\pi i\tau}. \tag{341}$$

Then we have

$$Z = \mathrm{Tr}\left[q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}\right]. \tag{342}$$

To do the trace, we need the spectrum of L_0. The action is

$$S = \frac{1}{8\pi}\int dxdt\,[(\partial_t X)^2 - (\partial_x X)^2]. \tag{343}$$

Now we expand in Fourier modes, momentarily ignoring winding number issues,

$$X = \sum_n X_n e^{ixn/r}, \tag{344}$$

where r is the radius of the spatial circle—for us, $\omega_1 = 2\pi r$. In what follows, all sums over roman letter variables will be sums over $\mathbb{Z}$. The momentum is $\pi_n = \partial_t X_{-n} r/2$, and the Hamiltonian is

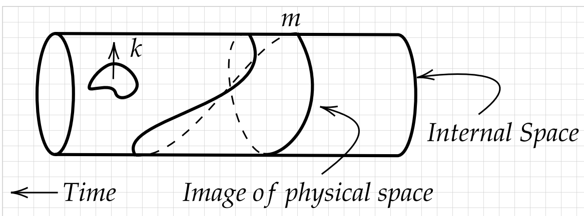
$$H = \frac{1}{r}\sum_n\left(\pi_n\pi_{-n} + \frac{n^2}{4}X_n X_{-n}\right). \tag{345}$$

This is a bunch of harmonic oscillators—from the momentum term we see that $m = 1/2$, so that the frequency of each oscillator is just $\omega_n = |n|$.

We will use this result in a bit, but first we will need to remember topological issues. Let us treat the zero mode X_0 separately from the other modes in the sum. From the commutator $[X_0, H] = 2i\pi_0/r$, we can get the time evolution of the zero mode. This further simplifies, since the compact nature of the boson forces $\pi_0 = k/R$ for $k \in \mathbb{Z}$.[36] We also need to add in a term that keeps track of the winding number of X: it is Rmx/r (with $X \sim X + 2\pi R$ defining the boson radius), which shifts as $Rmx/r \mapsto Rmx/r + 2\pi Rm$ around the spatial circle. Thus the decomposition for X

$$X(x,t) = X_0(0) + \frac{2k}{r}\tau + \frac{Rm}{r}x + \sum_{n\neq 0} X_n(t)X_{-n}(t). \tag{346}$$

As a picture, the setup is



[36] Remember that π_0 is the canonical momentum, not the physical kinematic momentum (the latter being defined through T_{01}).

Here the circle is the internal coordinate of the field, while the axial direction of the cylinder is time. Different images of the physical spatial circle under X are shown, one with winding number 0 and canonical momentum $\pi_0 \sim k$, and the other with winding number m (note that in a string theory context, we would replace "physical spacetime" with "worldsheet" and "internal space" with "(one compact direction of the) spatial manifold".

Now we need to go to complex coordinates. There are many options, but it seems like the best choice are

$$z = e^{(\tau+ix)/r}, \qquad \bar{z} = e^{(\tau-ix)/r}. \tag{347}$$

This is kind of unpleasant since $\tau = it$ is imaginary time, but oh well—it ends up giving the answer in the form written in the problem statement. The $1/r$ in the exponents is so that r disappears from the final expression for X. With this choice of coordinates

$$x = -\frac{ir}{2}(\ln z - \ln \bar{z}), \qquad t = -\frac{ir}{2}(\ln z + \ln \bar{z}). \tag{348}$$

Putting this into $X(t)$:

$$X(z, \bar{z}) = X_0 - i(\ln z)p_L - i(\ln \bar{z})p_R + i\sum_{n\neq 0} \frac{1}{n}(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}), \tag{349}$$

where the momenta are

$$p_L = \left(\frac{k}{R} + \frac{mR}{2}\right), \qquad p_R = \left(\frac{k}{R} - \frac{mR}{2}\right). \tag{350}$$

We can now get expressions for $L_0, \bar{L}_0$ by using $T = -\frac{1}{2} : \partial X \partial X :$. So taking the derivative, we have

$$L_0 = \frac{1}{4\pi i}\oint dz\, z \left(p_L^2 z^{-2} + \sum_{i,j\neq 0} a_i a_j z^{-i-j-2} + 2p_L \sum_{j\neq 0} a_j z^{-j-2}\right). \tag{351}$$

The last term dies while in the second term i gets set to $-j$, so

$$L_0 = \frac{1}{2}p_L^2 + \sum_{j>0} a_{-j} a_j. \tag{352}$$

Similarly,

$$\bar{L}_0 = \frac{1}{2}p_R^2 + \sum_{j>0} \bar{a}_{-j}\bar{a}_j. \tag{353}$$

Here the sum over $j > 0$ means a sum from $\mathbb{Z} \ni j = 1$ to $j = \infty$.

We can now finally get the partition function. For each oscillator mode j, the sum over occupation numbers gives $1/(1 - q^j)$. This is because as we saw earlier, the frequency of the jth mode is simply $|j|$. So the oscillator contribution to q^{L_0} is $\prod_{j>0}(1 - q^j)^{-1}$. The zero modes are accounted for just by summing over all momenta

k and winding numbers m, and so since the antiholomorphic oscillator contribution is the conjugate of the holomorphic contribution,

$$Z(q) = \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}, \tag{354}$$

where

$$\eta(q) \equiv q^{1/24} \prod_{j>0} (1-q^j). \tag{355}$$

Note that the theory is self-dual at the radius $R = \sqrt{2}$ (not $1/\sqrt{2}$ like earlier because of how we defined the coupling constant for the action). In the following we will consider the radius $R = 1$, which from earlier diary entries we know to be a value for which a fermion description works. Explicitly, at this radius we have

$$Z(q; R=1) = \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m/2)^2} \bar{q}^{\frac{1}{2}(k-m/2)^2}. \tag{356}$$

Our goal now is to relate this to fermions. First we break up the sum into m even and m odd:

$$Z(q; R=1) = \frac{1}{|\eta(q)|^2} \sum_{k,m} \left(q^{\frac{1}{2}(k+m)^2} \bar{q}^{\frac{1}{2}(k-m)^2} + q^{\frac{1}{2}(k+m+1/2)^2} \bar{q}^{\frac{1}{2}(k-m-1/2)^2} \right). \tag{357}$$

The first term is actually

$$\begin{aligned} \frac{1}{2|\eta(q)|^2} \left(\left| \sum_k q^{k^2/2} \right|^2 + \left| \sum_k (-1)^k q^{k^2/2} \right|^2 \right) &= \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}k^2} \bar{q}^{\frac{1}{2}m^2} \frac{1+(-1)^{k+m}}{2} \\ &= \frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m)^2} \bar{q}^{\frac{1}{2}(k-m)^2}, \end{aligned} \tag{358}$$

since the term $\frac{1+(-1)^{k+m}}{2}$ projects onto configurations where $k = m \bmod 2$, which is exactly fulfilled by the pair $k+m, k-m$ (we've re-labeled the summation variables—the point is that for any integers k, m, the combination $k+m, k-m$ survives the projection by $(1+(-1)^{k+m})/2$).

We can use a similar trick for the second term in $Z(q; R=1)$: since the members of the combination $k+m, k-m$ always have the same parity, we can change the sum to run over all integers $x = k+m, y = k-m$ such that $x = y \bmod 2$. We can then instead sum over *all* x, y, provided that we insert the projector $(1+(-1)^{x+y})/2$. Doing this, and then relabeling $x \to k$, $y \to m$ for consistency of notation, we have

$$\frac{1}{|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+m+1/2)^2} \bar{q}^{\frac{1}{2}(k-m-1/2)^2} = \frac{1}{2|\eta(q)|^2} \sum_{k,m} q^{\frac{1}{2}(k+1/2)^2} \bar{q}^{\frac{1}{2}(m+1/2)^2} (1+(-1)^{k+m}), \tag{359}$$

Consider the second term, proportional to $(-1)^{k+m}$. The exponential part is symmetric under the shift $k \mapsto -k-1$, but the $(-1)^{k+m}$ part picks up a minus sign, and so the second term is zero. Thus we can only keep the first term of this bit. It too factors between holomorphic and antiholomorphic contributions, and so the full partition function is

$$Z(q;R=1)=\frac{1}{2|\eta(q)|^2}\left(\left|\sum_k q^{k^2/2}\right|^2+\left|\sum_k(-1)^k q^{k^2/2}\right|^2+\left|\sum_k q^{\frac{1}{2}(k+1/2)^2}\right|^2\right). \quad (360)$$

This is starting to look more fermiony! The three terms are theta functions, and so we can write Z more compactly as (preserving the order of the terms as in the last equation)

$$Z(q;R=1)=\frac{1}{2|\eta(q)|^2}\left(|\theta_3(\tau)|^2+|\theta_4(\tau)|^2+|\theta_2(\tau)|^2\right). \quad (361)$$

The theta functions can be written as infinite products—see e.g. Polchinski. The product form for the theta functions all contain a factor that cancels the product in the $1/|\eta(q)|^2$ in the denominator, and leaves us with

$$Z(q;R=1)=\frac{1}{2|q^{1/24}|^2}\left(\prod_{j>0}|(1+q^{j-1/2})^2|^2+\prod_{j>0}|(1-q^{j-1/2})^2|^2+|q^{1/8}|^2\prod_{j\geq 0}|(1+q^j)^2|^2\right). \quad (362)$$

Our goal is in sight, since we are seeing the different boundary conditions for the fermions appearing.

To keep going, we will need the fermion partition functions. The Hamiltonian for the fermions is derived in the same way as the boson Hamiltonian, which we already did above. It will give us an oscillator contribution coming from the $L_0,\bar{L}_0$ operators, as well as a central charge piece that appears when we switch from the cylinder to the plane. Let's focus on a single real fermion with antiperiodic boundary conditions around the spatial circle: the central charge is 1/2, while the oscillator expansion gives

$$L_0=\sum_{k>0}(k-1/2)\lambda_k\lambda_{-k}, \quad (363)$$

where the λ_k's are Majorana operators and the sum is offset by 1/2 to get the boundary conditions right (note: our convention is such that the action for a single real fermion is $\frac{1}{4\pi}\int\psi\gamma^0\partial\!\!\!/\psi$). So, using the same logic that we used for the compact boson, the partition function for antiperiodic boundary conditions around both cycles is (NSNS / BB spin structure; B for “bounding”)

$$Z_{NSNS}(q)=\mathrm{Tr}_A[q^{L_0-1/48}\bar{q}^{\bar{L}_0-1/48}]=\frac{1}{|q^{1/48}|^2}\prod_{k>0}|1+q^{k-1/2}|^2. \quad (364)$$

When we work with periodic boundary conditions around the temporal cycle, we need to insert $(-1)^F$ to implement the supertrace. This sends $q\to -q$ in the above and so

$$Z_{NSR}(q)=\mathrm{Tr}_A[(-1)^F q^{L_0-1/48}\bar{q}^{\bar{L}_0-1/48}]=\frac{1}{|q^{1/48}|^2}\prod_{k>0}|1-q^{k-1/2}|^2. \quad (365)$$

When we have periodic boundary conditions in space, the form for L_0 changes in two ways: first, the momenta live in $\mathbb{Z}$ rather than $\mathbb{Z}+\frac{1}{2}$, and second, they change by a constant since the vacuum energy on the cylinder depends on the boundary conditions (see e.g. chapter 6 of the big yellow book). In particular,

$$L_0 = \sum_{k \geq 0} \lambda_{-k}\lambda_k + \frac{1}{16}. \tag{366}$$

Thus for the RNS (periodic in space, antiperiodic in time) spin structure, the partition function is

$$Z_{RNS}(q) = \mathrm{Tr}_P[q^{L_0 - 1/48}\bar{q}^{\bar{L}_0 - 1/48}] = \frac{1}{|q^{1/48-1/16}|^2}\prod_{k\geq 0}|1+q^k|^2 = |q^{1/24}|^2\prod_{k\geq 0}|1+q^k|^2. \tag{367}$$

Finally, when we do the RR torus, we get zero because of the zero mode: when we change the thing in the product to $1-q^k$, the $k=0$ term kills the partition function. So $Z_{RR}(q) = 0$.

A massless Dirac fermion for us is the same as two real fermions which are completely independent expect for the requirement that their boundary conditions match. Thus to get $Z_{\mathrm{Dirac}}(q)$ (with all spin structures counted), we just need to sum the squares of the real fermion partition functions over spin structures. This gives

$$\begin{aligned}
Z_{\mathrm{Dirac}}(q) &= \sum_{r,s\in\{NS,R\}} Z_{rs}(q)^2 \\
&= \frac{1}{2}\left(\frac{1}{|q^{2/48}|^2}\prod_{k>0}|(1+q^{k-1/2})^2|^2 + \frac{1}{|q^{2/48}|^2}\prod_{k>0}|1-q^{k-1/2}|^2 + |q^{2/24}|^2\prod_{k\geq 0}|1+q^k|^2 + 0^2\right) \\
&= \frac{1}{2|q^{1/24}|^2}\left(\prod_{j>0}|(1+q^{j-1/2})^2|^2 + \prod_{j>0}|(1-q^{j-1/2})^2|^2 + |q^{1/8}|^2\prod_{j\geq 0}|(1+q^j)^2|^2\right) \\
&= Z(q;R=1).
\end{aligned} \tag{368}$$

Thus we've shown that the compact boson on the torus (at $R=1$) is the same as a Dirac fermion, which in turn is a pair of Ising models coupled together in a certain way.

Note that for this correspondence to work, we have to sum over all spin structures for the fermion (of course, in order to have modular invariance we needed to sum over all spin structures since they [except RR] are permuted into one another by modular transformations. Checking the modular invariance of the final expression can be done by looking up the transformation properties of the theta functions, see e.g. Polchinski). Note that we can see here why people talk about modular invariance giving you information about how the holomorphic and anti-holomorphic sectors talk to each other, information which isn't available on the plane. Indeed, if the two sectors didn't talk to each other then we would have $Z_{\mathrm{Dirac}}(q) = Z_{\mathrm{Dirac}/2}(q)Z_{\mathrm{Dirac}/2}(\bar{q})$ for some function $Z_{\mathrm{Dirac}/2}(q)$. This factorization property is true for a particular spin structure, but not for the whole partition function, and so the boundary conditions introduced by the torus and the requirement of modular invariance let us see the constraints on the ways that the two sectors can talk to each other.

A more suggestive way to write the sum over spin structures is to write the fermion partition function as

$$Z_{\text{Dirac}}(q) = \text{Tr}_{NS}\left[\frac{\mathbf{1}+(-1)^F}{2} q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}\right] + \text{Tr}_R\left[\frac{\mathbf{1}+(-1)^F}{2} q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}\right], \tag{369}$$

where the subscript on the trace indicates the spatial boundary conditions[37]. The point of writing it this way is that it takes the form of fermions coupled to a $\mathbb{Z}_2$ gauge field, with the projectors $(\mathbf{1}+(-1)^F)/2$ enforcing that all states in the Hilbert space be a singlet under the $\mathbb{Z}_2$ symmetry sending $\psi \mapsto -\psi$. That is, we can view the temporal part of the spin structure as coming from a $\mathbb{Z}_2$ gauge field (modular invariance then forces us to sum over both spatial boundary conditions and trace over the whole $\mathcal{H}_{NS} \oplus \mathcal{H}_R$ Hilbert space). The point of this remark is that if we are writing down a mapping between a bosonic theory and a fermionic one, since we never have local fermion operators in the bosonic Hilbert space—only operators which are pairs of fermions. Because of this, the sign of any fermion operators we write down must be unphysical, and so which expect there to be a $\mathbb{Z}_2$ gauge redundancy in any putative fermionic dual model we write down. In this case, we see that this thinking is correct.

Vertex correlators

Today we have an exercise from the big yellow book, chapter 9. We will be considering a (non-compact) boson with action

$$S = \frac{1}{4\pi}\int dz d\bar{z}\, \partial\phi\bar{\partial}\phi. \tag{370}$$

Our goal will be to derive the correlation functions of vertex operators in a careful way. Here's the problem statement:

To define the vertex operator, do a mode expansion and separate out the zero mode as follows:

$$\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\alpha\Phi} : V'_\alpha(z)\bar{V}'_\alpha(\bar{z}), \tag{371}$$

and show that the zero mode is

$$\Phi(z,\bar{z}) = \phi_0 - ia_0 \ln(z\bar{z}) \tag{372}$$

[37]We could instead use $\mathbf{1}-(-1)^F$ in the second trace as well. This $\pm$ ambiguity is due to the fact that the term it appears in is the RR spin structure term, which is zero (the ambiguity relates to the two degenerate states on the RR torus which differ by whether or not the zero mode is filled)

while the $V'_\alpha(z)$'s are

$$V'_\alpha(z) = :e^{i\alpha\phi'(z)}: = \exp\left(-\alpha\sum_{n>0}\frac{1}{n}a_{-n}z^n\right)\exp\left(\alpha\sum_{n>0}\frac{1}{n}a_n z^{-n}\right), \tag{373}$$

and likewise for the $V'_\alpha(\bar{z})$'s. Here, $\phi'(z)$ denotes the holomorphic part of ϕ with the zero mode removed.

Find the $\langle\phi'(z)\phi'(w)\rangle$ correlator, and find the n-pt correlation function of the $V'_\alpha(z)$'s. Find the n-point correlation function of the $:e^{i\alpha\phi'(z,\bar{z})}:$ zero mode vertex operators, and use these results to find the correlators for the full vertex operators $\mathcal{V}_\alpha(z,\bar{z})$.

First lets review the mode expansion. On a cylinder with circumference L, we do $\phi = \sum_n \phi_n e^{2\pi i x/L}$. Finding the Hamiltonian is straightforward:

$$H = \frac{2\pi}{L}\sum_{n\in\mathbb{Z}}(\pi_n\pi_{-n} + \frac{n^2}{4}\phi_n\phi_{-n}), \quad \pi_n = \frac{L}{4\pi}\partial_t\phi_{-n}. \tag{374}$$

We can solve the Hamiltonian by introducing oscillators a_n. We will work with the conventions in the big yellow book, so that for $n \neq 0$,

$$\phi_n = \frac{i}{n}(a_n - \bar{a}_{-n}), \qquad [a_n, a_m] = n\delta_{n+m}. \tag{375}$$

Thus ϕ is decomposed as

$$\phi(x) = \phi_0 + i\sum_{n\neq 0}\frac{1}{n}(a_n - \bar{a}_n)e^{2\pi i n x/L}. \tag{376}$$

We commutate this with the Hamiltonian to get the time dependence, which is straightforward. In particular, the time dependence of the zero mode is $\phi_0 + \frac{4\pi}{L}\pi_0 t$. As we have done in the last couple of days, we let $z = e^{2\pi(\tau - ix)/L}$, $\bar{z} = e^{2\pi(\tau+ix)/L}$ where $\tau = it$. With these conventions,

$$t = -i\frac{L}{4\pi}\ln(z\bar{z}). \tag{377}$$

Putting this in and writing a_0 for π_0, the decomposition for ϕ is

$$\phi(z,\bar{z}) = \Phi(z,\bar{z}) + i\sum_{n\neq 0}\frac{1}{n}(a_n z^{-n} + \bar{a}_n\bar{z}^{-n}), \tag{378}$$

where the zero mode part is

$$\Phi(z,\bar{z}) = \phi_0 - ia_0\ln(z\bar{z}). \tag{379}$$

Now define the field

$$\phi'(z) = \sum_{n\neq 0}\frac{1}{n}a_n z^{-n} \tag{380}$$

to be the holomorphic part of $\phi(z,\bar{z})$ with the zero mode removed. We write its two point function as

$$\langle \phi'(z)\phi'(w)\rangle = \left\langle \sum_{m<0,n>0} \frac{1}{nm} a_n a_m z^{-n} w^{-m} \right\rangle, \tag{381}$$

since a_n with $n>0$ annihilates the vacuum (remember that a_n with $n<0$ act as the usual creation operators). Using the commutator, we can set $m=-n$ and get

$$\langle \phi'(z)\phi'(w)\rangle = \sum_{n>0} \frac{1}{n}\frac{w^n}{z^n} = -\ln\left(1-\frac{w}{z}\right). \tag{382}$$

Now we can get the correlator of a product of $V'_\alpha(z) =: e^{i\alpha\phi'(z)}:$ operators. To do the normal ordering we just need to move the creation operators (a_n with $n<0$) to the left of the annihilation operators, and so

$$V'_\alpha(z) =: e^{i\alpha\phi'(z)} := \exp\left(-\alpha \sum_{n>0} \frac{1}{n} a_{-n} z^n\right) \exp\left(\alpha \sum_{n>0} \frac{1}{n} a_n z^{-n}\right), \tag{383}$$

To get the correlators involving a product of $V'_\alpha(z) =: e^{i\alpha\phi'(z)}:$ operators, we need to know how to normal-order the product. To do this, we use the CBH formula to write

$$e^A e^B = e^B e^A e^{[A,B]}, \qquad \text{if } [A,B] \in \mathbb{C}. \tag{384}$$

The commutator of the terms appearing in the exponentials in the definition of the $V'_\alpha(z)$ is a c-number, so we can use this formula. Applying this to move all the $a_{-n}, n>0$ operators to the left in the product lets us figure out how to do the normal ordering. A bit of algebra gives (see e.g. the big yellow book, appendix 6.A)

$$\langle \prod_j V'_{\alpha_j}(z_j)\rangle = \langle : \prod_j V'_{\alpha_j}(z_j) :\rangle \exp\left(-\prod_{i<k} \alpha_i\alpha_k \langle \phi'(z_i)\phi'(z_k)\rangle\right). \tag{385}$$

Using the correlator that we just derived and the fact that the fully normal-ordered term is equal to 1, we get

$$\langle \prod_j V'_{\alpha_j}(z_j)\rangle = \prod_{j<k} \left(1-\frac{z_k}{z_j}\right)^{\alpha_j\alpha_k}. \tag{386}$$

Putting this together with the antiholomorphic piece, we have

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j,\bar{z}_j)\right\rangle = \left\langle \prod_j : e^{i\alpha_j\Phi(z_j,\bar{z}_j)} : \right\rangle \prod_{j<k} |z_j-z_k|^{2\alpha_j\alpha_j} |z_j|^{-2\alpha_j\alpha_k}. \tag{387}$$

So, we just need the zero mode part. Remembering that $\Phi(z,\bar{z})$ is formed from a linear combination of position and momenta operators for the zero mode, we can use the same approach we used above to do the normal-ordering. We regard a_0 as the annihilation operator and ϕ_0 as the creation operator, so that to normal-order we need to shuffle the a_0's to the left. This gives

$$\begin{aligned}\left\langle \prod_j : e^{i\alpha_j \Phi(z_j,\bar{z}_j)} : \right\rangle &= \left\langle : \prod_j e^{i\alpha_j \Phi(z_j,\bar{z}_j)} : \right\rangle \exp\left(\sum_{i<k} [-ia_0, \phi_0] \alpha_i \alpha_k \ln(z_i \bar{z}_i) \right) \\ &= \left\langle : \prod_j e^{i\alpha_j \Phi(z_j,\bar{z}_j)} : \right\rangle \prod_{i<k} |z_i|^{2\alpha_i\alpha_k}.\end{aligned} \tag{388}$$

Unlike with the non-zero-mode operators, the fully normal-ordered part actually does something. The vacua are parametrized by the value of the zero mode. So we work with the coherent vacua $|\beta\rangle$ such that $a_0|\beta\rangle = \beta|\beta\rangle$. Then we see that

$$a_0 e^{-i\alpha\phi_0}|\beta\rangle = i\frac{\delta}{\delta\phi_0} e^{-i\alpha\phi_0}|\beta\rangle = e^{-i\alpha\phi_0}(\alpha + a_0)|\beta\rangle = (\alpha+\beta)e^{-i\alpha\phi_0}|\beta\rangle, \tag{389}$$

so that $e^{-i\alpha\phi_0}|\beta\rangle = |\beta+\alpha\rangle$ shifts us between different vacua (think of electric flux operators). In the fully normal-ordered piece above, all the ϕ_0 operators stand to the left of the a_0 operators, and so we can let them act directly on the left vacuum bra. If our vacuum state is $|\beta\rangle$, then

$$\left\langle : \prod_j e^{i\alpha_j \Phi(z_j,\bar{z}_j)} : \right\rangle = e^{i\sum \alpha_i \beta} \langle \beta|\gamma\rangle = \delta_{\sum_i \alpha_i, 0}, \qquad \gamma = \beta + \sum_j \alpha_j, \tag{390}$$

and so the zero mode implements the charge-neutrality condition for us.

Putting everything together, we have

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \delta_{\sum_i \alpha_i, 0} \prod_{j<k} |z_j - z_k|^{2\alpha_j\alpha_k}. \tag{391}$$

We can use this result to figure out the OPE of two vertex operators. We have

$$: e^{i\alpha\phi(z)} :: e^{i\beta\phi(w)} :=: e^{i(\alpha\phi(z)+\beta\phi(w))} : |z-w|^{2\alpha\beta}. \tag{392}$$

Note that the RHS is non-zero even when $\alpha + \beta \neq 0$. It needs to be non-zero since we need to be able to take OPEs of vertex operators that don't satisfy charge neutrality and produce a non-zero result, since e.g. the product $\mathcal{V}_1\mathcal{V}_1\mathcal{V}_{-2}$ has a non-zero vev but can be evaluated by first performing the OPE on the first two factors. Charge neutrality enters when we take the vev of the above equation, as the vev of the fully-normal-ordered part on the RHS actually vanishes when charge neutrality is not satisfied:

$$\langle : e^{i(\alpha+\beta)\phi} : \rangle = \langle e^{i\Phi(\alpha+\beta)} \rangle \langle : e^{i(\alpha+\beta)\sum_{n\neq 0} \phi_n e^{2\pi i n x/L}} : \rangle = \langle e^{i\Phi(\alpha+\beta)} \rangle = \delta_{\alpha+\beta,0}, \tag{393}$$

where we have used that the normal-ordered exponential of the non-zero modes of ϕ is equal to 1, and the fact that Φ and $\phi_{n\neq 0}$ commute.

Now, what happens when we make the boson compact? My favorite way of writing the action is to take

$$S = \frac{R^2}{4\pi} \int dz d\bar{z}\, \partial\phi\bar{\partial}\phi, \qquad \phi \sim \phi + 2\pi. \tag{394}$$

We could also change the coefficient in front of the action at the expense of changing the compactification condition on ϕ. In either case, the only real change in the above calculation is that the COM momentum π_0 takes on discrete values. Now the only change coming from compactification is in the zero mode, but changing the COM momenta to be discrete doesn't change how the zero mode implements charge-neutrality in the correlator. Therefore the correlator of the vertex operators has the same value as it does for the non-compact boson. The only aspect where the compactification radius of the boson enters is in selecting which vertex operators are allowed. In the conventions above the allowed vertex operators are $:e^{in\phi}:$ with $n \in \mathbb{Z}$; if instead $\phi \sim \phi + 2\pi\gamma$ then they are $:e^{i\lambda\phi}:$ with $\lambda \in \gamma^{-1}\mathbb{Z}$. Either way, the correlator of the vertex operators is computed using the same formula as in the non-compact case.

Vertex correlators II

Today we have another quick and easy exercise from the big yellow book, chapter 9, which is the functional way of getting to the result of yesterday's diary entry.

Consider the real-space propagator

$$K(x,y) = -\ln\left(m^2[(x-y)^2 + a^2]\right), \tag{395}$$

which is the free boson correlator regulated by a mass m (long-distance cutoff) and a lattice spacing a (short-distance cutoff).

We will compute the vertex correlation function with functional methods and show that we get the result we obtained yesterday at the conformal point where $m, a \to 0$.

We use

$$\left\langle \prod_j \mathcal{V}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = Z[J] = Z[0] \exp\left(-\frac{1}{2}\int_{x,y} J(x)K(x,y)J^\dagger(y)\right), \tag{396}$$

where the relevant current for us is

$$J(x) = i\sum_j \alpha_j \delta(x - x_j). \tag{397}$$

Thus

$$Z[J] = \exp\left(\frac{1}{2}\sum_{j,k}\alpha_j\alpha_k\left(\ln(ma)^2 + \ln\left[\frac{|z_j - z_k|^2}{a^2} + 1\right]\right)\right). \tag{398}$$

Since we are interested in sending $a^2 \to 0$, we will take $|z_j - z_k|^2 \gg a^2$ for all $j \neq k$, allowing us to get rid of the $+1$ in the ln unless $j = k$, in which case the ln vanishes. So then

$$\begin{aligned} Z[J] &= (ma)^{\left(\sum_j \alpha_j\right)^2} \prod_{i<k} \left(\frac{|z_i - z_k|^2}{a^2} \right)^{\alpha_i \alpha_k} \\ &= m^{\left(\sum_j \alpha_j\right)^2} a^{\sum_l \alpha_l^2} \prod_{i<k} |z_i - z_k|^{2\alpha_i \alpha_k}. \end{aligned} \tag{399}$$

We see that when we take $m \to 0$, $Z[J]$ vanishes unless $\sum_j \alpha_j = 0$. Even if we have charge neutrality, we still have the prefactor of $a^{\sum_l \alpha_l^2}$. This isn't a problem though, since it factors as $\prod_l a^{\alpha_l^2}$, so that if we renormalize the vertex operators with the short-distance cutoff by

$$\widetilde{\mathcal{V}}_\alpha(z, \bar{z}) \equiv a^{-\alpha^2} \mathcal{V}_\alpha(z, \bar{z}), \tag{400}$$

then we have

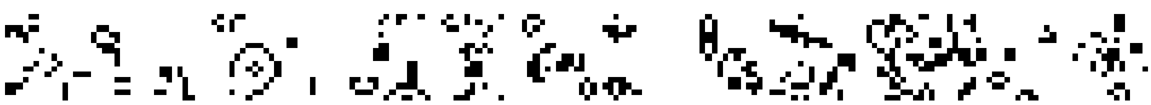
$$\left\langle \prod_j \widetilde{\mathcal{V}}_{\alpha_j}(z_j, \bar{z}_j) \right\rangle = \delta_{\sum_i \alpha_i, 0} \prod_{j<k} |z_j - z_k|^{2\alpha_j \alpha_k}, \tag{401}$$

which is the same correlator that we found yesterday. In fact, our need to renormalize the vertex operators in the above way is not surprising, and is equivalent to the statement that the vertex operators have anomalous dimension α^2. From the 2-point function $\langle \widetilde{\mathcal{V}}_\alpha(z_1, \bar{z}_1) \widetilde{\mathcal{V}}^\dagger_{\underline{\alpha}}(z_2, \bar{z}_2) \rangle \sim |z_1 - z_2|^{-2\Delta_\alpha}$ we see that the scaling dimension of the vertex operator $\widetilde{\mathcal{V}}_\alpha$ is $\Delta_\alpha = \alpha^2$. Thus the renormalization defined above lets us go between operators with dimensionless two-point functions ($\mathcal{V}$'s) to those with two point functions whose dimensions give the scaling dimension.

Before closing, I think it's worth elaborating a bit on the usage of the phrase "charge neutrality". The free boson has two $U(1)$ symmetries coming from the two conserved currents $J \sim i\partial\phi = i\partial\phi_+$, $\bar{J} \sim i\bar{\partial}\phi = i\bar{\partial}\phi_-$, which are conserved by virtue of the eom $\Box\phi = 0$ that allows us to split $\phi = \phi_+(z) + \phi_-(\bar{z})$. While we have focused on non-chiral vertex operators above, we can also restrict our attention to (anti)holomorphic vertex operators. Looking at e.g. the holomorphic sector, we can use the correlator $\langle \partial\phi(z) e^{i\alpha\phi_+(w)} \rangle \sim \alpha \frac{1}{z-w}$ to write

$$\left\langle J(z) \prod_j e^{i\alpha_i \phi_+(w_j)} \right\rangle \sim \sum_j \frac{\alpha_j}{z - w_j} \left\langle \prod_k e^{i\alpha_k \phi_+(w_k)} \right\rangle, \tag{402}$$

which when acted on both sides by $\oint_C dz$ for some contour C gives the usual Ward identity. Therefore the statement that the correlator on the RHS vanishes unless $\sum_j \alpha_j = 0$ is equivalent to the statement that all non-zero vertex correlators must be neutral under the two $U(1)$ symmetries.



Vacuum energy and boundary conditions

Today is another quickie: deriving a statement made in Ginzbarg's lectures on CFT about vacuum energies.

Consider a free Dirac fermion, with boundary conditions on the cylinder such that

$$\psi(\sigma) = e^{2\pi i\gamma}\psi(\sigma + 2\pi), \qquad \gamma \in \mathbb{R}/\mathbb{Z}, \tag{403}$$

where σ is the spatial coordinate of the (radius 1) cylinder. We will show that the vacuum energy has a γ dependence given by

$$E_0 = \frac{1}{12} - \frac{1}{2}\gamma(1-\gamma), \tag{404}$$

which is consistent with $\gamma \in \mathbb{R}/\mathbb{Z}$ and $\gamma \sim 1-\gamma$ (the vacuum energy only cares about the absolute value of the amount of twisting modulo 2π).

There are a few ways to do this problem. The first way is to use ζ function regularization to normal-order the L_0 operators on the cylinder. First, we note that since our complex fermion has the OPEs

$$\psi^\dagger(z)\psi^\dagger(w) \sim \psi(z)\psi(w) \sim 0, \qquad \psi^\dagger(z)\psi(w) \sim \frac{1}{z-w}, \tag{405}$$

if we write $\psi(z) = \frac{1}{\sqrt{2}}(\lambda(z) + i\eta(z))$ then we must have

$$\lambda(z)\lambda(w) \sim \eta(z)\eta(w) \sim \frac{1}{z-w}, \qquad \lambda(z)\eta(w) \sim 0, \tag{406}$$

i.e. the OPE forces the two Majoranas making up the Dirac fermion to decouple. Thus the energy momentum tensor and importantly for us the vacuum energy contribution to the dilitation generator can be obtained just by taking the answer for a single Majorana fermion and multiplying by 2. Thus on the cylinder we have

$$L_0 = \frac{1}{2}\sum_n n(:\lambda_{-n}\lambda_n: + :\eta_{-n}\eta_n:) = \sum_{n>0}(\lambda_{-n}\lambda_n + \eta_{-n}\eta_n) - \sum_{n>0} n, \quad n \in \mathbb{Z}+\gamma. \tag{407}$$

The last constant part is what shifts the vacuum energy density. We evaluate it with ζ function regularization. Letting

$$\zeta(q,r) = \sum_{n=0}^{\infty}(n+r)^q, \tag{408}$$

we see that we need to evaluate $\zeta(-1,\gamma)$. Luckily this is easily looked up:

$$\zeta(-1,\gamma) = -\frac{1}{2}(\gamma^2 - \gamma + 1/6), \tag{409}$$

and so the vacuum energy is evidently

$$E_0 = -\zeta(-1, \gamma) = \frac{1}{12} - \frac{1}{2}\gamma(1-\gamma), \tag{410}$$

as predicted.

The second, more rigorous way that doesn't use the ζ function is to find $\langle T \rangle$ directly using the mode expansion on the Dirac fermion. We decompose the Dirac fermion in a mode expansion as

$$\psi(w) = \sum_{k \in \mathbb{Z}^{>0}+\gamma} \left(\alpha_k e^{-kw} + \beta_k^\dagger e^{kw} \right), \tag{411}$$

where $w = \tau - ix$. This mode expansion comes from doing the expansion for the fermion on the circle and then getting the time dependence by commuting with the Dirac Hamiltonian. Notice that we are only summing over *positive* k in the above: here $\alpha_k = c(-k_F - k)$ destroys a left-moving particle while $\beta_k^\dagger = c(-k_F + k)$ creates a left-moving hole. The corresponding anitholomoprhic guy is the same thing but with right-moving momenta and coordinates:

$$\bar{\psi}(w) = \sum_{k \in \mathbb{Z}^{<0}+\gamma} \left(\alpha_k e^{k\bar{w}} + \beta_k^\dagger e^{-k\bar{w}} \right). \tag{412}$$

The signs in the exponents come from requiring the time dependence $e^{-\tau|k|}$, so that when $k < 0$ we need to invert the sign of $\bar{w} = \tau + ix$ in the exponent.

The modes satisfy the algebra $\{\alpha_k^\dagger, \alpha_l\} = \delta_{k,l}$; same for the β modes. Note that unlike the real fermions, there is no $\psi_0^2 = 1/2$ mode to worry about since α and $\alpha^\dagger$ are distinct. When we map these guys into the plane, we take $z = e^w$ and multiply by a factor of $(dz/dw)^{-h} = z^{-1/2}$ since the fermions have conformal dimension $h = 1/2$ to get

$$\psi(z) = \sum_{k \in \mathbb{Z}^{>0}+\gamma} \left(\alpha_k z^{-k-1/2} + \beta_k^\dagger z^{k-1/2} \right). \tag{413}$$

We can now calculate the expectation value $\langle \psi^\dagger(z)\psi(w) \rangle$:

$$\begin{aligned} \langle \psi^\dagger(z)\psi(w) \rangle &= \sum_{k,l \in \mathbb{Z}^{>0}+\gamma} z^{-k-1/2} w^{l-1/2} \langle \beta_k \beta_l^\dagger \rangle = \sum_{k \in \mathbb{Z}^{>0}+\gamma} z^{-k-1/2} w^{k-1/2} \\ &= \frac{1}{\sqrt{wz}} w^\gamma z^{-\gamma} \frac{z}{z-w}, \end{aligned} \tag{414}$$

since when we complex conjugate z^x we get z^{-x} for $x \in \mathbb{R}$ since z is actually the exponential of a purely imaginary number (and so $z^* \neq \bar{z}$ with these conventions unfortunately, only when we pretend that τ is real). Also, here w is now a coordinate on the plane, and is *not* the earlier w, which was the cylinder coordinate. Sorry! Also note that regardless of γ, unlike with $\mathbb{R}$ fermions we don't have to worry about treating a zero mode separately.

Anyway, we have

$$\partial_z \langle \psi^\dagger(z)\psi(w) \rangle = (-\gamma + 1/2) \frac{w^{\gamma-1/2} z^{-\gamma-1/2}}{z-w} - \frac{w^{\gamma-1/2} z^{-\gamma+1/2}}{(z-w)^2}, \tag{415}$$

and similarly for the derivative wrt w. Now the holomorphic stress tensor has the expectation value

$$T(w) = \frac{1}{2}\lim_{z\to w}\left\langle \partial_z\psi^\dagger(z)\psi(w) - \psi^\dagger(z)\partial_w\psi(w)\right\rangle + \frac{1}{\epsilon^2}, \tag{416}$$

where we've subtracted $2\cdot\frac{1}{2}\partial_z\frac{1}{z-w}$ evaluated at $z = w+\epsilon$ in line with the usual normal ordering prescription. Putting in our expressions for the derivatives,

$$\langle T(w)\rangle = \frac{1}{2}\lim_{z\to w}\left(\frac{1/2-\gamma}{z-w}(z^{-1/2-\gamma}w^{\gamma-1/2} + z^{1/2-\gamma}w^{\gamma-3/2}) - 2\frac{z^{1/2-\gamma}w^{\gamma-1/2}}{(z-w)^2}\right) + \frac{1}{\epsilon^2}. \tag{417}$$

Evaluating the term in the parenthesis for $z = w+\epsilon$ we find that it is equal to

$$\frac{1}{2}\lim_{z\to w}(\dots) = -\frac{1}{\epsilon^2} + \frac{1-4\gamma-4\gamma^2}{8w^2} + O(\epsilon). \tag{418}$$

The singular part cancels the $1/\epsilon^2$ introduced by the normal ordering, and since we are dealing with free fields there are no more singular parts leftover to be cancelled, leaving only the $1/w^2$ piece. Thus

$$\langle T(z)\rangle = \frac{1-4\gamma-4\gamma^2}{8z^2}. \tag{419}$$

Sanity check: when $\gamma = 1/2$ so that we have the normal anti-periodic boundary conditions for the fermions on the cylinder, we have $\langle T(z)\rangle = 0$ ✓.

Now let's go over to the cylinder. The holomorphic (left-moving) part of the Hamiltonian is found by (now w is the cylinder coordinate again—jeez this is awful notation)

$$H_L = \frac{1}{2\pi i}\int dw\, T(w) = \frac{1}{2\pi i}\int dz\, z^{-1}T(w) = \frac{1}{2\pi i}\int dz\, z^{-1}(z^2T(z) - 1/24), \tag{420}$$

where we used the transformation rule for T we derived earlier (with coordinates $z = e^w$) and put in the central charge $c = 1$. We see that the $n = 0$ mode of the Laurent expansion for T is selected out, which picks up the extra piece contributing to $\langle T(z)\rangle$ that we found above. So

$$E_0 = \langle L_0\rangle + \frac{1-4\gamma-4\gamma^2}{8} - \frac{1}{24} = \frac{1}{12} - \frac{1}{2}\gamma(\gamma-1), \tag{421}$$

since $\langle L_0\rangle = 0$ as L_0 is a sum of normal-ordered oscillators. We see that this approach gives exactly the same vacuum energy that we derived using ζ function regularization!

Fermion partition functions on the torus: the functional approach

Today is a check on our understanding of fermion path integrals on the torus: the goal is to confirm some statements in Ginzbarg's CFT lectures. Here's the problem statement:

For a free Majorana fermion on a torus with a given spin structure, find $Z(q)$, where $q = e^{2\pi i\tau}$ and τ is the modular parameter. Do so using functional methods and ζ function regularization methods (the preferred regulator since it respects modular invariance) rather than using the operator approach. You only need to reproduce the q dependence; don't worry about constants and stuff.

We start from

$$Z_{XY}(q) = \mathrm{Pf}_{XY}(\partial)\mathrm{Pf}_{XY}(\bar\partial) = \sqrt{\det{}_{XY}(\partial\bar\partial)}, \tag{422}$$

where $X, Y \in \{NS, R\}$ are spin structure labels.The (un-normalized) eigenfunctions of $\partial\bar\partial$ are

$$\psi_{n,m}(\alpha,\beta) = \exp\left(\frac{2\pi i}{2i\mathrm{Im}(\tau)}\left[(n+\alpha)(z-\bar z) + (m+\beta)(\tau\bar z - \bar\tau z)\right]\right), \qquad \alpha,\beta \in \{0, 1/2\}, \tag{423}$$

where we have assumed that the spacelike edge of the torus stretches from 0 to 1 in the complex plane. One checks that

$$\begin{aligned} (z\mapsto z+1):\ \psi_{n,m}(\alpha,\beta) &\mapsto (-1)^{2\beta}\psi_{n,m}(\alpha,\beta) \\ (z\mapsto z+\tau):\ \psi_{n,m}(\alpha,\beta) &\mapsto (-1)^{2\alpha}\psi_{n,m}(\alpha,\beta), \end{aligned} \tag{424}$$

so that α sets the timelike boundary conditions and β sets the spacelike boundary conditions.

From the eigenvalues we get that

$$\det{}_{XY}(\partial\bar\partial) = \prod_{n,m}\frac{\pi^2}{\mathrm{Im}(\tau)^2}|n+\alpha-(m+\beta)\tau|^2. \tag{425}$$

We will ignore the $\pi^2/\mathrm{Im}(\tau)^2$ part, which by using ζ function regularization ends up contributing something with τ dependence $\sqrt{\mathrm{Im}(\tau)}$ to the partition function. This is important for maintaining modular invariance—we won't worry about it now, but will just remember that we need to re-instate this bit if we want to get something modular invariant.

First consider the case where $X, Y = R, R$, i.e. $\alpha = \beta = 0$. Then from the zero mode in the partition function, we get $Z_{RR}(q) = 0$ as expected. What about

$X, Y = NS, R$, i.e. $\alpha, \beta = 1/2, 0$? This is periodic in space and anti-periodic in time. We have

$$\det_{NS,R}(\partial\bar{\partial}) \propto \prod_{n\in\mathbb{Z}}(n+1/2)^2 \prod_{n,m\in\mathbb{Z},m\neq 0} |n+1/2-m\tau|^2. \tag{426}$$

Now we need

$$\prod_{n\in\mathbb{Z}}(n+x) = e^{i\pi x} - e^{-i\pi x}, \tag{427}$$

so that the first product is just a constant, and so after some algebra (combining the products for $m > 0$ and $m < 0$),

$$\det_{NS,R}(\partial\bar{\partial}) \propto \prod_{m\in\mathbb{Z}^{>0}} |q^{-m}(1+q^m)^2|^2. \tag{428}$$

Now we use

$$\prod_{m>0} q^{-m} = e^{-\zeta(-1)\ln q} = q^{1/12}, \tag{429}$$

so that, taking the square root to get the partition function,

$$Z_{NS,R}(q) \propto (q\bar{q})^{1/24} \prod_{m\in\mathbb{Z}^{\geq 0}} |1+q^m|^2. \tag{430}$$

Now for $X, Y = R, NS$ i.e $\alpha, \beta = 0, 1/2$. Then

$$\det_{R,NS}(\partial\bar{\partial}) \propto \prod_{n\in\mathbb{Z}} |n-\tau/2|^2 \prod_{n,m\in\mathbb{Z},m\neq 0} |n-(m+1/2)\tau|^2. \tag{431}$$

The first product gives us

$$\prod_{n\in\mathbb{Z}} |n-\tau/2|^2 = |q^{-1/4}(1-q^{1/2})|^2. \tag{432}$$

The second product is

$$\prod_{n,m\in\mathbb{Z},m\neq 0} |n-(m+1/2)\tau|^2 = \prod_{m\in\mathbb{Z}^{>0}} |q^{-(m+1/2)/2}q^{-(m-1/2)/2}(1-q^{m+1/2})(1-q^{m-1/2})|^2. \tag{433}$$

We can change the $(1-q^{m-1/2})$ to a $(1-q^{m+1/2})$ by also including a factor of $(1-q^{1/2})$, which combines with the factor in the previous product to produce an $m = 0$ term in the full product. Taking everything together and taking the square root, we get

$$Z_{R,NS}(\partial\bar{\partial}) \propto (q\bar{q})^{-1/48} \prod_{m\in\mathbb{Z}^{\geq 0}} |1-q^{m+1/2}|^2, \tag{434}$$

which is exactly what we expect: the periodic boundary conditions in time give us a supertrace by sending $q \to -q$, while we have the half-odd-integer momenta needed for antiperiodic boundary conditions in space.

Finally for $\alpha = \beta = 1/2$, the $NSNS$ spin structure. We split up the product into $m = 0$ and $m \neq 0$ parts as before. The $m = 0$ part is

$$\prod_{n\in\mathbb{Z}} |n + 1/2 - \tau/2|^2 = |q^{-1/4}(1 + q^{1/2})|^2. \tag{435}$$

The $m \neq 0$ part is dealt with as before: the fact that $\alpha = 1/2$ means that we get a trace instead of a supertrace, and the same sort of algebra leads to

$$Z_{NS,NS}(\partial\bar{\partial}) \propto (q\bar{q})^{-1/48} \prod_{m\in\mathbb{Z}^{\geq 0}} |1 + q^{m+1/2}|^2. \tag{436}$$

Orbifolding basics

Today is an elaboration on part of a problem in the big yellow book, chapter 10. We will show that on the torus,

$$Z^2_{\text{Ising}} = Z_{\text{orb}}(R = 1), \tag{437}$$

where Z_{Ising} is the partition function of a single majorana fermion and $Z_{\text{orb}}(R = 1)$ is the free boson at compactification radius $R = 1$ orbifolded under the $\mathbb{Z}_2$ $X \mapsto -X$ symmetry.

Let's first write down Z_{Ising}, which is easy using our knowledge from the previous few diary entries. In addition to the trace of the $q^{L_0}\bar{q}^{\bar{L}_0}$ part, we need to know the vacuum energy contribution to partition function (i.e. the part that is $(q\bar{q})^{-c/24}$ if boundary conditions are not an issue). We find this contribution with the regularization

$$-\frac{1}{2}\sum_{n\in\mathbb{Z}+\alpha} = \frac{1}{24} - \frac{\alpha}{4}(1 - \alpha), \tag{438}$$

which we essentially derived a few days ago (it comes from doing the normal-ordering in the oscillator sum in the Hamiltonian. This is done *on the torus*, not on the plane, and so the modding is not shifted by 1/2). If the boundary conditions in space are anitperiodic then the fermion is modded in $\mathbb{Z} + 1/2$ on the torus, and hence we can take $\alpha = 1/2$ for NS spatial boundary conditions, giving a contribution of $-1/48$. If the boundary conditions are periodic, then we just set $\alpha = 0$ so that we get a $+1/24$

contribution. So then since each spin structure factors as holomorphic and antiholomorphic parts, we can just write down (remember that the the RR spin structure gives zero)

$$Z_{\text{Ising}} = \frac{1}{2}\left(\left|\frac{\theta_2}{\eta(q)}\right| + \left|\frac{\theta_3}{\eta(q)}\right| + \left|\frac{\theta_4}{\eta(q)}\right|\right), \tag{439}$$

where

$$\begin{aligned}\frac{\theta_2}{\eta(q)} &= \frac{1}{\sqrt{2}} q^{1/24} \prod_{n\in\mathbb{Z}^{\geq 0}} (1+q^n),\\ \frac{\theta_3}{\eta(q)} &= q^{-1/48} \prod_{n\in\mathbb{Z}^{\geq 0}} (1+q^{n+1/2}),\\ \frac{\theta_4}{\eta(q)} &= q^{-1/48} \prod_{n\in\mathbb{Z}^{\geq 0}} (1-q^{n+1/2}),\end{aligned} \tag{440}$$

and the η function is

$$\eta(q) = q^{1/24} \prod_{n\in\mathbb{Z}^{>0}} (1-q^n). \tag{441}$$

The $1/\sqrt{2}$ factor is the quantum dimension of the σ primary field and is inserted to have the usual expression for the modular S matrix.

Now for the orbifolded boson under the $\mathbb{Z}_2$ symmetry taking $X \to -X$. Let's recall how we arrive at the partition function. We want to "gauge" the $\mathbb{Z}_2$ symmetry (or rather, we want to do a Fourier transformation) by projecting the Hilbert space onto states which are singlets under the $\mathbb{Z}_2$, which we do with the operator $(\mathbf{1} + (-1)^X)/2$, where $(-1)^X$ is our dumb way of writing the operator which does the symmetry action on X (it inserts a $\mathbb{Z}_2$ symmetry defect that wraps around the spatial cycle of the torus and changes the temporal boundary conditions). Inserting this operator (the tube algebra Hamiltonian) ruins modular invariance since it's asymmetrical under S, and so we have also have to sum over spatial boundary conditions by inserting defects that wrap the temporal cycle. We end up summing over all (four) topologically distinct way to place $\mathbb{Z}_2$ symmetry defect lines on the torus (for us, they are added with no relative phases between them).

The torus with no symmetry defects just gives a contribution of $Z(R)/2$, the regular un-orbifolded version. The torus with a symmetry defect inserted which wraps the spatial cycle gives the holomorphic contribution

$$\frac{q^{-1/24}}{2} \text{Tr}_P[(-1)^X q^{L_0}]. \tag{442}$$

Here the trace is done with periodic boundary conditions in space, and so we have to take into account zero modes. The basis states for $\mathcal{H}_0$ (the zero-mode part of the Hilbert space) that diagonalize the action of $(-1)^X$ are $|k,w\rangle \pm |-k,-w\rangle$ (k is momentum, w is winding) for $k,w \in \mathbb{Z}$, since $(-1)^X$ acts as -1 on the the momenta and winding numbers. However, recall that the k,w dependence of the spectrum of L_0 comes from the term $\frac{1}{2}p_L^2$, where $p_L = (k/R + wR/2)$ (for $\bar{L}_0$, send $w \mapsto -w$). Thus the spectrum of L_0 is unchanged under the action of $(-1)^X$, and so all the terms in the trace above with the exception of $|0,0\rangle$ die in pairs. Thus we don't even have to worry about the zero modes, and so taking the trace over the nonzero oscillator modes in the usual way, we get the holomorphic part

$$\frac{q^{-1/24}}{2} \prod_{n \in \mathbb{Z}^{>0}} \frac{1}{1+q^n}. \tag{443}$$

Note that the + sign in the denominator comes from the action of $(-1)^X$ on the left bra in the trace.

The torus with a symmetry defect inserted along the temporal cycle just changes the modding of the boson modes to be in $\mathbb{Z}+1/2$, since it changes the spatial boundary conditions to be antiperiodic. Thus there are no zero modes to worry about, and the only subtlety is the change in the vacuum energy contribution due to the altered boundary conditions. The relevant normal-ordering result for bosons is

$$\frac{1}{2} \sum_{n \in \mathbb{Z}+\alpha} n = \frac{1}{2}\left(-\frac{1}{12} + \frac{\alpha}{2}(1-\alpha)\right), \tag{444}$$

which gives us the $-1/24$ for periodic boundary conditions that we used before, and which gives $+1/48$ for antiperiodic boundary conditions (just like for the fermions, but with opposite signs). So then we get the holomorphic term

$$\frac{2q^{1/48}}{2} \prod_{n \in \mathbb{Z}^{>0}} \frac{1}{1-q^n}. \tag{445}$$

Here, the factor of 2 in the numerator comes from the fact that we have two vacua which have identical L_0 eigenvalues (± 1 eigenvalues under $(-1)^X$). The 2 in the denominator comes from the $1/2$ in the projector onto $\mathbb{Z}_2$-invariant states.

Finally, the torus with defects wrapped around both cycles: this is just like the above, expect with $+q^n$ in the denominator, with the sign coming from the $(-1)^X$ acting on the oscillator modes in the trace. So we get the term

$$\frac{2q^{1/48}}{2} \prod_{n \in \mathbb{Z}^{>0}} \frac{1}{1+q^n}. \tag{446}$$

Adding everything together, remembering to square everything because of the antiholomorphic parts, and using the definitions of the θ functions and the η function, we get (note: I think the corresponding formula in the big yellow book has some incorrect factors of 2?)

$$Z_{orb}(R) = \frac{1}{2}\left(Z(R) + 2\left|\frac{\eta(q)}{\theta_2}\right| + 2\left|\frac{\eta(q)}{\theta_3}\right| + 2\left|\frac{\eta(q)}{\theta_4}\right|\right). \tag{447}$$

Note that only the first of the four terms actually depends on the compactification radius R, since it is the only term in which the presence of the zero modes contributed something nonzero to the partition function.

Specializing to $R = 1$, and recalling our result from a few days ago about the duality between Dirac fermions and the $R = 1$ boson, we have

$$Z(R=1) = Z_{\text{Dirac}} = \frac{1}{2}\left(\left|\frac{\theta_2}{\eta}\right|^2 + \left|\frac{\theta_3}{\eta}\right|^2 + \left|\frac{\theta_4}{\eta}\right|^2\right). \tag{448}$$

Now we need to make use of

$$\eta(q)^3 = \frac{1}{2}\theta_2\theta_3\theta_4, \tag{449}$$

which we use to substitute $\eta = \theta_2\theta_3\theta_4/(2\eta^2)$ in for the η's in $Z_{orb}(R=1)$. We get

$$Z_{orb}(R=1) = \frac{1}{4}\left(\left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right|\right)^2 = Z_{\text{Ising}}^2, \tag{450}$$

which is what we wanted to show. Thus orbifolding lets us "decouple" the two majorana fermions making up a Dirac fermion from one another and allows them to have separate spin structures.

One comment on orbifolds of orbifolds (for G finite and Abelian). The orbifold of an orbifold is the original theory, essentially because the Fourier transform is an involution (well, up to a sign). To orbifold an orbifold, we sum over all ways of twisting the orbifold, where an orbifold twisted by some function $\beta : G \to U(1)$ means

$$Z_{orb}^{\beta} = \frac{1}{|G|}\sum_{g,h}\beta(h)Z_g^h, \tag{451}$$

where Z_g^h denotes the torus with a g twist in the spatial direction and an h twist in the temporal direction. To get something modular invariant, we then need to write

$$Z_{orb}^{\omega} = \frac{1}{|G|}\sum_{g,h}\omega(g,h)Z_g^h, \tag{452}$$

where $\omega(g,h) = \beta(g)\beta(h)$. To orbifold the orbifold we sum over all such functions $\omega(g,h)$, which is the dual version (in the sense of Rep $\leftrightarrow$ Vec duality) of projecting onto a G-singlet state. Thus the procedure looks like

$$Z \xrightarrow{\text{orbifold}} \frac{1}{|G|}\sum_{g,h} Z_g^h \xrightarrow{\text{orbifold the orbifold}} \frac{1}{|G|^2}\sum_{\omega\in\text{Rep}^2(G)}\sum_{g,h}\omega(g,h)Z_g^h = Z, \tag{453}$$

so that orbifolding squares to the identity (remember that we are assuming G is finite and Abelian so that $\text{Rep}(G) \cong G$). Recall that since we are summing over genuine representations here, the 2-cocycle $\omega(g,h) = \beta(g)\beta(h)$ for $\beta \in \text{Rep}(G)$ is exact. That this works is easy to explicitly verify e.g. for the $\mathbb{Z}_2$ case we've been considering, with the last equality in the above chain holding because the Fourier transform is involutive up to a sign. If $G = \mathbb{R}$ then we can telegraphically illustrate this as

$$Z(x=0) \xrightarrow{\text{orbifold}} \int dx\, Z(x) \xrightarrow{\text{orbifold the orbifold}} \int dx \int dk\, e^{ikx} Z(x) = Z(x=0), \tag{454}$$

showing that orbifolding is an involution. Here the orbifold is $Z(k=0)$ (which is a singlet under translations).

The Cardy formula

This is a basic result that I'd seen a few times but had never derived. We will show that at high energies, modular invariance implies that for a 2D CFT, the density of states satisfies

$$\ln \rho(E) = \sqrt{\frac{2\pi Ec}{3}}, \tag{455}$$

where c is the central charge.

This is derived by using modular invariance on a stretched torus.

We will work on a torus with modular parameter $\tau = i\beta \in i\mathbb{R}$, with the real leg of the torus stretching from 0 to 1 as usual. We will first take (for a reason that will be clear in a sec) the "low temperature" limit of $\beta \gg 1$ so that the torus is very stretched out.

Since $\beta \gg 1$ and $\bar{\tau} = -\tau$, we have $q = \bar{q} = e^{2\pi i\tau} = q^{-2\pi\beta} \to 0$. The partition function is (assuming $c = \bar{c}$)

$$Z(q) = q^{-c/12}\mathrm{Tr}[q^{L_0+\bar{L}_0}]. \tag{456}$$

Since $q \to 0$, we can approximate the partition function by only the terms where the modes are unoccupied, so

$$Z(q) \approx q^{-c/12}. \tag{457}$$

Now we use modular invariance—the modular S transformation inverts τ and adds a minus sign, so that $S(q) = e^{-2\pi i/\tau} = e^{-2\pi/\beta} \to 1$. Since $S(q) \to 1$, we now have a partition function dominated by *high* energy modes. Thus by considering this super-stretched torus, modular invariance means that the high and low energy parts of the partition function are linked together. Since in the S-transformed partition function the weight of higher energy modes in the sum falls off very slowly we can change the sum over states to an integral and accordingly write

$$S[Z(q)] \approx \int dE\, \rho(E)e^{-E/\beta}, \tag{458}$$

since $H = 2\pi(L_0 + \bar{L}_0 - c/12)$.

Now we make the (a priori un-justified?) assumption that we can use a saddle-point approximation. Putting the ρ in the exponential, we see that the saddle-point condition on $E = E(\beta)$ is

$$\left.\frac{\partial \ln \rho(E)}{\partial E}\right|_{E=E_*} = \beta^{-1}. \tag{459}$$

So then if we use this to evaluate the parition function, then by modular invariance,

$$\ln Z(q) = \ln\left(S[Z(q)]\right) \implies \frac{\pi c\beta}{6} = \ln \rho(E_*) - \beta^{-1}E_*. \tag{460}$$

Hitting both sides of the RHS with ∂_β so that we can use our saddle point condition, we get

$$\frac{\pi c}{6} = \partial_\beta E_* \partial_E \ln \rho(E)|_{E=E_*} + \frac{E_*}{\beta^2} - \beta^{-1}\partial_\beta E_* = \frac{E_*}{\beta^2} \implies \beta = \sqrt{\frac{6E_*}{\pi c}}. \tag{461}$$

We can then use the relation that we took the derivative of to get

$$\ln \rho(E_*) = \frac{\pi c \beta}{6} + \frac{E_*}{\beta} = \sqrt{\frac{2\pi E_* c}{3}}. \tag{462}$$

This provides us with a sort of a posteriori justification for the saddle point, since the at the saddle point the effective "Hamiltonian" in the parition function is $E_*/\beta - \ln \rho(E_*) \propto \beta$, which is indeed large and thus perhaps deserving of a saddle-point treatment.

Anyway, the point is that since the saddle point energy E_* is large, we see that at high energy, the (logarithm of the) density of states is controlled by the central charge in the way promised in the introduction.

Alternate route to WZW central charge

This is a slightly elobarated version of yet another problem in the big yellow book, chapter 15. Here's the problem statement:

Consider the WZW theory at level k with lie algebra obtained from the group G. The current algebra is captured by the OPE

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \frac{if^{abc}J^c}{z-w}. \tag{463}$$

Show that the central charge is

$$c = \frac{k \dim G}{k+g}, \tag{464}$$

where g is the dual coxeter number, satisfying

$$g\delta_{cd} = \sum_{a,b} f_{abc}f_{abd} = C_2(G)\delta_{cd}. \tag{465}$$

In the expression for c, $\dim G$ is the dimension of the Lie algebra itself, not the dimension of the particular representation that the currents act on the fields with.

Show that this is the central charge *without* taking the TT OPE. You will need to find how the Virasoro algebra fits into the current algebra and will need to consider the action of various currents and Virasoro generators on a WZW primary field.

First we need to figure out the form of the energy momentum tensor (we are just trying to motivate the Sugawara construction). The action is (at the conformal fixed point)

$$S[g] = \frac{k}{8\pi}\int d^2x\,\mathrm{Tr}[\partial_\mu g^{-1}\partial^\mu g] - \frac{ik}{12\pi}\int_{M_3}\mathrm{Tr}[\omega\wedge\omega\wedge\omega], \tag{466}$$

where as usual $\omega = g^{-1}dg$. When we write this in complex coordinates the second term doesn't change since we've written it with differential forms, but the first term changes to $\frac{k}{4\pi}\int \mathrm{Tr}[\partial g^{-1}\bar\partial g]$ since $\partial = (\partial_0 - i\partial_1)/2$ while $|d^2x| = \frac{1}{2}|d^2z|$. Now for a scalar field we have

$$T^{\mu\nu} = -g_{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\sigma\phi g^{\sigma\nu}. \tag{467}$$

The only thing that changes for us is that we need to take the trace on the RHS. Now in our conventions $T = -2\pi T_{zz} = -\frac{\pi}{2}T^{\bar z\bar z}$ since in complex coordinates $g^{\mu\nu} = 2X$. So then for us,

$$T = -\pi\mathrm{Tr}\left[\frac{\partial\mathcal{L}}{\partial\bar\partial g\phi}\partial g\right]. \tag{468}$$

Now for the WZW Lagrangian, only the first kinetic term will contribute to T since the second term doesn't contain the metric (it's built out of wedge products, and so we use ϵ to contract indices rather than the metric).[38] This means we can write the classical stress tensor (i.e. the one obtained within classical field theory) as

$$T_c = -\frac{k}{2}\mathrm{Tr}[\partial g^{-1}\partial g] = \frac{1}{2k}\mathrm{Tr}[k(\partial g)g^{-1}k(\partial g)g^{-1}] = \frac{1}{2k}\sum_a J_aJ_a, \tag{469}$$

where the current is

$$J = J^at^a = -k(\partial g)g^{-1}. \tag{470}$$

So, this is the stress tensor we expect classically. Quantumly we try an ansatz with the same Sugawara form, except with a different coefficient, so that the ansatz for the actual stress tensor is

$$T = \frac{1}{\gamma}\sum_a (J^aJ^a), \tag{471}$$

where $(\dots)$ denotes normal-ordering, which is the full taking-off-all-singular-parts normal ordering. Since the theory is interacting, the normal-ordering is more complicated than for free fields, and means that unlike for free theories, the coefficient γ will receive quantum corrects (double contractions from interactions) to the classical value of $\gamma_c = 2k$.

Now we will figure out the actual value of γ. First, some preliminary work. Since the J^a's are $(1,0)$ currents, we define the mode expansion as

$$J^a(z) = \sum_n z^{-n-1}J^a_n. \tag{472}$$

[38] Alternatively, since it is linear in time derivatives it doesn't contribute to the Hamiltonian, and then by symmetry we can argue that it doesn't contribute to T.

We get the commutator of the current modes by doing the usual double contour integral with the help of the J^aJ^b OPE provided by the current algebra:

$$\begin{aligned}
[J_n^a, J_m^b] &= \frac{1}{(2\pi i)^2}\oint_0 dw \oint_w dz\, z^n w^m \left(\frac{k\delta_{ab}}{(z-w)^2} + \frac{if^{abc}}{z-w}\sum_l w^{-l-1}J_l^c\right) \\
&= nk\delta_{n+m}\delta_{ab} + \sum_l \frac{1}{2\pi i} if^{abc}\oint dw\, w^{m+n-l-1}J_l^c \\
&= nk\delta_{n+m}\delta_{ab} + if^{abc}J_{m+n}^c.
\end{aligned} \tag{473}$$

Sticking the expressions for the current modes into the stress tensor we find that the Virasoro generators (defined as usual with $T = \sum_n L_n z^{-n-2}$ so that L_n has conformal dimension n) can be written in terms of the current as

$$L_n = \frac{1}{\gamma}\sum_m : J_m^a J_{n-m}^a : \tag{474}$$

Here the normal-ordering means just what it does as if the J modes were oscillator modes of a free field: the operator with the larger mode index gets put to the right.

We also need the commutation relation between the Virasoro generators and the current modes:

$$[L_n, J_m^a] = -mJ_{n+m}^a, \tag{475}$$

which just comes from the fact that J^a, a $(1,0)$ current, is a primary with conformal dimension $h = 1$ (again, this is easy to check from the double-contour method of computing the commutator). Note that the J_0^a modes generate a symmetry corresponding to the Lie algebra G, as they commute with the Hamiltonian.

In the following, we will let $|\phi\rangle$ be a WZW primary state. This means the same thing as it does for primary states of the Virasoro algebra, namely

$$J_0^a|\phi\rangle = -t^a|\phi\rangle, \qquad J_{n>0}^a|\phi\rangle = 0. \tag{476}$$

Here, t^a is the representation matrix assigned to the generator a (it lives in the Lie algebra of G). We won't need to specify the exact irrep in what follows. That J_n^a annihilates $|\phi\rangle$ if $n > 0$ just means that $|\phi\rangle$ is a highest weight (or maybe better, "lowest weight" state for the current algebra. Also, note that $|\phi\rangle$ is also automatically a Virasoro primary (this isn't always the case for current algebras), since the normal-ordering in the expression of L_n in terms of the currents means that if $n > 0$, in the action of L_n on $|\phi\rangle$, the right-most current operator always has a mode number which is greater than zero.

Now on one hand, we have

$$[J_1^a, L_{-1}]|\phi\rangle = J_0^a|\phi\rangle = -t^a|\phi\rangle. \tag{477}$$

On the other hand, we have

$$[J_1^a, L_{-1}]|\phi\rangle = J_1^a L_{-1}|\phi\rangle = \frac{1}{\gamma}J_1^a(2J_{-1}^b J_0^b)|\phi\rangle = -\frac{2}{\gamma}[J_1^a, J_{-1}^b]t^b|\phi\rangle, \tag{478}$$

since only the $m = 0, m = -1$ terms in L_{-1} act nontrivially on $|\phi\rangle$. From the commutations we derived earlier, this turns into

$$-\frac{2}{\gamma}\left(k\delta_{ab}t^b + i\frac{1}{2}f_{abc}[t^c, t^b]\right)|\phi\rangle. \tag{479}$$

Using the definition of the dual coxeter number g, this becomes

$$[J_1^a, L_{-1}]|\phi\rangle = -\frac{2}{\gamma}(k+g)t^b|\phi\rangle. \tag{480}$$

Reconciling these two ways of writing the action of the commutator lets us conclude that

$$\gamma = 2(k+g), \tag{481}$$

so the correct quantum stress tensor is

$$T = \frac{1}{2(k+g)}\sum_a (J^a J^a). \tag{482}$$

Now we can get the central charge. As one does when computing unitarity constraints, we will compute the norm of the state $L_{-2}|\phi\rangle$. We will do this in two different ways: using the Virasoro algebra commutation relations and using the expression of L_{-2} in terms of the currents. We choose L_{-2} since it is the smallest weight Virasoro generator that lets us access the central charge: the central charge appears in the Virasoro algebra relations together with $\delta_{n+m}(n^3 - n)$, which gives us zero for $n = 0, 1$ but not $n = 2$.

Using the Virasoro commutation relations, we have

$$||L_{-2}|\phi\rangle||^2 = \langle\phi|[L_2, L_{-2}]|\phi\rangle = 4\langle\phi|L_0|\phi\rangle + \frac{c}{2}, \tag{483}$$

so that

$$\frac{c}{2} = -\frac{4}{\gamma}\langle\phi|t^a t^a|\phi\rangle + \langle\phi|[L_2, L_{-2}]|\phi\rangle. \tag{484}$$

Now we need to evaluate $\langle\phi|[L_2, L_{-2}]|\phi\rangle$ using the current modes, which is actually kind of gross. Keeping the terms in the mode expansion that survive, (namely $m = -2, -1, 0$) we have (repeated indices are summed)

$$\langle\phi|[L_2, L_{-2}]|\phi\rangle = \frac{1}{\gamma^2}\langle\phi|(-2t^a J_2^a + J_1^a J_1^a)(-2J_{-2}^b t^b + J_{-1}^b J_{-1}^b)|\phi\rangle. \tag{485}$$

We now must painstakingly commute the terms in the left group through until they act on the ket on the right. After doing the first series of commutations, we have

$$\begin{aligned}\langle\phi|[L_2, L_{-2}]|\phi\rangle = \gamma^{-2}\langle\phi|\Big[&4t^a(if^{abc}J_0^c + 2k\delta_{ab})t^b - 2t^a if^{abc}J_1^c J_{-1}^b - 2J_1^a if^{abc}J_{-1}^c t^b \\ &+ if^{abc}if^{acd}J_1^d J_{-1}^b - it^c f^{abc}J_1^a J_{-1}^b + k\delta_{ab}J_1^a J_{-1}^b + (k\delta_{ab} + if^{abc}J_0^c)(k\delta_{ab} + if^{abd}J_0^d)\Big]|\phi\rangle.\end{aligned} \tag{486}$$

Now we let the J_0^a's act on the primary fields and replace the remaining $J_1^a J_{-1}^b$'s with $if^{abc}J_0^c + k\delta_{ab}$. A few of the resulting terms die by the antisymmetry of the structure constants. A few comtain three t^a generators, but they always appear with some f^{abc}'s so we can antisymmetize them into two t^a's plus an extra f^{abc}. Doing this and again using the definition of the dual coexter number, we find

$$\langle\phi|[L_2, L_{-2}]|\phi\rangle = \frac{1}{\gamma^2}\left[\langle\phi|t^a t^a|\phi\rangle(8k+8g) + (2kg+2k^2)\dim G\right], \tag{487}$$

with $\dim G = \delta_{aa}$. Now the first bit on the RHS is $\gamma^{-2}\langle\phi|t^a t^a|\phi\rangle(8k+8g) = 4\gamma^{-1}\langle\phi|t^a t^a|\phi\rangle$, which cancels against the $-4\gamma^{-1}\langle\phi|t^a t^a|\phi\rangle$ term in (484). Thus we have found that the central charge is given by

$$\frac{c}{2} = \frac{2kg+2k^2}{\gamma^2}\dim G \implies c = \frac{k\dim G}{k+g}. \tag{488}$$

As examples, for $\mathfrak{su}(n)$ (for $n \geq 2$), the dual coxeter number is $g = n$, so that

$$c_{\widehat{\mathfrak{su}}(n)_k} = \frac{k(n^2-1)}{k+n}. \tag{489}$$

For the familiar case of $n = 2, k = 1$ we have $c = 1$, which agrees with the calculation we did earlier showing how the compact boson at the self-dual radius could be mapped onto the $\widehat{\mathfrak{su}}(2)_1$ CFT. Another interesting case is $n = k = 2$, for which

$$c_{\widehat{\mathfrak{su}}(2)_2} = \frac{3}{2}, \tag{490}$$

which hints at representations in terms of either three Majoranas or in terms of a free Majorana and a free boson (both are possible). Likewise, for $\widehat{\mathfrak{u}}(n)_k$ we have

$$c_{\widehat{\mathfrak{u}}(n)_k} = 1 + c_{\widehat{\mathfrak{su}}(n)_k}, \tag{491}$$

where the $+1$ is from the decoupled Abelian $U(1)$ factor. For example, when $k = 1$ we have

$$c_{\widehat{\mathfrak{u}}(n)_k} = n, \tag{492}$$

which is compatible with a free-fermion realization in terms of n Dirac fermions.

For $\widehat{\mathfrak{so}}(n)$ we have $g = n - 2$ so that

$$c_{\widehat{\mathfrak{so}}(n)_k} = \frac{\frac{k}{2}(n^2-n)}{(n-2)+k}. \tag{493}$$

In particular, for $k = 1$ we have

$$c_{\widehat{\mathfrak{so}}(n)_1} = \frac{n}{2}, \tag{494}$$

which is compatible with $\widehat{\mathfrak{so}}(n)_1$ being realized by n free Majorana fermions. For $k = g = n - 2$, we have

$$c_{\widehat{\mathfrak{so}}(n)_1} = \frac{n^2-n}{4}, \tag{495}$$

compatible with a realization in by $(n^2-n)/2$ free Majoranas.

Free fermion representation of $\widehat{\mathfrak{so}}(N)_g$ current algebra

This is a problem from the big yellow book, chapter 15. Here's the problem statement:

Consider N Majorana fermions transforming in the adjoint representation of $SO(N)$. Show how to build currents with these fermions that satisfy the $\widehat{\mathfrak{so}}(N)_g$ current algebra, where g is the dual coxeter number (see yesterday's diary entry). Compute the central charge. Answer:

$$c = \frac{1}{4}N(N-1), \tag{496}$$

which is precisely what one would expect from $\dim[\mathfrak{so}(N)] = N(N-1)/2$ (the dimension of the adjoint representation) flavors of real free fermions.

The fermions have dimension $1/2$ while WZW currents have dimension $(1,0)$, and so the J^a's we write down will need to be bilinear in the fermions. The natural choice, since we are woking in the adjoint representation, is to construct the currents with the structure constants:

$$J^a(z) = \alpha i f^{abc}\psi_b(z)\psi_c(z) \tag{497}$$

where α is some yet-to-be-determined constant. We can fix it by computing the OPE for the current. Using $\psi_a(z)\psi_b(w) \sim \delta_{ab}(z-w)^{-1}$, this is straightforward: remembering the minus signs when moving fermions around, we have

$$\begin{aligned} J^a(z)J^d(w) &\sim -\alpha^2 f^{abc}f^{def}\left(\frac{\delta_{ec}\delta_{bf}-\delta_{cf}\delta_{be}}{(z-w)^2} + \frac{1}{z-w}[\psi_b\psi_f\delta_{ce} - \psi_b\psi_e\delta_{cf} - \psi_c\psi_f\delta_{be} + \psi_c\psi_e\delta_{bf}]\right) \\ &\sim -\alpha^2\left(-\frac{4g\delta_{ad}}{(z-w)^2} + \frac{2}{z-w}f^{abc}(\psi_b\psi_f f^{dcf} + \psi_c\psi_e f^{deb})\right) \\ &\sim \frac{4g\alpha^2}{(z-w)^2}\delta_{ad} + i\left(\frac{4\alpha^2 i}{z-w}f^{abc}f^{dcf}\psi_b\psi_f\right). \end{aligned} \tag{498}$$

Now the anticommuting property of the fermions lets us use the identity

$$f^{abc}f^{dcf}\psi_b\psi_f = \frac{1}{2}f^{ade}f^{efg}\psi_f\psi_g. \tag{499}$$

To derive this, we use the Bianchi identity in the form

$$f^{ade}f^{efg}\psi_f\psi_g = -(f^{aeg}f^{dfe} + f^{deg}f^{fae})\psi_f\psi_g. \tag{500}$$

Plugging this in, using the antisymmetry of the ψ_a's and relabelling a bunch of variables gives the sought-for identity. Putting (499) in to the J^aJ^d OPE, we see that we get the OPE for an affine $\widehat{\mathfrak{so}}(N)$ algebra provided that $\beta = 1/2$ (because this choice of β fixes the $if^{ade}J^e/(z-w)$ term appearing in the OPE to have a coefficient of unity). So then recapitualating, the properly normalized currents are

$$J^a = \frac{1}{2}if^{abc}\psi_b(z)\psi_c(w), \tag{501}$$

and they have the OPE

$$J^a(z)J^b(w) \sim \frac{g\delta_{ab}}{(z-w)^2} + if^{abc}\frac{J^c(w)}{z-w}. \tag{502}$$

Thus the current algebra is the $\widehat{\mathfrak{so}}(N)_g$ algebra.

The stress tensor is constructed using the Sugawara strategy described in yesterday's diary entry, and we can use the formula derived yesterday to compute the central charge. Since we are at level $k = g$, we have

$$c = \frac{k \dim \mathfrak{so}(N)}{k+g} = \frac{1}{2}\dim \mathfrak{so}(N). \tag{503}$$

Now for $\mathfrak{so}(N)$ we have $\dim \mathfrak{so}(N) = \frac{1}{2}N(N-1)$,[39] meaning that

$$c = \frac{1}{4}N(N-1), \tag{505}$$

which is equal to 1/2 (central charge of each Majorana) times the number of fermion fields. This hints that even though from the Sugawara construction the theory doesn't look free, it actually is. This can be verified by carefully checking the equivalence of the seemingly interacting Sugawara (J^aJ^a) stress tensor and $N(N-1)/2$ copies of $T_\psi = -\frac{1}{2}\psi\partial\psi$. Showing this is straightforward; see the big yellow book chapter 15 for hints.

Sphere partition functions, the central charge, and the Weyl anomaly

Today we will show that the partition function of any 2d CFT on a sphere of radius a obeys the relation

$$\frac{d\ln Z_{S^2}}{d\ln a} = \frac{c}{3}. \tag{506}$$

To derive this, we will need to think about the trace anomaly and the central charge.

[39] Why? Consider the symmetrizer map $S : M \mapsto MM^T - \mathbf{1}$. Now $O(N) = \ker(S)$, while $\dim \mathrm{im}(S) = \sum_{i=1}^N i = \frac{1}{2}N(N+1)$ is the dimension of all symmetric matrices. So then

$$\dim O(N) = \dim GL(N) - \dim \mathrm{im}(S) = N - \frac{1}{2}N(N+1) = \frac{1}{2}N(N-1). \tag{504}$$

Then since $\dim \mathfrak{so}(N) = \dim \mathfrak{o}(N) = \dim O(N)$, $\dim \mathfrak{so}(N) = \frac{1}{2}N(N-1)$.

The metric on the sphere, in coordinates stereographically projected onto the plane, is $g_{\mu\nu} = \mathbf{1}_{\mu\nu} 4a^2/(1+|x|^2)^2$, so that the line element is $ds^2 = \frac{4a^2}{(1+|r|^2)^2}(dx^2+dy^2)$. This means that when we vary the radius of the sphere, the variation of the metric is proportional to the metric itself: for $a \mapsto a + \delta a$ we use $g_{\mu\nu}(x) \propto a^2$ to write

$$g_{\mu\nu} \mapsto g_{\mu\nu} + 2(\delta a)a^{-1}g_{\mu\nu}. \tag{507}$$

So then applying the Ward identity (with no operator insertions), we have

$$\delta \ln Z = -\frac{1}{2}\int d^2x \, \langle T^{\mu\nu}(x)\delta g_{\mu\nu}(x)\rangle = -\int d^2x \, \sqrt{g}\langle T^{\mu\nu}(x)\rangle \delta a a^{-1} g_{\mu\nu}(x), \tag{508}$$

and so

$$\frac{d\ln Z}{d\ln a} = -\int d^2x \, \sqrt{g}\langle \mathrm{Tr}\, T_{\mu\nu}\rangle. \tag{509}$$

Because of the trace anomaly (which is relevant since we're on a sphere), this will be non-zero.

We will calculate $\langle \mathrm{Tr} T_{\mu\nu}\rangle$ by working infinitesimally and finding $\delta\langle T\rangle$, where the variation is a Weyl transformation of the metric. We can get away with doing this because we know what the form of the answer will be. Indeed, since $\langle \mathrm{Tr} T_{\mu\nu}\rangle = 0$ classically and since we are working with a CFT, the only way in which $T_{\mu\nu}$ could fail to be traceless is for there to be some of local anomaly coming from a contact term (the kind of thing which ruins the tracelessness of T in e.g. QED in four dimensions comes from scales generated during the RG flow: since we are working with a genuine CFT this sort of thing cannot happen). Since this involves UV physics and can only depend on the spacetime metric, $\langle \mathrm{Tr} T_{\mu\nu}\rangle$ must be proportional to R (the Ricci scalar), since this is the only metric-dependent, local, mass-dimension-2 scalar function that could fit the bill. We then just have to determine the coefficient β in $\langle \mathrm{Tr} T_{\mu\nu}\rangle = \beta R$, and so finding the variation of the trace of $T_{\mu\nu}$ under a Weyl transformation is good enough for our present goal.

We will choose coordinates where the metric takes the form $g_{\mu\nu} = e^{\phi(x)}\eta_{\mu\nu}$, so that an infinitesimal variation of the metric away from flat space is $\delta g_{\mu\nu}(x) = \delta\phi(x)\eta_{\mu\nu}$. The Ricci scalar can be found after some pretty heinous algebra which I'd rather not type up to be

$$R = -e^{-\phi}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi. \tag{510}$$

This implies a fact that we will need later, namely that

$$\sqrt{g}R(x) = -\Box\phi(x) = -4\partial\bar{\partial}\phi(x), \tag{511}$$

where the factor of 4 comes from our conventions where e.g. $\partial = \frac{1}{2}(\partial_0 - i\partial_1)$.

Now we return to finding the trace of T. Upon varying $g_{\mu\nu}$ away from flat space, we have (going to be using σ, σ' instead of x, y for Cartesian coordinates from now on)

$$\delta\langle T_\mu{}^\mu(\sigma)\rangle = -\frac{1}{2}\int d^2\sigma' \, \langle T_\mu{}^\mu(\sigma) T_\nu{}^\nu(\sigma')\rangle \delta\phi(\sigma'), \tag{512}$$

since the variation of the metric is proportional to $\eta_{\mu\nu}$. Now we need the OPE between the two traces of the stress tensor on the RHS of this equation.

Getting this in a precise way is a bit tricky: we will hybridize an argument in some notes by Komargodski and and an argument in Tong's CFT notes. Going back to Cartesian coordinates, we examine the two point function $\langle T_{\mu\nu}(q)T_{\alpha\beta}(p)\rangle$, where q, p are momenta. By energy conservation, this two-point function needs to be killed by $\partial_\mu, \partial_\nu, \partial_\alpha$, and ∂_β. It also needs to contain a δ function enforcing momentum conservation, and so it needs to have the form

$$\langle T_{\mu\nu}(q)T_{\alpha\beta}(p)\rangle = \delta(p+q)\left[\frac{f(q^2)}{2}(\Pi^T_{\mu\alpha}(q)\Pi^T_{\nu\beta}(q) + \Pi^T_{\mu\beta}(q)\Pi^T_{\nu\alpha}(q)) + g(q^2)\Pi^T_{\mu\nu}(q)\Pi^T_{\alpha\beta}(q)\right], \tag{513}$$

where of course the transverse projectors are $\Pi^T_{\alpha\beta}(q) = q_\alpha q_\beta - q^2\eta_{\alpha\beta}$. We can fix the form of the functions f, g by requiring scale invariance: thus they must be algebraic in q^2, and they have to go as $1/q^2$ because of requiring the two-point function to be invariant under $q \mapsto \lambda q$ for $\lambda \in \mathbb{R}$ (remember that the delta function also transforms!). So, we can write $f(q^2) = a/q^2, g(q^2) = b/q^2$. Anyway, by taking $\mu = \nu$ and summing, we get

$$\langle T_\mu{}^\mu(p)T_{\alpha\beta}(q)\rangle = \delta(p+q)\left[\frac{a}{q^2}\Pi^T_{\mu\alpha}(q)\Pi^{T\mu}_{\beta}(q) + \frac{b}{q^2}\Pi^{T\mu}_{\mu}(q)\Pi^T_{\alpha\beta}(q)\right] = -\delta(p+q)(a+b)\Pi^T_{\alpha\beta}(q). \tag{514}$$

Contracting one more time and Fourier transforming, we have

$$\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \propto \Box\delta(\sigma-\sigma'), \tag{515}$$

where the proportionality constant $(a+b)$ is something that we can't determine with this method. The point of these couple of steps is to show that the two-point function of the traces has to be proportional to a double-derivative of a δ function. To find the proportionality constant, we need another argument.

In complex coordinates, conservation of energy is $\partial T^{z\bar{z}} + \bar{\partial}T^{\bar{z}\bar{z}} = 0$, or $\partial T_{z\bar{z}} - \bar{\partial}T/(2\pi) = 0$. Now with our conventions $T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma') = -16T_{\bar{z}z}(z)T_{\bar{w}w}(w)$. Using conservation of energy then, we have

$$\langle\partial_z T_{z\bar{z}}(z)\partial_w T_{w\bar{w}}(w)\rangle = -\frac{1}{4\pi^2}\bar{\partial}_z\bar{\partial}_w\langle T(z)T(w)\rangle. \tag{516}$$

We know what the RHS is, since we know the TT OPE. We have

$$\langle\bar{\partial}_z T_{z\bar{z}}(z)\bar{\partial}_w T_{w\bar{w}}(w)\rangle \sim \frac{1}{4\pi^2}\bar{\partial}_z\bar{\partial}_w\frac{c/2}{(z-w)^4} + \cdots = \frac{c}{48\pi^2}\partial_z^2\partial_w\bar{\partial}_z\bar{\partial}_w\frac{1}{z-w} + \ldots, \tag{517}$$

where $\ldots$ represents things that contain $\bar{\partial}_z T(z)$ and derivatives thereof. Now we need to use

$$\bar{\partial}_z\frac{1}{z-w} = 2\pi\delta(z-w, \bar{z}-\bar{w}), \tag{518}$$

which one can prove by using $\int_R d^2z\bar{\partial}f(z,\bar{z}) = -i\oint_{\partial R} f(z,\bar{z})$ with the function $f = 1/(z-w)$ and taking R to be a region containing the point w. Thus the OPE we've been looking at is

$$\langle\bar{\partial}_z T_{z\bar{z}}(z)\bar{\partial}_w T_{w\bar{w}}(w)\rangle \sim -\partial_z\partial_w\left(\frac{c}{24\pi}\partial_z\bar{\partial}_z\delta(z-w, \bar{z}-\bar{w})\right) + \ldots, \tag{519}$$

so then using the earlier relation we found between the various stress energy tensor two-point functions, we have

$$\frac{1}{16}\partial_z\partial_w\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \sim -\partial_z\partial_w\left(\frac{c}{24\pi}\partial_z\bar{\partial}_z\delta(z-w,\bar{z}-\bar{w})\right)+\dots \tag{520}$$

This is where the earlier analysis we did of the two-point function of T will come in handy: we know that the two point function of the traces has to be proportional to $\Box\delta(\sigma-\sigma')$. This lets us do two things: first, it lets us drop the ... (terms that go as $\bar{\partial}_z T$; of course classically this is zero anyway) on the RHS, since we know that no operators (only c-numbers) can appear on the RHS (actually, I guess the extra terms also die because they have nonzero spin, and we have rotational invariance). It also lets us strip away the derivatives and conclude that

$$\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \sim -\frac{2c}{3\pi}\partial_z\bar{\partial}_z\delta(z-w,\bar{z}-\bar{w}). \tag{521}$$

Naively this conclusion could only be reached modulo singular terms in the kernel of $\partial_z\partial_w$, but since we know that the LHS has to be proportional to $\Box\delta(x-y)$, such terms will not appear. Now we can change the δ function and the derivatives over to σ,σ' coordinates at the cost of a factor of 1/8 (2 from each derivative and 2 from the δ function since $dzd\bar{z}\to 2dxdy$), so that

$$\langle T_\mu{}^\mu(\sigma)T_\nu{}^\nu(\sigma')\rangle \sim -\frac{c}{12\pi}\Box\delta(\sigma-\sigma'). \tag{522}$$

Finally, putting this into our expression for $\delta\langle T_\mu{}^\mu\rangle$, we have

$$\delta\langle T_\mu{}^\mu(\sigma)\rangle = \frac{c}{24\pi}\int d^2\sigma'\,\Box\delta(\sigma-\sigma')\delta\phi(\sigma') = \frac{c}{24\pi}\Box\delta\phi(\sigma). \tag{523}$$

Using our earlier result for the variation of the Ricci scalar, we get

$$\delta\langle T_\mu{}^\mu(\sigma)\rangle = -\frac{c}{24\pi}\delta R(\sigma). \tag{524}$$

Therefore, integrating over the variation, we can conclude that

$$\langle T_\mu{}^\mu(\sigma)\rangle = -\frac{c}{24\pi}R(\sigma), \tag{525}$$

which holds in all geometries, not just those infinitesimally close to flat space (as we discussed earlier, the form $\langle T_\mu{}^\mu(\sigma)\rangle\propto R$ is required, and the coefficient of proportionality of course is geometry-independent).

Thus using our expression for $\delta\ln Z$ under a change in a (the radius of the sphere), we finally have

$$\frac{d\ln Z}{d\ln a} = -\frac{c}{24\pi}\int d^2\sigma\,\sqrt{g}R(\sigma). \tag{526}$$

On a sphere the Ricci scalar is $2/a^2$[40], so since $\int d^2\sigma\,\sqrt{g} = 4\pi a^2$,

$$\frac{d\ln Z}{d\ln a} = \frac{c}{3}, \tag{528}$$

which is what we wanted to show.

One final comment is that this whole derivation relied on determining the contact term of $\mathrm{Tr}\,T_{\mu\nu}$ with itself. It's often the case in QFT that contact terms like this are non-universal and depend on our regularization scheme. This isn't the case here, since the contact term in question is determined by the TT OPE, i.e. by correlation functions of the stress tensor at *separated* points. This information is universal, and ensures that the contact term is universal as well. These kind of universal contact terms usually show up in the OPEs of currents and in the context of anomalies; more on this in a future diary entry.

The central charge and entanglement entropy

Today we're deriving another classic result that I'd seen quoted a bunch but had never seen the derivation of. We will show that, for a 2D CFT in flat space with central charge c, the entanglement entropy for an interval $[a, b]$ is

$$S \sim c \ln |a - b|. \tag{529}$$

The strategy will be to use the replica trick and think about twist operators.

We use the replica trick in the usual way, writing for $A = [a, b]$,

$$S_A = -\lim_{n\to 1} \mathrm{Tr}[\rho_A \ln \rho_A^n] = -\lim_{n\to 1} \partial_n \mathrm{Tr}[\rho_A^n] = -\lim_{\varepsilon\to 0} \frac{\mathrm{Tr}[\rho_A^{1+\varepsilon}] - \mathrm{Tr}[\rho_A]}{\varepsilon} = \lim_{n\to 1} \frac{\mathrm{Tr}[\rho_A^n] - 1}{1-n}, \tag{530}$$

[40] A handy formula for the Ricci scalar is $R = 2\theta/A$, where θ is the angle between a vector and the image of itself parallely-transported around the boundary of a small geodesic ball (a ball bounded by geodesics), and A is the area of the loop (I think the factor of 2 is correct; some places seem to not have it). For the sphere, parallel transporting a vector halfway around the equator and then back along the prime meridian rotates the vector by π, so that the Ricci scalar is

$$R = 2\frac{\pi}{4\pi a^2/4} = 2/a^2. \tag{527}$$

where $\varepsilon = n-1$ (the limit is $n \to 1^+$) and we've used $\mathrm{Tr}[\rho_A] = 1$. As usual, we calculate the nth power of the reduced density matrix by calculating the partition function on the n-sheeted replica manifold. If we let $\Sigma(a)$ denote an n-fold twist operator (which inserts the endpoint of a $z^{1/n}$ branch cut at a), then traveling around a with $\Sigma(a)$ inserted is equivalent to moving between sheets of the Riemann surface. Since each of the n sheets (each a copy of $\mathbb{C}$) has the twist operators inserted, the trace in the expression for S_A is the same as the nth power of the two-point function of the twist operators:

$$\mathrm{Tr}[\rho_A^n] = \langle \Sigma(a)\Sigma(b)\rangle \sim \frac{1}{|a-b|^{2n\Delta_\Sigma}}. \tag{531}$$

We can get the scaling dimension Δ_Σ of the twist operator by the following argument. Let w be the coordinate on the n-sheeted Riemann surface used in calculating ρ^n, and let z be a coordinate on $\mathbb{C}$. Now zoom in to one of a, b, and consider a map which locally takes the Riemann surface to $\mathbb{C}$ in a region around this point. We can map the Riemann surface to the plane by taking $z = w^{1/n}$, which "unwinds" the Riemann surface and which is single valued when acting on w (we can e.g. think about w living in a space where $e^{2\pi i n} = 1$ but $e^{2\pi i} \neq 1$). Then we can use the transformation law for the stress tensor to write

$$\langle T(w)\rangle = \left\langle \partial_w z T(z) + \frac{c}{24}\{z; w\}\right\rangle = \frac{c}{24}\{w^{1/n}; w\}, \tag{532}$$

since on the plane the vev of the stress tensor vanishes (here we are thinking about the $T(w)$ insertion as occurring only on a single sheet of the replica manifold). The Schwartzian derivative is easily computed:

$$\{w^{1/n}; w\} = (n^{-1}-1)(n^{-1}-2)w^{-2} - \frac{3}{2}(n^{-1}-1)^2 w^{-2} = \frac{1}{2}(1-n^{-2}), \tag{533}$$

so that

$$\langle T(w)\rangle = \frac{1}{w^2}\frac{c}{24}(1-n^{-2}). \tag{534}$$

Thus we read off the conformal dimension of the twist operators as[41]

$$h_\Sigma = \frac{c}{24}(1-1/n^2), \tag{536}$$

so that the scaling dimension is $\Delta_\Sigma = c(1-n^{-2})/12$. Some sanity checks: when $n \to 1$ the twist operator becomes trivial, and $\Delta_\Sigma \to 0$ as required. Secondly, as $n \to \infty$ we have $\Delta_\Sigma \to c/12$. This makes sense because $n \to \infty$ gives a replica manifold

[41] Note that we have cheated somewhat here by working locally around one of the branch cut points, essentially assuming that the branch cut runs from $z = 0$ to $z = \infty$, when in reality it runs from a to b. We can correct for this by mapping the branch cut to one stretching from 0 to ∞ by using the conformal mapping $z = (w-a)^{1/n}/(w-b)^{1/n}$ (instead of $z = w^{1/n}$). Now the resulting OPE with the stress tensor needs to have singularities at each of the termination points of the branch cuts and needs to vanish when $a = b$, and so the general form of the vev of $T(w)$ is in fact

$$\langle T(w)\rangle = \frac{c}{24}(1-n^{-2})\frac{(a-b)^2}{(w-a)^2(w-b)^2}. \tag{535}$$

This of course gives us the same conformal dimension as derived in the non-footnoted text.

that becomes a cylinder: the two ends of the branch cut become the two ends of the cylinder, with $n \to \infty$ meaning that wrapping around the twist operators never returns one to one's starting point (the cohomology of the cylinder is $\mathbb{Z}$, which has no elements of finite order—this is the limit $\mathbb{Z}_n \to \mathbb{Z}$ for $n \to \infty$). This agrees with $\Delta_\Sigma \to c/12$ since we know that on the cylinder, $\langle T_{cyl}(z)\rangle = cz^{-1}/24$ (see previous diary entry), giving $h_\Sigma = c/24$ and $\Delta_\Sigma = c/12$.

Anyway, using the scaling dimension we get

$$\mathrm{Tr}[\rho_A^n] = \alpha|a-b|^{-\frac{c}{6}(n-n^{-1})}, \tag{537}$$

where α is some constant that we won't be able to determine. This gives

$$S_A = -\lim_{n\to 1}\partial_n \mathrm{Tr}[\rho^n] = -\lim_{n\to 1}\partial_n e^{-\frac{\alpha c}{6}(n-n^{-1})\ln|a-b|} = \alpha\frac{c}{3}\ln|a-b|, \tag{538}$$

as required (the argument of the log will be made dimensionless with some short-distance cutoff that we haven't been writing).

Basic conformal perturbation theory

Today we will look at how we can use the algebraic structure of a CFT to describe the RG flow in the vinicity of its fixed point (I read about this problem in a review on bosonization somewhere; sadly I forget which reference. This is pretty standard stuff, though). Anyway, consider perturbing a CFT by adding to the action the term

$$S = S_{CFT} + \int d^dx\, g_\alpha \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha}, \tag{539}$$

where the sum on α is implied and g_α are *dimensionless* couplings, with the $a^{-d+\Delta_\alpha}$ present to make the dimensional of the integrand correct. For studying perturbations around a fixed point, the most interesting operators to choose will be marginal ones, for which $\Delta_\alpha = d$.

Working perturbatively in the couplings g_α, we will be finding the $O(g_\alpha g_\beta)$ β functions for the g_α couplings in terms of the OPE data.

There are two ways of getting the result, which are actually rather similar: one uses dimensional analysis and block spin RG, while the other does the RG in a more symmetric way. Both methods take place in $\mathbb{R}$ space, due to the fact that we will need to use the OPEs coming from the CFT, which are awkward to formulate in momentum space.

Both methods start by writing the partition function as (here ϕ is some stand-in for an arbitrary collection of fields)

$$Z = \int \mathcal{D}\phi e^{-S_0[\phi]} \left(1 - g_\alpha \int d^d x\, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} + \frac{1}{2} g_\alpha g_\beta \int d^d x d^d y\, \mathcal{O}^\alpha(x)\mathcal{O}^\alpha(y) a^{-2d+\Delta_\alpha+\Delta_\beta} - \dots \right). \tag{540}$$

The plan will be to do some sort of real space reornmalization group step, and see how the effective coupling constants in front of each term in the expansion change.

First for the block spin method. This one is rather hueristic, so forgive the hand-waving in what follows. Let's first look at the linear term, which we won't need any CFT to deal with. One one hand, we can write it as (this is not to best way of getting the first-order beta function, but it is instructive for what will follow[42])

$$g_\alpha \int d^d x\, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = \sum_I \sum_{i\in I} g_\alpha \widetilde{\mathcal{O}}^\alpha(x_i). \tag{542}$$

Here I denotes a block spin site consiting of lattice sites x_i, and $\widetilde{\mathcal{O}}^\alpha(x_i)$ is the dimensionless operator defined by $\widetilde{\mathcal{O}}^\alpha = a^{\Delta_\alpha}\mathcal{O}^\alpha$. Now let the linear size of each block be L. Then just by dimensional analysis, we have

$$g_\alpha \int d^d x\, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = \sum_I a^{-d+\Delta_\alpha} \int_{\text{block}} d^d x\, g_\alpha \mathcal{O}^\alpha = \sum_I g_\alpha (L/a)^{d-\Delta_\alpha} \widetilde{\mathcal{O}}^\alpha(x_I). \tag{543}$$

We could write the result of the integration as being proportional to $L^{d-\Delta_\alpha}$ since because we are perturbing around a CFT, there is no dimensionful parameter from the theory (like a correlation length) to use in place of L to obtain the required dimensionality (other than I guess a, but this can't appear since we need the beta functions to not be explicitly dependent on a). Taking the block size L to be infinitesimally larger than the lattice spacing, $L = a + da$, we get

$$g_\alpha \int d^d x\, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} \to \sum_I g_\alpha (1 + (d - \Delta_\alpha) d\ln a) \widetilde{\mathcal{O}}^\alpha(x_I). \tag{544}$$

Thus we see that we get the same thing as the original term (which was $\sum_i g_\alpha \widetilde{\mathcal{O}}^\alpha(x_i)$), just with a different coupling cosntant $g_\alpha(L)$. Then defining the β function as $\beta_\alpha = +d_{\ln a} g_\alpha$, we see that to first order in the couplings, we get the expected answer:

$$\beta_\alpha = (d - \Delta_\alpha) g_\alpha + O(g_\alpha^2). \tag{545}$$

[42]Recall a simpler way: we rescale $a \mapsto a + da, g \mapsto g + dg$ and ask what dg needs to be such that the partition function is preserved to first order in the couplings. We have

$$g_\alpha \int d^d x\, \mathcal{O}^\alpha(x) a^{-d+\Delta_\alpha} = (g_\alpha + dg_\alpha) \int d^d x\, \mathcal{O}(x) a^{-d+\Delta_\alpha} (1 + (-d+\Delta_\alpha) d\ln a) \implies dg_\alpha = d\ln a (d - \Delta_\alpha) g_\alpha, \tag{541}$$

which gives in our (high energy) convention $\beta_\alpha = (\Delta_\alpha - d) g_\alpha$ to first order in the coupling.

There are several things to object to about this, some of which may be fixable. First, what do you mean "take L to be infinitessimally larger than a? You're on a lattice!". Indeed, taking $L = a + da$ is very formal. Secondly, the dimensional analysis part was kind of lame, since we actually had no way of computing any sort of geometry-dependent factors that are associated with the block spin procedure. The kind of wonky combination of mixing integrating in the continuum with the block spin aproach is also ugly. Worst of all, when we evaluated the integral with dimensional analysis, we should have obtained a sum $\sum_\beta \widetilde{\mathcal{O}}^\beta$ of different dimensionless operators that could be produced during the RG step, but we only kept the original $\beta = \alpha$ operator.

Now we look at the second order piece. We write it as

$$\frac{1}{2}g_\alpha g_\beta \int d^dx d^dy\, \mathcal{O}^\alpha(x)\mathcal{O}^\alpha(y)a^{-2d+\Delta_\alpha+\Delta_\beta} = \frac{1}{2}\sum_{I\neq J} g_\alpha g_\beta(1+(2d-\Delta_\alpha-\Delta_\beta))$$
$$\times(1-d\ln a)\widetilde{\mathcal{O}}^\alpha(x_I)\widetilde{\mathcal{O}}^\beta(x_J) + \frac{1}{2}g^\alpha g^\beta\sum_I \int_{\text{block}} d^dx d^dy\, \mathcal{O}^\alpha(x_I+x)\mathcal{O}^\beta(x_I+y)a^{-2d+\Delta_\alpha+\Delta_\beta}. \tag{546}$$

The first term becomes the quadratic part of the expansion of the exponential of the action after performing the RG step, and so will not be important in what follows (plus for the case we are most interested in, where both $\mathcal{O}^\alpha, \mathcal{O}^\beta$ are marginal, this term has no affect on the β function).

Now since the distances between different sites in a single block are "below the resolution" of the theory after doing the blocking, we will take the distance $|x-y|$ to be small enough for us to profitably use the OPE. Of course this is a bit hand wavy (what about all those neighboring lattice sites whose connecting links cut a block boundary?), but we will stick with it. Using this and then doing the integral over the intra-block coordinates using dimensional analysis (again, the only dimensionful scale we have by virtute of perturbing around a CFT is L), the second term in the previous equation is,

$$\begin{aligned}\frac{1}{2}g_\alpha g_\beta\sum_I\int_{\text{block}} d^dx d^dy\, C^{\alpha\beta}_\gamma \frac{1}{|x-y|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}}a^{-2d+\Delta_\alpha+\Delta_\beta}\mathcal{O}^\gamma(x_I+x)\\ = \frac{1}{2}g_\alpha g_\beta\sum_I C^{\alpha\beta}_\gamma\widetilde{\mathcal{O}}^\gamma(x_I)(L/a)^{2d-\Delta_\alpha-\Delta_\beta}\\ \to \frac{1}{2}g_\alpha g_\beta\sum_I C^{\alpha\beta}_\gamma\widetilde{\mathcal{O}}^\gamma(x_I)(1+(2d-\Delta_\alpha-\Delta_\beta)d\ln a),\end{aligned} \tag{547}$$

where we have taken $L \to a + da$ in the last step and assumed that $2d - \Delta_\alpha - \Delta_\beta \neq 0$. Acutally, the more relevant case (or in light of the subject, maybe I should say the more interesting case) is when all the perturbing operators are marginal. In this case dimensional analysis produces a $\ln(L/a)$, and the second term instead becomes

$$\frac{1}{2}g_\alpha g_\beta\sum_I C^{\alpha\beta}_\gamma\widetilde{\mathcal{O}}^\gamma(x_I)d\ln a. \tag{548}$$

We see that this term gives a contribution to the g_γ beta function, since it has the effect of just changing the $\widetilde{\mathcal{O}}^\gamma$ coupling constant in the block spin theory. Remembering the sign difference between this and the linear term in the expansion (540), we see that this gives

$$\beta_\gamma = -\frac{1}{2} g_\alpha g_\beta C_\gamma^{\alpha\beta}, \tag{549}$$

where we have assumed that the associated operator $\mathcal{O}_\gamma$ is marginal so that no linear part appears (again, this is the most interesting case for perturbing about a CFT fixed point).

Of course, all the gripes about the non-rigorous nature of this method that we raised when deriving the linear part of the β funciton can be raised here. In order to feel better about our result, we briefly discuss a way to make it a bit more precise by using a more symmetric approach for dealing with the second order term which I just learned about from Cardy's book (Renormalization and Scaling in Statistical Physics).

The basic idea is that we can remain in the continuum and treat the cutoff not as a lattice spacing per se, but rather just as the closest distance that operators are allowed to get from one another. So, we picture the operators as hard spheres, with the radii of the sphere being set by the cutoff. An RG step then proceeds by integrating out all terms that include operators separated a distance between a and $a + \delta a$ from one another, and then re-scaling coordinates in the usual way.[43] This makes dealing with the second order term super easy:

$$\begin{aligned}\frac{1}{2} g_\alpha g_\beta \int_{|x-y|\geq a} d^d x d^d y\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} = \frac{1}{2} g_\alpha g_\beta \Big(&\int_{|x-y|\geq L} d^d x d^d y\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} \\ &+ \int_{a\leq|x-y|<L} d^d x d^d y\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta}\Big).\end{aligned} \tag{550}$$

Doing the renormalization step means changing the effective cutoff to L by doing the second integral on the RHS and abosrbing the result into a rescaling of the coupling constants. We do this by using the OPE: taking $L = a + da$ and using translation invariance we get

$$\begin{aligned}\frac{1}{2} g_\alpha g_\beta &\int_{a\leq|x-y|<L} d^d x d^d y\, \mathcal{O}^\alpha(x)\mathcal{O}^\beta(y) a^{-2d+\Delta_\alpha+\Delta_\beta} \\ &= \frac{1}{2} g_\alpha g_\beta C_\gamma^{\alpha\beta} \int d^d x\, \frac{1}{a^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}} da A(S^{d-1}) a^{-d-1+\Delta_\alpha+\Delta_\beta} \mathcal{O}^\gamma(x) \\ &= \frac{1}{2} g_\alpha g_\beta C_\gamma^{\alpha\beta} A(S^{d-1}) d\ln a \int d^d x\, \mathcal{O}^\gamma(x) a^{-d+\Delta_\gamma},\end{aligned} \tag{551}$$

where $A(S^{d-1})$ is the area of the unit S^{d-1}. This gives essentially the same contribution to the β function as we got with the rather hand-waving block spin method, up to a factor of $A(S^{d-1})$. Indeed, assuming again that $\mathcal{O}^\gamma$ is marginal, we get

$$\beta_\gamma = -\frac{1}{2} A(S^{d-1}) g_\alpha g_\beta C_{\alpha\beta}^\gamma. \tag{552}$$

[43] Only the operators in the perturbations we added behave as hard spheres; the operators in the unperturbed CFT action don't need to be dealt with in this way because their correlation functions are already scale-invariant to begin with.

The unasthetic factor of the sphere area can be gotten rid of by absorbing into the coupling constants.

As an example, consider e.g. $\mathfrak{su}(2)_k$. The current operators $J, \bar{J}$ for the $SU(2)_L$ and $SU(2)_R$ symmetries, respectively) have dimension 1, and so current-current interactions are marginal (they also must be dimension 2 since the stress tensor is built out of $J^a J^a$ terms via the Sugawara construction). Consider deforming a WZW CFT with an anisotropic "Thirring model" type current-current interaction:

$$\mathcal{L}_{int} = \sum_a \lambda_a J^a \bar{J}^a. \tag{553}$$

Since the $J^a \bar{J}^a$ terms are all marginal, the beta functions $\beta_a = -d_{\ln a} \lambda_a(L)$ (sorry for the bad notation) is determined to lowest order by the quadratic term in our expression for the beta function. Now recall that the OPE is

$$J^a J^b \sim i\epsilon^{abc} \frac{J^c(z)}{z-w} + \frac{k\delta^{ab}}{(z-w)^2}. \tag{554}$$

Thus we have (no implicit summation)

$$(J^a \bar{J}^a)(z) \cdot (J^b \bar{J}^b)(w) \supset \sum_c \frac{1}{|z-w|^2} |\epsilon^{abc}| (J^c \bar{J}^c)(w). \tag{555}$$

We can then conclude that the $O(\lambda^2)$ β functions for the various interactions are (the numerical prefactor isn't important)

$$\beta_a = -\frac{\pi}{2} \sum_{b,c} |\epsilon^{abc}| \lambda_b \lambda_c. \tag{556}$$

As another application of this, we can do an easy check of the one-loop β function in the ϕ^4 model. We can only check up to the one-loop result since we have only kept terms quadratic in the $\mathcal{O}^\alpha$ in the expansion of the partition function. We will fix notation by

$$S = \int \left(\frac{1}{2} (\partial\phi)^2 + r\phi^2 + u\phi^4 \right). \tag{557}$$

In e.g. dimension $d = 4 - \epsilon$, the dimension of r is 2 while the dimension of u is ϵ (really 2ϵ, but we are taking $\epsilon \to 0$). This gives us the first-order terms in the β functions. Then we need the OPEs (schematic notation and writing a lot of numbers to make the combinatorics transparent)

$$\phi^2 \cdot \phi^2 \sim \frac{2 \cdot 2}{x^2} \phi^2, \qquad \phi^2 \cdot \phi^4 \sim \frac{4 \cdot 2}{x^2} \phi^4 + \frac{\frac{4!}{2 \cdot 2} \cdot 2}{r^4} \phi^2, \tag{558}$$

and

$$\phi^4 \cdot \phi^4 \sim \frac{\left(\frac{4!}{3!}\right)^2 \cdot 3!}{r^6} \phi^2 + \frac{\left(\frac{4!}{2!^2}\right)^2 \cdot 2}{r^4} \phi^4, \tag{559}$$

where we have ignored the most singular parts where all of the legs have been contracted and ingorned the ϕ^6 part in the last term. Then we can read off the OPE coefficients needed for calculating the β functions from the above formula. We will assume the coupling constants have been rescaled to get rid of the annoying factor of $A(S^{d-1})/2$ in our expression for the second-order contribution to the β functions. We get (still in the high-energy convention where we get β by differentiting wrt $\ln\Lambda$ and not $\ln a$)

$$\beta_r = 2r - 96u^2 - 24ur, \qquad \beta_u = \epsilon u - 72u^2 - 16ru. \tag{560}$$

From here one can compare to e.g. Peskin and Schroeder after traking down how the conventions differ. One can also use these to solve the for the WF fixed point, etc etc.

Duality in the Ising model

Today's problem came from wanting to understand a statement in [18] about duality in the Ising model. Our goal is to explain / elaborate on the content of the mini-section on page 38 of the just-cited paper.

Let X be a Riemann surface equipped with a choice of spin structure η. Let Z_+ denote a partition function for a spin structure chosen so that $\mathrm{Arf}(\eta) = 0$ (the spin structure can be extended to a bounding three-manifold), and Z_- a partition function for a spin structure with $\mathrm{Arf}(\eta) = 1$ (the spin structure is non-bounding). We want to examine what happens when the theory is pushed away from the self-dual conformal point by adding in a perturbation given by the energy operator $m\int\epsilon$.[44] Since $\epsilon \sim \psi\bar\psi$ is a fermion mass, this is indeed the right perturbation for tuning the theory away from the critical point.

Write the perturbed partition function as

$$Z_f[\eta] = \left\langle 1 - m\int_z \epsilon(z,\bar z) + \frac{m^2}{2}\int_{z,w}\epsilon(z,\bar z)\epsilon(w,\bar w) - \dots \right\rangle_\eta, \tag{561}$$

where the expectation value is computed in the CFT (i.e. at the critical point). We claim that Z_+ is even under $m \mapsto -m$, while Z_- is odd, that is, we claim that $\langle\epsilon^k\rangle$ for k even is only nonzero when $\mathrm{Arf}(\eta) = 0$, while for k odd it is only nonzero for $\mathrm{Arf}(\eta) = 1$.

We now take a look at why this is true.[45] We will work on the torus for simplicity.

[44] ϵ is an "energy operator" since $\epsilon \sim \psi\bar\psi$ where $\psi \sim X\prod Z, \bar\psi \sim Y\prod Z$ means that $\epsilon_j \sim \psi_j\bar\psi_j + \psi_j\bar\psi_{j+1} + h.c. \sim X_jX_{j+1} + Z_j$ contains the terms that appear in the Hamiltonian $H \sim \sum(XX+Z)$.

[45] Many thanks to Wenjie Ji for long discussions on this!

First, we note that $\langle\epsilon\rangle_\eta = 0$ if the spin structure η has antiperiodic boundary conditions in the spatial direction. This is because we are computing $\langle 0|\psi\bar{\psi}|0\rangle$ or $\langle 0|(-1)^F\psi\bar{\psi}|0\rangle$, both of which vanish since the ψs are primary fields which create states orthogonal to $|0\rangle$ when acting on $|0\rangle$. Since $\psi\psi \sim 1$, inserting an even number of ϵs results in something non-zero, while for the above reason inserting an odd number of ϵs gives zero.

Now for the spin structures with periodic boundary conditions in space. These boundary conditions are created by computing the expectation value in the states $|\sigma\rangle$ and $|\mu\rangle$, which differ by the occupation number of the fermion zero modes, and hence differ in their fermion parity (here the zero mode is a zero mode of the Hamiltonian, not a zero of the action [which doesn't exist if the time direction is antiperiodic]). More precisely, the two states differ in their $(-1)^F = (-1)^{F_L+F_R}$ eigenvalue, where $(-1)^{F_L}$ counts whether the holomorphic zero mode is filled, and $(-1)^{F_R}$ counts whether the antiholomorphic zero mode is filled. Since we are assuming the ∂ conditions for both ψ and $\bar{\psi}$ are the same, the ground states only carry a representation of the Clifford algebra associated to the total (non-chiral) zero mode algebra.

So, to do the trace, we need to sum over the two different ground states $|\sigma\rangle$ and $|\mu\rangle$, which differ in the occupation number of the zero modes. To compute e.g. $\langle\sigma|\epsilon|\sigma\rangle$, we use the operator-state correspondence to write $|\sigma\rangle = \sigma(0)|0\rangle$. Thus to get the expectation value of ϵ, we need to know the OPEs between $\psi, \bar{\psi}$, and σ, μ. Looking this up in the Big Yellow Book, we see that

$$\epsilon(z,\bar{z})\sigma(0) = i\psi(z)\bar{\psi}(\bar{z})\sigma(0) = i\psi(z)\frac{e^{-i\pi/4}}{\sqrt{2\bar{z}}}\mu(0) = \frac{1}{2|z|}\sigma(0). \tag{562}$$

Similarly,

$$\epsilon(z,\bar{z})\mu(0) = i\psi(z)\bar{\psi}(\bar{z})\mu(0) = i\psi(z)\frac{e^{i\pi/4}}{\sqrt{2\bar{z}}}\mu(0) = -\frac{1}{2|z|}\mu(0). \tag{563}$$

Note that the phase factors, which aren't always written, are very important here! Now we see that

$$\langle\epsilon(z,\bar{z})\rangle_{RN} = \langle\sigma|\epsilon(z,\bar{z})|\sigma\rangle + \langle\mu|\epsilon(z,\bar{z})|\mu\rangle = 0. \tag{564}$$

More generally, we see that $\langle\epsilon^k\rangle_{RN} = 0$ for odd k, while it is non-zero for even k. Note another rather slick way of reaching this conclusion would have been do just perform an S transformation on $\langle\epsilon\rangle_{NR} = 0$, which we already knew is true.

On the other hand, for the RR spin structure, we have

$$\begin{aligned}\langle\epsilon(z,\bar{z})\rangle_{RR} &= \langle\sigma|(-1)^F\epsilon(z,\bar{z})|\sigma\rangle + \langle\mu|(-1)^F\epsilon(z,\bar{z})|\mu\rangle = \langle\sigma|\epsilon(z,\bar{z})|\sigma\rangle - \langle\mu|\epsilon(z,\bar{z})|\mu\rangle \\ &= \frac{1}{|z|}\langle 0|\mathbf{1}|0\rangle \neq 0,\end{aligned} \tag{565}$$

where we took $|\mu\rangle$ to have an odd number of total zero modes and in the last step used $\sigma\otimes\sigma \sim \mu\otimes\mu \sim \mathbf{1}+\epsilon$ and $\langle\epsilon\rangle = 0$ (here $\langle\rangle$ without any subscript denotes the standard NN boundary conditions expectation value). So we conclude that on the RR torus, odd powers of ϵ are the ones that give nonzero expectation values. From a Lagrangian point of view, we could also argue that since $\int \mathcal{D}\psi_0 \mathbf{1} = 0$ (Grassmann integration is the same as differentiation), if there exists some ψ_0 such that $S[\psi_0] = 0$, then $Z = \int \prod_k \mathcal{D}\psi_k e^{-S} = 0$ (such a zero-action mode only exists on the RR torus). However, if we insert an ϵ into the partition function then we can do a mode expansion on it, with the term $\langle \epsilon \rangle_{RR} \supset \int \mathcal{D} \prod_k \mathcal{D}\psi_k \, \psi_0 \bar{\psi}_0 e^{-S}$ surviving and giving a non-zero expectation value.

Summing up, if $\mathrm{Arf}(\eta) = 0$ the series expansion for the parition function only includes even powers of m, while if $\mathrm{Arf}(\eta) = 1$ the series only includes odd powers. Now when we sum over spin structures, the full partition function for the bosons is

$$Z[m] = Z_+[m] \pm Z_-[m], \tag{566}$$

where the $\pm$ sign can be chosen freely (see e.g. the Big Yellow Book, chapter 11). This $\pm$ sign corresponds to projecting onto different (total) fermion parity sectors for the torus with periodic spatial ∂ conditions, which projects onto either the $|\sigma\rangle$ states or the $|\mu\rangle$ states. Since μ and σ are order-disorder duals of one another, we expect that this $\pm$ sign is switched under duality. Indeed, based on our comments above, we see that doing duality by taking $m \leftrightarrow -m$ is equivalent to flipping the $\pm$ sign, due to the evenness / oddness of the two partition functions.

We claim that doing duality, i.e. flipping the sign in the linear combination of Z_+ and Z_- is equivalent to tensoring with a Kitaev chain. Indeed, we will see that the partition function of the Kitaev chain in the topological phase is just a sign which depends on the spin structure in the right way to change the sign of Z_-.

Now let's explain this, starting with some more general comments inspired by reading [5] and other papers by Ryan + Kapustin. On a manifold with nontrivial topology, a bosonic theory can only be dual to a fermionic one if the duality relates a bosonic partition function to a sum over spin structures of fermionic partition functions, so that the bosonic theory has no spin structure dependence. A particular spin structure η, or a particular gauge field α, can be selected out by putting the analogue of e^{ikx} in the sum, in accordance with the relation between the two theories being related by a Fourier transform. Now recall that flat $\mathbb{Z}_2$ gauge fields are identified with elements of $H_1(X;\mathbb{Z}_2)$, so that the gauge field is defined the $\mathbb{Z}_2$ twists across each cycle α of the manifold in question (a $\mathbb{Z}_2$ twist along the cycle α changes the boundary conditions for cycles β such that $\alpha \cap \beta = 1$). So,

$$Z_f[\eta] = \frac{1}{2} \sum_\alpha (-1)^{\eta(\alpha)} Z_b[\alpha], \tag{567}$$

and[46]

$$Z_b[\alpha] = \frac{1}{2} \sum_\eta (-1)^{\eta(\alpha)} Z_f[\eta]. \tag{569}$$

[46] The normalisation factors here are specific to a torus. If we are not on the torus, we have to write the more cumbersome normalization factor

$$Z_f[\eta] = \frac{1}{\sqrt{2}^{\dim H_1(X;\mathbb{Z}_2)}} \sum_\alpha (-1)^{\eta(\alpha)} Z_b[\alpha]. \tag{568}$$

Here the action of η on $\alpha \in H_1(X;\mathbb{Z}_2)$ is given by $\eta(\alpha)=0$ if η assigns anti-periodic (N) boundary conditions to α, and $\eta(\alpha)=1$ if it assigns periodic (R) boundary conditions. This jives with the fact that antiperiodic boundary conditions are the "natural" ones, so that they are the identity in the $\mathbb{Z}_2$ group law. For example, if we label the nontrivial cycles on the torus as a_1, a_2, and a_1a_2, then e.g.

$$\eta_{NN}(0)=\eta_{NN}(a_1)=\eta_{NN}(a_2)=0, \quad \eta_{NN}(a_1a_2)=1, \tag{570}$$

while

$$\eta_{NR}(0)=\eta_{NR}(a_1)=\eta_{NR}(a_1a_2)=0, \quad \eta_{NR}(a_2)=1, \tag{571}$$

and

$$\eta_{RR}(0)=0, \quad \eta_{RR}(a_1)=\eta_{RR}(a_2)=\eta_{RR}(a_1a_2)=1. \tag{572}$$

Here the first label in the spin structure is the spatial boundary conditions and the second is the temporal ones. For us, spin structures will be identified with elements of $H^1(X;\mathbb{Z}_2)$. Here the 1-cocycleness comes from the failure of η to be a homomorphism on 1-chains:

$$\delta\eta(a,b)=\eta(a)+\eta(b)-\eta(a+b)=[a]\cap[b]. \tag{573}$$

One can check the consistency of this with the assignments of $\eta_{XY}(a)$ above.

Another way of writing $\eta(\alpha)$ for a gauge field α is by computing the integral $\eta(a)=\int_{\widehat{\eta}}\alpha$. Here, the submanifold $\eta \in H_1(X;\mathbb{Z}_2)$ is determined by the location of the branch cuts needed to determine the spin structure (each branch cut twists the boundary conditions of the fermions, with the "default" boundary condition [no branch cuts passed] being anti-periodic). Thus on T^2, the spin structure NR has a Poincare dual $\widehat{NR}$ given by the cycle which wraps the spatial cycle of the torus (so that the temporal boundary conditions are periodic).

As a sanity check, we check that the Fourier transform is involutive (it has order 2 and not order 4 since we're in $\mathbb{Z}_2$):

$$\begin{aligned}
Z_b[\alpha] &= \frac{1}{2}\sum_\eta(-1)^{\eta(\alpha)}Z_f[\eta] = \frac{1}{4}\sum_\eta\sum_{\alpha'}(-1)^{\eta(\alpha)+\eta(\alpha')}Z_b[\alpha'] \\
&= \frac{1}{4}\sum_{\alpha'}\sum_\eta(-1)^{\eta(\alpha+\alpha')+\int\widehat{\alpha}\cup\widehat{\alpha}'}Z_b[\alpha'] \\
&= \frac{1}{4}\sum_{\alpha'}\sum_\eta(-1)^{\eta(\alpha')+\int\widehat{\alpha}\cup\widehat{\alpha}'}Z_b[\alpha'+\alpha] \\
&= \frac{1}{2}\sum_{\alpha'}\delta(\alpha')(-1)^{\int\widehat{\alpha}\cup\widehat{\alpha}'}Z_b[\alpha'+\alpha] \\
&= Z_b[\alpha]
\end{aligned} \tag{574}$$

where we have used $\int\widehat{\alpha}\cup\widehat{\alpha}=0$. We have also used

$$\frac{1}{2}\sum_\eta(-1)^{\eta(\alpha)}=\delta(\alpha). \tag{575}$$

This is easy to check explicitly: if $\alpha=0$, then $(-1)^{\eta(\alpha)}=1$ for all spin structures. If $\alpha\neq0$, then two spin structures have $(-1)^{\eta(\alpha)}=-1$, while the other two have $(-1)^{\eta(\alpha)}=+1$, and so the sum gives zero.

We can also check that this Fourier transform formula is consistent with what we know from CFT. Recall that the Virasoro characters for the Ising CFT are given by

$$\chi_{\mathbf{1}} = \frac{1}{2}(Z_f[NN] + Z_f[NR]), \quad \chi_\epsilon = \frac{1}{2}(Z_f[NN] - Z_f[NR]) \tag{576}$$

and

$$\chi_{\sigma/\mu} = \frac{1}{2}(Z_f[RN] \mp Z_f[RR]), \tag{577}$$

where in the spin structure XY, X labels the spatial boundary conditions and Y the temporal ones. The choice of $\mp$ sign depends on what phase of the Ising model we are in (more on this later). The partition function for the Ising model with periodic boundary conditions for the spins, i.e. for a trivial gauge field $\alpha = 0$, is

$$Z_b[0] = |\chi_{\mathbf{1}}|^2 + |\chi_\epsilon|^2 + |\chi_\sigma|^2. \tag{578}$$

Now consider putting antiperiodic boundary conditions around the temporal cycle for the spin in the Ising model: this means working with a gauge field $\alpha = a_t$ which has nontrivial holonomy around the temporal cycle. The energy operator ϵ and the identity are not affected, but σ gets mapped to minus itself, and so we might guess that the partition function in the presence of a_t is

$$Z_b[a_t] = |\chi_{\mathbf{1}}|^2 + |\chi_\epsilon|^2 - |\chi_\sigma|^2. \tag{579}$$

In terms of the different spin structures, this is

$$Z_b[a_t] = \frac{1}{2}(Z_f[NN] + Z_f[NR]) - \frac{1}{2}(Z_f[RN] \mp Z_f[RR]). \tag{580}$$

Since $(NN)(a_t) = (NR)(a_t) = 0$ (the Poincare dual of the former spin structure is trivial while the dual of the second one is the spatial cycle, along which a_t has trivial holonomy) while $(RN)(a_t) = (RR)(a_t) = 1$, from our Fourier transform formula we see that we should assign minus signs to the RN and RR sectors. Indeed, this is exactly what happens from flipping the sign of $|\chi\sigma|^2$ in the expression for Z_b. One can then use modular invariance of the Ising spin partition function to confirm the Fourier transform formula for the other choices of Ising spin boundary conditions.

Now let's see what the partition functions are in the different phases. In the symmetry-broken phase, nontrivial gauge field twists are not allowed: they create domain walls in Z, and since the symmetry-broken phase ground state is an eigenstate of Z, a domain wall which wraps a nontrivial cycle creates an inconcsistency in the spin configuration, and so $Z_b[\alpha] = \delta_{\alpha,0}$. This means that the fermionic partition function is just

$$Z_f[\eta] = 1. \tag{581}$$

So, the symmetry-broken phase for the bosons maps to the trivial phase for the fermions.

For the bosons, we expect that in the paramagnetic (symmetric) phase, the partition function will be insensitive to the presence of twists caused by the gauge field: in an eigenstate of X, the operator which creates a domain wall in Z acts trivially. So, we have $Z_b[\alpha] = 1$ in the symmetric phase, and thus

$$Z_f[\eta] = \frac{1}{2}\sum_\alpha (-1)^{\eta(\alpha)}. \tag{582}$$

Now if $\eta \neq RR$, $\eta(\alpha)$ is equal to 1 for only one cycle, and the sum produces $(3-1)/2 = 1$. On the other hand, if $\eta = RR$ then three of the cycles are assigned 1, and the sum produces $(1-3)/2 = -1$. So then in the symmetric phase, the fermion partition function is the Arf invariant:

$$Z_f[\eta] = (-1)^{\text{Arf}(\eta)}. \tag{583}$$

This is precisely the partition function for the topological phase of the Kitaev chain. As a check that these work, we can use these fermionic partition functions to reproduce the bosonic ones. Taking $Z_f[\eta] = 1$ gives

$$Z_b[\alpha] = \frac{1}{2}\sum_\eta (-1)^{\eta(\alpha)} = \delta_{\alpha,0}, \tag{584}$$

which is the partition function for the symmetry-breaking phase[47]. Likewise, for the Kitaev chain in the topological phase,

$$Z_b[\alpha] = \frac{1}{2}\sum_\eta (-1)^{\eta(\alpha)+\text{Arf}(\eta)} = 1, \tag{585}$$

which in uniform in α and thus also matches the symmetric-phase partition function for the bosons (just check the last equality explicitly: for all spin structures, there is always one -1 term in the sum).

This confirms that tensoring with a Kitaev chain impliments the duality in the Ising model, since on the bosonic side interchanges the partition functions of the symmetric and symmetry-broken phases.

Now we briefly discuss how to see the different fermion parity results from a Hamiltonian viewpoint: we've seen that in the symmetric phase for the Ising spins (the topological phase for the fermions), there is a $1-(-1)^F$ projection in the path integral in the R sector, so that with periodic boundary conditions the fermion parity is odd (while it is even in the symmetry-breaking phase). The sanity check on this result is as follows. From the JW transformation, the Majorana femrions ψ satisfy $\psi_i = \prod_{j=i}^{i+L} Z_j \psi_{i+L}$ for a chain of length L (here the Hamiltonian is $H \sim -J\sum XX - h\sum Z$). The product $\prod_j Z_j = G$ is the generator of the global $\mathbb{Z}_2$ symmetry. So $\psi_i|0\rangle = G\psi_{i+L}|0\rangle = -\psi_{i+L}G|0\rangle = -\lambda_G\psi_{i+L}|0\rangle$, where λ_G is the $\mathbb{Z}_2$ charge of the ground state and we have used that the Majorana fermions anticommute with the G operator since it is a string of Z's. If the ground state has trivial $\mathbb{Z}_2$ charge then we are in the N sector and if it has -1 $\mathbb{Z}_2$ charge we are in the R sector. But G is also equal to the fermion parity

[47] Note how summing over spin stuctures produces a δ function for the gauge field α (since if $\alpha \neq 0$ then two spin structures give a -1 phase and two give a $+1$), but that summing over gauge fields does not produce a δ function for the spin structure: $\frac{1}{2}\sum_\alpha (-1)^{\eta(\alpha)} \neq \delta(\eta)$ (we would have to decide what exactly we mean by $\delta(\eta)$ also). Instead, this sum gives the Arf invariant.

operator, and so we learn that in the symmetric phase the N sector has even parity while the R sector has odd parity.

This can also be understood by actually computing H in momentum space. It is (for complex fermion operators c_k; the derivation is straightforward)

$$H = \sum_k (2h - 2J\cos(k))c_k^\dagger c_k - iJ\sin k(c_k c_{-k} + c_k^\dagger c_{-k}^\dagger). \tag{586}$$

In the N sector the fermion modes always come in pairs symmetric about $k = 0$ and so $(-1)^F = 1$. In the R sector, we have a mode at $k = 0$ and a mode at $k = \pi$. The latter is always un-filled since it is always at high energy. The former is filled, and hence the ground state has odd parity, provided that $J > h$, i.e. provided we are in the paramagetic phase. This provides another sanity check. A similar computation can be done for Majorana fermions γ_i, where in momentum space we have two unpaired modes, $\gamma_0^\dagger = \gamma^0$ and $\gamma_\pi^\dagger = \gamma_\pi$. Only the former is filled for the R spin structure and gives us the required odd parity. Also note that here duality does $\gamma_0 \mapsto \gamma_0, \gamma_\pi \mapsto -\gamma_\pi$, since duality in the Majorana language is equivalent to translation through half a unit cell ($i \mapsto i+1$ for the Majorana index). At the self-dual point this symmetry prevents us from hybridizing the unpaired modes with a term like $\gamma_0\gamma_\pi$. Also note that this $\mathbb{Z}_2$ duality symmetry is anomalous: it acts as a $\mathbb{Z}_2$ symmetry on the Majoranas but it actually squares to a translation, so that it cannot be implimented in an on-site way in terms of the original Ising (spin) variables. Indeed, there is no operator that we can write down in terms of the spin variables that is the charge operator for this symmetry.

Finally, we expand on why duality acts as translation by one site, and explain why this is equivalent to changing the sign of the fermion mass m. Recall that at the critical point, the Majorana chain is

$$H = iJ\sum_j^{2N} \eta_j \eta_{j+1}, \tag{587}$$

where the chain has N physical sites. Let us define the Majorana fields ξ, γ by

$$\eta_j = \frac{1}{\sqrt{2}}(\xi_j + (-1)^j \gamma_j). \tag{588}$$

The factor of $1/\sqrt{2}$ ensures that they have the usual $\{\xi_i, \xi_j\} = \{\gamma_i, \gamma_j\} = 2\delta_{ij}$ Clifford algebra relation. The $(-1)^j$ is needed since ξ, γ are the slowly varying fields which represent linearizations about the two points where the dispersion touches zero (the chemical potential for the Majoranas vanishes). Writing H in momentum space gives $H \sim \sum_{k>0} \eta_k^\dagger \eta_k \sin(k)$, so that the dispersion has zeros at $k = 0$ (the ξ mode) and at $k = \pi$ (the γ mode). Hence γ_j comes with a factor of $e^{\pi i j}$.

Anyway, putting this expansion into the Hamiltonian, we see that the $(-1)^j$ factors kill the off-diagonal terms, and so in the continuum we get a massless Majorana as expected:

$$H = \frac{iJ}{2}\sum_j (\xi_j \xi_{j+1} + \gamma_{j+1}\gamma_j) \implies S = \frac{iJ}{2}\int \bar{\Xi}\partial\!\!\!/ \Xi, \tag{589}$$

where $\Xi = (\xi, \gamma)^T$. Here we are taking $\gamma^0 = X, \gamma^1 = Y$.

Now let's add the term

$$\delta H = im \sum_j (-1)^j \eta_j \eta_{j+1}. \tag{590}$$

Since the coupling is alternating on each bond, the coupling strength changes within physical sites and between physical sites. Thus we expect that this term should be the one which drives us away from the critical point, where all the hopping strengths are equal. Indeed, if we put in our expansion for η, we get

$$\delta H = im \sum_j \left((-1)^j [\xi_{j+1}\xi_j - \gamma_{j+1}\gamma_j] + \xi_{j+1}\gamma_j - \gamma_{j+1}\xi_j\right). \tag{591}$$

The terms that vary as $(-1)^j$ die when we go to the continuum since they oscillate fast and cancel out, and so we have

$$\delta H = im \int dx \; (-\gamma - \partial_x \gamma)\xi + (\xi + \partial_x \xi)\gamma) \implies \delta S = 2 \int im\xi\gamma = -\int \bar{\Xi} imZ\Xi, \tag{592}$$

where the derivative terms have canceled. So δH is indeed a Majorana mass term. Now we know from above that $m \mapsto -m$ should be equivalent to doing duality. And indeed, from the definition of δH we see that sending $m \mapsto -m$ is equivalent to $j \mapsto j + 1$, i.e. it is equivalent to translation by one site (half of a physical lattice constant). This is another check that duality is realized by translation through half a physical lattice site. In the continuum, it can be implemented via the transformation $\Xi \mapsto Z\Xi$, which leaves the kinetic term invariant but changes the mass term as $m \mapsto -m$. That Z is the right matrix to use here makes sense if we look at (588), where we see that duality (translation by a lattice site) should take $\xi \mapsto \xi$ and $\gamma \mapsto -\gamma$, which is exactly what Z does.

Yet more on the $SU(2)$ point of the $c = 1$ CFT

This is a fast one, and comes from a problem in a pset assigned in Ashvin Vishwanath's fall 2018 class on quantum matter. The problem statement is as follows:

Consider the fermionized description of the anisotropic XXZ chain, with Hamiltonian

$$H = \sum_i \left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + g S_i^z S_{i+1}^z\right]. \tag{593}$$

Write down an expression for $S^z(r)$ in terms of the low energy fermion fields, including components both at zero momentum and at momentum $q = \pi$.

What value of g corresponds to the self-dual $SU(2)$ point? Find out by requiring that the $q = \pi$ component of the S^z spin density have the same power law exponent as the 2-point function of the S^+ operator. We will work in conventions where the bosonized Lagrangian is written as $\mathcal{L} = (\partial_\mu \theta / 2\pi)^2/(2K)$.

First of all,

$$S_r^{\pm} \to (-1)^r (-1)^{\sum_{r'<r} n_{r'}}, \tag{594}$$

where the $(-1)^r$ is needed to cancel a minus sign from moving a string operator past a creation operator in $S_i^+ S_{i+1}^-$ in order to produce a correct-sign hopping. Therefore we have

$$S_r^z = S_r^+ S_r^- - \mathbf{1}/2 = n_r - \mathbf{1}/2. \tag{595}$$

Thus in terms of the low energy fields,

$$\begin{aligned} S_r^z = c_r^\dagger c_r - 1/2 &= (e^{-ik_F r} R^\dagger(r) + e^{ik_F r} L^\dagger(r))(e^{ik_F} R(r) + e^{-ik_F r} L(r)) - 1/2 \\ &= n_R(r) + n_L(r) + (-1)^r (L^\dagger(r) R(r) + R^\dagger(r) L(r)) - 1/2, \end{aligned} \tag{596}$$

since $k_F = \pi/2$. While $(-1)^r$ carries momentum π so does $L^\dagger(r)R(r) + h.c.$, so that $\sum_j S_j^z$ has net zero momentum as required.

The part of the S^z 2-point function that goes as $(-1)^r$ is the part involving scattering from one of the Fermi points to the other:

$$\langle S_r^z S_0^z \rangle \ni (-1)^r \langle L^\dagger(r) R(r) R^\dagger(0) L(0) \rangle \to (-1)^r \langle e^{-i\theta(r)} e^{i\theta(0)} \rangle, \tag{597}$$

since in our conventions $L^\dagger(x) \to e^{i\phi_L(x)}$, $R(x) \to e^{-i\phi_R(x)}$ and $\theta = \phi_R - \phi_L$. From the action we read off that the propagator for the θ field is

$$G_\theta(r) = -2\pi K \ln |r|, \tag{598}$$

and so

$$\langle S_r^z S_0^z \rangle \to \frac{1}{|r|^{2\pi K}}. \tag{599}$$

Now we look at the $S^\pm$ correlator. We can figure out the image of $S_r^\pm$ under bosonization from its commutation relation with Z_r, which we know maps to $n_r - 1/2 = \partial_x \theta/(2\pi) - 1/2$. Since $[Z_r, S_r^\pm]] = \pm S_r^\pm$, we guess that $S_r^\pm \to e^{\pm i\phi(r)}$. This identification is natural since we know $\phi(r)$ gets shifted by the $U(1)$ symmetry of rotations about the z axis, in the same way that S^+ does. Indeed, from the number-phase relation $[\phi(x), \partial_y \theta(y)/2\pi] = i\delta(x - y)$, we check that

$$[Z_r, S_r^\pm] \to [\partial_r \theta, e^{\pm i\phi(r)}] = \pm e^{i \pm \phi(r)} \to \pm S_r^\pm, \tag{600}$$

as required. Actually this doesn't completely fix the bosonization of $S_r^\pm$. We will actually include an explicit factor of $(-1)^r$ in its bosonization (as we wrote down above), so that

$$S_r^\pm \to (-1)^r e^{\pm i\phi(r)}. \tag{601}$$

(the reason for doing this is to get the correct sign for the fermion kinetic term).

Anyway, this means that

$$\langle S_r^+ S_0^- \rangle = (-1)^r \langle e^{i\phi(r)} e^{-i\phi(0)} \rangle = (-1)^r e^{G_\phi(r)}. \tag{602}$$

The propagator $G_\phi(r)$ for ϕ is determined by T duality: if the coefficient of the free θ action is $R^2/4\pi$, then the coefficient for the free ϕ action is $1/(4\pi R^2)$. For us $R^2 = 1/2\pi K$, and so the coefficient for the ϕ action will be $K/2$. Thus we have

$$G_\phi(r) = -\frac{1}{2\pi K} \ln |r| \implies \langle S_r^+ S_0^- \rangle \to (-1)^r \frac{1}{|r|^{1/2\pi K}}. \tag{603}$$

If we require that the power law exponents in the $(-1)^r$ parts of the $S^\pm$ and S^z two point functions match, which is a necessary requirement if the theory is to have $SU(2)$ symmetry, then we require that $2\pi K = 1/(2\pi K)$, so that we predict the $SU(2)$ point to be located at $K = 1/2\pi$. We know from above that the operator $\cos 2\theta$ becomes marginal when $K = 1/2\pi$, so that the $SU(2)$ point is characterized by the radius at which the $\cos 2\theta$ Umklapp term crosses over between relevance and irrelevance.

Hermitian conjugation and inner products for Euclidean CFTs

Today we will essentially be doing a collection of several exercises in David Simmons-Duffin's class notes on CFT which tell us how to think about Hilbert space structures in Euclidean-space fomulations of CFTs.

We relate spin-1 operators in Lorentzian and Euclidean signatures via

$$\mathcal{O}_E^l(\tau, \mathbf{x}) = (-i)^{\delta_{l,0}} \mathcal{O}_L^l(t, \mathbf{x}), \tag{604}$$

where l is a vector index. The reason for the prefactor is because it is required to ensure that $O(d)$ or $O(d-1,1)$ transformations commute with the processes of continuing between signatures. We can see that this prefactor is correct by mapping a spin-1 operator to real time, doing a Lorentz tform, and then mapping back to E time:

$$\mathcal{O}_E^l(\tau, \mathbf{x}) \mapsto (-i)^{\delta_{l,0}} \mathcal{O}_L^l(t, \mathbf{x}) \mapsto \Lambda^l_m (-i)^{\delta_{m,0}} \mathcal{O}_L^m(t, \mathbf{x}) \mapsto i^{\delta_{l,0} - \delta_{m,0}} \mathcal{O}_E^m(\tau, \mathbf{x}). \tag{605}$$

If this is to be a sensible way of going between E and L signatures, then an $O(d)$ rotation R^{lm} must be related to the $O(d-1,1)$ rotation Λ^l_m via (our use of is here is legit because we are considering the complexification of the orthogonal groups)

$$R^{lm} = \Lambda^l_m i^{\delta_{l,0} - \delta_{m,0}}. \tag{606}$$

Let's check to see that this works: if it works, then the LHS needs to preserve δ_{lm}. Being slightly callous about index placement, and working in mostly positive signature, we have

$$R^{lm}R^{ln} = \Lambda^l_m\Lambda^l_n i^{2\delta_{l,0}-\delta_{m,0}-\delta_{n,0}} = \Lambda^l_m\Lambda_{ln} i^{-\delta_{m,0}-\delta_{n,0}} = \eta_{mn} i^{-\delta_{m,0}-\delta_{n,0}} = \delta_{mn}, \tag{607}$$

and so the RHS of (606) is indeed orthogonal. A general spin operator then maps as

$$\mathcal{O}_E^{l_1\ldots l_n}(\tau,\mathbf{x}) \mapsto (-i)^{\sum_i \delta_{l_i,0}}\mathcal{O}_L^{l_1\ldots l_n}(t,\mathbf{x}). \tag{608}$$

Now in Euclidean signature there is no natural choice of inner product. Since we want to ground the whole formalism in real time where physics is defined, we then use the inner product in real time to construct one in Euclidean time. The Euclidean Hermitian conjugation must then reverse the Euclidean time τ—different choices of what τ is lead to different Hilbert spaces, and hence to different quantizations of the theory. In addition to reversing τ, conjugation also changes the signs of vector indices with time components, since these have an extra i associated to them in Euclidean signature, as described above. Therefore the Euclidean Hermitian conjugation acts as

$$[\mathcal{O}_E^{l_1\ldots l_n}(\tau,\mathbf{x})]^\dagger = (-1)^{\sum_i \delta_{l_i,0}}\mathcal{O}_E^{l_1\ldots l_n}(-\tau,\mathbf{x}). \tag{609}$$

This is actually very sensible: since $\dagger$ flips the τ coordinate, vector indices pointing in the τ direction get minus signs. An easy mistake to make is to apply this formula blindly to everything with a vector index. If do this, we would conclude that e.g. $P_1^\dagger = -P_1$ and $P_l^\dagger = P_l$ for $l > 1$ (here P_i is the physical momentum operator on e.g. the cylinder; we are not (yet) working in radial quantization, where of course the momentum is conjugated as $P^\dagger = K$). However, this is a bit distressing, since we know that P_1 is the Hamiltonian, which should be Hermitian. In fact, the transformation rule is the opposite: P_1 is Hermitian, and the rest are anti-Hermitian (which can be seen e.g. by taking the $\dagger$ of $[P_1,\phi(0)] = \partial_\tau\phi(0)$, with $\phi(0)^\dagger = \phi(0)$). Likewise, one can show that $M_{lm}^\dagger = -M_{lm}$ for $l,m > 1$, while $M_{1l}^\dagger = M_{1l}$. So, one should only apply our Hermitian conjugation rule to primaries.

When we quantize on the cylinder $\mathbb{R}\times S^{d-1}$, τ is just the z-coordinate of the cylinder, and we can use the above formula for Hermitian conjugation. However, since we will more often work in radial quantization on the plane, we need to port this definition of Hermitian conjugation into radial quantization conventions. The zero of Euclidean time is the unit sphere in $\mathbb{R}^d$, and so $\tau \mapsto -\tau$ corresponds to the inversion $x^\mu \mapsto x^\mu/x^2$. The minus sign in the mapping (609) is accounted for in radial quantization by contracting all the indices of $\mathcal{O}_E$ with the matrix $I^\mu_\nu(x) = \delta^\mu_\nu - 2x^\mu x_\nu/x^2$, taking $x^\mu \mapsto -x^\mu$. Finally, we need to remember that the cylinder and the plane have metrics differing by a Weyl rescaling. Since $r \to e^\tau$, the Weyl rescaling is determined via $r^2(dr^2/r^2 + d\Omega^2_{d-1}) \to e^{2\tau}(d\tau^2 + d\Omega^2_{d-1})$, so that the rescaling factor is $e^{2\tau}$. Now we need to remind ourselves that if we have two different coordinate systems with metrics such that $ds_1^2 = e^{2w}ds_2^2$, then an operator with scaling dimension Δ is given in the two coordinate systems by $\mathcal{O}(x_1) = e^{-w\Delta}\mathcal{O}(x_2)$. This means that $\mathcal{O}_E(x^\mu) = e^{-\tau\Delta}\mathcal{O}_E(\tau,\mathbf{x})$. Therefore we can determine how $\dagger$ acts on scalar operators in radial quantization via

$$\mathcal{O}(x^\mu)^{\dagger_r} = e^{-\Delta\tau}\mathcal{O}(\tau,\mathbf{x})^\dagger = e^{-\Delta\tau}\mathcal{O}(-\tau,\mathbf{x}) = e^{-\Delta\tau}(e^{\Delta(-\tau)}\mathcal{O}(x^\mu/x^2)) = r^{-2\Delta}\mathcal{O}(x^\mu/x^2). \tag{610}$$

For operators with spin we just add in the I matrices we mentioned above to this equation, and hence

$$[\mathcal{O}_E^{l_1\dots l_n}(x^\mu)]^{\dagger_r} = r^{-2\Delta} I^{l_1}_{m_1}(x)\cdots I^{l_n}_{m_n}(x)\mathcal{O}_E^{m_1\dots m_n}(x^\mu/x^2). \tag{611}$$

The factor of $r^{-2\Delta}$ is crucial here, since it is needed to make inner products like

$$\langle\mathcal{O}|\mathcal{O}\rangle = \langle 0|\mathcal{O}(0)^{\dagger_r}\mathcal{O}(0)|0\rangle = \lim_{r\to\infty} r^{2\Delta}\langle 0|\mathcal{O}(r)\mathcal{O}(0)|0\rangle \tag{612}$$

finite in radial quantization (and properly dimensionless).

Unitarity bounds on CFT scaling dimensions

Today we'll be looking at unitarity constraints on scaling dimensions, and will essentially be doing an elaboration on an exercise in David Simmons-Duffin's class notes on CFT showing how various bounds on scaling dimensions arise from unitarity constraints.

First we will prove a bound on the scaling dimension of all primary operators with nonzero spin, namely that if $\mathcal{O}_n$ is a primary operator with spin $n > 0$, then

$$\Delta_{\mathcal{O}_n} \geq d - 2 + n. \tag{613}$$

In particular, for $n = 1$, this tells us that the minimal possible scaling dimension is $d - 1$, which is the dimension that a conserved current has. In fact we will see that a spin-1 operator is a conserved current iff its scaling dimension saturates the bound: conserved currents have no anomalous dimensions, and any spin-1 operator with no anomalous dimension is a conserved current.

First we have to define what we mean by "an operator of spin n" in dimensions other than 3. In here and what follows, spin n will always refer to an operator which transforms in the n-index symmetric traceless tensor representation of $SO(d)$. While for $SO(3)$ these representations exhaust all irreps,[48] this is of course not true for $SO(d > 3)$, when irreps assigned to antisymmetric tensors become possible to construct. However, conserved currents and energy momentum tensors and the like are

[48] Since any tensor with a pair of antisymmetric indices can be reduced contracting with the invariant symbol ϵ_{ijk}. This reduces the number of indices in the tensor and lets us turn antisymmetric tensors into symmetric ones. This fails for larger $SO(d)$ since contracting with the invariant ϵ symbol does not decrease the number of indices (for $d > 4$ it increases the number of indices).

usually all associated to symmetric traceless representations, so we will restrict our attention to them in what follows.

The proof goes by considering the inner product between two descendants in the conformal multiplet of a spin-n conformal primary $\mathcal{O}$:

$$\left(P^{\mu}|\mathcal{O}^{\boldsymbol{\beta}}\rangle\right)^{\dagger} P_{\nu}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = \langle\mathcal{O}^{\boldsymbol{\beta}}|K_{\nu}P^{\mu}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle. \tag{614}$$

Here the notation is such that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ label multi-indices in accordance with the symmetric and traceless nature of $\mathcal{O}$, so that e.g. $\mathcal{O}^{\boldsymbol{\alpha}} = \mathcal{O}^{\alpha_{(1},\ldots,\alpha_{n)}}$ (note the symmetrization). Note that this is not the inner product between two identical vectors and hence is not necessarily positive. Also note that we are not assuming any contraction between the μ, ν indices and those carried by $\mathcal{O}$. Using the commutator

$$[K_{\mu}, P_{\nu}] = 2\delta_{\mu\nu}D - 2M_{\mu\nu}, \tag{615}$$

where $M_{\mu\nu}$ is the $SO(d)$ generator for the rotation parametrized by μ, ν in the spin-n representation, we have (since $\mathcal{O}$ is primary)

$$\left(P_{\mu}|\mathcal{O}^{\boldsymbol{\beta}}\rangle\right)^{\dagger} P_{\nu}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = 2\langle\mathcal{O}^{\boldsymbol{\beta}}|\delta_{\nu\mu} - M_{\mu\nu}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle. \tag{616}$$

Now we use a clever trick: since $M_{\mu\nu} = -M_{\nu\mu}$, we can write

$$M_{\mu\nu} = \sum_{i<j=1}^{d} [A^{ij}]_{\mu\nu} M_{ij}, \tag{617}$$

where $[A^{ij}]_{\mu\nu} = \delta^i_{\mu}\delta^j_{\nu} - \delta^j_{\mu}\delta^i_{\nu}$ is the generator matrix of the rotation matrix parametrized by the tuple ij, represented in the vector representation. Let us normalize the states by

$$\langle\mathcal{O}^{\boldsymbol{\beta}}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = \delta^{\boldsymbol{\beta}\boldsymbol{\alpha}}. \tag{618}$$

Therefore we may write

$$\frac{1}{2}\left(P_{\mu}|\mathcal{O}^{\boldsymbol{\beta}}\rangle\right)^{\dagger} P_{\nu}|\mathcal{O}^{\boldsymbol{\alpha}}\rangle = \delta_{\mu\nu}\delta^{\boldsymbol{\alpha}\boldsymbol{\beta}}\Delta - \sum_{a}(A^a \otimes M^a)^{\boldsymbol{\alpha}\boldsymbol{\beta}}_{\mu\nu}, \tag{619}$$

where a runs over the generators of $\mathfrak{so}(d)$.

Hence we have obtained the inner product as a matrix element of a matrix constructed from a $\otimes$ of a matrix in the F irrep and one in the S_n irrep, where F is the fundamental (vector) irrep and S_n is the symmetric, traceless, n-index tensor irrep. To deal with this expression, we should reduce this $\otimes$ to its constituent irreps. This happens by writing

$$\begin{aligned}\sum_{a}(A^a \otimes M^a)^{\boldsymbol{\alpha}\boldsymbol{\beta}}_{\mu\nu} &= \frac{1}{2}\sum_{a}\left[(A^a \otimes \mathbf{1} + \mathbf{1} \otimes M^a)^2 - (A^a)^2 \otimes \mathbf{1} - \mathbf{1} \otimes (M^a)^2\right) \\ &= -\frac{1}{2}\left(C_2(F \otimes S_n) - C_2(F) \otimes \mathbf{1} - \mathbf{1} \otimes C_2(S_n)\right),\end{aligned} \tag{620}$$

since the generators for the $R_1 \otimes R_2$ rep are $T^a_{12} = T^a_1 \otimes \mathbf{1} + \mathbf{1} \otimes T^a_2$. Here the minus sign is because our generators are anti-Hermitian (e.g. A^a is real and antisymmetric), and since the quadratic Casimir is positive definite and is hence $\sum_a (T^a)^\dagger T^a = -\sum_a T^a T^a$ (e.g. $L^2 = -\partial_\mu \partial^\mu$ since ∂_μ is anti-Hermitian).

The quadratic Casimir[49] for the traceless symmetric nth-rank tensor representation of $SO(d)$ is allegedly

$$C_2(S_n) = n(n+d-2). \tag{621}$$

I tried for a stupidly long time to check this, but didn't quite figure it out. The thing I did figure out was the dimension of the representation S_n—to get C_2 we would then need the index, which I couldn't couldn't compute. The computation of the dimension is preserved for posterity's sake in the following footnote.[50]

Let's now go into a basis appropriate for the $\oplus$ decomposition of $F \otimes S_n$ into constituent irreps. In this basis, we have

$$(P|\mathcal{O}\rangle)^\dagger \, P|\mathcal{O}\rangle = 2\Delta \mathbf{1} - C_2(F) \otimes \mathbf{1} - \mathbf{1} \otimes C_2(S_n) + \bigoplus_{R \in F \otimes S_n} C_2(R), \tag{623}$$

where we are now thinking of the LHS as a matrix, rather than a matrix element. Now since the LHS is the matrix whose entries are the inner products of basis vectors for the first descendants of $\mathcal{O}$, it must be positive definite. The potential for it to have negative eigenvalues is contained within the choices for the different irreps in the $\oplus$ term: to get the strongest bound on Δ we should look for representations $R \in F \otimes S_n$ such that $C_2(R)$ is minimized. Since C_2 is larger for larger irreps, we should choose the smallest irrep appearing in the $\oplus$ decomposition. The smallest irrep is S_{n-1}, which we get from contracting the vector index of F with any of the indices of the symmetric tensors in S_n. Therefore we must have

$$2\Delta \geq (d-1) + n(n+d-2) - (n-1)(n-3+d) \implies \Delta \geq n+d-2, \tag{624}$$

as claimed. Note that this bound only applies to primary operators. For example, one shouldn't worry that the dimension of A^μ in free Maxwell theory is $(d-2)/2 < d-1$, since A^μ is not a primary (it is not even an operator in the physical Hilbert space).

[49] Just "Casimir" is imprecise terminology since for general groups there are usually many different Casimirs we can construct (involving kth powers of the generators—each k-index invariant symbol gives us such a Casimir).

[50] We want to figure out the dimension of the n-index symmetric tensor irrep of $SO(d)$. A tensor transforming under this irrep has n indices, each of which can take d values: therefore counting the number of such tensors amounts to the number of ways we can place n different objects (the indices of the tensor) in d different buckets. Only the total number of objects that gets placed in each bucket matters—the objects being placed in the buckets are all identical, on account of the symmetric property of the tensor meaning that all the indices can be freely exchanged. Recall from stat mech class that the best way to think about this is to consider counting the number of ways to arrange n objects and $d-1$ dividers between the different buckets. Since the dividers and objects are all identical, this number is $\binom{n+d-1}{n}$. Thus the number of symmetric tensors is $(n+d-1)!/(n!(d-1)!)$.

However, we need to take care of the traceless condition. This reads $T^{\mu\mu\nu_1\dots\nu_{n-2}} = 0$, where by symmetry it doesn't matter where in the index structure the two μs are. Note that the remaining indices $\nu_1, \dots, \nu_{n-2}$ are symmetric, and so the traceless condition gives us a number of constraints equal to the number of rank-$(n-2)$ symmetric tensors. By the above this is $\binom{n+d-3}{n-2}$. Therefore the dimension of S_n is

$$\dim[S_n] = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}. \tag{622}$$

For $d=3$ we get $2n+1$ as expected, while for e.g. $d=4$ we get $(n+1)^2$. From the form of the quadratic Casimir, it's clear that the formula for the index $T(S_n)$ is going to be complicated.

Now we will prove that all scalar operators have scaling dimension bounded from below by $\Delta \geq (d-2)/2$, which is saturated for the free scalar. Consider the inner product

$$0 \leq (P_\mu P^\mu |\mathcal{O}\rangle)^\dagger P_\mu P^\mu |\mathcal{O}\rangle = \langle \mathcal{O}| K_\mu K^\mu P_\nu P^\nu |\mathcal{O}\rangle. \tag{625}$$

Getting to the bound on Δ is now just a matter of algebra. Using the commutators

$$[D, K_\mu] = -K_\mu, \qquad [M_{\mu\nu}, P_\lambda] = \delta_{\nu\lambda} P_\mu - \delta_{\mu\lambda} P_\nu, \tag{626}$$

we have, using $M_{\mu\mu} = 0$,

$$\begin{aligned}
\langle \mathcal{O}| K_\mu K^\mu P_\nu P^\nu |\mathcal{O}\rangle &= \langle \mathcal{O}| K^\mu (2\delta_{\mu\nu} D - 2M_{\mu\nu} + P_\nu K_\mu) P^\nu |\mathcal{O}\rangle \\
&= \langle \mathcal{O}| 2(1+\Delta) K_\nu P^\nu - 2K^\mu (dP_\mu - \delta_{\mu\nu} P_\nu - P^\nu M_{\mu\nu}) + 4(\delta_{\mu\nu} D - M_{\mu\nu})^2 |\mathcal{O}\rangle \\
&= \langle \mathcal{O}|\mathcal{O}\rangle (4\Delta d(1+\Delta) - 4d^2\Delta + 4d\Delta + 4d\Delta^2) \\
&= 4d\Delta \langle \mathcal{O}|\mathcal{O}\rangle (2\Delta + 2 - d).
\end{aligned} \tag{627}$$

In order for this to be positive, we need either $\Delta = 0$ (which is the case if $\mathcal{O} = \mathbf{1}$, or $\Delta \geq (d-2)/2$, proving the bound. Note that since the inner product is positive-definite, if $\mathcal{O}$ is a scalar saturating the bound, then we necessarily have $\partial^2 \mathcal{O} = 0$ since in that case $\langle \partial^2 \mathcal{O} | \partial^2 \mathcal{O}\rangle = 0$, so that any scalar saturating the bound necessarily obeys the free-particle wave equation.

The point here is that the effects of interactions on a free scalar always increase its scaling dimension, meaning that interactions make correlations decay faster. This quasi makes sense—adding interactions means that a disturbance in a field ϕ will have a harder time propagating from x to y than it would if ϕ were free, since the disturbance can interact with fluctuations and decay faster as it propagates. This is also somewhat corroborated by imagining adding weak interactions to a given CFT, and using CPT to compute new beta functions like $\beta_g \sim y_g^{UV} g - C_{gg}^g g^2/2$—if the interaction is repulsive so that $g_* > 0$, then with positive OPE coefficients we will have $y_g^{IR} < y_g^{UV}$, meaning that interactions have increased Δ_g.

However, it is not always the case that turning on interactions increases the scaling dimensions of all operators. ethan: *to do: add the example of the $\mathbb{CP}^N$ model and the scaling dimension of $z^\dagger \sigma z$*

Inversions, reflections, and SCTs

This is a short problem posed by Rychkov during his 2019 TASI lectures: show that a CFT has inversion symmetry iff it has reflection symmetry by relating the two symmetries with conjugation by an element of $SO(d+1,1)$.

Since reflections (and inversions) aren't in the identity component of $O(d+1,1)$, CFTs don't necessarily have to have them as symmetries—the only symmetry a CFT must for sure have is $SO(d+1,1)$.

Our goal is to show that

$$R_\mu = O_\mu^{-1} I O_\mu, \qquad O_\mu \in SO(d+1,1), \tag{628}$$

where $I : x^\mu \mapsto x^\mu/x^2$ is the inversion and $R_\mu : x^\nu \mapsto (-1)^{\delta_{\mu,\nu}} x^\nu$ is the reflection of the μ coordinate. If this is true, then we will have proved that reflections and inversions are continuously connected—they are in the same connected component of $O(d+1,1)$—and so a CFT which has one as a symmetry necessarily has the other as a symmetry.

The inversion exchanges zero and infinity, and since R_μ leaves both zero and infinity invariant, any candidate O_μ will need to act nontrivially at infinity. The only generator of $SO(d+1,1)$ that acts at infinity is the SCTs K_ν, and so we will need to make use of the SCTs in constructing O_μ.

First let's try to understand what the hell the SCTs actually do. A finite SCT acts on the coordinates as

$$x^\mu \mapsto \frac{x^\mu - a^\mu x^2}{1 - 2a\cdot x + a^2 x^2}. \tag{629}$$

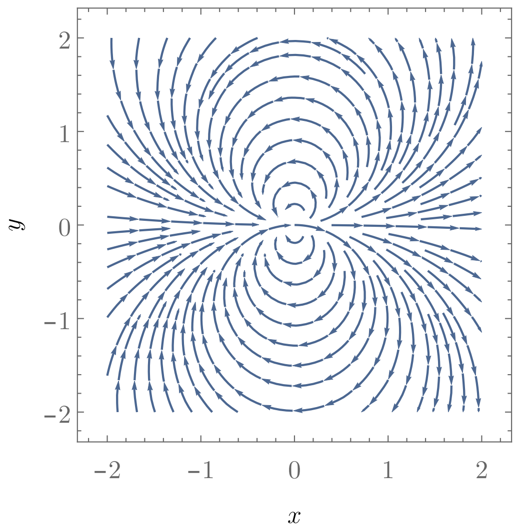
We can already see that this will move infinity, since it sends infinity to $-a^\mu/a^2$, and a^μ/a^2 to infinity. It also leaves zero invariant. In order to visualize better how it works, we can find the vector field associated with infinitesimal SCTs. Taking $|a|$ infinitesimal in the above equation, we find

$$x^\mu \mapsto 2(a\cdot x)x^\mu - a^\mu x^2 = a^\lambda(2x_\lambda x^\nu \partial_\nu - x^2\partial_\lambda)x^\mu, \tag{630}$$

and so K_μ is associated with the vector field

$$K_\mu = 2x_\mu(x\cdot\partial) - x^2\partial_\mu. \tag{631}$$

From this equation, we can see that K_μ acts by "swirling" around the origin. For example, in two dimensions, the operator K_x acts to move points along the following flow lines:



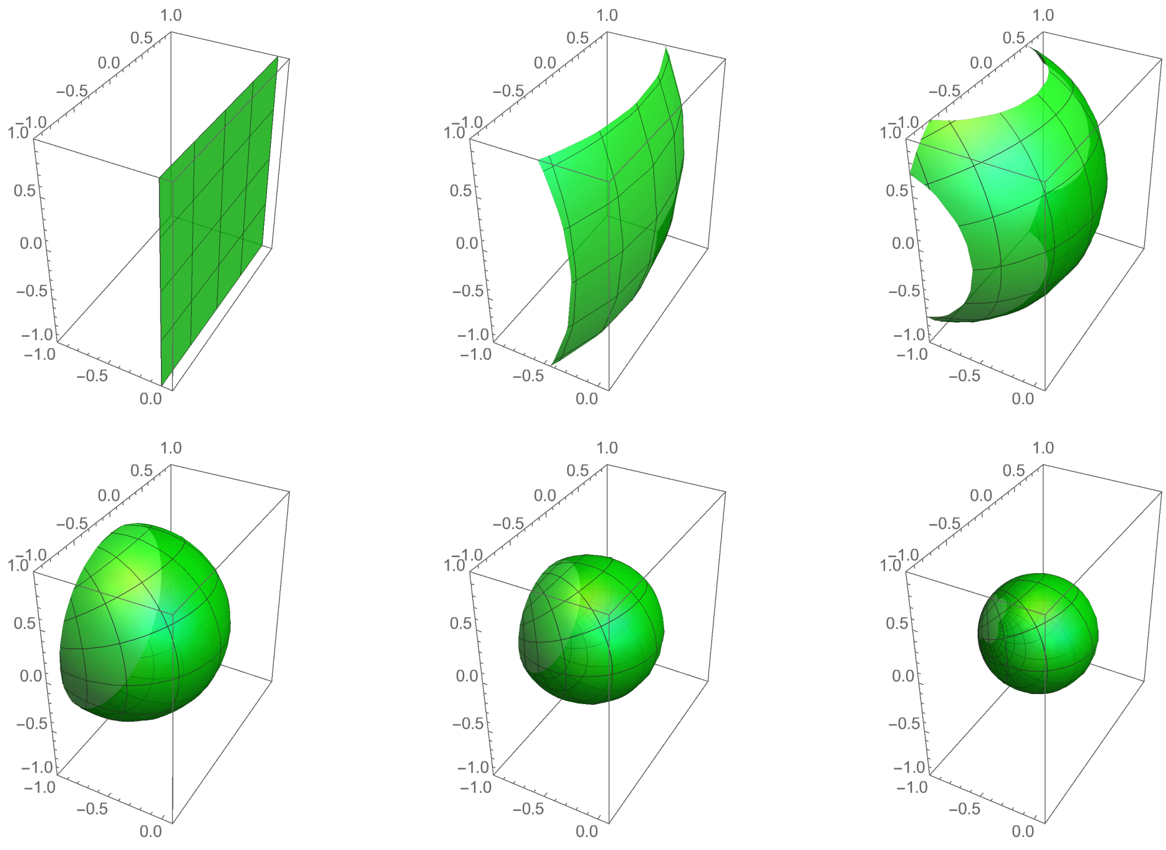
. (632)

I wish this plot had been included in e.g. the big yellow book; it was only after seeing this that I finally got some intuition for what K does.

Now, in order for $O_\mu^{-1} I O_\mu$ to be a reflection, O_μ must send the $x^\mu = 0$ hyperplane to the unit sphere, since the unit sphere is what I reflects about. The O_μ^{-1} will then map the sphere back to the $x^\mu = 0$ plane, and we will have accomplished a reflection about the plane. To see why the SCT can help do this, consider e.g. the image of the y axis in $\mathbb{R}^2$ under the SCT $K_x(1)$ (here our notation is that $K_\mu(t)$ is the SCT given by (629) with $a^\mu = t\hat{x}^\mu$):

$$K_\mu(1) : (0, y) \mapsto \frac{1}{1+y^2}(-y^2, y). \tag{633}$$

One then checks that the RHS defines a circle of radius 1 centered at $(-1/2, 0)$. This also happens in higher dimensions, with $K_\mu(1)$ mapping the $x^\mu = 0$ plane to the unit sphere centered on $x^\mu = -1/2$. This is perhaps best illustrated by some graphics. In the following figure, as me move along the grid, we show the image of the yz plane under $K_x(t)$ for t ranging from 0 to 1. By the time we're done, the plane is indeed compactified to a unit sphere centered at $(-1/2, 0, 0)$:



(634)

Anyway, summing up, we see that the reflection R_μ is smoothly connected to the reflection through the homotopy

$$R_\mu(t) = [T_{-\mu}(t/2)K_\mu(t)]^{-1} I T_{-\mu}(t/2)K_\mu(t), \tag{635}$$

where $t \in [0, 1]$ with $t = 1$ the reflection and $t = 0$ the inversion, and where $T_{-\mu}(t/2)$ is the translation by $t/2$ in the $-\hat{x}^\mu$ direction.

I think that this also provides us with a way to argue that any theory which is reflection-positive with respect to reflection about a plane is also reflection-positive in radial quantization (this works for primaries, but non-primary operators transform in more complicated ways under the mapping described above, though).

Scale invariance, conformal invariance, and algebras of charge operators

Today we're going to do an elaboration on an exercise given by Slava Rychkov during his 2019 TASI lectures. The goal is to explicitly compute the algebra of the charge operators for conformal transformations by showing that

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = Q_{[\eta,\varepsilon]}(\Sigma), \tag{636}$$

where $Q_\varepsilon(\Sigma)$ is the operator which generates the conformal transformation given by the vector field ε along the submanifold Σ, and where $[\varepsilon, \eta]$ is the Lie bracket. We will be working in Euclidean spacetime of arbitrary dimension.

First, we will only assume translation and rotation symmetry. We want to calculate $Q_\varepsilon(\Sigma) \cdot T^{\mu\nu}(0)$, for Σ a surface enclosing 0, which we can do by general reasoning. The only two ϵs we have access to for translations and rotations are $\varepsilon_\mu = a_\mu$ with a_μ constant (translations) and $\epsilon_\mu = \omega_{\mu\nu} x^\nu$ with $\omega_{\mu\nu}$ constant and antisymmetric (rotations). Now since $T^{\mu\nu}$ is conserved, the Ward identity tells us that its dimension is exactly 2. Since $Q_\varepsilon(\Sigma) \cdot T$ needs to be linear in ε, dimensional analysis then tells us that each term appearing in $Q_\varepsilon(\Sigma) \cdot T$ should contain one ε, one T, and one ∂. Therefore we may write

$$Q_\varepsilon(\Sigma) \cdot T_{\mu\nu} = \epsilon^\rho \partial_\rho T_{\mu\nu} + A \partial_\mu \epsilon^\rho T_{\rho\nu} + B \partial_\nu \epsilon^\rho T_{\rho\mu} \tag{637}$$

for some constants A, B. The coefficient of the first term has been fixed to one because we want momentum to act on all operators as $P_\varepsilon = \varepsilon^\mu \partial_\mu$. By $\mu \leftrightarrow \nu$ symmetry, we have $B = A$, and taking the divergence of this then tells us that $A = 1$.[51] This means we can write

$$Q_\varepsilon(\Sigma) \cdot T_{\mu\nu} = \varepsilon^\rho \partial_\rho T_{\mu\nu} - \partial^\rho \varepsilon_\mu T_{\rho\nu} + \partial_\nu \varepsilon^\rho T_{\rho\mu}, \tag{638}$$

where we've used the Killing equation for ε in the second term to change the sign.

[51]Use $\partial_\mu \partial^\mu \varepsilon^\rho = -\partial_\mu \partial^\rho \varepsilon^\mu = -\partial^\rho (\partial \cdot \varepsilon) = 0$.

Now we want to ask what algebra the charges obey. The commutator of charges is computed in the usual way:

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = \int_\Sigma d^{d-1}x^\mu \int_{S_x^{d-1}} d^{d-1}y^\alpha \, T_{\mu\nu}(x)\eta^\nu(x)T_{\alpha\beta}(y)\varepsilon^\beta(y), \tag{639}$$

where S_x^{d-1} is a sphere centered on x. Shrinking down this sphere and then using (638), we have

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = \int_\Sigma d^{d-1}x^\mu \, \eta^\nu(\varepsilon^\rho\partial_\rho T_{\mu\nu} - \partial^\rho\varepsilon_\mu T_{\rho\nu} + \partial_\nu\varepsilon^\rho T_{\rho\mu}). \tag{640}$$

The annoying thing about this is that the derivatives are derivatives in the full $\mathbb{R}^d$, and not covariant derivatives for the metric restricted to the surface Σ. This makes integrating by parts annoying if Σ is chosen to be some generic curved surface. However, since the charge operators are topological, we can choose any representative choice of Σ that we like—in order for integration by parts to be done easily, we will therefore choose Σ to be a hypercube. Integrations by parts can then be performed on each cube face, with the boundary terms all canceling out pairwise. Let us focus on the face F_1 of the cube with unit normal in the x^1 direction, just for concreteness. Then the first integral in (640) is

$$\int_{F_1} d^{d-1}x^1 \, \left[\eta^\nu\epsilon_1\partial^1 T_{1\nu} - \epsilon_\rho\partial^\rho\eta^\nu T_{1\nu} + \epsilon_1\partial^1\eta^\nu T_{1\nu}\right] + \dots, \tag{641}$$

where $\dots$ is a boundary term coming from integrating by parts on F_1—these terms will end up canceling when we sum over cubes, so we will avoid writing them explicitly. The first and last terms above can be combined as

$$\int_{F_1} d^{d-1}x^1 \, \left[\partial^1(\eta^\nu\epsilon_1 T_{1\nu}) - (\nabla_\varepsilon\eta)^\nu T_{1\nu}\right], \tag{642}$$

since ε is Killing (here $\nabla_\varepsilon = \varepsilon^\rho\partial_\rho$).

The next term we will deal with is the second term in (640). We can integrate by parts and write this as

$$-\int_{F_1} d^{d-1}x^1 \, (\partial_1\eta^\nu\varepsilon_1 T_{1\nu} + \eta^\nu\partial^1(\varepsilon_1 T_{1\nu}) - \partial_\rho\eta^\nu\varepsilon_1 T_{\rho\nu}). \tag{643}$$

By the symmetry of $T_{\mu\nu}$ and the Killing equation for η, the last term vanishes, and so this becomes

$$-\int_{F_1} d^{d-1}x^1 \, \partial_1(\eta^\nu\varepsilon_1 T_{1\nu}). \tag{644}$$

Note that this is precisely the right term to cancel the total ∂^1 derivative in (642). Therefore there are only two terms in (640) which survive, which together give

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = \int_\Sigma d^{d-1}x^\mu \, T_{\mu\nu}[(\nabla_\eta\varepsilon)^\nu - (\nabla_\varepsilon\eta)^\nu] = Q_{[\eta,\varepsilon]}(\Sigma). \tag{645}$$

Note the "backwards" order on the Lie bracket!

Now let us add the possibility that the theory also possesses scale invariance: therefore we can allow for transformations with $\partial_{(\mu}\varepsilon_{\nu)} \propto \delta_{\mu\nu}$, provided that $\mathrm{Tr}[T] = 0$. When we put this condition on T, we can allow a new term in the $Q_\varepsilon \cdot T_{\mu\nu}$ OPE, viz.

$$Q_\varepsilon(\Sigma) \cdot T_{\mu\nu} = \varepsilon^\rho \partial_\rho T_{\mu\nu} - \partial^\rho \varepsilon_\mu T_{\rho\nu} + \partial_\nu \varepsilon^\rho T_{\rho\mu} + \partial_\rho \varepsilon^\rho T_{\mu\nu}, \tag{646}$$

where the coefficient of the last term is fixed by requiring that the RHS be conserved. Now when we compute the charge commutator, we get, on each face, the integral

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] \ni \int_{F_1} d^{d-1}x^1\, \eta^\nu [\varepsilon^\rho \partial_\rho T_{1\nu} - \partial^\rho \varepsilon_1 T_{\rho\nu} + \partial_\nu \varepsilon^\rho T_{\rho 1} + \partial_\rho \varepsilon^\rho T_{1\nu}]. \tag{647}$$

Integrating by parts and dropping the boundary terms that will cancel, the first term in the above is

$$\int_{F_1} d^{d-1}x^1 \left[-\nabla_\varepsilon \eta^\nu T_{1\nu} - \eta^\nu (\partial \cdot \varepsilon) T_{1\nu} + \partial_1(\eta^\nu \varepsilon^1 T_{1\nu}) \right]. \tag{648}$$

The first term contains the part of the Lie bracket that we want, while the second term cancels with the third term in (647). Therefore we get

$$[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] \ni \int_{F_1} d^{d-1}x^1 \left[(\nabla_\eta \varepsilon^\nu - \nabla_\varepsilon \eta^\nu) T_{1\nu} + \partial_1(\eta^\nu \varepsilon^1 T_{1\nu}) - \eta^\nu \partial^\rho \varepsilon_1 T_{\rho\nu} \right]. \tag{649}$$

The last term is however

$$-\int_{F_1} d^{d-1}x^1\, [\partial_1(\eta^\nu \varepsilon_1 T_{1\nu}) - \varepsilon_1 \partial_\rho \eta^\nu T_{\rho\nu}]. \tag{650}$$

The first term cancels the second term in (649), while the second term vanishes for the usual reasons of symmetry and tracelessness of T. Therefore we again get $[Q_\varepsilon(\Sigma), Q_\eta(\Sigma)] = Q_{[\eta,\varepsilon]}(\Sigma)$.

ethan: *to-do: discuss why the $d = 2$ case is special, and global vs local conformal transformations*

Free Maxwell is not conformally invariant in $d \neq 4$

Today I'm going to write down some stuff I learned that was inspired during questions in one of Slava Rychkov's 2019 TASI lectures. We will see that free Maxwell theory, while scale invariant in any dimension, is only conformally invariant in $d = 4$. Update: turns out a good chunk of our discussion is contained in Rychkov's earlier paper [8].

There are a few ways to show that Maxwell is only conformally invariant in $d = 4$. One is through the gauge invariance of the improved stress tensor, which I won't go into. The other involve showing that the field strength F is not a primary: since F is used to create all the gauge-invariant local operators, if F is not a primary then the theory won't have any primaries, and we won't get a CFT. I've heard some people say "Maxwell can't be conformally invariant in $d \neq 4$ since only then is e^2 dimensionless", but this argument doesn't work, since it is prejudiced to a certain scaling dimension for A: for pure Maxwell the scaling dimension of A can be the scale-invariant choice of $(d-2)/2$ (no conflict with unitarity here since A isn't a primary), which gives a dimensionless e^2 (free Maxwell *is* scale invariant after all, just not conformally invariant).

F is not a primary because its 2-point functions don't have the right tensor structure

Now for our first way of showing that F isn't a primary. Since the action is free we can explicitly compute the correlator of A, and then that of F follows after some algebra. Since we just want the F correlator, we can choose any gauge fixing we like. We will make the choice where the action becomes $A \wedge \star \Box A$. With this, straightforward calculation gives (ignoring numerical prefactors of 2 and π and so on)

$$\langle \partial_\mu A_\lambda \partial_\nu A_\sigma \rangle = g_{\lambda\sigma} \partial_\mu \frac{(d-2)x_\nu}{x^d} = \frac{(d-2)g_{\lambda\sigma}}{x^d} \left(g_{\mu\nu} - d\frac{x_\mu x_\nu}{x^2} \right). \tag{651}$$

Getting the result of the correlator is then straightforward. If we let

$$\Gamma_{\mu\nu} \equiv g_{\mu\nu} - d\frac{x_\mu x_\nu}{x^2}, \tag{652}$$

then we get

$$\langle F_{\mu\nu} F_{\lambda\sigma} \rangle = \frac{d-2}{x^d} \left(\Gamma_{\mu\lambda} g_{\nu\sigma} - \Gamma_{\nu\lambda} g_{\mu\sigma} - \Gamma_{\mu\sigma} g_{\nu\lambda} + \Gamma_{\nu\sigma} g_{\mu\lambda} \right). \tag{653}$$

If we similarly define

$$\Gamma'_{\mu\nu} \equiv g_{\mu\nu} - \frac{d}{2}\frac{x_\mu x_\nu}{x^2}, \tag{654}$$

then we can write this as

$$\langle F_{\mu\nu} F_{\lambda\sigma} \rangle = \frac{2(d-2)}{x^d} (\Gamma'_{\mu\lambda}\Gamma'_{\nu\sigma} - \Gamma'_{\mu\sigma}\Gamma'_{\nu\lambda}), \tag{655}$$

since the x^μ terms with four different indices cancel and the factor of 2 out front gives the right number of terms quadratic in the metric.

Actually, the form of this correlator really follows without doing any calculations (up to the overall prefactor). Since the theory is free and F is conserved, the dimension of the expression is fixed. Since the F operators are normal-ordered, the components in the tensor structure will never pair up μ with ν or λ with σ; however, the correlator must be antisymmetric under the exchange of both pairs of indices. This basically fixes the correlator to have the structure given above (even the factor of $(d-2)$ can be motivated from our knowledge of the fact that in $d = 2$ there are no propagating degrees of freedom).

Why is this 2-point function incompatible with conformal invariance when $d \neq 4$? Because it doesn't have the right tensor structure. Indeed, the tensor structure of the 2-point function of any conformal primary transforming in some irrep of $SO(d)$ must be built entirely from the "inversion tensors" $I^{\mu\nu} = \delta^{\mu\nu} - 2x^\mu x^\nu/x^2$, which can be seen from looking at the action on the correlator by the (nontrivial) conformal transformations which fix the origin and the point x^μ—a good reference for this is DSD's class notes on CFT, section 8.2. Anyway, since we see that this is only true when $d = 4$, when Γ' becomes the inversion tensor, we can conclude that $F_{\mu\nu}$ can only be a primary in $d = 4$. $F_{\mu\nu}$ can't be a descendant since it is the "smallest" gauge-invariant field; hence free Maxwell is only a CFT in $d = 4$.

F is not a primary because it doesn't transform like one under conformal transforamtions

We now give a different "proof" for why $F_{\mu\nu}$ isn't a primary, by looking at the way it transforms under conformal transformations. This proof is premised on the assumption that A transforms in the same way as a primary would under conformal transformations, i.e. that it transforms by the pull-back of a diffeomorphism up to a conformal re-scaling factor, with scaling dimension $\Delta_A = (d-2)/2$, so that the theory is scale-invariant. Of course A really isn't a primary (its scaling dimension would violate the unitarity bound, and anyway it doesn't appear in the Hilbert space), but it also isn't gauge invariant so this doesn't bother us—we only care about F. Nevertheless, since $F = dA$, the transformation properties of F will be fixed by those of A, so in order for F to stand a chance of being a primary, A is still forced to transform in the same way that a primary would.

We will now give a general analysis of when the derivative of a vector primary is itself a primary.[52] A general CKV ξ_μ can be written as

$$\xi_\mu = a_\mu - \omega_{\mu\nu}x^\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b \cdot x. \tag{656}$$

The above terms represent translations, rotations, scale transformations, and SCTs, respectively. We calculate (in Euc. signature, for simplicity)

$$\partial_\mu \xi_\nu = \omega_{\mu\nu} + \delta_{\mu\nu}(\lambda - 2b \cdot x) + 2b_{[\mu}x_{\nu]} \equiv \delta_{\mu\nu}\sigma_\xi + \widetilde{\omega}_{\xi\mu\nu}. \tag{657}$$

Therefore we may write the transformation of A_μ under an infinitesimal conformal transformation as

$$\delta_\xi A_\mu = -\xi^\nu \partial_\nu A_\mu - \sigma_\xi \Delta_A A_\mu + \widetilde{\omega}_{\xi\mu}{}^\nu A_\nu. \tag{658}$$

This lets us figure out how $\partial_\mu A_\nu$ transforms: using the above expression for $\partial_\mu \xi_\nu$,

$$\begin{aligned} \delta_\xi(\partial_\mu A_\nu) &= -\xi^\alpha \partial_\alpha \partial_\mu A_\nu - \sigma_\xi \partial_\mu A_\nu - \Delta_A(\partial_\mu \sigma_\xi A_\nu + \sigma_\xi \partial_\mu A_\nu) - \widetilde{\omega}_{\xi\mu}{}^\alpha \partial_\alpha A_\nu - \widetilde{\omega}_{\xi\nu}{}^\alpha \partial_\mu A_\alpha + \partial_\mu \widetilde{\omega}_{\xi\nu}{}^\alpha A_\alpha \\ &= [\delta_\xi(\partial_\mu A_\nu)]_{CP} + \partial_\mu \widetilde{\omega}_{\xi\nu}{}^\alpha A_\alpha - \Delta_A \partial_\mu \sigma_\xi A_\nu, \end{aligned} \tag{659}$$

where the $[]_{CP}$ indicates the terms that would be present if $\partial_\mu A_\nu$ were a conformal primary (the σ_ξ part appears as $\sigma_\xi(\Delta_A + 1)\partial_\mu A_\nu$ which is correct since $\Delta_F = \Delta_A + 1$).

[52] Note that $F = dA$ does *not* imply that F is a descendant, even if A is a primary! For ϕ a scalar, $V = d\phi$ means that V is a descendant, but this isn't true if ϕ transforms in a nontrivial irrep of $SO(d)$.

Therefore the remaining two terms are the potential obstruction to F being a primary. We can simplify them by noting that

$$\partial_\mu \widetilde{\omega}_{\xi\nu\alpha} = 2\delta_{\mu\alpha} b_\nu - 2\delta_{\mu\nu} b_\alpha = -\delta_{\mu\alpha}\partial_\nu \sigma_\xi + \delta_{\mu\nu}\partial_\alpha \sigma_\xi. \tag{660}$$

This means

$$\delta_\xi(\partial_\mu A_\nu) = [\delta_\xi(\partial_\mu A_\nu)]_{CP} + \partial_\lambda \sigma_\xi (\Delta_A \delta^\lambda_\mu \delta^\rho_\nu + \delta^\rho_\mu \delta^\lambda_\nu - \delta_{\mu\nu}\delta^{\lambda\rho}) A_\rho. \tag{661}$$

Now consider taking the antisymmetric combination to find $\delta_\xi F_{\mu\nu}$. The antisymmetry in μ, ν means that the last term in the non-primary part dies. The first two terms in the non-primary part are related to one another by the exchange $\mu \leftrightarrow \nu$, and so they cancel after antisymmetrization if and only if $\Delta_A = 1$. Therefore we have the condition

$$F_{\mu\nu} \text{ is a conformal primary only if } \Delta_A = 1. \tag{662}$$

This means $F_{\mu\nu}$ is a primary only if $\Delta_F = 2$, which saturates the unitarity bound on an antisymmetric tensor. Anyway, the point is that scale invariance forces us to take $\Delta_A = (d-2)/2$, and so $F_{\mu\nu}$ is only a primary in $d = 4$. Again, in $d \neq 4$ conformal invariance would then imply that $F_{\mu\nu}$ is a descendant, but there is no local field that it could possibly be the descendant of. So as before we conclude that in $d \neq 4$, free Maxwell is not a CFT.

The $\mathbb{C}P^N$ model and emergent electromagnetism

The $\mathbb{C}P^N$ model is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}\left(|\partial_\mu z|^2 - |z^\dagger \partial_\mu z|^2\right), \tag{663}$$

where z is a $N+1$ component field, subject to the relations

$$|z|^2 = 1, \qquad (z_1, \ldots, z_{N+1}) \sim (e^{i\alpha(x)} z_1, \ldots, e^{i\alpha(x)} z_{N+1}). \tag{664}$$

Today we will do the following: first, we will make some brief mathematical comments on $\mathbb{C}P^N$ which didn't feel important enough to put as a standalone diary entry. Next we will go to two spacetime dimensions and do the standard mean field analysis at large N to show that in the mean field approximation, the theory is gapped with mass

$$m = \Lambda e^{-\pi/(g^2 N)}. \tag{665}$$

We will then show that at long distances, the effective action for the theory is Maxwell electrodynamics.

Edit: after writing this, I realized that the large-N analysis is essentially textbook material. The nontrivial parts of this diary are thus the discussion on kinematic issues and the expression for the effective charge in terms of N and m.

Mathematical preliminaries

First, a few general comments on $\mathbb{C}P^N$, which is the space of complex lines through the origin in $\mathbb{C}^{N+1}$. More often we see $\mathbb{C}P^N$ written as the space $(z_1, \dots, z_{N+1})/\sim$, where $(z_1, \dots, z_{N+1}) \sim (\lambda z_1, \dots, \lambda z_{N+1})$, $\lambda \in \mathbb{C}$, and $(z_1, \dots, z_{N+1}) \neq 0$. If we take

$$\lambda = \frac{1}{\sum_{i=1}^{N+1} |z_i|^2} \tag{666}$$

and multiply through, we get the conditions written in the problem statement, which shows us that[53]

$$\mathbb{C}P^N = S^{2N+1}/U(1). \tag{667}$$

For example, $\mathbb{C}P^1 = S^3/U(1)$, as realized via the Hopf map.

A way to construct $\mathbb{C}P^N$ as a quotient which is often more useful is

$$\mathbb{C}P^N \cong \frac{SU(N+1)}{U(N)}, \tag{668}$$

where the $U(N)$ factor is embedded via the map

$$U(N) \ni u \mapsto u \oplus \frac{1}{\det u} \in SU(N+1), \tag{669}$$

which ensures that the image of $U(N)$ in $SU(N+1)$ has determinant 1. Alternatively, one might write this as

$$\mathbb{C}P^N \cong \frac{SU(N+1)}{S[U(N) \times U(1)]}, \tag{670}$$

where $S[\dots]$ denotes the part of $\dots$ with unit determinant.

To prove this, we note that $SU(N+1)$ acts transitively on $\mathbb{C}P^N$: it acts transitively on the unit sphere in $S^{2N+1} \subset \mathbb{C}^{N+1}$ and therefore acts transitively on $\mathbb{C}P^N$, which as we saw is a quotient of the aforementioned sphere by $U(1)$. Thus we can identify

$$\mathbb{C}P^N \cong SU(N+1)/\mathrm{Stab}_x, \tag{671}$$

[53] A similar thing occurs for $\mathbb{R}P^N$: each line through the origin in $\mathbb{R}^{N+1}$ can be identified with a pair of points on the unit sphere S^N, and so $\mathbb{R}P^N \cong S^N/\mathbb{Z}_2$. Random stream-of-conciousness fact: $\mathbb{R}P^N$ is non-orientable when N is even, and orientable when N is odd. To see this, realize that the antipodal identification in the $\mathbb{Z}_2$ quotient of S^N is performed by sending a vector $\mathbb{R}^{N+1} \ni v \mapsto -v$. For N odd the map $v \mapsto -v$ is equivalent to a rotation, since it has determinant 1, while for even N it is an inversion, since it has determinant -1. Thus the S^N in the definition of $\mathbb{R}P^N$ is identified with itself in an even / odd way depending on N, and the claim follows.

where $\text{Stab}_x \subset SU(N+1)$ is the stabilizer subgroup of an arbitrary point $x \in S^N_{\mathbb{C}}$. This is basically the first isomorphism theorem: the image of a point x under the group action is isomorphic to the group modulo the kernel of the action, which is Stab_x. Anyway, to figure out Stab_x, we can just pick a convenient $x \in \mathbb{C}P^N$, e.g. $x = \{(e^{i\theta}, 0, \dots, 0) \,|\, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. This is clearly fixed under the $U(N)$ action on the last N coordinates, and also fixed under the $U(1)$ action on the first coordinate. We can combine these two actions to get something in $SU(N+1)$ if we shift the first coordinate by $\det^{-1}$ of the matrix acting on the last N coordinates, and so $\text{Stab}_x = U(N)$, with the embedding into $SU(N+1)$ as written above. This then proves the claim.

Symmetries and alternate path integral representation

Showing that the Lagrangian $\mathcal{L}$ is invariant under gauge transformations $z \mapsto e^{i\alpha(x)}z$ (which it must be, since these generate the $U(1)$ in the quotient) is straightforward after making use of $|z|^2 = 1$ and $\partial|z|^2 = 0 \implies z^\dagger \partial z = -z\partial z^\dagger$. One can emphasize this point by noting that $|z^2| = 1$ means that the linear combination

$$A_\mu \equiv \frac{1}{2i}(z^\dagger \partial_\mu z - z\partial_\mu z^\dagger) \tag{672}$$

transforms under a gauge transformation as $A \mapsto A + d\alpha$, meaning that $\partial_\mu - iA_\mu$ is an appropriate covariant derivative. One is then led to write down an alternate representation of the theory as (now in $i\mathbb{R}$ time)

$$Z = \int \mathcal{D}A\mathcal{D}\lambda\mathcal{D}z \exp\left(-\frac{1}{g^2}\int \left((D_A z)^\dagger D_A z - \lambda(|z^2| - 1)\right)\right), \tag{673}$$

where now A is an independent field (which just gets set to the definition given above when it's integrated out, which can be done exactly since we haven't included any kinetic terms for it) and λ enforces the $|z|^2 = 1$ constraint. One can confirm that this representation works by integrating out λ (since we're in $i\mathbb{R}$ time, the contour is along $i\mathbb{R}$) to get the sphere constraint on z, and then shifting A by

$$A \mapsto A + iz\partial z^\dagger, \tag{674}$$

which eliminates the coupling between A and z, and results in an action like $|A|^2 + |\partial z|^2 - |z\partial z^\dagger|^2$. Integrating out A then produces the original $\mathbb{C}P^1$ Lagrangian.

`<digression>`

A brief digression on the $N = 1$ case and its relation to the $O(3)$ non-linear sigma model: the claim is that these two models are related through the Hopf map $S^3 \to S^2$:

$$n^i = z^\dagger \sigma^i z. \tag{675}$$

Here the RHS is a vector since the zs are in the fundamental of $SU(2)$, and $1/2 \otimes (1/2)^* \cong 1/2 \otimes 1/2 = 0 \oplus 1$: choosing $\mathbf{1}$ instead of σ^i would project onto 0, while choosing σ^i gives us the vector (adjoint) rep. n^i is also normalized properly.[54]

Now the z spinors live in $S^3 \cong SU(2)$, but because of the $U(1)$ redundancy the target space for the $\mathbb{C}P^1$ model is actually $SU(2)/U(1) = S^2$, the same as the $O(3)$ nlsm. As a sanity check, we can check the global symmetries on each side. The $\mathbb{C}P^1$ has a global $U(2)$ symmetry but a local $U(1)$ symmetry, so the actual global symmetry is[55]

$$PU(2) = U(2)/U(1) = PSU(2) = SU(2)/\mathbb{Z}_2 = SO(3). \tag{677}$$

This matches the global symmetry of the nlsm if we pretend the global symmetry is $SO(3)$ instead of $O(3)$. We can get the full symmetry by taking into account the reflection that extends $SO(3)$ to $O(3)$, which is represented by the matrix $-\mathbf{1}$ and hence sends $n \mapsto -n$. On the $\mathbb{C}P^1$ side, this reflection is implemented by charge conjugation, which is unitary and acts through the invariant antisymmetric form of $SU(2)$:

$$\mathbb{Z}_2^C : z \mapsto Jz^*, i \mapsto i, \qquad J = -iY. \tag{678}$$

Indeed, n is odd under this symmetry:

$$\mathbb{Z}_2^C : z^\dagger \sigma^j z \mapsto z^\dagger (J^T \sigma^j J)^T z = -z^\dagger (J[\sigma^j]^T J) z = z^\dagger (\sigma^j J^2) z = -z^\dagger \sigma^j z. \tag{679}$$

One similarly checks that the $\mathbb{C}P^1$ Lagrangian is invariant under $\mathbb{Z}_2^C$, and that $A \mapsto -A$, which is how a gauge field is expected to transform. Now $O(3) = SO(3) \times \mathbb{Z}_2^C$ is a simple direct product, since $\mathbb{Z}_2^C$ is represented by the central element $-\mathbf{1}$. This means that $\mathbb{Z}_2^C$ should commute with the $PSU(2)$ symmetry on the $\mathbb{C}P^1$ side, which it does:

$$JU^* = UJ \implies J(Uz)^* = UJz^*, \quad \forall\, U \in PSU(2), \tag{680}$$

where the first equality holds since $J\mathcal{K}$ with $\mathcal{K}$ complex conjugation is the antilinear automorphism from the spin 1/2 representation of $SU(2)$ to itself, which is the manifestation of that representation's pseudoreality.

Let us now show the equivalence between the nlsm and $\mathbb{C}P^1$ actions explicitly. The kinetic term for the nlsm maps as

$$\frac{\rho}{2}|\nabla \mathbf{n}|^2 = \rho \left(\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ij}\delta_{kl} \right) (z_i^* \overset{\leftrightarrow}{\nabla} z_j)(z_k^* \overset{\leftrightarrow}{\nabla} z_l) = 2\rho \left((z^\dagger \nabla z)^2 + |\nabla z|^2 \right). \tag{681}$$

Other derivatives map similarly.

[54] The matrix $\sigma^\mu |z\rangle\langle z| \sigma^\mu$ is Hermitian and has trace 3 (since $\mathrm{Tr}[|z\rangle\langle z|] = 1$), so we can write it as $3\mathbf{1}/2 + c_i\sigma^i$. Also, since $n^i n_i$ is rotation-invariant, we can perform an $SU(2)$ rotation on the zs to any convenient spinor: we will choose $|z\rangle = (1, 0)^T$. Then $|z\rangle\langle z| = \mathbf{1}/2 + Z/2$, and conjugating by σ^j and summing over j tells us that $c_z = -1/2$, and so

$$n^i n_i = \langle z|(3\mathbf{1}/2 - Z/2)|z\rangle = 1 \tag{676}$$

as required.

[55] To get this, we have used the second isomorphism theorem: if H, N are subgroups of G with N normal, then $(NH)/N \cong H/(H \cap N)$. This just says that one can't "cancel the Ns" in $(NH)/N$ since if $N \cap H \neq 0$ then some elements of H will also be killed by the quotient. We have used the special case where $N = SU(n), H = Z(U(n)) = U(1)$ to show the obvious fact that $PU(n) = (SU(n)U(1))/U(1) \cong SU(n)/(U(1) \cap SU(n)) = PSU(n)$.

Depending on the context, we will also need to find a map for the WZW term. We will show that the WZW term at level S for the nlsm goes to $2Sz^\dagger\partial_\tau z$. This is rather obvious if one simply identifies the generators of $H_2(S^2) = \mathbb{Z}$ on both sides. However I've never seen the algebra actually worked out, so here we will give an explicit proof. It first helps to write the action density of the WZW term as

$$s_{WZW} = 2S\int \mathrm{Tr}[\widehat{n}\, d\widehat{n} \wedge d\widehat{n}], \tag{682}$$

where $\widehat{n} \equiv \mathbf{n}\cdot\boldsymbol{\sigma}/2$ so that $n^a = \mathrm{Tr}[\sigma^a\widehat{n}]$, and the integral is over a two-manifold which bounds the thermal circle. Note that another way of writing $\widehat{n}$ is as $\widehat{n} = |z\rangle\langle z| - \mathbf{1}/2$, which follows from "$1 = 1/2 \otimes 1/2 - 0$". Using this representation, we have

$$\begin{aligned}
s_{WZW} &= S\int \mathrm{Tr}\left[(2|z\rangle\langle -|\mathbf{1})\left(|dz\rangle\langle z| + |z\rangle\langle dz|\right)\wedge\left(|dz\rangle\langle z| + |z\rangle\langle dz|\right)\right]\\
&= S\int \mathrm{Tr}\left[(2|z\rangle\langle z|dz\rangle\langle z| + |z\rangle\langle dz| - |dz\rangle\langle z|)\wedge\left(|dz\rangle\langle z| + |z\rangle\langle dz|\right)\right]\\
&= S\int \left(2\langle z|dz\rangle\wedge\langle z|dz\rangle + 2\langle z|dz\rangle\wedge\langle dz|z\rangle + 2\langle dz|\wedge|dz\rangle + \langle dz|z|\rangle\wedge\langle dz|z\rangle - \langle z|dz\rangle\wedge\langle z|dz\rangle\right)\\
&= 2S\int \langle dz|\wedge|dz\rangle\\
&= 2iS\int da,
\end{aligned} \tag{683}$$

where $a = -i\langle z|dz\rangle$ and we have used the supercommutativity of $\wedge$ and the fact that $\langle dz|z\rangle = -\langle z|dz\rangle$. Hence the map for the WZW term is

$$S_{WZW} = 2iS\int d^dx\, d\tau\, a_\tau = 2S\int d^dx\, d\tau\, z^\dagger\partial_\tau z. \tag{684}$$

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Effective action at large N and emergent electromagnetism

We now integrate out z to get an effective action ($\mathbb{R}$ time[56])

$$S_{eff}[A,\lambda] = iN\ln\det(-D_A^2 + \lambda) - \frac{1}{g^2}\int \lambda. \tag{685}$$

Note that the D_A^2 is really D_A^2, and not $|D_A|^2$. As a first step, consider the case where $A = 0$ and where we approximate λ by a constant. In the large N limit, we can figure out what this constant is by taking the saddle point of the effective action with respect to λ. This produces

$$\frac{1}{Ng^2} = i\mathrm{Tr}\frac{1}{-\partial^2 + \lambda}, \tag{686}$$

[56]It's usually better to do this kind of stuff in $\mathbb{R}$ time since then the integration contour for λ runs along the $\mathbb{R}$ axis, which makes keeping track of signs and is eaiser.

where the trace is to be carried out in momentum space (the factor of the spacetime volume has canceled on both sides). We rewrite this as

$$\frac{1}{Ng^2} = \int_{\mathbb{R}^2} \frac{1}{k^2+\lambda}, \tag{687}$$

where the integral is over Euclidean momenta. Doing the integral up to a cutoff Λ gives a saddle point value of

$$\frac{4\pi}{Ng^2} = \ln(\Lambda^2/\lambda_*) \implies m_* = \sqrt{\lambda_*} = \Lambda e^{-4\pi/(g^2N)}, \tag{688}$$

which is the usual dimensional transmutation result: even though the theory was conformally invariant at the classical level, quantum effects produce a non-zero beta function through the Λ dependence of λ. The fact that we get a massive theory is also interesting because there is no obvious symmetric mass term we can write down in the UV (e.g. $m^2\sum_i |z_i|^2 = m^2$ doesn't work because of the constraint), so that this mass must be dynamically generated.

Note that if the dimension of spacetime were greater than two, then the presence of the cutoff would make the integral on the RHS of (687) bounded for arbitrarily small λ, and so if the t' Hooft coupling Ng^2 were small enough, there would be no nonzero solution to the mean-field equation for λ. This would indicate spontaneous symmetry breaking[57] and a Higgsing of the gauge field, and we'd have to go back and expand about the correct vacuum (however, since we're focusing on two dimensions, this need not concern us). Spontaneous symmetry breaking here can be argued by looking at the mean-field equations (which are exact as $N\to\infty$) for a configuration where z is constant. The mean-field equation for z is just $z\lambda = 0$. Hence if $\lambda_* \neq 0$ then we must be in the symmetric phase (which is the case in two dimensions), while if $\lambda_* = 0$ then we can have $\langle z\rangle \neq 0$, giving SSB. Hence the lower critical dimension for this theory is two.

Now we will take λ to be a constant and expand about small A, deriving the effective action for A at one-loop order. We write the $\ln\det$ as, now in $i\mathbb{R}$ time (note to self: the sign in the $-\lambda$s is annoying; should see if this can be fixed),

$$-N\mathrm{Tr}\ln(-D_A^2-\lambda) = -N\mathrm{Tr}\ln\left[(-\partial^2-\lambda)\left(1+\frac{A^2+2iA\partial+i(\partial A)}{-\partial^2-\lambda}\right)\right], \tag{689}$$

where the ∂ is understood to act on the z's. The overall factor of $(-\partial^2-\lambda)$ in the logarithm is an unimportant constant, so we drop it. We now expand the logarithm to second order. The first order contribution yields a term like G_zA^2 (G_z is the z propagator), plus things which vanish after integration; this first term doesn't depend on the momentum of A and hence can be dropped. To $O(A^2)$, the next diagram that contributes is a polarization bubble for the A propagator. Reading the Feynman rules off from the interaction vertex $(2iA\partial + i(\partial A))^2$, we get (still in Minkowski space)

$$-N\mathrm{Tr}\ln(-D_A^2-\lambda) \approx -\frac{N}{2}\int_{p,q} A^\mu(p)A^\nu(-p)\frac{(2q+p)_\mu(2(q+p)-p)_\nu}{(q^2-\lambda)((q+p)^2-\lambda)}. \tag{690}$$

[57] Santiy check: SSB occurs at small t' Hooft coupling since small Ng^2 means small fluctuations and hence increased tendency to order.

We are interested in the IR properties of this action, so we will take the small p limit. By gauge invariant we already know the form of the effective action, but we need to go through the details to figure out what the effective electric charge is. Since the first term in the integrand has no p-dependence, we focus on the second term, since this is the term that will produce the Maxwell term. The integral is done with the usual Feynman parameters: we use the Feynman parameters to simplify the denominator, shift the q momentum to eliminate the $q \cdot p$ term in the denominator, and then drop integrals which vanish because their integrands are odd. This yields (here $\Delta = -((x^2+x)p^2 - \lambda))$

$$\begin{aligned}\int_{p,q} A^\mu(p)A^\nu(-p)\frac{1}{2}\frac{(2q+p)_\mu(2q+p)_\nu}{(q^2-\lambda)((q+p)^2-\lambda)} &= \frac{1}{2}\int_{p,q} A^\mu(p)A^\nu(-p)\frac{(2(q-xp)+p)_\mu(2(q-xp)+p)_\nu}{(q^2-\Delta)^2} \\ &= \frac{1}{2}\int_{p,q} A^\mu(p)A^\nu(-p)\frac{4q_\mu q_\nu + p_\mu p_\nu(1-2x)^2}{(q^2-\Delta)^2}.\end{aligned} \tag{691}$$

We can now look up the integrals. To get a nice answer, we need to focus on small p (i.e. $O(p^2)$), so that e.g.

$$\frac{p_\mu p_\nu}{\Delta} \approx \frac{p_\mu p_\nu}{\lambda}, \qquad \ln\Delta \approx \ln\lambda - \frac{(x^2+x)p^2}{\lambda}. \tag{692}$$

Working in this approximation, the p-dependent parts of the effective action for A become

$$-iN\frac{1}{12\pi\lambda}\int_p A^\mu(p)A^\nu(-p)(g_{\mu\nu}p^2 - p_\mu p_\nu), \tag{693}$$

and so we have generated an emergent Maxwell theory with effective charge

$$\frac{1}{e_{\text{eff}}^2} = \frac{N}{6\pi\lambda}. \tag{694}$$

Here the g^2 dependence of the charge is contained within the saddle-point value for λ. Just like the Schwinger model this theory confines, with the spectrum consisting of z-$\bar{z}$ bound states. We can argue that this theory is asymptotically free by computing the β function for g; if this is asympotically free then from the mean field solution for λ we know that e_{eff}^2 will be as well. The beta function for g can be found by requiring that the effective potential for λ be independent of Λ. The calculation of the effective potential is essentially contained in the diary entry on the Gross-Nevau model, and so we won't repeat it in detail here. The gist is that one basically does the integral $\int dk\, k\ln(\lambda+k^2)$ by use of the replica trick and dim reg; doing so produces an effective potential like

$$V_{\text{eff}}(\lambda) \sim \lambda/g^2 + N\lambda\ln(\lambda/\Lambda^2), \tag{695}$$

which should remind us of the characteristic log found in the effective potentials for the Higgs fields in four-dimensional gauge theories (see the entries on the CW potential and fluctuation-induced first-order Higgs transitions). Requiring this to be independent of Λ gives the β function for the t' Hooft coupling as

$$\beta_\gamma \sim \gamma^2, \qquad \gamma \equiv g^2N, \tag{696}$$

which indeed demonstrates asympotitic freedom.

Dynamic generation of topological photon mass and domain wall anomalies

Consider QED$_3$, namely

$$S = \int d^3x \left(i\bar{\psi}\not{D}_A\psi + m(x)\bar{\psi}\psi - \frac{1}{2}F \wedge \star F \right). \tag{697}$$

Today's diary entry has two parts. First, we will show that radiative corrections from the fermions induce a Chern-Simons term and hence a topological mass for the photon (at momentum scales smaller than the fermion mass). Secondly, we will consider a domain wall where $m(x)$ changes sign and will show that such an object hosts chiral fermions with a gauge anomaly, with the anomaly being canceled by the dynamically generated CS terms away from the domain wall.

Chern-Simons terms: To find the CS term induced for the gauge field, we just have to compute the one-loop contribution to the effective action for the gauge fields after integrating out the fermions.[58] The relevant integral is

$$\text{bubble}^{\mu\nu} = (-1)i^2(-ie)^2 \int_p \text{Tr}\left[\gamma^\mu \frac{\not{p} + \not{q} + m}{(p+q)^2 - m^2}\gamma^\nu \frac{\not{p} + m}{p^2 - m^2}\right]. \tag{698}$$

After doing Feynman parameters to simplify the denominator there are two contributions to the integral in the large m limit: one proportional to $m\gamma^\mu\not{q}\gamma^\nu$, and another which contains terms like $g^{\mu\nu}m^2$ which do not depend on $\text{sgn}(m)$ (the others vanish under $\not{p} \to -\not{p}$ or under the trace). The latter terms will get renormalized away when we regularize e.g. a la PV, so we will ignore them in what follows. In mostly-negative signature $\mathbb{R}$-time, our γ matrices are

$$\gamma^0 = X, \qquad \gamma^1 = iY, \qquad \gamma^2 = iZ. \tag{699}$$

Thus they satisfy

$$\text{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda] = -2i\epsilon^{\mu\nu\lambda}. \tag{700}$$

[58] Recall that when the chiral anomaly is derived using the Feynman diagram approach, the one-loop calculation is exact, due to the fact that the anomaly is computed using the index theorem, and so the coefficient of the $F \wedge F$ term is quantized, implying one-loop exactness. The quantization of the CS term means that similar one-loop exactness applies in this setting.

Using this, we get (there are terms in the numerator involving the Feynman parameter and $\not{q}$ coming from the shift in p, but these end up cancelling due to the spin sum)

$$\text{bubble}^{\mu\nu} = 2ie^2\epsilon^{\mu\lambda\nu}\int_{p,x}\frac{q_\lambda m}{(p^2-\Delta)^2}, \tag{701}$$

where Δ is a function of m, q and the Feynman parameter x. Doing the integral,

$$\begin{aligned}\text{bubble}^{\mu\nu} &= -2ei\epsilon^{\mu\nu\lambda}q_\lambda m\frac{i}{(4\pi)^{3/2}}\Gamma(1/2)\Delta^{-1/2}\\ &= \text{sgn}(m)\frac{e^2}{4\pi}\epsilon^{\mu\lambda\nu}q_\lambda,\end{aligned} \tag{702}$$

where we have taken the long-wavelength limit where $m^2 \gg q^2$ so that $\Delta \to m^2$. This diagram appears in the effective action for the gauge field with a coefficient of $-1/2$ since it is the quadratic term in the expansion of the $\ln\det$, and so it thus gives us the CS term at level $-\text{sgn}(m)/2$. The CS term violates parity (i.e. either reflection or time-reversal) and so our theory violates parity if $m \neq 0$—but we already knew this, since fermion mass terms are parity-odd in odd dimensions (this provides another derivation of the oddness of m: under T the CS level is odd, so that m must be odd as well). The fractional level here arises since we didn't properly regulate the theory using e.g. PV regularization. If we did and chose e.g. the mass of the PV field to be large and positive, we would get a level of $k = [\text{sgn}(m) - 1]/2 \in \{0, -1\}$, which is well-defined. We could also have chosen the PV field to have a large negative mass, in which case the level would be valued in $\{0, 1\}$. We have a freedom of changing the sign of m and also the sign of the mass of the PV regulators, but the relative sign between the two masses is physical and determines what CS level we get (the two choices are related by time reversal). Another comment is that as we mentioned in an earlier footnote, since the IR CS level is quantized, our calculation must be one-loop exact—diagrams with l loops scale with the coupling constant as $(e^2)^{l-1}$, so if $l \neq 1$ made a contribution we could tune e^2 continuously and get a non-integral level.

Generalizing slightly to N_f fermions of masses m_i, we have

$$k_{IR} = k_{UV} - \sum_i \frac{\text{sgn}(m_i)}{2}, \tag{703}$$

where k_{UV} is the level of the sum of the PV regulator fields. In particular if $N_f \in (2\mathbb{Z}+1)$, we always have a fractional CS level in the UV. Also note that when $m_i = 0$, if $N_f \in (2\mathbb{Z}+1)$ the UV theory always breaks parity symmetry, since we have an odd number of PV fields—this is the parity anomaly (if $N_f \in 2\mathbb{Z}$, we could choose $N_f/2$ positive-mass PV fields and $N_f/2$ negative-mass PV fields, and the effective CS level would be zero). Note that for odd N_f parity is broken in the UV; it is not an infrared effect associated with the CS level generated by integrating out the fermions and instead indicates our inability to regularize the theory in a symmetry-preserving way.

Domain wall and anomaly cancellation: Now we consider a domain wall where $m(x)$ changes sign. For concreteness, let $x^3 = z$ be the direction normal to the domain wall. Then the CS terms generated by the fermions are by themselves not gauge-invariant, since under $A \mapsto A - d\alpha$ the action changes as

$$S \mapsto S + \frac{2e^2}{4\pi} \int_w \alpha F, \tag{704}$$

where $\int_w$ is an integral over the domain wall and we have chosen the mass to be positive on the $z > 0$ side of the domain wall wolog. This gauge-non-invariance must be canceled by something living on the wall.

Indeed it is; let's solve the Dirac equation to get the relevant anomaly-cancelling zero modes. We have, say on the $z > 0$ side of the domain wall,

$$i(\not{D}_A^w + \gamma^2(\partial_z - ieA_z))\psi = -m(z)\psi, \tag{705}$$

where $\not{D}_A^w$ denotes the Dirac operator restricted to the wall. We choose an ansatz where $\psi = \eta(x^0, x^1) f(z = x^2)$, with η a zero mode of the Dirac operator restricted to the wall. We can choose it to have definite chirality under $\bar{\gamma}_w = i\gamma^0\gamma^1$ since the wall is two-dimensional. Let $\bar{\gamma}_w \eta = c_\eta \eta$. We can match the $m(z)$ on the RHS with the usual exponential factor, so we take

$$\psi = i\eta \exp\left(-\int_0^z dz'(m(z') - ieA_z)\right). \tag{706}$$

For this to work, we need

$$i\gamma^2\eta = \eta. \tag{707}$$

But we have

$$i\gamma^2\eta = -i\gamma^0\gamma^1\eta = -\bar{\gamma}_w\eta = -c_\eta\eta, \tag{708}$$

and so if we choose η to be of negative chirality on the wall, we'll get a solution to the Dirac equation. A similar story gets told if we focus on $z < 0$: the two minus signs from $m(-z) = -m(z)$ and $\partial_{-z} = -\partial_z$ cancel out, and in the end we get two chiral modes on the domain wall, propagating in the *same* direction, with action

$$S_w = 2\int_w i\bar{\eta}\not{D}_A\eta. \tag{709}$$

The fact that the two modes propagate in the same direction along the wall (and so propagate with opposite handedness in their respective half-planes) is because a region with $m(z) > 0$ is essentially the time-reversed version of a region with $m(z) < 0$, since the Dirac mass is odd under time reversal in three dimensions.

Since the wall modes have a net chirality and coupled to the gauge field, they will have a gauge anomaly. Under $A \mapsto A - d\alpha$, the action thus shifts as

$$S \mapsto S - \frac{e^2}{2\pi} \int_w \alpha F. \tag{710}$$

This cancels the gauge anomaly coming from the bulk CS terms, and so the full action is well-defined.

Checking the chiral anomaly for non-Abelian gauge theory with operator relations

Today we will see how the chiral anomaly in YM theory can be derived in a UV way by explicitly computing the divergence of the axial current by way of a point-splitting approach used to define the current operator (the simpler $U(1)$ case is in P&S). In cond-mat language, this way of computing the anomaly with point-splitting is analogous to determining the anomaly by testing whether or not the action of the symmetry can be realized in an on-site manner.

To this end, consider a non-Abelian gauge theory coupled to Dirac fermions in four dimensions. We will first find the terms in the divergence of the chiral vector current which are quadratic and cubic in the gauge field, and show that they match with the usual $F \wedge F$ form for the anomaly. We will do the calculation by explicitly computing the divergence of j_5^μ and splitting the fermion two point function as

$$\bar{\psi}(x-\epsilon/2) P \exp\left(i\int_{x-\epsilon/2}^{x+\epsilon/2} A_\mu \, dx^\mu\right)\psi(x+\epsilon/2), \tag{711}$$

and taking $\epsilon \to 0$ in a symmetric limit.

The chiral vector current comes from the symmetry $\psi \mapsto \exp(-\gamma^5\alpha)\psi$, where by γ^5 we really mean $\gamma^5 \otimes \mathbf{1}_G$, where $\mathbf{1}_G$ is the identity matrix for the representation of the gauge group that the fermions live in. From the $i\bar{\psi}\not{D}\psi$ part of the action (with the convention $D = \partial + ieA$), we get

$$\partial_\mu \langle j_5^\mu\rangle = \partial_\mu \langle \bar{\psi}(x-\epsilon/2)\gamma^\mu\gamma^5 W \psi(x+\epsilon/2)\rangle, \tag{712}$$

where we've written W for the Wilson line connecting the two fermions. We take the derivatives and use the equations of motion

$$\not{\partial}\psi = ie\not{A}^a t^a \psi, \qquad \partial_\mu\bar{\psi}\gamma^\mu = -ie\bar{\psi}\not{A}^a t^a \tag{713}$$

on the terms containing $\partial_\mu\psi$ and $\partial_\mu\bar{\psi}$. This produces

$$\partial_\mu\langle j_5^\mu\rangle = ie\langle\bar{\psi}(x+\epsilon/2)\gamma^\mu\gamma^5\left(\partial_\mu A_\nu^a t^a \epsilon^\nu W - A_\mu^a(x+\epsilon/2)t^a W + W A_\mu^a(x-\epsilon/2)t^a\right)\psi(x-\epsilon/2)\rangle, \tag{714}$$

where the first term comes from the derivative of the Wilson line. We now need to move all the Wilson lines to stand to the right of all the t^a matrices. To this end, we expand the Wilson line to first order in ϵ and write

$$Wt^a \approx \left(1 + ie\int A^b t^b\right) t^a = t^a + ie\int A^b(t^a t^b - [t^a, t^b]) \approx t^a W + eA_\nu^b \epsilon^\nu f^{abc} t^c. \tag{715}$$

Then to first order in ϵ,

$$\partial_\mu\langle j_5^\mu\rangle = ie\langle\bar{\psi}(x+\epsilon/2)\gamma^\mu\gamma^5\left((\partial_\mu A_\nu^a - \partial_\nu A_\mu)t^a\epsilon^\nu + A_\mu^a A_\nu^b \epsilon^\nu f^{cab} t^c\right)\psi(x-\epsilon/2)\rangle, \tag{716}$$

which we obtained by doing some re-arranging of the gamma matrices and expanding the A's about x (two terms coming from expanding the Wilson line cancel). We will see momentarily that the leading singularity in the fermion contraction will go as $1/\epsilon$, which will justify our first-order expansion of the terms in the previous equations.

Now we need to contract the fermions. To leading order, the fermion contraction is just a propagator connecting $x-\epsilon/2$ with $x+\epsilon/2$. However, this gives us something like $\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu]$, which is always zero. In order to get something nonzero, we are going to need four gamma matrices to give a nonzero trace when they get hit by the γ^5. The required four gamma matrices come from a process in which the background A field interacts once with the fermion line connecting the two points, forming a T-shaped Feynman diagram (see the picture in chapter 19 of P&S). The Feynman diagram for the $\partial_{[\mu}A_{\nu]}$ term gives

$$\partial_\mu\langle j_5^\mu\rangle = (ie)^2\int_{p,q}\mathrm{Tr}\left[\frac{\not p+\not q}{(p+q)^2}\gamma^5\gamma^\mu\partial_{[\mu}A^a_{\nu]}t^a\epsilon^\nu A^b_\sigma(p)t^b\gamma^\sigma\frac{\not q}{q^2}e^{iq\epsilon-ip(x-\epsilon/2)}\right]+\ldots. \quad (717)$$

Note that here that the brackets simply mean $\partial_{[\mu}A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (there is no 1/2! normalization). In our signature $\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho] = -4i\epsilon^{\mu\nu\sigma\rho}$ and $\mathrm{Tr}[t^at^b] = C(r)\delta^{ab}$ (where r is the representation the fermions are in, e.g. $C(N) = 1/2$ for the fundamental of $SU(N)$), so

$$\text{relevant diagram} = 4e^2C(r)i\epsilon^\nu\epsilon^{\mu\lambda\sigma\rho}\partial_{[\mu}A^a_{\nu]}\int_{p,q}\frac{(q+p)_\lambda A^a_\sigma(p)q_\rho}{(p+q)^2q^2}e^{iq\epsilon-ip(x-\epsilon/2)}. \quad (718)$$

Since we are sending $\epsilon\to 0$, the large q limit is what will be relevant, so we can take the denominator to just be q^4. This is essentially because we are computing the divergences coming from splitting the product $\bar\psi\psi$, which come from high-momentum UV physics. Then the two integrals factor as a product: one is just $i\partial_\lambda A^a_\sigma$, while the other goes to

$$\int_q q_\rho\frac{e^{iq\epsilon}}{q^4} = -i\partial_{\epsilon_\rho}\left(\frac{i}{16\pi^2}\ln(1/\epsilon^2)\right) = -\frac{1}{8\pi^2}\frac{\epsilon_\rho}{\epsilon^2}, \quad (719)$$

where the other factor of i came from Wick rotation. This has a $1/\epsilon$ divergence, as promised earlier. The integral over p gave us a $\partial_\lambda A_\sigma$ term, and due to the antisymmetry of the ϵ tensor up front we can turn this into a $\partial_{[\lambda}A_{\sigma]}$ at the cost of a factor of 1/2. This means that after contracting the fermions, the first term in (716) gives us

$$-\frac{C(r)}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial_\lambda A^a_\sigma - \partial_\sigma A^a_\lambda). \quad (720)$$

Now we need to contract out the fermions in the other term that came from commuting the t^a through the Wilson line. Tracing out the spin and gauge indices using the rules described earlier, we get (the Feynman diagram looks the same, just with a different interaction vertex in between the fermions)

$$\text{relevant diagram} = 4iC(r)e^3\epsilon^{\lambda\mu\sigma\rho}A^a_\mu A^b_\nu f^{cab}\epsilon^\nu\int_{p,q}\frac{(q+p)_\lambda}{(q+p)^2}A^c_\sigma(p)\frac{q_\rho}{q^2}e^{iq\epsilon-ip(x-\epsilon/2)}. \quad (721)$$

We do the integrals in the same way as before, and this term adds to the previous one to give us the expression for the divergence of the chiral current up to terms cubic in the gauge field.

Recapitulating, we have

$$\partial_\mu \langle j_5^\mu \rangle = -\frac{C(r)e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}\left((\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial_\lambda A_\sigma^a - \partial_\sigma A_\lambda^a) + e(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(f^{abc}A_\lambda^b A_\sigma^c)\right) + \dots, \tag{722}$$

where $\dots$ is the second term with $\mu, \nu \leftrightarrow \lambda, \sigma$ and also the A^4 terms that we didn't write out due to laziness. This is exactly what we get from writing down the familiar $F^a \wedge F^a$ formula obtained with e.g. Fujikawa's method. In the common case where the gauge group is $SU(N)$ and the fermions are in the fundamental, the $C(r)$ becomes a $1/2$ and we get the $F^a \wedge F^a$ term with a $1/(32\pi^2)$ in front.

Phase diagrams of Abelian lattice gauge theory

Today we're compiling a list of facts / intuitions about Abelian lattice gauge theories in 3 and 4 spacetime dimensions, both for quantum Hamiltonians in d spatial dimensions and their $(d+1)$-dimensional classical counterparts. We will be working at $T = 0$ throughout.[59] Most good references for background material are from back in the 70s (Fradkin + Shenker, Kogut + coworkers, etc.) and are all very classic + well-known, and so I haven't bothered to add citations. Thanks also to Bi Zhen and Ji Wenjie for discussions.

The quantum Hamiltonians we will be considering are variants of

$$H = -\frac{1}{2}\left(\frac{1}{g}\sum_{\square} B_\square + g\sum_l \mathcal{X}_l + \lambda \sum_l Z_i^\dagger \mathcal{Z}_l Z_{i+1} + \frac{1}{\lambda}\sum_i X_i\right) + h.c. \tag{723}$$

for the $\mathbb{Z}_N$ case and

$$H = -\frac{1}{2}\left(\frac{1}{g}\sum_{\square} \cos(B_\square) + g\sum_l E_l^2 + \lambda \sum_l e^{-i\phi_i} U_l e^{i\phi_{i+1}} + \frac{1}{\lambda}\sum_i \pi_i^2\right) + h.c. \tag{724}$$

for the $U(1)$ case. Here the notation is that $\mathcal{Z}_l$ and $U_l = e^{iA_l}$ are gauge link variables, $\mathcal{X}_l$ and E_l are the conjugate momentum variables to $\mathcal{Z}_l$ and A_l, and Z, X (ϕ, π) are the

[59] Thinking about $T > 0$ and the associated finite-temperature transitions in the phase diagrams of gauge theories is usually done with distinct methods (mapping to a spin model one dimension down via Polyakov loops) and hence will be relegated to a separated diary entry.

matter fields / momenta in the discrete (continuous) case.[60] The $B_\square$s are defined in the usual way: for each plaquette $\square$ we take a positively (wrt a reference orientation) oriented path around the plaquette, and add a factor of $\mathcal{Z}_l$ (or U_l) if the orientation of the path agrees with the orientation of the link (fixed by the orientation of the cubic lattice, which we will always be working on), or a factor of $\mathcal{Z}^\dagger/U^\dagger$ if the link disagrees with the orientation of the path. The generator of gauge transformations in the discrete case is $X_v \prod_{l\in\partial v} \mathcal{X}_l^{\pm_l}$, where $\mathcal{X}_l^{\pm_l}$ is $\mathcal{X}_l$ if l points into the vertex v and $\mathcal{X}_l^\dagger$ if l points out of v. For $U(1)$, it is the usual $\exp(i(\nabla\cdot E-\rho))$. In the $U(1)$ case, the matter fields we're working with are pure phase variables—their magnitudes are assumed to be frozen out by a large Higgs potential, regardless of what actual phase we are working in. The hope is that this can be done without unduly modifying the phase diagram.[61] If the full matter field is $\psi=|\psi|e^{i\phi}$, the above kinetic term is equal to (up to unimportant constants) $|\psi_i - U_l\psi_{i+1}|^2$ after the $|\psi|$s are frozen out.

Now numerical studies of gauge theories are done on D+0 dimensional classical stat mech models, rather than with quantum models that have continuous time. Therefore to compare with numerics, it's best to do the usual $\mathcal{QC}$ mapping on the above quantum Hamiltonian. This procedure is a bit ambiguous and different presentations of the Hamiltonian are more or less well-suited to doing the $\mathcal{QC}$ mapping. In fact the above Wilsonian form of the quantum Hamiltonian does *not* have a simple classical counterpart, and to do the $\mathcal{QC}$ mapping in a simple way it is necessary to work with the modified Villain form of the Hamiltonian. The classical partition function obtained

[60] Recall why we write the coefficients of the dynamical terms (g,λ^{-1}) as the inverses of the coefficients of the kinetic terms (g^{-1},λ): this is a schematic way of indicating that in the Euclidean $D+0$ dimensional classical stat-mech model ("Lagrangian path integral"), the couplings appearing in the action are approximately the kinetic ones g^{-1},λ. For situations where the fields are continuous (like $U(1)$), this relation (of the coefficient of the canonical momentum term getting inverted when passing to the stat mech model) is exact (up to the usual complications caused by the spacetime anisotropy of the corresponding classical model): this is basically due to the relation that

$$\int dx'\, e^{-(x-x')^2/2K} f(x') = e^{-K\nabla^2/2} f(x), \tag{725}$$

which follows from writing f in fourier components and doing the quadratic integral (basically what's happening when going from Hamiltonians to Lagrangians—integrating out the momentum, which we assume to appear quadratically in the action). Note here that the coefficient of the canonical momentum K becomes K^{-1} when we write things in terms of the integral over "time" (the x coordinate in the above). Hence e.g. for Maxwell theory, the Hamiltonian $g^{-1}B^2+gE^2$ is equivalent to a stat-mech model in one dimension up with Hamiltonian $g^{-1}F^2$.

Now when the fields aren't continuous (or even if they are but we aren't in the continuum limit), this nice relation between the coefficients of the stat mech model and the quantum Hamiltonian doesn't hold—recall from the example of the transfer matrix in the Ising model that the coefficients in the classical model are written in terms of hyperbolic trig functions and such. So while in general it is incorrect to find the corresponding stat mech model by inverting the coefficient of the dynamic term, this rule is schematically correct, and at any rate when discussing stat mech models we will always assume that after doing the quantum $\to$ classical mapping, the model can be made homogeneous in spacetime and the precise coupling constants adjusted slightly without taking us out of the universality class in question.

[61] Since the gauge fields only couple to the phase degree of freedom, this seems like a reasonable expectation.

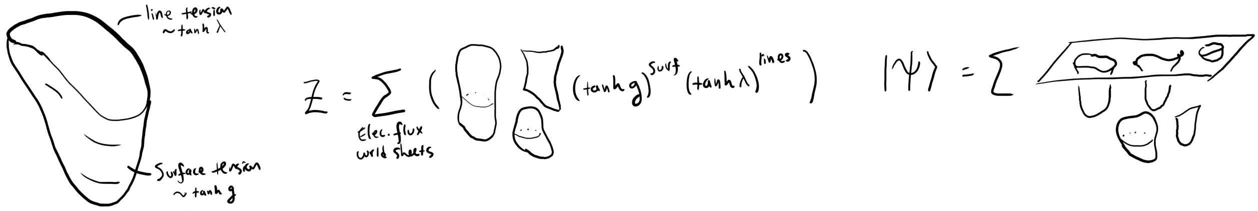


Figure 1: From left: an open string worldsheet, a schematic representation of the partition function, and a schematic representation of the ground-state wavefunction.

from the Villain modification of the previous quantum H is the isotropic model

$$Z = \sum_{\substack{\{p_\square\}\in\mathbb{Z}^{N_\square},\\ \{p_l\}\in\mathbb{Z}^{N_l}}} \sum_{\substack{\{z_l\}\in\mathbb{Z}_N^{N_l}\\ \{\phi_v\}\in\mathbb{Z}_N^{N_v}}} \exp\left(-\frac{1}{2g}\sum_\square\left(\frac{2\pi}{N}\mathcal{H}_\square - 2\pi p_\square\right)^2 - \frac{\lambda}{2}\sum_l\left(\frac{2\pi}{N}((d\phi)_l - z_l) - 2\pi p_l\right)^2\right) \tag{726}$$

in the $\mathbb{Z}_N$ case and

$$Z = \sum_{\substack{\{p_\square\}\in\mathbb{Z}^{N_\square},\\ \{p_l\}\in\mathbb{Z}^{N_l}}} \int\prod_l \mathcal{D}z_l \exp\left(-\frac{1}{2g}\sum_\square(\mathcal{H}_\square - 2\pi p_\square)^2 - \frac{\lambda}{2}\sum_l((d\phi)_l - z_l - 2\pi p_l)^2\right) \tag{727}$$

in the $U(1)$ case. Here the ps are integers used to write the Villain approximation to the cosines appearing the Wilsonian form of the action, and the d is a lattice differential. The $\mathcal{H}_\square$s are the gauge holonomies around $\square$; they are valued in $\mathbb{Z}_N$ or $[0, 2\pi)$ (and not $\mathbb{R}$!), as appropriate. While the Villain form looks complicated it is actually much simpler than the usual Wilsonian form (which is hoped to be in the same universality class as the above) due to the fact that all the fields appear quadratically. Note that the coefficients g, λ here are *not* the same as the ones in the quantum Hamiltonian (they don't even have the same dimension as the ones in the quantum Hamiltonian, nor is the relation between the two sets of coefficients very simple in general). That said the g, λ here control the same physics as the g, λ in the quantum Hamiltonian, and we will as such continue with this abuse of notation.

In what follows we will mostly discuss the quantum (Wilsonian) Hamiltonian, but when comparing with numerics will refer to the above classical action. Again, the hope is that the two approximations made (changing from the Wilsonian quantum Hamiltonian to a Villain approximation, and then approximating the resulting classical model by one that's homogeneous in spacetime) don't qualitatively affect the structure of the phase diagrams.

We will make some general comments on the 2+1D and 3+1D cases first, and then do a more systematic discussion of the phase diagrams. Until discussing the phase diagrams, all matter will be assumed to have unit charge.

To get an intuitive picture on the various phases, it's helpful to look at the D-dimensional stat mech model (alias the path-integral representation) in order to get some geometric intuition. For simplicity, consider the $\mathbb{Z}_2$ case, and (for the rest of this section only) consider the Wilsonian version of the classical model, viz.

$$S = -\frac{1}{2g}\sum_{\Box} B_{\Box} - \frac{\lambda}{2}\sum_{\langle ij\rangle} Z_i \mathcal{Z}_{\langle ij\rangle} Z_j, \tag{728}$$

where the $Z_i, \mathcal{Z}_l$ are now just numbers. The specification to $\mathbb{Z}_2$ makes writing the partition function easy since $e^{\alpha s} = \cosh\alpha + s\sinh\alpha$ for $s \in \mathbb{Z}_2$. Hence

$$Z \propto \sum_{\{\mathcal{Z}_l, Z_i\}} \prod_{\Box}(1+\tanh(g^{-1})B_{\Box}) \prod_{\langle ij\rangle}(1+\tanh(\lambda) Z_i \mathcal{Z}_{\langle ij\rangle} Z_j). \tag{729}$$

The terms which survive this sum are then terms that are formed by surfaces built from $B_{\Box}$s, which are either closed or which terminate on $Z\mathcal{Z}Z$ lines. The tendency for these surfaces to proliferate, as well as whether or not they tend to be open or closed, depends on the surface tension $\sim [\tanh(g^{-1})]^{-1} - 1$ and the line tension $\sim [\tanh(\lambda)]^{-1} - 1$. The ground-state wavefunction $|\Psi\rangle$ is found by cutting open the path integral along a spatial slice; hence the ground state is made from a combination of open and closed strings. The case of $D = 3$ is shown in the top row of Figure 1.

2+1D

Pure gauge theory

First we consider the $\mathbb{Z}_N$ case with $\lambda = 0$. First consider the deconfined regime of the quantum model. At $g = 0$ we need $B_{\Box}$ to act as $\mathbf{1}$ on every plaquette. The ground state here is a gas of strings of all sizes; this follows from cutting open the partition function (729) and noting that at $g = 0$ the surface tension vanishes while the line tension is infinite.

We can construct the $g = 0$ ground state $|\Omega\rangle$ explicitly either by applying all possible t'Hooft lines[62] to the $\otimes|\uparrow\rangle$ product state, or by applying all possible Wilson lines to the $\otimes|+\rangle$ product state:

$$|\Omega\rangle = \sqrt{N}^{N_{\Box}} \prod_{\Box} \Pi_{B_{\Box}} |\otimes +\rangle = \sqrt{N}^{N_{\Box}} \prod_{v} \Pi_{A_v} |\otimes \uparrow\rangle \tag{732}$$

where the projector $\Pi_{\mathcal{O}}$ for an operator $\mathcal{O}$ with eigenvalues in the Nth roots of unity is defined as the projector onto the $+1$ eigenspace:

$$\Pi_{\mathcal{O}} = \frac{1}{N}\sum_{k\in\mathbb{Z}_N} \mathcal{O}^k. \tag{733}$$

[62] In two spatial dimensions, t' Hooft lines are defined by

$$T_\gamma = \prod_{l\in\gamma} \mathcal{X}_l \tag{730}$$

for γ any path in $C_1(\Lambda^\vee; \mathbb{Z}_N)$, where $\Lambda^\vee$ is the dual of the spatial lattice Λ. To talk about dynamical questions, which is all we're really going to be caring about, we can focus on spatial manifolds with trivial topology. In this case we can always write

$$T_\gamma = \prod_{v\in\bar{\gamma}} A_v, \tag{731}$$

where $\bar{\gamma}$ is the collection of all vertices enclosed by γ.

Here also A_v is the usual $\prod_{l\in\partial v}\mathcal{X}_l$ vertex operator (the generator of gauge transformations in the pure gauge theory). Just to check the normalization and that the two ways of writing the state are indeed the same, we check

$$\langle\Omega|\Omega\rangle = N^{N_\square}\frac{1}{N^{2N_\square}}\langle\otimes\uparrow|\prod_{v,\square}\sum_{k,l\in\mathbb{Z}_N}A_v^k B_\square^l|\otimes +\rangle. \tag{734}$$

Since $B_\square$ is gauge invariant the A_vs and $B_\square$s commute; hence

$$\langle\Omega|\Omega\rangle = N^{N_\square}\langle\otimes\uparrow|\otimes +\rangle = N^{N_\square - N_l/2} = 1, \tag{735}$$

where N_l is the number of links.

First, note that W_γ has unit expectation value for all choices of γ. A perimeter law only sets in for W_γ when we add some line tension. When we turn on a small $g > 0$, we add in processes which measure electric flux, i.e. which create magnetic flux. Since the string of the Wilson line detects magnetic flux, dynamically created magnetic flux leads to nonzero line tension (said another way, the Wilson line creates an electric flux tube, which is now explicitly tensionful). Near the Ω state, we determine the vev of the Wilson line by doing first order perturbation theory: the eigenstates of the $g = 0$ Hamiltonian which enter the expansion are those which contain a single pair of fluxes on adjacent plaquettes, which can be written as $\mathcal{X}_l|\Omega\rangle$ for some l. Therefore if $|\Omega_g\rangle$ is the perturbed ground state,

$$\langle W_\gamma\rangle = \frac{\langle\Omega_g|W_\gamma|\Omega_g\rangle}{\langle\Omega_g|\Omega_g\rangle} = 1 + \frac{1}{2N}\sum_{l,l'} g^2\langle\Omega|\mathcal{X}_l^\dagger W_\gamma \mathcal{X}_{l'}|\Omega\rangle + h.c. + \dots, \tag{736}$$

where $\dots$ come from fluxes separated by multiple lattice spacings and configurations with more than two fluxes. Therefore

$$\ln\langle W_\gamma\rangle = -\frac{g^2}{4}|\gamma| + \dots, \tag{737}$$

with $\dots$ higher order in g. This is the expected perimeter law.

The t Hooft lines T_γ also have unit expectation value in $|\Omega\rangle$. In fact they have exactly unit expectation value even when the line tension is turned on—this is because the string of the t Hooft line is moved by applying gauge transformations A_v; hence the location of the string is unphysical and it cannot possibly have any tension. Therefore in the pure gauge theory, the t' Hooft lines are essentially trivial and hence there is no meaningful magnetic operator. This is in keeping with the fact that for lattice $\mathbb{Z}_N$ gauge theories, the magnetic 1-form symmetry is always explicitly broken by the lattice (it only emerges when we work strictly at $g = 0$), and so the t' Hooft loops can never be used as a diagnostic of what phase we're in.

Now let's go to the confining regime. The $g = \infty$ groundstate $|\Gamma\rangle$ is just a $\otimes$ of $+1$ eigenstates of $\mathcal{X}_l$ on every link. That the Wilson line obeys an area law is seen just from the usual high temperature expansion: expanding the $g^{-1}B_\square$ exponential[63] we see that the first term that survives has a product of $B_\square$ filling the loop—therefore

$$\frac{\langle\Gamma_g|W_\gamma|\Gamma_g\rangle}{\langle\Gamma_g|\Gamma_g\rangle} \sim e^{A(\gamma)\ln g^{-1}}, \tag{738}$$

[63] I guess since the two gauge terms in H don't commute, it's probably better to first go to the classical model one dimension up and do the expansion of the exponential there.

giving the expected area law.

The $U(1)$ case is different, since in this case we have monopoles, which prohibit a deconfined / topological phase. The intuition here is that in the 3+0D stat mech model (path integral), the operators which disorder the Wilson loops are monopoles, which are 3+0D are point-like instantons. It is therefore always favorable (in the free-energy sense) to have a macroscopic number of monopoles, which disorder the system and prevent the Wilson lines from getting a perimeter-law expectation value. This is basically the same as the free energy argument for why domain walls always disorder models with discrete symmetry in 1+0D—if the disordering operator is point-like, free energy considerations always favor proliferating it. If the disorder operator acting in a putative ordered phase costs energy E, then the free energy for configurations with a single disorder operator in a volume L goes as $F \sim E - \alpha \ln L$ for some constant α. When $L \to \infty$ we therefore always get a disordered phase at long distances (although we have to go to distances exponentially large in E [i.e. flowing for an RG time $\propto E$] in order to see the disordering). This argument also applies to $\mathbb{Z}_N$ gauge theory in 2+0D: in this case the disorder operators are just $\mathcal{X}_l$, which being local mean that this theory is always confined.

The Wilson line changes from a marginally confining QLRO form of $\langle W_\gamma \rangle \sim e^{-g^2 \ln|\gamma|}$ in the free-field approximation at small g to the regular area law at large g. Given that the Wilson line scales as a power of $|\gamma|$ in the free-field approximation, this seems to suggest that there's a KT transition between the power law and the exponential decay at large g. However in fact unlike the XY model, at any $g > 0$ the theory is massive, with $\star F$ having a mass $\propto e^{-1/g^2}$ (the action of the instanton)—this means that the KT transition occurs at $g = 0$ (although perturbatively in g the mass is always zero). It is useful to contrast this with the case of the XY model—in both cases, the action can be dualized and written as

$$S = \int d^d x \left(\frac{R^2}{4\pi} |d\phi|^2 + y \cos\phi \right), \tag{739}$$

where y depends on the action of the instanton and $R^2 \sim 1/g^2$ for the gauge theory. The difference is that in $D = 2$ the relevance of the cosine depends on the value of R^2, while for $D = 3$ the cosine is always relevant. Hence $U(1)$ gauge theory is massive at all nonzero g, while the XY model has a nonzero regime of masslessness.

With matter

Now turn on the coupling to matter, and again focus on the $\mathbb{Z}_N$ case. A useful tool for understanding the phase diagram is EM duality. This works in the quantum model via (as usual $\vee$ denotes cells on the dual lattice)

$$B_\square \leftrightarrow X_{\square^\vee} \qquad \mathcal{X}_l \leftrightarrow Z^\dagger_{(\partial l^\vee)_0} \mathcal{Z}_{l^\vee} Z_{(\partial l^\vee)_1}. \tag{740}$$

That is, the kinetic term for the gauge fields goes to the dynamic (momentum) term for the matter fields, and vice versa. Therefore in 2+1D the discrete lattice theory has a natural notion of EM duality sending $g \leftrightarrow \lambda$, implying a symmetry of the phase diagram. Under the duality, Wilson lines map to electric flux operators:[64]

$$\prod_{l\in\gamma} \mathcal{Z}_l \leftrightarrow \prod_{l\in\gamma^\vee} \mathcal{X}_l, \tag{741}$$

so that electric and magnetic fluxes are exchanged, as usual. Note that the RHS can be re-written as $\prod_{v\in\bar{\gamma}} X_v$ with $\bar{\gamma}$ the region enclosed by γ. Hence the Wilson line (the order parameter for the gauge theory) maps to the disorder operator on $\bar{\gamma}$ in the spin system; this is also what we expect from EM duality.

Since electric matter can screen the Wilson line, the latter follows a perimeter law at any nonzero λ. It is obviously still a perimeter law for small g, λ, while for large g we can do the high temperature expansion and expand the exponential in both g^{-1} and λ: there is always a term coming from $(Z_i^\dagger \mathcal{Z}_{\langle ij\rangle} Z_j + h.c.)^{|\gamma|}$ that contributes to $\langle W_\gamma\rangle$, and so in the limit of $|\gamma| \to \infty$, the leading behavior of the Wilson line will be $\langle W_\gamma\rangle \sim e^{-\ln(\lambda^{-1})|\gamma|}$. Nevertheless there is still an open Wilson line that can be used to distinguish the two phases at $\lambda > 0$; this is discussed in a subsequent section.

Just as dynamical magnetic flux is responsible for giving tension to Wilson lines, dynamical electric matter gives tension to t' Hooft lines; this is possible since since now the operator which moves the t' Hooft lines, viz. A_v, no longer generates gauge transformations. The fact that electric matter makes the t' Hooft lines tensionful is another way of explaining why charged matter fights confinement. Anyway, we check the tension at small λ by computing $\langle T_\gamma\rangle$ in an expansion in powers of λ. The ground state has all the matter fields in the $+1$ eigenstate of all the X_v operators, and the perturbations to this state come from states where pairs of electric charges are excited across a given link. This gives a finite line tension $\propto \lambda^2$ by the same calculation as (736).

Since we always have dynamic magnetic flux (created by the $g\sum_l \mathcal{X}_l$ term), the t' Hooft line can always be screened, and hence will always follow a perimeter law, even in the deconfined regime when $g \to 0$.[65] This again follows from a high temperature expansion, which tells us that $\langle T_\gamma\rangle \sim e^{-|\gamma| g}$ in the deconfined phase.

For the $U(1)$ case, adding (unit charge) matter doesn't really do anything. As we said, the fact that the pure gauge theory always confines is due to the presence of a photon mass induced by instantons. The mass term is relevant, and so in order for matter to do anything, its presence would need to change the scaling dimension of the gauge field considerably. This may be possible at intermediate coupling (or in large N), but I think the expectation is that for a single species of matter, the spectrum is still confined and massive at all λ.

[64]For economy of notation we aren't bothering to keep track of orientations of links and possible †s that need to be added; if the reader is bothered by this just pretend we are in the $N = 2$ case.

[65]Now in 3+1D, a phase in which both W and T are P-law implies the existence of gapless degrees of freedom—the correlation of linked W and T operators differs from that of unlinked ones, even though both operators are perimeter law, and this is only possible if there is a long-ranged force that mediates communication between the two operator insertions. In 2+1D this argument doesn't apply to the deconfined phase where both W and T are P-law, since there are no configurations of W and T loops that link / have a nontrivial signed intersection number (we are on a space of trivial topology).

3+1D

Pure gauge theory

In 3+1D the $\mathbb{Z}_N$ pure gauge theory has an EM duality, since the dual of a link (electric field) is a plaquette (magnetic field). This duality does $B_\square \leftrightarrow \mathcal{X}_l$, and hence does $g \leftrightarrow g^{-1}$.

This duality means that e.g. if there is only a single phase transition between the confined and topological phases, it must occur at $g = 1$. Furthermore if there are two transitions, they must be related by inversion about $g = 1$. The former possibility is observed to occur for $N = 2, 3, 4$, where 4+0D lattice calculations find a first-order transition at the coupling corresponding to the choice $g = 1$ in the quantum model.[66]

The duality is simple to see in the Hamiltonian model, but most of our knowledge of the phase diagrams comes from numerics, which are done on 4+0D lattice models. As shown in another diary entry, the classical pure gauge theory with action (726) is dual to a copy of itself with gauge coupling

$$\frac{1}{\widetilde{g}} = \frac{gN^2}{4\pi^2}. \tag{742}$$

Therefore if there is only one transition, in the classical gauge theory it must occur at

$$g_* = \frac{2\pi}{N}, \tag{743}$$

so that larger N theories have "more" confinement. In particular, the self-dual point moves to weaker coupling as N gets large.

The behavior of the Wilson lines deep in both phases is calculated using the same expansions as in the 2+1D case. The t' Hooft line operator is now defined as

$$T_\gamma = \prod_{l \in \partial \Sigma^\vee} \mathcal{X}_l, \tag{744}$$

where $\Sigma^\vee$ is a surface in the dual of the spatial lattice, and $\partial\Sigma^\vee$ contains the boundary links of those links which intersect $\Sigma^\vee$ transversely. This definition means that W_γ and $T_{\gamma'}$ cannot be freely unlinked as long as $\gamma \cap \Sigma^\vee \neq 0$.[67]

In the $U(1)$ case, there is no duality to help us—the theory has dynamic magnetic matter built in (by virtue of the fact that we're on a lattice) but lacks electric matter,

[66] This is actually rather surprising since the quantum-to-classical mapping only rigorously works when g is small.

[67] Another definition of the t' Hooft operator might be as the surface operator

$$T_{\Sigma^\vee} = \prod_{l \pitchfork \Sigma^\vee} \mathcal{X}_l. \tag{745}$$

Despite appearances this is a line operator in the pure gauge theory, since its interior is tensionless for all g and can be moved by applying gauge transformations. The reason that we might prefer this definition is that it is the natural discrete representation of the continuum $U(1)$ t' Hooft operator $e^{i\int_{\Sigma^\vee} \star F}$, and because it doesn't commute with W_γ if $\gamma \cap \Sigma^\vee \neq 0$. When electric matter is turned on this operator always has a vev which decays with the exponential of the area of $\Sigma^\vee$, while the t' Hooft line operator defined before this footnote continues to follow a perimeter law due to screening by magnetic fluxes.

and as such cannot possibly be self-dual[68] (this is indicated by the fact that the classical gauge theory duality just discussed becomes $g \overset{?!}{\leftrightarrow} 0$ in the $N \to \infty$ limit).

Anyway, unlike in 2+1D, magnetic monopoles fail to disorder the system at all g. This is basically just because the mangetic monopoles now form strings in spacetime, and so the free energy arguments about their proliferation are modified. Indeed, consider a t' Hooft line T of length L, with L macroscopically large, i.e. of the same order as the size of the system. We want to ask when such lines proliferate and disorder the system; this is done by computing their free energy in the 4+0D stat mech model. Since there is always magnetic matter around to screen them, their average energy will behave as a perimeter law, with $E_T = CL$ for some $C(\lambda, g, \dots)$. The configurational entropy that's relevant here is the log of the number N_L of loops of length $\sim L$ in a box of size L^4. This is known to scale exponentially in L, i.e. $N_L \sim (C')^L$ for some constant C'. The relevant free energy is then

$$F_T \sim CL - \alpha \ln(C')^L \sim L(C - \alpha \ln(C')) \tag{746}$$

for some α which again is a function of g, λ. Whether or not the t' Hooft lines proliferate is then a question of what the functions C, C', α are. Therefore a Coulomb phase is possible unlike in the 2+1D case where the sign of the free energy is always negative at large L.[69]

So, we know that $U(1)$ lattice gauge theory in 4+0D has a Coulomb phase at small g and a confined phase at large g. The simplest possibility is that there is a single transition between the Coulomb phase at $g \to 0$ and the confined phase, which is indeed borne out by numerics. Therefore if the $N \to \infty$ limit of the $\mathbb{Z}_N$ theories is to mimick the $U(1)$ theory, there cannot always be just a single transition in the $\mathbb{Z}_N$ theories, since the phase diagrams of the $\mathbb{Z}_{N\to\infty}$ and $U(1)$ cases wouldn't match up. As N increases there must then be a third Coulomb phase which springs up and eventually subsumes the topological phase. If we assume that there are just these three phases, then the phase boundaries are related by (743): the confined-Coulomb phase boundary moves to the critical coupling for the $U(1)$ transition, and the Coulomb-topological boundary moves to $g = 0$ as $N \to \infty$. The value N_* of N for which the Coulomb phase appears can be bounded by looking for when the self-dual point moves past the critical point of the $U(1)$ theory; it turns out that $N_* = 5$.[70]

With matter

Again, the computations of the scaling properties of the Wilson and t' Hooft lines is essentially the same as in the 2+1D case, so we won't elaborate any further. With

[68] Since we're on a lattice we always have to sum over configurations of U_ls with $\prod_{\square \in \partial \text{cube}} B_\square \neq 1$ around certain cubes. Such cubes would map to dual vertices for which the gauge transformation generator A_v doesn't act as $\mathbf{1}$, and so in order to get the duality to work we'd need electric matter as well.

[69] This is the same argument used to show that $\mathbb{Z}_N$ theories in 2+1D can have a deconfined phase at finite g, λ, since in that case the disorder operators are also one-dimensional (in that case they are open lines; closed t' Hooft loops commute with all Wilson lines and hence do not disorder the system like they do in 3+1D).

[70] This argument guarantees that we are not in the pathological situation where the behavior of $\mathbb{Z}_{N\to\infty}$ is different from that of $U(1)$.

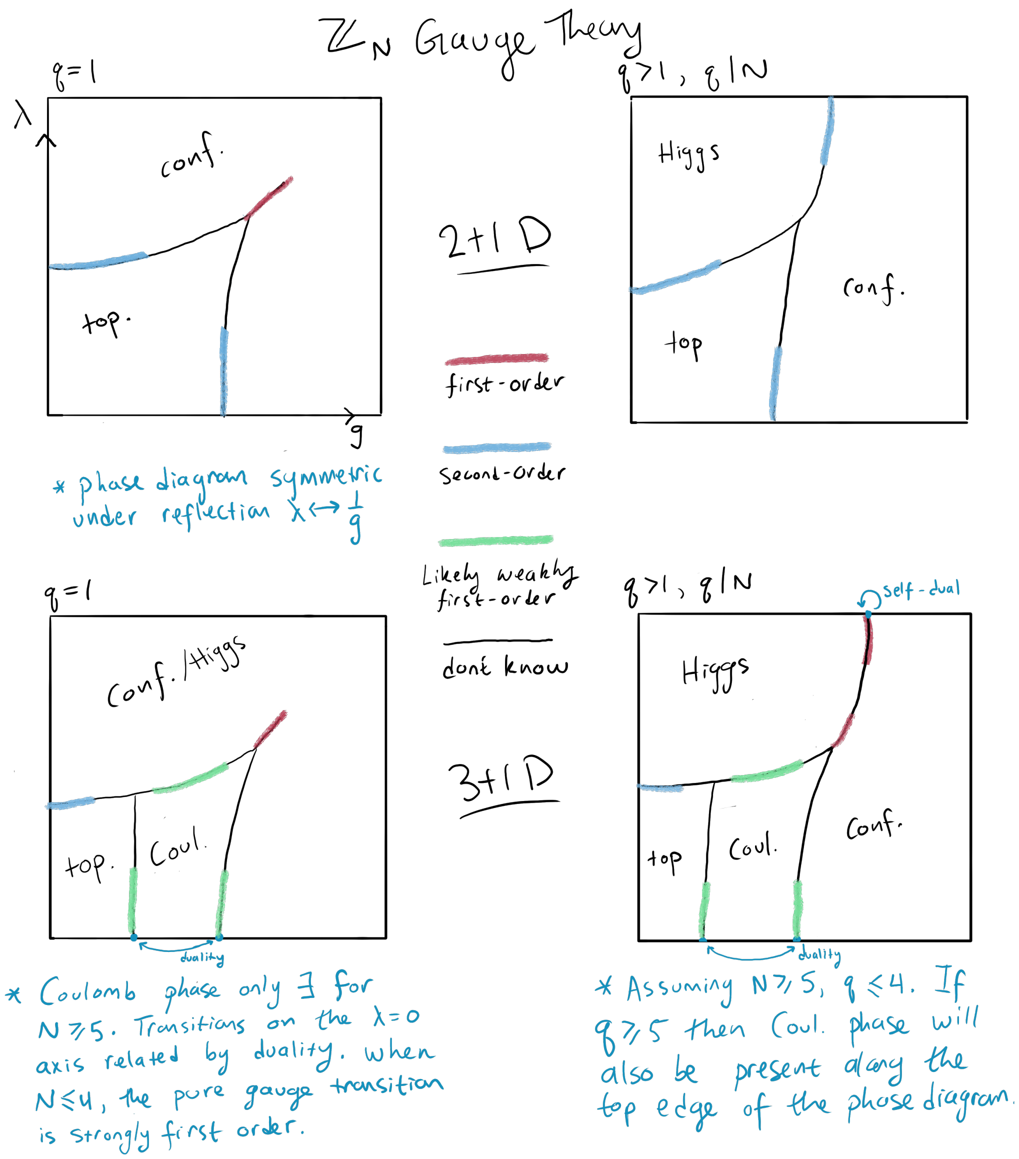


Figure 2: See text for comments. Top left should say reflection $\lambda \leftrightarrow g$.

matter turned on the duality that we used in the $\mathbb{Z}_N$ case isn't so helpful, since it maps the theory onto something with 2-form gauge fields coupled to 1-form matter. The $U(1)$ theory now admits EM duality, but this is not so helpful since the treatment of magnetic matter in this formulation is a bit awkward.

Phase diagrams

The phase diagrams for the $\mathbb{Z}_N$ case are shown in Figure 2. We label the deconfined region as "topological" instead of deconfined to distinguish it from the Coulomb phase, which also has deconfined charges but is massless, unlike the topological phase.[71]

[71]Looking at correlators of $B_\Box$ operators shows that the $\mathbb{Z}_N$ theories are always massive in the large g and small g limits.

First look at the top left phase diagram. Here the symmetry of the phase diagram is determined by the duality $\lambda \leftrightarrow g$.[72] The second order lines and the first order part past the tricritical point are known from numerics, and their existence and exponents can be established near the edges by using the known properties of the $\mathbb{Z}_N$ matter model at $g = \infty$. The curvature of the lines is determined in perturbation theory: integrating out the matter at small λ adds gauge plaquette terms to the action, hence producing $1/g_{eff} = 1/g + C\lambda^4 + \dots$ with $C > 0$; the transition is therefore shifted to larger g. This makes sense—matter fields fight confinement. Likewise, the effect of gauge fluctuations in the spin model is to promote disorder, since they reduce the energy cost of having non-parallel spins due to the fact that they make the sign of the spin-spin coupling fluctuate slightly. Hence they increase the critical value of λ, in accordance with the duality.

The top right shows the phase diagram when the matter has $q \in \mathbb{Z}_N$, with $\gcd(q, N) > 1$. The top part of the phase diagram is the bottom part of the $\mathbb{Z}_q$ phase diagram, since there we have pure $\mathbb{Z}_q$ gauge theory. Since $q < N$, the Higgs transition meets the $\lambda = \infty$ axis at a larger value of g. Not sure what happens to the formerly first-order part after the tricritical point.

The bottom left is in 3+1D, with unit charge matter, and for $N \geq 5$ where a Coulomb phase is present. As $N \to \infty$ this Coulomb phase subsumes the topological phase; for $N < 5$ only the topological and confined phases are present. The two vertical weakly first-order transitions are related by duality and are likely first-order due to the fact that the transition from Coulomb to Higgs near $\lambda = 0$ involves condensing unit strength monopoles, which by HLM we expect to be weakly first order (and hence by duality the transition into the topological phase is weakly first-order as well). Similarly the upper green line involves the condensation of electric charges a la the usual Higgs mechanism, which is weakly first order for the same reason. The blue line is a guess: I think that the absence of a massless photon in the topological phase means the HLM mechanism won't be effective, allowing the second order transition in the pure matter theory to survive the presence of the gauge field.

The bottom right is for $q > 1$ matter. Nothing really new here: we just take the bottom part of the $\mathbb{Z}_q$ phase diagram and paste it on top.

The case with gauge group $U(1)$ is shown in Figure 3. The top left is fundamental matter in 2+1D, which is boring as Polyakov taught us. When $q > 1$ we know we need to get a $\mathbb{Z}_q$ phase diagram on the top of the diagram; presumably the situation is something like the one drawn.

The bottom left shows 3+1D. Here the HLM mechanism (either for electric charges or monopoles, as the case may be) implies that both transitions near the edges are weakly first order. For $q > 1$ as in the bottom right diagram, the story is as with $q = 1$ except with a copy of the $\mathbb{Z}_q$ gauge theory phase diagram along the top part.

[72] This is how the couplings get mapped in the quantum Hamiltonian, but *not* how they are mapped in the classical lattice model, where the transformation involves a factor of N (see the diary entry on $\mathcal{QC}$ mappings).

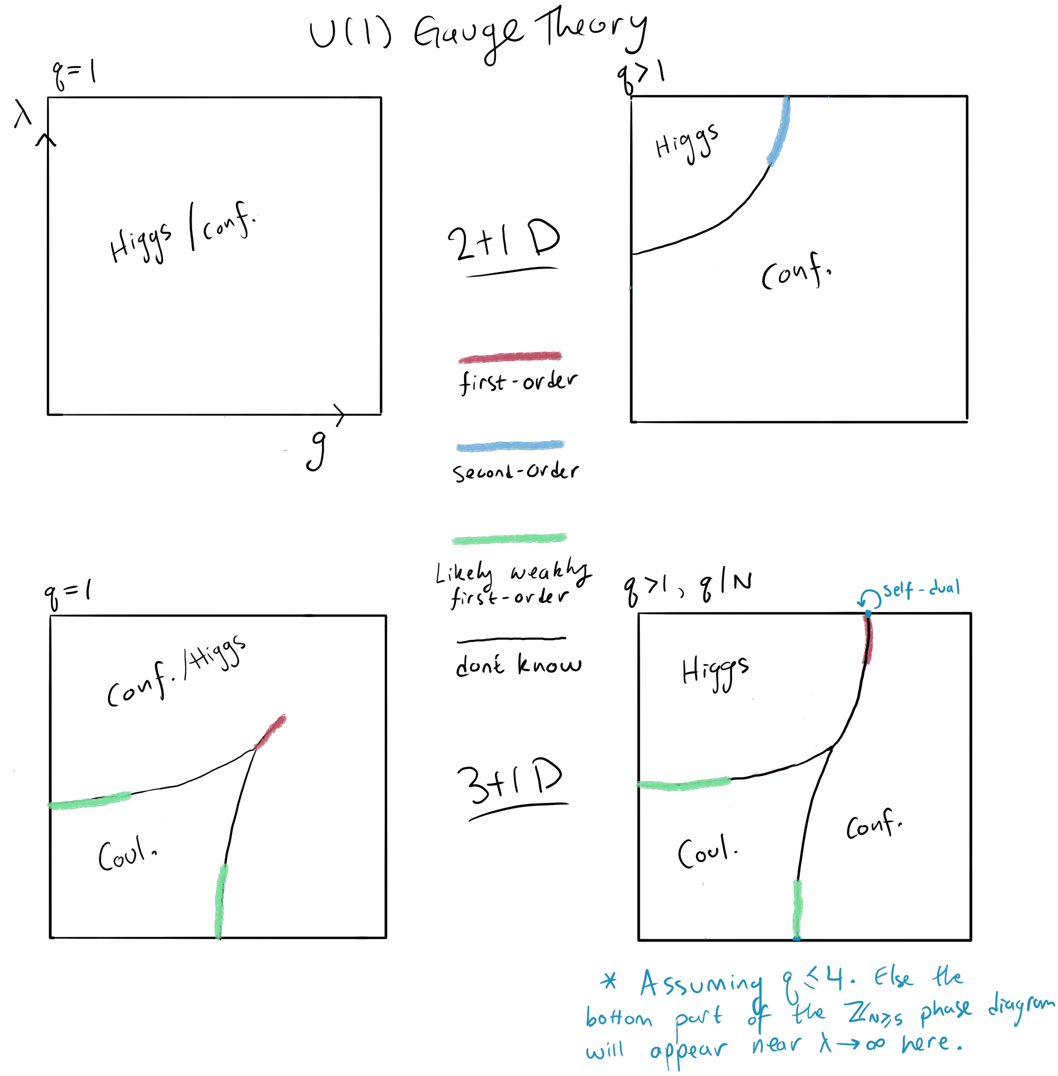


Figure 3: See text for comments.

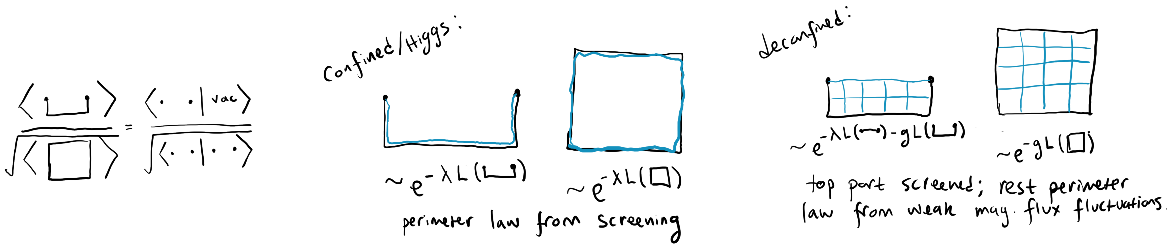


Figure 4: From right to left: a picture for the scaling of the cup operator in the deconfined phase, the same thing in the confined / Higgs phase, and a schematic rewriting of the ratio of Wilson lines as an inner product between a state with two charges and the vacuum.

Distinguishing the phases with open Wilson lines and the better "definition" of confinement

Given that when $\lambda \neq 0$ both W and T are always P-law, how do we distinguish the confined / Higgs phase from the deconfined one?[73] One answer was provided in the very cool paper [16]; this paper was sufficiently enlightening for me that I figured it would be worth taking a subsection to summarize it.

The point is that one should use a different order parameter, which measures where the line tension is coming from: is it coming from fluctuating magnetic fields around the path of the Wilson line (deconfinement), or is it coming from the energy needed to create the electric charges which screen the Wilson line (confinement)? There are many such operators which can tell the difference between these two types of line tension; the simplest one is the expectation value of the "cup operator" (my dumb name for it), defined by

$$C(L) = \frac{\langle Z_i^\dagger \left(\prod_{l\in\sqcup} \mathcal{Z}_l\right) Z_{i+L}\rangle}{\sqrt{\langle \prod_{l\in\square} \mathcal{Z}_l\rangle}}. \tag{747}$$

Here $\sqcup$ is a path in the shape of a $\sqcup$ with ends at the lattice sites $i, i+L$ (the addition is along some arbitrary direction), and is such that the two sides with ends are of length $L/2$, with the bottom segment having length L. The path $\square$ is a square with all sides of length L.

The meaning of this order parameter is in the left panel of figure 4. If we orient the vertical half-links of the cup in the time direction, then thinking in terms of a cut-and-glue approach to the partition function, we see that $\langle\sqcup\rangle$ represents the overlap between the vacuum (what you get when you cut open the $i\mathbb{R}$ time path integral) and the state with two charges separated by a distance L. The division by the closed Wilson loop is just to normalize the two-charge state correctly.

[73]Remember that we know that the phase boundary has to extend into the $\lambda > 0$ region since the effect of small λ is just to slighly renormalize the gauge coupling; this has the effect of shifting the value of the microscopic parameter g at which the transition occurs, but cannot eliminate the existence of the transition. Hence some sort of order parameter should remain well-defined upon adding in nonzero λ.

If $C(L) = 0$ as $L \to \infty$, then the two-charge state is orthogonal to the vacuum $|0\rangle$. Since any state can be expanded in terms of the spectrum of the Hamiltonian, and since $|0\rangle$ is in the spectrum, $C(L) = 0$ means that the two-charge state $|\bullet\bullet\rangle$ is part of the excitation spectrum of the theory. Since we have taken $L \to \infty$, this means that the spectrum of the theory includes free isolated charges—this is what we expect from the deconfined / topological phase.

On the other hand if $C(L) \neq 0$ in the $L \to \infty$ limit, then $|\bullet\bullet\rangle$ cannot be in the excitation spectrum, since it has overlap with the ground state. In this case free charges don't appear in the spectrum, which is what we expect from the confined phase.

This expectations are indeed borne out, at least at small λ where we can calculate the dominant L behavior of $C(L)$. First of all, we know from earlier that for large L, we have asymptotically (redefining g^2 by a factor of 4 so that the exponent is prettier)

$$\langle \prod_{l \in \square} \mathcal{Z}_l \rangle \sim \begin{cases} e^{-4L\ln(\lambda^{-1})} & \text{confined} \\ e^{-4Lg^2} & \text{deconfined} \end{cases} \tag{748}$$

Now for the $\sqcup$ part. The two Z_i, Z_{i+L} variables mean that the first term to survive in a small-λ expansion of the action is a string of $Z^\dagger \mathcal{Z} Z$s connecting the two $\sqcup$ endpoints. In the confined phase, and to lowest order in λ, this string will follow the path of the $\sqcup$; that way it completely screens the electric flux and gives a nonzero contribution to the expectation value at order λ^{2L} ($2L$ is the length of the $\sqcup$). In the deconfined phase, the first term appears at lower order in λ: we can connect the endpoints of the $\sqcup$ with a straight line of $Z^\dagger \mathcal{Z} Z$s at order λ^L, and then add in a contribution of e^{-2Lg^2} from the line tension of the electric flux line. This is larger than screening the whole line with electric charges provided that $\lambda < g$, which we will assume. So then

$$\langle Z_i^\dagger \left(\prod_{l \in \sqcup} \mathcal{Z}_l \right) Z_{i+L} \rangle = \begin{cases} e^{-2L\ln(\lambda^{-1})} & \text{confined} \\ e^{-L\ln(\lambda^{-1}) - 2Lg^2} & \text{deconfined.} \end{cases} \tag{749}$$

We then see that

$$C(L) \sim \begin{cases} \# > 0 & \text{confined} \\ e^{-L\ln(\lambda^{-1})} \to 0 & \text{deconfined,} \end{cases} \tag{750}$$

where $\#$ is some number that does not vanish as $L \to \infty$ (literally using our formulae this number would be 1, but we have only been calculating the leading L dependence; in reality $0 < \# < 1$ as $L \to \infty$). This then confirms that $C(L)$ serves as an OP to distinguish the two phases even when $\lambda > 0$ (at least, for perturbatively small λ).

Symmetries

Finally let's talk about the symmetries in the pure gauge theory. With no electric matter we of course have a $\mathbb{Z}_N^{(1)}$ or $U(1)^{(1)}$ symmetry. Somewhat confusing is the fact that the magnetic symmetry is different in the two cases: in the $\mathbb{Z}_N$ case it is a $\mathbb{Z}_N^{(D-2)}$ symmetry (which on the lattice is only ever just an emergent symmetry), while for

$U(1)$ it is a $U(1)^{(D-3)}$ symmetry (if it exists).[74] In fact, while both these symmetries have a magnetic character, their origins are pretty different, and it is probably best to not discuss them in the same context.

In discrete gauge theories, the magnetic $\mathbb{Z}_N^{(D-2)}$ symmetry comes emerges in the IR when we work below the gap of the $B_\Box$ term. At these low energies, we can work in a constrained subspace where the t' Hooft operators

$$T_\Sigma = \prod_{l \pitchfork \Sigma} \mathcal{X}_l, \qquad \Sigma \in C_{D-2}(\Lambda^\vee; \mathbb{Z}) \tag{751}$$

can't end (that is, they can only be defined for Σ with $\partial\Sigma = 0$), because they create energetically costly magnetic flux along $\partial\Sigma$. Therefore the conserved objects responsible for the $\mathbb{Z}_N^{(D-2)}$ symmetry are the lines / surfaces of the t' Hooft operators, which being $(D-2)$-dimensional give us a $\mathbb{Z}_N^{(D-2)}$ symmetry.

On the other hand, in $U(1)$ gauge theory t' Hooft operators can always be defined on manifolds with $\partial\Sigma \neq 0$, regardless of the value of g: this is because they can be used to create arbitrarily small amounts of magnetic flux along $\partial\Sigma$, which incurs arbitrarily small energetic cost. Therefore t Hooft operators for $U(1)$ are always breakable, unlike for $\mathbb{Z}_N$.

Instead, the magnetic symmetry for $U(1)$ comes from something totally different—the absence of magnetic monopoles. Since the operator that counts monopoles is always two-dimensional, this symmetry is indeed a $(D-3)$-form symmetry. This symmetry and the absence of monopoles is something that's well-defined in the continuum limit.

However, on the lattice I think this symmetry is always non-existent, at least if we use the standard Wilsonian form of the action. Indeed, I don't even think that the concept of a magnetic monopole on the lattice is really meaningful / useful, except at long distances and in the $g \to 0$ limit where we can use the weak-coupling continuum action anyway, and do away with the lattice.[75] The naive way of defining monopoles on the lattice would be to sum the "field strengths" $\mathcal{H}_\Box$ around the ⧉. Since modding by 2π is taken into account in the definition of $\mathcal{H}_\Box$, this sum indeed produces something in $2\pi\mathbb{Z}$. However, with this definition monopoles may be created at arbitrarily small energy cost, since we may have e.g. one $\Box$ contributing a holonomy of $2\pi - \varepsilon$ and another contributing ε, for an energy that vanishes as ε^2. In this sense monopoles are not really well-defined charges, since the field configurations they set up can be arbitrarily close to the vacuum as far as energetics goes (as long as the magnetic flux is collimated along a certain direction).

Part of this problem is that assigning $\mathcal{H}_\Box$ to the field strength at $\Box$ is only a reasonable thing to do in the continuum limit where all $\mathcal{H}_\Box$s are small. More field strength should mean more energy, which is not true if we assign field strength to $\mathcal{H}_\Box$

[74] Mathematically, the shift in the degrees comes from the fact that the group cohomology satisfies $H^k(G; U(1)) \cong H^{k+1}(G; \mathbb{Z})$ as derivable from $1 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$.

[75] $g \to 0$ isn't the only way to take the continuum limit, since there's a critical point at nonzero g_* for the lattice theory. Here the action we should write down is unknown (at least to me), and there's no obvious way of finding the continuum theory from the lattice model. Presumably though the holonomies are not slowly varying on the scale of a lattice, so that expanding the cosine is not allowed.

because of the cosine. Instead, field strength can be defined through the analogue of the continuum equation $d^\dagger F = J$, which is obtained by varying the action with respect to the A_ls. When we do this on the Wilsonian action we get

$$d^\dagger \sin \mathcal{H}_\square = J \implies F_\square = \sin \mathcal{H}_\square, \tag{752}$$

where $d^\dagger$ is the lattice divergence. $\sin \mathcal{H}_\square$ is a reasonable field strength since it vanishes for holonomies which cost zero energy and reduces to $\mathcal{H}_\square$ in the continuum limit of small $\mathcal{H}_\square$. One could then define the monopole charge by doing $\sum_{\square \in \Sigma} \sin \mathcal{H}_\square$ as a tentative monopole charge. While this is not quantized, it reduces to the usual monopole charge in the continuum limit, and in my opinion better encapsulates the energetical / dynamical properties of magnetic monopoles for all g.

Anyway, the point of the above rambling discussion is that in the weak coupling continuum limit fixed point, we can consider a regime where the monopoles are massive and a $U(1)^{D-3}$ symmetry emerges. Away from $g \to 0$ though this symmetry is explicitly broken as the monopoles start to condense upon approaching the confinement phase transition.

Transfer matrices and quantum-to-classical heuristics

The quantum-to-classical mapping for lattice models a la Kogut used to seem rather ad-hoc to me—in today's entry we'll make a few comments that made me feel a little more confident about the whole procedure.

Coupled springs

A good first example is a 1d classical model of springs:

$$Z = \int \prod_i dr_i \, e^{-\frac{1}{2K} \sum_i (r_i - r_{i+1})^2}, \tag{753}$$

where we e.g. fix ∂conds to be $r_0 = r_I$ and $r_N = r_F$, or else work with periodic ∂ conds. The goal is to rewrite this as a quantum problem in $0+1$D. This entails writing the partition function as $\langle r_F | e^{-\beta_Q H_Q} | r_I \rangle$ for some parameter β_Q and some 0d Hamiltonian H_Q. Therefore we need to collapse the product over dr_is into a trace of a power of a single matrix.

The key to doing this is to realize that for any function $f(r)$ (that can be Fourier transformed), we have

$$e^{K\nabla^2/2}f(r) = \int dr'\, e^{-(r-r')^2/2K} f(r'). \tag{754}$$

To prove this, it is enough to consider the case where $f(r) = e^{iqr}$, in which case

$$\int dr'\, e^{-(r-r')^2/2K} e^{iqr'} = e^{iqr}\int dr'\, e^{-(r')^2/2K+iqr'} = e^{iqr}e^{-q^2K/2} = e^{K\nabla^2/2}(e^{iqr}). \tag{755}$$

The sign of the exponent can be remembered by recalling that ∇^2 is negative-semidefinite.

Therefore the matrix $e^{K\nabla_r^2/2}$ is a "contraction matrix" that does the needed integral in the partition function over the coordinate r. Hence we can write the partition function as

$$Z = \langle r_F| \left(e^{K\nabla^2/2}\right)^N |r_I\rangle. \tag{756}$$

Evidently this represents time evolution by the operator

$$\beta_Q H_Q(p) = \frac{NK}{2}p^2, \tag{757}$$

where now $p = -i\nabla$. So indeed, we can rewrite the partition function as time evolution for a 0+1D quantum problem.

It is then straightforward to port this concept of transfer matrices up to higher dimensional versions of the spring model. Let us consider the classical model with Hamiltonian

$$\beta H_C = \frac{1}{2K_\perp}\sum_{\langle ij\rangle_\perp}(r_i - r_j)^2 + \frac{1}{2K_\parallel}\sum_{\langle ij\rangle_\parallel}(r_i - r_j)^2. \tag{758}$$

Here $\langle ij\rangle_\perp$ are all the links perpendicular to a certain distinguished axis (which will become time), and $\langle ij\rangle_\parallel$ are all the links parallel to the same axis. The matrix that does the contracting between different "time slices" is the matrix that contracts the $\perp$ hyperplanes together by doing the integrals over the $\parallel$ links. We see that this "contraction matrix" is just $C = \exp\left(\sum_i K_\parallel \nabla_i^2/2\right)$, and that it acts on the matrix $M = \sum_{\langle ij\rangle}(2K_\perp)^{-1}(r_i - r_j)^2$, where the sum over nns only occurs within one hyperplane. Therefore the integral over all coordinates in a given hyperplane is accomplished with the matrix CM, and so

$$Z = \langle \partial_F|(CM)^N|\partial_I\rangle, \tag{759}$$

where again N is the (dimensionless) length of the "time" direction and ∂_I, ∂_F are boundary conditions. Then if we have PBC in the $\parallel$ direction this gives us a quantum problem with the Hamiltonian

$$H_Q \propto -\frac{K_\parallel}{2}\sum_i \nabla_i^2 + \frac{1}{2K_\perp}\sum_{\langle ij\rangle}(r_i - r_j)^2, \tag{760}$$

which is exactly what we expect. The general point is that the contraction over the time direction can be implemented by new non-commuting momentum variables, and when this is done we get a quantum Hamiltonian.

Actually we've cheated a bit here—since ∇^2 and $(r_i - r_j)^2$ don't commute, we can't combine the C and M matrices into a single exponential. We can only do this if the commutator of the logs of C and M is small enough to be dropped in the exponential. This means that we need the limit[76]

$$\frac{K_\parallel}{K_\perp} \to 0. \tag{761}$$

Noting that the Ks have dimensions of length, we interpret this to mean that the quantum partition function is strictly speaking only obtained in the limit where the lattice spacing in the time ($\parallel$) direction is taken to be much smaller than the spacing in the spatial directions. That is, the theory becomes that of a quantum system only when we take the continuum limit in the time direction.[77] If we write the number of lattice spacings in the $\parallel$ direction as $N = 1/Ta$ where T^{-1} and a have dimensions of "time",

$$Z = \mathrm{Tr}[e^{-T^{-1}H_Q}], \qquad -H_Q = \frac{K_\parallel}{2a}\sum_i \nabla_i^2 + \frac{K_\perp}{2a}\sum_{\langle ij\rangle}(r_i - r_j)^2, \tag{763}$$

which is what we expect.

The Ising model

In the case of e.g. the Ising model, we use the same procedure, by introducing operators which don't commute with the spins to write the spin sum as a matrix product. In two dimensions with uniform couplings (for simplicity of notation; generalizing is easy), we start with $H_C \propto \sum_{\langle ij\rangle} Z_i Z_j$. The needed contraction identity is

$$\mathrm{Tr}_j\left[e^{-JZ_iZ_j}f(Z_j)\right]|i\rangle = e^{\alpha X_i}f(Z_i)|i\rangle, \tag{764}$$

where we will find α shortly, and where $|i\rangle$ can be either $|\uparrow\rangle$ or $|\downarrow\rangle$. Writing $f(Z) = f_e + f_o Z$, the sum is, after some algebra,

$$\mathrm{Tr}_j\left[e^{-JZ_iZ_j}f(Z_j)\right]|i\rangle = \left[\begin{pmatrix}\zeta^{-1} & \zeta \\ \zeta & \zeta^{-1}\end{pmatrix}\begin{pmatrix} f_e + f_o \\ f_e - f_o\end{pmatrix}\right]_i = \left[(\mathbf{1}\zeta^{-1} + X\zeta)\begin{pmatrix} f_e + f_o \\ f_e - f_o\end{pmatrix}\right]_i \tag{765}$$

where $\zeta \equiv e^J$. Therefore we need

$$\cosh\alpha = \zeta^{-1}, \qquad \sinh\alpha = \zeta \implies \alpha = \tanh^{-1}[\zeta^2]. \tag{766}$$

Each 1d line in the classical partition function is then integrated out with the matrix CM, where now $C = e^{\tanh^{-1}[\zeta^2]\sum_i X_i}$ and $M = e^{-J\sum_{\langle ij\rangle} Z_iZ_j}$, which lets us write Z as time evolution with the 1d TFIM Hamiltonian

$$H_Q \propto J\sum_{\langle ij\rangle} Z_iZ_j - \tanh^{-1}[\zeta^2]\sum_i X_i. \tag{767}$$

[76] Really, we want this ratio to be small compared to the typical eigenvalues of the commutator of the ∇^2 and $(r_i - r_j)^2$ terms that appear in the trace.

[77] Strictly speaking we can always write the classical partition function as a quantum system with Hamiltonain

$$T^{-1}H_{\rm awful} = -N\ln(CM), \tag{762}$$

where T^{-1} is the temperature of the quantum system. But this expression is horrendous, and certainly not something we want to map our classical model to, unless the assumptions on the Ks hold.

$U(1)$ gauge theory

Now let's do $U(1)$ gauge theory. We will start with the quantum theory and map to the classical one, for reasons that will become clear after we get the answer. Therefore we start with (probably should have relabeled g as g^2; oh well)

$$H = -\frac{1}{2g}\sum_{\Box}\mathcal{F}_{\Box}(U_{l\in\partial\Box}) - \frac{g}{2}\sum_{l} E_l^2. \tag{768}$$

Here $\mathcal{F}_{\Box}$ is some function of the link variables on $\Box$—it could e.g. be $\cos(\mathcal{H}_{\Box})$ where $\mathcal{H}_{\Box}$ is the holonomy around $\Box$, but we will leave it unspecified for now. We will do the $\mathcal{QC}$ mapping first (which given the usual Trotterization procedure is essentially unambiguously defined), and then see what types of plaquette terms are generated on the new temporal plaquettes—this will inform us about which form of $\mathcal{H}_{\Box}$ is "best" to work with.

Again we will be splitting up into "time" steps of size $\delta^{-1} \gg E_{\text{typical}}$. At each step, we will need to insert the gauge invariance projector, which is

$$\Pi = \prod_{v}\int \mathcal{D}\theta_v^0\, e^{i\theta\sum_{l\in\partial v}E_l}, \tag{769}$$

where θ_v^0 will be thought of as proportional to the logaraithm of a link variable U_l living on a link extending out of v and up into the "time" direction. Inserting Π at each timestep is needed since in the resolutions of $\mathbf{1}$ that are inserted at each step, we will be summing over *all* configurations for the link variables, not just the ones that obey Gauss's law.[78] One step in the Trotterized partition function, giving the matrix element between slices at timesteps j and $j+1$, is then

$$\prod_{v.l}\int \mathcal{D}E_l\,\mathcal{D}\theta_v^0\,\langle U_{j+1}|E\rangle\langle E|U_j\rangle \exp\left(\frac{g\delta}{2}\sum_l E_l^2 + \frac{\delta}{2g}\sum_{\Box}K(U_{l\in\partial\Box}) + i\theta_v^0\sum_{l\in\partial v}E_l\right). \tag{770}$$

In order to write down the above, we had to make an assumption: if $\langle E|e^{-\delta H}|U_j\rangle$ is to be simple, we need to split up the E and U parts of H, so that they can act on their respective eigenstates. One assumption that allows us to do this (but this may be slightly more strong than we need) is to assume that $\delta H \ll 1$, by which we mean that $\delta E \ll 1$ for any typical energy E that makes a non-negligible contribution to the path integral. [79] We will be sticking with δ small enough that this assumption is valid, which is equivalent to taking the continuum limit in the time direction.

Now we want to do the sum (despite the notation it's a sum, since the spectrum of the Es is $\mathbb{Z}$) over all values of the E_ls. We do this with Poisson resummation: for any function $f(x) \in L^1(\mathbb{R})$, the sums of f over $\mathbb{Z}$ and the Fourier transform $\widetilde{f}$ over $2\pi\mathbb{Z}$ are equal:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{p\in\mathbb{Z}}\widetilde{f}(2\pi p). \tag{771}$$

[78]We want to sum over all configurations since matrix elements like $\langle U_j|E\rangle$ are easier to deal with when the sum is over all U_js and Es, not just those that obey Gauss's law.

[79]Since H is unbounded we really need $\delta = 0$, but presumably the super high eigenstates of H aren't all that important for the questions we want to know. We will have in mind taking $\delta < \Lambda^{-1}$ where Λ is some finite but "large" cutoff scale.

A special application of this is to the function $f(x) = e^{-\frac{1}{2}x^2+ix\theta}$, for which we get

$$\sum_n e^{-\alpha n^2/2} e^{in\theta} = \sqrt{\pi/\alpha} \sum_p e^{-\frac{1}{2\alpha}(\theta-2\pi p)^2}. \tag{772}$$

Now since E_l is conjugate to the log of the U_j link variables, we have $\langle E_l|U_l\rangle = e^{iE_l A_l}$, where $U_l = e^{iA_l}$.[80] Each E_l therefore appears in the exponent as

$$g\delta E_l^2 + iE_l(A_{l;j} - A_{l;j+1} + \theta^0_{vL} - \theta^0_{vR}), \tag{773}$$

where $\theta^0_{vL/R}$ are the Lagrange multiplier variables at the end / beginning of the link L. For obvious reasons, let us denote the quantity in parenthesis as $\mathcal{H}_{\square_t}$. Then using the Poisson summation formula, and taking the product over all timesteps, the partition function becomes

$$Z = \sum_{\{p\}} \int \prod_l \mathcal{D}U_l \prod_v \mathcal{D}\theta^0_v \, \exp\left(-\frac{1}{2g\delta}\sum_{\square_t}(\mathcal{H}_{\square_t} - 2\pi p_{\square_t})^2 - \frac{\delta}{2g}\sum_{\square}\mathcal{F}_{\square}(U_{l\in\partial\square})\right), \tag{774}$$

where $\square_t$ are the temporal plaquettes and l the spatial links. Now the usual Wilsonian action would have a $\cos(\mathcal{H}_{\square_t})$ instead of the sum over all ps. This shows us that the Wilsonian lattice action is slightly less "natural", in the sense that it is not what we get when we do the above $\mathcal{QC}$ mapping on a quantum Hamiltonain (regardless of what we choose for $\mathcal{F}_\square$). To make this symmetric in spacetime then (despite the fact that the image of the $\mathcal{QC}$ mapping is very asymmetric in the time direction because of the annoying δ), it is nice to similarly choose the $\mathcal{F}_\square$ spatial part of the action to have the same Villain form. Therefore the classical partition function we get is, now writing the θ^0_vs as link variables on a $D+0$ dimensional spacetime and writing the couplings in plane and along the time direction as $K_s = \delta/2g$ and $K_t = 1/(2\delta g)$ respectively,

$$Z = \sum_{\{p\}}\sum_{\{z_l\}} \exp\left(-K_t\sum_{\square_t}(\mathcal{H}_{\square_t} - 2\pi p_{\square_t})^2 - K_s\sum_{\square}(\mathcal{H}_{\square} - 2\pi p_{\square})^2\right). \tag{775}$$

The usual Wilsonian cosine form is recovered in the limit where both the K_t and K_s couplings are small, so that we may use the Villain approximation

$$e^{\frac{1}{2g}\cos(\theta)} \mathrel{\approx\!\!\!\!\propto} \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2g}(\theta-2\pi k)^2}, \tag{776}$$

where we don't care about getting the normalization factors exactly right. This approximation gets better as θ gets forced to lie in $2\pi\mathbb{Z}$, i.e. when we are at weak coupling.[81]

This has been for pure gauge theory, but adding (bosonic, for simplicitly) matter is straightforward. The Wilsonian way to do this would be to add the minmial coupling

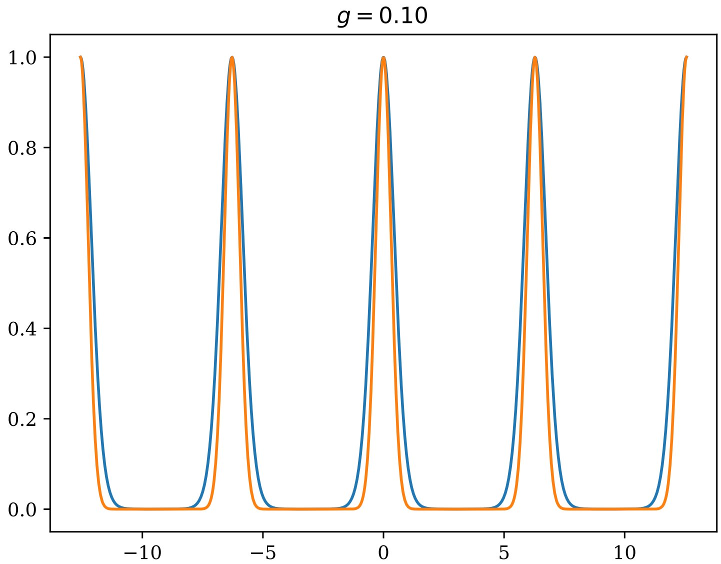
[80] The state $|U_j\rangle$ is the whole state at timestep j, so $|U_j\rangle = \bigotimes_l |U_l\rangle$. This is a bigger $\mathcal{H}$ space than the physical one, but that's okay, since we've added the Πs in between each resolution of the identity in the Trotterization.

[81] Aside: how well does the Villain approximation work? Let's plot the two functions and find out:

term $\lambda|\psi_i - U_{\langle ij\rangle}\psi_j|^2/2$. The manipulations become simpler if the magnitude of the matter field is frozen out as is the case near the Higgs phase. This simplifies things since then the only dynamical matter is the part of ψ directly responsible for the $U(1)$ symmetry, viz the phase ϕ of ψ. This in turn means that the expression for the generator of gauge transformations G is particularly simple, since the canonical momentum of ϕ is equal to the density appearing in G. We therefore will take the matter part of the Hamiltonian to have the villain form

$$H \supset \frac{\lambda}{2}\sum_l ((d\phi)_l - A_l - 2\pi m_l)^2 + \frac{1}{2\lambda}\sum_v \pi_\phi^2 \tag{780}$$

for several values of g,



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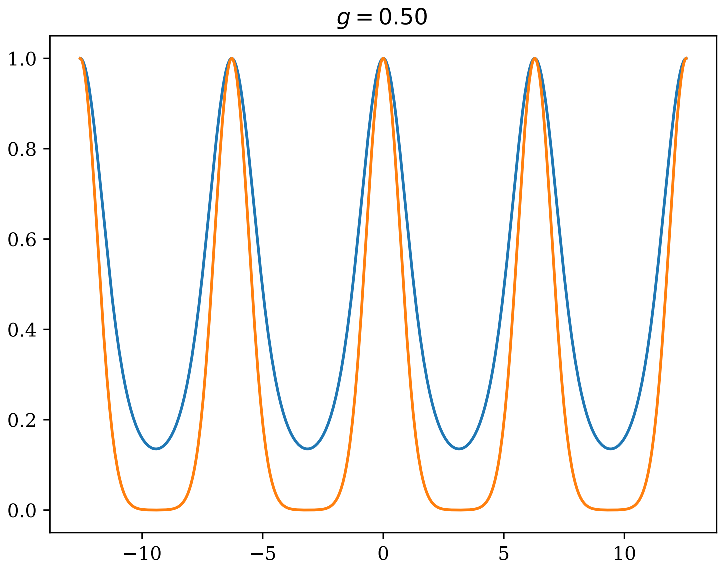


(778)

where $m_l \in \mathbb{Z}$ is summed over in the path integral and d is the lattice derivative. The projectors that get inserted at each timestep are now

$$\Pi = \prod_v \int \mathcal{D}\theta_v^0 \exp\left[i\theta_v^0 \left(\sum_{l\in\partial v} E_l - \pi_v\right)\right]. \tag{781}$$

We then Poisson-resum the integers appearing in the kinetic terms for the matter fields and gauge fields. The summations over E_l and π_ϕ are identical in form (since π_ϕ is the momentum for a phase variable, it too is valued in $\mathbb{Z}$), and we end up with the classical partition function

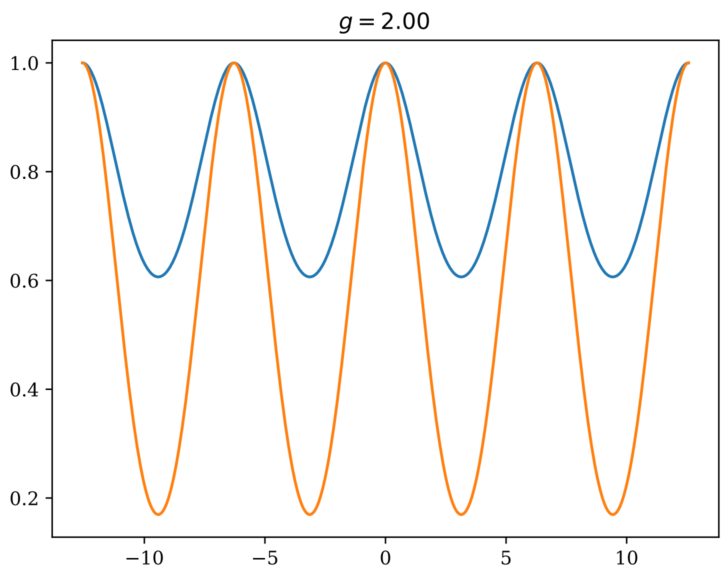
$$Z = \sum_{\{p\},\{q\}} \sum_{\{z_l\},\{\phi_v\}} \exp\left(-S_{\text{gauge}} - \frac{\lambda}{2\delta}\sum_{l_t}((d\phi)_{l_t} - A_{l_t} - 2\pi q_{l_t})^2 - \frac{\lambda\delta}{2}\sum_l((d\phi_l) - A_l - 2\pi q_l)^2\right), \tag{782}$$

with S_{gauge} the action appearing in the pure gauge partiton function above and l_t, l are temporal and spatial links, respectively.

$\mathbb{Z}_N$ gauge theory

As a more complicated example, consider quantum $\mathbb{Z}_N$ lattice gauge theory. One way to write the action is in the Wilsonian form

$$H = -\frac{1}{2g}\sum_\Box B_\Box - \frac{g}{2}\sum_l X_l + h.c. \tag{783}$$



(779)

So even when $g = 2$ it's not unreasonable to assume that the universal behaviors of models with the typical $\cos(B_\Box)$ action will still be in the same universality class as the "exact" model coming from the quantum-to-classical mapping.

where $ZX = \zeta_n XZ$. Again it's a bit easier to go from quantum to classical here, so that's what we'll do.[82] At a technical level, the $\mathbb{Z}_2$ case is a bit different from the $N > 2$ case. We will first treat the $\mathbb{Z}_2$ case, using the above Wilsonian form of the action, and then come back to the more general case.

When we trotterize we do the usual thing of inserting both position (Z basis) and momentum (X basis) resolutions of the identity. Again, let the time step interval be δ, so that the inverse temperature in the quantum model is $\beta = N\delta$. Then

$$\langle X|\Pi e^{-\delta H}|Z\rangle \approx \frac{1}{2}\prod_v \left(1 + \prod_{l\in\partial v} X_l\right) e^{\delta g^{-1}\sum_\square B_\square(Z) + \delta g \sum_l X_l}\langle Z|X\rangle, \tag{784}$$

where now the Xs and Zs are supposed to be thought of as numbers, not matrices. Now $\langle Z|X\rangle$ gives the matrix elements of the discrete Fourier transform, namely

$$\langle Z|X\rangle = \zeta_2^{zx}, \tag{785}$$

where we will use lowercase letters to denote the elements of $\mathbb{Z}_N$ with the group law being addition, i.e. $z, x \in \mathbb{Z}_N \subset \mathbb{N}$ (here $\mathbb{Z}_2$). This means that e.g. $z = N(\ln Z)/(2\pi i)$. We can then write the gauge invariance projector with a Lagrange multiplier as

$$\Pi = \frac{1}{2}\prod_v \sum_{k_v} \zeta_2^{k\sum_{l\in\partial v} x_l}. \tag{786}$$

Finally the X term in the Hamiltonian can be re-written as $\delta g(1-2x)$, so that the timestep is (dropping constants)

$$Z_{j\to j+1} = \prod_{l,v}\sum_{x_l,k_v} \zeta_2^{\sum_l x_l(z_{l;j} - z_{l;j+1} + k_{v0} - k_{v1})} \exp\left(\delta g^{-1}\sum_\square B_\square(z_j) + \delta g \sum_l (1-2x_l)\right). \tag{787}$$

Now we need to do the sum over the x_ls. Because we are only summing over $\mathbb{Z}_2$, this is easy to do: the relevant sum gives

$$e^{\delta g} + e^{-\delta g}\zeta_2^{\mathcal{H}_{\square_t}(z,k)}, \tag{788}$$

where $\mathcal{H}_{\square_t} \in \mathbb{Z}$ is the obvious notation for the discrete holonomy around the temporal plaquette $\square_t$. Since $e^{az} = \cosh a + s\sinh a$ if $s \in \pm 1$, we can define a new coupling g_t^{-1} in terms of which the sum is

$$e^{\delta g} + e^{-\delta g}\zeta_2^{\mathcal{H}_{\square_t}(z,k)} = \exp\left(g_t^{-1}\delta\zeta_2^{\mathcal{H}_{\square_t}(z,k)}\right) \tag{789}$$

where

$$g_t^{-1}\delta = \tanh^{-1}[e^{-2g\delta}]. \tag{790}$$

[82] In general quantum $\to$ classical is easier than classical $\to$ quantum. This is because when doing the latter there is a canonical procedure for Trotterizing, inserting resolutions of $\mathbf{1}$ in the momentum basis, etc. To do classical $\to$ quantum we have to make smart guesses about how to do the contraction in the transfer matrix, which is harder.

Doing the division another way gives instead $g_t^{-1}\delta = -\frac{1}{2}\ln\tanh[g\delta]$, but this is actually self-consistent due to the curious fact that

$$-\frac{1}{2}\ln\tanh\left(-\frac{1}{2}\ln\tanh(s)\right) = s. \tag{791}$$

Now the exponential of $\zeta_2 = -1$ with the $\mathcal{H}$ is just a pretentious way of writing the plaquette operator for the temporal plaquettes, hence we get an anisotropic classical model with partition function

$$Z = \prod_l \sum_{z_l} \exp\left(\frac{\delta}{g}\sum_{\square_s} B_{\square_s}(z) + \frac{\delta}{g_t}\sum_{\square_t} B_{\square_t}(z)\right). \tag{792}$$

As usual, if we hold g fixed then the $\mathcal{QC}$ correspondence is only exact in the limit that the temporal coupling diverges, since $\delta/g_t \to \tanh^{-1}(1) = \infty$.

Now we address the situation for $N > 2$. Things are more complicated now since the $+h.c.$ term in H is nontrivial. Adding in the conjugate gives cosines $\cos(\mathcal{H}_\square), \cos(x_l)$. The latter of these makes doing the sum over x intermediate states in the Trotterization difficult. So like with the $U(1)$ case, we can really only get anywhere if we work in the small-g regime where Villain-ization of the action is appropriate—this also has the technical bonus of allowing us to work directly with the x, z variables rather than the X, Z ones, the former being better suited to doing computations. Anyway we first perform Poisson resummation by doing the sum over m in the term appearing as $(2\pi x_l/N - 2\pi m)^2$. This gives us something linear in x, with x only appearing in the exponents of ζ_Ns, so that x can then be integrated out to give a mod N delta function. This sets $\mathcal{H}_{\square_t} = Np$ for $p \in \mathbb{Z}$, for every temporal plaquette. These manipulations are all basically the same as they were in the $U(1)$ case; when we're done we get

$$Z = \sum_{\{p\}} \int \mathcal{D}U_l \, \exp\left(\frac{\delta}{2g_t}\sum_{\square_t}\left(\frac{2\pi}{N}\mathcal{H}_{\square_t} - 2\pi p_{\square_t}\right)^2 + \frac{\delta}{2g}\sum_{\square}\left(\frac{2\pi}{N}\mathcal{H}_{\square} - 2\pi p_{\square}\right)^2\right), \tag{793}$$

where $g_t^{-1}\delta = \frac{N}{4\pi^2 g\delta}$.

Lattice duality for generalized classical $\mathbb{Z}_N$ spin models with gauge fields

Today we'll derive some dualities involving classical stat mech models of discrete matter + gauge fields on a lattice. Several specific cases of the general formula below appear in various places in the cmt literature, where their derivations take up much more space than is needed. Differential form notation is a huge time-saver!

We will be considering an isotropic D-dimensional stat mech model with partition function

$$Z = \sum_{\substack{m\in C^{q+1}(\Lambda;\mathbb{Z}) \\ n\in C^q(\Lambda;\mathbb{Z})}} \sum_{\substack{a\in C^q(\Lambda;\mathbb{Z}_N) \\ \phi\in C^{q-1}(\Lambda;\mathbb{Z}_N)}} \exp\left[-\frac{\beta}{2}\left(\frac{2\pi}{N}da - 2\pi m\right)^2 - \frac{\lambda}{2}\left(\frac{2\pi}{N}(d\phi - a) - 2\pi n\right)^2\right], \tag{794}$$

where Λ is some D-dimensional lattice. The sums over m and n can be thought of as coming from making the Villain approximation (large β,λ) to a discrete Wilsonian action written in terms of cosines like $\cos(2\pi da/N)$. Since the cosines are infinite order in the fields but the above representation is quadratic in everything, the Villain form will need to be used to do dualities. The differentials are of course lattice differentials.

The duality works by doing Poisson resummation on the m and n variables. We use

$$\sum_{m\in\mathbb{Z}} \widetilde{f}(2\pi m) = 2\pi \sum_{k\in\mathbb{Z}} f(k) \tag{795}$$

applied to each $q+1$, q cell of Λ. This means that e.g. (dropping proportionality constants)

$$\sum_{m\in C^{q+1}(\Lambda;\mathbb{Z})} \exp\left[-\frac{\beta}{2}\left(\frac{2\pi}{N}da - 2\pi m\right)^2\right] = \sum_{l\in C^{D-q-1}(\Lambda^\vee;\mathbb{Z})} e^{-l^2/2\beta}\zeta_N^{l\wedge da} \tag{796}$$

which we get after doing the fourier transform. Here $\Lambda^\vee$ is the dual lattice; writing l as a $D-q-1$ chain in this way is purely for notation's sake. If we preferred we could take $l \in C^{q+1}(\Lambda;\mathbb{Z})$ and replace the $l\wedge da$ with $\star l \wedge da$.

Doing the same Poisson resummation on the term with the n field, we see that the partition function can equivalently be written as

$$Z = \sum_{\substack{l\in C^{D-q-1}(\Lambda^\vee;\mathbb{Z}) \\ k\in C^{D-q}(\Lambda^\vee;\mathbb{Z})}} \sum_{\substack{a\in C^q(\Lambda;\mathbb{Z}_N) \\ \phi\in C^{q-1}(\Lambda;\mathbb{Z}_N)}} \exp\left[-\frac{l^2}{2\beta} - \frac{k^2}{2\lambda} + \frac{2\pi i}{N}\left(l\wedge da + k\wedge(d\phi - a)\right)\right]. \tag{797}$$

Now we can integrate out ϕ and a. Summing over ϕ tells us that

$$k = d\widetilde{a} + N\widetilde{m}, \qquad \widetilde{m}\in C^{D-q}(\Lambda^\vee;\mathbb{Z}), \quad \widetilde{a}\in C^{D-q-1}(\Lambda^\vee;\mathbb{Z}_N). \tag{798}$$

Summing over a then tells us that we can parametrize l as

$$l = d\widetilde{\phi} + \widetilde{a} + N\widetilde{n}, \qquad \widetilde{n}\in C^{D-q-1}(\Lambda^\vee;\mathbb{Z}), \quad \widetilde{\phi}\in C^{D-q-2}(\Lambda^\vee;\mathbb{Z}_N). \tag{799}$$

Therefore we can equivalently write the partition function in the dual form

$$Z = \sum_{\substack{\widetilde{m}\in C^{D-q}(\Lambda^\vee;\mathbb{Z}) \\ \widetilde{n}\in C^{D-q-1}(\Lambda^\vee;\mathbb{Z})}} \sum_{\substack{\widetilde{a}\in C^{D-q-1}(\Lambda;\mathbb{Z}_N) \\ \widetilde{\phi}\in C^{D-q-2}(\Lambda;\mathbb{Z}_N)}} \exp\left[-\frac{\widetilde{\beta}}{2}\left(\frac{2\pi}{N}d\widetilde{a} - 2\pi\widetilde{m}\right)^2 - \frac{\widetilde{\lambda}}{2}\left(\frac{2\pi}{N}(d\widetilde{\phi} - \widetilde{a}) - 2\pi\widetilde{n}\right)^2\right] \tag{800}$$

where the dual couplings are defined as

$$\widetilde{\beta} \equiv \frac{N^2}{\lambda(2\pi)^2}, \qquad \widetilde{\lambda} \equiv \frac{N^2}{\beta(2\pi)^2}. \tag{801}$$

Thus a $\mathbb{Z}_N$ theory of $(q-1)$-form variables coupled to a q-form gauge field with gauge and Higgs couplings (β, λ)[83] is the same as that of a theory of $(D-q-2)$-form matter on the dual lattice, coupled to a $(D-q-1)$-form gauge field, with couplings $(N^2/4\pi^2\lambda, N^2/4\pi^2\beta)$. The most useful examples of this are the strong-weak self-duality of gauge fields and matter in three dimensions, and the self-duality of the pure gauge theory in four dimensions, both of which of course have counterparts in the 2+1D and 3+1D quantum models.

BF theory basics

Today we will carefully go through sections 1, 2, and 3 of "Coupling QFTs to TQFTs" [18] , explain how the various presentations of BF theory are constructed, and explain what the global symmetries are.

First, a note on notation: I will largely only write down dA in an integrand when A is actually a well-defined form, so that dA is exact and d actually acts as the exterior derivative. For example, given a connection A on a nontrivial $U(1)$-bundle, I will try to only write integrals over F_A, the curvature of A, and will try to avoid writing dA. Also, as a warning, I will generally be sloppy about keeping track of various minus signs coming from the supercommutativity of the exterior derivative and from interchanging the order of wedge products; in any case these are easy to figure out post facto. Another thing to keep in mind vis-a-vis factors of 2π: if $B \in H^p(X;\mathbb{Z})$ and $A \in H^q(X;\mathbb{Z})$, then $B \wedge A \in H^{p+q}(X;\mathbb{Z})$, i.e. the wedge product is a product operation on cohomology, meaning that the wedge product of two forms with integer periods is also a form with integer periods. We will thus alwaye divide by 2π-s in such a way that we are only wedging together forms in $H^*(X;\mathbb{Z})$, so as to avoid writing cohomology groups like $H^*(X;2\pi\mathbb{Z})$.[84]

First formulation: The first way of writing BF theory is

$$\frac{i}{2\pi}\int_X H \wedge (F_a + nA), \tag{802}$$

[83] Okay, β is more like the inverse of the square of the gauge coupling.

[84] This is because $2\pi\mathbb{Z}$ is not a ring with the usual multiplication (e.g. $4\pi^2 \notin 2\pi\mathbb{Z}$), and so the cup product / wedge product operation no longer works.

where A is a q-form gauge field and a is a $q-1$ form gauge field. Both are defined up to large gauge transformations[85]. For example, if $q=1$ then a is a scalar, and $a \sim a + 2\pi\mathbb{Z}$. If $q=2$, then a is a 1-form, with $a \sim a+\alpha$ for all α such that $\int_{M_1} \alpha \in 2\pi\mathbb{Z}$ for all closed 1-manifolds M_1. In the above action, H is a $(D-1)$-form Lagrange multiplier field.[86] If $\partial X \neq \emptyset$, we take $H|_{\partial X} = 0$. This type of action comes from a generalized $U(1) \to \mathbb{Z}_n$ Higgs transition. Indeed, start with the Lagrangian

$$\mathcal{L}_{Higgs} = \frac{\rho^2}{2}|F_a + nA|^2, \qquad \rho \to \infty \tag{804}$$

which describes the deep IR theory for a charge n $(q-1)$-form field which has been Higgsed. Because we are interested in the $\rho \to \infty$ limit, we can freely add a term $\frac{\rho^*}{2}|H|^2$ to $\mathcal{L}$, where $\rho^* = \frac{1}{2\pi\rho}$. Now make the shift $\delta H = (\rho^*)^{-1} i \star (F_a + nA)$ (notice that we couldn't make such a shift if we were imposing a quantization condition on H's periods). This gives

$$\mathcal{L} = \frac{\rho^*}{2}|H|^2 + \frac{i}{2\pi} H \wedge (F_a + nA), \tag{805}$$

which goes over to the action we wrote above for $\rho \to \infty$.

Anyway, using this action, we can first do the integral over a: this tells us that H must be closed, and it sets the periods of H to be in $2\pi\mathbb{Z}$. Since the field strength of H vanishes (and we are assuming the spacetime manifold is torsion-free), we can globally Hodge-decompose it as

$$H = d\alpha_H + \omega_H, \tag{806}$$

where ω_H is harmonic and where the absence of a co-exact part comes from the fact that H must be closed. Up to constants relating $\mathcal{D}H$ to $\mathcal{D}\alpha_H$ and prefactors depending on $\dim H^{D-q}(X;\mathbb{Z})$, the path integral is

$$Z = \int \mathcal{D}A\mathcal{D}\alpha_H \sum_{\omega_H \in 2\pi H^{D-q}(X;\mathbb{Z})} \exp\left(\frac{i}{2\pi}\int_X (d\alpha_H + \omega_H) \wedge nA\right). \tag{807}$$

If $\partial X \neq 0$, the cohomology group becomes the relative cohomology $H^{D-q}(X, \partial X; \mathbb{Z})$.

[85] Large gauge transformations on a q-form gauge field are transformations $A \mapsto A + \alpha$, where α integrates to an element of $2\pi\mathbb{Z}$ on every closed q-manifold (and as such is a closed form). Note that α is a globally well-defined q form, although it may not be exact. Large gauge transformations of course never change the field strength and do not change the global well-definedness of A, despite some statements in the literature along the lines of "large gauge transformations take you between different magnetic flux sectors" when discussing th.

[86] In [18], H is taken to be quantized as

$$\int_{M_{D-q}} H \in 2\pi\mathbb{Z}, \tag{803}$$

for any closed $D-q$ submanifold M_{D-q}. Thus in this case, we can think of H as the field strength for some $U(1)$ $(D-q-1)$-form gauge field. I think we actually don't want to put this restriction on H though—if we do then the a field serves no purpose in the action (its job is to turn H into a field strength), and if we make this restriction H cannot come from a field arising from manipulating a Higgs theory (more on this in a sec).

The integral over α_H eliminates the local degrees of freedom (it sets the curvature of A to zero), while because the periods of ω_H over any closed manifold are in $2\pi\mathbb{Z}$, the sum over cohomology classes acts as a δ function setting

$$\int_{M_q} A \in \frac{2\pi}{n}\mathbb{Z} \tag{808}$$

for all closed M_q[87]. We can also get to this conclusion by integrating over H first: this tells us that n copies of A need to be exact; this is another way of saying that A is a $\mathbb{Z}_n$ gauge field.

Thus the Lagrangian of the theory is zero after integrating out H and a, and A is set to be a flat $\mathbb{Z}_n$ connection. If the Lagrangian is just zero, why are we going through all this trouble of writing down 0 in a bunch of different ways? The point is that as physicists we like to work with continuum fields and like to draw intuition from actions, and so hence prefer to re-write things like $Z \sim \sum_{\alpha\in H^p(X;G)} 1$ in terms of path integrals (additionally, writing things in terms of actions lets us more easily generalize away from the topological limit, where the gauge fields are allowed to have nonzero field strength).

Symmetries: What are the gauge transformations and the symmetries of the action? We obviously have the gauge transformation $a \mapsto a + d\lambda_{q-2}$. We also need to have gauge transformations on A as well, but in order for these to leave the action invariant, we need F_a to shift as well. So, the fields are tied together in the way they transform, and gauge transformations act as

$$a \mapsto a + d\lambda_{q-2} - n\lambda_{q-1}, \qquad A \mapsto A + d\lambda_{q-1}. \tag{811}$$

This is a local symmetry of the action, and so it really is a gauge symmetry. But notice that it shifts a by something which is not an exact form! This has consequences for what the global symmetries are.

Now for the global symmetries. First, we see that we can shift A by

$$A \mapsto A + \frac{1}{n}\epsilon_q, \qquad \epsilon_q \in 2\pi H^q\left(X;\mathbb{Z}\right), \tag{812}$$

so that the integral of ϵ_q over any q-manifold is valued in $2\pi\mathbb{Z}$ (and hence ϵ_q is closed). Due to the n in the action, such a shift leaves the action invariant modulo $2\pi\mathbb{Z}$ provided we also shift the cohomology class of F_a appropriately, which is allowed since we are

[87] To see this more carefully, by Poincare duality we can write any $\omega_H \in 2\pi H^{D-q}(X;\mathbb{Z})$ as

$$\omega_H = \sum_{c\in H_q(X;\mathbb{Z})} 2\pi m_c \widehat{c}, \tag{809}$$

where $m_c \in \mathbb{Z}$ and the hat indicates the Poincare dual. When we put this in the path integral, we get something like (assuming the homology of X is torsion-free for simplicity)

$$\sum_{\{m_c\}\in\mathbb{Z}^{\dim H_q(X;\mathbb{Z})}} \prod_c \exp\left(i\int_c nA\right), \tag{810}$$

where the product is over all homology classes in $H_q(X;\mathbb{Z})$. This acts as a bunch of δ functions which set $n\int_c A \in 2\pi\mathbb{Z}$ for all closed q-manifolds c.

summing over all $[F_a]$ in the path integral.[88] This means that we have a global $\mathbb{Z}_n$ q-form symmetry.

It also looks like we have a global $(q-1)$-form symmetry, since if we shift $a \mapsto a+\epsilon_{q-1}$, where ϵ_{q-1} is any flat form (with periods equal to arbitrary elements of $\mathbb{R}/2\pi\mathbb{Z}$), then S is left invariant, since it only depends on F_a. What would be the charged objects under this symmetry? Of course, they would be the Wilson loops for a, namely $W_a(M_{q-1}) = \exp(i\int_{M_{q-1}} a)$. Normally, Wilson loop operators are gauge invariant, since they contain integrals over closed surfaces and since gauge transformations shift $U(1)$ gauge fields by exact forms. This is no longer true however, since we have a gauge transformation $a \mapsto a + n\lambda_{q-1}$, where λ_{q-1} is not exact and has no quantization conditions on its periods. Thus the $W_a(M_{q-1})$ Wilson loops are actually not gauge invariant, and the $U(1)$ higher symmetry actually does not exist.

A brief aside: one might be tempted to make $W_a(M_{q-1})$ gauge invariant by attaching a q-manifold M_q to it (with $\partial M_q = M_{q-1}$), and adding an integral of A over M_q to the Wilson operator. There are two problems with this: first, this is only possible if M_{q-1} is homologically trivial, in which case $W_a(M_{q-1})$ can not be charged under a higher symmetry in the first place. Second, the upgraded Wilson operator would be a trivial operator since it would be constructed using an integral of $F_a + nA$ over M_q, but $F_a + nA$ is trivial (which one sees by integrating out H).

So far we have only identified a single $\mathbb{Z}_n$ q-form symmetry, but it turns out that there is another hidden higher $\mathbb{Z}_n$ symmetry. It's easiest to see in the second formulation, so we'll come back to it after we've discussed the second formulation.

Second formulation: To get the second presentation of the action, we can "dualize" a from a $q-1$ form gauge field to a $D-(q-1)-2 = D-q-1$ form gauge field B.

Verbose way: This way is longer, but is what usually goes down during dualization. We'll go through it here just to make sure that the second formulation is actually obtained by dualization.

We do the dualization by adding in a new q form gauge field G and a $D-q-1$ form gauge field B, so that we get the hard-to-look-at expression

$$Z = \int \mathcal{D}A\mathcal{D}H\mathcal{D}G\mathcal{D}B\mathcal{D}a \exp\left(\frac{i}{2\pi}\int_X (H \wedge (F_a - G + nA) + F_B \wedge G)\right), \tag{813}$$

where in the path integral we are summing over all possible bundles for all the fields[89] except for G, which is a globally well-defined form (i.e. dG really is trivial in cohomology). Adding all these fields hasn't actually done anything, which we can see by integrating out B: the globally well-defined part of B gives a δ function setting $dG = 0$, while the sum over cohomology classes of $F_B \in 2\pi H^{D-q}(X;\mathbb{Z})$ sets $\int_{M_q} G \in 2\pi\mathbb{Z}$ for all M_q. This means that G is the exterior derivative of some $q-1$ form gauge field, and so we can send $F_a - G \mapsto F_a$ by a field redefinition on a, recovering the original action.

[88] Note that we wouldn't also have to shift F_a if we had originally taken the periods of H to be quantized in $2\pi\mathbb{Z}$.

[89] e.g. all $U(1)$-bundles for A if A is a 1-form, all $U(1)$ gerbes for A if A is a 2-form, etc.

Now we Hodge decompose G as

$$G = d\alpha_G + d^\dagger \beta_G + \omega_G. \tag{814}$$

We gauge-fix G by setting the exact component of the Hodge decomposition to be the exact part of F_a. Since we are summing over all cohomology classes for G in the path integral, the cohomologically nontrivial part of F_a can be absorbed by shifting ω_G. [90] This eliminates a from the theory entirely.

We then do the path integral over H, which sets $G = nA$. So finally we get the BF action in its usual form, namely

$$S = \frac{in}{2\pi}\int_X B \wedge F_A. \tag{815}$$

Again, in this presentation, both F_A and F_B have periods in $2\pi\mathbb{Z}$. Technically, to write it like this we have integrated by parts, trading the integral over $\frac{i}{2\pi}F_B \wedge A$ for one over $\frac{i}{2\pi}B \wedge F_A$. This can be done since although the two integrals are not equal, they differ by an element of $2\pi\mathbb{Z}$ (this is best thought about with DB cocycles — more on this later).

Fast way: Starting from the first formulation (802), just integrate out a directly! As we have explained, the integral over the globally well-defined part of a sets $dH = 0$, while the sum over cohomology classes for F_a enforces the quantization of the periods of H. Thus after integrating out a we can write H as the field strength of a $U(1)$ gauge field, $H = F_B$, which after integrating by parts gives (815).

Symmetries: Let's now check the symmetries in this formulation. The gauge symmetries are just shifts in B and A by exact forms, and we have no gauge symmetries that act on both fields as we had in the first formulation.

As for the global symmetries, we still have the $\mathbb{Z}_n$ q-form symmetry coming from shifting A, as we must. Note that naively looking at the action, we might conclude that we in fact have a $U(1)$ q-form symmetry, since the action only contains F_A which doesn't change under shifting A by a flat q-form, no matter what the holonomy of the flat q-form is. This isn't true though, and the actual symmetry is discrete: one of the ways to see this is to integrate by parts and write the integrand as $F_B \wedge A$, in which the $\mathbb{Z}_n$ character of the shift symmetry on A is manifest. But really, one should formulate the integral using DB cohomology (more on this to come). The advantage of this presentation is that the other global symmetry manifests itself as a $(D-q-1)$-form $\mathbb{Z}_n$ symmetry coming from shifting B as

$$B \mapsto B + \frac{1}{n}\epsilon_{D-q-1}, \qquad \epsilon_{D-q-1} \in 2\pi H^{D-q-1}(X; 2\mathbb{Z}). \tag{816}$$

This is a symmetry because of the quantization on F_A.

Third formulation: The final presentation of the action is a "magnetic" one, in which we treat F_B as an independent field (i.e. not necessarily flat), and enforce the

[90] Such a shift is fine since although the shift changes the $F_B \wedge G$ term, it changes it by something in $2\pi\mathbb{Z}$ due to the quantization condition on F_B.

fact that is actually the curvature of B with a Lagrange multiplier $\widetilde{B}$. So we write the partition function as

$$Z = \int \mathcal{D}B\mathcal{D}F\mathcal{D}\widetilde{A} \exp\left(\frac{i}{2\pi}\int_X F \wedge (F_{\widetilde{A}} + nB)\right). \tag{817}$$

Hopefully the notation here is clear: $F_{\widetilde{A}}$ is the field strength of $\widetilde{A}$, which is locally $d\widetilde{A}$, while F is its own independent field (not necessarily the field strength of any q-connection). As we have seen several times already, the integral over the globally-defined part of $\widetilde{A}$ sets $dF = 0$, while the summation over the cohomology classes for $F_{\widetilde{A}}$ enforce Dirac quantization on F. This means that integrating out $\widetilde{A}$ sets F to be the curvature of a q-form gauge field, and so indeed this presentation is equivalent to the usual BF action (note that this is similar in form to, but not exactly the same as, the first presentation).

Symmetries: Let's now check the symmetries in this formulation. The gauge symmetries, like in the first formulation, act on two fields simultaneously. First, we can shift $\widetilde{A}$ by an exact form. Second, we can shift B by $d\lambda_{D-q-2}$ while also shifting $\widetilde{A}$ by $-n\lambda_{D-q-2}$.

Similarly to our analysis of the first formulation, we see that B has a $(D-q-1)$-form $\mathbb{Z}_n$ global symmetry. One might think that we have a higher symmetry corresponding to constant shifts in $\widetilde{A}$, but since $\exp(i\int\widetilde{A})$ is not gauge-invariant, this is not so. We also have the $\mathbb{Z}_n$ q-form symmetry identified earlier, but it is "hidden" in this presentation.

Careful explanation of CS level quantization

Today we will explain why the Chern-Simons level is quantized in $U(1)$ gauge theory. Our argument will hold on any manifold (e.g. even $\mathbb{R}^3$), and in particular will work even when $\int_{M_2} F = 0$ for all 2-submanifolds M_2 of spacetime. Thus the usual story about manifolds like $S^2 \times S^1$ and large gauge transformations isn't the whole story. I learned about this approach to CS terms from Alvarez's nice paper [3]; here we will follow these ideas and spell everything out in detail.

The usual explanations which everybody always repeats for why the CS coefficient is quantized are either "place the theory on $S^2 \times S^1$ with a unit of flux through the S^2 and look at large gauge transformations" or "realize it by integrating over a bulk 4-manifold and require independence of the extension". These are both somewhat unsatisfying to me since I want to know why even on an open manifold (like e.g. all the ones we are interested in the real wold; at least time will be $\mathbb{R}$), the CS term is quantized. That is, what is the reason that CS theory on say $\mathbb{R}^3$ only makes sense when the level is quantized? In what follows we will answer this question for the case when the gauge group is $U(1)$ for simplicity; the extension to non-Abelian gauge groups actually looks to be nontrivial, at least as far as notation is concerned, and may be revisted again in a future diary entry.

We can answer this by looking more carefully at what $\int_X A \wedge dA$ really means. It is often stated that the integrand is only well-defined up to a total derivative, but in fact the ambiguity in the integrand is much more serious than that.

The correct way to think about things is by using DB cohomology (differential characters), which is essentially a way of defining gauge fields within the framework of the Cech-de-Rham bicomplex. Recall that a $U(1)$ gauge field actually consists of three pieces of data: for a given decomposition of X into patches U_α, this data includes A_α (1-forms on each patch), $\Lambda_{\alpha\beta}$ ($\mathbb{R}$-valued 0-forms on each double overlap), and $n_{\alpha\beta\gamma}$ ($\mathbb{Z}$-valued 0-forms on each triple overlap). They relate with one another by

$$\delta_0 A_{\alpha\beta} = A_\alpha - A_\beta = d_0 \Lambda_{\alpha\beta}, \qquad \delta_1 \Lambda_{\alpha\beta} = 2\pi d_{-1} n_{\alpha\beta\gamma}, \tag{818}$$

where d_{-1} is just a suggestive way of writing the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and the δ's are the Cech differentials. The transition functions in the bundle are $g_{\alpha\beta} = \exp(i\Lambda_{\alpha\beta})$, so that sending $\Lambda_{\alpha\beta} \mapsto \Lambda_{\alpha\beta} + 2\pi m_{\alpha\beta}$ for $m_{\alpha\beta}$ valued in $\mathbb{Z}$ does nothing (one can also check that this changes $n_{\alpha\beta\gamma}$ by a coboundary, and so doesn't affect the cohomology class of $n \in H^2_C(X;\mathbb{Z})$ (the C is for Cech cohomology). We can sum this up by writing A as the triple

$$A = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma}), \tag{819}$$

with

$$\delta_{-1} A = A_\alpha, \quad \delta_0 A = d\Lambda_{\alpha\beta}, \quad \delta_1 \Lambda_{\alpha\beta} = 2\pi d_{-1} n_{\alpha\beta\gamma}, \tag{820}$$

where $(\delta_{-1}A)_\alpha = A|_{U_\alpha} = A_\alpha$ is the restriction.

Morally speaking, this is kind of like doing a Hodge decomposition. The A_α part (a $(1,0)$ form, i.e. de Rham degree 1 and Cech degree 0) keeps track of the local curvature (the field strength), while the $\Lambda_{\alpha\beta}$ part (a $(0,1)$ form) keeps track of the holonomy of the gauge field around non-contractible loops. We can see this because the holonomy is captured by a flat 1-form, i.e. an element $\lambda \in H^1(X;\mathbb{R})$. Since λ is globally well-defined, we can write it simply as

$$\lambda = (\delta_{-1}\lambda, 0, 0). \tag{821}$$

Alternately, since we are only interested in its holonomy, we can just as well write it as

$$\lambda = (0, f_{\alpha\beta}, 0), \tag{822}$$

where the $f_{\alpha\beta}$ are real-valued functions. The holonomy of λ around a given loop can be computed by summing up the $f_{\alpha\beta}$'s along 2-fold intersections of patches along the loop, as we will see later. Note that there is no $n_{\alpha\beta\gamma}$ term here because $\delta_1(f_{\alpha\beta}) = \delta_1(\delta_0\lambda)_{\alpha\beta} = 0$.

The $n_{\alpha\beta\gamma}$ part (a $(-1,2)$ form) in the decomposition of A keeps track of the topology of the bundle (i.e. the Chern class). This is because, as we will see, the integral of F_A (the curvature of A) over a closed 2-manifold is given by a sum of the $n_{\alpha\beta\gamma}$'s.

Why is all this data needed in order to be able to do integrals? The philosophy is basically "we want the integrals we write down to be independent of the way in which we choose to decompose X into coordinate patches". With that in mind, consider integrating the gauge field along a 1-cycle that starts at a point $a \in U_\alpha \setminus (U_\alpha \cap U_\beta)$ and ends at $b \in U_\beta \setminus (U_\alpha \cap U_\beta)$, with $U_\alpha \cap U_\beta$ non-empty. To define the integral, we need to integrate part of the way with A_α, and then the rest of the way with A_β. Suppose the transition point between these two is at $p \in U_\alpha \cap U_\beta$. Then tentatively our integral is

$$I(a,b;p) = \int_a^p A_\alpha + \int_p^b A_\beta. \tag{823}$$

The problem is that I is not independent of p! Indeed, one can check that, for $q \in U_\alpha \cap U_\beta$, we have

$$I(a,b;p) - I(a,b;q) = \Lambda_{\alpha\beta}(p) - \Lambda_{\alpha\beta}(q). \tag{824}$$

The fix is to just add this transition function term into the integral. Thus, the following integral is independent of p:

$$I(a,b) = \int_a^p A_\alpha - \Lambda_{\alpha\beta}(p) + \int_p^b A_\beta. \tag{825}$$

However, recall that we need the shift $\Lambda_{\alpha\beta} \mapsto \Lambda_{\alpha\beta} + 2\pi m_{\alpha\beta}$ to not do anything. But, this shift changes the value of $I(a,b)$ by something in $2\pi\mathbb{Z}$! The only way to fix this is to ensure that the only time we write $I(a,b)$ is in exponentials as $\exp(iqI(a,b))$, where $q \in \mathbb{Z}$ (really we should be taking the integration to be over a closed cycle, but of course the same $2\pi\mathbb{Z}$ ambiguity still occurs). This is just another way of saying that the Wilson loop operators must be taken with integer charge. We know that if the Wilson loop wraps a nontrivial cycle then $q \in \mathbb{Z}$ is required by invariance under large gauge transformations, but here we are saying that *even for topologically trivial cycles*, the charge in the Wilson loop must be taken to be in $\mathbb{Z}$, a fact dictated only by the topology of the gauge group and not by the topology of any particular Wilson loop.

On a related note, this formulation lets us see why flat connections can be either specified as collections $(\lambda_\alpha, 0, 0)$, or entirely in terms of transition functions $(0, f_{\alpha\beta}, 0)$. Indeed, for the first formulation, we write the integral $\int_C \lambda$ for some cycle C as

$$\int_C \lambda = \sum_\alpha \int_{C_\alpha} \lambda_\alpha, \tag{826}$$

since the transition functions vanish. Hopefully the notation is clear: the C_α are the segments of C that lie in the patch U_α. On the other hand, since λ is closed and each U_α is contractible, we can write $\lambda_\alpha = d\omega_\alpha$ for some 0-forms ω_α, and so

$$\int_C \lambda = \sum_{p \in U_\alpha \cap U_\beta} (\omega_\beta(p) - \omega_\alpha(p))). \tag{827}$$

Thus if we define the transition functions $f_{\alpha\beta} = \omega_\alpha - \omega_\beta$, we see that if we only care about the holonomy of λ, we can just as well replace it will the collection $(0, f_{\alpha\beta}, 0)$ — the transition functions entirely determine the holonomy of closed forms.

The next step up in complexity comes from integrating F_A over a surface S. We need to do so in a way that doesn't depend on what sort of way we choose to cover the spacetime manifold with patches. Consider at first the case where S is closed. F_A is closed, and so on each patch it is exact (we can and will always take our patches, as well as their n-fold intersections, to be topologically trivial). So then we have

$$\int_S F_A = \sum_\alpha \int_{C_\alpha} dA_\alpha = \sum_\alpha \int_{\partial C_\alpha} A_\alpha. \tag{828}$$

Here, the $C_\alpha \subset U_\alpha$ are non-overlapping 2-chains contained in each of the patches, such that $\cup_\alpha C_\alpha = S$. Note that there are many ways of choosing the C_α, but different choices do not affect the integral, since they differ by integrals of the form $\int d(A_\alpha - A_\beta) = \int d^2\Lambda_{\alpha\beta} = 0$. Additionally, we can always choose the C_α so that at most C_α meet at any given point (we can always choose the boundaries of the C_α to be a triangulation of S).

Returning to the integral of F_A over S, and assuming that S is orientable, we see that

$$\int_S F_A = \sum_{\alpha\beta} \int_{\partial C_\alpha \cap \partial C_\beta} (A_\alpha - A_\beta) = \sum_{\alpha\beta} \int_{\partial C_\alpha \cap \partial C_\beta} d\Lambda_{\alpha\beta}. \tag{829}$$

Each of the integrals in the above sum is over a line segment, and so each integral contributes a term like $\Lambda_{\alpha\beta}(b) - \Lambda_{\alpha\beta}(a)$. When we sum over all such line segments, we get three $\Lambda_{\alpha\beta}$ terms at each vertex (where three C_α's meet), and they appear in the form $\delta_1 \Lambda_{\alpha\beta\gamma} = 2\pi n_{\alpha\beta\gamma}$. Thus we have

$$\int_S F_A = \sum_{\alpha\beta\gamma} 2\pi n_{\alpha\beta\gamma}. \tag{830}$$

This is why we said that the $n_{\alpha\beta\gamma}$ determine the topology of the bundle (if $\partial S \neq \emptyset$ then the only thing that changes is that we get an additional integral of A over the boundary of S). Note that in order to get a non-zero Chern class, the transition functions $\Lambda_{\alpha\beta}$ could not all be constant. Thus in order to create bundles which are twisted, it is not enough to just twist the transition functions by constants: we have to have "twisting" inside of double overlaps as well[91].

Let us finally now turn to Chern-Simons theory. Our naive guess of what the relevant integral is would be

$$\sum_\alpha \int_{C_\alpha} A_\alpha \wedge dA_\alpha, \tag{831}$$

where each C_α is now a 3-cycle, the C_α are all non-overlapping, and their union is the full spacetime X. This is not invariant under moving around the boundaries of the C_α

[91] Another way to say this is that if the transition functions are constants, we can choose a gauge in which the connection is flat: $A_\alpha = 0$ is a connection which obeys $A_\alpha = g_{\alpha\beta}^{-1}(A_\beta - d)g_{\alpha\beta}$. Flat connections can't have non-zero Chern class, and so we conclude that the transition functions need to not be constant if we are to get $\int F_A \neq 0$.

though, which is a problem. When we wiggle one of the C_α boundaries, the difference in the integral as written above is an integral like

$$\int_{\delta C_{\alpha\beta}} (A_\alpha \wedge dA_\alpha - A_\beta \wedge dA_\beta) = \int_{\partial\delta C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_\beta, \tag{832}$$

where $\delta C_{\alpha\beta}$ is the volume enclosed by the two different choices of the boundary between C_α and C_β (I may add pictures at some point to make this clearer). Thus, we can take care of this ambiguity by adding in this term to the definition of the Chern-Simons integral, like like how we added $\Lambda_{\alpha\beta}(p)$ into the definition of the Wilson line integral. So, our improved integral for the CS action now looks like

$$\sum_\alpha \int_{C_\alpha} A_\alpha \wedge dA_\alpha - \sum_{\alpha\beta} \int_{C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_\beta, \tag{833}$$

where the $C_{\alpha\beta}$ are the 2-cells where the C_α 3-cells meet. Notice that to find the correction term to the naive $A_\alpha \wedge dA_\alpha$ term, we computed $\delta(A_\alpha \wedge dA_\alpha) = d\Lambda_{\alpha\beta} \wedge dA_\beta$, which we found to be a total derivative (we also used that $dA_\alpha = dA_\beta$). Thus when we took the Cech differential, we got something that was exact in de Rham cohomology. This is in keeping with the general Cech-de-Rham bicomplex structure of this whole construction.

Sadly, even the improved integral is not invariant under re-arranging the patches. Now we have to consider what happens when we wiggle one of the 1-cells $C_{\alpha\beta\gamma}$, which is a common boundary of three of the C_α's. Drawing some pictures, one can convince oneself that for two choices of $C_{\alpha\beta\gamma}$ that differ by the surface $\delta C_{\alpha\beta\gamma}$, the term that we added to the naive CS integral changes by the term

$$-\int_{\delta C_{\alpha\beta\gamma}} (\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha}) \wedge dA_\gamma = -2\pi \int_{\partial\delta C_{\alpha\beta\gamma}} n_{\alpha\beta\gamma} \wedge A_\gamma, \tag{834}$$

where we used that dA is the same on all three of the patches. Again, we see that the change in the integral is computed by taking a Cech differential, and that taking the differential gives us something exact in de Rham cohomology. Thus to cancel out *this* variation, we modify the CS action to

$$\sum_\alpha \int_{C_\alpha} A_\alpha \wedge dA_\alpha - \sum_{\alpha\beta} \int_{C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_\beta + 2\pi \sum_{\alpha\beta\gamma} \int_{C_{\alpha\beta\gamma}} n_{\alpha\beta\gamma} \wedge A_\gamma. \tag{835}$$

Even this isn't good enough, since we haven't looked at what happens when we wiggle around the 0-cells where four 3-cells meet. At this point, the pattern about how to fix the ambiguity should be clear: we take the Cech differential of the last term we added to the integral, find that we get an exact form, and then add that term back to the integral, but with opposite sign. Doing so gives the final form of the Lagrangian, and so the correct CS action is

$$S = \frac{k}{4\pi} I_{CS};$$
$$I_{CS} = \sum_\alpha \int_{C_\alpha} A_\alpha \wedge dA_\alpha - \sum_{\alpha\beta} \int_{C_{\alpha\beta}} \Lambda_{\alpha\beta} \wedge dA_\beta + 2\pi \sum_{\alpha\beta\gamma} \int n_{\alpha\beta\gamma} \wedge A_\gamma - 2\pi \sum_{\alpha\beta\gamma\sigma} \int n_{\alpha\beta\gamma} \wedge \Lambda_{\gamma\sigma}. \tag{836}$$

The integrals are evaluated on 3, 2, 1, and 0 cells, in turn.

Now we can note several things about this expression that aren't clear if we were to think of it as $A \wedge dA$. First, notice that I_{CS} is ambiguous up to elements of $(2\pi)^2\mathbb{Z}$, even if the spacetime manifold is completely trivial and F_A is globally exact. This is because we have an equivalence $\Lambda_{\alpha\beta} \sim \Lambda_{\alpha\beta} + 2\pi m_{\alpha\beta}$ where $m_{\alpha\beta}$ takes values in $\mathbb{Z}$, which e.g. shifts the $2\pi \int n \wedge \Lambda$ term by something in $(2\pi)^2\mathbb{Z}$ (the $\int n \wedge \Lambda$ term is really a sum over discrete points in the manifold). If we want S to be invariant modulo $2\pi\mathbb{Z}$, this forces the quantizaton of k even in topologically trivial scenarios. In fact, it forces $k \in 2\mathbb{Z}$ to be an *even* integer. We know that even levels describe bosonic systems, and this construction is only able to directly handle this case. For the fermionic case where we have a genuine spin TQFT and k is odd, we need a little bit more data: the spin structure needs to be introduced explicitly into the procedure described above, with minus signs coming from the spin structure cancelling out the extra minus signs that come from the above presentation of the action not being completely invariant under re-arrangements of the patches when k is odd. This is essentially the framing anomaly: the theory looks topological, but it actually retains a hidden dependence on the spin structure.

Also, note that changing A by a flat connection (a flat connection can be captured purely by transition functions, i.e. it can be written in the form $(0, f_{\alpha\beta}, 0)$) does *not* leave I_{CS} invariant (even if the spacetime is closed), contrary to what we would expect from writing the action as $A \wedge dA$. Indeed, we see that shifting A by a flat connection only leaves the action invariant if that flat connection is in $\Omega^1_{2\pi\mathbb{Z}/k}(X)$, i.e. only if the holonomy of the action is quantized in units of $2\pi/k$ for all 1-cycles of the spacetime. This is why CS theory has a $\mathbb{Z}_k$ 1-form symmetry, and not a $U(1)$ 1-form symmetry. One again we stress that this argument works on topologically trivial spacetimes with globally exact field strengths.

The same argument shows that e.g. in the three-dimensional BF theory with action

$$\frac{in}{2\pi} \int B \wedge F_A, \tag{837}$$

the action is *not* invariant under shifting A by a flat form λ, because of the presence of the $\int n_{\alpha\beta\gamma} \wedge \Lambda_{\gamma\sigma}$ correction term. If λ has holonomy $\exp(i \int_C \lambda) = \exp(2\pi i k/n)$ for all 1-cycles C and for $k \in \mathbb{Z}$ then the action changes only by an element in $2\pi i\mathbb{Z}$, and so we have a global $\mathbb{Z}_n$ 1-form symmetry, but not a global $U(1)$ 1-form symmetry.

Quantization for Chern-Simons

The goal of today's problem is to try to understand the connection between CS theory and the WZW action through the quantization of the former (motivated by reading the paper by Jackiw and others back in 1989 on the quantization of Chern-Simons theory). We will also try to work out the details of some of the results in [9].

We will be working with CS theory / WZW actions for $SU(2)_k$ for concreteness:

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{838}$$

We will quantize the theory using holomorphic quantization, by first quantizing the fields and then imposing the constraint of gauge-invariance. We will then check that the generators of gauge transformations form a linear representation of the gauge group, and find the wavefunctional for the quantized fields. Our answer will involve the WZW action.

When dealing with a system with constraints (for us, Gauss' law), we can either solve the constraint and then quantize, or the other way around.[92] In what follows we will adopt the latter strategy: this means we will quantize the fields in the normal way, and solve the constraint afterwards by requiring that the generator of the constraint act trivially on the wavefunction. As usual with these types of quantization problems, it will be helpful to do everything in complex coordinates.

From the $A \wedge \partial_t A$ term in the CS action, we can write $A = t^a A^a$ and take the trace and directly read off the commutator[93]

$$[A_i^a(z), A_j^b(w)] = \epsilon_{ij} \frac{4\pi i}{k} \delta_{ab} \delta(z - w). \tag{839}$$

To get the right coefficient, we need to remember that $t^a = \sigma^a/2$, so that $\mathrm{Tr}(t^a t^b) = \delta^{ab}/2$, turning the part of the action that survives in the $A_0 = 0$ gauge into $(k/8\pi) \int \epsilon^{ij} (\partial_t A_i^a) A_j^a$.

Slightly more carefully, we can find the symplectic form by varying the action. We get, for a spacetime X,

$$\delta S = \frac{k}{2\pi} \int \mathrm{Tr}[\delta A \wedge (dA + A \wedge A)] - \frac{k}{4\pi} \int_{\partial X} \mathrm{Tr}[A \wedge \delta A] \tag{840}$$

where the sign of the last term comes from the signs in the product rule for d (we take the two differentials d and δ to commute). The boundary term gives the symplectic current, and so varying it again gives

$$\Omega = -\frac{k}{4\pi} \int_{\partial X} \mathrm{Tr}[\delta A \wedge \delta A]. \tag{841}$$

The fact that the symplectic form is a wedge product of δA with itself means e.g. that Wilson loops in A don't commute with themselves and is responsible for all the stuff

[92] However it is actually not always clear that these two procedures give the same results; I remember hearing from Greg Moore that in fact there are some examples where they indeed give different results—they may be rather pathological examples, though. Should come back to this.

[93] In the commutator, the δ function is normalized to have unit integral with the measure $dx \wedge dy$, not with $dz \wedge d\bar{z}$ (sorry)

we know and love about CS theory. Also note that $\delta A \wedge \delta A \neq 0$ even in the $U(1)$ case—this may look at first glance like the wedge product of two one-forms, but it is actually two $(1,1)$ forms (one degree in de Rham and one variational degree), and is non-vanishing (the degree relevant for the graded commutation rules is the total degree in the $d\delta$ bicomplex).

As a sanity check, this gives a commutator of $[A_i(z), A_j(w)] = \varepsilon_{ij}\frac{2\pi i}{k}\delta(z-w)$ in the $U(1)_k$ case (note that it's 2π and not 4π because of the absence of the 1/2 factor from the trace). This means that linked wilson lines with charges p, q can be unlinked at the cost of a phase factor $e^{2\pi i pq/k}$. This gives us the correct result that $q = k$ lines have trivial braiding, and that for $k \in 2\mathbb{Z}+1$ the $q = k$ line is a fermion.

Anyway, we see from the symplectic form that we can choose e.g. either A_x or A_y as the canonical momentum. However as usual when doing geometric quantization it will be better to work with a holomorphic polarization, choosing A_z as the coordinate and $\bar{A} \equiv A_{\bar{z}}$ as the momentum. So then we have

$$[A^a(z), \bar{A}^b(w)] = \frac{4\pi i}{k}\delta_{ab}\delta(z-w). \tag{842}$$

As in Maxwell theory, A_0 has no momentum and imposes a Gauss' law constraint. Varying the action with respect to A_0^a, the first $A \wedge dA$ part gives us two copies of $\partial_{[i}A^a_{j]}/2$ (the 1/2 from the trace), while the second A^3 part gives us three copies of $\epsilon^{ij}A_i^b A_j^c i f^{abc}/4$, where various factors of 2 from the matrix structure and trace have cancelled. Thus after a bit more algebra, we see that the constraint from A_0 means that for any physical state $|\Psi\rangle$,

$$(\partial_z \bar{A}^a - \partial_{\bar{z}} A^a + i f^{abc} A^b \bar{A}^c)|\Psi\rangle = 0. \tag{843}$$

Thus the field strength will be the generator of gauge transformations on a given Cauchy slice (for us, the $z, \bar{z}$ plane). This is in contrast with Maxwell theory, where the generator of gauge transformations is the electric field $\star F$, rather than the magnetic field F. The fact that the momentum is magetic rather than electric is responsible for flux attachment and at the technical level just comes from the fact that the action has a $\wedge F$ rather than a $\wedge \star F$.

To check that the field strength generates gauge transformations when it acts on the gauge field, define the operator

$$U(\lambda) = \exp\left(-i\frac{k}{4\pi}\int \lambda^a \wedge F^a\right). \tag{844}$$

Using the commutation relations we have

$$\bar{A}^a = -\frac{4\pi i}{k}\frac{\delta}{\delta A^a}, \tag{845}$$

and so we can write $U(\lambda^a)$ as (after integrating by parts)

$$U(\lambda) = \exp\left(\int \left[\partial_z \lambda^a \frac{\delta}{\delta A^a} - i\frac{k}{4\pi}\partial_{\bar{z}}\lambda^a A^a - i f^{abc}\lambda^a A^b \frac{\delta}{\delta A^c}\right]\right). \tag{846}$$

Then expanding the exponentials and a little bit of algebra shows that

$$U(\lambda)A^a[U(\lambda)]^\dagger = A^a + \partial_z \lambda^a + i f^{abc} A^b \lambda^c, \tag{847}$$

which is exactly what we want.

When deriving the action of $U(\lambda)$ on A^a, the part proportional to $\partial_{\bar{z}}\lambda^a$ canceled out. It is still essential to keep though, since it generates gauge transformations for the antiholomorphic part. Indeed, a little bit of algebra gives

$$U(\lambda)\frac{\delta}{\delta A^a}[U(\lambda)]^\dagger = \frac{\delta}{\delta A^a} + i\frac{k}{4\pi}\partial_{\bar{z}}\lambda^a + if^{abc}\lambda^c\frac{\delta}{\delta A^b}, \tag{848}$$

so that

$$U(\lambda)\bar{A}^a[U(\lambda)]^\dagger = \bar{A}^a + \partial_{\bar{z}}\lambda^a + if^{abc}\bar{A}^b\lambda^c, \tag{849}$$

as required.

Now let's find the wavefunctionals, which will be holomorphic functionals of A. In what follows, it will be convenient to be able to work with group elements $g = e^\lambda, h = e^\gamma$, as well as Lie algebra elements. We find the wave functionals by requiring that they are invariant under the action of $U(g)$. This means we have to know how the exponentiation of $\int \mathrm{Tr}[\lambda \wedge F]$ acts on arbitrary functionals of A. This is slightly tricky, since the exponentiation of F is difficult—it contains operators that don't commute among themselves.

Let us break up $U(g)$ into two parts: the part which implements the gauge transformation on the holomorphic part (the part with the $\delta/\delta A$'s), and the part containing $\mathrm{Tr}[\partial_{\bar{z}}\lambda A]$ needed for the gauge transformations of the antiholomorphic part. Hence we will write

$$-\frac{ik}{2\pi}\int \mathrm{Tr}[\lambda \wedge F] = \mathcal{G}(g) + \frac{ik}{2\pi}\int \mathrm{Tr}[g^\dagger \partial_{\bar{z}} g A], \tag{850}$$

where

$$\mathcal{G}(g) \equiv \int \left[\partial_z \lambda^a \frac{\delta}{\delta A^a} - if^{abc}\lambda^a A^b \frac{\delta}{\delta A^c}\right] \tag{851}$$

is the logarithm of the part of $U(g)$ which does the gauge transformation.

Let $|\Psi[A]\rangle$ be a candidate gauge-invariant wavefunctional, and write the action of $U(g)$ on it craftily as

$$U(g)|\Psi[A]\rangle = U(g)e^{-\mathcal{G}(g)}|\Psi[{}^gA]\rangle. \tag{852}$$

If $U(g)$ only performed gauge transformations, then we would have $U(g)e^{-\mathcal{G}(g)} = \mathbf{1}$. Because of the extra part in $U(g)$ though, this operator is nontrivial.

Now we need to find out what $U(g)e^{-\mathcal{G}(g)}$ is. Since manipulating stuff in the exponentials is difficult, let us bring down the stuff in the exponentials using the "fake one-parameter evolution" trick that comes up often when e.g. talking about anomalies. Namely, introduce a homotopy parameter $\phi \in [0,1]$ and consider $U(g^\phi)e^{-\mathcal{G}(g^\phi)}$. At $\phi = 0$ the g field is just the identity on all of spacetime, while it becomes equal to the value of interest at $\phi = 1$. Geometrically, what we are doing is extending the spatial manifold to be realized as the boundary of a three-manifold, where the added direction is the ϕ direction. Since $g^\phi = \mathbf{1}$ at $\phi = 0$ we can compactify space to a point at $\phi = 0$, and so this three-manifold looks like a cone. Anyway, we use $g^\phi = e^{\phi\lambda}$ to compute the derivative

$$\partial_\phi(U(g^\phi)e^{-\mathcal{G}(g^\phi)}) = U(g^\phi)\left(\frac{ik}{2\pi}\int \mathrm{Tr}[g^\dagger\partial_{\bar{z}}gA] + \mathcal{G}(g)\right)e^{-\mathcal{G}(g^\phi)} - U(g^\phi)e^{-\mathcal{G}(g^\phi)}\mathcal{G}(g), \tag{853}$$

since the exponents are just linear in ϕ. When we bring the integral through the $e^{-\mathcal{H}(g^\phi)}$ it gets gauge-transformed, and so since the $\mathcal{G}(g)$'s cancel,

$$\partial_\phi(U(g^\phi)e^{-\mathcal{G}(g^\phi)}) = U(g^\phi)e^{-\mathcal{G}(g^\phi)}\frac{ik}{2\pi}\int \mathrm{Tr}[g^\dagger\partial_{\bar{z}}g(g_\phi^\dagger A g_\phi + g_\phi^\dagger\partial_z g_\phi)], \tag{854}$$

where g_ϕ is the same as g^ϕ but is used since $(g^\phi)^\dagger$ looks uglier.

Now to simplify this mess. The first term on the RHS is a is a total derivative since

$$\frac{d}{d\phi}\mathrm{Tr}[(\partial_{\bar{z}}g_\phi)g_\phi^\dagger A] = \mathrm{Tr}[\partial_{\bar{z}}(g_\phi\lambda)g_\phi^\dagger A - (\partial_{\bar{z}}g_\phi)\lambda g_\phi^{-1}] = \mathrm{Tr}[(\partial_{\bar{z}}\lambda)g_\phi^\dagger A g_\phi], \tag{855}$$

as $\partial_\phi g_\phi = g_\phi\lambda$. We break the second term up as

$$\mathrm{Tr}[g^\dagger\partial_{\bar{z}}g(g_\phi^\dagger\partial_z g_\phi)]dz\wedge d\bar{z} = \frac{1}{2}\mathrm{Tr}[g_\phi^\dagger\partial_z g_\phi\partial_{\bar{z}}\lambda + g_\phi^\dagger\partial_{\bar{z}}g_\phi\partial_z\lambda]dz\wedge d\bar{z} - \frac{1}{2}\mathrm{Tr}[g_\phi^\dagger dg_\phi\wedge g^\dagger dg], \tag{856}$$

where the wedge product (taken only on the spatial slice; not involving the time coordinate) treats ∂_z as coming first and $\partial_{\bar{z}}$ as coming second. The second term on the RHS is

$$\frac{1}{2}\mathrm{Tr}[g_\phi^\dagger\partial_z g_\phi\partial_{\bar{z}}\lambda + g_\phi^\dagger\partial_{\bar{z}}g_\phi\partial_z\lambda] = \frac{d}{d\phi}\mathrm{Tr}[g_\phi^\dagger\partial_z g_\phi g_\phi^\dagger\partial_{\bar{z}}g_\phi] \tag{857}$$

since two of the terms after taking the derivative on the RHS cancel. Finally, in the last term with the wedge products, we can use $\lambda = g_\phi^\dagger\partial_\phi g_\phi$ to plug in for λ. Then we can antisymmetrize the three derivatives and divide by a factor of 3 to get

$$\frac{1}{2}\int \mathrm{Tr}[g_\phi^\dagger dg_\phi\wedge g^\dagger dg] = \frac{1}{2\cdot 3}\int d^2z\,\epsilon^{\alpha\beta\gamma}\mathrm{Tr}[g_\phi^\dagger\partial_\alpha g_\phi g_\phi^\dagger\partial_\beta g_\phi g_\phi^\dagger\partial_\gamma g_\phi], \tag{858}$$

where on the RHS α,β,γ run over $z,\bar{z}$, and ϕ. Look at how WZW-like this is! Since the integral on the RHS is only over space, the RHS is also a total ϕ derivative, and it equals

$$\frac{d}{d\phi}\frac{1}{6}\int_{B_{3,\phi}}\mathrm{Tr}[g_{\phi'}^\dagger dg_{\phi'}\wedge g_{\phi'}^\dagger dg_{\phi'}\wedge g_{\phi'}^\dagger dg_{\phi'}], \tag{859}$$

where $B_{3,\phi}$ is a bounding 3-ball extending from $\phi' = 0$ to $\phi' = \phi$.

Recapitulating, the whole integral on the RHS of (854) is a total derivative. This means that $U(g^\phi)e^{-\mathcal{G}(g^\phi)}$ is actually the exponential of the argument of the total derivative, which means that after setting $\phi = 1$, we have found $U(g)e^{-\mathcal{G}(g)}$. Keeping track of the various factors of $k/2\pi$, we get

$$U(g)e^{-\mathcal{G}(g)} = \frac{ik}{4\pi}\int \mathrm{Tr}[g^\dagger(\partial_{\bar{z}}g)A + g^\dagger\partial_z g g^\dagger\partial_{\bar{z}}g] + \frac{ik}{24\pi}\int_{B^3}\mathrm{Tr}[g^\dagger dg\wedge g^\dagger dg\wedge g^\dagger dg]. \tag{860}$$

Thus when acting on wavefunctionals, $U(g)$ both implements gauge transformations and multiplies the wavefunctionals by this exponential factor. We write it as

$$U(g)|\Psi[A]\rangle = e^{i\Omega[g,A]}|\Psi[{}^gA]\rangle \equiv \exp\left(\frac{ik}{4\pi}\int \mathrm{Tr}[A\bar{J}_g]\right)e^{iS[g]}|\Psi[{}^gA]\rangle, \tag{861}$$

where we have suggestively written the current as $\bar{J}_g = g^\dagger\partial_{\bar{z}}g$ and where $S[g]$ is the WZW action (both kinetic and topological terms).

Since $U(g)$ is a representation of the gauge group, we need it to satisfy

$$U(h)U(g) = U(gh). \tag{862}$$

Note the perverse ordering of the group elements on the RHS. Such a perversion is needed since gauge transformations conventially act adjointly as

$$A \mapsto g^{-1}Ag + g^{-1}\partial_z g, \tag{863}$$

so that for the product gh, it is g which acts first, and h which acts second.

Do the $U(g)$ form a linear representation? Consider the product $U(h)U(g)$ acting on $|\Psi[A]\rangle$. If this is equal to the action of $U(gh)$, then we need

$$\Omega[g, A] + \Omega[h, {}^g A] = \Omega[gh, A] \qquad \text{mod } 2\pi\mathbb{Z}. \tag{864}$$

That is, we need Ω to have a coboundary which vanishes mod $2\pi\mathbb{Z}$. After some algebra, this condition means that we need

$$\frac{ik}{4\pi}\int \mathrm{Tr}[g^\dagger \partial_z g h^\dagger \partial_{\bar{z}} h] + S[g] + S[h] = S[gh]. \tag{865}$$

In a previous diary entry on the WZW term, we showed that this is indeed true. The coboundary of the WZW action is precisely equal to the term needed to ensure that the $U(g)$ form a linear representation of the gauge group.

Now that we know how the $U(g)$'s act on candidate wavefunctionals, we need to actually find a particular solution for $|\Psi[A]\rangle$. But this is easy: we want it to be invariant under the action of the gauge group, and so we can project onto the trivial representation of the gauge group by integrating over all gauge transformations, that is, by acting with $\int \mathcal{D}g U(g)$ on any candidate wavefunctional. For example, we may just take

$$\Psi[A] = \int \mathcal{D}g \exp\left(iS[g] + \frac{ik}{4\pi}\int \mathrm{Tr}[A\bar{J}_g]\right). \tag{866}$$

We see that the wavefunctional for the CS gauge fields is obtained just by plugging them in as sources for the WZW theory.

Finally we briefly touch on some stuff we have brushed over. What happens if the spatial manifold X has a boundary? Recall that

$$\delta S = \frac{k}{2\pi}\int \mathrm{Tr}[\delta A \wedge F_A] - \frac{k}{4\pi}\int_{\partial X} \mathrm{Tr}[A \wedge \delta A], \tag{867}$$

We will fix our boundary conditions by fixing $A|_{\partial X}$ to be some specific (not for sure zero) function. Then the boundary term in δS is an integral over $\mathrm{Tr}[A\delta\bar{A}]$. In order to receive no boundary corrections to the equations of motion, we have to add the counterterm

$$S_\partial = \frac{1}{4\pi}\int_{\partial X} \mathrm{Tr}[A\bar{A}]. \tag{868}$$

This counterterm then also shows up in the action of $U(g)$ that we use to find the wavefunctional, in order to ensure that gauge invariance is maintained.

Secondly, what if we are on a spatial manifold with nontrivial 1-cycles? In this case, since the constraint from A_0 merely fixes the gauge field to be flat, we can have fields with nontrivial holonomy. I think the best way to deal with the problem in this case is to adopt the “solve the constraint and then quantize” approach, whereby we first decompose $A = U^{-1}\alpha U + U^{-1}dU$, and then quantize. Here U is single-valued and α keeps track of the holonomy.

Chern-Simons Propagator

Today is a quickie: a simple calculation I realized I’d never seen done before. We will find the propagator for Abelian CS + Maxwell theory[94]

$$S = \frac{1}{2e^2}\int F_A \wedge \star F_A - i\frac{k}{4\pi}\int A \wedge F_A \tag{869}$$

and show that the theory describes massive excitations. We will also explain how a long-ranged statistical interaction between particles is possible in a massive theory.

Let’s find the answer for the mass in an easy way first so we can check our work. First we find the equations of motion, which we write as

$$\partial_\mu F^{\mu\nu} - \frac{\kappa}{2}\epsilon^{\nu\mu\lambda}F_{\mu\lambda} = 0, \qquad \kappa \equiv \frac{e^2 k}{2\pi}. \tag{870}$$

This is equivalently written as[95]

$$(d^\dagger - \kappa\star)F = 0. \tag{871}$$

Now we act on both sides with the operator $\star d + \kappa$, so that

$$(\star d + \kappa)(\star d - \kappa) \star F = (d^\dagger d - \kappa^2) \star F = 0. \tag{872}$$

Now since $F = dA$, $d^\dagger \star F = 0$. Thus

$$(d^\dagger d + dd^\dagger - \kappa^2) \star F = 0 \implies (\partial_\mu \partial^\mu - \kappa^2)(\star F)^\nu = 0, \tag{873}$$

[94] This is in Euclidean time. If we start from $\mathbb{R}$ time with a $+ik$ for the CS term, then continuing to $i\mathbb{R}$ time the CS term appears as $-ik$.

[95] $d^\dagger = -\star d \star$ in Euclidean signature in three dimensions when acting on 1-forms while $d^\dagger = +\star d \star$ when acting on 2-forms. The hodge star satisfies $\star^2 = 1$ when acting on any degree form.

indicating that $\star F$ is a massive vector field with mass

$$m = \kappa = \frac{e^2 k}{2\pi}. \tag{874}$$

When $e^2 \to \infty$ we have $m \to \infty$, which means that at strong coupling (in the deep IR if we use the scaling from the CS term), we have an infinite mass and we can focus only on the Chern-Simons term.

Now let's find the (Euclidean-time) progagator and check this. We will choose to use Feynman gauge, by adding the gauge-fixing term

$$S_{gf} = \frac{1}{2e^2}\int d^\dagger A \wedge \star d^\dagger A \tag{875}$$

to the action. This turns the Maxwell term into $\int A \wedge \star(d^\dagger d + dd^\dagger)A$. Since the thing in the parenthesis is the Hodge Laplacian, we just get a k^2 term. So then in momentum space, we need to find

$$G_{\mu\nu} = \left(C\epsilon^{\mu\lambda\nu}k_\lambda + Dg^{\mu\nu}k^2\right)^{-1}, \qquad C \equiv \frac{k}{2\pi}, \qquad D \equiv \frac{1}{e^2}. \tag{876}$$

The strategy for inverting this guy is to try to break it up into projectors, as usual. We let

$$\Pi_L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}, \qquad \Pi_T = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}. \tag{877}$$

The extra piece $C\epsilon^{\mu\lambda\nu}k_\lambda$ is in the image of Π_T (since it's orthogonal to Π_L), but it is not itself a projector. We look for a solution of the form

$$G^{\mu\nu} = X\Pi_T^{\mu\nu} + Y\Pi_L^{\mu\nu} + Z\epsilon^{\mu\lambda\nu}k_\lambda. \tag{878}$$

We find the constants X, Y, Z just by multiplying this ansatz by $iC\epsilon^{\mu\lambda\nu}k_\lambda + Dg^{\mu\nu}k^2$ and setting the result equal to $g^{\mu\nu}$. This gives three equations (the coefficients of the three types of terms in G) in three unknowns, which we solve for in terms of C, D. The algebra is kind of boring and gives (sign of the last term needs to be checked again)

$$G^{\mu\nu} = \frac{e^2}{k^2+m^2}\left(g^{\mu\nu} + \frac{m^2}{k^2}\frac{k^\mu k^\nu}{k^2} + \frac{m}{k^2}\varepsilon^{\mu\nu\lambda}k_\lambda\right), \tag{879}$$

where we have taken

$$m = \frac{C}{D} = \frac{ke^2}{2\pi}, \tag{880}$$

which agrees with our previous result. As a sanity check, we note that we get the pure Maxwell answer (in the Feynman gauge) when $m = 0$.

A similar computation is possible in other gauges. I won't write out the algebra, but in Lorentz gauge ($d^\dagger A = 0$) one obtains

$$G^{\mu\nu} = \frac{e^2}{k^2+m^2}\left(\Pi_T^{\mu\nu} + \frac{m}{k^2}\varepsilon^{\mu\nu\lambda}k_\lambda\right), \tag{881}$$

which is transverse as required and is somewhat nicer due to the absence of the $1/k^4$ longitudinal term.

We can now resolve the question posed at the beginning: if the theory is massive, how can it lead to nontrivial statistical interactions between particles which are infinitely long ranged? We see that the answer to this is that although we have a pole (after continuing to $\mathbb{R}$ time) at $k^2 = m^2$ rendering the propagator massive, we also have a pole at $k^2 = 0$. This zero-momentum pole is what allows us to transmit the information needed to transmute the statistics of the particles.

Maxwell in two dimensions

Today is a little brain-warmer that I realized I'd never done. We will be working though and elaborating on the appendix D of [12]. That is, we will be considering pure $U(1)$ gauge theory in two dimensions, with theta angle θ. We will put it on $\mathbb{R} \times S^1$, where the S^1 has circumference L and the $\mathbb{R}$ is time. We will find the spectrum as a function of θ and discuss issues related to superselection sectors and the role of Wilson lines in this theory.

First let us fix a gauge. We will choose a gauge in which A_1 is constant. This means we need to find an α such that

$$\begin{aligned}\partial_x^2\alpha(x,t) =& \partial_x A_1(x,t) \implies \partial_x\alpha(x,t) = A_1(x,t) - A_1(0,t) + c \\ &\implies \alpha(x,t) = \int_0^x dx'\,(A_1(x',t) - A_1(0,t)) + cx + d.\end{aligned} \tag{882}$$

This ensures that $A' = A - d\alpha$ will satisfy $\partial_x A_1' = 0$ for all t. Note that α is time-dependent! We can get the constants by requiring that α be well-defined, at least modulo 2π. This doesn't allow us to fix d (as d parametrizes the global $U(1)$ symmetry, it cannot be fixed), but it allows us to fix c so that (setting $d = 0$)

$$\alpha(x,t) = \int_0^x dx'\,(A_1(x',t) - A_1(0,t)) + x\frac{LA_1(0,t) - \int_0^L dx'\,A_1(x',t)}{L}. \tag{883}$$

One sees that at both $x = 0$ and $x = L$ we have $\alpha = 0$. This means that in the following, we can work with the variable

$$\phi(t) \equiv \int_0^L dx\,A_1(x,t), \tag{884}$$

so that the gauge-fixed $A_1(t)$ is $\phi(t)/L$.

Now we integrate out A_0 to enforce

$$\partial_x \star F = 0, \tag{885}$$

so that the electric field is a constant. We are left with the action (we have done the dx integral since everything is independent of x)

$$S = \frac{1}{2g^2 L} \int dt\, (\partial_t \phi)^2 + \frac{\theta}{2\pi} \int dt\, \partial_t \phi. \tag{886}$$

The canonical momentum is

$$p = \frac{1}{g^2 L} \partial_t \phi + \frac{\theta}{2\pi}. \tag{887}$$

Dimensionality check: $[g] = 1$, so p is dimensionless as it should be. After some aglebra we then find the Hamiltonian

$$H = \frac{g^2 L}{2} \left(p - \frac{\theta}{2\pi} \right)^2. \tag{888}$$

Now we are in $U(1)$ gauge theory, not $\mathbb{R}$ gauge theory (this is slightly artifical since there are no charges), so that $\phi \sim \phi + 2\pi$ as a result of large gauge transformations being gauged (that they are gauged is what we mean by a $U(1)$ gauge theory). So since ϕ is periodic, the eigenfunctions p are just $e^{in\phi}$ for $n \in \mathbb{Z}$, and thus the spectrum is

$$E_n = \frac{g^2 L}{2} \left(n - \frac{\theta}{2\pi} \right)^2, \qquad n \in \mathbb{Z}. \tag{889}$$

This has the expected twofold ground state degeneracy at $\theta = \pi$, exhibits the periodicity $\theta \sim \theta + 2\pi$, and so on. Note that the energy levels are liniearly proportional to the circumference of the circle, so that all the energy in the different states comes from the energy density of the vacuum. That is, there are no particles, which of course we know must be the case for gauge theory in two dimensions. Also note that the spectrum is dependent on θ, even though the added term $\theta \int F$ is topological, and hence it is independent of the metric and doesn't contribute to $T_{\mu\nu}$. In particular, it doesn't contribute to T_{00}, the Hamiltonian. But through its modification of the canonical momentum, it still has an effect on the spectrum.

What do the different n levels represent? They essentially represent the different values that the quantized electric flux $\star F$ can take on. We see that the electric flux is determined via

$$\star F = g^2 (n - \theta/2\pi). \tag{890}$$

The θ term contributes to the electric flux in the usual way, with $\theta \mapsto \theta + 2\pi$ equivalent to changing the electric flux by one unit (this is one of the reasons why θ acts as a background electric field). Thus the different n levels are distinguished by the value of the electric flux which is threaded around the circle. We can go between different n by appling the Wilson line operator, since

$$[e^{i\int dx\, A_1(t)}, (\star F)(t)] = -g^2 e^{i\int dx\, A_1(t)}, \tag{891}$$

which means that the operator $e^{i\star F/g^2}$ generates the 1-form symmetry which shifts the holonomy. Since $\star F$ is constant, the charge operator $e^{i\star F/g^2}$ is independent of position, which is the statement of "current conservation $\implies$ topological charge operator" for a point charge operator. Since of course this 1-form symmetry cannot be broken in two dimensions, we know we will be able to label the states by their 1-form charges, which is just the obvious statement that we can label states by their electric fluxes. Anyway, if $W = e^{i\phi}$ is the Wilson loop, at equal times we have

$$W(\star F/g^2 + 1) = \star FW. \tag{892}$$

Thus we get the obvious statement that acting with the Wilson line increases the electric flux by g^2. This takes $n \to n+1$, and so the Wilson loop takes us between the different E_n. We can also phrase this in terms of the easily-checked similarity transform (here $H(p;\theta)$ is the Hamiltonian)

$$WH(p;\theta)W^\dagger = H(p;\theta + 2\pi), \tag{893}$$

which again demonstrates $\theta \sim \theta + 2\pi$. However, since there are no charges in the theory, the Wilson loop is kind of a pathological operator, since there is no way to apply it "gradually". In order to actually have it at our disposal, we would need to have charges (very massive ones would be fine) that we could pair-create and use to make W. Since this option is not available to us (pure gauge theory in two dimensions has no particles, as we have said), the different E_n levels are actually disconnected from one another, regardless of L. Also note that if we were to put our theory on T^2 instead of $\mathbb{R} \times S^1$, we would have to be more careful with the quantization procedure, since the operator $e^{i\int A_1}$ would not make sense: the electric flux needs to jump by 1 when crossing the Wilson line, but the Wilson line does not divide T^2 into disjoint pieces, so we get a contradiction (for a similar reason, $e^{i\int A_0}$ would not make sense). This is a trivial example of a higher-symmetry-enforced selection rule.

Alternate approach to Wilson line expectation values in Chern-Simons

Today we will go over a functionally-flavored method for computing expectation values of Wilson line correlators in Abelian CS theories, focusing on the correlators $\langle W^q(C)\rangle$ and $\langle W^q(C)W^p(C')\rangle$, where $W(C) = \exp(i\int_C A)$ and where C, C' are homologically trivial but may have nontrivial linking. The usual way to do this is to solve the classical equations of motion in the presence of a source of charge q so that e.g. $F = (2\pi q/k)\star j$, where j is the source worldline.

The strategy is the same routine of using shifts of integration variables and Poincare duality that we know and love so well by now. For the insertion of two Wilson loops, for $U(1)_k$ we have

$$\langle W^q(C)W^p(C')\rangle = \frac{1}{Z}\int \mathcal{D}A \, \exp\left(i\frac{k}{4\pi}\int A\wedge dA + i\int A\wedge(q\widehat{C}+p\widehat{C}')\right). \tag{894}$$

To get rid of the Wilson loop insertion, we perform the shift

$$A \mapsto A - \frac{2\pi}{k}d^{-1}(q\widehat{C}+p\widehat{C}'). \tag{895}$$

Note that $\widehat{C}$ and $\widehat{C}'$ are both 2 forms, but taking the d^{-1} turns them into 1 forms. One can check using some integrations by parts that this shift kills the term in the exponent that is linear in A. This produces

$$\begin{aligned}\langle W^q(C)W^p(C')\rangle = \frac{1}{Z}\int \mathcal{D}A \, \exp\Big(i\frac{k}{4\pi}\int A\wedge dA - i\frac{2\pi}{k}pq\int \widehat{C}\wedge\frac{1}{d}\widehat{C}' - i\frac{\pi}{k}q^2\int \widehat{C}\wedge\frac{1}{d}\widehat{C} \\ - i\frac{\pi}{k}p^2\widehat{C}'\wedge\frac{1}{d}\widehat{C}'\Big).\end{aligned} \tag{896}$$

Now writing $C=\partial D$ and $C'=\partial D'$ so that $\widehat{C}=dD$ and $\widehat{C}'=dD'$, we have

$$\langle W^q(C)W^p(C')\rangle = \exp\left(-i\frac{2\pi}{k}pq\int \widehat{D}\wedge d\widehat{D}' - i\frac{\pi}{k}q^2\int \widehat{D}\wedge d\widehat{D} - i\frac{\pi}{k}p^2 D'\wedge d\widehat{D}'\right). \tag{897}$$

The terms with the self-CS interaction of $\widehat{D}$ and $\widehat{D}'$ are the expectation values of single Wilson loops, which we see by setting e.g. $p=0$:

$$\langle W^q(C)\rangle = \exp\left(-i\frac{q^2\pi}{k}\int \widehat{D}\wedge d\widehat{D}\right). \tag{898}$$

This is ill-defined since the integral computes the intersection of ∂D with D, which doesn't make sense. We can regulate it by using a framing of the curve C, following the strategy in Witten's original Jones Polynomial paper. Given such a framing, we replace $d\widehat{D}$ in the above integral with $d\widetilde{D}'$, where $\partial D'=C'$ is a copy of C displaced infinitesimally along the vector field defined by the framing. The integral $\int \widehat{D}\wedge d\widehat{D}'$ then becomes the linking number of C and C', which depends only on the topological class of the framing (how many times the framing winds as it travels around C). Thus this framing-assisted regularization is just a choice of how to do point-splitting regularization for the Wilson operator. In $\mathbb{R}^3$ or S^3 we can always choose the framing so that the linking number of C and C' is zero, but for more general manifolds this may not be possible. In what follows we will actually choose a framing that winds by 2π along C if C is homologically trivial in the ambient spacetime (which we will assume to be the case). The reason for doing this will become clear in a second. Note that we are not loosing much by doing this, since we have a controlled way of determining how the answer changes upon changing the framing.

Anyway, doing the framing regularization so that the self-intersection number is equal to 1, we obtain

$$\langle W^q(C)\rangle = (-1)^{q^2/k}. \tag{899}$$

This means that with this convention, a lone Wilson loop computes the topological spin $s = q^2/(2k) \mod 1$ of the relevant anyon[96]. This comes from the fact that with our convention, the ribbon formed by C and its deformed copy has a 2π twist in it, so that unlinked loops compute the topological spin. In another convention where the framing is topologically trivial, unlinked loops would simply have expectation value 1. Now we have

$$\langle W^q(C)W^p(C')\rangle = \langle W^q(C)\rangle\langle W^p(C')\rangle \exp\left(-2\pi i\frac{pq}{k}\mathcal{L}(C,C')\right), \tag{900}$$

where $\mathcal{L}(C,C') = \int \widehat{D}\wedge d\widehat{D}'$ is the linking number of C and C'. In particular, note that a line with charge k is transparent with respect to all other lines. If k is odd this transparent line has spin $(-1)^k = -1$, and so odd k theories contain a transparent fermion—this is why they are spin TQFTs.

If we were to repeat this exercise with e.g. the Abelian CS theory with K matrix kX, then we would start with

$$\langle W^q(C)W^p(C')\rangle = \frac{1}{Z}\int \mathcal{D}A \,\exp\left(i\frac{k}{4\pi}\int(A\wedge dB + B\wedge dA) + i\int(qA\wedge C + pB\wedge C')\right), \tag{901}$$

where we have assumed that C is an A line and C' is a B line. If they were both A lines or both B lines, then we see we could perform a shift on just one of the fields so that $\langle W^q(C)W^p(C')\rangle = 1$. This is a check that the A and B fields are bosons (there is no self-interaction to change their statistics). Also note here that the lack of a self-interaction in the action means that the single loop expectation values $\langle W^p(C)\rangle$ don't need to be renormalized: they are equal to 1 identically. Anyway, we can perform the shifts

$$A \mapsto A - \frac{2\pi}{k}q\frac{1}{d}\widehat{C}', \qquad B \mapsto B - \frac{2\pi}{k}p\frac{1}{d}\widehat{C}', \tag{902}$$

which produces the familiar formula

$$\langle W^q(C)W^p(C')\rangle = \exp\left(-2\pi i\frac{pq}{k}\mathcal{L}(C,C')\right). \tag{903}$$

[96] The topological spin is defined only modulo 1 since a Maxwell term (which we always imagine to be included in the action; it's just less relevant than the CS term in the IR using the standard scaling) leads to massive spin-1 particles (photons) that don't have any braiding phase with the sources. Hence by computing Wilson line vevs we can't distinguish a given anyon from the same anyon with a massive photon attached to it, and so the spin of the anyons is only well-defined modulo 1.

The Witten effect

For some reason I found Witten's original paper on θ terms and monopole / dyon statistics rather hard to understand, and so the task for today is to explain the Witten effect in detail / with a slightly more modern presentation. We will be interested both in pure $U(1)$ gauge theory and in a situation where some larger non-Abelian gauge group is Higgsed down to $U(1)$.

We'll first do the easy part of looking at $U(1)$ gauge theory in four dimensions. We write the action as (in Minkowski signature)

$$S = \frac{1}{2e^2}\int F\wedge\star F - \frac{\theta}{8\pi^2}\int F\wedge F. \tag{904}$$

Here $\frac{1}{8\pi^2}\int F\wedge F = \frac{1}{2}\int(F/2\pi)\wedge(F/2\pi)$, which is in $\mathbb{Z}$ if the spacetime X is spin, so that we have the correct normalization of the θ term, with $\theta\sim\theta+2\pi$. This is checked by remembering that the "instanton number" for $U(1)$ gauge theory is the second Chern character, which is

$$\mathrm{ch}_2 = \frac{1}{2}(c_1\wedge c_1 - 2c_2), \tag{905}$$

where the c_i's are the Chern classes. Since $c_2 = 0$ for Abelian theories and $c_1 = F/(2\pi i)$, we see that the θ term is $\theta\int\mathrm{ch}_2$, which is the proper normalization.

Anyway, back to the problem. One way to motivate the Witten effect is to look at the equation of motion near a domain wall where θ jumps by some amount $\Delta\theta$. Not paying attention to getting the numbers right, this gives

$$d^\dagger F \propto d^\dagger\star(F\wedge\theta) = \star(dF\wedge\theta)+\star(F\wedge d\theta), \tag{906}$$

which means that the effective magnetic ($j_m = \star dF$) and electric currents ($j_e = d^\dagger F$) look like

$$j_e^\mu \sim j_m^\mu\theta + \Delta\theta\epsilon^{\mu\nu\lambda z}F_{\nu\lambda}\delta(z), \tag{907}$$

where we have taken the domain wall to lie in the xy plane. In particular, the $\mu=0$ component says that

$$\rho_e\sim\theta\rho_m+\Delta\theta B^z\delta(z). \tag{908}$$

From the first term on the LHS, we see that monopoles in a $\theta\neq 0$ medium get electric charge attached to them. Alternatively, we can consider a spherical shell of material at θ, surrounded by a vacuum at $\theta = 0$. If there is magnetic flux B^r leaving the surface of the shell (this could be due to a magnetic monopole inside the shell or could come from a nontrivial 1st Chern class), then although $\rho_m = 0$ at the interface of the outer part of the shell with the vacuum, the second term on the RHS means that after integration we have $Q_e = \Delta\theta Q_m$ (where we have used Gauss' law for the magnetic field). This again shows how sources of magnetic field (be they monopoles or Chern classes) pick up electric charge when θ is turned on.

Now for a more precise justification. Consider the 1-form symmetry

$$A \mapsto A + \lambda\epsilon, \qquad \epsilon \in H^1(X;\mathbb{Z}), \tag{909}$$

where λ is a constant. Here large gauge transformations mean we identify $\lambda \sim \lambda + 2\pi$. The charge operator which generates the symmetry in the Hamiltonian formalism is

$$Q^{(1)}_{[\epsilon]} = \int_\Sigma \epsilon \wedge \frac{\delta}{\delta A}, \tag{910}$$

where Σ is space. Since $e^{2\pi Q^{(1)}_{[\epsilon]}}$ acts as the identity, the charges of this symmetry are quantized in $\mathbb{Z}$. To get the canonical momentum of A, we just differentiate the action and get

$$\frac{\delta}{\delta A} = i\left(\frac{\star F}{e^2} - \frac{\theta}{2\pi}F\right). \tag{911}$$

Now the electric charge operator is

$$Q_e(M) = \frac{1}{e^2}\int_M \star F \tag{912}$$

while the magnetic one is

$$Q_m(M) = \frac{1}{2\pi}\int_M F, \tag{913}$$

where the $1/2\pi$ normalization comes from normalizing the eigenvalues of Q_m to be integers. So we get

$$\begin{aligned} Q^{(1)}_{[\hat\epsilon]} &= \int_\Sigma \epsilon \wedge \left(\frac{\star F}{e^2} - \frac{\theta}{2\pi}F\right) = \int_{\hat\epsilon\subset\Sigma}\left(\frac{\star F}{e^2} - \frac{\theta}{2\pi}F\right) \\ &= Q_e([\hat\epsilon]) - \frac{\theta}{2\pi}Q_m([\hat\epsilon]). \end{aligned} \tag{914}$$

In particular, we see that the electric charge of a dyon is

$$Q_e([\hat\epsilon]) = Q^{(1)}_{[\hat\epsilon]} + \frac{\theta}{2\pi}Q_m([\hat\epsilon]). \tag{915}$$

Now $Q_{[\hat\epsilon]}$ and $Q_m([\hat\epsilon])$ are both quantized in $\mathbb{Z}$, so for $\theta \notin 2\pi\mathbb{Z}$ the electric charge is not integral. We also see that $T : \theta \mapsto \theta + 2\pi$ acts on the charge lattice (q,m) by $T : (q,m) \mapsto (q+m, m)$. Of course, in order to actually have nonzero charges we need $H_2(X;\mathbb{Z})$ to be nontrivial so that we can have nontrival choices for $\hat\epsilon$ (although this homology group can always be made nontrivial but excising small balls from spacetime).

Now for the non-Abelian version. We will do the usual example where we have $SU(2)$ broken down to $U(1)$ by giving a scalar ϕ in the fundamental of $SO(3)$ a vev (if we gave a vev to a scalar in the fundamental of $SU(2)$, the gauge group would be broken down completely). The symmetry breaking allows us to get dyons with charge assignements as in the $U(1)$ case but on spacetimes that have $H_2(X;\mathbb{Z}) = 0$, e.g. $\mathbb{R}^4$.

We will find it helpful to work with ϕ in the adjoint of $SU(2)$, rather than the fundamental of $SO(3)$ (of course they are the same, only the notation is different). This means we will write ϕ as a matrix in $\mathfrak{su}(2)$ as $\phi = \sigma^a\phi^a$. This is slightly more convinent compared to writing ϕ as a three-vector and having it transform under three-dimensional matrices.

The action is (Minkowski signature)

$$S = \int \left(-\frac{1}{2g^2}\mathrm{Tr}[F \wedge \star F] - \frac{\theta}{8\pi^2}\int \mathrm{Tr}[F \wedge F] + \mathrm{Tr}[d_A\phi \wedge \star d_A\phi] + \lambda\left(\mathrm{Tr}[\phi^2] - \frac{v^2}{2}\right)^2\right), \tag{916}$$

where the covariant derivative is $d_A\phi = d\phi - i[A,\phi]$. Locally then the potential makes ϕ want to go like e.g. $v\sigma^3/2$, provided $\lambda \neq 0$. We will look for a monopole solution where at infinity ϕ goes to $\phi = \frac{r^a\sigma^a v}{2r}$ at infinity. If $\lambda = 0$ then we can have $\phi = \frac{r^a\sigma^a v}{2r} + O(1/r)$, but if $\lambda \neq 0$ then this leads to an infinite potential energy. We also need to choose asymptotic falloff conditions on the gauge field so that the kinetic term $|d_A\phi|^2$ is finite when integrated over space: i.e., we need $d_A\phi \sim O(1/r^2)$ as $r \to \infty$. We can ensure that this is the case provided that we choose A as follows:

$$A = 2\frac{i}{v^2}[\phi, d\phi] + \frac{2}{v}\mathcal{A}\phi, \tag{917}$$

where $\mathcal{A}$ is a $U(1)$ gauge field. This works since

$$\begin{aligned} d_A\phi &= d\phi + \frac{2}{v^2}[[\phi, d\phi],\phi] = d\phi - \frac{2}{v^2}\epsilon^{abc}\epsilon^{cde}\phi^a(d\phi)^b\phi^d\sigma^e \\ &= d\phi - \frac{2}{2v^2}(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})\phi^a(d\phi)^b\phi^d\sigma^e = d\phi - \frac{2}{v^2}(\mathrm{Tr}[\phi^2]d\phi - \phi d\mathrm{Tr}[\phi^2]) \\ &\sim \frac{1}{v^2r^2}(\phi/r - d\phi), \end{aligned} \tag{918}$$

where in the last step we have kept the leading terms as $r \to \infty$. Since this goes as $1/r^2$, the kinetic term $|d_A\phi|^2$ has a finite integral over $\mathbb{R}^3$. Note that we don't actually need to know the functional form of ϕ for this to work (we can only solve for ϕ analytically when $\lambda = 0$).

The important part here is that the abelian gauge field $\mathcal{A}$ that we tacked on doesn't contribute to $d_A\phi$. It represents the gauge freedom in the inbroken $U(1)$ subgroup at infinity (rotations about the radial axis). When we multiply A by ϕ and take the trace the $[\phi, d\phi]$ part gets killed, and so $\mathcal{A}$ is proportional to the projection of A onto the radial direction:

$$\mathcal{A} = \frac{1}{v}\mathrm{Tr}[\phi A](1 + \dots), \tag{919}$$

where if $\lambda = 0$ we can have $\dots \sim O(r^{-1})$; otherwise since we need $\mathrm{Tr}[\phi^2] \to v^2/2 + O(r^{-2})$ we have $\dots \sim O(r^{-2})$. The identification of the $U(1)$ gauge field makes sense since projecting onto the $\hat{r}$ direction in $SU(2)$ by tracing with ϕ selects out the generator of rotations about the radial direction, which leave $\langle\phi\rangle$ invariant (the structure group is reduced to $U(1)$ at infinity). Similarly, the $U(1)$ field strength is defined asymptotically as

$$\mathcal{F} = \frac{1}{v}\mathrm{Tr}[\phi F]. \tag{920}$$

Since $F \sim O(1/r^2)$ at infinity in order for the gauge field kinetic term to have finite energy, the parts of ϕ which go as negative powers of r can be ignored (thus we have not written any $(1+O(r^{-1}))$ factor on the RHS of the above equation), and the $U(1)$ field strength becomes exactly the projection of F onto the radial direction (locally we can write $\phi \to v\sigma^3$ and then $\mathcal{F} \to F^3$).

Now we need to examine the residual "zero mode" gauge transformations that act on the $U(1)$ gauge field $\mathcal{A}$. We need to look for "gauge transformations" which act nontrivially on the fields at infinity (and hence are not gauged). We want to leave the scalar field configuration invariant, and since it transforms in the adjoint, our gauge transformation parameter should be something built out of ϕ so that its action on ϕ is trivial. We also need the gauge transformation parameter to be purely radial, so that it only affects $\mathcal{A}$. It also needs to not mess with the falloff conditions we've imposed on e.g. $d_A\phi$ so that the energetics are unchanged. A transformation which fits the bill is given by $U_\alpha = \exp(i\alpha\phi/v)$, where $\alpha \in \mathbb{R}$. Actually, the fact that this is the generator is kind of obvious: it performs rotations about the $\hat{r}$ direction, which is exactly what the unbroken $U(1)$ does. This maps $A \mapsto A + \alpha d_A\phi/v$ (this has to be nontrivial since we can't have $d_A\phi = 0$ identically: if we did, we would have a global $U(1)$ symmetry [global symmetries are parametrized by covariantly constant things]. But we know this can't happen, since the structure group does not globally reduce to $U(1)$). Under the transformation, $\mathcal{A}$ changes by

$$\delta\mathcal{A} = \frac{1}{v}\mathrm{Tr}[\phi d_A\phi] = \frac{1}{2v}d\mathrm{Tr}[\phi^2]. \tag{921}$$

If $\lambda = 0$ this can have an $O(r^{-1})$ contribution, and we get the familiar $\mathcal{A} \sim r^{-1}$ falloff behaviour of a gauge field in the Coulomb phase. Now since when $\alpha \in 2\pi\mathbb{Z}$ we have $U_\alpha \to \mathbf{1}$, the operators $U_{2\pi k}, k \in \mathbb{Z}$ act as gauged gauge tranformations. Therefore the "physical gauge transformations" (I know, this terminology is awful) are parametrized by $\alpha \in [0, 2\pi)$ and thus give a $U(1)$ symmetry as expected.

Since the symmetry is $U(1)$, the charges associated to the asymptotic $U(1)$ symmetry will be integral. The charge operator for the symmetry is

$$\mathcal{U}(\alpha) = \exp\left(-i\frac{\alpha}{v}\int_{\mathbb{R}^3}\mathrm{Tr}[d_A\phi \wedge \delta_A]\right). \tag{922}$$

From the Lagrangian we read off

$$\delta_A = \frac{1}{g^2}\star F - \frac{\theta}{4\pi^2}F, \tag{923}$$

so that $\mathcal{U}(\alpha) = e^{-i\alpha Q_\mathcal{A}}$, where

$$\begin{aligned} Q_\mathcal{A} &= \frac{1}{v}\int_{\mathbb{R}^3}\mathrm{Tr}\left[d_A\phi \wedge \left(\frac{1}{g^2}\star F - \frac{\theta}{4\pi^2}F\right)\right] \\ &= \frac{1}{v}\int_{S^2_\infty}\left(\frac{1}{g^2}\mathrm{Tr}[\phi\wedge\star F] - \frac{\theta}{4\pi^2}\mathrm{Tr}[\phi\wedge F]\right) - \frac{1}{v}\int_{\mathbb{R}^3}\mathrm{Tr}\left[\phi\wedge\left(\frac{1}{g^2}d_A\star F - \frac{\theta}{4\pi^2}d_AF\right)\right], \end{aligned} \tag{924}$$

where we've used the fact that ϕ being in the adjoint means that e.g. $\mathrm{Tr}[\phi \wedge F]$ is $SU(2)$-neutral, so that we may write $\mathrm{Tr}[d_A\phi\wedge F] = d\mathrm{Tr}[\phi\wedge F] - \mathrm{Tr}[\phi\wedge d_AF]$. Also in the above, the hodge star is taken with respect to the full spacetime. The Bianchi identity means $d_A F = 0$, while we have

$$\mathrm{Tr}[\phi \wedge d_A \star F] \to \mathrm{Tr}[\phi (d_A)_i F^{0i}] \propto \mathrm{Tr}[\phi \sigma^a \epsilon_{abc} \phi^b (d_A \phi)^c_0] = 0, \tag{925}$$

where we have assumed our monopole solution is such that ϕ is covariantly constant in time (of course, we see from this formula that a moving monopole will produce an electric field, just like a moving electric charge produces a magnetic field). Thus only the surface integrals at infinity contribute, and we have

$$\begin{aligned} Q_{\mathcal{A}} &= \frac{1}{v} \int_{S^2_\infty} \left(\frac{1}{g^2} \mathrm{Tr}[\phi \wedge \star F] - \frac{\theta}{4\pi^2} \mathrm{Tr}[\phi \wedge F] \right) \\ &= Q_e - \frac{\theta}{2\pi} Q_m. \end{aligned} \tag{926}$$

Now $Q_{\mathcal{A}}$ is quantized in $\mathbb{Z}$ and so is Q_m (it is the first Chern class of the $U(1)$ bundle at infinity, and in fact is valued in $2\mathbb{Z}$, because of the factor of 2 from the trace or alternatively because the thing getting the vev was in the adjoint of $SU(2)$ i.e. the fundamental of $SO(3)$, instead of the fundamental of $SU(2)$. This is in fact the minimal possible magnetic charge since in these conventions, if we introduced a field charged in the fundamental of $SU(2)$ to do a Dirac string experiment, the field would have electric charge $1/2$). This implies, as in the Abelian case, that the electric charge of a monopole is non-integral, and dependent on the value of θ.

More on the Schwinger model, its phases, and the theta term

Today's problem came from wanting to understand a statement made by Zohar Komargodski in his lecture on 1-form symmetries at a Stony Brook workshop; a good reference looks to be [?].

Consider the massive Schwinger model on the spacetime $S^1 \times \mathbb{R}$, where the fermions carry charge $q \in \mathbb{Z}$ and have a real mass $m \in \mathbb{R}$ (just for simplicity—axial rotations allow us to interchange the phase of the fermion mass with the θ angle, so this is done without loss of generality):

$$S = \int_{S^1 \times \mathbb{R}} \left(\frac{1}{2\pi} \bar{\psi} (\partial\!\!\!/ - iqA\!\!\!/) \psi + \frac{m}{2\pi} \bar{\psi}\psi + \frac{1}{2e^2} F \wedge \star F - i \frac{\theta}{2\pi} F \right). \tag{927}$$

Because the fermion has charge q, this theory has a $\mathbb{Z}_q$ 1-form symmetry.

We will explain what happens in the limits $m \ll e$ and $m \gg e$ and discuss the differences between $q = 1$ and e.g. $q = 2$. As usual, most of the action happens at $\theta = \pi$.

$m \gg e$. First we do the case of large mass. It is then reasonable to throw the fermions away, and look at the pure gauge theory. We already analyzed QED_2 with a θ term in a previous diary entry, where we showed that the spectrum was labeled by the different quantized values of the electric flux:

$$E_n = \frac{e^2}{2}\left(n - \frac{\theta}{2\pi}\right)^2 = \frac{e^2}{2}F_{01}^2, \tag{928}$$

where we have set the circumeference of the circle to 1 for simplicity. When $\theta = \pi$ we have a degeneracy corresponding to the choice of $F_{01} = \pm 1/2$. This degeneracy can get lifted by a mxing between the two ground states of order e^{-mL} if $q = 1$, but is exact if $q > 1$.

We can also look at this from the boson side. The action bosonizes to (using the quantum fields and strings part II conventions and working in Euclidean time)

$$S = \int \left(\frac{1}{8\pi}d\phi \wedge \star d\phi - \frac{m}{\pi}\cos\phi + \frac{1}{2e^2}F \wedge \star F - i\frac{\theta + q\phi}{2\pi}F\right). \tag{929}$$

Note that changing $m \mapsto -m$ is the same as changing $\phi \mapsto \phi + \pi$, which from the fact that ϕ appears in the θ term reminds us of why TIs occur inside regions of spacetime where the fermion mass has a sign opposite to the sign it has in vacuum. Anyway, sending $m \to \infty$ freezes out the boson and we get pure QED at $\theta = \pi$, which as we have said has two orthogonal degenerate ground states (and as we said the orthogonality is exact even in finite systems when $q > 1$).

$m \ll e$. Now we look at small mass. In fact, we start with $m = 0$. Here the θ dependence can be completely removed by a shift in ϕ: this is the chiral anomaly, since a shift in ϕ is generated by the vector current for ϕ, which is dual to the axial current for the fermions. The action for $m = 0$ is quadratic and thus easy to solve. We can integrate out the gauge field and use the results of previous diary entries to write the effective action as

$$S = \int \left(\frac{1}{8\pi}d\phi \wedge \star d\phi + \frac{e^2}{2}\min_{k\in\mathbb{Z}}(k - q\phi/2\pi)^2\right), \tag{930}$$

which holds as long as ϕ is slowly varying (if ϕ were non-compact, we would just have a quadratic mass term). Thus we see that we end up with a massive scalar (the soliton), and so the massless Schwinger model ("massless" in the sense that the action doesn't contain an explicit mass term for the fermions) is in fact actually massive. Since ϕ is valued in $[0, 2\pi)$, there is only one minumum of the potential if $q = 1$, but if $q > 1$ then we have q distinct minima.

Now we turn on a small mass. After shifting ϕ to kill the θF coupling, the potential for ϕ is

$$V(\phi) = -\frac{m}{\pi}\cos(\phi - \theta/q) + \frac{e^2}{2}\min_{k\in\mathbb{Z}}(k - q\phi/2\pi)^2. \tag{931}$$

Now the effect of θ is more important. Let's take $\theta = \pi$. If $q = 1$ then θ has the effect of shifting the minimum of the oscillating part of the potential to π. Superimposing the cosine on top of the quadratic potential has the effect of creating two distinct minima if the mass is large enough. We thus get the attractive picture of an Ising-type phase transition where two minima merge into one as m is varied, although where exactly this happens is hard to say, since it relies on us trusting this form of the effective potential for ϕ beyond the regime of parameters for which it was derived. Since the Ising transition is described by free fermions, we see that at some value of $m \sim e$ the confinement disspaears. The picture here is a line of alternating ± 1 charges, which since they change the flux by $\Delta F_{01} = 1$ are the domain walls for the Ising order parameter. When the domain walls start to proliferate confinement goes away at the massless Ising point since the electric fields of the domain walls cancel the background electric field coming from the θ term. Thus the physics is that of a tradeoff between the energy needed to create the screening charges (coming from the explicit $m\bar{\psi}\psi$ mass term) and the electrostatic energy saved when the screening fields are set up. One is led to think about this confinement transition as some sort of a toy model for the confinement transition occurring in $D = 4$ YM as the fermion masses are tuned.

If $q = 2$ then the quadratic part of $V(\phi)$ has two minima. Upon adding the mass term, there are still two minima. They are not equally spaced in ϕ because the cosine part of the potential is not symmetric about $\phi = 0$ while the quadratic part is. A similar picture holds for $q > 2$: the q distinct minima have their positions shifted, but for small m there remain q different minima.

The different minima are distinguished by the value of the electric flux around the ring, which comes spaced in integer units. Shifting the flux by q units is the same as shifting ϕ by 2π, and since we identify ϕ with $\phi + 2\pi$, different vacua that differ in their electric fluxes by q are connected. This is of course due to the fact that we have charge q particles which can propagate around the circle and change the flux by q. However, there is no process which can change the electric flux by $p < q$ units. Thus, even though we are on a circle, different vacua related by $\Delta F < q$ do not mix—they do not even mix in a way which is exponentially small in the particle mass / size of the circle. They only mix through a Wilson operator $e^{ip\oint A}$, which is nonlocal. Thus we genuinely have q distinct superselection sectors, even when space is compact.

So, at $\theta = \pi, q = 1$ we have the following picture: for $m \gg e$ we have two degenerate ground states distinguished by the electric flux, with an exponentially small mixing between them. Then at $m \ll e$ we only have one ground state, and so at some finite $m \sim e$ we have an Ising transition. If we e.g. change to $q = 2$, then the $m \gg e$ story is the same (although the two ground states do not mix), but for $m \ll e$ we still have two ground ground states (again with no mixing). Thus it is natural to guess that for $q = 2$ the Ising transition is eliminated. If $q > 2$, then it seems like as m is increased, there will be phase transitions where pairs of distinct minima merge into each other, eventually pairing down to leave behind the two minima of the $m \gg e$ limit (note to self: come back and think about this).

Allowed spectrum of charges in Abelian and non-Abelian gauge theory and generalizations of the Dirac quantization condition

Today the goal is to figure out what types of electric and magnetic charges are allowed to be possessed by line operators in non-Abelian gauge theory.

For two dyons of electric and magnetic charges (q,m), (q',m'), we will show that for $U(1)$ gauge theory the quantization condition is

$$qg' - q'g \in \mathbb{Z}, \tag{932}$$

and will determine the non-Abelian analogue of this formula.

We will then describe the allowed line operators in a non-Abelian gauge theory based on a general compact Lie group G (here by "line operators", we mean operators that are literally supported on a line, and do not come with any surface operator [topological or otherwise] attached).

Let's first do $U(1)$ gauge theory, which is the easiest to understand. The charge quantization condition we want to derive is

$$qg' - q'g \in \mathbb{Z}. \tag{933}$$

The minus sign is the tricky part, which is somewhat unintuitive if one just looks at a given braiding process and thinks about AB phases. Basically, the minus sign is the minus sign in $\star\star = -1$, which holds on 2-forms in four-dimensional spacetime in Lorentzian signature ($\star^2 = (-1)^{p(D-p)+1}$ on a Lorentzian-signature D manifold when acting on p-forms, while for Euclidean signature the exponent is shifted by one).

From the Lorentz force law and EM duality $E \mapsto B, B \mapsto -E$, the force on a dyon (q,g) moving at velocty v in the field of another (motionless) dyon (q',g') is (ignoring factors of $1/4\pi$)

$$m\frac{dv^i}{dt} = (qq' + gg')\frac{r^i}{r^3} + (qg' - q'g)\epsilon^{ijk}v_j\frac{r_k}{r^3}, \tag{934}$$

where the all-important minus sign comes from doing $B \mapsto -E$ in the Lorentz force term. We want to get an angular momentum out of this so that we can find something which is quantized, and so we cross both sides with r: $r \times d_t(v) = d_t(r \times v)$ since $v \times v = 0$, so

$$m\frac{d(r\times v)}{dt} = (qg' - q'g)\frac{r \times (r \times v)}{r^3}. \tag{935}$$

Using the identity for two epsilon symbols with one index contracted between them, the cross product goes to $[r \times (r \times v)]^i = r^2 v^i - (r\cdot v) r^i$ (maybe the sign is wrong). When we divide by r^3 we get $v^i/r - (r\cdot v)r^i/r$, which is exactly the time derivative of r^i/r^2. So then we have total time derivatives on both sides, and we conclude that

$$(r\times p)^i = (qg' - q'g)\hat{r}^i. \tag{936}$$

Thus from quantization of angular momentum, we see that $qg' - q'g$, with the minus sign, is the correct thing to put a quantization condition on.

Now we go to looking at the spectrum of line operators in a general non-Abelian gauge theory. We want to examine the quantization condition on the magnetic charge carried by a given dyonic line operator $\mathcal{O}$. Consider a (small) S^2 linking $\mathcal{O}$. We can take the magnetic field on this S^2 to be uniform, with the gauge field on the S^2 being

$$A_\pm = \frac{B}{2}(\pm 1 - \cos\theta)d\phi, \tag{937}$$

where B is a covaraintly constant matrix (it should be covariantly constant since F is covariantly constant; it can't be an actual constant since F isn't gauge-invariant) determined by the magnetic field[97]. A proof of why the field is convariantly constant is in the following footnote[98]. Here the coordinates A_+ are used for the northern hemisphere, and A_- is used for the southern hemisphere. The two expressions differ at the equator by $d(B\phi)$, which needs to be $-ig^{-1}dg$ for some well-defined g since the gauge fields on different patches are glued together with exterior derivatives of transition functions. We see that $g = e^{Bi\phi}$ on the equator, and so in order for g to be

[97] More precisely, it is a covaraintly constant section of the adjoint bundle on the S^2. Recall that sections of the adjoint bundle $\mathrm{Ad}P$ are gauge-invariant things (like field strengths). The adoint bundle is given by taking the product of a principal G bundle P over the relevant spacetime (sub)manifold with $\mathfrak{g}$, and then quotienting by the adjoint action Ad, so $\mathrm{Ad}P = (P \times \mathfrak{g})/\sim_{\mathrm{Ad}}$. The identification here is $(g\cdot\phi, F) \sim (\phi, \mathrm{Ad}_{g^{-1}}F)$, or $(g\cdot\phi, \mathrm{Ad}_g F) \sim (\phi, F)$, which is telling us to mod out by gauge transformations.

[98] An S^2 linking the line operator in question sees a magnetic field

$$F_{ij} = \epsilon_{ijk}\frac{x^k}{4\pi|x|^3}B(x) = \mathrm{vol}_{S^2}\wedge B(x), \tag{938}$$

since in this configuration there is a monopole at the center of the S^2. Note that B has x dependence since setting B to be a constant would not be a gauge-invariant thing to do in the non-Abelian case. However, B is covariantly constant.

To show this, consider the equations of motion on the S^2, namely $D_iF^{ij} = 0$. This reads

$$0 = (d^\dagger \mathrm{vol}_{S^2})^j B(x) + (\mathrm{vol}_{S^2})^{ij} D_i B(x). \tag{939}$$

Since the volume form on S^2 is co-closed (it is harmonic), we get that

$$\epsilon_{ijk}x^j D_k B(x) = 0. \tag{940}$$

Now consider the Bianchi identity, $D_{(i}F_{jk)} = 0$. This reads

$$0 = d\mathrm{vol}_{S^2}\wedge B(x) + \mathrm{vol}_{S^2}\wedge DB(x). \tag{941}$$

Again the first term vanishes, and this implies that

$$x^k D_k B(x) = 0, \tag{942}$$

since the antisymmetrizations from the wedge product and the definition of the volume form cancel out.

So together with the previous equation derived from the eom, we see that the vector $D_kB(x)$ is orthongonal to both the radial direction and to the directions tangent to the S^2. Thus it vanishes identically, and so $D_kB(x) = 0$ as claimed.

well-defined we need

$$e^{2\pi iB} = \mathbf{1} \tag{943}$$

to hold as a matrix equation.

Another essentially equivalent way to get the quantization is as follows. Gauge transformations act on B adjointly by conjugation. We now need a math fact: for any $B \in \mathfrak{g}$, we can always find a $B' = S^\dagger BS$ such that $B' \in \mathfrak{t}$, where $\mathfrak{t}$ is a maximal torus in $\mathfrak{g}$. Basically, we can always make a rotation in the gauge group to diagonalize the uniform field B (all elements in $\mathfrak{g}$ lie in *some* maximal torus). Since the components of the gauge field now commute with eachother, we can use stokes' theorem and write the Wilson loop as an integral over either the southern or northern hemispheres of the S^2. Demanding that these two integrals given consistent Wilson loops gives us the same quantization condition as before.

Now since $B \in \mathfrak{t}$, we can write $B = \beta^i H_i$ for some coefficient vector $\beta \in \mathbb{R}^r$, with $r = \dim \mathfrak{t}$ and where H_i are the generators of $\mathfrak{t}$. When we take the Wilson line to be in some representation, since the H_i can be simultaneously diagonalized, we can replace them with their eigenvalues μ_i, which are the weights of the given representation. Thus the quantization condition $e^{2\pi iB} = \mathbf{1}$ reads

$$\beta_i \mu^i \in \mathbb{Z}, \qquad \mu \in \Lambda_w(G). \tag{944}$$

Looking at the diary entry on weights and root lattices, we see that we can satisfy this condition by taking

$$\beta^i = 2\frac{\alpha_i}{\alpha^2} = \alpha_i^\vee, \qquad \alpha \in \Lambda_r(G),\, \alpha^\vee \in \Lambda_r^\vee(G) \tag{945}$$

so that the quantization condition is satisfied by virtue of quantization of angular momentum in $\mathfrak{su}(2)_\alpha$. This is why we often think of the allowed t'Hooft lines as coming from representations that are created with $\otimes$s of the adjoint representation, in contrast to Wilson lines which can be in any representation: if the β^i can always be written in terms of a root in $\Lambda_r(G)$, then since $\Lambda_r(G)$ is the root lattice for the adjoint representation, all allowed magnetic charges must come from $\otimes$s of adjoint representations (the $\otimes$ operation corresponding to the fusion of t'Hooft lines).

This is not strictly true though, since while $\beta \in \Lambda_r^\vee(G)$ is sufficient for satisfying the quantization condition, it is not always neccessary. The allowed values for β actually come from a sublattice of the co-root lattice. The most general choice of β would be to take $\beta \in (\Lambda_w(G))^*$, where the dual indicates functions into $\mathbb{Z}$. Looking back at the previous problem, we see that this is precisely what it means for β to be a weight of the dual group $G^\vee$. Thus $\Lambda_w(G^\vee)$ parametrizes the allowed values of magnetic charge.

The diagonalization we made to rotate the magnetic field so that $B \in \mathfrak{t}$ was made with a gauge transformation that was constant on the S^2 which we were using to study the quantization condition. This doesn't completely fix B though, since there are still rotations we can do within the Cartan subalgebra which represent residual gauge redundancies. These redundancies are precisely given by the action of the Weyl group for the dual Lie algebra $\mathfrak{g}^\vee$ (remember that the Weyl group is given by reflections about the roots, so that it only depends on the Lie algebra, and not on the choice of Lie group). A math fact is that if $WBW^\dagger \in \mathfrak{t}$ for $B \in \mathfrak{t}$, then W impliments a Weyl transformation. So, the Weyl group contains all the residual gauge transformations not fixed by our choice of magnetic field $B \in \mathfrak{t}$. Now the roots for the dual group are the co-roots of the original group, which means that the Weyl group acts in the same way on both the lattice of G and the lattices of $G^\vee$, since

$$\text{Weyl} : \mu \mapsto \mu - \alpha(\alpha_i^\vee \mu^i) = \mu - \alpha^\vee(\alpha_i \mu^i), \tag{946}$$

so that Weyl and Weyl$^\vee$ act in the same way. Thus our tentative classification scheme for magnetic charges is to label them by elements of the quotient $\Lambda_w(G^\vee)/\text{Weyl}$.

A slighty better way to classify the charges is to realize that no matter what the exact Lie group and dual Lie group are (given a particular $\mathfrak{g}$), magnetic lines in the dual root lattice $\Lambda_r(\mathfrak{g}^\vee) = \Lambda_r^\vee(\mathfrak{g})$ will always be allowed. Indeed, for $\mu \in \Lambda_w(G)$ and $H \in \Lambda_r^\vee(\mathfrak{g})$ we have $\mu(H) \in \mathbb{Z}$ regardless of the exact choice of $G = \widetilde{G}/\Gamma_G$, so that magnetic charges in $\Lambda_r(\mathfrak{g}^\vee)$ are always allowed. Thus to obtain a classification which distinguishes between the line operators that are allowed for different choices of Γ_G, we can quotient the lattice of all possible magnetic charges (viz. $\Lambda_w(G^\vee)$) by the lattice of those that will be there no matter what (viz. $\Lambda_r(G^\vee)$. Thus we propose to classify magnetic charges by the quotient

$$\Lambda_w(G^\vee)/\Lambda_r(G^\vee) = \Lambda_w^*(G)/\Lambda_r^\vee(G) = \Lambda_w(\widetilde{G})/\Lambda_w(G), \tag{947}$$

where we have used various manipulations derived in the last diary entry. Using the last section of that diary entry, we see that

$$\Lambda_w(G^\vee)/\Lambda_r(G^\vee) = \Gamma_G = \pi_1(G), \tag{948}$$

where again $\widetilde{G}/\Gamma_G = G$. That we get such a classification for the magnetic charges of 't Hooft operators in terms of $\pi_1(G)$ makes perfect sense, since the 't Hooft operators can be defined by the holonomy (valued in $\pi_1(G)$) they induce in gauge field configurations. So the full class of allowed magnetic charges is $\Lambda_w(G^\vee)$ (up to Weyl invariance, more on this in a sec), and once we mod out by the lines which always appear regardless of the Lie group, we see that they are classified by $\Gamma_G = \pi_1(G)$.

A similar statement can be made for the electric operators. Since the allowed electric operators are determined by the allowed representations that we can take the trace in, they are classified by $\Lambda_w(G)$ (up to Weyl invariance). But the lines in the adjoint representation, corresponding to points in the $\Lambda_r(\mathfrak{g})$ lattice, always appear (they are the "worldlines of the gauge fields", since the gauge fields transform adjointly under constant gauge transformations) regardless of the choice of Lie group ($\Lambda_r(\mathfrak{g}) \subset \Lambda_w(G)$ for all G with Lie algebra $\mathfrak{g}$). Thus we propose to classify electric operators by the quotient

$$\Lambda_w(G)/\Lambda_r(G) = \Lambda_w^*(G^\vee)/\Lambda_w^*(\widetilde{G}) = \Lambda_w(\widetilde{G}^\vee)/\Lambda_w(G^\vee) = \Gamma_G^\vee = \pi_1(G^\vee), \tag{949}$$

where we used that e.g. $\Lambda_r^\vee(G) = \Lambda_r^\vee(G^\vee)$ and $\Lambda_w(\widetilde{G}) = (\Lambda_r^\vee(\widetilde{G}))^*$. Note the nice symmetry of this quotient with the quotient for the magnetic charges!

So far we have been considering lines that were either purely electric or purely magnetic. What happens if we have dyonic lines? Some of the details for this are in [?], so we will be brief. Basically, you can only have a consistent dyonic line if the electric and magnetic fields commute with one another (so that they can be fused together on a line in an unambiguous way). This is already ensured if we take the charges to be given by the classification scheme above, since everything done above involved only the Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{g}^\vee$. In more detail, one first fixes a magnetic field B, and then chooses an electric field in the centralizer of B in G. The centralizer G_B gives rise to a Lie algebra $\mathfrak{g}_B$, whose Cartan algebra is still the Cartan algebra of G since the elements in the Cartan algebra commute with one another. The Weyl group acts on B, and we also have a redundancy coming from Weyl_B acting on the electric sector, where Weyl_B is the subgroup which fixes B. But the combined action of Weyl_B and $\text{Weyl}/\text{Weyl}_B$ on the electric sector is just the same as having the full Weyl act, and so we just get a single (diagonal) action of Weyl on the electric and magnetic sectors. Thus the whole analysis goes through unchanged for dyonic operators.

If we propose to classify line operators by $\Gamma_G^\vee \times \Gamma_G$, what about Weyl invariance? It turns out that this is already accounted for, since the Weyl group acts trivially on $Z(\widetilde{G})$, and hence on the above assignments of both magnetic and electric charges ($\Gamma_G, \Gamma_G^\vee \subset Z(\widetilde{G})$). Proof: recall that a weight vector μ under the action of the Weyl group changes by $\delta\mu = \alpha(\alpha_i^\vee \mu^i)$, where $\alpha^\vee$ is some co-root. The inner product here is just $\mu(H_\alpha)$, which since H_α is an $\mathfrak{sl}(2,\mathbb{C})$ generator, is an integer. Thus $\delta\mu \in \Lambda_r(G)$, and so the Weyl group acts on vectors by adding integer multiples of roots to them (see the $SU(3)$ figure from the last diary entry for a nontrivial example of how this plays out). In the quotient $Z(\widetilde{G}) = \Lambda_w(\widetilde{G})/\Lambda_r(\widetilde{G})$ this action is trivial, and so Weyl acts trivially on $Z(\widetilde{G})$.

Summarizing, we can classify the lattice L of line operators by

$$L = (\Lambda_w(G)/\Lambda_r(G)) \times (\Lambda_w(G^\vee)/\Lambda_r(G^\vee)) = \Gamma_G^\vee \times \Gamma_G = \pi_1(G^\vee) \times \pi_1(G) \subset Z(\widetilde{G})^2. \tag{950}$$

Since $\Gamma_G^\vee \times \Gamma_G$ is an Abelian group, multiplying equivalence classes of line operators is done easily by using addition in the group. The simplest cases are when one of Γ_G, $\Gamma_G^\vee$ is the center of $\widetilde{G}$. If $\Gamma_G = Z(\widetilde{G})$ then $\Lambda_w(G) = \Lambda_r(G)$ and $\Gamma_G^\vee = \mathbb{Z}_1$, so that the spectrum of line operators modulo gauge field world lines only includes magnetic line operators. Likewise, if the gauge group is simply connected so that $\Gamma_G = \mathbb{Z}_1$ then we have no 't Hooft lines (since $\pi_1(G) = 0$), and the spectrum has only electric line operators. Also note that this implies that no matter what the gauge group is, the number of line operators is always equal to the order of the center:

$$|L| = |\Gamma_G||\Gamma_G^\vee| = |Z(\widetilde{G})|. \tag{951}$$

Now we take another slightly approach to identifying L by looking at the analogue of the quantization condition on $qm' - q'm$ for the non-Abelian case. The same angular momentum argument goes through unmodified (I think—the argument for the $U(1)$ unfortunately took S-duality for granted, though), but the charges involved are now matrices and need to get turned into numbers with the help of an inner product. Since the electric (magnetic) charges are in $\mathfrak{t}^*$ ($\mathfrak{t}$), the inner product is just the evaluation map $\langle , \rangle : \mathfrak{t}^* \otimes \mathfrak{t} \to \mathbb{C}$. For two dyons with electric / magnetic field strengths (Q, B) and (Q', B'), we thus require

$$\langle Q, B' \rangle - \langle Q', B \rangle \in \mathbb{Z}. \tag{952}$$

Let us pretend we didn't know about the $\Gamma_G, \Gamma_G^\vee$ groups calculated previously, and just wanted to go about solving the quantization condition directly. By the physical arguments given earlier, we know that the lattice of line operators has to sit inside $Z(\widetilde{G})^2$:

$$L \subset (\Lambda_w(\widetilde{G})/\Lambda_r(\widetilde{G})) \times \Lambda_w(\widetilde{G}^\vee)/\Lambda_r(\widetilde{G}^\vee) = Z(\widetilde{G})^2, \tag{953}$$

where the group $\widetilde{G}^\vee$ is defined so that its roots are the co-roots of $\widetilde{G}$.

Solving the quantization condition by calculating the inner product is simple in our reduced classification scheme in terms of $\Gamma_G^\vee \times \Gamma_G \subset Z(\widetilde{G})^* \times Z(\widetilde{G})$ (we have been ignorning the difference between $Z(\widetilde{G})$ and $Z(\widetilde{G})^*$ since $Z(\widetilde{G})$ is Abelian), since we can use the group law in $Z(\widetilde{G})$. For simplicity, let us first assume that $Z(\widetilde{G}) = \mathbb{Z}_N$. We can thus write a given electric line in $\Lambda_w(\widetilde{G})/\Lambda_r(\mathfrak{g})$ as $\frac{q}{N}R$, where $R \in \Lambda_r(\mathfrak{g})$ is some root that we fix and $q \in \mathbb{Z}_N$. Similarly, we can choose a given magnetic line to be mH, where $H \in \Lambda_w(\widetilde{G}^\vee) = \Lambda_r^*(\widetilde{G})$ is a fundamental magnetic weight that has inner product 1 with R (this is possible since the fundamental weights are the dual of the roots), and $m \in \mathbb{Z}_N$. Thus the quantization condition for two dyons $((q/N)R, mH), ((q'/N)R, m'H)$ is

$$\frac{1}{N}(qm' - q'm)\langle R, H\rangle = \frac{qm' - q'm}{N} \in \mathbb{Z}. \tag{954}$$

More generally, if $Z(\widetilde{G})$ is an Abelian group with n $\mathbb{Z}_k$ factors, then we can proceed as above but assign electric lines as $q_1R_1 + \cdots + q_nR_n$, and likewise for magnetic lines. One then ends up with the same quantization condition as above in each $\mathbb{Z}_k$ factor. Thus if we specify two dyons by $(q, m), (q', m') \in Z(\widetilde{G})^2$, the quatization condition is

$$qm' - q'm = 0 \text{ in } Z(\widetilde{G}). \tag{955}$$

Let's now again specialize to the case where $Z(\widetilde{G}) = \mathbb{Z}_N$, which is basically the only case of interest (all simple Lie groups have $Z(\widetilde{G})$ cyclic expect $\mathrm{Spin}(4n)$, where it is $\mathbb{Z}_2^2$). How do the solutions to the quantization condition (955) relate to the subgroups $\Gamma_G, \Gamma_G^\vee$ that we identified earlier? Hopefully, both the solutions to (955) and the choices of Γ subgroups enumerate the same list of sublattices of the "full" charge lattice $\mathbb{Z}_N^2$. Let's now see why this is indeed the case.

Without loss of generality we can write

$$\Gamma_G = \mathbb{Z}_a, \quad \Gamma_G^\vee = \mathbb{Z}_b, \tag{956}$$

for some $a, b \in \mathbb{Z}$ such that $ab = N$. The most obvious way of embedding these groups into the full lattice $\mathbb{Z}_N^2$ is to take $\Gamma_G = (0, \frac{N}{a}k) \subset \mathbb{Z}_N^2$ for $k \in \mathbb{Z}_a$, and likewise to take $\Gamma_G^\vee = (\frac{N}{b}l, 0) \subset \mathbb{Z}_N$ for $l \in \mathbb{Z}_b$ (here the first factors are electric charges and the second factors are magnetic charges). Then the lattice of allowed operators is

$$L = \{(N/b)l, (N/a)k \,|\, (l, k) \in \mathbb{Z}_b \times \mathbb{Z}_a\} \tag{957}$$

This of course satisfies the quantization condition (955) for all l, k. However, depending on the choice of groups involved, this lattice will not be the only lattice allowed, as there may be multiple ways of embedding the $\Gamma_G, \Gamma_G^\vee$ groups into the full lattice $\mathbb{Z}_N^2$. This is related to the Witten effect.

In our conventions, the emedding of the group $\Gamma_G^\vee$ into $\mathbb{Z}_N^2$, which determines the allowed electric lines, will always be uniquely defined as

$$\Gamma_G^\vee = \mathbb{Z}_b = \{((N/b)k, 0) \,|\, k \in \mathbb{Z}_b\} \subset \mathbb{Z}_N^2. \tag{958}$$

That is, the group $\Gamma_G^\vee$ determines the purely electric operators in the charge lattice. This is because $\Gamma_G^\vee = \Lambda_w(G)/\Lambda_r(G)$ classifies the allowed representations of the gauge group (modulo those constructed from $\otimes$s of the adjoint). Each representation R always defines a purely electric line operator via $W_C = \mathrm{Tr}_R[\exp(\int_C A)]$.

By contrast, we have some freedom when it comes to the magnetic operators and the embedding of Γ_G. This freedom essentially comes from our ability to make a re-definition of what we mean by a magnetic charge. For example, when we were deriving the constraints on allowed magnetic charges for line operators, we probed a magnetic line operator $\mathcal{O}$ with a purely electric line, and then constrained the allowed magnetic lines as a function of the different representations the electric line could be taken in. But since electric lines don't have statistical phases with other electric lines, the quantization conditions on the magnetic charge of $\mathcal{O}$ would be unchanged if we replaced $\mathcal{O}$ with $\mathcal{O} \otimes W$, where W is a purely electric line. Thus we could define our magnetic operators to come attached with electric lines, and the whole story would go through unchanged. Theories with different types of W's attached to the magnetic operators $\mathcal{O}$ are related by the Witten effect (i.e. usually by a shift of some θ angle), and correspond to different ways of embedding Γ_G in the full $\mathbb{Z}_N^2$ charge lattice. For example, if $\Gamma_G = \mathbb{Z}_N$, we may choose to embed it in $\mathbb{Z}_N$ as any of the N distinct subgroups $\{(nl, l)\}$, with $n, l \in \mathbb{Z}_N$.

In general, we can say that the lattice L of line operators fits into an exact sequence

$$0 \to \Gamma_G^\vee \to L \to \Gamma_G \to 0. \tag{959}$$

The injection is unique and the choice of $\Gamma_G^\vee$ completely determines the purely electric operators we have access to. There are usually mutliple ways of projecting onto the magnetic group Γ_G though, each of which gives us a distinct solution to the quantization condition. Note that this sequence will not always be split, which means that there may be no subset of lines in the full charge lattice whose magnetic charges fuse in a Γ_G subalgebra (if there were, there would exist a splitting homonorphism $\Gamma_G \to L$ with that subalgebra as its image).

Again returning to $Z(\widetilde{G}) = \mathbb{Z}_N$, we see that if $\Gamma_G = \mathbb{Z}_1$, there is only one theory, whose charge lattice is the purely electric $\mathbb{Z}_N \times 0$ sublattice of $\mathbb{Z}_N^2$. As already mentioned, if $\Gamma_G = \mathbb{Z}_N$, there are N distinct theories, which differ in the charge assigned to the fundamental magnetic line and which are permuted by the modular T operation. Also note that regardless of the choice of Γ_G, if there is a single line (q, m) with $m = 1$, then there are no purely electric lines in the $\mathbb{Z}_N^2$ charge lattice (this follows from $|L| = N$). Likewise, if there is a line $(1, 0)$ then there are no magnetically charged lines (and as previously mentioned we must have $\Gamma_G = \mathbb{Z}_1$).

For $Z(\widetilde{G}) = \mathbb{Z}_N$, we can take $\Gamma_G^\vee = \mathbb{Z}_b$, $\Gamma_G = \mathbb{Z}_a$ for $N = ab$. The different equivalence classes of extensions are given by

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_a, \mathbb{Z}_b) = \mathbb{Z}_a \otimes_{\mathbb{Z}} \mathbb{Z}_b = \mathbb{Z}_{\gcd(a,b)}. \tag{960}$$

In this classification, the split extensions are trivial. The number of different split extensions is classified by different ways of taking semidirect products, i.e. different maps $\mathbb{Z}_b \to \mathrm{Aut}(\mathbb{Z}_a) = \mathbb{Z}_a^*$, where $\mathbb{Z}_a^*$ is the multiplicative group (of order $|\mathbb{Z}_a^*| = \phi(a)$). Thus the number of split extensions is $|\mathrm{Hom}(\mathbb{Z}_b \to \mathbb{Z}_a^*)|$ (note to self: where was I going with this paragraph?)

Two last things worth re-iterating before wrapping up. First, we have classifed the allowed line operators in the theory given $\widetilde{G}$ and Γ_G by picking out a certain subset of the charge lattice. Operators with charges running over all values of the charge lattice exist no matter what Γ_G is, but unless they are part of the given subset, they must come with surfaces (which may be topological depending on the phase) attached to them, and cannot be defined on homologically non-trivial cycles. Also, it is good to remember the distinction between magnetic charge (i.e. "GNO magnetic charge") and t' Hooft flux. The former is basically a lattice point in $\Lambda_w(G^\vee)$, while the latter is an element in $\pi_1(G)$. In particular, we have no operators with t Hooft flux in theories where the gauge group is simply connected, while we still have operators with nonzero GNO magnetic charge.

Higher symmetries in non-Abelian gauge theories

Today we will consider non-Abelian gauge theory for some gauge group G, obtained from a simply-connected covering space $\widetilde{G}$ by $G = \widetilde{G}/\Gamma_G$. We will determine the higher symmetries possessed by this theory, and show how theories with different choices of Γ_G are related from a higher-symmetry point of view. We will be basically be trynig to understand the comments in [18, 12] and explain what these papers are doing in detail. Throughout this diary entry we will be using notation as if we were dealing with differential forms even when we are really dealing with cochains with coefficients in a discrete group. Therefore things like $\wedge$ will always implicitly stand for their appropriate discrete analogues. Pointryagin squares and the like will also be implicitly understood but will be written with $\wedge$.

The difference between gauge theories based on different quotients $G = \widetilde{G}/\Gamma_G$ for finite Γ_G comes in the line operators that are allowed in the theory, as we saw in an earlier diary entry. The difference between the different Gs is the types of transition functions that they allow: a given set of transition functions may satisfy the cocycle condition (closed under the Cech differential) for one choice of Γ_G, but not another.

How does this affect the allowed line operators? Naively the Wilson lines operators do not care about the transition functions and the different choices of Γ_G (at least if their topology is trivial), since they are integrals of a Lie-algebra-valued quantity. But this is too hasty. To see why, we write the Wilson line by splitting it up into a bunch of patches. We take the path C to lie in the union of a collection of patches $\{U_\alpha\}$, with the segment of the Wilson line W_C lying in U_α denoted by W_α. The naive formula for $W(C)$ is then

$$W(C) = \mathrm{Tr}\left[\prod_\alpha W_\alpha\right], \tag{961}$$

but this is not quite correct. Indeed, it is not invariant under changing the local trivializations on each U_α. Under a change in trivialization g_α which is constant on each patch, we have $W_\alpha \mapsto g_\alpha^\dagger W_\alpha g_\alpha$, and so if we e.g. change the trivialization on a single patch, our fomula for $W(C)$ is not invariant. The correct thing to do is to glue each W_α together with transition functions $t_{\alpha\beta}$:

$$W(C) = \mathrm{Tr}\left[W_\alpha t_{\alpha\beta} W_\beta t_{\beta\gamma} W_\gamma \cdots\right]. \tag{962}$$

Under a change in transition functions of $\{g_\alpha\}$, we have $t_{\alpha\beta} \mapsto g_\alpha^\dagger t_{\alpha\beta} g_\beta$, and $W(C)$ is left invariant. This construction is related to the DB cohomology approach for integrating gauge fields, which is easier to spell out in more detail in the Abelian case.

Anyway, we now see how the choice of transition functions affects the Wilson loop. For example take $SU(N)$, and consider twisting the transition functions such that a single transition function appearing in $W(C)$ gets twisted by $t_{\alpha\beta} \mapsto e^{2\pi i/N} t_{\alpha\beta}$ (we are not changing any of the trivializations, just a transition function—this can be done without creating extra field strength only if C is homologically nontrivial). Then under this change we see that $W(C) \mapsto e^{2\pi i/N} W(C)$, desipte the fact that we haven't actually changed the coordinate-patch realizations of the gauge field $\{A_\alpha\}$ at all. Thus it is good to keep in mind that although $W(C)$ only involves the gauge field, the gauge field is really a principal bundle, which carries more information (viz. information about the transition functions) than just the 1-forms A_α.

The example we will be focusing on primarily is $\widetilde{G} = SU(N)$. The fundamental Wilson lines depend on the transition functions, and may not be good line operators when Γ_G is taken to be nontrivial. However, Wilson lines in the adjoint are always good line operators, regardless of Γ_G. We know this from the previous diary entries since the root lattice always gives legit line operators, but now we can see it in a different way.

For N the fundamental of $SU(N)$, we have

$$N \otimes \bar{N} = 1 \oplus A, \tag{963}$$

where A is the adjoint. Thus for all $SU(N)$ we can write a matrix in the adjoint as follows:

$$[U_A]^{ij}_{kl} = [U_N]^i_j [U_N^\dagger]^k_l - \frac{1}{N}\delta^i_l \delta^j_k. \tag{964}$$

In this expression, i, j are fundamental indices and k, l are antifundamental indices. Here the second factor projects out the $\mathbf{1}$ in the direct sum, and the $1/N$ is so that when we put in $U_N = \mathbf{1}$ then we get $\mathrm{Tr}[\mathbf{1}_A] = N^2 - 1 = \dim(A)$. The index structure is fixed to be $\delta^l_i\delta^j_k$, since this is the $\otimes$ of invariant symbols (δ functions) for the two pairs of $N, \bar{N}$ indices.

This means that the Wilson loop in the adjoint can be computed from the Wilson loop in the fundamental by

$$W_A(C) = \mathrm{Tr}_A e^{i\oint_C A} = \mathrm{Tr}\left\{\left[e^{i\oint_C A}\right]^i_k \left[e^{-i\oint_C A}\right]^j_l - \frac{1}{N^2}\delta^i_k\delta^j_l\right\}. \tag{965}$$

The trace sets $i = k$ and $j = l$, so

$$W_A(C) = \left|\mathrm{Tr}_f e^{i\oint_C A}\right|^2 - 1. \tag{966}$$

Thus we see that since the center symmetry changes the fundamental Wilson line by a phase, it leaves $W_A(C)$ invariant. Thus no matter what choice of $\Gamma_G \subset \mathbb{Z}_N = Z(SU(N))$ we make, the adjoint Wilson lines will always be well-defined well defined operators, blind to the allowed t' Hooft line operators.

With these introductory comments out of the way, let's now see how this works in a more detailed way. We'll focus on the simple example of the relation between $SU(N)$ and $PSU(N)$ gauge theory, although quotienting $SU(N)$ by other subgroups of $\mathbb{Z}_N$ can be done analogously.

Of course, the real difference between $SU(N)$ and $PSU(N)$ gauge theories is in the transition functions, and passing from $SU(N)$ to $PSU(N)$ means changing the transition functions. This isn't something that's easy to do in a transparent way as far as the variables naturally appearing in a QFT are concerned, so we will try to encode the changed transition functions into a field that appears in the action.

We want to "gauge" the $\mathbb{Z}_N$ part of the transition functions to obtain $PSU(N)$ gauge theory, which we will do by coupling the $SU(N)$ theory to a $\mathbb{Z}_N$ gauge field. Recall that discrete gauge fields are basically just $\mathbb{Z}_N$ transition functions: they have no local degrees of freedom on patches; all of their physical content is in transition functions between patches. Coupling $SU(N)$ to a $\mathbb{Z}_N$ gauge field will then allow us to identify two states that differ by twisting the transition functions by something in $\mathbb{Z}_N$, and we will have obtained a $PSU(N)$ theory. Since we are in the continuum, we will write down the $\mathbb{Z}_N$ gauge field using the BF theory approach, where we only deal with $U(1)$ fields. Since the $\mathbb{Z}_N$ symmetry we want to gauge is a 1-form symmetry, the gauge field will be a $\mathbb{Z}_N$ 2-form field B. we use the presentation of the BF action where NB gets set to be the field strength of a $U(1)$ gauge field though a Lagrange multipler coupling:

$$S \supset \frac{i}{2\pi}\int H \wedge (F_{\mathcal{A}} - NB), \tag{967}$$

where $F_{\mathcal{A}}$ is the field strength for a $U(1)$ gauge field $\mathcal{A}$ (we are avoiding writing it as $d\mathcal{A}$ since $\mathcal{A}$ may not be a 1-form). Here H is quantized to have $2\pi\mathbb{Z}$ periods around all closed 2-manifolds.

If F_A is the $SU(N)$ field strength, the naive thing to do would be to take $F_A \mapsto F_A - B\mathbf{1}$ in the action. This is what we would do if we were trying to gauge a $U(1)$ 1-form symmetry, since gauging the 1-form symmetry that acts on A means that F_A changes by total derivates under gauge transformations. But this isn't right: locally we can replace B with $\frac{1}{N}F_{\mathcal{A}}$, which means that the action $||F_A - \frac{1}{N}F_{\mathcal{A}}||^2$ now has more local degrees of freedom than it did when we started, which is definitely not what we want. The correct thing to do is to cancel out these extra local degrees of freedom with the $\mathcal{A}$ field. We will do this by taking a $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$ gauge theory, and killing off the $U(1)$ factor to produce an $SU(N)/\mathbb{Z}_N = PSU(N)$ theory.

To this end, consider the $U(N)$ field

$$\mathcal{A} = A + \frac{1}{N}\mathcal{A}\mathbf{1}. \tag{968}$$

Here A is traceless and $\mathcal{A}$ is the $U(1)$ part of the $U(N)$, with $\text{Tr}\mathcal{A} = \mathcal{A}$. By saying that this is a $U(N)$ field, we really just mean that the transition functions of A and those of $\mathcal{A}$ are correlated in a way such that their combination gives legit $U(N)$ transition functions. In particular, we can let the transition functions for the $SU(N)$ part fail to be closed in $\check{C}^1(M;\mathbb{Z})$, provided that the transition functions for $\mathcal{A}/N$ compensate this ($\mathcal{A}$ can do this while still remaining a legit $U(1)$ gauge field because of the $1/N$ in front of it in the definition for $\mathcal{A}$: $\mathcal{A}$ is a well-defined $U(1)$ gauge field, but $\mathcal{A}/N$ is not). This identification is the quotient by $\mathbb{Z}_N$ in $[SU(N) \times U(1)]/\mathbb{Z}_N$.

Since we now can shift the transition functions of the A bundle by Nth roots of unity through a gauge transformation, we almost have a $PSU(N)$ gauge field. The only problem is that it has an extra $U(1)$ local degree of freedom that the $PSU(N)$ theory doesn't have. This extra local degree of freedom is eliminated by adding the background field B to the action though, via

$$S \supset \frac{1}{2g^2}\int \text{Tr}\left[(F_{\mathcal{A}} - B\mathbf{1}) \wedge \star(F_{\mathcal{A}} - B\mathbf{1})\right]. \tag{969}$$

This is the proper way to couple the $SU(N)$ theory to a $\mathbb{Z}_N$ gauge field: neither B nor $\mathcal{A}$ by themselves is the $\mathbb{Z}_N$ gauge field; the $\mathbb{Z}_N$ gauge field involves both of them. This is the price we pay for wanting to work in the continuum. Note that A is not really the $PSU(N)$ field: while the bundle for A is allowed to have transition functions which fail the cocycle condition by something in $\mathbb{Z}_N$, we need the gauge fields $\mathcal{A}, B$ to make sure that the places where the cocycle condition fails can be removed by gauge transformations, so that such places do not contribute to a physical field strength. So it's really the whole package that constitutes the $PSU(N)$ field.

Now it may seem like we've actually just gotten back to our starting point by adding B, since it appears in the combination $\frac{1}{N}F_{\mathcal{A}}\mathbf{1} - B\mathbf{1}$, which seems to vanish upon integrating out H. But integrating out H only says that $F_{\mathcal{A}} = NB$ and does *not* imply that $\frac{1}{N}F_{\mathcal{A}} = B$. Locally it does, and so we have indeed added no new local degrees of freedom. But globally, knowing NB does not let you know B. Knowing NB means that you know B locally, and means that you know $(e^{i\int_\Sigma B})^N$ for all closed 2-manifolds Σ. In fact since $NB = F_{\mathcal{A}}$, we have $(e^{i\int_\Sigma B})^N = 1$ for all Σ, and so knowing NB doesn't give you any information about the holonomies of B around closed 2-manifolds (which are always Nth roots of unity). It may help to again explain exactly what happens when we integrate out H. Doing a Hodge decomposition $F_{\mathcal{A}} = d\alpha + \omega_{\mathcal{A}}$, $B = d\lambda + d^\dagger\epsilon + \omega_B$, $H = d\gamma + \omega_H$, $\omega_{\mathcal{A}} \in 2\pi H^2(M;\mathbb{Z})$, $\omega_H \in H^2(M;\mathbb{R}/2\pi\mathbb{Z})$, $\omega_H \in 2\pi H^2(M;\mathbb{Z})$ (recall that $\int_\Sigma H \in 2\pi\mathbb{Z}$ for all closed 2-manifolds Σ), we get a δ function setting $\epsilon = 0$, so that the non-cohomological degrees of freedom are all pure gauge and thus dissapear. Upon summing over $\omega_H \in 2\pi H^2(M;\mathbb{Z})$ we get the term

$$\sum_{\omega_H \in 2\pi H^2(M;\mathbb{Z})} \exp\left(\frac{i}{2\pi}\int \omega_H \wedge (\omega_{\mathcal{A}} - N\omega_B)\right) = \sum_{\widehat{\omega}_H \in H_2(M;\mathbb{Z})} \exp\left(i\int_{\widehat{\omega}_H} (\omega_{\mathcal{A}} - N\omega_B)\right). \tag{970}$$

Given the quantization of $\omega_{\mathcal{A}}$, this means that we can constrain

$$\omega_B \in \frac{2\pi}{N} H^2(M;\mathbb{Z}), \tag{971}$$

but ω_H is free to take on any cohomology class in this cohomology group (classes in $H^2(M;\mathbb{Z})$ are gauged under the 1-form gauge symmetry though, so only the classes in $(2\pi)/NH^2(M;\mathbb{Z}_N)$ are physically distinct). In particular, integrating out H does not actually set $B = \frac{1}{N}F_{\mathcal{A}}$, since the cohomology class of B is not fixed after integrating out H. This class is the sole degree of freedom carried by the $\mathbb{Z}_N$ gauge field.

Thus, the combination $\frac{1}{N}F_{\mathcal{A}} - B$ carries no local degrees of freedom, but carries global $\mathbb{Z}_N$ degrees of freedom: it is the $\mathbb{Z}_N$ 2-form gauge field that want to couple to the $SU(N)$ fields. As a sanity check, if we have a situation where $\int_\Sigma B \in 2\pi\mathbb{Z}$ for all Σ, then B is pure gauge and we can make a gauge transformation to eliminate $\mathcal{A}$ and B from the theory entirely. This gives us back the pure $SU(N)$ theory as required.

We can now look at the operators in the theory. The Wilson line in the fundamental of $SU(N)$ now is not gauge invariant under 1-form gauge transformations $\mathcal{A} \mapsto \mathcal{A}+N\lambda$, and so we must write it as a surface operator by attching a B surface:

$$W_f(C) \mapsto \mathrm{Tr}\left[\exp\left(i\oint_C \mathcal{A}\right) e^{-i\int_\Sigma B\mathbf{1}}\right] = \mathrm{Tr}\left[\exp\left(i\oint_C A\right)\right] e^{\frac{i}{N}\oint_C \mathcal{A}} e^{-i\int_\Sigma B}, \tag{972}$$

where $\partial\Sigma = C$ and we used $e^{x\mathbf{1}} = \mathbf{1}e^x$ for $x \in \mathbb{C}$. Because of the attached surface operator the fundamental Wilson lines are no longer part of the lines operators in the theory. The adjoint lines are the new "smallest charge" electric line operators: as we saw earlier they depend on the fundamental Wilson lines through the product $|W_f(C)|^2$. Since the Abelian part relating to the holonomy of $\mathcal{A}$ cancels when the square is taken, the adjoint Wilson loops don't see the $\mathcal{A}$ field and are consequently gauge-invariant well-defined line operators. Note that similar things like this are discussed in Tong's gauge theory notes, but unlike in his notes we are saying that the line operators in the $PSU(N)$ theory are the adjoint representation Wilson lines, not N-fold powers of the fundamental line. N-fold powers don't get you anything since you still need a surface operator attached to the line to ensure gauge invariance[99]. Also e.g. for $SU(3)$ we have

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \tag{973}$$

Since 10 cannot be built from $\otimes$s of the adjoint 8, the RHS is not invariant under the $\mathbb{Z}_N$ 1-form symmetry and thus taking three fundamental Wilson lines is not quite the right way to get something invariant under the 1-form symmetry.

[99]The surface operator is now $e^{-iN\int_\Sigma B}$. If Σ is closed then this is always equal to 1, and so the attached surface operator is topological (independent of the exact choice of Σ. But the surface operator still needs to be there, and if C does not bound a surface then the N-fold power of $W_f(C)$ can't be defined in a gauge-invariant way.

Now, are the global symmetries in the $PSU(N)$ theory right? As we've seen before, a pure $\mathbb{Z}_N$ 2-form gauge field comes with a $\mathbb{Z}_N$ 2-form electric symmetry and a $\mathbb{Z}_N$ $(D-2-1)$-form magnetic symmetry. For us $D=4$ so the magnetic symmetry is a 1-form symmetry. The 2-form electric symmetry sends $B \mapsto B+\gamma$ for a flat 2-form γ with periods in $\frac{2\pi}{N}\mathbb{Z}$. This symmetry is broken by the coupling to the $U(N)$ field $\mathcal{A}$, which is good since the $PSU(N)$ theory shouldn't have any 2-form symmetries. However, in keeping with our discussion the other day about $\widetilde{G}$ and $\widetilde{G}/\Gamma_G$ gauge theories, we know that the $PSU(N)$ theory should have a $\mathbb{Z}_N$'s worth of t' Hooft operators, which are the charged objects for a $\mathbb{Z}_N$ 1-form symmetry. This is precisely the 1-form magnetic symmetry of the B field. The t' Hooft loop is constructed with $\widetilde{A}$, where $F_{\widetilde{A}} = H$ (writing H like this is possible since we took H to be a closed 2-form with periods in $2\pi\mathbb{Z}$). So the t' Hooft operator is

$$T(C) = \exp\left(i \oint_C \widetilde{A}\right). \tag{974}$$

The last type of operator we have is

$$\mathcal{W}(\Sigma) = \exp\left(i \int_\Sigma B\right), \tag{975}$$

where Σ is closed. From the commutation relation between B and $\widetilde{A}$ (roughly $[\widetilde{A}, B] = i/N$), we see that $\mathcal{W}(\Sigma)$ is the charge operator for the magnetic 1-form symmetry, and that it has the correct commutation relations with the fundamental Wilson line $W_f(C)$ (the nontrivial commutation relation is due to the B surface operator attached to $W_f(C)$). It thus computes the integral of ω_2 over Σ, where $\omega_2 \in H^2(M;\mathbb{Z}_N)$ is the 2nd Stiefel-Whitney class for the $PSU(N)$ bundle restricted to Σ.

Now given this, how would we get back to the $SU(N)$ theory? We would need to gauge the $\mathbb{Z}_N$ 1-form magnetic symmetry with a $\mathbb{Z}_N$ 2-form gauge field, whose magnetic 1-form symmetry would become the electric symmetry of the $SU(N)$ theory. This new gauge field needs to have the effect of forcing $F_\mathcal{A}$ to be quantized in periods of $2\pi N\mathbb{Z}$, since as we saw this turns the $PSU(N)$ transition functions into $SU(N)$ transition functions. So we can add some fields with the action

$$\frac{i}{2\pi}\int H' \wedge (F_\mathcal{A} - NF_{\mathcal{A}'}), \tag{976}$$

where H' is yet another Lagrange multiplier 2-form gauge field and $\mathcal{A}'$ is a $U(1)$ gauge field. The periods of H are not quantized, so that integrating out H sets the cohomology classes of $F_\mathcal{A}$ and $NF_{\mathcal{A}'}$ to be equal. This makes $F_\mathcal{A}$ pure gauge under the original 1-form $\mathbb{Z}_N$ gauge symmetry and so $\mathcal{A}$ and B can be eliminated from the action, leaving us with an $SU(N)$ field. The electric symmetry of $SU(N)$ in this presentation is the 1-form symmetry that shifts $\mathcal{A}'$ by a closed form that has periods in $\frac{2\pi}{N}\mathbb{Z}$ (again, the fact that this is the symmetry can be most easily seen by integrating by parts and writing the relevent term in the action as $\frac{iN}{2\pi}\int F_{H'} \wedge \mathcal{A}'$, and recalling that H' is a $U(1)$ 2-form gauge field so that $F_{H'}$ has periods in $2\pi\mathbb{Z}$). The electric 2-form symmetry of the added $\mathbb{Z}_N$ gauge field that we used to get back to $SU(N)$ is broken by the coupling to $F_\mathcal{A}$, and so all the global symmetries are properly accounted for.

So, summarizing, we've seen that $SU(N)$ has an electric $\mathbb{Z}_N^{(1)}$ symmetry, while $PSU(N)$ as a magnetic $\mathbb{Z}_N^{(1)}$ symmetry, and that these two symmetries can be related to one another through a gauging procedure. We know that the electric symmetry in $SU(N)$ is only $\mathbb{Z}_N$ since we always have gluons in the adjoint representation, so Wilson lines in the adjoint can end and hence can't carry a 1-form charge. How do we know that we haven't missed e.g. a magnetic symmetry in $SU(N)$? One way to argue is to say that we can couple the theory to a Higgs field in the adjoint: this leads to dynamical t'Hooft Polyakov monopoles as we have seen in an earlier entry, which carry magnetic charge N in the magnetic lattice. Since adding the Higgs field doesn't break any symmetries, the symmetries of the Higgsed theory should be the same as the un-Higgsed one, and so the un-Higgsed one can't have any magnetic 1-form symmetries. Actually, apparently the pure glue theory actually does have dynamical charge N monopoles, created at the intersection of N Wilson lines (should find the reference for this).

We now briefly cover θ angles. After gauging to $PSU(N)$, the θ term is (remember that the $1/8\pi^2$ comes from expanding $\exp(F/2\pi)$)

$$S \supset \frac{i\theta}{8\pi^2}\int \mathrm{Tr}\left[(F_{\mathcal{A}} - B\mathbf{1}) \wedge (F_{\mathcal{A}} - B\mathbf{1})\right]. \tag{977}$$

Since F_A is traceless, this is

$$S \supset \frac{i\theta}{8\pi^2}\int \mathrm{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] + \frac{i\theta N}{8\pi^2}\int B \wedge B - \frac{i\theta}{4\pi^2}\int F_{\mathcal{A}} \wedge B. \tag{978}$$

Since $\mathrm{Tr}[F_{\mathcal{A}}] = F_{\mathcal{A}}$, we can write this as

$$S \supset i\theta c_2 + \frac{i\theta N}{8\pi^2}\int B \wedge B - \frac{i\theta}{4\pi^2}\int F_{\mathcal{A}} \wedge B + \frac{i\theta}{8\pi^2}\int F_{\mathcal{A}} \wedge F_{\mathcal{A}}. \tag{979}$$

Here the second Chern class is

$$c_2 = \frac{1}{2(2\pi)^2}\left(\mathrm{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] - \mathrm{Tr}[F_{\mathcal{A}}] \wedge \mathrm{Tr}[F_{\mathcal{A}}]\right), \tag{980}$$

and is always integral. Recall that we get this from the second-order contribution to $\det(\mathbf{1} + F_{\mathcal{A}}/2\pi)$, which we compute as $\exp(\mathrm{Tr}\ln(\mathbf{1} + F_{\mathcal{A}}/2\pi))$ by using the Taylor series for the log. Using the constraint from integrating out H, we can eliminate $\mathcal{A}$ and write everything in terms of B (we do this by writing $F_{\mathcal{A}} = NB$. This is not the same as writing $B = N^{-1}F_{\mathcal{A}}$, which is not a replacement that we are making):

$$S_\theta = i\theta\int c_2 - (N^2 - N)\frac{i\theta}{8\pi^2}\int B \wedge B, \tag{981}$$

where now B is constrained to have periods in $2\pi/N$ around all 2-submanifolds. We see that shifting θ by 2π is now nontrivial: where we dropped $2\pi i\int c_2 \in 2\pi\mathbb{Z}$. The N term is certainly non-trivial because of the quantization on B, while the $+1$ part is in $2\pi\mathbb{Z}$ if we are working on a spin manifold. If we are not on a spin manifold, this factor can contribute a $\pm$ sign to the path integral[100].

[100]The claim is that on a spin manifold M,

$$\int_M \frac{F}{2\pi} \wedge \frac{F}{2\pi} \in 2\mathbb{Z}, \tag{982}$$

The $\int B \wedge B$ term here prompts us to go back to our original action and add a counter-term of this form. We have several options for counterterms to add, given by an integer p. The counter-term we add is

$$S_{ct} = \frac{ipN}{4\pi} \int B \wedge B. \tag{986}$$

Under a gauge transformation that shifts B by F_λ (the curvature of a $U(1)$ gauge field), this counter-term changes by

$$\delta S_{ct} = ipN\pi \int \frac{F_\lambda}{2\pi} \wedge \frac{F_\lambda}{2\pi} + \frac{ipN}{2\pi} \int B \wedge F_\lambda. \tag{987}$$

The first term is in $2\pi\mathbb{Z}$ if we are on a spin manifold, but is non-trivial in general. There is no other term in the action that will give us a quadratic term in F_λ, so in order for S_{ct} to make sense we must have $pN \in 2\mathbb{Z}$ (if we are on a spin manifold, just $pN \in \mathbb{Z}$ is okay).

The second $B \wedge F_\lambda$ term is trivial if we use the constraint that B has periods in $\frac{2\pi}{N}\mathbb{Z}$ coming from integrating out H, since then the term is valued in $2\pi p\mathbb{Z}$. Integrating out H gets rid of the operator responsible for the t' Hooft lines though (recall $H = F_{\widetilde{A}}$), so it would be better if we could keep H around (we want to talk about what happens to the t' Hooft lines when θ is shifted, which is tricky to do if we have to integrate it out to insure gauge invariance. I admit this argument is a little shaky). Basically, we want to ensure that the action is gauge invariant without needing to integrate out any of the Lagrange multipliers. We see that we can cancel the second term directly if we take

$$H = F_{\widetilde{A}} \mapsto F_{\widetilde{A}} - pF_\lambda \tag{988}$$

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for any 2-form $F \in 2\pi H^2(M;\mathbb{Z})$ while on a non-spin manifold, the RHS is replaced by $\mathbb{Z}$. Why does the (non)admitance of a spin structure determine how gauge fields integrate? This is because if a four-manifold M is spin, the intersection form

$$H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z} \tag{983}$$

is even. This is because mod 2 we have, for any $A \in H^2(M;\mathbb{Z})$,

$$A \smile A = \omega_2 \smile A \qquad \text{mod } 2, \tag{984}$$

where ω_2 is the second Stiefel Whitney class. Thus if M is spin then ω_2 lifts to an even class in $H^2(M;\mathbb{Z})$, and so since $(F/2\pi) \in H^2(M;\mathbb{Z})$, the integral above must indeed be even if M is spin.

To prove the last equation, one notes that $\omega_2 \smile A = \omega_2(TM|_A)$, i.e. $\omega_2 \smile A$ meansures the Stiefel-Whitney class of A embedded in M (using Poincare duality to think of A as a 2-manifold). Now $TM|_A = TA \oplus TN$, where TN is the component of the tangent bundle of M normal to A. Then since A is an orientable manifold, the Whitney product formula reads

$$\omega_2(TM|_A) = \omega_2(TA) + \omega_2(TN). \tag{985}$$

The first term on the RHS is the mod 2 Euler class of A, which is trivial for A a 2-manifold, since $\chi(A) = 2 - 2g \in 2\mathbb{Z}$. The second term is the mod 2 Euler class of TN, which is precisely the self-intersection number of A mod 2. This is beacuse $\omega_2(TN)$ measures the zeros of vector fields in TN mod 2, and the zeros in TN precisely come from self-intersection points of A (the intersection at each self-intersection point can be made transverse, so at these points the tangent space of A generates the full tangent space of M; hence any vector field in TN must vanish at these points).

under the 1-form gauge transformation. This adds an extra term $ipN\int F_{\mathcal{A}}\wedge F_\lambda/2\pi$, but this is trivial since both $F_{\mathcal{A}}$ and F_λ are quantized in $2\pi\mathbb{Z}$ (by definition of $\mathcal{A}$ and λ, not because of Lagrange multipliers). Finally, note that if we take $p\mapsto p+2N$ then the action changes by $\frac{iN^2}{2\pi}\int B\wedge B$, which is trivial (after integrating out H). So we have the periodicity $p\sim p+2N$ (for generic parity N). On a spin manifold, or if N is even, we can do better and write $p\sim p+N$. This is the same as the fact that in Abelian CS theory with odd level, there are $2k$ particles with the kth one a fermion, while for k even the periodicity is smaller and there are only k particles. Adding this counter-term means that doing $\theta\mapsto\theta+2\pi$ is the same as doing $p\mapsto p+1-N$ (if spacetime is spin then it is just $p\mapsto p+1$).

Let's now check that the line operators we predicted to exist in the last diary entry are indeed the ones that are realized. As expected, for $\theta\neq 0$, a pure charge-1 t' Hooft line doesn't exist, since $e^{i\oint\widetilde{A}}$ is not gauge invariant (recall that we made $F_{\widetilde{A}}\mapsto F_{\widetilde{A}}-pF_\lambda$ under the 1-form gauge symmetry). However, we see that its gauge variance is exactly canceled by the gauge variance of p copies of the antifundamental Wilson line (since $\delta F_{\mathcal{A}}=NF_\lambda$ under the 1-form gauge transformation), so that the t' Hooft line becomes

$$T(C;p)=e^{\frac{i}{N}\oint_C\widetilde{A}}\mathrm{Tr}\left[\exp\left(-ip\oint_C\mathcal{A}\right)\right],\tag{989}$$

demonstrating that shifting the θ angle by 2π attaches electric charge 1 (really, -1) to the magnetic lines. By changing θ and thus changing p, we can realize the full range of $PSU(N)$ charge sublattices that we found in the previous diary entry, generated by the lines $(k,1)$ for $k\in\mathbb{Z}_N$ the electric charge of the minimal magnetically charged line.

Before we end, we briefly mention that some quotient theories have charge lattices that are not related by shifting θ. To find examples, we need Γ_G to be a proper subgroup of the center. The simplest example where we can quotient by a subgroup of the center is for $G=SU(4)/\mathbb{Z}_2$. Recall that the charge lattice is given by the exact sequence

$$1\to\Gamma_G^\vee\to L\to\Gamma_G\to 1.\tag{990}$$

For this example we have $\Gamma_G=\mathbb{Z}_2$ (and hence $\Gamma_G^\vee=\mathbb{Z}_2$), and so there are two possible choices for L: the split extension $L=\mathbb{Z}_2\rtimes\mathbb{Z}_2\cong\mathbb{Z}_2^2$, or the non-trivial extension in $H^2(\mathbb{Z}_2;\mathbb{Z}_2)$, namely $L=\mathbb{Z}_4$. In the former case we have two generators for the charge lattice, namely $(0,2)$ and $(2,0)$. In the later case, we have a single generator with magnetic charge 2, namely $(1,2)$. Changing θ can only relate theories that differ in the way in which they define the electric charge of the generator(s) of the charge lattice; it cannot relate two theories with cohomologically distinct extensions L. Indeed, we see that the higher symmetries in the problem are not given by two 1-form symmetries $\Gamma_G^\vee\times\Gamma_G$, but rather are determined by L. In the case of $L=\mathbb{Z}_4$, there is only a single mixed electromagnetic 1-form symmetry, while for $L=\mathbb{Z}_2^2$ we have both electric and magnetic 1-form $\mathbb{Z}_2$ symmetries. This gives us an understanding of why the two $SU(4)/\mathbb{Z}_2$ theories are not related by a shift in a θ angle: they have different global symmetries.

All about (fractional) instanton numbers

Today's entry is a collection of things about instantons and their normalizations in different gauge groups that I thought would be handy to have around as a reference. It was inspired by reading [2] and wanting to understand + extend the results.

Mathematical prelude

We begin this diary entry with definitions and useful math facts. For us, the instanton number I for a gauge bundle E over a four-dimensional spacetime X will be defined as

$$I = \int_X \mathrm{ch}_2(E), \tag{991}$$

where $\mathrm{ch}_2(E)$ is the second chern character. Recall that the Chern characters are obtained in the context of Chern-Weil theory from the expansion of $\mathrm{Tr}[e^{F/2\pi}]$ as

$$\mathrm{ch}_k(E) = \frac{1}{k!}\mathrm{Tr}\left[(F/2\pi)^{\wedge k}\right]. \tag{992}$$

Note that the Chern characters involve only a single trace, unlike the Chern classes.

We define I as the integral of $\mathrm{ch}_2(E)$ and not of $\mathrm{c}_2(E)$ (the second chern class), since it is the chern character, not the chern class, that appears in the index formula (and since we e.g. definitely want I to be nonzero when we choose the gauge group to be $U(1)$). For gauge groups like $SU(N)$ with traceless generators, $\mathrm{c}_2(E)$ and $\mathrm{ch}_2(E)$ are *almost* the same. The difference comes from torsion phenomena: the Chern classes can have torsion contributions, so that even when $\mathrm{Tr}[F] = 0$, we can have e.g. $c_1 \neq 0$, provided that c_1 is pure torsion. This can happen when the gauge group is the quotient of some simply connected group, like in the case of $SU(N)/\mathbb{Z}_N$. These torsionful contributions are ignored by Chern-Weil theory, but are important to keep track of.

In contrast, the Chern characters are defined as classes in $H^*(X;\mathbb{Q})$[101], and as such *never* have any torsionful elements. They are calculated purely from the local curvature, and are only sensitive to data about the gauge group's Lie algebra (whereas the Chern classes care about the full Lie group).

[101] The rational coefficients here are simply because the mod 1 contributions to the Chern characters come from the coefficients in the expansion of the exponential. Since the cohomology groups are isomorphic, we may equivalently just write $H^*(X;\mathbb{R})$.

An important difference between the Chern characters and the Chern classes is that Chern classes always integrate to integers. The example relevant to us is that the integral of c_2 is an integer on any manifold, spin or otherwise. In contrast, the integral of ch_2 is not generically integral on a non-spin manifold, since the intersection form on a non-spin manifold is not for sure even. Thus we should remember that the Chern classes are good $\mathbb{Z}$ characteristic classes, while the Chern characters are not.

The chern characters satisfy

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F), \qquad \mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F). \tag{993}$$

The former can be seen by plugging in $F_{E\otimes F} = \mathbf{1}_G \otimes F_H + F_G \otimes \mathbf{1}_H$ into the formulae for the chern characters, while the latter is straightforward to see since the chern characters involve only a single trace. On the other hand, the Chern class of the direct sum is the wedge of the Chern classes, instead of the sum:

$$\mathrm{c}(E \oplus F) = \mathrm{c}(E) \wedge \mathrm{c}(F). \tag{994}$$

This is the Whitney sum formula and can be seen from the definition of the Chern classes in terms of the expansion of $\det(\mathbf{1} + F_A/2\pi)$, and the fact that $\det(A \oplus B) = \det(A)\det(B)$. I'm unaware of any simple formula for $\mathrm{c}(E\otimes F)$, unless $E \cong \bigoplus_i \mathcal{L}_i, F \cong \bigoplus_j \mathcal{L}'_j$ for line bundles $\mathcal{L}_i, \mathcal{L}'_j$. In that case, we have

$$\mathrm{c}(E\otimes F) = \mathrm{c}\left(\bigoplus_{i,j} \mathcal{L}_i \otimes \mathcal{L}'_j\right) = \bigwedge_{i,j} \left(1 + \mathrm{c}_1(\mathcal{L}_i \otimes \mathcal{L}'_j)\right) = \bigwedge_{i,j}(1+\mathrm{c}_1(\mathcal{L}_i)+\mathrm{c}_1(\mathcal{L}'_j)), \tag{995}$$

where we used the Whitney sum formula and the fact that $\mathrm{c}_1(\mathcal{L}\otimes\mathcal{L}') = \mathrm{c}_1(\mathcal{L})+\mathrm{c}_1(\mathcal{L}')$. This can be seen by recalling that the first Chern class can be defined by the Euler class of the underlying real bundle. Since the expression for the Euler class involves the log of the transition functions, and since the transition functions of $\mathcal{L}\otimes\mathcal{L}'$ are the product of the transition functions for $\mathcal{L}$ and $\mathcal{L}'$, the Euler class of $\mathcal{L}\otimes\mathcal{L}'$ splits as a sum of the Euler classes of each line bundle—hence $\mathrm{c}_1(\mathcal{L}\otimes\mathcal{L}') = \mathrm{c}_1(\mathcal{L}) + \mathrm{c}_1(\mathcal{L}')$.

Something else we sometimes need to do is to determine characteristic classes / instanton numbers for product bundles $E \otimes F$, where E is a principal G-bundle and F is a principal H-bundle. The answer for the instanton number is what you would expect: for theories not involving a $U(1)$ factor so that their Lie algebra generators are traceless, we have

$$I_{E\otimes F} = \mathrm{ch}_2(E \otimes F) = I_G \dim H + I_H \dim G, \tag{996}$$

where the dimension means the dimension of the defining representaiton of the associated Lie algebras. This follows from $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F)$: taking the second order terms, we have

$$\mathrm{ch}_2(E \otimes F) = \mathrm{Tr}_G[\mathbf{1}_G]\frac{1}{8\pi^2}\mathrm{Tr}_H\left[F_H \wedge F_H\right] + \mathrm{Tr}_H[\mathbf{1}_H]\frac{1}{8\pi^2}\mathrm{Tr}_G\left[F_G \wedge F_G\right], \tag{997}$$

which gives us what we want.

Instantons

Normal instantons come from transitions between pure gauge field configurations in different homotopy classes of $\pi_3(G)$, where the 3 in $\pi_3(G)$ is a spatial slice (or region thereof) where the gauge fields asymptote to a constant (the elements in $\pi_3(G)$ are the glueing data for nontrivial bundles on S^4). These instantons can live in any $\mathbb{R}^4$-like region of a given 4-manifold, regardless of its topology. Furthermore they will exist for all choices of (non-Abelian) gauge groups, since $\pi_3(G) = \mathbb{Z}$ for all simple compact non-Abelian Lie groups G. These instantons are common to all gauge groups G that descend from some simply connected group $\widetilde{G}$ by quotienting by some finite Γ_G (which may be $\mathbb{Z}_1$). To show this, one uses the long exact sequence coming from $1 \to \Gamma_G \to \widetilde{G} \to G \to 1$. This sequence contains

$$\ldots \to \pi_4(\widetilde{G}/G) \to \pi_3(G) \to \pi_3(\widetilde{G}) \to \pi_3(\widetilde{G}/G) \to \cdots . \tag{998}$$

Now $\pi_{k>0}(\widetilde{G}/G) = \pi_{k>0}(\Gamma_G) = 0$ since Γ_G is discrete and the homotopy groups are basepoint preserving (the basepoint is fixed to be a given element of the target space for the definition of the homotopy group, so we don't get a $|\Gamma_G|$'s worth of constant maps, we just get a $\mathbb{Z}_1$'s worth). Thus we have an isomorphism $\pi_3(G) \cong \pi_3(\widetilde{G})$, and so the "small" instantons associated with $\pi_3(G)$ have the same instanton number no matter what Γ_G is.

As in [6], we will normalize the instanton number so that the minimal "small" instanton has instanton number $I = 1$. This minimal small instanton can always be taken to be a minimal $SU(2)$ instanton, on an S^3 around which $A \sim U^\dagger dU, U \sim e^{ix^a T^a}$, for an appropriately chosen trio of generators $T^z, T^+, T^- \in \{T^a\}$, with T^z, T^+, T^- generating an $\mathfrak{su}(2)$ Lie algebra. Recall that this embedding of $\mathfrak{su}(2)$ can always be done: we pick a pair of roots T^+, T^- that are eigenvalues under the action of Ad_A where A is such that Ad_A has maximal kernel, and then from these generators we can construct a T^z in the Cartan subalgebra of $\mathfrak{g}$ that together with the $T^\pm$ generates an $\mathfrak{su}(2)$. Thus for all choices of (compact, simple) Lie group G, we can always embed an $SU(2)$ instanton through a choice of $\mathfrak{su}(2) \to \mathfrak{g}$. This induces a map $SU(2) \to G$, and the normalization of the instanton number depends on the index of this map.

One foolproof way to find the normalization for the instanton number is to compute the instanton number by requiring that for a minimal small instanton field configuration F, we have

$$1 = \frac{1}{N_\mathfrak{g}} \int \mathrm{Tr}_{Ad_\mathfrak{g}} \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right). \tag{999}$$

Here $\mathrm{Tr}_{Ad_\mathfrak{g}}$ is taken in the adjoint representation of $\widetilde{G}$, which is always a representation for all $G = \widetilde{G}/\Gamma_G$, and $N_\mathfrak{g}$ is a normalization constant that fixes the equality. For example, consider $SU(2)$. We know the bundle E with a minimal $SU(2)$ instanton is such that

$$1 = \int \mathrm{c}_1(E) = \frac{1}{2} \int \mathrm{Tr}_f \left(\frac{F}{2\pi} \wedge \frac{F}{2\pi} \right). \tag{1000}$$

In the fundamental, we have the normalization

$$\mathrm{Tr}(T^a_f T^b_f) = \frac{\delta^{ab}}{2}. \tag{1001}$$

On the other hand, in the adjoint we have

$$\mathrm{Tr}(T^a_{Ad} T^b_{Ad}) = N\delta^{ab}, \tag{1002}$$

so that if we take the minimal $SU(2)$ instanton F^a but change the representation to the adjoint, the answer changes by a factor of $2N$. Thus we have

$$N_{\mathfrak{su}(N)} = 4N. \tag{1003}$$

When Γ_G is nontrivial we can have "large" instantons that contribute to the instanton number I but which make rational contributions to I instead of integral contributions. This is because if the topology of spacetime is nontrivial, we can have G bundles which are not $\widetilde{G}$ bundles. This is not as contrived a scenario as it seems, since nontrivial spacetime topologies can be created by inserting t' Hooft operators with nontrivial t' Hooft flux, which exist if $\pi_1(G) = \Gamma_G$ is nontrivial. To visualize the types of processes that give nonzero "large" instanton number, we can think about the $U(1)$ case, where our intuition is aided by the fact that the integral in $\int \mathrm{ch}_2(E)$ can be interpreted as a self-intersection number of the Poincare dual of F (such an interpretation is only possible in the non-Abelian case if $E = \bigoplus_i \mathcal{L}_i$ is a direct sum of line bundles so that F is diagonal). For example, if we consider a process in which two initially separated magnetic flux loops pass through each other to form a Hopf link and then later re-separate, then the self-intersection number of $\widehat{F}/2\pi$ is 2, and we get $I = 1$.

An important comment is that the fractional part of the instanton number is due entirely to w_2, as noted in [25] (temporarily assuming that the gauge group is semisimple; $U(1)$ factors can of course give fractional contributions not associated with a torsionful class). That is, we have small instantons, which give integral contributions to the instanton number, and large instantons caused by w_2 which can be fractional, and no other types of instantons. We see this as follows: to examine different ways to construct a $G = \widetilde{G}/\Gamma_G$ bundle, we start from a trivialization over the 1-skeleton of X. This is always possible for orientable X and connected G. Then, we try to extend this bundle over the 2-skeleton. If $\pi_1(G) = \Gamma_G$ is nontrivial, this may not be possible: if the trivialization on the 1-skeleton winds by an element of $\pi_1(G)$ along the boundary of a given 2-cell, the trivialization is not extendable into that 2-cell (this is a global obstruction if the product of all such holonomies in $\pi_1(G)$ accross all 2-cells is nontrivial). 2-cells where there is an obstruction to extending the trivialization are determined by w_2. Now we try to extend the trivialization over the 3-cells. This may be obstructed by an element in $\pi_2(G)$. However, since G is a topological group, we have the magical fact that $\pi_2(G) = \mathbb{Z}_1$, and so there is no obstruction at this level. Finally we try to extend into the 4-skeleton: this is obstructed by $\pi_3(G)$. But as we have seen $\pi_3(G) = \pi_3(\widetilde{G})$ parametrizes the "small" instantons, which are the same for both G and its universal cover, and so the contribution of $\pi_3(G)$ elements to I is always integral in our normalization. Thus the only possible contribution to the fractional part of I is w_2.

Examples

Now we will compute examples for a few classes of groups of interest.

As we mentioned above, if our gauge group is simply connected it can have no "large" instantons, and we will never have fractional instanton numbers. Fractional values of I ("large instantons") can arise from two things:

- $U(1)$ factors in the gauge group corresponding to $\mathbb{Z}$ factors in $\pi_1(G)$ (this is no surprise; we know that $I \in \frac{1}{2}\mathbb{Z}$ for $U(1)$ theories on general 4-manifolds)
- G bundles that cannot be lifted to $\widetilde{G}$ bundles because of a discrete obstruction, corresponding to torsion factors in $\pi_1(G)$.

In what follows we will derive expressions for the fractional part of the instanton number I for various gauge groups, written in terms of the discrete characteristic class w_2 that describes the gauge bundle (as well as the $U(1)$ part of the field strength, if it exists). We emphasize that the instanton number is never actually coming from doing a calculation with any discrete objects—the instanton number is the (integral of the) second Chern character, which never has any torsionful contributions. However, as we will see, the choice of the characteristic class w_2 puts constraints on what the instanton number can be, and so that the fractional part of I is indeed a function of w_2. This function will always turn out to be the Pontryagin square $P(w_2)$.

We will be interested in the question of whether or not $P(w_2)$ constitutes a topological action that is linearly independent from the instanton number. Recall that for $w_2 \in H^2(X;\mathbb{Z}_n)$,[102] we have $P(w_2) \in H^4(X;\mathbb{Z}_k)$, where $k = 2n$ if n is even, and $k = n$ if n is odd. We may therefore write the topological terms

$$S_{top}/2\pi \supset \frac{p}{2n} \int P(w_2), \tag{1004}$$

where $p \in \mathbb{Z}_{2n}$ if $n \in 2\mathbb{Z}$, and $p \in 2\mathbb{Z}_{2n}$ if $n \in 2\mathbb{Z}+1$.

Suppose the fractional part of the instanton number which depends on $P(w_2)$ is $(q/2n)\int P(w_2)$, so that the part of S_{top} involving w_2 is

$$S_{top} = \theta\left(\frac{q}{2n}\int P(w_2) + \dots\right) + \frac{p}{2n}\int P(w_2), \tag{1005}$$

where $\dots$ are the other contributions to the instanton number. If we can equivalently write this as

$$S_{top} = (\theta + \delta\theta)\left(\frac{q}{2n}\int P(w_2) + \dots\right), \tag{1006}$$

where $\delta\theta \in 2\pi\mathbb{Z}$ so as not to affect the integer part of I, then the discrete $P(w_2)$ class will not be independent from the instanton number. This is important because again, despite appearances, the stuff within the ()s in the above equation comes from integrating a local density, allowing the θ angle to take on a continuum of values.[103]

[102] As elsewhere in the diary, we are being slightly imprecise as writing $w_2 \in H^2(X;\mathbb{Z}_n)$. In reality w_2 is a characteristic class $w_2 : b_G(X) \to H^2(X;\mathbb{Z}_n)$, i.e. a map from the isomorphism classes of G-bundles over X to the cohomology on X, in this case with $\mathbb{Z}_n$ coefficients. When we write something like $w_2 \in H^2(X;\mathbb{Z}_n)$, we are using w_2 as a standin for the cohomology class one gets when evaluating w_2 on the gauge bundle E, which is left implicit in the notation.

[103] The term in ()s is well defined since a shift in $P(w_2)$ by a 4-form valued with periods in $2n\mathbb{Z}$ can be compensated for by a shift in the integer-valued part of I (the part coming from small instantons).

We will be able to write S_{top} as (1006) for any p provided that q generates all of $\mathbb{Z}_{2n}$ or $2\mathbb{Z}_{2n}$, depending on the parity of n. The condition for this to happen is that

$$\gcd(q, 2n) = \begin{cases} 1 & n \in 2\mathbb{Z} \\ 2 & n \in 2\mathbb{Z}+1 \end{cases}. \tag{1007}$$

When this condition is satisfied, there is no independent torsionful characteristic class that we can add to the topological action, and hence the whole topological action will appear with a continuously tunable coefficient θ.

$SU(N)$ and $PSU(N)$

We will now specialize to the case where $G = PSU(N)$. The degree to which a given $PSU(N)$ bundle E does not lift to an $SU(N)$ bundle is determined by a class

$$w_2(E) \in H^2(X; \mathbb{Z}_N), \tag{1008}$$

where X is spacetime. We can construct E by taking an $SU(2)$ bundle $\widetilde{E}$ and relaxing the cocycle condition on the transition functions to only hold modulo an Nth root of unity: $[g_{ij}g_{jk}g_{ki}]_{ab} = \delta_{ab}e^{2\pi i f_{ijk}/N}$, where the f_{ijk} are integers. The choice of f_{ijk} determines the $w_2(E)$ class, which when integrated over a given closed 2-submanifold tells us the fractional flux passing through that manifold.

A naive first guess would be that the instanton number for $PSU(N)$ bundles can be (in our normalization) an element of $\frac{1}{N}\mathbb{Z}$. This is because if E is a $PSU(N)$ bundle, then $E^{\otimes N}$ is a bundle whose transition functions those of an $SU(N)$ bundle, since the transition functions of $E^{\otimes N}$ are N-fold $\otimes$s of the transition functions for E, which ensures that the cocycle condition holds exactly in $E^{\otimes N}$ (i.e., not just up to an Nth root of unity). The instanton number for $E^{\otimes N}$ is found from

$$\mathrm{ch}(E^{\otimes N}) = \mathrm{ch}(E)^{\wedge N} = 1 + N\mathrm{ch}_2(E) + \dots, \tag{1009}$$

where we used $\mathrm{ch}_1(E) = 0$ on account of the tracelessness of the $SU(N)$ generators, and so one then might think that $\int \mathrm{ch}_2(E^{\otimes N}) \in \mathbb{Z}$ on account of its transition functions satisfying the cocycle condition. This is not quite true however, and in fact $\int \mathrm{ch}_2(E^{\otimes N}) \in \frac{1}{2}\mathbb{Z}$, a situation which is possible due to the fact that for line bundles, $I \in \frac{1}{2}\mathbb{Z}$ on non-spin manifolds (we will never be restricting to spin manifolds).[104] We'll see how this works in a second.

Now let us see how such a fractional instanton number can be realized. In what follows we will basically be working out in gory detail a computation described in [25] for the case of a spin manifold. The goal is to explicitly construct a $PSU(N)$ bundle that will get us the minimal possible I of $1/2N$.

First, let us fix a class w_2. Let $\mathcal{L}$ be the line bundle over X with first Chern class reducing to w_2 mod N:

$$w_2 = \mathrm{c}_1(\mathcal{L}) \mod N. \tag{1010}$$

[104] A better but more mathematical way to say this would be to say that we can have instanton numbers for $E^{\otimes N}$ that are in $\frac{1}{2}\mathbb{Z}$ because of the existence of the Pontryagin square operation, which lets us consistently "divide" torsionful intersection numbers by two. See the diary entry on the Pontryagin square for details.

Here the LHS is viewed as an element in $H^2(X;\mathbb{Z})$, but we will usually use the correspondence between elements of $H^*_{dR}(X;\mathbb{R})$ with quantized periods and those in $H^*(X;\mathbb{Z})$ to think of it as an actual 2-form in the de Rham sense. From $\mathcal{L}$ we can form the bundle $\mathcal{L}^{-1/N}$, defined to have transition functions which are $1/N$th roots of the transition functions of $\mathcal{L}$. In particular, the cocycle conditions in $\mathcal{L}^{-1/N}$ are only satisfied up to Nth roots of unity. We can then construct a $PSU(N)$ bundle E as follows:

$$E = \mathcal{L}^{-1/N} \otimes \left(\mathcal{L} \oplus \bigoplus_{i=1}^{N-1} T_i \right), \tag{1011}$$

where T_i is a trivial line bundle. The $\mathcal{L}^{-1/N}$ means that E is not an $SU(N)$ bundle. However, the $\mathcal{L}^{-1/N}$ factor does not turn the thing in parenthesis from an $SU(N)$ bundle into a $PSU(N)$ bundle, since the thing in the parenthesis is not an $SU(N)$ bundle: it has nonzero first Chern character, which precludes it from being an $SU(N)$ bundle. Indeed, (vector bundles associated to) $SU(N)$ principal bundles always have zero first Chern character, simply because $\mathrm{Tr}(F) = 0$ (the Chern characters never have torsion; they are defined totally within the context of Chern-Weil theory). Note that this doesn't necessarily mean that the first Chern *class* must vanish though, since the Chern classes can have torsionful contributions.

Anyway, if E is to be a $PSU(N)$ bundle then since at the Lie algebra level $PSU(N)$ and $SU(N)$ are identical, E must also have a first Chern character which vanishes. This indeed is true, and is the reason for the choice of powers of $\mathcal{L}$ appearing in E: we first use

$$\mathrm{ch}(E) = \mathrm{ch}(\mathcal{L}^{-1/N}) \wedge \left(\mathrm{ch}(\mathcal{L}) + \sum_{i=1}^{N-1} \mathrm{ch}(T_i) \right). \tag{1012}$$

Taking the first degree component gives

$$\mathrm{ch}_1(E) = -\frac{1}{N} c_1(\mathcal{L}) \cdot N + 1 \cdot c_1(\mathcal{L}) = 0 \tag{1013}$$

as required.

The construction of building E from "fractional" line bundles makes it clear that it is a $PSU(N)$ bundle. If $\lambda_{ij} = e^{i2\pi g_{ij}}$ are the transition functions for $\mathcal{L}$, then the transition functions for E are the matrices

$$\Lambda_{ij} = \mathrm{diag}(e^{i2\pi g_{ij}(1-\frac{1}{N})}, e^{-i2\pi g_{ij}/N}, \dots, e^{-i2\pi g_{ij}/N}). \tag{1014}$$

Note that while we still have $\det(\Lambda_{ij}) = 1$, $\delta\Lambda$ is no longer trivial:

$$(\delta\Lambda)_{ijk} = e^{-2\pi i f_{ijk}/N} \mathbf{1}, \tag{1015}$$

where the $f_{ijk} \in \mathbb{Z}$ are as before determined by the class w_2. This means that the Λ_{ij} are transition functions for a $PSU(N)$ bundle, but not for an $SU(N)$ bundle[105].

Now we will compute the instanton number of E, working modulo integral classes (i.e., just focusing on the fractional part). For some reason I chose to first do this by

[105]Here it is very important that $\delta\Lambda$ is a constant Nth root of unity times $\mathbf{1}$: having different Nth roots of unity along each entry of the diagonal would be no good, since only the diagonal $\mathbb{Z}_N$ is moded out by when passing to $PSU(N)$.

computing the second Chern class of E, which gives the instanton number since the second Chern class and second Chern character are equal in this case (the calculation of the Chern character is a little later on). We use the Whitney sum formula to write

$$\mathrm{c}(E) = (1 + \mathrm{c}_1(\mathcal{L}^{-1/N}) + \mathrm{c}_1(\mathcal{L})) \wedge \bigwedge_{i=1}^{N-1} (1 + \mathrm{c}_1(\mathcal{L}^{-1/N})). \tag{1016}$$

Taking the degree-2 part, we have

$$\mathrm{c}_2(E) = \frac{N^2 - N}{2}\mathrm{c}_1(\mathcal{L}^{-1/N}) \wedge \mathrm{c}_1(\mathcal{L}^{-1/N}) + (N-1)\mathrm{c}_1(\mathcal{L}) \wedge \mathrm{c}_1(\mathcal{L}^{-1/N}). \tag{1017}$$

Now the wedge product of the chern classes is, using the Pontryagin square to take the wedge product so as to properly count the self-intersections of the w_2 surface (see the diary on the Pontryagin square for more),

$$\mathrm{c}_1(\mathcal{L}^{-1/N}) \wedge \mathrm{c}_1(\mathcal{L}^{-1/N}) = \frac{1}{N^2}P(w_2). \tag{1018}$$

Then we get

$$\mathrm{c}_2(E) = \frac{P(w_2)}{2}\left(1 - \frac{1}{N} - 2 + 2\frac{1}{N}\right) = -\frac{1}{2}\left(1 - \frac{1}{N}\right)P(w_2). \tag{1019}$$

We can also do the computation by computing $\mathrm{ch}_2(E)$, which should agree with $\mathrm{c}_2(E)$ since $\mathrm{c}_1(E) = 0$. The calculation goes as follows:

$$\mathrm{ch}(E) = \mathrm{ch}(\mathcal{L}^{-1/N}) \wedge (\mathrm{ch}(\mathcal{L}) + N - 1)\,. \tag{1020}$$

Since for line bundles $\mathrm{ch}_2(\mathcal{L}) = \frac{1}{2}\mathrm{c}_1(\mathcal{L}) \wedge \mathrm{c}_1(\mathcal{L})$ we have, working modulo terms that are integral classes,[106]

$$\begin{aligned}
\mathrm{ch}_2(E) &= N\mathrm{ch}_2(\mathcal{L}^{-1/N}) + \mathrm{ch}_2(\mathcal{L}) + \mathrm{ch}_1(\mathcal{L}^{-1/N}) \wedge \mathrm{ch}_1(\mathcal{L}) \\
&= P(w_2)\left(\frac{N}{2N^2} + \frac{1}{2} - \frac{1}{N}\right) \\
&= \frac{(N-1)}{2N}P(w_2).
\end{aligned} \tag{1021}$$

These results tell us that the instanton number is valued in $\frac{1}{N}\mathbb{Z}$ on a spin manifold, and $\frac{1}{2N}\mathbb{Z}$ on a non-spin manifold. Something that's kind of interesting here is that as mentioned above, on a non-spin manifold we can get an instanton number l such that $NI \notin \mathbb{Z}$! Again, this is interesting because from the point of view of transition functions, we might be led to expect that the fractional part of the instanton number is always valued in $\frac{1}{N}\mathbb{Z}$, given that the transition functions of an $PSU(N)$ bundle can always be made into the transition functions for an $SU(N)$ bundle by raising them to their Nth powers. This becomes a little less surprising if we consider the rather dumb (since it's just an issue of normalization) example of $U(1)/\mathbb{Z}_N$: here the instanton

[106] We write $\mathrm{c}_1(\mathcal{L}) = w_2 + N\alpha$, where α has integral periods. We can then throw away a term $\frac{N^2-N}{2}w_2 \wedge \alpha$ in the second step since it is an integral class, on account of $(N^2 - N)/2 \in \mathbb{Z}$ for all N.

number in the $U(1)/\mathbb{Z}_N$ theory is valued in $\frac{1}{2N^2}\mathbb{Z}$, even though the transition functions all only fail the cocycle conditions by Nth roots of unity. Looking at this example, we see that the reason for N copies of a $PSU(N)$ bundle not necessarily giving a $I \in \mathbb{Z}$ just boils down to the fact that I is nonlinear in the field strength and that the $PSU(N)$ bundles are built from line bundles, which can have fractional instanton numbers on non-spin manifolds.[107]

To be pedantically explicit, we can do the construction of the minimal $PSU(N)$ bundle for $SU(2)$. Let A_m be the $U(1)$ gauge field for a 2π monopole whose worldline wraps some nontrivial cycle in spacetime, and let

$$A_{SO(3)} = \begin{pmatrix} A_m/2 & 0 \\ 0 & -A_m/2 \end{pmatrix}. \tag{1022}$$

Note that this has zero first Chern character as required,[108] and that it is constructed as a $\oplus$ of "fractional" line bundles: the total $SO(3)$ bundle is $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$, where $\mathcal{L}$ is a legit $U(1)$ line bundle. The instanton number for this field configuration is

$$I = \frac{1}{16\pi^2} \int F_{A_m} \wedge F_{A_m} \in \frac{1}{4}\mathbb{Z} \tag{1023}$$

on general manifolds, while it is in $\frac{1}{2}\mathbb{Z}$ on spin manifolds. This is in keeping with us being able to get strength $1/2N$ instantons for $PSU(N)$ gauge theories on a generic manifold, and strength $1/N$ instantons on spin manifolds. Note how this instanton picks out a particular direction in $SU(2)$ space, namely the σ^z direction, unlike the small instantons which wrap about the entire internal $SU(2)$ space.

A final way to see all of this is to write the $PSU(N)$ field in terms of a $U(N)$ field and a $\mathbb{Z}_N$ 2-form gauge field, as was done in the previous diary entry. Using the same notation as in that diary entry, we have

$$I = \frac{1}{8\pi^2} \int \mathrm{Tr}[(F_{\mathcal{A}} - B\mathbf{1}) \wedge (F_{\mathcal{A}} - B\mathbf{1})], \tag{1024}$$

where $\mathcal{A} = A_{SU(N)} + \mathcal{A}\mathbf{1}/N$ is a $U(N)$ gauge field, and $\mathcal{A}$ is a properly quantized $U(1)$ gauge field, with $F_{\mathcal{A}} = NB$ enforced through a Lagrange multiplier constraint. Then

$$I = \mathrm{c}_2(E_{U(N)}) + \frac{i}{8\pi^2} \int (F_{\mathcal{A}} \wedge F_{\mathcal{A}} + NB \wedge B - 2F \wedge B) \to \mathrm{c}_2(E_{U(N)}) + \frac{i}{8\pi^2}(N^2 - N) \int B \wedge B, \tag{1025}$$

[107]It's also kind of interesting that we can get the minimal instanton number by working with a direct sum of line bundles regardless of the spin-ness of the background manifold, since for $SU(N)$ bundles this isn't the case: consider e.g. an $SU(2)$ bundle $E = \mathcal{L} \oplus \mathcal{L}^*$. Then $\mathrm{ch}_2(E) = 2\mathrm{ch}_2(\mathcal{L})$, which on a spin manifold is in $2\mathbb{Z}$, twice the minimum allowed value. This tells us that on a spin manifold the minimal instantons for $SU(N)$ are the ones that involve twisting around more than just two axes (and hence cannot be composed into a $\oplus$ of line bundles). This is then to be contrasted with the $PSU(N)$ case, where bundles formed from direct sums of line bundles can always give the minimal instanton number.

[108]However, its first Chern class is a nontrivial element in $\mathbb{Z}_2$ cohomology—the second SW class of the $SO(3)$ bundle—which is torsionful (recall that the second SW class for a complex vector bundle is the mod-2 reduction of that bundle's first Chern class).

where we used the Lagrange multiplier constraint. The first term, the second Chern class of the $U(N)$ bundle, is in $\mathbb{Z}$. However, the second term is in $\frac{1}{2N}\mathbb{Z}$, since $B/2\pi$ has periods in $1/N$. Thus the instanton number for $PSU(N)$ theories is valued in $\frac{1}{2N}\mathbb{Z}$. B here is the 2-form that measures w_2 of the bundle, and so as above, we see that the fractional part of the instanton number comes from the "large" instantons (the small ones are determined by the $\mathrm{c}_2(E_{U(N)})$ factor).

Finally we ask whether the instanton number I is linearly independent from the discrete torsionful term $p/2n \int P(w_2)$. To answer this we calculate

$$\gcd(N-1, 2N) = \gcd(N-1, 2), \tag{1026}$$

which according to (1007) means that in this case, the discrete class $P(w_2)$ not independent from the instanton number for any N.

$SU(N)/\mathbb{Z}_M$

The computation for this case is similar to the computation for $PSU(N)$, but it's not in the literature so it's worth doing (we will be very brief, though). We can assume that $M|N$ with $1 < M < N$ wolog. We form an $SU(N)/\mathbb{Z}_M$ bundle by

$$E_{SU(N)/\mathbb{Z}_M} = \mathcal{L}^{(1-N)/M} \oplus (\mathcal{L}^{1/M})^{\oplus(N-1)}, \tag{1027}$$

which has zero first Chern character and has transition functions which fail the cocycle condition by powers of $e^{2\pi i/M}\mathbf{1}_N$, as required. The second Chern character is, after a little algebra,

$$\mathrm{ch}_2[E_{SU(N)/\mathbb{Z}_M}] = \frac{N(N-1)}{2M^2}\mathrm{c}_1(\mathcal{L})^{\wedge 2}, \tag{1028}$$

where $\mathrm{c}_1(\mathcal{L})$ mod M reduces to w_2 of the bundle, which is a $\mathbb{Z}_M$ valued form because $\pi_1[SU(N)/\mathbb{Z}_M] = \mathbb{Z}_M$. This result is pretty easy to understand from the $PSU(N)$ case: the Ns in the denominator come from the number of line bundles in the direct sum (which of course doesn't change when we change the group we're quotienting by), while the N^2 in the denominator changes to M^2 because the extent to which the constituent line bundles are allowed to be fractional changes when we change the quotient group to $\mathbb{Z}_M$. Therefore the instanton number is

$$I_{E_{SU(N)/\mathbb{Z}_M}} = \frac{2\pi N(N-1)}{2M^2}\int P(w_2) + \dots, \tag{1029}$$

where $\dots$ are the integer parts.

The condition for the discrete class $P(w_2)$ to not be independent from the instanton number is

$$\gcd(N(N-1)/M, 2M) = \begin{cases} 1 & M \in 2\mathbb{Z} \\ 2 & M \in 2\mathbb{Z}+1 \end{cases}. \tag{1030}$$

$U(N)/\mathbb{Z}_M$

In another diary entry, we show that

$$\pi_1[U(N)/\mathbb{Z}_M] = \mathbb{Z} \times \mathbb{Z}_g, \qquad g \equiv \gcd(N, M). \tag{1031}$$

Therefore $Q \equiv U(N)/\mathbb{Z}_M$ bundles will be labeled by a $\mathbb{Z}_g$ characteristic class w_2, along with the regular $\mathbb{Z}$-valued class for the "small" instantons (and, since we have a $U(1)$ factor, a fractional contribution from large instantons in the $U(1)$ part of the group).

We now need to ask how we can form Q bundles which are not $U(N)$ bundles. We claim that we can always obtain the minimal instanton number with a bundle

$$E_Q(r,q) = \mathcal{L}_d^{r/M} \otimes \begin{pmatrix} \mathcal{L}_t^{q(1-N)/g} & & & \\ & \mathcal{L}_t^{q/g} & & \\ & & \ddots & \\ & & & \mathcal{L}_t^{q/g} \end{pmatrix}, \qquad r,q \in \mathbb{Z}. \tag{1032}$$

Here the line bundle $\mathcal{L}_d$ (d for "diagonal") keeps track of the fractional $U(1)$ part of the instanton number, with the transition functions in $\mathcal{L}_d^{r/M}$ failing by powers of $\zeta_M^r \mathbf{1}_N$, where $\zeta_M = e^{2\pi i/M}$. The Chern character of $\mathcal{L}_d$ does *not* reduce to anything relating to the torsionful class w_2 when modded out by some integer: the element of $\pi_1[Q]$ defined by a triple patch overlap where the transition functions of $\mathcal{L}_d^{r/M} \otimes \mathbf{1}_N$ fail the cocycle condition by $\zeta_M^r \mathbf{1}_N$ is an element of the $\mathbb{Z}$ factor in $\pi_1[Q]$ (coming from topologically nontrivial maps $\det : U(N) \to S^1$), and hence is unrelated to torsionful w_2 class.

The dependence of the instanton number on the w_2 of the bundle $E_Q(r,s)$ is instead determined by the term involving $\mathcal{L}_t$ (t for "torsion"). $\mathcal{L}_t^{q/g}$ is a line bundle that fails the cocycle conditon by $\zeta_g^q \mathbf{1}_N$ on each triple overlap; this is allowed since ζ_g is a power of ζ_M. The structure of the direct sum of the $\mathcal{L}_t$s is such that around a triple patch overlap where the cocycle condition fails, one traces out a loop in the finite $\mathbb{Z}_g$ factor of $\pi_1[Q]$ (see the diary entry on $\pi_1[U(N)/\mathbb{Z}_M]$ for details). This means that the Chern class of $\mathcal{L}_t$ can be taken to reduce to w_2 mod g. Note that the factor involving the $\mathcal{L}_t$s gives us a minimal $SU(N)/\mathbb{Z}_g$ bundle when $q = 1$, which is what we expect from general considerations of how the quotient in Q acts on the $U(1)$ and $SU(N)$ factors.

Now we should actually compute the instanton number I. The total Chern character is

$$\mathrm{ch}(E_Q(r,q)) = (N-1)\mathrm{ch}(\mathcal{L}_d^{r/M} \otimes \mathcal{L}_t^{q/g}) + \mathrm{ch}(\mathcal{L}_d^{r/M} \otimes \mathcal{L}_t^{q(1-N)/g}). \tag{1033}$$

Taking the degree 2 part and simplifying modulo integer terms, we get

$$I = \int \left(\frac{Nr^2}{2M^2} \mathrm{c}_1(\mathcal{L}_d) \wedge \mathrm{c}_1(\mathcal{L}_d) + \frac{pq^2}{2g} P(w_2) \right) + \dots, \qquad p \equiv \frac{N(N-1)}{g} \in \mathbb{Z}, \tag{1034}$$

again with $\dots$ representing integer contributions. Note the absence of mixed terms between the $\mathcal{L}_d$ and $\mathcal{L}_t$ factors; this is because the bundle associated with the $\mathcal{L}_t$ factors has zero first Chern class. Also note that the term $P(w_2)$ is well-defined mod $2\pi\mathbb{Z}$: its well-definedness mod $2\pi\mathbb{Z}$ for arbitrary q depends on $pg \in 2\mathbb{Z}$, but this is always the case since $pg = N(N-1) \in 2\mathbb{Z}$. The minimal instanton number is then found by choosing either r or q to be zero and the other to be 1, depending on the choices of M and N.

From the above expression, we see that the condition for the instanton number to reproduce all possible discrete $P(w_2)$ terms is that

$$\gcd\left(\frac{N(N-1)}{g}, 2g \right) = \begin{cases} 1 & g \in 2\mathbb{Z} \\ 2 & g \in 2\mathbb{Z}+1 \end{cases}. \tag{1035}$$

Actually there's a small subtlety here: to "absorb" a possible discrete term into the instanton number, we have to shift θ by something in $2\pi\mathbb{Z}$. In the previous examples this hasn't been a problem, since the part of I that can be written in terms of $P(w_2)$ is the only contribution to I mod 1. However in the $U(1)$ case, we also have a fractional part of I which comes from the $U(1)$ part: the term $\text{Tr}[F] \wedge \text{Tr}[F] \propto \text{c}_1(E)^{\wedge 2}$ appears in I, and is fractional in general. Therefore the shift of θ to cancel the discrete $P(w_2)$ term is not as innocuous as it seems, since the phase contributed by this fractional $U(1)$ part will change.

However, in the $U(N)$ case we should really be considering a more general topological action with two distinct θ angles, since in the $U(N)$ case the second Chern class and second Chern character give distinct topological terms:

$$S_{top} = \theta_1 \int \text{ch}_2[E] + \theta_2 \int \frac{1}{2}\text{ch}_1[E] \wedge \text{ch}_1[E] + \frac{2\pi p}{2g} \int P(w_2), \tag{1036}$$

where again, $\text{ch}_1[E] \wedge \text{ch}_1[E]$ is just a pretentious way of writing $\text{Tr}[F] \wedge \text{Tr}[F]/4\pi^2$. The precise statement to make is then that if (1035) is satisfied, then the torsionful $P(w_2)$ term can be absorbed into the continuous θ terms by adjusting both θ_1 *and* θ_2. Just for fun, the condition that the discrete $P(w_2)$ term can be absorbed into the continuous theta term is shown as a function of N and M in figure 5.

$Sp(N)$ and $PSp(N)$

First, let's disambiguate the notation: here, by $Sp(N)$, we mean the *compact* group

$$Sp(N) \equiv U(2N) \cap Sp(2N; \mathbb{C}), \tag{1037}$$

where $Sp(2N; \mathbb{C})$ is the *non-compact* group of complex $2N$-$2N$ matrices that preserve $J \otimes \mathbf{1}_{N\times N}$, in the sense that

$$U \in Sp(N) \implies U^\dagger U = 1, \qquad U^T J U = J \otimes \mathbf{1}_{N\times N}, \tag{1038}$$

where

$$J \equiv (-iY) \otimes \mathbf{1}_{N\times N}, \qquad J^2 = -\mathbf{1}_{2N\times 2N} \tag{1039}$$

is our choice of symplectic form. The Lie algebra for the compact symplectic group [109] can be obtained by writing a general Lie algebra element T as a linear combination

$$\mathfrak{sp}(N) \ni T = i\mathbf{1} \otimes A + X \otimes B_1 + Y \otimes B_2 + Z \otimes B_3, \tag{1040}$$

where A is traceless and antisymmetric, and the B_i's are symmetric. Both A and the B_i's are real (they have to be (anti-)Hermitian in order for $e^{i\alpha T}$ to be unitary, and they have to be (anti-)symmetric in order for $e^{i\alpha T}$ to preserve the symplectic form). In this presentation we see clearly how $\mathfrak{su}(2)$ is embedded in $\mathfrak{sp}(N)$, viz. as the first factors in the $\otimes$. Additionally, we see that $Sp(1) = SU(2)$ and the center of $Sp(N)$ is $\mathbb{Z}_2$, as can be easily checked by looking for diagonal things that preserve $J \otimes \mathbf{1}$.

[109] Why's it called symplectic? Since it preserves iY, which is the antisymmetric form used in the commutation relations for the symplectic form on phase space: if $v = (x,p)^T$, then $v^T J v = i$ is the CCR, and we can send $v \mapsto Rv$ for any $R \in Sp(N)$ preserving the CCR.

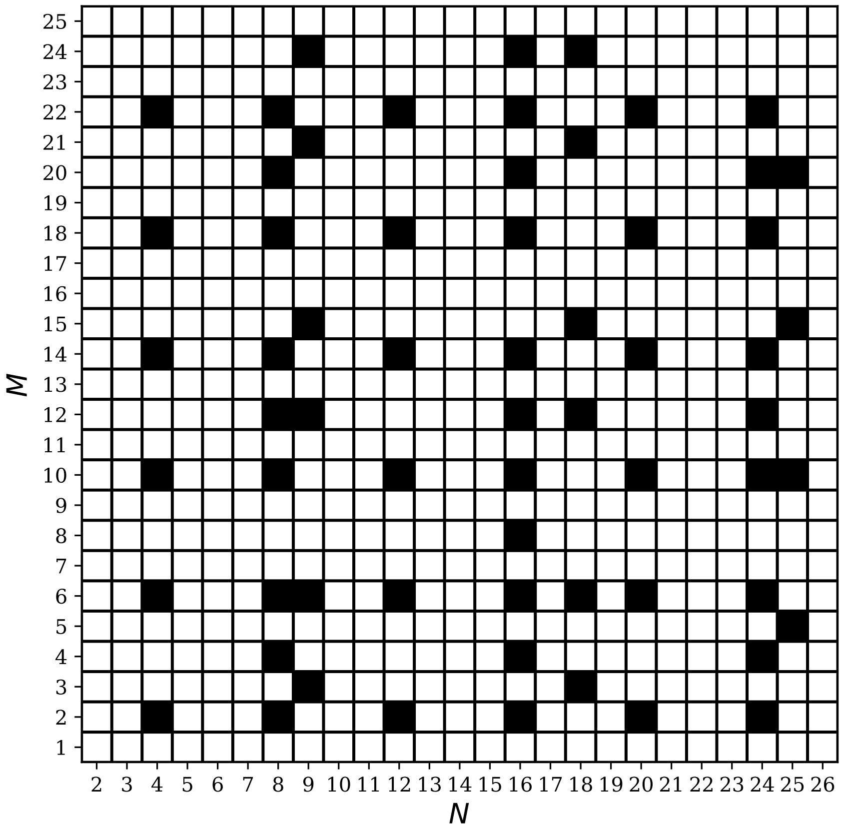

Figure 5: When the instanton number and discrete theta angle are independent for $U(N)/\mathbb{Z}_M$. A black filling means that they are independent. Going to larger values of N and M gives you something that looks like its nearly periodic, but actually isn't.

To get the normalization for the instanton number straight, we need to look at how $SU(2)$ embeds into $Sp(N)$. First, note that there can only be a single full $SU(2)$ factor in $Sp(N)$, since $Z(Sp(N)) = \mathbb{Z}_2 = Z(SU(2))$ means that we can't have multiple copies without having a quotient by their centers as well. We can also find such a full $SU(2)$ just by looking at matrices of the form $U \otimes \mathbf{1}_{N\times N}$, where $U \in SU(2)$. These are obviously unitary, and a quick check shows that they are also in $Sp(2N;\mathbb{C})$. Furthermore setting $U = -\mathbf{1}_{2\times 2}$ gives the center of $Sp(N)$, so we know that $Sp(N)$ really does have a full $SU(2)$ inside of it (i.e., the $SU(2)$ doesn't appear in a form where it's quotiented by $\mathbb{Z}_2$ in some way). Thus minimal $SU(2)$ instantons have instanton number 1 in $Sp(N)$, which we might write as

$$I_{Sp(N)} = \int p_1(Sp(N)). \tag{1041}$$

Explicitly, we can write the gauge field $A^{Sp(N)}$ for a minimal instanton in terms of the $SU(2)$ minimal instanton gauge field $A_{SU(2)}$ as e.g.

$$A_\mu^{Sp(N)} = A_\mu^{SU(2)} \otimes E_{11}, \tag{1042}$$

where E_{11} is the matrix with a 1 in the upper leftmost entry, and zeros everywhere else. Since E_{11} is symmetric, $A_\mu^{Sp(N)}$ is indeed in the Lie algebra $\mathfrak{sp}(N)$. Writing it like this, it's clear that $I_{Sp(N)} = p_1(Sp(N))$.

Now for the quotient groups $PSp(N) = Sp(N)/\mathbb{Z}_2$. How might we obtain a $PSp(N)$ bundle that's not an $Sp(N)$ bundle? We consider the bundle $E_{SO(3)} = \mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$, which is an $SO(3)$ bundle that does not lift to an $SU(2)$ bundle. Similarly to as in our discussion of $SU(N)$, $\mathcal{L}$ is the line bundle whose first Chern class reduces mod 2 to some class $w_2 \in H^2(X;\mathbb{Z}_2)$. Since roughly the transition functions fail the cocycle condition on triple overlaps by the value of w_2 on the triple overlap, the cocycle conditions of $\mathcal{L}^{1/2}$ fail by an amount controlled by $w_2/2$. We then use the diagonal embedding $SU(2) \to Sp(N)$ to use $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$ to create a $PSp(N)$ bundle that doesn't lift to an $Sp(N)$ bundle. Since the diagonal $SU(2) \to Sp(N)$ embedding sends

$$SU(2) \ni U \mapsto U \otimes \mathbf{1}_{N\times N} \in Sp(N), \tag{1043}$$

the $PSp(N)$ bundle we get is a direct sum of N copies of $E_{SO(3)}$[110]:

$$E_{PSp(N)} = E_{SO(3)}^{\oplus N} = (\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2})^{\oplus N}. \tag{1044}$$

Using manipulations like the ones used for looking at $PSU(N)$ bundles, we see that the instanton number mod 1 (i.e. the part of the instanton number that doesn't come from small instantons) is

$$[\mathrm{ch}_2(E_{PSp(N)})]_1 = \frac{N}{2}[\mathrm{ch}_2(\mathcal{L})]_2 = \frac{N}{2}\frac{P(w_2)}{2}, \tag{1045}$$

where $P(w_2)$ is again the Pontryagin square (a $\mathbb{Z}_4$ class, since it's acting on $\mathbb{Z}_2$ cochains), and where $[]_k$ denotes the mod k reduction. Since $P(w_2)/2$ is an integer

[110] This is more obvious if we write the embedding as $\mathbf{1}_{N\times N} \otimes U$, and change our definition so that the elements in $Sp(N)$ preserve $\mathbf{1}_{N\times N} \otimes J$.

class on a spin manifold by the even-ness of the intersection form, on spin manifolds we can have fractional instantons for $PSp(N)$ if N is odd, but not if N is even. From this we see that the discrete class $P(w_2)$ is independent from I provided that $N \in 2\mathbb{Z}+1$.

Before moving on, let's just clarify why we needed to choose the diagonal embedding of $SU(2)$ into $Sp(N)$, instead of e.g. the embedding $U \mapsto E_{11} \otimes U$ used to compute the normalization of $I_{Sp(N)}$ (I'm writing the tensor product in the opposite order since I find it slightly easier to visualize). Indeed, suppose we chose this embedding for the $SO(3)$ bundle. Then we would end up with a bundle whose transition functions could fail the cocycle condition by the matrix $-\mathbf{1}_{2\times 2} \oplus \mathbf{1}_{2N-2\times 2N-2}$. In a $PSp(N)$ bundle, the transition functions are only allowed to fail the cocycle condition by the matrix $-\mathbf{1}_{2N\times 2N}$, since this is the thing that gets quotiented out by upon passing to $PSp(N)$. In contrast, if we choose the diagonal embedding $U \mapsto \mathbf{1} \otimes U$, then we get a bundle whose transition functions fail the cocycle condition by $-\mathbf{1}_{2N\times 2N}$, which is what we want. Thus, we must choose the diagonal embedding.

$SO(N)$

We will now briefly look at the normalization of the instanton number for $SO(N)$. Some time in the future I may come back and discuss $\text{Spin}(N)$ and quotients of $SO(N)$.

First for $SO(3)$, which we've already mentioned above. To find the normalization, we compute the value that a minimal $SU(2)$ instanton has when lifted to the adjoint representation. This is easy: we can take the same $U^\dagger dU$ with $U \sim e^{ir^a T^a}$ type of instanton, we just have to change the T^a's. Now for $SU(N)$ we have $\text{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$, while for $SO(3)$ we have $\text{Tr}[T^a T^b] = 2\delta^{ab}$. So $\int p_1(E_{SO(3)})$ for an $SO(3)$ bundle with a minimal $SU(2)$ instanton is $4\int p_1(E_{SU(2)})$. Thus for $SO(3)$,

$$I_{SO(3)} = \frac{1}{4}\int p_1(SO(3)). \tag{1046}$$

The notation $p_1(SO(3))$ has the hopefully obvious meaning "$p_1(E)$ for some $SO(3)$ bundle E". Note that this conclusion was reached for an arbitrary manifold, spin or not spin. If we restrict ourselves to spin manifolds, the Pontryagin class is even, so that $I_{SO(3)} \in \frac{1}{2}\mathbb{Z}$ on spin manifolds. This can be proved decomposing the $SO(3)$ bundle as $\mathcal{L}^{1/2} \oplus \mathcal{L}^{-1/2}$ and realizing that the second chern character depends on the even-ness of the intersection form, or by using the relation

$$p_1(E) \mod 2 = P(w_2), \tag{1047}$$

where P is the Pointryagin square. Again we see an example of the fact that the quantization of the instanton number for simply connected Lie groups doesn't depend on whether the base manifold is spin (since there the instanton number is also the Chern class, which is integral on any manifold), but that for quotients of simply connected Lie groups, the quantiation of the instanton number does depend on whether the base manifold is spin.

Now for $SO(N \geq 4)$. We use

$$SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2, \tag{1048}$$

where the quotient is the diagonal $\mathbb{Z}_2$ (some people write this with $\otimes$ instead of $\times$, which I don't like: the tensor unit is $\mathbb{C}$, which means that we would already be making the $/\mathbb{Z}_2$ identification!). Note that since $Z(SU(2)\times SU(2)) = \mathbb{Z}_2^2$, taking the quotient leaves behind a factor of $\mathbb{Z}_2$ in the center, which is just right to match with $Z(SO(4)) = \mathbb{Z}_2$.

Regular (non-fractional) instantons are created in $SO(N>3)$ through embedding a minimal $SU(2)$ instanton into one of the $SU(2)$ factors in the decomposition for the subgroup $SO(4)\subset SO(N)$. Now, we can form fractional instantons in $SO(N)$ by embedding an $SO(3)$ instanton inside of $SO(N)$. The way this embedding works is also through the $SO(4)$ subgroup (note to self: can we show there are no other ways to do the embedding?), but it is the embedding into the diagonal subgroup of $[SU(2)\times SU(2)]/\mathbb{Z}_2$. The reason that the embedding must be done though the diagonal subgroup is because $SO(3)$ has trivial center, and so we need to embed $SO(3)$ in the diagonal subgroup so that the quotient by $\mathbb{Z}_2$ gives us something without a $-\mathbf{1}$ central element. Anyway, the point of this is that the minimal fractional instanton number in $SO(N)$ will be *twice* that in $SO(3)$, since both $SU(2)$ factors contribute. So

$$I_{SO(N)} = \frac{1}{2}\int p_1(SO(N)), \qquad N \geq 4. \tag{1049}$$

Again, this holds over arbitrary manifolds, be they spin or not spin. If the manifold is spin, we can conclude that $I_{SO(N)} \in \mathbb{Z}$ since in that case $p_1(SO(N))$ is an even class, as discussed earlier.

Dyon spin, statistics, and statistical transmutation from θ angles

Today's problem statement is as follows: consider $U(1)$ gauge theory in four dimensions. Explain why, if the 2π monopoles are bosonic at $\theta = 0$, they become fermionic when $\theta = 2\pi$. What if the theory is coupled to fermionic matter, so that the $q = 1$ electric charges are fermions? Derive the spin and statistics for the dyons in the charge lattice for all values of θ, and explain why the spectrum depends on the value of θ, even though $\theta\int F\wedge F$ is a topological term which doesn't contain the metric and hence doesn't contribute to $T_{\mu\nu}$. Why does the induced electric charge created by the θ term not contribute to the statistics of the monopoles? Also, what is the periodicity of θ on (non)-spin manifolds?

Now consider a discrete analogue, namely $\mathbb{Z}_N$ BF theory. What is the analog of the θ term, and how does it affect the statistics of the line / surface operators in the theory?

We start with yet another derivation of the Witten effect, prompted by reading an old paper by Wilczek that seemed cool but which made absolutely no sense to me when I read it. Consider a single monopole in spatial $\mathbb{R}^3$ of unit magnetic charge, with the magnetic field being set up by a vector potential A_0. Consider a change of the gauge field A which takes place over a time Δt:

$$A(t) = A_0 + \left[\frac{t-t_0}{\Delta t}\Theta(t-t_0)\Theta(t_0+\Delta t - t) + \Theta(t-(t_0+\Delta t))\right](0,\, \alpha f'(r),\, 0,\, 0), \tag{1050}$$

where we are in (t, r, θ, ϕ) coordinates. Here α is any real number and $f(r)$ is a smoothened step function with $f(\infty) = 1, f(0) = 0$. Since $A(t) - A_0 = 0$ for r near the origin, adding this change to the gauge field can be thought of as a change only of the gauge field on the coordinate patch that does not encompass the origin—thus it is well-defined even though the full $A(t)$ must be constructed by gluing patches. This $A(t)$ leads to a radial electric field

$$E_r(t) = \frac{\alpha f'(r)}{\Delta t}\Theta(t-t_0)\Theta(t_0+\Delta t - t). \tag{1051}$$

This electric field gives zero contribution to the energy if we take $\Delta t \to \infty$, since $\int d^3r\, dt\, E_r(t)^2 \propto \frac{1}{\Delta t} \to 0$. However, it does contribute to the θ term:

$$\frac{\theta}{8\pi^2}\int F\wedge F = \frac{\theta}{8\pi^2}\int \frac{1}{4}\cdot 4\cdot 2B_iE^i = \frac{\theta\alpha}{4\pi^2}4\pi\int_0^\infty dr\, r^2 B_r f'(r) = \frac{\alpha\theta\Phi_B}{4\pi^2}, \tag{1052}$$

where in the last step we integrated by parts and used $f(0) = 0$ to kill the $\nabla\cdot B$ term. Here the magnetic flux Φ_B is measured in units where a unit monopole has 2π flux. Thus the θ term contributes a phase to the path integral.

What is the physical interpretation of this phase? The final and initial gauge configurations differ by

$$A(t > t_0+\Delta t) - A(t < t_0) = \alpha\partial_r f. \tag{1053}$$

This is a function that goes to the constant α at spatial infinity. Thus it is a "large" gauge transformation, better called an asymptotic symmetry, which rotates the boundary conditions of the sections of the $U(1)$ bundle in question by a phase $e^{i\alpha}$ (if $\alpha = 2\pi$ then this is a legit gauge transformation). The electric charge of a system is defined as the representation that the system transforms under when acted on by asymptotic symmetries like this, and so we identify the relative phase between these two configurations (the phase picked up by the θ term in the path integral) with $e^{i\alpha q}$, where q is the electric charge of the monopole. Thus since this holds for all α, we have

$$e^{i\theta\Phi_B\alpha/2\pi} = e^{i\alpha q}, \implies q = \frac{\theta\Phi_B}{4\pi^2}. \tag{1054}$$

In particular, a unit monopole comes attached with an electric charge of $\theta/2\pi$.

Since for any current j such that $\langle e^{i\int A\wedge\star j}\rangle \neq 0$ we have that $\star j \in d\Omega^1(X\ \mathbb{R})$[111], we can invert the d and write $d^{-1}\star j = D$ for some $D \in \Omega^2(X;\mathbb{R})$. In the case where

[111] This is just because there does not exist a solution to the classical eom otherwise: $d\star F = \star j$

$\partial X \neq 0$, D may have support on ∂X. D is basically just the worldsheet swept out by the electric fluxes that link the two points created when we take the intersection of the Poincare dual of $\star j$ with any constant time slice.

Let's remind ourselves of why dyons can have fermionic statistics, thinking classically without a θ term. The angular momentum for a configuration with electromagnetic fields E, B is (I derived this by finding $T_{\mu\nu} \sim g_{\mu\nu} F \wedge \star F - F_{\mu\sigma} F^{\sigma}{}_{\nu}$ and taking T_{0i} to get the momentum, but surely there's a better way)

$$L_i = \int d^3 r \epsilon_{ijk} r^j \epsilon^{klm} E^l B^m. \tag{1056}$$

Now let B^i be a monopole of strength g at the origin and E^i be sourced by an electric point charge q at position r_0^i. Using the triple product $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, we get

$$L_i = \frac{1}{4\pi} \int d^3 r \frac{r_{0,i} r^2 - r_i (r_j r_0^j)}{r^3 |r - r_0|^3}. \tag{1057}$$

Now we get crafty and use

$$\partial_i \widehat{r}^j = \frac{\delta_{ij}}{|r|} - \frac{r_j r_i}{|r|^3}. \tag{1058}$$

Using this we can re-write the integrand above as $E^j \partial_j \widehat{r}_i$, and then integrate by parts. The boundary term is like $\int d^2 \Omega_{S^2_\infty} \widehat{r}^j \to 0$ since we are integrating over all angles.

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tells us that $\star j$ is exact, since $\star F$ is always a globally well-defined 2-form. Define the Poincare dual $\widehat{\mathcal{J}}$ of $\mathcal{J} = \star j$. Then

$$\star j \in d\Omega^3(X; \mathbb{R}) \implies \widehat{\mathcal{J}} \in B_1(X, \partial X; \mathbb{Z}). \tag{1055}$$

Here $B_1(X, \partial X; \mathbb{Z})$ are the 1-chains which are relative boundaries, i.e. for each element $C \in B_1(X, \partial X; \mathbb{Z})$ there is a 2-manifold M such that ∂M consists of C and 1-submanifolds of ∂X. This follows from Poincare duality applied to relative cohomology groups (we are being a bit caviler about switching between $\mathbb{R}$ and $\mathbb{Z}$ coefficients: we usually want to think of j as being in $C^1(X; \mathbb{Z})$, but we usually want to think of F as being in $\mathbb{R}$-valued dR cohomology, so we are sloppily mixing the two).

What sorts of current loops $\mathcal{J}$ are allowed by this condition? All contractible current loops are of course allowed. Non-contractible loops are only allowed on non-compact manifolds: this is because on non-compact manifolds we can have non-contractible loops that are in $B_1(X, \partial X; \mathbb{Z})$ (think of the loop on a cylinder). This is just another way of saying that we can't have a single charge on a compact manifold (think of a current line wrapping a temporal circle), since the flux can't be well-defined everywhere. On a non-compact manifold, we can have a single charge since flux can end on the boundary: thus if we are on a non-compact manifold we can choose $\mathcal{J}$ to be a non-contractible element of $B_1(X, \partial X; \mathbb{Z})$, and this implies that $d \star F$ must be trivial in $H^3(X; \mathbb{R})$ and must be non-trivial in $H^3(X, \partial X; \mathbb{R})$ (since in this case $(d\star F)|_{\partial X} = 0$ but $(\star F)|_{\partial X} \neq 0$; flux lines are ending on the boundary). A corollary of this / another way of saying the same thing is that $\langle e^{i \oint_C A} \rangle = 0$ if the curve C is nontrivial in $H_1(X, \partial X; \mathbb{Z})$: the only Wilson loops that can have vevs are ones integrated around curves which are relative boundaries (otherwise there is a trivial shift in integration variables [one which doesn't affect the boundary conditions] which the Wilson line transforms nontrivially under). A quick comment on a common current confusion: if we add background matter through $A \wedge \star j$, it seems like the 1-form symmetry on A is unbroken, since $\star j = dD$ means $\delta(A \wedge \star j) = \lambda \wedge dD$ for a flat 1-form λ, which seems to vanish upon integration by parts. The key is that for any allowed $\star j = dD$, we have $D|_{\partial X} \neq 0$, and since λ is flat we also have $\lambda|_{\partial X} \neq 0$ (provided $\lambda \notin H^1(X, \partial X; \mathbb{R})$ in which case no Wilson lines can have vevs under the symmetry coming from shifting A by λ). Thus an integration by parts actually gives $\int_{\partial X} \lambda \wedge D \neq 0$, and so the coupling to the current indeed breaks the symmetry.

This gives an integral over $(\nabla \cdot E)\widehat{r}^j$. Then using $\nabla \cdot E = 4\pi\delta(r - r_0)$, we have

$$L_i = -qg\widehat{r}_i. \tag{1059}$$

Note that L_i is pointed along the vector from the charge to the monopole, and that $|L|$ is *independent* of the distance between the charge and the monopole.

This electromagnetic angular momentum needs to be added on to the usual angular momentum of charged particles in order to get a conserved quantity. Using $\partial_t\widehat{r} = |r|^{-3}(\dot{r} - r(r \cdot \dot{r}))$, one can check that the conserved angular momentum for a particle of charge q moving in a monopole field of strength g is now ($m = 1$ units)

$$L_i = \epsilon_{ijk}r^j\dot{r}^k - eg\widehat{r}_i. \tag{1060}$$

If these angular momentum generators are correct, they need to satisfy the correct commutation relations. When we quantize, we write the angular momentum generators as

$$L_i = \epsilon_{ijk}r^j\pi^k - eg\widehat{r}_i, \tag{1061}$$

where $\pi^k = p^k - eA^k$ is the kinetic momentum and $p^k \leftrightarrow -i\partial^k$ is the canonical momentum. Since π^k parallel transports in the presence of A, its commutator picks up the field strength:

$$[\pi_i, \pi_j] = ieF_{ij}. \tag{1062}$$

One can then check, with the help of the identity $-\epsilon^{inl}\epsilon^{ilm} + \epsilon^{ilm}\epsilon^{jnl} = \epsilon^{ijk}\epsilon^{knm}$, that

$$[\epsilon_{ijk}r^j\pi^k, \epsilon_{joq}r^o\pi^q] = i\epsilon_{ijl}\epsilon_{lmn}r^m\pi^n + ieF_{ij}. \tag{1063}$$

The first term on the RHS is what we want if we want the usual $SU(2)$ relations for angular momentum, while the (uniform by assumption) F_{ij} term is some vaguely central-extensiony thing that screws up the commutation relations. So if our angular momentum generators are correct, the commutators with the $-eg\widehat{r}$ term need to cancel the field strength term above. This is indeed what happens: we compute

$$[\epsilon^{ilm}r_l\pi_m, -eg\widehat{r}^j] = ir_leg\epsilon^{ilm}\frac{1}{|r^3|}(\delta_{jm}r^2 - r_jr_m). \tag{1064}$$

When we subtract the $i \leftrightarrow j$ counterpart to get the full expression in the commutator, we find that these commutators contribute a total of $-2i\epsilon^{ijk}eg\widehat{r}_k$. One of these goes into the definition of L_k, and so

$$[L_i, L_j] = i\epsilon_{ijk}L^k + ieF_{ij} - ieg\epsilon_{ijk}\widehat{r}^k. \tag{1065}$$

We've defined g so that $\int F = 4\pi g$ on a small sphere surrounding the monopole, and so F is given by $F_{ij} = \epsilon_{ijk}g\widehat{r}^k$. Thus the two extra terms precisely cancel each other, and we recover the correct $SU(2)$ commutation relations.

Anyway, let's go back and look at the extra $-eg\widehat{r}$ term added on to L_i. In these conventions the quantization condition on the magnetic charge is $4\pi ge \in 2\pi\mathbb{Z}$, where e is the minimal electric charge. Thus $g \in \frac{1}{2}\mathbb{Z}$. Taking $g = 1/2$ for the minimal monopole and $q = 1$, we see that the minimal dyon has $L_i = -\frac{1}{2}\widehat{r}_i$. So, if we do a rotation about the $\widehat{r}$ axis through an angle of 2π, we get a phase of $e^{2\pi i\widehat{r}_i L^i} = -1$. Thus the minimal dyon is a fermion (via spin-statistics; more on this in a moment), as we would expect.

This same argument also works for deriving the quantization condition for a pair of dipoles $(q_1, g_1), (q_2, g_2)$: we put the electric and magnetic fields generated by the two dyons into the integral for angular momentum, and use the same trick described above. This gives $L_i = -(q_1 g_2 - q_2 g_1)\widehat{r}_i$, with r_i the separation vector between the monopoles. Note the relative minus sign between the two terms! This is kind of counter-intuitive, since the two dyons are being braided around one another with the same handedness.

Let's now check that the spin-statistics argument is correct by computing the self-statistics of a dyon. We do this by moving one (e, g) dyon in a π semicircle around the other (the two-dyon system has translation invariance so we can scoot the rotated system back to the original one after the π rotation for free). The electric charge of the moving dyon picks up a phase $e^{ie\int_C A(r)}$, where A is sourced by the magnetic monopole at the origin and C is the semicircular contour. By electromagnetic duality $(e, g) \mapsto (-e, g)$ the moving magnetic monopole picks up a phase $e^{-ig\int_C \widetilde{A}(r)}$. We find $\widetilde{A}(r)$ by noting that the Lagrangian for a single dyon contains the couplings $e\dot{r}_e^i A_i(r)$ and $-g\dot{r}_g^i \widetilde{A}_i(r)$. Since the total dyon system is translationally invariant, the total canonical momentum $p_e + p_g$ is conserved, $p_e + p_g = 0$. From varying the action, we have

$$p_e + p_g = m_e \dot{r}_e + m_g \dot{r}_m + eA(r_e - r_g) - g\widetilde{A}(r_g - r_m) = 0. \tag{1066}$$

Since this has to hold when e.g. the electric and magnetic charges are at rest, we require

$$eA(r) = g\widetilde{A}(-r). \tag{1067}$$

Now for a monopole we can use the solution

$$A_i(r) = \frac{g}{r}\frac{\epsilon_{ijk} r^j \widehat{n}^k}{r - r_l \widehat{n}^l}, \tag{1068}$$

where $\widehat{n}^j$ is some unit vector that we may choose freely. This solution is valid on a large enough patch on S^2 for our purposes, and we will take $\widehat{n} = \widehat{z}$ to be the unit vector in the plane normal to the movement of the dyon, so that $r^i \widehat{n}_i = 0$. For motion in the xy plane then, we have

$$A(r) = -\frac{g}{r}d\phi, \qquad \widetilde{A}(r) = \frac{e}{r}d\phi. \tag{1069}$$

Thus the total phase accumulated during the exchange is

$$\exp\left(-ieg\int_C r d\phi \frac{1}{r} + ieg\int_C r d\phi \frac{-1}{r}\right) = e^{2\pi i eg} = (-1)^{2eg}, \tag{1070}$$

which indeed gives us fermionic statistics if $eg \in \frac{1}{2}(2\mathbb{Z}+1)$, which agrees with spin-statistics.

Another way to see this is to make a "gauge transformation" to get rid of the potentials, at the expense of making the wavefunction not single-valued. I really don't like this way of doing things since wavefunctions should always be single valued and one should never do singular gauge transformations, but because a lot of other people seem to do similar things it's good to understand what the argument is. If we let r denote the vector pointing from dyon 1 to dyon 2, the Hamiltonian is (setting the mass of the dyons to 1/2 for simplicity)

$$H = (-i\partial_1 - eA(r) + g\widetilde{A}(r))^2 + (-i\partial_2 - eA(-r) + g\widetilde{A}(-r))^2 + (e^2+g^2)\frac{1}{|r|}. \quad (1071)$$

Now note that $eA(r) - g\widetilde{A}(r) = 2egd\phi/|r|$. Thus we can eliminate the gauge fields in H if we make a "gauge transformation"

$$\psi(|r|,\phi) \mapsto \widetilde{\psi}(|r|,\phi) \equiv \exp\left(-2ige\phi\right)\psi(|r|,\phi), \quad (1072)$$

where ϕ is the angular coordinate in the plane of the dyon's motion, with one of the dyons fixed at the origin. So the Schrodinger equation is now

$$\left(-\partial_1^2 - \partial_2^2 + \frac{e^2+g^2}{|r|}\right)\widetilde{\psi}(|r|,\phi) = E\widetilde{\psi}(|r|,\phi), \quad (1073)$$

where $\widetilde{\psi}$ has a non-single-valued part $e^{-2ige\phi}$. Changing $\phi \mapsto \phi + \pi$ exchanges the dyons, and does

$$\widetilde{\psi} \mapsto (-1)^{2ge}\widetilde{\psi}. \quad (1074)$$

Thus we again find that the dyons are fermions if $2ge \in 2\mathbb{Z}+1$.

Recapitulating, we have seen that bound states of a unit charge and a unit monopole are fermions, both in terms of their spin and in terms of their statistics. We have also seen that turning on a θ term changes the electric charge of the monopoles. This raises the question: are the statistics of the monopoles affected by θ? This seems reasonable because of the charge attachment, but definitely can't be true since we are in three dimensions and can only have bosons and fermions, but can tune θ continuously, implying that if the statistics did depend on θ then the statistics would vary continuously, which is a contradiction. Indeed, we will see that the electric field induced on the monopole is a "polarization effect" and makes no contribution to either the spin or the statistics of the monopole. This means that only the "microscopic" charge and monopole number are relevant for determining dyon spins and statistics. For example, suppose at $\theta = 0$ the pure minimal monopole is a boson. When we increase θ, the charge of the monopole increases, but it remains a boson. At $\theta = 2\pi$, it becomes a bosonic $(1,1)$ dyon. This means at $\theta = 2\pi$ the new charge-neutral monopole is really a $q = -1$ "microscopic" charge bound to the $(1,1)$ bosonic dyon. Since only the microscopic charge is relevant for determining the statistics, the new charge-neutral monopole is a fermion. More on this to follow.

Let's now verify the claim that the induced electric charge doesn't contribute to spin or statistics. For the spin, there is a very simple argument: applying Gauss' law $d(\star F/e^2 + \theta F/4\pi^2) = 0$ around a monopole of flux $\Phi = 2\pi m$ tells us that the induced electric field is

$$E^i_{ind} = \frac{m\theta}{2\pi}\frac{r^i}{r^3}. \quad (1075)$$

In particular, the induced electric field is purely radial, and parallel to the magnetic field (since the θ term is an $E \cdot B$ term). Now as we recalled earlier the angular momentum goes like $L \sim \int r \times (E \times B)$, and so the contribution to the total angular momentum from the induced field vanishes. Thus the spin of the monopoles is independent of θ.

Now we look at the statistics. We could do a wavefunction approach like we did previously, but here we will do something more field-theory-centric. Write the action in the presence of sources for monopoles and electric charges as (we're in $\mathbb{R}$ time since it seemed to be easiest to keep track of signs that way—can't promise that all the signs are correct though)

$$S = \int \left[-\frac{1}{2e^2} F \wedge \star F + \frac{\theta}{8\pi^2} F \wedge F + qA \wedge \star j \right], \tag{1076}$$

where $\star j$ is dual to the electric worldlines. We will keep track of the monopoles by doing the decomposition

$$F = dA + 2\pi\beta + \omega. \tag{1077}$$

Here β is the magnetic monopole part such that $\star d\beta$ is the monopole current, normalized so that its Poincare dual has $\mathbb{Z}$ periods (we are *not* assuming β is coexact; this is not exactly a Hodge decomposition. The reason why is so that we can more easily deal with monopoles). Here ω is a harmonic component that will only be activated when we have flux threading 2-cycles of spacetime (the monopoles are treated as locations where $dF \neq 0$, rather than excised balls in spacetime). The harmonic component decouples from the rest of the action and gives $S_\omega \sim \int \omega \wedge \star\omega + \theta \int \omega \wedge \omega$. We will avoid talking about it any further, and will thus only be dealing with non-harmonic forms in what follows. Thus the Hodge Laplacian will always be invertible on the forms we'll be working with.

The action is then re-written as

$$S = \int \left[-\frac{1}{2e^2} A \wedge \left((\star d^\dagger d + \star d d^\dagger) A + 4\pi d \star \beta - 2e^2 \frac{\theta}{2\pi} d\beta - 2e^2 q \star j \right) - \frac{2\pi^2}{e^2} \beta \wedge \star\beta + \frac{\theta}{2} \beta \wedge \beta \right], \tag{1078}$$

where we have taken boundary conditions so that $A|_{\partial X} = 0$ and inserted a gauge-fixing term in Feynman gauge. Now we make the shift

$$A \mapsto A - \frac{1}{2} \Box^{-1} \star \left(4\pi d \star \beta - 2e^2 \frac{\theta}{2\pi} d\beta - \frac{q}{2e^2} \star j \right). \tag{1079}$$

Here the $-$ sign is needed since $\star^2 = +1$ on 1-forms in Lorentzian signature. If the worldines of the monopoles and electric sources meet ∂X transversely, which we will assume, then this shift preserves the boundary conditions on A. This renders the A part of the action to just be $\int F \wedge \star F$, which we absorb into the normalization of the measure. This leaves us with

$$S = \int \left[\frac{1}{8e^2} \left(4\pi d \star \beta - 2e^2 \star \mathcal{J}_\theta \right) \wedge \Box^{-1} \star \left(4\pi d \star \beta - 2e^2 \star \mathcal{J}_\theta \right) - \frac{2\pi^2}{e^2} \beta \wedge \star\beta + \frac{\theta}{2} \beta \wedge \beta \right], \tag{1080}$$

where we've defined the current

$$\mathcal{J}_\theta \equiv qj + \frac{\theta}{2\pi} \star d\beta = qj + \frac{\theta}{2\pi} m, \tag{1081}$$

which is a linear combination of the charge and monopole currents j and m, in accordance with the induced electric charges stuck onto the monopoles because of the θ term. The terms involving β but not dependent on θ are

$$S \supset -\frac{2\pi^2}{e^2}\int \left(-d^\dagger\beta\wedge\star\Box^{-1}d^\dagger\beta + \beta\wedge\star\beta\right) = -\frac{1}{2\widetilde{e}^2}\int d\beta\wedge\star\Box^{-1}d\beta, \tag{1082}$$

where $\widetilde{e} = e/2\pi$ is the dual charge. Here we've used $\Box = dd^\dagger + d^\dagger d$ to write

$$\begin{aligned}\int \left(-d^\dagger\beta\wedge\star\Box^{-1}d^\dagger\beta + \beta\wedge\star\beta\right) &= \int \left(-\beta\wedge\star\Box^{-1}dd^\dagger\beta + \beta\wedge\star\Box^{-1}(dd^\dagger + d^\dagger d)\beta\right) \\ &= \int \beta\wedge\star d^\dagger\Box^{-1}d\beta = \int d\beta\wedge\star\Box^{-1}d\beta,\end{aligned} \tag{1083}$$

since $d, d^\dagger$ commute with $\Box$ and hence with $\Box^{-1}$.

We can also see this by going to momentum space: ignoring constants coming from combinatorial factors, we take $\beta\wedge\star\beta \to \frac{1}{q^2}\beta_{\mu\nu}q^2\beta^{\mu\nu}$, and then use

$$\beta\wedge\star\beta = \beta\wedge\star\Box\Box^{-1}\beta = -\partial^\sigma\beta_{\sigma\lambda}\partial^\gamma\frac{1}{-\partial^2}\beta_{\gamma\lambda} - \partial_\sigma\beta^{\rho\omega}\epsilon^{\sigma\lambda\omega\rho}\partial_\lambda\frac{1}{-\partial^2}\beta^{\lambda\alpha} \to \frac{1}{q^2}\left(q^\sigma q^\gamma\beta_{\sigma\lambda}\beta_{\gamma\lambda} + q_\sigma q_\lambda\epsilon^{\sigma\lambda\omega\rho}\beta^{\rho\omega}\beta^{\lambda\alpha}\right. \tag{1084}$$

The first term cancels with

$$-d^\dagger\beta\wedge\star\Box^{-1}d^\dagger\beta \to -\frac{1}{q^2}q^\sigma q^\gamma\beta_{\sigma\lambda}\beta_{\gamma\lambda}, \tag{1085}$$

which indeed leaves only the second term.

Since $d, d^\dagger$ commute with the Hodge Laplacian $dd^\dagger + d^\dagger d$, we can write the term containing β but not θ as

$$S \supset -\frac{1}{2\widetilde{e}^2}\int d\beta\wedge\star\Box^{-1}d\beta = \frac{1}{2\widetilde{e}^2}\int m\wedge\Box^{-1}\star m, \tag{1086}$$

where as before $m = \star d\beta$ is the monopole current. This is the electromagnetic dual of the electric current-current Coulomb interaction for the monopoles. The fact that such an interaction was induced could also have been argued in the following way: due to the fact that $dA + 2\pi\beta$, the effective action for β needs to be invariant under the shift $\delta\beta = d\lambda$ for λ a 1-form, since it can be compensated by a shift in A. Thus the effective action for β should involve the projector onto the coexact forms. Indeed, we have

$$\int d\beta\wedge\star\Box^{-1}d\beta = \int \beta\wedge\star\frac{d^\dagger d}{\Box}\beta, \tag{1087}$$

with $d^\dagger d/\Box$ the projector onto the coexact forms: $\Box^{-1}d^\dagger dd\lambda = 0$, while

$$\frac{d^\dagger d}{\Box}d^\dagger\omega = d^\dagger\frac{\Box - d^\dagger d}{\Box}\omega = d^\dagger\omega, \tag{1088}$$

so that it acts as $\mathbf{1}$ on coexact forms (it is not defined on the harmonic forms since they are in $\ker\Box$).

The θ dependence of the action is

$$S \supset \int \left[\frac{e^2}{2} \mathcal{J}_\theta \wedge \Box^{-1} \star \mathcal{J}_\theta - 2\pi d^\dagger \beta \wedge \Box^{-1} \star \mathcal{J}_\theta + \frac{\theta}{2} \beta \wedge \beta \right], \tag{1089}$$

where we've used $\star^2 = 1$ on 1-forms in real time in four dimensions. The first term is the usual current-current interaction for electrically charged sources: here the current is upgraded to include a term proportional to θ and the monopole current, reflecting the fact that the electric fields of the monopoles contribute to the usual electric interaction between Wilson lines. The second two terms are

$$\int \left[-2\pi d^\dagger \beta \wedge \Box^{-1} \star \mathcal{J}_\theta + \frac{\theta}{2} \beta \wedge \beta \right] = \int \left[-2\pi q d^\dagger \beta \wedge \Box^{-1} \star j - \theta d^\dagger \beta \wedge \Box^{-1} d\beta + \frac{\theta}{2} \beta \wedge \beta \right] \tag{1090}$$

The last two terms actually cancel since (I think the signs are correct) in momentum space

$$0 = A[q_\alpha \beta_{\mu\nu} \beta_{\lambda\sigma}] q^\alpha \epsilon^{\mu\nu\lambda\sigma} \implies \beta_{\mu\nu} \beta_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} = \frac{1}{q^2} 4 q^\alpha \beta_{\alpha\mu} q_\nu \beta_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma}. \tag{1091}$$

Here the first equality simply follows from being in four dimensions (A represents anti-symmetrization on the indices) and the factor of 4 comes from our ability to contract the q outside the antisymmetrizer with any four β indices. In the final equality, the left term corresponds to the $\beta \wedge \beta$ piece, while the right term corresponds to the $d^\dagger \beta \wedge \Box^{-1} d\beta$ piece. Another way to see this is (okay, I'm really just having fun at this point) to Hodge-decompose β as $\beta = d\alpha + d^\dagger \gamma$ (recall β had no harmonic component), and then write

$$\int (d\alpha + d^\dagger \gamma) \wedge (d\alpha + d^\dagger \gamma) = 2 \int d\alpha \wedge d^\dagger \gamma, \tag{1092}$$

and

$$-\int d^\dagger (d\alpha + d^\dagger \gamma) \wedge \Box^{-1} d(d\alpha + d^\dagger \beta) = \int d\star d\alpha \wedge \Box^{-1} \star d d^\dagger \gamma = \int d\alpha \wedge \Box^{-1} (d^\dagger d + d d^\dagger) d^\dagger \gamma = \int d\alpha \wedge d^\dagger \gamma. \tag{1093}$$

The θ-dependent β terms are the last expression minus half the second-to-last expression, and so they indeed cancel.

Recapitulating, the full action is

$$S = \int \left[\frac{e^2}{2} \mathcal{J}_\theta \wedge \Box^{-1} \star \mathcal{J}_\theta + \frac{1}{2\tilde{e}^2} m \wedge \Box^{-1} \star m - 2\pi q \beta \wedge \Box^{-1} \star dj \right]. \tag{1094}$$

The first two terms are the electric and magnetic current-current interactions, respectively, while the last bit is the AB term. Why is it an AB term? We can write it as

$$S_{AB} = -2\pi q \int \Box^{-1} dj \wedge \star \beta = -2\pi q \oint dx^\mu \int d^4 y \, \epsilon_{\mu\nu\lambda\sigma} \frac{x^\nu - y^\nu}{|x-y|^4} (\star \beta)^{\lambda\sigma}(y), \tag{1095}$$

where the $\oint$ is over the Poincare dual of $\star j$. This is a linking number between the current loop and the surface dual to $\star\beta$, which turns out to be the AB phase. This is seen a bit more explicitly by writing $\star j = dD$ (if j is not of this form, $\langle e^{i\int A\wedge\star j}\rangle = 0$), and using current conservation to write

$$\Box j = d^\dagger dj \implies \Box^{-1} dj = d(d^\dagger d)^{-1} \star dD = dd^{-1} \star d^{-1} dD = \star D, \tag{1096}$$

where we haven't bothered to keep track of potential minus signs. Thus S_{AB} is

$$S_{AB} = -2\pi q \int D \wedge \beta = -2\pi q \int_{\widehat{D}} \beta. \tag{1097}$$

Here the integral is over a disk bound by the current loop. For a geometry where a spatial current loop is drawn in spatial $\mathbb{R}^3$ with a monopole sitting at the origin, this just becomes

$$S_{AB} = -q \int_{\widehat{D}} dA_{mag} = -qm\Omega(\widehat{D}), \tag{1098}$$

where m is the monopole strength, A_{mag} the monopole part of the vector potential, and $\Omega(\widehat{D})$ the solid angle enclosed by the loop ($d\beta = 0$ away from the origin of $\mathbb{R}^3$ since the monopole current is $m \propto \delta(\vec{x})dt$, and so the dependence on $\widehat{D}$ is topological). This is precisely the AB phase we expect.

The important thing here is that the calculation of the AB phase involved only j, and not $\mathcal{J}_\theta$. This means that the monopoles have statistical interactions only with genuine microscopic electric currents, and they do not have any statistical interactions with the electric charge bound to them by the θ term. Thus the statistics of a dyon is calculated solely through its microscopic charge assignments, and its statistics are unchanged as θ is varied.

Now we will discuss the periodicity of θ, which is a bit subtle. We can see the difference in the periodicity of θ for fermionic / bosonic theories even without talking about spin structures and the evenness of the intersection form. For bosonic theories (both $(1,0)$ and $(0,1)$ are bosons, with notation (e,m)), the $(1,1)$ dyon is a fermion, as explained above. More generally, (e,m) is a fermion whenever e and m are both odd. Thus the $m=0$ row of the charge lattice is totally bosonic, the $m=1$ has alternating bosons / fermions, the $m=2$ row has all bosons, etc. Now when $\theta \mapsto \theta + 2\pi$, the $m=0$ row is invariant, the $m=1$ row moves to the right by one unit, the $m=2$ row moves to the right by two units, and so on (increasing θ changes the charge, but not the statistics, of a given dyon, since the induced charge doesn't enter in to the statistical calculation). Thus the $m \in (2\mathbb{Z}+1)$e rows are not invariant under the shift in θ, but are invariant under a 4π shift. So when e, m are bosons, $\theta \sim \theta + 4\pi$.

This is equivalent to the statement that on a non-spin manifold, the quantization of $\int (F/2\pi) \wedge (F/2\pi)$ is in $\mathbb{Z}$, not in $2\mathbb{Z}$. On a spin manifold, we have transparent fermions that we can bind to any of the particles in the charge lattice, changing their statistics. Thus on a spin manifold, the statement that the statistics of the $m=1$ row gets changed when shifting θ by 2π is not meaningful. On a non-spin manifold, (on which the theory for which both e, m are bosons can be defined) we don't have these transparent fermions, and so the shift of the $m=1$ row is meaningful.

Hold on, one might say, if the $(1,1)$ dyon is a fermion, how can we define the theory on a non-spin manifold? We can't use a spinc connection since the charge $(1,0)$ object is a boson, and therefore we have no spin-statistics relation. So what's going on? The answer is[112] that even though $(1,1)$ is a fermion, it doesn't need a spin structure on the spacetime manifold to have a well-defined framing. The point is that there is no choice of fundamental fields for which $(1,1)$ is a local operator: it is always a non-local object. We can give it a framing (a spin structure on its worldline) by using e.g. the vector that points from the electric charge to the magnetic charge. If it were a local operator we couldn't use its internal structure to give it a framing, so we'd have to give it a framing by using the one induced by the framing of the tangent space of the ambient manifold. If the ambient manifold is non-spin then this would be impossible. So non-spin manifolds preclude defining neutral *local* fermions, but not nonlocal (emergent) ones.

By contrast if e is a fermion and m is a boson, things change. Now the $m \in 2\mathbb{Z}$ rows of the lattice are alternating fermion-boson, while the $m \in (2\mathbb{Z}+1)$ rows are all bosonic. Shifting θ by 2π only shifts the $m \in (2\mathbb{Z}+1)$ rows of the lattice, which is trivial in this case since the statistics of all the dyons in the odd rows is bosonic. Thus $\theta \sim \theta + 2\pi$ when e is a fermion (regardless of whether we are on a spin manifold, or a non-spin manifold with a spinc connection). The case where both e and m are fermions is anomalous and probably warrants its own diary entry, to be written sometime in the future.

Discrete case

Now to briefly discuss the discrete case of $\mathbb{Z}_N$ BF theory. The action with sources included is

$$S = \int \left(\frac{n}{2\pi} F_A \wedge B + \frac{kn}{4\pi} B \wedge B + j \wedge \star A + \Sigma \wedge \star B \right), \tag{1099}$$

where the second term is the discrete θ term (of course, the appropriate squaring operation is really the Pontryagin square since $\wedge$ is not dual to the proper intersection product, but we will continue to abuse notation by writing $\wedge$). Now we normally have a gauge transformation on B where $\delta B = F_\lambda$ for F_λ the field strength of a $U(1)$ gauge field. We want the action to be gauge invariant without having to integrate any fields out to impose quantization conditions (i.e. we want the action to be invariant without using our knowledge that the periods of B will be quantized in $2\pi\mathbb{Z}/n$), and so we require that $\delta A = -k\lambda$ under this shift. Imposing this invariance in the presence of the currents means

$$k\lambda \wedge \star j - \lambda \wedge d \star \Sigma = 0 \implies j = \frac{1}{k} d^\dagger \Sigma. \tag{1100}$$

This forces $\Sigma = \Sigma_j + \Sigma_c$, where $\partial \widehat{\star\Sigma_j} = \widehat{\star j}$ and $\partial \widehat{\star\Sigma_c} = 0$. Here $\star\Sigma_j$ is dual to the surfaces attached to the Wilson lines to render them gauge invariant, while $\star\Sigma_c$ is dual to the worldsheets of closed strings that are charged under B.

Integrating out A says that

$$\star j = -\frac{n}{2\pi} F_B. \tag{1101}$$

[112] Thanks to Ryan Thorgren to helping me understand this! :D

Since the current is conserved, working in a gauge where $d^\dagger B = 0$ we can take $d^\dagger$ of both sides and write (we are in Euclidean time in four dimensions, so $d^\dagger = -\star d\star$, and $\star^2 = (-1)^{p^2}$ on p-forms)

$$\star\, dj = -\frac{n}{2\pi}\Box B \implies B = -\frac{2\pi}{n}\Box^{-1}\star dj. \tag{1102}$$

This is only a solution for B up to elements of $H^2(X) = \ker\Box$. We will assume for simplicity that this cohomology group vanishes, since what we really care about are the correlation functions of the various operators, which we can study on e.g. $\mathbb{R}^4$. Relaxing this assumption is no big deal, since the harmonic parts of B decouple from most other things in the action. Also, since we always take $\star j = dD$ for $\widehat{D}$ some disk (or more precisely, any 2-manifold with $\partial D \setminus \partial X = \widehat{\star j}$, where X is spacetime—recall that if j is not of this form, the path integral vanishes [also, in our case, the associated Wilson line would not be gauge invariant]), one may also write $dD = -\frac{n}{2\pi}F_B \implies B = -\frac{2\pi}{n}D$. This solution for B is only defined up to elements in $\ker(d)$. If we impose $d^\dagger B = 0$ then the exact part drops out, and we get a solution for B up to elements of $H^2(X)$. As before we assume this vanishes, and so we get the same result. In BF theory, we know that the periods of B are quantized, Is this reproduced by our solution? Let M_2 be any closed 2-chain in $H_2(X;\mathbb{Z})$. Then

$$\frac{2\pi}{n}\int_{M_2}\Box^{-1}\star dj = \frac{2\pi}{n}\int \star\widehat{M_2}\wedge\Box^{-1}dj = \frac{2\pi}{n}\int M_2\wedge D \in \frac{2\pi}{n}\mathbb{Z}, \tag{1103}$$

since we take $\star j$, and hence D, to be in the image of $H^*(X;\mathbb{Z})$ under the inclusion into de Rham cohomology. So indeed, B has the periods we expect.

Now we put this solution back into the action:

$$S = \frac{1}{2\pi}\int\left(\frac{4\pi^2 k}{2n}\Box^{-1}dj\wedge\Box^{-1}dj - \Sigma\wedge\frac{2\pi}{n}\Box^{-1}dj\right). \tag{1104}$$

We see that the discrete θ parameter k determines the self-statistical properties of the electric matter (sources for A), while the statistics between the magnetic (sources for B) and electric matter is determined just by n and is unaffected by the θ term. The fact that a θ term involving the "magnetic" field B affects the statistics of the matter coupled to the opposite field A is just like what happens in $U(1)$ gauge theory, whose action we might write as $d\widetilde{A}\wedge F/2e^2 + \theta F\wedge F/8\pi^2$, where $d\widetilde{A} = \star F$. Here the θ term involving the field F affects the charges of the sources coupled to the opposite field $\widetilde{A}$ ($d\widetilde{A}\wedge F$ is supposed to remind one of $F_A\wedge B$).

Basics of large N Yang-Mills

Toady we will consider $SU(N)$ YM theory at large N. We will explain why the t' Hooft limit is essential for getting analytic control on the theory, rather than just the

$N \to \infty$ limit, and derive the matrix structure of the propagator and the Feynman rules. Finally, we relate the Wilson line in the adjoint to the Wilson line in the fundamental and explain how this relationship simplifies in the large N limit. This material is of course all in the literature; I just wanted to understand things myself and make a reference I could refer to later.

First, we claim that $SU(N)$ YM theory we never want to take $N \to \infty$ limit without also taking $g \to 0$. To see why, recall that the $SU(N)$ beta function is

$$\frac{dg^2}{d\ln\mu} = -\frac{11g^4}{24\pi^2}C(SU(N)) = -\alpha g^4 N, \tag{1105}$$

where α is a positive number and we've used the fact that the quadratic casimir of the adjoint representation (aka twice the dual coxeter number) is $f_{abc}f_{dbc} = \delta_{ad}C(SU(N)) \implies C(SU(N)) = N$. If we integrate the β function, then we get

$$g^2(\mu) = \frac{Ng_0^2}{1 - \alpha N g_0^2 \ln(\Lambda_0/\Lambda_\mu)}, \tag{1106}$$

where the 0 subscripts are at some reference scale (like the UV). Λ_{QCD} is the scale at which perturbation theory breaks down, found by setting the denominator to zero. This gives

$$\Lambda_{QCD} = \Lambda_0 e^{-(\alpha N g_0^2)^{-1}}. \tag{1107}$$

Thus we see that if we take $N \to \infty$ without also keeping Ng^2 fixed, we get $\Lambda_{QCD} \to \Lambda_0$, and so the strongly-coupled scale becomes equal to the UV cutoff scale, leaving us unable to say anything useful.

So with this in mind, we define the finite coupling constant of interest as $\lambda = g^2 N$. Thus the action is

$$S = \frac{N}{\lambda}\int \mathrm{Tr}[F \wedge \star F]. \tag{1108}$$

Despite the N in front, we cannot just take the classical saddle point. This is essentially because while a non-classical field configuration will be suppressed by the N out front, it will be amplified by the fact that as N gets large there are many many more non-classical field configurations to have the fields in. So the theory is still very quantum.

To determine the Feynman rules, we will deal with the matrices A directly, rather than their components A^a in some representation. Now A transforms adjointly under the (global part of the) gauge group:

$$A^a_b \mapsto [U^\dagger]^{ad} A^d_e U_{eb} = [U^*]^{da} U_{eb} A^d_e. \tag{1109}$$

Thus the upper index of A transforms in $\bar{N}$ while the lower index transforms in N, reflecting the fact that $N \otimes \bar{N} = Ad \oplus \mathbf{1}$.

Now just as how we can write something transforming with spin 1/2 under $SU(2)$ as $a|\uparrow\rangle + b|\downarrow\rangle$, we can write

$$A^a_b = \sum_{\mathcal{A}=1\ldots N^2} A^{\mathcal{A}}[T^{\mathcal{A}}]^a_b. \tag{1110}$$

Here we are thinking of A as a vector transforming under the adjoint of $SU(N)$: the tuple (a,b) is a composite vector index. The $T^{\mathcal{A}}$'s are basis vectors, and the $[T^{\mathcal{A}}]^a_b$'s are their components. Maybe the notation $v^{\mathcal{A}}_{a,b}$ would be slightly better to emphasize that we are thinking of the $T^{\mathcal{A}}$ as vectors, rather than generator matrices. Indeed, $\mathcal{A}$ runs over N^2 different values, since the vectors being represented live in an N^2 dimensional space (the space of $N\times N$ unitary matrices).

Now since the adjoint representation has dimension N^2-1, we need to remove one of the $T^{\mathcal{A}}$'s from the basis, in order to get a vector space of the right dimensionality, as $\dim\mathfrak{su}(N) = N^2-1$. We want to remove the generator corresponding to the trivial representation, which is proportional to the identity matrix. Let us work in the normalization

$$\langle T^{\mathcal{A}}|T^{\mathcal{B}}\rangle = \mathrm{Tr}[T^{\mathcal{A}}T^{\mathcal{B}}] = \delta^{\mathcal{AB}}. \tag{1111}$$

Then the generator proportional to $\mathbf{1}$ that we need to remove is $[T^{N^2}]^a_b = \frac{1}{\sqrt{N}}\delta^a_b$. With this generator taken out, the completeness relation is now

$$\sum_{\mathcal{A}=1}^{N^2-1}[|T^{\mathcal{A}}\rangle\langle T^{\mathcal{A}}|]^{ac}_{db} = \sum_{\mathcal{A}=1}^{N^2-1}[T^{\mathcal{A}}]^a_b[T^{\mathcal{A}}]^c_d = \left(\delta^a_d\delta^c_b - \frac{1}{N}\delta^a_b\delta^c_d\right). \tag{1112}$$

If we were working with $U(N)$ so that we could have generators with nonzero trace, we would have the full N^2 generators and we would have $\mathbf{1} = \delta^a_d\delta^c_b$ on the RHS. Taking out the generator for the $SU(N)$ case means that $\mathrm{Tr}[\sum_{\mathcal{A}}|T^{\mathcal{A}}\rangle\langle T^{\mathcal{A}}|] = \frac{1}{2}(N^2-1)$.

Anyway, the point of writing A like this is that it allows us to figure out what the index structure of the propagator is. The kinetic term in the action looks like

$$\mathrm{Tr}[dA\wedge\star dA] = \sum_{\mathcal{A},\mathcal{B}}\langle T^{\mathcal{A}}|T^{\mathcal{B}}\rangle dA^{\mathcal{A}}\wedge\star dA^{\mathcal{B}} = \sum_{\mathcal{A}} dA^{\mathcal{A}}\wedge\star dA^{\mathcal{A}}, \tag{1113}$$

so that the propagator is only non-zero when it connects two A's with the same generator $T^{\mathcal{A}}$. Thus

$$\begin{aligned}\langle A^a_{\mu b}(x)A^c_{\nu d}\rangle &= \sum_{\mathcal{A},\mathcal{B}}[T^{\mathcal{A}}]^a_b[T^{\mathcal{B}}]^c_d\langle A^{\mathcal{A}}_\mu(x)A^{\mathcal{B}}_\nu(y)\rangle = \sum_{\mathcal{A}}[|T^{\mathcal{A}}\rangle\langle T^{\mathcal{A}}|]^{ac}_{db}\langle A^{\mathcal{A}}_\mu(x)A^{\mathcal{A}}_\nu(y)\rangle \\ &= D_{\mu\nu}(x-y)\left(\delta^a_d\delta^c_b - \frac{1}{N}\delta^a_b\delta^c_d\right).\end{aligned} \tag{1114}$$

Here $D_{\mu\nu}(x-y)$ is the regular vector propagator in whichever gauge fixing condition we feel like adopting. Together with this propagator, the Feynman rules are easy to write down in double-line notation.

Now we can look at general correlation functions. All operators of interest will be gauge invariant and hence will involve traces (they will have no free indices). We can focus on operators with a single trace, since operators with more traces can be built form single-trace ones. We find connected correlation functions for single-trace operators $\mathcal{O}_i$ by adding $\sum_i N\int J_i\mathcal{O}_i$ to the action, and then finding $\prod_j(N^{-1}\delta_{J_j})W[J]$, where $Z[J] = e^{-W[J]}$ as usual. The factors of N here are just so that $W[J]$ is the same order in N as the vacuum partition function. This is order $O(N^2)$, which can be seem from evaluating the simplest planar vacuum-to-vacuum graphs. In general, we see from the action that the amplitude of a given graph is determined by

$$\mathcal{A} \sim \left(\frac{N}{\lambda}\right)^{v-e} N^{f+s} = \lambda^{e-v} N^{\chi+s}, \tag{1115}$$

where v, e, f, s are the vertices, edges, faces, and sources of the Feynman diagram. We can then use $\chi = 2-2g-s$, where g is the genus and s is the number of holes (sources), to write

$$\mathcal{A} \sim \lambda^{e-v} N^{2-2g}. \tag{1116}$$

Thus no matter how many (pure glue) sources we insert, the leading-order in N diagrams that contribute to $W[J]$ will be planar and go as N^2 (drawing some pictures to "experimentally" test this is fun).

Now from the way we are computing connected correlation functions, we see that every functional differentiation with respect to a source that we need to perform multiplies the correlation function by $1/N$. Thus $\langle \mathbf{1} \rangle$ goes as $O(N^2)$, $\langle \mathcal{O}_1 \rangle$ goes as N^1, and in general an n-point connected correlation function goes as $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_c \sim N^{2-n}$. This means that as $N \to \infty$ all the 2-point functions of single-trace operators factorize:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle_c + \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \to (\langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle \sim O(N^2)) + O(1). \tag{1117}$$

In particular, $\langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle / \langle \mathcal{O} \rangle^2 = \langle \mathcal{O}\mathcal{O} \rangle_c / \langle \mathcal{O} \rangle^2 \sim N^{-2}$, so that as $N \to \infty$ the fluctuations become small.

Now we turn to the computation of the Wilson loop. We are interested in computing the Wilson loop in the adjoint representation. In general, we have

$$W_{R_1 \otimes R_2}(C) = P \exp\left(i \oint_C dx^\mu A^\alpha_\mu (T^\alpha_{R_1} \otimes \mathbf{1} + \mathbf{1} \otimes T^\alpha_{R_2})\right) = W_{R_1}(C) \otimes W_{R_2}(C). \tag{1118}$$

Here we have used that $T^\alpha_{R_1 \otimes R_2} = T^\alpha_{R_1} \otimes \mathbf{1} + \mathbf{1} \otimes T^\alpha_{R_2}$ (it's an easy check to see that the Lie bracket holds, and that $e^{i\theta^\alpha T^\alpha_{R_1 \otimes R_2}} = e^{i\theta^\alpha T^\alpha_{R_1}} \otimes e^{i\theta^\alpha T^\alpha_{R_2}}$), and then used that things which commute can be separately path-ordered.

We are most interested in the case of $SU(N)$, for which $N \otimes \bar{N} = Ad \oplus \mathbf{1}$. We need to project out the trivial representation from $W_{N \otimes \bar{N}}(C)$. We do this by (using the notation $a\bar{a} \in \mathbb{Z}_N^2$ for an adjoint index)

$$[W_{Ad}(C)]^{a\bar{a}}_{b\bar{b}} = [W_N(C)]^a_b [W_{\bar{N}}(C)]^{\bar{a}}_{\bar{b}} - \frac{1}{N} \delta^a_{\bar{b}} \delta^b_{\bar{a}}. \tag{1119}$$

The index structure on the last term is such that it only activates for indices that are diagonal in both the N and $\bar{N}$ factors, and is determined by taking the $\otimes$ of the intertwiners (the δ functions) for the indices in the N and $\bar{N}$ representations (delta functions like δ^b_a that connect two indices transforming in the same representation are not invariant symbols unless $N \cong \bar{N}$ which only happens if $N = 2$, so we need to have δ functions with an index structure that connects N and $\bar{N}$ indices). The $1/N$ normalization is to ensure that when we take the gauge field to vanish, we get $\mathrm{Tr}[W_{Ad}(C)|_{A=0}] = \dim(Ad) = N^2 - 1$.

Taking the trace and using the Hermiticity of the generators in the fundamental representation to write

$$W_{\bar{N}}(C) = P\exp\left(i\oint_C A^\alpha(-T_N^\alpha)^*\right) = P\exp\left(i\oint_C A^\alpha T_N^\alpha\right)^* = W_N(C)^*, \qquad (1120)$$

we see that

$$\langle \operatorname{Tr} W_{Ad}(C)\rangle = \langle |\operatorname{Tr} W_N(C)|^2\rangle - 1. \qquad (1121)$$

Since $W_N(C)$ is a single-trace operator, in the $N \to \infty$ limit the two point function is dominated by the disconnected part[113] , Thus at large N, we can move the square outside of the expectation value:

$$\langle \operatorname{Tr} W_{Ad}(C)\rangle \approx_{N\to\infty} |\langle \operatorname{Tr} W_N(C)]\rangle|^2. \qquad (1122)$$

Thus the coefficient of the adjoint area law is twice that of the fundamental line area law. This is actually kind of crazy, since we know that for small N the adjoint Wilson line always has perimeter law, since adjoint strings can break and end on gluons (adjoint sources can be screened by gluons). Looking through the old QCD literature, apparently the adjoint line goes as (schematically)

$$\langle \operatorname{Tr} W_{Ad}(C)\rangle \approx N^2 e^{-\sigma A} + e^{-\sigma P}, \qquad (1123)$$

where σ is a string tension, A is the area and P is the perimeter. Since normally $A \gg P$, the perimeter law piece dominates. However if we make $N \to \infty$, the area-law-scaling piece can actually win out—and it does if the fundamental lines are confined, as we saw above. Note however that for large enough Wilson lines, there is always a cross over to perimeter law for any finite N.

[113] Since the Wilson lines are nonlocal, this might be a little bit subtle to see. Consider first the 1-point function for the fundamental Wilson line. To zeroth order in the t' Hooft coupling, it just looks like a single fundamental line, drawn in the shape of C. This is $O(N)$, since there is one trace. To next order, we have to integrate over all ways for a propagator to connect two points on the fundamental line together (the Abelian Wilson line is the exponential of this double integral). The propagator is a double line, and so we are integrating over diagrams that look like two loops, which are parallel along the propagator line. This diagram thus has N dependence of λN: two gluon-quark-quark vertices that go as 1 in our choice of coupling, one propagator that goes as λ/N, and two sums over N for the two loops. Higher order terms have more propagators connecting the loop to itself, but adding a propagator in a planar way increases the number of propagators by one and the number of faces by one, resulting in an extra power of λ but the same $O(N)$ N-dependence (as usual non-planar diagrams are suppressed). Thus the 1-point function for the Wilson line is $O(N)$.

The connected correlation for two fundamental Wilson lines is $O(1)$ however. Indeed, consider the $O(\lambda)$ contribution to the connected part: it looks like two single (fundamental) lines, with a single double-line propagator connecting them. This diagram has one propagator and one loop, so it goes as $\lambda^1 N^0$. Adding further propagators cannot increase the N-dependence, and only increases the λ dependence. So $|\langle W_N(C)\rangle|^2$ is larger than $\langle |W_N(C)^2\rangle_c$ by a factor of N^2, in line with what we expect for two-point functions of single-trace operators.

When are CS theories spin TQFTs?

In today's diary entry we will answer the question in the title by working through a few representative examples. I'm sure this is in the literature somewhere, but thought it would be worthwhile to work things in detail.

One way to examine whether a CS theory is spin or not is to carefully define the CS action by breaking up the manifold into patches and defining the action in the style of DB cohomology; see a previous diary entry on this. This approach is kind of subtle for non-Abelian gauge groups though, so we will take the bounding 4-manifold approach, which is computationally simpler.

$U(1)_k$

As usual, define the CS action on a closed 3-manifold X by integrating an $F \wedge F$ term over some 4-manifold Y with $\partial Y = X$. The exponential of the action is independent of the choice of bounding 4-manifold Y provided that

$$\frac{k}{8\pi^2}\int_X F \wedge F \equiv \frac{k}{2} I \in \mathbb{Z} \tag{1124}$$

for all closed 4-manifolds M. Now, $F/2\pi \in H^2(M;\mathbb{Z})$, so we know for sure that $I \in \mathbb{Z}$ since the cup product of $F/2\pi$ with itself is then in $H^4(M;\mathbb{Z})$. Now if $k \in 2\mathbb{Z}$ then the (exponential of the) above integral is independent of M, regardless of whether M is spin or not. Thus if $k \in 2\mathbb{Z}$, the CS theory is insensitive to the spin structure and hence is bosonic. However, suppose $k \in 2\mathbb{Z}+1$. Then the CS action is only well-defined if $I \in 2\mathbb{Z}$. The constraint $I \in 2\mathbb{Z}\ \forall M$ can only be satisfied if we restrict our attention to M such that M is spin. If M is spin then $\omega_2(TM) = 0 \mod 2$ and the intersection form is even, meaning that I is always even. So, for odd k, the theory can only be defined using spin bounding 4-manifolds, and hence the original 3-manifold needs to come equipped with a spin structure as well. Thus odd k theories are spin TQFTs.

$SU(N)_k$

Now consider $SU(N)$. Now the relevant integral over a closed 4-manifold is

$$\frac{k}{8\pi^2}\int_M \mathrm{Tr}[F \wedge F] = k\mathrm{ch}_2(F) \in k\mathbb{Z}, \tag{1125}$$

since the second Chern character is the second second Chern class for $SU(N)$ on account of the tracelessness of the $SU(N)$ generators, it is quantized on account of the second Chern class being a $\mathbb{Z}$ characteristic class (for $U(1)$, the integral is just the second Chern character, which is not a class in $\mathbb{Z}$ cohomology). Note that the quantization of the integral does not depend on whether M is spin or not: the second

Chern class's integrality doesn't depend on the spin nature of M, since it does not (in general) compute an intersection form. Indeed, the minimal $\mathrm{ch}_2(F) = 1$ instantons are the "small" instantons that can exist on any manifold, regardless of its topology. They are constructed from bundles which are not tensor products of line bundles (if they were their quantization would be sensitive to $\omega_2(TM)$), and since they are "small" they can exist equally happily on spin- and non-spin manifolds. So, all the $SU(N)$ CS theories are bosonic.

$$U(N)_{k,q}$$

Now for $U(N)_{k,q}$, which is defined though

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathcal{A} \wedge d\mathcal{A} - \frac{2i}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathcal{A}] \wedge d\mathrm{Tr}[\mathcal{A}]. \tag{1126}$$

As explained before, the notation is done like this because q is ($1/N$ times) the effective $U(1)$ level, while k is the effective $SU(N)$ level.

Now we use the decomposition $U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N$. At the level of actions, we simply write $\mathcal{A} = A + \boldsymbol{\mathcal{A}}\mathbf{1}$, where A is an $SU(N)$ field (whose transition functions may fail by Nth roots of unity), $\boldsymbol{\mathcal{A}}$ is a $U(1)$ field (with transition functions failing in the inverse way). The quotient comes from the correlation of the transition functions between A and $\boldsymbol{\mathcal{A}}$. In terms of these fields, we have

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[A \wedge dA - \frac{2i}{3}A \wedge A \wedge A\right] + \frac{qN}{4\pi}\boldsymbol{\mathcal{A}} \wedge d\boldsymbol{\mathcal{A}}, \tag{1127}$$

so that qN is indeed the "effective $U(1)$ level". The scare quotes here are because $\boldsymbol{\mathcal{A}}$ isn't really a $U(1)$ field, because of the quotient: only $N\boldsymbol{\mathcal{A}}$ is a legit $U(1)$ field. So the legit $U(1)$ part is really

$$S \supset \frac{2\pi q/N}{8\pi^2}\int_Y d(N\boldsymbol{\mathcal{A}}) \wedge d(N\boldsymbol{\mathcal{A}}), \tag{1128}$$

where Y is a bounding 4-manifold. This would seem to indicate that we require $q \in N\mathbb{Z}$ in order for the action to be well-defined (independent of Y). But this is not quite the case, since the term in $\mathcal{L}$ involving A also stands a chance of being ill-defined on its own, due to the $\mathbb{Z}_N$ quotient. Indeed, from our previous diary entry on instanton numbers in $PSU(N)$ gauge theory, we saw that $\frac{k}{2}\int \mathrm{Tr}(dA/2\pi \wedge dA/2\pi)$ was quantized in $k/N\mathbb{Z}$. Thus the ill-defined-ness of the A part of the action alone is captured by $k/N \mod 1$. Since the transition functions of A and $\boldsymbol{\mathcal{A}}$ fail the cocycle condition in opposite senses at each triple overlap of patches, the fractional part of the instanton number for the A field is the negative of that for the $\boldsymbol{\mathcal{A}}$ field. Thus the total parameter measuring the ill-defined-ness of the action is actually $(k-q)/N \mod 1$. So, for a consistent theory, we need

$$k - q \in N\mathbb{Z}. \tag{1129}$$

Another way to say this is that since $\mathrm{Tr}[\mathcal{A}]$ is a well-defined $U(1)$ gauge field (but not $\mathcal{A}$ itself), the appearance of the term $(k-q)\mathrm{Tr}\mathcal{A} \wedge d\mathrm{Tr}\mathcal{A}/4\pi N$ in the action means that in order for this to be well-defined we need to have $(k-q)/N \in \mathbb{Z}$.

Yet another way to say it is that the theory needs to be invariant under simultaneous shifts in the transition functions of A and $\mathcal{A}$ by elements in $\mathbb{Z}_N$, which is realized on $\mathcal{A}$ through the shift $\delta\mathcal{A} = \frac{1}{N}d\phi$ for some 2π-periodic scalar ϕ. Since we are shifting both A and $\mathcal{A}$, $\mathcal{A}$ is invariant, and the action changes by

$$\delta S = \frac{(q-k)}{2\pi}\int d\phi \wedge F_{\mathcal{A}} \tag{1130}$$

(for the derivation of the fact that the prefactor is $1/2\pi$ and not $1/4\pi$, see the previous diary entry). Now since only $N\mathcal{A}$ is a $U(1)$ gauge field, the flux of $F_{\mathcal{A}}$ is quantized in $\overline{\mathbb{Z}}/N$. Thus in order for $\delta S \in \overline{\mathbb{Z}}$, we need $(q-k) \in N\mathbb{Z}$.

Anyway, when are these theories spin? Returning to the original formulation in terms of the $\mathcal{A}$ field, the appropriate four-dimensional integral to compute is

$$I = \frac{1}{8\pi^2}\int \left(k\mathrm{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] + \frac{q-k}{N}\mathrm{Tr}[F_{\mathcal{A}}] \wedge \mathrm{Tr}[F_{\mathcal{A}}] \right). \tag{1131}$$

Using the definition of the second Chern class,

$$I = 2\pi \int \mathrm{c}_2(E) + 2\pi \frac{k + (q-k)/N}{8\pi^2}\int d\mathrm{Tr}\mathcal{A} \wedge d\mathrm{Tr}\mathcal{A}, \tag{1132}$$

where E is the total $U(N)$ bundle. Since $\int \mathrm{ch}_2(E) \in \mathbb{Z}$ on any closed 4-manifold (spin or not), whether or not the theory is spin is determined by the second term. In particular, we get

$$k + \frac{q-k}{N} \in \begin{cases} 2\mathbb{Z} & \implies \text{not spin} \\ (2\mathbb{Z}+1) & \implies \text{spin} \end{cases}, \tag{1133}$$

where these are the only two options since as we said before, $(q-k) \in N\mathbb{Z}$.

$PSU(N)_k$

As we saw in a previous diary entry, on spin manifolds, minimal $PSU(N)$ bundles have instanton numbers that are in $\frac{1}{N}\mathbb{Z}$, and thus they are only defined when the level satisfies $k \in N\mathbb{Z}$. Since the fractional part of the instanton number came from the intersection number $\int B \wedge B$ of a 2-form $\mathbb{Z}_N$ gauge field, the fractional part of the instanton number will indeed depend on the existence of a spin structure: on non-spin manifolds we only have $I \in \frac{1}{2\mathbb{Z}}$. Thus $PSU(N)_k$ is spin if the level is an odd multiple of N ($k \in 2N\mathbb{Z} + N$), and non-spin if the level is an even multiple of N ($k \in 2N\mathbb{Z}$).

For example, take $PSU(2)_2 = SO(3)_2$: we obtain this from $SU(2)_2$ by identifying the representation 1 with the trivial representation. Now $SU(2)_2$ is the Ising theory, and 1 is the fermion. So, in order to identify 1 with 0, we need a spin structure. Thus $PSU(2)_2$ is a spin CS theory.

More generally, we know that the spin j line in $SU(2)_k$ has spin

$$\theta_j = \frac{j(j+1)}{k+2}. \tag{1134}$$

When $k \in 2\mathbb{Z}$, we can take the quotient to $PSU(2)_k$. The maximal spin line with $j = k/2$ is the generator of the $\mathbb{Z}_2^{(1)}$ symmetry we need to quotient by, and from the above we see that it has spin $\theta_{k/2} = k/4$. Therefore for $k \in 4\mathbb{Z}+2$ the generator is a fermion, and so $PSU(2)_k$ is spin for $k \in 4\mathbb{Z}+2$. On the other hand, when $k \in 4\mathbb{Z}$ the generator is a boson, and so for such values of k, $PSU(2)_k$ is not spin.

$DW_{p,q}$ theory

In the notation of last time, the $DW_{p,q}$ theory is

$$\mathcal{L} = \frac{p}{4\pi} a \wedge da + \frac{q}{2\pi} a \wedge db. \tag{1135}$$

Writing the action as an integral over a bounding 4-manifold tells us that these theories are spin when p is odd, and non-spin when p is even. This matches with the discussion of the 1-form symmetries of the theory in the previous diary entry: the generator for the $\mathbb{Z}_q^{(1)}$ symmetry shifting b is a boson and not anomalous, while the generator U_a for the $\mathbb{Z}_l^{(1)}$, $l \equiv \gcd(p,q)$ symmetry shifting a has spin

$$s[U_a] = \frac{p}{2l^2} \mod 1. \tag{1136}$$

This means that the spin of l copies of the charge operator is $s[U_b^l] = p/2 \mod 1$. Since l copies of the charge operator gives a line that has trivial statistics with everything, we see that if $p \in 2\mathbb{Z}$ we have no problem, while if $p \in 2\mathbb{Z}+1$ then the theory has a transparent fermion. However since the theory is spin if $p \in 2\mathbb{Z}+1$ the transparent fermion is trivial, and so U_b^l is a trivial line, as required.

$SO(N)_K$

The CS action for $SO(N)_K$ is written as

$$S = \frac{k}{8\pi} \int_M \mathrm{Tr}[F_A \wedge F_A], \tag{1137}$$

where the trace is taken in the vector representation. Note the factor of $1/8\pi$ in front, which differs from the usual $1/4\pi$ we've seen so far—the reason for this is ultimately that the reality of the $SO(N)$ representations ensures a doubling of the index of the Dirac operator on M, which by the index theorem lets us relate the η invariant and the CS action with an extra factor of $1/2$ compared to the normal definition—more on this in another diary entry.

Anyway, requiring that the integral be independent of the bounding 4-manifold means that for all closed M, we need

$$2\pi k \frac{1}{2 \cdot 8\pi^2} \int_M \mathrm{Tr}[F_A \wedge F_A] = \pi k \int p_1(A), \tag{1138}$$

where $p_1(A)$ is the first Pontryagin class. Now this is a legit $\mathbb{Z}$ characteristic class, but unlike the second Chern class, its quantization *does* depend on the type of manifold that it's on. In particular, the relation

$$p_1(A) = P(w_2(A)) + 2w_4(A) \mod 4 \tag{1139}$$

tells us that $\int p_1(A) \in 2\mathbb{Z}$ on spin manifolds. Thus $k \in 2\mathbb{Z}$ theories make sense on any manifold and are not spin, while $k \in 2\mathbb{Z}+1$ theories are spin.

So in general, the coefficient in front of the CS Lagrangian ($k/4\pi$, $k/8\pi$, etc.) can be determined by looking at how the relevant characteristic class (Chern or Pontryagin) is quantized on different types of manifolds. We should pick it so that for all k, the 2+1D CS action is well-defined on spin manifolds (but may require special choices for k to be defined on non-spin manifolds).

Before wrapping up, note how we never needed to compute the spectrum of line operators to make these statements, although that's certainly one way to figure our whether they are spin or not. However, just knowing whether they are spin already tells us a nonzero amount about their spectrum: we already know that e.g. a transparent fermion cannot appear in the spectrum for $SU(N)_k$ or $U(1)_{2k}$, but that one must appear in $U(1)_{2k+1}$.

Gauge (in)variance of non-abelian CS action and building instantons

Today is something simple that I hadn't done before. We will compute the gauge variation of the non-Abelian CS action explicitly, and use the result to show how one can build instantons on S^4.

We will work in math conventions where the gauge transformations act as

$$A \mapsto g^{-1}(A+d)g = g^{-1}Ag + \omega. \tag{1140}$$

The Lagrangian in these conventions is then

$$\mathcal{L} = \frac{ik}{4\pi}\text{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \tag{1141}$$

If we had $A \mapsto g^{-1}(A+id)g$ instead, we'd need to tack an i onto the 2/3 (which can be seen by tracking the i through the following manipulations). The gauge variation of the first part is

$$\text{Tr}[A \wedge dA] \mapsto \text{Tr}\left[(A^g+\omega) \wedge (-\omega A^g + (dA)^g - A^g\omega - \omega \wedge \omega)\right], \tag{1142}$$

where $A^g \equiv g^{-1}Ag$ and we've used $d\omega = -\omega \wedge \omega$. Now we use

$$\text{Tr}[X \wedge Y] = (-1)^{|X||Y|}\text{Tr}[Y \wedge X] \tag{1143}$$

to write this term as

$$\mathrm{Tr}[A\wedge dA]\mapsto \mathrm{Tr}\left[A\wedge dA-3\omega\wedge\omega\wedge A^g-\omega^{\wedge 3}+\omega(dA)^g-2A^g\wedge A^g\wedge\omega\right]. \quad (1144)$$

Clearly, the $A^{\wedge 3}$ term is going to be needed if we want to get something gauge invariant. This term changes as

$$\frac{2}{3}\mathrm{Tr}[A^{\wedge 3}]\mapsto\frac{2}{3}\mathrm{Tr}\left[A^{\wedge 3}+\omega^{\wedge 3}+3(A^g\wedge\omega\wedge\omega+A^g\wedge A^g\wedge\omega\right]. \quad (1145)$$

Adding these two contributions, we see that

$$\delta\mathcal{L}=\frac{k}{4\pi}\mathrm{Tr}\left[A^g\wedge\omega\wedge\omega(2-3)+\omega^{\wedge 3}(-1+2/3)+A^g\wedge A^g\wedge\omega(2-2)+\omega\wedge(dA)^g\right]. \quad (1146)$$

We can collect two of the surviving terms into a total derivative, so that

$$\delta\mathcal{L}=-\frac{k}{4\pi}d\mathrm{Tr}[\omega\wedge A^g]-\frac{k}{12\pi}\mathrm{Tr}[\omega^{\wedge 3}]. \quad (1147)$$

Now the first term doesn't contribute to δS, since $\omega|_{\partial X}=0$ if g is a gauge transformation and we fix ∂ conds on A ($X=$ spacetime). Since $\omega|_{\partial X}=0$, the second term in $\delta\mathcal{L}$ is 2π times the winding number density for a map from the (compactification of) X to the target Lie group. Together, these terms tell us what kind of WZW needs to live on ∂X in order for gauge invariance to be manifest with free boundary conditions on A. The winding number (okay, "winding" is probably best reserved for $S^1\to S^1$ situations—maybe "wrapping" would be pedantically better) term integrates to something in $\overline{\mathbb{Z}}$ (using (1143) it's straightforward to show that $\mathrm{Tr}[\omega^{\wedge 4}]=0$, so that the winding number density term is closed. Showing that the $1/12\pi$ coefficient is the correct normalization can be done by computing the integral for a fixed example field configuration; see one of the diary entries on WZW models for more detail. The winding number is $W=\frac{1}{24\pi^2}\int\mathrm{Tr}[\omega^{\wedge 3}]$.) Anyway, using this quantization on the integral of the $\omega^{\wedge 3}$ term, we see that the whole CS action is indeed gauge invariant modulo elements of $\overline{\mathbb{Z}}$.

Building instantonson S^4

Now we use this result to construct $SU(N)$ instantons on S^4. We will cover S^4 with two patches U_N and U_S, each homeomorphic to a 3-ball, with $U_N\cap U_S=S^3_{eq}$, the equatorial 3-sphere. We want to compute $I=\frac{1}{8\pi^2}\int\mathrm{Tr}[F\wedge F]$. Now on each patch U_N,U_S, the gauge field A is a well-defined 1-form, and so we can use Stoke's theorem. Thus

$$I=\frac{1}{8\pi^2}\left(\int_{U_S}\mathrm{Tr}[F_{A_S}\wedge F_{A_S}]+\int_{U_N}\mathrm{Tr}[F_{A_N}\wedge F_{A_N}]\right)=\frac{1}{2\pi}\int_{S^3_{eq}}(\mathcal{L}_{CS_1}[A_N]-\mathcal{L}_{CS_1}[A_S]), \quad (1148)$$

where $\mathcal{L}_{CS_1}[A]$ is the CS action at level 1.

Now to create an instanton we glue up the sections of the gauge bundle on U_N to those on U_S through a large gauge transformation[114]. The existence of nontrivial gauge transformations in this case is guaranteed from $\pi_3(SU(N)) = \mathbb{Z}$. So, we choose the transition function g_{NS} such that g_{NS} is a nontrivial homotopy class in $\pi_3(SU(N))$. Then the gauge fields get glued together as $A_N = g_{NS}^\dagger(A_S + d)g_{NS}$. So

$$I = \frac{1}{2\pi}\int_{S^3_{eq}}(\mathcal{L}_{CS_1}[g^\dagger_{NS}(A_S+d)g_{NS}] - \mathcal{L}_{CS_1}[A_S]). \tag{1151}$$

Now we can use our result for the gauge variation of $\mathcal{L}_{CS_1}$ to write

$$I = \frac{1}{2\pi}\int_{S^3_{eq}}\left(-\frac{1}{4\pi}d\mathrm{Tr}[\omega_{NS}\wedge g^\dagger_{NS}A_N g_{NS}] - \frac{1}{12\pi}\mathrm{Tr}[\omega_{NS}^{\wedge 3}]\right), \qquad \omega_{NS} = g^\dagger_{NS}dg_{NS}. \tag{1152}$$

The first term dies, and so we get

$$I = -\frac{1}{24\pi^2}\int \mathrm{Tr}[\omega_{NS}^{\wedge 3}] = -W \in \mathbb{Z}, \tag{1153}$$

which is (the negative of; sorry for the dumb sign choice) the winding number of g_{NS}.

Let's remind ourselves why the $1/24\pi^2$ coefficient is there, just for fun. We'll do the calculation for $SU(2)$ for simplicity. The winding number 1 map in $\pi_3(SU(N))$ is

$$g_{NS} = x_\mu\widetilde{\sigma}^\mu, \tag{1154}$$

with $x_\mu \in S^3$ a unit vector and $\widetilde{\sigma}^\mu = (\mathbf{1}, iX, iY, iZ)$. Note that as required, $g_{NS}g^\dagger_{NS} = x_\mu x^\mu \mathbf{1} = \mathbf{1}$, and $\det g_{NS} = x_\mu x^\mu = 1$.

To evaluate the winding number integral we can either go to spherical coordinates and do lots of algebra, or use a clever trick. The clever trick is as follows: since g_{NS} is uniform on the S^3, we just need to compute the winding number density at a particular point on the 3-sphere, and then multiply the result by $2\pi^2 = \mathrm{vol}\,(S^3)$. Let us choose the north pole, where the field points in the $\mathbf{1}$ direction. Now ω is

$$\omega = (x_\mu\widetilde{\sigma}^\mu)^\dagger d(x_\mu\widetilde{\sigma}^\mu) = (x_\mu\widetilde{\sigma}^\mu)^\dagger(\sigma_\nu - x^\lambda\sigma_\lambda x_\nu)\,dx^\nu. \tag{1155}$$

Evaluating this at the point $x_\nu = (1,0,0,0)$, the only derivatives that enter are ∂_i, where $i \in \{x,y,z\}$, since these are the coordinates in the tangent space at $(1,0,0,0)$.

[114] This is exactly the same as how we build e.g. mangetic monopoles on S^2 for $U(1)$ gauge theory: we take the gauge field on the northern / southern hemispheres to be e.g.

$$A_N = \frac{1-\cos\theta}{2}d\phi, \quad A_S = \frac{-1-\cos\theta}{2}d\phi, \tag{1149}$$

so that on the equator, $A_N - A_S = d\phi$, which means that on the equator, A_N and A_S differ by a large gauge transformation on the S^1 (also note how A_N is not well-defined at the south pole $\theta = \pi$, and A_S is not well-defined at the north pole $\theta = 0$). The "instanton" number is then

$$\frac{1}{2\pi}\left(\int_{U_N}F_{A_N} + \int_{U_S}F_{A_S}\right) = \frac{1}{2\pi}\int_{S^1_{eq}}(A_N - A_S) = 1, \tag{1150}$$

where $U_{N/S}$ is the northern / southern hemisphere.

Thus ω becomes just $\sigma_i dx^i$, and the integrand is

$$\frac{i^3}{24\pi^2}\mathrm{Tr}[\sigma^i\sigma^j\sigma^k]dx^i \wedge dx^j \wedge dx^k = \frac{i^3}{4\pi^2}\mathrm{Tr}[XYZ]d^3x = \frac{1}{2\pi^2}d^3x, \tag{1156}$$

which is just the pullback of the volume form on S^3. Multiplying this by the volume of S^3 we get 1, and so the $1/24\pi^2$ normalization was indeed correct.

What should we do if we want a winding number $W > 1$ map? To get winding number 1 we pulled back the volume form, so to get winding number W we should pullback W times the volume form. There are some physics books which say that for $\omega = g^{-1}dg$ we should keep the g configuration for $W = 1$ but replace g by g^W: this is wrong since then W appears in I as W^3 as I is cubic in the ω's. Instead, break up the spacetime S^3 as SS^2, where SS^{d-1} denotes the suspension of S^{d-1} by S^1. Let the coordinate on the S^1 that's doing the suspension be θ, and the coordinates on the S^2 be ϕ, ψ. Then if $g_1(\theta, \phi, \psi)$ is the winding number 1 configuration, the winding number W configuration is $g_W = g_1(W\theta, \phi, \psi)$. What's going on here is that the map completes winding number 1 during $\theta \in [0, \pi/W)$, and so the total winding number for $\theta \in [0, \pi]$ is W (we imagine composing W identity maps $S^3 \to S^3$ in a row, with the basepoint [$\theta = 0$ point] of each map being the terminal point [$\theta = \pi$ point] of the previous one). This is also clear from the integral formula: the wedge product means that the integrand contains one derivative for each of the coordinates on the S^3, so multiplying one of the coordinates by W will increase the integrand by a factor of W.

How do we make instantons for other gauge groups G? We do this by using a map $SU(2) \to G$ induced from a map $\mathfrak{su}(2) \to \mathfrak{g}$, which always exists because of roots. Specifically we can always get a winding number $W = 1$ map by taking the gauge configuration

$$g_{NS}^{SU(N)} = \mathbf{1}_{N-2} \oplus g_{NS}^{SU(2)}, \tag{1157}$$

which is in $SU(N)$ as required.

We can also get winding number $W > 1$ maps by embedding the $SU(2)$ instanton inside of $SU(N)$ with a different representation. For example, for the map $SU(2) \to SU(3)$, we can choose to embed the $SU(2)$ either in the fundamental, or in the adjoint. The difference in the instanton number just comes from the difference in the trace of the generators; in this case we can get winding number $W = \pm 4$.

Chirality of instanton-induced zero modes in four dimensions

Consider some massless fermions coupled a background gauge field. The index theorem tells us the net chirality ind $(\not{D}_A) = \nu_+ - \nu_-$ of the zero modes of the Dirac operator is determined by the instanton number (we are ignoring the gravitational contribution). However, it only tells us the difference in the left- and right-chirality zero modes; it does not tell us how many zero modes there are. Today we will argue however that in an instanton field such that $\nu_+ - \nu_- = n$, we actually generically have $\nu_+ = n, \nu_- = 0$.

Consider a zero mode of the Dirac operator with chirality $\pm$:

$$\not{D}_A(1 \pm \bar{\gamma})\psi_\pm = 0. \tag{1158}$$

Now hit this with $\not{D}_A$, and use (the $\circ$ notation here is meant to emphasize that the derivatives in the left $\not{D}$ act on the A in the right $\not{D}$)

$$\begin{aligned}\not{D}_A \circ \not{D}_A &= \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - A^a_\mu A^b_\nu\gamma^\mu\gamma^\nu T^a T^b - iA_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\} - i(\partial_\mu A_\nu)\gamma^\mu\gamma^\nu \\ &= \partial_\mu\partial^\mu - A_\mu A^\mu - i\partial_\mu A^\mu - \frac{1}{2}A^a_\mu A^b_\nu[\gamma^\mu,\gamma^\nu]T^aT^b - 2iA_\mu\partial^\mu - \frac{i}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\gamma^\mu\gamma^\nu \\ &= \partial_\mu\partial^\mu - A_\mu A^\mu - i\partial_\mu A^\mu - \frac{1}{2}if^{abc}A^b_\mu A^c_\nu T^a - \frac{i}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\gamma^\mu\gamma^\nu \\ &= (\partial_\mu - iA_\mu)^2 - \frac{i}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu,\end{aligned} \tag{1159}$$

where $(\partial_\mu - iA_\mu)^2$ means that the ∂_μ acts on the A_μ as well.

Using this, we have

$$0 = \not{D}_A \circ \not{D}_A(1 \pm \bar{\gamma})\psi_\pm = \left[(\partial_\mu - iA_\mu)^2 - \frac{i}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\right](1 \pm \bar{\gamma})\psi_\pm. \tag{1160}$$

Now in Euclidean signature, $\bar{\gamma} = \prod_\mu \gamma^\mu$ (all of the γs, including $\bar{\gamma}$, are Hermitian). Thus we have

$$\gamma^\mu\gamma^\nu\bar{\gamma} = -\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\gamma^\lambda\gamma^\sigma. \tag{1161}$$

We now multiply the field strength term in (1160) by $(1 \pm \bar{\gamma})/2$, which is allowable since it's a projector. Thus the putative zero mode satisfies

$$\left[(\partial_\mu - iA_\mu)^2 - \frac{i}{2}F_{\mu\nu}\Sigma^{\mu\nu}_\pm\right](1 \pm \bar{\gamma})\psi_\pm = 0, \tag{1162}$$

where we have defined

$$\Sigma^{\mu\nu}_\pm = \gamma^\mu\gamma^\nu \mp \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\gamma^\lambda\gamma^\sigma. \tag{1163}$$

Note that Σ_+ is anti-self-dual while Σ_- is self-dual (note to self: missed a sign?):

$$(\star\Sigma_\pm)^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\Sigma^{\lambda\sigma}_\pm = \mp\Sigma^{\mu\nu}_\pm. \tag{1164}$$

Let us write $F = \mathcal{F}_+ + \mathcal{F}_-$, where $\mathcal{F}_+ = (F + \star F)/2$ is self-dual and $\mathcal{F}_- = (F - \star F)/2$ is anti-self-dual (we are in Euclidean signature, with $\star^2 = (-1)^{p(4-p)}$ on p-forms). Now, the contraction of a SD with an ASD form vanishes, since $A \wedge \star B = B \wedge \star A$ means that $A \wedge \star B = -A \wedge \star B$ if only one of A, B is ASD. Thus we can write

$$\left[(\partial_\mu - iA_\mu)^2 - \frac{i}{2}\mathcal{F}_\mp^{\mu\nu}\Sigma_\pm^{\mu\nu}\right](1 \pm \bar{\gamma})\psi_\pm = 0. \tag{1165}$$

Now we consider a field for which

$$\frac{1}{8\pi^2}\int \mathrm{Tr}[F \wedge F] = n. \tag{1166}$$

Then

$$n = \frac{1}{8\pi^2}\int \left(\mathrm{Tr}[\mathcal{F}_+ \wedge \mathcal{F}_+] - |\mathrm{Tr}[\mathcal{F}_- \wedge \mathcal{F}_-]|\right). \tag{1167}$$

Here we have used $\int \mathcal{F}_+ \wedge \mathcal{F}_- = \int \star\mathcal{F}_+ \wedge \star\mathcal{F}_- = -\int \mathcal{F}_+ \wedge \mathcal{F}_- = 0$, and the fact that $0 < \int \mathcal{F}_- \wedge \star\mathcal{F}_- = -\int \mathcal{F}_- \wedge \mathcal{F}_-$. Thus we see that the self-dual part of the field strength contributes positively to the instanton number, while the anti-self-dual part contributes negatively. Both SD and ASD parts contribute positively to the $\int \mathrm{Tr}[F \wedge \star F]$ YM action. This means that if we want to look for a minimal-action configuration with a given instanton number, we can restrict ourselves to purely SD or purely ASD fields[115].

Let us suppose $n > 0$, so that the minimal action configuration has $\mathcal{F}_+ \neq 0, \mathcal{F}_- = 0$. Then we see that a putative ψ_+ zero-mode obeys

$$(\partial_\mu - iA_\mu)^2\psi_+ = 0, \tag{1169}$$

since there is no anti-self-dual field strength contribution. Now $(\partial_\mu - iA_\mu)$ is anti-Hermitian, so $(\partial_\mu - iA_\mu)^2$ is Hermitian with $\mathbb{R}$ eigenvalues. Furthermore, it is negative-definite, since the eigenvalue of an eigenspinor of $(\partial_\mu - iA_\mu)$ is purely imaginary (by anti-Hermitian-ness). Thus since all the eigenvalues of $(\partial_\mu - iA_\mu)$ have the same sign and only the non-normalizable choice $\psi_+ = 0$ has a zero eigenvalue, there are no normalizable solutions to the above equation, and we conclude that there are no $+$ zero modes. Similarly, if we were to choose $n < 0$ so that the minimal action configuration for the gauge fields resulted in a purely ASD field strength, we would find $(\partial_\mu - iA_\mu)^2\psi_- = 0$, meaning that there are no $-$ zero modes.

So, at least for minimal-action purely SD / ASD field configurations, not only does the instanton number determine the net difference in $+$ and $-$ chirality zero modes, but it also tells us that $\nu_- = 0$ if the instanton number is positive, while $\nu_+ = 0$ if the instanton number is positive, and so the chiral difference in zero modes is actually equal to the (signed) total number of zero modes. Now we can imagine slowly deforming

[115] This is being a bit glib, since there may be instantons with $n \neq 0$ and field strengths which are not purely SD or ASD, but which are still solutions to the equations of motion (just not minimal action ones). For example, consider $SU(N \geq 4)$ gauge theory. Then we can consider the field configuration

$$A^{SU(N)} = 0_{N-4\times N-4} \oplus A^{SU(2)}_{SD,k} \oplus A^{SU(2)}_{ASD,l}, \tag{1168}$$

where $A^{SU(2)}_{SD,k}$ is a configuration with self-dual field strength and $SU(2)$ instanton number k, and similarly for $A^{SU(2)}_{ASD,l}$. This configuration has instanton number $k-l$ and is a solution to the equations of motion, but is not purely SD or ASD.

the background fields away from the minimal action purely SD / ASD configuration, while keeping the instanton number fixed. Since the number of $\pm$ chirality zero modes cannot change continuously, we expect that all configurations with a given instanton number, not just the purely SD / ASD ones, have a total number of zero modes equal to the chiral difference in zero modes.

Finally, a miscellaneous comment on reflections that I didn't know where to put. We know that reflections take left-handed fermions to right-handed ones—therefore in order for the zero mode situation to be invariant under reflections, we must have that a reflection takes a SD 2-form to an ASD 2-form. Indeed, this is true: one can see this by realizing that if we consider a reflection of the α coordinate, then

$$F_{\mu\nu} \mapsto [I_\alpha]_\mu^\lambda [I_\alpha]_\nu^\rho F_{\lambda\rho} \qquad [I_\alpha]_\mu^\lambda \equiv \delta_\mu^\lambda - 2\delta_\mu^\alpha \delta_\alpha^\lambda. \tag{1170}$$

The result then follows after a little bit of algebra.

GSD for K matrix CS theory from phase space

Today is a quickie: we show a cool way that I hadn't seen in the literature (I'm sure it exists somewhere though) for how to get the $|\det K|^g$ GSD on a Riemann surface of genus G for a CS theory with K-matrix K.

The strategy we will take will be to compute the volume of phase space. First we need the symplectic form. We get this by taking a variation of $K(a, da) = a_i \wedge da_j K^{ij}$, integrating by parts, and looking at the boundary term. Choosing a Cauchy slice Σ_g on which to quantize, the sympletic potential is

$$\omega = \frac{1}{4\pi} \int_{\Sigma_g} K(a, \delta a). \tag{1171}$$

This gives us the symplectic potential as

$$\Omega = \frac{1}{4\pi} \int_{\Sigma_g} K(\delta a, \delta a) = \frac{1}{4\pi} \int_{\Sigma_g} K_{ij} \delta a^i \wedge \delta a^j. \tag{1172}$$

Here the wedge product takes place in both actual space and in variational space. Thus e.g.

$$\delta a^i \wedge \delta a^j = \delta_1 a_x^i \delta_2 a_y^j - \delta_2 a_x^i \delta_1 a_y^j - \delta_1 a_y^i \delta_2 a_x^j + \dots \tag{1173}$$

where δ_1, δ_2 are two (orthogonal) variations in variational space.

The space of solutions to the equations of motion is the space of flat connections on Σ_g. We can thus write

$$\delta a^i = \sum_{C_\mu \in H_1(\Sigma_g;\mathbb{Z})} \delta_\alpha \theta_\mu \widehat{C}_\mu, \tag{1174}$$

where the Poincare dual is taken in Σ_g, so that $\widehat{C}_\mu$ is a flat 1-form. Here the coefficients $\theta_\mu \in [0, 2\pi)$, since when $\theta_\mu \in \overline{\mathbb{Z}}$, $\theta_\mu \widehat{C}_\mu$ (no sum) is a large gauge transformation with $\overline{\mathbb{Z}}$ holonomy around the cycle C_μ.

Before plugging this in to the symplectic form, we note that for Σ_g of genus g, $H_1(\Sigma_g;\mathbb{Z}) = \mathbb{Z}^{2g}$, with generators $C_{0,\rho}, C_{1,\rho}$ for $\rho \in \mathbb{Z}_g$, such that

$$C_{\alpha,\rho} \cap C_{\beta,\sigma} = \delta_{\rho,\sigma}\delta_{\alpha,\beta+1}(-1)^\alpha, \tag{1175}$$

with $\alpha, \beta \in \mathbb{Z}_2$ and consequently where $\beta + 1$ is taken mod 2. The minus sign here is indeed because the $\widehat{C}_\mu$'s anticommute when wedged together.

Anyway, the point is that the homology of Σ_g is just g powers of the homology of the torus (since Σ_g is a connected sum). Thus, doing the integral, we can write Ω as

$$\Omega = \frac{1}{4\pi} K_{ij} \sum_{\rho=1,\dots,g} \sum_{\alpha=0,1} (-1)^\alpha \delta\theta^i_{\rho,\alpha} \wedge \delta\theta^j_{\rho,\alpha+1}, \tag{1176}$$

where now $\wedge$ only takes place in variational space. Since the K matrix is symmetric (it has to be so that the off-diagonal parts add pairwise to give mutual CS terms that are properly quantized as $\sum_{i<j} a^i da^j/2\pi$), the antisymmetry of the sum on α cancels the antisymmetry of the wedge product in variational space, and we can write

$$\Omega = \frac{1}{2\pi} K_{ij} \sum_{\rho=1,\dots,g} \delta\theta^i_{\rho,0} \wedge \delta\theta^j_{\rho,1}. \tag{1177}$$

Thus, for each torus ρ in the connected sum and for each flavor index i, the holonomy around the longitudinal cycle of the ρth torus, namely $\theta^i_{\rho,0}$, will constitute a position variable in the phase space. Its corresponding canonically conjugate momentum variable is then a linear combination (in flavor space) of the holonomies around the other cycle on the ρth torus, namely $\sum_j K_{ij}\theta^j_{\rho,1}$.

To find the GSD, we need to look at the symplectic volume of the ground state subspace. From the sum over ρ, we see that this factors into a product over each torus in the connected sum, each of which have the same phase space volume. Thus the GSD will be $GSD_{\Sigma_g} = (GSD_{T^2})^g$, where T^2 is the torus.

So, we just have to compute GSD_{T^2}. This is evidently

$$GSD_{T^2} = \int \bigwedge_{i=1,\dots,\dim K} \frac{K_{ij}}{4\pi^2} \delta\theta^i_0 \wedge \delta\theta^j_1, \tag{1178}$$

where the integral is in variational space. Here we have remembered to divide by a further factor of 2π since the phase space volume form for a single degree of freedom is $dq \wedge dp/h$, and in our units $h = 2\pi$.

To see how $\det K$ arises from this, we just have to use the antisymmetry of the variational wedge product. Since $\delta\theta^i_\alpha \wedge \delta\theta^i_\alpha = 0$, the only terms which survive the product are those which contain the full volume form

$$V = \bigwedge_{i\in 1,\ldots,\dim K} \delta\theta^i_0 \wedge \bigwedge_{j\in 1,\ldots,\dim K} \delta\theta^j_1. \tag{1179}$$

So, bringing the $\delta\theta$'s in our expression for GSD_{T^2} into this form,

$$GSD_{T^2} = \frac{1}{(4\pi^2)^{\dim K}} \int \bigwedge_{i=1,\ldots,\dim K} \delta\theta^i_0 \wedge \sum_{\{j_\lambda\}\in\mathbb{Z}^{\dim K}_{\dim K}} K_{1j_1}\delta\theta^{j_1}_1 \wedge K_{2j_2}\delta\theta^{j_2}_1 \wedge\cdots\wedge K_{\dim K j_{\dim K}}\delta\theta^{j_{\dim K}}_1. \tag{1180}$$

Moving all of the $\delta\theta_1$'s into order, which we do at the cost of an ϵ symbol, we have

$$GSD_{T^2} = \frac{1}{(4\pi^2)^{\dim K}} \int V \sum_{\{j_\lambda\}\in\mathbb{Z}^{\dim K}_{\dim K}} \epsilon^{j_1,\ldots,j_{\dim K}} K_{1j_1}K_{2j_2}\cdots K_{\dim K j_{\dim K}} = \frac{\det K}{(4\pi^2)^{\dim K}} \int V. \tag{1181}$$

Now since each of δ^i_0, δ^i_1 can be varied from 0 to 2π, the integral over V exactly cancels the factor in the denominator. Thus we get $GSD_{T^2} = |\det K|$, and hence $GSD_{\Sigma_g} = |\det K|^g$, as required.

Topological terms from integrating out fermions in four dimensions and some characteristic class relations for vector bundles

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Today's diary entry is a small compendium of results about what kind of θ terms are produced when integrating out massive fermions in four dimensions. A good reference for this diary entry is Witten's review article [26].

Complex / Dirac fermions

First we will look at the case where the fermions transform in a complex representation of the full symmetry group ("full" here meaning including spacetime symmetries) that we will assume to include a $U(1)$ fermion number conservation factor. In this case, there is no antisymmetric bilinear form we can use to construct a symmetric action involving a single fermion field, and so any symmetry-preserving Dirac operator appearing in the action will have to pair two independent fields $\bar{\psi}$ and ψ, with opposite charges under the $U(1)$ (this is what we mean when we say the fermion transforms in a complex representation: there is no antisymmetric bilinear form, invariant under the symmetry, that pairs a single fermion field with itself). Thus in the basis $(\psi,\,\bar{\psi})^T$, the Lagrangian will be purely off-diagonal:

$$\mathcal{L} = \begin{pmatrix} \psi & \bar{\psi} \end{pmatrix} \begin{pmatrix} 0 & i\not{D} \\ -i\not{D}^T & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \tag{1182}$$

where $i\not{D}$ is Hermitian. The minus sign and transpose are needed since the fermions need to be paired antisymmetrically: otherwise the action vanishes, since e.g.

$$\bar{\psi}_\alpha [i\not{D}]^{\alpha\beta} \psi_\beta = -\psi_\beta [i\not{D}^T]^{\beta\alpha} \bar{\psi}_\alpha. \tag{1183}$$

When doing manipulations like this, we should remember that $\partial^T = \overleftarrow{\partial} = -\partial$. So in this notation, the derivative in the matrix $[\not{D}]^{\alpha\beta}$ always acts on the second index: thus the ψ in $[\not{D}]^{\alpha\beta}\psi_\beta$ is differentiated, while the ψ in $[\not{D}]^{\beta\alpha}\psi_\beta$ is not.

Anyway, the point of this is just to note that this structure for the Lagrangian means that integrating out the fermions ψ and $\bar{\psi}$ produces a determinant (of $i\not{D}$) rather than just a Pfaffian. Working in Euclidean signature and adding a mass m and a gauge field A produces a partition function $Z[A;m] = \det(i\not{D}_A - m)$, where m is real (in Euclidean time γ^0 is Hermitian, so the Lagrangian is $\bar{\psi}(i\not{D}_A - m)\psi$).

Since $\not{D}_A$ anticommutes with $\bar{\gamma}$ (recall that we are working in four dimensions), if ψ is an eigenspinor of $i\not{D}_A$ with non-zero eigenvalue, then $\bar{\gamma}\psi$ is a linearly independent eigenspinor with an eigenvalue of the opposite sign (they are linearly independent since they have different eigenvalues: $\langle\psi, i\not{D}_A\psi\rangle = \lambda\langle\psi,\psi\rangle \implies \langle\lambda, i\not{D}_A\bar{\gamma}\psi\rangle = -\lambda\langle\psi,\bar{\gamma}\psi\rangle = \langle\bar{\gamma}\psi, i\not{D}_A\psi\rangle = +\langle\psi,\bar{\gamma}\psi\rangle \implies \langle\psi,\bar{\gamma}\psi\rangle = 0$). Since they are linearly independent, $\psi_\pm \equiv (1\pm\bar{\gamma})\psi/2$ must be nonzero for both choices of sign: non-zero modes have support on both chirality subspaces, and so every non-zero-mode comes as a member of a positive-negative eigenvalue pair. Remember that here "zero-mode" means a mode which is annihilated by $i\not{D}_A$, not a mode which is annihilated by the Hamiltonian.

Now for the partition function: we have

$$\det(i\not{D}_A - m) = \left(\prod_{\lambda_j>0} (\lambda_j - m)(-\lambda_j - m)\right) m^{N_+ + N_-}, \tag{1184}$$

where N_σ is the number of zero-modes with chirality σ. Note that when we say "number of zero-modes", we really mean "number of positive-charge zero modes": we are just computing the determinant of $i\not{D}_A$ as it acts on ψ, and not on $\bar{\psi}$. This number can be odd (and is the number relevant for computing the parition function), but the full number of zero modes, of both positive and negative charges, is always even. Indeed, if $\psi_\pm$ is a zero mode for the field ψ then its complex conjugate is a zero mode for the field $\bar{\psi}$[116] and so the full number of zero modes (for both the fields ψ and

[116]The complex conjugate zero mode has opposite chirality: the associated zero mode of $\bar{\psi}$ is $\bar{\psi}_\pm = \gamma^0\psi^*_\pm$, and $\bar{\gamma}\bar{\psi}_\pm = -\gamma^0\bar{\gamma}\psi^*_\pm = \mp\bar{\psi}_\pm$. Note how here we are treating $\bar{\psi}$ as a column vector like ψ, which is a slightly more transparent thing to do since it really is an independent field.

$\bar{\psi}$) is actually $2(N_+ + N_-)$.

The factor of $m^{N_+ + N_-}$ can also be understood from looking at how the zero modes get paired up by the mass term: each positively-charged zero mode ψ_+ appears in the path integral together with its negatively-charged partner as

$$\int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\int \psi_-^\dagger m \psi_+} = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \left(1 - \int \psi_-^\dagger m \psi_+\right) \propto m, \tag{1185}$$

because of how Grassmann integration works. Thus we get a factor of m for each positively-charged zero mode, rather than for zero modes of all charges.

Anyway, note how the product in the expression for $\det(i\not{D}_A - m)$ is independent of the sign of m due to the pairing of opposite-chirality eigenmodes — the only dependence on $\text{sgn}(m)$ comes from the zero modes. Thus we have

$$\frac{Z[A;m]}{Z[A;-m]} = (-1)^{N_+ + N_-} = (-1)^{N_+ - N_-} = e^{i\pi \text{ind }(i\not{D}_A)}. \tag{1186}$$

Now the index of the Dirac operator, for A a connection on a bundle E, is given by the index formula as

$$\text{ind }(i\not{D}_A) = \int \widehat{A} \wedge \text{ch}(E). \tag{1187}$$

Here $\text{ch}(E)$ is the Chern *character* of the bundle E, *not* the Chern class. So we can write this as

$$\text{ind }(i\not{D}_A) = \int \widehat{A} \wedge e^{F_A/2\pi}. \tag{1188}$$

Now the Dirac genus only involves Pontryagin[117] classes since it's a characteristic class in the real (involving traceless field strengths) tangent bundle. Thus only $4n$-dimensional classes contribute to $\widehat{A}$. For a 4-manifold M, we just need

$$\widehat{A} = 1 - \frac{1}{24}p_1(TM) + \dots, \qquad \text{ch}(E) = \text{Tr}[\mathbf{1}] + \text{Tr}[F_A/2\pi] + \frac{1}{2}\text{Tr}[F_A/2\pi \wedge F_A/2\pi] + \dots, \tag{1189}$$

with the trace taken in the fundamental representation. Then

$$\text{ind }(i\not{D}_A) = -\frac{\dim(E)}{24}\int p_1(TM) + \frac{1}{8\pi^2}\int \text{Tr}[F_A \wedge F_A]. \tag{1190}$$

Here $p_1(TM)$ is $\text{Tr}[R \wedge R]$ with some normalization that I can never remember. Writing the gravitational contribution in terms of the signature with $\int \widehat{A} = \sigma/8$, we have

$$\frac{Z[A;m]}{Z[A;-m]} = \exp\left(\frac{i\pi}{8\pi^2}\int \text{Tr}[F_A \wedge F_A] - i\pi \frac{\dim E}{8}\sigma\right), \tag{1191}$$

where σ is the signature. Note that this is a completely non-perturbative result.[118]

[117] Spelling?! I can never remember, although in any case it seems like there is no standard romanization.

[118] This is because locally, the integrand is a total derivative. If any Feynman diagram were to contribute to the effective action for A, it would then in momentum space contain a factor of p_{tot}, where p_{tot} is the sum of the momenta on all the external A legs attached to the diagram. Since momentum is conserved $p_{tot} = 0$, and therefore no Feynman diagram can contribute to this result. The only times when such topological terms can show up diagramatically is when there is an operator insertion (like j_A^μ) in the diagram to provide some extra momentum.

On a spin manifold $\sigma \in 16\mathbb{Z}$, and so the signature part makes no contribution. On a general non-spin manifold, σ can be an arbitrary integer, and so the fact that $\text{ind}\,(i\not{D}_A) \in \mathbb{Z}$ tells us that for a spinc connection A,

$$\frac{1}{2}\int \frac{F_A}{2\pi} \wedge \frac{F_A}{2\pi} - \frac{\sigma}{8} \in \mathbb{Z}, \tag{1192}$$

which means that for a spinc connection,

$$\frac{1}{2}\int \frac{F_A}{2\pi} \wedge \frac{F_A}{2\pi} \in \frac{1}{8}\mathbb{Z}. \tag{1193}$$

Of course, this makes total sense: if A is spinc then $2F_A/2\pi$ is an integer class, and so we can write the above integral as $\frac{1}{8}\int(2F_A/2\pi) \wedge (2F_A/2\pi)$, which is then manifestly in $\frac{1}{8}\mathbb{Z}$.

Pseudoreal / Majoranna fermions

So far we've seen what topological term gets generated upon integrating out a Dirac fermion. What about a Majorana fermion? Our fermion χ will be assumed to transform in a pseudoreal representation of the full symmetry group, so that there exists an antisymmetric bilinear form J invariant under the symmetry, which allows us to construct a symmetric mass term via $\chi_\alpha J^{\alpha\beta}\chi_\beta$. Since J is an invariant form then so too is $J(i\not{D}_A)$[119] and so the pairing for the kinetic term is then

$$\mathcal{L} \supset \chi^T J(i\not{D}_A)\chi. \tag{1194}$$

In what follows we will take J to be real, with $J^2 = -\mathbf{1}$, so that J can be thought of as a complex structure. I think that this can be done wolog (with this convention the Hermitian mass term is $i\chi^T J\chi$). Note that

$$\chi^T(Ji\not{D}_A)\chi = -\chi^T[Ji\not{D}_A]^T\chi \implies [Ji\not{D}_A]^T = -Ji\not{D}_A \implies [i\not{D}_A]^T J = Ji\not{D}_A. \tag{1195}$$

Since $J^2 = -\mathbf{1}$, we then have

$$J[i\not{D}_A]^T J = -i\not{D}_A, \tag{1196}$$

which indeed is telling us that J is a kind of complex structure. Now consider the operator JK, where K is complex conjugation. We have

$$JK[i\not{D}_A]^\dagger KJ = -i\not{D}_A. \tag{1197}$$

Since $(JK)^2 = -\mathbf{1}$ (this is true even if we chose J to be Hermitian instead of anti-Hermitian, since then J would be complex) and $i\not{D}_A$ is Hermitian (in Euclidean signature), we have

$$JKi\not{D}_A = i\not{D}_A JK, \tag{1198}$$

[119] Here it's best to think about $i\not{D}_A$ as being an operator rather than a bilinear form: J is used to raise / lower fermion indices, and $i\not{D}_A$ preserves the index placement. Thus J pairs two lower-index or two upper-index fermions, while $i\not{D}_A$ pairs and upper one with a lower one or vice versa.

i.e. $JK\gamma_j = -\gamma_j JK$. Therefore pseudoreal fermions come equipped with an antiunitary action JK that commutes with the Dirac operator. Since $(JK)^2 = -\mathbf{1}$, we can then conclude that all eigenspinors of $i\not{D}_A$ come in pairs (related by JK) with identical eigenvalues: each eigenspinor χ comes with a linearly independent eigenspinor $JK\chi$, with the same eigenvalue λ, so then in the basis $(\chi,\, JK\chi)$, $i\not{D}_A$ has a block $\lambda\mathbf{1}_{2\times 2}$. Multiplying this by J, we see that a single eigenvalue λ of the Dirac operator then appears in the Lagrangian as

$$\mathcal{L} \supset \begin{pmatrix} \chi & JK\chi \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} \chi \\ JK\chi \end{pmatrix}. \tag{1199}$$

Summing over all such pairs $\chi, JK\chi$, we get a big antisymmetric matrix. If we expand $e^{-\int \mathcal{L}}$ to the order at which the Grassmann integration gives something nonzero, we see that the partition function becomes

$$\mathrm{Pf}(Ji\not{D}_A) = \pm[\det(i\not{D}_A)]^{1/2}, \tag{1200}$$

since $\det J = 1$.

One other thing we will need to know is that the doubling of the spectrum because of JK also restricts to a doubling of each eigenspinor of definite helicity. So this means that all non-zero-mode eigenspinors of $i\not{D}_A$ are qudrupled (one for each chirality, and for each chirality two modes related by JK), while the zero-modes are merely doubled. To show this, we observe that since $(JK)^2 = -\mathbf{1}$ and since $\bar{\gamma}$ is a product of an even number of γ matrices,

$$JK\bar{\gamma}JK = JK\left(\prod_j \gamma_j\right)JK = -\prod_j (JK\gamma_j JK) = -\prod_j (-\gamma_j (JK)^2) = -\bar{\gamma} \implies JK\bar{\gamma} = \bar{\gamma}JK, \tag{1201}$$

and so

$$[JK, \bar{\gamma}] = 0. \tag{1202}$$

Thus each definite-chirality mode is doubled (note to self: do the Lorentzian-signature case as well).

Anyway, we can now compute the topological term induced by integrating out the fermions. Since the Pfaffian is the square root of the determinant, we can just naively take the square root of the partition function we found for the Dirac fermion (which did give a determinant), and write

$$\frac{Z[A;m]}{Z[A;-m]} = (-1)^{(N_+ + N_-)/2}. \tag{1203}$$

Now because of the doubling of the spectrum due to the JK operator we discussed above, and because it commutes with $\bar{\gamma}$, we know that (unlike in the Dirac case), both N_+ and N_- must separately be even. Thus we have $(-1)^{N_-/2} = (-1)^{-N_-/2}$, and so

$$\frac{Z[A;m]}{Z[A;-m]} = (-1)^{\mathrm{ind}\,(i\not{D}_A)/2} = \exp\left(\frac{1}{2}\left[\frac{i\pi}{8\pi^2}\int \mathrm{Tr}[F_A \wedge F_A] - i\pi\frac{\dim E}{8}\sigma\right]\right). \tag{1204}$$

This response is in keeping with the fact that a Majoranna is “half” a Dirac fermion.

For Majoranas, because the topological term involves $e^{\pi i p_1(B)/2}$ where B is either the gauge bundle or the tangent bundle, it evidently helps to have expressions for the Pontryagin classes mod 4. The most useful such relation is one derived in the entry on Pointryagin classes, namely

$$P(w_2(E)) = p_1(E) + 2w_4(E) \mod 4, \tag{1205}$$

for any vector bundle (real or complex) E. For $SO(3)$ the $w_4(E)$ term is trivial, but in general it makes a contribution[120]. In any case, this means that we can write the topological response as

$$\frac{Z[A;m]}{Z[A;-m]} = \exp\left(\frac{i\pi}{2}\left[P(w_2(E)) + 2w_4(E) - \frac{\dim E}{8}\sigma\right]\right). \tag{1206}$$

Here the second SW class of the gauge bundle is allowed to be non-trivial, as long as the second SW class of the tangent bundle is also non-trivial in the same way, so that $w_2(E) + w_2(TM) = 0 \mod 2$. Note that on a spin manifold, the even-ness of the intersection form means that $\int p_1(E) \in 2\mathbb{Z}$, and so the even-ness of the index of the Dirac operator means that $\frac{\dim E}{8}\sigma \in 2\mathbb{Z}$. In particular, taking just a single Majoranna fermion (not coupled to any gauge field) tells us that on a spin manifold, $\sigma \in 16\mathbb{Z}$. On the other hand, on a non-spin manifold, σ can be any integer, and this places constraints on how the Pointryagin term is quantized (although to work on a non-spin manifold, we need to be able to choose $w_2(E)$ in such a way that the full gauge- + spin-connection satisfies the cocycle condition).

The typical example for Majorana fermions is when A is a connection for an $SO(n)$ associated bundle. When n is even, $-\mathbf{1} \in SO(n)$ and $-\mathbf{1} \in \mathrm{Spin}(4)$ act in the same way on fermions, and so our fermion is really coupled to a $[\mathrm{Spin}(4) \times SO(n)]/\mathbb{Z}_2$ connection. Now for $n = \dim E$ even, the gravitational term is quantized in $\frac{1}{4}\mathbb{Z}$; thus in order to maintain the integrality of the index of the Dirac operator, the Pointryagin class must also be quantized in $\frac{1}{4}\mathbb{Z}$, in such a way that the gauge and gravitational contributions add to give something in $2\mathbb{Z}$. This quantization makes sense, since when passing from $SO(n) \to SO(n)/\mathbb{Z}_2$ for n even, the quantization of the instanton number (alias $\int p_1(E)$) changes by a factor of $1/4$ on a general non-spin manifold (see a diary entry in last year's diary for a discussion of why). By contrast when n is odd, there

[120] Recall that the kth SW class is the obstruction to finding $\mathrm{Rank}(E) - k + 1$ nowhere vanishing sections of E, and so they become trivial for $k > \mathrm{Rank}(E)$. As a consequence, any $SO(3)$ bundle has $w_4 = 0$, since the SW classes $w_k(E)$ with $k > \mathrm{Rank}(E)$ all vanish, and $\dim[SO(3)] = 3$.

An equivalent way to discuss w_k is to say that if the kth SW class is nonzero, then there is an obstruction to extending the trivialization of the bundle over the k-skeleton. But the converse is not true: there are plenty of cases where there is an obstruction to extending the trivialization, but the associated SW class vanishes. In general the obstruction to extend a G-bundle over the k-skeleton is captured by $\pi_{k-1}(G)$. This could fail to get detected by the SW classes either due to the fact that homotopy groups carry more data than cohomology groups, or because the obstructions always vanish mod 2. For example, the obstruction to extending an $SO(3)$ bundle over the 4-skeleton is non-zero as $\pi_3(SO(3)) = \mathbb{Z}$, even though $w_4 = 0$ because $4 > 3$. Moreover, no mod 2 class could detect this obstruction, since $\pi_3(SO(3))$ should really be thought of as $2\mathbb{Z}$. This is because elements in $\pi_3(SO(3))$ descend from elements in $\pi_3(S^3) = \mathbb{Z}$ from the map $S^3 \to SO(3)$, which is a double cover. Therefore a winding number 1 map in $\pi_3(S^3)$ maps onto a winding number 2 map in $\pi_3(SO(3))$; hence $\pi_3(SO(3)) = 2\mathbb{Z}$.

is no $\mathbb{Z}_2$ identification between the gauge and spin connections, and in order for our fermion to be well-defined, we need to work on a spin manifold.

How spin CS theory sees the spin structure

Consider a CS theory which is spin. How does the partition function $Z[\sigma]$ depend on the spin structure σ? That is, under $\sigma \mapsto \sigma + \beta$ with $\beta \in H^1(X; \mathbb{Z}_2)$, how does $Z[\sigma]$ change? Today we will answer this question by attempting to give a physicist's interpretation of the results in the very nice math paper [17]. I haven't fully understood [17], but I think the results below are likely to be a physicist's translation of a subset of its results.

To illustrate the idea, we will first show how $U(1)_1$ (and by extension, $U(1)_k$) depends on a spin structure; later we will generalize to non-Abelian groups. Our notation will be such that X is a closed 3-manifold, Y is an open 4-manifold with $\partial Y = X$, and Z is a closed 4-manifold.

First let us recall the usual story. The action $\int_Y \mathrm{ch}_2$ (with ch_2 the second Chern character) is independent of the choice of Y mod $\mathbb{Z}$ only if we restrict ourselves to choices of X and Y with spin structures, because of the resultant even-ness of the intersection form on any $Y \sqcup \bar{Y}'$ with $\partial Y = \partial Y'$. Making such a restriction is permissible since $\Omega_3^{\mathrm{Spin}}(pt) = 0$, so any choice of spin X always has a spin Y that it bounds.[121] In this way of thinking about things, the dependence of the action on the spin structure enters because the spin structure controls what sort of choices we are allowed to make for Y. Unfortunately though the bounding manifolds can become complicated, making explicit calculations of the spin structure rather difficult (consider e.g. the manifold with the RRR spin 3-torus as a boundary — understanding an explicit construction of this manfiold is prohibitively mathematical for most physicists).

However, I think a slightly different logic may be possible here. Namely, it may be possible to get away with only using a spin structure on X to define the action, and to still allow the extending manifolds Y to be *arbitrary* (i.e., not necessarily spin) 4-manifolds. If this is true, then when we say "spin CS theory", we mean that the

[121] Contrast this with e.g.

$$\Omega_2^{\mathrm{Spin}}(pt) = \Omega_1^{\mathrm{Spin}}(pt) = \mathbb{Z}_2, \tag{1207}$$

with the former generated by the RR torus and the later generated by the R circle.

theory needs a choice of spin structure on the 3-manifold X, but *not* on the various 4-manifolds Y that X bounds.

The claim is that we can *define* the spin CS action as

$$S[w_2] = \pi \int_Y \frac{F}{2\pi} \wedge \left(\frac{F}{2\pi} + w_2(TY) \right), \tag{1208}$$

where $w_2(TY)$ is the second SW class on Y,[122] which may be non-zero. The spin structure $\mathcal{S}$ on X is parametrized by the trivialization

$$w_2(TY)|_{\partial Y} = d\mathcal{S}, \qquad \mathcal{S} \in C^1(X;\mathbb{Z}_2) \tag{1209}$$

but as we said, the exactness (on the boundary) need not mean that Y is spin.

The reason for defining $S[w_2]$ in this ways is that this definition gives an $e^{iS[w_2]}$ that is independent of Y for *any* choice of Y (regardless of whether the spin structure on X extends into Y), since

$$S[w_2(TY)] - S[w_2(TY')] = \pi \int_{Y\sqcup \bar{Y}'} \frac{F}{2\pi} \wedge \left(\frac{F}{2\pi} + w_2 \right) \cong 2\pi \int_{Y \sqcup \bar{Y}'} \frac{F}{2\pi} \wedge \frac{F}{2\pi} \in \overline{\mathbb{Z}}, \tag{1210}$$

where the $\cong$ means working mod $\overline{\mathbb{Z}}$ (which is possible since $a \cup w_2 \cong a \cup a \bmod 2$). The independence wrt the choice of Y means that the action actually only cares about $w_2|_{\partial Y}$, i.e., only cares about the spin structure on X.

Now consider changing the spin structure. The difference between any two spin structures on X is given by an element in $H^1(X;\mathbb{Z}_2)$. So, let $\beta \in H^1(X;\mathbb{Z}_2)$, and consider extending β into Y. The claim is that making this shift induces the corresponding shift

$$w_2 \mapsto w_2 + d\beta. \tag{1211}$$

Of course if β is nontrivial we will not be able to keep β closed in Y if the Poincare dual cycle $\widehat{\beta}$ satisfies $\widehat{\beta} \sim 0$ in $H_{\dim Y - 1}(Y;\mathbb{Z}_2)$, so this will be a non-trivial change in w_2 in general.

But wait, even if $d\beta \neq 0$, why do we say that this shift is nontrivial? Don't we only care about the cohomology class of w_2? The answer to this is that the cohomology relevant to determining the spin structure (or lack thereof) of Y is in fact *relative* cohomology—w_2 is determined by a class in $H^2(Y, \partial Y; \mathbb{Z}_2)$, not $H^2(Y;\mathbb{Z}_2)$. So, while $d\beta$ is exact in $C^2(Y)$, it is *not* exact in $C^2(Y, \partial Y)$, since $\beta|_{\partial Y} \neq 0$, and thus shifting w_2 by $d\beta$ is nontrivial.

A more easily visualized way of describing how w_2 transforms is via Poincare duality (recall that $\widehat{w}_2$ is the submanifold on which the framing degenerates; fermions can be defined only in $Y \setminus \widehat{w}_2$). The shift of w_2 by $d\beta$ translates into

$$\widehat{w}_2 \mapsto \widehat{w}_2 + \partial_R \widehat{\beta}, \tag{1212}$$

where β is still the 1-cochain extension into Y and ∂_R is the relative boundary operaton

$$\partial_R : C_n(Y, \partial Y) = C_n(Y)/C_n(\partial Y) \to C_{n-1}(Y, \partial Y), \tag{1213}$$

[122] Or rather, its lift to a cocycle on $H^2(X;\mathbb{Z})$ so that it may cup with the first Chern class c_1; we will not bother to make this distinction.

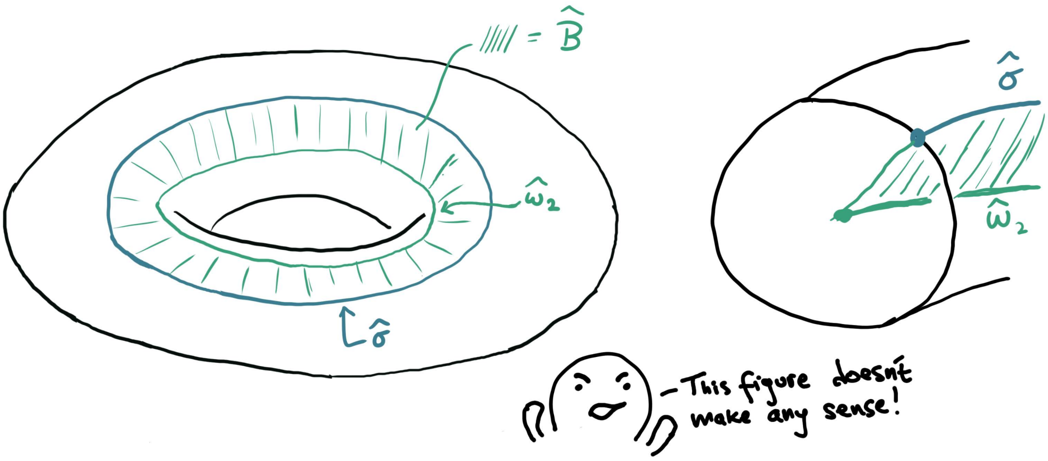


Figure 6: An example of the geometric picture of a spin torus that's been filled in "the wrong way". The left part is the full torus; the right is supposed to show a cut-out view; sorry for the rather poor artistry.

which selects out the boundary of the submanifold it acts on, modulo the part of that submanifold which is contained in ∂Y. That is,

$$\partial_R \widehat{\beta} = \partial \beta + \alpha, \qquad \alpha \subset \partial Y. \tag{1214}$$

Since Lefschetz duality gives the isomorphisms

$$H_p(Y, \partial Y; R) \cong H^{\dim Y - p}(Y; R), \qquad H_p(Y; R) \cong H^{\dim Y - p}(Y, \partial Y; R), \tag{1215}$$

we see that if we identify w_2 as a class in $H^2(Y, \partial Y; \mathbb{Z}_2)$ then Lefschetz duality relates it to a class in $H_{\dim Y - 2}(Y; \mathbb{Z}_2)$: since $\partial_R \widehat{\beta}$ can be nontrivial in the later group, this shift is indeed a nontrivial shift of w_2.

All this is perhaps best illustrated by a picture of an example. Consider figure 6. The left part shows a solid torus. The boundary of this torus has R spin structure around the cycle transverse to the cycle $\widehat{\sigma}$ marked in blue (which is drawn on the surface of the solid torus) and N spin structure around the other cycle. This spin structure cannot be extended into the solid torus; the obstruction is measured by the class w_2. The dual 1-cycle $\widehat{w}_2$ is drawn in green, and is supposed to be inside the torus: it represents a "vortex" where the spin framing can not be defined. Note that

$$\partial \widehat{\mathcal{B}} = \widehat{w}_2 + \widehat{\sigma}, \qquad \partial_R \widehat{\mathcal{B}} = \widehat{w}_2, \tag{1216}$$

where $\widehat{\mathcal{B}}$ is the "branch cut sheet" indicated in the figure. Now $\sigma \in H^1(T^2; \mathbb{Z}_2)$ can be thought of as a spin structure on T^2 (relative to the NN torus). When it is extended into the bulk, it can no longer be flat. In fact, when extended into the bulk, it becomes precisely $\mathcal{B}$. Accordingly, $d\mathcal{B} = w_2$,[123] and so even though w_2 is exact, Y is not spin. Also, we see that a transformation that changes the spin structure of the bounding

[123]This is not in contradiction with $\partial \widehat{\mathcal{B}} = \widehat{w}_2 + \widehat{\sigma}$ since $\partial \leftrightarrow d$ under Lefschetz duality only when ignoring contributions from boundary chains.

T^2 to NN can be realized by sending $w_2 \mapsto w_2 + d\sigma = 0$ (working mod 2 and yes, using σ to denote the extension of the boundary σ into the bulk, even though earlier we called it $\mathcal{B}$—sue me). We see in this case that w_2 is exact, and as such is entirely determined by its value on the boundary—or, since it is still exact on the boundary, it is entirely determined by the spin structure (choice of trivialization of w_2) on the boundary. Since the CS action we have defined only cares about $w_2|_{\partial Y}$, the situation where $w_2 = d\beta$ is determined just from the boundary spin structure is generic for our problem.

Just to be totally clear, when we say that a spin structure is a choice of trivialization $w_2 = d\mathcal{S}$, we are working on a closed manifold. On an open manifold, a spin structure is a choice of trivialization $w_2 = d\mathcal{S}_R$, where $\mathcal{S}_R \in H^1(Y, \partial Y; \mathbb{Z}_2)$ is a *relative* 1-cochain. In the example we above, we indeed saw that just having w_2 be exact is not enough to get a spin structure—instead, w_2 must be relatively exact.

Anyway, the whole point of this rather garrulous discussion is that it tells us how our CS action changes when we change the spin structure on X. If we change the 1-cocycle $\sigma \in H^1(X; \mathbb{Z}_2)$ parametrizing the spin structure[124] by $\sigma \mapsto \sigma + \beta$, then $w_2 \mapsto w_2 + d\beta$ means that the action for $U(1)_k$ changes as

$$(\delta S)[\beta] = k\pi \int_X c_1(E) \cup \beta, \tag{1217}$$

where E is the gauge bundle. The physical meaning of this $\int c_1(E) \cup \beta$ term is very intuitive: we may write it as $\int_{\widehat{c}_1(E)} \beta$, where $\widehat{c}_1(E)$ is the 1-chain corresponding to the worldlines of any magnetic monopoles that happen to be present. Therefore, we can interpret the (change in the) spin structure as producing a coupling of the current for the magnetic symmetry (namely $\star F/2\pi$) to the spin structure, where the spin structure is thought of as a background $\mathbb{Z}_2$ gauge field.[125] The fact that the spin structure of the manifold acts as a background $\mathbb{Z}_2$ gauge field tells us that when monopoles travel around cycles where the spin framing rotates, they pick up minus signs if k is odd—that is, for odd k, the monopoles are fermions. Of course, physically we know that this is what happens by flux attachment arguments; this procedure makes it rigorous.

Now we generalize to non-Abelian CS theory. This is quite straightforward—there is only one possible generalization of the result for the $U(1)$ case, viz. that the action changes by[126]

$$(\delta S)[\beta] = k\pi \int_X w_2(E) \cup \beta, \tag{1218}$$

where $w_2(E)$ is the second SW class of the gauge bundle (for complex vector bundles this is the mod-2 reduction of the first Chern class, and so reduces to the above result in the case when E is a complex line bundle). This may be derived by defining the

[124] Spin structures are really an $H^1(X; \mathbb{Z}_2)$ torsor; I will ignore this from now on.

[125] After all, to change the spin structure we may tensor either the spinor bundle or the gauge bundle with a real line bundle whose first SW class is β, so that spin structures indeed behave as background $\mathbb{Z}_2$ gauge fields.

[126] Again, we are being a bit fast and loose with the precise product operations between the various cochains appearing here—sorry.

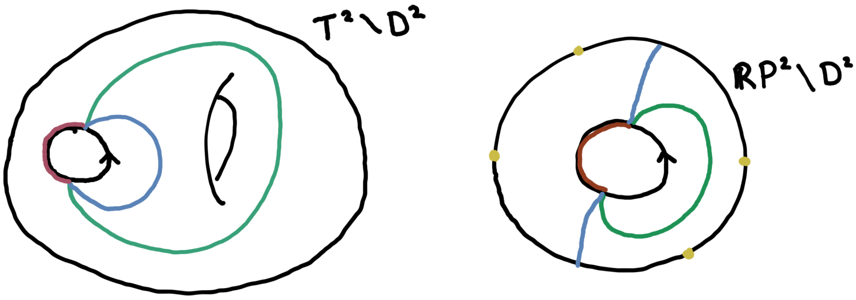


Figure 7

spin CS action as

$$S[w_2] = kS_{CS} + \pi k \int_Y w_2(E) \cup w_2(TY). \tag{1219}$$

This definition of the action works because as before, the extra term allows the action to be independent of the choice of Y. For example, suppose that the gauge bundle E is real (e.g. if the gauge group is $SO(N)$). Then $S_{CS} = \pi p_1(E)$ where $p_1 = \frac{1}{2}\text{Tr}(F/2\pi \wedge F/2\pi)$ is the Pointryagin density, and the independence of $S[w_2]$ on Y is demonstrated by (letting $Z = Y \sqcup \bar{Y}'$ as usual)

$$S[w_2(TY)] - S[w_2(TY')] = \pi k p_1(E_Z) + \pi k \int_Z w_2(E_Z) \cup w_2(TZ) \in \overline{\mathbb{Z}}, \tag{1220}$$

where in the last step we used

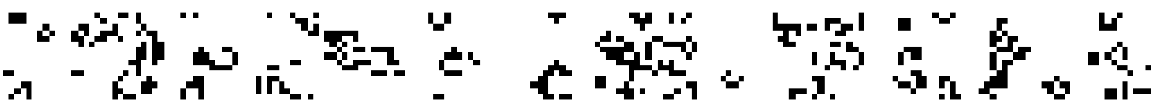
$$w_2(E_Z) \cup w_2(TZ) \cong w_2(E_Z)^2 \tag{1221}$$

and

$$p_1(E_Z) \cong w_2(E_Z)^2, \tag{1222}$$

where again $\cong$ means mod 2, and where we are not bothering to distinguish the Pointryagin square from the cup square for notational simplicity.

Note that as a byproduct we may conclude that all CS theories based on simply connected Lie groups (so that $w_2(E) = 0$—these theories do not have monopoles) are non-spin for any value of the level. *The spin structure dependence of CS theories thus comes entirely from needing to give a framing to monopole worldlines*, and so gauge groups without monopoles will never have spin CS actions. For example, $SU(N)_k$ is never spin. On the other hand, if $\pi_1(G) \neq 0$, then whether or not the theory is spin will depend on the level (in general, spin if the level is odd, non-spin if the level is even).



θ angles and deconfinement in two dimensions from the strong coupling expansion on the lattice

Today we're going to elaborate on some of the details implicitly contained in Seiberg's old paper on θ angles and lattice gauge theory [22]. The goal will be to consider Euclidean lattice $U(1)$ gauge theory with a θ term in two dimensions, and to find the free energy and Wilson loop vevs in the strong coupling limit. Results for the (weak-coupling) continuum limit can then be argued for schematically.

First we need to write down an appropriate Euclidean lattice action. We write it as

$$S = \beta S_{\text{matter}} + \frac{i\theta}{2\pi}\sum_{\square}\Phi_{\square}, \tag{1223}$$

where the flux $\Phi_\square$ is, for a plaquette $\square$ with bottom-left corner at the lattice site i,

$$\Phi_\square = -i\ln e^{i\oint_{\partial\square}A} = [A_x(i_x,i_y)+A_y(i_x+1,i_y)-A_x(i_x+1,i_y+1)-A_y(i_x,i_y+1)]_{[-\pi,\pi)}, \tag{1224}$$

where the subscript on the brackets indicates that we choose a branch of the logarithm such that $\Phi_\square$ is always valued in $[-\pi,\pi)$. In keeping with this branch, we also choose our lattice gauge fields $A_\mu(i)$ to be valued in $[-\pi,\pi)$;[127] in this notation the A_μ fields are just the logarithms of the $U(1)$ variables on each link for the aforementioned branch of the logarithm, and in particular they do not live in $\mathbb{R}$. Now with this convention $\sum_\square \Phi_\square \in 2\pi\mathbb{Z}$ when summed over the whole lattice. This means $\theta \sim \theta + 2\pi$, and in what follows we will always take $|\theta| \leq \pi$.

Let's first calculate the free energy $\mathcal{F}[\theta]$ in the infinite gauge coupling limit $\beta = 0$, where the action is purely the topological term. The free energy won't depend on the boundary conditions for the lattice in the thermodyamic limit, and so to be consist with the Wilson line calculations we'll do later, we take the lattice to be a cylinder of length L_x in the x direction, and circumference L_y in the y direction (the compact direction). We can then fix a gauge such that $A_x = 0$ (we can't choose $A_y = 0$ since $H^1(\text{Cyl};\mathbb{R}) \neq 0$; the holonomy $e^{i\oint dy\,A_y}$ is gauge invariant). The partition function is

$$Z[\theta] = \int \prod_{x,y=0}^{L_x,L_y} d\gamma_x^y \exp\left(\frac{i\theta}{2\pi}\sum_\square \Phi_\square\right), \tag{1225}$$

where we have written $\gamma_x^y = A_y(x,y)$ to save on the notation.

Note that the γ_x^y for different y are completely decoupled in this gauge: thus we can write

$$Z[\theta] = (Z_1[\theta])_y^L = \left(\int \prod_{x=0}^{L_x} d\gamma_x\, e^{i\bar\theta\sum_\square \Phi_\square}\right)^{L_y}, \qquad \bar\theta \equiv \theta/2\pi. \tag{1226}$$

[127] The reason that we choose the $[-\pi,\pi)$ branch instead of $[0,2\pi)$ is because it will make it easier to work with the θ angle, on which the free energy will depend on in a way that's symmetric about $\theta = 0$, not $\theta = \pi$.

We start with the integral over γ_0, since γ_0 only appears in one place. Thus

$$Z_1[\theta] = \int \prod_{x=0}^{L_x} d\gamma_x \, \exp\left(i\bar{\theta}[\gamma_1 - \gamma_0]_{[-\pi,\pi)}\right)(\dots), \tag{1227}$$

where $\dots$ involves things that don't contain γ_0.

The integral is easy to get confused about, so we will be pedantic. Suppose first that $\gamma_1 > 0$. Then the []s will come into affect when $\gamma_1 - \gamma_0 = \pi$, i.e. when $\gamma_0 = \gamma_1 - \pi$. Thus

$$\begin{aligned}\int_{-\pi}^{\pi} d\gamma_0 \, e^{i\bar{\theta}[\gamma_1-\gamma_0]_{[-\pi,\pi)}} &= \int_{-\pi}^{\gamma_1-\pi} d\gamma_0 \, e^{i\bar{\theta}(-2\pi+\gamma_1-\gamma_0)} + \int_{\gamma_1-\pi}^{\pi} d\gamma_0 \, e^{i\bar{\theta}(\gamma_1-\gamma_0)} \\ &= \frac{i}{\bar{\theta}}\left(e^{i(-\theta+\gamma_1\bar{\theta})}(e^{i\bar{\theta}(\pi-\gamma_1)} - e^{i\theta/2}) + e^{i\bar{\theta}\gamma_1}(e^{-i\theta/2} - e^{i\bar{\theta}(\pi-\gamma_1)})\right) \\ &= \frac{i}{\bar{\theta}}\left(e^{-i\theta/2} - e^{-i\theta/2+i\gamma_1\bar{\theta}} + e^{i\bar{\theta}\gamma_1 - i\theta/2} - e^{i\theta/2}\right) \\ &= \frac{2}{\bar{\theta}}\sin(\theta/2).\end{aligned} \tag{1228}$$

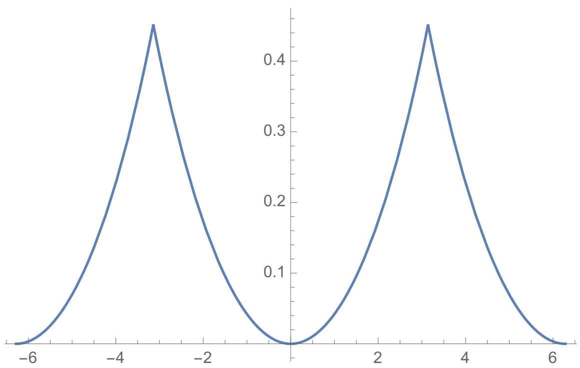
Since this is independent of γ_1, we of course get the same result if we take $\gamma_1 < 0$. The important thing here is that after γ_0 is integrated out, the resulting partition function looks *exactly* like the one for the partition function of a system with $L_x \mapsto L_x - 1$, multiplied by a factor of $2\sin(\theta/2)/\bar{\theta}$. Thus we can simply write[128]

$$Z[\theta] = \left(\frac{\sin(\theta/2)}{\theta/2}\right)^{L_x L_y}, \tag{1229}$$

so that the free energy per unit area is

$$\mathcal{F}[\theta] = -\ln\left(\frac{\sin(\theta/2)}{\theta/2}\right). \tag{1230}$$

Note that $\mathcal{F}[\theta] = \mathcal{F}[-\theta]$ as required. However, we also know that $\mathcal{F}[\theta] = \mathcal{F}[\theta \pm 2\pi]$: imposing this condition leads to a non-analyticity of $\mathcal{F}[\theta]$ at $\theta = \pm\pi$, which comes from the twofold GSD at the points where $\theta \in \pi(2\mathbb{Z}+1)$. $\mathcal{F}[\theta]$ looks like



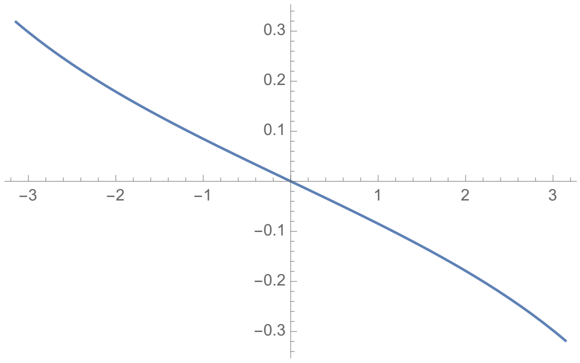
(1231)

[128] we are replacing $\bar{\theta} \to \theta$ here, since in retrospect it is nicer to have a normalization factor of $1/2\pi$ multiplying each $d\gamma_x^y$ in the partition function.

Now $\langle\Phi\rangle$ is odd under P and C (but unlike in 3+1D it is even under T if $A \mapsto -A$ as a differential form under T). The vev of the topological charge density is

$$\langle Q_{top}\rangle = -i\partial_\theta \mathcal{F}[\theta] = i\frac{\theta}{\sin(\theta/2)}\left(\frac{\cos(\theta/2)}{2\theta} - \frac{\sin(\theta/2)}{\theta^2}\right) = i\left(\frac{\cot(\theta/2)}{2} - \frac{1}{\theta}\right), \quad (1232)$$

which is non-vanishing (equal to $\mp i/\pi$) at the P symmetric points $\theta = \pm\pi$ (if we approach them from within $|\theta| \leq \pi$): thus, we have SSB for P (as well as for C). $\langle Q_{top}\rangle \neq 0$ just indicates the presence of a nonzero background electric field. As expected, the points $\pm\pi$ are the points of largest $|\langle Q_{top}\rangle|$: the plot of $\langle Q_{top}\rangle$ looks like



(1233)

Now let's diagnose confinement by looking at $\langle W_q(C_x)W_{-q}(C_{x+L})\rangle$, where $q \in \mathbb{Z}$ is a charge, C_x is a curve wrapping the y direction at x-coordinate x, and L is the lattice distance between the two Wilson lines. Because the partition function in the absence of Wilson lines factorizes as a product of partition functions on each x coordinate (due to the $\beta \to 0$ limit we've taken), when we calculate the expectation value we can, without loss of generality, take $x = 0$ and $L_x = L$. Therefore

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\int \prod_{x,y=0}^{L,L_y} \frac{d\gamma_x^y}{2\pi} \exp\left(iq\sum_{y=0}^{L_y}\gamma_0^y - iq\sum_{y=0}^{L_y}\gamma_L^y + i\bar{\theta}\sum_{x,y=0}^{L-1,L_y}[\gamma_{x+1}^y - \gamma_x^y]_{[-\pi,\pi)}\right). \quad (1234)$$

As before, the γ_x^y variables for different y are completely independent—the only variables that are linked together are the ones in the brackets. So then

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\left[\int \prod_{x=0}^{L}\frac{d\gamma_x}{2\pi} \exp\left(iq(\gamma_0 - \gamma_L) + i\sum_{x=0}^{L-1}\bar{\theta}[\gamma_{x+1}^y - \gamma_x^y]_{[-\pi,\pi)}\right)\right]^{L_y}. \quad (1235)$$

Let's look at just the part involving γ_0. Assume for simplicity that $\gamma_1 > 0$. Then similarly to before, the relevant integral is

$$\begin{aligned}\int_{-\pi}^{\gamma_1-\pi} d\gamma_0\, e^{iq\gamma_0 + i\bar{\theta}(-2\pi+\gamma_1-\gamma_0)} + \int_{\gamma_1-\pi}^{\pi} d\gamma_0\, e^{iq\gamma_0+i\bar{\theta}(\gamma_1-\gamma_0)} &= \frac{e^{i(-\theta+\bar{\theta}\gamma_1)}}{i(q-\bar{\theta})}\left(e^{i(q-\bar{\theta})(\gamma_1-\pi)} - e^{-i\pi(q-\bar{\theta})}\right) \\ &\quad + \frac{e^{i\bar{\theta}\gamma_1}}{i(q-\bar{\theta})}\left(e^{i\pi(q-\bar{\theta})} - e^{i(q-\bar{\theta})(\gamma_1-\pi)}\right) \\ &= -(-1)^q\frac{2e^{iq\gamma_1}\sin(\theta/2)}{q-\bar{\theta}}.\end{aligned} \quad (1236)$$

Therefore after integrating out γ_0, we get (assuming for simplicity that $L_y \in 2\mathbb{Z}$ to get rid of the $(-1)^q$)

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\left[\frac{2\sin(\theta/2)}{q-\bar{\theta}}\int\prod_{x=1}^{L}\frac{d\gamma_x}{2\pi}\exp\left(iq(\gamma_1-\gamma_L)+i\sum_{x=0}^{L-1}\bar{\theta}[\gamma^y_{x+1}-\gamma^y_x]_{[-\pi,\pi)}\right)\right]^{L_y}. \tag{1237}$$

Note that this looks, up to the multiplicative factor, exactly the same as what we had before, just with the left Wilson line moved one lattice spacing closer to the right one. Therefore we can easily iterate and integrate out the rest of the γ_xs:

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = \frac{1}{Z[\theta]}\left[\frac{2\sin(\theta/2)}{2\pi q-\theta}\right]^{LL_y}. \tag{1238}$$

This means that

$$\langle W_q(C_x)W_{-q}(C_{x+L})\rangle = e^{-L_yT}, \tag{1239}$$

where the line tension is

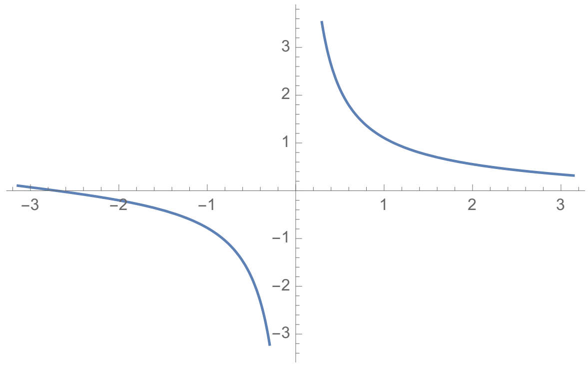
$$T = L\ln\left|\frac{2\pi q-\theta}{\theta}\right|, \tag{1240}$$

where as before $|\theta| \leq \pi$. Thus if $\theta \neq \pm\pi$, we will for sure have area-law confinement. However, suppose $\theta = \pi$: we get a completely tensionless and deconfined Wilson line, provided that $q = 1$. If we choose $q = -1$, we get a strong line tension, and an area law. The opposite is true if we take $\theta = -\pi$, with $q = -1$ giving a tensionless Wilson line: these two choices are related by C, which sends both $\theta \to -\theta$ and $q \to -q$. What's going on here is the following: the θ term sets up a background electric field of strength $\bar{\theta}$, pointing along the x axis. This electric field causes charges to be linearly confined, except when $\theta = \pm\pi$: in this case, the ± 1 strength electric flux created by the Wilson line insertions can screen the background electric field, leading to an electric field which is of uniform magnitude everywhere, but which has a reversal of sign in the domain enclosed by the two Wilson lines (the Wilson lines are domain walls for the spontaneously broken C symmetry, which is broken by the orientation of the electric field). The reason why one of the $q \pm 1$ Wilson line charges is confined while one is tensionless at $\theta = \pm\pi$ is just because only one choice of q allows for an electric field that is everywhere uniform in strength.

To back up this conclusion, we can calculate the vev of the topological charge density as a function of x. For x to the left of C_x or to the right of C_{x+L}, $\langle Q_{top}(x)\rangle = i(\cot(\theta/2)/2 - 1/\theta)$ as before. But for x in between the two loops, we find

$$\langle Q_{top}(x)\rangle = i\partial_\theta \ln\left[\frac{2\sin(\theta/2)}{2\pi q-\theta}\right] = i\frac{2+(2\pi q-\theta)\cot(\theta/2)}{4\pi q-2\theta}. \tag{1241}$$

The energetically favorable choices with P symmetry are $q = \pm1, \theta = \pm\pi$. For these choices we get $\langle Q_{top}(x)\rangle \to \pm i/\pi$. Note that this is *opposite in sign* to $\langle Q_{top}(x)\rangle$ for x not between the two Wilson lines: this is why the two Wilson lines are domain walls, across which the sign of $\langle Q_{top}(x)\rangle$ flips. At general θ, $\langle Q_{top}(x)\rangle$ for x in between the two Wilson lines (with $q = 1$) looks like, as a function of θ,



(1242)

The divergence at $\theta = 0$ comes from the fact that when $\theta = 0$, $\langle W_1(C_x)W_{-1}(C_{x+L})\rangle = 0$.

$SU(2) \times SU(2)$ chiral symmetry breaking

Today we're going through the details of the setup of the chiral lagrangian for the breaking of $SU(2) \times SU(2)$ in QCD (with the first generation of quarks only). This is standard stuff; I just wanted to have the details that were skipped over in the book I was reading (Zinn-Justin) spelled out somewhere.

In what follows we will be adding things to the free action

$$S_\psi = \int \bar{\psi} i \partial\!\!\!/ \psi, \qquad \psi = (u,\, d)^T, \tag{1243}$$

where u, d are the up and down quarks. This possesses a $U(2)^2$ flavor symmetry, where the two factors act separately on the left- and right-handed parts of the two quarks.

To discuss spontaneous chiral symmetry breaking phenomenologically, we will start by adding to the action the term

$$S_{\psi M} = -g \int \bar{\psi}(\Pi_+ M + \Pi_- M^\dagger)\psi, \qquad \Pi_\pm = \frac{1}{2}(\mathbf{1} \pm \bar{\gamma}), \tag{1244}$$

where M is some 2×2 matrix field that will serve as the order parameter for the spontaneously broken symmetry. We need to know how the discrete symmetries P and C are implemented on M (we are in Euclidean time, so we only care about C and

P). We will take P to act as reflection of a single spacetime coordinate for simplicity—following the discussion in the diary entry on fermions, we recall that P is implemented by $P : \psi \to \bar{\gamma}\gamma_i\psi, \bar{\psi} \mapsto \bar{\psi}\gamma_i$ for some i ($\bar{\psi}$ is a field independent from ψ, transforming in the inverse way under Spin(d) as ψ). Requiring that $S_{\psi M}$ is P-invariant means that

$$\Pi_+ M + \Pi_- M^\dagger = \gamma_i(\Pi_+ P(M) + \Pi_- P(M^\dagger))\gamma_i = \Pi_- P(M) + \Pi_+ P(M^\dagger) \tag{1245}$$

and so evidently P acts as

$$P(M) = M^\dagger. \tag{1246}$$

Charge conjugation symmetry gives us the constraint

$$[C\Pi_+ C^\dagger \otimes C(M) + C\Pi_- C^\dagger \otimes C(M^\dagger)]^T = \Pi_+ M + \Pi_- M^\dagger. \tag{1247}$$

Hopefully the notation here isn't too confusing: $C(M)$ is the charge-conjugated image of M, while the $\otimes$s are used since M and $\Pi_\pm$ act on different factors of the Hilbert space.[129] Now $C\gamma^\mu C^\dagger = -\gamma_\mu^T$, and thus $[C\bar{\gamma}C^\dagger]^T = \bar{\gamma}$ in $d = 4$ dimensions, while $[C\bar{\gamma}C^\dagger]^T = -\bar{\gamma}$ in $d = 2$. Thus $C\Pi_\pm C^\dagger = \Pi_\pm$ if $d = 4$, and $C\Pi_\pm C^\dagger = \Pi_\mp$ in $d = 2$. Therefore C symmetry tells us that

$$C(M) = M^T \quad (d = 4), \qquad\qquad C(M) = M^* \quad (d = 2). \tag{1248}$$

With these transformations, $S_{\psi M}$ is symmetric.

The flavor symmetry of the free term is $U(2)_+ \times U(2)_-$, which acts as

$$U(2)_+ \times U(2)_- : \psi \mapsto U_+^{\Pi_+} U_-^{\Pi_-}\psi, \qquad \bar{\psi} \mapsto \bar{\psi} P[(U_+^{\Pi_+} U_-^{\Pi_-})^\dagger] = \bar{\psi}(U_+^{\Pi_-} U_-^{\Pi_+})^\dagger, \qquad U_\pm = e^{i\theta_a^\pm t^a}, \tag{1249}$$

where t^a are the (Hermitian) generators for the $\mathfrak{u}(2)$ Lie algebra and the notation is $U_\alpha^{\Pi_\beta} = e^{i\theta_a^\alpha t^a \Pi_\beta}$, so that $U_\alpha^{\Pi_\beta}$ acts as $\mathbf{1}$ on spinors of chirality opposite to that of the index β. By looking at what happens when we decompose ψ as a sum of chiral spinors, we see that we can also write this as

$$U(2)_+ \times U(2)_- : \psi \mapsto (\Pi_+ U_+ + \Pi_- U_-)\psi, \qquad \bar{\psi} \mapsto \bar{\psi}(\Pi_+ U_-^\dagger + \Pi_- U_+^\dagger). \tag{1250}$$

Since $e^{i\Pi_\pm}\partial\!\!\!/ = \partial\!\!\!/ e^{-i\Pi_\pm}$, the free part of the action is invariant.

Under $U(2)_+ \times U(2)_-$, $S_{\psi M}$ transforms as

$$U(2)_+ \times U(2)_- : S_{\psi M} \mapsto -g \int \bar{\psi} \left(\Pi_+ U_-^\dagger M' U_+ + \Pi_- U_+^\dagger (M')^\dagger U_- \right) \psi, \tag{1251}$$

where M' is the image of M under $U(2)_+ \times U(2)_-$. Therefore this interaction will be symmetric provided that M transforms as

$$U(2)_+ \times U(2)_- : M \mapsto U_- M U_+^\dagger, \tag{1252}$$

which is the expected transformation law for a Goldstone field.

[129] We will of course usually omit the $\otimes$s, but here we've written them since e.g. $C\Pi_\pm C^\dagger C(M)$ is likely to cause confusion.

The next terms we need to add to the action are a kinetic term for M, and a potential to give M a vev:

$$S_M = \frac{1}{2\alpha} \int \left[\mathrm{Tr}[\partial_\mu M \partial^\mu M^\dagger] - V(MM^\dagger) \right], \tag{1253}$$

which preserves C and P. This is done at a purely phenomenological level, but microscopically we might imagine this as coming from the result of adding M to the action to decouple some type of fermion interaction, with expectation values of M being equal to expectation values of the corresponding chiral fermion bilinears that are condensed in the SSB regime. The above S_M terms are then assumed to be an EFT way of capturing the effective interactions for the order parameter M induced by the fermion dynamics.

We will also include a term that explicitly breaks the $U(2)_+ \times U(2)_-$ symmetry while preserving C and P. This is done with

$$S_B = -\frac{1}{\sqrt{2}} \int \mathrm{Tr}[B(M + M^\dagger)], \tag{1254}$$

where B is some fixed matrix, that we think of as a classical source / a "magnetic field" used to generate correlation functions of the order parameter. S_B preserves P for any choice of B since $M + M^\dagger$ is a scalar, while it preserves C if $B^T = B$. Note that $M - M^\dagger$ is a pseudoscalar though, so such a term would break P explicitly if added.

Let's now look at how all the terms we've introduced affect the axial and vector currents. The vector current is found by taking $U_+ = U_-$ and performing the variation $\psi \mapsto U\psi$, $M \mapsto UMU^\dagger$. Taking $U = e^{i\theta_a t^a}$ for θ_a small, we use $\partial_\mu M \mapsto i\partial_\mu \theta^a [t^a, M]$ to get (I'm not being super careful about signs and is)

$$\delta_V S = \int \left[\partial_\mu \theta^a \left(\bar{\psi}\gamma^\mu t^a \psi + i\mathrm{Tr}[t^a([M, \partial_\mu M^\dagger] + [M^\dagger, \partial_\mu M])] \right) + \frac{i}{\sqrt{2}} \theta_a \mathrm{Tr}[[t^a, B](M + M^\dagger)] \right], \tag{1255}$$

where we used the cyclicity of the trace and that $M^\dagger \mapsto UM^\dagger U^\dagger$. The part contracted with $\partial_\mu \theta^a$ is the vector current, and so the Ward identity tells us

$$\partial_\mu j_V^{\mu a} = \partial_\mu \left(\bar{\psi}\gamma^\mu t^a \psi + i\mathrm{Tr}[t^a([M, \partial_\mu M^\dagger] + [M^\dagger, \partial_\mu M])] \right) = -\frac{i}{\sqrt{2}} \mathrm{Tr}[[t^a, B](M + M^\dagger)]. \tag{1256}$$

The axial current is found by taking $U_- = U_+^\dagger$, so that $\psi \mapsto (\Pi_+ U + \Pi_- U^\dagger)\psi = e^{-i(\Pi_+ - \Pi_-)\theta_a t^a}\psi$ and $M \mapsto U^\dagger M U^\dagger$. This leads to a similar situation as the vector current, except with commutators replaced by anti-commutators:

$$\delta_A S = \int \left[\partial_\mu \theta^a \left(\bar{\psi}\gamma^\mu \bar{\gamma} t^a \psi + \mathrm{Tr}[t^a(\{M, \partial_\mu M^\dagger\} + \{M^\dagger, \partial_\mu M\})] \right) - \frac{i}{\sqrt{2}} \theta_a \mathrm{Tr}[\{t^a, B\}(M - M^\dagger)] \right], \tag{1257}$$

so that

$$\partial_\mu j_A^{\mu a} = \frac{i}{\sqrt{2}} \mathrm{Tr}[\{t^a, B\}(M - M^\dagger)]. \tag{1258}$$

Therefore the vector current is not conserved unless $B \propto \mathbf{1}$, while the axial current is not conserved for all $B \neq 0$.[130] Therefore B acts as a classical source for the axial current, and can also act as a source for the vector current if $B \not\propto \mathbf{1}$. Also note that as required, the divergence of j_V is a scalar, while the divergence of j_A is a pseudoscalar.

To discuss SSB, we will pick an explicit form for B and B — this is just the usual procedure of discussing the SSBroken state by adding a small background symmetry-breaking field to select out a particular ground state. For simplicity we will specialize to the case where $B = b\mathbf{1}$, which conserves j_V but explicitly breaks j_A. After M gets a vev from the potential, this choice of B will give (equal) masses to the u and d quarks. For the potential, we take the usual ($m^2 < 0$)

$$V(x) = \frac{1}{2}m^2 x + \frac{1}{24}\lambda x^2. \tag{1259}$$

Therefore we will parametrize M as

$$M = \frac{1}{\sqrt{2}}(\sigma\mathbf{1} + i\pi_a\sigma^a). \tag{1260}$$

We can now solve for $p = \langle\pi\rangle$ and $s = \langle\sigma\rangle$. If $b \neq 0$ then $p = 0$, which then after some straightforward algebra gives

$$s(m^2 + \lambda s^2/6) = b. \tag{1261}$$

Solving this to first order in b, we get

$$s = \sqrt{-6m^2/\lambda} + c\sqrt{-3/(2\lambda m^2)}. \tag{1262}$$

We can then plug this back into the potential by taking $\sigma \mapsto \sigma - s$ to find the masses of the σ and the π—we get, to first order in b,

$$m_\pi^2 = m^2 + s^2\lambda/6 = \frac{b}{4|m^2|}, \qquad m_\sigma^2 = m^2 + s^2\lambda/2 = 2|m^2| + \frac{3b}{4|m^2|}. \tag{1263}$$

As required, the π mass is zero when $b = 0$, since when $b = 0$ (i.e. when the external symmetry-breaking field is turned off) the π is a Goldstone boson. This vev for M also gives rise to a mass term for the fermions via the Yukawa coupling in $S_{\psi M}$; in this simple case where $B \propto \mathbf{1}$, at $m_\sigma \to \infty$ both quarks have the same mass, $m_u = m_d \propto gs$ (we don't really think of $\langle\pi\rangle$ as contributing to the mass since it only really does so when b is large, but this is outside of our approximation scheme).

A last comment is that this approach lets us easily deal with a possible θ term. If the quarks are coupled to an $SU(3)$ gauge field with $\theta \int c_2[A_{SU(3)}]$ in the action, then we can eliminate the θ term with a chiral rotation. This is the equivalent to performing the shift $M \mapsto e^{-2i\theta}M$, which then shows up equivalently as a shift $B \mapsto e^{i\theta Z}B$. This has the effect of doing $b \mapsto b\cos\theta$ for the purposes of computing $\langle\sigma\rangle$ and $\langle\pi\rangle$.

[130] This is because in order for $\partial_\mu j_A^{\mu a} = 0$ for all a with $B \neq 0$, we need B to anti-commute with all the t^a. This is impossible since this implies $B[t^a, t_b] = [t^a, t^b]B$, which cannot be true since $[t^a, t^b] = if^{abc}t^c$ is linear in the t^as, and hence B must anti-commute with it—a contradiction.

Gluon screening of Wilson lines in non-Abelian gauge theory and some useful representation theory computations

Today we're going over something that took me forever to finally understand: how exactly gluons can screen Wilson lines in certain representations to turn area law behavior into perimeter law behavior.

Since we are looking at pure gauge theory and are discussing screening, only non-Abelian gauge groups will be relevant, since only for non-Abelian groups are the gauge bosons charged. We will work on the lattice, since it will make the screening phenomenon most clear. We will show that for certain choices of representation R, the Wilson line $W_R(C)$ will have perimeter-law behavior in the strong-coupling expansion. Now in the weak coupling limit[131] the Wilson line always has perimeter law: we expand it to order g^2 and get $\langle W_R(C)\rangle \approx 1 + Ng^2 \oint\oint \langle A^a_\mu(x)A^a_\nu(y)dx^\mu \wedge dy^\nu\rangle$, where $N = \dim(R)C_2(R)/2$ and a is an arbitrary group index (no sum). We can replace the correlator to $O(g^2)$ with the Fourier transform of Π_T/p^2, which gives a perimeter law (in 3+1 D). This means that there are representations for which $W_R(C)$ has a perimeter law both at strong coupling and at weak coupling, and so the Wilson lines are tensionless throughout the phase diagram. From the conceptual point of view, this is rather obvious: we know that Wilson lines in the fundamental are perimeter-law when dynamical fundamental matter is included, due to screening effects. Since the gauge fields themselves are charged in the adjoint, they can screen Wilson lines in representations obtained from tensor products of the adjoint, and hence we expect that such Wilson lines should be perimeter-law even in the pure gauge theory. Actually showing results like this in detail is a bit technical, however.

Now for a given representation R, we want to compute

$$\langle W_R(C)\rangle = \left\langle \mathrm{Tr}_R\left[\prod_{l\in C} U_R(g_l)\right]\right\rangle = \int \prod_l \mathcal{D}g_l\, \mathrm{Tr}_R\left[\prod_{l\in C} U_l\right] \exp\left(-\beta\sum_{\square}\mathrm{Tr}_f\left[\prod_{l\in\partial\square} U_l\right]\right), \tag{1265}$$

where f is the fundamental representation of the gauge group, and the sum over plaquettes includes a sum over plaquette orientations.[132] Here the notation is such that

[131] The action is $\beta\sum_\square(\dots)$. For large β we can fix a gauge in which the product of Us around a plaquette goes to $e^{-a^2F_{\mu\nu}+\cdots}$, where $\mu\nu$ label the plane the plaquette is in and a is the lattice spacing (and $U_l = e^{i\int_l A} \approx \mathbf{1}$). This gives $S \approx \beta a^4\sum_\square \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$, and so (in four dimensions)

$$\frac{1}{g^2} \leftrightarrow \beta a^4. \tag{1264}$$

The continuum limit is thus the weak coupling ($\beta\to\infty$) limit. Note that we can't get access to the strong-coupling side of the gauge theory; β has to remain large for the continuum formulation to work.

[132] This is needed so that the action is real: if $\sum'_\square$ is a sum over all plaquettes without counting the

U_l is the representation of g_l in the fundamental or antifundamental, depending on the orientation of the link. A more verbose notation is $U_f(g_l)$, which we use for the representation matrices appearing in the Wilson loop. If a representation subscript on a $U(g)$ is omitted, it is implied that the representation is taken to be the fundamental. As usual, each plaquette term has two fundamental matrices and two antifundamental matrices: if the group elements on the links of a plaquette are $g_1, \dots, g_4$ labeled counterclockwise from the "bottom", then $U_\square \equiv \prod_{l\in\partial\square} U_l = U(g_1)U(g_2)U^*(g_3)U^*(g_4)$. Flipping the orientation of a plaquette is therefore equivalent to taking the trace in the conjugate representation f^*.

Since the weak-coupling limit gives a perimeter law, to address confinement we just need to look at strong coupling. For strong coupling, we expand the exponential in powers of β. Now $\int \mathcal{D}g_l\, U_l = 0$ by the properties of the Harr measure. If we were doing Abelian gauge theory, we would then derive an area law by expanding the exponential to Ath order (here A is the area enclosed by C), which would give us enough U_ls from the exponential to cancel those from the Wilson loop.

However, with non-Abelian gauge theories, we can do something different. Physically, this is because gluons are charged, so that in non-Abelian gauge theories, the gauge field itself can screen sources. This allows us to form tubes of glue around the Wilson loop, which can potentially screen it, depending on its charge.

For concreteness, consider a 3D (Euclidean) theory, with a Wilson loop inserted as above in a representation R. Now form a tube out of cubes, with the Wilson line located along one of the sharp edges of the tube. A section of this tube is shown in Figure 8. This tube will appear at order β^{4P} in perturbation theory, where P is the perimeter of the Wilson line (each corner contributes an extra β^4).

This tube will screen the Wilson line if

$$R^* \in f \otimes f^*. \tag{1267}$$

If the Wilson line weren't there, the tube would give a non-zero contribution to the partition function since each U_l on an edge appears with a corresponding U_l^* from a neighboring plaquette, allowing the integral over g_l to be nonzero. This is because $1 \in f \otimes f^*$. If $R^* \in f \otimes f^*$ then $1 \in R \otimes f^* \otimes f$, and so with a Wilson line with such an R can be added in the position shown without making the result vanish under integration.

To argue this precisely, we need to recall the orthogonality relation[133]

$$\int \mathcal{D}g\, [U_R(g)]^i_j [U_{R'}(g)]^k_l = \frac{1}{\dim R} \delta_{R^*,R'} \delta^i_l \delta^k_j. \tag{1271}$$

orientation separately, then

$$\sum_\square \mathrm{Tr}_f \prod_{l\in\partial\square} U_l = \sum_\square{}' \mathrm{Tr}_f \left(\prod_{l\in\partial\square} U_l + \prod_{l\in\partial\square} U_l^* \right), \tag{1266}$$

which is real (if the gauge group is real, like $SO(n)$, then a term like $\mathrm{Tr}_f(U_1U_2U_3U_4)$ has a partner $\mathrm{Tr}_f(U_4^{-1}U_3^{-1}U_2^{-1}U_1^{-1}) = \mathrm{Tr}_f[(U_1U_2U_3U_4)^T]$, and so the sum over orientations is redundant.

[133] The proof goes as follows: using the invariance of the Harr measure under shifts, one shows that

$$U_R(h) \int \mathcal{D}g\, U_R(g) E^k_l U_{(R')^*}(g^{-1}) = \left(\int \mathcal{D}G\, U_R(g) E^k_l U_{(R')^*}(g^{-1}) \right) U_{(R')^*}(h), \tag{1268}$$

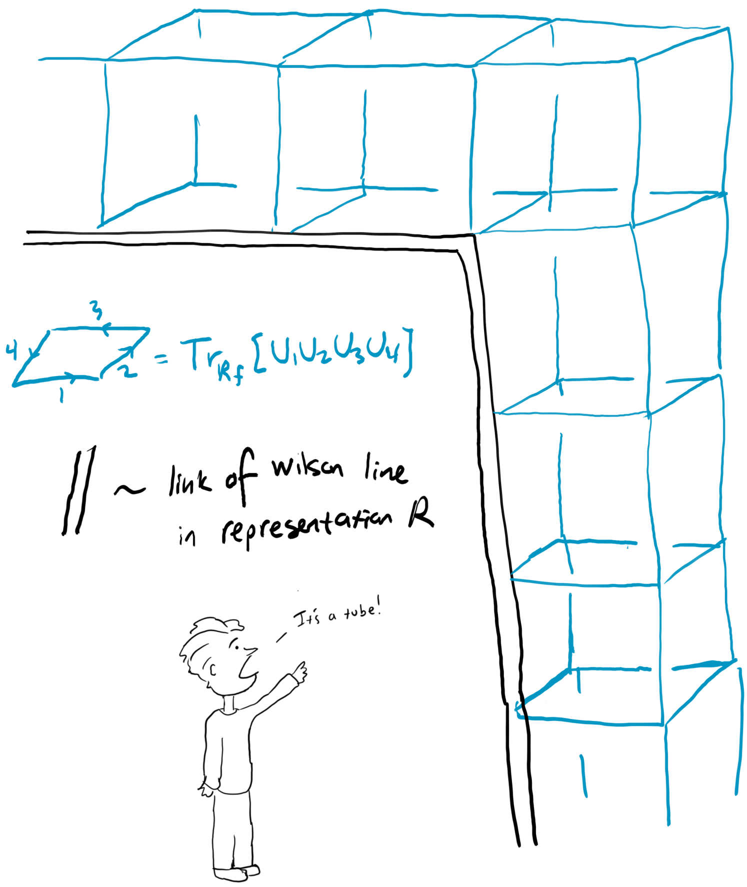


Figure 8: Geometry for how a tube of glue can screen a Wilson line.

The LHS is basically a group average over all similarity transforms of the matrix $[E^k_l]^i_j = \delta^i_l \delta^k_j$, and the RHS tells us that this average is zero if E^k_l has the 1 off of the diagonal, while E^k_l averages out to ($1/\dim R$ times) the identity if the 1 is on the diagonal. In particular, $\int \mathcal{D}g\, [U_R(g)]^i_j = 0$ unless $R = 1$.

Now consider expanding the expectation value to the order of β at which the tube geometry appears. We have

$$\langle W_R(C)\rangle = \int \prod_l \mathcal{D}g_l \, \mathrm{Tr}_R \left[\prod_{l\in C} U_R(g_l)\right] \prod_{\square \in T} \mathrm{Tr}[U_\square], \tag{1272}$$

where $[E^k_l]^i_j = \delta^i_l \delta^k_j$. Therefore using Schur's lemma, since both R, R' were assumed to be irreducible,

$$\int \mathcal{D}g\, [U_R(g) E^k_l U_{R'}(g)]^i_j = \delta_{R^*,R'} \delta^i_j C(E^k_l), \tag{1269}$$

where $C(E^k_l)$ is a constant. If we set $R^* = R'$, take the trace of both sides, and use the cyclicity of the trace, we get

$$\mathrm{Tr}[E^k_l] = \dim(R) C(E^k_l) \implies C(E^k_l) = \frac{1}{\dim R} \delta^k_l. \tag{1270}$$

where T is the tube. Combining these into one trace,

$$\langle W_R(C)\rangle = \int \prod_l \mathcal{D}g_l \operatorname{Tr}\left[\prod_{l\in C} U_R(g_l)\otimes \bigotimes_{\square\in T} U_\square\right]. \tag{1273}$$

Let us isolate just the terms that depend on g_1, where g_1 is the first link of the Wilson line:

$$\langle W_R(C)\rangle = \int \prod_l \mathcal{D}g_l \operatorname{Tr}\Bigg[(U_R(g_1)\otimes U(g_1)\otimes U^*(g_1)\otimes \mathbf{1})\cdot\Bigg(\prod_{l>1} U_R(g_l)\otimes \prod_{l\in\square_1\setminus l_1} U_l \otimes \prod_{l\in\square_1'\setminus l_1} U_l \otimes \bigotimes_{T\ni\square\neq\square_1,\square_1'} U_\square\Bigg)\Bigg]. \tag{1274}$$

Oh god, the notation. Sorry. Here, $\square_1, \square_1'$ are the two plaquettes in T that have the link l_1 as a side, and $\mathbf{1}$ is the identity acting on the tensor factors of every plaquette but these two.

Now let S be the matrix such that

$$S(U_R(g_1)\otimes U(g_1)\otimes U^*(g_1))S^{-1} = \bigoplus_{R_i\in R\otimes f\otimes f^*} U_{R_i}(g_1). \tag{1275}$$

For example, for $1/2\otimes 1/2$ in $SU(2)$, we have (see this footnote for a reminder about how to decompose $\otimes$s of irreps[134])

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}, \tag{1278}$$

[134] The general strategy is to take a tensor product of tensors transforming in two given representations, and then look for invariant subspaces among the collection of tensors in the tensor product, with each invariant space constituting a term in the direct sum decomposition. For example, we can work out $3\otimes 8$ in $SU(3)$, with 3 the fundamental and 8 the adjoint. Consider then $\chi^i A^j_k$. First, we can contract the i with the k, obtaining a single fundamental index and implying a 3 in the $\oplus$ decomposition. Next, after taking out the contracted piece, form $S^{ij}_k = \chi^{(i}A^{j)}_k$ and $A^{ij}_k = \chi^{[i}A^{j]}_k$. We can now contract both of these with ϵ_{lmn}. The former dies, and hence gives us an irrep. With two symmetrized upper indices and one lower index, we naively have an 18-dimensional irrep. But we have to remember that we have taken out the triplet that came from the contraction, and so $S^{kj}_k = 0\,\forall j$. This means that the S tensors define a $18-3=15$ dimensional irrep. When A^{ij}_k is contracted with the epsilon symbol, we get the tensor $\widetilde{A}_{kn} = \epsilon_{nij}A^{ij}_k$. Contracting this again with ϵ^{nlm}, we get

$$\epsilon^{nlm}\epsilon_{nij}A^{ij}_k = (\delta^l_i\delta^j_m - \delta^m_i\delta^l_j)A^{ij}_k = 0, \tag{1276}$$

since the trace part of A^{ij}_k has already been subtracted out. Thus $\widetilde{A}_{kn}$ is symmetric in its indices, and hence transforms as the 6^* irrep. Summarizing,

$$3\otimes 8 = 3\oplus 6^*\oplus 15. \tag{1277}$$

which takes the basis $\chi^i \otimes \psi^j$ to the basis $(\chi^0\psi^0, [\chi^0\psi^1 + \chi^1\psi^0]/\sqrt{2}, \chi^1\psi^1, [\chi^0\psi^1 - \chi^1\psi^0]/\sqrt{2})^T$. Actually, since we will want to calculate decompositions involving $f \otimes f^*$ and not $f \otimes f$ (in $SU(2)$ they happen to be isomorphic), we will want to work e.g. in the basis $\chi^i \otimes \psi_j$ instead. The matrix which takes the $\chi^i \otimes \psi_j$ basis to the $(\chi^0\psi_1, [\chi^0\psi_0 - \chi^1\psi_1]/\sqrt{2}, -\chi^1\psi_0, [\chi^0\psi_0 + \chi^1\psi_1]/\sqrt{2})^T$ basis is

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix} \tag{1279}$$

Note that the trivial representation appears symmetrically in this basis instead of representation, since it is just the trace.

Now consider the integration over g_1. Since all matrix elements of any representation other than the trivial one have zero average over the group, we will only get something nonzero if the trivial representation appears in the above $\oplus$. We can then (not worrying about constant factors) do the g_1 integral and write

$$\langle W_R(C) \rangle = \int \prod_{l>1} \mathcal{D}g_l \mathrm{Tr}\left[\left(1 \oplus \bigoplus_{R_i \in R\otimes f \otimes f^*} 0_{d_{R_i} \times d_{R_i}} \right) S(\cdots)S^{-1} \right], \tag{1280}$$

where $\cdots$ are the terms not containing g_1. We then write

$$S^{-1} \left(1 \oplus \bigoplus_{R_i \in R\otimes f \otimes f^*} 0_{d_{R_i} \times d_{R_i}} \right) S = \Pi_{\mathbf{1}}^{R\otimes f\otimes f^*}, \tag{1281}$$

where $\Pi_{\mathbf{1}}^{R\otimes f\otimes f^*}$ is the projector onto the trivial representation in $R \otimes f \otimes f^*$, expressed in the $\otimes$ basis. For example, in the $SU(2)$ example,

$$\Pi_{\mathbf{1}}^{1/2\otimes 1/2^*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{1282}$$

Therefore we have

$$\langle W_R(C) \rangle = \int \prod_{l>1} \mathcal{D}g_l \mathrm{Tr}\left[(\Pi_{\mathbf{1}} \otimes \mathbf{1}) \cdot (\cdots) \right], \tag{1283}$$

and we have successfully done the g_1 integral, getting a nonzero result.

Now we repeat this for all $g_l, l \in C$. Each integration gives us a factor of $\Pi_{\mathbf{1}}^{R\otimes f\otimes f^*}$ in the trace. We then do the integrals over the g_l for l a link in the tube not containing the Wilson line. These integrations make $\Pi_{\mathbf{1}}^{f\otimes f^*}$ matrices appear in the trace. When all is said and done, we get a trace over a bunch of products of things like $\mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \Pi_{\mathbf{1}}^{R\otimes f\otimes f^*} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$ and $\mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \Pi_{\mathbf{1}}^{f\otimes f^*} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$. These are giant matrices, and because of the projectors they are mostly made out of zeros. However, the trace does not vanish. Indeed, each matrix in the product always has a 1 in the upper-right hand corner: in the basis of the tensor product, the first basis vector in the Hilbert space that the matrix in the trace acts on is $T^{0\dots0}_{0\dots0}$, where T is the tensor product of the basis vectors for every single $\otimes$ factor. Since this entry always appears in the trace (the trivial representation), every matrix in the product appearing in the trace must not annihilate this vector; hence their product does not either. This gives us a positive, nonzero contribution to $\langle W_R(C)\rangle$. Since in this basis the matrix elements of the $\Pi_{\mathbf{1}}$ projectors are only either 0 or 1, this contribution cannot be canceled by any other parts of the trace, and so

$$\langle W_R(C)\rangle \sim \mathcal{N}\beta^{4P}, \tag{1284}$$

where $\mathcal{N}$ is some number depending on R, the gauge group, etc. Therefore the Wilson line has a perimeter law, and is screened.

So, the Wilson line will get screened if $R^* \in f \otimes f^*$, or equivalently if $R \in f \otimes f^*$, since $f \otimes f^*$ is invariant under conjugation. In particular, adjoint Wilson lines are always screened, since $f \otimes f^* \otimes A \ni 1$, which can be proved either by noting that $A \in f \otimes f^*$ and recalling that A is self-dual[135], or by noting that the generator matrices $[T^a]^i_j$ constitute invariant symbols with one adjoint index, one fundamental index, and one anti-fundamental index.[136] Physically this is intuitive since sources that come from tensor products of the adjoints should be able to be screened by a sufficient number of gluons, which are in the adjoint.

However, something more general is true: if R is any representation appearing in $(f \otimes f^*)^{\otimes N}$, for any N, then R is screened. Mathematically, this follows from simply repeating the above procedure, but going to β^{4NP} in perturbation theory, where an N-sheeted tube along the Wilson line appears. This N-sheeted tube will then be enough to screen the Wilson line.

The irreps that appear in the $\oplus$ decomposition of $(f\otimes f^*)^N$ are precisely those which transform trivially under the center of the gauge group. To put it another way, let G be the gauge group (and f its fundamental representation), $\widetilde{G}$ be its universal cover, and let $G' = \widetilde{G}/Z(\widetilde{G})$. Then the claim is that irreps appearing in the $\oplus$ decomposition of

$$\langle f \otimes f^*\rangle = \{(f \otimes f^*)^{\otimes N} \mid N \in \mathbb{N}\} \tag{1287}$$

are precisely the irreps of G'.

[135] The adjoint representation is always self-dual. Indeed, from two tensors $\chi^i_j \eta^l_m$, we can always contract both pairs of indices to form a singlet, and so $A \otimes A \ni 1$. Alternatively, one could note that the Killing form on the Lie algebra provides an isomorphism between A and its dual.

[136] Actually something more general is true, namely $R \otimes R^* \otimes A \ni 1$ for any irrep R. Indeed, consider an infinitesimal transformation by $U = \mathbf{1} + i\theta^b T^b_R$. Then thinking of a in T^a as an adjoint index, to $O(\theta)$ we have

$$\begin{aligned}[T^a]^i_j \mapsto &(\delta^i_k + i\theta^b [T^b_R]^i_k)(\delta^{ad} + i\theta^b [T^b_A]^{ad})[T^d_R]^k_l(\delta^l_j - i\theta^b[T^b_R]^l_j) \\ &= [T^a]^i_j + i\theta^b\left([T^b_R, T^a_R]^i_j + (-if^{bad})[T^d_R]^i_j\right) \\ &= [T^a]^i_j,\end{aligned} \tag{1285}$$

since $[T^a_A]^{bc} = -if^{abc}$. In particular, taking $R = A$ and using the reality of A tells us that

$$A \otimes A = \mathbf{1} \oplus A \oplus \dots. \tag{1286}$$

To show this pedantically, note that any irrep in $\langle f \otimes f^* \rangle$ transforms trivially under $Z(\widetilde{G})$, and hence is an irrep of G', and so $\langle f \otimes f^* \rangle \subset \text{Rep}(G')$. Conversely, any irrep of G' will appear in $\langle f \otimes f^* \rangle$. To show this, first note that the irreps of G' are a subset of the irreps of G, the later of which is generated by taking tensor powers of f. Thus all the irreps of G' will appear in the decomposition of some $f^{\otimes N}$ for some N. Let $m = |Z(G)|$. Then since irreps of G' are invariant under $Z(G)$, every irrep of G' appears in the decomposition of $f^{\otimes mk}$, for some $k \in \mathbb{N}$. Furthermore, $1 \in f^{\otimes m}$, and so likewise $1 \in (f^*)^{\otimes m}$. Thus $(f \otimes f^*)^{mk} \supset f^{\otimes mk} \otimes \mathbf{1}^{\otimes k}$, and so $\text{Rep}(G') \subset \langle f \otimes f^* \rangle$. Therefore we actually have that $\text{Rep}(G') = \langle f \otimes f^* \rangle$.

Summing up, we can say that a Wilson line in a representation R will be screened iff $R \in \text{Rep}(G')$. Again, physically this follows from the fact that the gluons are in the adjoint, and so the tube of glue can only screen things that transform trivially under $Z(G)$.

For example, if $G = SU(2)$, then integer-spin Wilson lines are screened, while half-odd-integer spin lines are not. In general, for $G = SU(N)$, $N \otimes N^* = \mathbf{1} \oplus A$, and so any irrep appearing in $\langle A \rangle$ will be screened. These are precisely the irreps coming from tensors with an equal number of upper and lower indices.

For example, in $SU(3)$, the $3, 3^*, 6, 6^*, 15, 15^*, \dots$ representations are un-screened, while e.g. the $A = 8, 10, 10^*, 27, \dots$ representations are screened. The adjoint representation $A = 8$ can be screened by a single tube of glue, but e.g. if $R = 10$ then a 2-sheeted tube will do the job: the two sheets provide[137]

$$(3 \otimes 3^*)^{\otimes 2} = (1 \oplus 8)^{\otimes 2} = 1^{\oplus 2} \oplus 8^{\oplus 4} \oplus 10 \oplus 10^* \oplus 27, \tag{1289}$$

and the 10^* on the RHS gives us the representation needed to screen the Wilson line. Of course, for $G = PSU(N)$, all Wilson lines are screened.

So, we see that confinement (at least to the extent to which confinement is captured by line tensions) is very much dependent on the topological properties of the gauge group! This is perhaps not too surprising, since some sort of monopole condensation is usually thought to be the mechanism whereby confinement occurs, and the different

[137] To derive this, we just need to calculate $8 \otimes 8$. From two adjoint tensors $A^i_j B^k_l$ we can take traces in two ways to create adjoints, and in one way to create a singlet, giving us $1 \oplus 8 \oplus 8$. Then we can take off the trace pieces and symmetrize / antisymmetrize the top two indices. Let $\widetilde{A}^{ij}_{kl}$ be antisymmetric in the upper two indices. Then form $\epsilon_{ijm}\widetilde{A}^{ij}_{kl}$, and contract with ϵ^{kln}; this gives zero since $\epsilon_{ijm}\epsilon^{kln}$ can be expanded as δ functions between the first and second triplet of indices, each term of which then vanishes by the tracelessness of $\widetilde{A}$. The same vanishing act happens if we instead contract with ϵ^{mkn} or ϵ^{mln}, and so $\epsilon_{ijm}\widetilde{A}^{ij}_{kl}$ is totally symmetric in mkl, and hence $\widetilde{A}^{ij}_{kl}$ gives us an irrep. A totally symmetric mkl gives us 10 independent tensors, and so this irrep is 10 dimensional. Antisymmetrizing the bottom two indices gives us another irrep, with three symmetric upper indices; this is also 10-dimensional. These two irreps are 10^* and 10, respectively. Note that 10 and 10^* are distinct, despite being invariant under the $\mathbb{Z}_3$ center of the gauge group (this is weird?); this is because the contraction $\widetilde{A}^{ij}_{kl}\widetilde{A}^{kl}_{ij} = 0$ due to the mixed symmetry / antisymmetry of the upper and lower sets of indices. Finally, we can symmetrize both the upper and lower indices: this gives 36 index choices, but there are 3×3 constraints on them coming from tracelessness, hence this irrep is 27-dimensional (and self-dual). Putting all of these together, we get

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10^* \oplus 27. \tag{1288}$$

types of monopoles which can exist depend on what the topology of the gauge group is.

(note to self: is the tube geometry really needed? Here's an alternative geometry: first take any "wall" of plaquettes that have the Wilson line on one edge, and the overlay the wall with its dual, forming a double-layer wall with a factor of $\mathrm{Tr}_{f\otimes f^*}[U_\square \otimes U_\square^*]$ for each plaquette. If $R^* \in (f\otimes f^*)^N$, then we take N copies of this wall, which provides enough glue to screen the Wilson line. This lets us do screening with slightly fewer plaquettes, and also gives us an approach which works in two dimensions)

Charge quantization and the weak mixing angle confusion

In various places (Ryder's QFT book, stack exchange, etc.) it is often stated that if the weak mixing angle is irrational, then electric charge is not quantized since the $U(1)$ of electromagnetism is embedded non-compactly in the full $SU(2)\times U(1)_Y$ electroweak gauge group. Today we will explain why this is incorrect (working as always under the assumption that the full gauge group is $SU(2)\times U(1)_Y$ and not $SU(2)\times \mathbb{R}$).

First we will show why the $U(1)$ of electromagnetism is compact, and afterwards explain why it is often stated otherwise in the literature. We choose the conventions where the covariant derivative and hypercharge gauge transformations look like

$$D_\mu\psi = (\partial_\mu - igW^a_\mu T^a - ig'B_\mu)\psi, \qquad \psi \mapsto e^{i\alpha(x)}\psi, \qquad B_\mu \mapsto B_\mu + \frac{1}{g'}\partial_\mu\alpha. \tag{1290}$$

Note that the transformation of the field is not $e^{ig'\alpha(x)}$ — g' is just a coupling constant, it is not the label of a $U(1)$ representation.

Upon Higgsing, the Higgs field (charged in the spinor of the $SU(2)$) gets a vev which we choose to be in the $\downarrow$ direction. Thus the $U(1)_e$ symmetry which we will identify with electromagnetism corresponds to rotations

$$\phi \mapsto e^{i\alpha T^3}e^{i\alpha/2}\phi, \tag{1291}$$

which leaves the vev of the Higgs invariant (the second factor is a hypercharge rotation). Note that in this convention, the Higgs has charge 1/2 under $U(1)_e$. We choose it to have charge 1/2 since the generators for the $SU(2)$ are $T^a = \sigma^a/2$. This $U(1)_e$ is obviously compact, since the rotations about the 3 axis are compact, the $U(1)_Y$ hypercharge was assumed to be compact, and in the above action we have $\alpha \sim \alpha + 4\pi$.

In this formulation, what are the volumes of the two $U(1)$ factors? In general, we have the representation q of $U(1)$, which acts as

$$q : \mathbb{R}/(2\pi/q)\mathbb{Z} \mapsto U(1)_{2\pi}, \quad x \mapsto e^{iqx}, \tag{1292}$$

where by $U(1)_{2\pi}$ we just mean the complex numbers of norm 1. Thus if our minimally charged field carries a representation q under $U(1)$, we are identifying $2\pi/q$ with 0, and so the gauge group is a circle with "volume" $\mathrm{vol}(G) = 2\pi/q$. Since for $SU(2)$ rotations we identify $\alpha \sim \alpha + 4\pi$, the $U(1)_{T^3}$ rotations come from the representation

$$1/2 : \mathbb{R}/(4\pi\mathbb{Z}) \to U(1)_{2\pi}, \quad x \mapsto e^{-ix/2}, \tag{1293}$$

and so $\mathrm{vol}(U(1)_{T^3}) = 4\pi$. Since we know we have quarks with charge 1/6 (the left handed ones), the field with minimal hypercharge has charge 1/6, and so $\mathrm{vol}(U(1)_Y) = 12\pi$. Note that we could rescale things so that the minimal hypercharge is 1, decreasing the volume of $U(1)_Y$ by a factor of 6, but then we'd also have to change the charge of the Higgs under T^3 rotations, which is less preferable since the normalization of these comes from their embedding in the $SU(2)$ factor. Anyway, if we did this, we see that the aspect ratio of the torus $U(1)_{T^3} \times U(1)_Y$ would be preserved by the rescaling. Since $U(1)_e$ is embedded compactly within this torus (one can think of it as a path wrapping the $U(1)_{T^3}$ cycle three times and the $U(1)_Y$ cycle once, since the ratio of the volumes of the two $U(1)$s is 3), it is embedded compactly no matter what our conventions regarding charge normalization are.

So, why do people say that $U(1)_e$ is non-compactly embedded inside $SU(2)\times U(1)_Y$? The argument is as follows: suppose we stick with the convention where the gauge couplings appear in the exponentials of the gauge transformations:

$$\phi \mapsto e^{ig\alpha T^3} e^{ig'\beta/2}\phi. \tag{1294}$$

ϕ is left invariant if we take $\beta = \alpha g/g'$. We then form the torus $U(1) \times U(1)$, and embed the $\beta = \alpha g/g'$ curve inside. If g/g' is irrational (which it generically is) then the $\beta = \alpha g/g'$ curve is dense in the torus $U(1) \times U(1)$, and hence electromagnetism is actually $\mathbb{R}_e$, a real line embedded in the full gauge group. Thus charge is not necessarily quantized, Polyakov monopoles cannot exist, and so on.

The problem with this is that the lengths of the sides of the torus are not 2π! Indeed, in the gauge transformations above, we are not identifying $\alpha \sim \alpha + 2\pi$ or $\beta \sim \beta + 2\pi$: the volumes of the gauge groups are not 2π, they are $2\pi/g$ and $2\pi/g'$. Thus the aspect ratio of the torus should actually be g/g', and the curve $\beta = \alpha g/g'$ is compactly embedded within such a torus.

There is another way to see that the quantization (or lack thereof) cannot possibly depend on the ratio of the gauge couplings: we just re-define $gW \mapsto W$ and $g'B \mapsto B$. Then the gauge couplings only appear in the kinetic terms for the gauge fields and cannot possibly affect the periodicity with which the fields transform. The gauge couplings only tell us about the dynamics of the gauge fields and how they couple to the matter fields: they have nothing to do with how the matter fields transform under the symmetry, or what representation of the symmetry group they carry. This is obvious but somehow still not discussed correctly in some hep textbooks!

Anomaly constraints in $SU(3) \times SU(2) \times U(1)$

Today we're working out what gauge anomaly constraints exist in the standard model, and how they end up being saturated. The problem statement is as follows:

Consider a gauge theory with gauge group $G = SU(3)\times SU(2)\times U(1)$, and consider k massless fermions coupled to G in various different representations. Assume that there is at least one fermion charged under each factor of the gauge group, and assume wolog that none of the fermions are completely neutral. Also note that we can take all the k fermions to be left handed wolog.[138] Assume that no two fermions exist which are conjugate, in the sense that one carries the representation (R, S, q) under $SU(3) \otimes SU(2) \otimes U(1)$ and the other carries $(\bar{R}, \bar{S}, -q)$. Also assume that there is at least one field with nontrivial charge under each factor (like the left-handed quark doublet in the SM).

First, we will find the minimal value of k such that all anomalies (including the gravitational anomaly) are cancelled. We will address how this change if you assume the fields are all in either the trivial or fundamental of $SU(3)$ and $SU(2)$? We will also look to see if there are any other combinations of 5 fields besides the standard model ones that are anomaly free (with these assumptions).

Let's start with the case $k = 2$. The only option we have is the pair

$$(R, S, q), \qquad (R, S, -q). \tag{1295}$$

The $U(1)^3$, $SU(2) \times U(1)^2$, and $SU(3) \times U(1)^2$ anomalies are zero, and the $SU(2)^3$ anomaly is zero since $SU(2)$ has no complex representations (recall that the anomaly indicator which diagnoses perturbative gauge anomalies vanishes if the representation R in question is isomorphic to R^*). Thus for this to work, we need a trivial $SU(3)^3$ anomaly from the R's. Unfortunately the only non-anomalous representations of $SU(3)$ are the real ones, which are of the form (i, i) (where the two indices label the values of the different diagonal generators). Since all the representations of $SU(2)$ are

[138] This works because the spinor representation is pseudoreal, with the pseudoreal structure being given by an antisymmetric matrix J. Then for any left-handed (say) fermion ψ_L appearing in the action and transforming in the representation R under the gauge group, we can change variables to work with $\psi_R = J\psi_L^\dagger$, which since $\bar{\gamma}J = -J\bar{\gamma}$ has the opposite chirality to ψ_L. The kinetic term for ψ_R then involves a coupling to the gauge field as $\bar{\psi}_R^\dagger A_\mu^a \gamma^\mu [T^a]^T \psi_R$, and by Hermiticity of the Lie algebra generators we see that ψ_R transforms in the R^* representation. So we can work e.g. change all right-handed fermions to left-handed ones as long as we remember to reverse their charges.

either real or pseudo-real, we in fact have that the conjugate of (R,S,q) is isomorphic to $(R,S,-q)$, which is a contradiction.

What about $k=3$? After playing around a bit, we can write down the following:

$$(\mathbf{6},\mathbf{3},1),\quad (\mathbf{6},\mathbf{3},-1),\quad (\overline{\mathbf{15}}',\mathbf{3},0), \tag{1296}$$

where $\mathbf{6}$ is the six-dimensional $SU(3)$ representation $(2,0)$, $\overline{\mathbf{15}}'$ is the 15-dimensional $SU(3)$ irrep $(1,2)$, and $\mathbf{3}$ is the vector representation (spin one) of $SU(2)$. The anomaly coefficients of these $SU(3)$ irreps are $A(\mathbf{6})=7$ and $A(\overline{\mathbf{15}}')=-14$ (see e.g. Cutler & Kephart 2000), and so the $SU(3)$ anomaly vanishes:

$$A_{SU(3)^3}((\mathbf{6},\mathbf{3},1)\oplus(\mathbf{6},\mathbf{3},-1)\oplus(\overline{\mathbf{15}}',\mathbf{3},0)) = 7\times 3+7\times 3-14\times 3=0. \tag{1297}$$

The $U(1)^3$ anomaly, $U(1)$-graviton anomaly, and the mixed $U(1)\times SU(2)^2$, $U(1)\times SU(3)^2$ anomalies are all zero since we have a symmetric coupling between 1 and -1 hypercharges to the rest of the representations. Since $\bar{\mathbf{6}}\not\cong\mathbf{6}$, there are no problems there. Finally, the global $SU(2)$ anomaly is also zero, since it requires the Dynkin index

$$t_2(S)=d(S)\frac{d^2(S)-1}{12} \tag{1298}$$

to be integral, which it is: $t_2(\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{3})=3\times 2$ (this amounts to requiring that $\mathrm{Tr}(T_3^2)$ be integral rather than half-integral, where we define $\mathrm{Tr}(T_3^2)=1/2$ for the fundamental representation. Essentially, we need there to be an even number of $SU(2)$ doublet fermions). So, evidently these three fields do the job.

This however made use of some big $SU(3)$ representations — what if we restrict ourselves to the trivial and fundamental representations? Cancellation of the global $SU(2)$ anomaly means that we need an even number of $SU(2)$ doublets. Can we do it with three fields? This would mean two doublets, and cancellation of the $SU(2)^2\times U(1)$ anomaly would mean that they would have opposite hypercharges, with the third field having zero hypercharge. Then cancellation of the $SU(3)^2\times U(1)$ anomaly would mean that both doublets are in the fundamental of $SU(3)$, but then the $SU(3)^3$ anomaly is non-zero. So at least four fields are needed.

For four fields, we either need two $SU(2)$ doublets or four for cancellation of the global $SU(2)$ anomaly. If we have two then they must have opposite hypercharges, and then must both be in $\mathbf{3}$ under $SU(3)$ by our assumptions about not having conjugate fields. But then the remaining two fields cannot possibly cancel the $SU(3)^3$ anomaly, so this doesn't work. So, can we have four $SU(2)$ doublets? Yes we can; consider the quadruplet

$$(\mathbf{3},\mathbf{2},a),\quad (\bar{\mathbf{3}},\mathbf{2},-a),\quad (\mathbf{3},\mathbf{2},b),\quad (\bar{\mathbf{3}},\mathbf{2},-b), \tag{1299}$$

where a,b are subject only to the constraint that $a\neq\pm b$. One sees that all the anomalies cancel.

Now we assume that there are five fields, as in the first generation of the Standard Model (of course in the SM, grouped according to their $SU(2)$ charges, these are $(e_L,\nu_L),(u_L,d_L),u_R,d_R,e_R$, with a possible sterile ν_R omitted). We can either have two or four $SU(2)$ doublets. Suppose first that there are four $SU(2)$ doublets. Then since we need at least one field charged under everything and we need the $SU(3)^3$

anomaly to vanish, we can either have two of these doublets in the fundamental of $SU(3)$ and two in the anti-fundamental, or one in the fundamental, one in the anti-fundamental, and two in the trivial. Consider first the former situation. Then the vanishing of the $SU(2)^2 \times U(1)$ anomaly in this case means that

$$\sum_{\text{doublets}} q_i = 0, \tag{1300}$$

where q_i is the hypercharge. But then the $U(1)$-graviton anomaly implies that $q_5 = 0$, where q_5 is the hypercharge of the last field. But then the last field is completely trivial, and we get the same answer as the $k = 4$ case. So, now consider the case where one of the doublets is in the fundamental of $SU(3)$ and the other is in the anti-fundamental. Then the $SU(3)^2 \times U(1)$ anomaly means that the hypercharges of the two fields coupled to $SU(3)$ are equal. Let their hypercharges be a, and let the charges of the other two $SU(2)$ doublets be b, c. Then we need $6a + c + d = 0$. Thus the $U(1)$-graviton anomaly reads $12a + 2c + 2d + q_5 = 0$, and so again we conclude that $q_5 = 0$ and the last field is completely decoupled.

So, we can assume wolog that there are only two $SU(2)$ doublets, as in the (first generation of the) SM. Suppose they are both charged under $SU(3)$. They cannot both be in the same $SU(3)$ irrep, since then there is no way to cancel the $SU(3)^3$ anomaly with the other $SU(2)$-neutral fields (the dimensions of the other fields are too small). So, one must be in $\mathbf{3}$ with the other in $\bar{\mathbf{3}}$. But then cancellation of the $SU(2)^2 \times U(1)$ anomaly means that the two doublets have opposite hypercharges, which gives a contradiction since we now have two fields that are conjugate to one another.

This means we can without loss of generality take the two doublets to be of the form

$$(\mathbf{3}, \mathbf{2}, a), \quad (\mathbf{0}, \mathbf{2}, b). \tag{1301}$$

Cancellation of the $SU(3)^3$ anomaly means that the other three fields have to be of the form

$$(\bar{\mathbf{3}}, \mathbf{0}, c), \quad (\bar{\mathbf{3}}, \mathbf{0}, d), \quad (\mathbf{0}, \mathbf{0}, e), \tag{1302}$$

which is starting to look increasingly like the Standard Model. Now one just needs to solve for the $U(1)$ hypercharges. We then have to solve the constraints

$$\begin{aligned} &6a + 2b + 3c + 3d + e = 0 \qquad 6a^3 + 2b^3 + 3c^3 + 3d^3 + e^3 = 0, \\ &2a - c - d = 0, \qquad 3a + b = 0. \end{aligned} \tag{1303}$$

I'm not going to write out all the algebra, which is straightforward. One finds that there are two solutions. One has all of the hypercharges equal to zero except for $c = -d$. Again, this leaves us with a completely decoupled field, and we are back to the case of $k = 4$ (and it violates our assumptions about there being at least one field charged under everything). The other solution is the Standard Model. So, given these assumptions, the SM is unique.

What other sorts of possible matter content can we have? There are many options, since there are many semisimple Lie groups with real or pseudo-real representations. There are also a few groups with complex representations that always have zero anomaly coefficients, like $SO(4n+2)$ with $n > 2$ (see Weinberg II). In particular, any gauge group that doesn't include factors of $SU(n)$ with $n \geq 3$ or factors of $U(1)$ will be automatically anomaly-free, regardless of the field content. If we have the group $SU(3) \times G$ where G is anomaly-free, then under the above restrictions we can have two fields $(R, g), (\bar{R}, g)$ where g is any complex irrep of G, or we can have $(R, g), (\bar{R}, h)$, where g, h are two distinct real representations of the same dimension. We can do a similar thing if the group is instead $U(1) \times G$. At this point, making concrete statements about what is allowed and what is not allowed has to be done on an increasingly case-by-case basis.

Chiral anomaly via Pauli-Villars

This is the first logical half of P&S problem 19.4, whose problem statement is as follows:

Derive the chiral anomaly for QED in four dimensions using a PV regulator, i.e. by adding a regulator fermion Ψ with mass M to the theory and showing that $\langle \partial_\mu j^{5\mu} \rangle$ is non-zero in the $M \to \infty$ limit (of course the current is explicitly not conserved when $M > 0$, but in the absence of an anomaly the extent of the non-conservation will vanish as $M \to \infty$). As usual, treat the gauge field as a background field. The relevant integral you will need to do should be UV-finite, even before the $M \to \infty$ limit is taken. Show that the anomaly is given by the limit

$$\langle d^\dagger j_5 \rangle = \lim_{M \to \infty} \left(\langle p, k | 2iM \bar{\Psi} \gamma^5 \Psi | 0 \rangle \right), \tag{1304}$$

where the state $|p, k\rangle$ contains the two photons of momenta p, q which appear as two of the legs in the usual triangle diagram (the current insertion is the third leg).

We will do the regularization by adding the massive fermion to the Lagrangian with the kinetic term $-\bar{\Psi}(i\not{D}_A - M)\Psi$. Consider now calculating the anomaly by e.g. point-splitting the fermion operators occuring in the expression for the chiral current and connecting the two fermions with a Wilson line. This calculation will receive contributions from both the original ψ fermion and the regulator fermion Ψ. As we saw in the diary entry on the anomaly in non-Abelian gauge theories, we will end up doing an integral like

$$\int_{k,q} e^{ik \cdot \eta - iq \cdot x} \frac{\epsilon^{\mu\nu\lambda\sigma}(k+q)_\nu A_\lambda k_\sigma}{(k^2 - m^2)((k+q)^2 - m^2)}, \tag{1305}$$

where η is an infinitesimal distance between two fermion operators which we are point-splitting and m is either 0 or M. The hope is that since we are sending $\eta \to 0$, we can get away with doing this before sending $M \to \infty$, and expand the integral at large k (viz. larger than M), getting a divergent contribution which is independent of M (and only the divergent contribution matters for the anomaly, since this integral is multiplied by an infinitesimal term proportional to η coming from the expansion of the operators in the OPE). So, this will result in the contributions from the two fermions cancelling, even though one is massive and the other isn't. The reason that we might expect this to work is basically that the anomaly is sensitive to UV physics (it comes from doing an OPE of two fermions defined at the same point), and so if we are allowed to take $M \to \infty$ at the end of the calculation in the spirit of PV regularization, the anomalous term should be independent of M.

Of course, we haven't actually gotten rid of the anomaly, since the Ψ fermion could still contribute to it directly. The chiral current for the massive fermion is

$$j^{5\mu} = \bar{\Psi}\gamma^\mu\gamma^5\Psi. \tag{1306}$$

We then take the divergence and use the Dirac equation for Ψ, so that

$$\langle \partial_\mu j^{5\mu} \rangle = \langle 2iM\bar{\Psi}\gamma^5\Psi \rangle. \tag{1307}$$

Since this vev can only comes from loops we'd naively expect it to be zero after we send $M \to \infty$, but as we will see this is not the case.

The first contributions to this expectation value come from triangle-diagram-like diagrams, with two outgoing photons radiating from a Ψ loop and with one insertion of $2iM\bar{\Psi}\gamma^5\Psi$ on the Ψ loop. These diagrams are derived in the usual way by adding $\int 2i\omega M\bar{\Psi}\gamma^5\Psi$ to the action, integrating out the fermions, expanding the ln det, and then differentiating with respect to ω by selecting out only those diagrams with a single insertion of $2iM\bar{\Psi}\gamma^5\Psi$. The relevant diagram (a fermion bubble with two photon legs and an insertion of $M\bar{\Psi}\gamma^5\Psi$) gives us

$$\begin{aligned}
\langle p,k|2iM\bar{\Psi}\gamma^5\Psi|0\rangle &= 2iMe^2 i^3 \int_q \mathrm{Tr}\left[\gamma^\nu \frac{1}{\not{q}-M}\gamma^\mu \frac{1}{\not{q}-\not{p}-M}\gamma^5\frac{1}{\not{q}+\not{k}-M}\right]\epsilon^*_\mu(p)\epsilon^*_\nu(k) \\
&= -8iM^2e^2\int_q \epsilon^{\nu\lambda\mu\sigma}\frac{q_\lambda(q-p)_\sigma - q_\lambda(q+k)_\sigma + (q-p)_\lambda(q+k)_\sigma}{(q^2-M^2)((q-p)^2-M^2)((q+k)^2-M^2)}\epsilon^*_\mu(p)\epsilon^*_\nu(k) \\
&= -8iM^2e^2\int_q\int_{x,y} 2\frac{q^2 g_{\lambda\sigma}/4 - p_\lambda k_\sigma}{(q^2-M^2+\dots)^3}\epsilon^*_\mu(p)\epsilon^*_\nu(k),
\end{aligned} \tag{1308}$$

where in the last step the 2 comes from passing to Feynman parameters and the $\dots$ are a standin for terms involving the Feynman parameters and the photon momenta. We do the shift to simplify the denominator in the usual way, but we don't have to really keep track of what happens since the q^2 term in the numerator dies by antisymmetry anyway. So we get

$$\langle p,k|2iM\bar{\Psi}\gamma^5\Psi|0\rangle = 16iM^2e^2\int_q\int_{x,y}\epsilon^{\mu\nu\lambda\sigma}p_\lambda k_\sigma\frac{\epsilon^*_\mu(p)\epsilon^*_\nu(k)}{(q^2-\Delta)^3}, \tag{1309}$$

where $\Delta = M^2 + \ldots$. This integral is perfectly UV finite, even without the $M \to \infty$ limit. We get

$$\int_q \frac{1}{(q^2 - \Delta)^3} = -\frac{i}{2(4\pi)^2}\Delta^{-1}. \tag{1310}$$

Taking now $M \to \infty$ the integral over the Feynman parameters just gives 1 since $\Delta \to M^2$, and so we get

$$\langle k, p|\partial_\mu j^{5\mu}|0\rangle = \frac{e^2}{2\pi^2}\epsilon^{\mu\nu\lambda\sigma}\epsilon^*_\mu(p)\epsilon^*_\nu(k)p_\lambda k_\sigma, \tag{1311}$$

which is the expected result. Evidently the somewhat dubious arguments we made in the beginning about terms cancelling worked out this time.

Trace anomaly via Pauli-Villars

This is the second logical half of the P&S problem that we started in the previous diary entry. The to-do list is:

Reproduce the trace anomaly for QED in four dimensions with a PV regularization scheme (you will also need to dimensionally-regularize one integral). In a similar spirit to the last problem, you should show that the anomaly is computed by taking the $M \to \infty$ limit of

$$\langle M\bar{\Psi}\Psi\rangle. \tag{1312}$$

Recall that the generator of dilitations is $D^\mu = T^{\mu\nu}x_\nu$, where $T^{\mu\nu}$ is the "improved" (symmetric, gauge-invariant) stress tensor. Now $\partial_\mu D^\mu = T^\mu{}_\mu$, so we get a trace anomaly when $\langle T^\mu{}_\mu\rangle$ is "unexpectedly" nonzero (once we learn about RG this is of course not surprising at all, since when $e \neq 0$ we are not at a fixed point).

Let's first write down what we expect to get for $\langle T^\mu{}_\mu\rangle$. In four dimensions e is dimensionless at the free fixed point, but since it is marginally irrelevant a mass scale is still introduced into the theory when doing renormalization. Performing a dilitation $x \mapsto e^{-\lambda}x$ for λ infinitesimal, the gauge coupling changes as (apologies for the profusion of e's — should change to g, but too late)

$$e \mapsto e + \partial_M e\delta M = e + \partial_M e(e^\lambda M - M) \implies \delta e = \beta(e). \tag{1313}$$

Performing this change of coordinates in the path integral, by the usual procedure we get

$$\langle \partial_\mu D^\mu\rangle = \langle T^\mu{}_\mu\rangle = \lambda\beta(e)\partial_e\mathcal{L} = \frac{\beta(e)}{2e^3}F_{\lambda\sigma}F^{\lambda\sigma} = \frac{1}{24\pi^2}F_{\lambda\sigma}F^{\lambda\sigma}, \tag{1314}$$

where in the last step we've used the first-order result for the β function in QED.

The goal now is to see how we can reproduce this with an explicit calculation. To find $\langle T^\mu{}_\mu \rangle$, we can formally add $\int \eta T^\mu_\mu$ to the action, integrate out the fermions, and then differentiate with respect to η. To do this we need the energy-momentum tensor for QED. Now we know the Hamiltonian is

$$H = \frac{1}{2e^2}(E^2 + B^2) - i\bar{\psi}(\gamma^i \partial_i - m)\psi, \tag{1315}$$

since when going from the Lagrangian to the Hamiltonian the ∂_t part in the Dirac operator gets cancelled. So then we just need to write down a more covariant expression of μ, ν that reduces to the above when $\mu = \nu = 0$. Such an expression is

$$T^{\mu\nu} = -\frac{1}{e^2} F^{\mu\sigma} F^\nu_\sigma + \frac{1}{4e^2} g^{\mu\nu} F^2 + i\bar{\psi}(\gamma^{\{\mu} D_A^{\nu\}})\psi - g^{\mu\nu}\bar{\psi}(i\not{D}_A - m)\psi. \tag{1316}$$

When this gets stuck in diagrams, it gives us source counterterms for the gauge field propagator, the fermion propagator, and the electron / photon vertex. If we hadn't added the regulator fermion, we would calculate the anomaly by computing $\langle T^\mu{}_\mu \rangle$ perturbatively in dimensional regularization, as outlined in P&S chapter 19. Using $g^\mu_\mu = g^{\mu\nu} g_{\nu\mu} = d$ in dimensional regularization, we have, for a massless fermion,

$$\langle T^\mu{}_\mu \rangle = F^2 \left(\frac{d}{4} - 1 \right) + (1 - d)\bar{\psi} i \not{D}_A \psi. \tag{1317}$$

The second term ends up not contributing to diagrams because of the equations of motion. However the first term does — it acts as a counterterm in photon propagators, and when these are stuck onto the legs of polarization bubbles with fermions running in the loop, the $(d/4 - 1)$ is rendered finite when it hits a divergent part of a $\Gamma(\epsilon/2)$ coming from the fermion integration. When we add in the regulator fermion, we get contributions to the polarization bubble from both the original fermion and the regulator. One can fairly quickly check (I won't write it out since it's similar to the calculation we'll do below) that the relevant one-loop contribution to the trace anomaly due to the regulator fermions comes from the integral

$$\frac{\epsilon}{4} \int_q \frac{f(k,p)}{(q^2 - \Delta)^2}, \tag{1318}$$

where p, k are external photon momenta and Δ goes to M^2 in the large M limit. In four dimensions, we get something like

$$\epsilon \left(\Gamma(\epsilon/2) - \ln \Delta + \dots \right), \tag{1319}$$

and so when we take $\epsilon \to 0$, we get something which is independent of M! Thus, the regulator fermions exactly cancel the contribution to anomalous terms in the polarization bubbles that the massless fermions make! This is similar to the reasoning in the last problem — we're still arguing that we can get away with waiting until the very end of the calculation to take $M \to \infty$.

Of course, this doesn't eliminate the anomaly, since the regulator fermions contribute to the trace of T explicitly through their mass term. So, we have

$$\langle T^\mu{}_\mu \rangle = \langle M \bar{\Psi} \Psi \rangle. \tag{1320}$$

Our task is to evaluate this in the limit $M \to \infty$ and see whether it is zero or not.

As usual, the first diagram that shows up is a triangle-type diagram with two photon legs and an insertion of the relevant operator (here $M\bar{\Psi}\Psi$) on the fermion loop. We get

$$\langle p,k|M\bar{\Psi}\Psi|0\rangle = iMe^2 \int_q \mathrm{Tr}\left[\gamma^\nu \frac{1}{\not{q}-M}\gamma^\mu \frac{1}{\not{q}-\not{p}-M}\frac{1}{\not{q}+\not{k}-M}\right]\epsilon^*_\mu(p)\epsilon^*_\nu(k) \tag{1321}$$

Only terms with even numbers of γ matrices contribute. Their contributions can be found using standard rules for tracing out products of γ matrices. Note that we cannot at this stage ignore linear in $\not{q}$ terms in the numerators, since we haven't Feynman-ized the denominator yet to make it a function of just q^2.

We need two Feynman parameters to do the integral. After a bit of algebra, one sees that the appropriate shift in momentum needed to eliminate the linear-in-q terms in the denominator is

$$q \mapsto q + kx - py, \tag{1322}$$

where x, y are the Feynman parameters. The numerator, before the shift, is

$$4g^{\mu\nu}M^3 + M\mathrm{Tr}\left[(\not{p}+\not{q})\gamma^\mu\not{q}\gamma^\nu + (\not{p}+\not{q})\gamma^\mu\gamma^\nu(\not{q}-\not{k}) + \gamma^\mu\not{q}\gamma^\nu(\not{q}-\not{k})\right], \tag{1323}$$

where we have dropped things with an odd number of gamma matrices. The possible terms that we get after the shift which have an even number of qs go like

$$g^{\mu\nu}, \quad g^{\mu\nu}q^2, \quad q^\mu q^\nu, \quad k^2 g^{\mu\nu}, \quad p^2 g^{\mu\nu}, \quad k^\mu p^\nu, \quad k^\nu p^\mu, \quad p^\mu p^\nu, \quad k^\mu k^\nu. \tag{1324}$$

The k^2 and p^2 terms are zero since we're dealing with photons. Anything with a p^μ or a k^ν is zero, since these momenta will be contracted with the polarizations and will thus be killed. To do the trace over the surviving terms, we use

$$\begin{aligned}\mathrm{Tr}[\not{r}\gamma^\mu\not{s}\gamma^\nu] &= 4(r^\mu s^\nu + r^\nu s^\mu - r\cdot s g^{\mu\nu}),\\ \mathrm{Tr}[\not{r}\gamma^\mu\gamma^\nu\not{s}] &= 4(r^\mu s^\nu - r^\nu s^\mu + r\cdot s g^{\mu\nu}).\end{aligned} \tag{1325}$$

I won't write out all the algebra for expediency's sake, since it is straightforward. One can check that the surviving $p^\nu k^\mu$ term appears as

$$4Mp^\nu k^\mu(1-4xy), \tag{1326}$$

and that the $p\cdot k$ appears with the same coefficient but with opposite sign (as dictated by gauge invariance).

What of the M^3 term in the numerator? This turns out to get killed by other things in the trace. We have

$$\begin{aligned}\mathrm{Tr}\left[\not{q}\gamma^\mu\not{q}\gamma^\nu + \not{q}\gamma^\mu\gamma^\nu\not{q} + \gamma^\mu\not{q}\gamma^\nu\not{q}\right] &= 16q^\mu q^\nu - 4g^{\mu\nu}q^2 \to g^{\mu\nu}\left(\frac{16}{4-\epsilon}-4\right)q^2,\\ &= g^{\mu\nu}\epsilon q^2\end{aligned} \tag{1327}$$

where we've anticipated the use of dimensional regularization in the integral. This term is integrated with a denominator of $(q^2-\Delta)^3$, where as usual Δ is a function of

x, y, p, k. This term then cancels the M^3 term:

$$\begin{aligned}\int_{q,x,y} \frac{4M^3 + \epsilon M q^2}{(q^2-\Delta)^3} &= \frac{-4iM^3}{32\pi^2\Delta} + \frac{Mi\epsilon}{16\pi^2}\left(\Gamma(\epsilon/2) + \text{finite}\right) \\ &= \frac{i}{8\pi^2}(-M^3/\Delta + M) \to 0,\end{aligned} \tag{1328}$$

since $\Delta \to M^2$ in the $M \to \infty$ limit. So, we have

$$\langle p, k | M\bar{\Psi}\Psi | 0\rangle = -2e^2 Mi \int_{q,x,y} \frac{4(4xy-1)(p\cdot k g^{\mu\nu} - p^\nu k^\mu)}{(q^2-\Delta)^3}\epsilon^*_\mu(p)\epsilon^*_\nu(k). \tag{1329}$$

The integral over the Feynman parameters is done over the range $\int_0^1\int_0^{1-x} dxdy$ since the integral is over the face of a 3-simplex in $\mathbb{R}^3$. Since we can set $\Delta \to M^2$ independent of x, y in the $M \to \infty$ limit, the integral over the Feynman parameters just yields $2/3$. So we get

$$\begin{aligned}\langle T^\mu{}_\mu\rangle &= -\frac{16e^2Mi}{3}\int_q \frac{p\cdot k g^{\mu\nu} - p^\nu k^\mu}{(q^2-M^2)^3}\epsilon^*_\mu(p)\epsilon^*_\nu(k) \\ &= \frac{e^2}{6\pi^2}\left(p\cdot k\epsilon^*(p)\cdot\epsilon^*(k) - p\cdot\epsilon^*(k) k\cdot\epsilon^*(p)\right).\end{aligned} \tag{1330}$$

Is this a sensible answer? Yes! We see that this is exactly equal to what we expected from the argument given earlier relating the trace anomaly to the beta function for the gauge coupling (note to self — did we drop a factor of two somewhere?)

Mixed t Hooft anomaly for the compact boson / Luttinger liquid

Consider the compact boson in two dimensions at radius R:

$$S = \frac{R^2}{4\pi}\int \partial_\mu\phi\partial^\mu\phi. \tag{1331}$$

Today's problem statement is as follows:

Working on the spacetime $S^1 \times \mathbb{R}$, write down the charge operators for the momentum and winding number global symmetries. Carry out the analysis using Abelian duality to define a field σ such that $d\phi \propto \star d\sigma$, and find the commutation relations between ϕ and σ.

Next, consider a field with non-zero charge under both momentum and winding number symmetries, and find the algebra of the symmetry charges. By considering subregion charge operators which act on submanifolds of the spatial circle, show that the symmetry is realized projectively. Demonstrate the nontrivial third $U(1)$ cohomology class which parametrizes the mixed t Hooft anomaly.

I first learned about this approach to anomalies from Alvarez's old paper [3]. A great reference for this subregion charge operator approach to anomalies is in [13]; see also the earlier work [10] for a good discussion of this way of thinking about anomalies as well as an example similar to today's where the symmetries in question are reduced to $\mathbb{Z}_2$ subgroups.

First let us write down the dual theory. This is done in the standard way by promoting $d\phi \mapsto D_B\phi$ for B a dynamical one-form, adding the term $\frac{i}{2\pi}\int B \wedge d\sigma$ for σ a zero-form, choosing unitary gauge to set $\phi \to 0$, and finally doing the Gaussian integral over B. This produces the dual action

$$S = \frac{1}{4\pi R^2}\int \partial_\mu\sigma\partial^\mu\sigma. \tag{1332}$$

We can find the relation between the field strengths by looking at where $d\phi$ goes under the mapping. We take $d\phi \to D_B\phi$ and then kill ϕ to get B. After we do the shift on B to eliminate the $B \wedge d\sigma$ coupling and then do the integral over B, we are left with $R^{-2} \star d\sigma$. Thus under the duality, we have

$$d\phi \leftrightarrow \frac{1}{R^2} \star d\sigma. \tag{1333}$$

From the original action, we see that the canonical momentum for ϕ is[139]

$$\pi_\phi = \frac{R^2}{2\pi}\partial_t\phi = \frac{R^2}{2\pi}\frac{1}{R^2}\epsilon^{tx}\partial_x\sigma = \frac{1}{2\pi}\partial_x\sigma, \tag{1334}$$

where x is the direction along the spatial S^1. Likewise, we could also do canonical quantization while choosing σ as the coordinate; in this case

$$\pi_\sigma = -\frac{1}{2\pi}\partial_x\phi. \tag{1335}$$

The symmetry here is part of the reason why we chose the factors of 2π in the action as we did.

From the commutation relation

$$[\phi(x), \partial_y\sigma(y)] = 2\pi i\delta(x-y), \tag{1336}$$

we get

$$[\phi(x), \sigma(y)] = -2\pi i\Theta(x-y). \tag{1337}$$

[139]Here we write an $=$ when replacing something with its image under T-duality. This is legit since we are considering a situation in which both fields appear in S only quadratically, and hence can be integrated out exactly, so that duality can be used as an equality inside partition functions.

We could also have chosen the RHS to be $-\pi i \text{sgn}(x-y)$, but the Θ function will be more convenient later on. This commutator shows that ϕ, σ are mutually very non-local, which makes sense due to them (or rather their derivatives) essentially being Fourier transforms of one another. Also note that we said we were working on a spatial S^1, which means that strictly speaking the Θ function above requires a basepoint. Technicalities related to this are no fun to keep track of and don't change the picture of the results we'll be getting, and so we will ignore these issues in what follows.

Using our expressions for the momenta of the fields, we see that the charges for the winding number and momentum symmetries are represented on charge w, n fields as

$$Q^w = \exp\left(\frac{iw}{2\pi}\int \partial_x \sigma\right), \qquad Q^n = \exp\left(-\frac{in}{2\pi}\int \partial_x \phi\right), \tag{1338}$$

where the integrals are over the whole S^1. Note that the charge operators involve exponentials of operators which don't commute, which will be important later.

To see if there are any anomalies, we should ask whether or not the symmetry is "splittable", i.e. whether or not the charge operators can be well-defined as operators acting on submanifolds of the spatial S^1 rather than on all of space. If the symmetry is splittable then we have a well-defined local current (obtained by making the submanifolds infinitesimally small); this can then be coupled to a gauge field and the symmetry can be gauged, implying the theory is anomaly free. If on the other hand the symmetry is not splittable then there must not be a well-defined local current, which prevents the symmetry from being gauged.[140]

To this end, consider the symmetry generator for an element $(\alpha, \beta) \in U(1)_{\text{momentum}} \times U(1)_{\text{winding}}$ on a given interval $I = [a, b]$:

$$U_I(\alpha, \beta) = \exp\left(-\frac{in[\alpha]}{2\pi}\int_a^b \partial_x \phi\right) \exp\left(\frac{iw[\beta]}{2\pi}\int_a^b \partial_x \sigma\right), \tag{1339}$$

where we have used the notation

$$[x] \equiv x \bmod 2\pi. \tag{1340}$$

We now want to know whether or not these subsystem symmetry generators realize a linear representation of the $U(1) \times U(1)$ symmetry group. Noting that the two exponentials in the definition of U_I commute,[141] one sees that

$$U_I(\alpha, \beta)U_I(\alpha', \beta') = \Omega_1^2((\alpha, \beta), (\alpha', \beta'); I)U_I(\alpha + \beta, \alpha' + \beta'), \tag{1342}$$

[140] Of course while we have used the word "current" these statements about splittability continue to hold when the symmetry group is discrete — indeed the failure of splittability (i.e. onsite-ness of the symmetry action) is usually how cond-mat people define the anomaly.

[141] This is because exponentials can be re-arranged at the cost of a factor

$$\exp\left(i\frac{nw[\alpha][\beta]}{2\pi}\int_a^b dx\, dy\, \partial_x \delta(x-y)\right) = 1, \tag{1341}$$

which only works because the two integrals over x and y are over the exact same domain; if one wiggles the support of one of the integrals this property disappears. This non-commutativity is another way to see the mixed anomaly and is elaborated on elsewhere. Different regularization conventions produce extra phase factors that don't affect the cohomology classes of the obstructions derived below, and so we won't bother keeping track of them.

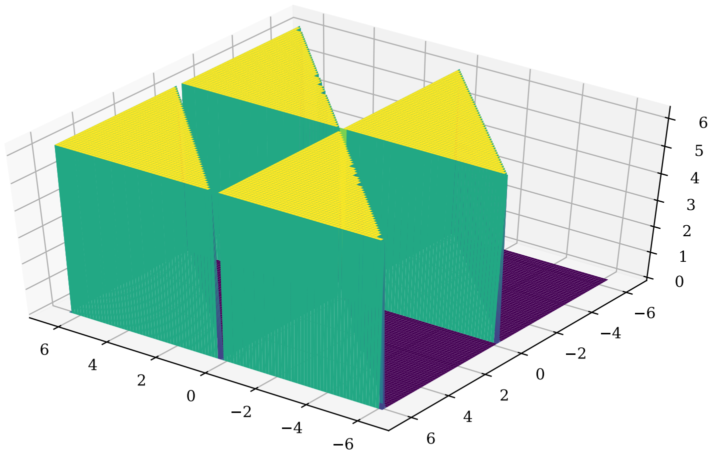
where the operator Ω_1^2 is a function of two group elements integrated over a 1-submanifold:

$$\Omega_1^2((\alpha,\beta),(\alpha',\beta');I) = \exp\left(-\frac{in\Delta(\alpha,\alpha')}{2\pi}\int_b^a \partial_x\phi\right)\exp\left(\frac{iw\Delta(\beta,\beta')}{2\pi}\int_b^a \partial_x\sigma\right), \tag{1343}$$

where we have defined the coboundary operator $\Delta : \mathbb{R}^2 \to 2\pi\mathbb{Z}$ by

$$\Delta(x,y) = [x] + [y] - [x+y]. \tag{1344}$$

$\Delta(x,y)$ is nontrivial for certain triangular regions; a plot looks like this:



(1345)

Since Δ maps into $\overline{\mathbb{Z}}$, Ω_1^2 is trivial when we take the interval I to extend over all of the spatial manifold Σ, since then the integrals are then $\int_\Sigma d\phi, \int_\Sigma d\sigma \in \overline{\mathbb{Z}}$ (assuming periodic boundary conditions on Σ).

Anyway, nontriviality of Ω_1^2 indicates that "symmetry fractionalization" occurs, at least in the extent that the split charge operators only act projectively on the Hilbert space of the theory (although here, the extent to which they are projective is caputred by a nontrivial operator rather than a simple phase factor). Note that the true global symmetry doesn't really get fractionalized, since the full charge operators still act in a linear representation ($\Omega_1^2 = 1$ when $I = \Sigma$). This is the same as saying that the symmetry can fractionalize on individual excitations, but only if those excitations obey a selection rule where they only come in configurations that transform in a linear representation.[142] This type of symmetry fractionalization is not sufficient to have an anomaly, although it is necessary, as we will see shortly.

Note that $\Omega_1^2((\alpha,\beta),(\alpha',\beta');I)$ only actually operates on ∂I (consistent with it being trivial when $I = \Sigma$), since the fact that $\phi(a)$ commutes with $\sigma(b)$ for $a < b$ means that we can write

$$\Omega_1^2((\alpha,\beta),(\alpha',\beta');I) = \Omega_0^2((\alpha,\beta),(\alpha',\beta');b)[\Omega_0^2((\alpha,\beta),(\alpha',\beta');a)]^*, \tag{1346}$$

[142] SPTs are thus basically the concept of fractionalization applied to group representations: we break up something transforming in a linear representation to several things transforming in projective ones, with an overall charge neutrality constraint. More on this in a separate diary entry.

where

$$\Omega_0^2((\alpha,\beta),(\alpha',\beta');x) = \exp\left(-\frac{in\Delta(\alpha,\alpha')}{2\pi}\phi(x)\right)\exp\left(\frac{iw\Delta(\beta,\beta')}{2\pi}\sigma(x)\right). \tag{1347}$$

For example, if we choose $(\alpha,\alpha') = (\beta,\beta') = (\pi,\pi)$, then

$$\Omega_0^2((\pi,\pi),(\pi,\pi)) = e^{-in\phi(x)}e^{iw\sigma(x)}. \tag{1348}$$

The fact that the commutation relations for the operators appearing in Ω_0^2 are very non-local implies that the symmetry is not realized in an "onsite" way on the boundary.

Also, as we will see this form for Ω_0^2 is in keeping with the descent equations for anomalies, which we will elaborate on later. Also, note that Ω_0^2 is only defined up to a coboundary of a group 1-cocycle, since we are allowed to re-define the U_I operators by phases via $U_I((\alpha,\beta)) \mapsto e^{ig((\alpha,\beta))}U_I((\alpha,\beta))$.

Now we need to figure out what the anomaly is, by going one level up in group cohomology (Ω_2^1 is a group 2-cochain, and the anomaly will be captured by a 3-cochain). To do this, we examine associativity of the symmetry operators, since the failure of associativity is captured by 3-cochains. The associativity of the U_I operators requires the cocycle condition on Ω_1^2, namely

$$(\delta\Omega_1^2)((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')) = 0, \tag{1349}$$

where δ is the coboundary operator. One can check that this will hold provided that

$$\Delta(\alpha,\alpha') + \Delta(\alpha+\alpha',\alpha'') - \Delta(\alpha',\alpha'') - \Delta(\alpha,\alpha'+\alpha'') = 0, \tag{1350}$$

which is indeed true. Note that we should really be writing e.g. $\Delta([\alpha+\alpha'],\alpha'')$, but we aren't since $\Delta([\alpha+\alpha'],\alpha'') = \Delta(\alpha+\alpha',\alpha'')$. We stress that if this associativity condition did not hold our theory would not make sense — it is merely a consistency check, not an anomaly test.

Now we ask if the action is realized associatively on individual points, i.e. whether or not the Ω_0^2 operators are annihilated by δ. This may or may not be true, and if not, it signals an anomaly. We define the cochain Ω^3 (which is just a phase factor — no ϕ or σ operators included) to measure the lack of associativity through

$$\Omega^3((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')) \equiv (\delta\Omega_0^2)((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')). \tag{1351}$$

This formula requires some explanation. Writing group elements momentarily as g,h,k, associativity tells us that

$$\begin{aligned} [U_I(g)U_I(h)]\,U_I(k) &= U_I(g)[U_I(h)U_I(k)] \\ \Omega_1^2(g,h)U_I(gh)U_I(k) &= U_I(g)\Omega_1^2(h,k)U_I(hk) \\ \Omega_1^2(g,h)\Omega_1^2(gh,k)U_I(ghk) &=^{U_I(g)} \Omega_1^2(h,k)\Omega_1^2(g,hk)U_I(ghk), \end{aligned} \tag{1352}$$

where $U_I(g)$ acts on Ω_1^2 in the way needed to allow us to pull it through to meet $U_I(hk)$. Thus as checked above, we require that $\delta\Omega_1^2 = 0$, where the coboundary operator includes the action of U_I (we didn't mention the action earlier since Ω_1^2 commutes with the U_I's). However, the individual operators Ω_0^2 which are localized at ∂_I do not need to be co-closed, and so in general we have

$$\Omega_0^2(g,h)\Omega_0^2(gh,k) = \Omega^3(g,h,k)\left({}^{U_I(g)}\Omega_0^2(h,k)\Omega_0^2(g,hk)\right). \tag{1353}$$

We can then use the commutation rules to derive the action

$${}^{U_I(\alpha,\beta)}\Omega_0^2((\alpha',\beta'),(\alpha'',\beta'')) = \exp\left(-\frac{inw}{4\pi}\left[\beta\Delta(\alpha',\alpha'')+\alpha\Delta(\beta',\beta'')\right]\right)\Omega_0^2((\alpha',\beta'),(\alpha'',\beta'')). \tag{1354}$$

Then using $\Theta(0)=1/2$ in the commutation relations for ϕ and σ operators at the same point, some algebra gives

$$\Omega^3((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')) = \exp\left(\frac{inw}{4\pi}f((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta''))\right), \tag{1355}$$

where

$$\begin{aligned} f((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')) = \Delta(\beta,\beta')\Delta(\alpha+\alpha',\alpha'') - \Delta(\beta',\beta'')\Delta(\alpha,\alpha'+\alpha'') \\ + \beta\Delta(\alpha',\alpha'') + \alpha\Delta(\beta',\beta''). \end{aligned} \tag{1356}$$

As with Ω_1^2, Ω^3 is only defined up to a 3-coboundary, since we can re-define the Ω_0^2 operators by group 2-cocycles without changing the value of Ω_1^2.

As an example, consider the case where we consider only the $\mathbb{Z}_2 \subset U(1)$ symmetry of translations of both ϕ and σ by π. Setting all of the group elements in the above to π, we find

$$\Omega^3((\pi,\pi),(\pi,\pi),(\pi,\pi)) = (-1)^{nw}. \tag{1357}$$

One can check that we would have obtained the form for Ω^3 had we used different conventions for the commutation relations and symmetry generators, e.g. if we had taken the ϕ,σ commutator to go like $-\pi i\text{sgn}(x-y)$ and taken the $\partial_x\phi$ integral in the symmetry generator to be from $a-\epsilon$ to $b+\epsilon$, where $\epsilon\to 0$ is used to ensure that the terms in U_I commute with one another.

Ω^3 represents the anomaly, and lives in $H^3(U(1)\times U(1);U(1))$ as expected. First, note that $\Omega^3=0$ if either $n=0$ or $w=0$. This signals the fact that the anomaly is a mixed anomaly between the two symmetries: either one by itself is non-anomalous, but together they become anomalous (essentially because their respective subsystem symmetry generators do not commute). Also, note that the way it was defined makes Ω^3 look like a coboundary and hence a trivial cohomology class, but this is not so since although $\Omega^3=\delta\Omega_0^2$, Ω_0^2 is not an element of $C^2(U(1)\times U(1);U(1))$ (group 2-cochains), since it contains the nonlocal ϕ,σ operators.

As a sanity check, we should make sure that $H^3(U(1)\times U(1);U(1))$ is non-zero using other methods. The Kunneth formula for group cohomology with $U(1)$ coefficients is a little dicey, so we instead calculate $H^4(U(1)^2;\mathbb{Z})$. We use that $H^p(U(1);\mathbb{Z})=\mathbb{Z}$ if p is even and $H^p(U(1);\mathbb{Z})=0$ else, which is true because the group cohomology of $U(1)$ (with $\mathbb{Z}$ coefficients) is the simplicial cohomology of the classifying space of $U(1)$, namely $\mathbb{CP}^\infty$, which has a single generator in each even degree. Thus we get

$$H^4(U(1)^2;\mathbb{Z}) \cong \bigoplus_{i=0}^4 H^i(U(1);\mathbb{Z})\otimes_{\mathbb{Z}} H^{4-i}(U(1);\mathbb{Z}) \cong \mathbb{Z}^3. \tag{1358}$$

In our example, we are only accessing the $\mathbb{Z}$ factor coming from $H^2 \otimes H^2$ (note to self: come back and pay more attention to group cohomology with a topological group — should probably be using the discrete topology.) Since group cohomology with topological groups is kinda weird, suppose as a consistency check that our symmetry group was instead $\mathbb{Z}_n \times \mathbb{Z}_n$. Then the Kunneth formula gives (recall that $H^p(\mathbb{Z}_n; \mathbb{Z})$ is $\mathbb{Z}$ if $p = 0$, $\mathbb{Z}_n$ if $p > 0$ is even, and 0 if p is odd)

$$H^3(\mathbb{Z}_n^2; U(1)) \cong \mathbb{Z}_n^3, \tag{1359}$$

which agrees with our $U(1)$ result in the limit $n \to \infty$.

Finally, we point out that even if we restrict the symmetry group to $\mathbb{Z}_2^2 \subset U(1)^2$ with $\phi \mapsto \phi + \pi, \sigma \mapsto \sigma + \pi$, we still have an anomaly, as Ω^3 becomes a nontrivial class in

$$H^3(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1)) \cong \mathbb{Z}_2^3. \tag{1360}$$

On a related note, even if only the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is preserved, the theory can still not be gapped without breaking the symmetry: for example, mass terms like $\cos\phi$ are not allowed since they are not symmetric, and while $\cos 2\phi$ is allowed, it is either irrelevant or leads to SSB since it has two inequivalent minima. This is in keeping with the rule that anomalous theories cannot be trivial in the IR unless the symmetry is explicitly broken.

Mixed t Hooft anomalies for free fields

In a previous diary entry, we saw that the $U(1) \times U(1)$ symmetry of the compact boson in 1+1D was realized projectively, signaling the existence of a mixed t Hooft anomaly. We did this by looking at the algebra of various split symmetry operators. Today we will understand this in a different light by explicitly showing that the $U(1) \times U(1)$ symmetry cannot be gauged, and will write down the anomaly polynomial (which in the 1+1D case is a 4-form). More generally, we will show this in pedantic detail for the $U(1) \times U(1)$ $(D/2-1)$-form symmetry possessed by $(D/2-1)$-form $U(1)$ gauge theory in D dimensions, where D is even.

We will write the action in D dimensions as

$$S = \frac{1}{2g^2} \int_M F \wedge \star F, \tag{1361}$$

where F is some $D/2$-form field which is locally $F = dA$ (depending on conventions there should potentially be some combinatorial prefactors dependent on D up front, but we will ignore them). Since the field strength F is a $D/2$ form, this theory is self-dual, as $\star F$ is also a $D/2$ form. The dual action is

$$S = \frac{g^2}{2(2\pi)^2} \int \mathcal{F} \wedge \star \mathcal{F}, \tag{1362}$$

where the dual field $\mathcal{A}$ (locally $\mathcal{F} = d\mathcal{A}$) is mapped to the original through $\mathcal{F} \leftrightarrow g^2 \star F/2\pi$ (with the meaning here that the correlation functions of $g^2 \star F/2\pi$ in the $F \wedge \star F$ action are the same as the correlation functions of $\mathcal{F}$ in the $\mathcal{F} \wedge \star \mathcal{F}$ action). In writing both of these actions, we are *not* assuming that A and $\mathcal{A}$ are globally well-defined. In fact, in the partition function we will need to sum over all possible line bundles for A and $\mathcal{A}$.

Suppose now that we try to gauge the two symmetries in the theory (the symmetries of shifting A and its dual $\mathcal{A}$ by flat $(D/2-1)$-forms). In order to gauge the shift symmetries, we introduce a gauge field B which lets us locally transform $A \mapsto A + \alpha$, where α is not necessarily in the kernel of d. There are many ways to see that doing so implies that the $\mathcal{A}$ shift symmetry cannot also be simultaneously gauged, signaling an anomaly. This is essentially because A and $\mathcal{A}$ are related to eachother by a “Fourier transform” and so the way in which a gauge field acts on them cannot be “simultaneously diagonalized” on both of the fields.

In order for the shift symmetry of A to be gauged, we must minimally couple to a background field B as[143]

$$S[B] = \frac{1}{2g^2} \int (F - B) \wedge \star (F - B). \tag{1363}$$

For example, if $D = 4$, then we have regular Maxwell theory. Under gauge transformations, both F and B shift by F', where F' is allowed to be the curvature of any connection on any line bundle (not necessarily trivial, since we are summing over all line bundles for the original gauge field A), and B is a 2-connection on a trivial 2-bundle ($U(1)$ gerbe).

Schematically, we can argue that this leads to a breaking of the shift symmetry for $\mathcal{A}$ as follows. The upgraded gauge-invariant current for the shift symmetry of A is $(B - F)/g^2$, which we get by computing $\delta S[B]/\delta B$. This is conserved provided that the sources of the A gauge field match up with $d^\dagger B$, i.e. provided that $d^\dagger F = d^\dagger B$. This holds as the equation of motion for A. We expect that the current for the dual shift symmetry on $\mathcal{A}$ is then

$$\mathcal{J} \sim \star(F - B). \tag{1364}$$

However, the conservation of $\mathcal{J}$ is broken by the curvature of B, since

$$d^\dagger \mathcal{J} \sim d^\dagger \star (F - B) \propto \star dB \neq 0. \tag{1365}$$

[143]In writing this expression, we are tacitly assuming that B is a connection on a trivial $(D/2)$-bundle, so that B is globally well-defined. Relaxing this assumption can be done but leads to keeping track of more details which aren’t super illuminating.

An analogous problem would have occurred had we started with the dual action, and gauged the shift symmetry on $\mathcal{A}$ by taking

$$\mathcal{F} \mapsto (\mathcal{F} - \mathcal{B}). \tag{1366}$$

Doing this would lead to a violation of the conservation of the J current of the form

$$d^\dagger J \propto \star d\mathcal{B}. \tag{1367}$$

Essentially, the point is that no matter which formulation we choose, one of the currents must not be conserved after we introduce a gauge field.[144]

Now for a more careful argument. To figure out what happens in the dual formulation, we just run Abelian duality on this in the standard way. If the coupling to the background gauge fields is not anomalous, we should get an action like $S[\mathcal{B}] \sim g^2 \int (\mathcal{F} - \mathcal{B})^2$. The usual duality recipe tells us to first add a gauge field, and then integrate out the original field variables. So, we add a $D/2$-form gauge field a to the action, and then add *another* $(D/2-1)$-form field $\mathcal{A}$ with curvature $\mathcal{F}$ whose job is to kill off a to reproduce the original non-gauged theory. Summing over all line bundles L for A and $\mathcal{L}$ for $\mathcal{A}$, we have the partition function

$$Z = \sum_{L,\mathcal{L}} \int \mathcal{D}A\mathcal{D}a\mathcal{D}\mathcal{A} \exp\left(\frac{1}{2g^2}\int (F - a - B)\wedge\star(F - a - B) + \frac{i}{2\pi}\int \mathcal{F}\wedge a\right). \tag{1372}$$

[144]We have been setting $\theta = 0$ so far for simplicity. With a θ term, it's easiest to work in terms of the self-dual and anti-self-dual field strengths. We write the action as

$$\frac{i}{4\pi}\int (\tau F_+ \wedge F_+ + \bar{\tau} F_- \wedge F_-) = \frac{i}{4\pi}\left(\tau\langle F_+, F_+\rangle - \bar{\tau}\langle F_-, F_-\rangle\right), \tag{1368}$$

where the modular parameter is $\tau = \frac{\theta}{2\pi} - \frac{2\pi i}{g^2}$. Running Abelian duality tells us that the appropriate dual fields should be identified as

$$\mathcal{F}_+ = i\tau F_+, \qquad \mathcal{F}_- = i\bar{\tau} F_-. \tag{1369}$$

So, without the θ term the duality between F and $\mathcal{F}$ is just Hodge duality plus an inverting of the coupling constant, while the θ term mixes F and its Hodge dual together.

Now we insert a background field for F, making the replacements $F_\pm \mapsto F_\pm - B_\pm$ in the action. We then run duality on this, which won't be written out explicitly for the sake of expediency. The appropriate dualized background field is

$$\mathcal{B} = \mathrm{Re}(\tau)B + \mathrm{Im}(\tau)\star B, \tag{1370}$$

and we find that running the duality produces the action we'd expect, up to a contact term for the dual background fields:

$$\begin{aligned} S_{dual} &= -\frac{i}{4\pi}\int\left(\frac{1}{\tau}(\mathcal{F}_+ - \mathcal{B}_+)\wedge\star(\mathcal{F}_+ - \mathcal{B}_+) - \frac{1}{\bar{\tau}}(\mathcal{F}_- - \mathcal{B}_-)\wedge\star(\mathcal{F}_- - \mathcal{B}_-)\right) - S_{\mathcal{B}}, \\ S_{\mathcal{B}} &= -\frac{i}{4\pi}\int\left(\frac{1}{\tau}\mathcal{B}_+\wedge\star\mathcal{B}_+ - \frac{1}{\bar{\tau}}\mathcal{B}_-\wedge\star\mathcal{B}_-\right). \end{aligned} \tag{1371}$$

As explained in the main text, this extra contact term signals the mixed anomaly.

Here we are being slightly sloppy and not writing factors of $1/\mathrm{vol}\, G$ for the various groups of gauge transformations.

In the present partition function A and $\mathcal{A}$ are independent fields. Thus the coupling $\mathcal{F} \wedge a$ is gauge-invariant since $\mathcal{F}$ does not transform under shifts in A (this can be corroborated by checking that under duality, both F and $F + d\alpha$ map to $g^2\mathcal{F}/2\pi$, instead of e.g. $F + d\alpha$ mapping to $\mathcal{F} + g^2 \star d\alpha/2\pi$). Also note that the $\mathcal{F} \wedge a$ coupling is gauge invariant under the $\mathcal{A} \mapsto \mathcal{A} + \alpha$ shift symmetry: since the integral over $\mathcal{A}$ sets a to be exact, after integrating out $\mathcal{A}$ we are left with $\int_M d\alpha \wedge db$ for some b with $a = db$, and so since $\alpha|_{\partial M} = 0$ (as α is a gauge transformation), then $\int_M d\alpha \wedge db = 0$.

To see that we haven't done anything by writing the partition function in this way, we see that the integral over the globally well-defined part of $\mathcal{A}$ sets $da = 0$, and the sum over $\mathcal{L}$ gives a delta function setting

$$\int_N a \in 2\pi\mathbb{Z} \to [a] \in H^{D/2}_{2\pi\mathbb{Z}}(M; \mathbb{R}), \tag{1373}$$

where $N \subset M$ is any closed $D/2$ manifold, and the subscript on the cohomology group indicates that a must integrate over any closed $D/2$ manifold to something in $2\pi\mathbb{Z}$. This quantization condition on a means that it is just the same as F: a closed $D/2$ form which is subject to Dirac quantization. Thus since we are summing over all line bundles for A, we can simply do a change of integration variables for A to absorb a into F, recovering the original action without a or $\mathcal{A}$.

Now we proceed by using the gauge freedom of a to kill off F, which is allowed since F is exact (this amounts to choosing "unitary gauge"). Then

$$Z = \sum_{\mathcal{L}} \int \mathcal{D}a\mathcal{D}\mathcal{A} \exp\left(\frac{1}{2g^2}\int (a+B) \wedge \star(a+B) + \frac{i}{2\pi}\int \mathcal{F} \wedge a\right). \tag{1374}$$

Now we kill off the $\mathcal{F} \wedge a$ term by shifting the integration as (the unsavory i here is just because we're working in $i\mathbb{R}$ time)

$$a \mapsto a - B - \frac{g^2 i}{2\pi} \star \mathcal{F}. \tag{1375}$$

Quietly absorbing the Gaussian integral into the integration measure, we get the dual action

$$S_{dual} = \frac{1}{2g_{dual}^2}\int (\mathcal{F} \wedge \star\mathcal{F} + \frac{i}{g}(B \wedge \mathcal{F} + \star\mathcal{F} \wedge \star B)), \tag{1376}$$

where the dual coupling constant is $g_{dual} = 2\pi/g$. To be a but more suggestive, we define

$$\mathcal{B} = \frac{2\pi i}{g^2} \star B. \tag{1377}$$

With this we get

$$S_{dual} = \frac{1}{2g_{dual}^2}\int ((\mathcal{F} - \mathcal{B}) \wedge \star(\mathcal{F} - \mathcal{B}) - \mathcal{B} \wedge \star\mathcal{B}) \tag{1378}$$

Note that this action is what we would expect to get if we had started by gauging the shift symmetry on $\mathcal{A}$, except for the anomalous $\mathcal{B} \wedge \star\mathcal{B}$ term, which looks like a mass for the background field. We see that this presentation is not gauge invariant no matter how we choose B to transform under the $\mathcal{A}$ shift symmetry, since the action transforms under gauge transformations as

$$\delta S \sim \int_M \alpha \wedge d \star \mathcal{B} \sim \int_M \alpha \wedge dB, \tag{1379}$$

and so the gauge invariance of the dual action is broken by the curvature of the background field.

Before we move on, we should point out that we have been using one higher $U(1)$ gauge field B to attempt to gauge both symmetries. In general, one might have thought that we should be allowed to use two higher gauge fields, since the full symmetry is a $U(1) \times U(1)$ 1-form symmetry. This turns out to not work, essentially because the two putative gauge fields have to be related to one another in an inconsistent way by duality. Let the two fields be denoted as B_e and $\star B_m$, with B_m neutral under the shift symmetry of A and B_e neutral under that of $\mathcal{A}$. In order to get a dual action which is electrically gauge-invariant, the B_m fields need to be included in the original action, and so we can write

$$S = \frac{1}{2g^2} \int (dA - B_e - \star B_m) \wedge \star (dA - B_e - \star B_m). \tag{1380}$$

In order for this to be magnetically gauge invariant, we need the terms with B_m to integrate to zero. But this is possible only if

$$\int dA \wedge B_m = 0, \tag{1381}$$

i.e. only if $dB_m = 0$, and as such we cannot gauge the $\mathcal{A}$ symmetry with a genuine background field (one which is allowed to be non-flat). Any way we look at it, there's a mixed t Hooft anomaly.

Simple example of a discrete 't Hooft anomaly

This was inspired by wanting to work through and elaborate on appendix D of "Theta, TR, and Temperature" [12]. I'm sure all of what follows exists in the literature somewhere; in any case it's just a slight elaboration on the TTrT paper. Our problem statement is as follows:

Consider the quantum mechanics of a free fermion with periodic boundary conditions in time, with Lagrangian $\mathcal{L} = i\psi^\dagger \partial_t \psi$. Show that this system has a mixed t Hooft anomaly between charge conjugation and $U(1)$. What is the corresponding bulk term that needs to be added to allow the system to be symmetric? Finally, briefly explain why this is equivalent to the bosonic particle-on-a-ring-with-flux model considered in Theta TR and Temperature, which is in turn equivalent to QED$_2$ at $\theta = \pi$. Consider this model with a cosine potential, and find out whether or not the GSD at $\theta = \pi$ is lifted.

Projective symmetry action

Formally, from the cohomology classification of anomalies (that can be captured this way), we know that in quantum mechanics, anomalies in a symmetry group will be classified by the group cohomology $H^2(G; U(1))$, which classifies central extensions of G by $U(1)$ (aka projective representations).

Let's first see that in quantum mechanics, the action of the classical symmetry group (in our case $O(2)$) must either be enlarged to a linear action of a central extension of the classical symmetry group, or be represented projectively on the Hilbert space of the theory (depending on the way we choose to look at things).

Let us write the group elements of $U(1)$ as α, the generator of $\mathbb{Z}_2^C$ as C, and their representations on the Hilbert space as R_α, R_C, which we will first assume to be unitary linear representations. They act on the ψ operators as

$$\alpha\psi\alpha^{-1} = e^{i\alpha}\psi, \qquad C\psi C^{-1} = \psi^\dagger. \tag{1382}$$

Thus from $R_\alpha \psi^\dagger R_\alpha^{-1} = e^{-i\alpha}\psi^\dagger$ (since the representation is assumed to be unitary), we have

$$e^{-i\alpha}\psi^\dagger = R_\alpha R_C \psi R_C^{-1} R_\alpha^{-1}, \tag{1383}$$

and so evidently

$$R_\alpha R_C = R_C R_\alpha^{-1}, \tag{1384}$$

and so the symmetry group we expect to get is $U(1) \rtimes \mathbb{Z}_2^C = O(2)$.

Is this symmetry realized linearly on the Hilbert space? The Hilbert space is $\mathcal{H} = \{|0\rangle, |1\rangle = \psi^\dagger|0\rangle\}$. Then

$$R_C|1\rangle = R_C\psi^\dagger R_C^{-1} R_C|0\rangle = \psi R_C|0\rangle. \tag{1385}$$

If $R_C|0\rangle \propto |0\rangle$ then $R_C|1\rangle = 0$, and so R_C is not a unitary representation. Thus we can take $R_C|1\rangle = |0\rangle$ (inserting a phase factor in this definition will not change the cohomological classification of the anomaly discussed below), and consequently $R_C|0\rangle = |1\rangle$.

Suppose $|0\rangle$ has charge q_0 under $U(1)$. Then

$$R_\alpha|1\rangle = R_\alpha\psi^\dagger R_\alpha^{-1} R_\alpha|0\rangle = e^{-i\alpha}e^{iq_0\alpha}|1\rangle. \tag{1386}$$

Thus, if q_1 is the charge of $|1\rangle$, we have

$$q_1 = q_0 - 1. \tag{1387}$$

Now consider evaluating this using C:

$$R_\alpha|1\rangle = R_\alpha R_C|0\rangle = R_\alpha R_C R_\alpha^{-1} R_\alpha|0\rangle = R_C R_\alpha^{-1}|0\rangle = e^{-iq_0\alpha}|1\rangle, \tag{1388}$$

and hence we also require

$$q_1 = -q_0. \tag{1389}$$

Evidently we must have $q_0 = 1/2, q_1 = -1/2$. But this is a contradiction to our assumption about how the symmetry is realized on the Hilbert space, since now

$$R_{\alpha=2\pi}|0\rangle = -|0\rangle, \tag{1390}$$

and so $R_{2\pi}$ is not represented trivially on the Hilbert space! In fact, it is represented as $R_{2\pi} = -1$. Thus the symmetry group actually acts projectively, with relations between representations of generators holding only modulo elements of $\mathbb{Z}_2$[145].

Group extensions:

Projective representations are classified by central extensions of the symmetry group. In quantum mechanics, we are allowed to extend the symmetry group by $U(1)$ — that is, we allow the representation of the symmetry to not be linear, as long as the non-linearity manifests itself purely as $U(1)$ phases. Thus in our case the full quantum symmetry group should fit into the exact sequence

$$1 \to U(1) \to G \to O(2) \to 1. \tag{1391}$$

Central extensions of the above form are classified by the cohomology group

$$H^2(O(2); U(1)) = \mathbb{Z}_2, \tag{1392}$$

which can be calculated with spectral sequences starting from the exact sequence $1 \to U(1) \to O(2) \to \mathbb{Z}_2 \to 1$. The nontrivial element of $H^2(O(2); U(1))$ is the central extension realized by our fermions, where rotations by 2π act as -1. We can write the extension realized by the fermions as

$$1 \to U(1) \to pin_+(2) \to O(2) \to 1, \tag{1393}$$

where $pin_+(2)$ is the double-cover of $O(2)$ for which $C^2 = 1$. The extension is central since the image of $\mathbb{Z}_2$ in $pin_+(2)$, namely the identity and the rotation by 2π, is central: $R_C R_{2\pi} = R_{-2\pi} R_C = R_{2\pi} R_C$. Note that since

$$H^2(U(1); U(1)) = H^2(\mathbb{Z}_2^C; U(1)) = 0, \tag{1394}$$

we need both the $U(1)$ and the $\mathbb{Z}_2^C$ symmetries to get the anomaly: this is why it is referred to as a mixed t Hooft anomaly.

Let us briefly digress by talking about the difference between $U(1)$ coefficients and $\mathbb{Z}_2$ coefficients. If we instead performed an extension by $\mathbb{Z}_2$, then our allowed quantum

[145]Re-defining $C|0\rangle = e^{i\beta}|1\rangle$ for some phase $e^{i\beta}$ would not have changed the fact that $R_{2\pi} = -1$.

symmetry groups would be given by $H^2(O(2);\mathbb{Z}_2)$. We can calculate this given the facts that

$$H^2(O(2);U(1)) \cong \mathbb{Z}_2, \qquad H^3(O(2);U(1)) \cong \mathbb{Z}\times\mathbb{Z}_2. \tag{1395}$$

We do this by applying the Kunneth formula for group cohomology to the cohomology $H^2(1\times O(2);\mathbb{Z}_2\otimes_{\mathbb{Z}} U(1))$. Since $\mathbb{Z}_2\otimes_{\mathbb{Z}} U(1) = \mathbb{Z}_2$ as $0 = (1+1)\otimes\alpha = 1\otimes(2\alpha)$ implies that taking the $\otimes$ with $\mathbb{Z}_2$ kills all of the elements in $U(1)$, this cohomology group is actually $H^2(1\times O(2);\mathbb{Z}_2\otimes_{\mathbb{Z}} U(1))$. Since $H^k(1;\mathbb{Z}_2)$ is $\mathbb{Z}_2$ if $k=0$ and is 0 else, we get the following exact sequence:

$$1 \to \mathbb{Z}_2\otimes H^2(O(2);U(1)) \to H^2(O(2);\mathbb{Z}_2) \to \text{Tor}\left[\mathbb{Z}_2, H^3(O(2);U(1))\right] \to 1. \tag{1396}$$

Plugging in the cohomology groups and using $\text{Tor}[\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2] = 0\oplus\mathbb{Z}_2$, we see that

$$H^2(O(2);\mathbb{Z}_2) \cong \mathbb{Z}_2^2. \tag{1397}$$

Thus, restricting the coefficients to $\mathbb{Z}_2$ leads to a doubling of the number of extensions. This is because with $\mathbb{Z}_2$ coefficients we can fractionalize the relation $C^2 = 1$ to $C^2 = -1$, as well as fractionalizing the 2π rotation. Thus we get $pin_\pm(2)$, $O(2)\times\mathbb{Z}_2$, and an extension where a 2π acts as the identity but $C^2 = -1$. When we use $U(1)$ coefficients instead this classification collapses to just two extensions, since the relation $C^2 = -1$ can be turned into $C^2 = +1$ by redefining $C \mapsto iC$. Thus since in quantum mechanics we need to choose $U(1)$ for the coefficient group, we can take $C^2 = +1$ and set the symmetry group realized by our fermions to be $pin_+(2)$ without loss of generality.

Diagnosing the anomaly by gauging: Now we will diagnose the anomaly by attempting to gauge the $U(1)$ symmetry which sends ψ to $e^{i\alpha}\psi$. Adding a background field A (since A only has a time component, we could also rename it as μ), the new Lagrangian is

$$\mathcal{L} = i\psi^\dagger(\partial_t - iqA)\psi. \tag{1398}$$

Charge conjugation sends $\psi \mapsto \psi^\dagger$, and $A \mapsto -A$.[146] The Lagrangian is invariant under charge conjugation because the integration by parts and fermion anticommutation give cancelling minus signs. It is also invariant under local $U(1)$ transformations by construction.

The Lagrangian is gauge invariant, but is the theory invariant under the full $O(2) = U(1)\rtimes\mathbb{Z}_2^C$ symmetry when quantized? To answer this, we look at the partition function. The Hamiltonian with $A=0$ is zero, while the A term adds a chemical potential to the Hamiltonian so that $H = -q\psi^\dagger A_0\psi$. The partition function is thus

$$\begin{aligned} Z[A] &= \text{Tr}_{\mathcal{H}} e^{-iH} = \langle 0|e^{iq\int\psi^\dagger A_0\psi}|0\rangle + \langle 1|e^{iq\int\psi^\dagger A_0\psi}|1\rangle \\ &= 1 + e^{iq\int A_0}. \end{aligned} \tag{1399}$$

Under charge conjugation, this maps to

$$C : Z[A] \mapsto 1 + e^{-iq\int A_0} = e^{-iq\int A_0} Z[A], \tag{1400}$$

[146]The usual discussion of A being a background field and hence not literally transforming under the symmetry action of course applies.

which is not equal to $Z[A]$ since in general the holonomy of A will not be in $(2\pi/q)\mathbb{Z}$. Thus, we conclude that when we try to couple the theory to a background $U(1)$ field, $\mathbb{Z}_2^C$ is broken—this indicates the presence of a t Hooft anomaly.

How might we cancel this anomaly? Observe that if we added the term $-\frac{q}{2}\int A$ to the Lagrangian, then the partition function would be equal to

$$\begin{aligned} \widetilde{Z}[A] &= \mathrm{Tr}_{\mathcal{H}} \exp\left(iq\int \psi^\dagger A\psi - \frac{iq}{2}\int A\right) \\ &= (1+e^{iq\int A_0})e^{-iq/2\int A_0}. \end{aligned} \tag{1401}$$

Under charge conjugation,

$$C : \widetilde{Z}[A] = 2\cos\left(\frac{q}{2}\int A_0\right) \mapsto 2\cos\left(-\frac{q}{2}\int A_0\right) = \widetilde{Z}[A], \tag{1402}$$

and so the partition function is invariant under $\mathbb{Z}_2^C$. The problem of course is that we have added an incorrectly quantized Chern-Simons term, which breaks the invariance of the Lagrangian under large gauge transformations which change the holonomy of A by $(2\pi/q)\mathbb{Z}$ (and is also not well-defined because of Cech-y reasons; see a later diary entry on the CS term). So this solution doesn't let us have a symmetric theory either, provided our background gauge field is a $U(1)$ gauge field and not an $\mathbb{R}$ gauge field.

Anomaly inflow then happens by simply allowing the time circle to bound a disk. This allows us to write the $-\frac{q}{2}\int A$ term in a gauge-invariant way as $-\frac{q}{2}\int_{D^2} F$. Simply letting the circle bound the disk dispenses with the large gauge transformation issue and lets the full theory, namely

$$S = \int_{\partial D^2} i\psi^\dagger(\partial_t - iqA)\psi - \frac{q}{2}\int_{D^2} F, \tag{1403}$$

which is now invariant under the full $U(1) \rtimes \mathbb{Z}_2^C$ symmetry.

Fermion nonconservation and the ABJ anomaly in 3+1D from Hamiltonian spectral flow

Just realized that there was still a problem in P&S that I wanted to do—it's from chapter 19. Here's the problem statement:

Examine the ABJ relation

$$\Delta N_L - \Delta N_R = -\frac{e^2}{16\pi^2}\int \epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{1404}$$

in four dimensions, where $N_{L/R}$ are the number of left- and right-handed fermions $\psi_{L/R}$ and the Δ measures differences between $t = \infty$ and $t = -\infty$. Take the background field to be

$$A^\mu = (0,\ 0,\ Bx,\ A), \tag{1405}$$

with B constant and A constant in space but possibly varying adiabatically in time. First, find the Hamiltonian. Then, solve the Schrodinger equation for the two fields $\psi_{L/R}(x)$. You should encounter a harmonic oscillator at some point during the calculation, just like when doing the analogous problem in two dimensions.

Consider putting the fermions in a box with sides of length L, with periodic boundary conditions. Find the degeneracy of the energy levels. Then change A adiabatically. What happens to the number of left- and right-handed fermions? Show that this checks out with the ABJ equation.

Now $\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} = 4\epsilon^{0ijk}F_{0i}F_{jk} = 8E_iB^i$, so that the ABJ formula is

$$N_L - N_R = -\frac{e^2}{2\pi^2}\int E_iB^i. \tag{1406}$$

The goal of this problem is to check this relation in a rather direct way.

First let's get the Hamiltonian. We have

$$\mathcal{H} = \pi_\psi D_0\psi - \mathcal{L}. \tag{1407}$$

Here π_ψ is the regular canonical momentum for ψ, namely $\pi_\psi = \bar{\psi}i\gamma^0 = i\psi^\dagger$. Note that the $p\partial_t q$ term is modified by replacing ∂_t with the covariant version D_0 (we are treating A^μ as a background field, not a dynamical one), which completely cancels the D_0 part of the Lagrangian.

We will work in mostly negative signature, so that the gamma matrices are $\gamma^i = iY\otimes\sigma^i$. When combined with the $\gamma^0 = X\otimes\mathbf{1}$ from $\bar{\psi}$, we get the matrix $-Z\otimes\sigma^i$. Thus in the basis $\psi = (\psi_L,\psi_R)^T$, we have

$$\mathcal{H} = -i\psi_R^\dagger\sigma^i(\partial_i - ieA_i)\psi_R + i\psi_L^\dagger\sigma^i(\partial_i - ieA_i)\psi_L. \tag{1408}$$

Now the only spatial coordinate that appears in the vector potential is x, so that we have translation symmetry for y and z. Thus if we have $\int\mathcal{H} = E$ for e.g. the right-handed fermions then we are led to consider the Eigenvalue problem

$$-i\sigma^i(\partial_i - ieA_i)\psi_R = E\psi_R, \qquad \psi_R = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix}e^{ik_yy+ik_zz}. \tag{1409}$$

Writing out the eigenvalue equation, we get the coupled equations

$$\begin{aligned}(E - k_z + eA)\phi_1 + (i\partial_x + ik_y - ieBx)\phi_2 &= 0\\ (E + k_z - eA)\phi_2 + (i\partial_x - ik_y + ieBx)\phi_1 &= 0.\end{aligned} \tag{1410}$$

So then after some algebra,

$$(\partial_x^2 + E^2 + B)\phi_1 = [(k_z - A)^2 + (k_y - Bx)^2]\phi_1, \tag{1411}$$

where we are temporarily letting $e = 1$. More suggestively,

$$(-\partial_x^2 + (k_y - Bx)^2 + (k_z - A)^2 - B)\phi_1 = E^2\phi_1, \tag{1412}$$

which is the Harmonic oscillator we were told we were going to find. Notice that k_y just sets the location of the center of the oscillator potential, but does not actually appear in the expression for the energy levels E (which we can get explicitly if we want, but we won't need the exact expressions).

What would have happened if we had done this with ψ_L instead? The only difference for ψ_L is that the eigenvalue equation is

$$+ i\sigma^i(\partial_i - ieA_i)\psi_L = E\psi_L, \qquad \psi_L = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} e^{ik_y y + ik_z z}, \tag{1413}$$

and so the only change is to replace E with $-E$, which won't affect conclusions about the degeneracy.

Now we put the Fermions in a box with side length L. The momenta are then $k_i \in (2\pi)/L\mathbb{Z}$. From the oscillator equation we saw that k_y sets the center of the oscillator potential through $x_c = k_y/B$ and doesn't affect the energy, while k_z sets the energy. So for a given energy (determined by k_z), different choices of k_y will lead to degenerate levels. However, we only have access to a certain number of k_y due to the finiteness of the box: taking the harmonic oscillator wavefunctions to be localized on a scale in the y direction much smaller than L, we require that $x_c < L$, or

$$k_y < BL \tag{1414}$$

since of course we need the wavefunction to be within the box. Since each value of k_y is spaced $\delta k_y = 2\pi/L$ apart, the degeneracy of each k_z level is

$$n = \frac{BL^2}{2\pi}, \tag{1415}$$

independent of k_z.

Now we consider an adiabatic change in A. Since this gives us a nonzero $E_z = \partial_t A$ and since the magnetic field is $B\hat{z}$, we have a nonzero E_iB^i during this process and expect $\Delta N_L - \Delta N_R \neq 0$ from the ABJ formula. We will consider a change in A so that the initial and final configurations are related by a gauge transformation with winding 2π around the x direction. So, we can set the initial A to be $A_i = 0$ and the finial A to be $A_f = 2\pi/L$, which is a large gauge transformation since the holonomy of the gauge field around the x direction in the final state is in $2\pi\mathbb{Z}$. This gives the minimal value we can have for $\int F \wedge F$, since in this context $\int F \wedge F$ measures the number of large gauge transformations (each a winding by 2π) that occur.

Since the initial and final states are related by a large gauge transformation, the solving of the eigenvalue problem proceeds in the same way at t_i and t_f, and we get the same harmonic oscillator spectrum. However, let's look at where the levels go during the evolution: solving for E in the harmonic oscillator Hamiltonian for ϕ_1, we see that E depends on A through $E \sim k_z - A$. Thus increasing A by $A \mapsto A + 2\pi/L$ is tantamount to *decreasing* k_z by $2\pi/L$, and so during the adiabatic evolution one level for the ψ_R fermions sinks below the Fermi level (wherever that may be). Conversely, we saw that when solving for the left-handed analogue φ_1, we just had to replace $E \mapsto -E$, and so for the left-handed fermions we have $E \sim A - k_z$, and so the change is equivalent to *increasing* k_z by $2\pi/L$, which pulls up a level's worth of ψ_L fermions out of the Fermi sea. Thus during the change in A, the vacuum looses R fermions but gains L fermions.

Since there are $BL^2/(2\pi)$ states in each level and since doing a large gauge transformation moves the ψ_R fermions down a level and the ψ_L fermions up a level, we have (now restoring e)

$$\Delta N_R - \Delta N_L = -\frac{BL^2}{2\pi} - \frac{BL^2}{2\pi} = -\frac{Be^2L^2}{\pi}. \tag{1416}$$

Does this agree with the anomaly formula? Let's check: during the time when A is changing (which is the only time during which $E \neq 0$), the non-zero part of E is $E_z = \frac{2\pi}{LT}$, where $T = t_f - t_i$ is the time of the adiabatic evolution. Thus according to the ABJ formula,

$$\Delta N_R - \Delta N_L = -\frac{e^2}{2\pi^2}\int d^4x E_i B^i = -\frac{e^2}{2\pi^2}\int d^3x B\frac{2\pi}{L} = -\frac{Be^2L^2}{\pi}, \tag{1417}$$

which is exactly what we got with the more direct approach. Yay!

Orbifolding orbifolds and gauging higher form discrete symmetries

Today we are revisiting an old diary entry which explained why orbifolding is an involution. Here we look at the same sort of result from a different perspective, as well as go into rather garrulous detail about the relation between orbifolding and gauging. A good reference to get inspired for this diary entry is the introduction of [5], specifically the part that talks about gauging 1-form $\mathbb{Z}_2$ symmetries.

We will work throughout mostly with a theory on a two dimensional spacetime with a discrete global 0-form symmetry G, which we will take to be $\mathbb{Z}_N$ for simplicity. Nothing prevents us from going to more general Abelian G and to higher form symmetries, other than a desire to make the notation simple. First we will discuss rather pedantically how this symmetry gets gauged in the continuum. Next we will explain orbifolding and its precise relation to gauging, and finally we will talk about what happens when these operations are done twice.

Gauging

Let us couple this theory to a flat background 1-form $\mathbb{Z}_N$ gauge field A, which we assume is done through some sort of minimal-coupling prescription in an action. The only gauge-invaraint data in the gauge field comes from its cohomology class. Letting the partition function on a D-manifold X in the presence of the background field A be $Z[X;A]$, the gauged partition function is obtained by path-integrating over A, i.e. by summing over all cohomology classes in $H^1(X;\mathbb{Z}_N)$ (coboundaries are pure gauge). The gauge partiton function $Z[X]$ is[147]

$$Z[X] = \frac{1}{\dim H^0(X;\mathbb{Z}_N)} \sum_{A\in H^1(X;\mathbb{Z}_N)} Z[X;A]. \tag{1418}$$

Here, the normalization out front is the volume of the group of global symmetries, viz. the $\mathbb{Z}_N$-valued functions that are constant on each path component of X. This is the "global part of the group of gauge transformations". Depending on the situation, we may or may not regard this as being gauged. Usually if we are in the setting where $X = \Sigma\times\mathbb{R}$ with $\partial\Sigma \neq 0$, these transformations are not gauged, since the boundary conditions on the fields are fixed and physical, with gauged gauge transformations required to vanish on $\partial\Sigma$ (think of CS theories and the physicality of the phase differences in Josephson junctions). We will elaborate more on this in the next subsection, but for now we will assume that X is closed (and we will continue to denote the slice on which we quantize by Σ). From here on we will assume X has only a single connected component, saving us the trouble of writing the normalization factor.

A Hamiltonian / wavefunction approach to the gauging procedure (i.e. operators and commutation relations on Σ rather than classical fields on X), which doesn't require thinking about an action as much, is as follows. First, we equip Σ with a cell decomposition (dual to a triangulation).[148] We then define the split charge operators $U_M(g) = e^{iQ_M(g)}$, where $M \subset \Sigma$ is a d-cell ($d = \dim\Sigma$) and $Q_M(g)$ is the charge operator $Q(g)$ for the element g restricted to M. $e^{iQ_M(g)}$ is defined to create an oriented domain wall on ∂M across which charged operators are acted on by R_g (in our case, a representation of $\mathbb{Z}_N$)[149]. Just for notation's sake, we might want to write this as $Q_M(g) = g\int_M \star j$ for j some 1-form conserved ($d^\dagger j = 0$) current. Here the notation is schematic; it is only meant to convey that $Q_M(g)$ is given by an integral

[147] Since we are choosing a specific representative element for each cohomology class, this is a gauge-fixed partition function.

[148] If we are working in the continuum, this should be thought of as a good cover of Σ. If we are working on a lattice, we should think of the cell decomposition as defining the lattice. We will largely use continuum notation in what follows, but this is only for concreteness.

[149] If our symmetry was anomalous, this step would be problematic — the charge operators would not be splittable in this way.

(or sum on the lattice) of something local (this may not be particularly nice, e.g. in the Ising model it would be $g\sum_{i\in M}\ln X_i$).

Now by assumption the Hamiltonian H is symmetric and local, and so $U_M(g)^\dagger HU_M(g)$ fails to be H by something supported on ∂M. To change this, we of course add a gauge field A to H which modifies the energies of G domain walls — one can think of this as promoting the couplings of the interaction / gradient terms to dynamical fields (best thought of on the lattice), but really what's going on is that the gauge field is just shorthand notation for the transition functions of the $\mathbb{Z}_N$ bundle the fields of the theory are defined on. Anyway, H then commutes with the operators

$$\mathcal{G}_M(g) = U_M(g)T_{\partial M}(g), \tag{1419}$$

where $T_{\partial M}(g)$ is an operator supported on ∂M which implements the change in the gauge field needed so that $[\mathcal{G}_M(g), H] = 0$ (this is just saying that a change in the sections plus a change in the transition functions is a trivial automorphism of the bundle). In the $\mathbb{Z}_N$ case, which we will be specializing to in the following for simplicity of notation, we have $T_{\partial M}(g) = e^{i\int_{\partial M}\star E}$, where E is the momentum of the gauge field (written as a 1-form).

To gauge the symmetry, i.e. to make the symmetry local, we need to work in a space where all of the $\mathcal{G}_M(g)$ with $M \subsetneq \Sigma$ act as the identity. We do this by cutting up Σ into a bunch of pieces (the cells in its cell decomposition) and sowing them back together with added g twists. That is, to gauge the symmetry we act with all possible split charge operators, by applying the operator

$$\Pi = \prod_{\text{d-cells } M}\sum_g \mathcal{G}_M(g) = \prod_{\text{d-cells } M}\Pi_{\mathbf{1}}(M) \tag{1420}$$

on some reference vacuum state $|0\rangle$, where $\Pi_{\mathbf{1}}(M)$ is the projector which projects onto states on which $\mathcal{G}_g(M)$ acts as $\mathbf{1}$. A particular term in the resulting sum for Π looks like a "g soap bubble foam operator". Note that any operator which is gauge variant has zero expectation value in the state $\Pi|0\rangle$, because of the phase interference accumulated during the sum over all possible soap bubble films.

The holonomy of the gauge field is determined by the net group element accumulated as one passes through a closed loop in the foam. Now products of $\mathcal{G}_M(g)$ cannot change the holonomy of the gauge field since all the M's are homologically trivial (they are patches in a good cover of Σ).[150] In order to change the holonomies, we act with the operators $T_N(g)$, where $N \in H_{d-1}(\Sigma, \mathbb{Z})$.

Now, do we want to sum over all holonomies, or not? If by "gauge G" we mean "make G local", i.e. "add a gauge field so that charge operators for G acting on submanifolds of space act as $\mathbf{1}$", then there is no a priori reason why the holonomies of the gauge field need to be summed over, since the holonomies are not shifted by any of the operators that we are projecting to $\mathbf{1}$. However, the natural thing to do is to sum over them: in the case of e.g. a $U(1)$ gauge field the holonomies are automatically summed over anyway since they are all continuously connected to one another, and

[150] Inserting these operators at a given time slice also doesn't change the holonomy of any temporal Wilson lines whose support transversely intersects M, since the time component of the gauge field has no canonical momentum.

for the discrete case the only gauge-invariant data in the field is its holonomy, and so if we don't sum over holonomies then after choosing a gauge-fixing condition we see that we have in fact done nothing at all.

Now we have been working in the Hamiltonian picture, and seen that we need to sum over all spatial holonomies of the gauge fields. When we switch to a path integral picture, where do the temporal holonomies of the gauge field come in? They come from a projection onto the trivial representation of the global symmetry. We have been assuming that the operators $\mathcal{G}_M(g)$ for $M \subsetneq \Sigma$ act as $\mathbf{1}$ on the physical Hilbert space but also that $\partial\Sigma = 0$, and so we need $\mathcal{G}_\Sigma(g)$ to act as $\mathbf{1}$. Inserting the appropriate projector turns out to implement a sum over temporal gauge field holonomies, as explained in the next subsection. Note however that when $\partial\Sigma \neq 0$ we again need to deal with boundary condition issues, and whether or not we insert this projector and sum over temporal holonomies is a question of what boundary conditions we choose.

Why is orbifolding the same as gauging?

We will now see why orbifolds and gauging are essentially the same thing. For notational simplicity, we will focus on the concrete example of orbifolding a $\mathbb{Z}_N$ subgroup of a $U(1)$ symmetry. We will also work in two spacetime dimensions, and will take the closed spatial manifold on which we quantize to be Σ (which in general is a disjoint union of S^1s). The length of the temporal circle will be T. All of this is really only for notational reasons; the physics ideas here are general.

Anyway, to do an orbifold we insert symmetry twist operators along the cycles of spacetime, and sum over all such possible insertions. Now when we choose a spatial slice and quantize, the symmetry twist operators along the spatial cycle are clear: they are the charge operators for the symmetry, since moving charge operators through them (by time-evolving) results in the appropriate $\mathbb{Z}_N$ action on the operator according to its charge, by virtue of the commutation relations. Therefore along the spatial cycle(s), we will want to insert $e^{in\frac{2\pi}{N}\int_\Sigma \star j}$ operators, for $n \in \mathbb{Z}_N$. We've written the $\star j$ here since again we are orbifolding a subgroup of $U(1)$, but in general this exponential would just be replaced by the operator $U_g(C)$ implementing the group action. The appropriate operators to insert along other cycles are then determined by modular invariance / freedom to choose the slice on which we quantize to be the same expression with $\star j$ integrated over the appropriate cycle. Then the orbifolded partition function is

$$Z_O[X] = \left\langle \sum_{C\in H_1(X;\mathbb{Z}_N)} e^{i\frac{2\pi}{N}\int \star j\wedge\widehat{C}} \right\rangle \tag{1421}$$

Now from this way of writing it, we can basically see why this is gauging: each operator insertion is the appropriate $\int j \wedge \star A$ term in the gauged action (where $A = \widehat{C}$), and the gauge field is integrated over by the sum over cohomology classes of $\widehat{C}$.[151]

Note that since we are only summing over a single representative element of each cohomology class, the above partition function is written in a gauge-fixed form. Indeed,

[151]This holds true for the discrete case, but in case you don't like the notation with $\int$s and $\wedge$s, you can write it in terms of $U_g(C)$ and realize that the $U_g(C)$s are just the transition functions for a $\mathbb{Z}_N$ bundle where the patch overlaps are along the curves C — this collection of transition functions is then the same thing as a $\mathbb{Z}_N$ gauge field.

form of the partition function changes if we do a gauge transformation by inserting e.g. $e^{i\frac{2\pi}{N}\int_I \star j}$ where $\partial I \neq 0$. Therefore there is no local symmetry in the above partition function, in that split charge operators don't act as the identity. To get a gauge-invariant formulation, we would want to actually quantize the gauge field so that we could introduce a canonical momentum for it, which would allow us to construct the gauge transformation generators from the split charge operators. Then a gauge-invariant partition function would be the same as the above, just with the sum over C running over all possible closed 1-dimensional submanifolds of spacetime.[152]

Now that we know why the above $Z_O[X]$ is the gauged partition function in a certain gauge-fixing choice, we will go into a little pedantic digression about why exactly we referred to it above as representing a partition function on a manifold with symmetry-twisted boundary conditions.

<digression>

Let's look at what the various operator insertions in $Z_O[X]$ do, from a Hamiltonian perspective. It's pretty clear what the spacelike $C \subset \Sigma$ insertions do: these operators appear in the form $\sum_{n\in\mathbb{Z}_N} e^{inQ}$, which obviously just implements the projector onto the trivial representation of the symmetry (since we have assumed $\partial\Sigma = 0$, no complications with boundary conditions here!). As we said, this can be thought of as twisting the boundary conditions in time, since time ordering means that we can relate $\mathcal{O}(t)$ with $\mathcal{O}(t+T)$ by commuting it through the charge operator, picking up a $\mathbb{Z}_N$ action on $\mathcal{O}$.

What do the other (timelike) C insertions do? From what we know about orbifolding, these are supposed to twist the boundary conditions on the fields in some way. But how do they do this? They are built from the spatial component of the current, which commutes with the fields, and so they are just numbers — they can't act on fields as nontrivial operators to implement any kind of twisting action.

The key here is to realize that these insertions modify the momentum operator (the real physical momentum operator, i.e. the 01 component of the stress tensor). Indeed, we see that they can be incorporated into the action by adding the term

$$\delta S = \int \frac{2\pi}{N} \star j \wedge \widehat{C}_t, \tag{1422}$$

where in this case $\widehat{C}$ is the 1-form Poincare dual to a 1-cycle homologous to the temporal circle. Varying this with respect to the metric (which appears in δS only

[152] For posterity's sake, something I was getting confused about: how does inserting the gauge transformation operators, given that they only act within a single spatial slice, implement a transformation which changes A by something which is exact on X, not just on Σ (of course restrictions to Σ of forms which are exact on X are exact while the converse is not true). Basically, the insertion of the gauge transformation operator $\mathcal{G}_I = e^{i\frac{2\pi}{N}\int_I(\star j - d\pi_A)}$ for $I \subset \Sigma_t$ (Σ_t a given time slice) projects the sum over A onto configurations where $A|_{\Sigma_t} = d\widehat{I}$. This happens because $\pi_A(t)$ appears in the path integral in the form $e^{i\pi_A(t)(d\widehat{I}-A(t))}$, which when integrated out sets $A(t) = d\widehat{I}$. Therefore when $\mathcal{G}_I$ is inserted into the path integral, in the case we're currently focusing on where the gauge field is flat, the sum over A gets projected onto configurations whose Poincare duals pass through the points $\partial I \subset \Sigma_t$. For the above orbifolded partition function the insertion of $\mathcal{G}_I$s sets the partition function to zero; hence it is not gauge invariant (but rather gauge-fixed).

implicitly via the Hodge star $\star$—explicitly, $\star j \wedge \widehat{C}_t = g^{\mu\nu} j_\mu (\widehat{C}_t)_\nu \, dt \wedge dx$), we see that this term modifies the stress tensor by the term

$$T_{\mu\nu} \supset \frac{2\pi}{N} j_{(\mu} (\widehat{C}_t)_{\nu)}. \tag{1423}$$

Now for a choice of C_t which is constant in time, we can write $\widetilde{C}_t = B(x)dx$ for some function B. Therefore the physical momentum is modified by a term[153]

$$\delta P(t) = \frac{2\pi}{N} \int_\Sigma dx \, j_0(t) B(x). \tag{1424}$$

Now if C_t is a thin line wrapping the temporal cycle then $B(x)$ is a bump function, and we see that the momentum receives a contribution from the charge density at that location. On the other hand, if we take C_t to be completely smeared out[154] so that $B(x)$ is constant, the contribution to the momentum is

$$\delta P = \frac{2\pi}{N} Q \, \mathrm{hol}(B) \in \frac{2\pi}{N} \mathbb{Z}, \tag{1425}$$

where Q is the global charge operator. This is why we say that the operator insertions in the partition function for timelike C twist the boundary conditions. While they don't actually do this, since our fields are always single-valued on spacetime, we do see that these operator insertions modify the momentum operator to give it the value it would have if the fields were appropriately multivalued, with twisted boundary conditions.

`</digression>`

Before moving on, we note another nice way of writing down the above orbifolded (aka gauge-fixed) partition function. We do this in a Hamiltonian perspective by writing

$$Z_O[X] = \left\langle \sum_{C \in H_1(X;\mathbb{Z}_N)} e^{i \frac{2\pi}{N} \int_C \star \pi_A} \right\rangle = \sum_{C \in H_1(X;\mathbb{Z}_N)} \langle T_C \rangle \,, \tag{1426}$$

where the T_Cs are the 't Hooft lines (see caveat below for what this notation actually means when C is spacelike). Here the expectation value is taken as $\langle \cdot \rangle = \sum_\phi \langle \phi, A = 0 | T(\cdot) e^{-iHT} | \phi, A = 0 \rangle$, where ϕ represents the matter fields. That is, we fix a definite configuration for the gauge fields, but sum over all the matter fields. Additionally, the Hamiltonian implicit in the expectation value contains the $\int_\Sigma \star j \wedge A$ term. The needed sum over A is implemented by the 't Hooft operators. For C timelike this is clear — then the insertion of T_C just modifies the Hamiltonian by $e^{i(2\pi n/N)\pi_{A_x}(y)}$ for the point $y = C \cap \Sigma$; when acting on the $|A = 0\rangle$ this translates the state to something which has

[153] Note that this change involves j_0, even though δS in this case involves only j_x! This is because we obtained P by considering the effect of adding in an infinitesimal off-diagonal component of the metric, which makes j_0 show up.

[154] So that technically $C \in H_1(X; \mathbb{R})_\mathbb{Z}$, where the subscript means that the elements have $\mathbb{Z}$ periods. The distinction between the continuum and the lattice here is really just a matter of notation, and I won't be too pedantic about talking explicitly about both cases.

a gauge field with $2\pi n/N$ holonomy around Σ, and hence summing over cohomology classes of timelike C reproduces the sum over all possible spatial holonomies for A.

The spacelike T_C insertions are slightly different — indeed, since A_0 has vanishing canonical momentum, the notation doesn't even make sense! Instead, when C is spacelike, we define $T_C = e^{i\frac{2\pi}{N}\int_C \star j}$ to be the global charge operator. Why does this operator create holonomy for A_0? Recall that A_0 appears in the path integral via $e^{i\int_\Sigma A_0(\star j - d\pi_A)}$ as a way of enforcing the gauge constraint at each time step — the momentum π_A is then integrated out, giving an action in terms of A_0. Therefore with T_C inserted at time t_0, and for C the n-fold multiple of Σ, the partition function will contain the term $e^{i\int_\Sigma(\star j - d\pi_A)(A_0 + 2\pi n/N)}$, since $\int d\pi_A 2\pi n = 0$. Therefore the insertion of T_C is exactly equivalent to shifting A by $\widehat{C} = (2\pi n/N)\delta(t-t_0)dt$, which indeed changes the holonomy of A around the temporal cycle by $2\pi n/N$. Therefore the insertions of the (properly defined) T_Cs as above indeed is equivalent to the orbifolded partition function.

Orbifolding the orbifold; gauging the "dual" symmetry

Now we will do another orbifold. We will see that this is equivalent to ungauging. Indeed, suppose we want to undo the gauging procedure. Then in the gauge-fixed presentation where we only sum over cohomology classes, the gauge field A can be frozen out if we project the holonomy of A around every cycle to be trivial, i.e. if we project onto states with no "magnetic flux" threading the cycles of X (killing the holonomy of A is tantamount to setting $A = 0$, since the gauge group is discrete). To this end, let

$$W_C \equiv e^{\frac{2\pi i}{N}\int_C A} \tag{1427}$$

be the Wilson loop around some cycle C. Then we can project onto the trivial gauge field by inserting a sum of W_C's raised to all possible powers in the parition function to act as the appropriate projector:

$$Z_{O^2}[X] = Z[X] = \langle \sum_{C\in H_1(X;\mathbb{Z}_N)} W_C\rangle_A = \sum_{A\in H^1(X;\mathbb{Z}_N)}\sum_{C\in H_1(X)} W_C Z[X;A]. \tag{1428}$$

Again note that we haven't had to write a sum over $n \in \mathbb{Z}_N$ of powers W_C^n, since this is automatically included in the sum over homology classes.

Why is this equivalent to orbifolding the orbifold? Recall that to do the first orbifold, we performed a weighted sum of partition functions, where each term was weighted by a phase determined by the charge along each cycle C of X, as measured by $\int \star j \wedge \widehat{C}$. We saw that this was equivalent to gauging the theory.

The claim is that inserting the Wilson lines as above does exactly the same thing, i.e. that the Wilson lines are the charge operators for a certain dual symmetry (viz. $W_C = \exp(\frac{2\pi}{N} i \int \star j^\vee \wedge \widehat{C})$), and that when this symmetry is gauged, we reproduce the original partition function. The duality here is nicely seen by looking at (1426), which has the exact same form as (1428).

We see from the formulae for the orbifold that it is basically doing a Fourier transform, while the second orbifold is basically doing the inverse Fourier transform. Thus the symmetry and its dual referred to above are essentially Fourier transforms of one another. To connect this with the notation that one usually sees in CFT books, recall that in the CFT example of a free boson with a $\mathbb{Z}_N$ orbifold on $X = T^2$, we have

$$Z \xrightarrow{orb} \frac{1}{N} \sum_{g,h\in\mathbb{Z}_N} Z_{g,h} \xrightarrow{orb} \frac{1}{N^2} \sum_{f,f'\in\text{Rep}(\mathbb{Z}_N)} \sum_{g,h\in\mathbb{Z}_N} f(g)f'(h)Z_{g,h} = Z, \tag{1429}$$

where $Z_{g,h}$ is the partition function on the torus with boundary conditions twisted or untwisted according to g, h. The above formulae are just a more general way of writing this.

The way to think about this dual symmetry, the fact that it always exists, the fact that its charge operators are the Wilson lines, and the reason why gauging it gets one back to the original theory are all explained in a separate diary entry on gauging higher symmetries, so we won't go into any further depth here.

Cooler proof of Goldstone's theorem for p-form symmetries

Today we looking at a proof of the generalized Goldstone's theorem which was presented in [?] and is complimentary to the one in my paper. Today we are just going to look through their arguments and fill in some details.

The proof of Goldstone's theorem will start from the Ward identity, and basically just uses dimensional analysis. In what follows, we will ignore factors of i and q (charge), as well as signs (these are all irrelevant for our purposes).

Let $\mathcal{O}_C$ be an operator charged under a p-form symmetry, with $\text{Supp}(\mathcal{O}_C) = C$ a p-dimensional submanifold of spacetime X (or an appropriately smeared bump-function version of a p-dimensional submanifold if we are being pedantic). We will assume that SSB occurs. If $p \geq 1$, the vacuum expectation value of the un-renormalized operator $\mathcal{O}$ is then allowed to vanish up to as fast as a "perimeter law", meaning that it may vanish as fast as $e^{-g^2 L^p/a}$, where g is a coupling constant and a is a UV cutoff. If this is the case, we will always renormalize $\mathcal{O}$ by subtracting off the UV divergence; this can be done with a simple multiplicative renormalization and ensures that the renormalized $\mathcal{O}$ has a finite, cutoff-independent vev. We will assume that such renormalization has been done in what follows.

Under an infinitesimal transformation, we let $\mathcal{O}$ transform as

$$\mathcal{O} \mapsto \mathcal{O} + \mathcal{O} \int_C \lambda, \tag{1430}$$

for some p-form λ. Since we have a p-form symmetry when $d\lambda = 0$, the action must vary as

$$e^{-S} \mapsto e^{-S}\left(1 + \int \star J \wedge d\lambda\right), \tag{1431}$$

where J is a $(p+1)$-form current which is classically conserved, $d^\dagger J = 0$. This conservation law ensures that the charge operator $Q(M_{D-p-1}) \sim \int_{M_{D-p-1}} \star J$ is topological. So then after integrating by parts (we assume that λ is compactly supported) the Ward identity reads

$$\langle \mathcal{O}_C \int_X \lambda \wedge \widehat{C}\rangle = \langle \mathcal{O}_C \int_X d \star J \wedge \lambda\rangle. \tag{1432}$$

We then conclude that

$$\langle \mathcal{O}_C\rangle \widehat{C}(x) = \langle \mathcal{O}_C (d\star J)(x)\rangle. \tag{1433}$$

Note that because we have assumed SSB, we can choose an $\mathcal{O}_C$ which is both charged under the symmetry and is such that the LHS is nonzero.

Now we pick an open $D-p$ manifold M_{D-p} which intersects C transversely at a point, and then integrate the ward identity over this manifold. We get

$$\langle \mathcal{O}_C\rangle \int_X \widehat{M}_{D-p} \wedge \widehat{C} = \int_{\partial M_{D-p}} \langle \mathcal{O}_C \star J\rangle. \tag{1434}$$

Now by our choice of M_{D-p}, this simplifies to

$$\langle \mathcal{O}_C\rangle = \int_{\partial M_{D-p}} \langle \mathcal{O}_C \star J\rangle. \tag{1435}$$

Note that the LHS is *independent* of the choice of M_{D-p}! In fact, it is just a constant. Thus we can make ∂M_{D-p} have support arbitrarily far away from the support of $\mathcal{O}_C$, and the RHS must remain a constant. This implies that we have the correlator

$$\langle \mathcal{O}_C \star J(r)\rangle \sim \frac{1}{r^{D-p-1}}, \tag{1436}$$

where r is some typical distance away from C. For example, for $p=0$ C is just a point, and we can take M_D to be a D-ball of radius r centered on C. For e.g. $p=1$ and $D=3$, we might take C to be the z axis and M_2 to be a solid disk in the xy plane centered at the origin and with radius r.

Anyway, the point is that this power law correlation function implies that we must have massless particles in the spectrum: if we had no massless particles, such a long-ranged correlation function would not be possible. Now we usually expect that for SSB the current will be realized as $J = dA$ for some p-form A[155]. The action for the Goldstones is then the usual

$$S = \frac{1}{2g^2}\int dA \wedge \star dA, \tag{1437}$$

[155]What about in electromagnetism with electric charges; $d^\dagger F = \rho$, $dF = 0$? We expect that in the Coulomb phase the $U(1)_m$ is still spontaneously broken, otherwise given the explicit breaking of $U(1)_e$, the massless photon would not generically be around at low energies. On the other hand, the $U(1)_m$ current is $\star F$, and if this were realized as $\star F = d\widetilde{A}$, then we would have $d^\dagger F = \star d^2\widetilde{A} = 0$, a contradiction. Alternatively, we could write $\star F = d\widetilde{A}$, but $\widetilde{A}$ would be a singular field, for which $d^2\widetilde{A} \neq 0$. This is just the electromagnetic dual of the statement that the vector potential is singular when magnetic monopoles are around. This isn't just a global issue since we are treating the matter as

where g^{-2} is the "superfluid stiffness".

Is this compatible with the Ward identity when SSB is assumed? Let us test it for the case where $\mathcal{O}_C = \exp\left(i \int_C A\right)$. The AA correlator, going as $1/k^2$, goes as $\langle A(r)A(0)\rangle \sim \int d^D k k^{-2} e^{ikr} \sim r^{2-D}$, so that

$$\langle e^{i\int_C A} \star J(r)\rangle \sim \partial_r \langle \int_C A(0)A(r)\rangle \sim \partial_r r^{2-D+p} \sim \frac{1}{r^{D-p-1}}, \tag{1438}$$

which is indeed what the Ward identity requires.

Anomalies and current OPEs

Today we try to get a better understanding of why central-extension-y terms which appear in current OPEs encode information about anomalies (in two dimensions throughout so that we can use CFT language).

Let us consider a theory with holomorphic and antiholomorphic currents, with OPEs

$$J(z)J(w) \sim \frac{k}{(z-w)^2} + \dots, \qquad \bar{J}(\bar{z})\bar{J}(\bar{w}) \sim \frac{\bar{k}}{(\bar{z}-\bar{w})^2} + \dots, \qquad J\bar{J} \sim 0, \tag{1439}$$

where the $\dots$ could include further terms linear in the current, like e.g. for WZW models / current algebras and stuff (the $\dots$ does *not* stand for nonsingular stuff). We assume the classical eom for the currents are $\bar{\partial} J = \partial \bar{J} = 0$.[156] By coupling the theory to background gauge fields, we will find information about the various anomalies in terms of the parameters $k, \bar{k}$ appearing in the OPEs. Finally we will take a slightly more general approach and deduce a more general form of the current-current OPEs. By keeping track of a non-singular contact term for the $J\bar{J}$ OPE, we will demonstrate the mixed anomaly between the current and the appropriately-defined axial current.

A good reference for background material to get inspired about this kind of stuff is Zohar Komargodski's notes on RG flows. This is standard stuff, but I just hadn't seen it written up anywhere.

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dynamical (i.e. we are not defining the electric charges by excising little bits that change the topology of spacetime). Thus, we have found a counterexample to the claim that the current is realized as $J = dA$ whenever SSB occurs. Note that in the generalized global symmetries + holography paper, the authors say that $\star F \neq d\widetilde{A}$ means that $U(1)_m$ is unbroken, but again this can't be the case since we know the 't Hooft operators have a perimeter law in the Coulomb phase.

[156] In situations where we have a non-chiral current J with $d^\dagger J = \bar{\partial} J + \partial \bar{J} = 0$ we can form linear combinations from J and $\star J$ to create such (anti-)holomorphic currents, so working with these eoms is wolog in this setting.

First let us remember what contact terms in the current-current correlators mean. We will be fast and schematic. Consider a two-point function with a contact term

$$\langle J(x)J(y)\rangle = f(x-y) + g\delta(x-y) + \dots . \tag{1440}$$

Since we get this by taking $\delta_{A(x)}\delta_{A(y)}F[A]$ for a background field which couples to J, the contact term must come from a counterterm like $\int A^2$ which has been added to the action. Similarly we could have a counterterm like $\int A\partial A$ which would give us a contact term like $\partial\delta(x-y)$, and so on; higher derivative counterterms give more singular contact terms, and counterterms higher order in A give contact terms to higher point functions of J. Often these contact terms are non-universal and can be modified at will without affecting the physics. Sometimes this is not the case though, e.g. when the contact terms are determined by an OPE in a CFT (since because of the limiting process involved in computing the OPE these terms are actually determined by correlation functions at *separated* points), or when they are required by gauge invariance (like the $A_\mu A^\mu$ contact term required by gauge invariance in scalar QED). Since the contact terms tell us about counterterms involving gauge fields, if we know the contact term structure we can learn about whether or not the theory has a gauge-invariant action when background gauge fields are added.

Actually before going into detail and discussing what happens when gauge fields are added, we can first argue from a Hamiltonian perspective why k and $\bar{k}$ determine the anomaly structure. Indeed, consider the chiral currents $J, \bar{J}$, which classically are conserved. The $k, \bar{k}$ terms in the OPE mean that the charge generators $Q = \oint \Sigma dz\, J$, $\bar{Q} = \oint_\Sigma d\bar{z}\, \bar{J}$ (here Σ is the spatial manifold — we will take it to be a unit S^1 for concreteness) do not commute with themselves (seen by doing the usual evaluation of the OPE inside the commutator), with $[Q,Q] = ik, [\bar{Q},\bar{Q}] = -i\bar{k}$ (with the commutator taken as usual in the radial quantization way). Now consider gauging e.g. the holomorphic symmetry by adding a gauge field A. The operators generating gauge transformations are then generated by the operators

$$\mathcal{G}(I) = \int_I (\partial_x \pi_A - \star J), \tag{1441}$$

with π_A the electric field operator and $I \subset \Sigma$ some interval in space. These operators don't commute with themselves, but this by itself isn't a problem — we are still allowed to work in a subspace where they act as $\mathbf{1}$ provided that either their commutator is also a gauge transformation (i.e. $[\mathcal{G}(I), \mathcal{G}(I')] \propto \mathcal{G}(I'')$, a structure that comes from the degree-1 simple poles in the current-current OPEs — this is what happens for non-Abelian symmetries) or unless their commutator is a c-number that can be done away with by a multiplicative renormalization. To make the calculation simple, examine the commutator $[e^{i\alpha\mathcal{G}(I)}, e^{i\beta Q}]$ for infinitesimal α. This produces (use the same trick for computing the commutator, viz. surrounding I by $\Sigma \cup \bar{\Sigma}$)

$$[e^{i\alpha\mathcal{G}(I)}, e^{i\beta \oint_\Sigma dz\, J}] \sim i\alpha\beta k|I|. \tag{1442}$$

Since the RHS isn't an $\mathcal{G}(I)$ when $k \neq 0$, we cannot work in a subspace where all the $\mathcal{G}(I)$ act trivially, and hence neither of the chiral symmetries can be gauged.[157] Note however that if $k = \bar{k}$, then the vector symmetry with charge $Q + \bar{Q}$ can be gauged, while if $k = -\bar{k}$ the axial symmetry with charge $Q - \bar{Q}$ can be gauged. If $\bar{k}$ isn't k or $-k$ though, nothing can be gauged.

Now we take a look at this from the perspective of explicitly adding in a background field. For concreteness we will be adding a gauge field for the vector current (and so we know from above that we will end up needing $k = \bar{k}$). To couple to the $U(1)_V$ gauge fields in a gauge-invariant way, we need to couple $\bar{A}$ with J and A with $\bar{J}$ (the gauge field A is not necessarily holomorphic, the notation just means that it is the z-component of the 1-form $A_\mu dx^\mu$). Now we expand the partition function in the background fields to quadratic order:

$$Z[A, \bar{A}] \approx \left\langle 1 - \int (\bar{J}A + J\bar{A}) + \frac{1}{2}\int_{z,w} (\bar{J}A + J\bar{A})(z) \cdot (\bar{J}A + J\bar{A})(w) \right\rangle . \tag{1443}$$

Let us now look at the gauge (in)variance of this expression. The linear term is gauge invariant since the equation of motion (viz. current conservation) holds on J with no other operators inserted. Any anomalous variation will thus come from the quadratic term. Using the OPEs (allowed since there are no other operator insertions; the $A, \bar{A}$s of course don't count since they are not dynamical), the variation of the partition function is (higher order terms in the OPE will vanish for $\oint dz\, z^n = 2\pi i \delta_{n,-1}$ reasons)

$$\delta Z[A, \bar{A}] = \delta \frac{1}{2}\int_{z,w} \left(\frac{k}{(z-w)^2} \bar{A}(z,\bar{z})\bar{A}(w,\bar{w}) + \frac{\bar{k}}{(\bar{z}-\bar{w})^2} A(z,\bar{z})A(w,\bar{w}) \right) . \tag{1444}$$

Let's just look at the first term. We have, for $A \mapsto A + d\gamma$, (and not being too careful about factors of π and 2, and using the same shitty notation where A and its z component are the same so that $A \mapsto A + d\gamma$ is in complex notation $A \mapsto A + \partial\gamma$, $\bar{A} \mapsto \bar{A} + \bar{\partial}\gamma$)

$$\begin{aligned} \delta \frac{1}{2}\int_{z,w} \partial_w \frac{k}{z-w} \bar{A}(w,\bar{w})\bar{A}(z,\bar{z}) &= \int \partial_w \frac{k}{z-w} \bar{A}(w,\bar{w}) \partial_{\bar{z}}\gamma(z,\bar{z}) \\ &= \int_{z,w} \partial_{\bar{z}} \frac{k}{z-w} \partial_w \bar{A}(w,\bar{w})\gamma(z,\bar{z}) \\ &= \int_{z,w} \delta^2(z-w, \bar{z}-\bar{w}) k \partial_w \bar{A}(w,\bar{w})\gamma(z,\bar{z}) \\ &= k\int_z \gamma \partial \bar{A}. \end{aligned} \tag{1445}$$

The second term is essentially identical and hence the variation of the partition function is

$$\delta Z[A, \bar{A}] = \int \gamma \left(k\partial\bar{A} + \bar{k}\bar{\partial}A \right) . \tag{1446}$$

[157] Note to self: can we show that the divergence of the current is zero if the curvature of the background field vanishes? There should be a Hamiltonian-centric way of seeing this based on operator commutators.

Now in the special case that $k = \bar{k}$, this can be canceled by adding the local counterterm

$$S_{ct} = k \int_z A\bar{A}, \tag{1447}$$

or since the variation in this case is the integral of $k\gamma F$, it can also be canceled by the variation of a Chern-Simons theory in three dimensions (note that I have not been keeping track of factors of 2π and stuff so the CS term won't have the proper normalization if we use the literal expression above—the needed numerical factors will enter e.g. from the step where we replaced $\partial_{\bar{z}}\frac{1}{z-w}$ with the delta function). Anyway, note that if $k \neq \bar{k}$, then no such local counterterm will do, and no anomaly cancellation is possible. Note that we would also derive an un-cancellable anomaly if one of the couplings $J\bar{A}$ or $\bar{J}A$ wasn't present in the gauged partition function. This further illustrates the general phenomenon of anomalies being tied to chirality: if the currents are intrinsically chiral, or if the gauging is done in a chiral way, there is an anomaly. This fits with our experience of gauge anomalies in hep-ph scenarios coming from fermions which are chirally coupled to gauge fields.

A rather simple example of a scenario in which the $\int A\bar{A}$ term needs to be employed is in scalar electrodynamics. For example, consider a compact scalar, with just a free action. The current associated with the shift symmetry is $J = \partial\phi$, $\bar{J} = \bar{\partial}\phi$. Suppose we want to gauge this symmetry via minimal coupling of a gauge field $(A, \bar{A})$ to the current. The OPEs are of course $JJ \supset 1/(z-w)^2$, $\bar{J}\bar{J} \supset 1/(\bar{z}-\bar{w})^2$, so that $k = \bar{k}$ and the anomaly can be canceled by a term $A\bar{A} = A_\mu A^\mu$. Of course such a term is present, since we know that the full gauged action has the kinetic term $(d\phi - A)^2$, which contains the A^2 term.

Now we back up a little bit and consider the current OPEs from a bit more general point of view. In momentum space, we have[158]

$$JJ = \frac{q_+^2}{q^2}k, \qquad J\bar{J} = -K, \qquad \bar{J}\bar{J} = \frac{q_-^2}{q^2}\bar{k}, \tag{1448}$$

where K is some arbitrary constant that we can choose by hand (as part of how we define the regularization procedure — it comes from $\delta_{A(z)}\delta_{\bar{A}(z)}$, which picks up the $\int A\bar{A}$ counterterm. As we said, these counterterms only affect correlation functions at coincident points, hence why they are part of our regularization procedure). Since the K term is momentum-independent, it only contributes a δ function contact term in real space and so doesn't affect any correlation functions at separated points—this is why we needen't worry about its arbitraryness. Here q_+ Fourier transforms to $\partial = (\partial_0 - i\partial_1)/2$ and q_- goes to $\bar{\partial} = (\partial_0 + i\partial_1)/2$. The $\pm$ sign indicates their chirality, so that ∂ kills right-movers (negative chirality) and $\bar{\partial}$ kills left-movers (positive chirality). In real space this works since schematically we have e.g. $q_+^2/q^2 = q_+/q_- \to \partial\frac{1}{\bar{z}} = \frac{1}{z^2}$.

Anyway, now let's look at current conservation. In our notation $\partial_\mu J^\mu$ is written as $\partial\bar{J} + \bar{\partial}J$. Using the OPEs, we find

$$\langle(\partial\bar{J} + \bar{\partial}J)\bar{J}\rangle = q_-(\bar{k} - K), \qquad \langle(\partial\bar{J} + \bar{\partial}J)J\rangle = q_+(k - K). \tag{1449}$$

[158]Here we are focusing just on the anomalous parts. More generally we can have terms like $JJ = \frac{q_+^2}{q^2}f(q^2)$ for $f(q^2)$ some dimensionless function. This won't appear in the examples we're interested in, and in any case we can absorb $f(\infty)$ into the k's so that in the UV the form for the OPEs below suffices.

Thus we can preserve current conservation for the $U(1)_V$ current only if $k = \bar{k}$ (as we saw before, we get a gauge anomaly if $k \neq \bar{k}$, which is compatible with this condition on current conservation), and provided that we also choose $K = k$.

Now let us define an axial current, whose holomorphic and anti-holomorphic components we will write as $\mathcal{J}, \bar{\mathcal{J}}$. The currents are related as[159]

$$\mathcal{J} = J, \qquad \bar{\mathcal{J}} = -\bar{J}. \tag{1450}$$

Thus current conservation for the axial current is

$$d^\dagger \mathcal{J} = \partial \bar{\mathcal{J}} + \bar{\partial} \mathcal{J} = -\partial \bar{J} + \bar{\partial} J = -i dJ, \tag{1451}$$

which is basically the usual Hodge duality formula for relating the currents of symmetries with mixed anomalies that arise in many free field theory contexts. Thus if both currents are conserved, then the regular (vector) current is both closed and co-closed. This in turn means that if both currents are conserved, we have

$$\partial \bar{J} = \partial \bar{\mathcal{J}} = 0, \qquad \bar{\partial} J = \bar{\partial} \mathcal{J} = 0, \tag{1452}$$

so that if both currents are conserved the un-barred currents really are holomorphic and the barred currents really are antiholomorphic.

Using the OPE, we can check that

$$\langle d^\dagger \mathcal{J} J \rangle = q_+(k + K), \qquad \langle d^\dagger \mathcal{J} \bar{J} \rangle = -q_-(\bar{k} + K). \tag{1453}$$

Thus the axial current is only conserved if we have both $k = \bar{k}$ (as usual), and if we choose $k = -K$. Now we see the mixed anomaly between J and $\mathcal{J}$ — for $k \neq 0$ it's impossible to choose our regularization conventions (alias K) in such a way that both $d^\dagger J = 0$ and $d^\dagger \mathcal{J} = 0$. Now the anomaly means that if we gauge the symmetry generated by J (as we usually do), $\mathcal{J}$ conservation will get broken, which we confirm by setting $K = -k = -\bar{k}$ and computing

$$\langle d^\dagger \mathcal{J} (\bar{J} A + J \bar{A}) \rangle = k(q_+ \bar{A} - q_- A) = k \star F. \tag{1454}$$

This tells us that $\langle d^\dagger \mathcal{J} \rangle = k \star F$, at least to lowest order. This comes from putting $d^\dagger \mathcal{J}$ in the path integral and expanding the $e^{-\int J_\mu A^\mu}$ term to first order, to create the usual

[159]One sees that the word axial is appropriate by e.g. by looking at the case of Dirac fermions: there we have $J = j_0 - ij_1 = 2L^\dagger L, \bar{J} = j_0 + ij_1 = 2R^\dagger R$, for $j^\mu = \bar{\Psi} \not{D}_A \Psi$ and $\Psi = (L, R)^T$. The chiral current is $\mathcal{J}_0 = L^\dagger L - R^\dagger R, \mathcal{J}_1 = i(L^\dagger L + R^\dagger R)$ for gamma matrices equal to the Pauli matrices, and so indeed $\mathcal{J} = 2L^\dagger L, \bar{\mathcal{J}} = -2R^\dagger R$.

bubble diagram with one photon leg. We see this in a slightly different way by writing

$$\begin{aligned}
\partial_\mu \langle \mathcal{J}^\mu(z)\rangle_{A,\bar{A}} &= -\partial \frac{\delta Z[A,\bar{A}]}{\delta A(z)} + \bar{\partial}\frac{\delta Z[A,\bar{A}]}{\delta \bar{A}(z)} \\
&= \left(-\partial \frac{\delta}{\delta A(z)} + \bar{\partial}\frac{\delta}{\delta \bar{A}(z)}\right)\Big\langle 1 - \int_u (J\bar{A} + \bar{J}A)(u) \\
&\quad + \frac{1}{2}\int_{u,v}(J\bar{A}+\bar{J}A)(u)(J\bar{A}+\bar{J}A)(v)\Big\rangle_{0,0} \\
&= \frac{1}{2}\left(-\partial \frac{\delta}{\delta A(z)} + \bar{\partial}\frac{\delta}{\delta \bar{A}(z)}\right)\int_{u,v}\Big(\frac{k}{(u-v)^2}\bar{A}(u)\bar{A}(v) \\
&\quad + \frac{\bar{k}}{(\bar{u}-\bar{v})^2}A(u)A(v) - \delta(u-v)K(\bar{A}(u)A(v) + A(u)\bar{A}(v))\Big) \\
&= \partial_z \int_u \left(\bar{\partial}_u \frac{1}{\bar{u}-\bar{v}}\bar{k}A(u) + K\delta(z-u)\bar{A}(u)\right) \\
&\quad - \bar{\partial}_z \int_u \left(\partial_u \frac{1}{u-z}k\bar{A}(u) + K\delta(z-u)A(u)\right) \\
&= -\bar{k}\bar{\partial}A(z) + K\partial\bar{A}(z) + k\partial\bar{A}(z) - K\bar{\partial}A(z),
\end{aligned} \tag{1455}$$

where we used that $\langle d^\dagger \mathcal{J}(z)\rangle_{0,0} = 0$. If we were to do this for $d^\dagger J$ instead of $d^\dagger\mathcal{J}$ we would have gotten

$$\langle d^\dagger J(z)\rangle_{A,\bar{A}} = \bar{k}\bar{\partial}A(z) - K\partial\bar{A}(z) + k\partial\bar{A}(z) - K\bar{\partial}A(z), \tag{1456}$$

meaning that as we saw before, we need $K = k = \bar{k}$ for conservation of the vector current. So making this choice, we get (remember we are not being careful with factors of 2 and stuff)

$$\langle d^\dagger \mathcal{J}(z)\rangle_{A,\bar{A}} = k(\partial\bar{A}(z) - \bar{\partial}A(z)) = k \star F(z). \tag{1457}$$

Thus we have completed our jillionth derivation of the chiral anomaly.

We have only been working up to the one-loop level. However, we know that this result is one-loop exact for the standard reason: by gauge invariance the only possibility for $d^\dagger\mathcal{J}$ is $f(e^2)\star F$, where e is the gauge coupling. But $\int d^\dagger\mathcal{J} \in \mathbb{Z}$ and $\frac{1}{2\pi}\int F \in \mathbb{Z}$, and so $f(e^2)$ cannot continuously depend on e. Thus it must be independent of e, barring super pathological counterexamples. Since the answer for $d^\dagger\mathcal{J}$ is thus independent of the gauge coupling, it is one-loop exact (the gauge coupling appears where $\hbar$ appears in the action, and diagrams with l loops go as $\hbar^{-1+l}$ by Euler characteristic reasons).

Subsystem symmetries and CMW-like constraints

Today is a short one. We will be looking at the paper [?] and their proof of a generalized Elitzer's theorem for subsystem symmetries (there were plenty of motivating examples for this even in pre-fracton days, e.g. Bose liquids, sliding Luttinger liquids, etc.). We will explain the logic behind the argument in the above paper that in a system with finite-ranged interactions, SSB of a discrete symmetry is impossible if the charge operators are one-dimensional, and SSB of a continuous symmetry is impossible if they are two-dimensional. Now of course this cannot strictly be true as stated, since we know that e.g. 1-form discrete symmetries can be broken in greater than or equal to two spacetime dimensions (at $T = 0$). In the course of the argument we will see what further qualifications are needed for the statement to hold.

The argument, which is basically a clever use of dimensional reduction plus the normal CMW theorem, goes as follows. Consider a q form symmetry, by which we mean a symmetry whose charge operators act on $D - q - 1$ manifolds. In this diary entry (and in this diary entry only) we will take this to be our definition of a q form symmetry — in particular, we will not stipulate that the charge operators be topological in the spacetime, only in time (since we are interested in generic, viz. non-relativistic, systems).

Consider a theory where the fields are schematically denoted by ϕ. Furthermore let $S[\phi]$ be some local action. Consider a charge operator $Q(M)$, where M is a closed $D - q - 1$ submanifold of spacetime X. Let us break up the fields as $\eta(x) = \phi(x)$ for $x \in M$ and $\bar\eta(x) = \phi(x)$ for $x \in X \setminus M$. Finally let $\mathcal{O}[\eta, \bar\eta]$ be some (not necessarily local) operator charged under $Q(M)$. Cheeckily rewrite the vev of $\mathcal{O}$ as

$$\langle \mathcal{O}[\eta,\bar\eta]\rangle = \frac{1}{Z}\int \mathcal{D}\bar\eta\,\mathcal{D}\eta\,\mathcal{O}e^{-S[\eta,\bar\eta]} = \int \mathcal{D}\bar\eta \left(\frac{\int \mathcal{D}\eta\,\mathcal{O}\,e^{-S[\eta,\bar\eta]}}{\int \mathcal{D}\eta\,e^{-S[\eta,\bar\eta]}}\right)\frac{\int \mathcal{D}\eta\,e^{-S[\eta,\bar\eta]}}{Z}, \tag{1458}$$

As usual, to get a vev for $\mathcal{O}$ we need to either add a symmetry-breaking field or fix boundary conditions appropriately — we will employ the latter approach.

Now we can bound the magnitude of the vev of $\mathcal{O}$ by

$$|\langle \mathcal{O}[\eta,\bar\eta]\rangle| \leq \int \mathcal{D}\bar\eta \left|\frac{\int \mathcal{D}\eta\,\mathcal{O}[\eta,\bar\eta_m]\,e^{-S[\eta,\bar\eta_m]}}{\int \mathcal{D}\eta\,e^{-S[\eta,\bar\eta_m]}}\right| \frac{\int \mathcal{D}\eta\,e^{-S[\eta,\bar\eta]}}{Z}, \tag{1459}$$

where $\bar\eta_m$ is the value of the field configuration on $X \setminus M$ such that the absolute value of the expression in parenthesis in (1458) is maximized (assuming that it's bounded). Then we see that we just have a factor of $Z/Z = 1$ in addition to the term evaluated at $\bar\eta = \bar\eta_m$, so

$$|\langle \mathcal{O}[\eta,\bar\eta]\rangle| \leq |\langle \mathcal{O}[\eta,\bar\eta_m]\rangle_{S[\eta,\bar\eta_m]}| \tag{1460}$$

where on the RHS the expectation value is taken with respect to the η fields while working in the fixed $\bar\eta_m$ background. This means that we can bound the expectation value of $\mathcal{O}$ by the expectation value it takes on in the presence of a fixed field configuration for the fields living on $X \setminus M$. Note that the $\bar\eta$ fields are simply frozen to $\bar\eta_m$ rather than integrated out, and so if the effective action for the η fields is local if the original full D-dimensional action is regardless of what the gap in the spectrum looks like. Likewise, the effective action for η will have short-ranged interactions if the original D-dimensional action does as well.

Now comes the catch: since $Q(M)$ acts as a global symmetry on M, the RHS of the above equation is essentially the expectation value that an operator in a $D - q - 1$ dimensional theory has in the presence of a global 0-form symmetry generated by $Q(M)$. Thus we can apply the regular CMW theorem (assuming short-ranged interactions) for 0-form symmetries to conclude that $|\langle \mathcal{O}[\eta, \bar{\eta}] \rangle| = 0$ if M is $d \leq 2$ dimensional and it generates a continuous symmetry, or if M is $d \leq 1$ dimensional and it generates a discrete symmetry.

Now we know examples of higher symmetries that are spontaneously broken, and yet have charge operators whose dimension comes into conflict with the above result. So what gives? The point is that these counter examples all occur (to my knowledge) when the q-form symmetry in question arises from a gauge theory. This means that the decomposition $\int \mathcal{D}\phi \to \int \mathcal{D}\eta \mathcal{D}\bar{\eta}$ is impossible, since the Hilbert space does not factorize as $\mathcal{H}_M \otimes \mathcal{H}_{X \setminus M}$. This is not just an issue of M not being "smooth" in X: we could thicken it up into a D-dimensional submanifold, or we could try to smoothly interpolate between η and $\bar{\eta}$: nothing we could do would let us do the field decomposition in this way. The dimensional reduction approach that this method uses doesn't work for gauge theories, since their nonlocal-ness means that the degrees of freedom in different directions are all inter-related and can't get separated in the way they would need to be to make this argument work.[160]

Yet another way to derive the chiral anomaly in two dimensions

Today is a short one: we'll be calculating the chiral anomaly / ABJ anomaly / mixed 't Hooft anomaly between vector and axial fields. We'll do this by looking at a Ward identity that gives the conservation of the axial current, which superficially (but only superficially) is a slightly different way compared to any that I've seen in books.

Since we already know the answer and have indeed derived it several times in previous diary entries, we won't worry too much about keeping numerical factors

[160] This splittability issue isn't helped by adding charged matter unless that matter explicitly breaks the higher symmetry. For example we can try adding electric matter to pure Maxwell, but this wouldn't help us make arguments about the $U(1)_m$ 1-form symmetry since the objects charged under it, viz. the 't Hooft lines, are non-local in the electric variables.

correct. The effective Euclidean action for the background vector field is

$$Z[A] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-\int [\bar{\psi} i \not{D} \psi + J_\mu A^\mu]\right). \tag{1461}$$

In two dimensions, we have (using the gamma matrices $\gamma^0 = X, \gamma^1 = Y, \bar{\gamma} = Z$) for a fermion $\psi = (L, R)^T$

$$J_0 = n_L + n_R, J_1 = i(n_L - n_R), \qquad \mathcal{J}_0 = n_L - n_R, \mathcal{J}_1 = i(n_L + n_R), \tag{1462}$$

where $\mathcal{J}$ is the axial current. This means that in two dimensions we have

$$\mathcal{J}_\mu = -i\epsilon_{\mu\nu} J^\nu \implies \star d^\dagger \mathcal{J} = -idJ \tag{1463}$$

where the i is a Euclidean signature artifact.

We can use this in a ward identity as follows: suppose we shift the gauge field by $\delta A = \star d\lambda$, where λ is a 0-form (with compact support). Then

$$\int \delta(J \wedge \star A) = \int dJ \wedge \lambda = i \int \star d^\dagger \mathcal{J} \wedge \lambda, \tag{1464}$$

and so evidently we have, taking λ to be infinitesimal so that we can expand to first order in λ,

$$Z[A + \star d\lambda] \approx Z[A]\left(1 - i\int \lambda \wedge \star \langle d^\dagger \mathcal{J}\rangle_A\right). \tag{1465}$$

On the other hand, we can get an explicit expression for the lower orders in the expansion for $Z[A]$. Putting the det $\not{D}_A$ in the exponent in the usual way, we get the usual representation of $Z[A]$ as a sum of bubbles with A lines sticking out of them. The first order tadpole graph gives zero, while the second gives the usual polarization bubble. One can evaluate this explicitly, or use gauge invariance to write down the answer (up to the coefficient). So, to second order,

$$Z[A] \approx \exp\left[-\frac{1}{2\pi}\int F \wedge \star\left(\frac{1}{\Box} F\right)\right]. \tag{1466}$$

This is the unique gauge-invariant dimension-2 thing we can build that's quadratic in A. Another way to write it uses

$$F \wedge \star(\Box^{-1} F) = A \wedge \star \frac{d^\dagger d}{\Box} A \to A_\mu (g^{\mu\nu} - q^\mu q^\nu / q^2) A_\nu, \tag{1467}$$

which is the usual projector onto the transverse modes. Anyway, varying this to first order in λ, gives

$$Z[A + \star d\lambda] \approx Z[A]\left(1 - \frac{1}{\pi}\int F \wedge \star \Box^{-1} dd^\dagger \star \lambda\right). \tag{1468}$$

Since it is acting on an exact form, $\Box^{-1} = (dd^\dagger)^{-1} = (d^\dagger)^{-1} d^{-1}$, and so after integrating by parts,

$$Z[A + \star d\lambda] \approx Z[A]\left(1 - \frac{1}{\pi}\int \lambda \wedge F\right). \tag{1469}$$

Now we can match up the two ways of calculating the partition function to obtain

$$\langle d^\dagger \mathcal{J} \rangle_A = -\frac{i}{\pi} \star F, \tag{1470}$$

which is the anomaly we wanted to show. The i is from our choice of Euclidean signature, and the $1/\pi$ (instead of $1/2\pi$) ensures that $\int \star d^\dagger \mathcal{J} \in 2\mathbb{Z}$ regardless of the A background, which is consistent with overall fermion number conservation and required since the $-1 \in U(1)_A$ is also the $-1 \in U(1)_V$, the latter of which we know is conserved in this approach.

C, R, T, and fermions in three dimensions

Today's diary entry is a careful compendium of various facts about fermions and their symmetries in three spacetime dimensions. This has to a large extent been superseded by the diary entry on general fermion spacetime symmetry actions in general dimensions, but the present diary entry goes into a fair bit more detail and hence has been kept.

In this diary entry we will be in 2+1 dimensions, in $\mathbb{R}$ time. We will use the Weyl basis for the γ matrices:

$$\gamma_0 = iY, \quad \gamma_1 = X, \quad \gamma_2 = Z. \tag{1471}$$

This has the advantage that all of the γ_μ's are real, which simplifies calculations with T. From the commutation relations of the γ's, we see that this choice works provided we use mostly positive signature.

Now we'll set conventions for what we mean by C and T. In QFT, T is an antiunitary operator that sends t to $-t$.[161] However, we have many options for what we mean by T, since we can compose T with any unitary transformation that commutes with the Lorentz group and still get something satisfying our definition of a T transformation. For a given situation, some of these choices for T will be symmetries, while others will not. In the following, by T, we will mean the antiunitary operator that acts on a Dirac fermion $\psi = \psi_1 + i\psi_2$ (here both ψ_1, ψ_2 are real Majorana fermions, and $\psi_i = (\psi_{i,L},\ \psi_{i,R})^T$) as

$$T : \psi(t,x) \mapsto \gamma_0 \psi(-t,x), \quad \psi_1(t,x) \mapsto \gamma_0 \psi_1(-t,x), \quad \psi_2(t,x) \mapsto -\gamma_0 \psi_2(-t,x). \tag{1472}$$

[161] By which we mean it acts on fields as $\phi(t) \mapsto \phi^*(-t)$. It doesn't actually act on t, which is just an integration variable in the action.

The γ_0 here switches L and R movers, which is something we want T to do. We will often write transformations like this as e.g. $T : \psi \mapsto \gamma_0\psi$, with the reversal of the time coordinate left implicit.

We could have also chosen to not put the minus sign in the transformation of ψ_2, and then we'd get a map $T : \psi \to \psi^\dagger$. This transformation will usually be denoted by CT, since we will define

$$C : \psi_i(t, x) \mapsto (-1)^{i+1}\psi_i(t, x), \tag{1473}$$

which flips the sign of the imaginary part of ψ. Finally we have a reflection, which we take to act on the x coordinate as

$$R : \psi_i(t, x, y) \mapsto \gamma_1\psi_i(t, -x, y). \tag{1474}$$

Note here we are being sloppy and writing x for either one spatial coordinate, or as shorthand for both spatial coordinates — context should prevent this from being unduly confusing.[162].

The final symmetry we'll be thinking about is the regular vector $U(1)$ symmetry. Written out explicitly, the $U(1)$ symmetry acts as a rotation on the Majorana fermions:

$$R_\alpha : \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \mapsto \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{1475}$$

The current $J = \bar\psi\gamma_\mu\psi\, dx^\mu$ is odd under T, so that J_0 is even while J_1 is odd. For example,

$$T(\bar\psi\gamma_1\psi) = \bar\psi\gamma_0^T\gamma_0\gamma_1\gamma_0\psi = \bar\psi\gamma_0^2\gamma_0\gamma_1\psi = -\bar\psi\gamma_1\psi. \tag{1476}$$

This means the charge operator $Q = \int J_0$ is even under T. Since $C(\psi) = \psi^\dagger$, J is odd under C, and so is Q. J_1 is odd under R while the other components are even, so $P(J) = J$ as a differential form. Anyway, from these definitions we see that we have the algebra

$$T^2 = (CT)^2 = \gamma_0^2 = (-1)^F, \quad C^2 = P^2 = \mathbf{1}, \quad Te^{iQ} = e^{-iQ}T, \quad Ce^{iQ} = e^{-iQ}C. \quad Re^{iQ} = e^{iQ}R. \tag{1477}$$

There are three types of masses we will consider for the fermions. They are defined as

$$\begin{aligned}\bar\psi M_D\psi &\equiv im_D(\bar\psi_1\psi_1 + \bar\psi_2\psi_2)\\ \bar\psi M_-\psi &\equiv im_-(\bar\psi_1\psi_1 - \bar\psi_2\psi_2)\\ \bar\psi M_m\psi &\equiv im_m\bar\psi_1\psi_2.\end{aligned} \tag{1478}$$

The second two break $U(1)$, and as we will see, are related by a $U(1)$ rotation to one another; hence they are not strictly speaking independent. We will look at each of the three masses in turn. As we will show in a second, they preserve or break the symmetries defined above in the following way:

[162]These assignments for the actions of C, R, T are specific to our choice of signature $(-,+,+)$. Unfortunately, the representation theory of the various pin groups means that these choices do not carry over to other signatures. For example, suppose we chose the signature $(+,-,-)$, with γ matrices (X, iY, iZ). Now suppose T acted as $T = T_U K$, with K complex conjugation and T_U unitary. Then in order to preserve $\bar\psi i\partial\!\!\!/\psi$, we need to have $T_U = \gamma^2$! Note that the Hermitian term $m\bar\psi\psi$ is T-odd. Similarly, for R to preserve the kinetic term, we can again take it to act as γ^2, so that T and R only differ by complex conjugation. What a mess!

M_D

First for the regular Dirac mass. One useful fact is that $\bar{\psi}_i\psi_i$ is even under T:

$$T(\bar{\psi}_i\psi_i) = \psi_i\gamma_0^T\gamma_0\gamma_0\psi_i = \bar{\psi}_i\psi_i. \tag{1479}$$

Thus

$$T(\bar{\psi}M_D\psi) = -\bar{\psi}M_D\psi, \tag{1480}$$

so that M_D is odd under T. Similarly, one shows that the Dirac mass is odd under R. On the other hand it preserves $U(1)$, since as a bilinear form for the vector (ψ_1, ψ_2), it is the identity. That it preserves $U(1)$ can also be checked explicitly, using the fact that $\bar{\psi}_1\psi_2 = \bar{\psi}_2\psi_1$ (there is no minus sign here, because of a minus sign picked up from the definition of γ_0). Finally, it also preserves C, since it is quadratic in ψ_2. Since it preserves $U(1)$ and C, there can be no anomalies involving combinations of just these two symmetries.

One perspective on why m_D breaks reflection is the following. Consider solving the Dirac equation in 2+1D: writing the Dirac spinor as $\psi = (\psi_+,\, \psi_-)^T$ (no more Majoranas until the next subsection), we can consider going into the $\mathbf{k} = 0$ rest frame, wherein we have

$$\partial_t\psi_+ = -m_D\psi_-, \qquad \partial_t\psi_- = m_D\psi_+. \tag{1481}$$

Solutions to this are $\psi_+ = e^{\pm i m_D t}$, with the $\pm$ sign free to be chosen at will. We need to fix a convention, and will choose the $+$ sign. This gives the solution $\psi = (\psi_+,\, \psi_-)^T = (e^{im_Dt},\, ie^{im_Dt})^T$, which is a $+$ eigenvector of J. What spin does this have? Spatial rotations are implemented in Spin(3) by $i[\gamma^1, \gamma^2]/4 = J/2$, and so we see that ψ has spin 1/2. Now consider changing the sign of m_D: this is equivalent to changing our convention about which sign to choose in $e^{\pm i m_D t}$, which changes the eigenvalue of ψ under J, and means that ψ now has spin $-1/2$. Now while the choice of spin $\pm 1/2$ is a convention, after fixing a convention, the difference in spins between positive and negative m_D is not. Since a definite spin is picked out for m_D nonzero, T and R must be broken by a nonzero Dirac mass.

M_m

Now for the Majorana mass. Since $\bar{\psi}_i\psi_i$ is even under T, $\bar{\psi}_1\psi_2$ is odd. Thus

$$T(\bar{\psi}M_m\psi) = +\bar{\psi}M_m\psi. \tag{1482}$$

However, since the Majoranna mass is linear in ψ_2, it is odd under C. By CRT symmetry it is thus odd under R as well (which is easily checked).

The Majorana mass also breaks $U(1)$, as is easily checked (as a bilinear form it is the matrix X, which has determinant -1 and thus can't transform in the trivial representation of $U(1)$). One also checks that under repeated applications of conjugation by the matrix representing a rotation $\pi/4$,

$$M_m \mapsto M_- \mapsto -M_m \to -M_- \to M_m. \tag{1483}$$

Since M_m goes to minus itself under a $\pi/2$ rotation, M_m transforms in the charge 2 representation of $U(1)$. Thus it breaks the $U(1)$ symmetry down to the $\mathbb{Z}_2$ of $(-1)^F$

symmetry, which can never be broken since $(-1)^F$ is part of the Loretnz group ($(-1)^F$ is the generator of the center of $SU(2) = \text{Spin}(3)$).

However, saying that M_m breaks charge conjugation is a little bit hasty. As mentioned earlier, we are free to modify any of the symmetry operators by the action of a unitary operator which commutes with the Lortenz group — our usual example of such an operator will be a rotation which performs the $U(1)$ symmetry. To this end, define a new charge conjugation operator by

$$C_m \equiv Ce^{i\pi Q/2}, \quad C_m : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto - \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}. \tag{1484}$$

With this definition we still have $C_m^2 = \mathbf{1}$, but now C_m no longer commutes with T (and so instead of $(C_m T)^2 = T^2 = (-1)^F$, we have $(C_m T)^2 = T^2 e^{i\pi Q} C^2 = (-1)^{2F} = \mathbf{1}$). The point of doing this is that M_m is even under this charge conjugation, since

$$C_m M_m C_m = Ce^{i\pi/2} M_m e^{-i\pi Q/2} C = -CM_m C = M_m, \tag{1485}$$

since as a bilinear form between $\bar{\psi}$ and ψ, M_m is X while C is Z.

CRT means that we must also be able to define an R that is preserved by M_m, since it preserves T and a C as well (CRT just means that there exists *a* choice of C, R, and T such that their product acts as the identity on the terms in the Lagrangian—a generic choice of such symmetry operators will not always have a product which acts as the identity). In this case, since the charge operator commutes with R, we define

$$R_m \equiv Re^{-i\pi Q/2}, \tag{1486}$$

which means that $C_m R_m T$ is a symmetry of the M_m mass. The price of realizing these symmetries is that we get more complicated relations among the symmetry generators, e.g. how now neither the reflection nor the charge conjugation operators commute with T.

M_-

Finally we turn to M_m, which is related to M_m by a $\pi/4$ $U(1)$ rotation, as we just saw. (thus it also has charge 2 under the $U(1)$). It is like the reverse of M_m: it breaks T (since the $\bar{\psi}_i \psi_i$ terms are T-invariant), but not C (since it is biliear in ψ_2). Even though it is odd under T, saying that it breaks time reversal is a bit hasty. Indeed, consider the time reversal operator

$$T_- \equiv Te^{i\pi Q/2}. \tag{1487}$$

It acts on the Majoranas as

$$T_- : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto -\gamma_0 \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}. \tag{1488}$$

Now since conjugating with $e^{i\pi Q/2}$ sends M_- to $-M_-$ (as M_- transforms in the charge-2 representation of $U(1)$), and since M_D is odd under T, we see that M_- is preserved by T_-. It's also easy to check that $T_-^2 = T^2 = (-1)^F$, and that

$$T_- C = CT_-(-1)^F \implies (CT_-)^2 = \mathbf{1}. \tag{1489}$$

As it stands M_- repsects a time reversal and a charge conjugation, but not a reflection. Thus by CRT we can find some new definition of R such that R is preserved. Indeed, we take

$$R_- \equiv Re^{i\pi Q/2}, \qquad R_- : \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \gamma_1 \begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix}. \tag{1490}$$

Then since M_- is in the charge-2 rep of $U(1)$, conjugating it by the $e^{i\pi Q/2}$ factor (which commutes with R) gives a factor of $e^{2\pi i/2} = -1$, which cancels its oddness under R. Thus M_- is even under R_-, and so is preserved by CR_-T_-.

Thus we conclude that identifying a mass term as odd or even under "time reversal" or "charge conjugation" or "reflection" is a bit subtle, since we have to specify exactly how these symmetries act. For some (legitimate) choices the symmetries may be broken, while for other (equally legitimate!) choices they may not be (nothing could be done to preserve a choice of each of C, R, and T in the case of the Dirac mass, though: since R commutes with e^{iQ} and since M_D is $U(1)$ invariant, nothing can change the fact that R is broken, and hence by CRT nothing can change the fact that at least one of C and T is broken).

Free energy singularities, contact terms, and anomalies

Today we will try to motivate an anomaly in a theory with symmetry group G implies singularities in the free energy $\mathcal{F}[A]$ of the theory to background gauge fields A for G, and will then try to understand how the various ways of saturating anomalies in the IR reproduce these singularities. We will also go over the physical meaning of background field counterterms, prompted by wanting to understand some remarks by Zohar Komargodski at the 2019 Jerusalem winter school.

First let us briefly recapitulate what a ('t Hooft) anomaly is. Imo a particularly good modern introduction to this is in [13].

An anomaly for a symmetry group G, *in the context of this diary entry*, will mean that if the partition function of the theory in the presence of background fields A for the symmetry G is $Z[A]$, then $Z[A]$ is not just a function of the gauge equivalence class of A, viz. $Z[A] \neq Z[{}^U A]$ for some gauge transformation $U(x)$. Here A is a schematic notation which stands collectively for all the background fields involved — we will use

A^a to refer to the field for a particular generator a of G, with current j^a. Specifically, if the theory is anomalous,[163] then we will have

$$Z[{}^U A] = e^{i\beta(A,U)} Z[A] \tag{1491}$$

where β is a *local functional* of A and U. In this diary we will be assuming that $\beta(A,\gamma) \in \mathbb{R}$, so that the anomaly only affects the imaginary part of the free energy. This is true for every $\mathbb{R}$-time anomalous theory that I know about (provably true for chiral anomalies and makes sense, since we want the vev of the current, which comes from differentiating $\mathcal{F}[A]$ with an extra factor of i, to be real), but when one goes to $\imath\mathbb{R}$ time (which we will be trying to avoid) one can get anomalies in the $\mathbb{R}$ part of $\mathcal{F}$, e.g. the trace anomaly / Weyl anomaly (although c.f. the diary entry on the meaning of the phrase "Weyl anomaly").

For the purposes of this diary entry, we will think of the global parts of the gauge transformations (for which $U(x)$ is a constant[164]) as being genuinely gauged (e.g. by working on a compact space), so that constant U is allowed in the above formula. In particular, this means that we do not have to turn on nontrivial backgrounds for all the factors in the symmetry group, e.g. if $G = H \times K$ we can turn on background fields for H only and then act with a generator of K — if a nontrivial β appeared here, it would signal a mixed anomaly between H and K. In this case, the anomaly comes from the fact that the background fields for H break the K symmetry, which can be seen without turning on a background for K (this approach is useful if K is some symmetry like T for which turning on backgrounds is annoying).

Since the gauge non-invariance occurs in the phase of the partition function, it is helpful to write things in terms of the free energy, in order to isolate this. In what follows we will specify to the case of Abelian G for simplicity (this is just for notational reasons, not physical ones — also, for discrete symmetries mentally replace $d, \wedge \mapsto \delta, \cup$, etc.), and write the gauge transformations as $A \mapsto A + d\gamma$. Let us write the free energy in terms of its real and imaginary parts as $F = \mathsf{F}[A] + i\mathcal{F}[A]$. Then we have[165]

$$\mathcal{F}[A + d\gamma] - \mathcal{F}[A] = -\beta(A,\gamma). \tag{1492}$$

We can be slightly more specific about what β is. First, it has to vanish when A is pure gauge. This is because when A is pure gauge it can be canceled by a change of variables in the path integral (at least perhaps up to local contact terms; these will be irrelevant for determining the anomaly anyway), and hence $Z[A = d\alpha]$ cannot have any α dependence. Since the expectation value of the current (i.e. $\delta S/\delta A$; not necessarily a continuous object) in the background field is $\langle j \rangle = i\delta F[A]/\delta A$, the functional β controls the expectation value of the divergence of the current as (taking the variation to be $\delta A = d\gamma$)

$$\langle d^\dagger j^a(x) \rangle = d^\dagger i \frac{\delta}{\delta d\gamma}(F[A + d\gamma] - F[A])|_{\gamma=0} = \frac{\delta}{\delta \gamma^a(x)} \beta(A,\gamma)|_{\gamma=0}. \tag{1493}$$

[163] Again, under the local definition of "anomaly" — more general responses to changes in the background fields are possible (e.g. they could induce operator-valued changes), but we will not discuss them here.

[164] We will also specifying to 0-form symmetries when the notation forces us to commit to a particular co-dimension for the charge generators.

[165] We're working in $\mathbb{R}$ time, but also in conventions where $F = -\ln Z$, without the i. In retrospect this may not have been the most aesthetically pleasing choice; oh well.

However, this being non-vanishing does not always mean there is anomaly, and indeed a nonzero β does not necessarily imply an anomaly — it may just be because we haven't defined the current properly. In particular, we might be able to re-define our current by some terms only involving the background field such that the RHS of the above, and hence the anomaly, vanishes. To explain this, we need a short digression on the role of contact terms in QFT.

`<digression>`

Normally in QFT one is free to add local counterterms in the background gauge field A at will to the action (if we don't plan on making the background fields dynamical, the counterterms don't even have to gauge invariant). This is because when they are non-dynamical such background fields are merely devices for computing correlation functions of the dynamical fields they couple to, and such local functionals of A don't affect the results of computing any of the physical correlation functions that we care about. Adding a local term $\alpha \int f(A)$ for some local functional f leads to a theory with α dependence which only appears when evaluating correlation functions of currents at coincident points. Since we normally don't care about such correlation functions (they are some non-universal UV stuff that we aren't trying to capture with the free energy), we can provisionally regard local functionals of the background fields as physically unimportant. This means that if we take the free energy $\mathcal{F}[A]$ for the field A written in momentum space and expand in the field momentum q, *only* the terms that go as negative integer or non-integer powers of q will be regarded as physical; all the positive-integer powers are beyond the purview of QFT. QFT is for the most part the study of things that are regulator-independent (well, expect that anomalies have to do with regulators, which is kind of the point), and almost by definition does not capture purely UV things.

Saying that all counterterms built from background gauge fields are unphysical is maybe going too far, though. After all, the CS term $A \wedge dA$ determines the Hall conductivity (here A is a *background* EM field, not the dynamical field which is integrated over in the field theory description of the Hall effect), which is physical and well-defined (that the CS term only affects correlation functions at coincident points can also be understood from the fact that CS theory has no radiation: the equations of motions have no derivatives, and the field strengths can be solved for as a local function of the sources). But how can the Hall conductivity be a universal thing to compute on the field theory level, if the Hall response is determined by a contact term in the background gauge fields? From a QFT perspective, we would say that only the Hall conductance mod 1 is universal information about the response of a given QFT. One way to argue on the contrary though is that counterterms that we add to change our regularization prescription should be able to be added locally, that is, we should be able to add them with a spatially varying coefficient. Of course, adding the term $\int \alpha(x) A \wedge dA$ is not allowed because it breaks gauge invariance unless α is a constant integer multiple of $1/4\pi$, and so the CS term is not a trivial change in regularization scheme in the same sense that e.g. $\int F \wedge \star F$ would be (which gives a very singular $\Box\delta(x)$ modification to the current-current correlation function).

`</digression>`

With these comments in mind, we see that since local parts of the free energy density are non-universal we are free to modify $\mathcal{F}[A]$ by any local functional $-i\int f(A)$, which may or may not be gauge-invariant (with the exception of those whose coefficients are quantized, as mentioned above — this issue won't actually come up until the last example, though). In particular if it is the latter, then we might be able to choose f such that the anomalous phase β is canceled, since now

$$i\delta\mathcal{F} = i(\mathcal{F}[A + d\gamma] - \mathcal{F}[A]) = \beta(A, \gamma) - [f(A + d\gamma) - f(A)], \tag{1494}$$

so that the variation of the free energy is $i\delta\mathcal{F} = \beta - \delta f$. Therefore if $\beta(A, \gamma)$ is a total variation, the free-energy can be made gauge-invariant by adjusting it with a local counterterm — doing so also re-defines the current (via $j = \delta_A S$), which gives us a new current which is conserved.

Anyway, from the above formula we see that we thus have a sort of cohomology problem: a nontrivial anomaly comes when β cannot be expressed as the gauge variation of a local functional of A, even though it is itself is a local functional. A good discussion of this is in [13].

We now attempt to give a more geometric picture to this cohomology problem, something that I first saw the idea of in [11]. Mathematically, if $\mathcal{A}$ is the moduli space of all possible gauge field configurations and if $\mathcal{G}$ is the group of gauge transformations,[166] then $Z[A]$ is a section of a line bundle over $\mathcal{A}/\mathcal{G}$, while $\mathcal{F}$ is a section of a $U(1)$ bundle over $\mathcal{A}/\mathcal{G}$.[167] $Z[A]$ is always a global section, but the bundle over $\mathcal{A}/\mathcal{G}$ may still be nontrivial, and its non-triviality will indicate an anomaly. Indeed, if there were no anomaly then we could add an appropriate counterterm $\delta^{-1}\beta$ to make $Z[A]$ a genuine function on $\mathcal{A}/\mathcal{G}$ and hence the bundle for the partition function (free energy) would be the trivial product $\mathcal{A}/\mathcal{G} \times U(1)$ $(\mathcal{A}/\mathcal{G} \times U(1))$. Taking the contrapositive means that a nontrivial bundle $\implies$ an anomaly. We will assume the converse is true in what follows; it is true at least for all the examples we will discuss.[168]

There are multiple ways in which the the bundle E of which $\mathcal{F}[A]$ is a section could be nontrivial. First, it may be nontrivial because it has a connection with curvature which gives invariants like Chern numbers that are nonzero. Secondly, it may be flat but with a connection that has nontrivial holonomy, so that the transition functions can not all be trivialized. We will see examples of both in a second.

We now want to elaborate on why anomalies imply singularities in the (imaginary part of) the free energy $\mathcal{F}$. The basic reason is from the issue of bundle topology above: $\mathcal{F}[A]$ is not a global section of E, and so when we write it in a particular coordinate system, it has branch cuts, which manifest themselves as singularities (this

[166] $\mathcal{G}$ includes all gauge transformations, viz. large ones as well, which are gauge transformations by functions $U(x)$ which cannot be continuously deformed to $\mathbf{1}$. An example of where we need to take into account such transformations is Witten's $SU(2)$ anomaly, where gauge transformations by $U(x)$ for $U(x)$ homotopic to the generator of $\pi_4(SU(2)) = \mathbb{Z}_2$ leave the partition function invariant if the theory involves an odd number of chiral fermions coupled to A.

[167] Thinking like this is probably useful only when the anomaly can be probed by turning on continuous background fields.

[168] As in [13], the catch is that the bundle being trivial just means that one can find a bundle automorphism that takes the bundle to the product bundle; the fact that this bundle automorphism may be non-local but that we are only working modulo local counterterms is the source of the difficulties.

doesn't mean that correlation functions of the currents are necessarily singular, since the variational derivative is always taken within a single coordinate patch on $\mathcal{A}/\mathcal{G}$). While these branch cuts are most clearly illustrated in the case with both discrete and continuous symmetries (parity anomaly; towards the end), we will first look at what this means for just $U(1)$ symmetries.

First, we will see that any theory with an anomaly involving products of $U(1)$s will necessarily be massless. This intuitively obvious fact (how could you saturate an anomaly coming from a diagram that can be derived perturbatively from a gapped theory which in the IR has only finitely many dof?) can be proven by looking at the formula for the divergence of the current. Indeed, write the vev of $d^\dagger j^a$ as $\langle d^\dagger j^a \rangle = P(F_A)$, where $P(F_A) = \delta_{\gamma^a}\beta(A,\gamma)|_{\gamma=0}$ is some functional of the field strength.[169] For example, consider a chiral anomaly-like case in D dimensions, where

$$\langle d^\dagger j^a \rangle = \alpha \varepsilon^a{}_b \star (dA^b)^{\wedge(D/2)}, \tag{1495}$$

with α a constant. Then, taking $D/2$ functional derivatives wrt A, we have

$$\varepsilon^{ab}\delta\left(\sum_{i=0}^{D/2} q_i\right) q_0^{\mu_0}\left(\prod_{i=1}^{D/2}\frac{\delta}{\delta A^b_{\mu_i}}\right)\langle j_a^{\mu_0}(q)\rangle \sim \varepsilon^{\mu_1\dots\mu_{D/2}}{}_{\nu_1\dots\nu_{D/2}}\prod_{i=1}^{D/2} q_i^{\nu_i}. \tag{1496}$$

Since the functional derivatives acting on the current produce more currents inside the expectation value, this becomes

$$\varepsilon^{ab}\delta\left(\sum_{i=0}^{D/2} q_i\right)\left\langle j_a^{\mu_0}(q)\prod_{i=1}^{D/2} j_b^{\mu_i}(q_i)\right\rangle \sim \varepsilon^{\mu_1\dots\mu_{D/2}}{}_{\nu_1\dots\nu_{D/2}}\prod_{i=1}^{D/2} q_i^{\nu_i}(q^{\mu_0})^{-1} \tag{1497}$$

Therefore the anomaly tells us that this $D/2+1$ point correlation function of the currents is long-ranged in $\mathbb{R}$ space — hence the theory cannot be massive, no matter how the anomaly gets saturated in the IR.

Vis-a-vis singularities in $\mathcal{F}[A]$: as we said above, the anomaly here is saying not just that the free energy in not invariant under local gauge transformations, but that a nontrivial background field actually breaks the global symmetry. Indeed, we can explicitly show by e.g. Fujikawa that the partition function changes under the action of a *global* symmetry (e.g. the fermion measure changes even when we perform a chiral rotation that is everywhere constant in spacetime). Getting a nonzero result for $\delta A = d\lambda$ in the limit where λ is a constant means there must be some sort of singularity coming from the inverse of a differential operator which allows the answer for $\lambda = const$ to be nonzero.

Let's look at one example in a little more detail, which of course is something we've seen a jillion times already, viz. the mixed anomaly between $U(1)^A$ and $U(1)^V$ for fermions in even dimensions. We know that e.g. in 1+1D, a $U(1)^A$ gauge transformation $\delta A^A = d\lambda$ produces, in the usual regularization scheme, a term

$$\delta\mathcal{F}[A^A, A^V] = \frac{i}{\pi}\int \lambda F^V = \delta_\lambda \frac{i}{2\pi}\int (d^{-1}A^A)F^V, \tag{1498}$$

[169] It can only depend on F_A in these examples since it must be local and must vanish when A is locally exact as we said above ethan: *come back and talk about $A_1 \wedge F_2$ and stuff* .

which is the variation of a term that is gauge invariant under $U(1)^V$, but which is singular at zero momentum (here by $d^{-1}A^A$ we just mean A^A_μ/q_μ). This means that in this regularization scheme,

$$\mathcal{F}[A^A, A^V] = \frac{i}{2\pi}\int A^A_\mu A^V_\nu \varepsilon^{\nu\lambda} q_\lambda/q_\mu + \cdots = \frac{i}{2\pi}\int (A^A_0 A^V_0 (q_1/q_0) - A^A_1 A^V_1 (q_0/q_1) + A^A_1 A^V_0 - A^A_0 A^V_1) + \ldots, \tag{1499}$$

where the dots are gauge-invariant (and in general non-local, i.e. involving field strengths and $\Box^{-1}$s) or built from local counterterms. We can always add $A^V \wedge A^A$ to get rid of the last two terms in parenthesis, but the claim is that the singular parts (which are responsible for the gauge non-invariance) will always be there, in any regularization scheme.

We can show this for example by looking at what happens to the integration measure under gauge transformations of both gauge fields. For $A^a \mapsto A^a + d\lambda^a$, standard manipulations with the Jacobian show that free energy changes as

$$\delta\mathcal{F}[A^A, A^V] \propto i \lim_{M^2\to\infty} \mathrm{Tr}\left[e^{(\not{D}_{A^V+\bar\gamma A^A})^2/M^2}(\lambda^A + \bar\gamma\lambda^V)\right], \tag{1500}$$

where we have chosen a heat-kernel regulator that is invariant under both gauge transformations (if we don't do this then we definitely won't be able to make the gauge fields dynamical). Now we write the square of the Dirac operator as (see another diary entry for the proof)

$$(\not{D}_{A^V+\bar\gamma A^A})^2 = D^2_{A^V+\bar\gamma A^A} + \frac{\gamma^\mu\gamma^\nu}{2}(F^V_{\mu\nu} + \varepsilon_\mu{}^\lambda F^A_{\lambda\nu}). \tag{1501}$$

When acting on the gauge-invariant Hilbert space the square of D just produces $-k^2$, where k is the momentum. When we do the integral over all k, we get a factor of M^2, which means in the limit that only the first term in the expansion of the exponential of the field strength survives. Then taking the trace over the spin indices, we find

$$\delta\mathcal{F}[A^A, A^V] = i\frac{1}{\pi}\int(\lambda^A F^V + \lambda^V F^A). \tag{1502}$$

This means that the general expression for the free energy is

$$\mathcal{F}[A^A, A^V] = \frac{i}{2\pi}\int(d^{-1}A^A F^V + d^{-1}A^V F^A + C A^A \wedge A^V) + \ldots, \tag{1503}$$

where the $\ldots$ are gauge invariant and C is some arbitrary coefficient.[170] By looking at this one sees that taking $C = \pm 1$ will render the theory gauge invariant under one of the two symmetries, but getting something that is invariant under both is impossible (if the relative sign between the F^A and F^V terms had turned out differently this would not be the case, and we would have had something non-anomalous).

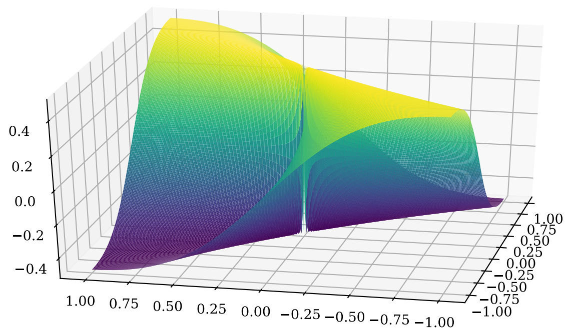
How are the singularities reproduced when SSB occurs? In the way one would expect — they are produced by integrating out the Goldstones. However since the Goldstones couple differently to the gauge field than the UV fields do, we'll briefly

[170] An $A^A \wedge \star A^V$ counterterm obviously doesn't help us.

discuss how this works explicitly. To illustrate this example we again look at an anomalous $U(1)$ symmetry (for notation's sake). Here a typical IR action includes schematically $(d\phi - A)^2$, with ϕ the goldstone. Now we know that $\mathcal{F}[A] = 0$ when A is pure gauge, while at the same time we know that for A a constant, the free energy must be $\int A \wedge \star A$, since the $d\phi \wedge \star A$ terms die after IBP (the nonzero free energy coming from the "velocity boost" of the SSBken state). These two facts are only compatible if there are some ∂^{-1}s in the free energy. Indeed, one can quickly check that the relevant part of the free energy is

$$\mathcal{F}[A] \sim \int F_A \wedge \star \Box^{-1} F_A = \int A_\mu (g^{\mu\nu} - q^\mu q^\nu / q^2) A_\nu. \tag{1504}$$

This is singular at zero momentum because of the second term, which cares about the way that we approach zero momentum; this singularity allows $\mathcal{F}[A]$ to go to zero for pure gauge configurations but still be nonzero when the gauge fields are constant. Just for fun, here's a plot of what the second term looks like (both the diagonal and off-diagonal parts are singular in the same way):



(1505)

One example where we have continuous gauge fields but an anomaly for a discrete symmetry is the mixed anomaly between T and $U(1)$ for Dirac fermions in odd spacetime dimensions. For example, in $D = 2 + 1$, we show in another diary on the parity anomaly that the partition function is

$$Z[A] = |Z[A]| \exp\left(\frac{ik}{4\pi}\int A \wedge dA + \frac{\pi i}{2}\eta(A)\right) \tag{1506}$$

with the definition

$$\eta(A) = \left(\frac{1}{4\pi^2}\int A \wedge dA + 1\right)_2 - 1, \tag{1507}$$

with the subscript meaning mod 2. Here k is a IQH response counterterm, which we will regard as physically meaningful in keeping with the comments above. From this we see that $\mathcal{F}[A]$ is a nontrivial $U(1)$ bundle over $\mathcal{A}/\mathcal{G}$. For example, when we're on the usual $S^2 \times S^1$ with $2\pi n$ flux through the S^2, we have (setting $k = 0$ for now)

$$\mathcal{F}[A] = \frac{n\pi i}{2}\left(\left(\frac{h}{2\pi} + 1\right)_2 + 1\right), \tag{1508}$$

where h is the holonomy of A around the S^1. The important part here is that as A winds around a closed (non-contractible) loop in the base space $\mathcal{A}/\mathcal{G}$, viz as h goes from 0 to 2π, we have $\mathcal{F}[A]$ go from 0 to $n\pi$, and then when $h = 2\pi + \varepsilon$, we suddenly have a jump to $\mathcal{F}[A] = -n\pi$. Now when $n \in 2\mathbb{Z}$, we see that there exists a choice of k such that $\mathcal{F}[A]$ is made to vanish. In a more hep-th way of thinking, where we regard the IQH responses as trivial counterterms which can be added at will to the free energy, this means that when $n \in 2\mathbb{Z}$ there is no anomaly. Our perspective will be slightly different, viz that when $n \in 2\mathbb{Z}$ we have a collection of theories parameterized by k, with one of them being anomaly free.[171] Regardless, when $n \in 2\mathbb{Z}+1$, the holonomy of $\mathcal{F}[A]$ as we move around the loop in $\mathcal{A}/\mathcal{G}$ is only in $\pi(2\mathbb{Z}+1)$, and can't be canceled by any local counterterm — hence, the "parity" anomaly, with $\mathcal{F}[A]$ giving a flat bundle over $\mathcal{A}/\mathcal{G}$ which is rendered non-trivial by the non-existence of a global section coming from the nontrivial transition function at the point where the sign of $\mathcal{F}$ flips (which, as it has to be, is also the point at which $Z[A] = 0$). This transition function becomes a branch cut singularity when we work in a single coordinate patch on $\mathfrak{a}/\mathcal{G}$. Thus finite sums can still produce singularities in the free energy (and in this case as well as the last, it is the zero modes which are responsible for them).

The parity anomaly revisited

Today's diary entry is a look at a way of understanding the parity anomaly, focused on looking rather explicitly at the zero modes realized in certain gauge field backgrounds. This diary entry was prompted by wanting to understand in detail some of the content presented by by Seiberg during his 2019 Jerusalem lectures.

The starting point is of course a Dirac fermion in 2+1D coupled to a background $U(1)$ field A. Beyond the usual minimal coupling, we can add any local gauge-invariant counterterms that are functions only of A, since such terms do not affect any fermion correlation functions beyond modifying contact terms. In 2+1D this means in particular that we can include a (properly quantized) CS counterterm in the background fields. So the action under consideration is (in $\mathbb{R}$ time)

$$S = \int \left[\bar{\psi} i \not{D}_A \psi + \frac{k}{4\pi} A \wedge dA\right], \tag{1509}$$

[171] In no perspective are we allowed to mod out by $\frac{1}{4\pi k}\int A \wedge dA$ FQH responses.

where the CS term is to be thought of as a counterterm that we're allowed to modify the free energy by. We can get the anomaly by specifying to a certain gauge field configuration. We will take space to be an S^2 with $\int_{S^2} F_A = 2\pi$, and will suppose that the spatial gauge fields are time-independent. The CS term then integrates to

$$\frac{k}{2\pi}\int A_0 F_{xy} = k\int dt A_0. \tag{1510}$$

Now the monopole background means the Hamiltonian for the fermions has a zero mode (see a previous diary entry; a monopole background of flux $2\pi n$ supports n zero modes). Since the Hamiltonian for the fermions is the spatial part of $\bar{\psi} i \not{D}_A \psi$, the action for this zero mode is

$$S_0 = \int dt\, \left(\bar{\psi}(i\partial_t + A_0)\psi + kA_0\right). \tag{1511}$$

Since the zero mode on the monopole is a two-level system, this is exactly equivalent to the problem of a single free fermion in quantum mechanics, coupled to a background $U(1)$ field.

We have already analyzed this in an earlier diary entry, where we saw that it had a mixed anomaly between $U(1)$ and C (or T). There are two degenerate states, viz. $|k\rangle, |k+1\rangle$. Time reversal doesn't exchange them, while we define C to act as

$$C : |l\rangle \mapsto |-l+2k+1\rangle, \tag{1512}$$

so that C interchanges $|k\rangle$ and $|k+1\rangle$.

Now we want to compute the partition function, as a matrix element of e^{-iHt} between initial and final states. To see an anomaly in C or T we will want to take symmetric boundary conditions. We will do the simplest thing, viz. summing over $|k\rangle$, $|k+1\rangle$ by working with periodic boundary conditions in time (we will avoid going to $i\mathbb{R}$ time though, since we want to think about T). We can then easily compute the partition function for this system in the deep IR, since we just have to sum over the two degenerate ground states.[172] The Hamiltonian is just $QA_0 = (\bar{\psi}\psi + k)A_0$, and so

$$Z[A] = e^{ik\oint A_0}(1 + e^{i\oint A_0}). \tag{1513}$$

This is not T or C invariant unless k is the (disallowed) value of $-1/2$, or if A is a special T-preserving configuration, like $A = 0$ or $A_0 = \pi/|S^1_t|$.[173]

Just to be pedantic: even though A is currently a classical background field, we are considering a transformation where A is in fact acted on by T, C. Technically this is a bit unfair, since A is just a real c-number function (proportional to **1** when

[172]Here "deep IR" means at energy scales below the gap between the zero modes and the rest of the spectrum, which is set by the curvature of the spatial S^2. Going to the deep IR like this is just done to simplify the formula for $Z[A]$ — including the contributions to $Z[A]$ from all the higher energy modes will just modify the real part of $Z[A]$ in a way that doesn't affect our conclusions about things related to anomalies (and indeed, the anomaly is an RG invariant, so ignoring everything going on above the degenerate ground states can be done wolog if we just want to know about the anomaly).

[173]In order for there to be no anomaly we must be able to maintain the symmetries while summing over all A, and so the fact that $Z[A]$ is symmetric for some values of A is irrelevant.

acting on the Hilbert space), and hence cannot really be acted on by any symmetry transformation. A more mathematically precise way of doing things would be to say that we have a symmetry if the partition function $Z[A]$ is covariant, namely that after applying the symmetry action for a group element g we get $Z[A] \mapsto Z[{}^gA]$ for some appropriate g-action. However this is equivalent to saying that $Z[A]$ is invariant under the symmetry as well as a combined action by g^{-1} on A, and since this is what we'd be doing anyway if A was made dynamical, we will be continuing to use this perspective. For example, T acts on the partition function as $\langle \phi_f | e^{-iHt} | \phi_i \rangle \mapsto \langle T\phi_f | e^{-iHt} | T\phi_i \rangle$, in addition to acting on A in the appropriate way. When we trotterize this to get the path integral, the symmetry action has the effect of replacing all states in the resolutions of the $\mathbf{1}$ by their T-reversed images: thus the action of T can be taken into account by replacing $\phi \mapsto T^\dagger \phi T$ in the action, *without* conjugating the i in e^{iS}. Now the appropriate action on the background field to maintain T symmetry at the classical level is $T : A \mapsto -A$ as forms (after an implicit change of variables). We see from the above expression that T is then broken[174] (the same arguments apply for $C : A \mapsto -A$), and this breaking of T and C is of course the mixed anomaly between $U(1)$ and T and C, which for historical reasons we call the parity anomaly.[175]

Now we write the partition function as

$$Z[A] = e^{ik\oint A_0} e^{i\pi\eta(A)/2} |Z[A]|, \qquad |Z[A]| = e^{i\pi\eta(A)/2} + e^{-i\pi\eta(A)/2}, \tag{1515}$$

where we have defined

$$\eta(A) \equiv \left(\frac{1}{\pi} \oint A_0 + 1 \right)_2 - 1, \tag{1516}$$

where the subscript means mod 2. This expression is gauge invariant and well-defined under changing transition functions on patches of the S^1 by things in $2\pi\mathbb{Z}$ (because of the mod 2), and so $Z[A]$ is manifestly gauge invariant. This way of defining the eta invariant means that $\eta(A)$ takes values in $[-1, 1)$, so that $\cos(\pi\eta(A)/2) = |Z[A]|$ is always positive.

Note that $\eta(A)$ has a discontinuity of 2 when the holonomy of A_0 crosses π. This means that the function $e^{i\pi\eta(A)/2}$ has a discontinuity at $\oint A_0 = \pi$, where it goes from i to $-i$. Note that when we square $e^{i\pi\eta(A)/2}$ we get the level-1 CS term, which is always well-defined — therefore the gauge invariant $e^{i\pi\eta(A)/2}$ is the correct way of writing the more schematic $e^{i\frac{1}{8\pi}\int A}$. Anyway, the point of writing things this way is that the (imaginary part of the) free energy now has a singular contribution as a function of the background fields, and as discussed in an earlier diary entry, such a singular behavior is a requirement for the existence of an anomaly.[176] Despite this singularity,

[174] This might actually make $Z[A]$ look T invariant as written since $A_0(t) \mapsto A_0(-t)$ and since $\oint dt$ is even under the change of variables, but remember that this form of the action came from using $\int F_A = 2\pi$. The time-reversed version of this is $T(\int F_A) = -2\pi$, which means that an extra minus sign appears in the T transformation of $i\oint A_0$.

[175] We've been looking at a path integral perspective, but this can also be seen in Hamiltonian language by noting that the algebra is actually (here R_α is the $U(1)$ rotation by α)

$$TR_\alpha = R_{-\alpha} T e^{-i(2k+1)}, \tag{1514}$$

which is only the expected algebra if $k = -1/2$ (same applies for C).

[176] Note how this singularity in the imaginary part of the free energy comes from the phase of the

the partition function $Z[A]$ is still continuous. This is because the singularity in the free energy happens exactly when $|Z| = 0$ (so the singularity is just in the derivative of $Z[A]$ wrt A, i.e. in the current correlation functions), due to a zero mode of the Dirac operator which occurs at the point where $\oint A_0 = \pi$.

That $|Z[A]|$ vanishes at this point is obvious from the above expression we wrote for it, but we can also see it by looking at the fermions. This goes as follows. On the S^1 of time, we work with the usual $\psi(t) = -\psi(t+\beta)$ boundary condition. Then the frequency is modded in $\frac{2\pi}{\beta}(l-1/2), l \in \mathbb{Z}$, and we can decompose

$$\psi(t) = \sum_{l\in\mathbb{Z}} e^{\frac{2\pi i}{\beta}(l-1/2)t}\psi_k, \tag{1517}$$

and so the condition for ψ_k to be a zero mode of $i\not{D}_A$ is that $-2\pi(l-1/2) = kA_0$. This happens if $k\int A_0 = \pi$, since then we can fix a gauge in which $A_0 = \pi/k\beta$. Basically, the π flux around the S^1 that is threaded in when $\oint A_0 = \pi$ "cancels" the AP boundary conditions for fermions, giving them effectively periodic boundary conditions and hence a zero mode. A Ramond spin structure would be handled by inserting $(-1)^F$ into the trace in the computation of $Z[A]$, giving

$$Z_R[A] = e^{i\pi\eta(A)/2 + ik\oint A_0}\sin(\pi\eta(A)/2), \tag{1518}$$

and so the zero mode of $\not{D}_A$ exists if $\oint A_0 \in 2\pi\mathbb{Z}$ (which one can also see from decomposing $\psi(t)$ in frequency modes).

Finally, note that the T and C non-invariance of the partition function depended on the boundary conditions we chose. For example, if we computed the partition function for a C breaking scenario where our boundary conditions had a definite particle number (e.g. either $|k\rangle$ or $|k+1\rangle$ entered Z, but not both), then the anomaly would be "gone", in the sense that the partition function could be rendered C and T symmetric (in fact, just $Z[A] = 1$) by an appropriate choice of counterterm k.

Anyway, let's return to a more general problem where we can't so easily take advantage of dimensional reduction to get a quantum mechanics problem. The general expression for the partition function has to still be expressed in terms of the eta invariant, by topological arguments. The general definition of $\eta(A)$ is such that it agrees with the one above after taking the specific case of a 2π flux through a spatial S^2: therefore we must have

$$\eta(A) = \left(\frac{1}{4\pi^2}\int A \wedge dA + 1\right)_2 - 1, \tag{1519}$$

partition function and that in particular, it comes from a discrete sum over two degenerate states. Therefore we do not have to integrate out a massless degree of freedom coupled to A in order to produce a singularity in the free energy $F[A]$, just summing over a finite number of degenerate states is sufficient. This is how singular free energies can arise from summations over degenerate ground states even in the context of thinking about e.g. anomalies in TQFTs where the theory in question is gapped and the sum over ground states is finite. In these scenarios the real part of the free energy is non-singular (it's a sum of a finite number of smooth exponentials), but as we have seen this does not preclude the imaginary part from being singular.

and the partition function is analogously

$$Z[A] = |Z[A]| \exp\left(\frac{ik}{4\pi}\int A \wedge dA + \frac{\pi i}{2}\eta(A)\right). \tag{1520}$$

Note that if we had taken the flux to be $2\pi n$ instead, we would get n zero modes. The sum over these in the IR limit of the partition function would then produce $e^{\pi i n\eta(A)/2}$, which can be killed by a local counterterm if n is even; hence there is no T-breaking in these backgrounds. However, when we say "gauge A", we (in this diary entry) include a sum over all flux sectors (as far as I can tell, this is not for sure required), and so in any case there is still a mixed anomaly, because if A is made dynamical then T and C are broken.

T, CT, and dualities

Today's diary entry is an elaboration on an exercise that Nati Seiberg assigned to the students at the 2018 / 2019 Jerusalem winter school on QFT. The problem was to explain why, in dualities, T and CT symmetries are often exchanged.

In the following, we will use a notation where $\mathcal{T}$, $\mathcal{CT}$ are the "duals" of T and CT under some "duality map" $\mathcal{D}$. They are defined by

$$\mathcal{D}[T\mathcal{O}] = \mathcal{T}\mathcal{D}[\mathcal{O}], \quad \mathcal{D}[CT\mathcal{O}] = \mathcal{CT}\mathcal{D}[\mathcal{O}], \tag{1521}$$

where $\mathcal{O}$ is any field that has an image under $\mathcal{D}$. The claim is that the usual story for dualities is $\mathcal{T} = CT, \mathcal{CT} = T$.

Particle on a ring

The first, simplest possible example is that of the duality (just a Fourier transform) between p and q for a particle on a ring, with Lagrangian

$$\mathcal{L}[q] = (\partial_t q)^2 - q^2. \tag{1522}$$

We define the symmetries T and C to act as

$$T : q(t) \mapsto q(-t), \qquad CT : q(t) \mapsto -q(t). \tag{1523}$$

From here on, the time dependence of variables will mostly be kept implicit.

The conjugate variable to q is p. When we write the path integral in the Hamiltonian formulation, we have the Berry phase term $S \supset \int dt\, \dot{q}p$. Under T, this goes to $-\int dt\, \dot{q}T(p(-t))$, while under CT we get $+\int dt\, \dot{q}CT(p(-t))$. Therefore invariance of this term under the symmetries tells us that p transforms as

$$T : p \mapsto -p, \qquad CT : p \mapsto p. \tag{1524}$$

Now duality here (alias Fourier transform) is

$$\mathcal{D} : q \mapsto p, \quad p \mapsto -q. \tag{1525}$$

Here the minus sign, which says that $\mathcal{D}^2 = -\mathbf{1}$, can be seen in several ways. One is that we require the symplectic form $dq \wedge dp$ to be invariant, with the antisymmetry of the $\wedge$ product necessitating the minus sign. Another way to see this is to note that the square of the Fourier transform is an inversion. That is, letting $\mathcal{F}$ be the Fourier transform,

$$\mathcal{F}^2[f(t)] = \mathcal{F}\int dt\, e^{i\omega t} f(t) = \int d\omega \int dt\, e^{i\omega t'} e^{i\omega t} f(t) = f(-t). \tag{1526}$$

This is just because of the fact that the Fourier transform performs a $\pi/2$ rotation in frequency-time space, with a π rotation then corresponding to a reversal of the time coordinate. Since dualities are often performed by a Fourier transform, and the one in the present context indeed is, we expect $\mathcal{D}^2 : q \mapsto -q$, which it does. As we will see, this holds even for more advanced kinds of dualities like Electromagnetic duality, which again is basically just a Fourier transform.

We can summarize the (now obvious) fact that $\mathcal{D} : T \leftrightarrow CT$ by drawing the following commutative diagram:

$$\begin{array}{ccc} q & \xrightarrow{\mathcal{D}} & p \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle \mathcal{T}} \\ q & \xrightarrow{\mathcal{D}} & p \end{array}, \tag{1527}$$

which tells us that $\mathcal{T} = CT$. A similar diagram shows that $\mathcal{CT} = T$.

Electromagnetic duality in four dimensions

A slightly more sophisticated example is electromagnetic duality in 3+1 dimensions. We will work with conventions where $T, C : A \mapsto -A$ (treating A as a 1-form and again suppressing the t argument). Thus the vector components E^i are even under time reversal while those of B^i are odd, with both E^i and B^i odd under C. This convention is subideal for many contexts, but we will stick with it due to it being the one most common in the literature.

As we have seen several times in previous diary entries, electromagnetic duality is (up to a constant of proportionality involving the gauge coupling), implemented by

Hodge duality: $\mathcal{D} : F \mapsto \star F$. Recall how this works: we implement $F = dA$ by the Lagrange multiplier term

$$S \supset \frac{i}{2\pi}\int F \wedge d\widetilde{A}, \tag{1528}$$

and then integrate out F by doing a shift of F by something proportional to $\star d\widetilde{A}$. If we were to insert F into the path integral, since $\langle F\rangle = 0$ when integrating out F, after the shift to eliminate the Lagrange multiplier term we'd be left with a path integral containing just an insertion of $\star d\widetilde{A}$; hence why $\mathcal{D}$ is basically Hodge duality.

On the components of the field strength, the duality is

$$\mathcal{D} : E \mapsto B, B \mapsto -E. \tag{1529}$$

The minus sign in the second map is a Lorentzian minus sign coming from lowering a time index on F, and ensures that $\mathcal{D}^2 = -\mathbf{1}$ (this is just because $\star^2 = (-1)^{1+p(D-p)}$ on p-forms in D-dimensional Minkowski space. In Euclidean signature this minus sign is still picked up since the proportionality constant between $\mathcal{D}(F)$ and $\star F$ is imaginary).

Now we can draw the following commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{D}} & B \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle \mathcal{T}} \\ E & \xrightarrow{\mathcal{D}} & B \end{array} . \tag{1530}$$

Thus we conclude that $\mathcal{T}$ must act trivially on B, from which we can make the identification $\mathcal{T} = CT$. Similarly,

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{D}} & B \\ \big\downarrow{\scriptstyle CT} & & \big\downarrow{\scriptstyle \mathcal{CT}} \\ -E & \xrightarrow{\mathcal{D}} & -B \end{array} , \tag{1531}$$

so that $\mathcal{CT} = T$ since B is odd under $\mathcal{CT}$. Thus duality exchanges T and CT. Another way to see this is to couple the theory to a background field for the $U(1)$ 1-form symmetry, and to use the transformation properties of this background field under T and CT to fix the identification of the symmetries on both sides. If H is the 2-form background field, we have (this was derived in a previous diary entry)

$$i\int ||F_A - H||^2 \leftrightarrow i\int (||F_{\widetilde{A}}||^2 + F_{\widetilde{A}} \wedge H), \tag{1532}$$

where we have omitted real constants (and are in $\mathbb{R}$ time). On the LHS, we see that H transforms in the same way under T, CT as F, and hence as A, does. On the RHS, because of the i, we see that H transforms in the opposite way as $F_{\widetilde{A}}$. Thus since the field strengths are odd under T and even under CT, duality exchanges C and CT. Using this type of $iF_{\widetilde{A}} \wedge H$ term (think of this as a field theory version of $ik \cdot x$) to show that $T \leftrightarrow CT$ is a typical strategy.

3d particle-vortex duality

We can also look at a slightly more complicated example, that of the duality between the 2+1 dimensional WF scalar and another 2+1 WF scalar coupled to a gauge field (and deformed away from their critical points by mass terms, with the mass terms defined to vanish at the critical point). The duality is (adding a background field A to both theories)

$$|D_A\phi|^2 + r|\phi|^2 + u|\phi|^4 \leftrightarrow |D_a\widetilde{\phi}|^2 + \widetilde{r}|\widetilde{\phi}^2| + \widetilde{u}|\widetilde{\phi}|^4 + \frac{i}{2\pi} a \wedge F_A. \tag{1533}$$

Since we are choosing $T(A) = -A$, we need $T(\phi) = \phi$. Then from the T-invariance of the CS term on the RHS (this is not a functional δ function since A is not integrated over and a is not a Lagrange multiplier, so the sign is meaningful), we see that $T(a) = a$, i.e. that T acts as CT on the dual gauge field. From the invariance of $|D_a\widetilde{\phi}|^2$, we get $T(\widetilde{\phi}) = \widetilde{\phi}^\dagger$, in keeping with T acting as CT on the dual fields.

T, CT, their anomalies, and bosonization in 1+1D

Today's diary entry is similar to yesterday's, and concerns the realization of the T and CT symmetries when doing bosonization of fermions coupled to a $U(1)$ gauge field in 1+1D. We will also show that when the fermions are Kramers doublets T and CT are anomalous, something that I hadn't seen in the literature before.

Since we have fermions and spacetime symmetries involved, we can already anticipate a panoply of different choices for how the symmetries act. To identify how the symmetries act, it is helpful to add a classical background field to both sides, which lets us set the standards for T, CT on both sides of the duality.

First we have to get the conventions straight for where the currents go under duality. We will work in $\mathbb{R}$ time, with conventions where $\mathcal{D}[\psi_\pm] = e^{-i\phi_\pm}$. These are the conventions in which the Dirac mass maps to $\cos(\theta)$, where $\theta = (\phi_+ - \phi_-)/2$ is the dual field to $\phi = (\phi_+ + \phi_-)/2$. In these conventions, the vector current for the fermions maps to the vector current for the bosons (*not* the topological current for the bosons), so that in the usual notation

$$\mathcal{D}[J_\pm] = \frac{1}{\pi}\partial_\pm\phi. \tag{1534}$$

Note that to write this we technically have had to use the equations of motion for the θ and ϕ fields (this is exact though since they appear quadratically in the action). More on this point and the (crucial!) factor of $1/\pi$ can be found in the diary entry on bosonizatoin conventions. Anyway, using this we find

$$\mathcal{D}\left[\int J \wedge \star A\right] = \mathcal{D}\left[\int (J_+ A_- + J_- A_+)\right] = \frac{1}{\pi}\int d\phi \wedge \star A = \frac{1}{\pi}\int \theta\, F_A, \tag{1535}$$

where we used $d\phi \overset{\text{eom}}{=} \star d\theta$, which again is legit since θ just appears quadratically in the action (the justification of the integration by parts and the well-definedness of these terms is in the diary entry on DB cohomology).

Summarizing, the duality map $\mathcal{D}$ thus does (for why the radius is $R = \sqrt{2}$, see the diary entry on bosonization conventions)

$$\mathcal{D} : \frac{1}{2\pi}\bar{\psi} i \not{D}_A \psi \leftrightarrow \frac{1}{2\pi} d\phi \wedge \star d\phi + \frac{1}{\pi}\theta F_A. \tag{1536}$$

If we had used the convention where $\mathcal{D}[\psi_\pm] = e^{\mp i\phi_\pm}$ we would have obtained the T-dual image of this action.

As a sanity check, the coefficient of $1/\pi$ is correct since a chiral transformation $\psi \mapsto e^{i\bar{\gamma}\alpha}\psi$ does $\theta \mapsto \theta + \alpha$. This reproduces the shift in the action of $(\alpha/\pi)\int F_A$ that comes from the chiral anomaly on the fermion side — it is α/π and not $\alpha/2\pi$ since integrating this term gives the divergence of the chiral current, and since the $\mathbb{Z}_2$ subgroup of $U(1)_A$ is preserved (as it is also part of $U(1)_V$), the integral of the divergence must be in $2\mathbb{Z}$.

Now for the symmetries. For posterity's sake, we record the easily proved facts that for any real differential form B,

$$T[B] = \zeta_B B \implies T[dB] = \zeta_B dB, \quad T[\star B] = -\zeta_B dB, \qquad \zeta_B = \pm 1, \tag{1537}$$

and likewise for T replaced by a reflection R. Here the notation again keeps the spacetime arguments implicit, and also tacitly includes a change of variables so that the spacetime arguments of the fields are unchanged at the end. For example, on a 1-form A, we have

$$T[A] = -\zeta_A A_t(-t,x)\, dt + \zeta_A A_i(-t,x)\, dx^i \to \zeta_A A, \tag{1538}$$

where the $\to$ means doing the change of variables $t \to -t$ (all differential forms will always appear under an integral sign so this can always be done). Thus in components, $A \mapsto \zeta_A A$ means that $A_t \mapsto -\zeta_A A_t$ while $A_i \mapsto \zeta_A A_i$. Because of this tacit change of variables, when working out the transformation properties of integrals, we can take $\int$ to be odd under any orientation-reversing spacetime symmetry, since the change of variables means the domain of integration gets its orientation reversed. Therefore we have e.g.

$$T : \int_X A \wedge B = -\zeta_A \zeta_B \int A \wedge B. \tag{1539}$$

We will fix conventions where C, R, T act on a $U(1)$ gauge field and the various components of its field strength as (from the above, we see that the differential form

dA transforms in the same way as A does)

$$\begin{aligned} T &: A \mapsto -A, \quad E^i \mapsto E^i \quad B^i \mapsto -B^i \\ C &: A \mapsto A, \quad E^i \mapsto -E^i, \quad B^i \mapsto -B^i \\ R &: A \mapsto A, \quad E^i \mapsto -E^i, \quad B^i \mapsto B^i \end{aligned} \tag{1540}$$

These are the old-school conventions and ideally T and CT would be swapped — we'll stick with these though since they're mostly what's used in the literature.

When coupling to the fermions, we need the $\int J_\psi \wedge \star A$ term to be T-invariant. Since $T(A) = -A$, we need (again as differential forms, so that $J_\psi = \bar{\psi}\gamma_\mu \psi dx^\mu$)

$$T(\star J_\psi) = \star J_\psi \implies T(J_\psi) = -J_\psi, \tag{1541}$$

so that the transformation of J_ψ correctly matches that of A. This transformation rule makes sense because T always takes $\psi^\dagger_\pm \psi_\pm \mapsto \psi^\dagger_\mp \psi_\mp$, and hence sends $J^0_\psi \mapsto J^0_\psi, J^1_\psi \mapsto -J^1_\psi$. Charge conjugation C sends $A \mapsto -A$, so that $CT(A) = A$, and likewise $CT(J_\psi) = J_\psi$. In $\mathbb{R}$ time we have $\mathcal{J}_\psi = \star J_\psi$ where $\mathcal{J}$ is the axial current, and so $T(\mathcal{J}) = \mathcal{J}, CT(\mathcal{J}) = -\mathcal{J}$.

Now we have to fix conventions for how the fermions transform under C, P, T. We will fix the signature to be $(+,-)$, with γ matrices $\gamma^0 = X, \gamma^1 = -iY = J$.[177] For time reversal, both choices $T = Y\mathcal{K}$ or $T = X\mathcal{K}$ are consistent: the former gives $T^2 = (-1)^F$ while the later gives $T^2 = \mathbf{1}$.[178] We will denote these two choices by T_- and T_+, respectively. We will take charge conjugation to be performed by $C : \psi \mapsto \bar{\psi}C^\dagger, \bar{\psi} \mapsto -C\psi$, which is a symmetry of the action if $C = Y$.[179] Componentwise, this is

$$C : \psi_\pm \mapsto \pm i\psi^\dagger_\pm. \tag{1542}$$

In our conventions P is the operation that appears in CPT (one could argue that this is philosophically not a very good approach to take, but my 2019 self thought otherwise), and hence our choices of C and T force us to choose $P = X$ if we work with T_+, and $P = J$ if we work with T_- (if we had chosen $(-,+)$ signature, the two actions of P would be reversed). More on the method to this madness is explained on the diary entry on fermions and spacetime symmetries. Working out the action on the field components, we see that

$$(CT_+)^2 = (-1)^F, \qquad (CT_-)^2 = +\mathbf{1}, \tag{1543}$$

so that the square of $CT_\pm$ is opposite to the square of $T_\pm$.

[177] If we change the signature, we will be able to change whether $P^2 = \mathbf{1}$ or $-\mathbf{1}$ on fermions: since we are mostly focused on time reversal, we won't worry about exploring all these options, and will just choose $(+,-)$ signature, in which $P^2 = \pm\mathbf{1}$ is determined from the choice of $T^2 = \pm\mathbf{1}$ and CPT invariance (this just means that we are working in conventions where we take P to be the thing appearing in CPT — this is of course not however the only option for P; other choices are consistent but less canonical).

[178] This is because for $T = U_T\mathcal{K}$, we need $U_T^\dagger Z U_T = -Z$.

[179] Note that the free action would also be preserved by $C : \psi_\pm \mapsto \psi^\dagger_\pm$. However, in this case, the Dirac mass $m_D\bar{\psi}\psi$ would be C-odd, which we don't want.

T_+: Consider first the case when fermions are Kramers singlets. Then applying T_+ to $\mathcal{D}[\psi_\pm] = e^{-i\phi_\pm}$, we get

$$\begin{tikzcd} \psi_\pm \arrow[r, "\mathcal{D}"] \arrow[d, "T_+"] & e^{-i\phi_\pm} \arrow[d, "\mathcal{T}_+"] \\ \psi_\mp \arrow[r, "\mathcal{D}"] & e^{-i\phi_\mp} \end{tikzcd} . \tag{1544}$$

Thus we have[180]

$$\mathcal{T}_+ : \phi_\pm \mapsto -\phi_\mp, \qquad \phi \mapsto -\phi, \quad \theta \mapsto \theta. \tag{1545}$$

Charge conjugation evidently acts on the boson side as $\mathcal{C} : \phi_\pm \mapsto -\phi_\pm \mp \pi/2$. However, one must be careful in deriving its action on θ and ϕ. θ and ϕ are fermion-parity even, which leaves room for a minus sign to enter in the action of charge conjugation which doesn't appear when C acts on single fermion operators. So for example,

$$C : \psi_+^\dagger \psi_- \mapsto -\psi_+ \psi_-^\dagger = +\psi_-^\dagger \psi_+ \implies \mathcal{C} : e^{i\theta} \mapsto e^{-i\theta}. \tag{1546}$$

If we had naively applied $\mathcal{C}$ to $\phi_+ - \phi_-$, we would have obtained $\mathcal{C} : \theta \mapsto -\theta - \pi$ instead. The difference between this transformation and the correct result comes down to the fact that when we bosonize, we only apply the mosonization map $\mathcal{D}$ to things that are normal-ordered, and so we have to keep track of possible normal-ordering signs that appear. Similarly, considering the action of C on $\psi_+^\dagger \psi_-^\dagger$, one finds the action of $\mathcal{C}$ on ϕ, so that

$$\mathcal{C} : \phi \mapsto -\phi, \quad \theta \mapsto -\theta. \tag{1547}$$

This means that the Dirac mass $\bar\psi\psi \to 2\cos\theta$ is $\mathcal{C}$-even while the chiral mass $i\bar\psi Z\psi \to 2\sin\theta$ is $\mathcal{C}$-odd, as expected for the present conventions.

Putting these together,

$$\mathcal{C}\mathcal{T}_+ : \phi_\pm \mapsto \phi_\mp \mp \pi/2, \qquad \phi \mapsto \phi, \quad \theta \mapsto -\theta. \tag{1548}$$

A sanity check is that $(\mathcal{C}\mathcal{T}_+)^2 = -\mathbf{1}$ when acting on $\phi_\pm$ (we will always have $(\mathcal{C}\mathcal{T}_\pm)^2 = \mathbf{1}$ when acting on ϕ or θ since these variables are fermion-parity even, so we just need to check the action of $(\mathcal{C}\mathcal{T}_\pm)^2$ on $\phi_\pm$). Indeed, keeping in mind that in this representation the complex conjugation in $\mathcal{T}_+$ sends scalars to minus themselves,

$$(\mathcal{C}\mathcal{T}_+)^2 : \phi_\pm \to -\phi_\mp \to -(-\phi_\mp \pm \pi/2) \to -\phi_\pm \pm \pi/2 \to \phi_\pm \pm \pi. \tag{1549}$$

Finally, with this choice of T_+, we have $P_+ = X$ on the fermions; thus

$$\mathcal{P}_+ : \phi_\pm \mapsto \phi_\mp, \qquad \phi \mapsto \phi, \quad \theta \mapsto -\theta. \tag{1550}$$

T_-: Now consider the case when the fermions are Kramers doublets. We take $T_- = Y\mathcal{K}$, so that $T : \psi_\pm \mapsto \pm i\psi_\mp$. Then we have

$$\begin{tikzcd} \psi_\pm \arrow[r, "\mathcal{D}"] \arrow[d, "T_-"] & e^{-i\phi_\pm} \arrow[d, "\mathcal{T}_-"] \\ e^{\pm i\pi/2}\psi_\mp \arrow[r, "\mathcal{D}"] & e^{-i\phi_\mp \pm i\pi/2} \end{tikzcd} . \tag{1551}$$

[180] When we look at the action of symmetries on ϕ, θ, which are related to fermion bilinears, we need to remember Klein factors, which are acted on by the γ matrices. For example, $\mathcal{D}[\psi_+^\dagger \psi_-] = \kappa_+\kappa_- e^{i\theta}$, $\mathcal{D}[\psi_-^\dagger \psi_+] = \kappa_-\kappa_+ e^{-i\theta}$, and so $\mathcal{T}_+ : \theta \mapsto \theta$ only makes sense if $\mathcal{T}_+ : \kappa_\pm \to \kappa_\mp$.

This identifies the bosonized action of T_- as

$$\mathcal{T}_- : \phi_\pm \mapsto -\phi_\mp \mp \pi/2, \quad \phi \mapsto -\phi, \quad \theta \mapsto \theta - \pi. \tag{1552}$$

Consequently,

$$\mathcal{CT}_- : \phi_\pm \mapsto \phi_\mp \pm \pi, \qquad \phi \mapsto \phi, \quad \theta \mapsto -\theta + \pi. \tag{1553}$$

Lastly, we have parity, which acts as $P_- = J$ on the fermions. Thus

$$\mathcal{P}_- : \phi_\pm \mapsto \phi_\mp + (1 \mp 1)\pi/2, \qquad \phi \mapsto \phi + \pi, \quad \theta \mapsto -\theta - \pi. \tag{1554}$$

We can summarize what we've derived so far in the following table:

	C	P_+	T_+	CT_+	P_-	T_-	CT_-
ϕ	$-$	$+$	$-$	$+$	$\phi+\pi$	$-$	$+$
θ	$-$	$-$	$+$	$-$	$-\theta-\pi$	$\theta-\pi$	$-\theta+\pi$
m_D	$+$	$+$	$+$	$+$	$-$	$-$	$-$
m_5	$-$	$-$	$+$	$-$	$+$	$-$	$+$
$\int \theta F_A$	$+$	$+$	$+$	$+$	$\frac{1}{2}\int F_A$	$-\frac{1}{2}\int F_A$	$-\frac{1}{2}\int F_A$

(1555)

Here the $\pm$ signs mean the fields get multiplied by ± 1, while the other entries indicate the amount to which each term shifts (e.g. ϕ shifts by π under $P-$). Note that $CP_\pm T_\pm$ acts trivially on θ and $\int \theta F_A$, as required.[181]

We see from the table that in the Kramers doublet case, all of the symmetries P_-, T_-, and CT_- are anomalous. Indeed, we can see this from the fact that neither of the fermion mass terms (which bosonize to $\cos\theta, \sin\theta$) are P_-, T_-, or CT_- invariant! Therefore we cannot do PV regularization without breaking these symmetries; hence the anomalies. On the other hand, the Kramers singlet case is non-anomalous (with the Dirac mass being a symmetry-preserving mass to give the PV regulator).

1-form anomalies in CS theory

In today's diary entry we will examine the anomalous nature of the 1-form symmetries present in various CS theories. I looked at [15] when writing this, and I think in the time since this was written there were a few more papers on the subject. I have not gotten around to comparing the calculations below to the existing literature (if it exists); any differences should assumed to be mistakes on my part.

[181] CP_-T_- doesn't act trivially on ϕ but that's okay since $e^{i\phi} \sim \psi_+^\dagger \psi_-^\dagger$ won't appear in by itself in a Lorentz-invariant theory (it will only appear as $\partial\phi$ and $\bar\partial\phi$). (note to self — come back to this!)

$U(1)_k$

Let's start with the simplest example, viz. $U(1)_k$. The 1-form symmetry acts as

$$\mathbb{Z}_k^{(1)} : A \mapsto A + \lambda, \qquad k\lambda \in 2\pi H^1(X;\mathbb{Z}). \tag{1556}$$

The charge operators are of course the Wilson lines. We can see that this is a symmetry by e.g. computing the spectrum of operators in the theory, but for posterity's sake let's see how it works from the action. Since λ is flat, a naive approach tells us that $\delta S = \frac{1}{4\pi}\int (k\lambda)\wedge F_A$ under the symmetry, which is only in $\frac{1}{2}\overline{\mathbb{Z}}$ (we are using the notation $\overline{\mathbb{Z}} \equiv 2\pi\mathbb{Z}$). Note that we cannot integrate this by parts to get zero by the flatness of λ, due to A not being strictly a well-defined form[182]. As usual, the confusion can be ameliorated by writing things in terms of the field strengths by using a bounding 4-manifold M. Then

$$\delta S = \frac{k}{2\pi}\int_M F_\lambda \wedge F_A + \frac{k}{4\pi}\int_M F_\lambda \wedge F_\lambda, \tag{1557}$$

where F_λ is the field strength of the extension of λ into the bulk 4-manifold M. Note that since the holonomy of λ may be nontrivial, although it is flat on the boundary, it will not be flat in M, and a priori, it will not be globally an exterior derivative, i.e. we may not have $F_\lambda = d\lambda$ globally on M.

Now we need to integrate by parts: we will get only boundary terms, since $dF_A = d(d\lambda) = 0$. However, doing so is slightly subtle, since λ might not be a globally well-defined form. Thus we cannot write e.g. $\int d\lambda\wedge B = \int \lambda\wedge dB + \int_{\partial M}\lambda\wedge B$ for a 2-form B (the sign is correct because of the supercommutativity of d). However, since λ is flat, we know that λ is a well-defined form on ∂M. Thus in the bulk, we may write

$$\lambda = \Lambda + B, \qquad F_B \in 2\pi H^2(M,\partial M;\mathbb{Z}), \tag{1558}$$

where Λ is a $U(1)$ gauge field which is globally well-defined so that $[F_\Lambda] = [d\Lambda] = 0$ in $2\pi H^2(M;\mathbb{Z})$, and B is a non-globally-well-defined part which vanishes on ∂M since $\lambda|_{\partial M}$ is globally well-defined (thus $\lambda|_{\partial M} = \Lambda|_{\partial M}$). Thus we can write

$$\delta S = \frac{k}{2\pi}\int_M [(d\Lambda + F_B)\wedge F_A + d\Lambda\wedge F_B] + \frac{k}{4\pi}\int_M (d\Lambda\wedge d\Lambda + F_B\wedge F_B). \tag{1559}$$

Assuming we choose M to be spin if k is odd, the last term vanishes modulo $\overline{\mathbb{Z}}$. Since Λ is globally well-defined, the $d\Lambda\wedge F_B$ term vanishes on account of the flatness of F_B and the fact that $F_B|_{\partial M} = 0$. Likewise the $F_B\wedge F_A$ part vanishes mod $\overline{\mathbb{Z}}$: we can see this by decomposing A in the same way that we decomposed Λ, and using that

[182] A similarly hasty use of integration by parts on the CS action leads to confusion in the usual way of showing that $k\in\mathbb{Z}$ in the CS action, namely by e.g. placing the theory on $S^1\times S^2$ with $\int F = 2\pi$ around the S^2. In the usual story one integrates by parts to get $S = \frac{k}{2\pi}\oint A_t \int_{S^2} F_{xy} = k\oint A_t$ (there is a factor of 2 here from the IBP), which says that $k\in\mathbb{Z}$ for invariance under large gauge transformations around the S^1. But what if we first did the large gauge transformation, and then did the integration by parts? Since the field strength of the large gauge transformation vanishes, the IBP fails to pick up a factor of 2, and we conclude that the change in the action is instead $(k/4\pi)\oint\lambda\int_{S^2}F_{xy}$ for $\oint\lambda = 2\pi$, which seems to imply that $k\in 2\mathbb{Z}$ is required. So, it is best to only integrate by parts when we really know that it is legit.

$\frac{1}{2\pi}\int F_C \wedge F_B \in \overline{\mathbb{Z}}$ for $F_C \in 2\pi H^2(M,\partial M;\mathbb{Z})$. So finally, we integrate the remaining two terms by parts and get

$$\delta S = \frac{1}{2\pi}\int (k\lambda)\wedge F_A, \tag{1560}$$

since $\Lambda|_{\partial M} = \lambda|_{\partial M}$ and since $d\lambda|_{\partial M}$ is flat. But since $k\lambda$ has periods in $\overline{\mathbb{Z}}$, we see that $\delta S \in \overline{\mathbb{Z}}$, and so indeed, the 1-form transformation $\delta A = \lambda$ *is* a symmetry of the action.

⋄ ⋄ ⋄

To gauge this symmetry, we want the "split" symmetry operators $U(g;C)$ (not only the full charge operators) to act trivially on the Hilbert space, where the split symmetry operators are defined on *open* submanifolds $C : \partial C \neq 0$ and implement a transformation by the group element g (here $g \in \mathbb{Z}_k$). Requiring the global (unsplit) charge operators to act trivially is equivalent to projecting onto the singlet sector of the Hilbert space, which can be done by inserting the operator

$$\Pi_{\mathbf{1}} = \sum_{C\in H_1(X;\mathbb{Z}_k)} e^{i\int_C A} \tag{1561}$$

into the path integral. This is orbifolding. This is not quite how we want to think of gauging, since we haven't made the symmetry local in any way, we've just projected onto a particular (trivial) value of the charge (when we gauge we want to work with genuinely gauge-invariant states, not ones that are in some particular gauge fixing). A hint of the anomaly can be seen by noticing that the split symmetry operator $U(q,C)$, which naively includes $e^{iq\int_C A}$ since the current for the 1-form symmetry is $j = \star A$, is not gauge invariant (under both the 0-form and 1-form gauge transformations) if $\partial C \neq 0$. This is a problem since after including the 2-form gauge field we expect $U(q,C)$ to be modified by the canonical momentum for the 2-form gauge field (think of $\nabla\cdot\mathbf{E}$), which we don't expect to transform in a way that would fix this issue, and hence the gauge variance of $U(q,C)$ will continue to be problematic.

Let's now see how this plays out. We will let B be the $\mathbb{Z}_k$ 2-form gauge field. To enforce the quantization of B's periods, we will as usual add to the action the BF term (we are in $\mathbb{R}$ time, so no factors of i are included)

$$S \supset \frac{k}{2\pi}\int B\wedge d\phi, \tag{1562}$$

where ϕ is a 2π-periodic scalar. Now we can fix the variance under 0-form gauge transformations of $e^{i\int_C \star j} = e^{i\int_C A}$ when $\partial C \neq 0$ by writing the operator $U(q,C)$ which implements the gauge transformation as

$$U(q,C) = e^{iq\int_C (A+d\phi)}, \tag{1563}$$

provided that under $A \mapsto A + d\gamma$ we have $\phi \mapsto \phi - \gamma$ (this preserves the 2π-periodicity of ϕ, since γ is itself a 2π-periodic scalar). This makes sense, since ϕ is the canonical momentum for B, and so this is exactly what we normally do when gauging the symmetry operators: the operators which perform the gauge transformations are the original charge operators defined on open submanifolds, with the canonical momentum for the gauge field integrated along their boundaries (again as an example, the generator of gauge tforms in QED is the integral of the matter current over an open volume, together with the integral of $\star F$, the canonical momentum for the gauge field, over the boundary of the volume).

Now by design, if $D \in C_1(X;\mathbb{Z})$ is such that $C \cap D \neq 0$, then $W(D) = e^{i\int_D A}$ is not gauge invariant under the $\mathbb{Z}_k^{(1)}$ gauge transformations, since it does not commute with $U(q, C)$. Note that no matter what D is, we can always find a C such that $W(D)$ is not invariant under $U(q, C)$: this is true even when $[D] = 0$ in $H_1(X;\mathbb{Z})$, in which case $W(D)$ is actually neutral under the original 1-form global symmetry.

We can make $W(D)$ gauge invariant by attaching a surface operator built out of B to it: if $[D] = 0$ in $H_1(X, \partial X;\mathbb{Z})$ (so that D either bounds a disk, is a linear combination of nontrivial classes in $H_1(X;\mathbb{Z})$ with total “charge” zero so that it bounds some other surface, or together with a submanifold of the boundary of spacetime bounds a surface) we can find some M such that $\partial M \setminus (\partial M \cap \partial X) = D$ (here X is spacetime, and gauge transformations always vanish at ∂X). The operator

$$\widetilde{W}(M) = \exp\left(i\int_D A + i\int_M B\right) \tag{1564}$$

is then gauge-invariant. Why? Because when we compute its commutation relation with $U(q, C)$ (with e.g. $C \cap D = 1$), we get one factor of $e^{2\pi iq/k}$ from the $[A, A] \sim i/k$ commutation relation, and another from the $[\phi, B] \sim i/k$ commutation relation, which occurs from the contact term between the ϕ inserted at the end of C and the B integrated over M. If $[D] \neq 0$ in $H_1(X;\mathbb{Z})$ then $W(D)$ can't be made gauge-invariant, and its vev vanishes (although this was true before gauging, since $\langle W(D)\rangle$ can then be shifted by a change in integration variables which doesn't affect the boundary conditions on A).

As we hinted at above, the anomaly is seen very simply from the fact that the operators $U(q, C)$ which perform the $\mathbb{Z}_k^{(1)}$ gauge transformations are not themselves invariant under the same $\mathbb{Z}_k^{(1)}$ transformations (although as we have seen they are at least invariant under the 0-form $U(1)$ gauge transformations on A). That is, they don't commute with themselves (because of $[A, A] \sim i/k$). Since $\partial C \neq 0$, it is impossible to attach a B surface to render $U(q, C)$ gauge invariant. Thus the $\mathbb{Z}_k^{(1)}$ symmetry can't actually be gauged.

We can also see this from the action. Basically, while F_A can be made gauge invariant by $F_A \mapsto F_A - B$, the CS term cannot be made gauge invariant since it involves more than just F_A. Indeed, let us write the variation of A under the 1-form gauge transformation as

$$\delta A = \lambda, \qquad \lambda \in \frac{1}{k} Z^1_O(X;\mathbb{Z}). \tag{1565}$$

Here we have defined $Z^1_O(X;\mathbb{Z})$ as the set of 1-forms such that their spatial Poincare duals are *open* codimension-1 submanifolds of space, which have integral intersection number with every element in $C_1(X;\mathbb{Z})$ that intersects them transversely. Thus the elements in $Z^*_O(X;\mathbb{Z})$ are not closed, but they are not closed in a very specific way (we are not letting λ be an arbitrary 1-form since we are gauging a $\mathbb{Z}_k$ 1-form symmetry, and not a $U(1)$ one). Another way to say this is that

$$\lambda \in \frac{1}{k} Z^1_O(X;\mathbb{Z}) \implies \int_C \lambda \in \frac{1}{k}\mathbb{Z} \quad \forall C \in C_1(X;\mathbb{Z}), \tag{1566}$$

where the value for the integral will generically depend on the exact choice of C, and not just its homotopy equivalence class. Connecting this with our earlier notation vis-a-vis the $U(q,C)$ gauge transformation operators, we would say that $U(q,C)$ shifts A by $\lambda = \frac{q}{k}\widehat{C}$, where $\widehat{C}$ is the spatial Poincare dual of $C \in C_1(X;\mathbb{Z})$.

Anyway, the CS term varies as

$$\delta \int A \wedge dA = 2\int A \wedge d\lambda + \int \lambda \wedge d\lambda. \tag{1567}$$

To cancel at least the first term we can try to introduce a $\mathbb{Z}_k$ gauge field B to the action and add

$$S \supset -\frac{1}{2\pi}\int A \wedge B, \tag{1568}$$

with $\delta B = d\lambda$ under the gauge transformation. However this a) cannot cancel the term in δS quadratic in λ and b) produces an extra piece linear in B. So, after adding this coupling, the total variation of S is

$$\delta S = \frac{k}{4\pi}\delta \int (A\wedge dA - 2A\wedge B) = \frac{k}{4\pi}\int (2\lambda \wedge B + \lambda \wedge d\lambda). \tag{1569}$$

Of course, we can make the action gauge invariant by letting B live in four dimensions, at the price of picking up an explicit dependence on a bounding 4-manifold M. This is just because F_A can always be made gauge-invariant, and we can write the terms in our modified action involving A as

$$S \supset \frac{k}{4\pi}\int_M (F_A - B)\wedge (F_A - B), \tag{1570}$$

which is manifestly gauge invariant (and still only depends on $A|_{\partial M}$). However, and this is where the anomaly comes in, it depends on the choice of M, since

$$\frac{k}{4\pi}\int_{N_4 | \partial N_4 = \emptyset} (F_A - B)\wedge (F_A - B) \in \frac{1}{k}\overline{\mathbb{Z}}, \tag{1571}$$

which is not valued in $\overline{\mathbb{Z}}$ except in the trivial case $k=1$ where there is no symmetry to begin with (here we have used the fact that the periods of B are valued in $k^{-1}\overline{\mathbb{Z}}$ — the periods of F_A are still in $\overline{\mathbb{Z}}$ though, since the 1-form gauge transformations only change A by forms which are globally well-defined up to elements in $2\pi H^2(N_4;\mathbb{Z})$). Since k copies of this bulk action integrate to something in $\overline{\mathbb{Z}}$ over all closed 4-manifolds, we have a $\mathbb{Z}_k$ anomaly.

To write the full gauged action for the four-dimensional B, we just need to include the term which makes B into a $\mathbb{Z}_N$ gauge field. Since B lives in four dimensions, the appropriate BF term is $(k/2\pi)\int_M B \wedge F_\Phi$, where Φ is a 1-form $U(1)$ gauge field.

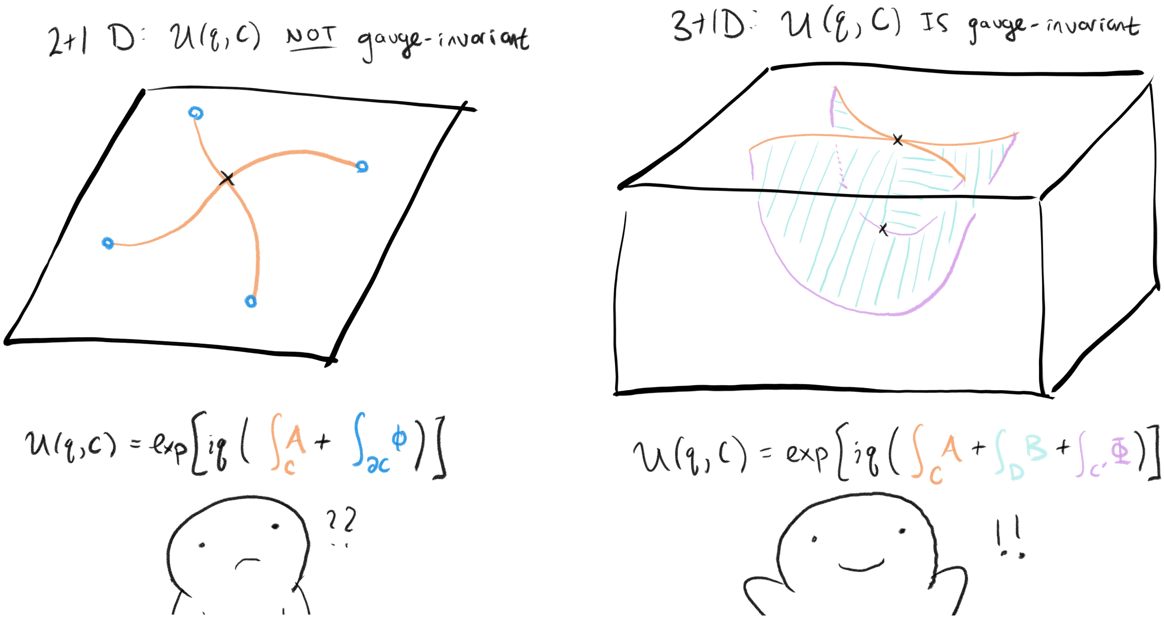


Figure 9: The generators of gauge transformations for the 1-form gauge symmetry in gauged $U(1)_k$. Contact terms that contribute to the commutator of the charge operators are marked with black x's. In 2+1D the $U(q,C)$ are charged and the symmetry can't be gauged, while in 3+1D, with B surfaces extending into the bulk, it can be.

But this term changes as $(k/2\pi)\int_M d\lambda \wedge F_\Phi = (k/2\pi)\int_{\partial M} \lambda \wedge F_\Phi$ under the 1-form gauge transformation , which is problematic. The way to get around this is to include a $-(k/2\pi)\int_{\partial M} B \wedge \Phi$ boundary term in the action. Together with the boundary term, the full part of the action involving Φ is $-(k/2\pi)\int_M F_B \wedge \Phi$, which is manifestly invariant under the 1-form gauge transformation . Recapitulating, the full action is

$$S = \frac{k}{4\pi}\int_{\partial M}(A \wedge F_A - 2A \wedge B - 2B \wedge \Phi) + \frac{k}{4\pi}\int_M (B \wedge B + 2B \wedge F_\Phi). \tag{1572}$$

The full generator of gauge transformations is now

$$U(q,C) = \exp\left(iq\left[\int_C A + \int_D B + \int_{C'} \Phi\right]\right), \tag{1573}$$

where D is a disk with $\partial D = C \cup C'$, $\partial C = \partial C'$, and where C' is entirely contained within the four-dimensional bulk. The $U(q,C)$'s commute with one another: the contact term between the A's on the surface is canceled between a contact term where the $\int_{C'} \Phi$ line intersects the $\int_D B$ surface. This is illustrated in Figure 9. On the left we show the $U(q,C)$ operators in the strictly 2+1D theory, which are not gauge invariant. On the right we show how, after attaching B surfaces and Φ lines to them, they become gauge invariant.

Twisted $\mathbb{Z}_N$ gauge theory

We now look at the DW theory which we will call $DW_{p,q}$, namely

$$\mathcal{L} = \frac{p}{4\pi} a \wedge da + \frac{q}{2\pi} a \wedge db. \tag{1574}$$

What are the global symmetries? First, there is clearly a $\mathbb{Z}_N^{(1)}$ symmetry that shifts b by $1/N$ times a large gauge transformation. Similarly, there is also a $l \equiv \gcd(p,q)$ symmetry shifting a: this is the best we can do, as e.g. the coupling between a and b means we don't have the full $\mathbb{Z}_p^{(1)}$ symmetry of the first term, unless p divides q, and we don't have the full $\mathbb{Z}_q^{(1)}$ symmetry from shifts on a in the second term, unless q divides p.

<diversion>

Let's pause for a momentum to discuss the spin and statistics of the lines in this theory. A naive reading on this would be as follows: the canonical momentum for a is $pa + qb$ and the canonical momentum for b is qa. Thus b lines commute with each other, while a lines have a self-linking phase determined by $1/p$. The mutual statistics of a and b is nonzero because they do not commute with each other, and is determined by $1/q$.

This naive reading is incorrect: even if the canonical momentum for a field ϕ as read off from $\partial\mathcal{L}/(\partial\partial\phi)$ does not involve ϕ itself, ϕ may still have nontrivial statistical interactions with itself. Indeed, the correct way to determine the commutation relations between Wilson lines is by using the inverse of the K matrix. Let's quickly remind ourselves of why: for $i \in \mathbb{Z}_{\dim K}$ and letting $\star q^\alpha \cdot J^\alpha = \star q_i^\alpha \cdot J_i^\alpha$ be the 2-form Poincare dual to a support of a particular configuration of Wilson lines $\prod_\alpha W_\alpha = \prod_\alpha e^{i\sum_j \oint_{C_\alpha} A_j}$, we have

$$\langle \prod_\alpha W_\alpha \rangle = \frac{1}{Z[J=0]} \int \prod_i \mathcal{D}A_i \exp\left(\frac{i}{4\pi} \int A_i [K]_{ij} \wedge dA_j + i \sum_{\alpha,i} q_i^\alpha \int A_i \wedge \star J_i^\alpha \right). \tag{1575}$$

Shifting A to kill off the AJ coupling, we get

$$\langle \prod_\alpha W_\alpha \rangle = \exp\left(2\pi i \sum_{\alpha,\beta} q_i^\alpha q_j^\beta \int \star J_i^\alpha [K^{-1}]_{ij} \wedge d^{-1} \star J_j^\beta \right). \tag{1576}$$

Taking all the Wilson loops to be supported on the boundaries of disks means that the $\star J^\alpha$ are not in $\ker(d)$, and so the above formula makes sense. Anyway, taking two linked loops, one with a unit charge for A_i and another with a unit charge for A_j (and taking the framing of each loop to be trivialized so that the diagonal in α terms in the above formula do not contribute) gives us the braiding matrix

$$[S]_{ij} = \exp\left(2\pi i [K^{-1}]_{ij}\right). \tag{1577}$$

This can also be derived just by looking at $[A_i, \pi_{A_m}] = \sum_j [A_i, \bar{K}_{mj} A_j] = i\delta_{im}$. Here spacetime indices are kept implicitly, with $[A_i, A_j] = A_i \wedge A_j - A_j \wedge A_i$. Also, $\bar{K} = K/2\pi$. Anyway, multiplying by $[\bar{K}^{-1}]_{mk}$ and summing over m:

$$\sum_j [A_i, A_j]\delta_{k,j} = i \sum_m \delta_{im} [\bar{K}^{-1}]_{mk} \implies [A_i, A_j] = i[\bar{K}^{-1}]_{ij}. \tag{1578}$$

Using this commutation relation to unlink any loops that are linked together in $\prod_\alpha W_\alpha$, one recovers the above expression for the S matrix (after choosing a framing).

In the present $DW_{p,q}$ example, the K matrix and its inverse are

$$K = \begin{pmatrix} p & q \\ q & 0 \end{pmatrix}, \qquad K^{-1} = \begin{pmatrix} 0 & 1/q \\ 1/q & -p/q^2 \end{pmatrix}. \tag{1579}$$

Thus even though a appears in the canonical momentum for b, we see that b still fails to commute with itself. So we see that the b line is *not* a boson, despite the fact that its canonical momentum does not involve itself. In fact, it has spin $-p/2q^2$! And similarly, despite the self-CS term for a, we see that a is actually a boson! Physically, what's going on here is that b lines carry flux for a, which by the self-CS term for a have nontrivial braiding with themselves, since this term tells us that a flux also carries a charge. This allows b lines to not commute with themselves. Likewise, a lines carry a flux, which makes them seem like they would not commute with themselves. But a fluxes also carry b charge, and b charge carries a flux, and this all works out in such away that the a lines actually carry net zero a flux.

A particularly transparent example of when this happens is the case when $q = p$. In that case, we can diagonalize the K matrix by something in $SL(2, \mathbb{Z})$ via

$$K \mapsto \Lambda^T K \Lambda = qZ, \qquad \Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1580}$$

This means that in terms of the variables b and $c = a + b$, the Lagrangian is that of $U(1)_q \otimes U(1)_{-q}$. In this formulation, it is clear that b has spin $-1/(2q)$ (from the $U(1)_{-q}$ factor), while a has spin 0 mod 1, since $a + b$ has spin $+1/(2q)$ and

$$e^{2\pi i s(a)} = e^{2\pi i s(c-b)}\sqrt{[S]_{c-b,c-b}} = [S]_{c,-b}\sqrt{[S]_{c,c}[S]_{-b,-b}} = 1\cdot\sqrt{e^{2\pi i/q}e^{-2\pi i/q}} = 1 \implies s(a) =_1 0. \tag{1581}$$

</diversion>

Now let's continue to look at the symmetries of the theory. The $\mathbb{Z}_q^{(1)}$ symmetry which shifts b by $d\phi/q$ is easy to identify: it is generated by the operator

$$\mathbb{Z}_q^{(1)} = \langle e^{i\oint a} \rangle, \tag{1582}$$

which has the right qth root of unity phase linking with the $e^{i\oint b}$ line needed to generate the symmetry. Since $e^{i\oint a}$ has trivial self-linking, this symmetry is not anomalous.

Now for the $\mathbb{Z}_l^{(1)}$ symmetry that shifts a lines by $e^{2\pi i/l}$ (recall $l \equiv \gcd(p, q)$). Since a has trivial self-linking, the operator generating this symmetry should include $\exp(iq/l \oint b)$, since the linking of a and b lines gives a phase $e^{2\pi i/q}$. But this operator also shifts b lines, which is bad since b lines are neutral under the $\mathbb{Z}_l^{(1)}$ symmetry. If we tack on a line $e^{i\beta\oint a}$ to the symmetry generator, imposing that the generator link trivially with b tells us that

$$-\frac{pq}{q^2 l} + \frac{\beta}{q} = 0 \implies \beta = p/l. \tag{1583}$$

This means that the $\mathbb{Z}_l^{(1)}$ symmetry is generated by the line

$$\mathbb{Z}_l^{(1)} = \langle \exp\left(i\frac{q}{l}\oint b + i\frac{p}{l}\oint a\right)\rangle. \tag{1584}$$

What is the anomaly of this symmetry? To find out, we need the self-linking phase of the charge operator. This phase determines the anomaly as

$$\text{Anomaly} = \frac{1}{2}\left(-\frac{p}{q}\left(\frac{q}{l}\right)^2 + 2\frac{1}{q}\frac{qp}{l^2}\right) = \frac{p}{2l^2} \mod 1, \tag{1585}$$

where the first term is the self-linking of b and the second is the a-b mutual phase (the factor of $1/2$ is because we want the spin of the charge operator. On spin manifolds, we should take this mod $1/2$ and not mod 1). This is indeed an anomaly appropriate for a $\mathbb{Z}_l^{(1)}$ symmetry, since it is a $\mathbb{Z}_l$ effect, in that $l(p/l^2) = p/l \in \frac{1}{2}\mathbb{Z}$ indicates that l copies of the charge operator is either trivial, or a transparent fermion. One special case that shows up often is when $p = -rq$ and the theory has two $\mathbb{Z}_q^{(1)}$ symmetries. In this case, the anomaly of the $\mathbb{Z}_q^{(1)}$ symmetry that shifts a is $-r/q$.

Finally, note that there's a mixed anomaly, of a $\mathbb{Z}_l$ character, between the two symmetries. This is just due to the fact that the generators for the $\mathbb{Z}_q^{(1)}$ and $\mathbb{Z}_l^{(1)}$ symmetries don't commute: the phase between them is $e^{2\pi i/l}$ (which is trivial if we take l copies of either generator, as it should be).

This conclusions can be corroborated by just going in and trying to gauge the symmetry directly. The symmetry that shifts b is clearly non-self-anomalous, since b only appears in the action by way of its field strength and we can just make the replacement $F_b \mapsto F_b - B_b$, where B_b is the background field for the $\mathbb{Z}_q^{(1)}$ symmetry. However, since the generator for the symmetry that shifts b carries charge under the $\mathbb{Z}_l^{(1)}$ symmetry , adding the B_b field will break the $\mathbb{Z}_l^{(1)}$ symmetry. Indeed, after adding the B_b field the action shifts by the following term under $a \mapsto a + \frac{1}{l}d\phi$:

$$\delta S = \frac{q/l}{2\pi}\int d\phi \wedge B_b \in \frac{1}{l}\overline{\mathbb{Z}}. \tag{1586}$$

Thus we recover the $\mathbb{Z}_l$ mixed anomaly between the two 1-form symmetries.

Basically because of the self-CS term for a, the $\mathbb{Z}_l^{(1)}$ symmetry shifting a has a self-anomaly. To find the appropriate characterization of the anomaly, we start from the gauge-invariant bulk action (omitting the Lagrange multipliers that make B_a, B_b quantized appropriately for simplicity)

$$\begin{aligned} S &= \frac{p}{4\pi}\int_M (F_a - B_a)\wedge(F_a - B_a) + \frac{q}{2\pi}\int_M (F_a - B_a)\wedge(F_b - B_b) \\ &= S_{\partial M} + S_{bulk}, \end{aligned} \tag{1587}$$

where M is some bounding 4-manifold, and

$$\begin{aligned} S_{\partial M} &= S_{DW_{p,q}} - \frac{1}{2\pi}\int_{\partial M}\left[a\wedge(pB_a + qB_b) + qb\wedge B_a\right], \\ S_{bulk} &= \frac{1}{4\pi}\int_{\partial M}\left[pB_a\wedge B_a + 2qB_a\wedge B_b\right]. \end{aligned} \tag{1588}$$

The second line in the above equation parametrizes the anomaly. If we consider the dependence on the choice of M by integrating S_{bulk} over a closed 4-manifold, we see that the first term is valued in $p\overline{\mathbb{Z}}/2l^2$ on a non-spin manifold, and $p\overline{\mathbb{Z}}/2l^2$ on a spin manifold, while the second term is valued in $\overline{\mathbb{Z}}/l$. The quantization of the second term confirms the $\mathbb{Z}_l$ nature of the mixed anomaly, while the quantization of the first term confirms our result for the anomaly of the $\mathbb{Z}_l^{(1)}$ symmetry.

$U(N)_{k,q}$

Our conventions will be such that $U(N)_{k,q}$ is defined though

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathcal{A}\wedge d\mathcal{A} - \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathcal{A}]\wedge d\mathrm{Tr}[\mathcal{A}]. \tag{1589}$$

The notation is done like this because q is ($1/N$ times) the effective $U(1)$ level, while k is the effective $SU(N)$ level. The reason why the effective $U(1)$ level is qN can be seen by starting with the decomposition

$$U(N)_{k,q} \cong [SU(N)_k \times U(1)_{qN}]/\mathbb{Z}_N, \tag{1590}$$

where the quotient identifies the center of $SU(N)$ with the appropriate Nth roots of unity in $U(1)$. Since the quotient here says that we can freely change transition functions in the $U(1)$ bundle to make the cocycle condition fail by Nth roots of unity so long as we change the transition functions in the $SU(N)$ bundle in the opposite way, the $\mathbb{Z}_N$ quotient is equivalent to gauging the diagonal $\mathbb{Z}_N^{(1)}$ symmetry which acts on both $SU(N)$ and $U(1)$ fields; for the $U(1)$ part to be well-defined its level then needs to be in $N\mathbb{Z}$, as indicated above.

At the level of manipulating actions, we start by decomposing the $U(N)$ field as

$$\mathcal{A} = A + \boldsymbol{\mathcal{A}}\mathbf{1}, \tag{1591}$$

where A is an $SU(N)$ field (whose transition functions may fail by Nth roots of unity), $\boldsymbol{\mathcal{A}}$ is a "$U(1)$ field" with transition functions failing in the inverse way—hence $N\boldsymbol{\mathcal{A}}$ is a properly-quantized $U(1)$ field, and $N\int F_{\boldsymbol{\mathcal{A}}} \in \overline{\mathbb{Z}}$. The quotient comes from the correlation of the transition functions between A and $\boldsymbol{\mathcal{A}}$ (more on this when we talk about $SU(N)_k$ in the next subsection). In terms of these fields, we have

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right] + \frac{qN}{4\pi}\boldsymbol{\mathcal{A}}\wedge d\boldsymbol{\mathcal{A}}, \tag{1592}$$

so that qN is indeed the "effective $U(1)$ level" as claimed above. To get this we've used that A is traceless and that

$$\mathrm{Tr}[A\wedge A\wedge\boldsymbol{\mathcal{A}}] = \boldsymbol{\mathcal{A}}\wedge\mathrm{Tr}[A\wedge A] = 0 \tag{1593}$$

on account of $\mathrm{Tr}[X\wedge Y] = (-1)^{|X||Y|}\mathrm{Tr}[Y\wedge X]$.

Now the $U(1)$ part started out with a $\mathbb{Z}_{qN}^{(1)}$ symmetry pre-gauging. After we gauge to perform the $\mathbb{Z}_N$ quotient though, the quantization condition on $\boldsymbol{\mathcal{A}}$ is modified, so that only $NF_{\mathcal{A}}$ has periods in $\overline{\mathbb{Z}}$. Now let us shift $\mathcal{A}$ by λ, with $d\lambda = 0$. The action changes by

$$\delta S = \frac{q}{2\pi}\int \lambda \wedge (NF_{\mathcal{A}}). \tag{1594}$$

Since $NF_{\mathcal{A}}$ is quantized in $\overline{\mathbb{Z}}$, we see that $\delta S \in \overline{\mathbb{Z}}$ provided that $\lambda = \frac{1}{q}d\phi$. Thus we see that the $U(N)_{k,q}$ theory has a $\mathbb{Z}_q^{(1)}$ symmetry, that acts by shifting $\mathcal{A}$.

Is it anomalous? Yes: the charge operator for the remaining $\mathbb{Z}_q^{(1)}$ symmetry is

$$U(p, C) = e^{iNp\oint_C \mathcal{A}}, \qquad p \in \mathbb{Z}_q, \tag{1595}$$

with the factor of N needed to perform the shift correctly, and ensures invariance under the gauged diagonal $\mathbb{Z}_N^{(1)}$ symmetry. Computing the braiding phase of the charge operator with itself, we find a phase of $N^2/(Nq)$ since Nq is the effective $U(1)$ level. Thus the anomaly is measured by $N/q \mod 1$. This means in particular that there is no anomaly if $q = N$ (in order for the the theory to be well-defined $q = N$ means $k \in N\mathbb{Z}$). Note that the anomaly of $U(N)_{k,q}$ is the same as the anomaly of N copies of $U(1)_q$.

$SU(N)_k$

Now we look at $SU(N)_k$ CS theory. For all k, this theory has a $\mathbb{Z}_N$ 1-form symmetry, coming from the center of the gauge group. What is the anomaly of the $\mathbb{Z}_N^{(1)}$ symmetry?

Four-dimensional perspective

The easiest way of figuring this out is probably by using what we know about regular four-dimensional pure YM at various values of θ. We know that $\exp(ik\int \mathcal{L}_{CS}[A]/4\pi)$ is the operator which implements the $\theta \mapsto \theta + 2\pi k$ similarity transformation in $SU(N)$ YM, where θ is 2π-periodic, and so if we know what the $\theta \mapsto \theta + 2\pi k$ shift does in the $PSU(N)$ theory, where the 1-form symmetry has been gauged, we'll be able to say something about the anomaly of the gauged CS theory.

Let us now go partway towards turning the theory into a $PSU(N)$ gauge theory by adding a background $\mathbb{Z}_N$ 2-form field B (we'd get the full $PSU(N)$ theory by path integrating over B). We went over how to do this in a previous diary entry, but I think the discussion there was a bit confused and long-winded. Here's how it works: we first write the $SU(N)$ theory as

$$\mathcal{L} = \frac{\theta}{8\pi^2}\left(\mathrm{Tr}[F_{\mathcal{A}} \wedge F_{\mathcal{A}}] - \mathrm{Tr}[F_{\mathcal{A}}] \wedge \mathrm{Tr}[F_{\mathcal{A}}]\right) + \frac{1}{2\pi}F_Y \wedge \mathrm{Tr}[\mathcal{A}]. \tag{1596}$$

Here Y is a Lagrange multiplier field, and $\mathcal{A}$ is a $U(N)$ gauge field.[183]

As in the last subsection, we will find it helpful to decompose $\mathcal{A}$ as

$$\mathcal{A} = A + \mathcal{A}\mathbf{1}. \tag{1597}$$

[183] The second term in the parenthesis ensures that, because the full term in parenthesis is the second Chern class of a $U(N)$ field, we have $\theta \sim \theta + 2\pi$ identically, without having to first integrate out Y. This is desired because $\theta \sim \theta + 2\pi$ in the $SU(N)$ theory (the $SU(N)_k$ CS theory is not spin; more on this later), while if the second term in parenthesis were absent we might not have such a periodicity.

Here A is $\mathfrak{su}(N)$-valued and $\mathcal{A}$ is $\mathfrak{u}(1)$-valued. However, A is not a connection on a $SU(N)$ bundle, and $\mathcal{A}$ is not a connection on a $U(1)$ bundle: rather, the transition functions g_A and $g_{\mathcal{A}}$ satisfy

$$\delta g_A \delta g_{\mathcal{A}} = \mathbf{1}, \qquad \delta g_A, \delta g_{\mathcal{A}} \in \mathbf{1} e^{\frac{2\pi i}{N}\mathbb{Z}}. \tag{1598}$$

In this description, we have a gauge transformation whereby the transition functions g_A and $g_{\mathcal{A}}$ change by opposite roots of unity. Note that this means that only $NF_{\mathcal{A}}$ is a legit $U(1)$ field strength.

Anyway, let's see why this is equivalent to the $SU(N)$ theory. We just integrate out Y: this sets $F_{\mathcal{A}} = 0$ and the sum over $[F_Y] \in H^2(X;\mathbb{Z})$ tells us that we can set $\mathcal{A} = \frac{1}{N}d\phi$, for $d\phi$ a large gauge transformation. The flatness constraint tells us that we will always have $\delta g_{\mathcal{A}} = \mathbf{1}$ (since a nontrivial $\delta g_{\mathcal{A}}$ would contribute to the 1st Chern class), and hence $\delta g_A = \mathbf{1}$ as well: now both $SU(N)$ and $U(1)$ factors are legitimate bundles. Additionally, such an $\mathcal{A}$ can be completely absorbed into a change of the $g_{\mathcal{A}}$ transition functions by Nth roots of unity (the transition functions change by *constants* on each double-overlap). These transition functions can then be absorbed into the g_A transition functions, and so the $\mathcal{A}$ field completely disappears, leaving us with an $SU(N)$ action, as required.

The theory has a $\mathbb{Z}_N^{(1)}$ symmetry that comes from twisting the transition functions in the $SU(N)$ bundle by Nth roots of unity. In our $U(N)$ formulation, this is equivalent to shifting the g_A by Nth roots of unity, which in turn is equivalent to keeping the $g_{\mathcal{A}}$ transition functions fixed, but making a shift $\mathcal{A} \mapsto \mathcal{A} + \frac{1}{N}d\phi$. To gauge this symmetry then, we should make the replacement $F_{\mathcal{A}} \mapsto F_{\mathcal{A}} - B\mathbf{1}$. In what follows we will take B to be some fixed background field with periods in $2\pi/N$ around all closed 2-cycles. This gives us the Lagrangian

$$\mathcal{L} = \frac{\theta}{8\pi^2}\left(\mathrm{Tr}[(F_{\mathcal{A}} - B\mathbf{1})^{\wedge 2}] - (\mathrm{Tr}[F_{\mathcal{A}} - B\mathbf{1}])^{\wedge 2}\right) + \frac{1}{2\pi} Y \wedge \mathrm{Tr}[F_{\mathcal{A}} - B\mathbf{1}]. \tag{1599}$$

Consider integrating out Y. This sets $F_{\mathcal{A}} = B$, which means that $F_{\mathcal{A}}$ becomes quantized in periods of $2\pi/N$. Because of the connection between the transition functions of the $SU(N)$ and $U(1)$ bundles, we then erase $F_{\mathcal{A}} - B\mathbf{1}$ from the action and get

$$\mathcal{L} = \frac{\theta}{8\pi^2}\mathrm{Tr}[F_A \wedge F_A], \qquad w_2(E_{PSU(N)}) = \frac{N}{2\pi}B \in H^2(X;\mathbb{Z}_N), \tag{1600}$$

where A is now a connection on a $PSU(N)$ bundle $E_{PSU(N)}$. Thus we see the role of B is to turn the $SU(N)$ connection into a $PSU(N)$ connection, with the topological class of the $PSU(N)$ bundle controlled by the cohomology class of B. When B gets integrated over, we perform a sum over all $PSU(N)$ bundles, and obtain a genuine $PSU(N)$ gauge theory.

In a previous diary entry we saw that the 2π periodicity in θ is lost in the $PSU(N)$ theory, and instead that changing θ by 2π induces a shift in the action given by a counterterm in B. Indeed, we can write $\mathcal{L}$ as

$$\mathcal{L} = \theta c_2(E_{U(N)}) + \frac{\theta}{4\pi}\left(-\mathrm{Tr}[F_{\mathcal{A}} \wedge B\mathbf{1}] + N\mathrm{Tr}[F_{\mathcal{A}}] \wedge B + \frac{N-N^2}{2}B \wedge B\right) + \frac{1}{2\pi}Y \wedge \mathrm{Tr}[F_{\mathcal{A}} - B\mathbf{1}]. \tag{1601}$$

The advantage of writing it this way is that the $SU(N)$ part of F_A has completely disappeared into the second Chern class of the $U(N)$ bundle. Now integrating out Y, we have (using $\mathrm{Tr}[F_A \wedge B\mathbf{1}] = 0$)

$$\mathcal{L} = \theta c_2(E_{U(N)}) + \frac{\theta}{4\pi}\left(-N + N^2 + \frac{N - N^2}{2}\right) B \wedge B, \tag{1602}$$

where now the $U(N)$ bundle $E_{U(N)}$ is constrained to have first Chern class $c_1(E_{U(N)}) = \frac{N}{2\pi}B \in H^2(X;\mathbb{Z})$. Since the second Chern class is integral, under $\theta \mapsto \theta + 2\pi$ the action shifts as (modulo elements of $2\pi\mathbb{Z}$)

$$S_\theta \mapsto S_\theta + \frac{1}{4\pi}(N^2 - N)\int B \wedge B, \tag{1603}$$

This is nontrivial, since $\int B \wedge B \in \mathbb{Z}/N^2$ ($\in 2\mathbb{Z}/N^2$) on generic (spin) closed 4-manifolds, and hence θ is actually not 2π periodic.

Anyway, the point is the following: consider a domain wall where θ jumps by 2π. We know that such a domain wall can be created by inserting $\exp(i\int_X \mathcal{L}_{CS}[A]/4\pi)$ into the path integral, where X is a 3-manifold defining the domain wall. By the above discussion, we know that the action differs on the two sides of the domain wall by a $B \wedge B$ counterterm in the background field. However, integrating $B \wedge B$ over an open submanifold of spacetime is not a gauge-invariant thing to do! Doing a gauge transformation on B produces an anomalous term, consisting of an integral over the codimension-1 submanifold X:

$$\delta S = \frac{i}{4\pi}(N-1)\int_X \mathrm{Tr}[2B \wedge \lambda + \lambda \wedge d\lambda], \tag{1604}$$

for $\delta B = d\lambda$ (and we are tacitly writing e.g. B for $\mathbf{1}B$). Since we know that $PSU(N)$ gauge theory in four dimensions is self-consistent, this anomaly must be canceled by an anomaly of the $SU(N)_1$ CS theory.

The anomaly is determined by looking at how the shift in S_θ depends on the bounding 4-manifold. Integrating it over a closed 4-manifold tells us that $e^{i\delta S_\theta} = e^{2\pi l \frac{N-1}{2N}}$ for some $l \in \mathbb{Z}_N$. Thus we can conclude that the CS theory $SU(N)_1$ has anomaly $(N-1)/2N \mod 1$. The anomaly for $SU(N)_k$ must then be $k(N-1)/(2N) \mod 1$, since $SU(N)_k$ is the theory defined by the similarity transform on the codimension-1 slice where the $\delta\theta = 2\pi k$ domain wall happens, and the gauge-non-invariance of the bulk action in the presence of the domain wall is exactly k times the result when the θ angle jumps by 2π. So, the theory has an anomaly given by

$$\text{Anomaly} = \frac{k(N-1)}{2N} \mod 1 \quad (\mathrm{mod}\, 1/2 \text{ if spin}). \tag{1605}$$

Here the reduced anomaly for the spin case comes from the fact that the intersection form is then even, which limits the phases that δS_θ in (1603) can take when integrated over closed 4-manifolds. Actually, we can do a bit better: if $k \in 2\mathbb{Z}$ then the N^2 part of (1604) is trivial on all manifolds, and so we can effectively say that the anomaly is just $-k/N$ if $k \in 2\mathbb{Z}$.

Three-dimensional perspective

Now let's look at this from the three-dimensional perspective directly. One naive way to write the $SU(N)_k$ theory is to write

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathcal{A}\wedge d\mathcal{A} + \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right] + \frac{1}{2\pi}y\wedge d\mathrm{Tr}[\mathcal{A}], \tag{1606}$$

where y is a Lagrange multiplier that roughly speaking turns the $U(N)$ field $\mathcal{A}$ (a $\mathfrak{u}(N)$-valued form) into an $\mathfrak{su}(N)$-valued 1-form. This is not completely correct, however, since if k is odd this theory is spin, while we know that $SU(N)_k$ is non-spin for any value of k (because the $SU(N)$ instanton number is equal to $2\pi\int \mathrm{c}_2(E)$ where $\mathrm{c}_2(E)$ is the second Chern class, which is integral on all closed manifolds, spin or not).

To fix this, we will add a $U(1)_p$ term using the $U(1)$ field $\mathrm{Tr}[\mathcal{A}]$. Note that we are free to shift the definition of the Lagrange multiplier field by

$$y \mapsto y \pm \mathrm{Tr}[\mathcal{A}] \tag{1607}$$

(since $\mathrm{Tr}[\mathcal{A}]$ is a properly quantized $U(1)$ field), which changes p by ± 2. So, to find out how to render the theory non-spin, we just need to find out the correct parity to use for p.

Anyway, to get the answer for the correct non-spin theory, we write the full Lagrangian as

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[\mathcal{A}\wedge d\mathcal{A} - \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right] + \frac{\eta_k}{4\pi}\mathrm{Tr}[\mathcal{A}]\wedge d\mathrm{Tr}[\mathcal{A}] + \frac{1}{2\pi}y\wedge d\mathrm{Tr}[\mathcal{A}], \tag{1608}$$

where η_k is to be determined. The integral needing to be done to check the quantization condition on η_k is

$$I = \frac{2\pi}{8\pi^2}\int_{M_4}\left(k\mathrm{Tr}[F_\mathcal{A}\wedge F_\mathcal{A}] + \eta_k\mathrm{Tr}[F_\mathcal{A}]\wedge\mathrm{Tr}[F_\mathcal{A}] + 2dy\wedge d\mathrm{Tr}[\mathcal{A}]\right) \tag{1609}$$

for some closed 4-manifold M_4. The last term is always in $\overline{\mathbb{Z}}$, while the first two can be written as

$$I = 2\pi\int c_2(E_{U(N)}) + \pi(k+\eta_k)\int \mathrm{Tr}[F_\mathcal{A}/2\pi]\wedge\mathrm{Tr}[F_\mathcal{A}/2\pi], \tag{1610}$$

where $\mathrm{c}_2(E_{U(N)})$ is the second Chern class of the $U(N)$ bundle. Since this is always an integral class regardless of the base space of the bundle, we conclude that we need $k+\eta_k$ to be even. Thus we can take e.g. $\eta_k = 0$ if $k \in 2\mathbb{Z}$, and $\eta_k = -1$ if $k \in 2\mathbb{Z}+1$. Another (simpler) choice (and the one we will adopt) is to simply set $\eta_k = -k$, which as we mentioned above is equivalent since η_k and $\eta_k \pm 2$ define equivalent theories. Adopting this choice, we have

$$\mathcal{L}_{SU(N)_k}[A] = \mathcal{L}_{U(N)_{k,k(1-N)}}[\mathcal{A}] + \frac{1}{2\pi}y\wedge d\mathrm{Tr}[\mathcal{A}]. \tag{1611}$$

Thus $SU(N)_k$ is realized as a constrained version of a $U(N)_{k,q}$ theory at $q = k(1-N)$. The freedom to shift y by $\pm\mathrm{Tr}[\mathcal{A}]$ manifests itself in the equivalence $q \sim 2N$.

As we saw previously, we can split up $\mathcal{A}$ into an $SU(N)$ part A and a diagonal part $\mathcal{A}$, provided that the cocycle conditions for the A and $\mathcal{A}$ parts fail in canceling ways. Recall the decomposition $\mathcal{A} = A+\mathcal{A}\mathbf{1}$. With this decomposition, in order to implement the matching-cocycle-conditions property, we require that the diagonal transformation shifting the transition functions for both A and $\mathcal{A}$ by opposite Nth roots of unity be a gauge transformation. Note that we can do such a shift while keeping A traceless, since we are only changing the transition functions by constants: the change in transition functions is done at the level of the glueing data between patches, not at the level of the 1-forms A defined on single patches. By contrast, when we perform such a shift on $\mathcal{A}$, we will do it by directly taking $\delta\mathcal{A} = \frac{1}{N}d\phi$ (ϕ as usual is 2π-periodic), without changing the transition functions for the $\mathcal{A}$ bundle. Either way we do it, the effect of this identification is to gauge a diagonal $\mathbb{Z}_N^{(1)}$ symmetry that shifts both A and $\mathcal{A}$. The transformation acts nontrivially on A Wilson lines since they are defined by $\text{Tr}[e^{i\int_{U_\alpha}A}e^{i\Lambda_{\alpha\beta}}e^{i\int_{U_\beta}A}\dots]$, with $\Lambda_{\alpha\beta}$ the transition functions between patches, and since the transformation shifts the $\Lambda_{\alpha\beta}$'s. Note that this gauge transformation, while not changing the field strength $F_{\mathcal{A}}$, *does* change the field strengths of A and $\mathcal{A}$: if we make the cocycle condition fail by an Nth root of unity on a given triple overlap of patches, then this induces fractional flux in both A and $\mathcal{A}$.

Now we can get a more precise understanding of what the Lagrange multiplier y is doing. Integrating out y tells us that $d\mathcal{A} = 0$, and that $\int \mathcal{A} \in \frac{1}{N}\overline{\mathbb{Z}}$ around all closed 1-manifolds. Thus we may write $\mathcal{A} = \frac{1}{N}d\phi$. But we see that this is gauge-equivalent to $\mathcal{A} = 0$ under the 1-form gauge symmetry. So, integrating out y leaves us with just the $SU(N)_k$ part of the action, which is what we want.

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Returning to $\mathcal{L}$, we have

$$\mathcal{L} = \frac{k}{4\pi}\left(\text{Tr}\left[A\wedge dA - \frac{2i}{3}A\wedge A\wedge A\right] + (N-N^2)\mathcal{A}\wedge d\mathcal{A}\right) + \frac{N}{2\pi}y\wedge d\mathcal{A}, \tag{1612}$$

again using the tracelessenss of A and the antisymmetry to kill the $A\wedge A\wedge\mathcal{A}$ contribution. Note the $k(N-N^2)$ level of the $\mathcal{A}$ CS term: peeking back at the analysis of the bulk gauge theory, we see that this is exactly the right number needed to cancel the bulk anomaly, and is a hint that we're on the right track.

Let's pause to figure out what the symmetry is. We started with a pure $SU(N)_k$ CS term, which as we know has a $\mathbb{Z}_N^{(1)}$ symmetry. We then wrote it in terms of a $U(N)_{k,q}$ theory plus a Lagrange multiplier, where for us we chose $q = k(1-N)$. As we saw earlier, the $U(N)_{k,q}$ theory by itself has a $\mathbb{Z}_q^{(1)}$ global symmetry. This symmetry is generically broken by the Lagrange multiplier term, since under it we have

$$\delta S = \frac{N}{2\pi k(1-N)}\int F_y\wedge d\phi \notin \overline{\mathbb{Z}}. \tag{1613}$$

So, does this mean that we have no 1-form symmetry? This would be a problem if so. But actually, the $\mathbb{Z}_N^{(1)}$ symmetry that we need to be there does exist. To see how it works, consider shifting $\mathcal{A}$ by some flat 1-form λ. The action changes as

$$\delta S = \frac{k(1-N)}{2\pi}\int \lambda\wedge(NF_{\mathcal{A}}) + \frac{N}{2\pi}\int F_y\wedge\lambda. \tag{1614}$$

In order for the last term to be in $\overline{\mathbb{Z}}$, we see that we need to take $\lambda = d\phi/N$. Then the variation in S is

$$\delta S = \frac{k(1-N)/N}{2\pi} \int \lambda \wedge (N F_{\mathcal{A}}). \tag{1615}$$

This is in general nontrivial, but we see that we can cancel it, if we take the symmetry transformation to involve an appropriate shift in y as well. This gives us a genuine $\mathbb{Z}_N^{(1)}$ symmetry, under which we have

$$\mathbb{Z}_N^{(1)} : \mathcal{A} \mapsto \mathcal{A} + \frac{1}{N} d\phi, \qquad y \mapsto y - \frac{k(1-N)}{N} d\phi. \tag{1616}$$

If this is the right symmetry, it should shift fundamental Wilson lines by Nth roots of unity. And indeed it does:

$$\mathbb{Z}_N^{(1)} : W_f(C) = \mathrm{Tr}_f[e^{i\oint_C (A + \mathcal{A}\mathbf{1})}] \mapsto e^{i\frac{1}{N}\oint_C d\phi} W_f(C) \tag{1617}$$

(note that $e^{i\oint \mathcal{A}}$ is not a gauge-invariant operator to consider the transformation properties of). Note that $e^{i\oint y}$ also shifts under the symmetry, so that it must also be electrically charged. More on this in a bit.

What is the operator which generates this symmetry? It turns out to be $\exp(i\oint y)$. This is rather surprising, since looking at the action one might be forgiven for thinking that the y line was bosonic.

To find the statistics of the y line, it is helpful to Higgs the theory down to $\mathbb{Z}_N$. In terms of the $SU(N)$ variables, the effect of the Higgsing is to leave the theory with only $\mathbb{Z}_N$ transition functions as degrees of freedom. In the continuum, it's easier to deal with this condition by writing the transition functions instead as diagonal $\mathbb{Z}_N$ 1-form matrices, with trivial transition functions. So to that end, Higgsing for us at the computational level means taking $A = 0$ and $\mathcal{A} = a$, with a a $\mathbb{Z}_N$ field. Since $A = 0$ and A has trivial transition functions, the cocycle condition will be satisfied exactly for a, and the flux of F_a will be quantized in the regular way. So, upon doing this, we get the $DW_{p,q}$ theory with $p = kN(1-N)$, $q = N$:

$$SU(N)_k \xrightarrow{Higgs} \frac{kN(1-N)}{4\pi} a da + \frac{N}{2\pi} y \wedge da. \tag{1618}$$

Note that in addition to the $\mathbb{Z}^{(1)}_{\gcd(kN(1-N),N)} = \mathbb{Z}_N^{(1)}$ symmetry, we also have a symmetry that shifts y by a $\mathbb{Z}_N$ gauge field. The appearance of this magnetic symmetry is expected after we move from $SU(N)$ (which has no t'Hooft line operators since $\pi_1(SU(N)) = 0$) to $\mathbb{Z}_N$ (which does have magnetic operators since we can have $\mathbb{Z}_N$ branch cuts in the transition functions).

We've already been through this theory in lots of detail, and we learned that the mutual statistics between the a and y lines are[184]

$$[S]_{a,a} = 1, \quad [S]_{a,y} = e^{2\pi i/N}, \quad [S]_{y,y} = e^{-2\pi i k(1-N)/N}. \tag{1619}$$

Recall from a ways back that we could perform a change of variables on y that shifted $k(1-N) \mapsto k(1-N) \pm 2N$. We see that this leaves the braiding phases invariant (and

[184]The N/N factor in the $[S]_{y,y}$ matrix element is important, since when square-rooted it contributes to the spin of the y line. However, it does not affect how y lines transform under the $\mathbb{Z}_N^{(1)}$ symmetry.

because of the factor of 2, it also leaves the spins invariant), and so reassuringly the shift indeed acts trivially on the modular data of the theory.

From the above entries of the S matrix, we see that the line $e^{i\oint y}$ generates the $\mathbb{Z}_N^{(1)}$ symmetry of $SU(N)_k$, since these braiding phases mean that wrapping lines with the line $e^{i\oint_C y}$ is equivalent to performing the shift (1616) (where $d\phi$ is determined by the topology of C).

We can now easily figure out the anomaly: from taking the square root of $[S]_{y,y}$ to get the spin of the generating line, we read off the anomaly as $k(1-N)/2N \mod 1$. If we are on a spin manifold then having the generating line be a fermion is okay, and so in that case the anomaly is $k(1-N)/2N \mod 1/2$. Note that this is exactly the right anomaly to cancel the bulk anomaly that we derived earlier in (1605)! Nice. Note that the anomaly of $SU(N)_k$ is the same as that of $[SU(N)_1]^{\otimes k}$, because of the constant k prefactor. Also note that since $N-1$ is coprime to N, we will only have a non-anomalous theory if $k \in 2N\mathbb{Z}$ (or $k \in N\mathbb{Z}$ if spin).

Can we say anything about this line in the $SU(N)$ context? Yes: under the $\mathbb{Z}_N^{(1)}$ symmetry we have

$$e^{i\oint y} \mapsto e^{2\pi i k/N} e^{i\oint y}. \tag{1620}$$

Since Wilson lines in the fundamental transform with a $e^{2\pi i/N}$ phase, this tells us that the generator $e^{i\oint y}$ can be identified with a Wilson line in a k index symmetric $SU(N)$ representation. This makes sense because, as noted in [14], $e^{i\oint y}$ is the operator we get when slicing open the 2-dimensional surface operator which implements the $\mathbb{Z}_N^{(1)}$ symmetry in the 3+1 D theory. Now the $SU(N)_k$ theory lives at an interface where the bulk θ angle changes by $2\pi k$. The Witten effect means that the t'Hooft operators on both sides of the surface (which are not genuine line operators) have electric charges differing by k. This k difference in electric charges is realized by the fact that the charge operator on the interface, namely $e^{i\oint y}$, carries electric charge k.

This is a manifestation of the mixed anomaly between the $\mathbb{Z}_N^{(1)}$ symmetry and time reversal at $\theta \in \pi(2\mathbb{Z}+1)$. Indeed, consider a 2π domain wall for θ, where θ jumps from $-\pi$ to π. The operator which inserts this domain wall is the charge operator for T, since it interpolates between the two ground states (which differ by $\theta \mapsto -\theta$). The mixed anomaly comes from the fact that this domain wall operator and the surface operator which implements the $\mathbb{Z}_N^{(1)}$ symmetry don't commute: indeed, they do not commute because of a contact term, and their lack of commutativity can be seen from the fact that along their intersection is a fundamental Wilson line (since we are in four dimensions, a 3-manifold and a 2-manifold intersect at a 1-manifold).

As we have seen, if we try to gauge the $\mathbb{Z}_N^{(1)}$ symmetry in the 2+1D theory, we run in to problems since the operators which perform the gauge transformations (the fundamental Wilson lines) do not commute with each other. This can be fixed by using the same procedure as in the $U(1)_k$ case. First, we write the action for the theory as

$$\begin{aligned} S = &\frac{k}{4\pi}\int_{\partial X} \mathrm{Tr}\left[\mathcal{A}\wedge d\mathcal{A} + \frac{2i}{3}\mathcal{A}^3\right] - \frac{k}{4\pi}\int_{\partial X}\mathrm{Tr}[\mathcal{A}]\wedge\mathrm{Tr}[F_\mathcal{A}] + \frac{k(N-1)}{2\pi}\int_{\partial X} B\wedge\mathrm{Tr}[\mathcal{A}] \\ &+ \frac{1}{2\pi}\int_{\partial X}(y\wedge\mathrm{Tr}[F_\mathcal{A} - B\mathbf{1}] + N\Phi\wedge B) - \frac{N}{2\pi}\int_X F_\Phi\wedge B + \frac{k(N-N^2)}{4\pi}\int B\wedge B. \end{aligned} \tag{1621}$$

To get this, we just took the four-dimensional gauged action (with Φ a 1-form Lagrange multiplier to make B a $\mathbb{Z}_N$ field), and integrated by parts. The extra $N\Phi \wedge B$ term is needed to make things gauge invariant, as we will see.

The transformation rules for the fields are as follows. First, we have a gauge transformation under which $\delta y = \delta\Phi = d\alpha$. Next, we have a $\mathbb{Z}_N^{(1)}$ gauge transformation, generated by $e^{i\oint_C y}$, which shifts

$$\mathcal{A} \mapsto \mathcal{A} + \frac{2\pi}{N}\widehat{C}, \quad B \mapsto B + \frac{2\pi}{N}d\widehat{C}, \quad y \mapsto y + \frac{2\pi k}{N}\widehat{C}, \quad \Phi \mapsto \Phi + \frac{2\pi k}{N}\widehat{C}. \tag{1622}$$

Here $\widehat{C}$ is the Poincare to a possibly open curve in ∂X, with the Poincare dual having some arbitrary extension into the bulk X. Here $\widehat{C}$ is such that $\int_{C'}\widehat{C} \in \mathbb{Z}$ for all $C' \in C_1(\partial X;\mathbb{Z})$, but where the value for the integral may depend on more than just the homotopy class of C'. One can check that the action is invariant up to the term $-\frac{k}{2\pi}\int \widehat{C}\wedge F_B$, which is in $\overline{\mathbb{Z}}$ because of the quantization on F_B. Thus, the whole action is gauge-invariant.

Anyway, these transformation laws let us write down the correct, gauge invariant, generator of gauge transformations for the gauged $\mathbb{Z}_N^{(1)}$ symmetry. It is

$$U(q,C) = \exp\left(iq\left[\int_C y + \int_{C'}\Phi + k\int_D B\right]\right). \tag{1623}$$

Here $C \cup C' = \partial D$, with C' only in the bulk; see Figure 9 for a similar setup. Do these operators commute with each other? Yes! $U(q,C)$ and $U(p,\widetilde{C})$, with C, $\widetilde{C}$ two intersecting curves in ∂X will have a contribution to their commutator of the form $e^{2\pi i qpk/N}$, which comes from the commutator of the two y lines. However, they will also have a compensating contribution from the commutator between the $\int_{\widetilde{C}'}\Phi$ line and the $k\int_D B$ surface (which intersect in the bulk), since Φ and B have a braiding phase of $e^{2\pi i/N}$. Thus the $U(q,C)$ are indeed legit generators of the $\mathbb{Z}_N^{(1)}$ gauge transformations.

Summary

The theories that we've looked at are

$$\begin{aligned}
&U(1)_k : \frac{k}{4\pi}A\wedge dA \\
&DW_{p,q} : \frac{p}{4\pi}a\wedge da + \frac{q}{2\pi}a\wedge db \\
&U(N)_{k,q} : \frac{k}{4\pi}\mathrm{Tr}[\mathcal{A}\wedge d\mathcal{A} + 2i/3\mathcal{A}^3] + \frac{q-k}{4\pi N}\mathrm{Tr}[\mathcal{A}]\wedge d\mathrm{Tr}[\mathcal{A}] \qquad (k-q)\in N\mathbb{Z} \\
&SU(N)_k : U(N)_{k,k(1-N)} + \frac{1}{2\pi}y\wedge d\mathrm{Tr}[\mathcal{A}].
\end{aligned} \tag{1624}$$

The symmetries and anomalies (on general, non spin manifolds, provided that the theory is not spin) are

	1-form symmetry	Anomaly (mod 1)	Spin?
$U(1)_k$	$\mathbb{Z}_k$	$1/k$	if $k\in 2\mathbb{Z}+1$
$DW_{p,q}$	$\mathbb{Z}_q$ on b, $\mathbb{Z}_{\gcd(p,q)}$ on a	0,$p/\gcd(p,q)$,$1/\gcd(p,q)$ (mixed)	if $p\in 2\mathbb{Z}+1$
$U(N)_{k,q}$,	$\mathbb{Z}_q$	N/q	if $k+(q-k)/N\in 2\mathbb{Z}+1$
$SU(N)_k$	$\mathbb{Z}_N$	$(k-Nk)/2N$	No

(1625)

Here the anomaly is determined by taking the mod 1 residue of the entry in the third column. In the last column we have indicated when the theories are spin, which will is determined in a different diary entry.

One interesting thing is to check how this is compatible with known level-rank dualities. For example, consider the duality $U(1)_N \leftrightarrow SU(N)_1$ (it is usually $U(1)_{-N}$, but in these conventions the anomalies are such that we write it as $U(1)_N$). This duality hold holds as spin TQFTs. Indeed, while they have the same $\mathbb{Z}_N^{(1)}$ symmetry, let's compare their anomalies: for $U(1)_{-N}$ we have $1/N$, while for $SU(N)_1$ we have $(1-N)/2N$. These are of course not the same. But, on a spin manifold, the anomaly of $SU(N)_k$ is actually $(k - Nk)/N \mod 1$ since the generator of the $\mathbb{Z}_N^{(1)}$ symmetry is allowed to be a fermion. Setting $k=1$ the anomaly becomes $1/N$, which matches that of the $U(1)_N$ theory.

The chiral anomaly in 2 dimensions and the fermion bubble

Today is a very basic calculation that can likely be found in a nonzero number of QFT texbooks but which was a good brainwarmer and which I figured couldn't hurt to keep around for posterity. We will be considering massless fermions coupled to a $U(1)$ gauge field in 1+1D and computing the two-dimensional analogue of the triangle diagram in order to derive the divergence of the chiral current.

The relevant diagram to compute when examining $\partial_\mu j_5^\mu$ is a fermion bubble with one insertion of j_5 and one gauge field leg. Thus to get $\partial_\mu j_5^\mu$ we need to compute the polarization bubble $\Pi_{\mu\nu}(q^2)$ with a $\bar{\gamma}$ inserted in the trace, and then contract it with a q_μ and a A_ν. For external momentum q, and making sure to order the matrices in the numerator correctly, the graph gives (the is from the propagators kill the minus sign from the fermion loop)

$$(\text{bubble graph})(q^2) = e^2 \int_p \mathrm{Tr}\left[\gamma^\mu \bar{\gamma} \frac{i\not{p}}{p^2} \gamma^\nu \frac{\not{p}+\not{q}}{(p+q)^2}\right]. \tag{1626}$$

Now use

$$\not{p}\gamma^\nu = 2p^\nu - \gamma^\nu \not{p}, \qquad \gamma^\mu \bar{\gamma} = \epsilon_{\mu\nu}\gamma^\nu, \tag{1627}$$

where we are in $\mathbb{R}$ time and have chosen X, iY as γ matrices. We also need

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 2g^{\mu\nu}, \qquad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 2(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) \tag{1628}$$

to simplify the trace. One gets

$$(\text{bubble graph})(q^2) = 2e^2 \int_p \frac{p^\nu (p+q)^\mu + p^\mu (p+q)^\nu - g^{\mu\nu} p^\sigma (p+q)_\sigma}{p^2 (p+q)^2}, \tag{1629}$$

which we evaluate with Feynman parameters in the usual way. We then use dimensional regularization to do the integrals and renormalize with minimal subtraction, yielding

$$i\Pi^{\mu\nu}(q^2) = \frac{ie^2}{\pi}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right), \tag{1630}$$

which is in the form required by gauge invariance. The divergence of the axial current to one-loop order is then

$$\begin{aligned} \partial_\mu j_5^\mu \to q_\mu j^\mu &= q_\mu \epsilon^{\mu\nu} \frac{e^2}{\pi}\left(g_{\nu\lambda} - \frac{q_\nu q_\lambda}{q^2}\right) A^\lambda \\ &= \frac{e^2}{\pi} q_\mu \epsilon^{\mu\lambda} A_\lambda. \end{aligned} \tag{1631}$$

In $\mathbb{R}$ space then,

$$\partial_\mu j_5^\mu = \frac{e^2}{2\pi} F_{\mu\nu}\epsilon^{\mu\nu} = 2e^2 \mathrm{c}_1, \tag{1632}$$

where c_1 is the first Chern class. The fact that the divergence of the axial current is *twice* the first Chern class is essential, since without the factor of 2 we'd conclude that the A_μ background field could lead to a non-conservation of fermion number (as c_1 is an integral class). This factor of two comes from the fact that each charge-1 chiral fermion contributes $e^2\mathrm{c}_1$ to the anomaly, and a single Dirac fermion has two such chiral fermions. As another sanity check, we see that if we didn't have the ϵ symbol (corresponding to computing the divergence of the vector current), we'd get a divergence of $d^\dagger \Box^{-1} d^\dagger dA = \Box^{-1} d^\dagger d^\dagger dA = 0$.

Note that we also could have regulated by imposing a hard cutoff in momentum space. The $q^\mu q^\nu$ term would be unchanged since it has no UV divergences, but the $g^{\mu\nu}q^2$ piece would pick up a $\ln\Lambda$ divergence, which we would regulate away. The form of the resulting Π would then imply that gauge invariance is violated (the divergence of the regular vector current $\bar\psi\gamma^\mu\psi$ is non-zero). Thus, courtesy of the mixed anomaly between $U(1)_A$ and $U(1)_V$, we may gauge one of them, but doing so breaks the other.

Topological defects in axion electrodynamics and anomaly inflow

Today's diary entry comes from wanting to understand in detail constructions involving axions and anomaly inflow as sketched in e.g. Jeffery Harvey's TASI lectures on anomalies.

We will be considering QED coupled chirally to a scalar with a $U(1)$-breaking potential:

$$\mathcal{L} = -\frac{1}{2}F \wedge \star F + i\bar{\psi}\not{D}_A\psi - \frac{1}{2}(\partial\sigma)^2 + \bar{\psi}(\mathrm{Re}(\sigma) + i\bar{\gamma}\mathrm{Im}(\sigma))\psi - V(\sigma), \tag{1633}$$

where $V(\sigma)$ has a classical minimum at $m \neq 0$. When the scalar gets a vev, it induces a mass term for the fermions, which in addition to the regular $m\bar{\psi}\psi$ mass term also has a chiral part.

We will show that when σ is Higgsed (viz. $\sigma \to me^{i\theta}$), the fermions are responsible for generating a spacetime-dependent theta angle for the gauge field. We will then consider a field configuration with a topological string defect in θ, and show that the action for such a field configuration suffers from a gauge anomaly by computing the divergence of the gauge current (this can be done with a one-loop calculation). By solving the Dirac equation on the string defect, we will show that the defect hosts chiral zero modes whose own gauge anomaly renders the full theory gauge invariant.

The anomaly

Throughout we will assume we are working in the symmetry-broken phase, where σ gets a vev, thereby giving a mass to the fermions. We assume the potential $V(\sigma)$ is such that the classical vacua form an S^1, and thus it makes sense to consider defect codimension-2 objects (strings) around which the phase of σ winds by something in $2\pi\mathbb{Z}$.

One (schematic) argument for why the fermions generate a θ term is as follows: for $\sigma = me^{i\theta}$, we can eliminate the $\bar{\gamma}$ coupling in $\mathcal{L}$ by performing the shift

$$\psi \mapsto e^{-i\theta\bar{\gamma}/2}\psi \tag{1634}$$

in the path integral. Then because of the chiral anomaly, the action should shift by

$$S \mapsto S + \frac{1}{8\pi^2}\int \theta F \wedge F, \tag{1635}$$

thus generating a (spacetime-dependent) θ angle (the shift is the integral of θ against the second Chern character and not twice the second Chern character since we've rotated by $\theta/2$). This is subtle for defect string configurations though, since θ is not well-defined on its own, which makes doing the usual Fujikawa method kind of tricky. It will turn out that the right answer is the integrated-by-parts version of this, namely

$$\frac{1}{8\pi^2}\int d\theta \wedge F \wedge A. \tag{1636}$$

To get this, we compute the divergence of the gauge current in the limit of large m and obtain the above expression. To leading order the divergence in the gauge current is caused by a diagram consisting of a single ψ loop with an external J_μ source, an A_ν gauge field leg, and an external σ leg. To evaluate this diagram and get a non-zero answer, we need to correctly take into account the fermion mass. This is done most easily by choosing a particular symmetry-breaking state to evaluate the diagram in. Let $\sigma = \sigma_1 + i\sigma_2$ with σ_i real. We will take the state where the vev of σ is real, with $i\sigma_2$ the Goldstone. We thus write $\langle\sigma\rangle = \langle\sigma_1\rangle = m$ for the fermion mass, and so the chiral coupling between σ and the fermions gives the fermions a mass and leaves them with a $\bar\psi i\bar\gamma\sigma\psi$ coupling. In this minimum, the contribution to the gauge current from this diagram on scales much larger than m^{-1} is

$$J^\mu(q+k) = e\int_p \mathrm{Tr}\left[\gamma^\mu \frac{\not p + m}{p^2-m^2}\gamma^\nu \frac{\not p - \not q + m}{(p-q)^2-m^2}(i\bar\gamma\sigma_2(k))\frac{\not p - \not q - \not k + m}{(p-q-k)^2-m^2}\right]A_\nu(q). \tag{1637}$$

In this expression, q is the momentum of the external photon, and k is the momentum of the axion field (the σ field, but in the chosen symmetry-breaking state only the imaginary part σ_2 is fluctuating).

We can simply this a fair bit by noting that terms with an odd number of γ matrices will vanish after being traced over, since $\mathrm{Tr}(\gamma^\mu) = 0$ for all μ. In order to survive the trace with $\bar\gamma$, we need exactly four compensating γ matrices, so only terms linear in m survive in the numerator. We then use (in minkowski, mostly negative signature)

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\bar\gamma] = -4i\epsilon^{\mu\nu\lambda\sigma}. \tag{1638}$$

Any numerator odd in the momentum p which is being traced out will vanish by symmetry, and because of the ϵ symbol in the above trace, no terms involving $\not p$ will survive. Thus the only term which survives is

$$J^\mu(q+k) = 4me\epsilon^{\mu\nu\lambda\sigma}\int_p \frac{\sigma_2(k)q_\lambda(q_\sigma+k_\sigma)}{(p^2-m^2)((p+q)^2-m^2)((p+q+k)^2-m^2)}A_\nu(q). \tag{1639}$$

We then simplify the denominator with Feynman parameters, and shift p to complete the square (which happily doesn't affect the numerator). This gives

$$J^\mu(q+k) = 4me\epsilon^{\mu\nu\lambda\sigma}\int_{x,y}\sigma_2(k)q_\lambda(q_\sigma+k_\sigma)A_\nu(q)\int_p\frac{1}{(p^2-\Delta)^3}, \tag{1640}$$

where Δ is a gross function of q, k, and the Feynman parameters x, y. We can then do the integral no problem, and we get

$$J^\mu(q+k) = -\frac{ime\epsilon^{\mu\nu\lambda\sigma}\sigma_2(k)q_\lambda(q_\sigma+k_\sigma)}{8\pi^2}\int_{x,y}\frac{1}{\Delta}. \tag{1641}$$

Now we have been assuming that the fermion mass m is much bigger than the momentum scales of interest, namely q and k. With this approximation Δ turns out to simply be m^2. We can also drop the quadratic in q term, since the gauge field is assumed to be smooth so that $[\partial_\mu, \partial_\nu]A = 0$, and so we get

$$J^\mu(q+k) = -\frac{ie\epsilon^{\mu\nu\lambda\sigma}\sigma_2(k)q_\lambda k_\sigma}{4\pi^2 m}A_\nu(q). \tag{1642}$$

In real space, we could write this as

$$J^\mu = -\frac{ie\epsilon^{\mu\nu\lambda\sigma}F_{\nu\lambda}\sigma_1\partial_\sigma\sigma_2}{8\pi^2\sigma_1}. \tag{1643}$$

Of course, this expression for J^μ is not general since we've chosen a particular vacuum to evaluate the diagram in. Getting the general expression is easy though: we just write down the appropriate expression which reduces to the above answer when $\langle\sigma\rangle$ is real and which is invariant under the chiral rotation $(\sigma_1, \sigma_2)^T \mapsto R_\alpha(\sigma_1, \sigma_2)^T$ (where $R_\alpha \in O(2)$ is a rotation by α under which $\psi \mapsto e^{-i\alpha\bar{\gamma}/2}$). Such a rotation preserves the matrix $iY\partial$, and so we write the general J^μ as

$$J^\mu = \frac{ie\epsilon^{\mu\nu\lambda\sigma}F_{\nu\lambda}}{8\pi^2|\sigma|^2}(\sigma_2\partial_\sigma\sigma_1 - \sigma_1\partial_\sigma\sigma_2), \tag{1644}$$

which indeed reduces to our previous expression when we choose the $\langle\sigma\rangle = \langle\sigma_1\rangle$ vacuum. One can also check that if we go back and work about the $\langle\sigma\rangle = i\langle\sigma_2\rangle$ vacuum with an $i\bar{\gamma}\sigma_2$ mass term for the fermions and a $\bar{\psi}\psi\sigma_1$ interaction vertex, then we get the other term in the above equation. Finally, parametrizing $\sigma = me^{i\theta}$, we find that the gauge current is

$$J^\mu = \frac{e}{8\pi^2}\epsilon^{\mu\nu\lambda\sigma}\partial_\nu\theta F_{\lambda\theta}, \tag{1645}$$

which ends up being independent of the fermion mass and matches the $d\theta \wedge F \wedge A$ answer we had guessed earlier for the shift in the action, since $J \propto \star(d\theta \wedge F)$. Since $dF = 0$, The divergence in the gauge current is

$$d^\dagger J = \frac{e}{8\pi^2}\star(d^2\theta \wedge F). \tag{1646}$$

This is not zero, since θ is not a well-defined function when we consider a string defect configuration, only $d\theta$ is. Since

$$\int_\gamma d\theta = 2\pi \tag{1647}$$

for any curve γ which links the defect (assuming the defect has 2π winding),

$$d^2\theta = 2\pi\widehat{s}, \tag{1648}$$

where s is a 2-chain parametrizing the string and $\widehat{s}$ is its Poincare dual. This works because

$$\int_\gamma d\theta = \int_D d^2\theta = 2\pi\int_D \widehat{s} = 2\pi \mathrm{int}(D, s) = 2\pi, \tag{1649}$$

where D is a disc bounded by γ and $\mathrm{int}(D, s) = 1$ is the intersection number. For example, if s lies along the z-axis in space and doesn't move in time, $d\theta = \frac{1}{r}\hat{\phi}$ in cylindrical coordinates. Thus the divergence in the gauge current

$$d^\dagger J = \frac{e}{4\pi}\star(\widehat{s} \wedge F) \tag{1650}$$

is non-zero only on the string defect, and so under gauge transformations $A \mapsto A - d\alpha$, the action shifts as

$$S \mapsto S + \frac{e^2}{4\pi}\int_s F\alpha, \tag{1651}$$

where the integral is over the string defect. In order for this theory to make sense, this anomaly needs to be canceled by something living on the string.

Anomaly cancellation

We expect that an anomaly of the above form will be canceled by a gauge anomaly from chiral fermions living on the string, and this is indeed the case. Suppose the string lives along the z axis, and work in cylindrical coordinates. The Dirac equation is

$$(i\not{D}_A + m(r)e^{i\bar{\gamma}\theta})\psi = 0, \tag{1652}$$

where $m(r)$ goes to zero at $r = 0$ and goes to the minimum of $V(\sigma)$ at $r \to \infty$. Since $\not{D}_A$ anticommutes with $\bar{\gamma}$, we can write

$$i\not{D}_A\psi_\mp + m(r)e^{\pm\theta}\psi_\pm = 0, \tag{1653}$$

where $\psi_\pm$ are eigenspinors of $\bar{\gamma}$. For simplicity, let us take only A_z, A_t to be non-zero (we just want to know the form of the fermion solution for a particular gauge field configuration). This still allows for non-zero field strength on the string (i.e. non-zero E_z), which gives us an F for which the anomaly is non-zero. Our ansatz for ψ_- is

$$\psi_- = \eta f(r), \tag{1654}$$

where η is a zero-mode of the Dirac operator restricted to the string:

$$i(\gamma^t\partial_t + \gamma^z\partial_z - ie(A_t + A_z))\eta = 0. \tag{1655}$$

Since the string worldsheet is two-dimensional, we can assign the zero-mode η a define parity, and thus it has the potential to contribute to a cancellation of the gauge anomaly (we will determine its parity shortly). Thus we need to solve

$$\eta i\not{D}_A^\perp f(r) = -m(r)e^{i\theta}\psi_+, \tag{1656}$$

where $\not{D}_A^\perp$ is the Dirac operator on the coordinates orthogonal to the string. Solving this is easy since A is zero for these coordinates. Thus we have

$$i(\cos\theta\gamma^x + \sin\theta\gamma^y)\eta\partial_r f(r) = -m(r)e^{i\theta}\psi_+. \tag{1657}$$

We can take care of the $m(r)$ factor with an $f(r)$ which is exponentially localized to the string. We write $f(r) = \exp(-\int_0^r dr' m(r'))$, thus

$$i(\cos\theta\gamma^x + \sin\theta\gamma^y)\eta\exp\left(-\int_0^r dr' m(r')\right) = e^{i\theta}\psi_+. \tag{1658}$$

We can cleverly re-write this as

$$i\gamma^x e^{i\theta\bar{\gamma}_{\text{ext}}}\eta\exp\left(-\int_0^r dr' m(r')\right) = e^{i\theta}\psi_+, \tag{1659}$$

where $\bar{\gamma}_{\text{ext}} = -i\gamma^x\gamma^y$ is the chirality operator on the components orthogonal to the string. To write the Dirac equation in this form, we have taken η to be an eigenspinor of $\bar{\gamma}_{\text{ext}}$, this is possible since $\bar{\gamma}$ and $\bar{\gamma}_{\text{ext}}$ commute. Thus can solve the problem by choosing η to be the positive-chirality eigenstate of $\bar{\gamma}_{\text{ext}}$. Since η came from ψ_- which is a negative chirality eigenstate of $\bar{\gamma}$, the fact that $\bar{\gamma}_{\text{ext}}\eta = +\eta$ implies $\bar{\gamma}_{\text{int}}\eta = -\eta$, so that η is a negative-chirality zero-mode on the string. Since this also determines ψ_+, we have

$$\psi_- = \eta \exp\left(-\int_0^r dr' m(r')\right), \qquad \psi_+ = i\gamma^x \eta \psi_-, \tag{1660}$$

where again, $\bar{\gamma}_{\text{int}}\eta = -\eta$. Since γ^x commutes with $\bar{\gamma}_{\text{int}}$, we importantly have that the chirality of the zero mode components of both the ψ_+ and ψ_- is *the same* (while the chirality of the components orthogonal to the string are opposite).

Recapitulating, the presence of the string leads to a pair of zero modes propagating along the string with the same chirality. Since they are chiral with negative chirality, we can write their contribution to the Lagrangian as

$$i\bar{\eta}\not{D}^s_A \frac{(1-\bar{\gamma}_{\text{int}})}{2}\eta, \tag{1661}$$

where $\not{D}^s_A$ is the Dirac operator restricted to the string. Thus we have a coupling between the gauge field and the chiral current, which leads to a gauge anomaly. When we do a gauge transformation, this shifts by the usual gauge anomaly in two dimensions, determined by the index theorem. For $A \mapsto A - d\alpha$, the action for the string zero modes changes by

$$\delta S_s = -\int_s \frac{e^2}{4\pi} F\alpha, \tag{1662}$$

which exactly cancels the anomaly in the gauge current caused by the current-fermion-gauge-field diagram we computed earlier! Thus anomaly inflow from the bulk onto the string defect renders the whole theory self-consistent and anomaly-free.

Flavor symmetries of fermions and $Sp(N)$ gauge theories

In this diary entry we will discuss global flavor symmetries of fermions—both in general terms and in the specific example of fermions coupled to an $SU(2)$ gauge field (inspired by wanting to understand the construction in [6]).

First, some notational housekeeping. In the following, we will let

$$J \equiv (-iY) \otimes \mathbf{1}_N \tag{1663}$$

be the symplectic form preserved by elements in $Sp(2n;\mathbb{K})$, where $\mathbb{K}$ is some field. The compact subgroup of $Sp(2n;\mathbb{C})$ will be denoted

$$Sp(n) \equiv U(2n) \cap Sp(2n;\mathbb{C}). \tag{1664}$$

The question that motivated this entry was the following. Consider

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i \not{D}_A \psi_i, \tag{1665}$$

where A is the connection on some gauge group (which may be trivial). What is the global internal symmetry group of the above theory? Naively the answer is $U(N)$ (plus a possible $\mathbb{Z}_2^C$ depending on the gauge group), but this is a little bit hasty. To elucidate what the full symmetry group is, break ψ apart in terms of two real fields via

$$\psi_i = \chi_i + i\eta_i, \tag{1666}$$

where χ_i and η_i are majoranas, and define

$$\Psi^T \equiv (\chi_1, \chi_2, \dots, \chi_N, \eta_1, \dots, \eta_N)^T. \tag{1667}$$

Then since the action of $O(2N)$ preserves the commutation relations of the Majorannas and leaves $\mathcal{L}$ invariant, the global flavor symmetry is clearly $O(2N)$. Since $U(N) \subset O(2N)$, the full symmetry group is bigger than the naive $U(N)$.

What sorts of constraints can break the $O(2N)$ down to the naive $U(N)$? As far as mass terms go, the Dirac mass is $\bar{\psi}\psi = \bar{\chi}\chi + \bar{\eta}\eta$, since $\bar{\eta}\chi = \bar{\chi}\eta$. Thus the Dirac mass is invariant under $O(2N)$ and hence also under $U(N)$. The fermion number operator however is $\psi^\dagger\psi = 2 + 2i\chi^T\eta$, which up to a constant is $\Psi^T J \Psi$, and therefore is *not* preserved by the full $O(2N)$. This term is of course preserved by the diagonal $U(1)$, since the action of $U(1)$ is by $\Psi \mapsto U\Psi$, with $U = S \otimes \mathbf{1}_{N\times N}$ and $S \in SO(2)$. Since $-iY \in SO(2)$ and $SO(2)$ is Abelian, we have

$$U^T J U = (S^T \otimes \mathbf{1})(-iY \otimes \mathbf{1})(S \otimes \mathbf{1}) = (\mathbf{1} \otimes \mathbf{1})(-iY \otimes \mathbf{1})(S^T S \otimes \mathbf{1}) = J. \tag{1668}$$

Now while $\psi^\dagger\psi$ is not preserved by $O(2N)$, it is preserved by the full $U(N)$ (as should be obvious from how it acts on the complex fermions). Moreover, $U(N) \subset O(2N)$ is the maximal subgroup that preserves $\psi^\dagger\psi$. Indeed, preserving $\psi^\dagger\psi$ means preserving $\Psi^T J \Psi$, which means that if $R \in O(2N)$ is to preserve $\Psi^T J \Psi$, we need $R^T J R = J \implies R \in Sp(2N;\mathbb{R})$. Thus the group of transformations that preserve complex fermion number is

$$O(2N) \cap Sp(2N;\mathbb{R}) \cong U(N). \tag{1669}$$

Here the last equality is a manifestation of the 2-in-3 property, namely that the intersection

$$O(2N) \cap GL(N;\mathbb{C}) \cap Sp(2N;\mathbb{R}) = U(N), \tag{1670}$$

and that actually $U(N)$ is equal to the intersection of any two of the three groups on the LHS. Why is this? Let $V \in Sp(2n;\mathbb{R})$. Then $V^T J V = J$. Alternatively, let $V \in GL(N;\mathbb{C})$. Then when viewed as a $2N \times 2N$ real matrix, in order to have a legit complex structure, we need V to commute with some matrix i, such that $i^2 = -\mathbf{1}$ and $Vi = iV$. If the complex structure and symplectic structure being considered are compatible,[185] then we need to take $i = J$. Finally, if $V \in O(2N)$, then $V^T V = \mathbf{1}$. Thus if $V \in O(2N) \cap GL(N;\mathbb{C}) \cap Sp(2N;\mathbb{R})$, then $V^T = V^{-1}, V^T JV = J$, and $V^{-1}JV = J$. Then we see that any two of these properties implies the third; hence the 2-in-3 property. We can realize the matrices in $U(N)$ in this way by using our knowledge of the Lie algebra of $Sp(N)$, and taking only the real part. So we claim that all the elements in $U(N)$ can be written as

$$O(2N) \cap Sp(2N;\mathbb{R}) = U(N) = \{\exp\left(\mathbf{1} \otimes A + iY \otimes S\right) \,:\, A^T = -A,\, B^T = B\}. \tag{1671}$$

It's easy to check that the above matrices are orthogonal and preserve J. Have we missed any? No, let's count dimensions: there are $(N^2 - N)/2$ choices for A and $(N^2 + N)/2$ choices for S, and all of these choices give distinct elements in $O(2N) \cap Sp(2N;\mathbb{R})$. This adds up to N^2 total elements, which is the same as the number of generators for $U(N)$. So indeed, all the elements in $U(N)$ can be written as real matrices in this way.

Anyway, enough with that digression. Returning to the fermion problem, we see that $O(2N)$ contains elements which do not preserve $\psi^\dagger\psi$. Thus if we restrict to transformations that preserve the fermion number, we get that the flavor part of the symmetry group is the naive $U(N)$. As example, consider the matrix $Z \otimes \mathbf{1} \in O(2N)$. In the Ψ basis this sends all the χ's to themselves, and it multiplies the η's by minus signs. Thus $Z \otimes \mathbf{1} : \psi_i \mapsto \psi_i^\dagger$, and so $Z \otimes \mathbf{1}$ is charge conjugation. This doesn't preserve $\psi^\dagger\psi$, and indeed, while $Z \otimes \mathbf{1} \in O(2N)$, $Z \otimes \mathbf{1} \notin Sp(2N;\mathbb{R})$ and so $Z \otimes \mathbf{1} \notin U(N)$.

Another way to understand how the $O(2N) \to U(N)$ restriction of the symmetry group can come about is to remember that complex numbers are not simply two copies of $\mathbb{R}$: there is a complex structure that relates the two copies. Consider multiplication by i, $\psi_i \to i\psi_i$. We see that in the Ψ basis, this acts as J. Thus $i = J$ when acting on the Majorana fermions. Now if our flavor symmetry transformation R does not involve complex conjugation, then $Ri\psi = iR\psi$. But when written in terms of Majoranas, this means that $RJ = JR$, and so from the orthogonality of R, we have $R \in Sp(2N;\mathbb{R})$, and thus from the 2-in-3 property we know that $R \in U(N)$ (another way to say this is that $RJ = JR$ is the requirement of the existence of a complex structure, and tells us that $R \in GL(N;\mathbb{C})$. But from the 2-in-3 property, the orthogonality of R then implies that R is in $Sp(2N;\mathbb{R})$ as well. Since R is then both orthogonal and symplectic / complex structure preserving, must have $R \in U(N)$). So if we want R to preserve the complex structure, i.e. for R to not be anti-unitary, then R must be in $U(N)$ (which admittedly sounds kind of tautological).

[185] A complex structure on a vector space $\mathcal{W}$ means we can realize it as a direct sum $\mathcal{W} \cong V \oplus V$, where V is real (so e.g. the tangent bundle TM has an [almost] complex structure). A symplectic structure on $\mathcal{W}$ means that we can decompose it as $\mathcal{W} \cong W \oplus W^*$ for W real (this means choosing coordinates and momentum; e.g. T^*M has a symplectic structure with the dimensions coming from M being the coordinates and those coming from the fiber being the momenta), with the symplectic form being given by $\omega(v \oplus f, u \oplus g) = f(u) - g(v)$. The compatibility of the complex and symplectic structures means that we can choose V (known as the real subspace) and W (known as the Lagrangian subspace) to be equal, with the symplectic form corresponding to multiplication by i.

$SU(2)$ gauge theory with N Dirac fermions

We now try to understand the global symmetry of N Dirac fermions, all coupled to an $SU(2)$ gauge field in the fundamental. This again comes from wanting to understand the construction in [6]. As in the previous subsection, the naive guess for what the internal part of the global symmetry should be, namely $U(N)/\mathbb{Z}_2$ (or maybe $[U(N)/\mathbb{Z}_2] \rtimes \mathbb{Z}_2$ for charge conjugation) is not correct. In fact the internal part of the global symmetry is actually $PSp(N)$!

Let's see how this comes about. Let $\psi_i = (\psi_{i\uparrow}, \psi_{i\downarrow})^T$ be one of the Dirac fermions in the fundamental of $SU(2)$. A single fermion in $SU(2)_f$ can be built from four majoranas. We can build it as a matrix field as follows:

$$\mathcal{X}_i = \frac{1}{\sqrt{2}}(\chi_i^1 \mathbf{1} + i\chi_i^a \sigma^a), \tag{1672}$$

with $a \in \{x, y, z\}$. We build the constituent complex fermions from the Majoranas so that (dropping the flavor index for simplicity)

$$\mathcal{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^1 + i\chi^z & i\chi^x + \chi^y \\ i\chi^x - \chi^y & \chi^1 - i\chi^z \end{pmatrix} = \begin{pmatrix} \psi_\uparrow & \psi_\downarrow \\ -\psi_\downarrow^\dagger & \psi_\uparrow^\dagger \end{pmatrix}. \tag{1673}$$

With this one can check that $\mathrm{Tr}[\bar{\mathcal{X}} \not{D}_A \mathcal{X}]$ gives the correct Dirac Lagrangian, with the $SU(2)$ gauge field A acting on the right in the covariant derivative. The mass term $\mathrm{Tr}[\bar{\mathcal{X}}\mathcal{X}]$ is $\sum_\alpha \bar{\chi}^\alpha \chi^\alpha$, as expected.

Consider the right action on $\mathcal{X}$ by $SU(2)$. Right multiplication by e.g. $e^{i\alpha Z}$ does

$$\mathcal{X} \mapsto \mathcal{X} e^{i\alpha Z} = \begin{pmatrix} e^{i\alpha}\psi_\uparrow & e^{-i\alpha}\psi_\downarrow \\ -e^{i\alpha}\psi_\downarrow^\dagger & e^{-i\alpha}\psi_\uparrow^\dagger \end{pmatrix}, \tag{1674}$$

which is just what a gauge rotation about the z axis in $SU(2)$ should do. So, we see that the $SU(2)$ we want to gauge is the right action on $\mathcal{X}$ by $SU(2)$.

The left action then parametrizes the system's global flavor symmetry. In order for $\bar{\mathcal{X}} \not{D}_A \mathcal{X}$ to be left invariant, the U in $\mathcal{X} \mapsto U\mathcal{X}$ must be unitary, and since there are N flavors of Dirac fermions, $U \in U(2N)$. However, there is an additional restriction. Indeed, consider the fact that

$$\mathcal{X}^\dagger = (Y \otimes \mathbf{1})\mathcal{X}^T (Y \otimes \mathbf{1}). \tag{1675}$$

Now take $\mathcal{X} \mapsto U\mathcal{X}$. Then we need

$$\mathcal{X}^\dagger U^\dagger = (Y \otimes \mathbf{1})\mathcal{X}^T U^T (Y \otimes \mathbf{1}) \implies \mathcal{X}^\dagger = (Y \otimes \mathbf{1})\mathcal{X}^T (Y \otimes \mathbf{1})^2 U^T (Y \otimes \mathbf{1}) U = \mathcal{X}^\dagger (Y \otimes \mathbf{1}) U^T (Y \otimes \mathbf{1}) U. \tag{1676}$$

In particular, this means that

$$U^T J U = J \implies U \in Sp(N) = U(2N) \cap Sp(2N; \mathbb{C}). \tag{1677}$$

So, the global symmetry on the left action is tentatively identifiable with $Sp(N)$.

Actually this is not completely true, since it may happen that elements of the global symmetry acting from the left act in the same way as elements of the gauge group acting from the left. Clearly this is true for $-\mathbf{1}$, which acts the same both as a $Sp(N)$ element from the left and an $SU(2)$ element from the right. In fact, this is the only common element shared by the two actions. Indeed, consider a given element of $Sp(N)$ acting from the left, and ask if it is equivalent to an element of $SU(2)$ acting from the right. Since the $SU(2)$ acts in the same way on each Dirac fermion, we just need to look for elements of $Sp(N)$ that are diagonal on the flavor index, and so we can restrict ourselves to a single flavor wolog, and take the left action to be that of $Sp(1) = SU(2)$. Then consider the $U(1)$ rotation $e^{i\alpha Z}$ acting from the left. This multiplies both $\psi_\uparrow$ and $\psi_\downarrow$ by the same phase. This can never be done by an element of $SU(2)$ acting on the left: the only element which just multiplies $\psi_\uparrow$ and $\psi_\downarrow$ by phases does so in a gauge-invariant way, namely by multiplying $\psi_\uparrow$ by $e^{i\alpha}$ and $\psi_\downarrow$ by $e^{-i\alpha}$. So the left action by $e^{i\alpha Z}$ is only equivalent by the right action of something in $SU(2)$ if $e^{i\alpha Z} = -\mathbf{1}$. Since every element in the left $SU(2)$ can be written as $e^{i\alpha Z}$ in the right choice of basis, every element of the left $SU(2)$ action (except $-\mathbf{1}$) must also not be expressible as the action of some $SU(2)$ element from the right. Thus only the $-\mathbf{1}$ gets modded out, and the global symmetry acting on the left is in fact $PSp(N) = Sp(N)/\mathbb{Z}_2$.

The fact that the global symmetry is so big is kind of surprising! For example, take $N = 1$: the internal part of the global symmetry is then $PSp(1) = SU(2)/\mathbb{Z}_2 = SO(3)$. If we just looked at the Lagrangian $\bar\psi \not{D}_A \psi$, we might have thought that the global internal symmetry was $U(1)$, or maybe $O(2) = U(1) \rtimes \mathbb{Z}_2$ after including charge conjugation. But in fact the real global symmetry is bigger! This is because the conclusion that the symmetry is $U(1)$ came from requiring the global symmetry to act identically on both of the components of the $SU(2)$ doublet. This is a natural thing to do, since the global symmetry has to commute with the action of the gauge group. But we see from this example that we can actually have the global symmetry act nontrivially on the different components in the $SU(2)$ doublet! For example, consider the left action by $e^{i\alpha Z}$. This is the diagonal $U(1)$ that we would have guessed to be the naive global symmetry. But what about the left action by $e^{i\alpha X}$? One checks that this sends e.g $\psi_\uparrow \mapsto i\psi_\downarrow^\dagger, \psi_\downarrow \mapsto i\psi_\uparrow^\dagger$: so it mixes the two components of the doublet, but also charge-conjugates them; this allows it to commute with gauge transformations. Thus the action of charge conjugation is built in to the $PSp(N)$ symmetry. Or more precisely, it's mixed up between the $PSp(N)$ and $SU(2)$ actions.

More abstractly, the fact that we get a $PSp(N)$ global symmetry can be understood through the decomposition

$$SO(4N) \supset \frac{SU(2) \times Sp(N)}{\mathbb{Z}_2}. \tag{1678}$$

The relevance of this is that N Dirac fermions in the fundamental of $SU(2)$ can be written as $4N$ Majorannas, which are acted on by $SO(4N)$. The $Sp(N)$ factor in the above decomposition is the largest subgroup which commutes with the $SU(2)$, and so after gauging the $SU(2)$ we are left with an $Sp(N)/\mathbb{Z}_2$'s worth of global symmetry.

Note that this inclusion is not an equality in general, as we check by computing dimensions: as a Lie algebra, $\dim \mathfrak{so}(4N) = (16N^2 - 4N)/2 = 8N^2 - 2N$, while

$$\dim[\mathfrak{su}(2) \times \mathfrak{sp}(N)] = 3 + (N^2 - N)/2 + 3(N^2 + N)/2 = 2N^2 + N + 3 \leq \dim \mathfrak{so}(4N). \tag{1679}$$

In fact the equality does hold when $N = 1$ for which both Lie algebras are 6-dimensional, which is just a manifestation of

$$SO(4) = \frac{SU(2) \times Sp(1)}{\mathbb{Z}_2}, \tag{1680}$$

since $Sp(1)$ has alias $SU(2)$.[186]

Let's see how this decomposition works explicitly, at the level of Lie algebra generators. $Sp(N)$ is complex, and so in order to embed it in $SO(4N)$, we will need to send $i \mapsto J_2$, where now we are using the notation $J_2 \equiv -iY$. Recalling a diary entry from last year wherein the generators for $\mathfrak{sp}(N)$ were written down, we have that when embedded in $\mathfrak{so}(4N)$, the $\mathfrak{sp}(N)$ generators are

$$\mathfrak{sp}(N) \ni (A_N \otimes \mathbf{1}_2 + S_N^y \otimes iY) \otimes \mathbf{1}_2 + (S_N^x \otimes X + S_N^z \otimes Z) \otimes J_2, \tag{1683}$$

where A_N is an antisymmetric $N \times N$ matrix, the S^a are symmetric $N \times N$ matrices, and $\mathbf{1}_2$ is the 2×2 unit matrix.

What about the $SU(2)$ factor? To embed this in $SO(4N)$, we need to turn the symmetric generators X, Z into antisymmetric matrices. We do this by tensoring with J_2. Since the exponential map is $\exp(\mathfrak{so}(4N)) = SO(4N)$ (no factor of i), we also want the $SU(2)$ generators to be i times the normal physicist-convention $SU(2)$ generators. One sees that the following three generators

$$\sigma^1 = \mathbf{1}_N \otimes J_2 \otimes X, \quad \sigma^2 = \mathbf{1}_N \otimes \mathbf{1}_2 \otimes iY, \quad \sigma^3 = \mathbf{1}_N \otimes J_2 \otimes Z \tag{1684}$$

obey $[\sigma^a, \sigma^b] = \epsilon^{abc}\sigma^c$, and hence generate an $SU(2) \subset SO(4)$ subgroup. Furthermore, we see that all of these generators commute with the $\mathfrak{sp}(N)$ generators, telling us that indeed, $[SU(2) \times Sp(N)]/\mathbb{Z}_2 \subset SO(4N)$, where the quotient is because $-\mathbf{1}_{2N}$ is in both factors. Furthermore we see that $Sp(N)$ is the largest subgroup that commutes with $SU(2)$, so that when e.g. $SU(2)$ is gauged, $PSp(N)$ is the global symmetry that remains.

[186] Here's a way to see $SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2$: let $r^2 = x^2 + y^2 + z^2 + t^2 = 1$ define S^4, and consider the matrix $U = t\mathbf{1} + i(xX + yY + zZ)$. We will think of the coordinates $v = t + iz, w = y + ix$ as equivalent coordinates for the sphere in $\mathbb{C}^2$. With this,

$$U = \begin{pmatrix} v & w \\ -\bar{w} & \bar{v} \end{pmatrix} \tag{1681}$$

Consider $[SU(2)_L \times SU(2)_R]/\mathbb{Z}_2$, with the first factor acting on U on the left and the second factor acting on the right. A generic element of $SU(2)$ is conjugate to $e^{i\theta Z}$. Acting on the left, these actions send

$$e^{i\theta Z_L} : v \mapsto e^{i\theta v}, \quad w \mapsto e^{i\theta w}, \qquad e^{i\theta Z_R} : v \mapsto e^{i\theta v}, \quad w \mapsto e^{-i\theta w}. \tag{1682}$$

This means that the left $SU(2)$ action, after being conjugated so that it only has a Z part, rotates the tz plane and the xy plane by equal angles, while the right $SU(2)$ action rotates them by opposite angles. Since both actions preserve $\det U = 1$, they are symmetries of S^4. Furthermore since they only rotate planes (pairs of basis vectors) instead of individual basis vectors, they are orientation preserving (okay, also since they have det $= 1$). We can generate all of $SO(4)$ by rotating arbitrary planes, and so after modding out by the common $-\mathbf{1}$ to both actions, we get the desired isomorphism.

Global symmetry that remains after gauging a subgroup

Suppose we have some fields transforming under a global symmetry group $\mathcal{G}$ (which includes spacetime symmetries, e.g. some pin group if they are fermions), and we gauge a subgroup $G \subset \mathcal{G}$. What is the surviving global symmetry group? Not $\mathcal{G}/G$: G may not be normal, and so it may "take out" more of $\mathcal{G}$ than just itself. Today we will answer the question, and discuss a few illustrative examples.

We can find the resulting global symmetry group by examining how various charge operators for the symmetries in $\mathcal{G}$ commute with the generator of gauge transformations in G. If U_θ generates the gauge transformation $\psi \mapsto e^{\theta^a T^a_G}\psi$ for the G gauge group (T^a are the generators of G; the case of G discrete is basically the same) and if e^{iQ_h} is the charge operator for a global symmetry acting with an element $h \in \mathcal{G}$, then we require that

$$e^{iQ_h} U_\theta = U_{\theta'} e^{iQ_h}, \tag{1685}$$

since then the action of e^{iQ_h} is well-defined when acting on physical states, for which U_θ acts as $\mathbf{1}$ for all choices of θ.

Let the surviving global symmetry group be denoted by $\mathcal{G}'$. Then the above means that if $h \in \mathcal{H}'$ then for any $g \in G$ we must have $h^{-1}gh = g'$ for some $g' \in G$. The means that h must be in the normalizer[187] of G with respect to $\mathcal{G}$, quotiented by G itself (which is trivially in the normalizer):

$$\mathcal{G}' = N_{\mathcal{G}}(G)/G. \tag{1686}$$

That $N_{\mathcal{G}}(G)/G$ is a subgroup[188] of $\mathcal{G}$ is easy to check (easy to see that the normalizer is a subgroup, and G is by definition normal in $N_{\mathcal{G}}(G)$, so we can consistently take the quotient). The simplest example is of course $\mathcal{G} = H \times G$, for which $N_{\mathcal{G}}(G) = \mathcal{G}$ and hence $\mathcal{G}' = \mathcal{G}/G = H$.

Gauging $U(1) \subset O(2n)$

As a slightly nontrivial example, think of $\mathbb{R}$ fermions and let $\mathcal{G} = O(2n)$ be a flavor symmetry with G the diagonal $U(1)$, a given element of which takes the form $R_\theta^{\oplus n}$, where R_θ is a 2×2 rotation matrix. Now $[R_\theta, x\mathbf{1} + yJ] = 0$ for all x, y, and therefore one can show that $U(n) \subset O(2n)$ commutes with the diagonal $U(1)$ (we embed $U(n)$ into $O(2n)$ by writing each complex entry $z = x + iy$ as $x\mathbf{1} + yJ$), and so we at least have $U(n) \subset N_{O(2n)}(U(1))$.

What about the reflection that extends $SO(2n)$ to $O(2n)$? If we take this reflection to be the generator R of $\mathbb{Z}_2^R$ such that $R = Z \oplus \mathbf{1}_{2n-2}$, then we get something that

[187] Recall that $N_{\mathcal{G}}(G) = \{h \in \mathcal{G} \,|\, hG = Gh\}$.

[188] We might call this the Weyl group of G in $\mathcal{G}$, or something like that, since the Weyl group in the context of Lie theory is defined as $N_G(T)/T$, where T is a maximal torus.

doesn't commute with the $U(1)$, since conjugation by R does

$$R_\theta^{\oplus n} \mapsto RR_\theta^{\oplus n}R = R_{-\theta} \oplus R_\theta^{n-1} \notin U(1). \tag{1687}$$

Hence, $\mathbb{Z}_2^R \notin N_{O(2n)}(U(1))$. However, consider the action of $Z^{\oplus n}$, which reflects every other axis. This performs the action that we would usually associate with charge conjugation, viz. $Z^{\oplus n}R_\theta^{\oplus n}Z^{\oplus n} = R_{-\theta}^{\oplus n} \in U(1)$. Therefore we define the charge conjugation matrix C by

$$C \equiv Z^{\oplus n} \in N_{O(2n)}(U(1)). \tag{1688}$$

Suppose $n \in 2\mathbb{Z}$. Then C has determinant 1, and is in fact part of $SO(2n)$—it is not an outer automorphism extending $SO(2n)$ to $O(2n)$. However, if $n \in 2\mathbb{Z}+1$, $\det Z^{\oplus n} = -1$, and it is a reflection outer automorphism that extends to $O(2n)$. Regardless of whether it is a reflection or not, it is not in $U(n)$, since when represented as a matrix in $O(2n)$, the only diagonal matrix in $U(n)$ is the identity. Now one checks that C is a good moniker for $Z^{\oplus n}$ by noting that $C^\dagger UC = U^*$ for any $U \in U(n)$. Therefore we have at least $N_{O(2n)}(U(1)) \supset U(n) \rtimes \mathbb{Z}_2^C$. In fact the normalizer is exactly $U(n) \rtimes \mathbb{Z}_2^C$: $U(n) \rtimes \mathbb{Z}_2^C$ is a maximal subgroup of $O(2n)$ (which can be worked out from the material in [4]), and since the normalizer is a proper subgroup of $O(2n)$, it must be $U(n) \rtimes \mathbb{Z}_2^C$. Taking the quotient by $U(1)$, we then get that the remaining global symmetry is

$$\mathcal{G}' = PSU(n) \rtimes \mathbb{Z}_2^C. \tag{1689}$$

In this case the $\mathbb{Z}_2^C$ really is a charge conjugation symmtey, since it corresponds to the group $\mathrm{Out}(PSU(n)) = \mathbb{Z}_2$.[189]

Gauging $SU(2) \subset O(4n)$

As another example relevant for fermions, considering gauging an $SU(2)$ subgroup of $O(4n)$, where the $SU(2)$ subgroup acts in a block-diagonal way, with each block a 4×4 orthogonal matrix. We will write basis vectors in $\mathbb{R}^{4n}$ suggestively as $v \equiv (\chi_1^\uparrow, \eta_1^\uparrow, \chi_1^\downarrow, \eta_1^\downarrow, \chi_2^\uparrow, \dots)^T$, where we think of the variables as Majorana fermions coming from n complex fermions in the fundamental of $SU(2)$; e.g. $\psi_{\sigma i} = \chi_i^\sigma + i\eta_i^\sigma$. The $SU(2)$ we're gauging is then realized as (the subscript g appears when needed to distinguish the gauge group from other groups floating around)

$$SU(2)_g \ni \begin{pmatrix} x\mathbf{1} + yJ & w\mathbf{1} + zJ \\ -w\mathbf{1} + zJ & x\mathbf{1} - yJ \end{pmatrix}^{\oplus n}, \qquad (x,\, y,\, w,\, z) \in S^3. \tag{1690}$$

Let's first work out the simple case of $n = 1$. We claim that the normalizer includes $Sp(1) \cong SU(2)$, which is realized as matrices of the form

$$Sp(1) \ni x\mathbf{1}\otimes\mathbf{1} + w\mathbf{1}\otimes J + yJ\otimes X + zJ\otimes Z, \qquad (x,\, y,\, w,\, z) \in S^3. \tag{1691}$$

Here the first tensor factors keep track of the way of representing complex numbers with real matrices, while the second factors keep track of the Pauli matrix structure

[189] We are tacitly assuming $n \neq 2$.

of the group elements. Indeed, the above matrices can be seen to commute with the gauged $SU(2)_g$, which is generated by matrices of the form

$$SU(2)_g \ni x\mathbf{1}\otimes\mathbf{1} + yJ\otimes\mathbf{1} + wZ\otimes J + zX\otimes J, \qquad (x,\, y,\, w,\, z) \in S^3. \tag{1692}$$

Here by contrast it is the second tensor factors that keep track of the representation of $1, i$ in terms of real matrices. The antisymmetric form preserved by the $Sp(1)$ in the normalizer is $J\otimes\mathbf{1}$. Therefore the normalizer is at least $N_{O(4)}(SU(2)) = [Sp(1)\times SU(2)]/\mathbb{Z}_2 = SO(4)$, with the quotient coming from $-\mathbf{1}$ being in both groups.[190]

We claim that the reflection that extends $SO(4)$ to $O(4)$ is not in the normalizer. Abstractly, this is because this reflection generates $\mathrm{Out}(SO(4)) = \mathbb{Z}_2$, which from experience we know exchanges the two $SU(2)$s in $SO(4) \cong (SU(2)\times SU(2))/\mathbb{Z}_2$—therefore conjugation by the reflection should not be in the normalizer of a single $SU(2)$ factor. Let's check this with an example: the reflection can be taken to be $R = (-Z)\oplus\mathbf{1}$. When we act on the matrix $J\otimes X \in SU(2)_g$ in the gauged $SU(2)$ we get

$$R^{-1}(J\otimes X)R = -(\mathbf{1}\otimes J) \in Sp(1), \tag{1693}$$

so that conjugation by the reflection indeed swaps $SU(2)_g$ and $Sp(1)$, meaning that the $Sp(1)$ is the reflected image of the gauge group. Therefore since the normalizer is a proper subgroup of $O(4)$ and contains $SO(4)$, it must be $SO(4)$. Another way of saying this is that since $SO(4)$ is a maximal subgroup of $O(4)$, $SO(4)$ must be the whole normalizer, with the global symmetry group thus being $\mathcal{G}' = PSp(1) = PSU(2)$.

This means "charge conjugation" is already included in the normalizer: indeed, if we let it act as $\chi^\sigma + i\eta^\sigma \mapsto \chi^\sigma - i\eta^\sigma$ then $C = \mathbf{1}\otimes Z$ is already included, and if we let it exchange spins then it is $C = J\otimes Z$, which is also included. But really, the point is that we shouldn't be calling such a thing charge conjugation: $\mathrm{Out}(PSU(2)) = \mathbb{Z}_1$ and $PSU(2)$ isn't a semi-direct product, and so there can't possibly be any type of charge conjugation symmetry remaining in the theory after the gauging occurs.

Now we return to the case of arbitrary n. It is easier in this case to make a change of basis and package the vector of χs and ηs as a $2n\times 2$ matrix with complex entries. Thought of as a n-component column vector V with matrix-valued entries, the ith entry is the "quaternonic fermion" matrix

$$V_i = \begin{pmatrix} \chi_i^\uparrow + i\eta_i^\uparrow & \chi_i^\downarrow + i\eta_i^\downarrow \\ -\chi_i^\downarrow + i\eta_i^\downarrow & \chi_i^\uparrow - i\eta_i^\uparrow \end{pmatrix}. \tag{1694}$$

The $SU(2)$ we are gauging is then realized as the right action on the above matrices, with the elements in $SU(2)$ written as 2×2 complex matrices, rather than 4×4 real

[190] A rather highbrow way of saying why the $Sp(1)$ commutes with the $SU(2)_g$ is that it uses the isomorphism coming from pseudoreality of the fundamental rep of $SU(2)$ together with complex conjugation to create a trivial action on the $SU(2)_g$. The matrices $\mathbf{1}\otimes\mathbf{1}$ and $\mathbf{1}\otimes J$ in $Sp(1)$ form a $U(1)$ that is the obvious one which commutes with $SU(2)_g$ (the diagonal particle-number $U(1)$ symmetry in the action). The matrices $\mathbf{1}\otimes X, \mathbf{1}\otimes Z$ both have the effect of complex-conjugating the $SU(2)_g$, since they both anti-commute with the way we've chosen to represent the number i in that group, namely as $\mathbf{1}\otimes J$. They can then be combined with the matrix $J\otimes\mathbf{1}$, which is the isomorphism establishing the pseudoreality of $SU(2)$, to produce $J\otimes X$ and $J\otimes Z$, which complete the $Sp(1)$ part of the normalizer.

matrices. The length of the vector v is determined by

$$|v|^2 = \frac{1}{2}\mathrm{Tr}[V^\dagger V], \tag{1695}$$

and as such is properly preserved by the $SU(2)$ right action. This form also makes it clear that the length is preserved by a left $SU(2n)$ action, which by construction commutes with the right $SU(2)$. However, the left $SU(2n)$ action is too big: the structure of the is in the above form of V_i needs to be preserved by any putative left action, so that the right $SU(2)$ acts properly. The structure of the is (the $\sqrt{-1}s$, not the flavor labels) is encapsulated in the relation $(\mathbf{1}_n \otimes J)^\dagger V J = V^*$, which needs to be preserved by the left action. Therefore if $U \in SU(2n)$ then we need

$$(\mathbf{1}_n \otimes J)^\dagger U(\mathbf{1}_n \otimes J)(\mathbf{1}_n \otimes J)^\dagger V J = (UV)^* \implies (\mathbf{1}_n \otimes J)^\dagger U(\mathbf{1}_n \otimes J) = U^*, \tag{1696}$$

which since U is unitary means $U^T(\mathbf{1} \otimes J)U = \mathbf{1} \otimes J$, and so in fact we must have $U \in Sp(n)$. If this is a bit too slick, one can also make the following (still rather slick) argument: whatever the global symmetry group that remains is, the full symmetry group (gauged $SU(2)$ + global) better have only real representations, since we started off with an $O(4n)$ symmetry. Since we gauged an $SU(2)$ acting in the fundamental, which is pseudoreal, the global symmetry group must also act via a pseudoreal representation, since the $\otimes$ of two ps$\mathbb{R}$ reps is $\mathbb{R}$. Therefore the global symmetry can't be $U(2n)$—$Sp(n)$ works though, since it acts in a ps$\mathbb{R}$ way.

Anyway, we now know that the normalizer is at least $N_{O(4n)} \subset [Sp(n) \times SU(2)]/\mathbb{Z}_2$. Is this the whole normalizer? This is in fact the whole normalizer, since a math fact [4] is that $(Sp(n) \times Sp(1))/\mathbb{Z}_2 = PSp(n) \times SU(2)$ is a maximal subgroup of $O(2n)$, and through the same reasoning as in the last example, this must be the full normalizer. Therefore the global symmetry remaining is found by taking a quotient by $SU(2)$, producing

$$\mathcal{G}' = PSp(n). \tag{1697}$$

Note that we do not get $U(n) \rtimes \mathbb{Z}_2$ or similar, which we might have naively concluded based on thinking about complex fermions.

Finally, what about charge conjugation? There actually is no real charge conjugation symmetry in this case: $\mathrm{Out}(PSp(n)) = \mathbb{Z}_1$ and so there's no type of charge conjugation that we're missing, and $PSp(n)$ can't be written as a semidirect product involving a $\mathbb{Z}_2$ factor,[191] and so there is no $\mathbb{Z}_2^C$ symmetry hiding in the $PSp(n)$. If we were to think about charge conjugation as sending $\chi_i^\sigma \mapsto \chi_i^\sigma$ and $\eta_i^\sigma \mapsto -\eta_i^\sigma$ then it is already included in $(Sp(n) \times SU(2))/\mathbb{Z}_2$, while if we were to have it acting with J in on the $SU(2)$ factor then it'd already be included in the $Sp(n)$ factor—but either of these actions wouldn't really be a charge conjugation symmetry, since neither of them are outer automorphisms. So, while our un-gauged symmetry group $O(4n)$ includes a charge conjugation symmetry since we can write it as $SO(4n) \rtimes \mathbb{Z}_2^C$, when we gauge $SU(2)$, the existence of a charge conjugation symmetry goes away.

[191] Because $PSp(n)$ is connected.

Charge conjugation, outer automorphisms, and the best definition of time reversal

In today's diary, we'll briefly go over the reasons for why what the definitions the community at large uses for T and for CT should really be swapped, and will try to elucidate the meaning of CPT in simpler terms.

This discussion should be prefaced by mentioning that the philosophy here is that symmetries aren't less of some fundamental thing that a theory is defined by; rather they are more like tools that we use to study a given theory. Consequently, the exact definition of T and CT and so on is rather subjective. We define symmetries in a way that suits our needs for doing calculations, rather than having them handed down to us from on high. Therefore as there is no real "natural" definition of these symmetries, one should try to work with conventions that are as pleasant as possible — hence the present diary entry.

"Problems" with the historical definition of T

Conceptually, I think that the most natural definition of a time-reversal symmetry is one which is antiunitary and reverses time, and does nothing else pertaining to other possible internal symmetries. In particular, T should be a part of the Lorentz group, in the sense that its action (modulo the $\mathbb{C}$ conjugation part) should be restricted to act only via an action of the Lorentz group on the Lorentz indices of the fields in the theory.[192]

To see an example of why the historically-used T isn't necessarily part of the Lorentz group and hence fails this criterion, consider fermions coupled to a gauge field A,[193] with the action containing the term $\bar{\psi}\not{A}^a T^a \psi$ for A^a_μ some real vector fields. We will focus on this term for clarity and since it is conceptually the simplest thing to look at (all other irreps of Lorentz can be obtained from $\otimes$s of (s)pinor reps; hence the focus on fermions). Anyway, this term is definitely something that from experience we "expect" to be T-invariant. Since A^a_μ has a vector index the natural action of time reversal would be something like

$$T : A_\mu \mapsto U_T^\dagger A_\mu U_T = (-1)^{\delta_{\mu,0}} A_\mu, \tag{1698}$$

[192] Terminology reminder: the Lorentz group is not connected, and only the **1** component of the Lorentz group needs to be a symmetry in relativistic QFT. So e.g. reflections are part of the Lorentz group, but may or may not be symmetries.

[193] This field may or may not be dynamical; we are just introducing it in order to be able to talk about the action of symmetries in a convenient way.

for some unitary U_T—note that there is no action on the gauge index of A^a_μ. However, this does not lead to T-invariance in general, because of the $\mathbb{C}$ conjugation. The general condition for the T invariance of this coupling is (since we are physicists, the T^as are Hermitian)

$$U_T^\dagger A^a_\mu (T^a)^* U_T = (-1)^{\delta_{\mu,0}+1} A^a_\mu T^a, \tag{1699}$$

which will not be satisfied for a general gauge group and representation.

Consider first the case when the fermions transform in a real representation. Then $(T^a)^* = -T^a$ on account of the group representations being $e^{i\theta^a T^a}$, and so in this case the expected transformation law for T, viz. (1698), leaves $\bar\psi \not{A} \psi$ invariant.[194]

Now consider the case when the fermions are in a pseudoreal representation R, with J the antisymmetric matrix relating R and R^* through conjugation. Then since our generators are Hermitian, $J^\dagger T^a J = -(T^a)^*$. Therefore we see that we can take A to transform with the sign in (1698) along with conjugation by J, so that $U_T = J$ when acting on the gauge indices.[195] This means that the action of time reversal actually involves a nontrivial action on the gauge indices, because of the conjugation by J. Therefore the action of T does not just depend on the Lorentz indices of the field in question, which in my opinion is not ideal for a definition of T.

The situation is even worse for fermions in a complex representation. Since there is no longer an isomorphism connecting R and R^*, we won't be able to find a choice of U_T that will work, and we have to map each of the different A^a individually with a different sign: we need to act on the generator index with a transformation via $A^a_\mu \mapsto R^{ab} A^b_\mu$. For example, for the fundamental of $SU(3)$ with the usual basis for T^a, we would need T to map A^a for $a = 2, 5, 7$ with one sign, and A^a for $a = 1, 3, 4, 6, 8$ with the other.

Anyway, the point here is that this definition of T necessitates adding in an action on the gauge indices which cancels the complex conjugation performed by T, except in the case where the fermions transform in a real representation. This means that this T does not generically act solely on Lorentz indices. In other words, if we write $U_T = U_{T,I} \otimes U_{T,L}$ where the first factor acts on the internal indices and the second factor acts on the Lorentz indices, then $U_{T,I} \neq \mathbf{1}$.[196] In fact the precise requirements for the two factors are that (assuming the Dirac adjoint is used)[197]

$$U_{T,I}^\dagger (T^a)^* U_{T,I} = -T^a, \quad U_{T,L}^\dagger \mathcal{K}[i\gamma^0(-\gamma^0\partial_0 + \gamma^i \partial_i)]\mathcal{K} U_{T,L} = i\not{\partial}. \tag{1701}$$

With this transformation, $\not{A}$ transforms in the same way as $i\not{\partial}$, and the fermion action is consequently invariant.

[194] In this case there may not be any nontrivial notion of charge conjugation anyway, and so T and CT might not be distinguishable to begin with.

[195] We can either have

$$T : A^a_\mu \mapsto (-1)^{\delta_{\mu,0}} J^\dagger A^a_\mu J, \tag{1700}$$

or we can strip off the Js and modify the transformation of the fermions ψ under T by an action of J on the flavor indices. Either way, $\bar\psi \not{A} \psi$ is invariant.

[196] We are as always ignoring weird things like SUSY where the internal and spacetime dof mix.

[197] Pedantic detail: remember that the sign in front of ∂_0 doesn't come from $U_{T,L}^\dagger \partial_0 U_{T,L} = -\partial_0$, since in the present way of thinking about things we are never *actually* acting on spacetime; the action of symmetries is entirely on the fields, and not on numbers like t (conceptually, I think it's best to always think of the action as being entirely performed through conjugating second-quantized operators). The minus sign instead comes when when change variables $t \mapsto -t$ in the action.

Why CT is better

First, recall what we mean by C: it is a $\mathbb{Z}_2$ outer automorphism of the symmetry group G,[198] which acts on a field ψ in a representation R of G as $\psi \mapsto C\psi$, with $C\psi$ transforming in the dual representation R^*. Sometimes we would write this as $\psi \mapsto \psi^*$ in the case of e.g. a $U(1)$ symmetry for a single complex fermion, but this could always be re-written as $\psi \mapsto C\psi$ where we think of ψ as two real fermions, and of C as acting via the matrix Z.

There are two possible conventions for the action of C: it can dualize the representation of only the *internal* symmetries involved, or it can include a dualization of the spacetime symmetry representation as well. For example, in the former definition, C would map a field which annihilates left-handed neutrinos to a field which annihilates left-handed antineutrinos, while in the latter definition it would map to a field which annihilates right-handed antineutrinos. Here an antiparticle is one whose *internal* symmetry quantum numbers are all the duals of the quantum numbers of the particle in question—therefore in the former definition C sends particles to antiparticles, while in the latter definition it does this plus an action on the Lorentz indices (usually by parity). Now the former definition of C may not even be a legit operation to perform in a given QFT (right-handed neutrinos [not anti-neutrinos] may not even exist!), while with the second definition, C is always a legit thing to do. Therefore, we will work with the later definition, as we have done throughout most of the diary.

In any case, in the discussion of C, P, and T (or better, C, R, T), C always feels like a bit of a misfit, since it's not part of the Lorentz group. In fact, with our definition of T, we have already seen that T is not always part of the Lorentz group either! However, we will now argue that the product CT always *is* part of the Lorentz group.[199]

If we include the action of C in the definition of time reversal, the problems found above for the case of a ps$\mathbb{R}$ or $\mathbb{C}$ representation go away. Indeed, assuming that the theory possesses a C symmetry and writing the charge conjugation matrix as $C_I \otimes C_L$, we see that C-invariance requires[200]

$$C_L^\dagger \gamma_\mu C_L = -\gamma_\mu^T, \qquad C_I^\dagger T^a C_I = -[T^a]^T. \tag{1702}$$

Now recall that we are in conventions where the T^a are Hermitian; this means that the equation for C_I can be written

$$C_I [T^a]^T C_I^\dagger = -T^a \implies C_I [T^a]^* C_I^\dagger = -T^a. \tag{1703}$$

But we see that this is the same as the equation for $U_{T,I}$, just with †s in different places! Hence we may in fact set $U_{T,I} = C_I^\dagger$, meaning that when we put C and T together as $CT = (CT)_I \otimes (CT)_L \mathcal{K}$, we have $(CT)_I = U_{T,I} C_I = C_I^\dagger C_I = \mathbf{1}$, so that in fact CT acts only on Lorentz indices as claimed.

[198] Where pedantically the "full" symmetry group is G', with $G'/\mathbb{Z}_2^C = G$, or $G' \cong G \rtimes \mathbb{Z}_2^C$.

[199] Again, by "part of the Loretnz group", we mean that its action on a field is determined entirely by the Lorentz indices of the field—it acts trivially on all other indices, like flavor indices. It still complex conjugates fields though, so the action is not solely through the action of the Loretnz group.

[200] We are using the same conventions as in the long diary entry on pinors and representation theory, where $\bar\psi M \psi \mapsto \bar\psi C^\dagger M^T C \psi$ under C.

Particles, antiparticles, and CPT

The exact transformation that "takes particles to antiparticles" is, judging from the internet, a source of great confusion. On account of QFT being unitary, one fuzzily expects there to be some sort of "particles to antiparticles / Hermitian conjugation" transformation that does some sort of charge conjugation-y thing and is a symmetry of all relativistic QFTs (this of course turns out to be CPT). But then what exactly "exchanges particles and antiparticles"? Some people say C, others say some perverse version where C is anti-unitary, etc. In cmt CT often acts as particle-hole, so is it CT? But when learning about leptogenesis / early universe cosmology, the matter / anti-matter asymmetry is usually couched in terms of CP violation, so is it CP? Making things worse, Green, Schwartz and Witten say that it is in fact CPT that exchanges particles and antiparticles. Agrh! What's going on?

As usual with this kind of stuff, it's a bit of a conceptual / terminological minefield. To me, the best discussions of this stuff are in Weinberg vol I (this is really the definitive reference I feel), Sidney Coleman's QFT book chapter 22, and Haag.

Basically, in situations where there are some particles and antiparticles which are distinct, C interchanges particles and antiparticles (again, an antiparticle only has opposite *internal* quantum numbers), and C invariance of a theory means that e.g. particles can be substituted for antiparticles in any process without changing the amplitude for that process to occur (this does not mean that C can be defined to act trivially on particles that are their own antiparticles if there are also other things in the theory—e.g. in order for C to be a symmetry in QED, we must define it so that the photon transforms with C-parity -1).[201] Just exchanging particles and antiparticles is not the same as doing the nebulous "Hermitian conjugation thing" that we expect to be a symmetry of all relativistic QFTs, and indeed of course not all physical theories are invariant under C. Consider e.g. the weak interaction. The coupling of the leptons to the gauge field looks like (ignoring prefactors)

$$\mathcal{L}_W \ni \bar{\Psi}\not{W}(1-\bar{\gamma})\Psi + h.c. = (j_V^{a\mu} - j_A^{a\mu})W^a_\mu + h.c., \tag{1704}$$

where e.g. $\Psi = (e,\, \nu_e)^T$ for the first generation. Since j_A and j_V transform with different signs in $d \in 4\mathbb{Z}$,[202] there is no way to render this term C-invariant, and so the weak interactions respect no type of charge conjugation symmetry.

On the other hand, this term does respect CP symmetry, and sometimes one hears that it is really CP which exchanges particles and antiparticles, e.g. as in discussions of cosmology (again, CP may really be C depending on how C is defined). CP defined with our definition of C is of course also not always a symmetry, and the real-life example is the weak coupling to the quarks provides an example with broken CP:

$$\mathcal{L}_W \ni \bar{d}_i(1-\bar{\gamma})\not{W}u_jV^{ij} + h.c. \tag{1706}$$

[201] And remember that we really are *defining* the action of C, *not* deriving it.

[202] When $d \in 4\mathbb{Z}$, we have $C^\dagger\gamma_\mu^T C = -\gamma_\mu$ so that the vector current is odd, while

$$C^\dagger(\gamma_\mu\bar{\gamma})^TC = -C^\dagger\bar{\gamma}C\gamma_\mu = -\bar{\gamma}\gamma_\mu = +\gamma_\mu\bar{\gamma}, \tag{1705}$$

and so the chiral current is even.

where V is the CKM matrix, $d_i = \begin{pmatrix} d & s & b \end{pmatrix}_i^T$ are the down-like quarks, and $u_i = \begin{pmatrix} u & c & t \end{pmatrix}_i^T$ are the up-like quarks. It turns out that in our universe the CKM matrix is such that CP is violated.

The real "Hermitian conjugation thing" that is always a symmetry in relativistic QFT is of course CRT (i.e, we can always choose a C, R, and T such that CRT is a symmetry). Actually, if we use $\mathcal{T} = CT$ for the "correct" time reversal transformation which only acts on Lorentz indices, then CRT becomes $R\mathcal{T}$; this is obviously always a symmetry, as can be seen by analytically continuing into Euclidean time (see Coleman's book for a good explanation of why analytically continuing Feynman diagrams is always legit) and noting that $R\mathcal{T}$ just does a rotation. Basically, $R\mathcal{T}$ takes a spatial slice and "flips it over", which involves reversing time and changing an odd number of spatial coordinates (since this operation preserves the spatial slice it has a well-defined action on the Hilbert space of the theory). Anyway, this is why some sources refer to CRT as the thing which establishes "a correspondence between particles and antiparticles". Whether or not this is the best terminology depends on whether antiparticles have opposite quantum numbers under spacetime symmetries as well as internal ones, but with our present definition of C (dualizes the whole representation, not just the internal part), I think this terminology is fair.

Real, complex, and chiral Majoranas—disambiguation

Today we will try to explain exactly what the classifier "Majorana" means when applied to fermions. In some contexts (usually CMT), a Majorana fermion is taken to mean a $\mathbb{R}$ fermion, and in others (usually particle physics) it's taken to mean a (perhaps $\mathbb{C}$) fermion which admits a Lorentz-invariant pairing with itself. The confusion between these definitions is particularly onerous in the many review articles out there that talk about Majoranas as e.g. neutrinos side-by-side with Majoranas as realized in solid-state physics—the two notions are different, but I've never really seen this clearly explained anywhere. Distinguishing between the various cases is what we'll do today.

Majoranas as real fermions

The most common definition (or at least the definition we will take as canonical) of a Majorana is a fermion which transforms in a real representation of the total symmetry group (spacetime + internal) of the theory.

Let's think about how this could arise. In general, the mode expansion for a fermion field is

$$\psi_\alpha(x) = \int_{\mathbf{p}} \left(c_{\mathbf{p}\sigma} u_\alpha(\mathbf{p},\sigma) e^{ip\cdot x} + d^\dagger_{\mathbf{p}\sigma} v_\alpha(\mathbf{p},\sigma) e^{-ip\cdot x} \right), \tag{1707}$$

with α the spinor index and σ an (implicitly summed over) spin index determined according to the representation theory of the little group of $\mathbf{p}$. In order for ψ to transform in a well-defined way under the symmetry groups involved, we must have that if c annihilates particles transforming under the internal symmetry group in a representation R, then d annihilates particles transforming under R^*.

The u and v vectors are there to do the following job: when we perform a Lorentz transformation, we do it by acting on the creation / annihilation operators as (ignoring translations)

$$U(\Lambda) c_{\mathbf{p}\sigma} U^{-1}(\Lambda) = \sqrt{(\Lambda p)^0/p^0} D_{\sigma\rho}(\Lambda, p) c_{\mathbf{p}_\Lambda \rho}, \tag{1708}$$

with D the representation matrix acting on spin (it depends on Λ and p since it's the representations of the little group of $\mathbf{p}$ that are relevant here, so e.g. in the massive case a boost into the particle's rest frame is involved). The u and v coefficients are "intertwiners" that convert this action into an action that results in a homogeneous transformation of the ψ field as a whole. That is, they allow us to write the transformed version of $\psi(x)$, after substituting in the RHS of the above expression into the mode expansion (and likewise for $d^\dagger$), as $R_\Lambda \psi(\Lambda^{-1}x)$, with R_Λ some chosen (s)pinor representation of the Lorentz transformation. In order for ψ to have a well-defined transformation under the spacetime symmetry group, it must therefore annihilate particles in the appropriate (s)pinor representation, and create particles in the dual of this representation (e.g. if it creates left-handed electrons, it must annihilate right-handed positrons).

If the full (spacetime + internal) representation that ψ transforms under is isomorphic to its dual,[203] then c and d destroy particles with the same quantum numbers, and it is possible to in fact cut down on the degrees of freedom and have $c = d$. The conjugate of ψ in this case is

$$\psi^* = \int_{\mathbf{p}} \left(c_{\mathbf{p}\sigma} v^*_\alpha(\mathbf{p},\sigma) e^{ip\cdot x} + c^\dagger_{\mathbf{p}\sigma} u^*_\alpha(\mathbf{p},\sigma) e^{-ip\cdot x} \right). \tag{1709}$$

If we can find a unitary matrix C such that[204]

$$u^*(\mathbf{p},\sigma) = \gamma^0 C^T v(\mathbf{p},\sigma), \quad v^*(\mathbf{p},\sigma) = \gamma^0 C^T u(\mathbf{p},\sigma), \tag{1710}$$

then we will have (I'm assuming mostly negative signature here for notational simplicity)

$$\psi^* = \gamma^0 C^T \psi \implies \bar{\psi} C^\dagger = \psi. \tag{1711}$$

Looking back at the long diary entry on pinors and representation theory, we see that $C : \psi \mapsto \bar{\psi} C^\dagger$ is exactly how we defined charge conjugation, and so this condition

[203] We will avoid calling such a representation real or pseudoreal; see the next section for more detail why.

[204] The pesky γ^0s here are why conventions for charge conjugation very often differ by an action of γ^0, i.e. by a parity transformation. Therefore it is important to disambiguate between C and CP when reading stuff.

reads $C : \psi \mapsto \psi$ — fermions which satisfy the above condition are their own charge-conjugates.

This condition is usually taken as the definition of a Majorana fermion in particle physics; see the next section for more discussion. However, the cond-mat (and math-ph) circles usually seem to define a Majorana fermion as a *real* field. The condition $\psi = \bar{\psi}C^\dagger$ is then not good enough, and we require the stronger condition of $\psi = \psi^*$, i.e. that the unitary matrix relating the conjugates of the u and v spinors is given precisely by $C = \gamma^0$ (equivalently, we require that the Dirac and Majorana adjoints agree).

This means that a Majorana, in the sense of a real fermion, must transform under a real representation of the full (spacetime + internal) symmetry group. When is this possible? Consulting the results in the big diary entry on the representation theory of pin groups, we see that we can have chiral Majorana spinors in spacetime dimension $d = 2$, Majorana spinors in $d = 3$, and Majorana spinors (but *not* chiral Majorana spinors) in $d = 4$. Similarly, we can have Majorana pinors in dimensions $d = 2, 3, 4$, but the existence of Majorana pinors depends on the choice of signature in $d = 3, 4$, e.g. we can have Majorana pinors that transform under $\mathrm{Pin}(2, 1)$ but not under $\mathrm{Pin}(1, 2)$, and under $\mathrm{Pin}(3, 1)$ but not under $\mathrm{Pin}(1, 3)$ (here the notation is $\mathrm{Pin}(s, t)$ with s positive signature and t negative signature).

Finally, note that second-quantized Majorana operators of the kind used in e.g. the Kitaev chain haven't shown up in the present discussion. The second-quantized operators appearing in the mode expansion for ψ are not these kinds of operators; they still create particles (and act on a Fock space of integer dimension), and even though the Majorana is in a real representation with $\psi^* = \psi$ and is hence its own antiparticle, $c^\dagger \neq c$.

Complex Majoranas

As we mentioned above, other people (usually particle physicists) use a more general definition of Majorana fermion, viz. a field which is self-dual in the sense of the last section: ψ^* is related to ψ through a unitary transformation built from the charge conjugation matrix C and γ^0. With this definition, a Majorana is any fermion that admits an invariant pairing with itself—I'll take this to mean any fermion such that $\chi^T C \not{\partial} \chi$ is invariant under Lorentz transformations and under any internal symmetry group action that the theory might come with; others might use the requirement that a Majorana mass term $\chi^T M_m \chi$ exist (these two definitions are slightly different; consider e.g. whether or not chiral Majorana spinors make sense in 1+1D, as $\nexists$ a mass term).

One might think that this implies that the χ can be taken to be real. Indeed, as we saw, it means that there must be a unitary transformation given by $\gamma^0 C$ relating the transposed matrices in the full representation R of the total symmetry group to their inverses. If the representation R were unitary, then this would mean that C provided a unitary transformation between R and R^*, since the conjugate would combine with the transpose in χ^T to produce the requisite inverse. Therefore if R were unitary, then we would conclude that Majoranas in the above sense would exist only for R a real representation.

However, being non-compact, the spacetime part $\mathrm{Spin}(d - 1, 1)$ (or $\mathrm{Pin}(d - 1, 1)$, or what have you) will *not* be represented with unitary matrices acting on the Lorentz indices![205] This means that while we may be able to construct an invariant pairing of the form above, this pairing may not necessarily connect R with R^*, since a field transforming under $R^\dagger$ will not necessarily pair with one transforming under R to give something invariant.

For example, consider Spin$(3,1)$. The matrices generating the action in the spinor representation can be chosen to be e.g.

$$S_{ij} = S_{ij}^+ \oplus S_{ij}^- = \frac{1}{2}\varepsilon_{ijk}(\sigma^k \oplus \sigma^k), \qquad S_{0i} = S_{0i}^+ \oplus S_{0i}^- = \frac{i}{2}(\sigma_i \oplus -\sigma_i). \tag{1712}$$

The chiral spinor representations $S_\pm$ are obviously complex. However, since they are all built out of σ matrices, we have

$$\Lambda^\pm_{\theta,\beta} Y = [e^{i(\theta^{ij}S^\pm_{ij}+\beta^k S^\pm_{0k})}]^T Y = Y e^{-i(\theta^{ij}S^\pm_{ij}+\beta^k S^\pm_{0k})} = Y[\Lambda^\pm_{\theta,\beta}]^{-1}. \tag{1713}$$

Therefore the term $\chi_\pm^T Y \chi_\pm$ is in fact Spin$(3,1)_\pm$ invariant, for $\chi_\pm$ transforming in the $S_\pm$ representation. Note how this gives a mass between two *Weyl* fermions—the $\chi_\pm$ are "Majorana-Weyl fermions". Also note that we can build a Dirac fermion from two of these Majorana-Weyl fermions—this wouldn't be possible if the Majoranas were real, simply by counting dof.

Because of the absence of complex conjugation, we will only be able to create an invariant out of a $\chi_\pm$ bilinear if $\chi_\pm$ transforms in a $\mathbb{R}$ or ps$\mathbb{R}$ representation under all internal symmetries, since all internal symmetries will be represented unitarily on the Hilbert space (we normally don't discuss infinite-dimensional internal symmetries). This means for example that it is still impossible to have a Majoranna (in the present section's definition of the term) that transforms in a complex representation of an internal symmetry group, even though the Majoranna field itself is allowed to be complex. Basically, all the subtleties and distinctions between real Majoranas and the more general type of $\mathbb{C}$ Majoranas considered here originate from the action of the spacetime symmetry group.

Neutrinos

These complex Majoranas are the ones encountered in neutrino physics—we know from colloquium talks that neutrinos might be Majoranas, and since they are chiral and live in 3+1 dimensions, they definitely cannot be $\mathbb{R}$ fermions. Actually saying that neutrinos can be Majoranas is kind of a cheat, as we shall see.

[205] The discussion here is a bit sloppy, since of course everything in QFT transforms under unitary representations. Non-compact Lie groups don't have any finite dimensional unitary representations, and so the action of Spin$(d-1,1)$ or Pin$(d-1,1)$ on the spinor indices is non-unitary, since there simply aren't any unitary representations on $\mathbb{C}^{2^{\lceil (d-1)/2\rceil}}$. However, the actual (s)pinors aren't vectors in $\mathbb{C}^{2^{\lceil (d-1)/2\rceil}}$—they carry position / momentum labels, and instead live in $\mathbb{C}^{2^{\lceil (d-1)/2\rceil}} \otimes L^2(\mathbb{R}^{d-1})$. Since this space is infinite-dimensional, the Lorentz group can (and does) act on it in a unitary way—after all, the representation in (1708), acting on the second quantized operators, *is* unitary. In our whole discussion we've completely forgot about position / momentum indices, and so for our purposes, the action of the Lorentz group appears non-unitary (and when we say that a term like $\chi^T C \not{\partial}\chi$ is "invariant", we mean as regarded in $\mathbb{C}^{2^{\lceil (d-1)/2\rceil}}$, i.e. forgetting about position indices; of course the position indices of the χ fields are permuted by the Lorentz transform.)

First let's recall some hep-ph and see how this works. We can write the neutrino field in general as

$$\nu_L = \int_{\mathbf{p}} \left(c_{\mathbf{p}\sigma} u_L(\mathbf{p}, \sigma) e^{ip\cdot x} + d^\dagger_{\mathbf{p}\sigma} v_R(\mathbf{p}, \sigma) e^{-ip\cdot x} \right). \tag{1714}$$

So, ν_L creates a left-handed neutrino and destroys a right-handed anti-neutrino. ν_L is part of an $SU(2)$ doublet along with the e_L field, and carries $U(1)_Y$ charge -1. How can neutrinos get a mass term? They can't have a Dirac mass for the same reason as the other leptons, viz. the fact that the weak interactions couple only via left-handed components, while Dirac masses mix left- and right-chiralities (and there is no $\bar{\nu}_R$ field in the SM to pair with anyway!). We can't use the usual Yukawa coupling to the Higgs, either; the coupling between $\bar{e}_R$ and the doublet (e_L, ν_L) gets broken down under SSB so as to only give the electron a mass.

Suppose that the neutrino field is a Majorana field, so that $\nu_{L\alpha} = \bar{\nu}_{L\beta} C^\dagger_{\beta\alpha}$. Can we have a Majorana mass term

$$\mathcal{L}_{SM} \supset m_M \nu_L^T C \nu_L \tag{1715}$$

in the SM? No unfortunately not; such a term violates both $SU(2)_W$ and $U(1)_Y$. This is because actually ν_L can *not* actually be a Majorana fermion—there's no way we can take $c = d$ in the ν_L field mode expansion while preserving $U(1)_Y$, since the representation is complex and unitary.

So, why do particle physicists talk about Majorana neutrinos? They don't seem to fit into the framework of complex neutrinos as outlined in the previous section. The reason that this terminology quasi makes sense is seen by considering what happens when $SU(2)_W \times U(1)_Y$ gets spontaneously broken. We may consider the term

$$\mathcal{L}_{SM} \supset \frac{1}{\Lambda} [\boldsymbol{\phi}^T J \boldsymbol{\nu}]^T C [\boldsymbol{\phi}^T J \boldsymbol{\nu}], \tag{1716}$$

where $\boldsymbol{\nu} = (e_L,\, \nu_L)^T$, the matrix multiplication of the two square brackets is over spinor indices, and $\boldsymbol{\phi}$ is the Higgs, with Λ an energy scale needed because of the irrelevance of the operator in question (irrelevant in the full UV theory where the Higgs isn't condensed; after the Higgs gets a vev you might say that it is relevant like any other fermion mass term). The J makes the expression $SU(2)_W$ invariant, and the opposite $U(1)_Y$ charges for the Higgs and the leptons ensure that $U(1)_Y$ is okay too. When SSB happens the first component of $\boldsymbol{\phi}$ gets a vev v, and we get the Majorana mass term

$$\mathcal{L}_{SM} \supset \frac{v^2}{\Lambda} \nu_L^T C \nu_L. \tag{1717}$$

So, *after* SSB occurs and the $SU(2)_W$ and $U(1)_Y$ symmetries disappear, we can indeed think of the ν_Ls as Majoranas, with a Majorana mass (this is made possible by their neutrality under $U(1)_{EM}$).

Anyway, from the above discussion, we can really see that neutrinos in cond-mat (real fermions) are rather conceptually distinct from neutrinos in hep-ph (complex fermions that are equal to their charge-conjugates; in the case of neutrinos only after SSB). This is what makes the "solid state physicsists realized Majorana's vision before particle physicists did" spiel so vexing—the two things are totally different.

Spatial diffeomorphisms to quadratic order

Our goal in this note is to determine the operator which generates spatial diffeomorphisms, viz. transformations affected by coordinate changes $x^i \mapsto x^i + \xi^i$. When ξ^i is *constant* in space, the answer is of course

$$T_\xi = e^{i\xi_i \int P^i}, \tag{1718}$$

where

$$P^i = \frac{1}{2i}\big((\partial^i \psi^\dagger)\psi - \psi^\dagger \partial^i \psi\big) \tag{1719}$$

is the momentum density. When ξ^i varies in space, more care is required.

Let us first be clear about how T_ξ should act on the field operators $\psi, \psi^\dagger$ (whether these are bosonic or fermionic is immaterial for all of the following). One might be tempted to go with

$$T_\xi^{\text{naive}} \, : \, \psi(x) \mapsto \psi(x+\xi). \tag{1720}$$

There is however a problem with T_ξ^{naive}, which can be illustrated by noting that, when acting on the total charge $Q \equiv \int d^dx\, \rho$, one has

$$T_\xi^{\text{naive}} \, : \, Q \mapsto \int d^dx\, \rho(x+\xi) = \int d^dx'\, \rho(x') \frac{1}{\det(\mathbf{1} + \partial\xi)} \neq Q, \tag{1721}$$

where by $\partial\xi$ we mean the $d \times d$ matrix with components $\partial_i \xi_j$. Thus simply translating the fields is not enough: we must additionally ensure that ρ transform like a density, i.e. we must require that

$$T_\xi \, : \, \rho(x) \mapsto \rho(x+\xi)\det(\mathbf{1} + \partial\xi). \tag{1722}$$

When acting on ψ, we thus require

$$T_\xi \, : \, \psi(x) \mapsto \psi(x+\xi)\sqrt{\det(\mathbf{1}+\partial\xi)}, \tag{1723}$$

and likewise for $\psi^\dagger$.[206]

In what follows we will obtain an expression for T_ξ valid to quadratic order in ξ, since this is what we will need for performing the phonon energetics calculation (and

[206] The correctness of this transformation law can also be seen by requiring that the LHS of $[\psi(x), \psi^\dagger(y)]_\pm = \delta(x-y)$ transform according to the Jacobian factor that the delta function on the RHS picks up under the appropriate coordinate transformation.

anway I am not sure I know the answer to all orders in ξ yet). For this it will be helpful to have an appropriate expansion of the determinant, which is

$$\begin{aligned}\det(\mathbf{1}+\partial\xi) &= \exp\left(\mathrm{Tr}[\ln(\mathbf{1}+\partial\xi)]\right)\\ &\approx \exp\left(\mathrm{Tr}\left[\partial\xi - \frac{1}{2}(\partial\xi)^2\right]\right)\\ &\approx 1+\partial_i\xi^i - \frac{1}{2}\partial_i\xi^j\partial_j\xi^i + \frac{1}{2}(\partial_i\xi^i)^2,\end{aligned} \tag{1724}$$

so that

$$\sqrt{\det(\mathbf{1}+\partial\xi)} \approx 1+\frac{1}{2}\partial_i\xi^i + \frac{1}{8}(\partial_i\xi^i)^2 - \frac{1}{4}\partial_i\xi^j\partial_j\xi^i. \tag{1725}$$

Therefore our desired transform of ψ is, to quadratic order in ξ,

$$T_\xi\,:\,\psi\mapsto\psi\left(1+\frac{1}{2}\partial_i\xi^i+\frac{1}{8}(\partial_i\xi^i)^2-\frac{1}{4}\partial_i\xi^j\partial_j\xi^i\right)+\xi^i\partial_i\psi\left(1+\frac{1}{2}\partial_j\psi^j\right)+\frac{1}{2}\xi^i\xi^j\partial_i\partial_j\psi. \tag{1726}$$

What then is the explicit form of T_ξ? One may check that the naive guess $\exp(i\int\xi_iP^i)$ does not work (nor does it generate T_ξ^{naive}). It is however easy to convince oneself that the form of the operators in P^i cannot be modified. We are then prompted to take the ansatz

$$T_\xi = \exp\left(i\int\gamma_iP^i\right), \tag{1727}$$

where γ_i, which equals ξ_i in the limit when ξ_i is constant, is a function to be determined. We determine it by explicitly calculating the transformation of ψ, working as above only to quadratic order in ξ. At this order,

$$T_\xi\psi T_\xi^\dagger \approx \psi + [\ln T_\xi,\psi] + \frac{1}{2}[\ln T_\xi,[\ln T_\xi,\psi]]. \tag{1728}$$

The first commutator is calculated using

$$\int f(x)[\psi^\dagger(x),(\partial_i\psi)(y)] = (\partial_i f)(y), \tag{1729}$$

and gives

$$[\ln T_\xi,\psi] = \frac{1}{2}(\partial_i\gamma^i)\psi + \gamma^i\partial_i\psi. \tag{1730}$$

Adding in the second commutator then gives, after some algebra,

$$T_\xi\psi T_\xi^\dagger \approx \psi\left(1+\frac{1}{2}\partial_i\gamma^i+\frac{1}{8}(\partial_i\gamma^i)^2+\frac{1}{4}\gamma^i\partial_i\partial_j\gamma^j\right)+\partial_i\psi\left(\gamma^i+\frac{1}{2}\gamma^i\partial_j\gamma^j+\frac{1}{2}\gamma^j\partial_j\gamma^i\right)+\frac{1}{2}\gamma^i\gamma^j\partial_i\partial_j\psi. \tag{1731}$$

It is then not hard to work out the explicit form of γ_i by staring at this equation. We must have $\gamma_i = \xi_i + O(\xi^2)$ to get the $\partial_i\partial_j\psi$ part right, and to kill the $\xi^i\partial_i\partial_j\gamma^j$ term the $O(\xi^2)$ piece must be $-\frac{1}{2}\xi^j\partial_j\xi^i$. A small amount of algebra demonstrates that this choice in fact exactly yields the desired transformation law (1726). Therefore the generator to quadratic order is

$$T_\xi = \exp\left(i\int\left(\xi^i-\frac{1}{2}\xi^j\partial_j\xi^i\right)P_i\right). \tag{1732}$$

In particular, the term linear in ξ is just ξ, with no contributions from derivatives of ξ. This means that T_ξ^{naive} can indeed be used in many applications, e.g. in the computation of the phonon elastic matrix below.

Non-relativistic duality in 2+1D and vortex motion

We've already seen examples of relativistic dualities a million times in previous diary entries, and so here we fill out things by treating a non-relativistic example of interacting bosons in 2+1D which are close to a SF state. The goal will be to find a presentation where the properties of the vortices are explicit in the action. As a bonus we will be able to derive why smoke rings move the way they do.

The duality we will be doing today starts from quartically interacting non-relativistic bosons in 2+1D (following this line of reasoning in 1+1D is one way rather formal way of arriving at the hydrodynamic description of the compact boson). Writing the boson field in terms of its phase ϕ and its density ρ, assuming that ρ has a vev of $\bar\rho$, and dropping terms that are irrelevant at the non-interacting fixed point as well as total derivatives, the Lagrangian is schematically

$$\mathcal{L} = i\rho\partial_t\phi - \frac{\bar\rho}{2m}|\nabla\phi|^2 - \frac{1}{2}(\rho-\bar\rho)V(\rho-\bar\rho), \tag{1733}$$

where the interaction V is translation invariant but may be nonlocal, i.e. $H \supset \frac{1}{2}\int d^2x\, d^2y\, (\rho(x)-\bar\rho)V(x-y)(\rho(y)-\bar\rho)$. We now do duality by integrating in a vector field X_μ. Only the X_i are new dynamical variables, while X_0 is defined to be ρ. This gives

$$\mathcal{L} = iX_0\partial_t\phi + \frac{m}{2\bar\rho}X_iX^i - iX_i\nabla^i\phi - \frac{1}{2}(X_0-\bar\rho)V(X_0-\bar\rho) \tag{1734}$$

We will break up ϕ as $\phi_s+\phi_v$, where $d^2\phi_s = 0$ and $d^2\phi_v = \sum_j q_j\delta(x-x_j)$, with x_j and q_j the positions and charges of the vorticies in the phase of ϕ, respectively.[207] Integrating out ϕ_s then sets $d^\dagger X = 0$, and so we may parametrize X as $X = \star dA$, where the $\star$ is in spacetime and A is a $U(1)$ gauge field. We will not be trying to integrate out ϕ_v, since our goal is to get an effective action for the vortex currents. Writing

$$J = \star d^2\phi_v \tag{1735}$$

[207] s stands for ”smooth”, not ”singular”. Sorry!

for the vortex current, we have

$$\mathcal{L} = iA_\mu J^\mu + \frac{m}{2\bar{\rho}} E_i E^i - \frac{1}{2}(B - \bar{\rho})V(B - \bar{\rho}), \tag{1736}$$

where B and E are the electric and magnetic fields of A. To make the second two terms more Maxwellian, we shift the field by $A \mapsto A + \bar{A}$, where $\varepsilon^{ij}\partial_j \bar{A}_i = \bar{\rho}$, $\bar{A}_0 = 0$, and $\partial_t \bar{A}_i = 0$. The action is then

$$S = \int d^3x \left(i(A_\mu + \bar{A}_\mu)J^\mu + \frac{m}{2\bar{\rho}} E_i E^i \right) + \frac{1}{2}\int d^3x\, d^3y\, B(x)V(x-y)B(y), \tag{1737}$$

with $V(x-y) \propto \delta(x^0 - y^0)$. If we assume the interaction is local in space as well, with strength V, then the Lagrangian is

$$\mathcal{L} = i(A_\mu + \bar{A}_\mu)J^\mu + \mathcal{L}_{\text{Maxwell}}[e^2, c], \tag{1738}$$

where the effective electric charge and speed of light are

$$e^2 = \frac{\bar{\rho}}{m}, \qquad c^2 = \frac{V\bar{\rho}}{m}. \tag{1739}$$

Thus we get a gas of charges (vorticies) interacting with a Coulomb interaction, as we knew we had to. Changing the nature of the density interaction evidently changes the dispersion of the photons. For example, if we consider $V(\mathbf{x}-\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|^n}$ then $V(k) \sim k^{n-2}$, the propagator (we will be in $d^\dagger A = 0$ gauge) goes like $(m\omega^2/\bar{\rho} - V(k)k^2)^{-1}$; hence if the interactions are logarithmic ($n = 0$) the gauge field does not disperse.

A_0 can be integrated out easily (since it doesn't appear in B, it can be integrated out independently of the form of $V(x-y)$[208]): its propagator is local in time and has $\ln|\mathbf{x}-\mathbf{y}|$ spatial dependence; hence

$$S = \int d^3x \,\left(i(A_i + \bar{A}_i)J^i\right) + \frac{1}{2}\int d^3x\, d^3y\, \left(B(x)V(x-y)B(y) - \frac{\bar{\rho}}{2\pi m} J_0(x)\delta(x^0 - y^0)\ln|\mathbf{x}-\mathbf{y}|J_0(y) \right) \tag{1740}$$

This is the logarithmic vortex interaction that we knew we needed to get.

If we let $J_0 = \sum_j q_i \delta(\mathbf{x} - \mathbf{x}_j)$ and $J^i = \sum_j q_j \partial_t x_j^i \delta(\mathbf{x} - \mathbf{x}_j)$, then we can get the equations of motion for the vortex positions $\mathbf{x}_j$ just in the way that we would for non-relativistic charged particles in a magnetic field. The (expectation value of the) equations of motion for $\mathbf{x}_j$ are then

$$\star\, \mathbf{v}_j = \frac{1}{2\pi m}\sum_k q_k \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|^2}, \tag{1741}$$

where we have ignored the coupling of the fluctuating magnetic field to the vortex currents, and where the $\star$ is in space. Note that the $\bar{\rho}$ from the background magnetic field and the $\bar{\rho}$ from the interaction strength have canceled, and that the eom for $\mathbf{x}_j$

[208] Since the interaction energy of the vorticies (found by integrating out A_0) comes from the gradient energy of ϕ, A_0 must be able to be integrated out in a way which is independent of the form of the interactions between the ρ, ϕ bosons.

is actually independent of the charge q_j; it instead only depends on the charges of the other vortices.

Now we can finally understand how smoke rings and other types of vortices move. A cross-section of a smoke ring consists of two vortices with opposite charge. According to the above, the vortices are attracted to one another through the Coulomb interaction. However, from the above eom, we see that they do not move towards each other; rather they move together in a direction perpendicular to their separation, with the sign of the direction determined by the sign of the vortex charges. Additionally, despite the fact that the interaction between two like charges is repulsive, we see that two like vorticies actually circle each other at constant separation, with the handedness of the motion determined by the sign of their charges. This behavior essentially comes from the fact that in this model the vortices are very massive, i.e. have no kinetic energy. Hence by energy conservation they cannot move closer or further from one another, as there is no kinetic energy term to make up for the change in potential energy that would occur as a result of such motion.

Path integrals for Majorana fermions in one dimension and spin structures

This is an elaboration on an exercise posed in Quantum Fields and Strings for Mathematicians. The problem statement is as follows:

Consider the quantum mechanics of $2n$ Majorana fermions η_i, with action

$$S = \int dt \; i \sum_j \eta_j \frac{d}{dt} \eta_j. \tag{1742}$$

Suppose time is an S^1. Compute the partition function for both spin structures on this S^1. Also compute the twisted partition functions $\mathrm{Tr}_{\mathcal{H}}(g)$ and $\mathrm{sTr}_{\mathcal{H}}(g)$, where g is a matrix in the spinor representation (containing both chiralities) of $\mathrm{Spin}(2n)$ which acts to permute the fermion flavors and which can be thought of as implementing a flavor-twisted boundary condition in the path integral.[209]

[209] The reason for choosing to twist the boundary conditions by an element of $\mathrm{Spin}(2n)$ rather than something else with the same Lie algebra (like $O(2n)$ will become clear while doing the problem. Note that the dimensions are right here since $\mathrm{Spin}(m)$ for $m \in 2\mathbb{Z}$ has dimension $2^{m/2}$ and $2n$ Majoranas have Hilbert space dimension 2^n.

Computing the Pfaffians

First we evaluate the partition functions by taking Pfaffians. There are two spin structures on S^1, which are permuted by $H^1(S^1;\mathbb{Z}_2) = \mathbb{Z}_2$. This means there are two spin bundles on S^1, which we will write as S_+ and S_-. S_+ is the trivial $\mathbb{Z}_2$ bundle and S_- is the mobius strip. We will label the trivial bundle (the double-cover) "non-bounding" NB and the non-trivial bundle "bounding" B. This is because the nontrivial bundle bounds a disk, while the trivial bundle does not. Why? Well, when we embed the S^1 into a 2-manifold equipped with a spin structure, we need homologically trivial closed loops along which the fermion framing is parallel-transported to act as -1 in $\mathrm{Spin}(2) = U(1)$. Thus the S_- bundle is the one which can be embedded in two dimensions, while the S_+ bundle can only be embedded in two dimensions if it's embedded along a non-contractible loop. See another diary entry for more details about this. Anyway, each of the majoranas η_j, being real fields, should be viewed as sections of either the associated spinor bundle $S_+ \otimes \mathbb{R}$ or the associated spinor bundle $S_- \otimes \mathbb{R}$, depending on the spin structure.

Will we also want to compute the partition function in the presence of a background field for the global flavor symmetry $O(2n)$. Writing the twisted Dirac operator as $\not{D}_R$ for $R \in O(2n)$, we need to compute $\mathrm{Pf}(\not{D}_R) \in \mathbb{R}$ as a function of R. Unfortunately the sign is ambiguous, since we compute $\mathrm{Pf}(\not{D}_R)$ by taking a square root. Now $\mathrm{Pf}(\not{D}_R)$ can be thought of as a section of a real line bundle, the Pfaffian line bundle, over $O(2n)$. The structure group is $O(1) = \mathbb{Z}_2$, corresponding to a choice of sign in taking the square root. Unfortunately since $\pi_1(O(2n)) = \pi_1(SO(2n)) = \mathbb{Z}_2$ (the first equality from the fact that the identity component of $O(2n)$ is $SO(2n)$), this line bundle can be nontrivial, which means that the sign of the partition function may not be well defined.

However, suppose we instead twist the fields by an element in $\mathrm{Spin}(2n)$. There are no nontrivial line bundles over $\mathrm{Spin}(2n)$,[210] and so the Pfaffian line bundle must be trivial, giving us a well-defined partition function. This is why we restricted to a twist by an element of $\mathrm{Spin}(2n)$ in the problem statement. We will henceforth let $Z_\pm(R)$ denote the partition function with $\pm$ spin structure, twisted by an additional possible element $R \in \mathrm{Spin}(2n)$.

Now since the twisting by $R \in \mathrm{Spin}(2n)$ can be done by inserting R into either $\mathrm{Tr}_{\mathcal{H}}[R]$ or $\mathrm{Tr}_{\mathcal{H}}[(-1)^F R]$ (depending on the spin structure) and since $(-1)^F$ is $-\mathbf{1}$ in $\mathrm{Spin}(2n)$,[211] we have

$$Z_\pm(R) = Z_\pm(V^{-1}RV), \quad \forall\, V \in \mathrm{Spin}(2n) \tag{1744}$$

so that $Z_\pm(R)$ is a class function on $\mathrm{Spin}(2n)$. This means we can, by conjugating,

[210] This is because $\pi_1(\mathrm{Spin}(2n)) = 0$. At a mathematical level, we would demonstrate this as

$$\mathrm{Line}_{\mathbb{R}}(M) = [M, BO(1)] = [M, B\mathbb{Z}_2] = H^1(M;\mathbb{Z}_2) = \mathrm{Hom}(\pi_1(M), \mathbb{Z}_2), \tag{1743}$$

where M is any manifold and $\mathrm{Line}_{\mathbb{R}}(M)$ are the different real line bundles over M. Thus if $\pi_1(M) = 0$, there are no nontrivial real line bundles over M.

[211] $(-1)^F$ is the nontrivial element in the $\mathbb{Z}_2 \subset Z(\mathrm{Spin}(2n))$ that gets projected out when taking the quotient to $SO(2n)$. $Z(\mathrm{Spin}(2n))$ is either $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$, depending on whether n is odd or even, respectively. The extra $\mathbb{Z}_2 \subset Z(\mathrm{Spin}(2n))$ that remains after taking the quotient to $SO(2n)$ constitutes the center of $SO(2n)$, via $Z(SO(2n)) = \mathbb{Z}_2$.

work in a basis where R is the exponential of an element in a maximal torus of $\mathfrak{spin}(2n)$. Letting the generators of the Clifford algebra Cliff$(2n)$ be γ_i, a generic element of the chosen maximal torus can be written as

$$MT(\mathfrak{spin}(2n)) \ni -i\ln R = \sum_{j=1}^{n} r_j \gamma_{2j-1}\gamma_{2j} \tag{1745}$$

for some coefficients r_j. As required, all of the $\gamma_{2j-1}\gamma_{2j}$ commute with each other, and they all square to $-\mathbf{1}$. Thus we can choose a basis in which each $\gamma_{2j-1}\gamma_{2j}$ is represented by a matrix with n 2x2 blocks on the diagonal, with the matrix $J = -iY$ in the jth block and 0s everywhere else. Thus R is the exponential (no factor of i!):

$$R = \exp\left(\bigoplus_{j=1}^{n} r_j J\right), \qquad J = -iY. \tag{1746}$$

To put this into the Lagrangian, we just need to construct the spin connection ω appearing in $\partial_\mu - \omega_\mu^{ij}\Sigma_{ij}$, with $\Sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$. From the above construction, normalizing so that the radius of the temporal S^1 to be 1, we see that we may take

$$\omega_t^{2j-1,2j} = r_j/2\pi. \tag{1747}$$

Indeed, with this choice we have $e^{i\oint dt\, \omega_t^{ij}\Sigma_{ij}} = e^{iR}$.

We now need to compute the Pfaffian of $\not{D}_R$. Since the spin connection breaks up into a direct sum, we can consider each 2-by-2 block individually. First, consider the non-bounding partition function $Z_+(R)$. The partition function $Z_+(r_j)$ for a single block (containing a single pair of Majoranas) is[212]

$$Z_+(r_j) = \mathrm{Pf}\left(\frac{d}{d\tau} + 2\omega_t^{12}\Sigma_{12}\right) = \mathrm{Pf}\left(\frac{d}{d\tau} + \frac{r_j}{2\pi}J\right). \tag{1748}$$

The relevant determinant is then (since we have chosen the S^1 to have radius 1, the eigenfunctions of $d/d\tau$ are $e^{i\omega\tau}$, $\omega \in \mathbb{Z}$)

$$\mathrm{Det}\left(\frac{d}{d\tau} + \frac{r_j}{2\pi}J\right) = \prod_{n\in\mathbb{Z}}(-n^2 + r_j^2/(2\pi)^2). \tag{1749}$$

We regularize this with the ζ function:

$$\begin{aligned}\prod_{n\in\mathbb{Z}}(-n^2 + r_j^2/4\pi^2) &= \frac{r_j^2}{4\pi^2}\prod_{n\neq 0} n^2 \prod_{n\neq 0}\left(1 - \frac{r_j^2}{4\pi^2 n^2}\right) \\ &= \frac{1}{\pi^2}\left(\prod_{n=1}^{\infty} n^2\right)^2 \left(\frac{r_j}{2}\prod_{n=1}^{\infty}\left(1 - \frac{r_j^2}{4\pi^2 n^2}\right)\right)^2 \\ &= 4\sin^2(r_j/2),\end{aligned} \tag{1750}$$

[212] We are in Euclidean signature. In real time, the Dirac operator is $i\not{D}_R = i\partial_t + \omega_t^{ij}\Sigma_{ij}$, since $\omega_t^{ij} = -\omega_t^{ji}$ and since Σ_{ij} is anti-Hermitian (again, just to avoid confusion: ω is the spin connection for the global Spin$(2n)$ symmetry, not the Spin(1) symmetry of the temporal circle). When we go to Euclidean time, the Dirac operator instead becomes $\partial_\tau + \omega_t^{ij}\Sigma_{ij}$.

where we used

$$\sin(x) = x\prod_{j=1}^{\infty}(1 - x^2/(\pi^2 n^2)) \tag{1751}$$

to deal with the second product. To deal with the first product, we used

$$\partial_s\zeta(s) = \sum_n \partial_s e^{-s\ln n} \implies [\partial_s\zeta](0) = -\sum_n \ln n \tag{1752}$$

to write

$$e^{-2[\partial_s\zeta](0)} = e^{\sum_{n=1}^{\infty}\ln(n^2)} = \prod_{n=1}^{\infty} n^2. \tag{1753}$$

Using the result $[\partial_s\zeta](0) = -\frac{1}{2}\ln 2\pi$ then tells us that

$$\prod_{n=1}^{\infty} n^2 = 2\pi, \tag{1754}$$

which allows us to take care of the first product. This means that[213]

$$Z_+(R) = \mathrm{Det}^{1/2}\left(\bigoplus_j^n\left[\frac{d}{d\tau} + \frac{r_j}{2\pi}J\right]\right) = 2^n\prod_j|\sin(r_j/2)|. \tag{1756}$$

Here the absolute value bars are in place because we are choosing the positive sign when taking the square root (so that $r_j \sim r_j + 2\pi$). As mentioned before, we can make a consistent choice of this sign for all r_j since $\pi_1(\mathrm{Spin}(2n)) = 0$.

Anyway, we see that if we don't twist by any gauge fields the partition function vanishes; $Z_+(\mathbf{1}) = 0$. This is to be expected, since choosing the $S_+\otimes\mathbb{R}$ bundle gives us a zero mode of the (untwisted) Dirac operator, causing the partition function to vanish. If we twist all the fermions by a 2π rotation, which amounts to choosing $r_j = \pi$ for all j (since this is what gives $R = -\mathbf{1}$ when exponentiated), then we effectively switch to AP boundary conditions and get $Z_+(R) = 2^n$, the dimension of the Hilbert space (each of the $2n$ Majoranas gives a contribution to the dimension of $\sqrt{2}$).

We can similarly compute partition functions for the bounding bundle $S_-\otimes\mathbb{R}$ twisted by some element $R\in\mathrm{Spin}(2n)$. We just take our result for the $S_+\otimes\mathbb{R}$ bundle, but shift $r_j\mapsto r_j+\pi$, since this adds an extra 2π twist and hence shifts the boundary conditions, as mentioned above. Thus we get

$$Z_-(R) = 2^n\prod_{j=1}^{n}|\cos(r_j/2)| \tag{1757}$$

[213]If we had wanted the result in $\mathbb{R}$ time, we'd have obtained (note to self: are you sure there is no factor of i in the spin connection?)

$$Z_-^2 = \det\left(i\frac{d}{dt} + \frac{1}{2}J\right) = \prod_{n\in\mathbb{Z}}(n^2 + 1/4) = 2^{2n}\sinh^2(\pi/2). \tag{1755}$$

for the bounding spin bundle. As expected, this counts the dimension of the Hilbert space (and in particular is non-zero) when there is no extra Spin($2n$) twist ($R = 0$).

We could also obtain this result by using the fact that the boundary conditions are AP to compute, for a single block,

$$\begin{aligned}
[Z_-(r_j)]^2 &= \prod_{n \in \mathbb{Z}-1/2} (-n^2 + r_j^2/(2\pi)^2) \\
&= \prod_{n \in \mathbb{Z}-1/2} n^2 \left[\prod_{n=1}^{\infty} \left(1 - \frac{r_j^2}{(n-1/2)^2(2\pi)^2} \right) \right]^2 \\
&= \left(\prod_{n \in \mathbb{Z}-1/2} n^2 \right) \cos^2(r_j/2)
\end{aligned} \tag{1758}$$

where we used the product formula

$$\cos(x) = \prod_{j=1}^{\infty} (1 - x^2/(\pi^2(n-1/2)^2)). \tag{1759}$$

Now we need to deal with the remaining product. We write

$$\prod_{n \in \mathbb{Z}-1/2} n^2 = \prod_{n \in \mathbb{Z}} \frac{1}{4}(2n-1)^2 = \prod_{n \in 2\mathbb{Z}+1} n^2. \tag{1760}$$

Here we have used that the ζ-regularized number of integers is zero:

$$\prod_{n \in \mathbb{Z}} \alpha = e^{\sum_{n \in \mathbb{Z}} \ln \alpha} = 1. \tag{1761}$$

This means that when doing ζ-function regularization,

$$\prod_{n \in \mathbb{Z}} \alpha f(n) = \prod_{n \in \mathbb{Z}} f(n) \tag{1762}$$

for any non-zero constant α. We can now write the product as

$$\prod_{n \in 2\mathbb{Z}+1} n^2 = \frac{\prod_{n \in \mathbb{Z} \setminus 0} n^2}{\prod_{n \in \mathbb{Z} \setminus 0} (2n)^2} = 4. \tag{1763}$$

Here we have used that the ζ-regularized number of *non-zero* integers is -1 (yes, this means $\sum_{n \in \mathbb{Z}} \neq \sum_{n \in \mathbb{Z} \setminus 0}$), so that

$$\frac{1}{\prod_{n \in \mathbb{Z} \setminus 0} 4} = e^{-\sum_{n \in \mathbb{Z} \setminus 0} \ln 4} = e^{\ln 4} = 4. \tag{1764}$$

Putting this into our expression for $Z_-(r_j)$ and taking the product over all blocks j, we recover (1757).

Let's briefly comment on why the two approaches (taking the momentum to be quantized in either $\mathbb{Z}$ or $\mathbb{Z} + 1/2$ and not changing the Spin($2n$) gauge field, or always taking the momentum to be in $\mathbb{Z}$ but modifying the $\mathrm{Spin}(2n)$ gauge field in the case of bounding spin structure) agree. Basically, they agree since the $\mathbb{Z}_2$ of $\mathrm{Spin}(1)$ is equal to the $(-1)^F$ of $\mathrm{Spin}(2n)$, so that we can absorb the spin structure into the $\mathrm{Spin}(2n)$ gauge field. At the formal level, this is just coming from the fact that we can permute spin structures by tensoring the spin bundle with the nontrivial line bundle ϵ (the mobius band) over the circle: $S_\pm \otimes \epsilon \cong S_\mp$. Since the Dirac operator is computed with a connection on the bundle $S_\pm \otimes E_{\mathrm{Spin}(2n)}$, we can write $(S_\pm \otimes \epsilon) \otimes E_{\mathrm{Spin}(2n)} = S_\pm \otimes (E_{\mathrm{Spin}(2n)} \otimes \epsilon)$, and transfer the spin structure onto the gauge connection. The point of this

$$Z_\pm(R) = Z_\mp(R_{2\pi} \cdot R), \tag{1765}$$

where $R_{2\pi}$ is a 2π rotation (alias $(-1)^F$) in $\mathrm{Spin}(2n)$. As we saw though, the fact that these expressions agree is not immediately obvious from the calculations.

Directly taking the trace

Let's see if we can reproduce these results by computing the trace directly. With no additional twisting going on, we know we need to get 0 for the non-bounding spin structure $S_+ \otimes \mathbb{R}$ and 2^n for the bounding spin structure $S_- \otimes \mathbb{R}$. This means that

$$Z_-(\mathbf{1}) = \mathrm{Tr}_{\mathcal{H}}(\mathbf{1}) = 2^n, \qquad Z_+(\mathbf{1}) = \mathrm{sTr}_{\mathcal{H}}(\mathbf{1}) = \mathrm{Tr}_{\mathcal{H}}[(-1)^F] = 0. \tag{1766}$$

Here the supertrace sTr is a trace weighted by the matrix $(-1)^F \in \mathrm{Spin}(2n)$.

Note that the *nontrivial bundle* corresponds to the *untwisted* (regular) trace, while the *trivial* bundle corresponds to the *twisted* trace (supertrace). Again, this is because fermions "naturally" have a minus sign associated to them when traveling around closed loops (they naturally have anti-periodic boundary conditions), so that taking the regular trace with the $S_- \otimes \mathbb{R}$ bundle just leads to a sum over states (the two minus signs cancel out). However, with the $S_+ \otimes \mathbb{R}$ bundle, we have given the fermions an extra twist, and the partition function vanishes since the even and odd parts cancel.

Because $Z_\pm(R)$ is a class function on $\mathrm{Spin}(2n)$, it can be written as a linear combination of the characters of $\mathrm{Spin}(n)$. In particular, $Z_\pm(R)$ will be a sum of characters of $\mathrm{Spin}(2n)$ weighted by $(-1)^{(1\mp 1)j}$, where j is a half-odd-integer or an integer according to whether the representation j of $\mathrm{Spin}(2n)$ restricts to a representation of $SO(2n)$ or not. This doesn't really help (me) compute the partition function, though.

Let's start with the bounding $S_- \otimes \mathbb{R}$ bundle. We need to know how to represent R in the basis that we're taking the trace in. For some reason, I found this a bit tricky, and I think the way I do it below could be improved.

First, we know that $\dim \mathcal{H} = 2^n$, since each pair of Majoranas gives us an $SU(2)$ algebra. As we discussed earlier, R can be chosen to lie in exponential of the maximal torus of $\mathfrak{spin}(2n)$. We can then represent R on $\mathcal{H}$ as a product of factors $\otimes_{j=1}^n R_j$, where each R_j factor is a 2x2 matrix acting on the Majoranas η_{2j-1}, η_{2j}. Then we have

$$Z_-(R) = \prod_{j=1}^n \mathrm{Tr} R_j. \tag{1767}$$

Therefore we just need to know how to represent the action of the spin group on a single 2x2 block. Now a single pair of Majoranas can be represented by the matrices

X, Z, so that the product $XZ = J$ generates $\mathrm{Spin}(2) = U(1)$. Elements of $\mathrm{Spin}(2)$ are written as $e^{rJ/2}$, with $r = \pi$ giving $e^{\pi J/2} = J$, which anti-commutes with the fermion-odd matrices X and Z: therefore we identify it with $(-1)^F$ (note to self: the factor of $1/2$ is needed to get right results but the derivation is suspect).

With our $\otimes$ decomposition of R, which is basically a reduction $\mathrm{Spin}(2n) \to [\mathrm{Spin}(2)]^n$ (made possible by the fact that $Z_\pm(R)$ is a class function, as mentioned above), we then have

$$\begin{aligned} Z_-(R) = \mathrm{Tr}_{\mathcal{H}}(R) &= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} \exp\left(\frac{1}{2} r_j J\right)\right] \\ &= \prod_{j=1}^{n} \mathrm{Tr}\begin{pmatrix} \cos(r_j/2) & -\sin(r_j/2) \\ \sin(r_j/2) & \cos(r_j/2)\end{pmatrix} \\ &= 2^n \prod_{j=1}^{n} \cos(r_j/2), \end{aligned} \tag{1768}$$

which matches our earlier result.

Now for the non-bounding bundle $S_+ \otimes \mathbb{R}$. We have

$$Z_+(R) = \mathrm{sTr}_{\mathcal{H}}(R) = \mathrm{Tr}_{\mathcal{H}}((-1)^F R). \tag{1769}$$

We saw above how to represent $(-1)^F$ on $\mathcal{H}$: just set $r_j = \pi$ for all j. Actually we will take $r_j = -\pi$ to match the sign conventions of the previous section; once we have fixed a choice there is no ambiguity since the Pfaffian bundle over $\mathrm{Spin}(2n)$ is trivial. With this choice,

$$(-1)^F = \bigotimes_{j=1}^{n} e^{-\pi J/2} = (-J)^{\otimes n}. \tag{1770}$$

Now we can calculate

$$\begin{aligned} Z_+(R) &= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} (-J)\exp\left(\frac{1}{2} r_j J\right)\right] \\ &= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} \exp\left(\frac{1}{2} (r_j - \pi) J\right)\right] \\ &= \mathrm{Tr}_{\mathcal{H}}\left[\bigotimes_{j=1}^{n} \begin{pmatrix} \sin(r_j/2) & \cos(r_j/2) \\ -\cos(r_j/2) & \sin(r_j/2)\end{pmatrix}\right] \\ &= 2^n \prod_{j=1}^{n} \sin(r_j/2), \end{aligned} \tag{1771}$$

which reproduces our calculation using the Pfaffians.

Fractional charge of fermion mass solitons in two dimensions

Today we will be doing a simple problem in the theme of charge fractionalization in 1+1D systems that I had nevertheless not previously seen a derivation of. We work in two spacetime dimensions, with the action

$$S = \int_{M_2} \left(\bar\psi \not{D}_A \psi + \bar\psi M e^{i\phi\bar\gamma}\psi \right), \tag{1772}$$

where ϕ is a scalar field and $\bar\gamma = Z$ is the chirality operator (we are working in Euclidean signature). We will assume M is large, and consider a time-dependent ϕ which asymptotes to the constants $\phi_\pm$ at $x = \pm\infty$ (note that this is not the same as a Dirac fermion with a real mass that goes to $\pm m$ at $\pm\infty$, but rather a sort of complex generalization thereof). The goal will be to derive the electric charge carried by this kink.

To find the charge induced by the kink, we need to integrate out the fermions. Doing so is slightly tricky, though: we don't want to use $(\not{p} + Me^{i\phi\bar\gamma})^{-1}$ as a free propagator since inverting the exponential is a pain. However, we can't do the usual expansion of the Tr ln while treating the mass term as a vertex, since we are interested in large M and so we won't be able to truncate the expansion. Furthermore, treating the mass term as a vertex is bad since it will turn out that the induced charge of the kink doesn't depend on M. Thus we need to find a way to include M in a propagator in a simple way such that we can do a $1/M$ expansion.

The way to most efficiently integrate out the fermions turns out to be to make use of the trick we learned in a previous diary entry: first compute the variation of the effective action and then integrate over the variation at the end. Taking the variation we get

$$\delta S = -\mathrm{Tr}\left(\delta D D^\dagger (DD^\dagger)^{-1}\right), \tag{1773}$$

where this time (note that in our notation $\not{D}_A = \not\partial + i\not{A}$ so that $\not{D}_A^\dagger = -\not{D}_A$)

$$D = \not{D}_A + Me^{i\phi\bar\gamma}, \qquad D^\dagger = -\not{D}_A + Me^{-i\phi\bar\gamma}. \tag{1774}$$

The reason for doing this is that the product $DD^\dagger$ has within it the term $G_f^{-1} = -\partial_\mu\partial^\mu + M^2$ which we can use to do an expansion in large M. If I did the algebra right, then

$$DD^\dagger = G_f^{-1} - \frac{M}{2i}\sin(\phi\bar\gamma)(\not\partial + i\not{A}) + A_\mu A^\mu - i(\not\partial\not{A}) - 2iA^\mu\partial_\mu - iM(\not\partial\phi)\bar\gamma e^{-i\phi\bar\gamma}, \tag{1775}$$

where we used $-i\not\partial\not{A} = (\not\partial\not{A}) - 2iA^\mu\partial_\mu + i\not{A}\not\partial$. This is a bit of a mess, but when we invert it the large M limit can save the day, since the largest factor of M comes in G_f^{-1} which is nice and simple. So after inverting $DD^\dagger$ in this way and dropping terms that are small in $1/M$, we get

$$\begin{aligned}\delta S = +\mathrm{Tr}\Big[&(i\delta\not{A} + i\bar\gamma M\delta\phi e^{i\phi\bar\gamma})(\not\partial + i\not{A} - Me^{-i\phi\bar\gamma})G_f\Big(\mathbf{1} + G_f\Big(\frac{M}{2i}\sin(\phi\bar\gamma)(\not\partial + i\not{A}) \\ &- A^2 + i\not\partial\not{A} - 2iA^\mu\partial_\mu + iM\not\partial\phi\bar\gamma e^{-i\phi\bar\gamma}\Big)\Big)\Big].\end{aligned} \tag{1776}$$

Just so we can find our way through this mess slightly easier, let us note that

$$\int \frac{d^2k}{(2\pi)^2} G_f^2 = \int \frac{dk}{2\pi} \frac{k}{(k^2+M^2)^2} = \frac{1}{4M^2\pi}, \tag{1777}$$

so that when we do the integrals we are going to be picking up factors of $1/M^2$ (the terms involving just a single G_f will die for spin trace / momentum oddness reasons).

The expression for δS above looks like a disaster, but actually a lot of things drop out under the trace or under the implicit momentum integration (by oddness). In fact, we are just interested in the charge of the kink, so we can restrict our attention to terms linear in A (since we want to find the current, by functionally differentiating wrt A and then setting $A \to 0$). If we take a critical look at δS, we see that the key surviving term is the guy coming from the $i\not\partial \not A$ piece. This part gives

$$\delta S = \mathrm{Tr}\left[\bar\gamma M^2 \delta\phi G_f^2 \not\partial \not A\right] + \dots. \tag{1778}$$

Doing the trace is easy since the derivative is just acting on $\not A$, and so

$$\delta S = \int d^2x \int \frac{d^2k}{(2\pi)^2} \mathrm{Tr}_\sigma[\bar\gamma M^2 \delta\phi \not\partial \not A] \frac{1}{k^2+M^2} + \dots, \tag{1779}$$

where the trace is just the spin trace. The trace against $\bar\gamma$ selects out an antisymmetric structure for ∂ and A which gives us the field strength, and since $\mathrm{Tr}_\sigma[\mathbf{1}] = 2$ we have

$$\delta S = \frac{1}{2\pi} \int d^2x \delta\phi F + \dots, \tag{1780}$$

which is a θ term controlled by the phase of the fermion mass, as expected.

Thus integrating over the variation, we see that the current is

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi, \tag{1781}$$

and so the charge induced by the kink is (assuming space is a line)

$$Q = \int_{\mathbb{R}} j^0 = \int_{\mathbb{R}} \partial_x \phi \frac{1}{2\pi} = \frac{1}{2\pi}(\phi(\infty) - \phi(-\infty)). \tag{1782}$$

Thus the charge is set by degree to which the phase of the fermion mass winds along the kink. The most common scenario, since we have two degenerate states at $\phi = \pi, 0$, is to have $\phi(\infty) - \phi(-\infty) = \pi$, so that the mass term goes from being $M \cdot \mathbf{1}$ at $-\infty$ to $M \cdot \mathbf{1}$ at ∞. In this case, the charge of the kink is $Q = 1/2$.

The kink also localizes a fermion zero mode. A low-tech way to see this is to just explicitly solve $H\psi = 0$ for some choice of $\phi(x)$. For example, suppose that $\phi(x) = \pi\Theta(x)$ so that the mass term is $M \cdot \mathbf{1}$ at $-\infty$ and $M \cdot \mathbf{1}$ at ∞. Then for example at $x > 0$ the equation $H\psi = 0$ reads

$$(i\partial_x - A_x)\psi_L = M\psi_R, \qquad -(i\partial_x - A_x)\psi_R = M\psi_L, \tag{1783}$$

so that $-(i\partial_x - A_x)^2\psi_L = M^2\psi_L$, which we can solve handily as

$$\psi_L(x) \propto \exp\left(i\int_0^x A - M|x|\right), \tag{1784}$$

which is localized around the kink and which could have been guessed from the fact that the phase of zero modes is given by the parallel transport formula (since $D_A\psi = 0$ is the requirement that ψ be parallel-transported) and their magnitude should be localized to the region where the mass changes.

Quantization in AdS

This is a problem taken from a pset assigned in Hong Liu's holography class. The problem is as follows: consider a scalar field in AdS_{d+1}, with action

$$S = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2). \tag{1785}$$

Here Greek indices run over time, $d-1$ "space" dimensions, and the radial coordinate z. Near the boundary at $z=0$, we have the asymptotic expression

$$\phi(z\to 0, x) = A(x)z^{d/2-\nu} + B(x)z^{d/2+\nu}, \qquad \nu \equiv \sqrt{d^2/4 + m^2R^2}, \tag{1786}$$

where R is the AdS radius, and A, B are some functions of the "spacetime coordinates". Our to-do list is as follows:

a) Define an inner product for wavefunctions and show its time independence. b) What is the condition on ν for a given ϕ to be normalizable? c) Find the stress tensor and show that it is covariantly conserved. d) Define the energy E as the integral of $\sqrt{-g}g^{tt}T_{tt}$ over a given Cauchy slice, and find the explicit form of $\partial_t E$. e) When does the energy flux at the boundary vanish? f) Show that E is finite if the chosen wavefunction is normalizable, and infinite otherwise.

a) We define the inner product as

$$\langle\phi,\psi\rangle_{\Sigma_t} = -i\int_{\Sigma_t} dz\,d\vec{x}\,\sqrt{-g}g^{tt}(\phi^*\partial_t\psi - \partial_t\phi^*\psi), \tag{1787}$$

where Σ_t is any Cauchy surface, and the g^{tt} is required to make the integral invariant under rescaling of t. Taking the difference of the inner products at different times, we have

$$\langle\phi,\psi\rangle_{\Sigma_{t'}} - \langle\phi,\psi\rangle_{\Sigma_t} = -i\int_M d^dx^\mu_\perp(\phi^*\nabla_\mu\psi - \nabla_\mu\phi^*\psi), \tag{1788}$$

where M is the timelike boundary of the spacetime volume bounded by the two Cauchy slices, located at $z=0$. Here we have used the fact that the integral over the bounded volume vanishes, on account of

$$\nabla_\mu(\phi^*\nabla^\mu\psi - \nabla^\mu\phi^*\psi)\sqrt{-g} = 0, \tag{1789}$$

by virtue of the equations of motion, viz. $\nabla^2 = m^2$ when acting on ϕ and ψ. The integral on the RHS of (1788) vanishes if ϕ and ψ are properly normalized at infinity, and so the inner product is time-independent.

b) ϕ has the asymptotic expansion

$$\phi(z \to 0) = A(x) z^{d/2-\nu} + B(x) z^{d/2+\nu}. \tag{1790}$$

Now since $\sqrt{-g} \propto z^{-(d+1)}$ and $g^{tt} \propto z^2$, we have

$$\langle \phi, \phi \rangle_\Sigma \sim -i \int_\Sigma dz\, d\vec{x}\, (AA' z^{1-2\nu} + BB' z^{2\nu+1} + 2AB' z). \tag{1791}$$

To get something finite, we need the total power of z to be greater than -1. This is always satisfied by the B mode and so the B mode is always normalizable. For the A mode to be normalizable we need $1 - 2\nu > -1$, so the A mode is normalizable only if

$$0 \leq \nu < 1. \tag{1792}$$

c) There are two terms that contribute to the stress tensor: the Lagrangian density and the $\sqrt{-g}$ in the measure. The variation of the former wrt the metric is simple, while the latter is found by using the usual

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta e^{\mathrm{Tr} \ln g} = -\frac{\sqrt{-g}}{2} \delta \mathrm{Tr} \ln g \implies \frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} = -\frac{\sqrt{-g}}{2} g^{\mu\nu}. \tag{1793}$$

Since the variation of the Lagrangian density wrt the metric components is just $\sqrt{-g}\partial^\mu\phi\partial^\nu\phi$, we have

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}} \left(\sqrt{-g}\partial^\mu\phi\partial^\nu\phi - \frac{\sqrt{-g} g^{\mu\nu}}{2} (\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2) \right) \\ &= \nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2} g^{\mu\nu} (\nabla_\lambda\phi\nabla^\lambda\phi + m^2\phi^2), \end{aligned} \tag{1794}$$

where in the last step we have replaced ordinary derivatives with covariant derivatives since they act in the same way on ϕ, which is a scalar.

Now let's verify that T is covariantly conserved. This is straightforward since the metric is covariantly constant, meaning that we don't have to worry about the differences between raised and lowered indices, and that the covariant derivatives pass straight through the $g^{\mu\nu}$:

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla^2\phi\nabla^\nu\phi + \nabla_\mu\phi\nabla^\mu\nabla^\nu\phi - g^{\mu\nu}\nabla_\mu\nabla^\lambda\phi\nabla_\lambda\phi - m^2\phi\nabla^\nu\phi \\ &= \nabla^\nu\phi(\nabla^2\phi - m^2\phi) = 0, \end{aligned} \tag{1795}$$

again by virtue of the equations of motion. Thus the stress tensor is covariantly conserved.

d) Define the energy as

$$E = \int_{\Sigma_t} dz\, d\vec{x}\, \sqrt{-g} g^{tt} T_{tt}. \tag{1796}$$

The g^{tt} here is needed so that under time rescalings it cancels the rescaling of T_{tt}, so that the whole action rescales like $\sqrt{-g} \mapsto \lambda\sqrt{-g}$ under $t \mapsto \lambda^{-1}t$, which is appropriate for an energy. Another way to write $\sqrt{-g}g^{tt}$ would be $\sqrt{-g_t}n^t$, where g_t is the induced metric on Σ_t and $n^t = (0, z/R, 0, \ldots, 0)$ is the temporal unit vector in coordinates $(z, t, \vec{x})$. Under time rescalings $\sqrt{-g}g^{tt}$ transforms as a vector; as does $\sqrt{-g_t}n^t$ since $\sqrt{-g_t}$ is a scalar under time rescalings (as the metric on Σ_t involves no dt^2 piece).

We can write the time derivative of E as

$$\begin{aligned}\partial_t E &= \frac{1}{\delta t}\left(\int_{V_{\delta t}} d^dx\, dz\, \sqrt{-g}\nabla^\mu T_{t\mu} - \int_{M_{\delta t}} d^dx\, \sqrt{-g_{M_{\delta t}}}n^z T_{tz}\right) \\ &= -\int_{\Sigma_t} d\vec{x}\, \sqrt{g_{\partial\Sigma_t}}n^t n^z T_{tz},\end{aligned} \tag{1797}$$

where the various manifolds are defined straightforwardly: $V_{\delta t}$ is the spacetime volume sandwiched between Σ_t, and $\Sigma_{t+\delta t}$, and $M_{\delta t}$ is the timelike component of its boundary. In the last step, we have used the fact that the time integral in the second term on the first line just produces a factor of δt.

e) First, we need

$$n^t = (0, z/R, \vec{0}), \quad n^z = (z/R, 0, \vec{0}), \tag{1798}$$

which comes from e.g. $n^z n_z = 1$. Now the induced metric on ∂_{Σ_t} has determinant $g_{\partial\Sigma_t} = +(R/z)^{2d-2}$, and so

$$F|_{z=0} = \partial_t E \sim \int_{\Sigma_t} d\vec{x}\, z^{-d+3}T_{tz} = \int_{\Sigma_t} d\vec{x}\, z^{-d+3}\nabla_t\phi\nabla_z\phi. \tag{1799}$$

Now suppose $\phi \sim z^\omega$. Then

$$F|_{z=0} \sim z^{-d+3+2\omega-1}, \tag{1800}$$

and so if the flux is to be zero we must have

$$F|_{z=0} = 0 \implies \omega > d/2 - 1. \tag{1801}$$

Now for the Bz^Δ mode this is always true, since $\Delta = d/2 + \nu$. For the $Az^{d-\Delta}$ mode, this condition reads $\nu < 1$, which is precisely the condition that the A mode be normalizable. So we see that (non)normalizable modes have (non)zero energy flux at infinity.

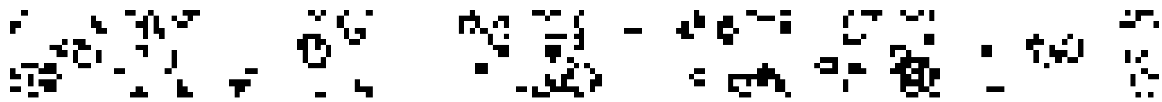
f) The part of T_{tt} with the smallest power of z is the $(\nabla_t\phi)^2$ part. Again, suppose $\phi \sim z^\omega$. Then the contribution to E with the smallest power of z is

$$E \sim \int_{\Sigma_t} dz\, d\vec{x}\, z^{-d-1}z^2 z^{2\omega} \sim z^{-d+2+2\omega}. \tag{1802}$$

If the energy is to be finite, we need $2(1+\omega) > d$. For the A mode we have $\omega = d/2+\nu$, and the energy is always finite. For the B mode we have $\omega = d/2 - \nu$, and

$$E < \infty \implies \nu < 1. \tag{1803}$$

Of course this is the same condition on ν for the B mode to be normalizable. So, normalizable modes, in either quantization scheme, are the ones with finite E.



AdS Propagators

This is another problem from Hong's holography class. The to-do list is:

a) How is Lorentzian AdS different from Euclidean AdS? We will use the latter spacetime in what follows. b) Let ϕ be a massive scalar field, and find the buik-to-∂ propagator $K(z,x;x')$. c) Find a relation between K and the bulk-to-bulk propagator G in terms of the limit of G as one of its arguments approaches the boundary. d) Write down a general boundary correlation function in terms of a limit of a bulk correlation function.

a) From the $1/z^2$ dependence of the metric, we see that in Euclidean AdS, the distances in the x coordinates vanish at $z=\infty$, and so $z=\infty$ is just a single point, unlike in Lorentzian AdS. Another way of seeing this is to recognize that Euclidean AdS is the Poincare disk, with $z\to\infty$ corresponding to the single point at the center of the disk.

b) We want to get the boundary-to-bulk propagator. Using the equations of motion, the propagator K at $z=\infty$ needs to satisfy

$$(\partial_M(\sqrt{-g}g^{MN}\partial_N\phi)-m^2\sqrt{-g})K=0. \tag{1804}$$

Since $z=\infty$ is a single point in the bulk, in the $z\to\infty$ limit $K(x,z;x')$ can only depend on z (not x since all x are the same at $z=\infty$, and not x' by rotational invariance of the Poincare disk). Thus, putting in the z dependence of the metric, we have

$$\left[\partial_z\left((R/z)^{d+1}(z/R)^2\partial_z\right)-m^2(R/z)^{d+1}\right]K(z\to\infty)=0. \tag{1805}$$

Assuming a power-law $K(z)\propto z^\alpha$, we have

$$(1-d)\alpha+\alpha(\alpha-1)-m^2R^2=0 \implies \alpha=\frac{d}{2}\pm\sqrt{d^2/4+m^2R^2}. \tag{1806}$$

We will see later that the requirement that K go to a δ function at the $z=0$ boundary requires us to select out the larger root (which we denote as Δ), and so

$$K(z\to\infty)=Cz^\Delta, \tag{1807}$$

for some $C\in\mathbb{R}$.

Now we can use the homogeneity of AdS (despite how it looks when drawn as a Poincare disk, no point is special) to get $K(x,z;0)$: we first perform the transformation

$$z\mapsto\frac{z}{z^2+x^2},\quad x^\mu\mapsto\frac{x^\mu}{z^2+x^2} \tag{1808}$$

on $K(z \to \infty)$. We then use translation invariance in the x^μ directions (rotational invariance of the Poincare disk) to get the Poisson form

$$K(x, z; x') = C \left(\frac{z}{z^2 + (x - x')^2} \right)^\Delta . \tag{1809}$$

Here we see that the power of Δ gives us a δ function when $x = x'$. This power is also correct since we have

$$K(z \to 0, x; x') = z^{d-\Delta} \delta^d(x - x'). \tag{1810}$$

In standard quantization, ϕ has z^Δ scaling, so that $\int d^dx' \, K(z, x; x')\phi_0(x')$ has the correct scaling.

c) We can relate the bulk-to-bulk propagator to K by using the bulk-to-boundary map and one of Greens identities, namely

$$\int_M d^{d+1}x \sqrt{-g} \left(\phi_1 G^{-1} \phi_2 - \phi_2 G^{-1} \phi_1\right) = \int_{\partial M} d^dx \sqrt{-g_\partial} \left(\phi_1 n^\mu \partial_\mu \phi_2 - \phi_2 n^\mu \partial_\mu \phi_2\right), \tag{1811}$$

where g_∂ is the induced metric on the boundary, G is the bulk propagator, and n^μ is the unit normal on the boundary. The trick is then to employ this identity with $\phi_1 = K(z, x; x'), \phi_2 = G(z, x; z'', x'')$. Now since the LHS is over the bulk and since $(\nabla^2 - m^2) = G^{-1}$ annihilates K in the bulk (the only place it doesn't annihilate K is at coincident points on the boundary), the LHS is

$$LHS = \int_M d^{d+1}x \sqrt{-g} K(z, x; x')(\nabla^2 - m^2)G(z, x; z'', x'') = K(x'', z''; x'), \tag{1812}$$

by definition of G. On the other hand, since $\sqrt{-g_\partial} = (R/z)^d$ and $n^\mu = z$, the RHS is

$$RHS = \int_{\partial M} d^dx \, z^{-d+1} K(z, x; x') \overleftrightarrow{\partial_z} G(z, x; z'', x''), \tag{1813}$$

where $\partial^\leftrightarrow$ denotes the antisymmetrized derivative. Here we have dropped the R dependence since it will cancel out in the end.

Using the asymptotic $z \to 0$ form for K as written above, we can explicitly take the derivative with respect to z and get

$$RHS = \int_{\partial M} d^dx \, z^{-d+1} \delta(x - x') \left(z^{d-\Delta} \partial_z G(z, x; z'', x'') - (d - \Delta) z^{d-\Delta-1} G(z, x; z'', x'')\right). \tag{1814}$$

Now since $G(z, x; z'', x'')$ is normalizable, we know that it has the same $z \to 0$ scaling as the bulk normalizable mode, namely z^Δ. Thus

$$RHS = z^{-d+1}(z^{d-\Delta} \Delta z^{-1} - (d - \Delta) z^{d-\Delta-1}) G(z, x; z'', x''), \tag{1815}$$

where we are implicitly taking the $z \to 0$ limit. In the notation we used in class, $\Delta = d/2 + \nu$, and so

$$RHS = \lim_{z \to 0} (2\Delta - d) z^{-\Delta} G(z, x; z'', x'') = 2\nu z^{-\Delta} G(z, x; z'', x''). \tag{1816}$$

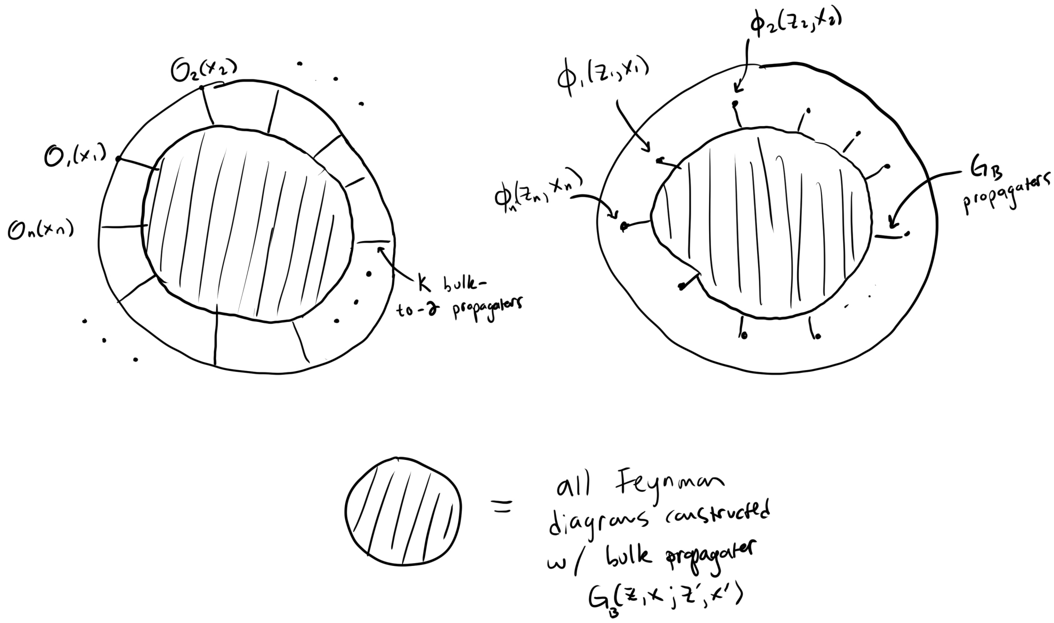


Figure 10: One of the ugliest figures I've ever made. Here, the shaded blob stands for all possible Feynman diagrams constructed from the bulk propagator.

Setting this equal to LHS and moving the $2\nu z^{-\Delta}$ over to the other side and re-labeling some dummy variables, we get

$$\lim_{z\to 0} G(z,x;z',x') = \frac{z'^{\Delta}}{2\nu} K(z,x;x'). \tag{1817}$$

d) Let ϕ_i be the bulk scalar dual to a boundary operator $\mathcal{O}_i$. The correlation function for a product of $\mathcal{O}_i$'s at various points on the boundary can be determined by computing all Feynman diagrams in the bulk that have external legs on the boundary. Thus, a correlation function of n $\mathcal{O}_i$'s involves n K propagators (which connect the boundary $\mathcal{O}_i$'s to the part of the Feynman diagrams that live in the bulk, plus a bunch of G propagators which constitute the bulk part of the Feynman diagrams. On the other hand, we can consider the same class of Feynman diagrams, but with the external legs all made up of G propagators which terminate at points that have some small value of z. This is a bulk correlation function of ϕ_i fields. Taking the $z \to 0$ limit then gets us back to the correlation function of the $\mathcal{O}_i$'s. So, the only difference between the two correlation functions is whether we use K or G for the external legs. If we use G's, then we need to take the $z \to 0$ limit for one of G's arguments—luckily the previous part told us how to do this. So, using our result from c, we have

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{CFT} = \lim_{\{z_i\}\to 0} \prod_i (2\nu_i z_i^{-\Delta_i}) \langle \phi_1(z_1,x_1)\cdots\phi_n(z_n,x_n)\rangle. \tag{1818}$$

All of this is illustrated in figure 10.

Massive vectors in AdS

Yet another problem from Hong's holography class. Consider a massive vector field in AdS:

$$S = -\int d^{d+1}x\,\sqrt{-g}\left(\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}m^2 A_M A^M\right), \tag{1819}$$

where M, N run over (z, x^μ). Do the following:

a) When $m^2 = 0$, find the asymptotic behavior of A_μ at the boundary $z \to 0$. b) What is the scaling dimension of the boundary current J^μ corresponding to the bulk gauge field A? c) When $m \neq 0$, what is the asymptotic behavior of A_M at the boundary? d) Now what is the scaling dimension of J^μ? e) What happens to A_z when $m \neq 0$? f) What are the differences between the massive and massless cases?

a) The equation of motion when $m = 0$ is just

$$\partial_M(\sqrt{-g}F^{MN}) = 0 \implies \partial_M(z^{-d-1}g^{ML}g^{NO}\partial_{[L}A_{O]}) = 0, \quad \forall N. \tag{1820}$$

Let us assume the asymptotic behavior $A_\mu(z \to 0) \sim z^\Delta$, and work in a gauge where $A_z = 0$. The only derivative in the equations of motion we then care about in the $z \to 0$ limit is the one with $L = z$, and so setting $N = \mu$ we have

$$\partial_z(z^{-d-1+2+2}\partial_z z^\Delta) = 0 \implies (-d+3)\Delta + \Delta(\Delta - 1) = 0, \tag{1821}$$

and so we have two options: $\Delta = 0$ or $\Delta = d - 2$. Thus we can write

$$A_\mu(z \to 0) = a_\mu(x) + b_\mu(x) z^{d-2}. \tag{1822}$$

The a_μ is the non-renormalizable piece, while the b_μ part is renormalizable.

b) The non-renormalizable piece a_μ is the part that is relevant for computing the scaling dimension of the current in the CFT dual to A_μ, since a_μ is the part which we interpret as a change in boundary conditions. To determine the scaling dimension of J^μ, we can look at the boundary integral

$$\int_{\partial AdS} d^d x\, a_\mu J^\mu. \tag{1823}$$

Now we consider performing the isometry $z \mapsto \lambda z$, $x^\mu \mapsto \lambda x^\mu$. a_μ is independent of z but it carries a covector index, so it transforms with a factor of λ. Writing $J^\mu(x/\lambda) = \lambda^{\Delta_J} J^\mu(x)$, we have

$$\int_{\partial AdS} d^d x\, a_\mu J^\mu \mapsto \lambda^{-d+1+\Delta_J}\int_{\partial AdS} d^d x\, a_\mu J^\mu. \tag{1824}$$

Since we need this term to be invariant, we find that $\Delta_J = d-1$, as expected of a current in a d-dimensional CFT.

Note that A_μ and A^μ have different scaling behaviors since they differ by e.g. $g^{\mu\mu}$, which scales as z^2. To determine the dimension of J^μ we need to integrate it against something with a covariant index, so it is the scaling of A_μ, not A^μ, which is needed.

c) When the vector field is massive, the equation of motion becomes

$$\partial_M(\sqrt{-g}F^{MN}) - \sqrt{-g}m^2 A^N = 0, \qquad \forall N. \tag{1825}$$

Again, let $A_\mu \sim z^\Delta$ near the boundary. Then we have

$$\partial_z(z^{-d-1}\partial_z A_\mu g^{zz}g^{\mu\nu}) - z^{-d-1}m^2 A_\mu g^{\mu\nu} = 0, \tag{1826}$$

so that

$$\Delta\partial_z(z^{-d+3+\Delta-1}) - z^{-d+1+\Delta}R^2m^2 = 0 \implies \Delta(-d+2+\Delta) - R^2m^2 = 0, \tag{1827}$$

where the R^2 comes from the inverse metric factors. There are thus two possible choices for the scaling behavior of A_μ which are compatible with the equations of motion, and we can write

$$A_\mu = a_\mu z^{\Delta_+} + b_\mu z^{\Delta_-}, \qquad \Delta_\pm = 1 - \frac{d}{2} \pm \sqrt{(d-2)^2/4 + m^2R^2}. \tag{1828}$$

Sorry for the profusion of Δ's! It's just entrenched as a theme by this point and there's no going back.

d) In standard quantization, the non-renormalizable part will be the Δ_+ piece. Looking at the boundary term $\int_{\partial AdS} b_\mu z^{\Delta_+} J^\mu$ and performing the re-scaling of x and z tells us that $-d+1+\Delta_J - \Delta_+ = 0$, and so in this case J^μ has scaling dimension

$$\Delta_J = d - 1 + \Delta_+ = \frac{d}{2} + \sqrt{(d-2)^2/4 + m^2R^2}. \tag{1829}$$

Sanity check: when $m = 0$ we recover $\Delta_J = d-1$, as required.

e) When $m \neq 0$, we can no longer use gauge invariance to fix $A_z = 0$. The z component of the equations of motion reads, focusing only on the z-dependence of A_z,

$$\partial_z(\sqrt{-g}[g^{zz}]^2\partial_z A_z) - \sqrt{-g}m^2 g^{zz}A_z = 0. \tag{1830}$$

Since all the components of the metric have the same z dependence, the z dependence of A_z is fixed in the same way as that of the A_μ.

f) In the massless case, we have gauge invariance under $A \mapsto A + d\chi$. Accordingly, the boundary J^μ operator must be divergenceless, and so it should be thought of as a conserved current. Since we integrate conserved currents over codimension 1 manifolds to get numbers, we need the dimension of J^μ to be $d-1$. By contrast, when $m \neq 0$, there is no gauge invariance, and J^μ is not a conserved current; hence its scaling dimension is not fixed at $d-1$.

Wilson Loop Vevs in $\mathcal{N} = 4$ SYM using AdS/CFT

Today is another problem from a pset in Hong's holography class. This time, we're computing Wilson loops in $\mathcal{N} = 4$ SYM with holography. The problem is as follows:

By evaluating the saddle-point of the NG action corresponding to a geometry in which two quarks have been inserted a distance L apart in the boundary CFT, find the potential energy $V(L)$ coming from the interaction between the two quarks (the relevant Wilson loop here is a rectangle of sides L, t, where $t \gg L$). How does the result behave in the high temperature and low-temperature limits?

We can compute Wilson loop vevs in $\mathcal{N} = 4$ SYM in the limit $g_s \to 0$ (no sum over different topologies) and $\alpha' \to 0$ (when we can use the saddle-point solution to the string path integral). On the CFT side this limit is nontrivial since it corresponds to $N, \lambda \to \infty$.

We just need to compute the classical string action, since

$$\langle W(C) \rangle = Z_{str}[\partial\Sigma = C] \approx e^{iS_{cl}[\partial\Sigma = C]}, \tag{1831}$$

where Σ is the string worldsheet.

At finite T (here T is temperature, not the temporal length of the Wilson loop, which we will write as t), the appropriate bulk geometry to use is an AdS-Schwarzchild black hole at temperature T. The metric is

$$ds^2 = \frac{R^2}{z^2}\left(-(1-\bar{z}^d)dt^2 + d\vec{x}^2 + \frac{1}{1-\bar{z}^d}dz^2\right), \tag{1832}$$

where we've defined

$$\bar{z} \equiv z/z_0, \qquad T = \frac{d}{4\pi z_0}. \tag{1833}$$

z_0 is the location of the horizon, which is closer to the $z = 0$ boundary at larger temperatures.

Let us choose the contour C to run in the $x^1 - t$ plane. We can then parametrize the worldsheet with coordinates $(\tau, \sigma) = (t, x^1)$. We are interested in the energy of two quarks a distance L apart. If the temporal length T of the curve C is much larger than L, then the shape of Σ is determined by a function $z(\sigma) = z(x^1)$, with boundary conditions $z(\pm L/2) = 0$. We will use the NG action (rather than the Polyakov action) to compute S_{cl}, since we aren't ever going to need to quantize anything. The induced metric on the worldsheet is then determined by

$$ds_w^2 = \frac{R^2}{z^2}\left(-dt^2(1-\bar{z}^d) + d\sigma^2\left[1 + \frac{z'^2}{1-\bar{z}^d}\right]\right), \tag{1834}$$

with $z' = \partial_\sigma z$.

The NG action is (again, assuming $t \gg L$ so that the Lagrangian on the classical solution can be treated as time-independent)

$$S_{NG} = -\frac{R^2 t}{\pi\alpha'}\int_0^{L/2}\frac{d\sigma}{z^2}\sqrt{1-\bar{z}^d + z'^2}, \tag{1835}$$

where we pulled out the R^4/z^4 from the determinant of the induced metric and used the symmetry $z(\sigma) = z(-\sigma)$ that must be satisfied by the classical solution.

We can eliminate the z' inside the square root by using the equations of motion. Since $\mathcal{L}$ is independent of σ, we have

$$- z' \frac{\partial \mathcal{L}}{\partial z'} + \mathcal{L} = c, \tag{1836}$$

where c is a constant. For us, this is

$$\frac{z'^2}{z^2 \sqrt{1 + z'^2 - \bar{z}^d}} = \frac{\sqrt{1 + z'^2 - \bar{z}^d}}{z^2} + c, \tag{1837}$$

or

$$\frac{1 - \bar{z}^d}{z^2 \sqrt{1 + z'^2 - \bar{z}^d}} = c. \tag{1838}$$

We can get c by noticing that at $\sigma = 0$, $z' = 0$ by symmetry. So, let $z_* \equiv z(0)$. Then

$$c = \frac{\sqrt{1 - \bar{z}_*^d}}{z_*^2}. \tag{1839}$$

The energy of the Wilson line configuration is then computed as

$$E(L) = \frac{\sqrt{\lambda}}{\pi} \int_0^{z_*} \frac{dz}{z^2 z'} \sqrt{1 + z'^2 - \bar{z}^d}, \tag{1840}$$

since $\lambda = R^4/\alpha'^2$. Solving for z' in terms of z and c, we have

$$z' = \sqrt{(1 - \bar{z}^d) \left(\frac{1 - \bar{z}^d}{c^2 z^4} - 1 \right)}. \tag{1841}$$

Putting this into the integral and doing some housekeeping, we get

$$E(L) = \frac{\sqrt{\lambda}}{\pi} \int_0^{z_*} \frac{dz}{z^2 \sqrt{1 - c^2 z^4 / (1 - \bar{z}^d)}}. \tag{1842}$$

Now $E(L)$ has a $z \to 0$ divergence, but this just corresponds to the diverging mass of the two quarks. Recalling that the quark mass goes as the inverse of their z-coordinates, we expect the quark mass to show up as a $1/\epsilon$ divergence if we cut the integral off below at ϵ.

Now in order to get $E(L)$, we need an expression for z_* in terms of L. We can get an integral equation which gets us part way there by solving for z' and integrating from $\sigma = -L/2$ to $\sigma = 0$:

$$\frac{L}{2} = \int_0^{z_*} dz \left[(1 - \bar{z}^d) \left(\frac{1 - \bar{z}^d}{c^2 z^4} - 1 \right) \right]^{-1/2}. \tag{1843}$$

To see what's happening here more clearly, there are two limits we can take. The first is the $T \to 0$ limit (or equivalently, the small L limit). In this limit we can send

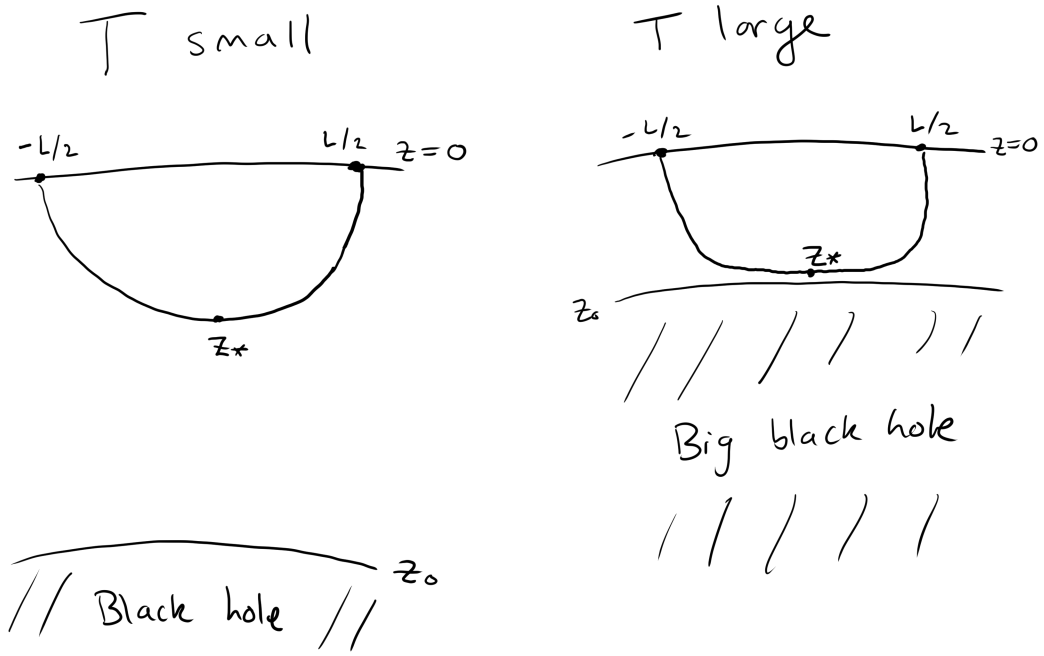


Figure 11: Another one of the ugliest figures I've ever made; explanation is in the text.

$\bar{z} \to 0$ and $c^2 \to z_*^{-2}$, since when $T = 0$ the horizon is pushed to $z_0 = \infty$. In this limit, our integral equation determining z_* is

$$\frac{L}{2} = \int_0^{z_*} dz \frac{1}{\sqrt{z_*^4/z^4 - 1}} = z_* \alpha, \tag{1844}$$

where $\alpha \approx 0.6$ is determined in terms of Elliptic integrals. This means that at $T = 0$, the maximal $z(\sigma)$ value on the worldsheet extends a distance into the bulk which grows linearly with L.

Now we can calculate $V(L)$ in this limit, by using $E(L) = 2M + V(L)$ and subtracting off the divergent mass piece:

$$V(L) = \frac{\sqrt{\lambda}}{\pi z_*} \left(\int_{\epsilon/z_*}^1 \frac{dx}{x^2\sqrt{1 - x^4}} - \frac{1}{\epsilon} \right). \tag{1845}$$

The integral can be evaluated in terms of hypergeometric functions. Doing this, sending $\epsilon \to 0$, and using our expression for z_*, we get

$$V(L) = -\frac{2\alpha\sqrt{\lambda/\pi}\Gamma(3/4)}{L\Gamma(1/4)}. \tag{1846}$$

The most important features here are the $1/L$ dependence (from scale invariance in the CFT), and the interesting $\sqrt{\lambda}$ coupling dependence.

Now we can look at what happens at higher T, or equivalently, at Wilson lines that have L large enough so that z_* approaches the horizon at z_0. By looking at (1843), we see that z_* is monotonically increasing with L (this is a bit gross to show, but it ultimately comes down to $\partial_{z_*} c < 0$).

As we keep increasing L, there reaches a point $L_{screening}$ where (1843) has no solution. Looking back, we see that this must mean that $c = 0$: this is when z_* "disappears" behind the black hole horizon. For $L > L_{screening}$, we no longer can have a Wilson line connecting the two quarks: the worldsheet ends up splitting apart, and terminating on the black hole. In the way we've been doing things, this corresponds to the trivial solution $V(L) = 0$, and the quarks are fully screened. The crossover between the $1/L$ dependence of the potential and the fully screened potential can be found numerically, but I'll be content with this simple understanding of the two limits.

Pion decay

This is essentially P&S, problem 19.2. Given the effective Lagrangian

$$\mathcal{L} = \frac{4G_F}{\sqrt{2}}(\bar{l}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L) + h.c, \tag{1847}$$

find the amplitude for the decay $\pi^+ \to \lambda^+\nu$, and compute the decay rate. Remember that the vector gauge currents are conserved, but the chiral currents are not. Parametrize them by

$$\langle 0|j^{\mu 5a}(x)|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ipx}, \tag{1848}$$

where f_π is a constant.

We get the matrix element for the decay by writing

$$i\mathcal{M}(2\pi)^4\delta(q+k-p) = i\int d^4x\,\frac{4G_F}{\sqrt{2}}\langle l,\nu|(\bar{l}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L)|\pi^+(p)\rangle, \tag{1849}$$

where k, q are the momenta of the lepton l and the neutrino. We can also write this as

$$i\mathcal{M}(2\pi)^4\delta(q+k-p) = i\bar{u}(q)\gamma^\mu\frac{1-\gamma^5}{2}v(k)\int d^4x\,e^{ix(q+k)}\frac{4G_F}{\sqrt{2}}\langle 0|(\bar{u}_L\gamma_\mu d_L)|\pi^+(p)\rangle, \tag{1850}$$

where $\bar{u}$, v are the usual spinors which we use to build the plane wave solutions to the Dirac equation.

It's helpful to first rewrite the matrix element a bit by using the currents

$$j^{\mu a} = \bar{Q}\gamma^\mu\tau^a Q, \qquad j^{\mu 5a} = \bar{Q}\gamma^\mu\gamma^5\tau^a Q, \tag{1851}$$

where $Q = (u, d)$ and $a = x, y, z$ is the $SU(2)$ index. First consider the combination

$$j^{\mu 1} + ij^{\mu 2}. \tag{1852}$$

This choice of $SU(2)$ index structure projects onto $\bar{u}d$ currents (there is no 1/2 needed since the generators are $\tau^a = \sigma^a/2$). We can then select out the left handed components by projecting with $(1 - \gamma^5)$, so that

$$\bar{u}_L \gamma^\mu d_L = \frac{1}{2}(j^{\mu x} + ij^{\mu y} - j^{\mu 5x} - ij^{\mu 5y}). \tag{1853}$$

The first two currents are conserved (they are gauge currents), so when we expand the $\bar{u}\gamma d$ term with the currents, these terms die. Using the parametrization written above for the matrix elements of the chiral currents and the fact that π^+ only has x and y $SU(2)$ indices (a π^+ is an up and an anti-down, so actually it is built as $\sigma^x - i\sigma^y$ in our basis), we get

$$i\mathcal{M}(2\pi)^4\delta(q + k - p) = i\bar{u}(q)\gamma^\mu \frac{1 - \gamma^5}{2} v(k) \int d^4x \, e^{ix(q+k)} \frac{4G_F}{2}(-ip_\mu f_\pi e^{-ipx}), \tag{1854}$$

and so

$$i\mathcal{M} = G_F f_\pi \bar{u}(q) \not{p}(1 - \gamma^5)v(k). \tag{1855}$$

Now we can get the decay rate. To square the matrix element $\mathcal{M}$, we need the spin sums

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m. \tag{1856}$$

Using the Dirac equation and momentum conservation, we have

$$\bar{u}(q)\not{p} = \bar{u}(q)(\not{q} + \not{k}) \approx \bar{u}(q)\not{k}, \tag{1857}$$

since the neutrino is essentially massless. Then we use the Dirac equation for the lepton to write

$$\not{k}(1 - \gamma^5)v(k) = (1 + \gamma^5)\not{k}v(k) = -(1 + \gamma^5)m_l v(k). \tag{1858}$$

So,

$$i\mathcal{M} = -G_F f_\pi m_l \bar{u}(q)(1 + \gamma^5)v(k). \tag{1859}$$

Squaring it and taking the trace with the help of the spin sums, this term becomes

$$G_F^2 f_\pi^2 m_l^2 \mathrm{Tr}[\not{q}(1 + \gamma^5)(\not{k} - m_l)(1 - \gamma^5)]. \tag{1860}$$

The terms odd in γ matrices will die, so the m_l piece doesn't contribute. Then since $2\mathrm{Tr}[\not{q}\not{k}] = 2(q_\mu k_\nu)\mathrm{Tr}[g^{\mu\nu}\mathbf{1}] = 8(q \cdot k)$,

$$|\mathcal{M}|^2 = 8G_F^2 f_\pi^2 m_l^2 (E_\nu E_l + k^2), \tag{1861}$$

where we went to the rest frame where $\mathbf{p} = 0$. Looking up the formula for the decay rate in P&S (and using $E_{cm} = m_\pi$), we get

$$\gamma(l^+\nu \to \pi^+) = 8G_F^2 f_\pi^2 \int \frac{d\Omega}{16\pi^2} \frac{1}{2m_\pi^2} |k| m_l^2 (E_\nu E_l + k^2). \tag{1862}$$

Taking the equation $m_\pi = E_\nu + E_l$ and raising it to the fourth power gives, after some algebra,

$$k^2 = \left(\frac{m_\pi^2 - m_l^2}{2m_\pi}\right)^2 \implies E_\nu = \frac{m_\pi^2 - m_l^2}{2m_\pi}. \tag{1863}$$

Then since $E_l^2 = k^2 + m_l^2$,

$$E_l = \frac{m_\pi^2 + m_l^2}{2m_\pi}. \tag{1864}$$

Now we can plug these into the formula for Γ. Since $\int d\Omega = 4\pi$, this produces after some algebra,

$$\gamma(l^+\nu \to \pi^+) = \frac{m_l^2 m_\pi}{4\pi} G_F^2 f_\pi^2 (1 - m_l^2/m_\pi^2)^2. \tag{1865}$$

Thus the ratio of the decay rates for $\mu^+\nu$ decay and $e^+\nu$ decay is

$$\frac{\gamma(e^+\nu \to \pi^+)}{\gamma(\mu^+\nu \to \pi^+)} = \frac{m_e^2}{m_\mu^2}\frac{(1 - m_e^2/m_\pi^2)^2}{(1 - m_\mu^2/m_\pi^2)^2} \ll 1. \tag{1866}$$

Plugging in numbers with the Fermi constant set at $G_F = \sqrt{2}g^2/(8m_W^2)$ and $\tau_\pi \approx 2.6 \times 10^{-8}$ sec, we get

$$f_\pi = \sqrt{\frac{4\pi}{\tau_\pi m_\mu^2 m_\pi}} \frac{1}{G_F(1 - m_\mu^2/m_\pi^2)} \approx 100\text{Mev}. \tag{1867}$$

We've used just the decay rate to the muon since the contribution of the electron decay channel is suppressed by a factor of $m_e^2/m_\mu^2 \ll 1$ as we saw above.

Higgs effective potential

This is essentially a problem from Schwartz's QFT book, chapter 34. Consider a theory where a scalar field ϕ is coupled to some other fields. We want to calculate, to 1-loop, the contribution of these other fields to the effective potential for ϕ. As an example, consider the example where the other field is a fermion ψ, with the action

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi - V(\phi) + i\bar{\psi}\partial\!\!\!/\psi - Y\phi\bar{\psi}\psi. \tag{1868}$$

We will find the effective potential for the ϕ field, including the contributions from the fermions and the self-coupling of the ϕ field, and will show how this generalizes when we couple ϕ to arbitrarily many different fields.

First let's recall how the effective action approach works in the usual QFT-flavored discussion. We will be working just to 1-loop order throughout.

We first include a current term $\int J\phi$ in the action, and then expand the action about the vev of ϕ, writing $\phi \mapsto \eta + \varphi$, with $\varphi = \langle \phi \rangle_J$. The generating functional of connected correlation functions is then

$$e^{-iW[J]} = \int \mathcal{D}\eta \exp\left(i \int \left[\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi+\eta] + (J_1 + \delta J)(\varphi + \eta) + \frac{\delta \mathcal{L}}{\delta\phi}\Big|_{\varphi} \eta + \frac{1}{2}\int \eta \frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_{\varphi} \eta \right] \right) \tag{1869}$$

Here $\delta\mathcal{L}$ contains counterterms, and we've stopped the expansion of the Lagrangian at quadratic order since we will only be interested in 1-loop effects when integrating out η. As in P&S, J_1 is defined to be the current such that $\delta_\phi \mathcal{L}|_\varphi = -J_1$, so that J_1 is the current which in the classical limit gives an expectation value of φ. We then adjust the "counterterm" δJ order-by-order in perturbation theory to ensure that the expectation value of ϕ in the presence of the current really is φ, i.e. that the expectation value is not changed by a nonzero $\langle \eta \rangle$. This means that δJ is chosen to precisely cancel the tadpole diagrams involving η, and so we can write

$$e^{-iW[J]} = \int \mathcal{D}\eta \exp\left(i \int \left[\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi+\eta] + (J_1 + \delta J)\varphi + \frac{1}{2}\int \eta \frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_{\varphi} \eta \right] \right) \tag{1870}$$

Doing the integral over η gives

$$-W[J] = \int \left(\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi] + J\varphi + \frac{i}{2}\mathrm{Tr}\left[\ln - \frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_{\varphi} \right] \right), \tag{1871}$$

Using the definition of the effective action as $\Gamma[\varphi] + W[J] = -\int J\varphi$, we get

$$\Gamma[\varphi] = \int \left(\mathcal{L}[\varphi] + \delta\mathcal{L}[\varphi] + \frac{i}{2}\mathrm{Tr}\left[\ln - \frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\Big|_{\varphi} \right] \right), \tag{1872}$$

which is independent of J as required (and which again is only correct to 1-loop).

Let's now consider what happens when we have a scalar coupled via a Yukawa coupling to a massless fermion, with the Lagrangian written in the problem statement. Taking $\phi \mapsto \varphi + \eta$ and then dropping the terms linear in η (since they get canceled in the effective action by the condition on the current), we see that after doing the integral over the fermions,

$$\Gamma[\varphi] = \int \left(\frac{1}{2}\varphi\Box\varphi - V(\varphi) + \delta\mathcal{L}[\varphi] + \frac{i}{2}\mathrm{Tr}\left[\ln\left(-\Box + \frac{\delta^2 V}{\delta\phi\delta\phi}\Big|_{\varphi} \right) \right] - i\mathrm{Tr}\left[\ln(-i\partial\!\!\!/ + Y\varphi) \right] \right). \tag{1873}$$

Taking φ to be a constant lets us evaluate the traces. Let

$$m_\varphi^2 \equiv V''(\varphi) \tag{1874}$$

be the effectivce mass-squared of the φ field (it may or may not be positive). Then the bosonic trace is (here V is the spacetime volume)

$$\mathrm{Tr}[\ln(-\Box + m_\phi^2)] = VI, \qquad I = \int_p \ln(-p^2 + m_\phi^2). \tag{1875}$$

The integral becomes convergent if we go to Euclidean time and differentiate it three times wrt m_ϕ^2:

$$\partial^3_{m_\phi^2} I \to i\frac{2\cdot 2\pi^2}{16\pi^4}\int_0^\infty dp\,\frac{p^3}{(p^2+m_\phi^2)^3} = \frac{i}{8\pi^2}\int_0^\infty du\frac{u}{(u+m_\phi^2)^3} = \frac{1}{16\pi^2 m^2}, \quad (1876)$$

where the $\to$ means that we rotated to $i\mathbb{R}$ time to do the integral. Integrating three times then gives

$$I = \frac{i}{16\pi^2}(Am_\varphi^4 + Bm_\varphi^2 + C + [m_\varphi^4/2]\ln(m_\varphi^2/\phi_R^2)), \quad (1877)$$

where A, B, C are some (infinite) constants—they will be dealt with using the counterterms. ϕ_R is a dimensionful scale introduced during the renormalization process, and is fixed as one of our renormalization conditions. Since the trace we just computed appeared in the effective action with a 1/2 coefficient, the effective potential is ($V_{eff}[\varphi] = \Gamma[\varphi]/V$)

$$V_{eff}[\varphi] = C + V_R[\varphi] + \frac{1}{64\pi^2}m_\varphi^4\ln(m_\varphi^2/\phi_R^2) + \dots, \quad (1878)$$

where C is a cosmological constant, $V_R[\varphi]$ is the renormalized potential (into which the terms $Am_\varphi^4 + Bm_\varphi^2$ have been absorbed), and the $\dots$ signify the contribution from the fermions (note that C and the renormalized parameters in $V_R[\varphi]$ will need to be adjusted further after we calculate the fermion contribution, which will have its own divergences as well).

Now we need to deal with the fermion contribution. We write (dropping an infinite constant)

$$\mathrm{Tr}[\ln(-i\partial\!\!\!/ + Y\varphi)] = V\int_p \mathrm{Tr}[\ln(1 - p\!\!\!/ Y\varphi/p^2)] = V\int_p\sum_{n=0}^\infty\frac{1}{n}\mathrm{Tr}[(p\!\!\!//p^2)^n](Y\varphi)^n. \quad (1879)$$

Now the trace can be evaluated as

$$\mathrm{Tr}[(p\!\!\!//p^2)^n] = \begin{cases} 4p^{-n}, & n\in 2\mathbb{Z} \\ 0 & n\in 2\mathbb{Z}+1\end{cases}. \quad (1880)$$

The "proof" of this is as follows: if n is odd then we are taking a trace of an odd number of γ matrices, which vanishes since we are in four dimensions. If n is even, we have to calculate the sum

$$\sum_{\mu_1\cdots\mu_n}\mathrm{Tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_n}]p_{\mu_1}\cdots p_{\mu_n}. \quad (1881)$$

Now this sum will only be nonzero if all of the μ_i group off in pairs, so that $\mu_i = \mu_j$ for some pair i, j. The trace will then produce $\pm\mathrm{Tr}[\mathbf{1}] = \pm 4$, depending on the way in which the indices get paired up. The number of ways N_n to pair up indices for a given n can be calculated inductively. For $n = 2$, there is only one way ($\mu_1 = \mu_2$), means $N_2 = 1$ and $\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}] = 4\eta^{\mu_1\mu_2}$. For $n = k+2$, the μ_1 index can be paired with $k+1$

different other indices. Once this decision is made, the remaining k indices can pair with each other in N_k ways, and so using the base case, we see that

$$N_{k+2} = (k+1)N_k \implies N_n = \prod_{i=0}^{n/2-1}(2i+1). \tag{1882}$$

Now most of these N_n different pairings give different signs for the trace, and cancel out. However, since N_n is always odd, this cancellation is never complete, and always leaves behind one term. So then

$$\sum_{\mu_1\cdots\mu_n} \mathrm{Tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_n}]p_{\mu_1}\cdots p_{\mu_n} = 4\eta^{\mu_1\mu_2}\cdots\eta^{\mu_{n-1}\mu_n}p_{\mu_1}\cdots p_{\mu_n} = 4p^n, \tag{1883}$$

which proves the claim.

Using this identity, we have

$$\mathrm{Tr}[\ln(-i\partial\!\!\!/+Y\varphi)] = 4V\int_p \sum_{n\in 2\mathbb{Z}^{\geq 0}} \frac{1}{n}(Y\varphi/p)^n = 2V\int_p \ln(1-Y^2\varphi^2/p^2) = 2V\int_p \ln(-p^2+Y^2\varphi^2), \tag{1884}$$

where we "un-dropped" the infinite constant from $\mathrm{Tr}\ln i\partial\!\!\!/$ in the last step (this whole rigamarole of expanding the log and then un-expanding it was just to deal with the spin trace). Thus we are back to calculating the same integral as we did when we integrated out the bosonic field: we get

$$\mathrm{Tr}[\ln(i\partial\!\!\!/ - Y\varphi)] = \frac{1}{16\pi^2}(Y\varphi)^4\ln\frac{(Y\varphi)^2}{\Lambda^2} + \dots, \tag{1885}$$

where the $\dots$ are polynomials in $Y\varphi$ with infinite coefficients that will go into determining the correct counterterms to be used for determining the renormalized coupling constants. Keeping track of the sign that the functional determinant entered into $\Gamma[\varphi]$ with, we then have

$$V_{eff}[\varphi] = C' + V_R[\varphi] + \frac{1}{64\pi^2}m_\varphi^4\ln(m_\varphi^2/\phi_R^2) - \frac{1}{16\pi^2}(Y\varphi)^4\ln(Y^2\varphi^2/\phi_R^2). \tag{1886}$$

Note how the contribution from the fermion has a numerical coefficient that is a factor of 4 greater than the boson one, and is negative. The factor of 4 ultimately comes from the fact that a Dirac fermion has 4 components; the minus sign is from the properties of fermionic functional integration. From these remarks, it is clear how to generalize the above potential to include arbitrarily many fields that couple to ϕ: let Ξ_i be a field that couples to ϕ through some interaction $\mathcal{L}_{\Xi_i\phi}$. Then after we expand about φ and drop tadpoles, the one-loop the integral over Ξ_i will produce a functional determinant, with the analogue of m_φ or $Y\varphi$ being defined as

$$m_{\Xi_i} = \frac{\delta^2\mathcal{L}_{\Xi_i\phi}}{\delta\Xi_i\delta\Xi_i}\Big|_{\phi=\varphi} \tag{1887}$$

if Ξ_i is a fermion (with Yukawa coupling to ϕ), or

$$m_{\Xi_i}^2 = \frac{\delta^2\mathcal{L}_{\Xi_i\phi}}{\delta\Xi_i\delta\Xi_i}\Big|_{\phi=\varphi} \tag{1888}$$

if Ξ_i is a boson, with a coupling to ϕ like $\mathcal{L}_{\Xi_i\phi} \sim \phi^2\Xi_i^2 + \dots$. Then the effective potential becomes

$$V_{eff}[\varphi] = C + V_R[\varphi] + \sum_i (-1)^{2\sigma_i} \frac{n_i}{64\pi^2} m_{\Xi_i}^4 \ln \frac{m_{\Xi_i}^2}{\phi_R^2}, \tag{1889}$$

where σ_i is the spin of Ξ_i and n_i is the number of real dof that Ξ_i carries (e.g. 1 for a boson, 8 for an $SU(2)$ fundamental Dirac fermion, etc.).

One more helpful remark to make about the effective potential. Recall that the physical meaning of $V_{eff}[\varphi]$ is the minimum energy density state of the theory, given that the expectation value of ϕ is fixed at φ. Depending on the choice of potential $V_R[\phi]$, there is no reason that m_φ^2 (going back to the case of a single scalar field) should always be positive—indeed, for the usual Mexican hat potential, it is negative near $\varphi = 0$. Then from the above, we see that the argument of the logarithm is negative, and we get an imaginary effective potential. What does this mean? Actually, the meaning is quite physical: an imaginary part means that the time evolution factor $e^{-iV_{eff}[\varphi]T}$ is exponentially damped[214], which means the state with $\langle\phi\rangle = \varphi$ is unstable. This is totally reasonable, since e.g. for the Mexican hat potential, this tells us that regions where $V''[\varphi] < 0$ are unstable: we can't have a theory where the field has zero expectation value, since such a theory is unstable. What's happening here is the ϕ^4 analogue of particle production by strong electric fields with $\langle E^2\rangle > 2m_ec^2$: forcing the vev of the electric field to be too high results in an unstable state, and screening will occur until the vev of E^2 is brought down. This is a quantum effect, so we needed to calculate the 1-loop contribution to $V_{eff}[\varphi]$ in order to see it.

As another super simple example of how this works, consider a quantum particle moving in a potential $V(x)$. Following the procedure above, we get an effective potential of (here x, like φ, is just a number, not a coordinate to be integrated over in the path integral)

$$V_{eff}(x) = V(x) + \frac{1}{2}\int dk\, \ln(k^2 + V''(x)). \tag{1890}$$

The integral is divergent but can be made convergent by differentiating once with respect to V''. Then integrating with respect to V'', we get (may or may not have gotten numerical factors right)

$$V_{eff} = V_R(x) + \frac{1}{16\sqrt{\pi}}\sqrt{V''(x)}, \tag{1891}$$

where we absorbed a (divergent; unimportant) constant into $V(x)$ and wrote the result as $V_R(x)$. The point is that if $V''(x) < 0$, the effective potential is imaginary (keeping track of the $i\epsilon$ would inform us that the imaginary part is negative), and tells us that the particle decays with a decay rate that goes as $\sqrt{V''(x)}$, since the time evolution of the particle at this position is damped by $\sim e^{-\sqrt{V''(x)}}$ if $V''(x) < 0$. This e.g. happens for a quantum particle in the Mexican hat potential when we consider small x. Classically

[214]to check that the sign is such that it is indeed damped, we need to go back and keep track of $i\epsilon$ factors. However clearly the opposite possibility, viz. that it grows exponentially, is obviously not physical.

the particle can balance on the maximum at $x = 0$; quantum mechanically it cannot. Of course this is obvious, but here we have actually computed the precise degree to which it cannot!

Finally, let us comment on what the computation of the effective potential means for SSB. We know from a previous diary entry that the Gaussian fixed point of scalar QED is rendered unstable by the effective potential induced by radiative corrections of the gauge field. Given the form of the effective potential above, wouldn't we also conclude that the Gaussian fixed point is rendered unstable in e.g. pure ϕ^4 theory? If we let $V(\varphi) = g\varphi^4/12$ be a nominally critical renormalized potential, then the effective potential we have derived reads

$$V_{eff}[\varphi] = g\varphi^4 + \frac{\varphi^4}{256\pi^2}L, \tag{1892}$$

where $L \equiv \ln(\varphi^2/2\phi_R^2)$. Now this effective potential *does* have a minimum at nonzero φ, which satisfies

$$L_{min} = -\frac{1}{2} - \frac{1}{256\pi^2 g}. \tag{1893}$$

The point here is that $|L_{min}|$ is not small, since we are performing a perturbative expansion in g (nevermind the $256\pi^2$). Therefore our use of the 1-loop potential is unreliable in the region of the naive potential minimum; if one does a proper RG analysis and goes to subleading order this unphysical minimum is evidently done away with.

In scalar QED, the situation is a bit different. There one finds a term proportional to $\varphi^4 e^4 L$. This is the leading term in the effective potential that depends on e^2; hence it is perturbatively consistent to take $g \sim e^4$, and the resulting minimum one gets has $L_{min} \sim g/e^4$, which is order 1.

Freedom of the Schwinger model without bosonization

Today we're reading the original Schwinger model paper [21] and seeing how he was able to derive the spectrum without using bosonization. The original paper is a bit abstruse in some aspects (as to be expected with Schwinger), so we will just elaborate on some things and provide details of calculations that aren't in the paper (we're also using different notation, so watch out).

The argument goes in two steps. First, we compute an exact expression for the current, and then we use this and the ward identity to derive the spectrum.

By considering the action coupled to sources as $S \ni \int(\bar{J}\psi + \bar{\psi}J)$, shifting ψ by $i\not{D}_A^{-1}J$ gives

$$Z[J] = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i\int_x \left[\bar{\psi}i\not{D}_A\psi + \frac{1}{2e^2}|F_A|^2\right] - \int_{x,y}\bar{J}(x)\mathcal{G}(x-y)J(y)\right), \tag{1894}$$

where the *exact* Green's function for ψ is $\mathcal{G}$, with

$$\not{D}_A(x)\mathcal{G}(x-y) = \delta(x-y). \tag{1895}$$

The thing that's special about 1+1D is that we can actually get a tractable expression for $\mathcal{G}$, at least at small $|x-y|$.

Indeed, take the ansatz

$$\mathcal{G}(x-y) = G_0(x-y)e^{i(\phi(x)-\phi(y))}, \tag{1896}$$

where $\phi(x)$ is a "holonomy function" such that

$$\not{\partial}\phi = \not{A} \tag{1897}$$

(ϕ is a matrix in spin space, i.e. transforms in the $\otimes$ of the [non-chiral] spinor representation and its dual), and $G_0(x-y)$ is the free propagator, i.e.

$$\not{\partial}_x G_0(x-y) = \delta(x-y) \implies G_0(x-y) = \int_p e^{ip\cdot(x-y)}\frac{-i\not{p}}{p^2}. \tag{1898}$$

Now the rewriting $\not{A} = \not{\partial}\phi$ is done completely wolog, and is just another way of parametrizing $\not{A}$.[215] To see this, we do a Hodge decomposition on A, writing $A = d\alpha + d^\dagger \star \beta$ (as we said, we don't care about a possible harmonic part). Now $d\alpha = \partial_0\alpha dx^0 + \partial_1\alpha dx^1$ in any signature, whereas

$$d^\dagger \star \beta = \begin{cases} \partial_1\beta dx^0 - \partial_0\beta dx^1 & \text{Euclidean} \\ \partial_1\beta dx^0 + \partial_0\beta dx^1 & \text{Lorentzian} \end{cases} \tag{1899}$$

This can be derived by tracking down the minus signs involved in the Hodge stars in $d^\dagger$, or more simply by realizing that $d^\dagger \star \beta$ needs to be co-exact, i.e. annihilated when contracted with ∂^μ. Therefore $\partial_0\beta_0 - \partial_1\beta_1 = 0$, which fixes the form of $d^\dagger \star \beta$ as above.

To show why (1897) is wolog, we will directly write down ϕ in terms of A. In $\mathbb{R}$ time with $\gamma^0 = J, \gamma^1 = X$ we have, using the Hodge decomposition,

$$\not{A} = \partial_0(\alpha J + \beta X) + \partial_1(\alpha X + \beta J) = \gamma^\mu\partial_\mu(\alpha\mathbf{1} + \beta Z), \tag{1900}$$

so that $\phi = \alpha\mathbf{1} + \beta Z$. In $i\mathbb{R}$ time with $\gamma^0 = Y, \gamma^1 = X$, we instead have

$$\not{A} = \partial_0(\alpha Y + \beta X) + \partial_1(\alpha X - \beta Y) = \gamma^\mu\partial_\mu(\alpha\mathbf{1} - i\beta Z), \tag{1901}$$

[215] Strictly speaking this is only true on a spatial $\mathbb{R}$; on a spatial S^1 there's a holonomy degree of freedom that can't be captured by this parametrization — this doesn't matter for us though, since we only care that $\not{\partial}\phi = \not{A}$ hold locally.

so that $\phi = \alpha \mathbf{1} - i\beta Z$.

Returning from this aside, we see that the ansatz (1896) for the exact propagator works since

$$(\partial\!\!\!/ - iA\!\!\!/)\left(G_0(x-y)e^{i(\phi(x)-\phi(y))}\right) = \left(\delta(x-y) + G_0(x-y)[i\partial\!\!\!/\phi(x) - iA\!\!\!/]\right)e^{i(\phi(x)-\phi(y))} = \delta(x-y), \tag{1902}$$

with the last equality following from the definition of ϕ and $\delta(x-y)e^{i(\phi(x)-\phi(y))} = \delta(x-y)$.

Now we can get an exact expression for the current, using point splitting. We have

$$j^\mu(x) = -\lim_{\epsilon\to 0}\langle\psi_\alpha(x+\epsilon/2)\bar{\psi}_\beta(x-\epsilon/2)\rangle\gamma^\mu_{\beta\alpha}\exp\left[-i\int_{x-\epsilon/2}^{x+\epsilon/2} d\lambda^\mu\, A_\mu(\lambda)\right], \tag{1903}$$

where the Wilson line has been inserted to maintain gauge invariance. So this is then

$$j^\mu(x) = \lim_{\epsilon\to 0}\left(\mathrm{Tr}[\mathcal{G}(\epsilon)\gamma^\mu](1 + i\epsilon^\mu A_\mu + \dots)\right). \tag{1904}$$

Using our expression for $\mathcal{G}$, and using that

$$G_0(x) = -\partial\!\!\!/_x \int_p \frac{e^{ip\cdot x}}{p^2} = -\frac{1}{2\pi}\partial\!\!\!/[\ln(|x|\Lambda) + \dots] = -\frac{1}{2\pi}\frac{x\!\!\!/}{|x|^2}, \tag{1905}$$

we find, taking the trace,

$$j^\mu(x) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\frac{\epsilon_\nu}{\epsilon^2}(2g^{\mu\nu}[1 - i\epsilon^\lambda A_\lambda(x)] + i\epsilon^\lambda \mathrm{Tr}[\gamma^\mu\gamma^\nu\partial_\lambda\phi(x)]\right). \tag{1906}$$

Taking the limit symmetrically, (and removing a factor of i that I must have goofed on)

$$j^\mu(x) = -\frac{1}{\pi}A^\mu(x) + \frac{1}{2\pi}\mathrm{Tr}[\partial^\mu\phi(x)]. \tag{1907}$$

We can deal with the ϕ term by hitting both sides of $\partial\!\!\!/\phi = A\!\!\!/$ with $\partial\!\!\!/$ and taking the trace. On one hand,[216]

$$\mathrm{Tr}[(\partial\!\!\!/)^2\phi] = \Box\mathrm{Tr}[\phi], \tag{1909}$$

while on the other hand,

$$\mathrm{Tr}[\partial\!\!\!/A\!\!\!/] = 2d^\dagger A. \tag{1910}$$

Therefore

$$\mathrm{Tr}[d\phi] = 2d\Box^{-1}d^\dagger A, \tag{1911}$$

[216] Here $\Box = -\partial_\mu\partial^\mu$ is positive-definite. In terms of differential forms, the positive-definiteness means the sign is fixed as $\Box = +(d^\dagger d + dd^\dagger)$, which is positive definite since, letting $A = d\alpha + d^\dagger\beta + \omega$ with ω harmonic,

$$\int A\wedge\star\Box A = \int(d\alpha + d^\dagger\beta)\wedge\star(d^\dagger dd^\dagger\beta + dd^\dagger d\alpha) = ||dd^\dagger\beta||^2 + ||d^\dagger d\alpha||^2 \tag{1908}$$

and so, as differential forms,

$$j = -\frac{1}{\pi}(1 - d\Box^{-1}d^\dagger)A = -\frac{1}{\pi}\Box^{-1}(\Box - dd^\dagger)A = -\frac{1}{\pi}\Box^{-1}d^\dagger dA \implies j^\mu = -\frac{1}{\pi}[\Pi_T]^{\mu\nu}A_\nu, \tag{1912}$$

where Π_T is the transverse projector. Because of the presence of the projector, this result is manifestly conserved and gauge-invariant. Now in this expression, A^μ is a dynamical field, and this expression only makes sense inside of $\int \mathcal{D}A$. However, we can pull it out the path integral by coupling the gauge field to a source current J_μ and then taking $A \mapsto -ie\delta_J$.[217] Therefore we can write

$$j^\mu(x) = i\frac{e}{\pi}[\Pi_T]^{\mu\nu}\frac{\delta}{\delta J}Z[J], \tag{1914}$$

where $j^\mu(x)$ is now the current in the presence of the source J, which may or may not be turned off after taking the functional derivative. Note that regardless, we will always need to have $d^\dagger J = 0$ so as to retain gauge invariance.

Anyway, while the form of the current looks simple, note that this is a *non-perturbative* result! It's also the result we'd have gotten if we computed the current using the usual 1-loop polarization bubble diagram (a bubble with a current insertion on one end and an A_μ field on the other) — evidently the 1-loop result is exact, something which is made possible by a miraculous (at least in this way of looking at things) cancellation between all the other diagrams. Also as a sanity check, note that this result is compatible with the chiral anomaly: taking $j \mapsto \star j$ gives

$$j_A = -\frac{1}{\pi}(d^\dagger)^{-1} \star F, \tag{1915}$$

giving the correct result for $d^\dagger j_A$. In fact, knowledge that the chiral anomaly is 1-loop exact, plus the above formula, would have also been a sufficient starting point to derive our expression for j^μ, since in two dimensions the vector and axial currents are just $\star$s of one another. This fact makes the cancellation of all the diagrams in the computation of j^μ less mysterious — it just comes down to the index theorem (not so mysterious in 1+1D dimensions because of thinking about electric fields and Fermi seas) and the simple relation between the chiral and vector currents in 1+1D.

Now consider the consequences of the Ward identity for changing variables in the $\mathcal{D}A$ measure. The Ward identity reads (I have a sign difference from the original paper, but I think this is due to different metric signature choices)

$$\left(-i\frac{1}{e}d^\dagger d\frac{\delta}{\delta J^\mu} + (J_\mu + j_\mu)\right)Z[J] = 0, \tag{1916}$$

where here $j_\mu(x)$ is to be understood as a function of $-i\delta/\delta J^\mu(x)$. Now since both currents are conserved, we can freely insert the projector Π_T in front of the $J + j$ term.

[217] The e here appears since in our conventions the JA coupling is

$$S \ni ie^{-1}\int J \wedge \star A. \tag{1913}$$

Plugging in our expression for the current, and writing the $d^\dagger d$ in the above equation as $\Pi_T\Box$, we have

$$[\Pi_T]^{\mu\nu}\left(-i\Box\frac{\delta}{\delta J^\nu}+i\frac{e^2}{\pi}\frac{\delta}{\delta J^\nu}+J_\nu\right)Z[J]=0, \tag{1917}$$

and so, defining the massive Greens function

$$\mathcal{G}_m=\frac{1}{\Box-m^2},\qquad m^2\equiv e^2/\pi, \tag{1918}$$

we can multiply by $\mathcal{G}_m$ (since $\Box$ commutes with Π_T) and conclude that the generating functional obeys

$$i[\Pi_T]^{\mu\nu}\frac{\delta}{\delta J^\nu(x)}Z[J]=\int_y[\Pi_T]^{\mu\nu}\mathcal{G}_m(x-y)J_\nu(y)Z[J]. \tag{1919}$$

Solving this, we conclude that the generating functional of connected correlation functions has the *exact* expression

$$W[J]=\frac{1}{2}\int_{x,y}J_\mu(x)[\Pi_T]^{\mu\nu}\mathcal{G}_m(x-y)J_\nu(y). \tag{1920}$$

Therefore the only nonzero connected correlation function in the theory is the two-point function, given by $\mathcal{G}_m(x-y)$ and therefore the model is exactly equivalent to a free massive scalar of mass m. Since $\mathcal{G}_m$ is a massive propagator, the above equation tells us that two (probe) charges stuck at positions x,y see an exponentially screened potential—this is kind of what happens after the photon is made massive through Higgsing, but of course here there was no photon to begin with, since we are in 1+1D (and likewise, there can be no Goldstone since the symmetry is explicitly broken by the chiral anomaly).

A natural question to ask is to what extent these features are changed when we add a mass (which ruins the soluability of the model). When we have a mass we can also have a θ term, which complicates things, but basically what happens is that the model ceases to be in a "Higgs" regime—the potential between incommensurate charges (like $J=\frac{1}{2}\widehat{C}-\frac{1}{2}\widehat{C}'$ for some curves C,C') *is* long-ranged, and leads to confinement of these charges—but also that there is screening, so that the potential between sources with integral charges is finite-ranged. More on this in a later diary entry.

The non-relativistic limit of a Dirac fermion coupled to a $U(1)$ gauge field and magnetic moments

Today we're doing a rather long elaboration on problem 10.1 in Schwartz, which is something basic that I'd never worked through before. The goal is to take the non-relativistic limit of the Dirac equation and derive how the electron couples to EM fields in this limit. Unfortunately I don't think the method which Schwartz suggests in the problem statement works when the background electric field is nonzero (due to a tricky sleight of hand involving differential operators and square roots). We will follow the outline of the problem but will use what I think are more careful (but longer) methods to get the electric field dependence right.

Our sign conventions for the Dirac equation will be

$$(i\hbar c\not{\partial} - ec\not{A} - mc^2)\psi = 0, \tag{1921}$$

so that $i\partial_t\psi = H_D\psi$ means[218]

$$H_D = -i\gamma^0\gamma^j D_j + eA^0 + \gamma^0 mc^2, \qquad D_j = \hbar c(\partial_j + i\hbar^{-1}eA_i). \tag{1922}$$

Schwartz tells you to subtract off eA_0 and square, but doing this makes you liable to forget about terms like $\partial_j A_0$ which need to be retained to get e.g. the SOC term and the $\nabla^2 A_0$ term. Choosing the non-Weyl basis (so that the rest energy $\gamma^0 mc^2$ is diagonal — we want this to be diagonal since we will be doing an expansion in the non-relativistic limit, where we project into the fixed-particle subspace)

$$\gamma^0 = Z\otimes\mathbf{1}, \qquad \gamma^j = J\otimes\sigma^j, \tag{1923}$$

we have

$$H_D = (Z\otimes\mathbf{1})mc^2 + \phi + i(X\otimes\sigma^j)D_j, \qquad \phi\equiv eA_0. \tag{1924}$$

From now on, all $\otimes$s and $\mathbf{1}$s will be omitted: X, Y, Z will be understood to live in the first $\otimes$ factor, and σ^j will be understood to live in the second.

The strategy is now to subject H to a SW transformation and systematically eliminate all the terms which are off-diagonal in the first tensor factor, so that the transformed Hamiltonian contains only things like $\mathbf{1}\otimes(\dots)$ and $Z\otimes(\dots)$. The offending off-diagonal terms will usually come in the form of $X\sigma^j\partial_j$s and $J\sigma^j\partial_j$s. Eliminating these terms will give us a Hamiltonian that we can easily find the spectrum of. We will work in the non-relativistic limit where $p^2/2m \ll mc^2$—this facilitates the diagonalization process described above because the off-diagonal terms will generally go as ratios of momenta to powers of mc^2.

To this end, rewrite the Schrodinger equation as

$$e^\Lambda i\hbar\dot\psi = e^\Lambda He^{-\Lambda}(e^\Lambda\psi) \implies i\hbar\partial_t\widetilde{\psi} = \left(e^\Lambda He^{-\Lambda} + i\hbar(\partial_t e^\Lambda)e^{-\Lambda}\right)\widetilde{\psi} = \widetilde{H}\widetilde{\psi}, \qquad \widetilde{\psi}\equiv e^\Lambda\psi. \tag{1925}$$

Here Λ is some anti-Hermitian matrix that we will need to solve for. In another diary entry we will prove the tools needed to show that when expanded in Λ, the Schrodinger equation for $\widetilde{\psi}$ is

$$i\hbar\partial_t\widetilde{\psi} = \sum_{k=0}^\infty\left(\frac{\mathcal{N}_k}{k!} + i\hbar\frac{\mathcal{C}_k}{(k+1)!}\right)\widetilde{\psi}, \tag{1926}$$

where we have defined the nested commutators

$$\mathcal{N}_k \equiv [\Lambda,[\cdots,[\Lambda,H]\cdots]], \qquad \mathcal{C}_k\equiv[\Lambda,[\cdots,[\Lambda,\dot\Lambda]\cdots]], \tag{1927}$$

[218]The only potentially hard-to-remember assignment of cs and $\hbar$s is the one for the vector potential. A telsa is a kg /(s C), so that eA_j is valued in kg m/s, meaning that ecA_j has units of energy.

where both terms contain k appearances of Λ.

We will now work out an expression for Λ, order-by-order in the non-relativistic limit. Because I'm feeling slightly masochistic at the time of writing, we will go to order $(pc)^4/(mc^2)^3$. This will entail performing three SW transformations—one to push all the p terms to order p^2, one to get to p^3, and then a final one to get to p^4. Now keeping all the terms that can possibly contribute, we have

$$\begin{aligned}\widetilde{H} = H + ([\Lambda, H] + i\hbar\dot{\Lambda}) + \frac{1}{2}([\Lambda,[\Lambda,H]] + i\hbar[\Lambda,\dot{\Lambda}]) + \frac{1}{6}([\Lambda,[\Lambda,[\Lambda,H]]] + i\hbar[\Lambda,[\Lambda,\dot{\Lambda}]]) \\ + \frac{1}{24}([\Lambda,[\Lambda,[\Lambda,[\Lambda,H]]]] + i\hbar[\Lambda,[\Lambda,[\Lambda,\dot{\Lambda}]]]).\end{aligned} \tag{1928}$$

Now split up the Hamiltonian as

$$H_D = H_0 + H', \qquad H_0 = \phi + Zmc^2, \qquad H' = iX\not{D}_A. \tag{1929}$$

To lowest order, we should look for a Λ such that $[\Lambda, H_0] = -H' + \dots$, where $\dots$ does not contain $\not{D}_A$. Such a Λ is

$$\Lambda_1 = \frac{i}{2mc^2}J\not{D}_A = -\frac{1}{2mc^2}ZH' \implies [\Lambda_1, H_0] = \frac{1}{2}[Z, iJ\not{D}_A] + \Omega = -iX\not{D}_A + \Omega, \tag{1930}$$

where we've defined

$$\Omega \equiv [\Lambda_1, \phi] = \frac{i}{2mc^2}J\hbar c\boldsymbol{\sigma}\cdot\nabla\phi. \tag{1931}$$

Note that Λ_1 is properly anti-Hermitian, so that e^{Λ} implements a unitary transformation on H. We will also need

$$[\Lambda_1, H'] = \frac{1}{mc^2}Z(\not{D}_A)^2 = 4mc^2 Z\Lambda_1^2 \tag{1932}$$

and

$$[\Lambda_1, Z\Lambda_1^2] = \frac{iX}{4(mc^2)^3}\not{D}_A^3 = -2Z\left(iJ\frac{\not{D}_A}{2mc^2}\right)^3 = -2Z\Lambda^3. \tag{1933}$$

Similarly,

$$[\Lambda_1, Z\Lambda_1^3] = -2Z\Lambda_1^4 \tag{1934}$$

To save space, define the rest energy as $\alpha \equiv mc^2$. Then we have

$$\begin{aligned}[\Lambda_1,[\Lambda_1,H]] &= -4\alpha Z\Lambda_1^2 - 8\alpha Z\Lambda_1^3 + [\Lambda_1,\Omega], \\ [\Lambda_1,[\Lambda_1,[\Lambda_1,H]]] &= 8\alpha Z\Lambda_1^3 + 16\alpha Z\Lambda_1^4 + [\Lambda_1,[\Lambda_1,\Omega]] \\ [\Lambda_1,[\Lambda_1,[\Lambda_1,[\Lambda_1,H]]]] &= -16\alpha Z\Lambda_1^4 + [\Lambda_1,[\Lambda_1[\Lambda_1,\Omega]]],\end{aligned} \tag{1935}$$

where in the last line we only kept terms of order Λ_1^4. To make the expansion less messy, we will work in the limit where the momenta of both the $\widetilde{\psi}$ field and the EM field are small compared to the rest energy α, so that we're basically doing an expansion in $1/\alpha$, and dropping everything that goes as $1/\alpha^{n\geq 4}$. Therefore in our order of approximation, we plug everything into $\widetilde{H}$ and get

$$\begin{aligned}\widetilde{H} &= H_0 + 2\alpha Z\left(\Lambda_1^2 - \frac{4}{3}\Lambda_1^3 + \Lambda_1^4\right) + \Omega + \frac{1}{2}[\Lambda_1, \Omega] + \frac{1}{6}[\Lambda_1, [\Lambda_1, \Omega]] \\ &\quad + i\hbar\left(\dot{\Lambda}_1 + \frac{1}{2}[\Lambda_1, \dot{\Lambda}_1] + \frac{1}{6}[\Lambda_1, [\Lambda_1, \dot{\Lambda}_1]]\right) \\ &= \widetilde{H}_0 + \widetilde{H}',\end{aligned} \tag{1936}$$

where $\widetilde{H}_0$ is diagonal in the first $\otimes$ factor and $\widetilde{H}'$ is off-diagonal.

We aren't done, because there are still terms which are off-diagonal in the first $\otimes$ factor. Therefore we must perform another SW transformation to get rid of these terms. For our first transformation, note that we chose $\Lambda_1 = -\frac{1}{2\alpha}ZH'$, where H' was the term that was off-diagonal and needed to be killed. This prompts us to try a second transformation with

$$\Lambda_2 = -\frac{1}{2\alpha}Z\widetilde{H}' \approx -\frac{1}{2\alpha}Z\left(-\frac{8}{3}\alpha Z\Lambda_1^3 + \Omega + i\hbar\dot{\Lambda}_1\right), \tag{1937}$$

where we dropped higher-order terms that are killed by the $1/\alpha$ in front.

To check that this works, one just needs to check that $[\Lambda_2, \widetilde{H}_0] = -\widetilde{H}' + \dots$, where $\dots$ don't involve bare $\not{D}_A$s. For example, let's check that the Λ_1^3 term in $\widetilde{H}'$ gets killed:

$$[\Lambda_2, \widetilde{H}_0] \ni [\Lambda_2, H_0] \ni \frac{4}{3}[\Lambda_1^3, Z\alpha] = \frac{8Z\alpha}{3}\Lambda_1^3, \tag{1938}$$

which is exactly the right term needed to kill the $-\frac{8\alpha Z}{3}\Lambda_1^3$ appearing in $\widetilde{H}'$. The full commutator is then checked to be (still to order $1/\alpha^3$)

$$[\Lambda_2, \widetilde{H}_0] \approx -\widetilde{H}' - \frac{4}{3}[\phi, \Lambda_1^3]. \tag{1939}$$

Since all the terms in Λ_2 go as at least $1/\alpha^2$, we have

$$e^{\Lambda_2}\widetilde{H}e^{-\Lambda_2} = \widetilde{H}_0 + \frac{1}{2}[\Lambda_2, \widetilde{H}'] - \frac{4}{3}[\phi, \Lambda_1^3]. \tag{1940}$$

The commutator is worked out to be, to this order[219]

$$\frac{1}{2}[\Lambda_2, \widetilde{H}'] = -\frac{1}{2\alpha}[Z(\Omega + i\hbar\dot{\Lambda}_1), \Omega + i\hbar\dot{\Lambda}_1] = \frac{Z}{\alpha^3}(\hbar ce)^2(\mathbf{e}\cdot\boldsymbol{\sigma})^2 = \frac{Z}{\alpha^3}(\hbar ce)^2|\mathbf{E}|^2. \tag{1942}$$

A final SW transform removes this, at the expense of generating further higher-order terms which are then thrown away. So in conclusion, to this order, the transformed Hamiltonian is

$$H_{\text{eff}} = \phi + Z\alpha + 2\alpha Z(\Lambda_1^2 + \Lambda_1^4) + \frac{1}{2}[\Lambda_1, \Omega] + \frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] + \frac{Z}{\alpha^3}(\hbar c)^2(\mathbf{e}\cdot\boldsymbol{\sigma})^2. \tag{1943}$$

[219] To get this, we used

$$\dot{\Lambda}_1 = -\frac{ec}{2\alpha}\boldsymbol{\sigma}\cdot\dot{\mathbf{A}}. \tag{1941}$$

Now we just need to calculate the remaining commutators. The first is

$$\frac{1}{2}[\Lambda_1, \Omega] = \frac{1}{2(2\alpha)^2}[\not{D}_A, \hbar c\boldsymbol{\sigma}\cdot\nabla\phi] = \frac{e(\hbar c)^2}{2(2\alpha)^2}\sigma^i\sigma^j\partial_i\partial_j A_0 + \frac{\hbar ce}{2(2\alpha)^2}[\sigma^i, \sigma^j]\nabla_i A_0 D_j. \tag{1944}$$

The second is

$$\frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] = \frac{ec\hbar}{2(2\alpha)^2}[\not{D}_A, \boldsymbol{\sigma}\cdot\dot{\mathbf{A}}] = \frac{(\hbar c)^2 e\sigma^j\sigma^i\partial_j\dot{A}_i}{2(2\alpha)^2} + \frac{\hbar ce}{2(2\alpha)^2}[\sigma^i, \sigma^j]A_i D_j. \tag{1945}$$

Adding these up and simplifying the σ commutators, we get

$$\begin{aligned}\frac{1}{2}[\Lambda_1, \Omega] + \frac{i\hbar}{2}[\Lambda_1, \dot{\Lambda}_1] &= \frac{e}{2}\left(\frac{\hbar c}{2\alpha}\right)^2\sigma^i\sigma^j\partial_j(\partial_i A_0 - \dot{A}_i) - i\frac{ec\hbar}{4\alpha^2}\sigma^k\epsilon_{ijk}(\partial_i A_0 - \dot{A}_i)D_j \\ &= \frac{e}{2}\left(\frac{\hbar c}{2\alpha}\right)^2(\nabla\cdot\mathbf{e} + i\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{e})) - i\frac{ec\hbar}{(2\alpha)^2}\boldsymbol{\sigma}\cdot(\mathbf{e}\times\mathbf{D}).\end{aligned} \tag{1946}$$

These terms, which we've worked so hard for, are the ones that we miss out on if we take the approach in e.g. Schwartz.

The last thing we need to do then is to calculate Λ_1^2. It is

$$2\alpha\Lambda_1^2 = \frac{1}{2\alpha}\left[(\hbar c\partial_i + iecA_i)^2 + \frac{i[\sigma^i, \sigma^j]}{4}\hbar c^2 eF_{ij}\right] = \frac{1}{2\alpha}\left[(\hbar c\partial_i + iecA_i)^2 - 2ec^2\mathbf{S}\cdot\mathbf{B}\right], \tag{1947}$$

with $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$. Therefore

$$2\alpha Z(\Lambda_1^2 + \Lambda_1^4) = -Z\left[\frac{|\boldsymbol{\pi}|^2}{2m} + 2\mu_B\mathbf{S}\cdot\mathbf{B}\right] + Z\frac{1}{8\alpha^3}(c^4|\boldsymbol{\pi}|^4 + 8\hbar^2e^2c^4|\mathbf{B}|^2), \tag{1948}$$

where $\boldsymbol{\pi} = \mathbf{p} + ec\mathbf{A}$ is the canonical momentum and $\mu_B = e/(2m)$ is the Bohr magneton. Note that we have dropped the $\boldsymbol{\pi}$-$\mathbf{B}$ cross terms in the expansion above, but retained the $\mathbf{B}^2$ term: this is just because the later will fit nicely in with the $\mathbf{E}^2$ term we derived above (notice that its coefficient is $\hbar^2e^2c^2/(m^3c^4)$, which is c^2 times the coefficient of the $\mathbf{E}^2$ part). In principle we should keep the mixed term; we're just dropping it cause it's relatively high-order, and ugly.

We have finally calculated everything we need to calculate. For aesthetic purposes we will rename $Z \mapsto -Z$, just so that the $|\pi|^2$ term has a coefficient $+Z$ instead of $-Z$. Adding everything together, we get our final Hamiltonian:[220]

$$\begin{aligned}H_{\text{eff}} = Zmc^2 + eA_0 + Z\left(\frac{|\boldsymbol{\pi}|^2}{2m} - \frac{|\boldsymbol{\pi}|^4}{8m^3c^2}\right) - \frac{e\hbar}{4m^2c^2}\left(\hbar\nabla\cdot\mathbf{E} + i\hbar\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{E}) + \sigma\cdot(\mathbf{E}\times\boldsymbol{\pi})\right) \\ - 2Z\mu_B\mathbf{S}\cdot\mathbf{B} + Z\frac{\hbar^2e^2}{m^3c^2}(c^2|\mathbf{B}|^2 + |\mathbf{E}|^2).\end{aligned} \tag{1949}$$

[220] I tried, but I think slightly failed, to get all the minus signs consistent in the above. I might come back and fix them later, but for now I'll just fix the signs in the answer by what we know they should be on physical grounds (of course, we can take $Z \mapsto -Z$ without affecting anything).

From this Hamiltonian, we can read off a lot. For example, we can find the gyromagnetic ratio g for the electron by choosing a rotationaly-symmetric background field and looking at the relative sizes of the spin $\mathbf{S}\cdot\mathbf{B}$ and orbital $\mathbf{L}\cdot\mathbf{B}$ interactions. The former appears just as $2\mu_B$. For the later, fix a gauge in which we have a uniform field along the z direction, viz. $\mathbf{A} = \frac{B^z}{2}(-y, x, 0)$. The lowest order term which involves $\mathbf{L}\cdot\mathbf{B}$ comes from the expansion of $|\pi|^2$, and we see that the coefficient in front of B^zL^z is $e/2m = \mu_B$. Therefore the ratio of the two couplings tells us that $g = 2$.

Another thing one can do is to find the spin angular momentum of the electron, by checking that $[\mathbf{L}+a\mathbf{S}, H_{\text{eff}}] = 0$ for $a = 1$, provided that the EM potential is spherically symmetric. We can also read off the SOC interaction for a Coulomb potential; choosing $\mathbf{E} = -\partial_r A_0\widehat{\mathbf{r}}$ the $\mathbf{E}\times\boldsymbol{\pi}$ term becomes

$$+\frac{e\hbar}{4m^2c^2}\sigma^i\left(r^{-1}\partial_r A_0[\mathbf{r}\times\boldsymbol{\pi}]_i\right) = \frac{e\hbar}{4m^2c^2}r^{-1}\partial_r A_0\boldsymbol{\sigma}\cdot\mathbf{L}. \tag{1950}$$

When we send $c\to\infty$ we get a spin $SU(2)$ symmetry that acts only on the spin indices of the spinors, but does not involve an actual action on spacetime; the $1/c^2$ effect here breaks this symmetry.

The remaining terms in H_{eff} include the relativistic $|\boldsymbol{\pi}|^4$ correction to the kinetic energy, the magnetic interaction between the electron and the magnetic field produced by $\nabla\times\mathbf{E}$, a quantum-mechanical correction to the potential energy caused by the zero-point motion of the charge density (the $\nabla\cdot\mathbf{E}$ term), and an induced kinetic term for the EM fields (the last term—I haven't seen it in any textbooks before so its coefficient may be suspect).

Added: A helpful shortcut to thinking about the structure of the terms we get is just to write the eigenvalue problem for the Dirac Hamiltonian as

$$\det\begin{pmatrix} m\mathbf{1}-\lambda\mathbf{1} & -\boldsymbol{\sigma}\cdot(\mathbf{p}+e\mathbf{A}) \\ \boldsymbol{\sigma}\cdot(\mathbf{p}+e\mathbf{A}) & -m\mathbf{1}-\lambda\mathbf{1}\end{pmatrix} = 0 \tag{1951}$$

where λ is the energy. This means that in the non-relativistic limit we'll get the term $\frac{1}{2m}[\boldsymbol{\sigma}\cdot(\mathbf{p}+e\mathbf{A})]^2$. Note the $\boldsymbol{\sigma}$ here, which usually isn't written, since when $\mathbf{A} = 0$ the mometa commute with eachother and the Pauli matrices just multiply to $\mathbf{1}$s. However since now $\mathbf{p}+e\mathbf{A}$ doesn't commute with itself, the $SU(2)$ structure is important. Using the identity

$$\sigma^a v_a\sigma^b u_b = v_a u^a + i\epsilon_{abc}v^a u^b\sigma^c = \mathbf{v}\cdot\mathbf{u} + i(\mathbf{v}\times\mathbf{u})\cdot\boldsymbol{\sigma}, \tag{1952}$$

where v, u are two vectors whose components may not commute with each other, we see that

$$\frac{1}{2m}[\boldsymbol{\sigma}\cdot(\mathbf{p}+e\mathbf{A})]^2 = \frac{1}{2m}((\mathbf{p}+e\mathbf{A})^2 + \hbar e\mathbf{B}\cdot\boldsymbol{\sigma}). \tag{1953}$$

Writing $\mathbf{A} = \mathbf{B}\times\mathbf{r}/2$, the first term gives us the angular momentum part of the paramagnetic term ($(\mathbf{r}\times\mathbf{p})\cdot\mathbf{B}$ from $\mathbf{A}\cdot\mathbf{p}$; $\nabla\cdot\mathbf{A} = 0$ in our gauge choice) and the diamagnetic term (from the $\mathbf{A}^2$ part), and the last term is just the usual coupling of the spin to the field that makes up the other half of the paramagnetic response. As we can see from the above result for H_{eff}, getting anything beyond this requires going at least to order $1/m^2$.

Current-current correlators for N scalar fields

Today we're doing a simple problem suggested by Pufu for the attendees of one of the bootstrap schools. Nothing complicated, but I thought it would be good to have around as a reference.

Consider a theory of N free scalars. Find an expression for the 2-point function of the currents, $\langle J^\mu_{ij}(x) J^\nu_{kl}(0)\rangle$. What aspects of this 2-point function are unchanged if $O(N)$ symmetry-preserving interactions are added?

First let us recall what the currents are. Taking $\phi_i \mapsto \phi_i + \epsilon_a(x) A^a_{ij}\phi_j$ where $A^a_{ij} = |e_i\rangle\langle e_j| - |e_j\rangle\langle e_i| \in \mathfrak{so}(N)$ tells us that

$$\delta S = \int d^dx\, \partial_\mu \epsilon_a (A^a_{ij}\phi_j \partial^\mu \phi_i + A_{ij}(\partial^\mu\phi_i)\phi_j) = \int d^dx\, (\partial_\mu \epsilon_{ij}) J^\mu_{ij}, \qquad J^\mu_{ij} = \phi_i \partial^\mu \phi_j - \phi_j \partial^\mu \phi_i. \tag{1954}$$

We want to compute the current-current correlators. Since we care about the coefficients, we need to remember exactly what the propagator is. We invert ∂^2 by requiring that $G_{ij}(r) = \langle \phi_i(r)\phi_j(0)\rangle$ go as $\alpha\delta_{ij}/|r|^\gamma$, where

$$\partial_\mu \frac{\alpha}{r^\gamma} = \frac{\widehat{r}^\mu}{A(S^{d-1})r^{d-1}}, \tag{1955}$$

so that $\partial^2 G_{ij}(r) = \delta_{ij}\delta(r)$. Therefore

$$G_{ij}(r) = \frac{1}{(d-2)A(S^{d-1})}\frac{1}{|r|^{d-2}} \equiv E r^{2-d}. \tag{1956}$$

To get the current 2pt function, we need to compute things like

$$\langle : \phi^i(x)\partial_\mu \phi^j(x) :: \phi_k(y)\partial_\nu \phi_l(y) : \rangle = (\partial^x_\mu G_{jk}(x-y))(\partial^y_\nu G_{il}(x-y)) + G_{ik}(x-y)\partial^x_\mu \partial^y_\nu G_{jl}(x-y), \tag{1957}$$

where the superscripts on the derivatives just indicate which variable they are being taken with respect to. Taking the derivatives,

$$\langle : \phi^i(x)\partial_\mu \phi^j(x) :: \phi_k(y)\partial_\nu \phi_l(y) : \rangle = \delta_{ik}\delta_{jl}\frac{E^2(d-2)}{r^{2d-2}}\left(\delta_{\mu\nu} - \frac{d r_\mu r_\nu}{r^2}\right) - \delta_{il}\delta_{jk}\frac{E^2(d-2)^2}{r^{2d}} r_\mu r_\nu. \tag{1958}$$

The full current-current correlator comes from taking the above and adding $-(i \leftrightarrow j) - (k \leftrightarrow l) + (i,j \leftrightarrow k,l)$. Therefore

$$\langle J^\mu_{ij}(r) J^\nu_{kl}(0)\rangle = \frac{2}{(d-2)A^2(S^{d-1})r^{2d-2}}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})\left(\delta^{\mu\nu} - 2\frac{r^\mu r^\nu}{r^2}\right). \tag{1959}$$

The dimensionality is just what is needed to ensure that the current has no anomalous dimension, and the factor of 2 in the last factor is just what is needed to ensure that the RHS is divergenceless.

We can also get the dimension of the current operator with a Ward identity. This is a slightly better way of finding A since it doesn't make any assumptions about the Lagrangian. The fact that this method recovers the same result as above means that conserved currents always have an anomalous dimension of zero.

Let $\mathcal{O}$ be any operator such that is charged under the symmetry generated by J. Let us consider varying the fields as

$$\phi \mapsto \phi + \eta(x)\delta_S\phi, \tag{1960}$$

where $\delta_S\phi$ is a symmetry and $\eta(x)$ is an indicator function equal to ϵ for $x \in R$ and 0 else. Letting $\mathcal{O}$ be supported at a point $y \in R$, we have (taking ϵ infinitesimal)

$$\langle \delta\mathcal{O}(y)\rangle = \int_X d^dx\,(\partial_\mu\eta(x))\langle J^\mu(x)\mathcal{O}(y)\rangle = \int_{\partial R} d^{d-1}x_\perp^\mu\,\langle J_\mu(x)\mathcal{O}(y)\rangle. \tag{1961}$$

Since this equation must hold regardless of what ∂R is, on dimensional grounds we must have $[J] = d-1$, so that J must have zero anomalous dimension.

When the current is associated with a non-Abelian symmetry as in the above context, the current itself provides us with such an $\mathcal{O}$. If the structure constant $f^{abc} \neq 0$, then $\delta_a\langle J^b_\mu(x)J^c_\nu(0)\rangle \neq 0$ (here δ_a is the variation which sends $\delta_a J^b_\mu = \eta i f_a{}^b{}_c J^c_\mu$, and the currents are such that $\langle J^a(x)J^b(0)\rangle \propto \delta^{ab}$), and so taking $\mathcal{O}(y) = J^b_\mu(y)J^c_\nu(0)$ gives (not keeping track of numerical factors)

$$f^a{}_{bc}\langle J^b_\mu(y)J^c(0)\rangle \sim \int_{\partial R} d^{d-1}x_\perp^\lambda\langle J^a_\lambda(x)J^b_\mu(y)J^c_\nu(0)\rangle. \tag{1962}$$

Requiring this to hold for arbitrary R again tells us the dimension of J and that $\langle J(x)J(0)\rangle \sim 1/|x|^{2(d-1)}$ exactly, as long as J is conserved.

Supersymmetric localization and the geometry of phase space path integrals

Today we're doing a computation that illustrates the basic features of supersymmetric localization in a very simple geometric way. This is a slight elaboration on a problem that was assigned in David Skinner's SUSY course; I found the problem statement on his webpage.

Consider a symplectic manifold (M, ω) of dimension $2m$ equipped with a Hamiltonian map $H : M \to \mathbb{R}$ which generates a $U(1)$ action on M. Consider the zero-dimensional action (the factor of $1/2$ isn't in the problem statement I found online, but I think it's necessary?)

$$S = \alpha \left(H(x) + \frac{1}{2}\omega_{ab}\psi^a\psi^b \right), \tag{1963}$$

where the ψ^a are a set of $2m$ Grassmann variables. Consider the supersymmetry generator

$$Q = \psi^a \frac{\partial}{\partial x^a} + V^a(x)\frac{\partial}{\partial \psi^a}, \tag{1964}$$

where V is the Hamiltonian vector field associated to the action of H. Use localization techniques to exactly compute the partition function

$$Z = \frac{1}{(2\pi)^m} \int d^{2m}x\, d^{2m}\psi\, e^{-S(x,\psi)}. \tag{1965}$$

First let's try to understand the SUSY operator Q better. We can get a better understanding of Q by computing its square:

$$Q^2 = V^a \frac{\partial}{\partial x^a} + \psi^a \frac{\partial V^b}{\partial x^a}\frac{\partial}{\partial \psi^b}. \tag{1966}$$

In what follows, it will be helpful to think geometrically. This is done by realizing that the ψ^a variables are really just a way of writing /basis covectors dx^a. In this notation, a 1-form V is written as $V_a\psi^a$, which has all the properties of a 1-form written as $V_a dx^a$ (the weird way that fermion measures transform allows the ψ^a to behave as basis covectors). Thinking about this, we realize that $\psi^a \frac{\partial}{\partial x^a}$ is really just $d = dx^a \wedge \partial_a$, and that $V^a(x)\frac{\partial}{\partial \psi^a}$ is really just i_V, which operates by contraction: $i_V(B_a dx^a) = V^a B_a$, $i_V(C_{ab}dx^a \wedge dx^b) = V^a C_{ab} dx^b$, and so on. Therefore, in geometric language, the SUSY operator is the equivariant derivative associated with the vector field V (think of the covariant derivative associated to the gauge field V), viz.

$$Q = d_V = d + i_V. \tag{1967}$$

Now let us identify the square of Q. In differential geometry language, converting ψ^as to dx^as, we claim that

$$Q^2 = \mathcal{L}_V = i_V d + d i_V, \tag{1968}$$

which is the Lie derivative along V. To show this, we can e.g. consider how Q^2 acts on a 1-form $\rho = \rho_a\psi^a = \rho_a dx^a$:

$$Q^2\rho = \left(\frac{\partial V^a}{\partial x^b}\rho_a + V^a \frac{\partial \rho_b}{\partial x^a} \right)\psi^b. \tag{1969}$$

Let us then compute

$$\mathcal{L}_V \rho = i_V(d\rho) + d(i_V\rho). \tag{1970}$$

The first term is

$$i_V(d\rho) = i_V\partial_a\rho_b\psi^a\psi^b = V^a\left(\frac{\partial\rho_b}{\partial x^a} - \frac{\partial\rho_a}{\partial x^b}\right)\psi^b = V^a\partial_{[a}\rho_{b]}dx^b. \tag{1971}$$

The second term is

$$d(i_V\rho) = \frac{\partial V^a}{\partial x^b}\rho_a\psi^b + V^a\frac{\partial\rho_a}{\partial x^b}\psi^b. \tag{1972}$$

Adding these two together, we get

$$\mathcal{L}_V\rho = \left(\frac{\partial V^a}{\partial x^b}\rho_a + V^a\frac{\partial\rho_b}{\partial x^a}\right)\psi^b = Q^2\rho, \tag{1973}$$

and so Q^2 indeed acts as the Lie derivative.

Let's now show that the action is invariant under SUSY transformations. The first term transforms simply as

$$QH = \psi^a\frac{\partial H}{\partial x^a}, \tag{1974}$$

which in differential geometry notation would read $QH = d_V H = dH$. The second term is

$$Q\left(\frac{1}{2}\omega_{ab}\psi^a\psi^b\right) = \frac{1}{2}\left(\psi^c\frac{\partial\omega_{ab}}{\partial x^c}\psi^a\psi^b + \omega_{ab}V^a\psi^b - \omega_{ab}\psi^a V^b\right) = \omega_{ab}V^a\psi^b, \tag{1975}$$

where we have used the antisymmetry of ω and the fact that ω is closed as a 2-form, so that the antisymmetrization enacted by the fermions in the first term makes the first term vanish. In terms of differential geometry language, the above computation reads

$$Q\omega = d_V\omega = d\omega + i_V\omega = V^a\omega_{ab}dx^b. \tag{1976}$$

Putting these two transformations together, we see that the action is indeed invariant under SUSY transformations, since

$$QS = \psi^a\frac{\partial H}{\partial x^a} + \omega_{ab}V^a\psi^b = \psi^a\omega_{ab}V^b + \omega_{ab}V^a\psi^b = 0, \tag{1977}$$

where we've used the definition of the Hamiltonian vector field V, namely that V is determined by dual to $\frac{\partial}{\partial x^a}H$ via the symplectic form, which sets up the isomorphism $\omega : TM \cong T^*M$:

$$V^a = \omega^{ab}\frac{\partial H}{\partial x^b}. \tag{1978}$$

Just for fun, let's recall why this is true: a Hamiltonian vector field V is a vector field in phase space along which the symplectic form is preserved, which means that

$$0 = \mathcal{L}_V\omega = d(i_V\omega), \tag{1979}$$

since $d\omega = 0$. Therefore locally $i_V\omega$ is exact, and we can write it as $i_V\omega = dH$, where H is of course the Hamiltonian.

In order to run the localization procedure, we will add a Q-exact term to the action, and then show that the path integral is unchanged by its presence. The natural Q-exact term to add is the equivariant derivative $d_V V = QV$. More formally, let g be a

$U(1)$-invariant metric on M.[221] Define $\Psi = g(V,\psi) = \psi^a V_a$, and consider adding $Q\Psi$ to the action, by defining[222]

$$S_\lambda(x,\psi) \equiv S(x,\psi) + \lambda Q\Psi, \tag{1980}$$

where

$$Q\Psi = V^a V_a + \psi^a\psi^b\partial_a V_b. \tag{1981}$$

Now we will show that $Q^2\Psi = 0$, which will mean that $QS_\lambda = 0$. Based on the geometric interpretation of Q that we just gave, the differential-geometric way to prove this is is to show that

$$Q^2\Psi = \mathcal{L}_V V = (i_V d + d i_V)V \tag{1982}$$

vanishes. Indeed, the two terms on the RHS are

$$i_V dV = i_V(\partial_a V^b dx^a \wedge dx^b) = V^a\partial_{[a}V_{b]}dx^b = (V^a\partial_a V_b - V^a\partial_b V_a)dx^b, \qquad d i_V V = 2V^a\partial_b V_a dx^b, \tag{1983}$$

and so

$$\mathcal{L}_V V = V^a(\partial_a V_b + \partial_b V_a)dx^b = 0, \tag{1984}$$

since V is Killing field.

Now let us consider the variation of the partition function Z_λ with respect to λ:

$$(2\pi)^m\partial_\lambda Z_\lambda = \int d^{2m}x\, d^{2m}\psi\, \partial_\lambda S_\lambda e^{-S_\lambda} = \int d^{2m}x\, d^{2m}\psi\, Q(\Psi)e^{-S_\lambda} = \int d^{2m}x\, d^{2m}\psi\, Q\left(\Psi e^{-S_\lambda}\right), \tag{1985}$$

since $QS_\lambda = 0$. However, both terms that are created when Q acts on something are total derivatives—either a total ψ derivative (in which case we get zero since $\int d\psi = 0$) or a total x^a derivative (in which case we get zero if $H(x) \to \infty$ at the limits of x integration, which we assume). Therefore $\partial_\lambda Z_\lambda = 0$, and so Z_λ is actually independent of λ. This means that we can send $\lambda \to \infty$ without changing the partition function. Now the exponential of the $\lambda\psi^a\psi^b\partial_a V_b$ term, being fermionic, has a power series expansion that truncates after a finite number of terms and ensures that its contribution to the partition function occurs as λ^m. The exponential of $-\lambda V^a V_a$ will thus dominate over the λ^m term and send the integrand to zero, unless x is such that $V^a V_a(x) = 0$. Let x_* denote a point where $V^a(x_*) = 0$. Since the symplectic form is non-degenerate, this must mean that at x_* we have $(\partial_a H)(x_*) = 0$, and so the points that the integral localizes around are the critical points of the Hamiltonian.

Another way to see that the integral localizes, which doesn't rely on adding the extra term to the action, is to realize that anything which is Q-closed is also Q exact everywhere except for the critical points where $V(x_*) = 0$. Since $Qe^{-S} = 0$ the integrand in Z is Q-closed, and so this means that the integrand is a total derivative everywhere except the critical points; by Stoke's theorem we then see that Z receives contributions only from the critical points.

[221] If we are given a metric that's not $U(1)$-invariant, we can get an invariant one just by averaging over the $U(1)$ action, so this is done wolog.

[222] I'm getting sick of the fractions, so from now on ∂_a means differentiation wrt x^a (fermionic derivatives will be expressed more verbosely).

To prove the above claim about Q-exactness, we proceed as follows. Consider the field (best thought of as an inhomogeneous differential form living in $\Omega^\bullet(M)$)

$$\Gamma = V \wedge (QV)^{-1}, \tag{1986}$$

which is defined on $M \setminus X_*$, where X_* is the set of critical points. Here the inverse of QV is defined as (note to self: from here until when I go back to discussing the $\lambda Q\Psi$ tactic, the operator Q is $d - i_V$ instead of $d + i_V$ as above—bleh, at least it doesn't affect the conclusions — will try to find the motivation to cleanup later)

$$(QV)^{-1} = (dV - V^2)^{-1} = -\frac{1}{V^2}(1 - dV/V^2)^{-1} = -\frac{1}{V^2}\sum_{i=1}^{m} V^{-2k}(dV)^{\wedge k}, \tag{1987}$$

where again the notation is $V = V_a\psi^a$, $dV = \partial_a V_b \psi^a\psi^b/2$. This is the proper way to take an inverse of an inhomogeneous differential form for the same reason that $\sum_j (-x)^j = (1+x)^{-1}$. In our case the sum is finite and so we get

$$(1 - dV/V^2)\sum_{j=1}^{m}(dV/V^2)^{\wedge j} = 1 - (dV/V^2)^{\wedge(m+1)}, \tag{1988}$$

but the last term vanishes since its dimension is too large. Note that the inverse is only well-defined away from the critical points.

A more pedestrian approach, which convinces us that this is the right way to take the inverse, is as follows. We want to find Γ such that $Q\Gamma = 1$. Since Q is odd, we can parametrize Γ as a series of odd terms:

$$\Gamma = -\frac{A_a\psi^a}{V^2} - \frac{B_{abc}\psi^a\psi^b\psi^c}{V^2} - \dots, \tag{1989}$$

where the minus signs and $1/V^2$s are just for convenience. Now the first term, when acted on by Q, is

$$Q\Gamma \ni \frac{V^a A_a}{V^2} - \frac{\partial_a A_b \psi^a\psi^b}{V^2} + \frac{2}{V^4}V^c\partial_a V_c A_b \psi^a\psi^b. \tag{1990}$$

Since the first term is the only one in $Q\Gamma$ with no ψs, we need $A_a = V_a$, so that

$$Q\Gamma \ni 1 - \frac{\partial_a V_b \psi^a\psi^b}{V^2} + \frac{2}{V^4}V^c\partial_a V_c V_b \psi^a\psi^b. \tag{1991}$$

The second term in Γ, when acted on by Q, contributes 2- and 4-fermion terms. If $Q\Gamma = 1$, then we need the 2-fermion terms to precisely the 2-fermion terms in the above equation. The 2-fermion terms produced by the B term in $Q\Gamma$ are

$$Q\Gamma \ni \frac{V^a}{V^2}\left(-B_{abc} + B_{bac} - B_{bca}\right)\psi^b\psi^c. \tag{1992}$$

Suppose we choose $B_{abc} = V^{-2}V_a\partial_b V_c$, which is what the above formula for the inverse tells us to do. Then the above terms become

$$Q\Gamma \ni \frac{1}{V^4}\left(V^2\partial_a V_b - V_a V^c\partial_c V_b + V_a V^c\partial_b V_c\right)\psi^a\psi^b. \tag{1993}$$

Therefore we have, re-naming some dummy indices and letting ... denote terms with 4 or more fermions,

$$Q\Gamma = 1 - \frac{1}{V^4}\left(V_a V^c \partial_b V_c + V_a V^c \partial_c V_b\right) + \cdots = 1 - \frac{V^a V_c}{V^4}\partial_{(b}V_{c)} = 1 + \ldots \tag{1994}$$

since V is Killing.

One can then check that choosing the next term to be $-V^{-2}C_{abcde}\psi^a \cdots \psi^e$ with $C = (dV \wedge dV)/V^4$ cancels the 4-fermion terms in the QB_{bcd} term, and that the 6-fermion terms in the QC_{abcde} term are canceled by the next order term, and so on. This cancellation occurs up until we reach products of m fermions, at which point all further terms die by antisymmetry.

Anyway, using Γ we can note that since $Q\Gamma = 1$,

$$QS = 0 \implies Q(\Gamma S) = S, \tag{1995}$$

and so S is Q-exact. Therefore all Q-closed forms are exact away from the critical points, proving our claim about localization.

Anyway, going back to the $\lambda Q\Psi$ approach which is easier to implement in practice, we expand about the critical point and then define y by $y = x\sqrt{\lambda}$ and $\widetilde{\psi}$ by $\widetilde{\psi} = \psi\sqrt{\lambda}$,[223] yielding

$$\begin{aligned} Z = Z_{\lambda\to\infty} = \frac{1}{(2\pi)^m} \sum_{x_*:V^a(x_*)=0} \int d^{2m}y\, d^{2m}\psi\, \exp\Big[\alpha\Big(&- H(x_*) \\ &- \frac{1}{2}(\partial_a\partial_b(V_cV^c))(x_*)y^a y^b - (\partial_{[a}V_{b]})(x_*)\widetilde{\psi}^a\widetilde{\psi}^b + \ldots\Big), \end{aligned} \tag{1996}$$

where ... are terms that vanish when $\lambda \to \infty$ and hence can be dropped. We can now do the integrals no problem:

$$Z = \sum_{x_*:V(x_*)=0} e^{-\alpha H(x_*)}\alpha^m \frac{\mathrm{Pf}[(\partial_{[a}V_{b]})(y_*)]}{\sqrt{\det[(\partial_a\partial_b V_c V^c)(y_*)]}}. \tag{1997}$$

Now around each critical point, we can choose coordinates where V has the canonical form of a Hamiltonian vector field, viz.

$$V(x_*) = \sum_{i=1}^m n_i(x_*)\left(p^i\frac{\partial}{\partial q^i} - q^i\frac{\partial}{\partial p^i}\right), \tag{1998}$$

where we have split up the coordinates near x_* into a set of m "coordinates" q_i and m "momenta" p_i. Here the $n_i \in \mathbb{Z}$ since H generates a $U(1)$ symmetry. With this choice of coordinates, about each critical point we have

$$\mathrm{Pf}[(\partial_{[a}V_{b]})(x_*)/2] = \prod_{i=1}^m n_i(x_*), \qquad \det[(\partial_a\partial_b V_c V^c)(x_*)/2] = \det[(\partial_a V_c \partial_b V^c)(x_*)] = \prod_{i=1}^m n_i^4(x_*), \tag{1999}$$

[223] The integration measure is invariant since the fermionic and bosonic measures transform oppositely.

and so

$$Z = \sum_{y_*:V(y_*)=0} e^{-\alpha H(y_*)} \frac{\alpha^{-m}}{\prod_{i=1}^m n_i(y_*)}. \tag{2000}$$

Finally, we can massage this by noting that the fermion path integral in the $\lambda = 0$ action can also be done exactly, giving

$$\int d^{2m}\psi \, e^{-\frac{1}{2}\omega_{ab}\psi^a\psi^b} = \frac{1}{m!}\omega^{\wedge m}, \tag{2001}$$

which is the usual phase-space measure (on the LHS $\omega = \omega_{ab} dx^a \wedge dx^b$). Therefore the localization formula is

$$\frac{1}{m!}\int \omega^{\wedge m} \, e^{-\alpha H(x)} = \sum_{x_*:V(x_*)=0} e^{-\alpha H(x_*)} \frac{(2\pi/\alpha)^m}{\prod_{i=1}^m n_i(x_*)}. \tag{2002}$$

Majorana quantum mechanics, path integrals, and traces of γ matrix exponentials

Today we're doing another problem from David Skinner's SUSY class. This one is pretty easy and shares a few features in common with a previous diary entry on Majorana path integrals, so we will be rather succinct.

Consider the QM action

$$S[\psi] = \frac{1}{2}\int_{S^1} d\tau \left(\psi_a \partial_\tau \psi^b + \omega_{ab}\psi^a\psi^b\right), \tag{2003}$$

where ω is antisymmetric and $a = 1, \ldots, 4$. Using the path integral obtained from this action, find expressions for

$$\mathrm{Tr}[\bar{\gamma} e^{-\omega_{ab}\gamma^a\gamma^b/4}], \qquad \mathrm{Tr}[e^{-\omega_{ab}\gamma^a\gamma^b/4}] \tag{2004}$$

in terms of sinh and cosh expressions. Here $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ are the 4d Gamma matrices.

First, some preliminaries. Consider an operator $\mathcal{O}$ acting on a single two-dimensional Fock space. Then we claim that

$$\mathrm{Tr}[\mathcal{O}] = \int d\bar{\eta} \, d\eta \, e^{-\bar{\eta}\eta} \langle -\bar{\eta}|\mathcal{O}|\eta\rangle, \tag{2005}$$

where the coherent states are, in our conventions,

$$|\eta\rangle = e^{-\eta\psi^\dagger}|0\rangle, \qquad \langle\bar{\eta}| = \langle 0| e^{\bar{\eta}\psi}, \tag{2006}$$

with $\psi^\dagger|0\rangle = |1\rangle$ (the ψs are operators; the ηs are Grassmann numbers). With these conventions, $\langle\bar\eta|\eta\rangle = e^{\bar\eta\eta} = 1 + \bar\eta\eta$. One can then check that the minus sign in the left bra in (2005) is needed to get the trace, and that if it's replaced with a $\langle\bar\eta|$, one instead gets the supertrace.

When we Trotterize the above expression for the trace with $\mathcal{O} = e^{-\beta H}$ (remembering the $e^{-\bar\eta_t\eta_t}$ factors!), we see that we get the usual $\bar\eta\partial_t\eta - H$ action, provided that we can identify $\bar\eta_N(\eta_1 + \eta_N)/\delta\tau$ with a time derivative (where there are N steps in the Trotterization). Therefore after Trotterizing, the trace computes the path integral with antiperiodic boundary conditions. Similarly, the supertrace $\text{Tr}[(-1)^F\mathcal{O}]$ computes the trace with periodic boundary conditions.

To connect with the gamma matrices, one just realizes that four majoranas, when quantized, obey the same algebra as 4-dimensional gamma matrices. More precisely, because of normalization, the identification is $\psi^a = \gamma^a/\sqrt{2}$. Since $\bar\gamma$ anticommutes with all of them, we have in $(-1)^F = \bar\gamma$ in this representation. Therefore the partition functions with the two choices of boundary conditions are

$$Z_P = \text{Tr}[\bar\gamma e^{-\omega_{ab}\gamma^a\gamma^b/4}], \qquad Z_A = \text{Tr}[e^{-\omega_{ab}\gamma^a\gamma^b/4}]. \tag{2007}$$

So, to compute these traces, we can alternately compute

$$Z_{A/P} = \det\left[\frac{1}{2}\frac{d}{d\tau} - \omega/2\right]^{1/2}, \tag{2008}$$

with the eigenvalues for d_τ being determined from the boundary conditions.

First consider the case when the fermions are periodic around the S^1. Then we have

$$Z_P = \det\left[\prod_{n\in\mathbb{Z}}(i\pi n + \omega/2)\right]^{1/2}, \tag{2009}$$

where now the determinant is taken only in the spinor tensor factor. Therefore

$$\text{Tr}[\bar\gamma e^{-\omega_{ab}\gamma^a\gamma^b/4}] = \det\left[\frac{\omega}{2}\prod_{n\in\mathbb{N}}(\pi^2n^2 + \omega/2)\right]^{1/2} = N\det[\sinh(\omega/2)]^{1/2}, \tag{2010}$$

where we've used

$$\sinh(x) = x\prod_{n\in\mathbb{N}}(1 + x^2/(\pi n)^2), \tag{2011}$$

and where N is an ω-independent constant (determined e.g. through ζ regularization).

The antiperiodic spin structure case is basically the same: we get

$$\text{Tr}[e^{-\omega_{ab}\gamma^a\gamma^b/4}] = \det\left[\prod_{n\in\mathbb{N}+1/2}(\pi^2n^2 + \omega/2)\right]^{1/2} = M\det[\cosh(\omega/2)]^{1/2}, \tag{2012}$$

where M is another ω-independent constant. Here the relevant product formula is

$$\cosh(x) = \prod_{n\in\mathbb{N}}\left(1 + \frac{x^2}{(\pi n - \pi/2)^2}\right). \tag{2013}$$

Large N matrix model quantum mechanics and eigenvalue distributions

This is an elaboration on a problem from a pset assigned in Hong Liu's 2018 class on AdS / CFT. We consider a matrix model with partition function

$$Z = \int \mathcal{D}M \, \exp\left(-\frac{N}{g}\mathrm{Tr}[V(M)]\right), \tag{2014}$$

where $V(M)$ is a polynomial potential (so that the action is a function only of the eigenvalues of M), and the integral over M runs over all $N \times N$ Hermitian matrices. We will eventually specialize to $V(x) = x^2/2 + x^4$. We will also denote the eigenvalue density by

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i), \tag{2015}$$

where $\{\lambda_i\}$ are the eigenvalues. Taking $N \to \infty$, we will assume $\rho(\lambda)$ approaches a continuous function supported on some interval $I \subset \mathbb{R}$.

Do several things:

- Find an expression for Z to leading order in the $N \to \infty$ limit.
- Define the complex function

$$F(\xi) = \int_I d\lambda \, \frac{\rho(\lambda)}{\xi - \lambda}. \tag{2016}$$

  Discuss the analytic properties of $F(\xi)$.

- Show that $F(\mu - i\varepsilon) \in \mathbb{R}$ if $\mu \in \mathbb{R} \setminus I$, and find $F(x \to \infty)$.
- Use the previous results to determine the form of F.
- Find $\rho(\lambda)$ explicitly.
- At what value of the coupling does the free energy have a non-analytic part? Find the leading non-analytic behavior of the free energy near the critical point.
- You should find that the critical coupling is negative, $g_c = -1/48$. Does this make sense? The fact that $g_c < 0$ means that this theory has a perturbative expansion about $g = 0$ that has a finite radius of convergence—unlike what happens in normal QFT, we get more than just an asymptotic expansion. Why is this?

First, we use the Vandermonde determinant (see other diary entry) to turn the measure $\mathcal{D}M$ into an integral over the eigenvalues, weighted by an exponential involving a Coulomb eigenvalue repulsion $\ln|\lambda_i - \lambda_j|$, which contributes to the action. To get the saddle point equation, we vary the action with respect to a particular eigenvalue λ_i and set the result to zero: this gives

$$\frac{N}{g}V'(\lambda_i) = 2\sum_{j:j<i}\frac{1}{\lambda_i - \lambda_j} - 2\sum_{j:j>i}\frac{1}{\lambda_j - \lambda_i} = 2\sum_{j:j\neq i}\frac{1}{\lambda_i - \lambda_j}. \tag{2017}$$

Turning the sum into an integral using the eigenvalue distribution $\rho(\lambda)$, we get

$$\frac{N}{2g}V'(\lambda) = P\int d\mu\,\frac{\rho(\mu)}{\lambda - \mu}, \tag{2018}$$

where the principal value has been taken since the sum avoids terms with $i = j$ where $\lambda_i - \lambda_j = 0$.

The saddle point value for the partition function is just obtained by evaluating the matrix exponential on the saddle point distribution of eigenvalues. So the free energy is

$$\mathcal{F} = -\ln Z \approx \frac{N}{g}\sum_i V(\lambda_i) - \ln\prod_{i\neq j}|\lambda_i - \lambda_j| \to \frac{N^2}{g}\int d\lambda\,\rho(\lambda)V(\lambda) - N^2 P\int d\lambda d\mu\,\rho(\lambda)\rho(\mu)\ln|\lambda - \mu|, \tag{2019}$$

where λ is a distribution of eigenvalues satisfying the saddle point equation. Here the logarithm comes from putting the vandermonde determinant in the exponential.

Now define the complex function

$$F(\xi) = \int_I d\lambda\frac{\rho(\lambda)}{\xi - \lambda}, \qquad \xi \in \mathbb{C}. \tag{2020}$$

Here $I = \mathrm{supp}(\rho) \subset \mathbb{R}$ is assumed to be a union of intervals in $\mathbb{R}$. Using the Dirac identity, we can take $\xi = \mu - i\epsilon$ for μ real and send $\epsilon \to 0$ to get

$$F(\mu - i\epsilon) = i\pi\int d\lambda\,\rho(\lambda)\delta(\mu - \lambda) + P\int d\lambda\frac{\rho(\lambda)}{\mu - \lambda}. \tag{2021}$$

Here we have used the equation of motion to replace the principal part of the integral above with the derivative of $V(\mu)$. Thus we see that

$$\mathrm{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu), \qquad \mathrm{Re}[F(\mu - i\epsilon)] = \frac{V'(\mu)}{2g}, \qquad \mu \in I. \tag{2022}$$

Note that these properties hold only for $\mu \in I$: if $\mu \in \mathbb{R}\setminus I$ then

$$\mathrm{Im}[F(\mu - i\epsilon)] = 0, \qquad \mathrm{Re}[F(\mu - i\epsilon)] = \int d\lambda\frac{\rho(\lambda)}{\mu - \lambda}, \qquad \mu \in \mathbb{R}\setminus I. \tag{2023}$$

In particular, the real part of $F(\mu - i\epsilon)$ needn't be related to $V'(\mu)$ if $\mu \notin I$, since the saddle-point equation relating $V'(\mu)$ to the principal part of the relevant integral was derived under the assumption that $\mu \in I$.

Note that $F(\xi)$ is analytic everywhere, and on $I \subset \mathbb{R}$ it has a branch cut ($\rho(\lambda)$ is assumed to be well-behaved in the $N \to \infty$ limit). We see that across the branch cut at $\mu \in I$, $F(\xi)$ changes by $2\pi i \rho(\mu)$.

Note that if we take $\xi \in \mathbb{R}$ and send $\xi \to \infty$, we have

$$F(\xi \to \infty) = \int_I d\lambda \, \frac{\rho(\lambda)}{\xi}(1 + \lambda/\xi + \dots) = \frac{1}{\xi} + O(\xi^{-2}), \tag{2024}$$

where we have used the normalization of $\rho(\lambda)$. Note that $\mathrm{Re}[F(\mu - i\epsilon)]$ does not go to $V'(\mu)/2g$ for $\mu \notin I$ (unless V is logarithmic, which we will assume to not be the case).

We can use this information to find out what $F(\xi)$ is. In the following we will assume for simplicity that I is a single connected interval centered on zero, so that $I = [-a, a]$ for some $a \in \mathbb{R}$. This will be the case if we have a potential $V(\Lambda)$ with a unique minimum at 0, like $V(\Lambda) = \frac{1}{2}\Lambda^2 + \Lambda^4$. We will determine a self-consistently using the constraints we've derived on F.

Since we know that $F(\xi)$ is analytic but has a branch cut at $I = [-a, a]$ on the $\mathbb{R}$ axis, we expect that $\sqrt{\xi^2 - a^2} = \sqrt{\xi - a}\sqrt{\xi + a}$ will show up in $F(\xi)$ in order to give us the right branch cut structure (viz. a branch cut connecting $\pm a$), and in order to make $\mathrm{Im}[F(\mu - i\epsilon)]$ nonzero only when $\mu \in I$. Since $F(\xi)$ is analytic, we expect $F(\xi) = g(\xi) + f(\xi)\sqrt{\xi^2 - a^2}$, where f, g are some polynomials in ξ with positive powers and real coefficients (as $\mathrm{Im}[F(\mu - i\epsilon)] = 0$ if $\mu \notin I$).

The requirement that the real part of $F(\mu - i\epsilon)$ go to $V'(\mu)/2g$ when $\mu \in I$ tells us that $g(\xi) = V'(\xi)/2g$. We can then get $f(\xi)$ by requiring $F(\xi \to \infty) \to 1/\xi + O(1/\xi^2)$:

$$F(\xi \to \infty) \approx \frac{V'(\xi)}{2g} + f(\xi)\left(\xi - \frac{a^2}{2\xi} + O(\xi^{-3})\right). \tag{2025}$$

Thus

$$f(\xi) = \frac{1}{\xi^2 - a^2/2} + \frac{\xi V'(\xi)}{2g(a^2/2 - \xi^2)}, \tag{2026}$$

with a^2 to be determined by requiring $f(\xi)$ to be a $\mathbb{R}$ polynomial with positive powers. We know that $\deg(f) = \deg(V') - 1$, which again follows from our knowledge of $F(\xi \to \infty)$.

We will now specialize to the case

$$V(\lambda) = \frac{1}{2}\lambda^2 + \lambda^4. \tag{2027}$$

So then $\xi V'(\xi) = \xi^2 + 4\xi^4$, and

$$f(\xi) = \frac{1}{2g(\xi^2 - a/2)}(2g - \xi^2 - 4\xi^4). \tag{2028}$$

Since $V'(\lambda)$ is third order, we know that $f(\xi)$ will be second order, which allows us to stop at the leading order expansion for the square root for now. Writing $f(\xi) =$

$A + B\xi + C\xi^2$ we see that $B = 0$, $C = -2/g$, and

$$\frac{1}{2g} + A = \frac{ca^2}{2}, \tag{2029}$$

so that

$$f(\xi) = -\frac{1}{2g}(1 + 2a^2 + 4\xi^2). \tag{2030}$$

This then determines the eigenvalue distribution to be, using $\mathrm{Im}[F(\mu - i\epsilon)] = \pi\rho(\mu)$ for $\mu \in [-a, a]$,

$$\rho(\mu) = -\frac{1}{\pi} f(\mu)\sqrt{a^2 - \mu^2} = \frac{1}{2\pi g}(1 + 2a^2 + 4\mu^2)\sqrt{a^2 - \mu^2}. \tag{2031}$$

As an aside, we can recover the Wigner distribution by looking at $V(\lambda) = \frac{1}{2}\lambda^2$. In this case, since we know that the degree of $f(\xi)$ is two less than the degree of $V(\xi)$, $f(\xi)$ must be a constant. Working it out and solving for a^2 in the manner described below gives

$$\rho(\mu)|_{V(\lambda)=\lambda^2/2} = \frac{1}{2\pi g}\sqrt{4g - \mu^2}, \tag{2032}$$

which is the famous Wigner distribution. This is already a rather nontrivial result: if there were no interaction effects between the eigenvalues due to the Vandermonde determinant then we'd have a Gaussian distribution of eigenvalues. Turning on the interactions leads to eigenvalue repulsion and hence intuitively to a smearing out of the distribution—and yet the tails of the Gaussian are cut off to form a semicircle with finite extent, which seems to be the opposite of what we'd expect if the eigenvalues were just getting smeared out! Very interesting.

Anyway, now we return to the quartic potential (2027). To get a, we need to get the $1/\xi$ piece of $F(\xi \to \infty)$, which requires expanding the square root to include the $-a^4/8\xi^4$ term. Setting the coefficient of the $1/\xi$ piece to 1 means that

$$3a^4 + a^2 - 4g = 0 \implies a^2 = \frac{1}{6}\left(-1 + \sqrt{1 + 48g}\right). \tag{2033}$$

Recapitulating, we have shown that

$$F(\xi) = \frac{1}{2g}\left(\xi + 4\xi^3 - \left[4\xi^2 + \frac{2}{3} + \frac{1}{3}\sqrt{1 + 48g}\right]\sqrt{\xi^2 + \frac{1}{6} - \frac{1}{6}\sqrt{1 + 48g}}\right). \tag{2034}$$

We can now get an explicit expression for the free energy

$$\mathcal{F}/N^2 \approx \frac{1}{g}\int d\lambda\, \rho(\lambda)V(\lambda) - P\int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu)\ln|\lambda - \mu|. \tag{2035}$$

The second term in the free energy with the ln is hard to integrate, but we have another option: we can integrate the equations of motion to obtain

$$\frac{V(\lambda) - V(0)}{2g} = P\int d\mu\, \rho(\mu)\left[\ln|\lambda - \mu| - \ln|\mu|\right], \tag{2036}$$

which means that (since $V(0) = 0$)

$$P \int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu) \ln|\lambda - \mu| = \frac{1}{2g}\int d\lambda\, \rho(\lambda) V(\lambda) + P\int d\mu\, \rho(\mu)\ln|\mu|. \tag{2037}$$

Putting this into the second integral in the expression for the free energy,

$$\mathcal{F}/N^2 \approx \int d\lambda\, \rho(\lambda)\left(\frac{V(\lambda)}{2g} - \ln|\lambda|\right). \tag{2038}$$

The first term is

$$\frac{1}{2g}\int_{-a}^{a} d\lambda\, \rho(\lambda)(\lambda^2/2 + \lambda^4) = -\frac{a^4}{128g^2}(2 + 10a^2 + 9a^4), \tag{2039}$$

while the second term is

$$-2\int_0^a d\lambda\, \rho(\lambda)\ln\lambda = \frac{a^2}{16g}\left(2 + a^2(3 + 6\ln 4) + \ln 16 - 4(1+3a^2)\ln a\right). \tag{2040}$$

Now we add these two together, and carry out an expansion in small $\epsilon = g - g_c = g + 1/48$. Here the critical point $g_c = -1/48$ is the coupling at which the free energy becomes singular. This point is supposed to mark the phase transition where complicated Feynman diagrams dominate and the Feynman diagrams go over to form a continuum geometry. I think the picture is that the quantity $\Delta = g - g_c$ is a "chemical potential for triangles in the triangulation of spacetime". In the Louiville model (gravity theory with just the dilaton) that is supposed to describe the other side of the phase transition, Δ appears as a cosmological constant: writing the metric in conformal gauge as $g = e^{\phi}\eta$, we get an action like

$$\int \left[\frac{1}{2}(\partial\phi)^2 + \Delta e^{\phi}\right]. \tag{2041}$$

Doing the expansion with Mathematica, we find that the free energy has an (imaginary) constant part, a term proportional to ϵ, one proportional to ϵ^2, and then one proportional to $\epsilon^{5/2}$, which is the leading singular part. So, the leading non-analytic behavior of $\mathcal{F}$ is a 5/2 power dependence on the distance from the critical point (n.b. the cancellation of the $\epsilon^{1/2}, \epsilon^{3/2}$ terms is nontrivial!).

The fact that $g_c < 0$ means that near the transition point, the potential for the eigenvalues is actually naively unstable, since $V(\pm\infty) = -\infty$. And yet, the theory we've been working with actually has a smooth well-behaved free energy for a finite range of negative g! Somehow, taking the $N \to \infty$ limit first is enough to stabilize what would otherwise be an unstable potential, allowing the $g_c < g < 0$ theory to make sense. The transition at g_c comes when the eigenvalues start "spilling over" the brim of the potential and running off towards $\pm\infty$, but this transition doesn't happen until well after the potential becomes unstable. This is actually pretty crazy—if the interactions between the eigenvalues generated by the Vandermonde determinant were positive, then one might expect that the interactions would "bind" the eigenvalues together and prevent them from spilling over the edge to the unbounded region of the potential. However, the forces between the eigenvalues are repulsive! This seems like it should make the problem even worse, with the eigenvalues getting "pushed over the edge" of the potential.

One way of understanding that the theory can be stable for $g < 0$ is to note that the perturbative diagrammatic expansion in this theory has a finite radius of convergence about $g = 0$. This is not true in typical QFTs, which have only an asymptotic expansion about the free point.[224] In QFT the number of diagrams usually grows factorially, which is the ultimate source for the vanishing of the radius of convergence for the perturbative expansion. The analytic nature of the expansion for the matrix model tells us that the number of diagrams grows more slowly—until we get to g_c the number of diagrams is controlled, and the non-analyticity doesn't hit us until the "continuum geometry" phase transition.

As a more physical take on this, let's review an old argument (due to Dyson) why QFTs should generically admit only asymptotic expansions in couplings. For example, consider QED. If the perturbative expansion in e^2 (and remember that it is e^2, and not e, which we are expanding in) has a finite radius of convergence, then QED would make sense for small negative e^2. So, take $e^2 < 0$, and consider nucleating N positron-electron pairs from the vacuum, by expending an energy $E \sim 2Nm$. Now separate the electrons and positrons into two groups, and separate the two groups from one another. The energy for this configuration is then schematically $E \sim 2Nm - e^2N^2$, which can always be made negative for N large enough. Therefore the vacuum is unstable, and the theory breaks down. Evidently $e^2 < 0$ doesn't make sense, and the expansion in e^2 has zero radius of convergence. The same argument can be made in ϕ^4 theory to show that the expansion in the ϕ^4 coupling cannot have a finite radius of convergence.

Large N theories, like the matrix model considered above, provide a way of getting around this argument, and they typically have a series expansion in the t' Hooft coupling gN with a finite radius of convergence. At the technical level, this is because the large-N diagrammatics allow us to resum infinitely many diagrams. Physically, we can also try running Dyson's argument: for example, consider large N YM: we nucleate N pairs of gluons from the vacuum, and then separate them in two groups. Since in the t' Hooft limit we are sending $g \to 0$, the attractive interactions between the two bunches of gluons vanishes in the large N limit, and so Dyson's argument doesn't apply.

Another subtlety needs to be dealt with: at $g \to g_c$ the saddle-point solution gives $a^2 < 0$, which seems problematic since the eigenvalues we're integrating over should always be real, due to the Hermiticity of the matrices in question. So, isn't $g < 0$ already ruled out? I guess the philosophy here is that the important thing to look at is really the singular behavior of the partition function: we used the WKB approximation to get the partition function, and while within this approximation $a^2 < 0$ strictly speaking doesn't make sense, after getting our expression for $\mathcal{F}$ we can just work with it directly, forgetting about the approximation where it came from.

[224]It is important here that we are talking about the free UV fixed point, which is generically right on the boundary of allowed parameter space, as we will argue for in a second. If we had some nontrivial interacting UV fixed point then this would generically not be a worry.

Another possible resolution of this is via analogy with mean-field solutions in other large-N models, like the $O(N)$ vector model. There we implement constraints like $\phi^2 = 1$ using a Lagrange multiplier λ. To do this exactly, λ needs to be integrated over $i\mathbb{R}$. However since we are just interested in doing mean field, we can instead choose λ to be real and nonzero (in the disordered phase), provided that $\langle \phi^2 \rangle = 1$. What's going on here may be similar, with the mean-field solution lying outside of the parameter space of the exact one.

Functional RG equations

Consider a scalar field theory with arbitrary interactions given by $\mathcal{L}_I[\phi]$. Impose a soft UV cutoff by modifying the action as (Euclidean signature)

$$Z[J] = \int \mathcal{D}\phi \, e^{-S+\int J\phi}, \qquad S = S_0 + S_I, \tag{2042}$$

with

$$S_0 = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} \phi_p (p^2 + m^2) e^{p^2/\Lambda^2} \phi_{-p}, \tag{2043}$$

and

$$S_I = \int \frac{d^4p}{(2\pi)^4} \mathcal{L}_I[\phi], \tag{2044}$$

where $\mathcal{L}_I$ contains all possible interactions for ϕ_p. Here, the current J only contains momentum modes up to some scale $\mu < \Lambda$ (the energy scale below which we are interested in calculating correlation functions), and the purpose of the $e^{p^2\Lambda}$ term is so that the propagator gets smoothly cutoff at very high momentum. The action S_I explicitly depends on Λ through the dimensionality of the coupling constants appearing in S_I.

The functional RG flow works by changing the cutoff Λ while simultaneously modifying S_I in such a way that correlation functions at scales below μ are left invariant; this is an (exact) way of integrating out high-momentum modes to get an effective action for the low-momentum ones. The requirement that the correlation functions blow μ be preserved under varying Λ is that the change of the generating functional of correlation functions with respect to Λ is independent of the current:

$$\frac{d}{d\ln\Lambda} Z[J] = \text{ independent of } J, \tag{2045}$$

In what follows, we will derive an expression for $d_{\ln\Lambda} S_I[\phi]$. Note: this is in Schwartz, but I think there are some typos in the problem, so don't worry about trying to derive what he tells you to derive. Polchinski's original paper may be a better reference.

Let's first see if we can guess the answer. When we change the cutoff, what needs to happen to $\mathcal{L}_I[\phi]$ so that the partition function is unchanged? Lowering the cutoff means that certain high energy modes get integrated out. In diagrams, this means that lowering the cutoff corresponds to "collapsing" certain high energy propagator lines. There are two types of lines we can collapse: one which is an open line connecting two different vertices, and one which joins back on itself, with both ends at the same vertex. We can select out lines of the first kind by computing $\delta_{\phi_p}\mathcal{L}_I\delta_{\phi_{-p}}\mathcal{L}_I$, and we can select out the latter types of lines by doing $\delta_{\phi_p}\delta_{\phi_{-p}}\mathcal{L}_I$. After we have selected out these lines, we need to integrate over the high energy modes, which we can do by multiplying with the propagator and integrating over p. We expect there to only be one propagator since we are looking at an infinitesimal variation in $\mathcal{L}_I$, and we want to select out the lowest-order contribution in $\delta\Lambda$. Thus we expect something like

$$\frac{d}{d\ln\Lambda}\mathcal{L}_I \sim \int_p \frac{e^{-p^2/\Lambda^2}p^2/\Lambda^2}{p^2+m^2}\left(\frac{\delta\mathcal{L}_I}{\delta\phi_p}\frac{\delta\mathcal{L}_I}{\delta\phi_{-p}} + \frac{\delta^2\mathcal{L}_I}{\delta\phi_p\delta\phi_{-p}}\right). \tag{2046}$$

The $e^{-p^2/\Lambda^2}p^2/\Lambda^2$ factor ensures that $\mathcal{L}_I$ only changes near the cutoff momentum, which will be corroborated soon.

To figure out what conditions $\mathcal{L}_I$ needs to satisfy, just differentiate Z with respect to Λ:

$$\frac{d}{d\ln\Lambda}Z = \int\mathcal{D}\phi\int_p\left(\frac{p^2}{\Lambda^2}e^{p^2/\Lambda^2}\phi_p(p^2+m^2)\phi_{-p} - \frac{d}{d\ln\Lambda}\mathcal{L}_I\right)e^{-S}. \tag{2047}$$

This must be independent of J, which lets us figure out how $\mathcal{L}_I$ changes. Note that $\frac{d}{d\ln\Lambda}\mathcal{L}_I$ should only have support near $p^2=\Lambda^2$.

To see what we should write for $\frac{d}{d\ln\Lambda}\mathcal{L}_I$, let's calculate the functional derivatives of e^{-S}. We get

$$\frac{\delta}{\delta\phi_p}e^{-S} = -\frac{1}{(2\pi)^4}\left((p^2+m^2)\phi_{-p}e^{p^2/\Lambda^2} + \frac{\delta\mathcal{L}_I}{\delta\phi_p}\right)e^{-S}, \tag{2048}$$

and

$$\begin{aligned}\frac{\delta^2}{\delta\phi_p\delta\phi_{-p}}e^{-S} = &-\frac{1}{(2\pi)^4}\left((p^2+m^2)e^{p^2/\Lambda^2} + \frac{\delta^2\mathcal{L}_I}{\delta\phi_{-p}\delta\phi_p}\right)e^{-S} \\ &+\frac{1}{(2\pi)^8}\left((p^2+m^2)\phi_{-p}e^{p^2/\Lambda^2} + \frac{\delta\mathcal{L}_I}{\delta\phi_p}\right)\left((p^2+m^2)\phi_p e^{p^2/\Lambda^2} + \frac{\delta\mathcal{L}_I}{\delta\phi_{-p}}\right)e^{-S}.\end{aligned} \tag{2049}$$

We can now make a better educated guess about the factors in $\frac{d}{d\ln\Lambda}\mathcal{L}_I$. We choose

$$\frac{d}{d\ln\Lambda}S_I = (2\pi)^4\int d^4p\,\frac{e^{-p^2/\Lambda^2}p^2/\Lambda^2}{p^2+m^2}\left(\frac{\delta S_I}{\delta\phi_p}\frac{\delta S_I}{\delta\phi_{-p}} - \frac{\delta^2 S_I}{\delta\phi_p\delta\phi_{-p}}\right). \tag{2050}$$

The reason why this ends up working is the following: first, we have, for any momentum p,

$$0 = \int\mathcal{D}\phi\,\frac{\delta}{\delta\phi_p}\left[\left(\phi_p e^{p^2/\Lambda^2} + \frac{(2\pi)^4}{2}\frac{1}{p^2+m^2}\frac{\delta}{\delta\phi_{-p}}\right)e^{-S+\int J\phi}\right], \tag{2051}$$

just because at the extremes of the functional integration, ϕ is infinite everywhere in spacetime, and so we get a factor of $e^{-\infty}$ from e^{-S}. Now we apply the above relation with $p > \mu$, where again μ is the scale above which the current vanishes (therefore the functional derivative is taken with respect to a high-energy mode). After some algebra that I won't write out, we evaluate the functional derivatives and get

$$\frac{e^{p^2/\Lambda^2}Z[J]}{2} = \int \mathcal{D}\phi \left(\phi_p \frac{p^2+m^2}{2} e^{2p^2/\Lambda^2}\phi_{-p} + \frac{1}{2(p^2+m^2)}\left(\frac{\delta^2 S_I}{\delta\phi_p\delta\phi_{-p}} - \frac{\delta S_I}{\delta\phi_p}\frac{\delta S_I}{\delta\phi_{-p}}\right)\right) e^{-S+\int J\phi}. \tag{2052}$$

Next, we calculate $d_{\ln\Lambda} Z[J]$:

$$d_{\ln\Lambda} Z[J] = -\int \mathcal{D}\phi \left(\int \frac{d^4p}{(2\pi)^4}\frac{p^2+m^2}{2}\phi_p\phi_{-p}\left(-\frac{2p^2}{\Lambda^2}e^{p^2/\Lambda^2}\right) + d_{\ln\Lambda} S_I\right). \tag{2053}$$

Now we plug in our ansatz for $d_{\ln\Lambda} S_I$ into this equation, and use (2052) to get

$$d_{\ln\Lambda} Z[J] = \int d^4p\, e^{-p^2/\Lambda^2}\frac{p^2}{\Lambda^2}(e^{p^2/\Lambda^2}Z[J]), \tag{2054}$$

which tells us that the change of the generating function under changes in the cutoff is

$$\frac{dW[J]}{d\ln\Lambda} = \int d^4p\,\frac{p^2}{\Lambda^2}, \tag{2055}$$

which as required is independent of J, and so if S_I changes according to the functional differential equation provided in our ansatz, correlation functions at energy scales below Λ are all independent of Λ.

Note that all of this has been non-perturbative and exact! The equation for $d_{\ln\Lambda} S_I$ gives a *linear* differential equation for how the coupling constants in S_I change as the cutoff varies, and so the effective coupling constants at any lower scale can be found in principle only using our knowledge of the coupling constants at a fixed UV scale. Note that if S_I is only quadratic in ϕ (e.g. $S_I = \sum_k g_k \int \phi_p p^{2k}\phi_{-p}$), then S_I remains quadratic in ϕ, but if it contains any interactions, then after integrating the equation for $d_{\ln\Lambda} S_I$ down from a cutoff of Λ to a cutoff of μ, we generically will generate all possible interactions allowed by symmetry.

Properties of momentum-shell propagators and details on momentum-shell RG

ethan: *under construction*

$$G_>(\mathbf{r}) \equiv \int_{\Lambda-\delta\Lambda<k<\Lambda} d^d k \, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^n}. \tag{2056}$$

In various dimensions up to three, this gives

$$G_>(r) = \begin{cases} \frac{1}{2\pi^2 r} \int dk \, k^{1-n} \sin(kr) & d=3 \\ \frac{1}{2\pi} \int dk \, J_0[kr], & d=2, \\ \frac{1}{\pi} \int dk \, k^{-n} \cos(kr), & d=1, \end{cases} \tag{2057}$$

where the integrals are over $k \in [\Lambda - \delta\Lambda, \Lambda]$. When $\delta\Lambda/\Lambda \to 0$, this becomes

$$G_>(r) \approx \begin{cases} \Lambda^{2-n} \frac{\delta\Lambda}{2\pi^2 \Lambda r} \sin(\Lambda r) & d=3 \\ \Lambda^{1-n} \frac{\delta\Lambda}{2\pi} J_0[\Lambda r] \xrightarrow{\Lambda r \gg 1} \Lambda^{1-n} \frac{\delta\Lambda}{\sqrt{2\pi^3 \Lambda r}} \cos(\Lambda r) & d=2, \\ \Lambda^{-n} \frac{\delta\Lambda}{\pi} \cos(\Lambda r) & d=1 \end{cases} \tag{2058}$$

Note how the power of r decreases by 1/2 as we move up in dimension. Also note that in accordance with the integral being over only a finite slice of momenta, all are finite for all values of Λr.

These are the propagators for stat mech models (no time)—what about when we have a time dimension as well? We will first assume that the inverse propagator is quadratic in frequency, and that the momentum dispersion goes as k^{2l} (since k here is of order of a cutoff, we can take the dispersion to be dominated by the largest power of k^2 that appears in the action). Therefore

$$G_>(r,t) = -i \int_{\omega\in\mathbb{R}} \int_{\mathbf{k}\in\text{shell}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{-\omega^2 + \alpha^2 k^{2l} - i\varepsilon}. \tag{2059}$$

Here the sign of the $i\varepsilon$ is determined by remembering that proper time-ordering means that the sign of $i\varepsilon$ matches the sign of the frequency part of $G^{-1}(k,\omega)$. Doing the ω integral of course gives

$$G_>(r,t) = \theta(t) G_>(r,t) + \theta(-t) G_>(r,-t), \tag{2060}$$

where

$$G_>(r,t) = \alpha^{-1} \int_{\mathbf{k}\in\text{shell}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-k^l t)}}{2k^l}. \tag{2061}$$

Therefore at finite time, when the shell is narrow, we just have to multiply the above results by $e^{-i\Lambda^l \alpha|t|}$ and divide by 2:

$$G_>(r,t) \approx \begin{cases} \Lambda^{2-l} \frac{\delta\Lambda}{4\pi^2 \Lambda \alpha r} \sin(\Lambda r) e^{-i\Lambda^l \alpha|t|} & d=3 \\ \Lambda^{1-l} \frac{\delta\Lambda}{4\pi\alpha} J_0[\Lambda r] e^{-i\Lambda^l\alpha|t|} \xrightarrow{\Lambda r \gg 1} \Lambda^{1-l} \frac{\delta\Lambda}{\sqrt{8\pi^3 \Lambda r \alpha}} \cos(\Lambda r) e^{-i\Lambda^l \alpha|t|} & d=2, \\ \Lambda^{-l} \frac{\delta\Lambda}{2\pi\alpha} \cos(\Lambda r) e^{-i\Lambda^l\alpha|t|} & d=1 \end{cases} \tag{2062}$$

In the case where the propagator is linear in frequency, we instead have

$$G_>(r,t)_{NR} = -i \int_{\omega\in\mathbb{R}} \int_{\mathbf{k}\in\text{shell}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{\omega + \alpha k^{2l} + i\varepsilon}. \tag{2063}$$

We then do the ω integral and get

$$G_>(r,t)_{NR} \approx \begin{cases} \Lambda^2 \dfrac{\delta\Lambda}{2\pi^2 \Lambda r} \sin(\Lambda r) e^{i\Lambda^{2l}\alpha t}\theta(t) & d=3 \\ \Lambda \dfrac{\delta\Lambda}{2\pi} J_0[\Lambda r] e^{i\Lambda^{2l}\alpha t}\theta(t) \xrightarrow{\Lambda r\gg 1} \Lambda \dfrac{\delta\Lambda}{\sqrt{2\pi^3 \Lambda r}} \cos(\Lambda r) e^{i\Lambda^{2l}\alpha t}\theta(t) & d=2, \\ \dfrac{\delta\Lambda}{\pi} \cos(\Lambda r) e^{i\Lambda^{2l}\alpha t}\theta(t) & d=1 \end{cases} \tag{2064}$$

Note how here we have a $\theta(t)$ in the correlator: since particle-antiparticle creating processes don't happen, the vacuum has no particle-antiparticle pairs, and so acting with an annihilation operator on the vacuum gives zero. Therefore for $t>0$ we have $\langle \phi(t)\phi^\dagger(0)\rangle \neq 0$, but must have $\langle \phi^\dagger(t)\phi(0)\rangle = 0$.

We now prove some properties about $G_>(r)$. First,

$$\int d^d r \, G_>(r) = 0. \tag{2065}$$

This is obvious just because $G_>$ has no Fourier components at $\mathbf{k}=0$, and can also be derived from the approximate expressions in (2064). Similarly, integrating $G_>(r)$ against any even power $2m \in 2\mathbb{N}$ of r gives zero, since the power can be turned into a derivative (has to be even else we get a non-local $\sqrt{\Delta^2_{\mathbf{k}}}$)[225]:

$$\int d^d r \, r^{2m} G_>(r) = (-1)^m \int d^d k \, \frac{\nabla^{2m}_{\mathbf{k}} \delta(\mathbf{k})}{k^n} = 0. \tag{2068}$$

This is basically just saying that diagrams with a single internal propagator connecting different points don't contribute to wavefunction renormalization—this is of course because they are just tree diagrams.

Using (2064), one would also conclude that $\int_{\mathbf{r}} G_>(r)^s r^{2m} = 0$ for all $s,m\in\mathbb{N}$. This is not correct however, since powers of $G_>$ contain modes and zero momentum / frequency. For example,

$$\int_{\mathbf{r}} G_>(r)^2 = \int_{\mathbf{k}} G_>(\mathbf{k}) G_>(-\mathbf{k}) \propto \Lambda^{d-1-2n}\delta\Lambda. \tag{2069}$$

[225] This is a bit blase, since

$$\int dx \int dy \, f(x,y) = \int dy \int dx \, f(x,y) \tag{2066}$$

only if

$$\int dx\, dy \, |f(x,y)| < \infty, \tag{2067}$$

which is certainly not generically the case for $f(x,y) = x^m y^{-n} e^{ikx}$ when integrated on the domain in question (since it includes all of $\mathbb{R}^d$ and the integral over $\mathbf{k}$ space is finite). However, we can always regulate the integrals so that they do converge, by adding a small imaginary momentum to make $G_>$ go as $e^{-r\varepsilon}$ at $r\to\infty$. This establishes convergence of the integral of the absolute value—we can then switch integration orders, and finally send $\varepsilon\to 0$ at the end.

In general, the integral of $G_>(r)^s$ will have $s-1$ sums over momenta variables, each of which lie in a shell of width $\delta\Lambda$. Therefore in general these heuristic phase space arguments tell us that

$$\int_{\mathbf{r}} G_>(r)^s \propto \Lambda^{(d-1)(s-1)-sn}(\delta\Lambda)^{s-1}. \tag{2070}$$

Including even powers of r in the integrand (odd powers still vanish by presumed $SO(d)$ symmetry) gives

$$\int_{\mathbf{r}} G_>(r)^s = \int_{\mathbf{k}} \tag{2071}$$

These types of integrals appear in diagrams with $s-1$ loops, so that an expansion in $\delta\Lambda/\Lambda$ is an expansion in the number of loops.

Having looked at the general properties of these correlators we will now put them to use to do momentum-shell RG. Let the action be S_0+S_I, where S_I is the interaction. Write $\phi = \phi_> + \phi_<$ and let $S_e[\phi_<]$ be the effective action after integrating over $\phi_>$. Then derivatives of S_e wrt $\phi_<$ (evaluated at $\phi_< = 0$) generate connected correlation functions for $\phi_>$ in a way determined by S_I. For example,

$$-\frac{\delta^2(S_e[\phi_<]-S_0[\phi_<])}{\delta\phi_<(x)\delta\phi_<(z)}\Big|_{\phi_<=0} = \left\langle \frac{\delta^2 S_I}{\delta\phi(x)^2}\Big|_{\phi=\phi_>}\right\rangle \delta(x-y) - \left\langle \frac{\delta S_I}{\delta\phi(x)}\frac{\delta S_I}{\delta\phi(y)}\Big|_{\phi=\phi_>}\right\rangle_c . \tag{2072}$$

Here the expectation values are wrt the fast fields, so that e.g. if $S_I = \phi^4$, $\langle \delta^2_{\phi(x)} S_I\rangle = \int \mathcal{D}\phi_>\, 12\phi^2(x)\, e^{-S_0[\phi_>]}$. A more precise way to say what $S_e[\phi_<]-S_0[\phi_<]$ is is that it's the generating functional for correlation functions of S_I. Functionally integrating this and its generalizations to higher derivatives over $\phi_<$, we get that

$$S_e[\phi_<] = S_0[\phi_<] - \sum_{k=1}(-1)^k\frac{1}{k!}\langle S_I^k\rangle_c. \tag{2073}$$

Explicitly, this is

$$S_e[\phi_<] = S_0[\phi_<] - \sum_{k=1}(-1)^k\frac{1}{k!}\sum_{m_1,\dots,m_k\geq 1}\int\prod_{i=1}^k dx_i\,\left\langle \prod_{l=1}^k \frac{\delta^{m_l}S_I}{\delta\phi(x_l)^{m_l}}\Big|_{\phi_<=0}\frac{\phi_<^{m_l}(x_l)}{m_l!}\right\rangle_c \tag{2074}$$

A word of caution here: the cumulants appearing in this expression (the connected correlation functions) are *not* in general evaluated as $\langle (S_I - \langle S_I\rangle)^k\rangle$! Even though the second and third cumulants are indeed expectation values about the mean (so that $\langle S_I^k\rangle_c = \langle (S_I-\langle S_I\rangle)^k\rangle$ for $k=2,3$), this is *not* true in general. The general expression for the cumulants in terms of the moments of S_I is in general actually quite messy after $k>3$.

Anomalous dimension at the $O(N)$ WF fixed point via the sunrise diagram and self energies

Today we're doing a rather long-winded elaboration on problem 7.2 from Sachdev's book. The goal is to compute the self energy to two-loop orderf at the WF fixed point in the $O(N)$ model in $4-\varepsilon$ dimensions in Euclidean signature, and then to use this calculation to compute the anomalous dimension of the ϕ field.

Our conventions for the action will be that the interaction term has a $1/4!$ term in it, viz.

$$\mathcal{L} \supset \frac{r^2}{2}\sum_i \phi_i^2 + \frac{g\mu^{4-d}}{4!}\left(\sum_i \phi_i^4 + 2\sum_{i<j}\phi_i^2\phi_j^2\right), \tag{2075}$$

where as usual μ is some energy scale introduced to do dim reg. Sachdev uses momentum-shell RG for most of his calculations (although maybe mixes in some dim reg when doing the 2-loop computation? The details aren't given), but we will stick to dim reg.

At the critical point, the 2-point function in the $O(N)$ model will have the scale invariant form

$$\chi(k) \sim \mu^{-\eta}\frac{1}{|k|^{2-\eta}}, \tag{2076}$$

where η is the anomalous dimension. At the same time, we may also write

$$\chi(k) \sim \frac{1}{k^2 - \Sigma(k)}, \tag{2077}$$

where the self energy has $\Sigma(0) = 0$ at the critical point.

We would now like to compute $\Sigma(k)$ to lowest order at the critical point. The first graph that gives nontrivial momentum dependence is the sunrise diagram, which is quadratic in the coupling g. Sachdev asks you to do it directly in momentum space in the problem statement, but I think this is prohibitively hard—the calculation is done in Ramond's QFT book, but it's enough of a mess that I didn't want to go through and check that there were no typos. Seriously, it's really bad. Happily though, there is another way to do it: in real space! In real space, we need to compute[226]

$$\text{Sunrise}(x) = g^2\frac{N+2}{18}\frac{1}{((2-d)A(S^{d-1})|x|^{d-2})^3}. \tag{2079}$$

[226] Remember that $\Sigma(k)$ contains the self-energy diagrams *with external propagators removed*, hence the sunrise diagram in $\mathbb{R}$ space only contains one integral over position. It appears in the real-space propagator via

$$G(x) = G_0(x) + \mathcal{O}(g) + g^2\int_{y,z} G_0(y)G_0^3(y-z)G_0(z-x). \tag{2078}$$

Amputating the external legs $G_0(y), G_0(z-x)$ on the last term gives us the Fourier transform of the g^2 contribution to $\Sigma(k)$.

To get this, we've just cubed the free propagator ($A(S^{d-1})$ is the area of the unit $(d-1)$-sphere), and tacked on the appropriate combinatorial factor. The symmetry factor comes from

$$g^2\frac{N+2}{18} = (N-1)(2g\cdot 2^2/4!)^2\cdot\frac{1}{2} + g^2\frac{1}{3!}. \tag{2080}$$

In the expression for the sunrise diagram, we are interested in the value that g takes on at the fixed point, which is $O(\varepsilon)$. Therefore since we are just interested in a lowest order in ε calculation we have dropped any r-dependence in the propagators above, since the fixed point of r in this scheme is also $O(\varepsilon)$.

Getting the momentum-space result is then easy, since we may use the identity (which comes from integrating Bessel functions)

$$\int_{\mathbb{R}^d} d^dx\,|x|^{-a}e^{ix\cdot k} = \frac{(2\pi)^d\Gamma((d-a)/2)}{\pi^{d/2}2^a\Gamma(a/2)}|k|^{-(d-a)}, \tag{2081}$$

and plug in $a=3d-6$. This tells us that, letting $d=4-\varepsilon$ so that $d-a=-2+2\varepsilon$,

$$\text{Sunrise}(k) = Cg^2(N+2)\Gamma(-1+\varepsilon)|k|^{2-2\varepsilon}, \tag{2082}$$

where $C>0$ is a constant that I don't see any point in keeping track of.[227] The N dependence we'll keep, though.

Now since the fixed point value of g is at $g_*\sim\varepsilon/(N+8)$ (the exact fixed point was worked out in the diary entry on the anisotropic $O(N)$ model), working to lowest order in ε means working to $O(\varepsilon^2)$. When we send $\varepsilon\to 0$, we expand the Γ function as

$$\Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon}+\dots, \tag{2083}$$

where the $\dots$ are finite. The term that cancels the pole in $1/\varepsilon$ comes from writing $|k|^{-2\varepsilon}=e^{-\varepsilon\ln k^2/\mu^2}\approx 1-\varepsilon\ln k^2/\mu^2$, and so when $\varepsilon\to 0$ we have

$$\text{Sunrise}(k)_{\varepsilon\to 0} = Cg^2(N+2)k^2\ln(k^2/\mu^2)+\dots, \tag{2084}$$

where $\dots$ are terms that are not divergent / are polynomial in k (recall that algebraic functions of momentum become contact terms in $\mathbb{R}$ space, and so we don't care about them—it's all about the logs!). It really is worth emphasizing that this derivation via $\mathbb{R}$ space was orders of magnitude easier than the direct momentum-space computation—so keep $\mathbb{R}$ space in mind next time!

Before going on to look at the anomalous dimension, we'll look at what happens to this in $\mathbb{R}$ time. Sending $k^2\to\mathbf{k}^2-\omega^2$, we see that we will get an imaginary contribution when the argument of the log goes negative, and so to this order (viz. quadratic in ε), we find (still with $r=0$)

$$\Sigma_I(0,\omega) = Cg^2(N-2)\theta(\omega)\omega^2. \tag{2085}$$

Computing the above sunrise diagram would be a bit harder if we wanted to do it for $r\neq 0$, but luckily we can basically just write down the answer: since when $r\neq$

[227] From now on, "C" will be a stand-in for any positive constant that I don't want to keep track of.

0 the threshold for particle production is $k^2 = (3r)^2$ (as there are only even-body interactions), the only possibility is to have

$$\Sigma_I(0,\omega) = Cg^2(N-2)\theta(|\omega| - 3r)(|\omega| - 3r)^2. \tag{2086}$$

Now we want to get the anomalous dimension. The simple way to find η is just to solve

$$\frac{1}{k^2 - \Sigma(k)} = \frac{\mu^{-\eta}}{k^{2-\eta}} \implies \Sigma(k) = k^2(1 - e^{-\eta \ln k/\mu}) = k^2 \sum_{n=1}^{\infty} \frac{1}{n!}\eta^n(-1)^{n+1}\ln^n(k/\mu). \tag{2087}$$

Let us expand η as a series,

$$\eta = \sum_{i=2}^{\infty} g^n \eta_n, \tag{2088}$$

where the sum starts at $n = 2$ because the first contribution to η is the sunrise diagram. Since we have already found the $O(g^2k^2 \ln k/\mu)$ contribution, we know that

$$\eta_2 = C(N+2). \tag{2089}$$

Note that this immediately tells us an *infinite* number of terms appearing in the self-energy, at all orders in the coupling g: specifically, our computation of the sunrise diagram has determined the coefficients of all of the leading logs (those that go as $[\ln(k/\mu)(g^2\eta_2)]^n$) in the expansion, and therefore at a fixed order in $\ln(k/\mu)$, we know the coefficient of the lowest-order contribution in g (at order g^{2n}). Similarly, if we were brave enough to compute the subleading log term arising at three loops and going as $g^3 \ln k/\mu$, we'd also know an infinite number of terms (the subleading terms in g at a fixed order in $\ln(k/\mu)^n$, going as g^{2n+1})—this type of resummation is exactly what we're used to seeing from RG analysis.

Summarizing, we've shown that at the order of the leading logs, we have shown that the anomalous dimension at the fixed point is

$$\eta = C(N+2)g_*^2 = C'\varepsilon^2 \frac{N+2}{(N+8)^2}. \tag{2090}$$

As expected from large N intuition, the anomalous dimension vanishes in the $N \to \infty$ limit.

From coupled Ising CFTs to the $O(2)$ model in three dimensions via conformal perturbation theory

Today we are doing an elaboration on an exercise given by Zohar Komargodski to the attendees of the 2017 Bootstrap school. The goal is to show that two 3d Ising models flow to the $O(2)$ fixed point when coupled through a deformation $\epsilon_1\epsilon_2$. First we'll derive some needed conformal perturbation theory results, and then we'll attack the problem both through an ϵ expansion and a direct perturbation away from the 3d Ising CFT.

General conformal perturbation theory

Consider a CFT perturbed by a collection of operators with dimensionless couplings λ_i and scaling dimensions Δ_i:

$$\delta S = \sum_i a^{-d+\Delta_i}\lambda_i \int \mathcal{O}_i, \tag{2091}$$

where a is the short-distance cutoff (from here on, summation over repeated indices is implied). We will use a derivation of the CPT beta functions inspired from reading a paper on disordered Ising models by Zohar and DSD [19].

The deformation will generically take the theory away from the fixed point. To get the β functions, we will choose an observable in the theory and require that the derivative of its expectation value with respect to the short distance cutoff vanish (again with the usual caveat that we are working modulo powers of a). There are many observables to choose from, but the one we will find most convenient is the overlap between the state $|0\rangle$ and $|\mathcal{O}_i\rangle$ in the presence of the perturbation to the action in a region R. Therefore we will need to compute[228]

$$\frac{d}{d\ln a}\langle \mathcal{O}_i| e^{-\lambda_j a^{-d+\Delta_j}\int_R \mathcal{O}_j}|0\rangle = 0. \tag{2092}$$

To evaluate this, we expand the exponential to quadratic order in the couplings, and using $\langle \mathcal{O}_i| = \lim_{x\to\infty}\langle 0|x^{2\Delta_i}\mathcal{O}(x)$, we have

$$\frac{d}{d\ln a}\left\langle 1 - V_R\lambda_i a^{-y_i} + \frac{\lambda_j\lambda_k}{2}a^{-y_j-y_k}\lim_{z\to\infty}\int_R d^dy\, d^dx\, \mathcal{O}^i(z)\mathcal{O}^j(x)\mathcal{O}^k(y)\right\rangle = 0, \tag{2093}$$

with $y_i \equiv d-\Delta_i$ and V_R the volume of R. Here we used the OPE to simplify the second term. If we take R to be bounded we can do the OPE between $\mathcal{O}^j$ and $\mathcal{O}^k$, and then the OPE between the resulting operator and $\mathcal{O}^i$. Since we will be doing the OPE between an operator at ∞ and one in R, it doesn't really matter where exactly in R the latter operator is located. Therefore we can always take the OPE to be taken with x located at the center of R (we'll take R to be a ball). The integral over x then

[228]Other similar choices of observables, like $\langle e^{-\lambda_j a^{-d+\Delta_j}\int \mathcal{O}_j}\rangle$ or $\langle \mathcal{O}_i|e^{-\lambda_j a^{-d+\Delta_j}\int \mathcal{O}_j}|\mathcal{O}_k\rangle$, do not give us, upon differentiation, formulas which are as nice. This is because they have β functions mixed up with annoying integrals. The merits of the observable we chose to study lie mainly in the fact that the linear term in the coupling λ_i is nonzero—after differentiating this gives us a nice factor of β_i all by itself, and this makes the resulting manipulations easier.

produces a factor of V_R. We then have remaining an integral $S^{d-1}\int dr\, r^{\Delta_i-\Delta_j-\Delta_k+d-1}$. We will absorb the S^{d-1} factor by re-scaling all the coupling constants, and so we then have, letting $d_t = \frac{d}{d\ln a}$ be the differential for RG time,

$$y_i a^{-y_i}\lambda_i - \beta_i a^{-y_i} + a^{-(y_j+y_k)}\left(\beta_j\lambda_k - \frac{\lambda_j\lambda_k}{2}(y_j+y_k-d_t)\right)C^i_{jk}\int dr\,\frac{1}{r^{-\Delta_i+\Delta_j+\Delta_k-d+1}} = 0 \tag{2094}$$

Note that to obtain this equation, we had to assume that the theory the expectation value was being taken with respect to was a CFT—otherwise the d_t pick up extra terms from the change in the action being used to construct the expectation value.

To satisfy this equation to first order in the couplings, we need $\beta_i = y_i\lambda_i + O(\lambda^2)$. In the term $\beta_j\lambda_k$, only the linear part of β_j contributes at quadratic order in the couplings, and this term cancels the $-\lambda_j\lambda_k(y_j+y_k)/2$ piece. Therefore, writing the quadratic piece of β_i as $\beta^{(2)}$, we have

$$\beta^{(2)}_i a^{-y_i} = C^i_{jk}a^{-y_j-y_k}\frac{\lambda_j\lambda_k}{2}d_t\int dr\frac{1}{r^{-\Delta_i+\Delta_j+\Delta_k-d+1}}. \tag{2095}$$

The differential of the integral gives $-d_t a^{\Delta_i-\Delta_j-\Delta_k+d}/(\Delta_i-\Delta_j-\Delta_k+d) = -a^{-y_i+y_j+y_k}$, and so

$$\beta^{(2)}_i = -\frac{\lambda_j\lambda_k}{2}C^i_{jk} \implies \beta_i = y_i\lambda_i - \frac{\lambda_j\lambda_k}{2}C^i_{jk}. \tag{2096}$$

This gives us the β functions for the couplings of the perturbations added to the action to deform the theory away from the fixed point. To see the extent to which other operators (viz. those corresponding to the scaling variables in the CFT) are modified by the perturbation, we need to make a further deformation and include these operators in the exponential. For example, we will want to compute the extent to which the perturbation modifies the scaling dimension of an operator $\mathcal{O}$ in the CFT. This is done by adding $\delta\lambda_{\mathcal{O}}\int\mathcal{O}$ to the exponential in the deformation, and then expanding the exponential to linear order in $\delta\lambda_{\mathcal{O}} = \lambda_{\mathcal{O}} - \lambda_{\mathcal{O}*}$, where $\lambda_{\mathcal{O}*}$ is the value of the coupling in the CFT. This is because in order to determine the scaling dimension of $\mathcal{O}$ in the CFT, we need to perturb away from the CFT by deforming by $\mathcal{O}$, and then examine how quickly the theory flows back to / away from the fixed point. To the extent that $\mathcal{O}$ remains a scaling variable in the presence of the $\lambda_i\mathcal{O}_i$ deformation, we then find that $\beta_{\mathcal{O}}$ is made nonzero by the term

$$\beta_{\mathcal{O}} = y^{(0)}_{\mathcal{O}}\delta\lambda_{\mathcal{O}} - (\delta\lambda_{\mathcal{O}})\lambda_j C^{\mathcal{O}}_{\mathcal{O}j}, \tag{2097}$$

where $d - y^{(0)}_{\mathcal{O}}$ is the scaling dimension of $\mathcal{O}$ in the un-deformed theory. Since $\delta\lambda_{\mathcal{O}}$ vanishes at the fixed point, this means that the scaling dimension of $\mathcal{O}$ is corrected by

$$\Delta_{\mathcal{O}} = \Delta^{(0)}_{\mathcal{O}} + \lambda_{j*}C^{\mathcal{O}}_{\mathcal{O}j}. \tag{2098}$$

ϵ expansion from $d = 4$

Now we consider two Ising models, coupled though the product of their energy operators. The action in $d = 4-\epsilon$ dimensions is

$$S = \int\left(\frac{1}{2}\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi} + ta^{-2}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + ga^{-\epsilon}(\phi_1^4+\phi_2^4) + \eta a^{-\epsilon}\phi_1^2\phi_2^2\right). \tag{2099}$$

We will evaluate the β functions around the Guassian fixed point, and use them to predict where the theory flows. The starting point of the flow is hence just free field theory, and so getting the OPE coefficients is but a simple matter of combinatorics. They are (normalizing the fields so that $\langle\phi_i(r)\phi_j(0)\rangle = \frac{1}{x^{d-2}}$)

$$C^{\eta}_{\eta\eta} = \binom{2}{1}^4 = 16, \quad C^{\eta}_{\eta g} = 2\cdot 2!\binom{4}{2} = 24, \quad C^{g}_{gg} = 2!\binom{4}{2}^2 = 72, \quad C^{g}_{\eta\eta} = 2. \tag{2100}$$

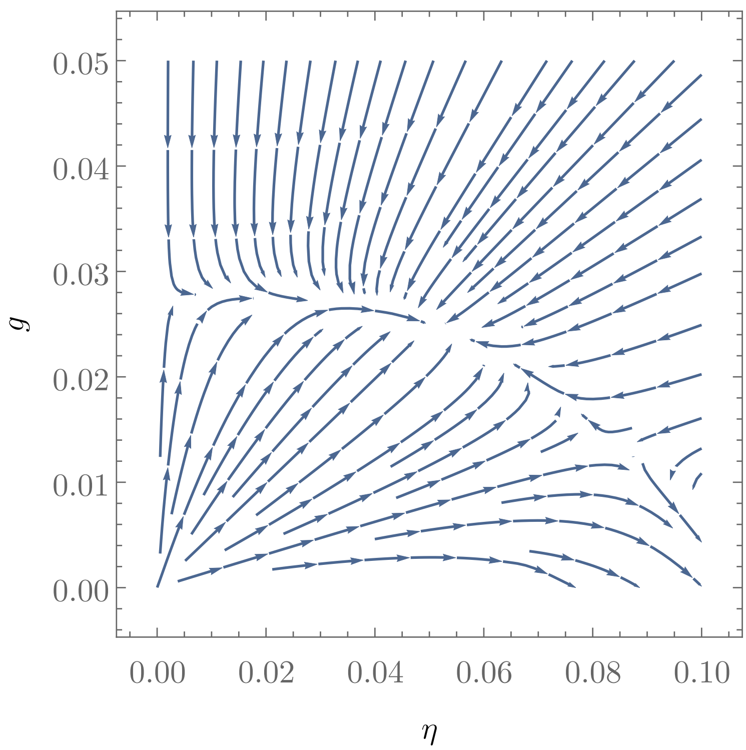
The ones involving t are

$$C^{t}_{tt} = \binom{2}{1}^2 = 4, \quad C^{t}_{gg} = \binom{4}{3}^2 3! = 96, \quad C^{t}_{gt} = \binom{4}{2}2! = 12, \quad C^{t}_{\eta\eta} = \binom{2}{1}^2 2! = 8, \quad C^{t}_{\eta t} = 2$$
$$C^{g}_{tt} = 1, \quad C^{\eta}_{tt} = 2, \quad C^{g}_{tg} = \binom{4}{1}2! = 8, \quad C^{\eta}_{\eta t} = 2\cdot 2^2 = 8. \tag{2101}$$

First let us ignore t—we tune it to zero in the UV, and then anticipate that the error in our fixed point value $t_* = 0$ will only be of order ϵ^2 (since the non-zero β function for t at $t=0$ comes only from η^2, g^2 terms). Then plugging the relevant OPE coefficients in to our formulae for the β functions (all of t, g, η are being treated as perturbations), we obtain

$$\beta_\eta = \epsilon\eta - 8\eta^2 - 24\eta g, \qquad \beta_g = \epsilon g - \eta^2 - 36g^2. \tag{2102}$$

The flow looks like



(2103)

There are four fixed points, only one of which is IR stable with respect to both η and g. We find that at this point,

$$\eta_* = \frac{\epsilon}{20}, \qquad g_* = \frac{\epsilon}{40}, \qquad y_\eta = -\epsilon, \qquad y_g = -\epsilon/5, \tag{2104}$$

while the fixed point value of the quadratic term vanishes: $t_* = 0$. Therefore at the fixed point we may write the action, to the extent that doing so is meaningful, as

$$S = \int \left(\frac{1}{2} \partial \boldsymbol{\phi} \cdot \partial \boldsymbol{\phi} + \frac{\epsilon}{40} a^{-\epsilon} (\boldsymbol{\phi} \cdot \boldsymbol{\phi})^2 \right). \tag{2105}$$

Note that the fixed-point values of η, g are such that the resulting action is $O(2)$-symmetric.

Now if we go back and include t, we have

$$\beta_t = 2t - 2t^2 - 2t\eta - 12tg - 48g^2 - 4\eta^2, \tag{2106}$$

and

$$\beta_\eta = \beta_\eta(t=0) - t^2 - 8\eta t, \qquad \beta_g = \beta_g(t=0) - 8tg - \frac{1}{2}t^2. \tag{2107}$$

When $\epsilon \ll 1$ we have $t_* \sim O(\epsilon^2)$, which can be dropped as anticipated earlier. We then get that at the fixed point, (this isn't quite correct since the presence of g^2, η^2 terms mean that t is not by itself a scaling variable at the fixed point, even to order ϵ)

$$y_t = 2 - 2\epsilon(1/20 + 6/40) + O(\epsilon^2) = 2(1 - \epsilon/5) + O(\epsilon^2). \tag{2108}$$

When $\epsilon \sim 1$ the solutions for the fixed points get very complicated, and saying anything concrete gets more difficult. However, we note that setting $\epsilon = 1$ in the above equation (and yet still dropping the $O(\epsilon^2)$ terms, lol) gives a scaling dimension of $\Delta_t = 7/5 = 1.4$ for the energy operator in the $O(2)$ model. In fact the actual scaling dimension is $\Delta_t \approx 1.51$, so this laughably crude approximation actually gets the right scaling dimension within 7% error. However, the fact that it under-estimates the scaling dimension is important, since it suggests that an energy-energy deformation of two coupled $O(2)$ models is relevant, when in fact it is (barely) irrelevant.

Perturbation theory direction from the Ising CFT

Now we'll take a more direct approach. Instead of starting from the Gaussian fixed point in 4 dimensions and flowing along the WF trajectory towards the putative $O(2)$ fixed point, we will start with two coupled Ising models in three dimensions, and flow along the trajectory generated by the inter-model coupling. Since the inter-model coupling term is only very slightly relevant, the flow will likely be a lot shorter than the flow from the Gaussian fixed point in 4 dimensions, and so will hopefully provide us with a better prediction for the $O(2)$ CFT data.

The Ising models will be coupled together through their energy operators:

$$S = S_{I,1} + S_{I,2} + \eta \int \epsilon_1 \epsilon_2. \tag{2109}$$

First let us recall some results for the 3d Ising model, namely [20]

$$\Delta_\epsilon \approx 1.41, \qquad \Delta_{\epsilon'} \approx 3.82, \qquad C^{\epsilon}_{\epsilon\epsilon} \approx 1.53, \qquad C^{\epsilon'}_{\epsilon\epsilon} \approx 1.54, \tag{2110}$$

where ϵ is the energy operator (schematically, ϕ^2) and ϵ' is the next lightest $\mathbb{Z}_2$-even scalar in the spectrum (schematically, ϕ^4). Note that an operator like ϵ^2, with scaling dimension the relevant ≈ 2.8, doesn't appear in the spectrum. Also note that $[\epsilon_1 \epsilon_2] \approx 2.82 < 3$, and so the deformation that couples the two Ising models at the decoupled fixed point $\mathcal{I}$ is slightly relevant: $y^{\mathcal{I}}_\eta \approx 0.18$. The beta function for the deformation is therefore

$$\beta_\eta = y^{\mathcal{I}}_\eta \eta - \frac{\eta^2}{2} C^\eta_{\eta\eta}, \qquad C^\eta_{\eta\eta} = (C^\epsilon_{\epsilon\epsilon})^2. \tag{2111}$$

Therefore the IR fixed point $\mathcal{F}$ in this approximation is at

$$\eta_* = 2 y^{\mathcal{I}}_\eta (C^\epsilon_{\epsilon\epsilon})^{-2} \approx 0.15, \tag{2112}$$

and the dimension of η at the IR fixed point $\mathcal{F}$ is, using $\beta_\eta \approx y^{\mathcal{I}}_\eta (1 - 2C^\eta_{\eta\eta}/(C^\varepsilon_{\varepsilon\varepsilon})^2 \eta = -y^{\mathcal{I}}_\eta \eta$ near the IR fixed point,

$$\Delta^{\mathcal{F}}_\eta = d + y^{\mathcal{I}}_\eta \approx 3.18 \implies y^{\mathcal{F}}_\eta = -y^{\mathcal{I}}_\eta = -0.18, \tag{2113}$$

Going back and calculating what we would expect for $y^{\mathcal{I}}_\eta$ at the fixed point with the ϵ expansion,[229] we obtain $y^{\mathcal{F}}_\eta = -2/5$, which is in pretty good agreement with the (presumably more accurate) result above.

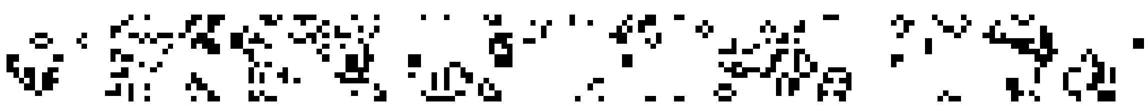
We can also calculate the extent to which the coupling modifies the scaling dimension of the total energy operator $\boldsymbol{\varepsilon}$, pretending that $\boldsymbol{\varepsilon} \equiv \epsilon_1 + \epsilon_2$ is still an exact scaling variable at the IR fixed point. Using the results obtained in the first section, we see that the scaling dimension at the IR fixed point is approximately

$$\Delta^{\mathcal{F}}_{\boldsymbol{\varepsilon}} = \Delta^{\mathcal{I}}_\epsilon + \eta_* C^{\boldsymbol{\varepsilon}}_{\eta\boldsymbol{\varepsilon}}. \tag{2114}$$

The scaling dimension of the energy operator thus increases along the flow. Since $C^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}\eta} = C^\epsilon_{\epsilon\epsilon}$, we have, using our earlier expression for η_*,

$$\Delta^{\mathcal{F}}_{\boldsymbol{\varepsilon}} = \Delta^{\mathcal{I}}_\epsilon + 2\frac{3 - \Delta^{\mathcal{I}}_\eta}{C^\epsilon_{\epsilon\epsilon}} \approx 1.63. \tag{2115}$$

The actual scaling dimension of $\boldsymbol{\varepsilon}$ in the $O(2)$ is in fact $\Delta^{O(2)}_{\boldsymbol{\varepsilon}} \approx 1.51$, which eh, isn't too bad for a leading-order approximation. Unfortunately it does about the same as the more crude ϵ expansion estimate, although it over-estimates $\Delta_{\boldsymbol{\varepsilon}}$ instead of under-estimating, like the ϵ expansion does (in fact, the ϵ expansion predicts $\Delta_{\boldsymbol{\varepsilon}} < \Delta_\epsilon$ (by a tiny bit), which we know can't be right).



[229] Again, because of the $g\eta$ term in β_η, η is technically speaking not a scaling variable.

The Gross-Neveu model

Consider the Gross-Neveu model, which is a theory of N Dirac fermions in 2 spacetime dimensions, interacting through a quartic term:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - g^2(\bar{\psi}_i\psi^i)^2, \tag{2116}$$

with $i = 1, \dots, N$. Show that a dynamical mass term (which spontaneously breaks the chiral symmetry) for the fermions is generated by dimensional transmutation at one-loop order. Note that the chiral symmetry here is just a discrete $\mathbb{Z}_2$ symmetry which acts as,

$$\psi_i \mapsto \bar{\gamma}\psi_i, \tag{2117}$$

and sends $\bar{\psi}_i\psi^i \mapsto -\bar{\psi}_i\psi^i$ (here $\bar{\gamma}$ has alias γ^5). In the limit $N \to \infty$, Ng^2 fixed, show that the one-loop result is exact, resulting in a mass term which is non-analytic in g^2, namely $m_{\text{eff}} = \Lambda e^{-\pi/(Ng^2)}$ (think of QCD and BCS theory).

In two dimensions, we write represent the Clifford algebra with $\gamma^0 = iY, \gamma^1 = X$, so that the chirality operator is $\bar{\gamma} = \gamma^0\gamma^1 = Z$. In the absence of a $\bar{\psi}_i\psi^i$ mass term, we have the $\mathbb{Z}_2$ chiral symmetry $\psi \mapsto \bar{\gamma}\psi$, while a mass term breaks the symmetry explicitly.

To start, we decouple the fermion interaction in the usual way. Letting the decoupling HS field be called σ, the decoupled partition function is

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp\left(i\int (i\bar{\psi}\partial\!\!\!/\psi - \sigma\bar{\psi}\psi - \frac{1}{2g^2}\sigma^2)\right). \tag{2118}$$

Now we integrate out the fermions to get an effective action for σ. Since there are N copies of fermions we get $\det^N(i\partial\!\!\!/ - \sigma)$, and so

$$Z = \int \mathcal{D}\sigma\, e^{iS_{\text{eff}}[\sigma]}, \tag{2119}$$

where

$$S_{\text{eff}}[\sigma] = -iN\ln\det(i\partial\!\!\!/ - \sigma) - \frac{1}{2g^2}\int \sigma^2. \tag{2120}$$

To evaluate the $\ln\det$, we go to momentum space in which $i\partial\!\!\!/ - \sigma$ is block-diagonalized, where each block is labeled by a different value of the momentum. One can think of the operator as becoming $\bigoplus_k(k\!\!\!/ - \sigma)$. Since $\det(\bigoplus_\alpha A_\alpha) = \det\prod_\alpha(A_\alpha \otimes \mathbf{1}_{\bar{\alpha}}) = \prod_\alpha \det A_\alpha$, we write

$$\det(i\partial\!\!\!/ - \sigma) = \prod_k \det(k\!\!\!/ - \sigma). \tag{2121}$$

Evaluating the determinant with the form of the γ matrices given above gives

$$\ln\det(i\partial\!\!\!/ - \sigma) = iV\int_k \ln(\sigma^2 + k^2), \tag{2122}$$

where we've rotated into Euclidean space, and V is the volume of the two-dimensional spacetime. We do the integral with the "replica trick", i.e. by using $\ln\rho = -\partial_n \rho^{-n}|_{n\to 0}$. Doing the integral (see e.g. P&S), one gets (working with the MF assumption where σ is a constant in order to do the integrals)

$$\begin{aligned}\ln\det(i\partial\!\!\!/ - \sigma) &= -iV \lim_{n\to 0}\partial_n \int_k \frac{1}{(\sigma^2+k^2)^n} \\ &= -iV\Gamma(-d/2)\left(\frac{\sigma^2}{4\pi}\right)^{d/2},\end{aligned} \tag{2123}$$

where we need to take the $d\to 2$ limit. For $d = 2+\epsilon$ we can use the expansion

$$\Gamma(-d/2) \approx \frac{2}{\epsilon} + \gamma. \tag{2124}$$

Then taking $\epsilon \to 0$, we get[230]

$$\ln\det(i\partial\!\!\!/ - \sigma) = -iV\frac{\sigma^2}{4\pi}\left(\frac{2}{\epsilon} + \gamma - \ln(4\pi) + \ln(\sigma^2/\Lambda^2)\right) \tag{2125}$$

for some renormalization scale Λ. We choose the counterterm for $1/g^2$ in the original Lagrangian to kill the $1/\epsilon$ terms and the constants. Then the effective potential is

$$\begin{aligned}V_{\text{eff}}[\sigma] &= -\frac{1}{V}\int \left[\mathcal{L} + \mathcal{L}_{ct} - iN\ln\det(i\partial\!\!\!/ - \sigma)\right] \\ &= \frac{1}{2g^2}\sigma^2 + \frac{\sigma^2}{4\pi}N\ln(\sigma^2/\Lambda^2).\end{aligned} \tag{2126}$$

With this renormalization scheme we can calculate the β function, which we do by requiring that the physical effective potential be independent of the renormalization scale Λ. We get

$$\beta(g) = -\frac{dg}{d\ln\Lambda} = +\frac{N}{2\pi}g^3, \tag{2127}$$

which tells us that the theory is asymptotically free: as we go to higher Λ scales, the theory becomes increasingly weakly coupled.

In any case, we can now minimize the effective potential to see if σ gets a vev in the mean-field approximation (we are ignoring IR symmetry restoration). Doing this yields

$$\sigma^2 = \Lambda^2 e^{-\frac{2\pi}{g^2 N}} \neq 0. \tag{2128}$$

Thus at 1-loop order, σ gets a vev. Since $\langle\sigma\rangle \propto \langle\bar\psi\psi\rangle$[231] and leads to the chiral symmetry being spontaneously broken in this approximation as a result of the fermion condensate which forms. Note that the expression for the mass of the fermions has

[230] Dim reg is one way to regulate the integral. Another is to simply stay in two dimensions and do the integral as $\sim \partial_n(\sigma^2+k^2)^{1-n}|_0^\Lambda$ and take the derivative—this only gives a finite result if $n>1$, but analytically continuing the exact result to $n\to 0$ reproduces the correct effective potential.

[231] σ appears quadratically and without derivatives, so it can be directly integrated out and we can replace σ with $\bar\psi\psi$ in correlation functions—of course this is exactly the point of decoupling the fermion interaction like this.

exactly the same form as the expression for the mass gap in a superconductor: it is non-analytic in the coupling, and linearly proportional to the cutoff (debeye frequency). Also note that we have two solutions for σ, consistent with the fact that the symmetry that is being spontaneously broken is a $\mathbb{Z}_2$ symmetry. This result is exact in the large N (but fixed t' Hooft coupling g^2N) limit; one can straightforwardly check that higher-loop corrections to the effective action are suppressed in powers of $1/\sqrt{N}$.

We can also check this by doing a saddle point analysis on the effective action for σ. Again in the large N limit, we expect this to be exact. Varying the effective action with respect to σ and setting the result equal to zero gives

$$\frac{\sigma(x)}{g^2} = -iN\frac{\delta}{\delta\sigma(x)}\ln\det(i\partial\!\!\!/ - \sigma). \tag{2129}$$

We use $\delta \ln A = A^{-1}\delta A$ and take the determinant over the spin indices explicitly to get

$$\frac{\sigma(x)}{g^2} = -2iN\int_{k=0}^{\Lambda}\langle x|\frac{\sigma}{k_\mu k^\mu + \sigma^2}|x\rangle \tag{2130}$$

where Λ is a cutoff. Going to Euclidean signature to do the integral over momentum gives

$$\frac{\sigma}{g^2} = \frac{N}{2\pi}\ln\left(\frac{\Lambda^2+\sigma^2}{\sigma^2}\right), \tag{2131}$$

which when solved leads to exactly the same vev for σ that we derived using the minimal subtraction renormalization scheme.

The Coleman-Weinberg potential and fluctuation-induced first order transitions

This is one of the final projects in P&S, and is basically the high-energy way of thinking about the HLM fluctuation-induced first order transitions in superconductors.

Consider scalar QED in four dimensions with a Mexican hat potential:

$$\mathcal{L} = -\frac{1}{2}F\wedge\star F + |D_A\phi|^2 + \frac{1}{2}\mu^2|\phi|^2 - \frac{\lambda}{6}|\phi|^4, \tag{2132}$$

where our sign conventions are $(D_A\phi)_\mu = (\partial_\mu - ieA_\mu)\phi$. There are several things to do:

- Compute the effective potential for ϕ to one-loop order. You should find that even at small values of $\mu^2 < 0$, where classically there is only one minimum, spontaneous symmetry breaking occurs for small e and small λ.
- Find the β functions for e and for λ.
- Use these to evaluate the ratio m_σ^2/m_A^2, where m_σ is the mass of the fluctuations about the minimum of the potential and m_A is the mass of the Higgsed photon. Show that after taking into account the RG flow, there is still a symmetry-breaking vev for ϕ even when μ^2 is negative, provided that $|\mu|$ is small.

Disclaimer: this problem is pretty long, and it would be rather onerous to type out all the algebra, and so I'll be a bit laconic in places.

Effective potential:

To get the effective potential, we need to integrate out the fluctuations in the ϕ field, as well as integrate out the gauge field. Following P&S we parametrize ϕ as

$$\phi = v + \frac{1}{\sqrt{2}}(\sigma + i\pi), \tag{2133}$$

where σ, π are real scalars that we will integrate out.

To one-loop order, the effective action is determined by the terms second order in the fluctuations. The relevant parts of the action are

$$\begin{aligned} S = \frac{1}{2}\int \Big(&(\partial_\mu \pi \partial^\mu \pi + \partial_\mu \sigma \partial^\mu \sigma + e^2 A^2(\pi^2+\sigma^2) + 2e^2 v^2 A^2 + 2\sqrt{2}e^2 A^2 v\sigma \\ &- (\lambda v^2/3 - \mu^2)\pi^2 - (\lambda v^2 - \mu^2)\sigma^2\Big). \end{aligned} \tag{2134}$$

The one-loop diagrams contributing to $V_{\text{eff}}(\phi)$ all have the same form: the $2n$-th order contribution to the effective potential is a graph with $2n$ external ϕ legs and a single loop connecting them (remember that the n-th order contribution to Γ are the n-point 1PI diagrams). Because of the interactions in the action, this loop can be a π loop, a σ loop, or an A loop, but there are no graphs with different types of fields running around the loop. This means that for the purpose of getting V_{eff}, we can ignore the interactions between the fields and just treat them as separate free fields.

In the transverse gauge $d^\dagger A = 0$, the gauge field contribution is

$$\frac{1}{2}A_\mu(g^{\mu\nu}\partial^2 + 2e^2v^2)A_\nu. \tag{2135}$$

This is a massive vector field, which due to the transverse constraint behaves just like three scalars, giving us a factor of

$$\int \mathcal{D}A \to [\det(\partial^2 + 2e^2v^2)]^{-3/2}. \tag{2136}$$

The σ and π contributions give us

$$\int \mathcal{D}\sigma \to [\det(\partial^2 - (\mu^2 - \lambda v^2))]^{-1/2}, \qquad \int \mathcal{D}\pi \to [\det(\partial^2 - (\mu^2 - \lambda v^2/3))]^{-1/2}. \tag{2137}$$

The effective action is thus the original Mexican hat for ϕ, plus the three $\ln\det$ terms. These are calculated through dimensional regularization in the regular way, see e.g. the appendix of P&S for the integrals. We get

$$\ln\det(\partial^2 + \alpha^2) = iV\int_k \ln(k^2+\alpha^2) = -\frac{i\alpha^2 V}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \ln\alpha^2 + \dots\right), \tag{2138}$$

where the ... are constants, $d = 4 - \epsilon$, and V is the spacetime volume. We choose the counterterms in the original Lagrangian to kill off the divergent parts, using the minimal subtraction scheme. The cutoff-dependent term that each ln det produces is

$$\ln\det(\partial^2 + \alpha^2) \to \frac{iV\alpha^4}{2(4\pi)^2}\ln(\alpha^2/\Lambda^2) \tag{2139}$$

where Λ^2 is the cutoff. We choose the counterterms to cancel the $2/\epsilon + \ldots$ divergence at all scales, and to cancel the logarithmic cutoff-dependent term at the renormalization scale $M^2 < \Lambda^2$. Thus the counterterms contain logarithms of the form $-\ln(\alpha^2/M^2)$. Adding up the contributions from the three fields, and using $\Gamma[\phi]/V = -V_{\text{eff}}$ for a uniform expectation value of ϕ, we we get

$$\begin{aligned} V_{\text{eff}} = -\mu^2\phi^2 + \frac{\lambda\phi^4}{6} + \frac{1}{4(4\pi)^2}\Big(&\ln[(-\mu^2 + \lambda\phi^2)/M^2](-\mu^2 + \lambda\phi^2)^2 \\ &+ \ln[(-\mu^2 + \lambda\phi^2/3)/M^2](-\mu^2 + \lambda\phi^2/3)^2 + 3\ln[2e^2\phi^2/M^2](2e^2\phi^2)^2\Big). \end{aligned} \tag{2140}$$

Working at the classical critical point $\mu^2 = 0$, let us simplify the potential in the region $\lambda \sim e^4$ very small, which we will see is a region through which RG flows always pass. We get

$$V_{\text{eff}} \approx \frac{\lambda\phi^4}{6} + \frac{3e^4\phi^4}{16\pi^2}\ln\left(\frac{2e^2\phi^2}{M^2}\right). \tag{2141}$$

Minimizing this, we find the following vev for ϕ^2:

$$\phi^2 = \frac{M^2}{2e^2}\exp\left(-\frac{8\pi^2\lambda}{9e^4} - \frac{1}{2}\right). \tag{2142}$$

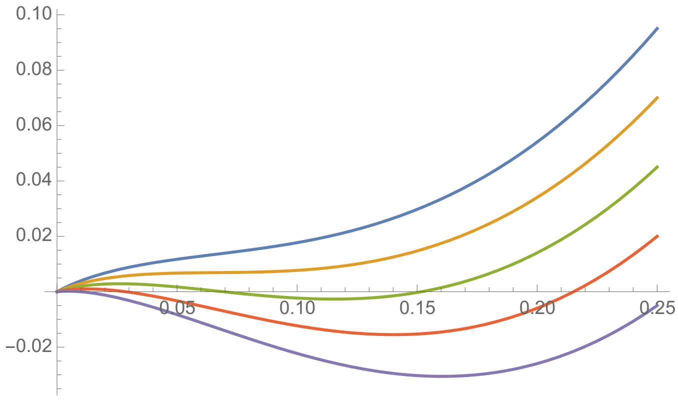
Thus the vev for ϕ is nonzero even when $\mu^2 = 0$, which could not happen classically.[232] For $\mu^2 \neq 0$, we can check by plotting the effective potential for different values of μ^2 (at fixed $\lambda \sim e^4$ and at fixed M^2), that a nontrivial minimum in V_{eff} survives even when $\mu^2 < 0$.[233] Note also that we can solve the above equation for λ as a function of e at a given renormalization scale (which we usually take to be $M^2 = 2e^2\phi^2$ for convenience). Thus we may trade two dimensionless parameters of the model (e and λ) for one dimensionless and one dimensionfull parameter — another example of dimensional transmutation.

Since the minimum of the potential becomes nonzero when the coefficient of the quadratic term is still positive, this provides us with an example of a (weakly) first

[232] The fact that the Higgs transition occurs when the quadratic term is positive is rather un-intuitive in my opinion: one would think that the effects of quantum fluctuations would *decrease* the tendency to order; the naive expectation would be that the Higgs transition wouldn't occur until some negative mass at a finite distance below zero. In fact, it is the opposite situation which occurs! Crazy.

[233] When doing this, one finds regions of parameter space where for certain values of ϕ, the arguments of the logarithms become negative—this is not a pathology, and indicates that in that case, there is no field configuration with expectation value ϕ: trying to force the expectation value of the field to be ϕ leads to an unstable state. This is like if we try to force the electric field in some region of space to be greater than $\sqrt{2m_e c^2}$: such an expectation value is not stable, since the vacuum will nucleate pairs to screen the electric field. This quantum effect is exactly what the imaginary parts of the effective potential capture.

order transition. This can be seen clearly by making a plot; choosing values of e^2, M^2, and λ so that the plot looks nice, the effective potential $V_{\text{eff}}(\phi)$ looks like



(2143)

where the curves are shown for large-ish positive mass squared ($\mu^2 < 0$; blue) down to zero mass squared $\mu^2 = 0$; purple). Here we can explicitly see the first-order nature of the transition. Since the first-order nature of the transition is caused by the logarithm, which is there to account for the 1-loop fluctuations about the classical potential, this is an example of a "fluctuation-induced first-order transition". As can be seen by the dependence of the effective potential on e^2, the interesting features of this potential are entirely due to the fluctuations of the gauge field; if the gauge field is treated classically then no such first-order transition occurs.

β **functions:**

Now for the β functions. We can go back to working with A and ϕ, which is much easier since it reduces the number of diagrams we have to compute. This is allowed since we only care about β_e and β_λ. e and λ are both dimensionless at the free fixed point in this problem, and so they cannot depend on the dimensionful parameter m.

The beta function for e is easier, so we turn to that first. We get β_e by looking at one-loop corrections to the photon propagator, i.e. by examining how charge renormalization occurs.

There are two one-loop diagrams to evaluate: the polarization bubble and a diagram where a ϕ loop intersects the A propagator line at a single vertex. The latter graph is independent of the photon momentum p^2, and as such won't contribute to the beta function, since it will be completely killed by the δ_e counterterm at any RG scale M. The polarization bubble is

$$\Pi^{\mu\nu}(p^2) = -e^2 \int_q \frac{i}{q^2}\frac{i}{(q-p)^2}(2q-p)^\mu(2p-q)^\nu, \tag{2144}$$

where q is the momentum flowing in the ϕ loop. The momenta in the numerator come from Fourier-transforming the vertex $eA^\mu(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger)$. We evaluate the bubble in the usual way using Feynman parameters and massaging the resulting expression into an integral that we can look up in the appendix of P&S (here $\int_x$ means $\int_0^1 dx$)

$$\begin{aligned}\Pi^{\mu\nu}(p^2) &= e^2\int_{q,x}\frac{(2q-p)^\mu(2q-p)^\nu}{(q^2+p^2x-2xp\cdot q)^2}\\ &= e^2\int_{q,x}\frac{(2q+p(2x-1))^\mu(2q+p(2x-1))^\nu}{(q^2+p^2(x+x^2))^2}\\ &= e^2\int_{q,x}\frac{g^{\mu\nu}q^2+p^\mu p^\nu(2x-1)^2}{(q^2-\Delta)^2},\end{aligned} \tag{2145}$$

where $\Delta = (x^2+x)(-p^2)$ and where we shifted integration over q to simplify the denominator in the second step. Looking up the integrals and doing dimensional regularization, and then doing the integral over x, we get

$$\Pi^{\mu\nu}(p^2) = \frac{ie^2}{3(4\pi)^2}\left(\frac{1}{\epsilon}-\ln(-p^2/\Lambda^2)+\dots\right)(p^2g^{\mu\nu}-p^\mu p^\nu), \tag{2146}$$

where $\dots$ are irrelevant constants and the momentum dependence is required by our choice of gauge.

We then figure out the charge renormalization by setting the propagator equal to a sum over all numbers of sequential polarization bubbles in the usual way:

$$e^2(M^2=-p^2)D^{\mu\nu}(p^2) = e^2D^{\mu\nu}(p^2)+e^2D^{\mu\alpha}(p^2)\frac{e^2}{3(4\pi)^2}\left[\ln(-p^2/\Lambda^2)(p^2g_{\alpha\gamma}-p_\alpha p_\gamma)\right]D^{\gamma\nu}(p^2)+\dots, \tag{2147}$$

where $\dots$ contains $n>1$ polarization bubbles, e.g. $D\Pi D\Pi D$, $D\Pi D\Pi D\Pi D$, and so on. This becomes a geometric series in the ln factor, since in our gauge $d^\dagger d\Box^{-2}d^\dagger d = d^\dagger\Box^{-1}d$, i.e. $(p^2g_{\mu\alpha}-p_\mu p_\alpha)(g^\alpha_\nu/p^2-p^\alpha p_\nu/p^4) = g^{\mu\nu}-p^\mu p^\nu/p^2$. Summing the geometric series, we get

$$e^2(M^2) = \frac{e^2}{1-\frac{e^2}{3(4\pi)^2}\ln(M^2/\Lambda^2)}, \tag{2148}$$

where e is as before the bare electric charge and $e(M)$ the charge at scale M. From this, we calculate the beta function

$$\beta_e = \frac{de(M)}{d\ln M} = \frac{e^3(M)}{48\pi^2}. \tag{2149}$$

Now for β_λ, which is more of a pain. Since there are many 1-loop diagrams to compute if we want to get β_λ directly, we take a different approach using the CS equation. First, we find the field-strength renormalization of ϕ.

To one-loop order the ϕ propagator correction due to ϕ itself is killed by the mass counterterm (this is true because we are taking $m^2=0$ in the Lagrangian, see P&S chapter 12), so we only need to worry about the gauge field contribution. There are two relevant diagrams: one with a straight ϕ line and a A bubble meeting it at a single vertex, and one with a polarization bubble consisting of one A line and one ϕ line. The former diagram gives a contribution independent of the external ϕ momentum, so we can ignore it in what follows (since we'll be setting the external momentum equal to the RG scale, and then differentiating wrt that scale). The bubble diagram gives

$$\begin{aligned} \text{bubble}^{\mu\nu} &= (-ie)^2 \int_q \frac{1}{(p-q)^2} \left(\frac{g^{\mu\nu}}{p^2} + \frac{p^\mu p^\nu}{p^4} \right) (2p-q)^\mu (2p-q)^\nu \\ &= I_1 + I_2, \end{aligned} \tag{2150}$$

where I_1 is the diagonal part and I_2 the $p^\mu p^\nu$ part. The former is

$$\begin{aligned} I_1 &= -e^2 \int_q \frac{(2p-q)^2}{q^2(p-q)^2} \\ &= -e^2 \int_{q,x} \frac{(p-q)^2}{(xq^2 + (1-x)(q+p)^2)^2} \\ &= -e^2 \int_{q,x} \frac{(p-q+xp)^2}{(q^2 + (x^2+x)p^2)^2} \\ &= -e^2 \int_{q,x} \frac{(1+x)^2 p^2 + q^2}{(q^2 - \Delta)^2}, \qquad \Delta = -(x+x^2)p^2. \end{aligned} \tag{2151}$$

Using the integrals in the appendix of P&S, and keeping only the logarithmic part, we get

$$I_1 \sim -\frac{2ie^2}{(4\pi)^2} \ln(-p^2/\Lambda^2). \tag{2152}$$

For I_2, we use the Feynman trick with $(A^2B)^{-1} = -\partial_A (AB)^{-1}$ to write

$$I_2 = -e^2 \int_{q,x} \frac{2x(p^2-q^2)^2}{(x(p+q)^2 + (1-x)q^2)^3}. \tag{2153}$$

We only care about divergent parts, so keeping only these, and dropping the odd-in-momentum terms that die under the integration,

$$I_2 = -e^2 \int_{q,x} \frac{2x((x^2-2)p^2q^2 + q^4)}{(q^2 + x(1+x)p^2)^3}. \tag{2154}$$

Once again we turn to the P&S appendix for the integrals. Keeping only the logarithmic parts, we get

$$I_2 = \frac{ie^2}{4\pi^2} p^2 \ln(-p^2/\Lambda^2), \tag{2155}$$

where the value of the numerical prefactor shouldn't be trusted up to a factor of 5 or so.

We now know the wavefunction renormalization γ, at least to one-loop order, since we now know what the δ_Z counterterm for ϕ should be. Using (see P&S chapter 12)

$$\frac{\partial \delta_Z}{\partial \ln M} = 2\gamma, \tag{2156}$$

we get

$$\gamma = -\frac{3e^2}{(4\pi)^2}, \tag{2157}$$

where again the numerical factor probably shouldn't be trusted since I wasn't being too careful.

Now that we have the wavefunction renormalization we can find β_λ using our knowledge of the effective potential. Just like the connected n-point functions, we can get constraints on the RG flow by evaluating $V_{\text{eff}}(\phi)$ at different renormalization scales. The effective potentials evaluated at two different scales M, M' are related by how the ϕ field scales under RG, and so taking $M' = M + \delta M$, we obtain

$$(M\partial_M + \beta_\lambda \partial_\lambda + \beta_e \partial_e - \gamma\phi\partial_\phi)V_{\text{eff}} = 0 \tag{2158}$$

(see chapter 13 of P&S, the minus sign in front of γ is because the effective potential is related to inverse propagators rather than propagators like the free energy). We know the effective potential and β_e, so plugging in and doing some algebra we get

$$\beta_\lambda = \frac{1}{24\pi^2}\left(-18e^2\lambda + 54e^4 + 5\lambda^2\right). \tag{2159}$$

In terms of the time $t = -\ln M$ along the RG flow, we have

$$d_t e = -\frac{e^3}{48\pi^2}, \qquad d_t\lambda = -\frac{1}{24\pi^2}\left(-18e^2\lambda + 54e^4 + 5\lambda^2\right). \tag{2160}$$

Note that $d_t e$ is always negative and that $d_t\lambda$ has a negative term proportional to e^4, so that we flow to smaller values of the couplings very quickly. Also note that $d_t\lambda < -(9e^2/2 - 2\lambda)^2$, so that $d_t\lambda$ is negative definite and the flow to small λ, e is inevitable.

We already know $e(M)$ from our computation of β_e, and we can get λ for $\lambda \ll e^2$ (a regime to which the RG flow will always pass through) by solving the β function by integrating from M_0 to M:

$$\lambda(M) = e^4(M)\left(\frac{\lambda(M_0)}{e^4(M_0)} + \frac{54}{24\pi^2}\ln(M/M_0)\right). \tag{2161}$$

Additionally, from the transformation

$$\phi \mapsto (1 - (\delta M/M))\phi \tag{2162}$$

under RG, we get $d\phi = -d\ln M\gamma\phi$ so that

$$\phi(M) = \phi(M_0)\left(\frac{M}{M_0}\right)^{-\gamma}. \tag{2163}$$

Using these expressions for e, λ, ϕ at the scale M, we can plug them into V_{eff} to figure out the effective potential at any given scale.

The mass (of the oscillatory mode about the potential minimum) is found by taking the second derivative of V_{eff} with respect to ϕ, and evaluating the result at $\langle\phi\rangle$ (which we will continue to lazily just write as ϕ). To simplify the resulting expression, we choose to evaluate it at the RG scale set by the vev, namely $M^2 = 2e^2\phi^2$. Taking the derivatives then gives

$$m_\sigma^2 = 3\phi^2\left(\frac{2\lambda}{3} - \frac{e^4}{4\pi^2}\right). \tag{2164}$$

The value of ϕ is still determined by the same exponential form as in our pre-RG analysis, so that this scale for M, we have

$$\lambda = \frac{9e^4}{8\pi^2}. \tag{2165}$$

This lets us write the mass more simply as

$$m_\sigma^2 = \frac{3e^4\phi^2}{2\pi^2}. \tag{2166}$$

The gauge boson mass is read off from the Lagrangian as $m_A^2 = 2e^2\phi^2$, and so we determine that the ratio of the masses is

$$\frac{m_\sigma^2}{m_A^2} = \frac{3e^2}{4\pi^2}. \tag{2167}$$

Finally, we look what happens if we take $\mu^2 < 0$, i.e. if we give ϕ a positive mass. Classically this would preclude symmetry breaking, but in quantum field theory this is no longer the case, because of the logarithm generated by the fluctuations. We add $m^2\phi^2$ to V_{eff}, find the second derivative of V_{eff} to get m_σ^2 as a function of ϕ, and then minimize V_{eff} to find ϕ in terms of e, λ, m, which then allows us to solve for m_σ^2. Mathematica gives the unilluminating

$$\begin{aligned}\phi &= \sqrt{\frac{2}{3}}\frac{m\pi}{e^2}\frac{2}{\sqrt{\text{ProductLog}(-16m^2\pi^2/(3e^2M^2))}} \\ &= \frac{M}{\sqrt{2}e}\exp\left(\frac{1}{2}\text{ProductLog}\left[\frac{-16m^2\pi^2}{3e^2M^2}\right]\right).\end{aligned} \tag{2168}$$

One can check that this reduces to our old result when $m = 0$, and that the symmetry-breaking minimum of the potential disappears at some $m_c^2 \sim M^2e^2 > 0$, so that even with the RG analysis included, there is still symmetry breaking for $m^2 > 0$ where classically there would be none.

Finally, note that all of this is true only perturbatively. We know that when we e.g. push the ε expansion out to $\varepsilon = 1$, we get a theory with a second order transition, because it becomes dual to the $O(2)$ WF theory in the IR.

β function for sine-Gordon in 2+1 dimensions at finite temperature and the KT transition for $U(1)$ gauge theory

Today we will try to understand and elaborate on the analysis of the KT transition that occurs in $U(1)$ gauge theory in three dimensions as studied in [24]. I was originally interested in this for higher-symmetry reasons, but we won't really talk about them in what follows.

An outline of this diary entry: first, we spend a few pages motivating why we expect there to be a KT-type transition in the $U(1)$ gauge theory and explaining how we can think of the critical properties of the gauge theory in terms of the XY model in two dimensions. We then will go through the RG calculation which shows the existence of the KT transition explicitly. This is technically interesting mostly since we will be at finite temperature.

Confinement phase transitions in gauge theories as regular phase transitions in scalar theories

Recall that the most useful diagnostic of confinement at finite temperature is the Polyakov loop

$$P(x) = \operatorname{Tr} \exp\left(i \int dt A_0(x,t) \right), \tag{2169}$$

where we are working in the convention where the gauge coupling appears in the action as $\frac{1}{2g^2} F \wedge \star F$. In what follows we will only be working with $U(1)$ theories, and so we can drop the Tr. If the vev of the Polyakov loop vanishes then the free energy of a single charge is infinite, and the theory confines. On the other hand, if the Polyakov loop is allowed to develop a vev, then we have symmetry breaking and the free energy of an external (non-dynamical) charged source is finite — this is the deconfined phase. Of course, we will be working with pure gauge theory throughout, since adding dynamical matter means that screening occurs, preventing the Polyakov loop from being a useful diagnostic of confinement (since it always has a finite vev after regularization, assuming we add a Wilson loop which is well-defined, i.e. is taken in a legit representation of $U(1)$ so that the charge of the Wilson loop is in $\mathbb{Z}$).

To come up with an effective action for the order parameter $P(x)$, we need to integrate out the other degrees of freedom, namely the spatial components of the gauge fields. We can argue that this procedure will always give us a local theory as follows: at $T \to 0$ we will assume the gauge coupling g^2 is such that the theory confines. This will be the case in the example of interest to us, since $U(1)$ gauge theory is confining at $T = 0$ in two spatial dimensions. Thus the spatial Wilson loops obey an area law, and the correlations between the spatial gauge fields are short-ranged, and so we can integrate them out while keeping the theory local.

At $T \to \infty$, we get what is effectively a zero-temperature version of the gauge theory in one less dimension, at coupling $g_{d-1}^2 = T g_d^2$, plus a field coming from the now-compactified A_0[234]. Since the lower dimensional theory is confining if the $T \to 0$ theory is (going to lower dimensions makes confinement stronger), the spatial gauge fields again have exponentially decaying correlations and can be integrated out. Note

[234]The coupling comes from setting a gauge in which A_0 is constant in time, and then doing the integral along the thermal circle. The relevant term in the action is then $(\beta/2g^2) \int \nabla A_0 \cdot \nabla A_0$, and so the effective coupling is $g_{\text{eff}} = \sqrt{T} g$.

that the fact that the spatial gauge fields have area-law decaying correlations does *not* mean that the full theory (confined spatial gauge fields + compactified A_0 degree of freedom) is confining! Confinement is only probed by measuring the free energy of charged sources (captured by $P(x)$), and *not* by measuring the spatial correlations of the gauge fields! Thus at finite T, the connection between Wilson line tension and confinement is only true for certain temporally-oriented Wilson lines. Confinement is diagnosed with temporal Wilson loops in the $T = 0$ theory, which become things related to the compactified A_0 field in the $T \to \infty$ theory—they are in principal very different objects than the spatial Wilson loops, which do not in this case carry any information about the energetics of charge sources.

Anyway, as was done in [24], we can then conjecture that at all temperatures, we can always integrate out the spatial gauge fields, producing an effective action for the Polyakov loop $P(x)$. Thus this theory can actually be described by a scalar field spin model in $d-1$ dimensions. In particular, it can be described with a global 0-form symmetry equal to the 1-form symmetry of the Polyakov loops in the gauge theory (namely, the center of the gauge group). For $U(1)$ gauge theory the dual spin model is thus an XY model, while for $\mathbb{Z}_N$ gauge theory (obtained by Higgsing a charge N scalar with the term $\lambda\cos(\partial_\mu\theta - iNA)$, taking λ to be large, and working in unitary gauge) we get an XY model with cosine-type interactions.

Let's look at the high-temperature and low-temperature limits of the gauge theory and see how they behave in the spin model. At $T \to 0$ we are in a confining phase, with a linear confining potential between free charges. Thus a two-point function of Polyakov loops goes as

$$\langle P(x)P^*(0)\rangle \sim e^{-\alpha|x|/T}, \tag{2170}$$

where α is the string tension — this is an area law. This then corresponds to the disordered phase of the associated spin model, where the spin model correlation length is read off to be

$$\xi = \frac{T}{\alpha}. \tag{2171}$$

So, we should think of the confining phase as the *disordered* phase.

Now we take $T \to \infty$. Generically we would expect the gauge theory to become deconfined—let's see why we would expect this. As $T \to \infty$, the thermal circle shrinks to become very small. Intuitively this means that nonzero momentum modes in the compactified direction all have very high energy, and so we can focus on field configurations that are constant in time. This means that the value of the Polyakov loop is approximately constant throughout space, meaning that the Polyakov loops have long range correlation functions, implying SSB of the 1-form symmetry and deconfinement.

More precisely, we can see this by thinking about the Wilson action for gauge theory, which looks something like

$$S \sim \sum_{\text{plaquettes}} \left(\beta_t E^2 + \beta_s B^2\right), \tag{2172}$$

where the E^2 and B^2 terms are the electric and magnetic fields at the plaquettes, realized by taking the product of Wilson lines around the edges of the (temporal and spatial, respectively) plaquettes. The ratio $\sqrt{\beta_t/\beta_s}$ represents the ratio of the spatial lattice spacing to the temporal (thermal circle) lattice spacing, and so fixing the number of lattice sites, we see that when we take T to be large, $\sqrt{\beta_t/\beta_s} \gg 1$. This means that at high T, configurations with nonzero electric flux (a definite value of the 1-form charge) are suppressed: this is deconfinement, and the electric field is "screened" (although again, we don't actually have dynamic charges).

Now consider two Polyakov loops separated by one spatial lattice spacing: $P(x)$ and $P(x+\hat{i})$. We can write

$$P(x) = \exp\left(i\int_{\text{strip}} F\right) P(x+\hat{i}), \tag{2173}$$

where the integral is over the thin cylinder enclosed by the two loops. Since F on this cylinder is the electric field strength and the electric field gets frozen out at high T, we have $P(x) \approx P(x+\hat{i})$, and thus the Polyakov loop must be slowly varying in space. This means that the free energy of two external sources separated by a distance x does not grow linearly with their separation: instead, we have the scaling

$$\langle P(x)P^*(0)\rangle \sim \text{const.} \tag{2174}$$

Thus at high T we are in the deconfined phase, and the system can screen charges. In the spin model, we see that this corresponds to symmetry breaking, and so the deconfined phase maps onto the ordered phase of the spin model.

It's worth emphasizing that the *high-temperature* phase of the gauge theory corresponds to the *ordered* phase of the spin model. Normally we think of going to higher temperatures as restoring symmetry, but with this example it's the opposite (and indeed, higher-form symmetries like this generically prefer to be broken at high temperature rather than low temperature). Zohar and co. have some nice recent papers emphasizing this perspective.

However, this symmetry-breaking argument is only true in large enough dimensions. Indeed, we have argued that in the case of $U(1)$ gauge theory in 2+1D at high T (which is what we will be most interested in), we can map the problem onto an XY model in two dimensions (zero temperature), which cannot actually have a symmetry-breaking phase by virtue of the Mermin-Wagner theorem. This means that at high T, this theory is actually never really deconfined at high temperature. Instead of a constant two-point function, we get

$$\langle P(x)P^*(0)\rangle \sim \frac{1}{|x|^\eta}, \tag{2175}$$

where η is some constant that depends on the gauge coupling. We might call this "quasi-long-ranged-deconfinement", but we do not have a genuinely deconfined phase at high T. Neither is it deconfined at low temperature because of the effects of monopoles a la Polyakov (this is also true in the continuum without monopoles), and so actually we never have a genuine deconfined phase. However, based on our experience with the XY model, we can guess that there will be some sort of KT-type transition, where the theory always is confined but the characteristics of how it is confined change with the coupling (or with T). This is corroborated by looking at how the two-point function of the Polyakov loop scales: it goes from an exponential decay at low T to a power-law at high T, and thus there must be some kind of phase transition in between. We have good reason to expect that it will be a KT transition, because of our experience with the XY model.

Getting to the sine-Gordon action

First we want to get to a sine-Gordon action starting from the gauge theory, which will make the RG analysis easier (this is standard stuff; see e.g. Altland + Simons for the next few paragraphs). The Polyakov loops, which let us build intuition for what to expect, will actually not play a role in what follows: the dual field will still be a scalar, but it will be related to A through Hodge duality as $d\phi \sim \star F$, and not by tracing A over a circle.

We start from the usual $U(1)$ gauge theory action at coupling g, and then dualize to a compact scalar field ϕ in the usual way. We will be working on a lattice, but will use continuum notation — hopefully this will not be unduly confusing.

If we are working on a manifold with trivial first cohomology[235] and the gauge bundle is trivial, then the action in terms of the scalar ϕ is

$$S = -\frac{g^2}{2(2\pi)^2}\int d\phi \wedge \star d\phi + \int \phi \wedge q. \tag{2176}$$

Here q parametrizes the locations of the instantons (which we will refer to as monopoles since we are in 2+1D): it is a three-form defined on the cubes of the spacetime lattice and is equal to

$$q = \sigma(x) d^3x, \tag{2177}$$

where $\sigma(x)$ is the monopole charge at each cube. Thus we have a situation quite similar to the XY model:

$$\text{spin waves and vortices} \sim \text{gauge fields and monopoles} \tag{2178}$$

To get the partition function, we just need to know how to sum over the configurations of monopole charges. We can sum over all possible charge configurations and assume that only charge $\sigma = \pm 1$ monopoles contribute (valid in the limit where the monopole fugacity, which we have to set by hand, is small) to get

$$Z = \left\langle \sum_{N=1}^{\infty} \sum_{\{\sigma_j\}\in\mathbb{Z}_2^N} y^N \frac{1}{(N/2)!^2}\binom{N}{N/2}^{-1} \prod_{i=1}^{N}\int d^D r_i \exp\left(i\sum_{j=1}^{N}\sigma_j \phi(r_j)\right)\right\rangle \tag{2179}$$

where the expectation value is taken with respect to the free ϕ action and we've defined the dimensionfull variable

$$y \equiv \frac{1}{a^D} e^{-\mu}, \tag{2180}$$

with $e^{-\mu}$ the monopole fugacity and a a short-distance cutoff. Some comments are in order: first, the expectation value will vanish for any configurations that have non-zero

[235] This won't be true for us since we'll be on a thermal cylinder. However, the zero modes that we would have to account for actually get killed off by a choice of boundary conditions on the ends of the cylinder at spatial infinity (which we must set non-symmetrically since we are interested in SSB), and so we don't have to care about them in what follows.

net charge. Neutral charge configurations will have $N/2$ positive and negative charges. Since the positive charges are indistinct from one another (as are the negative charges), we are required to put in the $[(N/2)!]^{-2}$ term. Furthermore, we need to mod out by the number of ways to partition N charges into a neutral configuration, which is the reason for the $\binom{N}{N/2}$ factor. Expanding out the exponential and then re-exponentiating, we get the sine-Gordon action

$$S = \int d^D x \, \left(\frac{\alpha}{2} d\phi \wedge \star d\phi - 2y\cos\phi\right), \tag{2181}$$

where we've defined

$$\alpha \equiv (g/2\pi)^2. \tag{2182}$$

Spatial RG procedure

Having obtained the sine-Gordon model, we now want to do an RG analysis and see whether we get a BKT-like RG flow. This is different from the standard analysis since we will work in arbitrary dimensions (for now), and since we're at finite temperature. Because of non-zero temperature, we will do RG in space, but not in the thermal direction. This anisotropic RG procedure is a bit awkward, but lets us capture the physics of the RG flow correctly.

We will follow [24] pretty closely for now. As usual, we split up ϕ as

$$\phi = \varphi + \chi, \tag{2183}$$

where φ is the low-momentum ($p < \Lambda'$) part and χ is the high-momentum ($\Lambda' < p < \Lambda$) part. The $(\partial\phi)^2$ part splits into $(\partial\varphi)^2 + (\partial\chi)^2$ since $(\partial\varphi)^2$ is block diagonal in momentum space, and so

$$Z' = \int D\varphi \exp\left(-\frac{\alpha}{2}\int \partial_\mu\varphi\partial^\mu\varphi\right) Z'[\varphi], \tag{2184}$$

where

$$Z'[\varphi] = \int D\chi \exp\left(-\frac{\alpha}{2}\int \partial_\mu\chi\partial^\mu\chi\right) \exp\left(2y\int \cos(\varphi+\chi)\right). \tag{2185}$$

The strategy now is to work at weak coupling, where the monopole fugacity y is small (when the field is fluctuating slowly, adding in a monopole is costly since one has to twist the value of the field at a large number of lattice sites, while while the field is rapidly fluctuating, this twisting is sort of already going on because of the fluctuations). Now, we know that $\ln Z[J]$ is the generating function for correlation functions with disconnected pieces subtracted. For example, we schematically have

$$\left.\frac{\delta^2}{\delta J(x)\delta J(y)}\right|_{J=0} \ln Z[J] = \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle, \tag{2186}$$

where $Z[J]$ includes an $\exp(\int \phi J)$ coupling. Taking n variational derivatives of $\ln Z$ produces n-point correlation functions with their disconnected pieces subtracted off. This allows us to easily write down a series expansion for $\ln Z'$ in terms of y, which appears as a current for the $\int \cos$ term. Thus

$$\ln Z' = 1+2y\int d^dx\langle\cos\phi\rangle+\frac{(2y)^2}{2}\int d^dx\, d^dy\ (\langle\cos\varphi(x)\cos\varphi(y)\rangle - \langle\cos\varphi(x)\rangle\langle\cos\varphi(y)\rangle)+\dots, \tag{2187}$$

where the expectation values are taken with respect to the free χ action.

To compute the expectation value $\langle\cos\rangle$, we have to compute $\langle e^{\pm i\chi(x)}\rangle$. We can do this by completing the square in the action in the usual way. Since the expectation value is independent of x, we get

$$\langle\cos(\varphi+\chi)\rangle = \exp\left(-\frac{1}{2}D(0)\right)\cos\varphi. \tag{2188}$$

where D is the scalar propagator for χ, $D(x-z) = \langle\chi(x)\chi(y)\rangle$. Note that in our conventions D is dimensionless, and so $D \neq \nabla^{-2}$. Instead,

$$D(x-z) = \frac{1}{2\alpha}\nabla^{-2}(x-z). \tag{2189}$$

Since the mass dimension of α is $[\alpha] = D-2$, the dimension of ∇^{-2} must also be $D-2$. This works out since we want $\nabla_x^2\nabla^{-2}(x-z) = \delta(x-z)$ to be a delta function we can integrate on a D-manifold, and so the mass dimension of the delta function must be D. Also note that D is the propagator for χ, which lives at high momentum. So when defining the real-space D, we have to define it as an integral of $D(k)$ only over a momentum shell $\Lambda' < k < \Lambda$.

When we do the mixed expectation values $\langle\cos\cos\rangle$, things are only marginally more complicated: now we need to compute $\langle\exp(i[\chi(x)\pm\chi(z)])\rangle$. We still do this by completing the square, but now we get a factor like $D(x-z)$ instead of $D(0)$, so that

$$\langle\exp(i[\chi(x)\pm\chi(y)])\rangle = \exp\left(-D(0)\mp D(x-z)\right). \tag{2190}$$

Putting this in and using $\cos a\cos b = (\cos(a+b)+\cos(a-b))/2$, the order y^2 term in $\ln Z'$ is

$$\int d^Dx\, d^Dz\, y^2 e^{-D(0)}\left[(e^{-D(x-z)}-1)\cos(\varphi(x)+\varphi(z)) + (e^{D(x-z)}-1)\cos(\varphi(x)-\varphi(z))\right]. \tag{2191}$$

Now we need to make some approximations to make this more tractable. If we are in a big enough dimension[236], $e^{\pm D(r)}-1\to 0$ for large r[237]. So we may be able to get away with an expansion in small $|x-z|$. So define the relative coordinate r by

$$\varphi(z) = \varphi(x) + \partial_\mu\varphi(x)r^\mu, \tag{2192}$$

[236] For us, this big enough dimension is three: one might worry about three becoming two after compactification at high temperature, but we will be assuming that the monopole fugacity is small enough that the average separation of monopoles is much larger than the length scale set by the thermal circle, so that for the purposes of figuring out the behavior of $D(r)$, we can think of working in $\mathbb{R}^3$.

[237] $D(r)$ is proportional to $\delta\Lambda/\Lambda$, but the possible divergence we need to worry about is the one for $r\Lambda \gg \delta\Lambda/\Lambda$. In 2d we'd get a log, which would diverge at sufficiently large r.

and substitute this in (this isn't so unreasonable—the higher-order terms are all irrelevant anyway). This generates two terms, one of which goes like $\cos(2\varphi(x)+\partial_\mu\varphi(x)r^\mu)$ and thus will be irrelevant compared to $\cos(\varphi(x))$ (and so we will drop it). The other term we expand in small r^μ, and after dropping an (infinite) φ-independent constant and integrating by parts, we get

$$\int -d^Dx\, d^Dr\, y^2 e^{-D(0)}(e^{D(x-z)}-1)(\partial_\mu\varphi(x)r^\mu)^2. \tag{2193}$$

Putting this into the full action, we obtain

$$Z=\int D\varphi \exp\left(\frac{\alpha}{2}\int d^Dx\,\varphi\partial_\mu K^{\mu\nu}\partial_\nu\varphi + 2ye^{-\frac{1}{2}D(0)}\int d^Dx\,\cos\varphi\right), \tag{2194}$$

where

$$K^{\mu\nu}=\delta^{\mu\nu}+\frac{y^2e^{-D(0)}}{\alpha}\int d^Dr\,(e^{D(r)}-1)r^\mu r^\nu. \tag{2195}$$

Note that K is diagonal. We should now re-scale so that the kinetic term has the same form as before we did the momentum-shell decomposition. Since $K^x\equiv K^{xx}=K^{yy}\neq K^t\equiv K^{tt}$, the scaling is anisotropic. This is of course expected since we are at finite T, and the propagator D is anisotropic. So define

$$\phi=\sqrt{K^x}\varphi \tag{2196}$$

and then scale the momentum as $p'=\Lambda p/\Lambda'$ so that Λ is again made the momentum cutoff. When we scale the momentum, we are only scaling the *spatial* momentum, and leaving the temperature fixed. This produces

$$Z=\int D\phi \exp\left(\frac{\alpha}{2}\int d^Dx\,\phi\left(\partial_i\partial^i+\gamma^2\partial_t^2\right)\phi + 2ye^{-\frac{1}{2}D(0)}\int d^Dx\,\cos(\phi/\sqrt{K^x})\right), \tag{2197}$$

where

$$\gamma^2=(\Lambda/\Lambda')^2K_t/K_x. \tag{2198}$$

Note that the mass dimension of y is $[y]=D$, and so the transformation of y cancels out the transformation of the integration measure in the second term. Also note that our RG step has changed the form of the $\cos\phi$ mass term, since there is now a factor of $1/\sqrt{K_x}$ inside the cosine. This means that we should go back to the original action and instead write the term as

$$\cos(\phi)\mapsto\cos(\sigma\phi), \tag{2199}$$

and then examine the RG flow of the dimensionless variable σ. This is the same as looking at the RG flow for the radius of the scalar.

These transformations change the form of the χ propagator D, which is the same as the propagator for ϕ but integrated over a different range of momenta. Namely the coefficient of the time derivative part changes, since we are only scaling in space and not time. Thus we need to go back to our original action and instead write

$$L=\frac{\alpha}{2}\phi(\partial_i\partial^i+\gamma^2\partial_t\partial^t)\phi, \tag{2200}$$

and then examine how γ flows under RG. This is essentially an anisotropic field strength renormalization procedure. The correct propagator for the high-momentum modes is thus (note the prefactor!)

$$D(x) = \frac{1}{2\alpha}\int_{\Lambda'}^{\Lambda}\frac{d^dp}{(2\pi)^d}T\sum_{n\in\mathbb{Z}}\frac{e^{ip\cdot x+i2\pi Tnt}}{p^2+(\gamma 2\pi Tn)^2}. \tag{2201}$$

Getting the β functions

So, all told we have three things that we need to keep track of under RG: the monopole fugacity (alias y), the period of the mass term (alias σ, alias the compactification radius for the boson), and the "effective temperature" or anisotropy between space and the thermal circle (alias γ; T is held fixed).

To do the RG, we let $\Lambda' = \Lambda - \delta\Lambda$ for small $\delta\Lambda$. Then for $dD = D|_{\Lambda'=\Lambda-\delta\Lambda} - D|_{\Lambda'=\Lambda}$, we get

$$dD(x,t) = \frac{1}{2\pi}\delta\Lambda\frac{\Lambda}{2\alpha(2\pi)^2T\gamma^2}J_0(\Lambda x)\sum_n\frac{e^{i2\pi nTt}}{(\Lambda/2\pi T\gamma)^2+n^2} \tag{2202}$$

in $d=2$ spatial dimensions and

$$dD(x,t) = \frac{1}{(2\pi)^2}\delta\Lambda\frac{\Lambda^2}{2\alpha(2\pi)^2T\gamma^2}\frac{\sin\Lambda x}{\Lambda x}\sum_n\frac{e^{i2\pi nTt}}{(\Lambda/2\pi T\gamma)^2+n^2} \tag{2203}$$

in $d=3$.

Now we use the summation

$$\sum_{n\in\mathbb{Z}}\frac{e^{ina}}{n^2+b^2} = \frac{\pi}{b}\frac{\sinh(ab)+\sinh[(2\pi-a)b]}{\cosh(2\pi b)-1}, \tag{2204}$$

which is valid for $0\le a\le 2\pi$ and arbitrary b (which holds for us since t is runs from 0 to $1/T$ and we are using $a = 2\pi Tt$).

This gives (with $b = \Lambda/2\pi T\gamma$)

$$dD(x,t) = \frac{1}{8\pi\alpha}\frac{\delta\Lambda}{\gamma}J_0(\Lambda x)\frac{\sinh(\Lambda t/\gamma)+\sinh[(\Lambda/T-\Lambda t)/\gamma]}{\cosh(\Lambda/T\gamma)-1} \tag{2205}$$

in two spatial dimensions. We can simplify this using

$$\frac{\sinh x+\sinh(y-x)}{\cosh y-1} = \frac{e^x+e^{-x+y}}{e^y-1}, \tag{2206}$$

so that

$$dD(x,t) = \frac{\delta\Lambda}{8\pi\alpha\gamma}J_0(\Lambda x)\frac{e^{\Lambda t/\gamma}+e^{-\Lambda(t-1/T)/\gamma}}{e^{\Lambda/\gamma T}-1}. \tag{2207}$$

Likewise in three dimensions,

$$dD(x,t) = \frac{\delta\Lambda\Lambda}{16\pi^2\alpha\gamma}\frac{\sin(\Lambda x)}{\Lambda x}\frac{e^{\Lambda t/\gamma}+e^{-\Lambda(t-1/T)/\gamma}}{e^{\Lambda/\gamma T}-1}. \tag{2208}$$

To get the RG equations, we need to compute dK^x, $d\gamma$, and dy. In two dimensions, the first is (one α^{-1} from the definition of $K^{\mu\nu}$ and one from the propagator)

$$dK^x = \frac{y^2}{\alpha}\int d^2r dD(r)x^2 = \frac{\delta\Lambda}{4\alpha^2}y^2\Lambda^{-5}I_3, \qquad I_n \equiv \int dr\, r^n J_0(r), \tag{2209}$$

while the second is obtained from

$$d\gamma = \frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x). \tag{2210}$$

and by using

$$dK^t = \delta\Lambda\frac{y^2\gamma^2 I_1}{4\alpha^2\Lambda^5}\left(4 + (\Lambda/T\gamma)^2 - \frac{2\Lambda}{T\gamma}\coth(\Lambda/2T\gamma)\right). \tag{2211}$$

Getting dy is easy: we see that $2y$ is replaced by $2ye^{-\frac{1}{2}D(0)}$ under rescaling, so that under an infinitesimal rescaling, we have

$$dy = -\frac{1}{2}dD(0). \tag{2212}$$

In three dimensions, we have the similar

$$dK^x = \frac{\delta\Lambda}{16\alpha^2}y^2\Lambda^{-5}\widetilde{I}_3, \qquad \widetilde{I}_n \equiv \int dr\, r^n \sin r \tag{2213}$$

and

$$dK^t = \delta\Lambda\frac{y^2\gamma^2 \widetilde{I}_1}{4\pi\alpha^2\Lambda^5}\left(4 + (\Lambda/T\gamma)^2 - \frac{2\Lambda}{T\gamma}\coth(\Lambda/2T\gamma)\right). \tag{2214}$$

$d\gamma$ is essentially the same:

$$d\gamma = \frac{3}{2}\frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x). \tag{2215}$$

Of course, the I integrals are infinite, but this won't be too much of a problem (this came from the fact that we're doing RG with a hard momentum cutoff instead of something better like using the CS equation).

RG: two spatial dimensions

Our dimensionless RG parameters in two spatial dimensions are (recall that the mass dimension of y is the dimension of spacetime)

$$b \equiv \sigma\frac{T}{4\pi^2\alpha} = \frac{T}{g^2}, \qquad \tau \equiv \frac{\Lambda}{T\gamma}, \qquad m \equiv \frac{2y}{\Lambda^2 T}. \tag{2216}$$

The definition of τ is used just because the ratio $\Lambda/T\gamma$ is common, and we have defined the dimensionless m (for "monopole") which is related to the instanton fugacity. The parameter $b = T/g^2$ is the coefficient that appears in front of E^2 in the gauge theory Hamiltonian and so b^{-1} gives the "temporal coupling" of the gauge theory. When b

becomes very large the electric field gets frozen out and pinned to a constant value throughout space, since as the radius of the thermal circle becomes small, non-zero-mode fluctuations of the electric flux are suppressed.

Now

$$d\gamma = \frac{\delta\Lambda}{\Lambda} + \frac{1}{2}(dK^t - dK^x), \tag{2217}$$

and so

$$\begin{aligned} d\ln\tau &= 2d\ln\Lambda - \frac{1}{2}(dK^t - dK^x) \\ &= d\ln\Lambda\left(2 + b^2m^2 f(\tau,\gamma)\right), \end{aligned} \tag{2218}$$

where $f(\tau,\gamma)$ is a function which as $\tau \to 0$ goes to $-I_3$ (a constant).

From our expression for dK^x we also get

$$d\ln b = d\ln\Lambda \frac{y^2 I_3}{8\Lambda^4\alpha^2} = d\ln\Lambda \frac{m^2b^2\pi^4 I_3}{2}. \tag{2219}$$

Finally, for dm we have

$$\begin{aligned} d\ln m &= -2d\ln\Lambda + \frac{d\Lambda}{16\pi\alpha\gamma}\coth(\Lambda/2\gamma T) \\ &= d\ln\Lambda\left(\frac{1}{4}\tau\pi b\coth(\tau/2) - 2\right), \end{aligned} \tag{2220}$$

where the first term comes from dy and the second from the factor of Λ^{-2} in the definition of m and we used

$$dD(0,0) = \frac{\delta\Lambda}{8\pi\alpha\gamma}\coth(\Lambda/2\gamma T). \tag{2221}$$

Absorbing the some unsightly prefactors by scaling $m \to m\pi^2\sqrt{I_3/2}$ and defining $dt = -d\ln\Lambda$ as time along the RG flow with $t = -\infty$ when no momenta have been integrated out and $t = \infty$ when all momenta have been integrated out, we can rewrite the RG equations as

$$\begin{aligned} db &= -m^2b^3dt, \\ dm &= -m\left(\frac{\pi}{4}b\tau\coth(\tau/2) - 2\right)dt \\ d\tau &= -\tau dt\left(2 - m^2b^2\widetilde{f}(\tau,\gamma)\right), \end{aligned} \tag{2222}$$

where $\widetilde{f}(\tau,\gamma)$ is a rather gross function that at $\tau \to 0$ goes to 1[238]. For small enough b^2m^2, $\ln\tau$ monotonically decreases along the flow, so that $\tau \to 0$ as $t \to \infty$. This is intuitive: if we zoom out of the cylinder, the cylinder becomes "longer" (we see more of the cylinder), but since we are holding the radius of the cricle fixed, it effectively becomes "skinnier", which means high "temperature", which means low τ.

Now it is but a hop and a skip to KT! In fact, we essentially already have the KT RG equations, just with an extra parameter τ related to the fact that we're at finite temperature. However, if we are at weak enough coupling, and are just limited

[238] For posterity's sake, $\widetilde{f}(\tau,\gamma) = 1 - I_3^{-1}\gamma^2(4 + \tau^2 - 2\tau\coth(\tau/2))$.

in the deep IR, we can take $\tau \to 0$. Heuristically, we can think that as we zoom out in real space, the aspect ratio of the cylinder changes and the thermal circle appears to shrink, thus taking us to higher temperatures (recall that $\tau \to 0$ as $T \to \infty$). That this happens is not a forgone conclusion because of the fact that γ scales under RG as well, but at weak coupling this turns out to be the case. Thus we may send $\tau \to 0$ and write

$$\begin{aligned} \frac{db}{dt} &= -m^2 b^3, \\ \frac{dm}{dt} &= -m\left(\frac{\pi}{2}b - 2\right) \end{aligned} \tag{2223}$$

as our RG equations. This is essentially the same as in the regular KT analysis, expect that b appears in the first equation as b^3 instead of b^2 and some of the numerical coefficients are different by various factors of π and 2. The difference in the numerical factors doesn't affect the existence of a KT transition, and the differing power of b only changes the transition quantitatively, not qualitatively (it changes the time that trajectories spend near the $y = 0$ fixed point).

To analyze our version of the KT equations, we measure the distance to the critical point with the variable x:

$$b = \frac{4}{\pi}(1 - x). \tag{2224}$$

To lowest order in x, then,

$$\frac{dx}{dt} \approx \frac{16}{\pi^2}m^2, \qquad \frac{dm}{dt} \approx 2mx. \tag{2225}$$

The fact that we have b^3 instead of b^2 only appears at order x^3, which is irrelevant for our discussion. We then use the above to write

$$\frac{dx^2}{dm^2} = \frac{8}{\pi^2} \implies m^2 = \frac{\pi^2}{8}(x^2 + C), \tag{2226}$$

which is the familiar KT hyperbola (the only difference from regular KT is the number on the RHS). The integration constant C measures the distance from the critical trajectory, since when $C = 0$ we get the critical line. Using the RG equation for x, we see that

$$\frac{dx}{dt} = 2(x^2 + C), \tag{2227}$$

which is exactly the same as the KT result. Since x is monotonically increasing, the system continues to get floppier and floppier (smaller spin stiffness) as we go to larger distances, since monopoles become more and more important at large distances. Proceeding from here, one can find the non-analyticities signifying the phase transition in the usual way.

β function for scalars coupled to a non-Abelian gauge field

This is from P&S, chapter 16. Consider a non-Abelian gauge theory coupled to a scalar field:

$$\mathcal{L} = -\frac{1}{2g^2}\text{Tr}[F \wedge \star F] - (D_\mu \phi)^\dagger D_\mu \phi - V(\phi), \tag{2228}$$

with the sign convention $D_\mu = \partial_\mu - iA_\mu^a t^a$. Here t is taken in the adjoint representation when acting on A and is taken in some representation r when acting on ϕ. We will be writing down the Feynman rules and computing the β function for g to lowest order. To speed up things, we will use the fact that for Yang-Mills with fermions, the β function is

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{n_f}{3}C(r)\right), \tag{2229}$$

where $t^a t^a = C_2(r)\mathbf{1}$ for the representation r (with $r = G$ indicating the adjoint) and $\text{Tr}[t^a t^b] = C(r)$.

First let's get the Feynman rules. We can just do this by inspection: we have the usual 3-point and 4-point gross gauge boson vertices from the kinetic term, plus the $(D_\mu\phi)^\dagger D_\mu \phi$ term. The relevant parts of this are

$$g^2 A_\mu^a A_a^\mu \phi^\dagger \phi, \quad igA^\mu(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger). \tag{2230}$$

The former term gives us a two-scalar-two-A interaction vertex. Since there are two different ways of attaching the gauge boson propagators onto the vertex, this vertex produces the Feynman rule

$$A^a A^b \phi^\dagger \phi \to ig^2\{t^a, t^b\}. \tag{2231}$$

For the trivalent vertex, we fix our conventions so that the derivative acts on the incoming ϕ propagator with momentum p, so that

$$A_\mu^a \phi^\dagger \phi \to -ig(p+p')_\mu t^a. \tag{2232}$$

To get β_g, we will use the CS equation on the gauge boson + two-scalars vertex (the three-point function between a gauge boson and two scalars). To do this, we need to know the counterterm δ_3 for this vertex, as well as the wavefunction renormalization counterterms δ_ϕ, δ_e for the scalar and the photon. Then varying the three-point function with respect to the RG scale, we get (to lowest order in g)

$$2\gamma_\phi + \gamma_A + \partial_{\ln M}\left(-i\delta_g - ig\sum_i(\cdots - \delta_{Z_i})\right) - i\beta_g = 0, \tag{2233}$$

where the ... are independent of the RG scale, the sum is over the external legs and comes from the external propagator corrections, and δ_g is the counterterm for the vertex we are considering. By varying the two-point functions for the gauge field and the scalar with respect to M, we get (as usual to lowest order)

$$\gamma_i = \frac{1}{2}\partial_{\ln M}\delta_{Z_i}, \tag{2234}$$

so that, if we denote $g\delta_v = \delta_g$ for the vertex counterterm,

$$\beta_g = g\partial_{\ln M}(-\delta_1 + \delta_\phi + \delta_A/2). \tag{2235}$$

In Abelian gauge theory the first two terms would have canceled for Ward identity reasons, but now they do not. Instead, they provide some gauge-variant divergent part which cancels with a gauge-variant divergent part of the δ_A correction for the gauge field (coming from ghosts and the gauge field self-interaction). The gauge-variant part of δ_A doesn't depend on the matter content, meaning that $-\delta_1 + \delta_\phi$ actually does not depend on the matter content either! See P&S chapter 16.5 for some more discussion about this.

This means that the only dependence of the β function on the chosen matter fields comes from their contribution to δ_A, which is very easy to compute since we only have to do a single polarization bubble integral (the $\phi^2 A^2$ vertex diagram doesn't contribute to the β function since it is momentum-independent). The diagram is (we can ignore a potential mass for the scalar since it can't contribute to the β function by dimensional analysis)

$$(\text{polarization bubble})^{\mu\nu}(q) = -\mathrm{Tr}[t^a t^b]g^2 \int_p \frac{(2p+q)^\mu(2p+q)^\nu}{p^2(p+q)^2}. \tag{2236}$$

We know the form of the bubble will have a momentum dependence dictated by being properly transverse, so we can just compute the coefficient of e.g. the q^2 piece. This gives

$$C(r)g^2\delta^{ab}\int_p \frac{4p^\mu p^\nu}{(p^2-\Delta)^2} = \frac{1}{3}\frac{g^2 C(r) g^{\mu\nu}\delta^{ab}}{(4\pi)^2} q^2 \ln(-q^2/\Lambda^2). \tag{2237}$$

This thus determines the counterterm for the wavefunction renormalization of A to be

$$\delta_A = \frac{1}{3}\frac{g^2 C(r)}{(4\pi)^2}\ln(M^2/\Lambda^2). \tag{2238}$$

Its contribute to the beta function is thus

$$\beta_g = \frac{1}{2}g\partial_{\ln M}\delta_A = \frac{g^3 C(r)}{3(4\pi)^2}. \tag{2239}$$

Now the only difference between the β function for the theory coupled to fermions and one coupled to scalars is the different values they contribute to δ_A by way of the polarization bubble. Both bubbles have the same color trace, but the fermions have an extra trace over spin indices. This ends up contributing an extra factor of 4 (see similar calculations done earlier), and so fermions in a representation t^a contribute four times as much to β_g as scalars in the same representation do. Thus we only have to turn to P&S chapter 16 and observe that the β function for fermions coupled to a non-Abelian gauge field is

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right). \tag{2240}$$

Thus we deduce that if the field is coupled to scalars instead, the beta function is

$$\beta_g = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}n_f C(r)\right). \tag{2241}$$

Another method to calculate the β function is the background field method, where we integrate out fluctuations of the gauge field and express the effective action in terms of a new effective charge, which is more cmt-y. This is essentially in P&S, so I won't write it out here.

Anomalous dimension in ϕ^4 theory

Today is short and simple: we will calculate the anomalous dimension of the operator ϕ^2 in ϕ^4 theory (in four spacetime dimensions). Recall that the anomalous dimension of an operator $\mathcal{O}$ is defined by

$$d_{\ln\mu}\mathcal{O} = \gamma_{\mathcal{O}}\mathcal{O}. \tag{2242}$$

We will calculate γ_{ϕ^2} to first order in λ.

To find the anomalous dimension, we can insert the term $\int J\phi^2$ into the Lagrangian. To one loop order, J gets corrected by a ϕ^4 vertex, with the relevant graph looking like a fish. If we take J at momentum k, then the two (incoming) parts of the tail of the fish will have momenta p and $-k-p$, with the two segments of the body of the fish having momenta q and $k-q$ (we take the k momentum of the current J to be "incoming" with respect to the head of the fish). There is a symmetry factor of 2 associated with the number of ways to order the lines on the tail of the fish, and so the fish diagram is

$$\text{fish} = \frac{i^2\lambda}{2}\int_q \frac{1}{(q^2-m^2)((k-q)^2-m^2)}. \tag{2243}$$

We could do Feynman parameters and dim reg, but we care only about the coefficient of the large logarithm, so we can also just directly select out the UV divergent part and do the integration: the surface area of an S^3 is $2\pi^2 R^4$, so

$$\text{fish} \sim -2\lambda \frac{\pi^2}{32\pi^4} \ln(\Lambda/\Delta), \tag{2244}$$

where Δ is an invariant built out of the external momenta and the mass.

We thus renormalize the current by writing

$$J_b = J_R Z_J = J_R(1+\delta_J), \tag{2245}$$

where

$$\delta_J = \frac{\lambda}{16\pi^2} \ln(\Lambda/\mu), \tag{2246}$$

where μ is the RG scale at which we want the one-loop correction to vanish. Since J_b is independent of μ,

$$\begin{aligned} 0 &= \frac{d}{d\ln\mu}(J_R(1+\delta_J)) \\ &= \frac{dJ_R}{d\ln\mu}(1+\delta_J) + J_R \frac{d}{d\ln\mu}\left(\frac{\lambda}{16\pi^2}\ln(\Lambda/\mu)\right). \end{aligned} \tag{2247}$$

Which means that

$$0 = \gamma_J\left(1+\frac{\lambda}{16\pi^2}\ln(\Lambda/\mu)\right) + \frac{\beta_\lambda}{16\pi^2}\ln(\Lambda/\mu) - \frac{\lambda}{16\pi^2}. \tag{2248}$$

Now β_λ is order λ^2, since it is computed by looking at the corrections to the ϕ^4 vertex which is marginal at the free fixed point. γ_J is also first order in λ, and so to first order we have $\gamma_J = \frac{\lambda}{16\pi^2}$. The anomalous dimension of ϕ^2 is the negative of this, so

$$\gamma_{\phi^2} = -\frac{\lambda}{16\pi^2}. \tag{2249}$$

Alternately, we could have obtained this by calculating $G_2(\phi(x),\phi(x))$ in an RPA-like way, by summing diagrams that look like chains with all possible number of beads on them (each bead is a ϕ^4 bubble attached to the chain at a single point). At the renormalization scale μ, we would sum the series and get

$$G_2 \sim G_{2,0}\frac{1}{1-\frac{\lambda}{16\pi^2}\ln(\Lambda/\mu)}. \tag{2250}$$

Taking the derivative wrt $\ln\mu$ and working to first order in λ reproduces the correct result. Thus the RG reproduces an infinite number of diagrams (subleading logs) in the usual way. In this ϕ^4 example directly summing the infinite number of diagrams is easier than calculating the beta functions, but in more complicated contexts the RG approach is much easier.

RG and the nonlinear σ model on a quotient space

This is a slight elaboration on a problem from Altland and Simons. Consider the nlσm in two dimensions, viz.

$$S = \frac{1}{\lambda}\int d^2x \, \mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}]. \tag{2251}$$

We will be considering a situation of a sigma model which describes an EFT for a phase in which an $SU(2)$ symmetry is spontaneously broken to a $U(1)$ subgroup, with the fields g being valued in the quotient space $SU(2)/U(1) \sim S^2$ (which of course is not a group, despite the use of g—sorry). The goal is to compute the beta function β_λ.

We will find the RG flow for λ by using the background field method. This proceeds by breaking up g as $g = g_s g_f$ into "slow" and "fast" degrees of freedom and integrating out g_f in the usual Polykovian way.

First let us do a bit of massaging to the action. Since working with S^2-valued fields is hard, we will instead work with $SU(2)$-valued fields and impose a local $U(1)$ gauge redundancy to recover the correct quotient space. We can then essentially guess the form of the action in this formulation in terms of $SU(2)$ fields based on the requirement of global $SU(2)$ invariance and local $U(1) \subset SU(2)$ invariance: we should get something that looks like a covariant derivative. The slow fields are being treated as background fields, so they should enter as the gauge field in the covariant derivative.

Let us write $g \in SU(2)$ by rotating away from the generator Z of the $U(1)$ unbroken subgroup as $g = g_s g_f Z g_f^{-1} g_s^{-1}$. We do this to make the fact that the action describes modes living in the coset space $SU(2)/U(1) = S^2$ manifest: Z is the generator of the unbroken $U(1)$, and $hZh^{-1} \subset U(1)$ for any h in the $U(1)$ subgroup. Let us also define

$$\Gamma = g_f Z g_f^{-1}, \qquad \Phi_\mu = g_s^{-1}\partial_\mu g_s. \tag{2252}$$

Then

$$g_s^{-1}(\partial_\mu g) g_s = \Phi_\mu \Gamma + \partial_\mu \Gamma - \Gamma \Phi_\mu. \tag{2253}$$

Putting this into the trace and using the cyclic property of the trace, we indeed get a Yang-Mills-y action, viz.

$$S = \frac{1}{\lambda}\int \mathrm{Tr}\left(d\Gamma + [\Phi, \Gamma]\right)^2. \tag{2254}$$

It will be helpful to choose a gauge in which Φ is fixed to a particular form, which we can do using invariance of Γ under local $U(1)$ transformations. Let us take $U(1)$ to act on the left, with $g_f \mapsto h g_f$ for $h \in U(1)$. Since Z is the generator of the $U(1)$, doing this transformation doesn't affect the $[\Phi, \Gamma]$ term (essentially by construction — the form of Γ was chosen so that it lives in the coset space). However, the derivative term changes:

$$d\Gamma \mapsto dh\Gamma - h\Gamma h^{-1} dh h^{-1}. \tag{2255}$$

Putting this into the action and using the cyclic property of the trace to conjugate by h, we see that the action shifts to

$$S \mapsto \frac{1}{\lambda}\int \mathrm{Tr}\left(d\Gamma + [h^{-1}(\Phi+d)h, \Gamma]\right)^2, \tag{2256}$$

which is exactly what we'd expect from a gauge transformation. We can then use this freedom to fix a gauge (at least locally) in which Φ is sort of in "unitary gauge", in that it has no Z component. That is, we can fix Φ so that

$$\{\Phi_\mu, Z\} = 0. \tag{2257}$$

This is accomplished I believe with the choice $h = \exp(i\psi(x)Z)$, where

$$\psi(x) = -i\int_\bullet dx^\mu \ \mathrm{Tr}[Z\Phi_\mu] + \psi(\bullet), \tag{2258}$$

for some basepoint $\bullet$.

Now let us finally break apart the action into a slow component, a fast component, and a mixed term that couples slow and fast degrees of freedom. We will parametrize g_f by

$$g_f = \exp(iW), \qquad W = \begin{pmatrix} 0 & z \\ \bar{z} & 0\end{pmatrix}, \quad z \in \mathbb{C}. \tag{2259}$$

Note that since W is built out of an X and a Y, this form of g_f is preserved by the action of the $U(1)$ subgroup by $h(x)$. One then sees that to second order in W,

$$g_f = Z(1 - 2iW - 2W^2). \tag{2260}$$

Since the form of W means that $ZWZ = -W$, the $(d\Gamma)^2$ term in the action becomes

$$S_f = \frac{(2i)^2}{\lambda}\int \mathrm{Tr}[\partial_\mu W Z \partial^\mu W Z] = \frac{4}{\lambda}\int \mathrm{Tr}[\partial_\mu W \partial^\mu W] = \frac{8}{\lambda}\int \partial_\mu z \partial^\mu \bar{z}. \tag{2261}$$

Now the commutator appearing in the action is, to second order in W,

$$[\Phi_\mu, \Gamma] = 2\Phi_\mu Z + 2iZ\{\Phi_\mu, W\} + 2Z\{\Phi_\mu, W^2\}, \tag{2262}$$

where we used $Z\Phi_\mu Z = -\Phi_\mu$ by virtue of our gauge choice. Squaring this and working to quadratic order in W, we see that the slow mode part is

$$S_s = \frac{4}{\lambda}\int \mathrm{Tr}[\Phi^2], \tag{2263}$$

while the slow-fast coupling part is

$$S_{sf} = -\frac{4}{\lambda}\int \mathrm{Tr}\left[Z(\Phi_\mu W + W\Phi_\mu)Z(\Phi_\mu W + W\Phi_\mu)\right] + \frac{8}{\lambda}\int \mathrm{Tr}\left[\Phi_\mu\{\Phi_\mu, W^2\}\right]. \tag{2264}$$

We can massage this into

$$S_{sf} = -\frac{8}{\lambda}\int \mathrm{Tr}\left[\Phi_\mu W Z \Phi^\mu W Z\right] + \frac{8}{\lambda}\int \mathrm{Tr}\left[W^2\Phi^2\right], \tag{2265}$$

or equivalently,

$$S_{sf} = -\frac{8}{\lambda}\int \mathrm{Tr}\left[\Phi_\mu W \Phi^\mu W - W^2\Phi^2\right]. \tag{2266}$$

The mixed $d\Gamma$ and commutator cross-terms won't contribute since they vanish after we go to momentum space.

Now we take advantage of the specific form of W and Φ. Since Φ anticommutes with Z, we can write it as

$$\Phi_\mu = \begin{pmatrix} 0 & \phi_\mu \\ \bar{\phi}_\mu & 0 \end{pmatrix}, \tag{2267}$$

Thus we get

$$S_{sf} = -\frac{8}{\lambda}\int \left(\phi^2 z^2 + \bar{\phi}^2\bar{z}^2 - 2|\phi|^2 z\bar{z}\right). \tag{2268}$$

To find the correction to the coupling constant induced by integrating out the fast modes, we will take $e^{-S_{sf}}$ and expand it as $1 - S_{sf} + \ldots$, keeping terms of order W^2. Thus we will be computing the expectation value of the above integral with respect to the free $\partial_\mu z \partial^\mu \bar{z}$ action. Since only the $\bar{z}_p z_p$ two-point function is non-zero, the first two terms in the above expression for S_{sf} will only have a non-zero expectation value at order $z^2\bar{z}^2$, which we are dropping. Thus, after we expand the exponential, we may ignore the first two terms. Thus the relevant expectation value to compute is

$$\int \mathcal{D}z\mathcal{D}\bar{z}\left(\frac{8}{\lambda}\int_p \mathrm{Tr}[\Phi^2]\bar{z}_p z_p\right)\exp\left(-\frac{8}{\lambda}\int_p p^2 z_p \bar{z}_p\right). \tag{2269}$$

We find this to be

$$\begin{aligned}\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d\ln p}{2\pi}\mathrm{Tr}[\Phi^2] &= -\frac{1}{2\pi}\ln\left(1+\frac{d\Lambda}{\Lambda}\right)\mathrm{Tr}[\Phi^2] \\ &= -\frac{\mathrm{Tr}[\Phi^2]}{2\pi}d\ln\Lambda,\end{aligned} \tag{2270}$$

where $\Lambda - \delta\Lambda$ is the boundary between the fast modes and the slow modes, and where $d\Lambda = -\delta\Lambda$ is negative, since the cutoff is being decreased.

Now we re-exponentiate to find that the new slow action is

$$S_s = \frac{4}{\lambda + d\lambda}\int \mathrm{Tr}[\Phi^2], \tag{2271}$$

where

$$\frac{4}{\lambda+d\lambda} = \frac{4}{\lambda} + \frac{d\ln\Lambda}{2\pi}. \tag{2272}$$

We re-write this as

$$\lambda + d\lambda = \frac{\lambda}{1+\frac{1}{8\pi}d\ln\Lambda} = \lambda - \frac{1}{8\pi}d\ln\Lambda, \tag{2273}$$

and so the β function is

$$\frac{d\lambda}{d\ln\Lambda} = -\frac{\lambda^2}{8\pi}. \tag{2274}$$

Thus we have asymptotic freedom, with weak coupling in the UV (as expected for a NLSM into a manifold with positive curvature, namely S^2) and with a strong-coupling disordered phase in the IR. In terms of the time flow along the RG trajectory, we can write

$$\frac{d\lambda}{dt} = \frac{\lambda^2}{8\pi}. \tag{2275}$$

How special is the choice of dimension 2? To get some insight into this, we can consider doing the same calculation in dimension $d = 2 + \epsilon$. The relevant integral is then schematically

$$\int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^2p}{4\pi^2} p^{-2} = \frac{1}{2\pi\epsilon}(\Lambda^{-\epsilon} - \Lambda^{-\epsilon}(1 - \epsilon d \ln \Lambda)). \tag{2276}$$

The affect of the extra $\Lambda^{-\epsilon}$ is to just add a $\epsilon d \ln \Lambda$ contribution the integral, which is opposite in sign to the $d = 2$ part. Thus the β function is upgraded to

$$\frac{d\lambda}{d\ln\Lambda} = -\frac{\lambda^2}{8\pi} + \epsilon. \tag{2277}$$

This tells us that while in $d = 2$ there is no phase transition and the theory just flows to a disordered state (as it must because of the CMW theorem), in $d > 2$ there is a phase transition between the ordered and disordered phases at some finite value of λ. Thus $d = 2$ is the lower critical dimension of the theory.

RG in the 1+1D $SU(N)$ WZW model

Like in yesterday's diary entry, consider the $SU(N)$ WZW model on a two-dimensional spacetime. The action is

$$S = S_{kin} + S_{wzw} = \frac{1}{\lambda}\int \mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}] - \frac{ik}{12\pi}\int_{B^3} \mathrm{Tr}[\omega \wedge \omega \wedge \omega], \tag{2278}$$

where g is a map from the spacetime X (which since we will take the fields to be constant at infinity is topologically an S^2) into $SU(N)$, B^3 is a three-ball which bounds spacetime, and where ω is the Maurer-Cartan form on $SU(N)$ pulled back to B^3.

Using the background field method with an explicit momentum-shell cutoff, find the beta function for λ. Show that $\lambda = 8\pi$ is a fixed point, the existence of which is made possible by the wzw term.

To do the RG, we split up $g = g_s g_f$ into low- and high-momentum parts. We find it helpful to parametrize g_f as $g_f = e^W$ for $W \in \mathfrak{su}(N)$. Since we are only interested in finding β_λ to one-loop order, we only need to keep terms quadratic in W (and to quadratic order, the measure $\mathcal{D}g_f$ is the same as $\mathcal{D}W$). Thus in the action we can make the replacement

$$\partial_\mu g_f \approx \partial_\mu W + \frac{1}{2}\{\partial_\mu W, W\}. \tag{2279}$$

Let us focus on S_{kin} first. After some straightforward algebra, we find

$$S_{kin}[g] = S_{kin,s}[g_s] + S_{kin,f}[W] + S_{kin,sf}[g_s, W], \tag{2280}$$

where

$$S_{kin,s}[g_s] = \frac{1}{\lambda}\int \mathrm{Tr}[\partial_\mu g_s \partial^\mu g_s^{-1}], \qquad S_{kin,f} = -\frac{1}{\lambda}\mathrm{Tr}[\partial_\mu W \partial^\mu W], \tag{2281}$$

and

$$\begin{aligned} S_{kin,sf}[g_s, W] &= \frac{1}{2\lambda}\int \mathrm{Tr}\left[g_s^{-1}\partial_\mu g_s[\partial_\mu W, W] + (\partial_\mu g_s^{-1})g_s[W, \partial_\mu W]\right] \\ &= \frac{1}{\lambda}\int \mathrm{Tr}\left[\star\omega_s \wedge [dW, W]\right], \end{aligned} \tag{2282}$$

where $\omega_s = g_s^{-1}\partial_\mu g_s\, dx^\mu$.

Now for the WZW term. The well-definedness of this term forces the coefficient to be an integer (times $i/12\pi$) which means the factor in the WZW term can't flow under RG—this does not preclude the WZW term from contributing to the RG flow of other couplings (viz. λ), however.

Let us first expand the Maurer-Cartan form ω appearing in the WZW term. Again, straightforward algebra gives

$$\omega \approx \omega_s(1 + W + W^2/2) - W\omega_s - W\omega_s W + \frac{1}{2}W^2\omega_s + \frac{1}{2}[\partial_\mu W, W] + \partial_\mu W. \tag{2283}$$

Now we take the wedge product of three copies of the above expression. We only want to keep terms that have at most two derivatives in the slow fields (since they are slowly varying), and we only need to keep up to quadratic order in W. The very last $\partial_\mu W$ parts contribute a factor of $3\mathrm{Tr}[dW \wedge dW \wedge \omega_s]$. The only other term that is not cubic in ω_s comes from the commutator, and so

$$S_{top}[g_s, W] = -\frac{ik}{4\pi}\int_{B^3} \mathrm{Tr}\left[\left(\frac{1}{2}[dW, W] \wedge \omega_s + dW \wedge dW\right) \wedge \omega_s\right]. \tag{2284}$$

The integrand is a total derivative:

$$d\left(-\frac{i}{8\pi}\mathrm{Tr}\left[[dW, W] \wedge \omega_s\right]\right) = -\frac{i}{8\pi}\mathrm{Tr}\left[-2dW \wedge dW \wedge \omega_s + [dW, W] \wedge \omega_s \wedge \omega_s\right], \tag{2285}$$

since $d\omega_s = -\omega_s \wedge \omega_s$ (we've also used the sign rules for the supercommutativity of the wedge product). So, we have

$$S_{top}[g_s, W] = -\frac{ik}{8\pi}\int_X \mathrm{Tr}[[dW, W] \wedge \omega_s]. \tag{2286}$$

To write the full slow-fast action succinctly, it is helpful to introduce

$$\widetilde{\omega} = \omega_s - \frac{i\lambda k}{8\pi} \star \omega_s, \tag{2287}$$

where the $\star$ is the Hodge star on X. With this, we have

$$S = S_f[W] + S_s[g_s] + S_{sf}[g_s, W], \tag{2288}$$

where

$$S_{sf}[g_s, W] = \frac{1}{\lambda} \int \mathrm{Tr}[\star\widetilde{\omega} \wedge [dW, W]]. \tag{2289}$$

Thus, while the wzw term can't get renormalized through a change of its coefficient (the level), it contributes towards the renormalization of the kinetic term.

Now we expand the exponential $\exp(-S_{sf})$ in the path integral. The linear term vanishes since it contains a single fast momentum from the dW piece, and so the first relevant contribution comes from the S_{sf}^2 term. Thus the effective action for the slow fields is

$$S_{eff,s}[g_s] = S_s[g_s] - \ln\left(1 + \frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle\right), \tag{2290}$$

where the expectation value is taken with respect to the $\lambda^{-1}\int \mathrm{Tr}(\partial W)^2$ action. The expectation value is

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = -\frac{4}{2\lambda^2}\int_{p,q,p',q'} p_\mu p'_\nu \langle \mathrm{Tr}[\widetilde{\omega}_q^\mu W_p W_{-q-p}]\mathrm{Tr}[\widetilde{\omega}_{q'}^\nu W_{p'} W_{-q'-p'}]\rangle, \tag{2291}$$

where we've gone to momentum space and we've used $2p + q \approx 2p$ since p is a fast momentum and q is a slow momentum (corrections to this are irrelevant).

We take the expectation value by contracting the various W's. Contracting two W's in the same trace yields zero: this is because the propagator for the W fields is diagonal in the $\mathfrak{su}(N)$ generators (expand the W's in terms of T^a's in the kinetic term for W and use the orthogonality of the T^a's under the trace inner product), and so contracting two W's in the same trace produces something like $\mathrm{Tr}[\widetilde{\omega}C_2]$, where $C_2 \sim \sum_a T^a T^a$ is the quadratic Casimir, which is central in the Lie algebra. Expanding $\widetilde{\omega}$ (which lives in $\mathfrak{su}(N)$ since it is built from the Maurer-Cartan form) in terms of the traceless T^a, one sees that the trace vanishes.

So, we just need to consider the two contractions between the W's in different traces. This gives (see Altland and Simons chapter 8 for some useful identities)

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = -\frac{N}{2\lambda^2}\int_{p,q} G_p G_{p+q} p_\mu p_\nu \mathrm{Tr}[\widetilde{\omega}_q^\mu \omega_{-q}^\nu], \tag{2292}$$

with the factor of N coming from the sum over the internal loop in the polarization diagram (use the double line notation to see). If we again approximate $p + q \approx p$ then we can do the integral over the fast momentum p easily:

$$\frac{1}{2}\langle S_{fs}^2[g_s, W]\rangle = -\frac{N}{8\pi}\int_q \ln\left(\frac{\Lambda}{\Lambda - \delta\Lambda}\right) \mathrm{Tr}[\widetilde{\omega}_q^\mu \widetilde{\omega}_{\mu,-q}]. \tag{2293}$$

Taking $d\Lambda = -\delta\Lambda$ and expanding,

$$\frac{1}{2}\langle S^2_{fs}[g_s, W]\rangle = -d\ln\Lambda \frac{N}{8\pi}\int \mathrm{Tr}[\widetilde{\omega}\wedge\star\widetilde{\omega}], \tag{2294}$$

where the integral is now in $\mathbb{R}$ space. From the definition of $\widetilde{\omega}$, one sees that this integral is

$$\frac{1}{2}\langle S^2_{fs}[g_s, W]\rangle = d\ln\Lambda \frac{N}{8\pi}\left(1-\left(\frac{\lambda k}{8\pi}\right)^2\right)\int \mathrm{Tr}[\omega_s\wedge\star\omega_s]. \tag{2295}$$

Adding this in with the $\omega_s \wedge \omega_s$ term, we see that the effective coupling $\lambda + d\lambda$ is

$$\frac{1}{\lambda + d\lambda} = \frac{1}{\lambda} + d\ln\Lambda\frac{N}{8\pi}\left(1-\left(\frac{\lambda}{8\pi}\right)^2\right), \tag{2296}$$

which gives the β function

$$\frac{d\lambda}{d\ln\Lambda} = -\frac{N\lambda^2}{8\pi}\left(1-\left(\frac{\lambda k}{8\pi}\right)^2\right). \tag{2297}$$

There are several things to note about this. First, we see that it is asymptotically free for small λ, which is what we expect based on our experience with σ models into spaces with positive curvature. However, we also see that

$$\lambda_* = 8\pi/k \tag{2298}$$

constitutes a fixed point of the RG flow at 1-loop, which is made possible only by the presence of the wzw term. Such a fixed point had to occur at this value of λ for $k = 1$, since it is precisely at this value of λ that the $k = 1$ theory admits a bosonization duality to a theory of N flavors of free fermions which, being free, does not flow under RG. Thus the presence of the wzw term is essential for making the bosonization work.

Also note that the fixed point λ_* is at weak coupling in the large k limit. Therefore as $k \to \infty$ the RG distance between the free point and the nontrivial CFT goes to zero, meaning that the 1-loop expansion becomes exact in the $k \to \infty$ limit. This means that an expansion in $1/k$ is similar to an ε expansion (which is a bit hard to conceive of doing here due to the WZW term).

While we've been in two dimensions, the generalization to the theory of a vector field in S^{d+1} in d dimensions is straightforward.[239] For example, in $d = 1$ the kinetic term is irrelevant and we just get the action for a spin $k/2$ (after remembering to adjust the normalization of the WZW term as $2\pi ik/A(S^{d+1})$), while in $d = 0$ we write $\omega = (\cos\theta, \sin\theta)$ to get $S = ik\int du\,(\cos\theta\partial_u\sin\theta - \sin\theta\partial_u\cos\theta) = ik\Theta$, where we have taken "spacetime" to be an S^0 with $\theta = 0$ on one component of the S^0 and $\theta = \Theta$ on the other (this describes the action of instantons in theories with topological terms).

[239] $\pi_{d+1}(S^{d+1}) = \mathbb{Z}$ means we always have a WZW term.

Asymptotic symmetry in an RG flow

This is from P&S, chapter 12. Consider two massless copies of ϕ^4 theory interacting quadratically:

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 + (\partial\theta)^2) - \frac{\lambda}{4!}(\phi^2 + \theta^2) - \frac{2\rho}{4!}\phi^2\theta^2. \tag{2299}$$

Compute the β functions for λ, ρ, and ρ/λ. Show that $\beta_{\rho/\lambda}$ has fixed points at $0, 1, 3$, and that the one at $\rho/\lambda = 1$ is stable, so that if e.g. we start with $\rho/\lambda < 3$ then we flow to a phase where $\rho = \lambda$, which is characterized by the model obtaining an $O(2)$ global symmetry in the IR. Note that this problem is a special case of that of computing the RG flow for a cubically-anisotropic $O(N)$ model which appears in another diary entry.

We just want the β functions to leading order in the couplings. Let us find the counterterm for ρ to one-loop order by examining the one-loop corrections to a vertex with two external ϕ lines and two external θ lines. There are six diagrams, counting the original ρ term and the δ_ρ counterterm. The four other diagrams are s and t channel $2\phi \to 2\theta$ diagrams, and two u channel diagrams (warning, I may have gotten mixed up with s and u), which differ by the type of field (θ or ϕ) which propagate in the loop. The basic integral is

$$\text{some typical diagram} = -C \int_q \int_x \frac{1}{(q^2 - \Delta)^2}, \tag{2300}$$

where x is a Feynman parameter, C is a constant that depends on the type of diagram, and Δ is some function of the square of the external momenta (e.g. the s, t, u channels $(p_1 + p_2)^2, (p_1 + p_4)^2$, etc.) and x. In dim reg, this gives

$$\text{some typical diagram} = -\frac{Ci}{4\pi^2} \int_x \left(\frac{2}{\epsilon} - \ln(\Delta/\Lambda) + \dots \right), \tag{2301}$$

with Λ the UV cutoff. Basically all we then need to do is compute the different Cs for the different diagrams.

The s channel diagram has two ρ vertices. Each vertex contributes a factor of $-i(2\rho/4!) \cdot 2 \cdot 2 = -i\rho/3$, where the extra factors of 2 are the symmetry factor. Thus the 1-loop corrections for the s and t channels are (just writing the ln terms and putting p_1, p_2 on the "bottom" of the diagrams and p_3, p_4 on the "top")

$$-(\rho/3)^2 \frac{i}{(4\pi)^2} \left(\ln \frac{-(p_1 + p_2)^2}{\Lambda^2} + \ln \frac{-(p_1 + p_4)^2}{\Lambda^2} \right). \tag{2302}$$

The u channel diagrams have one $-i(\lambda/4!) \cdot 4!$ vertex and one $-i\rho/3$ vertex, plus a symmetry factor of $1/2$ since the two internal lines are identical. These thus add and produce

$$-(\rho/3)\lambda \frac{i}{(4\pi)^2} \ln \frac{-(p_2 + p_4)^2}{\Lambda^2}. \tag{2303}$$

The ρ counterterm enters into the diagrams as $-i\delta_\rho/3$, since the original ρ interaction appeared with a 1/3 prefactor. We will fix our renormalization conditions so that the effective interaction between two θs and two ϕs is $\rho/3$ for the choice of momenta where $s = t = u = -M^2$ (the RG conditions are imposed at spacelike momenta as usual). Thus the renormalization-scale dependent part of the counterterm needs to be

$$\delta_\rho \sim \frac{1}{16\pi^2}\left(\lambda\rho + 2\rho^2/3\right)\ln(M^2/\Lambda^2). \tag{2304}$$

This gives us the β function for ρ:

$$\beta_\rho = \frac{d\rho}{d\ln M} = \frac{1}{8\pi^2}\left(\lambda\rho + 2\rho^2/3\right)\ln(M^2/\Lambda^2). \tag{2305}$$

Now for β_λ, which we evaluate by focusing on the correction to graphs with four external ϕ lines. In addition to the bare λ term and the $-i\delta_\lambda$ counterterm, there are six one-loop graphs. There are s, t, u channel graphs for graphs with an internal ϕ loop that go as λ^2, and likewise there are s, t, u channel graphs where the internal loop is a θ. The former three have a factor of $\lambda^2/2$ where the 1/2 is the symmetry factor of the internal loop, and the latter three similarly have a factor of $(\rho/3)^2/2$. Thus the M-dependent part of the counterterm needed to reduce the full term to $-i\lambda$ at our RG scale is

$$\delta_\lambda \sim \frac{1}{16\pi^2}\left(3\lambda^2/2 + \rho^2/6\right)\ln(M^2/\Lambda^2), \tag{2306}$$

where we've remembered to multiply by 3 since each s, t, u channel result is the same. Thus β_λ is

$$\beta_\lambda = \frac{1}{8\pi^2}\left(3\lambda^2 + \rho^2/6\right). \tag{2307}$$

The β function for the ratio is

$$\begin{aligned}\beta_{\rho/\lambda} &= \frac{1}{\lambda}\left(\beta_\rho - \frac{\rho}{\lambda}\beta_\lambda\right)\\ &= \frac{1}{8\pi^2\lambda}\left(\lambda\rho + 2\rho^2/3 - 3\rho\lambda/2 - \rho^3\lambda/6\right).\end{aligned} \tag{2308}$$

Note that this has fixed points at $\rho = 0$, $\rho = \lambda$, and $\rho = 3\lambda$. The $\rho = \lambda$ fixed point has an "emergent" global $O(2)$ symmetry, so we would like to know whether this fixed point is attractive or not. Indeed it is: writing $\rho = 3\lambda(1-\eta)$ for small η, one gets

$$\beta_{\rho/\lambda} \approx \frac{3\lambda\eta}{8\pi^2}, \tag{2309}$$

so that ρ/λ gets smaller at long distances, approaching the $\rho/\lambda = 1$ fixed point. Likewise if $\rho = \eta\lambda$ then $\beta_{\rho/\lambda} \approx -\eta\lambda/(16\pi^2)$, so that ρ/λ increases as we flow to larger distances, again approaching the $\rho/\lambda = 1$ symmetric fixed point.

ϕ^4 theory coupled to fermions and a natural relation

This is from P&S. We consider a nlsm with symmetry group $O(2)$ coupled to fermions by using the nlsm field to create a varying chiral mass term for the fermions:

$$\mathcal{L} = \frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + i\bar{\psi}\partial\!\!\!/\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi, \tag{2310}$$

where $i = 1, 2$.

Find the classical value for the fermion mass, assuming SSB occurs. Then show that this receives corrections at one-loop order, but that these corrections are *finite*.

First, note that while the fermion mass term means that the full $O(2) \times U(1)$ symmetry is broken, the diagonal subgroup consisting of the rotations

$$\phi \mapsto R_\theta\phi, \qquad \psi \mapsto e^{-i\gamma^5\theta/2}\psi \tag{2311}$$

is still a symmetry of the theory (checking this is straightforward). Secondly, assume that SSB occurs for ϕ. Wolog we can let $\langle\phi\rangle = (v, 0)^T$, so that the classical value of the fermion mass becomes

$$m_f = gv. \tag{2312}$$

Now we want to find quantum corrections to this. This is most easily accomplished when we write e.g. $\phi = (v + \sigma, \pi)$, where $v = \mu/\sqrt{\lambda}$ is the classical vev of ϕ in the SSB state. σ is massive and π is massless, and the Feynman rules are derived from the relevant terms

$$\mathcal{L} \supset -ig\bar{\psi}\gamma^5\pi\psi - g\bar{\psi}\sigma\psi - \lambda v\pi^2\sigma. \tag{2313}$$

There are further interactions like $-\frac{1}{2}\lambda\sigma^2\pi^2$, but these won't play a role in this problem.

To examine the corrections to m_f, we need to consider three things: the counterterm for g (which affects the mass due to $m_f = gv$) and the corrections to the propagator which come from polarization bubble diagrams with ψ and either π or σ.

The polarization bubble with σ for a ψ fermion with momentum p is (this diagram is a matrix with spinor indices for the ψ spins on the ends of the bubble)

$$\begin{aligned}
\sigma \text{ bubble} &= (ig)^2 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{q\!\!\!/ + m_f}{q^2 - m_f^2} \\
&= -g^2 \int_{q,x} \frac{xp\!\!\!/ + m_f}{(q^2 - \Delta_\sigma)^2} \\
&= -g^2 \int_x (xp\!\!\!/ + m_f) \frac{i}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma\right)(1 + \epsilon \ln\sqrt{4\pi/\Delta_\sigma}),
\end{aligned} \tag{2314}$$

where $d = 4 - \epsilon$ and Δ_σ is built out of x, p, and m_σ. The diagram involving π is similarly

$$\begin{aligned}\pi \text{ bubble} &= -(ig)^2 \int_q \frac{\gamma^5}{(p-q)^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \\ &= +g^2 \int_{q,x} \frac{-x\not{p} + m_f}{(q^2 - \Delta_\pi)^2} \\ &= -g^2 \int_x (x\not{p} - m_f) \frac{i}{16\pi^2} \left(\frac{2}{\epsilon} - \gamma\right) (1 + \epsilon \ln \sqrt{4\pi/\Delta_\pi}),\end{aligned} \tag{2315}$$

where Δ_π is built out of x and p. Adding these together and sending $\epsilon \to 0$ we get

$$\sigma\,\text{bubble} + \pi\,\text{bubble} = -g^2 \int_x x\not{p}(\text{divergent stuff}) - g^2 m_f \int_x \frac{i}{16\pi^2} \ln(\Delta_\pi/\Delta_\sigma). \tag{2316}$$

The coefficient of the m_f term is finite, and so these diagrams lead to a finite correction to the fermion mass (the correction is non-zero since $\Delta_\sigma \neq \Delta_\pi$ due to σ being massive and π being massless). The field strength renormalization term needed because of the $x\not{p}$ term is infinite, but the mass correction is finite.

Now we need to look at the correction to the coupling g which also affects the renormalized fermion mass: if the correction to m_f is to be finite then the counterterm δ_g better be finite as well.

We will work in an RG scheme suggested in P&S, where the $\psi\psi\pi$ vertex receives no radiative corrections when the π particle carries away zero momentum. To work out δ_g, the counterterm needed to ensure that the vertex has no radiative corrections, we need to compute four diagrams (one-loop corrections to the $\psi\psi\pi$ vertex). Two of the diagrams have only one ψ propagator in the loop (the other two guys in the loop are a π and a σ) and are finite:

$$\text{two finite diagrams} \sim g^2(-\lambda v) \int_q \frac{1}{(p-q)^2} \frac{1}{(p-q)^2 - m_\sigma^2} \frac{1}{q^2 - m_f^2}(2m_f), \tag{2317}$$

which goes like $\int dq\, q^3 q^{-6}$ and contains no divergences (the $\not{q}$'s in the numerator cancel since they anticommute with γ^5).

The two diagrams with two fermion propagators in the loop go as $\int dq\, q^3 q^{-4}$ and do contain a $1/\epsilon$ divergence. The one with an internal π propagator is

$$\begin{aligned}\pi\psi\psi\,\text{loop} &= (ig)^3 i^3 \int_q \frac{1}{(p-q)^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \\ &= -g^3 \int_q \frac{1}{(p-q)^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2}.\end{aligned} \tag{2318}$$

On the other hand, the one with an internal σ propagator is

$$\begin{aligned}\sigma\psi\psi\,\text{loop} &= g^2 (ig) i^3 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2} \\ &= +g^3 \int_q \frac{1}{(p-q)^2 - m_\sigma^2} \frac{\not{q} + m_f}{q^2 - m_f^2} \gamma^5 \frac{\not{q} + m_f}{q^2 - m_f^2}.\end{aligned} \tag{2319}$$

These two diagrams would entirely cancel if not for the fact that the second diagram has an m_σ in one of the propagators. When we evaluate this in dimensional regularization however, the only dependence on m_σ of the second diagram comes from the $\epsilon \ln \sqrt{4\pi/\Delta_\sigma}$ term, which is not divergent. Thus all the divergent parts are independent of m_σ, and hence cancel between the two diagrams, with their sum looking something like $g^3 \int_x \ln(\Delta_\pi/\Delta_\sigma)$. So, recapitulating, the counterterm δ_g is finite, as are the polarization bubble contributions to the renormalization of m_f, so that m_f only receives finite corrections.

The ϵ expansion and β functions in the anisotropic $O(N)$ model ✓

Today we're doing an extended version of a problem suggested by Pufu for the attendees of one of the bootstrap schools. We will be considering the $O(N)$ model deformed with an anisotropy v which reduces the symmetry to the "cubic" symmetry group $\mathbb{Z}_2^N \rtimes S_N$, acting by $\phi_i \mapsto -\phi_i, \phi_i \mapsto \phi_{\rho(i)}, \rho \in S_N$. The action is

$$S = \int d^dx \left(\frac{1}{2} \nabla_\mu \boldsymbol{\phi} \cdot \nabla^\mu \boldsymbol{\phi} + \frac{1}{2} t_0 \boldsymbol{\phi} \cdot \boldsymbol{\phi} + \frac{1}{4!} \sum_{i,j=1}^{N} (u_0 + v_0 \delta_{ij}) \phi_i^2 \phi_j^2 \right). \tag{2320}$$

The numbers in the action aren't the ones given in Pufu's assignment, but I think these factors lead to the simplest manipulations.

Anyway, our goal is to study the RG properties of this model in $d = 4 - \epsilon$. We will perturb in ϵ, and let N be arbitrary. Pufu provides students with the 1-loop beta functions and asks them to analyze the RG flow. We will do this but will instead also calculate the 1-loop beta functions ourself: it's fun and this way we know that the factors we get in the beta functions are correct (the result Pufu writes down is probably correct, but one has to make some variable-re-definitions to get to his result that he doesn't mention, and which I couldn't quite figure out).

First let's define the dimensionless couplings that we will be doing RG with.[240] The dimension of ϕ_i is $(d-2)/2 = 1 - \epsilon/2$. This means that we can define a dimensionless renormalized mass through $t_0 = a^{-2} Z_t t$, where a^{-1} is the UV cutoff. Since $[\phi_i]^4 =$

[240] We'll be doing dim-reg instead of conformal perturbation theory a la Cardy since the latter is much messier combinatorially.

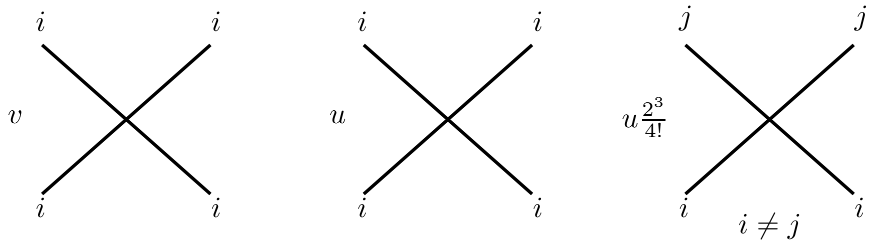
$4 - 2\epsilon$, the bare couplings for the quartic terms are determined through dimensionless couplings u, v via

$$u_0 = uZ_u a^{-\epsilon}, \qquad v_0 = vZ_v a^{-\epsilon} \tag{2321}$$

with $Z_u = 1 + \delta_u$ and likewise for Z_v (the dimensions work out since $(4 - 2\epsilon) + \epsilon = d$).

We will determine the beta functions by using the fact that the bare couplings are independent of the UV cutoff a^{-1}. Our mission is to find the appropriate counterterms δ_g, since they will carry cutoff-dependence. We will just go to 1-loop, and will work to quadratic order in all the couplings. N will be arbitrary.

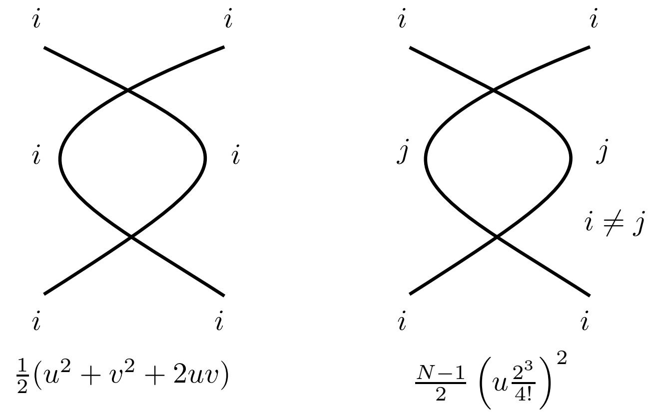
The basic 4-point vertices are



(2322)

The factor of $2^3/4! = 1/3$ comes from the symmetry factor of the diagram (2!2!) and the fact that $\phi_i^2\phi_j^2, i \neq j$ appears in the Lagrangian with coefficient $2u/4!$.

With all the outgoing legs labeled by the same index, the t-channel graphs are



(2323)

The s and u channel processes are the same. So all the 1-loop graphs with identical external legs give us the 4-point function

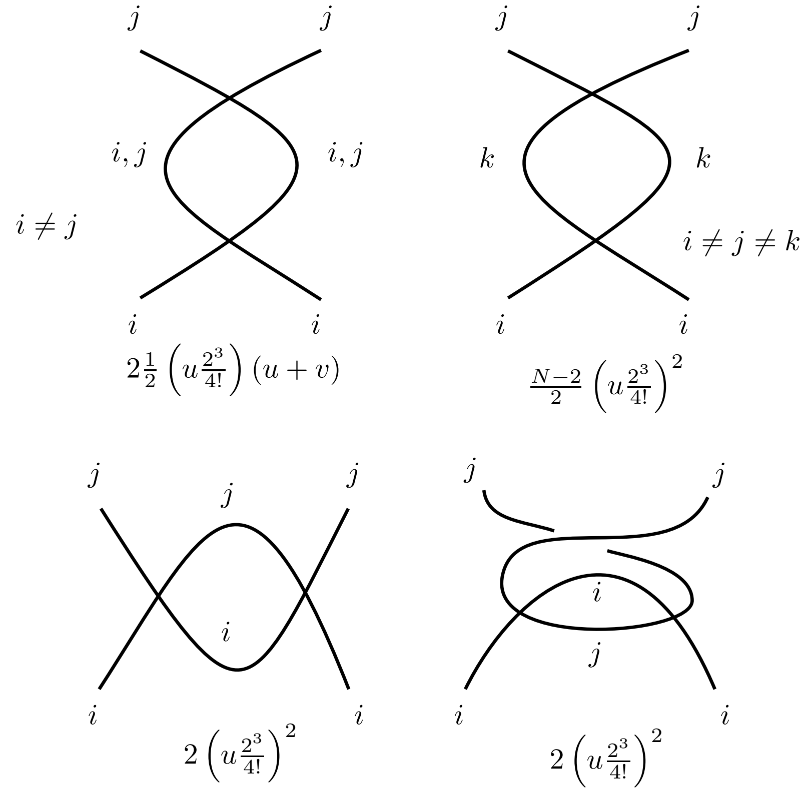
$$G^{(4)}_{iiii} = -i\left((1+\delta_v)v + (1+\delta_u)u + \frac{1}{2}\left(u^2 + v^2 + 2uv + \frac{N-1}{9}u^2\right)(V(s) + V(t) + V(u))\right)\prod_k \frac{i}{p_k^2}, \tag{2324}$$

where p_k are the external momenta and

$$V(p^2) = \int_k \frac{i^2}{k^2(k+p)^2}. \tag{2325}$$

Note that $V(p^2)$ does not include a 1/2 symmetry factor. Also note that there is no t appearing in the above propagator—we are going to be treating t as a vertex, i.e. as a (small) coupling in its own right. This is done wolog since summing up the geometric series for a straight propagator line with all possible t insertions recovers $i/(p^2 - t)$.

Anyway, the above formula for $G^{(4)}_{iiii}$ does not determine the counterterms δ_u, δ_v uniquely. To find them, we need to also renormalize the vertex with two i legs and two j legs, for $i \neq j$. The contributing diagrams are



(2326)

Adding these all up,

$$G^{(4)}_{iijj} = -i\left(\frac{1}{3}(1+\delta_u)u + \left(\frac{2}{2\cdot 3}(u^2+vu) + \frac{N-2}{2\cdot 3^2}\right)V(t) + \frac{1}{3^2}u^2(V(s)+V(t))\right)\prod_k \frac{i}{p_k^2}. \tag{2327}$$

Note the absence of the 1/2 symmetry factor in the last term, since the s and t channel processes don't have internal legs with the same index. As another example of how the counting works, look at e.g. the $(u^2 + vu)$ term: the 2 in the numerator comes from taking the internal loop to be either i or j, the 2 in the denomenator is a symmetry factor, and the 3 is $2^3/4!$.

Ensuring that the divergences in $G^{(4)}_{iijj}$ are canceled lets us determine the counterterm δ_u. Setting our renormalization conditions for momenta with $s = t = u = -M^2$, some algebra tells us that the divergences in $G^{(4)}_{iijj}$ are canceled, and that the 4-point function reduces to just $-iu/(\prod p^2)$ at the scale $-M^2$, provided that

$$\delta_u = -V(-M^2)\left(\frac{N+8}{6}u + v\right). \tag{2328}$$

We can now substitute this counterterm into the expression for the $G^{(4)}_{iiii}$ Greens function to solve for δ_v. Some algebra gives

$$\delta_v = -V(-M^2)\left(2u + \frac{3}{2}v\right), \tag{2329}$$

which ensures that $G^{(4)}_{iiii}$ reduces to $-i(u+v)/(\prod p^2)$ at the scale $-M^2$.

Calculating $V(-M^2)$ is standard. For simplicity we will set the RG conditions to be at zero momentum. Then the integral for $V(0)$ is logarithmically divergent: when we cut it off at the UV cutoff a^{-1}, the a-dependence is

$$V(0) \supset \frac{1}{16\pi^2}\ln(a^{-2}). \tag{2330}$$

We only care about the a dependence, since this is what gives the counterterms their a dependence, which is the thing that's needed to compute the beta functions.

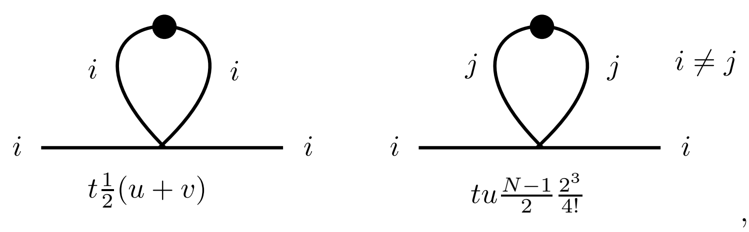
Now in pure $\lambda\phi^4/4!$ theory (i.e. if $N=1, v=0$), a standard calculation shows that the beta function is $\beta_\lambda = \epsilon\lambda - 3\lambda^2/(16\pi^2)$. Therefore it is helpful to re-define λ by absorbing the $3/16\pi^2$ factor. We will do the same thing, by defining the new variables

$$\mathsf{u} \equiv \frac{3}{16\pi^2}u, \qquad \mathsf{v} \equiv \frac{3}{16\pi^2}v. \tag{2331}$$

With these conventions, the a-dependent parts of the counterterms are, after some algebra, (note how no a^ϵs have been entering the expressions for the counterterms—both the δs and the couplings constants are dimensionless)

$$\delta_{\mathsf{v}} = \left(\frac{4}{3}\mathsf{u} + \mathsf{v}\right)\ln a, \qquad \delta_{\mathsf{u}} = \left(\frac{N+8}{9}\mathsf{u} + \frac{2}{3}\mathsf{v}\right)\ln a. \tag{2332}$$

Finally, we need the mass counterterm, working to quadratic order in all the couplings. Now the corrections to the propagator that go as u, v are zero in dimensional regularization, since they are $\sim \int_k k^{-2}$ and diverge as a power law. However, since we are treating the mass as an interaction vertex, we do get a logarithmic divergence from diagrams which go as tu, tv. There are only two such 1PI diagrams where the mass appears: they are[241]



, (2333)

where the dot is the mass insertion. The two-point function at momentum p then contains the terms

$$G^{(2)}_{ii} = -\frac{i^2}{p^2}t(1+\delta_t)\left(\frac{v}{2} + \frac{N+2}{6}u\right)\int_k \frac{i^2}{k^4}, \tag{2334}$$

[241] The δ_v and δ_u counterterms don't contribute to this order since their contributions would be third order in the couplings.

which is exactly the integral that we did above for the 1-loop calculations. Therefore the counterterm is determined as (going back to the u,v variables)

$$\delta_t = \left(\frac{1}{3}\mathsf{v} + \frac{N+2}{9}\mathsf{u}\right)\ln a. \tag{2335}$$

Now we can compute the beta functions, for example for u, as follows:

$$0 = \frac{d}{d\ln a}(a^{-\epsilon}Z_{\mathsf{u}}\mathsf{u}) \implies \beta_{\mathsf{u}} = \epsilon\mathsf{u} - \frac{\mathsf{u}}{Z_{\mathsf{u}}}\frac{d\delta_{\mathsf{u}}}{d\ln a}, \tag{2336}$$

and likewise for v. For t, the only change is in the first term:

$$\beta_t = 2t - \frac{t}{Z_t}\frac{d\delta_t}{d\ln a}. \tag{2337}$$

To quadratic order in the coupling constants we can take $1/Z_g \to 1$ in the above expressions for each coupling g, and therefore we derive[242]

$$\begin{aligned}
\beta_{\mathsf{u}} &= \epsilon\mathsf{u} - \frac{N+8}{9}\mathsf{u}^2 - \frac{2}{3}\mathsf{u}\mathsf{v},\\
\beta_{\mathsf{v}} &= \epsilon\mathsf{v} - \mathsf{v}^2 - \frac{4}{3}\mathsf{u}\mathsf{v},\\
\beta_t &= 2t - \frac{1}{3}t\mathsf{v} - \frac{N+2}{9}t\mathsf{u}.
\end{aligned} \tag{2338}$$

I found a reference for the above β functions in [7] (although the details of how they arrived at them aren't given), and amazingly, after converting to our normalization conventions, they agree with the above! What are the odds of that?! I can't tell you how amazed I was when I discovered this. Evidently we actually kept track of all the symmetry factors correctly—a small miracle.

Some preliminary things to notice about these beta functions: first, t only appears in the beta function for itself. This is general, and is simply because while n-valent vertices for $n > 2$ can combine to give larger-valence vertices or contract to give smaller valence vertices, 2-valent vertices can only combine to make more 2-valent vertices. This means that the mass flow won't affect where the interaction vertices flow.[243]

[242] Note that in this scheme (dimreg), all the fixed points have $t = 0$. If we were doing a different scheme, like conformal perturbation theory, there would be u^2 and v^2 terms in the beta function for t, since in conformal perturbation theory one works with the OPEs directly, which include contractions with more than 1 "loop" (in the $\mathbb{R}$ space Feynman diagrams). So, don't try to compare these β functions to the ones obtained in a different scheme; only the critical exponents are physical, and not the β functions, which are basis-dependent.

[243] This is part of a more general statement: a collection of dimensionless coupling constants g_β, with dimensionful coupling constants $g_{0\beta} = \Lambda^{[g_{0\beta}]}g_\beta$, will only appear in the beta function for a coupling g_α if

$$[g_{0\alpha}] \geq \sum_\beta [g_{0\beta}]. \tag{2339}$$

In particular, this means that marginal and irrelevant operators, for which $[g_{0\alpha}] < 0$, are never renormalized by relevant operators (for which $[g_{0\beta}] > 0$); hence the absence of t in the beta functions

Second, the beta function for v is independent of N, roughly because the v interaction is diagonal and identical to the one in $N = 1$ ϕ^4 theory. However, the beta function for u does depend on N, which tells us that the qualitative behavior of the RG will likely depend on how big N is. Also note that the coefficients of the uv terms in the beta functions for u and v aren't the same.

Now let's analyze the beta functions. First, we will determine the fixed points. This is easy enough to do, and we will do it with slightly more generality than we have to. Let us write the generalized beta functions as

$$\beta_{g_i} = d_i g_i - \sum_j \Gamma_{ij} g_i g_j, \tag{2340}$$

where the $\Gamma_{g_i g_j}$ are some numbers.[244] In the case of three couplings u, v, t where t doesn't appear in the beta functions for u, v (as above), we find four fixed points: all of them have $t_* = 0$, with the values of u and v being determined as

$$\begin{aligned}
&\mathcal{G}: \quad u_* = v_* = 0, \\
&\mathcal{U}: \quad u_* = \frac{d_u}{\Gamma_{uu}}, \quad v_* = 0, \\
&\mathcal{V}: \quad u_* = 0, \quad v_* = \frac{d_v}{\Gamma_{vv}}, \\
&\mathcal{M}: \quad u_* = \frac{d_v\Gamma_{uv} - d_u\Gamma_{vv}}{\Gamma_{uv}\Gamma_{vu} - \Gamma_{uu}\Gamma_{vv}}, \quad v_* = \frac{d_u\Gamma_{vu} - d_v\Gamma_{uu}}{\Gamma_{uv}\Gamma_{vu} - \Gamma_{uu}\Gamma_{vv}}.
\end{aligned} \tag{2341}$$

$\mathcal{G}$ is the Gaussian fixed point. In our model, $\mathcal{U}$ is the nontrivial $O(N)$-symmetric fixed point, $\mathcal{V}$ is an $O(N)$-breaking fixed point where the theory splits into a sum of N decoupled ϕ^4 theories, while at the mixed fixed point $\mathcal{M}$, $O(N)$ is broken but the theory is not diagonal.

To analyze each fixed point, we need the linearized beta functions, viz.

$$\bar{\beta}_{g_i} = d_i \bar{g}_i - \sum_j \Gamma_{ij}(g_{j*}\bar{g}_i + \bar{g}_j g_{i*}), \tag{2342}$$

where $\bar{g}_i \equiv g_i - g_{i*}$. Diagonalizing these equations determines the scaling variables at the appropriate fixed point.

At the Gaussian fixed point $\mathcal{G}$, we see that the (ir)relevance of the couplings are of course entirely determined by the d_g—for our $O(N)$ model this means that all of the couplings are relevant for $\epsilon > 0$, and so $\mathcal{G}$ is very unstable. At the $\mathcal{U}$ fixed point, we

for u, v. To prove this, note that since the counterterm δ_{g_α} is dimensionless, the counterterm will appear with in diagrams in the form $\delta_{g_\alpha}\Lambda^{[g_{0\alpha}]}$. On the other hand, a correction to the g_α interaction coming from operators with couplings g_β will have a Λ dependence of $\Lambda^{\sum_\beta [g_{0\beta}]+l}$, where $l > 0$ is a cutoff dependence coming from doing loop integrals. In order for the counterterm to cancel divergences coming from these operators, we then need $[g_{0\alpha}] = \sum_\beta [g_{0\beta}] + l$, proving the claim.

[244] This is done wolog since Γ_{kj} with $j, k \neq i$ will never appear in β_{g_i} at quadratic order—this is just because all terms in the expression for β_{g_i} must contain a g_i, as we saw in the derivation above.

have

$$\begin{aligned}\bar{\beta}_t^{\mathcal{U}} &= \left(2 - \frac{\Gamma_{ut}}{\Gamma_{uu}} d_u\right)\bar{t}, \\ \bar{\beta}_u^{\mathcal{U}} &= -d_u\bar{u} - d_u\frac{\Gamma_{uv}}{\Gamma_{uu}}\bar{v}, \\ \bar{\beta}_v^{\mathcal{U}} &= \left(d_v - \frac{\Gamma_{vu}}{\Gamma_{uu}} d_u\right)\bar{v},\end{aligned} \tag{2343}$$

and similarly for the $\mathcal{V}$ fixed point. At the $\mathcal{M}$ fixed point, well... I won't write it out, since we are going to be specializing back to the $O(N)$ model now.

For the $O(N)$ model, the fixed points are

$$\begin{aligned}\mathcal{G}: &\quad \mathsf{u}_* = \mathsf{v}_* = 0, \\ \mathcal{U}: &\quad \mathsf{u}_* = \frac{9\epsilon}{8+N}, \quad \mathsf{v}_* = 0, \\ \mathcal{V}: &\quad \mathsf{u}_* = 0, \quad \mathsf{v}_* = \epsilon, \\ \mathcal{M}: &\quad \mathsf{u}_* = \frac{3\epsilon}{N}, \quad \mathsf{v}_* = \frac{\epsilon(N-4)}{N}.\end{aligned} \tag{2344}$$

We are working with arbitrary N, but consider for a moment taking $N \to \infty$. We see then that the $\mathcal{U}$ fixed point merges with $\mathcal{G}$, while $\mathcal{V}$ merges with $\mathcal{M}$. Therefore the theory in the large N limit behaves as the decoupled sum of N copies of the ϕ^4 theory, up to $1/N$ corrections: this is exactly what we expect from the usual large N story, where in the $N \to \infty$ limit the different vector components decouple from one another.

Now we need to examine the stability of the different fixed points for different N. Notice that when $N < 4$, the $\mathcal{M}$ fixed point has $\mathsf{v}_* < 0$—therefore we should look at the stability of the model to make sure that this is okay. While either v, u can be negative, the potential must still be bounded from below. To find the region of stability for the potential, we look at its derivative wrt ϕ_k, where $\phi_k^2 \geq \phi_i^2\,\forall\, i$. Requiring that this be positive for positive ϕ_k means that

$$\mathsf{v}\phi_k^2 + \mathsf{u}\sum_j \phi_j^2 > 0 \tag{2345}$$

in the limit $\phi_k \to \infty$. If $\mathsf{v} < 0$, then we simply need $\mathsf{u} > -\mathsf{v}$. If $\mathsf{u} < 0$, then having $\mathsf{v} > -\mathsf{u}$ isn't enough: the strongest constraint comes from when all the ϕ_i^2 are equal, and tells us that in fact $\mathsf{v} > -N\mathsf{u}$. These conditions define the region of stability for our model.

Anyway, we see that $\mathsf{v}_* = \epsilon(N-4)/N$ is allowed when $\mathsf{u}_* = 3\epsilon/N$ for all $N > 1$, so that this fixed point is indeed always within the range of stability. Since the sign of v_* changes at $N = 4$, we expect that $N = 4$ might be a critical value across which the nature of the RG flow changes. Indeed, this suspicion is confirmed when we notice that at $N = 4$, the $\mathcal{U}$ and $\mathcal{M}$ fixed points become degenerate. To see exactly what happens, we linearize the beta functions, obtaining for $\mathcal{U}$ (omitting those for t since it's always relevant and hence not so interesting)

$$\bar{\beta}_{\mathsf{u}}^{\mathcal{U}} = -\epsilon\left(\bar{\mathsf{u}} + \frac{6}{N+8}\bar{\mathsf{v}}\right), \qquad \bar{\beta}_{\mathsf{v}}^{\mathcal{U}} = \epsilon\frac{N-4}{N+8}\bar{\mathsf{v}}, \tag{2346}$$

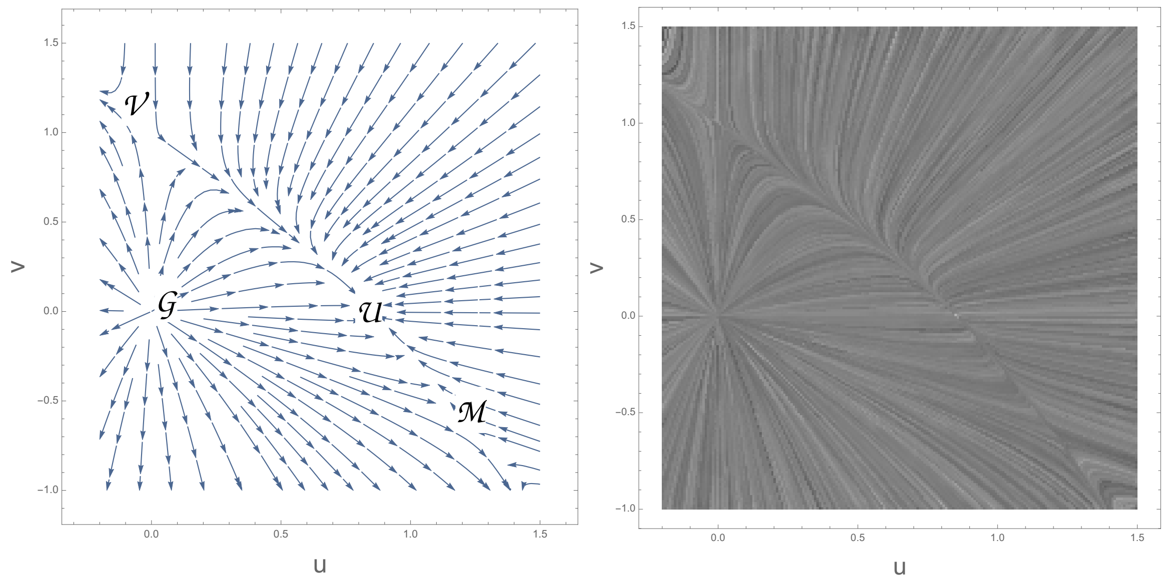


Figure 12: Left: RG flow for $N = 2, \epsilon = 1$. Right: same thing, but shown with a pretty convolution plot instead.

for $\mathcal{V}$

$$\bar{\beta}^{\mathcal{V}}_{\mathsf{u}} = \frac{\epsilon}{3}\bar{\mathsf{u}}, \qquad \bar{\beta}^{\mathcal{V}}_{\mathsf{v}} = -\epsilon\left(\bar{\mathsf{v}} + \frac{4}{3}\bar{\mathsf{u}}\right), \tag{2347}$$

and for $\mathcal{M}$,

$$\bar{\beta}^{\mathcal{M}}_{\mathsf{u}} = -\frac{\epsilon}{3}\bar{\mathsf{u}}(1+8/N), \qquad \bar{\beta}^{\mathcal{M}}_{\mathsf{v}} = \epsilon\left(\frac{4}{N} - 1\right)\left(\bar{\mathsf{v}} - \frac{4}{3}\bar{\mathsf{u}}\right). \tag{2348}$$

We could now proceed by diagonalizing these equations to get the scaling variables at each fixed point, but instead we'll just turn to pictures to better visualize things. However, we can at least see from the equations that the $\mathcal{V}$ fixed point is always unstable wrt adding u, and that the stability of the other two fixed points depends on whether N is bigger or less than 4, which in the 1-loop approximation is the critical N for which the behavior changes qualitatively.

For $N < 4$, the $\mathcal{U}$ point is the stable IR fixed point. The $\mathcal{M}$ point is located at negative v, and positive-v deviations away from it flow into $\mathcal{U}$. In figure 12, we show an example of the flow for $N = 2, \epsilon = 1$.

On the other hand, when $N > 4$, the $\mathcal{M}$ fixed point moves to positive v, u, and usurps the $\mathcal{U}$ point as the IR fixed point, with small positive v perturbations away from $\mathcal{U}$ now leading to $\mathcal{M}$. As an example, the flow with $N = 7$ is shown in figure 13.

The scaling dimensions of the various operators are easily calculated from the linearized β functions: for posterity's sake, they are

$$\begin{aligned} &\Delta^{\mathcal{G}}_{\mathsf{u}} = 2, \quad \Delta^{\mathcal{G}}_{\mathsf{v}} = \Delta^{\mathcal{G}}_{t} = 1 \\ &\Delta^{\mathcal{U}}_{\mathsf{u}} = -1, \quad \Delta^{\mathcal{U}}_{\mathsf{v}} = \frac{N-4}{N+8}, \quad \Delta^{\mathcal{U}}_{t} = \frac{N+14}{N+8}, \\ &\Delta^{\mathcal{V}}_{\mathsf{u}} = 1/3, \quad \Delta^{\mathcal{V}}_{\mathsf{v}} = -1, \quad \Delta^{\mathcal{V}}_{t} = 5/3, \end{aligned} \tag{2349}$$

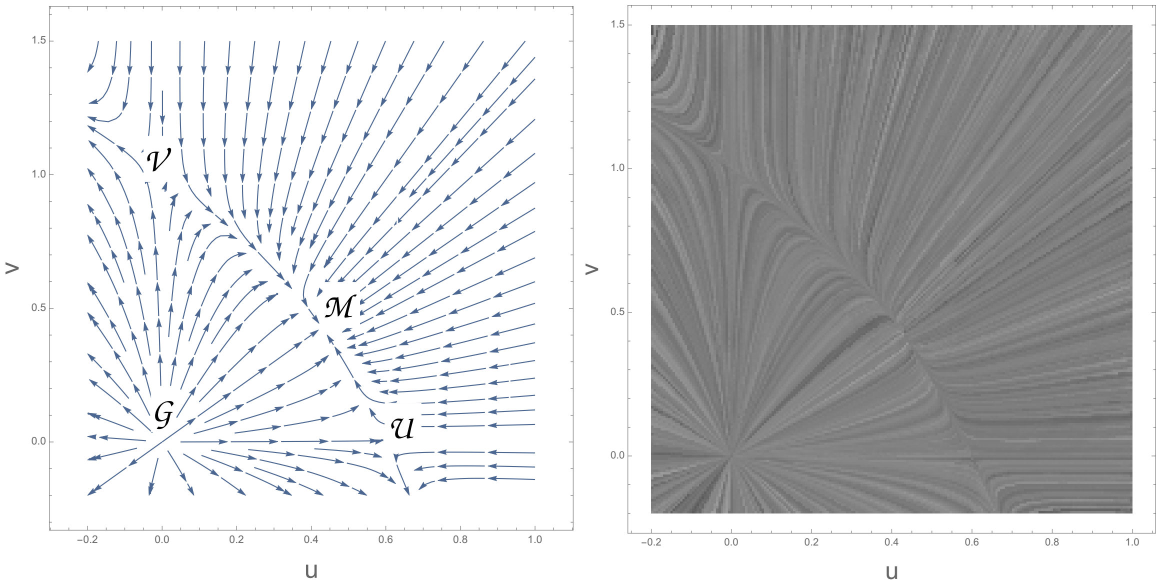


Figure 13: Same thing as the last figure, but now with $N = 7, \epsilon = 1$.

while the dimensions about $\mathcal{M}$ are complicated and I don't want to write them down.

Anyway, we see that for small $N < 4$, the IR fixed point is $O(N)$ symmetric, despite the fact that the microscopic UV Lagrangian contains an interaction which breaks the $O(N)$ symmetry explicitly. This is therefore an example of an emergent symmetry. In line with our intuition from large N, this no longer occurs when N is made sufficiently large ("fluctuations become too weak at large N to ensure emergent symmetry").

Another thing to note is that if $N > 4$ and the flow starts with either $\mathsf{u}, \mathsf{v} < 0$, or if $N < 4$ and the flow starts with either $\mathsf{u} < 0$ or with v below the line connecting $\mathcal{G}$ to $\mathcal{M}$, then the flow takes us to negative values of u, v that lie outside of the region of stability for the model. When we leave the stability region this means that a previously neglected ϕ^6 term needs to be kept, and that a first-order transition occurs due to the shape of the potential: this then provides us with an example of a "fluctuation-induced" first order transition.

Scale invariance + dimensional transmutation in quantum mechanics

Today we are mentioning a couple brief examples where dependence of various quantities on regulators occurs in scale-invariant quantum-mechanical problems. The point here is to illustrate that this is not just a phenomenon germane to QFT.

Consider a Schrodinger equation of the form

$$\left(-\nabla^2 - \frac{\alpha}{r^2}\right)\psi = i\partial_t\psi. \tag{2350}$$

This equation is left invariant under the re-scaling $r \mapsto \gamma r$, $t \mapsto \gamma^2 t$. Note that this scale invariance depends on the potential scaling as $1/r^2$, so that α is dimensionless. The scaling is generated by the operator $\mathcal{O} = \mathbf{r}\cdot\nabla + 2t\partial_t$, which does an anisotropic dilitaion in spacetime. The symmetry also gives rise to a conserved quantity, $\rho \equiv -\frac{1}{2}\mathbf{r}\cdot\nabla + t\partial_t$, since

$$[\rho, Ht] = 0, \tag{2351}$$

meaning that ρ commutes with the time evolution operator e^{-iHt} and hence is conserved.

Scale invariance here manifests itself in the fact that the S-matrix is independent of E. Indeed, this can already be seen in the WKB approximation, where the phase shift in a wavepacket passing through the potential relative to the shift in the absence of the potential is

$$\delta = 2\int_0^\infty dr\,\left(\sqrt{E+\alpha/r^2} - \sqrt{E}\right) = 2\int_0^\infty dr\,\left(\sqrt{1+\alpha/r^2} - 1\right) \stackrel{\sim}{\to} 2\alpha\sinh^{-1}(\alpha/a), \tag{2352}$$

where a is a (dimensionless, since we changed variables in the second step to a dimensionless r) short-distance cutoff. This is indeed independent of E. Furthermore, it is divergent and depends on a scale used to regulate the theory in much the same way as in examples of "dimensional transmutation" in QFT.

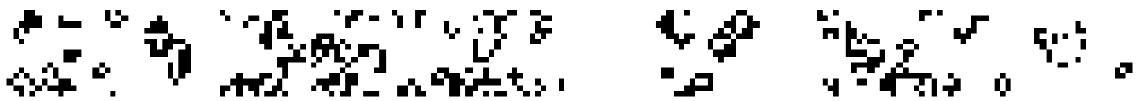
In two dimensions we have another option for a scale-invariant problem, since we can match the dimension of $-\nabla^2$ using a δ function potential:

$$\left(-\nabla^2 - \alpha\delta(\mathbf{r})\right)\psi = i\partial_t\psi. \tag{2353}$$

Here the binding energy of the bound state depends on a regulator in the same way that e.g. the BCS gap or the mass in $1+1D$ large N σ models does: we have

$$k^2\psi_k - \alpha\psi(0) = E\psi_k \implies \frac{1}{\alpha} = \int d^2k\frac{1}{k^2-E} = \pi\ln(\Lambda/E) \implies E = \Lambda e^{-1/\pi\alpha}, \tag{2354}$$

where Λ is our UV cutoff. Thus even in quantum mechanics, divergences can manifest themselves in ways similar to the ones in QFT.



Fixed points for QCD β function

Today we're taking a very simple look at the BZ fixed point. This was suggested as an exercise at one of the bootstrap schools; I found the problem statement written online (by Komargodski).

First, some trivial comments on why $\beta(\alpha_*) = 0$ means that α_* is an RG fixed point in a renormalizable QFT. For $\beta = \beta(\alpha, t)$ a general differential equation in terms of α and the RG time t, points where $\beta = 0$ are not in general fixed points. For example, suppose that $\beta = 2(t-1)$, so that $\alpha(t) = (t-1)^2$. Then $\beta = 0$ when $\alpha = 0$, but this is not a fixed point, since $\alpha''(t) \neq 0$ at $t = 1$. The reason why zeros of the β function are fixed points in practice is that the β function is always an autonomous DE—it never depends explicitly on the RG time (the RG flow only depends on the choice of coupling constants, not on what path was taken to get to them). Then since $\beta = \beta(\alpha)$, the root α_* with $\beta(\alpha_*) = 0$ has $\alpha(t) = \alpha_*$ as a constant solution; if β also depended on t then α_* would depend on t, and wouldn't be a fixed point. Note that the fact that we can take β to be autonomous isn't a priori obvious (and is only true up to terms that go to zero in Λ^{-1}), and is responsible for a lot of the utility of the RG.

Anyway, the beta function for non-Abelian gauge theory coupled to n fermions in a representation R is, to 2-loop, (in cond-mat conventions where $\beta > 0$ indicates that a coupling is relevant)

$$\beta(\alpha) = \frac{d\alpha}{dt} = 2(\beta_0 + \beta_1/\alpha^2), \tag{2355}$$

where $t = -\ln \Lambda/\Lambda_0$ is the RG time and $\alpha = 16\pi^2/g^2$. The first term is

$$\beta_0 = 4n\frac{T_2(R)}{3} - \frac{11}{3}C_2(G). \tag{2356}$$

The first order correction term is

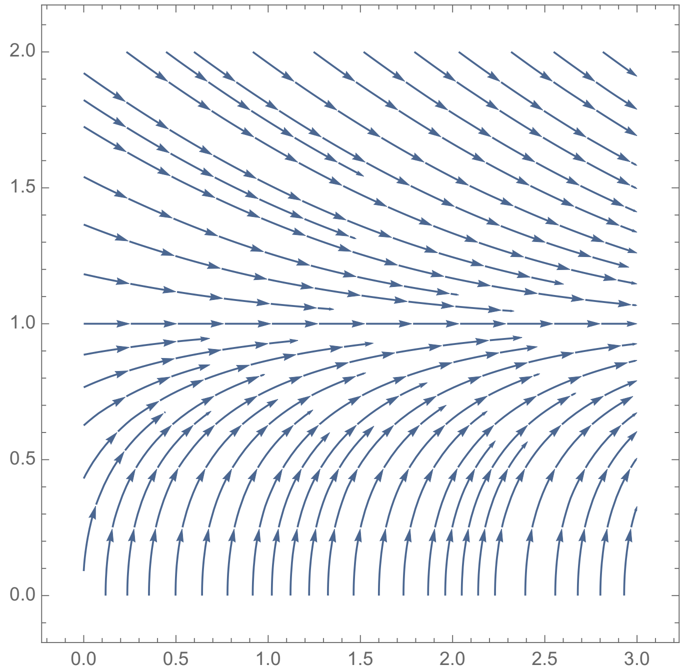
$$\beta_1 = \left(\frac{20}{3}C_2(G) + 4C_2(R)\right) nT_2(R) - \frac{34}{3}(C_2(G))^2. \tag{2357}$$

We will mainly be interested in having R be either then fundamental / anti-fundamental, or the adjoint. Note that $T_2(Ad) = C_2(G) > T_2(F)$, so that adjoint fermions push the theory closer to being IR-free (just because more fermion flavors mean more screening and hence less strongly coupled IR physics, and replacing fundamental fermions with adjoint fermions is roughly the same as increasing the number of flavors).

First take $\beta_0 < 0$. This is where the IR fixed point is stable. Mathematically, this is seen by nothing that for the fixed point to exist, $\beta_0 < 0 \implies \beta_1 > 0$. Then we see that

$$\frac{\partial\beta(\alpha)}{\partial\alpha} = -2\beta_1/\alpha^2 < 0. \tag{2358}$$

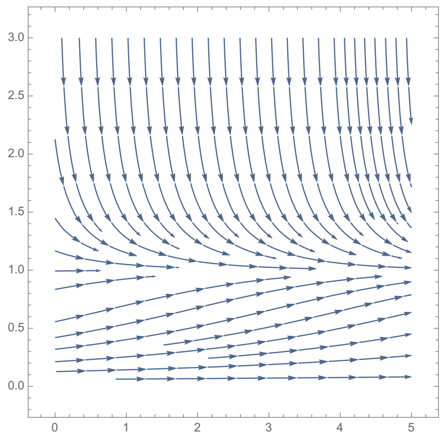
The fact that the derivative of the beta function is negative at the fixed point guarantees IR stability. Anyway, we can solve for $\alpha(t)$ explicitly in terms of a productlog function, but just plotting the β function is more illuminating: for e.g. $\beta_0 = -1/2, \beta_1 = 1/2$, (this is different from a usual RG flow diagram involving two different couplings—here the y axis is α and the x axis is RG time. This is redundant since the flow of α doesn't depend on RG time, but the plot looks nice so what the hell[245]



. (2360)

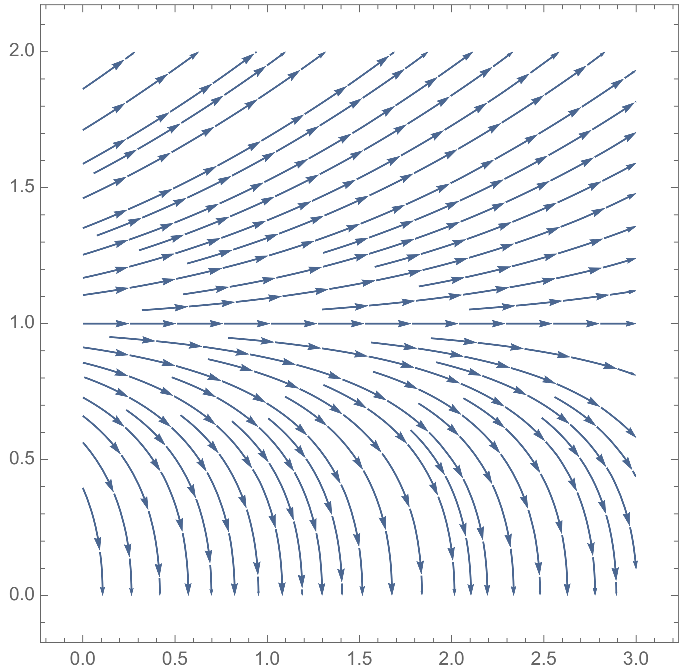
Now suppose $\beta_0 > 0$—this is where the fixed point is IR unstable, since then we have

[245]If you prefer to have a plot of g^2 instead of α, here it is, in mini-form:



(2359)

$\partial_\alpha \beta(\alpha) > 0$ at the fixed point. Indeed, the RG flow is (for $\beta_0 = 1/2, \beta_1 = -1/2$)



(2361)

Now we will focus on $G = SU(N)$, for which $C_2(G) = C_2(Ad) = N$. The beta function for $R = F$ the fundamental representation is

$$\beta_0^F = \frac{1}{3}\left(2n - 11N\right), \qquad \beta_1^F = -\frac{34}{3}N^2 + N\frac{13n}{3} - \frac{n}{N}, \tag{2362}$$

while for the adjoint,

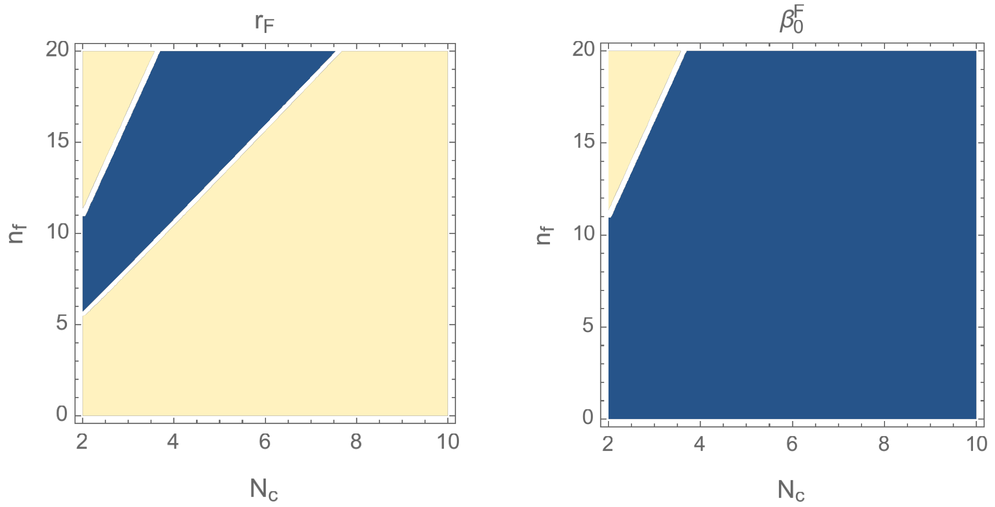
$$\beta_0^{Ad} = \frac{1}{3}(4Nn - 11N), \qquad \beta_1^{Ad} = N^2\left(-\frac{34}{3} + 26n\right) - 2n. \tag{2363}$$

To have a fixed point, we need $\beta_1/\beta_0 < 0$. Therefore we define the two ratios

$$r_F \equiv (\beta_1/\beta_0)_F = \frac{-34N^2 + 13nN - 3n/N}{2n - 11N}, \qquad r_{Ad} \equiv (\beta_1/\beta_0)_{Ad} = \frac{(26n - 34)N - 6n/N}{4n - 11}. \tag{2364}$$

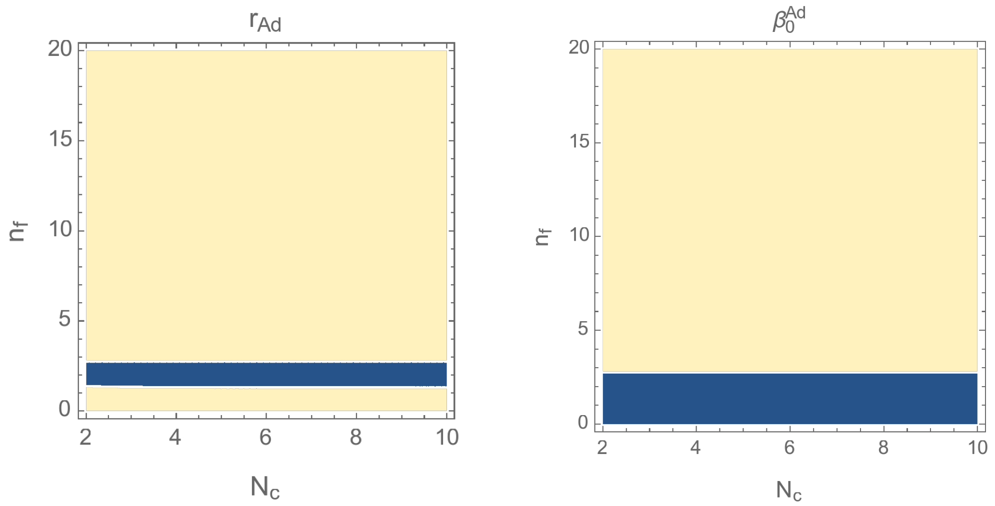
We can plot the sign of the above ratios to figure out when a fixed point will occur. The rs will change sign when either β_0 or β_1 passes through zero. For the fundamental representation, we have (here the plots are just the signs of the relevant quantities;

negative is blue and positive is cream)



$$(2365)$$

For the adjoint,



$$(2366)$$

Let N be fixed, let n_{*0}^R be the number of flavors at which $\beta_0^R = 0$ (this is when the theory passes from conformal to deconfined), and let n_{*1}^R be the number of flavors at which $\beta_1^R = 0$ (this is when the theory passes from being confined to being conformal). Then some algebra gives

$$n_{*0}^F = \frac{11N}{2}, \qquad n_{*1}^F = \frac{34N}{13 - 3/N^2} \tag{2367}$$

for the fundamental and

$$n_{0*}^{Ad} = \frac{11}{4}, \qquad n_{*1}^{Ad} = \frac{34}{78 - 6/N^2} \tag{2368}$$

for the adjoint. A sanity check is that $n_{0*}^R > n_{1*}^R$, since we know that when $n^R = 0$ we should have confinement.

Finally some comments on these two critical fermion numbers. The point n^R_{*0} is the exact location of the upper boundary of the conformal window, i.e. its location doesn't depend on our ignorance of the higher order terms in the expansion of β. To see this, consider n bigger than n^R_{*0}, so that $\beta_0 > 0$. Then if we start the flow at $1/\alpha$ infinitesimal, only β_0 contributes to the flow, and since here $\beta_0 > 0$, $1/\alpha$ is driven back to zero—the higher-order terms have no chance to contribute. On the other hand, suppose n is very slightly less than n^R_{*0}. If we start from $g^2 = 0$ then $\beta_0 < 0$ tells us that g^2 initially increases. This is true regardless of what the higher-order terms in β are. Now the fixed point we identified is at $(\alpha^*)^{-1} = -\beta_0/\beta_1$, which can be made arbitrarily small by tuning n arbitrarily close to n^R_{*0}. Therefore we can reach the fixed point after an arbitrarily short RG flow, which means that the fixed point can be reached, at least if we are infinitesimally close to n^R_{*0}, before any of the other terms in the β function expansion have a chance to contribute. Therefore n^R_{*0} indeed marks a sharp boundary between a region with a trivial free IR fixed point and a nontrivial (weakly) interacting one. This is similar to the WF fixed point in the ϵ expansion—we don't need to know the full expression for the β function in order to guarantee the existence of a nontrivial IR fixed point, since by tuning ϵ we can make the WF fixed point arbitrarily close to the Gaussian one.

In contrast, n^R_{*1} is not a sharp lower boundary for the conformal window. This is where $\beta_1 \to 0$, and so our expression for the fixed point is $(\alpha^*)^{-1} \to \infty$. If we start from $\alpha^{-1} = 0$ then the flow takes a long time to get to α_*, and by the time we reach it, the higher-order terms in the β function will contribute, shifting the location of the fixed point. Hence, the lower bound of the conformal window cannot be reliably calculated in perturbation theory.

Fluctuation-induced first-order transitions in $U(1)$ gauge theory a-la HLM

Today we're going to work out some of the calculations implicit in the classic HLM paper on the SCing transition being first order once the fluctuations of the gauge field are properly taken into account. This is something I'd heard about but never actually looked at before today.

Type I SCs

The first part of the HLM paper talks about type I superconductors. Here the argument for a first-order transition is fairly straightforward. We start from the usual

$$H = \int \left(K|(\nabla - iq\mathbf{A})\psi|^2 + t|\psi|^2 + \frac{u}{2}|\psi|^4 + \frac{1}{4}F_{ij}F^{ij} \right). \tag{2369}$$

The point is that type I SCs order so well that they are basically described by MFT, and the effect of fluctuations on the magnitude of the order parameter $|\psi|$ is essentially negligible—using the Ginzburg criterion (another diary entry) one can establish the irrelevance (in the colloquial sense) of fluctuations in $|\psi|$ (and hence in ψ, since as usual the HLM paper is rather caviler about issues of gauge invariance) for all reduced temperatures t expect those absurdly close to 0.

Given that we can take $\psi = \psi_0$ to be constant (and $\mathbb{R}$ wolog), all of the business is in the integral over the gauge field $\mathbf{A}$. Since $\mathbf{A}$ appears qudratically the integral is easy:

$$F[\psi_0] = \int \left(t\psi_0^2 + \frac{u}{2}\psi_0^4 \right) + V\frac{3}{2}\int_0^\Lambda d^3\mathbf{q}\, \ln[|\mathbf{q}|^2 + m^2], \qquad m^2 \equiv Kq^2\psi_0^2, \tag{2370}$$

with $V = \int_{\mathbf{x}}$. Here the divvying up of the scaling dimensions in the mass is that $[\psi_0] = 0$ since we are in a regime where ψ doesn't fluctuate, and $[\sqrt{K}q] = 1$. Physically, $m = \xi_L^{-1}$, with ξ_L the London penetration depth.

Anyway, the integral gives

$$\int dq\, q^2 \ln[q^2 + m^2] = -\frac{2}{3}m^3 \arctan[q/m]\big|_0^\Lambda + \dots, \tag{2371}$$

where the $\dots$ contains a constant (diverging with Λ) and a Λm term that can be killed off with a mass counterterm for ψ. In the limit where the cutoff $\Lambda \gg m$ (here we are at t small enough that the UV cutoff length is small compared to m^{-1}, but t large enough so that ψ can be accurately treated as fluctuationless) we then get

$$F[\psi_0] = \int \left(t\psi_0^2 + \frac{u}{2}\psi_0^4 - \frac{1}{4\pi}K^{3/2}q^3\psi_0^3 \right). \tag{2372}$$

The fact that the integral over $\mathbf{A}$ produces a cubic term with negative coefficient is the reason why the transition is secretly first-order. The fact that we'll get a term that goes as an odd power of ψ_0 is rather inobvious at first, since $\mathbf{A}$ only couples to ψ_0^2 in H, which means that the effective action after integrating out $\mathbf{A}$ will only contain diagrams with an even number of ψ_0s (the expansion of the $\ln[q^2 + m^2]$ term only contains terms that go as $\psi_0^{2n}, n \in \mathbb{N}$). Therefore it was important to actually do the log integral exactly (and indeed, there would have been no reason to expand the $\ln[q^2 + m^2]$ and truncate after the first few terms; we are not working at weak coupling or anything).

Type II SCs

Type II SCs are more difficult, since fluctuations in $|\psi|^2$ can no longer be ignored, forcing us to do a more careful RG analysis.

We will be working in the framework of the ε expansion, as in the original HLM paper. However, the existence of particle-vortex duality means that the conclusion of a fluctuation-induced first-order transition is *not* true for $\varepsilon = 1$. I'm not sure whether or not the conclusion remains true for $0 < \varepsilon < 1$.

To start, one can either break up ϕ into $\mathbb{R}$ and $i\mathbb{R}$ parts or work with $\phi, \phi^\dagger$ as separate variables. The advantage of the latter approach is the economy of notation, while its disadvantage is that I could never quite figure out the correct symmetry factors to use in diagrams involving lots of ϕs. Because of these combinatorial failures of mine, we will break up the field ψ_i as

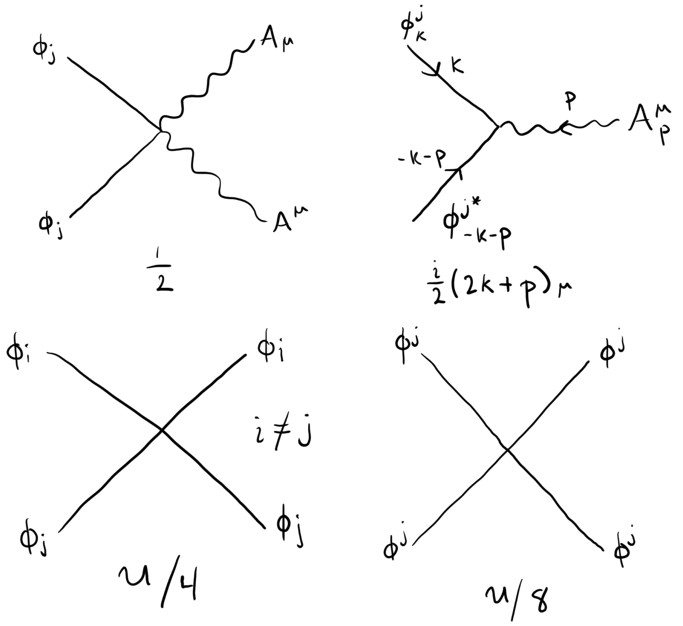
$$\psi_j = \frac{\phi_j + i\phi_{j^*}}{\sqrt{2}}, \tag{2373}$$

with ϕ_j, ϕ_{j^*} real fields and the $1/\sqrt{2}$ is there in order to get canonically normalized kinetic terms for the ϕs. We will adopt the (somewhat unusual) choice that $[\phi_j] = (d-2)/2 = 1 - \varepsilon/2$, but $[A_\mu] = 1$. This ensures that the coupling between A_μ and the bosons is exactly dimensionless, but it means that the kinetic term for A_μ, which in our conventions is where the gauge coupling g lives in the action, comes with an extra factor of $\Lambda^{-\varepsilon}$ in order that g^2 can remain dimensionless. The action is then, in $i\mathbb{R}$ time,

$$\begin{aligned} S = \int \Bigg(&\sum_{j=1}^{2n} \left[\frac{1}{2} d\phi_j \wedge \star d\phi_j + \frac{1}{2} A_\mu A^\mu \phi_j^2 + \frac{u}{8} \Lambda^\varepsilon \phi_j^4 \right] + \frac{u}{4} \Lambda^\varepsilon \sum_{i<j}^{2n} \phi_i^2 \phi_j^2 \\ &+ \sum_{j=1}^{n} \frac{1}{2} A_\mu (\phi_j \partial^\mu \phi_{j^*} + \phi_{j^*} \partial^\mu \phi_j) + \frac{\Lambda^{-\varepsilon}}{2g^2} F_A \wedge \star F_A \Bigg) \end{aligned} \tag{2374}$$

where u and g^2 are both properly dimensionless.

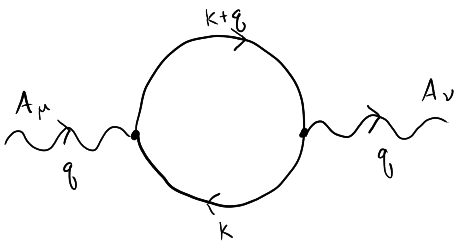
The Feynman rules are then (you need to actually do the Fourier tform to see how the $A\phi\phi$ vertex works)



(2375)

where we are not including symmetry factors in the displayed vertices. The momentum dependence of the $A\phi\phi$ vertex is assigned by taking all the momentum arrows to point inwards, and then taking the momentum of the ϕ_j minus the momentum of the ϕ_{j^*}.

The easiest beta function to get is the one for the gauge coupling. This is because the renormalization of g^2 is dictated by gauge invariance: $g^2 A_\mu A^\mu |\psi|^2$ must renormalize in the same way as $|\nabla\psi|^2$, meaning that g^2 is renormalized exactly in the opposite way as $A_\mu A^\mu$. This means the flow of g^2 is found to 1-loop just by computing the usual polarization bubble[246]:



$$= -\frac{ni^2}{4} A^\mu_q A^\nu_q \int_k \frac{(2k+q)_\mu (2k+q)_\nu}{k^2(k+q)^2}, \qquad (2376)$$

which is dealt with using Feynman parameters as usual; the manipulations to get this with momentum dependence of the form $\Pi^T_{\mu\nu}(q)$ are in e.g. P&S or an earlier diary entry. The minus sign on the RHS is unfortunately crucial to keep track of[247]: if we take the lower leg of the loop to be a ϕ_j line and the upper leg to be a ϕ_{j^*} line, the the left vertex contributes a factor of $-i(2k+q)$, while the right vertex contributes $+i(2k+q)$.

I find momentum shell RG to be conceptually the clearest scheme for dealing divergences; hence the ($\mathbb{R}$ time) divergent integrals over k that we will encounter be

[246] We don't need to consider the bubble coming from a single $A^2\phi^2$ vertex since that term is q-independent and just cancels against the q-independent part of the diagram below

[247] Although for this particular diagram the sign is fixed by unitarity, since it contributes to the A self energy.

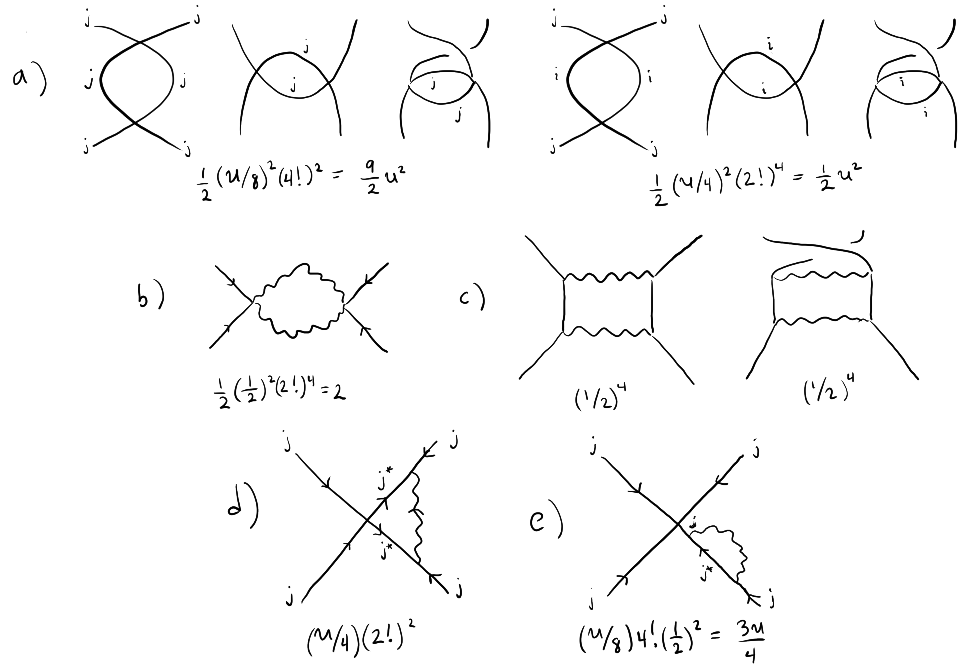


Figure 14: The symmetry factors and coefficients for the relevant diagrams for getting β_u; not shown are momenta and factors of $g^2\Lambda^\varepsilon$ from the gauge propagators.

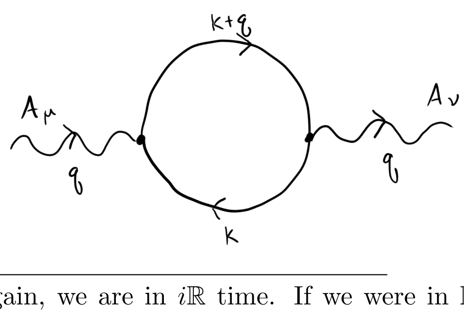
dealt with by[248]

$$\int_{\Lambda-d\Lambda<|k|<\Lambda} d^dk\,\frac{1}{k^4} = A(S^d)\int_{\Lambda-d\Lambda}^{\Lambda}\frac{dk}{(2\pi)^d}\,\frac{k^{d-1}}{k^4} = \frac{2\pi^{d/2}}{(2\pi)^d\Gamma(d/2)}\frac{\Lambda^{d-4}-(\Lambda-d\Lambda)^{d-4}}{d-4}$$
$$\equiv \Lambda^{-\varepsilon}\gamma(d)dt, \tag{2377}$$

where the RG time is $dt = d\ln\Lambda$ (here $d\Lambda > 0$ means that the cutoff is getting *smaller* in the way we've written stuff, so that indeed the sign is $dt = +d\ln\Lambda$), and we've defined

$$\gamma(d) \equiv 2\pi^{d/2}/((2\pi)^d\Gamma(d/2)). \tag{2378}$$

This means that the polarization bubble gives a contribution to the photon self-energy that looks like



$$\rightarrow -n\frac{\gamma(d)}{4\cdot 3}\Lambda^{-\varepsilon}dt\,A^\mu_q A^\nu_q \Pi^T_{\mu\nu}(q)q^2, \tag{2379}$$

[248] Again, we are in $i\mathbb{R}$ time. If we were in $\mathbb{R}$ time we'd need to do a Wick rotation $k^0_E = -ik^0_L$ (unfortunately the minus sign is important) in the first step.

where the internal scalars live at energies between $\Lambda - d\Lambda, \Lambda$ and the 1/3 comes from the Feynman parameter integrals that we've not written out.

The new gauge coupling is then found to 1-loop order by putting the contributions from these diagrams into the new effective action e^{-S} (remember that $[A_q] = [A(x)] + d$)

$$\begin{aligned} \int_0^\Lambda d^d q \, \frac{\Lambda^{-\varepsilon}}{g_{t+dt}^2} A_q^\mu A_q^\nu q^2 \Pi_{\mu\nu}^T(q) &= \int_0^{\Lambda - d\Lambda} d^d q \, \Lambda^{-\varepsilon} \frac{1}{g_t^2} \left(1 + g_t^2 n \gamma(d) dt/12\right) A_q^\mu A_q^\nu q^2 \Pi_{\mu\nu}^T(q) \\ &= \int_0^\Lambda d^d q \, \frac{\Lambda^{-\varepsilon}(1 - \varepsilon dt)}{g_t^2(1 - n g_t^2 \gamma(d) dt/12)} A_q^\mu A_q^\nu q^2 \Pi_{\mu\nu}^T(q) \\ &= \int_0^\Lambda d^d q \, \frac{\Lambda^{-\varepsilon}}{g_t^2(1 + [\varepsilon - n g_t^2 \gamma(d)/12] dt)} A_q^\mu A_q^\nu q^2 \Pi_{\mu\nu}^T(q) \end{aligned} \tag{2380}$$

From this we can read off the beta function, viz.

$$\beta_{g^2} = \frac{dg^2}{dt} = \varepsilon g^2 - \frac{n\gamma(d)}{12} g^4. \tag{2381}$$

Now when we expand in ε, we can use $\gamma(4-\varepsilon) = 1/8\pi^2 + O(\varepsilon)$ and drop the $O(\varepsilon)$ terms on the assumption that the fixed point we're looking for has $g^2 \sim O(\varepsilon)$ so that εg^4 terms are negligible in our approximation. Therefore

$$\beta_{g^2} = \varepsilon g^2 - \frac{n}{4 \cdot 24\pi^2} g^4. \tag{2382}$$

Or, in terms of the fine structure constant $\alpha \equiv g^2/4\pi$,

$$\beta_\alpha = \varepsilon \alpha - \frac{n}{4 \cdot 6\pi} \alpha^2, \tag{2383}$$

with the second term $n/4$ times what it is in QED (modulo the factor of 1/4, which is due to the $1/\sqrt{2}$ normalization in the decomposition of the complex field into its real constituent parts; the gauge coupling in bosonic QED would then usually be half of of the g^2 written here), just because we have n bosons that each act like 1/4 of a fermion when we are (close) to four dimensions.

Now we can work on β_u. Since we already saw how the manipulations for getting the effective coupling worked in the case of g^2 (finding the term generated by integrating out the high energy modes, putting it back into the action, shifting the cutoff, etc.) we will be more terse / schematic in what follows.

We will look at the renormalization of the four-point vertex where all the flavors on the outgoing legs are identical, wolog. First, consider the series of diagrams in row a) of Figure 14 (in the second trio, $i \neq j$). All of the s, t, u channels give the same result for each choice of flavors in the loop. We see that these sum to

$$\text{diagrams in row a)} = \left(\frac{3u^2}{2}(9 + (2n-1))\gamma(d)\Lambda^\varepsilon dt\right)\phi_j^4 \to \left(\Lambda^\varepsilon \frac{3u^2}{8\pi^2}(n+4)dt\right)\phi_j^4. \tag{2384}$$

Since the ϕ_j^4 vertex appears in diagrams with coefficient $(u/8)4! = 3u$, we see that the above series of diagrams makes a contribution to u_{t+dt}, and hence to β_u, of

$$\beta_u \supset -\frac{n+4}{8\pi^2} u^2, \tag{2385}$$

which matches with the result in the asymptotic freedom paper, a sign that our combinatorics have worked so far. As another confidence boost, this also matches the appropriate term in the beta function for the $O(2n)$ vector model, see e.g. Zinn-Justin page 649.[249]

The next diagram to tackle is b). This one is easy, since it's just the usual $\int k^{-4}$ integral:[250]

$$\text{b) diagrams } = \left(3 \cdot \frac{2}{2^2} g^4 \Lambda^\varepsilon \gamma(d) dt\right) \phi_j^4, \tag{2386}$$

where we have $1/2^2$ from the vertices, 3 from the associated crossed diagrams, and $(2!)^2/2$ from the photon propagators. This will give a contribution to the u beta function like

$$\beta_u \supset -\frac{1}{2 \cdot 8\pi^2} g^4. \tag{2387}$$

The other g^4 diagrams in c) are similar; they add up to give

$$\text{c) diagrams } = \frac{3}{16 \cdot 8\pi^2} g^4 \implies \beta_u \supset -\frac{1}{16 \cdot 8\pi^2} g^4, \tag{2388}$$

where the 3 comes from the crossed diagrams. *ethan: really? need to come back and be more vigilant about these pesky symmetry factors*

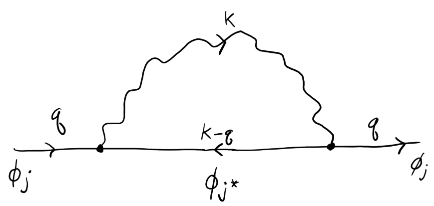
The last 1PI diagram is d). It evaluates to

$$\text{d) diagram } = -\left(6 \frac{ug^2}{4} \frac{1}{8\pi^2}\right) \phi_j^4, \tag{2389}$$

where the $6 = \binom{4}{2}$ is the number of ways of putting on the photon leg, and the crucial minus sign comes from $i^2 k_\mu (-k^\mu)(-1)^3 = -1$, where the first two factors are from the photon couplings and the $(-1)^3$ comes from the fact there are three vertices and that the action appears as e^{-S} in the path integral. This then gives a contribution

$$\beta_u \supset +\frac{ug^2}{2 \cdot 8\pi^2}. \tag{2390}$$

Lastly we need to do the scalar wavefunction renormalization (the diagrams in e). The scalar wavefunction renormalization is essentially the same as the photon wavefunction renormalization, since both polarization bubbles involve the same couplings (again, the $A^2\phi^2$ coupling is unimportant since it is momentum-independent and is killed by the momentum-independent part of the following diagram). Indeed, paying careful attention to signs,

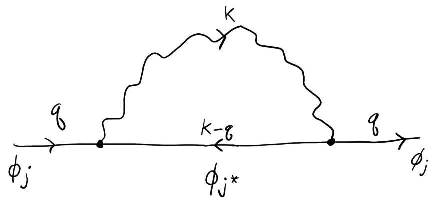


$$= +\frac{1}{4} g^2 \Lambda^\varepsilon \int \frac{(2q-k)^\mu (2q-k)_\mu}{(k-q)^2 k^2} \phi_{j;q}^2. \tag{2391}$$

[249] Just to check that this wasn't an accident, we can go through the combinatorics for the case where the vertex instead has two i legs and two j legs, with $i \neq j$. The diagrams give, working from left to right along the first trio of row a), a contribution to the new u vertex of in agreement with the beta function when all the outgoing legs are identical; this also serves as a sanity check of the obvious fact that if there is no $O(2n)$ anisotropy when we start the flow, no anisotropy will be generated.

[250] We'll be working in a gauge where the photon propagator is just $\delta_{\mu\nu}/k^2$.

The integral is the same as the one we did for the photons:



$$= -\frac{\gamma(d)g^2}{4\cdot 3}dt\,\phi_{j;q}^2 q^2 \tag{2392}$$

therefore the wavefunction renormalization happens by

$$\phi_{t+dt}^2 = \left(1+\frac{g^2\gamma(d)dt}{12}\right)\phi_t^2 \equiv Z_\phi \phi_t^2. \tag{2393}$$

Putting this all together and accounting for the εdt contribution from the $d^d q$ measure, we have

$$\frac{4!u_{t+dt}}{8}\phi_{j;t+dt}^4 = (1+\varepsilon dt)\phi_{j;t}^4\left(\frac{4!u_t}{8} - dt\left[\frac{3u_t^2}{8\pi^2}(n+4) - \frac{g^4}{8\pi^2}(3/2+3/16) + \frac{3g^2u}{2\cdot 8\pi^2}\right]\right) \tag{2394}$$

Plugging in for the wavefunction renormalization,

$$u_{t+dt} - u_t = \varepsilon u_t dt + \frac{dt}{8\pi^2}\left(-u_t^2(n+4) - g^4\frac{9}{16} + \frac{1}{3}g_t^2 u_t\right) \tag{2395}$$

Therefore the beta function is

$$\beta_u = \varepsilon u - \frac{1}{8\pi^2}\left(u^2(n+4) + \frac{9}{16}g^4 - \frac{1}{3}g^2 u\right). \tag{2396}$$

Note the crucial *positive* g^2u term on the RHS, which came from the diagram d (the scalar wavefunction renormalization also contributed to the g^2u with opposite sign[251], but was importantly smaller than the contribution from diagram d).

To present the beta functions together in a nice way, it is helpful to define

$$\lambda \equiv g^2/(4\cdot 8\pi^2), \qquad v \equiv u/8\pi^2. \tag{2397}$$

Then the beta functions are

$$\beta_\lambda = \varepsilon\lambda - \frac{n}{3}\lambda^2, \qquad \beta_v = \varepsilon v - (n+4)v^2 + 4v\lambda/3 - 9\lambda^2. \tag{2398}$$

The signs are all correct and β_λ and the first two terms in β_v are correct[252], but despite trying over and over and over to figure out where I went wrong with symmetry factors, the last two terms in β_v likely don't have the right numerical coefficients.

In any case, these algebraic quibbles don't have much bearing on the moral lessons to be taken away from this calculation. We would like to know when a second order

[251] Again, this sign is fixed by unitarity.

[252] Actually there seemed to be an annoying discrepancy of 1/2 between the $n+4$ terms in the beta functions quoted in the HLM paper and those from the asymptotic freedom paper—but comparison with e.g. the $O(2n)$ model means that the $n+4$ term in β_v is correct as written.

transition is possible. In the original HLM paper they were interested in three dimensions, so that $\varepsilon = 1$. To determine this we just solve the above β functions: besides the Gaussian and pure gauge fixed points there are two fixed points with $v \neq 0$; the one with smaller v is at

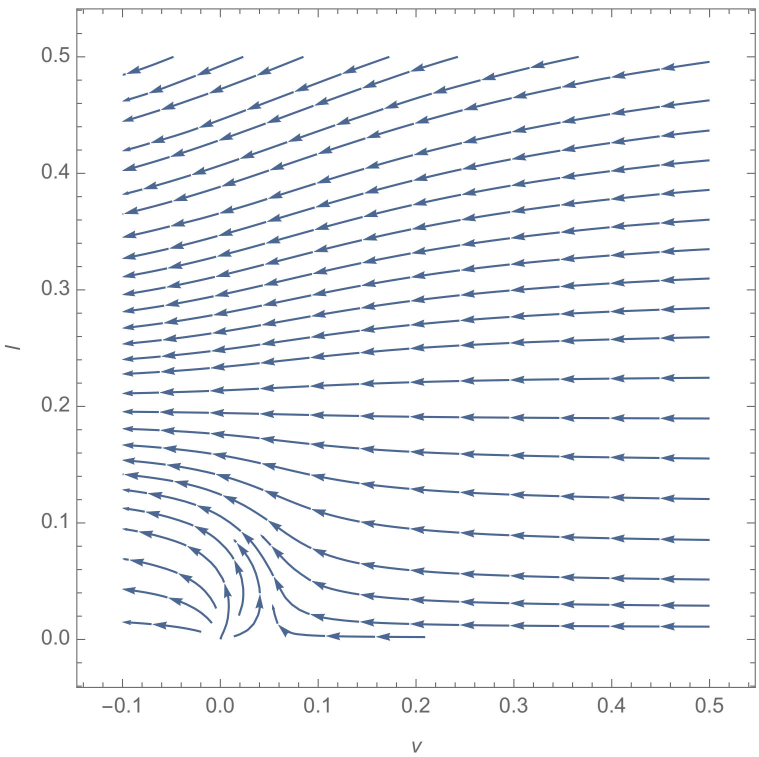
$$v_* = \frac{4n + n^2 - n\sqrt{n^2 - 316n - 1280}}{2(4n^2 + n^3)}, \tag{2399}$$

which of course does not exist until the thing in the square root is positive. The condition on this turns out to be that (again this is wrong because we didn't get the exactly correct β functions, but morally you get the point)

$$n > n_* = 320. \tag{2400}$$

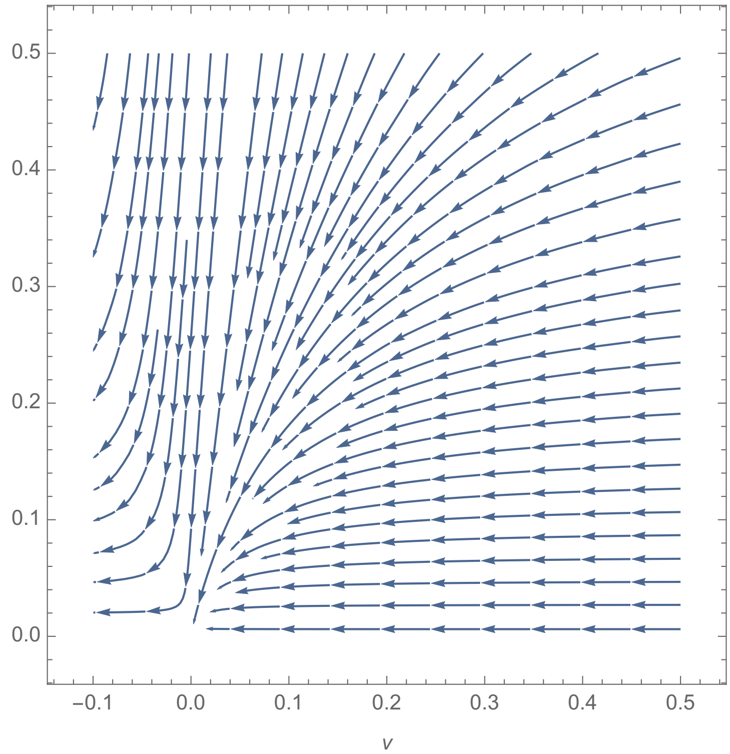
Therefore we need quite a lot of flavors to stabilize a second order transition!

The first order transition is of course manifested in the RG flows by a runaway flow towards negative v. Working in $\varepsilon = 1$, when $n \ll n_*$ the RG flow looks like



(2401)

where the y axis label is what Mathematica thinks λ is supposed to look like. Both the Gaussian fixed point and the gauge-less WF fixed point are unstable to a first-order transition. As n increases to near n_* the WF point on the $\lambda = 0$ axis gets closer and closer the Gaussian one (v_* for the gauge-less WF point goes down as $1/n$, asymptotically). When $n \gtrsim n_*$, we instead have



(2402)

which now shows that the Guassian fixed point is IR stable for a nonzero region of parameter space. The nontrivial fixed point that describes the second order transition that exists when $n > n_*$ is invisible in this picture; even at the minimal possible n of n_*, this fixed point is at $v_* = 1/640$.

Something worth noting here is that our determination of n_* strictly speaking does not work when $\varepsilon = 0$, since when $\varepsilon = 0$ there are no nontrivial fixed points (λ_* and v_* are both $\propto \varepsilon$ at every fixed point). However, it is again n_* which determines the IR stability of the Gaussian FP when $\varepsilon = 0$ (the Gaussian FP is always attractive along the $\lambda = 0$ line when $\varepsilon = 0$; we want to know when it is attractive for any non-zero value of λ). Indeed, the Gaussian FP will be IR stable provided that there exists some flow towards the origin with $\lambda = sv$ at late RG times, with $s > 0$. We plug this relation into the β functions and then solve for s: the resulting quadratic equation will have a positive solution only for certain n, and doing the calculation confirms that this n is exactly n_*. When $n > n_*$ the RG flow looks pretty much the same as it does when we had $\varepsilon = 1$; when $n < n_*$ the flow is boring: all initial points (except the v axis) just flow straight to negative v.

Note that this model is also often studied in the context of the large n expansion (with $g^2 n, un$ held fixed in a t' Hooft limit). The large n expansion is well suited for computing things like critical exponents and scaling dimensions, but extracting the above result about the first order phase transition is harder, since taking $n \to \infty$ doesn't help one to compute β functions at strong coupling. To find the first order transition in the large n expansion, one should presumably just directly calculate the free energy order by order in $1/n$, and look for a first-order transition directly in the singularity of the free energy (this is what we did in the diary entry on 2d matrix models). However, if one actually does this, it turns out (I think) that one needs to go beyond leading order in $1/n$—the crucial diagrams that led to the first-order transition were the ones in d) of Figure 14, which are suppressed in the t' Hooft limit relative to terms like the usual 1-loop all-scalar corrections to the $|\phi|^4$ vertex, which have an extra n.

Finally, we re-emphasize the remark made at the outset, namely that the conclusion from the above theoretical calculations is *not* actually reliable at $\varepsilon = 1$, since we know for a fact that PV duality in $d = 2 + 1$ exists (with $n = 1!$) and that the critical point is second order. All the same it is good to understand the calculation, and as far as I'm aware the conclusion may continue to hold for $0 < \varepsilon < 1$ (for $\varepsilon = 0$ we are definitely fine).

RG flow of $\mathbb{Z}_2^L$-symmetric coupled 2+1D Ising CFTs and gradient flow

Today we will be examining the RG flow in a sum of L coupled Ising models:

$$S = \int d^{4-\varepsilon}x \left(\frac{1}{2} \sum_i \left[(\partial\phi_i)^2 + \frac{t_i \Lambda^2}{2} \phi_i^2 \right] + \frac{\Lambda^\varepsilon}{8} \sum_{i,j} \phi_i^2 \phi_j^2 g_{ij} \right) \tag{2403}$$

where the sums are for $i, j = 1, \ldots, L$. We will be restricting to couplings that are invariant under a $\mathbb{Z}_2^L$ action of $\phi_i \mapsto f_i \phi_i$, where each $f_i \in \pm 1$ can be chosen independently. We will show that this theory is unstable to a fluctuation-induced first order transition (at least within the ϵ expansion).

The matrix g_{ij} can be taken to be a symmetric matrix, and for stability reasons we may take it to be positive definite. More precisely, the potential is stable as long as g_{ij} is positive semi-definite. However, if $\det g = 0$, then there exists a free direction in field space—by a change of basis, we get a decoupled flavor which does not interact. Therefore if such a g_{ij} is a fixed point it will for sure be unstable; since we are only interested in stable fixed points, degenerate g_{ij}s can be ignored.

Now enumerating all the RG fixed points for general L is essentially impossible (the symmetry group $\mathbb{Z}_2^L$ possesses too many quartic invariants). In fact even in the case of small L like $L = 4$, there are many fixed points—looking at the literature, it's actually unclear if there are any values of $N > 2$ for which the fixed points have been completely classified.

However, if we restrict our attention only to *stable* fixed points (i.e. stable with respect to all quartic perturbations respecting $\mathbb{Z}_2^L$), we will see that the situation becomes much more tractable. In what follows we will first do some direct calculations for the simplest choices of couplings, and then turn to some more powerful general statements.

First, we will assume that the g_{ij} couplings are translation-invariant and symmetric. In accordance with this, we will use the notation $g_{ij} = g_{i-j} = g_{j-i}$. Now from past experience we know that $L = 2$ critical Ising models coupled by their energy operators flows to the (relativistic) XY model, while a stack of $L = 3$ flows to the $O(3)$-symmetric fixed point; both fixed points are stable.

To find out what happens for general L, we need (at least) the 1-loop beta functions.[253] These are calculated in the usual way: the counterterms approach gives (no real need to keep track of the mass term since in dim reg t doesn't appear in the beta functions of the quartic couplings), with the notation $g_{i-j} \equiv g_{ij}$,

$$\beta_{g_j} = \varepsilon g_j - 2g_j^2 - 2g_0 g_j - \frac{1}{2}\sum_k g_k g_{j-k}. \tag{2404}$$

Note how g_0 is singled out; this occurs because of the different symmetry factors for vertices that involve four identical indices.[254] The matrix determining the scaling dimensions at a given fixed point is accordingly

$$\mathcal{B}_{jl} \equiv \frac{\partial \beta_{g_j}}{\partial g_l}\bigg|_{g_j = g_j^*} = \delta_{jl}(\varepsilon - 4g_j^* - 2g_0^*) - 2g_j^*\delta_{l0} - g_{j-l}^* \tag{2406}$$

First, we obviously have the usual $g_0 \neq 0, g_{j\neq 0} = 0 \;\forall\; j$ Ising$^{\oplus L}$ fixed point; here $g_0^* = \frac{2}{9}\varepsilon$, g_0 is irrelevant with $y_{g_0} = -\varepsilon$, and all of the $g_{j\neq 0}$s are equally relevant with eigenvalue $y_{g_{j\neq 0}} = \varepsilon/3$.

Because of the last term in (2404), it is impossible to have a fixed point with strictly finite-ranged couplings if any of the $g_{j>0}$ are non-zero, at least if we restrict to positive couplings. The solutions with maximally long-ranged coupling between the layers are those where the couplings are symmetric under the action of S_L: these are the cubic fixed points, with symmetry group $G = \mathbb{Z}_2^L \rtimes S_L$. Since G has only two quartic invariants, in this case there are only two distinct β functions to solve. Letting $h \equiv g_{j>0}$, the fixed points are determined by

$$\varepsilon g_0 = \frac{9}{2}g_0^2 + \frac{L-1}{2}h^2, \qquad \varepsilon h = 3g_0 h + \frac{L+2}{2}h^2. \tag{2407}$$

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[253] It will turn out that the 1-loop answers will be good enough for assessing stability within the context of small ε for all $L \neq 4$; for $N = 4$ there's a cancellation which necessitates going to two-loop.

[254] If we prefer, in Fourier space this is

$$\beta_{g_p} = \varepsilon g_p - \frac{2}{2\pi}\int_q g_q g_{p-q} - g_p \frac{2}{2\pi}\int_q g_q - \frac{1}{2}g_p^2. \tag{2405}$$

Unfortunately since the expression for β_{g_j} involves both multiplication and convolution in the layer index, Fourier transforming doesn't really help.

The fixed points are thus

$$(g_0^*, h^*) = \begin{cases} \mathcal{C} : & \left(\frac{2\varepsilon}{9}(1-1/L), \frac{2\varepsilon}{3L}\right) \\ \mathcal{S} : & \left(\frac{2\varepsilon}{8+L}, \frac{2\varepsilon}{8+L}\right) \end{cases} \tag{2408}$$

$\mathcal{S}$ is the usual $O(L)$-symmetric fixed point, while the cubic fixed point $\mathcal{C}$ goes over to Ising$^{\oplus L}$, up to $O(1/L)$ corrections. In this case the matrix determining the scaling dimensions is

$$\mathcal{B} = \begin{pmatrix} \varepsilon - 9g_0^* & -h^* & -h^* & \dots \\ -3h^* & \varepsilon - 4h^* - 3g_0^* & -h^* & \dots \\ -3h^* & -h^* & \varepsilon - 4h^* - 3g_0^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2409}$$

Diagonalizing this, we find three different eigenvalues (now temporarily writing $g = g_0^*$ and $h = h^*$ for notation's sake):

$$\text{Spec}(\mathcal{B}) = \begin{cases} \lambda_{L-2} = \varepsilon - 3g - 3h & m = L-2, \\ \lambda_\pm = \varepsilon - 6g - \frac{L+2}{2}h \pm \frac{1}{2}\sqrt{36g^2 - (24+12L)gh + (L^2+16L-8)h^2} & m = 1 \end{cases} \tag{2410}$$

where m denotes the multiplicity. The $\mathcal{S}$ fixed point is unstable: the eigenvalues of $\mathcal{B}$ are

$$\text{Spec}(\mathcal{B})|_{\mathcal{S}} : \lambda_{L-2} = \lambda_+ = \frac{\varepsilon(L-4)}{8+L}, \qquad \lambda_- = -\varepsilon, \tag{2411}$$

so that as $L \to \infty$ there are $L-1$ relevant parameters, with eigenvalues $y = +\varepsilon$. As a sanity check, these match exactly with the scaling dimensions for the $O(N)$-symmetric fixed point found in Cardy's book.

At $\mathcal{C}$, $\mathcal{B}$ has eigenvalues

$$\text{Spec}(\mathcal{B})|_{\mathcal{C}} : \lambda_{L-2} = \frac{\varepsilon(L-4)}{3L}, \qquad \lambda_+ = \frac{\varepsilon(4-L)}{3L}, \qquad \lambda_- = -\varepsilon. \tag{2412}$$

So this is only barely more stable than the $\mathcal{S}$ fixed point: we still have $L-2$ relevant parameters at large L, although the relevant deformations are less relevant than the ones at the $\mathcal{S}$ fixed point ($y \to \varepsilon/3$ versus $y \to \varepsilon$). Note that the scaling dimensions of the majority of the couplings have only changed with respect to the decoupled fixed point by an amount $-4\varepsilon/3L$, which vanishes as $L \to \infty$. This makes sense because $\mathcal{C}$ moves closer to the decoupled fixed point as $L \to \infty$, but it also means that we still have a long way to flow before we reach stability.

Anyway, at this point, attempting to brute-force solve the β functions and look for the stable points quickly gets out of hand. Additionally, restricting ourselves to translationally-invariant couplings is a bit fine-tuned—I don't see any reason a priori why translationally-non-invariant couplings would necessarily yield only unstable fixed points.[255] We could attempt an analysis of the β functions in the general case, [256] but this turns out not to be necessary, as long as we are only interested in addressing the question of stability. In fact, we can use some more general results about the beta functions in ϕ^4 theories to prove that *no stable fixed points exist* for any $L > 4$.

The most concise way of going about the following is to formulate the RG flows as gradient flows on a certain space of symmetric tensors. I learned about this from Zinn-Justin's QFT and critical phenomena book—in what follows we will derive the results that are important for us. For this, let us momentarily step back to the more general case of a L-component theory with interactions g_{ijkl}, where g_{ijkl} is a symmetric tensor that satisfies $v^i v^j v^k v^l g_{ijkl} > 0$ for any real vector $v \in \mathbb{R}^L$.[257] The β function is (re-scaling away numerical factors from the loop integration as usual)

$$\beta_{ijkl} = \varepsilon g_{ijkl} - \sum_{mn}(g_{ijmn}g_{mnkl} + g_{ikmn}g_{mnjl} + g_{ilmn}g_{mnkj}). \tag{2414}$$

The three quadratic terms are just the s, t, u channels (the ways of partitioning the $ijkl$ indices into unordered pairs). This simple form is why we've momentarily generalized away from the case with $\mathbb{Z}_2^L$-symmetric couplings—when we keep the couplings arbitrary the symmetry factors are much simpler to keep track of; if we worked directly with the $\mathbb{Z}_2^L$-symmetric couplings we'd have lots more δs floating around.

Anyway, the point of using this notation is that we can write

$$\beta_{ijkl} = \frac{\delta}{\delta g_{ijkl}}\mathcal{U}(g), \qquad \mathcal{U}(g) = \frac{\varepsilon}{2}\sum_{ijkl} g_{ijkl}^2 - \sum_{ijklmn} g_{ijkl}g_{klmn}g_{mnij}. \tag{2415}$$

To save space, we will use the multiindex notation $\beta_I = \delta_I \mathcal{U}(g)$, where $g_{ijkl} = g_I$. One technical point that Zinn-Justin glosses over: since we are working only with symmetric tensors, we need to use the variational derivative instead of the partial derivative, since we are in a constrained space where the components of g_{ijkl} are not independent variables—it makes no sense to take the derivative with respect to g_{ijkl} while keeping g_{ijlk} fixed ($\partial_I \mathcal{U}(g) \neq \beta_I$, the difference coming in combinatorial factors from the symmetry). We can still use the partial derivative, but we have to compensate for the fact that the partial derivative is operating in a bigger space by dividing out by the symmetry factor of the tensor in question. Therefore we can write

$$\frac{\delta}{\delta g_I} = M^{IJ}\frac{\partial}{\partial g^J}, \tag{2416}$$

[255] For example, a tetragonally-symmetric solution with non-zero couplings $g_{1,2} = g_{3,4} = g_{5,6} = \dots$ exists, although it turns out to be unstable (although if we cranked up these couplings and let the layers RG flow in pairs we would get a stack of XY models, which we know is stable).

[256] For posterity's sake, in this case the CPT approach gives (with $u = g_{ii}$)

$$\begin{aligned}
\beta_u &= \varepsilon u - 36u^2 - 4\sum_k g_{1k}g_{1k} - 4ut \\
\beta_{g_{ij}} &= \varepsilon g_{ij} - 12u g_{ij} - 8\sum_k g_{ik}g_{kj} - 4g_{ij}t \\
\beta_t &= 2t - 2t^2 - 6ut - \sum_j g_{1j}t - 48u^2 - \sum_j g_{1j}g_{1j}.
\end{aligned} \tag{2413}$$

[257] The $\mathbb{Z}_2^L$-symmetric case of interest would be $g_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl}g_{ik} + \delta_{ik}\delta_{jl}g_{ij} + \delta_{il}\delta_{jk}g_{ij})$.

where the metric is

$$M^{IJ} = \delta^{IJ}\frac{1}{N_I}, \tag{2417}$$

with N_I the number of distinct permutations of I (e.g. $N_{1234} = 4!$, $N_{1223} = \binom{4}{2}$, etc.). In what follows we will use ∂s, and raise / lower indices with this metric, so that e.g. $\beta^I = M^{IJ}\partial_J\mathcal{U}(g)$. It then follows that $\mathcal{U}(g)$ is an RG monotone: letting t be RG time,

$$\frac{d\mathcal{U}(g)}{dt} = \beta^J\partial_J\mathcal{U}(g) = M^{JK}\partial_K\mathcal{U}(g)\partial_J\mathcal{U}(g) > 0, \tag{2418}$$

since the metric is positive definite. We will now show that if there is a stable fixed point, it is the *unique* maximum of $\mathcal{U}(g)$ in coupling constant space. Hence if $\mathcal{U}(g) = \mathcal{U}(g')$ with $g \neq g'$, then neither g nor g' can be stable fixed points.

So, suppose g_1, g_2 are two distinct fixed points. We know that $\mathcal{U}$ is an RG monotone, and so we might be able to get an idea about the relative stability of these two fixed points by comparing $\mathcal{U}$ between them. To this end let $g(\lambda) = g_1\lambda + g_2(1-\lambda)$ be an interpolating path of couplings (this is not generically an RG flow, so $\mathcal{U}$ needn't be monotonic in λ). $\mathcal{U}$ varies along the path as

$$\partial_\lambda\mathcal{U}(g(\lambda)) = (g_1^I - g_2^I)\beta_I \equiv \Delta^I\beta_I. \tag{2419}$$

Since g_1, g_2 are fixed points, $\partial_\lambda\mathcal{U}(g)|_{\lambda=0,1} = 0$. Since β is quadratic in the g_Is and vanishes at the endpoints,$\beta_I = C_I\lambda(1-\lambda)$ for some λ-independent C_I. Then another derivative wrt λ gives

$$\partial_\lambda^2\mathcal{U}(g(\lambda)) = \Delta^I\Delta^J\partial_I\partial_J\mathcal{U}(g(\lambda)) = \Delta^I C_I(1-2\lambda). \tag{2420}$$

If we evaluate this at the endpoints of the interpolation, we get

$$\Delta^I\mathcal{B}_{IJ}(g_1)\Delta^J = \Delta^I C_I, \qquad \Delta^I\mathcal{B}_{IJ}(g_2)\Delta^J = -\Delta^I C_I, \tag{2421}$$

where again $\mathcal{B}_{IJ} = \partial_I\partial_J\mathcal{U}$ determines the scaling dimensions at a fixed point. Therefore

$$\Delta^I(\mathcal{B}_{IJ}(g_1) + \mathcal{B}_{IJ}(g_2))\Delta^J = 0, \tag{2422}$$

which since $\Delta^I \neq 0$ means that it is impossible for $\mathcal{B}(g)$ to be positive-definite at both g_1 and g_2—hence at least one of the two fixed points must be unstable, meaning that only at most one stable fixed point exists.

A corollary of this is the following. Let $g_I = R_I^J g_J$, with R a representation of some group element of $O(L)$ acting in the vector$^{\otimes 4}$ representation. Such $O(L)$ transformations map fixed points to fixed points, since β_I transforms covariantly. Therefore from the above result, any fixed point g_I acted on non-trivially by $O(L)$ *must* be unstable, and so the only stable fixed point is the $O(L)$ symmetric one, where $g_{ijkl} \propto (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Since we know the $O(L)$-symmetric is unstable for $L > 4$ by the calculation we did above, we thus conclude that there are no stable fixed points for $L > 4$.

This result was proved by looking at the behavior of $\mathcal{U}(g)$ with respect to arbitrary variations in the space of symmetric tensors g_I. Depending on the application though, we may only be interested in variations that preserve some symmetry group $G \subset O(L)$. Making this restriction can eliminate relevant directions in coupling constant space, and result in symmetry-protected stability.[258] However, since $\lambda g_1 + (1-\lambda) g_2$ is G-invariant if both g_i are, the above result means that there is at most one G-stable fixed point for each G. Furthermore, from the previous paragraph, if g_I is a G-stable fixed point and there exists some group element in $O(L)$ which maps g_I to any G-invariant coupling distinct from g_I itself, then g_I must be unstable.

Now we can finally apply this to the problem at hand to show that there are no $\mathbb{Z}_2^L$-stable fixed points for $L > 4$. Indeed, consider the action of $\sigma \in S_L \subset O(L)$ on a given $\mathbb{Z}_2^L$-invariant coupling g_{ij}. This maps $g_{ij} \mapsto g_{\sigma(i)\sigma(j)}$, which is of course also $\mathbb{Z}_2^L$ symmetric. Therefore a necessary condition for a g_{ij} to give a $\mathbb{Z}_2^L$-stable fixed point is for $g_{\sigma(i)\sigma(j)} = g_{ij}$ for all $\sigma \in S_L$, which means that g_{ij} must have the form $g_{ij} = g_0 \delta_{ij} + h$. But we have already solved this case and shown that there are no stable fixed points with this restricted class of couplings when $L > 4$. Therefore we conclude that as long as $L > 4$, the theory is unstable towards a fluctuation-induced first-order transition.[259]

Logarithmic singularities in free energies

Today's entry is basically a combination of a few different exercises in Cardy's book on RG and stat mech. We will be trying to understand when logarithmic singularities are expected to show up in free energies of theories near second order phase transitions.

Consider first for simplicity a phase transition involving only a single relevant parameter g. The first claim is that if the RG eigenvalue of g, viz. $y = d - \Delta_g$, satisfies $d/y \in \mathbb{Z}$, then the singular part of the free energy density scales as

$$f_{s\pm} \sim A_\pm |g|^{d/y} \ln |g|, \tag{2423}$$

where the $A_\pm$ are constants and the $\pm$ refers to g approaching 0 from above / below. Indeed, if $d/y \in \mathbb{N}$, the fact that a log of g must appear is fairly obvious: g can only appear in f in the form of $|g|^{d/y}$, and so unless we have logarithms it is impossible to produce a function that is singular in $|g|$ (the reason why terms like $|g|$ [which have

[258]A dumb example is $G = O(L)$, then the $O(L)$-symmetric fixed point is obviously stable. From the examples we worked out above, we also know that $G = \mathbb{Z}_2^L \rtimes S_N$ (where the two independent couplings were $g_{j>0}$ and g_0) is big enough to protect a stable fixed point.

[259]To build confidence, I also checked this by numerically searching for $\mathbb{Z}_2^L$-symmetric fixed points of the β functions for $L \leq 10$ (and of course found no stable ones).

derivatives which are infinite at $g = 0$] don't count as singular is explained in the following footnote).[260]

Blindly applying scaling relations can lead to an apparent contradiction, however. For example, we know that an RG step of size s leads to the scaling relation (no longer explicitly distinguishing the two sides of the critical point)

$$f_s(g) = s^{-d} f_s(s^y g). \tag{2424}$$

Note that as proved in another diary entry, provided g is not marginal we can always define g in a way such that it flows under each RG step as $g \to s^y g$, regardless of how large g is (the step size s still needs to approach 1 from above, however), and in what follows we will assume such a choice of g has been made. If we then apply n RG steps and flow to the point where $s^{ny} = 1/g$, we have

$$f_s(g) \sim g^{d\nu} f_s(1). \tag{2425}$$

This argument seems like it precludes a logarithmic dependence on g. The catch is that the $=$ in (2424) is an equality only up to non-singular terms, since only the singular part satisfies scaling. Indeed, we see that the form of f_s in (2423) satisfies instead

$$f_s(g) = s^{-nd} f_s(s^{ny} g) + ny|g|^{d/y} \ln s, \tag{2426}$$

which is indeed (2424) if we ignore the second non-singular piece.

As a quick example, consider the 2d Ising model, deformed by the energy operator. The energy operator has RG eigenvalue $+1$ (since it's the fermion mass), and so we expect a logarithmic singularity. Indeed, since the low energy excitations are massive fermions, we have

$$f \sim \int dk\, \sqrt{k^2 + g^2} \to \frac{1}{2}\left(\Lambda^2 + g^2 \ln(\Lambda/g)\right), \tag{2427}$$

with Λ the UV cutoff. The singular part of this is indeed $\sim |g|^{d/y} \ln(g)$, as expected.

The case where $d/y \in \mathbb{N}$ also has interesting consequences for the universality of critical amplitudes (the following is the solution to an exercise in Cardy's book). Performing n RG steps and keeping track of the non-singular part tells us that

$$f(g) = \sum_j^n s^{-jd} h(s^{jy} g) + s^{-nd} f_s(s^{ny} g), \tag{2428}$$

where $h(g)$ is the analytic-in-g term produced from integrating out the higher-energy modes. The second term on the right will vanish in the limit $n \to \infty$, since the coupling

[260]One pedantic comment on what we mean by "singular". By f being singular, we mean that there is some finite $n \in \mathbb{N}$ such that there exist nth derivatives of f with respect to the couplings which diverge. Since all scaling analyses always take place not exactly at the critical point but rather at a small distance away (i.e. g is always kept both small and non-zero), this divergence needs to be visible at finite g. For example, while the function $|g|^{2n+1}$ is strictly speaking singular at $g = 0$, this singularity only shows up in terms of the form $\partial^m \delta(g)$, which are not visible at finite g. Therefore f being singular means that it must contain a dependence on the couplings like $|g|^{\alpha \notin \mathbb{N}}$ or $\ln |g|$.

will eventually stop flowing once it reaches the value it takes on at the IR fixed point. Therefore for g small and positive and $s \to 1^+$, we can take the $n \to \infty$ limit and write

$$f(g) = \int_0^\infty dx\, s^{-xd} h(s^{xy} g). \tag{2429}$$

Now define the integration variable $t = s^{xy} g$. We then have

$$f(g) = g^{d/y} \int_g^\infty dt\, t^{-d/y-1} h(t). \tag{2430}$$

For generic values of d/y we evaluate this by integrating by parts enough times so that the power of t is large enough to allow the lower limit of the integral to be extended to zero (which is done up to analytic corrections). When $d/y \in \mathbb{Z}$

This means that, unlike the case of generic d/y, here the coefficient of the leading logarithm in the free energy is actually universal. Normally only ratios of amplitudes (like the ratio of the coefficients for initial trajectories starting from $g > 0$ and $g < 0$) are universal, not the amplitudes themselves.

Scaling dimensions for a two-component compact boson

Today we will calculate the scaling dimensions of the cosine operators in a general two-component compact boson model (a sigma model into T^2) in 1+1D.

Consider a two-component compact boson theory (with both fields having 2π periodicity) given by

$$S = \frac{1}{4\pi} \int g^{\alpha\beta} \partial_\mu \theta_\alpha \partial^\mu \theta_\beta. \tag{2431}$$

We can find the scaling dimensions of the cosines by decomposing the metric in veilbeins as $g_{\alpha\beta} = [E^T]_{\alpha a} \delta^{ab} E_{b\beta}$. Of course there is an $O(2)$ gauge redundancy in choosing the vielbeins; the choice we'll stick with is

$$g = \begin{pmatrix} g_1 & g' \\ g' & g_2 \end{pmatrix} \implies E_{\alpha a} = \begin{pmatrix} \sqrt{g_1} & g'/\sqrt{g_1} \\ 0 & \sqrt{g_2 - g'^2/g_1} \end{pmatrix}_{\alpha a}. \tag{2432}$$

Then defining new fields by $\vartheta_a \equiv E_{a\alpha}\theta^\alpha$ and letting $\boldsymbol{\theta} = (\theta_R, \theta_L)^T$, the Lagrangian is just $\mathcal{L}_0 = \frac{1}{4\pi} \partial \boldsymbol{\vartheta}^T \cdot \bar{\partial} \boldsymbol{\vartheta}$. However one must also keep in mind the periodicity of the fields, which is now modified: instead of $\boldsymbol{\vartheta} \sim \boldsymbol{\vartheta} + 2\pi \mathbf{n}$ with $\mathbf{n} \in \mathbb{Z}^2$, we instead have $\boldsymbol{\vartheta} \sim \boldsymbol{\vartheta} + 2\pi E\mathbf{n}$. This means that e.g. $e^{i\vartheta_\alpha}$ is not generically an allowed vertex operator in the theory.[261] The allowed vertex operators are instead $e^{i\mathbf{n}^T E^{-1}\boldsymbol{\vartheta}}$, and their correlation functions are computed using

$$\langle e^{i\gamma\vartheta(x)} e^{i\lambda\vartheta(y)} \rangle \sim |x-y|^{\gamma\lambda} \delta_{\gamma+\lambda} \tag{2433}$$

for a field ϑ whose free term has the coefficient $R^2/4\pi$ (for us of course the ϑ fields have $R = 1$; they are at the self-dual radius).

The Lagrangian in terms of the ϑ fields is

$$\mathcal{L} = \frac{1}{4\pi} \partial \boldsymbol{\vartheta}^T \cdot \bar{\partial} \boldsymbol{\vartheta} + \sum_{\mathbf{n} \in \mathbb{Z}^2} \alpha_{\mathbf{n}} \cos\left(\mathbf{n}_\alpha E^{\alpha a} \vartheta_a\right), \tag{2434}$$

where the index-up veilbein is the inverse. Using the above correlator for the ϑ vertex operators, we see that

$$\langle \cos(\mathbf{n}^T \cdot \boldsymbol{\theta}(x)) \cos(\mathbf{m}^T \cdot \boldsymbol{\theta}(0)) \rangle \sim \frac{\delta_{n_\alpha E^{\alpha 1} + m_\beta E^{\beta 1}} \delta_{n_\alpha E^{\alpha 2} + m_\beta E^{\beta 2}}}{|x|^{-(n_\alpha E^{\alpha 1} m_\beta E^{\beta 1} + n_\alpha E^{\alpha 2} m_\beta E^{\beta 2})}} + (\mathbf{m} \leftrightarrow -\mathbf{m}) \tag{2435}$$

Now the inverse veilbeins satisfy $E^{-1}[E^{-1}]^T = g^{-1}$, and so the exponent can be written as $n_\alpha g^{\alpha\beta} m_\beta$. The product of delta functions written out says that $(\mathbf{n}+\mathbf{m})^T E^{-1} = (0,0)$; since E is non-degenerate this means $\mathbf{n}+\mathbf{m} = (0,0)^T$. Accounting for the other possibility where the delta function sets $\mathbf{m} = +\mathbf{n}$ instead, we have

$$\langle \cos(\mathbf{n}^T \cdot \boldsymbol{\theta}(x)) \cos(\mathbf{m}^T \cdot \boldsymbol{\theta}(0)) \rangle \sim \frac{\delta_{\mathbf{n}+\mathbf{m}} + \delta_{\mathbf{n}-\mathbf{m}}}{|x|^{\mathbf{n}_\alpha g^{\alpha\beta} \mathbf{n}_\beta}}, \tag{2436}$$

and so the cosine (as well as the related sin) has dimension

$$\Delta_{\mathbf{n}} = \frac{1}{2} n_\alpha g^{\alpha\beta} n_\beta. \tag{2437}$$

Since g (and hence g^{-1}) are positive-definite forms if the Hamiltonian is positive-definite, $\Delta_{\mathbf{n}} > 0$ as required. Sanity check: suppose that $\mathbf{n} = (1,0)^T$ and $g = R^2 \oplus 0$. Then we get a scaling dimension of $1/2R^2$, which is exactly what we expect (recall that the free fermion point is $R = 1/\sqrt{2}$, which gives a scaling dimension of 1 as required by something which fermionizes to a Dirac mass).

The scaling dimensions of the vertex operators for the phase fields ϕ_α are computed in essentially the same way — duality inverts the metric, and so the dimension $\Delta^\vee_{\mathbf{n}}$ of the operator $\cos(\mathbf{n} \cdot \boldsymbol{\phi})$ is

$$\Delta^\vee_{\mathbf{n}} = \frac{1}{2} n_\alpha g_{\alpha\beta} n_\beta. \tag{2438}$$

[261] Just to be a bit garrulous here: it'd be wrong to say that the φ fields have radius 1—the ϑ fields really don't have independent radii, unlike the θ fields: when we do the linear transformation to the ϑ fields, we change the basis vectors in the lattice that generates the torus in the sigma model, and so now instead of both fields possessing independent periodicity requirements, they have a funky periodicity relation that is mixed up between the two fields. All of this just affects what kinds of vertex operators we can write down though, and if we start from vertex operators written in terms of the θ and then map them to ones in terms of the ϑ, we can never go wrong.

The same sanity checks can be done on this expression, and it passes them.

Consider first the case when $g_{LL} = g_{RR} = g$ (with $|g_{LR}| < g$ needed for stability). In this case the scaling dimensions of the θ cosines are

$$\Delta_{\mathbf{n}} = \frac{1}{2(g^2 - g_{LR}^2)}(g(n_1^2 + n_2^2) - 2g_{LR}n_1n_2). \tag{2439}$$

Now it is impossible to have all of the cosines $\cos(\mathbf{n}\cdot\mathbf{t})$ and $\cos(\mathbf{n}\cdot\boldsymbol{\phi})$ be simultaneously irrelevant. However, we can find regions of parameter space without relevant deformations if we impose some symmetries.

For example, consider a theory with $U(1)^2$ symmetry, which acts by shifting the ϕ_α fields. Then no cosines of the ϕ_α fields are allowed, and we can find a stable gapless phase by finding a region of parameter space where $\Delta_{\mathbf{n}} > 2$ for all $\mathbf{n}$. The boundary of this region can be found by setting $\Delta_{\mathbf{n}} = 2$, solving for n_1 in terms of n_2, and requiring that there be no real solutions. We find $n_1 = (g/g_{LR})n_2 \pm \sqrt{(1 - g_{LR}^2/g^2)(4g - n_2^2)}$, and so since $g_{LR}^2/g^2 < 1$ by assumption, we will have no relevant cosines if $4g - n^2 < 0$ for all nonzero n_2, i.e. provided that $g < 1/4$. Therefore the stable region of parameter space is given by $g_{LR} < g < 1/4$. In the more general case when $g_{LL} \neq g_{RR}$, we instead have

$$\Delta_{\mathbf{n}} = \frac{1}{2(g_{LL}g_{RR} - g_{LR}^2)}(g_{RR}n_1^2 + g_{LL}n_2^2 - 2g_{LR}n_1n_2). \tag{2440}$$

Fluctuation-induced first-order transitions in non-Abelian gauge theory

Today we're going to look at the stability of the free fixed point in Yang-Mills coupled to n flavors of quartically interacting bosons:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}[F_A \wedge \star F_A] + |D_A\phi|^2 + m|\phi|^2 + \frac{u}{2}|\phi|^4, \tag{2441}$$

where $|\phi|^2 = \langle \phi, \phi \rangle$, with $\langle \, , \, \rangle$ some invariant pairing for the representation R that ϕ transforms in. Instead of analyzing completely general choices of gauge groups and representations, we will actually just specialize to the case with gauge group $SU(N_c)$, with N_b bosons in the fundamental. viz.[262] Just as in the $U(1)$ case, we will see that

[262] In general, depending on the gauge group and the representation the bosons are in, there may be many different interaction terms we can write down: each term comes from an invariant symbol of the gauge group. The mass term and the quartic interaction come from $R \otimes R^* \ni \mathbf{1}$ and $(R \otimes R^*)^{\otimes 2} \ni \mathbf{1}$, but we could also have e.g. one from R^4 if $G = SU(4)$, etc. etc. Since we will be interested in the possibility of these theories describing critical points of second-order phase transitions, we will be assuming that no cubic invariant exists, so that $\mathbf{1} \notin R^{\otimes 3}, (R \otimes R^*) \otimes R$ (which rules out e.g. the fundamental of $SU(3)$ and all real representations).

at weak coupling the $m = 0$ point has an instability towards negative u unless n is larger than some (often rather large) critical value (which is of course always larger than the value needed to allow the theory to be free in the IR).

The beta function for the gauge coupling is the easiest to get, since as in the Abelian case, gauge invariance means that it can be calculated entirely in terms of the gauge field self energy. To one loop order, no quartic ϕ couplings appear in the diagrams for the A self energy; hence β_{g^2} will be a function of g^2 alone.

The first type of self-energy diagram to calculate is the one where a ϕ field flows in the loop. The only thing that makes this different from the $U(1)$ case is that there is now an extra factor of $\mathrm{Tr}[T^aT^b] = T(R)\delta^{ab}$, with $T(R)$ the index of the representation that the bosons are in.

The other contribution to the self energy comes when a gauge field flows in the loop (the "cactus diagrams" with scalar and gauge fields in the loop don't contribute since they vanish in dimensional regularization, and being momentum independent wouldn't contribute to the wavefunction renormalization anyway). The group theoretic factor comes from the f^{abc}s on the two three-point gauge boson vertices, giving $f^{abc}f^{bcd} = \mathrm{Tr}[T^a_A T^d_A] = T(A)\delta^{ad} = C_2(G)\delta^{ad}$. Since there are n bosons flowing in the polarization bubble, the beta function for the gauge coupling is therefore the standard

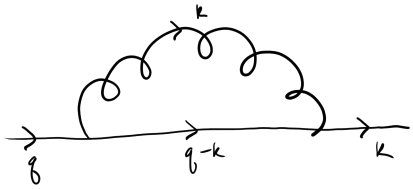
$$\beta_\alpha = \varepsilon\alpha + \alpha^2\left(\frac{11}{3}C_2(G) - \frac{n}{3}nT(R)\right), \tag{2442}$$

where $\alpha = g^2/8\pi^2$.

The beta function for u is quite a bit more complicated but not unassailably so, since the diagrams involved don't contain any three- or four-point gauge boson vertices, which are the real source of annoyance due to the complicated numerators they bestow upon Feynman diagrams.

The term of order u^2 can be obtained following the procedure outlined in the previous diary entry on the case when $G = U(1)$; it gives us the familiar $(n+4)u^2/8\pi^2$ term.

The next easiest diagrams to compute are the ones which involve only the $\phi^\dagger A\partial\phi$ vertices. For example, consider the self energy of the scalar (needed for getting the normalization of the ϕ fields). The relevant diagram is (schematically)



$$\phi^\dagger_j T^a_{jl} T^b_{mk}\phi_k \langle A^a_\mu A^b_\nu\rangle\langle\phi^\dagger_m\phi_l\rangle \to C_2(R)\,\delta_{jk}\,\phi^\dagger_j\phi_k\int\frac{(2q-k)_\mu(2q-k)_\nu}{k^2(q-k)^2} \tag{2443}$$

As of the time of writing I was not feeling like texing up the calculations; in any case they really are quite similar to those in the $U(1)$ case, which is in another diary entry. The result turns out to be

$$\beta_u = \varepsilon u - (N_b N_c + 4)u^2 + \frac{3(N_c^2-1)}{N_c}u\alpha - \frac{3(N_c-1)(N_c^2+2N_c-2)}{4N_c^2}\alpha^2, \qquad (2444)$$

with now $u = u/8\pi^2$ (in a computer's sense of '='). Note that N_b only appears in the first term, as this is the only one that comes from diagrams with a full ϕ loop.

In the diary entry on fluctuation-induced first-order transitions, we computed the β functions in the ϵ expansion to diagnose stability. We could do the same thing here, but for variety's sake we'll get at the answer from a different perspective.

For notation's sake, let

$$\beta_\alpha = -A\alpha, \qquad \beta_u = -au^2 + bu\alpha - c\alpha^2. \qquad (2445)$$

In our context, all of the above coefficients will be positive.

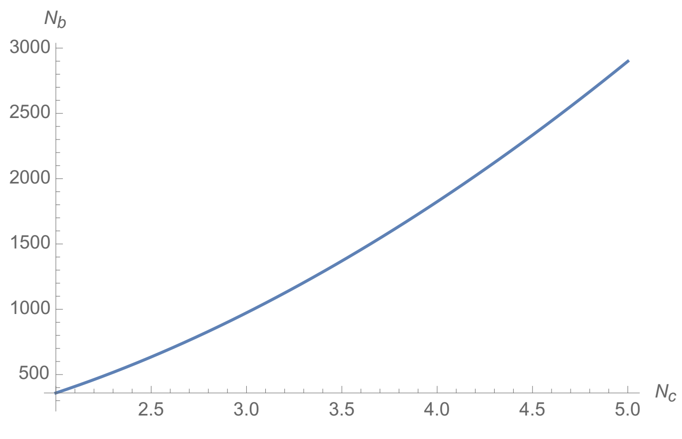
Suppose now that $(0,0)$ is an attractive IR fixed point, with some subset of flow lines that start at $\alpha > 0$ flowing onto $(0,0)$ (the flow lines that start on the u axis will always flow to $(0,0)$). If this is the case, then we must have a flow which flows into $(0,0)$ from a line with > 0 slope in the u-α plane. Therefore there must be a solution to the β function ODEs with $\alpha = su$ for some $s > 0$, at least for α, u very close to the origin. Close to the origin where this parametrization works, s will also be independent of RG time. Putting this into the beta functions, we see that we must have

$$-As = -a + bs - cs^2, \qquad (2446)$$

and since $s > 0$ we must then have

$$(A+b)^2 \geq 4ca. \qquad (2447)$$

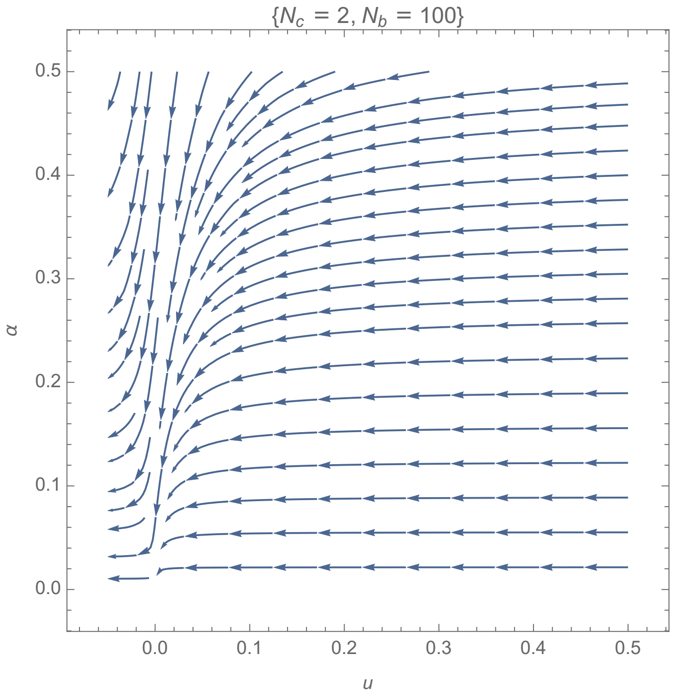
N_b^* is then found by making the above an equality and solving for N_b. When we do this for e.g. $N_c = 2$, we find $N_b^* = 359$, which as with the $U(1)$ case is rather large. Not surprisingly, N_b^* is a monotonically increasing function of N_c, e.g.



(2448)

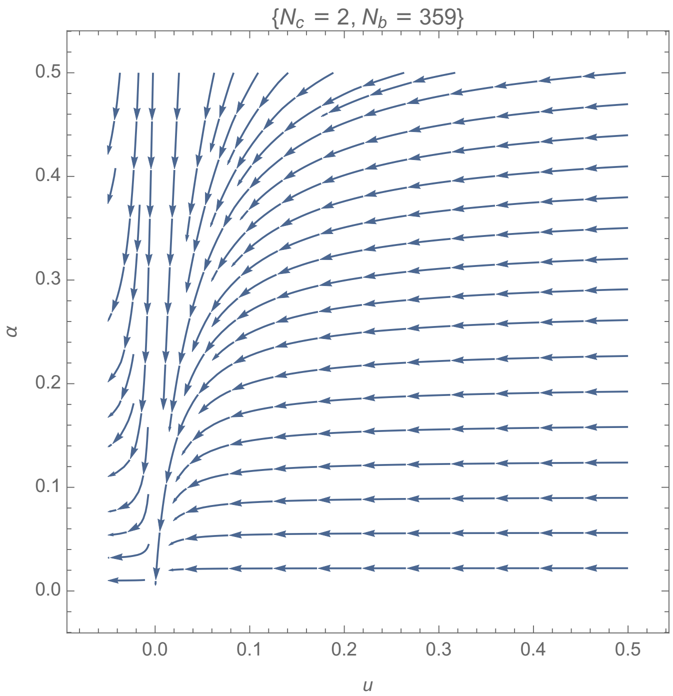
which makes sense because the threshold for potential IR freedom increases with N_c.

Lets see this visually for the case of $N_c = 2$, where $N_b^* = 359$. For $N_b < N_b^*$, all of the flows (expect for the ones that start at $\alpha = 0$) lead to a $u < 0$ instability:



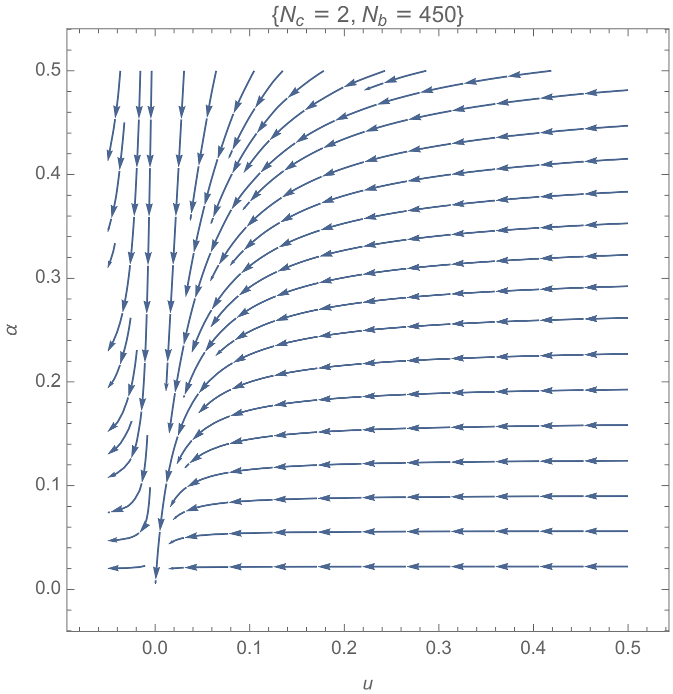
(2449)

Right at the critical number of bosons, we have



(2450)

while when we're above N_b^*, the flows look like



$$(2451)$$

In this plot there are lines that start at $\alpha, u > 0$ and flow onto the $\alpha = u = 0$ trivial fixed point (the separatrix can be seen as a flow line hitting the origin).

$SU(N)$ WZW basics

Consider the $SU(N)$ WZW model on a two-dimensional spacetime. The action is

$$S = S_{kin} + S_{wzw} = \frac{1}{8\pi}\int_X \mathrm{Tr}[\partial_\mu g \partial^\mu g^{-1}] + i2\pi \int_{B^3} f^*(\alpha), \tag{2452}$$

where g is a map from the spacetime X (which since we will take the fields to be constant at infinity is topologically an S^2) into $SU(N)$, $f : B^3 \to SU(N)$ where $\partial B^3 = X$ is a three-ball which bounds spacetime, and where α is some nontrivial form in $H^\bullet(SU(N); \mathbb{R})$.

If $\omega = g^{-1}dg$ is the Maurer-Cartan form on $SU(N)$ pulled back to X, then the forms

$$\lambda_j = \mathrm{Tr}(\omega^{\wedge j}), \qquad j = 3, 5, \dots, 2N-1 \tag{2453}$$

are classes in $H^\bullet(SU(N); \mathbb{R})$ pulled back to functions on X. We will mainly be interested in $SU(2)$, so we only have the $\mathrm{Tr}(\omega \wedge \omega \wedge \omega)$ form. So, take

$$f^*(\alpha) = C\mathrm{Tr}(\omega \wedge \omega \wedge \omega), \tag{2454}$$

where C is a normalization constant to be determined, which will ensure that the periods of $f^*(\alpha)$ lie in $\mathbb{Z}$ (so that S is independent of the choice of the bounding manifold B^3). We will focus on $SU(2)$ because we are interested in a model which lives in two dimensions, so that the WZW term is defined through the pullback of a cohomologically nontrivial 3-form from the group manifold to S^2. For $SU(N)$ the 3-form will always be $\mathrm{Tr}(\omega \wedge \omega \wedge \omega)$, and so there isn't much difference between the different N's. The case of $SU(2)$ is easiest since α is then proportional to the volume form on S^3, which makes things simple.[263]

First, show that α indeed defines a nontrivial cohomology class. Then calculate the equations of motion from varying g, and interpret them as current conservation equations. Finally, calculate the value of C.

First let us check that α is closed. By using the cyclic invariance of the trace, we have

$$d\lambda_3 = \mathrm{Tr}[d\omega \wedge \omega \wedge \omega]. \tag{2457}$$

Now

$$d\omega = -g^{-1}dg \wedge g^{-1}dg = -\omega \wedge \omega, \tag{2458}$$

so

$$d\lambda_3 = -\mathrm{Tr}[\omega \wedge \omega \wedge \omega \wedge \omega]. \tag{2459}$$

When we take the trace, we can apply the normal supercommutativity of the wedge product since

$$\mathrm{Tr}(A \wedge B) = \sum_{ij} A_{ij} \wedge B_{ji} = (-1)^{|A||B|} B_{ji} \wedge A_{ij} = (-1)^{|A||B|}\mathrm{Tr}(B \wedge A). \tag{2460}$$

Applying this to the above with $A = \omega$ and $B = \omega \wedge \omega \wedge \omega$, we conclude that $d\lambda_3 = 0$. We will confirm that λ_3 is not exact in a little bit, which will then show that α is indeed a nontrivial cohomology class.

[263] As a general comment,

$$H^\bullet(SU(N); \mathbb{Z}) \cong \mathbb{Z}[\alpha_3, \alpha_5, \dots, \alpha_{2N-1}], \tag{2455}$$

where the α_i are of degree i and come from traces of odd numbers of wedge products of ωs. This can be proved by noting that $SU(2) = S^3$ gives an easy base case, and then using the SES

$$1 \to SU(N-1) \to SU(N) \to S^{2N-1} \to 1. \tag{2456}$$

Now for the equations of motion. First, let's do the kinetic term. We have

$$\begin{aligned} S_{kin}[e^{iW}g] - S_{kin}[g] &= \frac{1}{8\pi}\int \mathrm{Tr}\left[\partial_\mu(g+iWg)\partial^\mu(g^{-1}-ig^{-1}W)\right] - S_{kin}[g] \\ &= \frac{i}{8\pi}\int \mathrm{Tr}\left[\partial_\mu W g\partial^\mu g^{-1} + W\partial_\mu g\partial^\mu g^{-1} + \partial_\mu g(-g^{-1}\partial^\mu W - \partial^\mu g^{-1}W)\right] \\ &= \frac{i}{8\pi}\int \mathrm{Tr}\left[W((\partial^2 g)g^{-1} - g\partial^2 g^{-1}\right], \end{aligned} \tag{2461}$$

where we integrated by parts in the last step.

Now for the wzw term. We first need the variation of the Maurer-Cartan form:

$$\begin{aligned} \delta\omega &= g^{-1}(\mathbf{1}-iW)d[(\mathbf{1}+iW)g] - \omega \\ &= -ig^{-1}Wdg + ig^{-1}(dWg + Wdg), \end{aligned} \tag{2462}$$

and so the variation in the WZW term is

$$\begin{aligned} S_{wzw}[e^{iW}g] - S[g] &= 6\pi i C\int_{B^3} \mathrm{Tr}\left[(-ig^{-1}Wdg + ig^{-1}(dWg+Wdg))\wedge\omega\wedge\omega\right] \\ &= -6\pi C\int_{B^3} \mathrm{Tr}[W\wedge dg\wedge dg^{-1}] \\ &= -6\pi C\int_X d^2x\; W\epsilon^{\mu\nu}\partial_\mu g\partial_\nu g^{-1}, \end{aligned} \tag{2463}$$

where we used the cyclicity of the trace.

In a little bit we will show that the normalization constant C needs to be

$$C = -\frac{1}{24\pi^2}. \tag{2464}$$

Putting this in, we arrive at the equations of motion:

$$2i\epsilon^{\mu\nu}\partial_\mu g\partial_\nu g^{-1} + (\partial^2 g)g^{-1} - g\partial^2 g^{-1} = 0. \tag{2465}$$

We can also re-write this as

$$i\epsilon^{\mu\nu}\partial_\mu g\partial_\nu g^{-1} + (\partial^2 g)g^{-1} - (\partial_\mu g)g^{-1}(\partial^\mu g)g^{-1} = 0, \tag{2466}$$

or finally, as

$$i\epsilon^{\mu\nu}\partial_\mu g\partial_\nu g^{-1} + (\partial^2 g)g^{-1} + \partial_\mu g\partial^\mu g^{-1} = 0. \tag{2467}$$

Now define the antiholomorphic current $\bar{J}_\mu$ by

$$\bar{J} = -(\partial_{\bar{z}} g)g^{-1}, \tag{2468}$$

where $z = x^0 + ix^1$ (depending on your preferences there may be a different constant out in front). Then we can compute

$$\partial_z\bar{J} = -(\partial^2 g)g^{-1} - \partial_{\bar{z}}\partial_z g^{-1} = -(\partial^2 g)g^{-1} - \partial_\mu g\partial^\mu g^{-1} - i\epsilon^{\mu\nu}\partial_\mu g\partial_\nu g^{-1} = 0, \tag{2469}$$

which vanishes by the equations of motion. Similarly, we also define a holomorphic current J, with

$$J = g^{-1}\partial_z g, \qquad \partial_{\bar{z}} J = 0. \tag{2470}$$

A few words to help us understand these currents: in keeping with the two currents being "conjugates" of one another, the action of time reversal (sending $g \mapsto g^{-1}$ and $z \mapsto \bar{z}$) exchanges the two currents, since

$$T : J \mapsto g\partial_{\bar{z}} g^{-1} = -(\partial_{\bar{z}} g)g^{-1} = \bar{J}. \tag{2471}$$

The currents also couple chirally to the symmetries in the product $SU(2)_L \times SU(2)_R$ (which act on g on the left and right respectively), since under $SU(2)_L$,

$$J \mapsto g^{-1}h^{-1}\partial_z hg = J, \qquad \bar{J} \mapsto -h^{-1}(\partial_{\bar{z}} g)g^{-1}h^{-1} = hJh^{-1}, \tag{2472}$$

while similarly under $SU(2)_R$,

$$J \mapsto h^{-1}Jh, \qquad \bar{J} \mapsto \bar{J}. \tag{2473}$$

Finally, note that the wzw term here is essential for getting the "right" equations of motion (at least, the right ones if we have bosonization in mind as a physical context): without it, the equations of motion would be a conservation of a different kind of current, and would read

$$d^\dagger \omega = 0. \tag{2474}$$

Now we check that C is indeed given by $-1/24\pi^2$. We do this by requiring that the action be independent of the exact choice of B^3, modulo elements of $2\pi i\mathbb{Z}$. This will be the case if

$$2\pi i C \int_{M_3} \mathrm{Tr}[\omega \wedge \omega \wedge \omega] \in \mathbb{Z} \tag{2475}$$

for all closed 3-manifolds M_3. We need only check this for a particular manifold, like S^3, since as we saw earlier, the integrand is closed. The fact that the integrand is closed also tells us that the above integral will be quantized: we just have to figure out what the correct normalization is. One can also check this by computing the variation of the integrand under $g \mapsto e^{iW}g$: as we saw earlier, the variation is exact, and so the above integral is stationary under any infinitesimal variation of g, meaning that it must be quantized.

Anyway, let's find the normalization coefficient. We parametrize the S^3 by

$$g = e^{ix^3 n_a \sigma^a} = \cos(x^3) + in_a\sigma^a \sin(x^3), \tag{2476}$$

where x^3 is an angular coordinate running from 0 to π and $n : X \to S^2$ is a field which maps spacetime to S^2 (this is the usual way of building a three-sphere out of an S^2 and an S^1). Then we have

$$\omega_\mu \, dx^\mu = g^{-1}\partial_\mu g \, dx^\mu = in_a\sigma^a \, dx^3 + i\partial_j n_a \sigma^a \, dx^j, \tag{2477}$$

where i runs through the spacetime coordinates and we've used $(n_a\sigma^a)^2 = 1$.

Putting this in, the integral is

$$I = 6\pi i C \int_{M_3} d^3x \, \epsilon^{3jk} \sin^2(x^3) \, \mathrm{Tr}[n_a \sigma^a \partial_j n_b \sigma^b \partial_k n_c \sigma^c]. \tag{2478}$$

Doing the x^3 integral and taking the trace,

$$I = 6\pi^2 C \int_X d^2x \, \epsilon^{jk} \epsilon^{abc} n_a \partial_j n_b \partial_k n_c. \tag{2479}$$

Because of the double-antisymmetrization, this is

$$I = 12\pi^2 C \int_X d^2x \, \epsilon^{abc} n_a \partial_1 n_b \partial_2 n_c. \tag{2480}$$

We recognize the $n \cdot \partial_i n \times \partial_j n \, dx^i \wedge dx^j$ as the pullback of the volume form on S^2 to X through the map n (it is just the Jacobian for the map: $\partial_i n \times \partial_j n$ lies perpendicular to the S^2, parallel to n, and has magnitude equal to the local magnification of the mapping $n : X \to S^2$). Thus we have

$$I = 12\pi^2 C \int_{n(X)} \mathrm{vol}_{S^2}. \tag{2481}$$

Since this can be non-zero, $\mathrm{Tr}[\omega^3]$ must not be exact and hence it must indeed represent a nontrivial cohomology class.

Since the volume form is closed, the integral is a topological invariant which is of course the winding number. So we have

$$I = 48\pi^3 C w, \qquad w \in \mathbb{Z}. \tag{2482}$$

Since we have to consider maps with winding number $w = 1$, we see that we need (the minus sign is just convention)

$$C = -\frac{1}{24\pi^2} \implies 2\pi i C \int_{M_3} \mathrm{Tr}[\omega^3] \in \mathbb{Z}. \tag{2483}$$

We also see a little bit of the relation between WZW terms and topological terms: after we integrated over the x^3 coordinate, the WZW term became a topological θ term on the compactified spacetime $X \sim S^2$.

σ model in two dimensions

This is a very standard calculation, but was new to me at the time I did it. Consider a σ model in two dimensions, where the target space is the sphere S^N. The action, after introducing the Lagrangian multiplier in the usual way, is

$$S = \frac{1}{2\lambda}\int d^2x\,\big(\partial_\mu\phi\partial^\mu\phi - i\lambda\sigma(\phi^2-1)\big)\,. \tag{2484}$$

Working in the large N limit, find the two point function for σ, namely $\langle\sigma(-q)\sigma(q)\rangle$. It will help to expand about the equilibrium constant value for σ. The expression should simplify in the large q limit.

The equation of motion for ϕ is

$$\phi\Delta\phi = i\lambda\sigma, \tag{2485}$$

where we have used $\phi^2=1$. Here, $\Delta = -\partial_\mu\partial^\mu$ is the Hodge Laplacian. From this, we expect on dimensional grounds that

$$\langle\sigma(x)\sigma(y)\rangle \sim \frac{1}{|x-y|^4}. \tag{2486}$$

Fourier-transforming, we expect

$$\langle\sigma(-q)\sigma(q)\rangle \sim |q|^2. \tag{2487}$$

Let's get the exact propagator. Integrating out ϕ, we have

$$Z = \int\mathcal{D}\sigma\exp\left(-\frac{N}{2}\mathrm{Tr}\ln\left(\frac{\Delta}{\lambda}-i\sigma\right)-\frac{i}{2}\int\sigma\right). \tag{2488}$$

Again, Δ is the Hodge Laplacian. Since $N\to\infty$ we can minimize with respect to $\sigma=\text{const}$ to find the equilibrium value of σ. Doing this tells us that

$$\frac{i}{2} = \frac{iN}{2}\int_p\frac{1}{p^2/\lambda - i\sigma}, \tag{2489}$$

i.e. that

$$1 = \frac{N}{4\pi}\ln\left(-\Lambda^2/i\lambda\sigma\right), \tag{2490}$$

so that the equilibrium value for the mass of ϕ is determined by

$$\sigma = i\frac{M^2}{\lambda}, \qquad M^2 = \Lambda^2 e^{-4\pi/N\lambda}. \tag{2491}$$

To get the inverse propagator for σ, we just need to find the term in the effective action quadratic in σ. We vary σ as $\sigma = iM^2/\lambda + \delta\sigma$, and isolate the piece in the effective action quadratic in $\delta\sigma$. This comes from the usual polarization bubble type of Feynman diagram. Let Γ be the effective action for σ that we wrote down above. Then

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta\sigma(-q)\delta\sigma(q)}\Big|_{\sigma=iM^2/\lambda} &= -\frac{\delta^2}{\delta\sigma(-q)\delta\sigma(q)}\frac{N}{2}\mathrm{Tr}\left[\frac{1}{\Delta^2+M^2}i\lambda\delta\sigma\frac{1}{\Delta^2+M^2}i\lambda\delta\sigma\right] \\ &= \frac{\delta^2}{\delta\sigma(-q)\delta\sigma(q)}\frac{N\lambda^2}{2}\int_{x,y} G(x,y)\delta\sigma(x)G(y,x)\delta\sigma(y), \end{aligned} \tag{2492}$$

where G is the Greens function for the massive scalar. This gives

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta\sigma(-q)\delta\sigma(q)}\Big|_{\sigma=iM^2/\lambda} &= N\lambda^2\int_p \frac{1}{(p^2+M^2)((p-q)^2+M^2)} \\ &= N\lambda^2\int_p\int_0^1 dx\frac{1}{[p^2+M^2+x(1-x)q^2]^2} \\ &= \frac{N\lambda^2}{4\pi}\int_0^1 dx\frac{1}{M^2+x(1-x)q^2} \\ &= \frac{N\lambda^2}{4\pi}\frac{\ln\left(1+\frac{q(q+\sqrt{4M^2+q^2})}{2M^2}\right)}{q\sqrt{4M^2+q^2}}. \end{aligned} \tag{2493}$$

If we take the $q^2 \gg M^2$ limit, things simplify, and the RHS goes to $N\lambda^2\ln(q^2/M^2)/(4\pi q^2)$. This is the inverse propagator, so the propagator in this limit is thus

$$\langle\sigma(-q)\sigma(q)\rangle = \frac{4\pi}{N\lambda^2}\frac{q^2}{\ln(q^2/M^2)}. \tag{2494}$$

The point of this is that the dynamically generated mass gives us a new scale that can appear in the propagator (viz. M), and that even in the deep UV this scale is still present in the two point function, which does not reduce to that of a free field.

Even more on the $O(N)$ and nonlinear σ models in two dimensions

This problem is a way to try to understand some of the content in Polyakov's book — the goal is to understand what's written there and to fill in the details that are omitted.

Consider the $O(N)$ vector model which maps spacetime into the $N-1$ sphere, so that the symmetry group is $O(N)$. As we have seen earlier, this model admits a saddle point solution in $N \to \infty$ limit: using the saddle point, one can see how dimensional transmutation occurs and defines a mass scale.

Using the saddle point method, what is the propagator for the massive excitations? What kind of divergences arise and what kind of renormalization must be done? After looking at this (you should compute the field strength renormalization), go back and find the exact propagator and beta function to order g^2 (without using large N) by employing a background field method with a Wilsonian-picture momentum integration. Show how to do the renormalization and compare to the saddle point results. Show that as $N \to \infty$, the model describes a free theory.

Now consider the case of the $SU(N)$ nlσm, still in 2 dimensions. Find the propagator and the beta function for g^2. In the large N limit, is the theory described by the saddle point, like the $O(N)$ model is? Why not?

The $O(N)$ vector model: The action, with the Lagrange multiplier λ to enforce that the vector n lives on S^{N-1}, is

$$S = \frac{1}{2g^2} \int \left(\partial_\mu n_a \partial^\mu n^a + \lambda(n^2 - 1)\right). \tag{2495}$$

Assuming that a massive solution exists and integrating out n,

$$S = -\frac{1}{2g^2} \int \lambda + \frac{N}{2} \ln\det(-\partial^2 + \lambda). \tag{2496}$$

We have already seen how to do the saddle point multiple times: it gives the mass

$$\lambda = m^2 = \Lambda^2 \exp\left(-\frac{4\pi}{Ng^2}\right). \tag{2497}$$

Let's find the 2-point function for the saddle point solution, which we will use for comparison later. We will primarily be interested in short-distance behavior (or small mass) in the following so for $m \ll r^{-1}$ we find

$$\langle n(r) \cdot n(0) \rangle = \frac{Ng^2}{2\pi} \int dp\, p \frac{e^{ipr}}{p^2 + m^2} \approx \frac{Ng^2}{2\pi} \int_0^{1/mr} d\alpha \frac{\alpha e^{i\alpha m r}}{1 + \alpha^2} \tag{2498}$$

where $\alpha = p/m$ and the N comes from the N components of the n vector. Thus in the small mr limit we get the log which indicates that as $N \to \infty$ the n field is free:

$$\langle n(r) \cdot n(0) \rangle \approx -\frac{Ng^2}{2\pi} \ln(mr) = 1 - \frac{Ng^2}{2\pi} \ln(r/a), \tag{2499}$$

where we have used the saddle point expression for m and where $\Lambda = a^{-1}$. Now as stated, this result cannot be correct for small N, since for $N = 2$ it predicts that the propagator for the n field has nontrivial field strength renormalization, which cannot be true since in that case the n field is a map into S^1 and therefore has no interactions and is free. Later on we will see that the correct thing to do is to replace N by $N-2$.

Now we want to consider fluctuations about the saddle point. To this end, define the field ϕ through

$$\lambda = m^2 + \sqrt{\frac{2}{N}}\phi, \tag{2500}$$

where the $q = 0$ momentum component of ϕ vanishes. Putting this in the $\mathrm{Tr}\ln$ and expanding to third order gives

$$S_{eff} = S_{eff,\phi=0} - \frac{N}{2}\left(\frac{1}{N}\int_q \phi_q\phi_{-q}\Pi_2(q^2) - \frac{2}{N}\sqrt{\frac{2}{N}}\int_{q,r}\phi_q\phi_r\phi_{-q-r}\Pi_3(q,r) + \dots\right), \tag{2501}$$

where the first term is the saddle point, $\Pi_2(q^2)$ is the polarization bubble with two external ϕ legs, $\Pi_3(q^2)$ is the analogous diagram with three external ϕ legs, and so on. In these diagrams, the internal propagators are $G(p) = 1/(p^2+m^2)$. Note that there is no bubble with a single leg, since we are expanding around a saddle point. The $1/\sqrt{N}$ factor in front of the 3-legged bubble tells us that in the large N limit we can only worry about the polarization bubble as for as renormalizability goes. From this expression, we see that the propagator for the ϕ field is

$$G_\phi(q^2) = \frac{1}{\Pi_2(q^2)}. \tag{2502}$$

Thus the λ field has gained some dynamics from the n field. Let us look at the large q limit of this expression to see how renormalization should work. We have

$$\Pi_2(q^2) = \int_p \frac{1}{(p^2+m^2)((p-q)^2+m^2)} = \int_p\int_x \frac{1}{(p^2+q^2(x^2+x)+m^2)^2}. \tag{2503}$$

We can estimate this in the large q limit by noting that only small x will be important to the integral, so that defining $dy = q^2dx$,

$$\Pi_2(q^2) \approx \int_p\int_y \frac{q^{-2}}{(p^2+y+m^2)^2} \approx \int_p \frac{q^{-2}}{p^2+m^2}, \tag{2504}$$

where the integral over y was done from 0 to q^2. So then

$$\Pi_2(q^2) \approx \frac{1}{2\pi}\int dp \frac{pq^{-2}}{p^2+m^2}. \tag{2505}$$

In the $q^2 \gg m^2$ limit then,

$$\Pi_2(q^2) \approx \frac{1}{2\pi q^2}\ln(q^2/m^2), \tag{2506}$$

and so the propagator for the ϕ field is

$$G_\phi(q^2) = \frac{2\pi q^2}{\ln(q^2/m^2)}, \tag{2507}$$

which diverges as $q \to \infty$, which is bad. This divergence will also appear in the propagator for the n field, which one can see by considering the first 1-loop correction to the n field two-point function.

The reason for this divergence is essentially due to the fact that we haven't renormalized the mass m^2 of the elementary excitations. As Polyakov points out in the book, the divergence arises because we are confining the particles to the sphere with the λ field, which because of the uncertainty relation leads to the above divergences. So we can fix this by adding a mass term for λ:

$$\mathcal{L} \mapsto \frac{1}{2g^2}\left[(\partial n)^2 + \lambda(n^2-1)\right] - \frac{\lambda^2}{4\beta}. \tag{2508}$$

Solving the equation of motion for λ gives us a Lagrangian with a term $\beta(n^2-1)/g^4$, and so in the large β limit (small mass for λ) we recover the σ model. When we do the large N expansion with this term added, we just have to add the new mass term into the ϕ propagator:

$$G_\phi(q^2) \mapsto \frac{1}{\Pi_2(q^2) + \frac{1}{\beta N}}, \tag{2509}$$

which resolves the power law divergences at large q.

So, we want to do mass renormalization. The way to do it is to ensure that the inverse propagator has no power divergences like the one appearing above for the naive ϕ propagator. To this end we subtract off the zero momentum part of the self energy to define the mass renormalization, so that the n field Greens function is

$$G_n(q^2) = \frac{1}{q^2 + m^2 + \Sigma(q^2) - \Sigma(0)}. \tag{2510}$$

Let us now check that subtracting off $\Sigma(0)$ cancels the power divergences. The first term in $\Sigma(q^2)$ is a diagram with a straight n field line of momentum q that has a ϕ field arc attached to it at momentum p. There are two factors of $\sqrt{2/N}$ coming from the two vertices, and so we can write (we are just interested in the structure of the divergences, so we will take $q^2 \gg m^2$ in what follows)

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N}\int_p \frac{1}{\Pi_2(p^2)}\left(\frac{1}{(p-q)^2} - \frac{1}{p^2}\right). \tag{2511}$$

As we can see, if we didn't have the second $\Sigma(0)$ term, the power law divergence of the ϕ field propagator would infect the self energy of the n field, and we would not get something that was logarithmically divergent. So indeed, mass renormalization is the correct procedure for dealing with the power divergence.

Now combine the two fractions in the last integrand into one and then multiply the numerator and denominator by $(p+q)^2$:

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N}\int_p \frac{1}{\Pi_2(p^2)}\left(\frac{(2q\cdot p - q^2)(q^2 + p^2 + 2q\cdot p)}{p^2(p^4 + q^4 - 4(q\cdot p)^2)}\right). \tag{2512}$$

Since the denominator is even under $p \mapsto -p$ we can simplify this to

$$\Sigma(q^2) - \Sigma(0) = \frac{2}{N}\int_{q<p<\Lambda} dp \frac{p}{2\pi\Pi_2(p^2)} \frac{4(q\cdot p)^2 - p^2q^2}{p^6(1 + q^4/p^4 - 4(p\cdot q)^2/p^4)} + \text{finite}. \tag{2513}$$

We can drop the terms other than the p^6 in the denominator and absorb them into the +finite part. Then using $\int p_\mu p_\nu \to \frac{1}{2}\delta_{\mu\nu}\int p^2$ since we are in two dimensions, we get

$$\Sigma(q^2) - \Sigma(0) = \frac{1}{N}\int_{q<p<\Lambda} dp \frac{2}{p\ln(p^2/m^2))} + \text{finite}. \tag{2514}$$

Now $d\ln(\ln(p^2/m^2)) = 2/[p\ln(p^2/m^2)]$, so

$$\Sigma(q^2) - \Sigma(0) = \frac{1}{N}\ln\left(\frac{\ln(\Lambda^2/m^2)}{\ln(q^2/m^2)}\right) + \text{finite}. \tag{2515}$$

As promised, we see that the new propagator has no power law divergences, but it does have a weird nested log structure. Thus mass renormalization has led to logarithmic divergences which can subsequently be cleaned up with field strength renormalization for the n field.

Now let us calculate the propagator directly (i.e. not assuming large N but assuming small $g^2(q)$), without using the saddle point approximation, and see to what extent the saddle point results are reproduced. We do this essentially by the background field method. We split up the field into slow and fast components by letting the slow component be some vector $n_0(x)$ and letting the fast components be fluctuations about that vector. Thus the relevant decomposition is

$$n = \sqrt{1-\psi^2}n_0 + \psi_a e^a, \tag{2516}$$

where the $\{e^a\}$ are a collection of vectors orthogonal to $n_0(x)$ on the sphere[264] Note that $n_0(x)$ has unit length and is orthogonal to all of the frames e^a. Now since n lies on the sphere, we know that $\partial_\mu n_0(x)$ will lie in the tangent space at x and hence be orthogonal to $n_0(x)$ and expressible in terms of the e^a's. Thus for some 1-form ω^a and some $\mathfrak{o}(N-1)$-valued 1-form A (since $O(N-1)$ is the group which acts on the basis vectors normal to n_0), we may follow Polyakov and write

$$\partial^\mu n_0 = \omega^a_\mu e^a, \qquad \partial_\mu e^a = [A_\mu]^{ab} e_b - \omega^a_\mu n_0. \tag{2518}$$

The minus sign in the second term is needed since n_0 being $\perp$ to all the e^a implies that the change of n_0 in the e^a direction is equal to the change of e^a in the $-n_0$ direction (draw a picture to check, or notice that the fact that the whole ensemble of frames rotates rigidly implies $n_0 \cdot \partial_\mu e^a + e^a \cdot \partial_\mu n_0 = 0$). Now we put this into the action for the n field (the $n^2 = 1$ constraint is explicitly built into our parametrization, so no need for Lagrange multipliers). The quadratic parts in the fields are

$$S_{\psi^2} = \frac{1}{2g^2}\int \left[(\delta_{ab}\partial_\mu\psi_b - A^{ab}_\mu\psi_b)^2 + B^a_\mu B^{b\mu}(\psi_a\psi_b - \psi_c\psi^c\delta_{ab}) + B^a_\mu B^{a\mu}\right], \tag{2519}$$

[264] This is an expansion which is essentially tantamount to going over to Riemann normal coordinates. The action is $\partial_\mu n^a \partial^\mu n_a$, which for our present decomposition becomes schematically

$$\partial_\mu(n_0+\psi)^a\partial^\nu(n_0+\psi)^b(\delta_{ab} - \frac{1}{3}R_{acbd}\psi^c\psi^d), \tag{2517}$$

where R_{acbd} is the Riemann curvature tensor for the target space S^{N-1}.

where we have used the skew symmetry of A by virtue of the fact that it lives in $\mathfrak{o}(N-1)$. The $B_\mu^a B_a^\mu$ term is the slow $(\partial_\mu n_0)^2$ part.

Let us now compute the β function, the field strength renormalization, and the propagator. Because of the decomposition of the n field we have chosen we will work in the Wilsonian point of view where we change the high energy cutoff by a small amount. Note that A_μ has dimension 1 and appears in the action only in the covariant derivative. A_μ really is a gauge field, since it is a connection on a frame bundle which just tells us how to relate one arbitrary choice for the e^a frame at one point to the arbitrary choice made at another. Thus the only way that A can appear in what follows is in the Maxwell term $F_A \wedge \star F_A$, but since we are in two dimensions this is irrelevant. Thus we can ignore the A field in matters concerning renormalization.

Now let us compute the correlation function

$$C_\psi^{ab} = \langle \psi^a \psi^b - \psi_c \psi^c \delta^{ab} \rangle. \tag{2520}$$

As mentioned above, we only need to care about the ω fields. Since the ω fields couple quadratically to the ψ's we have a single tetravalent vertex, and so to one loop order our only diagrams are a ψ loop and a ψ loop with an ω loop glued on. Since the free term for the ψ's is diagonal in a, b, we get

$$C_\psi^{ab} = (\delta^{ab} - (N-1)\delta^{ab}) \int_{\Lambda - \delta\Lambda}^{\Lambda} \frac{dp}{2\pi} \frac{g^2}{p}, \tag{2521}$$

since there are $N-1$ fields in the $\psi_c \psi^c$ contraction. So then taking $d\Lambda = -\delta\Lambda$ to be infinitesimal,

$$C_\psi^{ab} = \delta^{ab}(N-2) \frac{g^2}{2\pi} d \ln \Lambda. \tag{2522}$$

When we expand the exponential of the action in small ψ, do the integral over the fast fields, and then re-exponentiate, we thus get a term that goes as

$$\frac{1}{2g^2} \int B_\mu^a B^{a\mu} (N-2) \frac{g^2}{2\pi} d \ln \Lambda. \tag{2523}$$

Therefore the effective charge is

$$g_{eff}^2 = \frac{g^2}{1 + g^2 \frac{N-2}{2\pi} d \ln \Lambda}. \tag{2524}$$

Writing $g_{eff}^2 = g^2 + dg^2$ we obtain the beta function

$$\beta(g^2) = \frac{2-N}{2\pi} g^4, \tag{2525}$$

which as we have seen several times so far is asymptotically free when the symmetry group $SO(N)$ is non-Abelian (or when the n field maps into a space with positive curvature). We will need the expression for $g^2(p)$, which we can get by integrating the β function from a reference scale μ to p:

$$g^2(p) = \frac{g^2(\mu)}{1 - \frac{2-N}{2\pi} g^2(\mu) \ln(p/\mu)}. \tag{2526}$$

Now we want to get the 2-point function for the n fields and compare it to what we got with the saddle point approximation. We have

$$\begin{aligned}\langle n(r)n(0)\rangle &= \langle n_0(r)n_0(0)\sqrt{(1-\psi^2(r))(1-\psi^2(0))}+\psi_a(r)\psi^a(0)\rangle \\ &\approx \langle n_0(r)n_0(0)(1-\psi^2(0))\rangle \\ &= \langle n_0(r)n_0(0)\rangle_{p<\Lambda-\delta\Lambda}(1-\langle\psi_c(0)\psi^c(0)\rangle_{\Lambda-\delta\Lambda<p<\Lambda}),\end{aligned} \tag{2527}$$

where we dropped the $\psi(r)\psi(0)$ term since the ψ's are rapidly fluctuating, and expanded to quadratic order in ψ. We know the ψ^2 expectation value, since $\langle\psi_c\psi^c\rangle = -C^{aa}_\psi + \langle\psi_a\psi_a\rangle$ (no sum over a). So then

$$\langle n(r)n(0)\rangle = \langle n_0(r)n_0(0)\rangle_{p<\Lambda-\delta\Lambda}\left(1+\frac{N-1}{2\pi}g^2(\Lambda)d\ln\Lambda\right), \tag{2528}$$

since the two-point function for a single ψ component goes as $-d\ln\Lambda$.

From here we can read off the field strength renormalization: in order to ensure that the log divergences in the 2 point functions get canceled to one loop order, we need to have

$$\gamma(g^2) = \frac{N-1}{2\pi}g^2 \tag{2529}$$

(the positive sign comes from the fact that the free propagator goes to minus of the log term).

Knowing this, we can finally get the expression for the n-field propagator which is not limited to the large N saddle point. Let the propagator be $G_n(p^2)$. Then we have

$$\frac{\partial(G_np^2)}{\partial\ln(p/\mu)} = \gamma(g^2(p))(G_np^2), \tag{2530}$$

with μ the RG scale (momentarily going to a continuum RG picture). This equation is most easily proven graphically: the field strength renormalization counter terms appear in the full expression for the propagator in a geometric series of the form $\frac{1}{p^2}\gamma(g^2)\ln(p/\mu)\frac{1}{p^2}+\dots$, which gives rise to the above equation. We can also write this as

$$\frac{\partial\ln(p^2G_n)}{\partial\ln(p/\mu)} = \gamma(g^2(p)). \tag{2531}$$

Thus to order g^2 we can use this and our knowledge of γ to get

$$\begin{aligned}G_n(p^2) &= \frac{1}{p^2}\exp\left(\frac{N-1}{2\pi}g^2(\mu)\int\frac{d\ln(p/\mu)}{1+\frac{N-2}{2\pi}g^2(\mu)\ln(p/\mu)}\right) \\ &= \frac{1}{p^2}\left(1+\frac{N-2}{2\pi}g^2(\mu)\ln(p/\mu)\right)^{\frac{N-1}{N-2}}.\end{aligned} \tag{2532}$$

This would look like what we would expect from a free field provided that the term in the big parenthesis was not raised to a power. Thus as $N\to\infty$ we have

$$G_n(p^2;N\to\infty) = \frac{1}{p^2}\left(1+\frac{N-2}{2\pi}g^2(\mu)\ln(p/\mu)\right), \tag{2533}$$

which indeed indicates that the n field becomes free in the $N \to \infty$ limit.

Now we can compare this to the saddle point propagator we derived earlier. We see from the correlation function (2499) that the difference between the saddle point answer and the actual result is just a replacement of N with the correct coefficient $N-2$ (and so in particular we get the correct behavior in the Abelian $N=2$ case).

$SU(N)$ **nlσm:** Now we want to look at the case of the $SU(N)$ nonlinear σ model (here the fields are sections of $SU(N)$ bundles [though we will actually use $U(N)$], unlike the case of the $O(N)$ model where $O(N)$ was the symmetry group, not the target space. Sorry for the bad but standard terminology), which is very similar to the $O(N)$ model technically, but which must give us different answers since as we will see the $SU(N)$ version is *not* described by the saddle point in the large N limit.

To get the propagator we need the β function (so that we can get the effective coupling constant at an arbitrary momentum scale) and the anomalous dimension. We already found the β function back in (2274). There we had $\lambda = 2g^2$, and we had taken $N=2$. Getting the β function for general N isn't hard, though: we just have N terms in the trace rather than 2, and so we can translate our old result to

$$\beta(g^2) = -\frac{Ng^4}{4\pi}, \tag{2534}$$

which holds to order g^4. Just like the $O(N)$ model we have asymptotic freedom, with it becoming "more" asymptotically free at large N. Integrating this gives the charge at a given momentum scale in terms of the RG scale μ:

$$g^2(p) = \frac{g^2(\mu)}{1+\frac{N}{4\pi}g^2(\mu)\ln(p/\mu)}. \tag{2535}$$

Now we need to know the anomalous dimension. This is also easy to get by looking back at the previous problem we did. We first write $T^aT^a = C_2\mathbf{1}$ where the T^a are the $SU(N)$ generators. The quadratic casimir here is $C_2 = (N^2-1)/2N$ (check with $C_2 = 3/4$ for $SU(2)$, which works since the generators are $X/2, Y/2, Z/2$). By looking back at the previous problem, we find the anomalous dimension

$$\gamma = \frac{N^2-1}{2\pi N}g^2. \tag{2536}$$

Then we can use the earlier formula for $\partial \ln(Gp^2)$ to get the propagator for the $SU(N)$ field:

$$\begin{aligned} G_U(p^2) &= \frac{1}{p^2}\exp\left(\int d\ln(p/\mu)\ \frac{C_2}{\pi}\frac{g^2(\mu)}{1+\frac{N}{4\pi}g^2(\mu)\ln(p/\mu)}\right) \\ &= \frac{1}{p^2}\left(1+\frac{N}{4\pi}g^2\ln(p/\mu)\right)^{4C_2/N}. \end{aligned} \tag{2537}$$

When we integrate this over p to get the correlator in real space, the exponent gets another power since we are integrating the above expression in parenthesis against $d\ln p$. In the $N\to\infty$ limit $4C_2/N + 1 \to 3$, so we get

$$G_U(r, N\to\infty) \approx \left(1 - \frac{Ng^2}{4\pi}\ln(r/a)\right)^3. \tag{2538}$$

This does *not* go to just a single ln in the $N \to \infty$ limit, and so the $SU(N)$ theory does not look like a free theory as $N \to \infty$ (it actually *does* go to a free theory as $N \to \infty$, but the propagators are still not free and it only becomes free in a way that is hard to see with the present variables).

Given that the propagator is not free, the saddle-point approximation should not be good as $N \to \infty$. Why is this? Proceeding along the lines of the previous analysis, it would seem like we could take the Tr ln in the effective action for the Lagrange multiplier and do a saddle point analysis on it, since the coefficient of the Tr ln will contain an N.

This is not the case though, essentially because the matrix nature of the principal fields mean that the number of Lagrange multiplier fields grows with N, which destroys the saddle point. To see this, let us write the action as

$$S = \frac{1}{g^2}\int \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger], \tag{2539}$$

where $U(x)$ is a $U(N)$-valued field. There is really no difference between this and the $SU(N)$ case since the determinant part (the Abelian $U(1)$) decouples and doesn't affect issues relating to renormalizability.

As with the $O(N)$ model, we enforce that $UU^\dagger = \mathbf{1}$ using a Lagrange multiplier (note the differing factor of 1/2 since we now are working with complex fields):

$$S = \frac{1}{g^2}\int \left(\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \mathrm{Tr}[\lambda(\mathbf{1} - UU^\dagger)]\right). \tag{2540}$$

Integrating over the U's, which is possible since the addition of λ loosened the unitary constraint on the U's and turned the integral into a Gaussian, gives

$$S_{eff}[\lambda] = -\frac{1}{g^2}\int \mathrm{Tr}\lambda + N\mathrm{Tr}\ln(-\partial^2\mathbf{1} + \lambda). \tag{2541}$$

Here we get the N from e.g. diagonalizing and seeing that after taking the trace we are left with N complex scalars.

The effective action for λ looks essentially the same as it did for the $O(N)$ vector model, so it is natural to expect that λ gets an expectation value given by some saddle point condition:

$$\langle\lambda\rangle = m^2\mathbf{1}. \tag{2542}$$

Indeed, doing the saddle point analysis by varying with respect to λ_{aa} gives the same answer for $\langle\lambda_{aa}$ as it did for λ in the $O(N)$ vector model, just with slightly different numerical coefficients accounting for the complex nature of the field.

Just as we defined the ϕ fields in the $O(N)$ vector model, we define Φ fields as

$$\lambda = m^2\mathbf{1} + \frac{1}{\sqrt{N}}\Phi, \tag{2543}$$

where the trace of Φ has no $q = 0$ component, i.e. $\int \mathrm{Tr}\Phi = 0$, so that Φ doesn't affect the expectation value. Now put this into the effective action:

$$\begin{aligned} S_{\text{eff}} = &-\int \mathrm{Tr}\left[\mathbf{1}(\frac{m^2}{g^2} - p^2 - m^2)\right] - \frac{1}{2}\int_q \Phi_{ab}[\Pi_2]_{ab,cd}(q^2)\Phi_{cd} \\ &+ \frac{1}{3\sqrt{N}}\int_{q,r}[\Pi_3]_{ab,cd,ef}(q,r)\Phi_{ab,q}\Phi_{cd,r}\Phi_{ef,-q-r} + \dots, \end{aligned} \tag{2544}$$

where e.g.

$$[\Pi_2]_{ab,cd}(q^2) = \frac{\pi_2(q^2)}{2}[\delta_{ac}\delta_{bd}+\delta_{bc}\delta_{ad}], \qquad [\Pi_3]_{ab,cd,ed}(q^2) = \frac{\pi_3(q,r)}{6}[\delta_{ae}\delta_{fd}\delta_{cb}+\delta_{af}\delta_{ed}\delta_{bc}+\dots], \tag{2545}$$

where the ... indicates all ways to pair the indices in Kronecker delta functions so that a does not appear in the same δ as b and likewise for the pairs cd and ef. Here π_2 is the regular polarization bubble (with a fixed color structure) with internal propagators computed using the mass m^2 and π_3 is the three-legged version.

In the $O(N)$ model the saddle point was stable and we got a free theory since the nonlinear part with the Π_3 kernel made an insignificant contribution as N went to infinity, allowing the corrections to the free Φ propagator to vanish. This is not so in the $SU(N)$ model, though, and the saddle-point does not work. We see this by writing the Π_3 term in double-line notation as a sum of three-tentacled amoeba, where the sum is over the various ways to assign matrix indices to the six ends of the lines in the amoeba. The first order correction to the free Φ propagator caused by the Π_3 term thus looks like $N^{-1} \times \mathbb{P}$, where $\mathbb{P}$ is a double-line version of the regular polarization bubble. Importantly, it has a completely internal circular line that when summed over produces a factor of N, cancelling the N^{-1} that came from the two factors of $1/\sqrt{N}$. Thus this contribution can not be ignored as $N \to \infty$, and so the nonlinearities are important even at large N, corroborating our finding that the $SU(N)$ model was not free at large N. Proceeding in the standard way and using the Gauss-Bonnet theorem, one can show that all planar diagrams do not vanish as $N \to \infty$, while diagrams that glue up on higher-genus surfaces become increasingly small.

WZW Miscellanea

Today's problem is a smattering of little things about the WZW term. The format will be a series of mini-questions. I found these questions listed as exercises in Abanov's lecture notes on topological terms in QFT.

Preliminaries: For a vector n that lives in S^2, define

$$W = \frac{1}{16\pi i}\int_D \mathrm{Tr}[\widehat{n} \wedge d\widehat{n} \wedge d\widehat{n}], \tag{2546}$$

where D is a two-disk and $\widehat{n} = n^a\sigma^a$. Show that the variation of W only depends on the values that the n field takes on ∂D. Also show that

$$S_{WZW} = 4\pi S W[n^a] \tag{2547}$$

is well-defined as an action on the manifold ∂D provided that $S \in \frac{1}{2}\mathbb{Z}$.

First we rewrite W as

$$W = \frac{1}{16\pi i}\int_D i\epsilon^{abc}\mathrm{Tr}[n^a dn^b \wedge dn^d \sigma^c\sigma^d] = \frac{1}{8\pi}\int_D \epsilon^{\mu\nu} n\cdot(\partial_\mu n \times \partial_\nu n), \tag{2548}$$

where we've written out the derivative explicitly since $dn \wedge \times dn$ is bad notation. Now we vary this, sending $n \mapsto n + \delta n$. Since $n^1 = 1$, $\delta n \cdot n = 0$. But since $\partial_\mu n$ is also orthogonal to n, $\partial_\mu n \times \partial_\nu n$ is parallel to n, and so $(\delta n)\cdot(\partial_\mu n \times \partial_\nu n) = 0$. Thus the variation is

$$\delta W = \frac{1}{8\pi}\int_D \epsilon^{abc} n^a (d\delta n^b \wedge dn^c + dn^b \wedge d\delta n^c). \tag{2549}$$

Integrating both terms by parts and again using that $(\delta n)\cdot(\partial_\mu n \times \partial_\nu n) = 0$ as well as the supercommutativity of $\wedge$, we get

$$\delta W = \frac{1}{4\pi}\int_{\partial D} dx^\mu \ \epsilon^{abc}\delta n^a \partial_\mu n^b n^c, \tag{2550}$$

and so indeed, the variation only cares about the fields on the boundary. This is because W is measuring the area on the target S^2 swept out by the n field. This area only depends on the trajectory of the n field as one moves around the circle ∂D. Indeed, the boundary values of n on ∂D trace out some region on the target S^2, and the values of the field on the interior of D "fill in" the area enclosed by this trajectory in a smooth way. By thinking of this visually, it's clear that changing the way that D "fills in" this region cannot change the total (signed) area swept out by the n field.

Now for the well-definedness of S_{WZW}. The difference in two extensions to the 2-manifold D from the given ∂D is the integral of S_{WZW} over a closed 2-manifold M. Since δW only depended on the values of n on the boundary, since $\partial M = 0$ we know that the values of W evaluated over M must be quantized. All we need to do is check the coefficient. We've got

$$S_{WZW} = S\int_M \epsilon^{abc}\frac{1}{2}n^a(dn^b \wedge dn^c). \tag{2551}$$

But the integrand on the RHS is precisely the pullback of one half of the volume form on S^2 by the map $n : M \to S^2$ (the $1/2$ because of the antisymmetrization and since the volume form is $\epsilon^{abc}n^a\partial_x n^b \partial_y n^c$ if the coordinates are x, y). So then

$$S_{WZW} = S\int_{n(M)\subset S^2} \mathrm{vol} = 4\pi S w, \tag{2552}$$

where $w \in \mathbb{Z}$ is the winding number. Thus we get something well-defined provided that $S \in \frac{1}{2}\mathbb{Z}$. The most suggestive way to write S_{WZW} is probably

$$S_{WZW} = 2S\frac{2\pi}{\Omega_2}\frac{1}{2}\int_D \epsilon^{\mu\nu} n\cdot(\partial_\mu n \times \partial_\nu n), \tag{2553}$$

where $\Omega = 4\pi$. If we let D be a sphere, then the integral gives us an integer multiple of $2\Omega_2$ (where the 2 comes from the antisymmetrization of the derivatives in spacetime—if we used a $\wedge$ for the spacetime derivatives we wouldn't need the 2).

Spin precession: Now add a magnetic field h, so that

$$S[h] = S_{WZW}[n] - S \int dt\ h^a n^a. \tag{2554}$$

Assume h is independent of time for simplicity. Derive spin precession in an $SU(2)$ invariant way.

We just compute the equations of motion. To do this, add a Lagrange multiplier to enforce $n^2 = 1$ so that we can vary n freely:

$$S[h, \lambda] = S_{WZW}[n] - S \int dt\ h^a n^a + \frac{1}{2}\lambda(n^2 - 1). \tag{2555}$$

We can use our knowledge from the first part to write

$$\delta S = \int dt\ \left(S\epsilon^{abc}\partial_t n^b n^c + \lambda n^a - Sh^a\right)\delta n^a. \tag{2556}$$

Taking the term in the parenthesis and dotting it with n,

$$\lambda = Sh^a n^a - S\epsilon^{abc} n^a \partial_t n^b n^c = Sh^a n^a. \tag{2557}$$

Thus the equation of motion gives

$$\epsilon^{abc}\partial_t n^b n^c + n^a(h^b n^b) - h^a = 0, \tag{2558}$$

or as a vector equation,

$$\partial_t n \times n = h - (h \cdot n)n. \tag{2559}$$

Note that as expected, if we set n parallel to h then n doesn't want to move, while if we set n normal to h then n wants to precess about the h axis.

Quantization: Derive the commutation relations for the spin operator from S_{WZW}. Use this to check that the equations of motion you derived earlier correspond to the Heisenberg equations of motion.

To derive the spin commutation relations we should figure out what the symplectic form is. Since we only have the ϕ and θ variables, they will label the phase space. Since the phase space is a sphere, we expect a closed but not exact symplectic form. We want to start from the term

$$S_{WZW} = \frac{S}{2}\int_D \epsilon^{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n). \tag{2560}$$

We can proceed in kind of a dumb way by varying this to find the symplectic current and then taking a second variation to find what the symplectic form is, after which the Poisson brackets can be read off. Luckily we already computed the variation in terms of the n field. Plugging in the coordinate representation for n, doing a fair bit of algebra (use Mathematica as it's fairly heinous), and using the equations of motion gives the symplectic current. Taking another variation gives the symplectic form, which if I did the algebra correctly is

$$j = -\delta(S\cos\theta) \wedge \phi \implies \Omega = -\delta(S\cos\theta) \wedge \delta\phi, \tag{2561}$$

where the expression is evaluated at some given initial time (symplectic currents are integrated over codimension 1 Cauchy slices, which in our case are just points). Thus from the symplectic form we conclude that the symplectic partner for ϕ is $S\cos\theta$, and so

$$\{\phi, S\cos\theta\} = 1. \tag{2562}$$

Thus when we take the Poisson bracket of two quantities we do it as follows:

$$\{X, Y\} = -\frac{1}{S\sin\theta}(\partial_\phi X\partial_\theta Y - \partial_\phi Y\partial_\theta X), \tag{2563}$$

which indeed gives $\{\phi, S\cos\theta\} = 1$.

As expected, these expressions only make sense within a coordinate patch, and the symplectic form is not globally exact, since it is the field strength for a monopole configuration on S^2.

Anyway, now we can use this to compute the Poisson bracket of the different n vectors. The relevant ones are $S^x = S\cos\phi\sin\theta,\ S^y = S\sin\phi\sin\theta,\ S^z = S\cos\theta$. For example,

$$\{S^y, S^z\} = -\frac{S}{\sin\theta}(-\cos\phi\sin^2\theta + 0) = S^x. \tag{2564}$$

Of course, the general rule is $\{S^a, S^b\} = \epsilon^{abc}S^c$. When we pass to quantum mechanics we change to commutators and add in the i, and as a result derive the spin commutation relations. Note that we did not start from anything involving Pauli matrices to do this.

Now we check the Heisenberg equations of motion. The WZW term doesn't enter into the Hamiltonian since it is linear in time derivatives (but of course it still affects the equations of motion, unlike e.g. a theta term). Then we can compute

$$[H, S^a] = [S^a, S^b]h^b = i\epsilon^{abc}S^c h^b \implies \partial_t S^a = \epsilon^{abc}S^b h^c \implies \partial_t n^a = \epsilon^{abc}n^b h^c. \tag{2565}$$

Is this equivalent to the equations of motion we derived earlier? Yes: putting this into the equations of motion implies

$$h^a - (h^b n^b)n^a = \epsilon^{abc}\epsilon^{bde}n^d h^e n^c = (\delta_{cd}\delta_{ae} - \delta_{ad}\delta_{ce})n^d h^e n^c, \tag{2566}$$

which is a trivial equality.

Relation to the θ term: Now suppose that the value of n on the 1-manifold is constrained to lie on a circle of constant latitude, with θ a constant. Find S_{WZW} in this case (note that we do not impose this restriction on the extension of n into the disk [and it will in general not be possible to impose such a condition]). Show that making this restriction on θ turns the WZW term into a θ term.

Also, in an AFM chain, show how the WZW term gives rise to a θ term that is nontrivial when $2S$ is odd. Sort of conversely, motivate why an $S = 1$ chain with a θ term placed on an open line has two spin 1/2's at the boundary.

First let's compute δW with this restriction. In spherical coordinates,

$$\delta n = (-\delta\phi\sin\phi\sin\theta,\ \delta\phi\cos\phi\sin\theta,\ 0)^T. \tag{2567}$$

Similarly, $\partial_\mu n$ also has no z component. Thus the integrand of δW goes like $n^z(\delta n \times \partial_t n)^z$. But from the form of δn, the cross product vanishes, and so $\delta W = 0$. This means that with this restriction on θ, W is quantized. Thus in order to compute W, we are free to choose a convenient extension of n into the 2-manifold as well as a convenient shape for the 2-manifold, since W cannot change if we change either of these choices. So we will choose the bounding 2-manifold to be the unit disk, with coordinates ϕ, r. We are even allowed to choose the ϕ coordinate of the disk to be an integer multiple of the ϕ coordinate on the S^2 that n maps into, where the integer is the winding number of the spacetime (really just time) S^1 into the $S^1 \subset S^2$ defined by the constant value of θ. If this winding number is w, then with this restriction we can choose n to be

$$n = (-\sin(w\phi)\sin(r\theta),\ \cos(w\phi)\sin(r\theta),\ \cos(r\theta))^T\,. \tag{2568}$$

The first two components reduce to w times the volume form on S^1 at $r = 1$ where the field is constrained to lie on the constant θ circle. This field extension we've chosen looks like a cowlick on the head of a particularly curly-haired person, with the hair all standing straight up at the center of their head.

Putting this into (no Jacobian changing from Cartesian coordinates since our integrand is a differential form)

$$S_{WZW} = S\int_{D^2} drd\phi\ \epsilon^{abc}n^a(\partial_r n^b \partial_\phi n^c) \tag{2569}$$

and doing some algebra gives

$$S_{WZW} = 2\pi i S w(\cos\theta - 1), \tag{2570}$$

which is just measuring the area on the sphere defined by the line of constant latitude at the given value of θ. So, we get a θ term (where the action is proportional to the volume form on the temporal S^1) with coefficient iSw. As it must, when $\theta = \pi$ (the polar angle of the n vector, not the θ of the θ term—sorry!) this gives us something in $2\pi i\mathbb{Z}$. Also note that when the vector is restricted to lie on the equator, we get a theta term at theta-angle $\theta = \pi$ (relevant for te AFM spin chain).

For the AFM spin chain, the action has a term which is just $\sum_i S_{WZW}[n_i]$, where n_i is the n-field at the site i. Since we expect a staggered configuration to be a good "mean field", we can as a first pass examine $\sum_i S_{WZW}[(-1)^i n_i]$, where now n_i is slowly varying. Since $S_{WZW}[n]$ measures the area on the target S^2 enclosed by the trajectory of the n field, $S_{WZW}[n_j] + S_{WZW}[-n_{j+1}] = S_{WZW}[n_j] - S_{WZW}[n_{j+1}]$ measures a difference in the areas swept out by the two curves along their trajectories. Drawing a picture shows that this area difference is proportional to $\int dt\ \epsilon^{abc}n^a\partial_x n^b \partial_t n^c$, where x is the coordinate along the chain. Integrating this over the length of the chain, we get

$$\frac{1}{2}S\int \epsilon^{abc}n^a dn^b \wedge dn^c. \tag{2571}$$

Note that this term now maps the spacetime S^2 into the target S^2, and is defined without reference to a bounding 3-manifold. Since it integrates to $2\pi S w$ where w is the winding number, it is a θ term which only contributes if $2S$ is odd. Thus the WZW terms sum up and interfere with eachother to produce a θ term.

Now suppose we are given the nlσm description of an open AFM chain with the θ term, for $2S$ even. Suppose we already know that the chain is gapped for this choice of S, so that we can focus on the θ term. Now the θ term is the integral of the volume form over the chain. The volume form is closed and is also exact since spacetime has trivial H^2, so the θ term must reduce to something localized on the ends of the chain. Indeed, plugging in the form for n means the θ term is, after some algebra,

$$\frac{S}{2}\int dxdt \ \sin\theta(\partial_t\theta\partial_x\phi - \partial_x\theta\partial_t\phi). \tag{2572}$$

Let's assume that we're working with periodic time. Then we can integrate by parts and write the term as

$$-\frac{S}{2}\int_{\partial C}\cos\theta\partial_t\phi, \tag{2573}$$

where the integral is over the ends of the chain. This means that we get WZW terms on the chain ends. In particular if $S = 1$ then we get WZW terms with spin 1/2, suggesting that the critical $S = 1$ AFM spin chain hosts spin 1/2's at its edges.

WZW action from fermions

This problem is one of the problems I found listed as an exercise in Abanov's lecture notes on topological terms in QFT.

Consider a fermion in 0+1 dimensions coupled to a vector:

$$S = \int d\tau \ \bar{\psi}(\partial_\tau + Mn^a\sigma^a)\psi, \tag{2574}$$

where M is a mass and n maps into S^2. Integrate out the fermion and find the leading two terms in a $1/M$ expansion. To do this, it is helpful to first compute δS and then un-do the variation at the very end.

Integrating over the fermions,

$$\delta S = -\delta\mathrm{Tr}\ln(\partial_\tau + M\widehat{n}), \tag{2575}$$

where we have used the notation $\widehat{n} = n^a\sigma^a$. Let $D = \partial_\tau + M\widehat{n}$ be the "covariant derivative". Then

$$\delta S = -\mathrm{Tr}(\delta D D^{-1}) = -\mathrm{Tr}(\delta D D^\dagger(DD^\dagger)^{-1}), \tag{2576}$$

which we have written in this way since expanding $DD^\dagger$ in $1/M$ is easier. Now since $\widehat{n}^2 = n^2\mathbf{1} = \mathbf{1}$,

$$DD^\dagger = (\partial_\tau + M\widehat{n})(-\partial_\tau + M\widehat{n}) = -\partial_\tau^2 + M^2 + M\partial_\tau\widehat{n}. \tag{2577}$$

We then expand the inverse of this in the following way, writing $G_f = (-\partial_\tau^2 + M^2)^{-1}$ for the free propagator:

$$(DD^\dagger)^{-1} = \frac{G_f}{\mathbf{1} + G_f M \partial_\tau \widehat{n}} = \frac{G_f(\mathbf{1} - G_f M \partial_\tau \widehat{n})}{1 - (G_f M \partial_\tau n)^2}, \tag{2578}$$

where the denominator on the RHS is now just a number. To the leading orders in M we can actually just replace the denominator by 1, and so

$$\delta S = -\mathrm{Tr}\left[(M\delta\widehat{n})(-\partial_\tau + M\widehat{n})G_f(\mathbf{1} - G_f M \partial_\tau \widehat{n})\right]. \tag{2579}$$

We do the trace by

$$\mathrm{Tr}[\mathcal{O}] = \int d\tau \int \frac{d\omega}{2\pi} e^{i\omega t} \mathrm{Tr}_\sigma[\mathcal{O}] e^{-i\omega t}, \tag{2580}$$

where Tr_σ indicates a trace over the spin degrees of freedom. We wrote the trace like this (by inserting a resolution of the identity in frequency space) since the operator $\mathcal{O}$ in question isn't local in time (it involves a bunch of G_f's), but is local in frequency space. Writing this out,

$$\delta S = -\frac{M}{2\pi} \int d\tau d\omega \, \mathrm{Tr}_\sigma \left[\delta\widehat{n}(i\omega + M\widehat{n}) \frac{1}{\omega^2 + M^2} \left(\mathbf{1} - \frac{1}{\omega^2 + M^2} M \partial_\tau \widehat{n} \right) + \delta\widehat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau^2 \widehat{n} \right]. \tag{2581}$$

Dropping things that will die after taking the spin trace and things that are odd in frequency,

$$\delta S = \frac{M}{2\pi} \int d\tau d\omega \, \mathrm{Tr}_\sigma \left[\delta\widehat{n} M \widehat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau \widehat{n} - \delta\widehat{n} \frac{1}{(\omega^2 + M^2)^2} M \partial_\tau^2 \widehat{n} \right]. \tag{2582}$$

The relevant integral is

$$\int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + M^2)^2} = \frac{1}{4M^3}, \tag{2583}$$

and so

$$\delta S = \int d\tau \, \mathrm{Tr}_\sigma \left[\frac{1}{4} \delta\widehat{n}\widehat{n} \partial_\tau \widehat{n} - \frac{1}{4M} \delta\widehat{n} \partial_\tau^2 \widehat{n} \right]. \tag{2584}$$

Taking the trace and integrating the second term by parts,

$$\delta S = \int d\tau \left(-\frac{i}{2} \epsilon^{abc} \delta n^a \partial_\tau n^b n^c + \frac{1}{2M} \delta(\partial_\tau n)^a \partial_\tau n^a \right). \tag{2585}$$

Looking at yesterday's problem, we see that the first term is precisely the variation of the WZW term for a single spin 1/2 in 0+1 dimensions (spin 1/2 because of the prefactor), while the second term is the variation of a kinetic term for the vector. So the effective action is

$$S_{eff} = S_{kin} + S_{WZW} = \frac{1}{4M} \int d\tau \, (\partial_\tau n^a)^2 - \frac{1}{8\pi i} \int_D \mathrm{Tr}[\widehat{n} \wedge d\widehat{n} \wedge d\widehat{n}], \tag{2586}$$

where D is any two-disk that bounds the temporal circle.

QED_3 as a σ model

The goal of this problem is to work out the details of a description of QED_3 as a σ model that appears in a very nice paper by Senthil and Fisher [23].

The starting point is QED_3 with $N = 2$ flavors of Dirac fermions and the square of an $SU(2)$ vector that we want to use as a potential order parameter field. The action is

$$S = \int \left(\bar{\psi} i \not{D}_A \psi + g(\bar{\psi}\sigma^a \psi)^2 \right), \tag{2587}$$

where σ^a is a Pauli matrix in flavor space (not spin space!) and we've omitted the $F \wedge \star F$ term. We will show that at long distances,[265] this behaves like an $O(4)$ model with a theta term at $\theta = \pi$.

To show this, we will decouple the $g(\bar{\psi}\sigma^a\psi)^2$ term with a 3-component vector N^a. After this is done, we write with $N^a = Mn^a$ for n a unit vector in S^2. Working with M large, we integrate out the fermions and find the current. We then comment on how the skyrmions are endowed with spin.

We then switch to a $\mathbb{CP}^1$ representation for the vector, and show that we get a mixed CS term between the electromagnetic gauge field and the $\mathbb{CP}^1$ gauge field. This then in turn reduces to the $O(4)$ model with a θ term at $\theta = \pi$.

First we do the decoupling. We add the term $N^2/2g$ to the action where N is a 3-component vector, and then shift

$$N^a \mapsto N^a + ig\bar{\psi}\sigma^a\psi. \tag{2588}$$

We then take $N = Mn$ for n a unit vector (this restriction is supposed to not affect the phase diagram), and so

$$S = \int \bar{\psi} \left(i\not{D}_A + iMn^a\sigma^a \right) \psi. \tag{2589}$$

Let's first find the current after integrating out the fermions, which we will see gives electric charge to the solitons. Using the trick of two days ago, we have

$$J^\mu = i\frac{\delta}{\delta A_\mu} \mathrm{Tr} \ln \left(i\not{D}_A + iMn^a\sigma^a \right) = i\mathrm{Tr}\left(\gamma^\mu (D^\dagger D)^{-1} D^\dagger \right), \tag{2590}$$

with

$$D = i\not{\partial} + iMn^a\sigma^a. \tag{2591}$$

[265] And long distances are really all we can talk about: from the kinetic term, we see that the dimension of ψ is 1, so that $[g] = -1$ and the model is (naively) non-renormalizable. We will be relating this to the $O(4)$ nlσm in three dimensions, which is also non-renormalizable, and we will only be able to make statements about IR physics in the following.

Here γ^μ are the Pauli matrices acting on spin space (we are in Euclidean signature), and we have set $A = 0$ in D since to find the current we can take the functional derivative with respect to J and then set the gauge field to zero. We then expand

$$(D^\dagger D)^{-1} = \frac{1}{-\partial^2 + M - M(\partial\!\!\!/ n^a)\sigma^a} = \frac{G_f}{\mathbf{1} - G_f M(\partial\!\!\!/ n^a)\sigma^a}, \tag{2592}$$

where as usual $G_f = (-\partial^2 + M^2)^{-1}$. To write it in this form, we have to remember that γ^μ and σ^a are both Pauli matrices, but commute since they act on different tensor factors. We expand this as

$$(D^\dagger D)^{-1} \approx G_f(\mathbf{1} + G_f M(\partial\!\!\!/ n^a)\sigma^a)(\mathbf{1} + G_f M(\partial\!\!\!/ n^b)\sigma^b). \tag{2593}$$

After putting this into our expression for J^μ and dropping things which get obviously killed by the momentum integration or the spin trace,

$$J^\mu = i\mathrm{Tr}\left(\gamma^\mu\left[2G_f^2 M^2(\partial\!\!\!/ n^a)\sigma^a + G_f^3 M(\partial\!\!\!/ n^a)\sigma^a(\partial\!\!\!/ n^b)\sigma^b\right] iMn^c\sigma^c\right). \tag{2594}$$

The first term goes like $(\partial_\mu n^a)n^a$ (with the index structure required if it wants to survive the flavor trace) which dies since n being a unit vector means that it's orthogonal to ∂n. In order to survive the traces over the spin and flavor indices, we need a $\epsilon^{\mu\nu\lambda}\epsilon^{abc}$ index structure. Thus

$$J^\mu = M^3\mathrm{Tr}(\mathbf{1}_{4\times 4})\int d^3x\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+M^2)^3}\epsilon^{\mu\nu\lambda}\epsilon^{abc}n^a\partial_\nu n^b\partial_\lambda n^c, \tag{2595}$$

which is starting to look a lot like a skyrmiony thing. Indeed, the integral gives $1/(32\pi|M|^3)$ and so

$$J^\mu = \frac{\mathrm{sgn}(M)}{8\pi}\int\epsilon^{abc}n^a[\star(dn^b\wedge dn^c)]^\mu. \tag{2596}$$

Thus, we have shown that skyrmions carry electric charge! The current that couples to electromagnetism is precisely the topological charge density.

There are a few more parts of the effective action that we need to get. One is the kinetic term for n. We get the kinetic term by taking a variation with respect to n^a. Writing the variation of the trace as $\mathrm{Tr}[\delta DD^\dagger(DD^\dagger)^{-1}]$ and expanding, to leading order in the large M expansion the relevant term comes when the $i\partial\!\!\!/$ in $D^\dagger$ hits the term in the expansion for $(DD^\dagger)^{-1}$ that is linear in M. We get

$$\begin{aligned}\delta S &= -\mathrm{Tr}[M^2\delta n^a\sigma^a G_f^2\partial\!\!\!/^2 n^b\sigma^b] + \dots \\ &= -4M^2\int d^3x\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+M^2)^2}\delta n^a\partial_\mu\partial^\mu n^a + \dots.\end{aligned} \tag{2597}$$

The integral is $1/8\pi M$, and so after integrating by parts over spacetime and integrating over the variation (we have $(\delta\partial_\mu n^a)\partial^\mu n^a$ which is a total variational derivative) we get

$$\delta S = \frac{M}{4\pi}\int d^3x(\partial_\mu n^a)^2. \tag{2598}$$

The last thing in the effective action we need is a topological term, which bestows spin upon the skyrmions. Since it's non-perturbative we have to find a clever way of getting to it. Such a way can be found in e.g. [1] (or in the classic paper by Wilczek and Zee), which we now go through. Basically, the idea is to consider a particular field history for a skyrmion and find what the topological term is in that case, and then write down the general topological term by writing down a covariant version of the specific topological term.

We consider a field configuration on e.g. $S^2 \times S^1$ with a skyrmion that rotates by 2π along the S^1 factor. This trajectory measures the spin angular momentum of the skyrmion, since the phase acquired during the trajectory is $e^{iS} = e^{2\pi i J}$. If the skyrmion is to be rotated around z axis as we pass around the S^1, then we can consider the configuration where

$$n^a\sigma^a = e^{i\sigma^z\alpha(t)/2} n_0^a \sigma^a e^{-i\sigma^z\alpha(t)/2}, \tag{2599}$$

where t is the S^1 coordinate, $\alpha(2\pi) = \alpha(0) + 2\pi$, and n_0 is a static reference vector.

To get at the angular momentum of the skyrmion in a perturbative way, we can perturb in $\partial_t\alpha$ since the evolution is adiabatic. To bring a time derivative of α into the functional determinant, we consider performing the transformation $\psi \mapsto e^{i\alpha(t)\sigma^z/2}\psi$. This is not single-valued though, and it changes the boundary conditions of the fermions around the S^1 (from periodic to antiperiodic or vice-versa). We can fix the boundary conditions by doing a further transformation which only affects the phase by $\psi \mapsto e^{i\alpha(t)/2}\psi$, which cancels out the change in boundary conditions. Doing this transformation kills the exponentials in the $e^{i\sigma^z\alpha(t)/2} n_0^a \sigma^a e^{-i\sigma^z\alpha(t)/2}$ term, and picks up a time derivative term from the Dirac operator. Thus we have to evaluate

$$\operatorname{Tr}\ln\left(i\not{D}_A + \frac{i\gamma^0}{2}(\mathbf{1}+\sigma^z)\partial_t\alpha + iMn_0^a\sigma^a\right). \tag{2600}$$

The angular momentum of the skyrmion is obtained by differentiating the effective action with respect to the angular frequency with which the skyrmion rotates (think $E = m\omega^2 r^2/2 \implies \partial_\omega E = mvr = J$), and so to get the spin of the skyrmion we need to functionally differentiate the above with respect to $\partial_t\alpha$. This is easy though, since we can just look back at how we computed the variation of the effective action with respect to the gauge field—$\partial_t\alpha$ appears in the functional determinant in the exact same way as A_0, except with an extra $(\mathbf{1}+\sigma^z)/2$ tacked on. Computing the variation in the same way as before and then integrating over the variation gives the topological term (a Hopf term, as we will see shortly) on this manifold:

$$S_H = \frac{i}{2}\int_{S^2\times S^1} \partial_t\alpha J^0, \tag{2601}$$

where J^0 is the time component of the topological current given earlier (selected out because of the γ^0 in the trace). Note that since the topological charge is conserved, the Hopf term vanishes if α is topologically trivial, i.e. if it does not wind by some element of $2\pi\mathbb{Z}$ (since if it does not wind we can integrate by parts and get zero).

So, we've found a topological term that is constructed exactly in the same way as the coupling of A_μ to the topological current, except that A_μ is replaced by parts of the n field (in our example it is replaced by $\partial_t\alpha$). Because of the normalization of the topological current and the factor of 1/2 in front, we see that the above term gives the skyrmions spin 1/2.

To find the general presentation of the Hopf term, we just have to "covariantize" the particular form of S_H found above. One way to do this is to write it in terms of the $SU(2)$ matrix U which rotates between σ^z and $n^a\sigma^a$: with some algebra one can check that we get

$$S_H = -i\pi\frac{1}{24\pi^2}\int \mathrm{Tr}[U^\dagger dU \wedge U^\dagger dU \wedge U^\dagger dU]. \tag{2602}$$

Actually, a slightly cooler presentation of this action is as a linking number a la the usual interpretation of the Hopf invariant. To write it down in this way we need to recast stuff in the $\mathbb{CP}^1$ language.

Let's then switch over to $\mathbb{CP}^1$ variables. The kinetic term for the n vector becomes the $|D_a z|^2$ term (here $a_\mu = -iz^\dagger\partial_\mu z$), where $n^a\sigma^a = 2zz^\dagger - \mathbf{1}$. What about the $A^\mu J_\mu$ term coupling the skyrmions to the electromagnetic field? The topological current J_μ maps to $i\star da/2\pi = i\star(dz^\dagger\wedge dz)/2\pi$ in the $\mathbb{CP}^1$ variables, which one can see either with a fair bit of algebra or with a "what else could it be" argument (the coefficient is fixed by the integrality of the topological charge).[266]

Finally, the Hopf term is just like the $A_\mu J^\mu$ coupling, except it has no A and only involves the n field. This means that in the $\mathbb{CP}^1$ language, it becomes a CS term for the $\mathbb{CP}^1$ gauge field, $a\wedge da$ (this is another way to see that it computes the Hopf invariant). Thus our new action is

$$S = \frac{M}{4\pi}\int |(\partial_\mu - ia_\mu)z|^2 + \frac{i}{2\pi}\int A\wedge da + \frac{1}{2e^2}\int F_A\wedge\star F_A + S_H, \tag{2606}$$

where S_H is now the CS term for a.

Since the $\mathbb{CP}^1$ variables were introduced to deal with a theory involving a vector living in S^2, we do not yet have the $O(4)$ model we were promised. To get an $O(4)$ model, we have to access the full $O(4)$ symmetry of the z variables, instead of the $O(3)$ symmetry we get when acting on the quotient $S^3/S^1 = S^2$. Thus to get an $O(4)$ symmetry, we need to "eliminate the $\mathbb{CP}^1$ gauge field a", since it is responsible for quotienting out by S^1.

We can see how this might happen by doing the integral over A and checking what the resulting action for the z fields is. The parts in the action involving A are

$$S_A = \frac{1}{2e^2}\int F_A\wedge\star F_A + \frac{i}{2\pi}\int a\wedge F_A. \tag{2607}$$

[266] More carefully, working in a parametrization where

$$z = \begin{pmatrix}\cos(\theta/2)e^{i(\gamma+\phi)/2}\\ \sin(\theta/2)e^{i(\gamma-\phi)/2}\end{pmatrix}. \tag{2603}$$

Then (factors of 2 and minus signs?!)

$$da = -idz^\dagger\wedge dz = -\sin(\theta/2)\cos(\theta/2)\,d\theta\wedge d\phi = \sin(\theta/2)\,d\theta\wedge d\phi, \tag{2604}$$

which is the the pull-back of the volume form on S^2, and hence $\star da$ is indeed the topological current. This can also be written as

$$[F_a]_{\mu\nu} = (D_{[\mu}z)^\dagger D_{\nu]}z, \tag{2605}$$

which can be checked with a little bit of algebra.

To integrate over A carefully, we first add an extra field ϕ in the path integral that enforces the exactness of F, and then treat F as an unconstrained field that we path-integrate over independently. The new action for the gauge fields reads

$$S_A = \frac{1}{2e^2}\int F\wedge\star F + \frac{i}{2\pi}\int a\wedge F_A + \frac{i}{2\pi}\int \phi\wedge dF. \tag{2608}$$

We can now perform the shift

$$F\mapsto F+\alpha\star a, \tag{2609}$$

where α is some constant to be determined. A little algebra shows that choosing $\alpha=-ie^2/2\pi$ cancels out the CS coupling of F to a. After doing this shift, we have

$$S_A = \frac{1}{2e^2}\int F\wedge\star F + \frac{i}{2\pi}\int \phi\wedge dF + \frac{e^2}{4\pi^2}\int \phi\wedge d\star a + \frac{e^2}{8\pi}\int a\wedge\star a. \tag{2610}$$

If we work in transverse gauge $d^\dagger a=0$ things become simpler since then $d\star a=0$ and the extra $\phi\wedge d\star a$ term dies. Then we can do the integral over ϕ to set $F=dA$, and so

$$S_A = \frac{1}{2e^2}\int F_A\wedge F_A + \frac{e^2}{8\pi}\int a\wedge\star a. \tag{2611}$$

Thus we have traded the CS coupling between the gauge fields for a mass term for a. The Maxwell field is now totally decoupled, and just gives a constant in the path integral which we can drop. Furthermore, the massiveness of a induced by the electromagnetic field means that we can integrate it out and get

$$S=\frac{M}{4\pi}\int|\partial_\mu z|^2+S_H+\dots, \tag{2612}$$

where $\dots$ include terms that are not $O(4)$ symmetric (note that S_H is $O(4)$ symmetric). In this way of writing it, the $O(4)$ symmetry comes from the fact that we can rotate among the four components of the complex spinor z (recall $|z|^2=1$). All the terms in $\dots$ are seemingly less relevant at long distances than the $|\partial_\mu z|^2$ term. Whether the $\dots$ terms can really be ignored or not is a tricky dynamical question that apparently no one knows the answer to. However, it is reasonable to hypothesize (based on the present understanding of the "duality web; more on this is a future diary entry) that these terms are indeed irrelevant, which frees up the full $O(4)$ symmetry of the model.

So accepting this somewhat shaky conclusion, we have what we wanted to find: an $O(4)$ model with a theta term at $\theta=\pi$ (it's a θ term since we started out with a Hopf term for the $n\in S^2$ vector, but after switching to z variables and killing off the gauge fields it becomes a θ term for an $O(4)$ field since we are now mapping $S^3\to S^3$ rather than $S^3\to S^3/S^1$). Oh yeah, one final thing: do we have an $O(4)$ symmetry in the formulation in terms of the U's? Yes, it acts as

$$O(4)\cong(SU(2)_L\times SU(2)_R)/\mathbb{Z}_2, \tag{2613}$$

with the two $SU(2)$ factors acting on the U matrices on the left and on the right, respectively (this is a symmetry since in S_θ all the U's are sandwiched by $U^\dagger$'s, and vice versa—the quotient by $\mathbb{Z}_2$ avoids double-counting the center $Z(SU(2))=\mathbb{Z}_2$).

β function for general two-dimensional nonlinear σ models and Ricci flow

Consider a nlσm in two dimensions, with action[267]

$$S = \frac{1}{4\pi\alpha}\int_\Sigma d^2\sigma\,\gamma^{\mu\nu}g_{ij}(X)\partial_\nu X^i\partial_\mu X^j. \tag{2614}$$

Here $X : \Sigma \to M$ for some Riemannian manifold M with metric g_{ij} and some Riemann surface Σ with metric $\gamma_{\mu\nu}$, and where $g_{ij}(X) = X^* g_{ij}$ is the pullback of the metric on M by X.

Show that the beta function(al) for g_{ij} is, to one loop order, given by the Ricci flow equation

$$\beta_{ij} = \frac{dg_{ij}}{d\ln\mu} = R_{ij} = \frac{1}{2}Rg_{ij}, \tag{2615}$$

where R_{ij} is the Ricci tensor (of the target space M), R is the Ricci scalar, and we have used

$$R_{ij} = \frac{1}{2}g_{ij}R, \tag{2616}$$

which holds in two dimensions (see the previous diary entry on the linear dilaton CFT). This is a weak-coupling result, which you should derive by assuming that the geometry of M varies slowly compared to $\sqrt{\alpha}$. Note that the coupling for the theory is roughly $\sim 1/r$ for r the radius of curvature, so that if $R_{ij} > 0 \implies \beta_{ij} > 0$, then since$r$ increases as we flow to the UV, we see that $R_{ij} > 0$ implies the theory is asymptotically free.

Note that unlike a free boson in two dimensions, here we are letting X be *dimensionful*, with mass dimension -1, and hence $[\alpha] = -2$ so that $\sqrt{\alpha}$ is a length which can be compared with a radius of curvature, which means that weak coupling is when M is "big" compared to the "string scale". This is the more string theory oriented way of writing things down. Alternatively we could let X be dimensionless and write the action as $l^2\int \partial X^i\partial X^j g_{ij}$, where now the invariant distance is $ds^2 = l^2 g_{ij}dX^i dX^j$, with ds^2 and l both dimensionless. The small parameter in this case is then l^{-1}.

First let's assess the sensibility of such a result. We see that for target manifolds with positive curvature, the theory is asymptotically free, since large radii of curvature

[267] In string theory, it is more natural to let X have Greek indices since the target space is spacetime, and to let γ have Roman indices since γ lives on the worldsheet. Here we are thinking of Σ as spacetime, and so the conventions are reversed.

translate to weak coupling (we've also derived this earlier for the case where $M = S^n$). This makes perfect sense, since the strength of the coupling at a given point in M is given by the inverse radius of curvature at that point: going to the UV by "zooming in" on a positive curvature region means increasing the radius of curvature locally, which leads to a smaller coupling constant, agreeing with asymptotic freedom. Geometrically, this is Ricci flow: positive curvature regions get smaller and negative curvature regions get larger. Also, we could have got the answer by using dimensional analysis to write, to leading order,

$$\beta_{ij} = aR_{ij} + bg_{ij} + cRg_{ij}. \tag{2617}$$

However, $b = 0$ since we know that when the target space is flat, the theory is totally conformal and $\beta_{ij} = 0$. Moreover, we know that $R_{ij} = \frac{1}{2}g_{ij}R$ in two dimensions, so that $\beta_{ij} \propto R_{ij}$. Then using remarks similar to those above, we could fix the sign of the coefficient by physical reasoning: we see that these general arguments get us nearly all the way to the answer!

To do things more carefully, we will adopt the background field approach. We will let the "slow" degree of freedom (the background field) be denoted by ϕ, so that the full field is $X^i = \phi^i + \gamma^i$, where γ^i is the "fast" degree of freedom representing fluctuations away from the background field. We want to integrate out the γ^i and see how this changes the coupling constant (the metric) for the slow degrees of freedom. The problem with this is that γ^i is defined as the difference between X^i and ϕ^i, which for a given spacetime coordinate map to different points on the target manifold M. Thus γ^i is not actually a vector, since it transforms non-covariantly under coordinate transformations on M.

Instead of the object γ^i, we can work with vectors by using the following prescription. Let $\lambda^i(s)$ be the geodesic (we assume there is only one) passing between ϕ^i and X^i, with $\lambda^i(0) = \phi^i$ and $\lambda^i(1) = X^i$. Furthermore, define

$$\zeta^i \equiv \frac{d\lambda^i(s)}{ds}\Big|_{s=0} \tag{2618}$$

as the tangent to the geodesic at ϕ^i. ζ^i is of course a vector, and we will use it as the integration variable instead of γ^i. We can expand X^i about the slow field ϕ^i in terms of ζ^i by flowing along the geodesic from ϕ^i with the help of $e^{s\zeta}$:

$$\lambda(s) = e^{s\zeta}\phi = \phi + s\nabla_\zeta X|_\phi + \frac{1}{2}s^2\nabla_\zeta^2 X|_\phi + \dots \tag{2619}$$

with X^i obtained by evaluating this at $s = 1$.

We now want to find an explicit expansion for $(e^{s\zeta}\phi)^i$ in terms of ζ^i (i.e. in terms of s). We write

$$(e^{s\zeta}\phi)^i = \phi^i + s\zeta^i + \frac{1}{2}s^2C_2^i + \frac{1}{3!}s^3C_3^i + \dots \tag{2620}$$

for some as-yet-undetermined coefficients C_n^i. We also need to expand the Christoffel symbols about ϕ:

$$\Gamma^k_{ij}(e^{s\zeta}\phi) = \Gamma^k_{ij}(\phi) + s\partial_l\Gamma^k_{ij}(\phi)\left(s\zeta^i + \frac{1}{2}s^2C_2^i + \frac{1}{3!}s^3C_3^i + \dots\right)^l + \dots \tag{2621}$$

Now recall the geodesic equation, which we get by requiring that the covariant derivative of the tangent vector along the geodesic (namely ζ^i) vanish:

$$\frac{d^2\zeta^i}{ds^2} + \Gamma^i_{jk}\frac{d\zeta^j}{ds}\frac{d\zeta^k}{ds} = 0. \tag{2622}$$

We plug the above power series into the geodesic equation (after plugging in the expansion of $e^{s\zeta}\phi$ into the series for the Christoffel symbols), and equate powers in s. The s^0 term tells us that

$$C^i_2 = -\Gamma^i_{jk}\zeta^j\zeta^k. \tag{2623}$$

The next order term is

$$C^i_3 = \left(2\Gamma^i_{km}\Gamma^m_{lj} - \frac{1}{3}\partial_{(j}\Gamma^i_{kl)}\right)\zeta^k\zeta^l\zeta^j, \tag{2624}$$

and the higher order terms won't be important to us.

So, putting this in for C^i_2, we can write the expansion of $e^{\zeta}\phi$ about ϕ as

$$X^i = (e^{\zeta}\phi)^i = \phi^i + \zeta^i - \frac{1}{2}\Gamma^i_{jk}\zeta^j\zeta^k + O(\zeta^3). \tag{2625}$$

Now suppose we had chosen coordinates about ϕ^i in which the geodesics were straight lines passing through ϕ. These are Riemann normal coordinates, in which geodesics passing through ϕ are used to construct a coordinate system which is locally $\mathbb{R}^{\dim M}$. In Riemann normal coordinates then, the linear order approximation $\phi^i + s\zeta^i$ is actually exact, since the geodesic is a straight line. So then looking at the series expansion for X^i, we see that in Riemann normal coordinates, all of the C^i_n must vanish. This means in particular that

$$\Gamma^k_{ij}(\phi) = \partial_{(i}\Gamma^j_{kl)}(\phi) = 0 \quad \text{(in Rieman normal coordinates)}. \tag{2626}$$

This is consistent with the fact that the normal coordinate system is locally $\mathbb{R}^{\dim M}$. Note that this statement is made at the origin of the coordinate system, and does not hold at other points (Christoffel symbols without an argument will always be assumed to be evaluated at ϕ). One upside is that the Riemann tensor in normal coordinates is (just by definition)

$$R^i{}_{jkl} = \partial_k\Gamma^i_{lj} - \partial_l\Gamma^i_{kj} \quad \text{(in Rieman normal coordinates)}, \tag{2627}$$

since the terms quadratic in Γ die. By writing $\frac{1}{3}(R^i{}_{jkl} + R^i{}_{lkj})$ in terms of the Christoffel symbols in this way, and then adding and subtracting $(1/3)\partial_k\Gamma^i_{lj}$ and using that $\partial_{(i}\Gamma^j_{kl)} = 0$, the derivative of the Christoffel symbol at the origin is related to the curvature tensor as (still in Rieman normal coordinates)

$$\frac{1}{3}(R^i{}_{jkl}(\phi) + R^i{}_{lkj}(\phi)) = \frac{1}{3}(3(\partial_k\Gamma^i_{lj})(\phi) - \partial_{(l}\Gamma^i_{jk)}(\phi)) = (\partial_k\Gamma^i_{lj})(\phi). \tag{2628}$$

Finally we need to know how to expand more general tensors about the origin ϕ in terms of the ζ's. In normal coordinates, this is just the normal Taylor expansion since we locally are in $\mathbb{R}^{\dim M}$. However, if we write it in terms of covariant things, the expansion will work for any coordinate system. So, we will presently work in normal coordinates and re-write the taylor expansion in terms of covariant derivatives.

Consider a $(2,0)$ tensor T_{ij} (two covariant indices). Since the Christoffel symbols at the origin vanish in normal coordinates, the first order term in the expansion is

$$\partial_k T_{ij}(\phi)\zeta^k = \nabla_k T_{ij}(\phi)\zeta^k. \tag{2629}$$

The second order term is trickier though. We have

$$\frac{1}{2}\nabla_k\nabla_l T_{ij}\zeta^k\zeta^l = \frac{1}{2}\nabla_k(\partial_l T_{ij} - \Gamma^m_{il}T_{mj} - \Gamma^m_{jl}T_{im})\zeta^k\zeta^l. \tag{2630}$$

After we take the ∇_k and evaluate at ϕ (the origin), all the un-differentiated Christoffel symbols will die. Thus, using our expression for the derivative of the Christoffel symbols in terms of the Riemann curvature tensor,

$$\frac{1}{2}(\nabla_k\nabla_l T_{ij})(\phi)\zeta^k\zeta^l = \frac{1}{2}\left(\partial_k\partial_l T_{ij}(\phi) - \frac{1}{3}(R^m{}_{lki} + R^m{}_{ikl})T_{mj}(\phi) - \frac{1}{3}(R^m{}_{lkj} + R^m{}_{jkl})T_{im}(\phi)\right)\zeta^k\zeta^l. \tag{2631}$$

Now we use the Bianchi identity in the form

$$R^m{}_{lki} + R^m{}_{ikl} = -R^m{}_{kil} \tag{2632}$$

on the curvature tensors in the above expansion. After solving for $\frac{1}{2}\partial_k\partial_l T_{ij}(\phi)$ in terms of the other covariant stuff, we can finally rewrite the Taylor expansion for T as

$$T_{ij}(e^\zeta\phi) = T_{ij}(\phi) + \nabla_k T_{ij}(\phi)\zeta^k + \frac{1}{2}\zeta^k\zeta^l\left(\nabla_k\nabla_l T_{ij}(\phi) - \frac{1}{3}R^m{}_{kil}T_{mj}(\phi) - \frac{1}{3}R^m{}_{kjl}T_{im}(\phi)\right) + O(\zeta^3). \tag{2633}$$

Behold, everything is covariant! Thus, this expansion holds in any coordinate system, not just in Riemann normal coordinates. Also note that if T is taken to be the metric, then $\nabla_k g_{ij} = 0$ means

$$g_{ij}(e^\zeta\phi) = g_{ij}(\phi) - \frac{1}{3}R_{ikjl}\zeta^k\zeta^l + O(\zeta^3). \tag{2634}$$

Again, this holds in any coordinate system.

With this preparatory work out of the way, we can start massaging the action. We need to expand both the metric and the derivatives. We know how to do the former; the latter is, in Riemann normal coordinates centered on ϕ^i, (note to self: convince yourself that the derivatives on ϕ don't need to be covariant ones)

$$\begin{aligned}\partial_\mu\left(\phi^i + \zeta^i - \frac{1}{2}\Gamma^i_{jk}\zeta^l\zeta^k\right) &\approx \partial_\mu\phi^i + \partial_\mu\zeta^i - \frac{1}{6}(\partial_\mu\phi^m)(R^i_{lmk} + R^i_{kml})\zeta^k\zeta^l, \\ &= \partial_\mu\phi^i + \nabla_\mu\zeta^i - \frac{1}{3}\partial_\mu\phi^m R^i_{lmk}\zeta^k\zeta^l,\end{aligned} \tag{2635}$$

where the covariant derivative acting on ζ is

$$\nabla_\mu\zeta^i = \partial_\mu\zeta^i + \partial_\mu\phi^m\Gamma^i_{mj}\zeta^j. \tag{2636}$$

The derivatives on ϕ just convert the derivative into the $i, j, k, \ldots$ index space[268]. Again, we derived this in normal coordinates, but the answer holds in a generic coordinate system since it's gauge-invariant under gauge transformations (coordinate transformations) on ζ^i. This is good news for us, since we can use the above expression in the action, the fields of which generically cannot be written in normal coordinates for more than one point in the target manifold (unless of course if $R_{ijkl} = 0$).

Now we put the above into the action, along with the expansion of the metric, and keep everything below third order in ζ. This produces

$$S = \frac{1}{4\pi\alpha}\int \sqrt{\gamma}\gamma^{\mu\nu}\Big[g_{ij}(\phi)\left(\partial_\mu\phi^i\partial_\nu\phi^j + \nabla_\mu\zeta^i\nabla_\nu\zeta^j + 2\nabla_\mu\zeta^i\partial_\nu\phi^j\right) + R_{ijkl}(\phi)\partial_\mu\phi_i\partial_\nu\phi^k\zeta^j\zeta^l\Big] \tag{2638}$$

Unfortunately, we cannot do Feynman diagrams with this action. The reason is that the kinetic term for ζ is dependent on ϕ, since the kinetic term involves the full metric $g_{ij}(\phi)$. This makes doing calculations with the ζ propagator essentially impossible.

To fix this, we need to change the metric in the ζ kinetic term to a flat one. We do this by switching over to vielbeins, with $\zeta^i = e^i_a\zeta^a$. The e^i_a are orthonormal frames and $g_{ij} = e^a_i e^b_j\eta_{ab}$, so that the kinetic term is

$$\gamma^{\mu\nu}g_{ij}\partial_\mu\zeta^i\partial_\nu\zeta^j = \eta_{ab}\gamma^{\mu\nu}\partial_\mu\zeta^a\partial_\nu\zeta^b, \tag{2639}$$

so that the ζ^a propagator is the usual propagator which can now be used in Feynman diagrams. From now on, we will assume a flat spacetime for simplicity, so that $\gamma^{\mu\nu} = \eta^{\mu\nu}$.

To translate the rest of the action into the vielein formulation, we just need to note how the covariant derivatives change. Let us denote the spin connection (a matrix-valued 1-form on Σ) as ${\omega_\mu}^a{}_b$. The spin connection is built out of the regular affine connection with Γ, plus a term which keeps track of how the basis frames rotate as we travel around Σ, which gives it the properties of an $SO(\dim M)$ gauge field over Σ (in Euclidean signature). This extra term is the analogue of the Mrauer-Cartan form $g^{-1}dg$ in gauge theory, and is written as $e^i_b\partial_\mu e^a_i$. Using the vielbeins to deal with the mixed indices in the Christoffel symbols, the spin connection is

$${\omega_\nu}^a{}_b = e^a_i e^k_b\Gamma^i_{\nu k} - e^a_i\partial_\nu e^i_b = \partial_\nu\phi^j e^a_i e^k_b\Gamma^i_{jk} - e^a_i\partial_\nu e^i_b. \tag{2640}$$

The minus sign in front of the $e^a_i\partial_\nu e^i_b$ term is chosen so that the vielbein frames are covariantly constant: taking the covariant derivative of the framing e^i_a and using the Christoffel symbols to deal with the lower index and the spin connection to deal with the upper index, we have (recall that covariant indices get a minus sign in the covariant derivative and contravariant indices get a plus sign)

$$D_\mu e^a_j = \partial_\mu e^a_j - \Gamma^k_{ij}e^a_k\partial_\mu\phi^i + {\omega_\mu}^a{}_b e^b_j = 0, \tag{2641}$$

[268] This is made possible because the Christoffel symbols transform as a $(2,1)$ tensor (two co-vector indices and one vector index) up to an extra term

$$\Gamma^k_{ij} = \partial_a x^k\left[\Gamma^a_{bc}\partial_i x^b\partial_j x^c + \partial_i\partial_j x^a\right], \tag{2637}$$

where the transformation is from x^i coordinates to x^a coordinates. This can be checked using algebra and $\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. This means that if we want to only transform one of the bottom indices of the Christoffel symbol, then we can treat Γ as a tensor, essentially since the non-tensorial piece is $\partial_i\partial_b x^a = \partial_i\delta_{b^a} = 0$; this allows us to write $\nabla_\mu\zeta^i$ as above.

which follows from the definition of the spin connection and the definition of the inverse framing $e^i_b e^b_k = \delta^i_k$. In what follows, we will use D_μ to denote covariant derivatives in the vielbein (gauge) formalism, and ∇_μ for covariant derivatives in the regular coordinate basis.

We can use these properties to convert the action into the vielbein formalism. After swapping out the covariant derivatives ∇_μ for the gauge covariant derivatives D_μ, we get

$$S = \frac{1}{4\pi\alpha}\int \eta^{\mu\nu}\Big[g_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j + \eta_{ab}\left(D_\mu\zeta^a D_\nu\zeta^b + 2D_\mu\zeta^a e^b_j\partial_\nu\phi^j\right) + e^j_a e^l_b R_{ijkl}(\phi)\partial_\mu\phi_i\partial_\nu\phi^k\zeta^a\zeta^b\Big]. \tag{2642}$$

Now the kinetic term for ζ is nice and ready for doing Feynman diagrams. The relevant vertices are a $\omega\omega\zeta\zeta$ vertex, a $k^\mu\omega\zeta\zeta$ vertex, and a $\zeta\zeta\partial\phi\partial\phi R$ vertex. The gauge field ω won't actually contribute to any β functions, since gauge theory in two dimensions has no divergences[269].

All this means that getting the β function for the metric is incredibly simple. We can do it either by figuring out what counterterms need to be added or by letting the fast fields ζ^i be defined over an energy range from Λ to $\Lambda + d\Lambda$. Adopting this approach, there is only one diagram that contributes to the renormalization of $g_{ij}(\phi)$, namely the one with two outgoing $\partial_\mu\phi^i$ legs joined at a bubble formed from a single ζ^a propagator. This diagram gives

$$(\text{diagram})^{ik} = \int_k e^j_a e^l_b\langle\zeta^a\zeta^b\rangle(k^2)R_{ijkl} = \ln\left(\frac{\Lambda + d\Lambda}{\Lambda}\right)\eta^{ab}e^j_a e^l_b R_{ijkl} = d\ln\Lambda R_{ik}. \tag{2644}$$

In the last step, we have used (basically, the trace is the same in all coordinates)

$$\eta^{ab}e^j_a e^l_b R_{ijkl} = R^m{}_{ilk}g_{mj}e^j_a e^l_n\eta^{ab} = R^m{}_{ilk}g_{mj}g^{jl} = R^m{}_{imk} = R_{ik}. \tag{2645}$$

So finally, we differentiate the effective g_{ij} (the one which includes the above radiative correction) with respect to $\ln\Lambda$ and find that

$$\beta_{ij} = R_{ij} \tag{2646}$$

as required.

[269] Divergences for gauge theories are two degrees lower than naive power counting suggests, I think roughly since we loose one degree of freedom to the non-dynamical A_0 and another to gauge invariance. One can check e.g. that the two diagrams contributing to the renormalization of the ω propagator give a contribution

$$(\text{two diagrams})^{\mu\nu}(q) \sim \int_k\left(\frac{\delta^{\mu\nu}}{k^2} - \frac{2k^\mu(k-q)^\nu}{k^2(k-q)^2}\right) \sim \int_{k,x}\frac{\delta^{\mu\nu}q^2}{(k^2-\Delta_q)^2}, \tag{2643}$$

which indeed has an integrand going as $1/k^3$ rather than the naive $1/k$. Here the μ,ν are the spacetime indices for the external ω legs.

Skrymion energy in $O(3)$ nslm

Today's diary entry is a fast one: calculating the energy of the minimally-charged Skyrmion in the two-dimensional (classical) $O(3)$ nlsm, with energy

$$E = \frac{\rho}{2} \int d^2r \, |\nabla \mathbf{n}|^2, \tag{2647}$$

with $|\mathbf{n}| = 1$. I couldn't find the derivation in any freely available stuff online (though I didn't look that hard), but the calculation is fun and quick.

We can do the calculation by thinking about a 2π flux background for the $\mathbb{CP}^1$ representation, but we will instead stick with the n^a variables.

By symmetry, the minimal-charge skrymion solution $\mathbf{n}(\mathbf{r})$ will be rotationally invariant. Since we are in two dimensions the skyrmion action will satisfy $S[\mathbf{n}(\mathbf{r}/\xi)] = S[\mathbf{n}(\mathbf{r})]$ for any ξ; hence we must have a one-parameter family of solutions related by scale transformations. Since the generator of $\pi_2(S^2)$ is the unit map, the skyrmion solution will therefore take the form of a conformal map from S^2 onto the plane.

As is the case with any sigma model, the action can be written as

$$S = \int_\Sigma \mathrm{Tr}[\mathbf{n}^*(g)\eta], \tag{2648}$$

with η the metric on space(time) and g the metric on the target space T, with $\mathbf{n}^*(g)$ denoting the pullback by $\mathbf{n} : \Sigma \to T$. Since in our case $\mathbf{n} : \mathbb{R}^2 \to S^2$ must be a conformal map depending , the pullback metric $\mathbf{n}^*(g)$ must have the conformally flat form

$$[\mathbf{n}^*(g)]_{ij} = \frac{4\xi^2}{(\xi^2 + r^2)^2}\delta_{ij}, \tag{2649}$$

for some scale parameter ξ. We then have

$$S[\mathbf{n}] = \frac{\rho}{2} \int_0^\infty dr \, 2\pi r \mathrm{Tr}[\mathbf{n}^*(g)] = \frac{\rho}{2} \int_0^\infty dr \, 2\pi r \frac{8}{(1 + r^2)^2} = 4\pi\rho. \tag{2650}$$

Note that we never had to know the explicit coordinate representation for $\mathbf{n}$!

Most papers I have read quote the skyrmion energy as $8\pi\rho$, but note that this only is consistent if the spin stiffness appears as $\rho|\nabla \mathbf{n}|^2$ in the action, without the factor of $1/2$ that we have in the present setting.

We can check this result by writing out the explicit form of the conformal map. The conformal maps from $S^2 \to \mathbb{R}^2$ correspond to doing stereographic projections with different scale factors ξ. Therefore the explicit coordinate expressions for the components of $\mathbf{n}$ can be obtained from

$$x^i/\xi = \frac{n^i}{1 + n^z}, \tag{2651}$$

where $i = x, y$. Letting $r^2 = x^2 + y^2$, the fact that $|\mathbf{n}| = 1$ tells us that, after a bit of algebra, the skyrmion solution has components

$$n^z = \frac{\xi^2 - r^2}{\xi^2 + r^2} \implies (n^x,\, n^y) = \frac{2}{1 + r^2/\xi^2}(x/\xi,\, y/\xi). \tag{2652}$$

Now we only need to plug this into the action, and see what we get. Since we know all choices for ξ will have the same action, we will now set $\xi = 1$. The Laplacian of the above solution is

$$-\nabla^2 \mathbf{n} = \frac{8}{(1+r^2)^3}\left(2r\cos(\phi),\, 2r\sin(\phi),\, 1 - r^2\right). \tag{2653}$$

This then gives

$$-\mathbf{n} \cdot \nabla^2 \mathbf{n} = \frac{8}{(1+r^2)^2}, \tag{2654}$$

which then is integrated with the same integral as above. ✓

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