Learning with Error and GSW's Homomorphic Encryption

June 26, 2019

Intuition of LWE

$$14s_1 + 15s_2 + 5s_3 + 2s_4 \approx 8 \pmod{17}$$
$$13s_1 + 14s_2 + 14s_3 + 6s_4 \approx 16 \pmod{17}$$
$$6s_1 + 10s_2 + 13s_3 + s_4 \approx 3 \pmod{17}$$
$$10s_1 + 4s_2 + 12s_3 + 16s_4 \approx 12 \pmod{17}$$
$$9s_1 + 5s_2 + 9s_3 + 6s_4 \approx 9 \pmod{17}$$
$$3s_1 + 6s_2 + 4s_3 + 5s_4 \approx 16 \pmod{17}$$
$$\vdots$$
$$6s_1 + 7s_2 + 16s_3 + 2s_4 \approx 3 \pmod{17}$$

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- Output $(\vec{a}, \langle \vec{a}, \vec{s} \rangle + e)$

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Given samples from $A_{\vec{s},\chi}$,

- Search version: Find $\vec{s}$.

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- Decision isn't much easier than search

Search to decision reduction

Works when $q \in O(poly(n))$ is prime.

- Inductive process; solve variable-by-variable

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- Maps to the uniform distribution otherwise.
- Check using the blackbox for decision version. Try another k until the guess is correct.

GSW's LWE formulation

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- $\mathrm{Dec}_{sk} \circ \mathrm{Enc}_{pk}(x, r) = x$ with overwhelming probability over r

Asymmetric cryptography cont.

Typical use of such a scheme:

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- Ciphertext indistinguishability (TODO)

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- Application includes quantum computing

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- private key $\vec{v}$ is a vector

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- $(C_1 C_2)\vec{v} \approx (\lambda_1 \lambda_2)\vec{v}$
- When plaintexts are booleans, $I_N - C_1 C_2$ encodes NAND.

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- $\text{Flatten} = \text{BitDecomp} \circ \text{BitDecomp}^{-1}$

GSW's tools (Flatten)

$$\mathsf{Flatten}(110_2, 101_2, 1_2, 11_2,$$
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$$= (1, 0, 0, 1, 1, 0, 0, 1)$$

GSW's tools cont.

Some basic properties

- $\langle \text{BitDecomp}(\vec{a}), \text{Powersof2}(\vec{b}) \rangle = \langle \vec{a}, \vec{b} \rangle$

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- Modulus $q = 2^l$ (to simplify some proofs)

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- $m \in O(n \log q)$

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2. Output sk as 1 on the first coordinate, followed by $-\vec{s}$.

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2. Sample $\vec{e} \leftarrow \chi^m$
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4. Observe that $pk \cdot sk = \vec{e}$

GSW's Construction - Encryption

Input: μ

1. Sample $R \in \{0, 1\}^{N \times m}$

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GSW's Construction - Decryption

$$\text{Flatten}(\mu \cdot I + \text{BitDecomp}(R \cdot pk)) \cdot \text{Powersof2}(sk)$$

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$$
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&= \mu\text{Powersof2}(sk) + R \cdot pk \cdot sk
\end{aligned}
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GSW's Construction - Decryption cont.

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- Recover μ's least significant bit by $LSB(\mu) = 2^{l-1}\mu$

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- Recover μ's least significant bit by $LSB(\mu) = 2^{l-1}\mu$
- Recover μ's next bit by $2^{l-2}(\mu - LSB(\mu))$
- Similar for all other bits of μ.
- Decryption breaks down when the error reaches $q/4$.

GSW's Construction - Security

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- If $C = \text{Enc}_{pk}(\mu, r)$ hides μ, so does $\text{BitDecomp}^{-1}(C)$, since C can be derived from it.
- $\text{BitDecomp}^{-1}(C) = \mu \cdot \text{BitDecomp}^{-1}(I) + R \cdot A$
- Fact: The joint distribution $(A, R \cdot A)$ is indistinguishable from uniform, if $m > 2nl$

GSW's Construction - NAND

- $(I - C_1 \cdot C_2)\vec{v} = (1 - \mu_1\mu_2)\vec{v} - \mu_2\vec{e}_1 - C_1\vec{e}_2$

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- Final error increase by a factor of $(N + 1)^L$

GSW's Construction - Multiplying by constant

Input: C, α

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- Observe $M_\alpha \cdot C\vec{v} = M_\alpha \cdot (\mu\vec{v} + \vec{e}) = \alpha\mu\vec{v} + M_\alpha \cdot e$

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- Error increases by a factor of N

GSW's Construction - Addition

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- Error increases by a factor of 2

GSW's Construction - Multiplication

- $C_1 \cdot C_2 \vec{v} = C_1(\mu_2 \vec{v} + \vec{e}_2) = \mu_1 \mu_2 \vec{v} + \mu_2 \vec{e}_1 + C_1 \vec{e}_2$

GSW's Construction - Multiplication

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- Error increase depends on what's being encrypted
- May need to assume bounds on the values being computed

References

- O. Regev. The Learning with Errors Problem.
- C. Gentry, A. Sahai, B. Waters. Homomorphic Encryption from Learning with Errors: Conceptually-Simpler, Asymptotically-Faster, Attribute-Based.