# 2-local Hamiltonian construction

June 3, 2019

# Notation

- Let $\lambda(H)$ denote the smallest eigenvalue of $H$.
- Let $H\big|_S = \prod_S H \prod_S$.
- Let $\widehat{t}$ denote $t$ represented in unary.

# Change of definition

An operator $H : \mathcal{B}^{\otimes n} \to \mathcal{B}^{\otimes n}$ is a $k$-local Hamiltonian if $H = \sum_{j=1}^{r} H_j$ where $H_i$ are hermitian acting on at most $k$ qubits.

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**note**

The positive semidefinite condition is no longer there.

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- There are $T = (|T_2| + 1)L - 1$ gates in total.
- Let $T_1 = [T] \setminus T_2$ be time corresponding to 1-qubit gates

# Projection Lemma

Let $H_1, H_2$ be Hamiltonians where $H_2 \geq 0$, and $S = \ker H_2$. Then $\exists J \in \mathbb{R}$ s.t. $\lambda(H_1\big|_S) - \frac{1}{8} \leq \lambda(H_1 + JH_2) \leq \lambda(H_1\big|_S)$.

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**idea**

Construct $H_2$ to "cut out" vectors outside $\ker H_2$

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$$H = H_{out} + J_{in}H_{in} + J_2 H_{prop2} + J_1 H_{prop1} + J_{clock} H_{clock}$$

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- $H_{prop1}$ corresponds to correct propagations of 1-qubit gates
- $H_{prop}$ corresponds to correct propagation
- $H_{in}$ corresponds to the input qubits being initialized properly
- $S_{legal} \supset S_{prop1} \supset S_{prop} \supset S_{in}$

# Restriction to legal clock states

$$H_{clock} = \sum_{1 \leq i < j \leq T} I \otimes |01\rangle\langle 01|_{ij}$$

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# Restriction to $S_{prop1}$

$$H_{prop1} = \sum_{t \in T_1} H_{prop,t}$$

$$H_{prop,t} = I \otimes |\widehat{t}\rangle\langle\widehat{t}| + I \otimes |\widehat{t-1}\rangle\langle\widehat{t-1}| - U_t \otimes |\widehat{t}\rangle\langle\widehat{t-1}| - U_t^{\dagger} \otimes |\widehat{t-1}\rangle\langle\widehat{t}|$$

# Restriction to $S_{prop1}$ cont.

Let $t \in T_2$. Then the following are true:

- $|0\rangle\langle 0|_{f_t} \otimes |\widehat{t}\rangle\langle\widehat{t}| = (|\widehat{t}\rangle + |\widehat{t+1}\rangle)(\langle\widehat{t}| + \langle\widehat{t+1}|)$

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Similarly,

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# Restriction to $S_{prop}$

Naive construction:

$$H_t = I \otimes |\widehat{t}\rangle\langle\widehat{t}| + I \otimes |\widehat{t-1}\rangle\langle\widehat{t-1}| - C_\phi \otimes |\widehat{t}\rangle\langle\widehat{t-1}| - C_\phi^\dagger \otimes |\widehat{t-1}\rangle\langle\widehat{t}|$$

$$H' = \sum_{t \in T_2} H_t$$

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Equivalently, $H_t =$
$|00\rangle\langle 00|_{f_t,s_t} \otimes (|\widehat{t}\rangle - |\widehat{t-1}\rangle)(\langle\widehat{t}| - \langle\widehat{t-1}|)+$
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# Restriction to $S_{prop}$ cont.

$$H_{prop2} = \sum_{t \in T_2} H_{prop2,t}$$

$$\begin{aligned}
&H_{prop2,t} = \\
&|00\rangle\langle 00|_{f_t,s_t} \otimes 4(|\widehat{t}\rangle - |\widehat{t-1}\rangle)(\langle\widehat{t}| - \langle\widehat{t-1}|)+ \\
&|01\rangle\langle 01|_{f_t,s_t} \otimes 2(|\widehat{t}\rangle - |\widehat{t-1}\rangle)(\langle\widehat{t}| - \langle\widehat{t-1}|)+ \\
&|10\rangle\langle 10|_{f_t,s_t} \otimes 2(|\widehat{t}\rangle - |\widehat{t-1}\rangle)(\langle\widehat{t}| - \langle\widehat{t-1}|)+ \\
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$$(|\widehat{t-1}\rangle\langle\widehat{t-1}| + |\widehat{t}\rangle\langle\widehat{t}|)$$

# Restriction to $S_{in}$

$$H_{in} = \sum_{i=m+1}^{N} |1\rangle\langle 1|_i \otimes |0\rangle\langle 0|$$

$$H_{out} = (T+1)|0\rangle\langle 0|_1 \otimes |T\rangle\langle T|$$

Then $\langle \eta | H_{out} | \eta \rangle$ is the eigenvalue, as well the probability of rejection.

# Conclusion

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- Otherwise the circuit should reject with probability greater than $1 - \varepsilon$. Which means $\lambda(H) > 1 - \varepsilon - \frac{4}{8} = \frac{1}{2} - \varepsilon$.

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- Otherwise the circuit should reject with probability greater than $1 - \varepsilon$. Which means $\lambda(H) > 1 - \varepsilon - \frac{4}{8} = \frac{1}{2} - \varepsilon$.
- So 2-local Hamiltonian is QMA-hard.