Delegation of Quantum Computations

Based on Local Hamiltonians

October 20, 2019

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- Verifier can store qubits and apply X/Z measurements.
- Amplification can be done. (This isn't trivial.)

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- Special case:
  $H_j \in \mathcal{G}_{XZ} = \{ U_0 \otimes U_1 \otimes \ldots \otimes U_n : U_i \in \{ I, X, Z \} \}$

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Goal: Given certificate ϕ, estimate

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$$\Rightarrow p = \frac{1}{2} - \frac{1}{2D}\langle\phi|H|\phi\rangle$$

Preliminary facts

Hadamard and Toffoli gates are:

- universal (with real states)
- in span $\mathcal{G}_{XZ}$

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Goal: construct H such that

$$|\phi\rangle = \sum_{t=0}^{T} U_t \ldots U_1 |x\rangle \otimes |\hat{t}\rangle$$

is low energy on a yes-instance.

Checking for valid outputs

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$$= \frac{1}{4}(Z_1 - Z_T) + \frac{1}{4}\sum_{t=1}^{T-1}(I - Z_t Z_{t+1})$$

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$$H_{prop,t} = I \otimes |\widehat{t}\rangle\langle\widehat{t}| + I \otimes |\widehat{t-1}\rangle\langle\widehat{t-1}| - U_t \otimes |\widehat{t}\rangle\langle\widehat{t-1}| - U_t^\dagger \otimes |\widehat{t-1}\rangle\langle\widehat{t}|$$

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$$H_{prop,1} = \frac{1}{2}(I + Z_2) - U_1 \otimes \frac{1}{2}(X_1 + X_1 Z_2)$$

$$H_{prop,T} = \frac{1}{2}(I - Z_{t-1}) - U_T \otimes \frac{1}{2}(X_T - Z_{T-1} X_T)$$

Analysis of H_{prop}

By induction,

$$\Rightarrow K_{clock} \cap K_{prop} = \{ \sum_{t=0}^{T} U_t \ldots U_1 \mid y\rangle \otimes \mid \hat{t}\rangle : \mid y\rangle \in \mathcal{B}^{\otimes n} \}$$

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Claim:

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$$\Rightarrow W^\dagger H_{prop,t} W = I \otimes \left(|\widehat{t}\rangle \langle \widehat{t}| + |\widehat{t-1}\rangle \langle \widehat{t-1}| - |\widehat{t}\rangle \langle \widehat{t-1}| - |\widehat{t-1}\rangle \langle \widehat{t}| \right)$$

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$$\Rightarrow W^\dagger H_{prop} W = 2 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & \ddots & & \\ & & \ddots & \ddots & -\frac{1}{2} & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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$\Rightarrow \lambda(W^{\dagger} H_{prop} W) = 0$;
$\lambda_2(W^{\dagger} H_{prop} W)$ is inverse polynomially bounded away from 0

Projection Lemma

Let H_1, H_2 be local Hamiltonians where $H_2 \geq 0$. Let $K = \ker H_2$.

$$\exists J = \frac{\mathrm{poly}(\|H_1\|)}{\lambda_2(H_2)}$$

$$\lambda(H_1\big|_K) - \frac{1}{8} \leq \lambda(H_1 + JH_2) \leq \lambda(H_1\big|_K)$$

Use of projection lemma

By applying the projection lemma iteratively,

$$\lambda(H_{out}\big|_K) - \frac{3}{8} \leq \lambda(H_{out} + J_{in}H_{in} + J_{clock}H_{clock} + J_{prop}H_{prop}) \leq \lambda(H_{out}\big|_K)$$

where $K = \text{span}\{\sum_{t=0}^{T} U_t \ldots U_1 \ |x\rangle \otimes \ |\hat{t}\rangle\}$

$\square$

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$$\Rightarrow \left\langle\eta\right|\left(I \otimes \left|1\right\rangle\left\langle1\right|\right)\left|\eta\right\rangle = \sum_s \lambda_s \overline{y_s} y_s$$