

# Properties of exp

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When the power series  $\sum_{n=0}^{\infty} \frac{X^n}{n!}$  is well-defined (including when  $X$  is a real number, complex number, or matrix), it is equal to  $\exp(X)$ . This truth is so ubiquitous that it is often used as a definition of  $\exp$ . Here, I provide some more precise arguments on this subject.

1.  $\exp(tX) = y(t)$

As mentioned in a footnote, I wrote that  $\exp(tX) = y(t)$  in the definition of  $\exp$ , but this shouldn't be part of the definition. Rather, it follows from the definition  $\exp(X) = y(1)$ . To see why, define  $z(s) = y(ts)$ . Then  $\exp(tX) = z(1) = y(t)$ , so we just need to show the tangent vector  $\frac{dz}{ds}(0)$  is  $tX$

$$\frac{dz}{ds} = \frac{dz}{d(ts)} \frac{d(ts)}{ds} = \frac{dy}{dt} \cdot t$$

which, evaluated at 0, gives  $tX$ .

2.  $\exp(X)\exp(Y) = \exp(X+Y)$

This one isn't always true, in fact it's true only when  $X$  and  $Y$  commute, i.e.  $XY = YX$ . We can prove it by matching terms of the power series. Computing the degree- $n$  term of  $\exp(X)\exp(Y)$  involves multiplying all terms with degrees that sum to  $n$  (since exponents are additive), so

$$\exp(X)\exp(Y) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \sum_{\ell=0}^{\infty} \frac{Y^\ell}{\ell!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{X^k Y^{n-k}}{k!(n-k)!}$$

Recalling the binomial theorem, we may conclude

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{X^k Y^{n-k}}{k!(n-k)!} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} X^k Y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{(X+Y)^n}{n!} = \exp(X+Y) \end{aligned}$$

It immediately follows that  $\exp(0) = 1$  by considering  $\exp(0) = \exp(0+0)$ .

$$3. \exp(X) = \lim_{n \rightarrow \infty} (1 + \frac{X}{n})^n$$

The binomial theorem tells us that the coefficient of  $X^k$  in  $(1 + \frac{X}{n})^n$  is  $\binom{n}{k}/n^k$ . The coefficient of  $X^k$  in  $\exp(X)$  is  $\frac{1}{k!}$ , so we just need to show that

$$\lim_{n \rightarrow \infty} \frac{\binom{n}{k}}{n^k} = \frac{1}{k!}, \text{ or } \lim_{n \rightarrow \infty} \frac{k! \binom{n}{k}}{n^k} = 1$$

Observe that the above expression simplifies to

$$\lim_{n \rightarrow \infty} \frac{\frac{n!}{(n-k)!}}{n^k} = \lim_{n \rightarrow \infty} \frac{n(n-1) \cdots (n-k+1)}{n^k} = \lim_{n \rightarrow \infty} \prod_{\ell=0}^{k-1} \frac{n-\ell}{n}$$

Then, it's a straight shot to the finish line

$$\lim_{n \rightarrow \infty} \prod_{\ell=0}^{k-1} \frac{n-\ell}{n} = \prod_{\ell=0}^{k-1} \lim_{n \rightarrow \infty} \frac{n-\ell}{n} = \prod_{\ell=0}^{k-1} 1 = 1$$

$$4. e^x = (1 + \frac{x}{n})^n$$

$e^x$  is only defined for  $x \in \mathbb{C}$ , so we can leverage properties of complex numbers not present in matrices, namely division.

From the previous proof, we know  $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ . Using exponent rules, we have

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{nx}$$

Let  $k = nx$ , so  $k \rightarrow \infty$  when  $n \rightarrow \infty$ . Then

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{nx} = \lim_{k \rightarrow \infty} \left(1 + \frac{x}{k}\right)^k$$

Easy!