YOU MUST USE THE CONSTRUCTIONS GIVEN IN CLASS

1. Construct a regular expression over {a,b,c} for the language accepted by this nfa:

	a	b	c	
→ A	/	B	/	0
B	B	/	A,C	1
C	/	B,C	/	1

STUDY HIS LECTURES

2. Prove that the language L(G) is not regular where G is the following cfg:

$G = (\{S,A,B,C\}, \{a,b,c\}, \{S \to Aa|B|C, A \to aS, B \to a, C \to b\}, S)$.

Note: You must first determine L(G).

3. Construct a reduced dfa for the following extended regular expression over {0,1,2}:

$[(100^*)^* \cap \overline{1^*}]$

Note: You must first determine nfas for (100*)* and 1*, then do the intersection. The answer must then be reduced.

4. Construct a Chomsky normal form grammar for L(G) for the following cfg G:

$G = (\{S,B\}, \{a,b,c,d\}, \{S \to SaSbc|Ba, B \to cBd|S|\varepsilon\}, S)$.

Note: You must first remove all ε- and all unit productions.

5. Construct a Greibach normal form grammar for L(G) for the following CNF G:

$G = (\{S,A\}, \{a,d\}, \{S \to AS|A|d, A \to SS|a\}, S)$.

Note: You must first remove all unit productions. You must derive all the productions for S and A; indicate how the result looks for S' and A' as applicable.

6. Prove that the following language L is not contextfree: $L = \{ 0^{n+1} 1^{n-1} 0^n \mid n>0 \}$. $\{0^{n+2} 1^n 0^{n+1} \mid n \geq 0\}$

7. Consider the class $\mathcal{L}_A$ of all contextfree languages over the fixed alphabet A.

(a) Is $\mathcal{L}_A$ countable? Yes

(b) Is the class $\mathcal{M}_A$ countable where $\mathcal{M}_A$ consists of all languages over A that are not in $\mathcal{L}_A$? No.

(c) Is the class $\mathcal{L}_A \cap \mathcal{M}_A$ countable? Yes.

This is just a guide.

For each question, you must give a precise argument substantiating your answer.

8. Construct a Turing machine for the language in Question 6, $L = \{ 0^{n+1} 1^{n-1} 0^n \mid n>0 \}$.

Note: Describe first the process in English; then translate this into moves of the Turing machine.

9. Let L_1 and L_2 be arbitrary languages, subject to the specification in either (i) or (ii).

Consider the following four general questions:

$\alpha - \beta = \alpha \cap \overline{\beta}$

(Q1) Does $\overline{L_1} - L_2$ contain a given fixed word w? rec (Q2) Is $\overline{L_1} - L_2$ non-empty? r.e. but not rec/not re

(Q3) Does $L_1 \cap L_2$ contain a given fixed word w? rec/r.e. but not rec (Q4) Is $L_1 \cap L_2$ non-empty? r.e. but not rec.

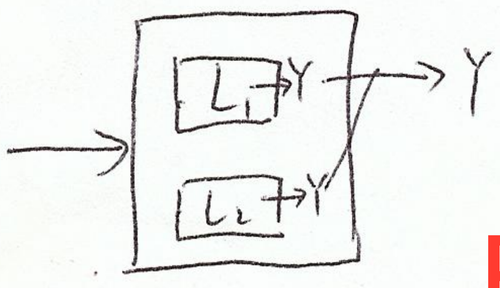
For each of these four questions **explain with reasons** whether the problem is recursive, not recursive but r. e., or non-r. e., provided r.e. but not rec.

(i) Both L_1 and L_2 are recursive. (ii) Both L_1 and L_2 are r. e., but not recursive.

Points: 1: 6 2: 8 3: 14 4: 12 5: 12 6: 12 7: 13 8: 8 9:15

L_1 rec $\Rightarrow \overline{L_1}$ rec.

Final Exam



Yes $\Leftrightarrow \neq \emptyset$

Final

Final Study Guide

The Power Set of a Countably Infinite Set is Uncountable

Theorem 1
If S is a countably infinite set, 2^S (the power set) is uncountably infinite.
Proof: We show 2^S is uncountably infinite by showing that $2^{\mathbb{N}}$ is uncountably infinite. (Given the natural bijection that exists between $2^{\mathbb{N}}$ and 2^S –because of the bijection that exists from $\mathbb{N}$ to S– it is sufficient to show that $2^{\mathbb{N}}$ is uncountably infinite.)

Assume that the set $2^{\mathbb{N}}$ is countably infinite.

The subsets of $\mathbb{N}$ can be listed $A_0, A_1, A_2, \ldots$ so that every subset is A_i for some i.

We define another set $A = \{i | i \geq 0 \text{ and } i \notin A_i\}$ which contains those integers i which are not members of their namesake set A_i.

But A is a subset of $\mathbb{N}$, and so $A = A_j$ for some j. But this means

1. If $j \in A$, then $j \notin A$.
2. If $j \notin A$, then $j \in A$.

We have a contradiction, since j must either be in the set A or not in the set. Therefore $2^{\mathbb{N}}$ is not countably infinite. $\diamond$

The Theorem that the power set of a countably infinite set is an uncountable set indicates that the set of all languages over any alphabet Σ, $|\Sigma| \neq 0$ is uncountable. (2^{Σ^*} is an uncountable set.)

1.

	a	b	c	
A	/	B	/	0
B	B	/	A,C	1
C	/	B,C	/	1

$$\Rightarrow \begin{cases} L_A = bL_B \\ L_B = aL_B \cup cL_A \cup cL_C \cup \varepsilon \\ L_C = bL_B \cup bL_C \cup \varepsilon \end{cases}$$

$L_C = bL_C \cup (bL_B \cup \varepsilon) \Rightarrow L_C = b^*(bL_B \cup \varepsilon) = b^*bL_B \cup b^*$

$L_B = aL_B \cup (cL_A \cup cL_C \cup \varepsilon) \Rightarrow L_B = a^*(cL_A \cup cL_C \cup \varepsilon)$

$= a^*cL_A \cup a^*cL_C \cup a^*$

$= a^*cbL_B \cup a^*c(b^*bL_B \cup b^*) \cup a^*$

$= (a^*cb \cup a^*cb^*b)L_B \cup a^*cb^* \cup a^*$

$\Rightarrow L_B = (a^*cb \cup a^*cb^*b)^*(a^*cb^* \cup a^*)$

$\Rightarrow L_A = bL_B = b(a^*cb \cup a^*cb^*b)^*(a^*cb^* \cup a^*)$

$\Rightarrow L_C = b^*b(a^*cb \cup a^*cb^*b)^*(a^*cb^* \cup a^*) \cup b^*$

2. $S \to Aa \mid B \mid C$ $\quad 2n+1$

$A \to aS$

$B \to a$

$C \to b$

$\Rightarrow S \to Aa \mid a \mid b$

$S \to a$, $S \to b$, $S \to Aa \to aSa \to aaa$, aba, $aAa^2 \to a^4$, aba^2

$a^2Sa^2 \to a^5$, a^2ba^2

$a^2Aa^3 \to a^6$, a^2ba^3

a^3Sa^3 - -

$L_1 = \{a^n \mid n \geq 0\}$ $\quad L_2 = \{a^{n-1}ba^n \mid n \geq 0\}$ $\quad L_3 = \{a^nba^n \mid n \geq 0\}$

$L = L_1 \cup L_2 \cup L_3$ Proving any of the L_1, L_2, L_3 is not regular will prove L is <u>not regular</u>. e.g. L_2 . . .

3. $(100^*)^* \cap 1^* = \overline{(100^*)^* \cup 1^*}$

	0	1	2
0	/	1	/ 0
1	2	/	/ 0
2	/	/	/ 1

	0	1	2
0	3	/	/ 0
3	/	/	/ 1

$\downarrow *$

	0	1	2
0	3	/	/ 1
3	3	/	/ 1

	0	1	2
0	/	5	/ 0
5	/	/	/ 1

$\xrightarrow{*}$

	0	1	2
0	/	5	/ 1
5	/	5	/ 1

	0	1	2
0	4	15	4 1
1	2	4	4 1
2	3	1	4 ~~1~~ 0
3	3	1	4 0
4	4	4	4 1
5	/	5	/ 1

$\cup$

	0	1	2
0	/	1	/ 0
1	2	/	/ 0
2	3	/	/ 1
3	3	/	/ 1

$\downarrow *$

	0	1	2
0	/	1	/ 1
1	2	/	/ 0
2	3	1	/ 1
3	3	1	/ 1

DFA $\longrightarrow$

	0	1	2
0	/	1	/ 1
~~1~~	/	/	/ 0
~~1~~	2	/	/ 0
~~2~~	3	1	/ 1
3	3	1	/ 1

Complement $\longrightarrow$

	0	1	2
0	4	1	4 0
4	4	4	4 1
1	2	4	4 1
2	3	1	4 0
3	2	1	4 0

	0	1	2	
0	4	15	4	1
1	2	4	4	1
2	3	1	4	0
3	3	1	4	0
4	4	4	4	1
5	/	5	/	1

$\xrightarrow{dfa}$

	0	1	2	
0	4	15	4	1
4	4	4	4	1
15	2	45	4	1
2	3	1	4	0
45	4	45	4	1
3	3	1	4	0
1	2	4	4	1

$\xrightarrow{complement}$

		0	1	2	
A	0	4	15	4	0
B	4	4	4	4	0
C	15	2	45	4	0
D	2	3	1	4	1
E	45	4	45	4	0
F	3	3	1	4	1
G	1	2	4	4	0

$\Rightarrow$

	0	1	2	
A	B	C	B	0
B	B	B	B	0
C	D	E	B	0
D	F	G	B	1
E	B	E	B	0
F	F	G	B	1
G	D	B	B	0

$\Rightarrow$

				F
				F
DF	ABCEG			F
DF	CG	ABE		F
DF	A	CG	BE	F
①	②	③	④	F
2	1	3	4	①

A	B	C	D	E	G	
D	ABCEG					
CG	D	ABE				
B	CG	D	AE			
A	B	C	D	E	G	
②	③	④	⑤	⑥	⑦	

7 states.

4. $S \to SaSbc \mid Ba$

$B \to cBd \mid S \mid \underline{\varepsilon}$.

$\xrightarrow{B \to \varepsilon}$ $S \to SaSbc \mid Ba \mid a$

$B \to cBd \mid cd \mid \underline{S}$

	0	1	2	
1	4	3	4	0
2	2	3	4	1
3	2	4	4	0
4	4	4	4	0

$\cancel{\Rightarrow S \to SS_1 \mid BXa}$

$\xrightarrow{S \to a}$ $S \to SaSbc \mid aaSbc \mid aaabc \mid Saabc \mid Ba$

$B \to cBd \mid cd \mid S \mid \underline{a}$

$\xrightarrow{B \to a}$ $S \to SaSbc \mid aaSbc \mid aaabc \mid Saabc \mid Ba \mid aa$

$B \to cBd \mid cad \mid cd \mid S$

$\xrightarrow{B \to S}$ $S \to SaSbc \mid Ba \mid Sa \mid a$

$B \to cBd \mid cSd \mid cd$

$\Rightarrow S \to SS_1 \mid BXa \mid SXa \mid a$

$S_1 \to XaS_2$

$S_2 \to SS_3$

$S_3 \to XbXc$

$B \to CB_1 \mid CB_2 \mid XcXd$

$B_1 \to BXd$

$B_2 \to SXd$

$Xa \to a$

$Xb \to b$

$Xc \to c$

$Xd \to d$

5. $S \to AS \mid A \mid d$

$A \to SS \mid a$

$\xrightarrow{S \to A} A_1 \quad S \to AS \mid AA \mid d$

$A_2 \quad A \to SS \mid SA \mid AS \mid AA \mid a$.

renaming. $A_1 \to A_2A_1 \mid A_2A_2 \mid d$

$A_2 \to A_1A_1 \mid A_1A_2 \mid A_2A_1 \mid A_2A_2 \mid a$

$i=1$, we are done.

$i=2$, $A_2 \to \underbrace{A_2\overbrace{A_1A_1}^{\alpha_1}}\mid \underbrace{A_2\overbrace{A_2A_1}^{\alpha_2}}\mid \overbrace{dA_1}^{\beta_1} \mid \underbrace{A_2\overbrace{A_1A_2}^{\alpha_3}} \mid \underbrace{A_2\overbrace{A_2A_2}^{\alpha_4}} \mid \overbrace{dA_2}^{\beta_2}$

$\mid \underbrace{A_2A_1}_{\alpha_5} \mid \underbrace{A_2A_2}_{\alpha_6} \mid \underset{\beta_3}{a}$

$\Rightarrow A_2 \to dA_1 \mid dA_2 \mid a \mid dA_1A_2' \mid dA_2A_2' \mid aA_2'$

$A_2' \to A_1A_1 \mid A_2A_1 \mid A_1A_2 \mid A_2A_2 \mid A_1 \mid A_2 \mid A_1A_1A_2' \mid A_2A_1A_2'$

$\mid A_1A_2A_2' \mid A_2A_2A_2' \mid A_1A_2' \mid A_2A_2'$

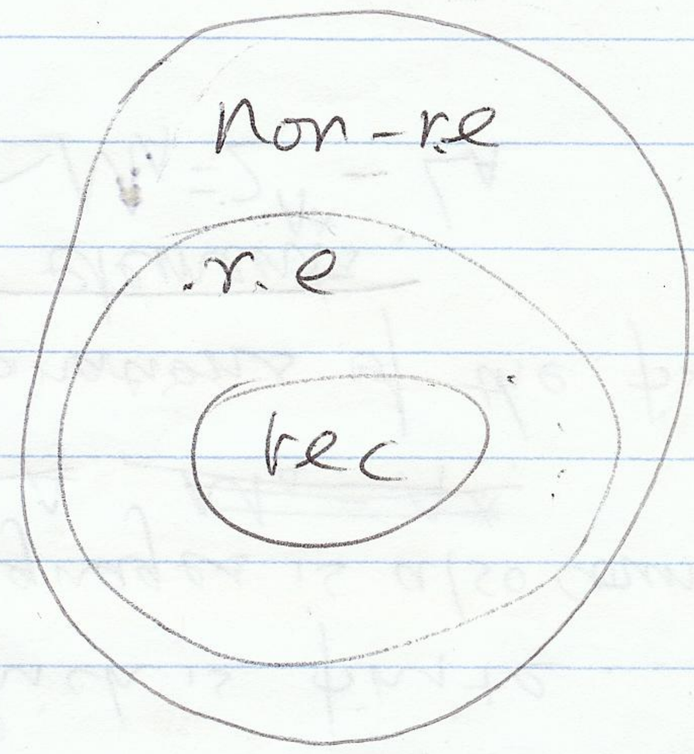
$\Rightarrow A_1 \to dA_1A_1 \mid dA_2A_1 \mid aA_1 \mid dA_1A_2'A_1 \mid dA_2A_2'A_1 \mid aA_2'A_1$

$\mid dA_1A_2 \mid dA_2A_2 \mid aA_2 \mid dA_1A_2'A_2 \mid dA_2A_2'A_2 \mid aA_2'A_2 \mid$

change back to SA.

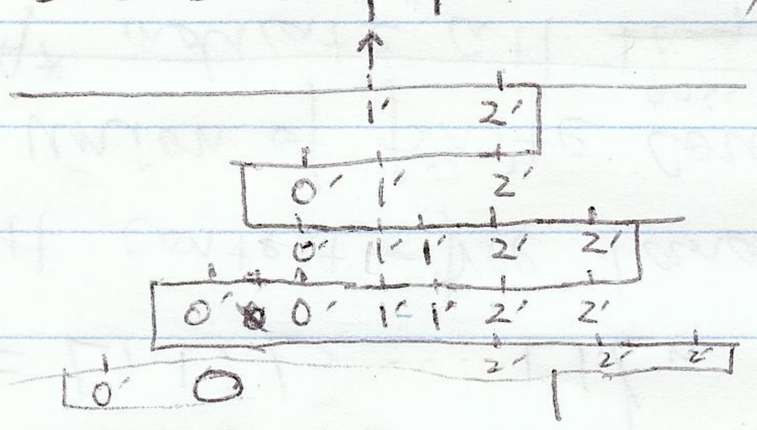
6. $L = \{0^{n+1}1^{n-1}0^n \mid n>0\}$

$\Rightarrow L = \{0^{n+2}1^n0^{n+1} \mid n \geq 0\}$ - - - -



8. Turing machine $L = \{0^{n+2}1^n0^{n+1} \mid n \geq 0\}$

$\not b\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ \not b$



	0	1	0'	1'	2'	⊬
q_0	$(q_0, 0, R)$	$(q_1, 1', R)$				$(q_{10}, \not b, L)$
q_1	$(q_2, 2', L)$	$(q_1, 1, R)$			$(q_2, 2', R)$	
q_2	$(q_3, 0', R)$	$(q_2, 1, L)$	$(q_2, 0', L)$	$(q_2, 1', L)$	$(q_2, 2', L)$	
q_3		~~$(q_3, 1', R)$~~	$(q_3, 0', R)$	$(q_4, 1', R)$		
q_4		$(q_1, 1', R)$			$(q_5, 2', R)$	
q_5	$(q_6, 2', L)$				$(q_5, 2', R)$	
q_6	$(q_7, 0', R)$		$(q_6, 0', L)$	$(q_6, 1', L)$	$(q_6, 2', L)$	
q_7	$(q_6, 2', L)$		$(q_7, 0', R)$	$(q_7, 1', R)$	$(q_5, 2', R)$	$(q_8, \not b, L)$
q_8	$(q_9, 0', L)$		$(q_8, 0', L)$	$(q_8, 1', L)$	$(q_8, 2', L)$	
q_9	$(q_9, 0', L)$					$(q_f, \not b, L)$
q_{10}	$(q_{11}, 0', L)$					
q_{11}	$(q_{12}, 0', L)$					
q_{12}	$(q_{13}, 0', L)$					$(q_f, \not b$
q_{13}						

need one more state if no ⊬ on left.

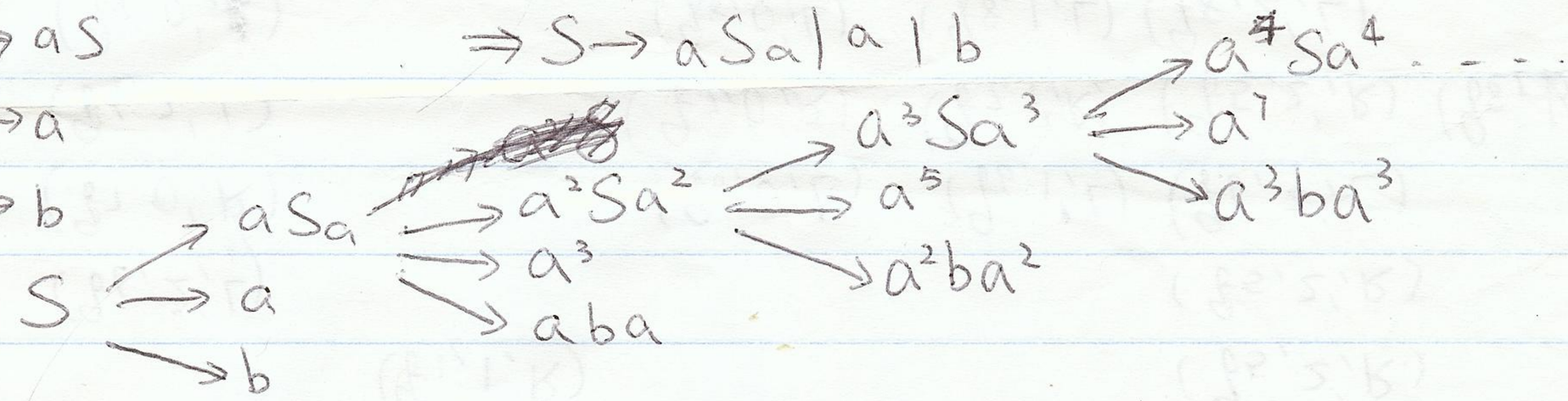
2. $S \to Aa \mid B \mid C$ $\qquad$ $S \to Aa \mid a \mid b$

$A \to aS$ $\qquad$ $\Rightarrow S \to aSa \mid a \mid b$

$B \to a$

$C \to b$

$\Rightarrow$



$\Rightarrow L_1 = \{a^n b a^n \mid n \geq 0\}$ $\quad L_2 = \{a^{2n+1} \mid n \geq 0\}$

and $L = L_1 \cup L_2$

7.(1) L_A is a union of context free language if it is accepted by a PDA, and PDA is finite automata, thus countable.

$\Rightarrow L_A = L_1 \cup L_2 \cdots \cup L_n$, where n is the ~~set~~ number of all context-free lanaguages, which is finite.

$\Rightarrow$ the union of finite countable lanaguages is also countable

(2) ~~Let A^* indicate all languages over A.~~ ~~$M_A = A^*$~~

$\Rightarrow$ Let A^* indicate all possible combinations of the fixed alphabets. Then ~~M_A includes 2^{A^*} elements.~~

$\because A^*$ is countably infinite. $\qquad M_A = 2^{A^*} - L_A$

$\therefore 2^{A^*}$ is uncountable

$\therefore 2^{A^*} - L_A$ is uncountable.

(3) $L_A \cap M_A = \{\phi\}$ so it's countable.

example: Is $\bar{L_1} \cdot \bar{L_2} \neq \phi$?

① rec. if L_1, L_2 are rec, then we can go through all the word until we find a word that in the language. if $\bar{L_1} \cdot \bar{L_2} = \phi$, then TM won't halt, so it's r.e, not rec.

② r.e. not rec: It's not r.e.

$\overline{L_1} - L_2 = \overline{L_1} \cap \overline{L_2}$

9. (i) Both L_1 and L_2 are recursive.

(Q1) L_1 is rec.

$\Rightarrow \overline{L_1}$ is also rec.

Since $\overline{L_1}$ and L_2 are recursive

$\therefore \overline{L_1} - L_2$ is recursive, which means always halt for a given fixed word w belongs to $\overline{L_1}$ but not L_2

(Q2) $\overline{L_1} - L_2 \neq \emptyset$?

$\overline{L_1}$ is recursive $\quad$ $\overline{L_1}$ is recursive, $\overline{L_2}$ is recursive.

L_2 is recursive

For decide $\overline{L_1} - L_2 = \emptyset$, the TM for $\overline{L_1} - L_2$ has to run forever to enumerate all the possibilities with no guarantee to stop. But for $\overline{L_1} - L_2 \neq \emptyset$, once we find a word in $\overline{L_1} - L_2$ and not empty, the TM will halt on the input and say yes.

$\Rightarrow \overline{L_1} - L_2 \neq \emptyset$? is r.e. but not recursive.

(Q3) $L_1 \cap L_2$ contain a given fixed word w ?

Since L_1 and L_2 are recursive

TM for $L_1 \cap L_2$ can always answer yes for word w in $L_1 \cap L_2$ and say no when $w \notin (L_1 \cap L_2)$

$\therefore L_1 \cap L_2$ is recursive.

(Q4) $L_1 \cap L_2 \neq \emptyset$?

TM for $L_1 \cap L_2$, simulate input word w for $L_1 \cap L_2$, when $w \in L_1 \cap L_2$ and $w \notin \emptyset$, the TM halts and answer yes. but for $L_1 \cap L_2 = \emptyset$, the TM has to go through all the cases for $L_1 \cap L_2$ and runs forever with no guarantee to stop and say no.

$\therefore L_1 \cap L_2 \neq \emptyset$? is r.e but not recursive.

(ii) Both L_1 and L_2 are r.e. but not recursive.

(Q1) $\overline{L_1} - L_2$

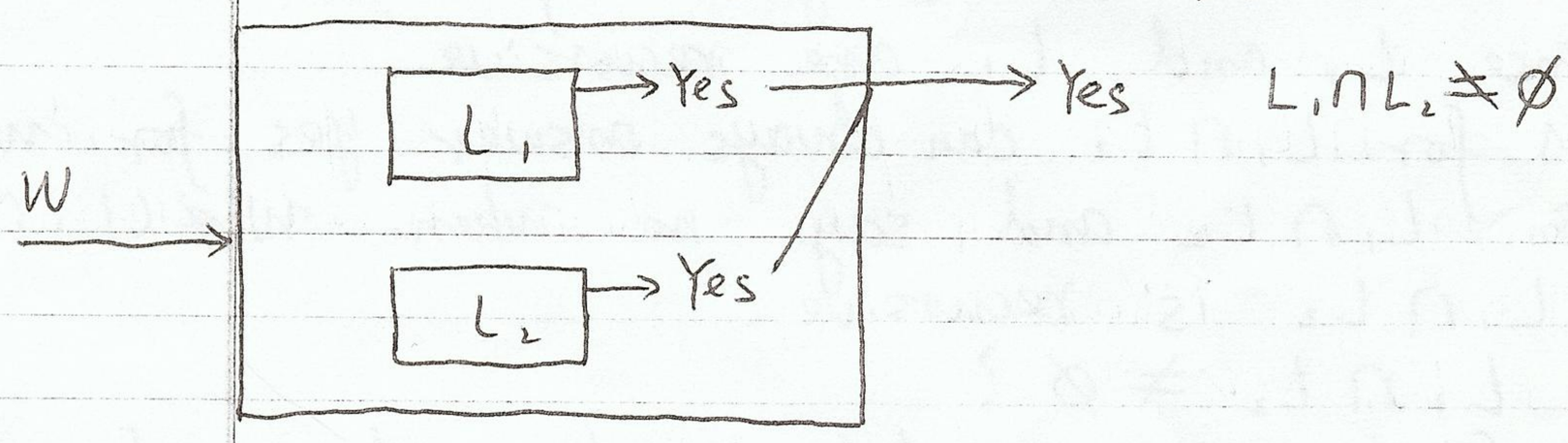
$\because L_1$ is r.e but not recursive

$\therefore$ there is no way for $\overline{L_1}$ halts on input w
and no such TM for $\overline{L_1} - L_2$
$\therefore$ $\overline{L_1} - L_2$ is not r.e.

(Q2) $\overline{L_1} - L_2 \neq \emptyset$?
$\because$ L_1 is r.e. but not recursive
$\therefore$ no TM for $\overline{L_1}$
$\therefore$ $\overline{L_1} - L_2$ is non r.e.

(Q3) $L_1 \cap L_2$ contain a given fixed word w.
$\because$ L_1 and L_2 is r.e. but not recursive
$\therefore$ when $w \in L_1$ and $w \in L_2$, the TM for $L_1 \cap L_2$
always answers yes on input w.
$\therefore$ $L_1 \cap L_2$ is r.e. but not recursive.

(Q4) $L_1 \cap L_2 \neq \emptyset$?
L_1 and L_2 are r.e. but not recursive



$\therefore$ $L_1 \cap L_2 \neq \emptyset$ is r.e but not recursive.