PHY 180 - Computational Physics - Spring 2023

Project 4: Pendulum and Chaos
Due: Friday, February 24th
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1 Introduction

The simple pendulum is a fairly tame system that can be explored in an introductory physics course. It uses the "small angle" approximation of $\sin\phi \approx \phi$ so that analytical solutions are possible. Even some more complicated systems like the driven and damped oscillators still maintain their analytic solutions.

However, the physical pendulum brings physics students back to reality, in which $\sin\phi \not\approx \phi$. This system, even without driving and damping forces, can only be solved numerically. Furthermore, the introduction of driving and damping forces together can result in a chaotic system, where a tiny change in the initial conditions yields radically different behavior.

In the regime between chaotic motion and predictable motion, we can find a series of period-doubling bifurcations, as described by McLaughlin [1]. These can be seen best using the Poincaré map. This is a plot of the angular speed ω versus the angle ϕ (also called the phase space) in which the points are collected at the end of each period. The expectation for periodic motion is a single point in this Poincaré map. However, for certain initial conditions, the period can bifurcate, in which two points show up instead of one point.

The goal of this project is to examine the physical pendulum system studied by McLaughlin. The differential equation describing this is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + k\frac{\mathrm{d}\phi}{\mathrm{d}t} + [a + 2q\cos(\Omega t)]\sin\phi = 0 \tag{1}$$

where k is the damping force, a is the natural frequency of the pendulum, q is proportional to the driving force amplitude, and Ω is the driving force frequency. The following parameters are set by McLaughlin: $k = 0.2$, $a = 1$, and $\Omega = 2$. The previously mentioned bifurcation points are determined by adjusting the value of q. We will attempt to determine these points to greater accuracy and precision than McLaughlin.

2 Modifications to the Code

On top of translating the given code from FORTRAN 77 to FORTRAN 90 and defining all variables explicitly, several other changes were made to the code. Starting in the main program, all of the initialization code was moved to a separate `initialize` subroutine. The output file is also opened in the main program to avoid potential problems. Besides this, the cosmetic print statement at the end including the CPU time was added after the main code ran smoothly.

As mentioned, the initialization subroutine was created to assign initial values to all variables and to prepare the output file with comments including these values. Some of the variables were renamed for more clarity. This includes changing `th()` to `phi()`, `om()` to `w()`, and `xk` to `kappa`. Further changes to the sizes of these arrays and the removal of the `nmax` and `ndim` variables are explained next. Additionally, the formatting of the output file was modified to make it more readable.

Finally, the `calculate` subroutine had a fair number of changes. First, the size the arrays `t()`, `phi()`, and `w()` were changed from `ndim` to `4`. This is because the `write` statement to the output file occurs within the main time loop of the calculation, so the storage of previous values is unnecessary. The particular reason for the change of the dimension to `4` is to clean up the number of variables with essentially the same name. Rather than having variables `phi1`, `phi2`, `phi3`, and `phi4` all defined for the 4th-order Runge-Kutta (RK4) method, we let the index of the array take care of the labelling. Thus, `phi(1)` is used for the first step of the RK4 method and is updated at the end of the full timestep. The other three indices of these arrays are used for the middle steps of the RK4 method. Likewise, `dw` and `dphi` were changed to four-dimensional arrays. After this, the single loop ranging from `i = 1,nmax` was modified into a nested loop with `j = 1,npp` inside of `i = 1,nperiod`. This loops over the points per period inside of the a loop over the total number of periods. This modification makes the stroboscopic measurements at the end of each period much easier and allows us to focus on the end behavior of the pendulum with the last 64 periods.

3 Results

The modified code is able to reproduce the first three figures in McLaughlin's paper, as seen in Figures 1-3. With this confidence, we proceeded to the determination of the q-values for period bifurcation. These values were found via a binary search by hand, which was carried out to the 5th decimal place.

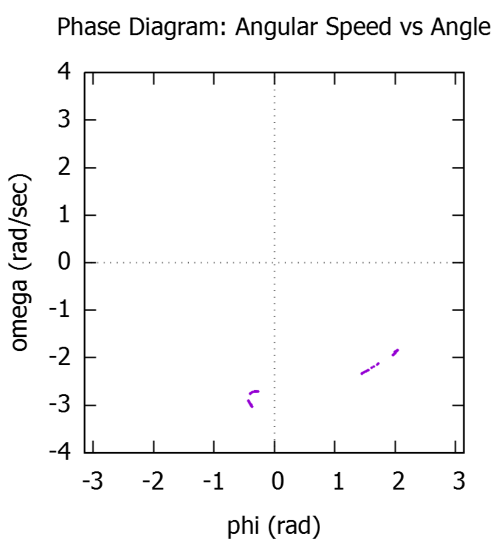

Figure 1: Phase diagram of the last 200 periods. Initial values: $\phi(0) = 1.036$, $\omega(0) = -2.150$, $k = 0.2$, $q = 1.036$

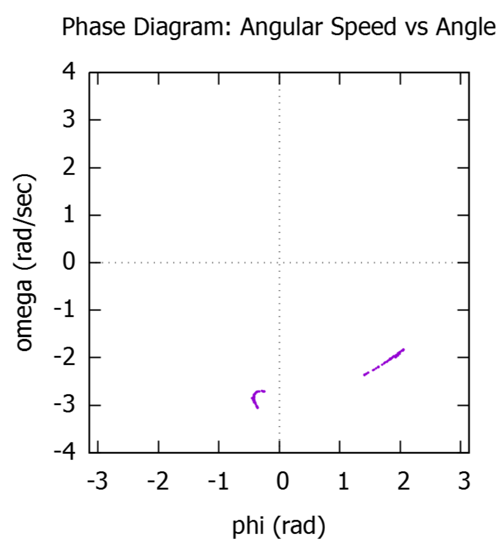

Figure 2: Phase diagram of the last 200 periods. Initial values: $\phi(0) = 2.063$, $\omega(0) = -1.819$, $k = 0.2$, $q = 1.0375$

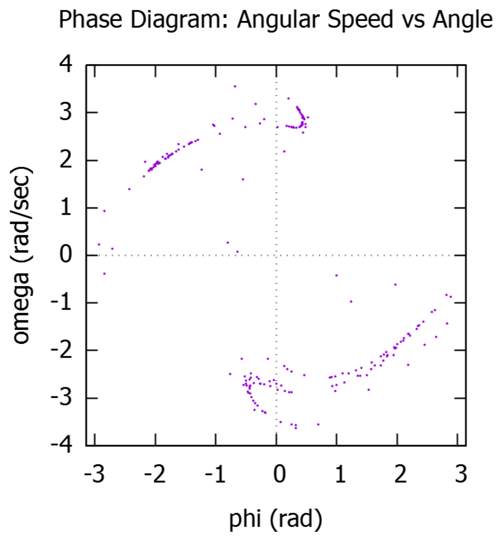

Figure 3: Phase diagram of the last 200 periods. Initial values: $\phi(0) = 0.3456$, $\omega(0) = -3.104$, $k = 0.2$, $q = 1.05$

The q-value was then taken to be the average of the q_n, which produced n points, and q_{2n}, which produced $2n$ points. For larger n, the window was restricted to the point near $(1.6, -2.2)$ on the phase diagram, as the bifurcation here was often largest even for small changes to q.

$$\begin{gathered}
q_2 = 0.790085 \pm 0.000005 \qquad q_4 = 0.989205 \pm 0.000005 \\
q_8 = 1.024175 \pm 0.000005 \qquad q_{16} = 1.031175 \pm 0.000005 \\
q_{32} = 1.032685 \pm 0.000005
\end{gathered} \tag{2}$$

The ratios of differences between q-values, denoted δ, are calculated to be:

$$\begin{aligned}
\delta_4 &= \frac{q_4 - q_2}{q_8 - q_4} = 5.694023 \pm 0.001914 \\
\delta_8 &= \frac{q_8 - q_4}{q_{16} - q_8} = 4.995714 \pm 0.008565 \\
\delta_{16} &= \frac{q_{16} - q_8}{q_{32} - q_{16}} = 4.63576 \pm 0.037323
\end{aligned} \tag{3}$$

4 Conclusions

The precision and accuracy of the q-values are much better than those produced by McLaughlin. Further, a $2 \to 1$ combination point was found between $q = 0.655$ and $q = 656$, which McLaughlin fails to mention in his paper. We also see a better convergence to the Feigenbaum constant $\delta_\infty = 4.669201$, but δ_{16} overshoots this value.

References

[1] John B. McLaughlin. Period-doubling bifurcations and chaotic motion for a parametrically forced pendulum. *Journal of Statistical Physics*, 24(2):375–388, February 1981.