1552 Unofficial Exam 4 Practice Solutions

- I recommend you study the other material first, then once you feel ready, try this practice exam out. You should time yourself and give yourself 50 minutes.
- Even though this is just a practice, you should still be practicing good notation. You can always email me at ethanphan@gatech.edu if you want me to check your work.
- Solutions will be posted `https://ethanphan.me/f24-exam4-sol.pdf` on Saturday evening.
- Made by Ethan Phan. Good luck!

Question	Points	Score
1	8	
2	30	
3	30	
4	32	
Total:	100	

Good luck!!

You got this :)

1. (8 points) True/False. [NOTE: For practice, you should go back after the exam and determine *why* each of the statements is true/false.]

   (a) Let $f(x) = xe^x$. Then $f^{(5)}(0) = 4$

   A. True $\boxed{\text{B. False}}$

$$f(x) = xe^x = x \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$n=4$

$$f(x) = \sum_{K=0}^{\infty} \frac{f^{(K)}(0)}{K!} x^K$$

$K=5$

$$\frac{f^{(5)}(0)}{5!} x^5 = \frac{x^5}{4!}$$

$$f^{(5)}(0) = \frac{5!}{4!} = 5$$

   (b) $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \dots$

   $\boxed{\text{A. True}}$ B. False

$$1+2+\frac{2^2}{2!}+\dots = \sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$$

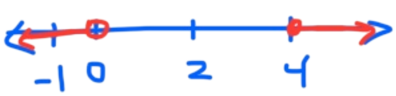
   (c) If the power series $\sum_{n=1}^{\infty} a_n(x-2)^n$ diverges at $x = 4$, then the power series diverges at $x = -1$

   $\boxed{\text{A. True}}$ B. False

↳ center @ $x=2$

diverge @ $x=4$ $\Rightarrow$ radius of conv. ≤ 2

$\Rightarrow$ red must diverge



   (d) Let $\sum_{n=1}^{\infty} a_n(x+2)^n$ be a power series and suppose $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 2$. Then the power series converges at $x = 4$.

   A. True $\boxed{\text{B. False}}$

↳ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x+2)^{n+1}}{a_n(x+2)^n} \right|$$

$$= \frac{1}{2}|x+2| < 1$$

$|x+2| < 2$

$-2 < x+2 < 2$

$0 < x < 4$

inconclusive @ $x=4$

2. (30 points) (a) Find the radius of convergence of the power series. [NOTE: On the actual exam, you should make your final answer very clear - write something like $\boxed{\text{radius} = \ldots}$ and circle it.]

$$\sum_{n=0}^{\infty} \frac{3^n(2x-1)^n}{2^{2n+1}\sqrt{n+2}} = \sum_{n=0}^{\infty} \frac{3^n(2x-1)^n}{4^n \cdot 2\sqrt{n+1}}$$

Ratio Test

don't forget absolute value and proper limit notation!

$$\lim_{n\to\infty} \left| \frac{3^{n+1}(2x-1)^{n+1}}{2\cdot 4^{n+1}\sqrt{n+3}} \cdot \frac{2\cdot 4^n\sqrt{n+2}}{3^n(2x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3\sqrt{n+2}}{4\sqrt{n+3}} \right| \cdot |2x-1|$$

$$= \frac{3}{4}\sqrt{\lim_{n\to\infty} \frac{n+2}{n+3}} \cdot |2x-1|$$

$$= \frac{3}{4}|2x-1| < 1$$

$$|2x-1| < \frac{4}{3}$$

$$-\frac{4}{3} < 2x-1 < \frac{4}{3}$$

$$-1/3 < 2x < 7/3$$

$$\to -1/6 < x < 7/6$$

distance is 8/3 so $R = \frac{8/3}{2} = 4/3$

$$\boxed{\text{Radius} = 4/3}$$

(b) What is the *open* interval of convergence of the power series?

$$-1/6 < x < 7/6$$

(c) Check the endpoints for convergence

never say "b_n diverges" when you mean "$\sum_{n=1}^{\infty} b_n$ diverges"

$$\sum_{n=0}^{\infty} \frac{3^n(2x-1)^n}{2^{2n+1}\sqrt{n+2}}$$

@ $x = 7/6$: $\sum_{n=0}^{\infty} \frac{3^n(\frac{7}{3}-1)^n}{4^n \cdot 2\sqrt{n+2}} = \sum_{n=0}^{\infty} \frac{3^n(4/3)^n}{4^n \cdot 2\sqrt{n+2}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+2}}$

LCT to $\sum_n b_n = \frac{1}{\sqrt{n}}$, $\sum_n b_n$ div, p-series $p = \frac{1}{2} \le 1$.

$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+2}} = \sqrt{\lim_{n\to\infty} \frac{n}{n+2}} = \sqrt{1} = 1.$

Since 1 is pos. and finite, by LCT both series diverge

@ $x = -1/6$: $\sum_{n=0}^{\infty} \frac{3^n(-\frac{1}{3}-1)^n}{4^n \cdot 2\sqrt{n+2}} = \sum_{n=0}^{\infty} \frac{3^n(-4/3)^n}{4^n \cdot 2\sqrt{n+2}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$

AST: $\sum_n \frac{(-1)^n}{\sqrt{n+2}}$ is alternating. $|a_n| = \frac{1}{\sqrt{n+2}}$

1) $\lim_{n\to\infty} \frac{1}{\sqrt{n+2}} = 0$ ✓

2) $\sqrt{n+2} < \sqrt{n+3}$

$\Rightarrow |a_{n+1}| = \frac{1}{\sqrt{n+3}} < \frac{1}{\sqrt{n+2}} = |a_n|$. $\{|a_n|\}$ decreasing ✓

By AST, convergence @ $x = -1/6$

(d) What is the interval of convergence of the power series?

$[-1/6, 7/6)$ OR $-1/6 \le x < 7/6$

3. (30 points) (a) Find a second degree Taylor polynomial for $f(x) = \ln(x)$ centered at $x = 1$. Show all your work (i.e., do not use the Taylor polynomial you memorized).

$f(x) = \ln x$ $\qquad f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\qquad f'(1) = 1$

$f''(x) = -x^{-2} = \frac{-1}{x^2}$ $\qquad f''(1) = -1$

$f(x) \approx T_2(x) = 0 + 1(x-1) - \frac{1}{2!}(x-1)^2$

$= (x-1) - \frac{1}{2}(x-1)^2$

(b) Approximate $\ln(1.5)$ using part (a).

$\ln(1.5) \approx T_2(1.5) = \frac{1}{2} - \frac{1}{2}(1/2)^2$

$= \frac{4}{8} - \frac{1}{8} = 3/8$

(c) Using Taylor's Remainder Theorem, estimate the error of your answer in part (b).

$$|R_n(x)| \le \max\left|f^{(n+1)}(c)\right| \frac{|x-a|^{n+1}}{(n+1)!}$$

$x = 1.5 \quad a = 1 \quad n = 2$

$|f^{(3)}(x)| = |2x^{-3}| = \frac{2}{x^3}$

$\frac{d}{dx}|f^{(3)}(x)| = -6x^{-4} < 0$

$\frac{d}{dx}|f^{(3)}(x)| < 0$ on $[1, 1.5]$

So $|f^{(3)}(x)|$ is decr. on $[1, 1.5]$

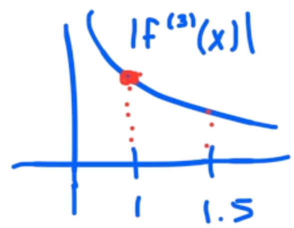
showing $|f^{(3)}(x)|$ is decr., so take the left endpoint's value

Choose $M = f^{(3)}(1) = 2$

$|R_2(1.5)| \le 2 \cdot \frac{|1.5-1|^3}{3!}$

$= 2 \cdot \left(\frac{1}{2}\right)^3 \cdot \frac{1}{6} = \frac{1}{24}$

try to always simplify your answers



4. (32 points) (a) Find a series expansion for $g(t)$. Write your series in sigma notation and simplify exponents as far as possible.

do not forget to write "Σ"!

$$g(t) = \sin\left(\frac{t^2}{2}\right)$$

keep variables consistent, g(t) is a function of t so your sum should use "t"s

$$\sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin\left(\frac{t^2}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(t^2/2\right)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(t^2)^{2n+1}}{2^{2n+1}(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{2^{2n+1}(2n+1)!}$$

(b) Integrate your series in part (a) to find a series expansion for $f(x)$. Write your series in sigma notation.

$$f(x) = \int_0^x \sin\left(\frac{t^2}{2}\right) dt$$

$$\int_0^x \sin\left(\frac{t^2}{2}\right) dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{2^{2n+1}(2n+1)!}\, dt$$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+3}}{2^{2n+1}(2n+1)!(4n+3)}\right]_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{2^{2n+1}(2n+1)!(4n+3)}$$

(c) Using part (b), find a series expansion for $f(1)$.

$$f(1) = \int_0^1 \sin\left(\frac{t^2}{2}\right) dt$$

$$f(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!(4n+3)}$$

(d) Using the Alternating Series Error Theorem, estimate the value of $f(1)$ within an error of at most 0.05. [NOTE: You should simplify your answer as far as possible, even if the exam instructions don't specify to!]

$$f(1) = \frac{1}{2(1)(3)} - \underbrace{\frac{1}{8 \cdot 6 \cdot 7}}_{< 0.05} + \ldots$$

Stop @ first value which is less than the error and sum all terms before it.

$$f(1) \approx \frac{1}{6}$$