- topological mfld of dim n is Hausdorff and 2^{nd} countable and locally Euclidean
  - Hausdorff: $\forall x, y \in M$, $\exists$ open $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$
    - ↳ subspaces and products of Hausdorff are Hausdorff
    - ↳ if Hausdorff, then compact $\Rightarrow$ closed. Singletons closed
  - Second ctbl: topology has ctbl basis
    - ↳ cont, open image of 2^{nd} is 2^{nd}. subspaces, ctbl product, open quotients
  - locally Euc: $\forall p \in M$, $\exists U \subseteq M$ open, $\varphi : U \to V \subseteq \mathbb{R}^n$ open (coord chart) homeomorphism
- smoothly compatible: $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ smooth
  $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$

- smooth atlas: $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ coord charts, 1) U_α covers 2) all charts sm. comp.
  - ↳ need to use maximal sm atlas to give sm structure on M.
- two sm mflds $(M, \mathcal{A})$, $(N, \mathcal{B})$ diffeomorphic if $\exists f : M \to N$ homeo s.t. $\varphi \in \mathcal{B}$ iff $\varphi \circ f \in \mathcal{A}$

$\underline{S^n}$: $\sigma_N : S^n \setminus \{N\} \to \mathbb{R}^n : (x^1, \dots, x^{n+1}) \mapsto \frac{1}{1 - x^{n+1}}(x^1, \dots, x^n)$

$\sigma_S : S^n \setminus \{S\} \to \mathbb{R}^n : x \mapsto -\sigma_N(-x) : (x^1, \dots, x^{n+1}) \mapsto \frac{1}{1 + x^{n+1}}(x^1, \dots, x^n)$

$\sigma_N^{-1}(x) = \frac{1}{1+|x|^2}(2x^1, \dots, 2x^n, |x|^2 - 1)$ $\quad \sigma_S^{-1}(x) = \frac{1}{1+|x|^2}(2x^1, \dots 2x^n, 1 - |x|^2)$

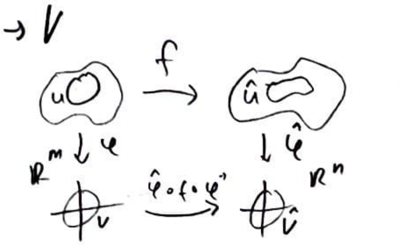
$\underline{\mathbb{RP}^n}$: $U_i = \{[x^0 : \dots : x^n] \in \mathbb{RP}^n : x_i \neq 0\}$, $\varphi_i : U_i \to \mathbb{R}^n : [x^0 : \dots : x^n] \mapsto \frac{1}{x_i}(x^0, \dots, \hat{x}_i, \dots, x^n)$

$\varphi_i^{-1} = (x^0, \dots, \hat{x}_i, \dots, x^n) \mapsto [x^0 : \dots : x^{i-1} : 1 : x^{i+1} : \dots : x^n]$

$\underline{\mathbb{CP}^n}$: same as $\mathbb{RP}^n$ except with $\mathbb{C}$.

- open subsets U of sm mflds: restrict $\varphi_\alpha : U_\alpha \to V$ to $\varphi_\alpha|_U : (U_\alpha \cap U) \to V$

$M(n, m; \mathbb{R}) = \mathbb{R}^{n \cdot m}$



$GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ is open (preimage of cont map)

- $f : M \to N$ is smooth if $\forall p \in M$, $\exists \varphi : U \to V$, $\hat{\varphi} : \hat{U} \to \hat{V}$ with $\hat{\varphi} \circ f \circ \varphi^{-1} : V \to \hat{V}$ is sm @ $\varphi(p)$ ($p \in U$, $f(p) \in \hat{U}$)
  - ↳ sm $\Rightarrow$ cont, composition of sm is sm
- diffeomorphism: bijective, smooth, inverse smooth
- bumpfunctions

- a derivation at $p \in \mathbb{R}^n$ is a linear map $D: C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ s.t. $D(fg) = (Df)g(p) + f(p)(Dg)$ ← product rule
- Lemma 1: 1) $\forall v \in \mathbb{R}^n_p$, $(D_v f)(p) = \lim_{h \to 0} \frac{f(p+hv) - f(p)}{h} = \frac{d}{dt} f(p+tv)\big|_{t=0}$

  2) If $D: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation @ p, $\exists v \in \mathbb{R}^n_p : D = D_v$.

  * directional derivatives in $\mathbb{R}^n_p$ are the same as derivations. $\boxed{T_x \mathbb{R}^n \cong \mathbb{R}^n_x}$
- derivation @ $p \in M$ is a linear map $D: C^\infty(M) \to \mathbb{R}$ s.t. $D(fg) = (Df)g(p) + f(p)(Dg)$.
- $T_pM = \{\text{derivations @ } p\}$ - tangent space of M @ p. (set of maps $D: C^\infty(M) \to \mathbb{R}$)
- $f: M \to N$ smooth. induces map $df_p: T_pM \to T_{f(p)}N$ (differential of f @ p) by ($v \in T_pM$)

  $df_p(v)(g) = v(g \circ f) \in \mathbb{R}$ (df_p takes in derivations (maps $C^\infty M \to \mathbb{R}$) and outputs derivs. ($C^\infty(N) \to \mathbb{R}$) $(C^\infty(M) \to \mathbb{R})$,

  these act on fns $g \in C^\infty(N)$, $g \circ f : M \to \mathbb{R} \in C^\infty(M)$, the domain of v).

  ↳ $df_p(v): C^\infty(N) \to \mathbb{R}$ so is in $T_{f(p)}N$. ← chain

  ↳ df_p is linear. $\boxed{d(g \circ f)_p = (dg_{f(p)}) \circ (df_p)}$. id_M induces $d(id_M)_p = id_{T_pM}$

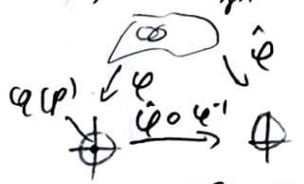
  $f: M \to N$ diffeo $\Rightarrow$ df_p is an isomorphism
- $U \subseteq M$ open. $i: U \hookrightarrow M$ inclusion induces isomorphism $di_p: T_pU \to T_pM$ $\forall p \in U$

  ↳ so $T_pM \cong T_pU \cong T_{\varphi(p)}V \overset{V \subseteq \mathbb{R}^n \text{ open}}{\cong} T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\varphi(p)} \cong \mathbb{R}^n$. $\dim(T_pM) = n$

  ↑ coord charts are diffeo
- local coords: $v \in T_pM$ can be rep'd by $v = \sum_{i=1}^n v^i \underbrace{(d\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\right)}_{\text{"}\partial/\partial x^i\text{"}}$ $\boxed{\left\{\frac{\partial}{\partial x^i}\right\}_i \text{ is a basis for } T_pM}$ ← basis for $\mathbb{R}^n_p$, identified with

  in other coords, $v = \sum_{j=1}^n w^j \frac{\partial}{\partial y^j}$ with $w^j = \sum_{i=1}^n \partial y^j / \partial x^i \, v^i$

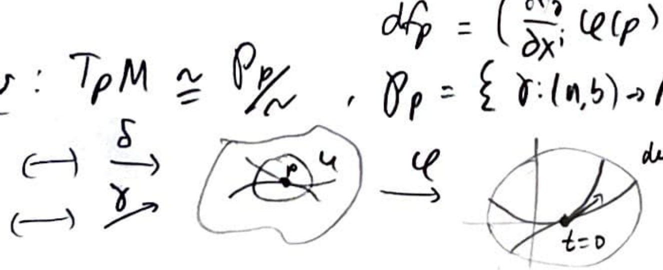
  ↳ $\begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix} = \left[\partial y^j / \partial x^i\right] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ Jacobian of $\hat{\varphi} \circ \varphi^{-1}$



in local coords, differential df_p is given by total derivative: $df_p = Dg(p) = \left(\frac{\partial f_j}{\partial x^i}\right)$, $(f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix})$

$df_p = \left(\frac{\partial f_j}{\partial x^i} \varphi(p)\right)$

- curves: $T_pM \cong \mathcal{P}_p/\sim$, $\mathcal{P}_p = \{\gamma: (a,b) \to M \mid \gamma(0) = p\}$, $\delta \sim \gamma$ if $\exists$ coord chart at p $\frac{d}{dt}(\varphi \circ \gamma)\big|_{t=0} = \frac{d}{dt}(\varphi \circ \delta)\big|_{t=0}$



- **Linearization**: $f: M \to N$ sm, if df_p has maximal rank then we can choose coord charts s.t.
  $(m \le n)$: $\hat{\varphi} \circ f \circ \varphi^{-1}(x^1 \dots x^m) = (x^1 \dots x^m, 0, \dots 0)$ (inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$)
  $(m \ge n)$: $\hat{\varphi} \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$ (projection $\mathbb{R}^m \to \mathbb{R}^n$)
- **Rank thm**: df_p has const rank k in open set, then can choose coord charts [on U] ($\subseteq M$ o.m.)
  $\hat{\varphi} \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots 0)$ ($\mathbb{R}^m \xrightarrow{proj} \mathbb{R}^k \xhookrightarrow{inclusion} \mathbb{R}^n$)
- **Inverse fn thm**: If f sm, df_p iso, then f is local diffeo, $(df^{-1})_{f(p)} = (df_p)^{-1}$
  (f diffeo $\Rightarrow$ df_p iso, converse not exactly true.. only local diffeo, e.g. $\mathbb{R} \to S^1 : t \mapsto (\cos t, \sin t)$)
  - bijective local diffeo is a diffeo (everywhere).
- **Immersion**: $f: M \to N$ sm, df_p injective $\forall p$.
- **(smooth) embedding**: f immersion and $f: M \to f(M)$ is a homeomorphism
  - if f injective immersion, and f open / f closed / f compact then f embedding
- $S \subseteq M$ **submfld** dim k: $\forall p \in S$, $\exists \varphi: U \to V$ in M $\varphi(S \cap U) = V \cap (\{0\} \times \mathbb{R}^k)$ - slice chart
  - sm str. is inherited by intersecting w/ S.
  - inclusion of submfld is embedding
  - ☆ submflds are the same as images of smooth embeddings ☆
- **Submersion**: $f: M \to N$ sm., df_p surj $\forall p$ (all $p \in N$ are regular values)
- **Preimage thm**: $f: M \to N$ sm, $q \in N$ regular value, then $S = f^{-1}(q)$ is submfld of dim $m-n$.
  $T_p S = \ker df_p$.
- **Lie groups**: sm mfld G w/ group structure s.t. mult and inverse are smooth
  - G Lie gp $\Rightarrow$ $T_I G$ is a Lie Algebra: vector space V w/ $[\cdot, \cdot]$ operation
    1) bilinearity (linear in each) 2) anticommutative $[v, u] = -[u, v]$
    3) Jacobi id: $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$
- **Sard's thm**: set of critical values has measure zero in N (regular values are dense)
- **Whitney Embedding**: M (compact) embeds in $\mathbb{R}^k$ for some k. In fact, into $\mathbb{R}^{2n+1}$ ($\mathbb{R}^{2n}$ in fact)

- fiber bundle $\begin{matrix} F \to E \text{ total} \\ \text{fiber} \quad \downarrow p \\ \quad B \text{ base}\end{matrix}$ E, B, F top spaces. $p: E \to B$ cont. surjection.

$$\begin{tikzcd} U \times F \arrow[r, "\varphi_\alpha"] \arrow[d, "p_1"'] & p^{-1}(U) \arrow[dl, "p"] \\ U \end{tikzcd}$$
projection

$\forall x \in B$, $\exists U$ nbh x, homeo $\varphi_U : U \times F \to p^{-1}(U)$ (local trivialization)

- a section is a map σ $E \underset{p}{\overset{\sigma}{\rightleftarrows}} B$ st. $p \circ \sigma = id_B$. Set of sections is $\Gamma(E)$
- bundle map: pair $(f, \bar{f})$ st. $\begin{matrix} E & \xrightarrow{\bar f} & E' \\ p\downarrow & \circ & \downarrow p' \\ B & \xrightarrow{f} & B' \end{matrix}$ (bundles to bundles)

bundle isomorphism. $E \xrightarrow{f} E'$, $p \searrow \circ \swarrow p'$ over B, and f is a diffeo (restriction to fibers is iso)

- transition maps: $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \text{Homeo}(F)$, $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$, $g_{\alpha\alpha} = id_F$
$\varphi_\beta^{-1} \circ \varphi_\alpha (p, x) = (p, g_{\beta\alpha}(p)(x))$
- vector bundle: $E \xrightarrow{p} B$ st. $p^{-1}(x)$ is a vector space $\forall x \in B$, local triv. $\varphi : U \times \mathbb{R}^n \to p^{-1}(U)$ that cover B st $\varphi|_{\{x\} \times \mathbb{R}^n} : \mathbb{R}^n \to p^{-1}(x)$ is linear iso $\forall x \in U$
- Lemma: vector bundle $\iff$ 1) $F = \mathbb{R}^n$ (fibers are $\mathbb{R}^n$) 2) $\exists \{U_\alpha\}$ cover of B st $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ $\forall \alpha, \beta$
- tangent bundle: $TM = \bigsqcup_{x \in M} T_x M$, p projection
  - ↳ gets sm str. inherited from M: $\{d\varphi_\alpha : TU_\alpha \to TV_\alpha\}$

$\begin{matrix} TM & \xrightarrow{df} & TN \\ p\downarrow & \circ & \downarrow p \\ M & \xrightarrow{f} & N \end{matrix}$ bundle map

- vector field: sm section $v : M \to TM$ (a choice of vectors in TM for each M, "continuous").
  - ↳ $\mathfrak{X}(M) = \Gamma(TM)$ = set of vector fields (is a vector space)
- vector fields in 1-1 w/ derivations: $v \in \mathfrak{X}(M)$, $f : M \to \mathbb{R}$, $v \cdot f : M \to \mathbb{R} : x \mapsto \underset{\in TM, \text{ derivation @ } x}{v(x)} \cdot f$, v derivation on M $v : C^\infty(M) \to C^\infty(M)$.
  - ↳ $df_v : M \to \mathbb{R} : x \to df_x(v(x)) \in T_{f(x)}\mathbb{R} \cong \mathbb{R}$
  $df(v)(x) = v \cdot f(x)$
- Lie bracket: $[v, w]f = v \cdot (w \cdot f) - w \cdot (v \cdot f)$ $f, g \in C^\infty(M)$, $v, w, u \in \mathfrak{X}(M)$
1) skew sym, 2) $[fv, gw] = fg[v, w] + f(v \cdot g)w - g(w \cdot f)v$, 3) linear 4) Jacobi id
- Flows: $v \in \mathfrak{X}(M)$ vec field. $\Phi : \underset{\subseteq M \times \mathbb{R}}{W_{\varepsilon, S}} \to M$: $\Phi(p, 0) = p$, $d\Phi_{(p,t)}\left(\frac{\partial}{\partial t}\right) = v(\Phi(p, t))$
$\Phi(\Phi(p, s), t) = \Phi(p, s+t)$
$\gamma^p(t) = \Phi(p, t)$ is a flow line (fix p, run thru t)
$\gamma^p(0) = p$, $\gamma^{p\prime}(t) = (d\gamma^p)_{\gamma(t)} = v(\gamma(t))$

flows give vector fields. vec fields give flows

- ODE thm: F smooth on open, then $\exists$ unique sol to ODE
- Lie derivative: $\mathcal{L}_v f = v \cdot f = df(v) = (\text{at } x)$, $\lim_{t \to 0} \frac{f \circ \Phi(x, t) - f(x)}{t} = \frac{d}{dt}\left(f \circ \Phi(x, t)\right)\big|_{t=0}$. $\mathcal{L}_v w = [v, w]$

transverse: $T_{f(p)} N = \operatorname{im} df_p \oplus T_{f(p)} S \quad \forall p \in f^{-1}(S)$: $f: M \to N$ transverse to $S \subset N$ ①

$F: C \times C' \to \mathbb{R}^3 : F(x,y) = y - x$.

$C + v$ intersects C' when $\underset{\exists (x,y) \in C \times C'}{F(x,y)} = y - x = v \Rightarrow \overset{\in C}{x} + v = \overset{\in C'}{y}$

F is sm. the set of critical vals in $\mathbb{R}^3$ has ms 0. (Sard)

(where dF_p is not surj. for any $p \in F^{-1}(q)$)

$dF_p : T_p C \times T_p C' \to \mathbb{R}^3$

$\quad \cong \mathbb{R} \quad \cong \mathbb{R}$

i.e. when F is surj. Let $A = \{ v \in \mathbb{R}^3 : \exists (x,y) : F(x,y) = v \} \subseteq \mathbb{R}^3$

for any $q \in A$, $F^{-1}(q)$ is non empty and for any $p \in F^{-1}(q)$,

dF_p is surjective.

---

the set of v for which $C+v$ intersects C' is the image of F.

Since $dF_p : T_p C \times T_p C' \to \mathbb{R}^3$ is a map from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$,

it cannot be surjective for any $p \in C \times C'$, hence

$\operatorname{im} F$ has ms. 0 since it lies entirely in the critical pts. ✓

exact: $\omega = d\eta$ for some η.   closed: $d\omega = 0$

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$d\omega = \frac{d}{dx}\left(\frac{-y}{x^2+y^2}\right)\underbrace{dx\wedge dx}_{=0} + \frac{d}{dy}\left(\frac{-y}{x^2+y^2}\right)dy\wedge dx + \frac{d}{dx}\left(\frac{x}{x^2+y^2}\right)dx\wedge dy + \frac{d}{dy}\left(\frac{x}{x^2+y^2}\right)\underbrace{dy\wedge dy}_{=0}$$

$$= \frac{(x^2+y^2)(-1)-(2y)(-y)}{(x^2+y^2)^2}\,dy\wedge dx + \frac{(x^2+y^2)(1)-(2x)(x)}{(x^2+y^2)^2}\,dx\wedge dy$$

$$= -\frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}\,dx\wedge dy + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\,dx\wedge dy$$

$$= \frac{+x^2-y^2-x^2+y^2}{(x^2+y^2)^2}\,dx\wedge dy = 0 \quad ✓ \quad \omega \text{ is closed.}$$

Suppose ω is exact, so $\omega = df$. Then $\int_{S^1}\omega = \int_{S^1} df \overset{\text{Stokes}}{=} \int_{\partial S^1} f = 0$ since $\partial S^1 = \emptyset$.

But: $\int_{S^1}\omega = \int_{[0,2\pi]} \omega_{(\cos t, \sin t)}\left(\langle -\sin t, \cos t\rangle\right) dt = \int_0^{2\pi} \frac{-\sin t}{\cos^2 t+\sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t+\sin^2 t}(\cos t)\,dt$

parametrize S^1 by $\gamma(t) = (\cos t, \sin t)$, $\gamma'(t) = \langle -\sin t, \cos t\rangle$
$[0,2\pi] \to \mathbb{R} - \{0\}$

$$= \int_0^{2\pi} dt = 2\pi \neq 0.$$

S23

1) M, N sm mflds then $M \times N$ is sm manifold

Pf: let $\{U_\alpha, \varphi_\alpha\}_\alpha$, $\{V_\beta, \psi_\beta\}$ be coord chs for M, N.

Then we claim $\{U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta\}_{\alpha,\beta}$ give a sm maximal atlas for $M \times N$. They clearly cover $M \times N$ as U_α cov M and V_β cov N.

For sm compatible: If $\alpha, \alpha', \beta, \beta'$, have $(U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) \neq \emptyset$, then

$$(\varphi_\alpha \times \psi_\beta) \circ (\varphi_{\alpha'} \times \psi_{\beta'})^{-1} = \underbrace{(\varphi_\alpha \circ \varphi_{\alpha'}^{-1})}_{sm} \times \underbrace{(\psi_\beta \circ \psi_{\beta'}^{-1})}_{sm}.$$

$\square$

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4) M compact sm, α closed 1-form. Show if α is never zero then $H^1_{DR}(M)$ is non trivial.

Pf: We show the contrapositive: suppose $H^1_{DR}(M)$ is trivial. Then α is also exact, $\alpha = df$ for some f 0-form. $f \in C^\infty(M)$, i.e., $f: M \to \mathbb{R}$ but M compact so f has a critical pt by extreme value thm, i.e., $df = 0$ at a pt, but then α vanishes somewhere. $\square$

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7) M compact oriented sm n-mfld, w/ ∂M. Stokes to show no sm retraction $r: M \to \partial M$.

~~Pf: $\exists$ n-form ω that is never vanishing (M oriented), so $\int_M \omega \neq 0$~~

~~Now $r^*\omega$ is a~~

Pf: let ω be a $n-1$ form on ∂M s.t. $\int_{\partial M} \omega \neq 0$ (∂M oriented)

Then $r^*\omega$ is a $n-1$ form on M.

$$\int_M d(r^*\omega) = \int_{\partial M} r^*\omega = \int_{\partial M} \omega \neq 0$$ since r is id on ∂M.

Since $d\omega = 0$ (since $H^n_{DR}(\partial M) = 0$ as ∂M $n-1$ dim),

$$\int_M d(r^*\omega) = \int_M r^*(d\omega) = \int_M 0 = 0.$$

$\square$

Sp24 1) M, N submfds of $\mathbb{R}^n$, $\dim(M) + \dim(N) < n$. Show M, N disj after arb small translation of N.

Pf: Let $\varepsilon > 0$. define $f: M \times N \to \mathbb{R}^n$ by $f(x,y) = x - y$. is smooth

Since for any $(x,y) \in M \times N$, $df_{(x,y)} : T_{(x,y)}(M \times N) \to T_{f(x,y)}\mathbb{R}^n$

$$T_{(x,y)}(M \times N) \cong T_xM \times T_yN$$

cannot be surjective as $\dim T_xM + \dim T_yN = \dim M + \dim N < n = \dim \mathbb{R}^n$.

By Sard's thm, $\operatorname{im} f$ has ms zero in $\mathbb{R}^n$ (it lies in the critical values),

But then $\exists v \in B_\varepsilon(0) \setminus \operatorname{im} f$ (i.e. a translation with magnitude less than ε, not in $\operatorname{im} f$), as $B_\varepsilon(0)$ has positive measure.

Since $v \notin \operatorname{im} f$, $\forall (x,y) \in M \times N$, $y + v \neq x$, hence $N + v$ and M are disjoint. But $\varepsilon > 0$ was arbitrary ✓ , □

Sp 24 7) S, M sm mflds, $f, g : S \to M$ homotopic. ω closed k-form on M, S is k-dim. Show $\int_S f^*\omega = \int_S g^*\omega$

Pf) $\exists F : S \times I \to M$ s.t. $F(x,0) = f(x)$, $F(x,1) = g(x)$, F cont but we can homotope F so that it is smooth. (i.e. f, g are sm homotopic also). relative to $S \times \{0,1\}$.

$F^*\omega$ is a k-form on $S \times I$ and $dF^*\omega = F^* \underbrace{d\omega}_{0} = 0$ since ω closed.

$$0 = \int_{S \times I} dF^*\omega \overset{\text{Stokes}}{=} \int_{\partial S \times I} F^*\omega = \int_{-S \times \{0\}} F^*\omega + \int_{S \times \{1\}} F^*\omega = -\int_S f^*\omega + \int_S g^*\omega$$

F24, 3) M compact n-fld no boundary, smoothly embedded in $\mathbb{R}^{n+1}$. Suppose origin O not in M. Show $\exists$ line passing through O which intersects M finitely often.

Pf: define $F: S^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ by $F(p,t) = pt$. F smooth and $dF_{(p,t)}$ is injective when $(p,t) \in S^n \times (\mathbb{R} \setminus \{0\})$, so F immersion on here.

We can homotope F to be $\pitchfork$ to M. By Transversality Thm, except for a set of ms 0, $f_x: S^n \to \mathbb{R}^{n+1}: p \mapsto F(p,x)$ is $\pitchfork$ to M.

Choose such a p_0 (so that $f_{p_0} \pitchfork M$).

and choose $L = F(p_0, \mathbb{R})$ i.e. the line through O and p_0.

$\underbrace{\dim L + \dim M = 1+n = \dim \mathbb{R}^{n+1}}_{\Rightarrow L \cap M \text{ are points}}$, $\underbrace{M \text{ compact}}_{|L \cap M| \text{ finite}}$ $\Rightarrow L \cap M$ is finite.

F24 4) $M = (S^1 \times S^{n-1}) \setminus \{p\}$ $n \geq 2$. Intersection theory to prove no sm embedding of $M \hookrightarrow \mathbb{R}^n$.

Pf: S^1 and S^{n-1} can be embedded into M which intersect transversally and only once ($\dim S^1 \times S^{n-1} = 1+n-1 = n = \dim \mathbb{R}^n$). If M can be embedded in $\mathbb{R}^n$ then so can S^1 and S^{n-1} with a single transverse intersection, a contradiction as we can homotope S^1 to a single pt, hence mod 2 intersection would be 0, but this should be preserved by sm. homotopy.

$\square$

S submfld of M, codim >2. Show if M simp. conn, so is $M\setminus S$. (8)

Pf: let $\gamma: S^1 \to M\setminus S$ be a sm map, [WTS $\gamma \simeq$ const map.]

Similarly, γ is nullhomotopic to a const map @ $x_0 \in M\setminus S$.

let $H: S^1 \times I \to M$ be this homotopy:

$H(p,0) = x_0$, $H(p,1) = \gamma(p)$

and also so that H is sm.

Now we can perturb H to be transverse to S, so we may just assume that $H \pitchfork S$.

But since $\dim M - \dim S > 2$, then $\dim(S^1 \times I) + \dim(S) = \dim S + 2 < \dim M$,

so $\mathrm{im}(H)$ and S actually has empty intersection and

H will be a homotopy between γ and a const map in $M\setminus S$.

$v = y\frac{\partial}{\partial x} - (x+1)\frac{\partial}{\partial y}$ $\quad \alpha = x\,dy - y\,dx$

$i_v(\underbrace{dx\wedge dy - dy \wedge dx}_{2\,dx\wedge dy}) = y\,dy - 0 \; + (x+1) - ($ ✓

F21: 3) $f_0, f_1 : S \to M$

Pf: f_0, f_1 homotopic and smooth maps $\Rightarrow$ f_0, f_1 smoothly homotopic.

$\exists H : S \times I \to M$ smooth $H(x,0) = f_0(x)$, $H(x,1) = f_1(x)$.

ω closed k-form on M

$H^*\omega$ is a k-form on $S \times I$

$$0 = \int_{S\times I} H^* \underbrace{d\omega}_{0} = \int_{S\times I} dH^*\omega = \int_{\partial S\times I} H^*\omega = \int_{-S\times\{0\}} H^*\omega + \int_{S\times\{1\}} H^*\omega = -\int_S f_0^*\omega + \int_S f_1^*\omega$$

4) $v = \frac{\partial}{\partial w}$ $u = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, $f \in C^\infty$

$$[v,u]\cdot f = v\cdot u f - u\cdot v\cdot f = \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}\right) - \left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\frac{\partial f}{\partial w}$$

$$= \cancel{\frac{\partial^2 f}{\partial w \partial x}} + z\cancel{\frac{\partial^2 f}{\partial w \partial y}} + \left(\cancel{w\frac{\partial^2 f}{\partial w \partial z}} + \frac{\partial f}{\partial z}\right) - \cancel{\frac{\partial^2 f}{\partial x \partial w}} - \cancel{z\frac{\partial^2 f}{\partial y \partial w}} - \cancel{w\frac{\partial^2 f}{\partial z \partial w}}$$

$$[v,u] = \frac{\partial}{\partial z}$$

$$[[v,u],u]f = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}\right) - \left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\frac{\partial f}{\partial z}$$

$$= \cancel{\frac{\partial^2 f}{\partial z \partial x}} + \left(\cancel{z\frac{\partial^2 f}{\partial z \partial y}} + \frac{\partial f}{\partial y}\right) + \cancel{w\frac{\partial^2 f}{\partial z \partial z}} - \cancel{\frac{\partial^2 f}{\partial x \partial z}} - \cancel{z\frac{\partial^2 f}{\partial y \partial z}} - \cancel{w\frac{\partial^2 f}{\partial z \partial z}}$$

$$[[v,u],u] = \frac{\partial}{\partial y}$$

WTS: $u, v, \frac{\partial}{\partial z}, \frac{\partial}{\partial y}$ span $T_p\mathbb{R}^4$ $\forall p$.

$u - z[[v,u],u] - w[v,u] = \frac{\partial}{\partial x}$ so span of these 4 contain all the coord. vectors at any pt in $\mathbb{R}^4$, so span $T_p\mathbb{R}^4$ //.

5) X dim 1 $\subseteq W$ dim 4. Show $f, g: S^1 \to W$ $\text{im} f$, $\text{im} g$ disj from X are homotopic in W iff homotopic in $W - X$

Pf: ($\Leftarrow$): If f, g homotopic in $W \setminus X$ then clearly they are homotopic in W.

($\Rightarrow$): Sps f, g homotopic in $W \Rightarrow f, g$ sm. homotopic.

$\exists\ H: S^1 \times I \to W$ smooth s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$.

H is trivially transverse w/ X @ endpts $S^1 \times \{0\}$, $S^1 \times \{1\}$ (they're disjoint)

hence by transversality thm, we can homotope H rel $\partial S^1 \times I = S^1 \times \{0,1\}$ so that $\bar{H}$ is transverse to X

Then $\bar{H}^{-1}(X)$ is a submfld of codim $4-1 = 3$ but $\dim(S^1 \times I) = 2$

so $\bar{H}^{-1}(X)$ is empty. Thus $\bar{H}$ is a homotopy from f, g in $W - X$.

7) M n-fld. ω closed k-form M, η closed ℓ-form on M, show $\omega \wedge \eta$ closed $k+l$ form and $[\omega \wedge \eta]$ only depends on $[\omega]$, $[\eta]$.

Pf: $d(\omega \wedge \eta) = \underbrace{d\omega}_{0} \wedge \eta + (-1)^k \omega \wedge \underbrace{d\eta}_{0} = 0.$

If $[\omega'] = [\omega]$ and $[\eta] = [\eta']$ then $\exists\ \alpha, \beta$ s.t. $\omega' = \omega + d\alpha$, $\eta' = \eta + d\beta$

$\omega' \wedge \eta' = (\omega + d\alpha) \wedge (\eta + d\beta) = \omega \wedge \eta + \omega \wedge d\beta + d\alpha \wedge \eta + d\alpha \wedge d\beta$

$= \omega \wedge \eta + (-1)^k d(\omega \wedge \beta) + d(\alpha \wedge \eta) + d(\alpha \wedge d\beta)$

$= \omega \wedge \eta + d\left((-1)^k \omega \wedge \beta + \alpha \wedge \eta + \alpha \wedge d\beta\right)$

so $[\omega' \wedge \eta] = [\omega \wedge \eta]$

$d(\omega \wedge \beta) = \underbrace{d\omega \wedge \beta}_{0} + (-1)^k \omega \wedge d\beta$

$d(\alpha \wedge \eta) = d\alpha \wedge \eta + \underbrace{(-1)^{k-1} \alpha \wedge d\eta}_{0}$

$d(\alpha \wedge d\beta) = d\alpha \wedge d\beta + \underbrace{(-1)^{k-1} \alpha \wedge dd\beta}_{0}$

$i_v \omega = \sum_{i=0}^{k} (-1)^{i-1} \alpha^i(v)\, \alpha^1 \wedge \dots \wedge \hat{\alpha^i} \wedge \dots \alpha^k$

1) $f: M \to N$ sm. M compact nonempty, N connected. Show df_x inv $\Rightarrow f$ surj | F22

Pf: Let $S = f(M)$. S is compact in N, since N Hausdorff, S closed.

Let $y \in S$, let $x \in f^{-1}(y)$. Then by Inverse Funct Thm, f is a local diffeo: $\exists U \ni x$, $U \subseteq M$ open st. $f|_U : U \to f(U)$ is a diffeo, so $f(U)$ is an open set in S containing y. Hence S is open.

S clopen in $N \Rightarrow S = N$ or $S = \emptyset$. Since M nonempty, $S = N$ and f is surjective. □

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2) K submfld of $\mathbb{R}^3$ diffeo to S^1. $\forall \varepsilon > 0$, show $\exists v$, $|v| < \varepsilon$ & $K, K+v$ disj

Pf: Let $\varepsilon > 0$, $f: K \times K \to \mathbb{R}^3 : (x,y) \mapsto x - y$. Then f is sm and $\operatorname{im} f$ is the pts for which if $v \in \operatorname{im} f$, then $\exists (x,y)$ st. $x - y = v \Rightarrow x = y + v$ i.e., $K, K+v$ are not disj. But $df_{(x,y)} : T_{(x,y)} K \times K \to \mathbb{R}^3$ which cannot be surj for any $(x,y) \in K \times K$ since dim of domain is 2 and $\dim \mathbb{R}^3 = 3$. So by Sard's, $\operatorname{im} f$ lies in the critical values which have measure 0. So $B_\varepsilon(0) \setminus \operatorname{im} f \neq \emptyset$, take any v in here, so $K, K+v$ are disjoint. □

Pf 2: $f : K \times \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x,v) = x + v$. $df_{(x,v)}$ is surj b/c fixing x, $a \mapsto f(x,a)$ is a translation and so deriv. is surj and $T_{(x,v)} K \times \mathbb{R}^3 \cong T_x K \times T_v \mathbb{R}^3$ so $df_{(x,v)}$ as a whole is surj.

Then f is transverse to any submfd, in particular, K. Transversality thm says $F_v : K \to \mathbb{R}^3$ is transverse on a dense set of $v \in \mathbb{R}^3$, hence on some v in ε-ball.

$x \mapsto f(x,v) = x + v$

Then K is codim 2 in $\mathbb{R}^3$, so $F_v^{-1}(K)$ is codim 2 in K, but $\dim K = 1$, so $F_v^{-1}(K)$ is empty, hence $K, K+v$ are disjoint. □

4) $f: \mathbb{R}^3 \to \mathbb{R}^2$ $f(x,y,z) = (x, yz)$. $\omega = \sin x\, dy$ 1-form on $\mathbb{R}^2$.

c) let (u,v) be coords on $\mathbb{R}^2$ so $\omega = \sin u\, dv$ and

for f: $u = x$, $v = yz$

a) $f^*\omega = \sin x\, d(yz) = \sin x\,(z\,dy + y\,dz) = z\sin x\, dy + y \sin x\, dz$

b) $d(f^*\omega) = \frac{\partial}{\partial x}(z \sin x)\, dx \wedge dy + \frac{\partial}{\partial z}(z\sin x)\, dz \wedge dy$

$+ \frac{\partial}{\partial x}(y \sin x)\, dx \wedge dz + \frac{\partial}{\partial y}(y \sin x)\, dy \wedge dz$

$= z\cos x\, dx \wedge dy + \sin x\, dz \wedge dy + y \cos x\, dx \wedge dz + \sin x\, dy \wedge dz$

$=$ " $- \sin x\, dy \wedge dz$ " "

c) $V = \frac{\partial}{\partial x}$, $\mathcal{L}_V f^*\omega = d\underbrace{\iota_V f^*\omega}_{0} + \iota_V d f^*\omega$ $\quad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{k+1}\alpha \wedge d\beta$

$= \iota_V (z\cos x\, dx \wedge dy + y \cos x\, dx \wedge dz)$

$= z\cos x\, dy + y\cos x\, dz$

---

5) S sm compact conn 2 Mfd of $\mathbb{R}^3$. Show $\exists$ plane in $\mathbb{R}^3$ st intersects in union of circle.

Pf: let $\pi: \mathbb{R}^3 \to \mathbb{R}: (x,y,z) = x$, sm, and $\pi|_S$ is sm. $\pi(S)$ is compact & conn so is an interval. $\pi|_S$ has regular values are dense by Sard,

so pick $q \in \operatorname{im}(\pi|_S)$. ~~then for all $(x,y,z) \in (\pi|_S)^{-1}(q)$, $d\pi_{(x,y,z)}$ is surj~~

then $(\pi|_S)^{-1}(q)$ is a submfd of S of codim 1, i.e., a 1-mfd.

(S no boundary??) then $(\pi|_S)^{-1}(q)$ is a nonempty union of circles: q is in the image so it must be nonempty and sub 1-mfds on a surface w/o ∂ is circles, $\square$

$\pi^{-1}(q)$ is a plane in $\mathbb{R}^3$.

$$\alpha = (x^2+y^2+z^2)^{-3/2}(x\, dy\wedge dz - y\, dx\wedge dz + z\, dx\wedge dy) \quad \mathbb{R}^3 - \{0\}$$

7

a) $d\alpha = \frac{d}{dx}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dx\wedge dy\wedge dz + \frac{d}{dy}\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right) dy\wedge dx\wedge dz + \frac{d}{dz}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) dz\wedge dx\wedge dy$

$$= \left(\frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)(x)}{(x^2+y^2+z^2)^3}\right) dx\wedge dy\wedge dz + \frac{(x^2 \ldots) - \frac{3}{2}(\ldots)^{1/2}(2y)(y)}{(\ldots)^3} dx\wedge dy\wedge dz + \frac{(\ldots) - (\ldots)(2z)(z)}{(\ldots)^3} dx\wedge dy\wedge dz$$

$$= \left(\frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}x^2 - 3(x^2+y^2+z^2)^{1/2}y^2 - 3(x^2+y^2+z^2)^{1/2}z^2}{(x^2+y^2+z^2)^3}\right) dx\wedge dy\wedge dz$$

$$= 3\left(\frac{(x^2+y^2+z^2)^{1/2}\left((x^2+y^2+z^2) - x^2 - y^2 - z^2\right)}{(x^2+y^2+z^2)^3}\right) dx\wedge dy\wedge dz = 0 \quad ✓$$

α is closed

b) Parametrize S^2:

c) If α exact, then $\alpha = d\eta$, but $\int_{S^2}\alpha = \int_{S^2} d\eta = \int_{\partial S^2}\eta = 0$ since $\partial S^2 = \emptyset$.

α a 1-form. M compact m. v vecfield on M, $\varphi_t : M \to M$ flow. (3)

a) $\varphi_t^* \alpha = e^t \alpha$

$$\mathcal{L}_v \alpha(x) = \lim_{t \to 0} \frac{\varphi_t^* \alpha_{\varphi_t(x)} - \alpha_x}{t} = \lim_{t \to 0} \frac{e^t \alpha_{\varphi_t(x)} - \alpha_x}{t} =$$

$$\mathcal{L}_v \alpha(x) = \frac{d}{dt}\left(\varphi_t^* \alpha\right)\Big|_{t=0} = \frac{d}{dt}\left(e^t \alpha\right)\Big|_{t=0} = e^t \alpha\big|_{t=0} = \alpha \quad ✓$$

$\cancel{\mathcal{L}_v \alpha = d\iota_v \alpha + \iota_v d\alpha}$

b) $d\alpha = 0$. so α is closed. Assume $\mathcal{L}_v \alpha = \alpha$.

$$\mathcal{L}_v \alpha = \alpha = d(\iota_v \alpha) + \underbrace{\iota_v d\alpha}_{0} = d(\iota_v \alpha) \quad ✓$$

M sm. compact nonempty mfld. Show no submsn $F: M \to \mathbb{R}^k$ for any $k > 0$.

Pf: We showed on HW. submersions are open maps.

$F(M)$ is open since F open. (and M open)

$F(M)$ is compact since F cont, so $F(M)$ closed since $\mathbb{R}^k$ Hausdorff.

$\mathbb{R}^k$ is connected, so $F(M)$ is either $\emptyset$ or $\mathbb{R}^k$.

M is nonempty, so $F(M) = \mathbb{R}^k$ which is not compact. $\lightning$. $\square$

$\underset{M}{\mathbb{R}} \xrightarrow{F} \underset{N}{\mathbb{R}^2}$ $\quad$ $G: \underset{N}{\mathbb{R}^2} \to \underset{P}{\mathbb{R}}$ constant rank, but $G \circ F : \mathbb{R} \to \mathbb{R}$ is not const rank

$F(t) = (\cos t, \sin t)$ $\quad$ $G(x,y) = x$ $\quad$ $G \circ F(t) = \cos t$

$dF_p = \begin{bmatrix} -\sin p \\ \cos p \end{bmatrix}$ has const rank 1 (for any $p \in \mathbb{R}$)

$dG_{(x,y)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ for any $(x,y) \in \mathbb{R}^2$, const rank

$d(G \circ F)_t = \begin{bmatrix} -\sin t \end{bmatrix}$ This is rank 0 for $t = 2\pi n, n \in \mathbb{Z}$. rank 1 otherwise. so nonconst. $\square$

8 2021 practice

$[y\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}]f = y\frac{\partial}{\partial x}(x^2\frac{\partial f}{\partial y} + y\frac{\partial f}{\partial z}) - (x^2\frac{\partial}{\partial y}(y\frac{\partial f}{\partial x}) + y\frac{\partial}{\partial z}(y\frac{\partial f}{\partial x}))$

$[u,v] = uv - vu$

$$= y(2x\frac{\partial f}{\partial y} + x^2\frac{\partial^2 f}{\partial x\partial y}) + \cancel{y^2\frac{\partial^2 f}{\partial x\partial z}}$$

$f \in C^\infty$

$$- (x^2(\frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial y\partial x}) + \cancel{y^2\frac{\partial^2 f}{\partial z\partial x}})$$

$$= 2xy\frac{\partial f}{\partial y} + \cancel{x^3 y\frac{\partial^2 f}{\partial x\partial y}} - x^2\frac{\partial f}{\partial x} - \cancel{x^2 y\frac{\partial^2 f}{\partial x\partial y}} = 2xy\frac{\partial f}{\partial y} - x^2\frac{\partial f}{\partial x}.$$

$$[y\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}] = -x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}.$$

7, 2021 Practice $f: \mathbb{R}P^n \to \mathbb{R}$ diffble, $n>0$. $\exists p, q \in \mathbb{R}P^n$, $p \neq q$. s.t. $df_p = df_q = 0$.

Pf: $\mathbb{R}P^n$ is compact, so by extrem value thm, f attains a min and max @ p, q respectively. (if $p = q$ then f is const map and we can choose distinct $p \neq q$ anyways).

$f(p) = \min f$, $f(q) = \max f$.

So $df_p = 0$, $df_q = 0$.

$\to$ im f is a compact conn. subset of $\mathbb{R}$, hence a closed interval. $\square$

205C Final #5: N embedded subfld of M, X, Y sm. vec. fields on M.

If X, Y restrict to N are tangent to N, then $[X, Y]|_N$ is tangent to N.

(in fact, $[X|_N, Y|_N] = [X, Y]|_N$)

$$\left[\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right]f = \frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}f\right) - y\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right) = y\frac{\partial^2}{\partial x\partial y}f - y\frac{\partial^2}{\partial y\partial x}f = 0$$

Pf. let $\varphi: N \to M$ be a sm. embedding, so φ is an immersion, hence $d\varphi_p: T_pN \to T_{\varphi(p)}M$ is injective $\forall p$.

We know X, Y restricted to N is tangent to N, i.e.:

$\forall p \in N$, $X_{\varphi(p)}, Y_{\varphi(p)} \in d\varphi_p(T_pN) \subseteq T_{\varphi(p)}M$,

Since $d\varphi_p$ is injective $\forall p$, define new vector fields $\tilde{X}, \tilde{Y}$ on N by:

$$\tilde{X}_p = (d\varphi_p)^{-1}(X_{\varphi(p)}), \quad \tilde{Y}_p = (d\varphi_p)^{-1}(Y_{\varphi(p)}).$$

$$\begin{tikzcd}
 & X \arrow[dl, "f \in C^\infty(M)"'] \arrow[d, "p \in M"] & \\
D_Xf = Xf \in C^\infty(M) \arrow[d, "p \in M"'] & X_p \arrow[dl, "f \in C^\infty(M)"] & \text{derivation = tangent vector @ } p \\
X_pf \in \mathbb{R} & &
\end{tikzcd}$$

dir. derivative @ X_p.

[WTS: $[X, Y]_{\varphi(p)} \in d\varphi_p(T_pN) (\subseteq T_{\varphi(p)}(M))$ $\forall p \in N$

i.e. $[X, Y]|_N$ is tangent to N]

We will check for any $f \in C^\infty(M)$, $p \in N$:

$$[X, Y]_{\varphi(p)}f = X_{\varphi(p)} \cdot (Y \cdot f) - Y_{\varphi(p)} \cdot (X \cdot f)$$

def of $\tilde{X}, \tilde{Y}$

$$= d\varphi_p(\tilde{X}_p) \cdot (Y \cdot f)) - d\varphi_p(\tilde{Y}_p)(X \cdot f)$$

def of differential

$$= \tilde{X}_p((Y \cdot f) \circ \varphi) - \tilde{Y}_p((X \cdot f) \circ \varphi)$$

$$= \tilde{X}_p(\tilde{Y} \cdot (f \circ \varphi)) - \tilde{Y}_p(\tilde{X} \cdot (f \circ \varphi))$$

$$= [\tilde{X}, \tilde{Y}]_p(f \circ \varphi) = (d\varphi_p)[\tilde{X}, \tilde{Y}]_p(f)$$

$\square$

So $[X, Y]_{\varphi(p)} \in d\varphi_p(T_pN)$.

Prove any (cont) map $f: S^n \to S^m$, $n<m$ is nullhomotopic.

Pf: By Whitney Approximation, f is homotopic to a smooth map, so we may assume f is smooth. By Sard's thm, since $\dim(S^n) = n < m = \dim(S^m)$, $\operatorname{im} f$ has measure 0 (lies within critical values). So f is not surj, take $p \in S^m \setminus \operatorname{im} f$ and so f can be thought of a map from $S^n \to S^m \setminus \{p\}$.

But $S^m \setminus \{p\} \cong \mathbb{R}^m$ (diffeo) by g, and $\mathbb{R}^m$ is contractible (e.g., $h: \mathbb{R}^m \times I \to \mathbb{R}^m$ $h(x,t) = tx$ gives a homotopy between id $\simeq$ constant map @ 0) so the composition $g^{-1}(h(g(f(x)), t))$ is a homotopy between f and a constant map so f is nullhomotopic. ◻

F23 1) $X: x^2+y^2-z^2=1$ $\quad Y: x+y+z=0$. Show $X \cap Y$ is sm subfld

Pf: let $F: \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x,y,z) = (x^2+y^2-z^2, x+y+z)$.

F is clearly sm, if we can show $(1,0)$ is a regular val,

we're done by preimage thm as $F^{-1}(1,0) = X \cap Y$ will be a sm subfld

(of codim = dim $\mathbb{R}^2 = 2$, i.e., of dim $= 3-2=1$).

$dF_{(x,y)} = \begin{bmatrix} 2x & 2y & -2z \\ 1 & 1 & 1 \end{bmatrix}$ has full rank if $(x,y,z) \neq (0,0,0)$

and two of $x, y, -z$ are distinct

For any $(x,y,z) \in F^{-1}(1,0)$, $x^2+y^2-z^2=1$, so $(x,y,z) \neq (0,0,0)$ ✓. □

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4) M^m, N^n sm, M nonorientable, Show $M \times N$ nonorientable.

Pf: Sps $M \times N$ orientable. Then fix $p \in N$ and a basis $\{v_1, \dots, v_n\}$

for T_pN, let ω be a nowhere zero $m+n$ form on $M \times N$, and

identify M with $M \times \{p\}$ in $M \times N$. Then define η on M (a m form)

by $\eta(u_1, \dots, u_m) = \omega(u_1, \dots, u_m, v_1, \dots, v_n)$ which is nowhere vanishing. ↯ □

($\Leftarrow$): If M, N orientable, take ω, η m-form, n-form nowhere vanishing. Then

$\pi_M^*\omega \wedge \pi_N^*\eta$ is nowhere vanishing $m+n$ form on $M \times N$. □

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7) α closed 2-form on 4-sphere. Show $\alpha \wedge \alpha$ vanishes at some pt.

Pf: $d\alpha = 0$ b/c α is closed. $d(\alpha \wedge \alpha) = d\alpha \wedge \alpha + (-1)^2 \alpha \wedge d\alpha = 0$

If $\alpha \wedge \alpha$ is nowhere vanishing then $\int_{S^4} \alpha \wedge \alpha \neq 0$.

But since $H^2_{DR}(S^4) = 0$, it must be that α is also exact, so $\alpha = d\eta$ for

some 1-form η. $d(\eta \wedge \alpha) = d\eta \wedge \alpha + (-1)^1 \eta \wedge \underbrace{d\alpha}_{0} = \alpha \wedge \alpha$.

Then $\int_{S^4} \alpha \wedge \alpha = \int_{S^4} d(\eta \wedge \alpha) = \int_{\partial S^4} \eta \wedge \alpha = \int_{\emptyset} \eta \wedge \alpha = 0$. ↯ □

8) Show $S^1 \times S^1$ not simply conn by int mod 2 | F23

Pf: $T = S^1 \times S^1$ Choose $(x,y) \in T$, then the embedded circles $\{x\} \times S^1$ and $S^1 \times \{y\}$ intersect transversally exactly once (mod 2).

But if T is simply conn, we can (smoothly) homotope one curve to a pt not on the other, hence the intersection is now 0, contradiction, as int mod 2 is invariant under smooth homotopy. $\square$

$x, y \in M$ connected, sm., then $\exists$ compactly supported isotopy from $\varphi_0 : M \to M$ to $\varphi_1 : M \to M$ st $\varphi_0 = id_M$, $\varphi_1(x) = y$

Pf: Take a sm injective path $\gamma : [0,1] \to M$ st $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma'(t) \neq 0$ $\forall t$. ~~Then we can take $[0,1]$ as an embedded submfld of M~~, and by Tubular Nbhd

~~Thm, $\exists$~~

Then $\exists$ an open set w/ compact closure, U in M which contains $\gamma([0,1]) =: I$:
Since I compact, it can be covered in a finite # of open sets w/ compact closure, the union of which covers I. Now create a vec. field V s.t. on I, if $p = \gamma(t)$ then $V(p) = \gamma'(t)$. Next this can be extended so that vector field is 0 on $M \setminus U$, and arbitrary elsewhere. Now V has compact support so the flow exists $\forall$ time: $\exists$ $\Phi : M \times \mathbb{R} \to M$ sm st. $\Phi(p,0) = p$ and $d\Phi_{(p,t)}\left(\frac{\partial}{\partial t}\right) = V(\Phi(p,t))$. Remember $\varphi_t : M \to M$ by $\varphi_t(p) = \Phi(p,t)$ gives a diffeo on M, and φ_0 is id_M.

$\gamma'(t) = V(\gamma(t))$ by def and $\gamma(0) = x$ so γ is a flow line starting @ x. Flow lines are unique so $\gamma(t) = \Phi(x,t)$. Hence
$\varphi_1(x) = \Phi(x,1) = \gamma(1) = y$. $\square$

$f: \mathbb{R}^5 \to \mathbb{R}^3$ sm. Show $\exists$ sphere S centr 0 in $\mathbb{R}^3$ st $f^{-1}(S)$ is a sm (possibly empty) 4-dim submfld in $\mathbb{R}^5$. (4)

2016 Sp

Pf: Defn $g: \mathbb{R}^3 \to \mathbb{R}$ by $g(x,y,z) = x^2+y^2+z^2$. $g \circ f$ sm. So by Sard's thm, as regular values are dense, $\exists$ a reg value of $q \in \mathbb{R}$. Then $g^{-1}(q)$ is a submfld of $\mathbb{R}^3$, and by construction, in fact a sphere centered at 0 in $\mathbb{R}^3$, call it S. Also

$$f^{-1}(S) = f^{-1} \circ g^{-1}(q) = (g \circ f)^{-1}(q).$$

so $f^{-1}(S)$ is a submfld of $\mathbb{R}^5$ (also by reg. value thm) of codim 1, hence of dim 4 in $\mathbb{R}^5$.

2016 Sp 8: $v \in S^1$, $w \in S^2$. $U \subseteq S^1 \times S^2$ open that contains $S^1 \times \{w\}$ and $\{v\} \times S^2$. Show U is not diffeo to an open subset of a simply conn 3-mfld.

Pf. Suppose U diffeo to M simply conn 3-mfld. Then $S^1 \times \{w\}$ and $\{v\} \times S^2$ can be embedded in M which intersect transversally and only once (at image of (v,w)). M simply connected means the embedded S^1 is null homotopic, so the mod 2 intersection # with any complementary dim submfld (2), in particular the embedded S^2, should be 0. ↯. □

9) 2021 Practice

a) $F: M \to N$ sm, $c \in N$, $F^{-1}(c)$ embedded submfld codim $= \dim N$, $\Rightarrow$ c a reg. val?

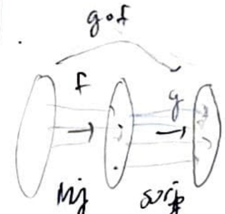
<u>False:</u> $F: \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = x^3$. $F^{-1}(0) = \{(0,y) : y \in \mathbb{R}\}$ is a codim 1 space in $\mathbb{R}^2$, $\dim \mathbb{R} = 1$, ✓. For any $(x,y) \in F^{-1}(0)$, $dF_{(x,y)} = [3x^2 \ \ 0]$ which is rank 1 iff $x \neq 0$, so $c=0$ is not a regular value as $(0,0) \in F^{-1}(0)$ and $dF_{(0,0)} = [0 \ \ 0]$ and so not surjective. □

$\forall p \in M$, $\exists \sigma : N \to M$ st. $p \in \operatorname{Im} \sigma$.

b) $\pi: M \to N$ sm, $\exists \sigma : N \to M$ a sm section of π $(\pi \circ \sigma = id_N)$. Show π is submersion.

$$\begin{tikzcd} M \arrow[d, "\pi"', shift right] \\ N \arrow[u, "\sigma"', bend right] \end{tikzcd}$$

Pf: let $q \in N$, $p \in f^{-1}(q)$. [WTS: $d\pi_p$ is surj].

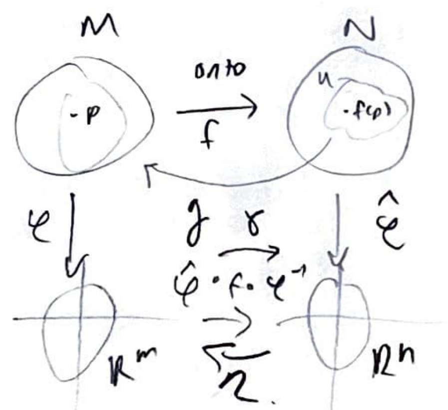


$\exists \sigma : N \to M$ sm section st $p \in \operatorname{Im}(\sigma)$.

$$\underset{\substack{\| \\ id.}}{(d\, id_N)} = d(\pi \circ \sigma)_p = d\pi_{\sigma(p)} \circ d\sigma_p$$ implies $d\pi_p$ is surjective. □

$f: M^m \to N^n$ surj, submersion, then $\forall p \in M$, $\exists\, U \subseteq f(p)$ in N open, (5)

$g: U \to M$ sm s.t. $f \circ g = id_U$.



Pf: Let $p \in M$. We know $df_p: T_pM \to T_pN$ is surjective (f submersion). By submersion thm,

$\exists$ coords $\varphi: U \to V$ @ p, $\hat{\varphi}: \hat{U} \to \hat{V}$ @ $f(p)$ s.t.

$\hat{\varphi} \circ f \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n) = \gamma(x^1, \ldots, x^m)$

i.e. f is a projection in local coords.

a section of this map is $\eta: \mathbb{R}^n \to \mathbb{R}^m$

by $\eta(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$.

$g = \varphi^{-1} \circ \eta \circ \hat{\varphi}$ $\qquad\qquad$ $f = \hat{\varphi}^{-1} \circ \gamma \circ \varphi$

$f \circ g(a) = \hat{\varphi}^{-1} \circ \gamma \circ \varphi \circ \varphi^{-1} \circ \eta \circ \hat{\varphi}(a) = \hat{\varphi}^{-1} \circ \gamma \circ \eta(x^1(a), \ldots, x^n(a))$

$= \hat{\varphi}^{-1} \circ \gamma(x^1(a), \ldots, x^n, 0, \ldots, 0) = \hat{\varphi}^{-1}(x^1(a), \ldots, x^n(a)) = a$