CΔLCULUS $\oint$ciety

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Presentation 2

Contents

Edexcel IGCSE Further Mathematics Specification

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9 Calculus

What students need to learn	Notes
A Differentiation and integration of sums of multiples of powers of x (excluding integration of $\frac{1}{x}$), $\sin ax, \cos ax, e^{ax}$	No formal proofs of the results for ax^n, $\sin ax$, $\cos ax$ and e^{ax} will be required.
B Differentiation of a product, quotient and simple cases of a function of a function	
C Applications to simple linear kinematics and to determination of areas and volumes	Understanding how displacement, velocity and acceleration are related using calculus. The volumes will be obtained only by revolution about the coordinate axes.
D Stationary points and turning points	
E Maxima and minima	Maxima and minima problems may be set in the context of a practical problem. Justification of maxima and minima will be expected.
F The equations of tangents and normals to the curve $y = f(x)$	$f(x)$ may be any function which the students are expected to be able to differentiate.
G Application of calculus to rates of change and connected rates of change	The emphasis will be on simple examples to test principles. A knowledge of $dy \approx \frac{dy}{dx} dx$ for small dx is expected.

Applications of Derivatives

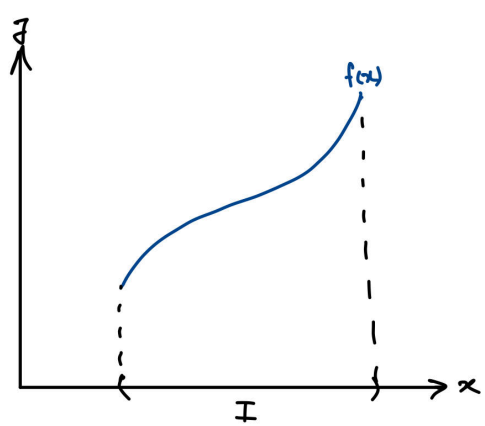
Maxima and Minima

Let f be a function defined over an interval I and let c be a point in I.

Definition 1: We say that:

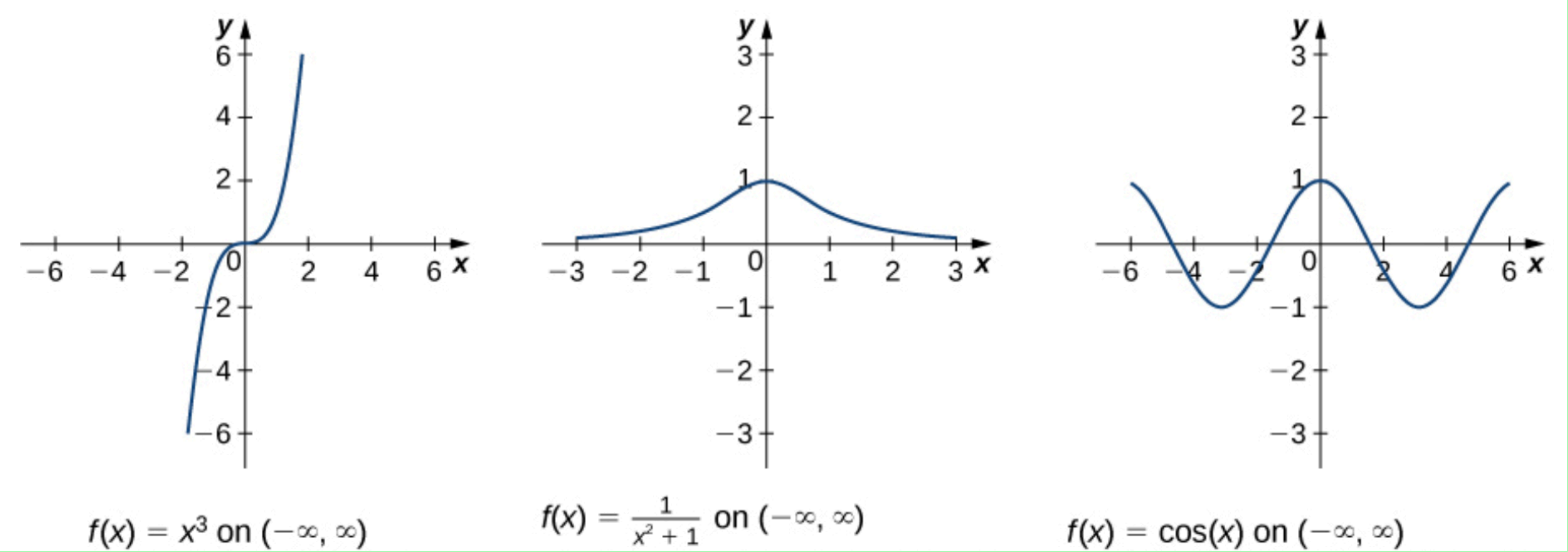
- f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$.
- f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$.

If f has an *absolute maximum* or an *absolute minimum* on I at c, we say f has an **absolute extremum** on I at c.



Maxima and Minima

Example 1:



Maxima and Minima

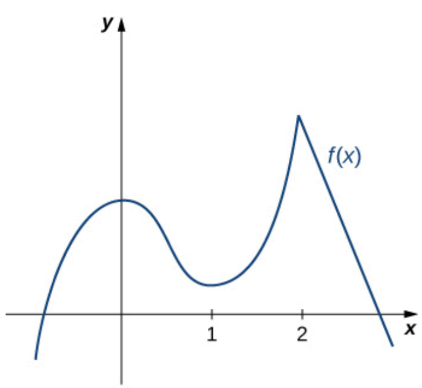
On the other hand, let I be an *open interval*, which contains a point c, and is in the domain of f. Then:

Definition 2: We say that:

- f has a **local maximum** at c if there exists I and $f(c) \geq f(x)$ for all $x \in I$.
- f has a **local minimum** at c if there exists I and $f(c) \leq f(x)$ for all $x \in I$.

A function f has a **local extremum** at c if f has a *local maximum* or a *local minimum* at c.

How is this different from absolute extrema?



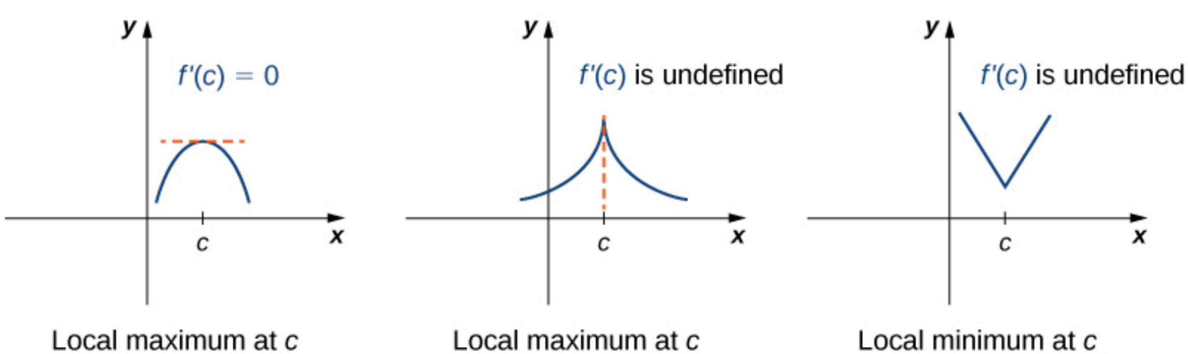
Maxima and Minima

Definition 3: We say that c is a **critical point** of f if $f'(c) = 0$ or $f'(c)$ is undefined. *Note:* c is **stationary point** of f if $f'(c) = 0$.

As can be observed,

Theorem 1: If f has a *local extremum* at c and f is differentiable at c, then $f'(c) = 0$.

Proof by basic principles and local extrema.



Maxima and Minima

Example 2: Find the critical points of the following functions (use desmos to determine what type of local extrema they are):

1. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
2. $f(x) = \left(x^2 - 1\right)^3$
3. $f(x) = \frac{4x}{1+x^2}$

Maxima and Minima

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Answer:

- local maxima at $x = 1$; local minima at $x = 4$
- critical points at $x = 0, \pm 1$; local/absolute minimum at $x = 0$
- absolute maximum at $x = 1$; absolute minimum at $x = -1$

Maxima and Minima

Example 4: Find the critical points of the following functions (use desmos to determine what type of local extrema they are):

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Now you can easily find the maximum/minimum value of quadratic functions without having to complete the square (while also demonstrating ur superior skills 💪).

Maxima and Minima

How would you know if a critical point is a (absolute) maximum or minimum point without graphing?

Theorem 2 (Second Derivative Test): If $f'(c) = 0$ then:

- f has an *absolute maximum* at c if $f''(x) < 0$.
- f has an *absolute minimum* at c if $f''(x) > 0$.

If $f''(x) = 0$, then the function *may* have a **stationary point of inflection** at $x = c$.

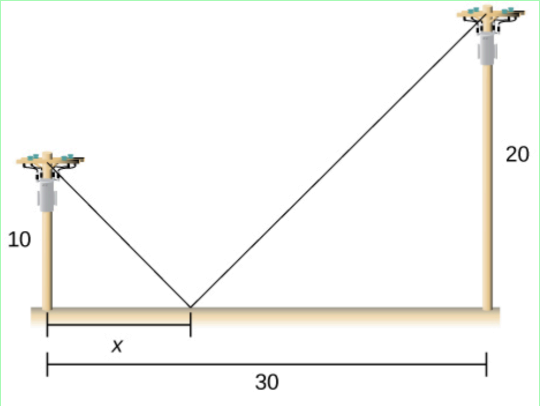
Example 5: Find the critical point(s) of the following functions and determine if they are *maximum, minimum* or *inflection points*:

1. $y = 4x^3 - 3x$
2. $y = \ln(x - 2)$
3. $y = \frac{1}{x-1}$

Maxima and Minima

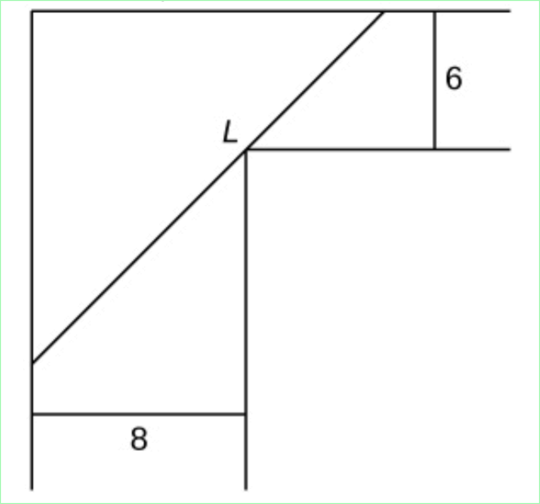
We can also apply extrema to optimisation problems.

Example 6: Two poles are connected by a wire that is also connected to the ground. The first pole is 20 ft tall and the second pole is 10 ft tall. There is a distance of 30 ft between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



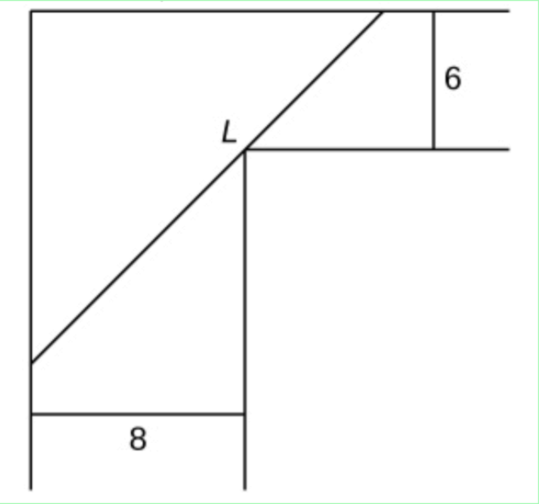
Maxima and Minima

Example 7: You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



Maxima and Minima

Example 8: You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



Answer: 19.73 ft

Kinematics

Remember that:

- velocity is the ***rate of change*** of displacement
- acceleration is the ***rate of change*** of velocity

What does this mean?
What is Calculus?

Kinematics

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What is Calculus?

Kinematics

Remember that:

- velocity is the *rate of change* of displacement
- acceleration is the *rate of change* of velocity

What does this mean?
What is Calculus?

Theorem 5 (Kinematics): If the displacement $s(t)$ is a function of time:

$$v(t) = \frac{ds}{dt}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Kinematics

Example 9: A particle P is moving on the x-axis and its displacement s m/s , t s after a given instant, is given by:

$$s(t) = t^3 - \frac{1}{4}t^4$$

1. Find the *instantaneous velocity* of the particle at time $t = 2$.
2. Find the *instantaneous acceleration* of the particle at time $t = 2$.

Kinematics

Example 10: A particle P is moving on the x-axis and its displacement s m/s , t s after a given instant, is given by:

$$s(t) = t^3 - \frac{1}{4}t^4$$

1. Find the *instantaneous velocity* of the particle at time $t = 2$.
2. Find the *instantaneous acceleration* of the particle at time $t = 2$.

Answer: $v(2) = 4$ m/s ; $a(2) = 0$ m/s^2.

Now you can also flex ur superior skill in physics 💪.

Asymptotes and Shape of Graphs

Definition 4: (Informal)

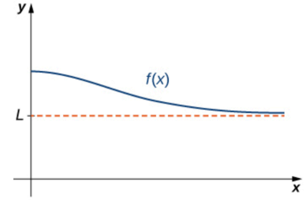
- If the values of $f(x)$ become *arbitrarily close* to L as $x \to \infty$, then:

$$\lim_{x \to \infty} f(x) = L$$

- If the values of $f(x)$ becomes *arbitrarily close* to L as $x \to -\infty$, then:

$$\lim_{x \to -\infty} f(x) = L$$

Definition 5: If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then the line $y = L$ is a **horizontal asymptote** of f.



Asymptotes and Shape of Graphs

Example 11: Find the horizontal asymptote of

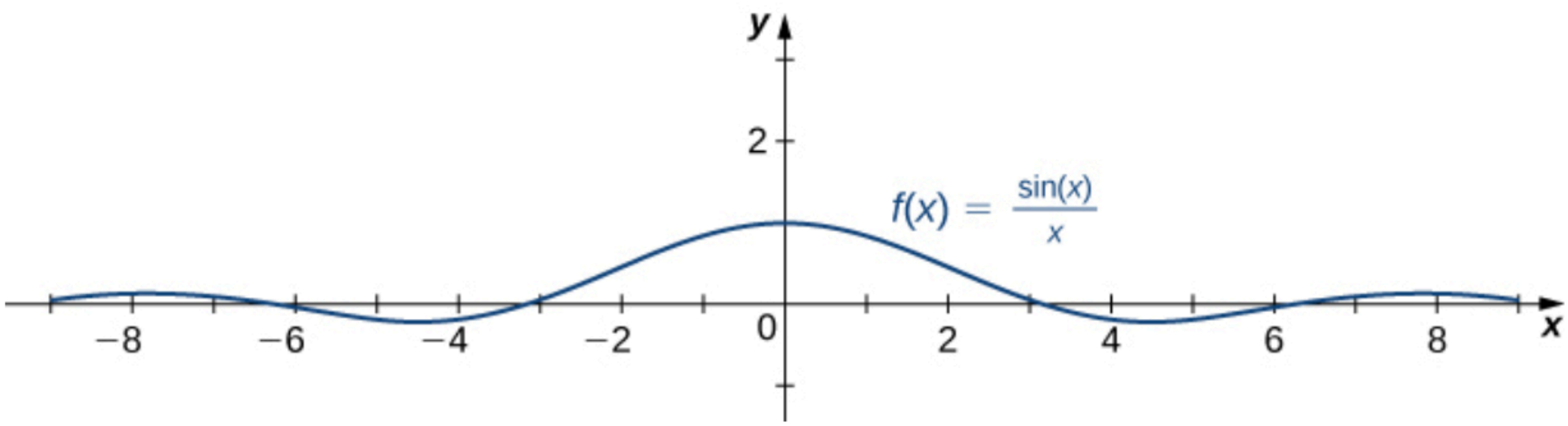
$$f(x) = \frac{\sin x}{x}$$

Asymptotes and Shape of Graphs

Example 12: Find the horizontal asymptote of

$$f(x) = \frac{\sin x}{x}$$

Answer: horizontal asymptote of $y = 0$ as $x \to \pm\infty$.



L'Hôpital's Rule

Suppose f and g are *differentiable functions* over an open interval containing a, except possibly at a.

Theorem 6 (L'Hôpital's Rule): If $\frac{f(x)}{g(x)}$ is indeterminate (i.e, $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$), and $g'(x) \neq 0$ then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

L'Hôpital's Rule

Example 13: Determine:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

L'Hôpital's Rule

Example 14: Determine:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Maclaurin and Taylor Series Expansion

We can also use derivatives to create an approximation for functions.

Theorem 7 (Taylor Series Expansion): If f has derivatives of all orders at $x = a$, the **Taylor series** of f centered at a is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Definition 6 (Maclaurin Series): A **Maclaurin series** is a *Taylor series expansion* of a function, centered at 0.

Maclaurin and Taylor Series Expansion

Example 15: Determine the Taylor series expansion of:

1. $\sin x$
2. $\cos x$
3. e^x

Maclaurin and Taylor Series Expansion

Example 16: Determine the Taylor series expansion of:

1. $\sin x$
2. $\cos x$
3. e^x

Notice that you have now expressed the above functions as polynomials. Differentiate each of the taylor series expansions you have found (power rule).
What do you notice?

Review

Differentiation Bee!

$$\frac{d}{dx}\left(x^{x}\right)$$

Differentiation Bee!

$$\frac{d}{dx}(x^x)$$

Answer: $x^x + x^x \ln(x)$

Differentiation Bee!

Dirichlet eta function:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

PPQ

3 Given that $y = \mathrm{e}^{2x}(x^2 + 1)$

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(3)

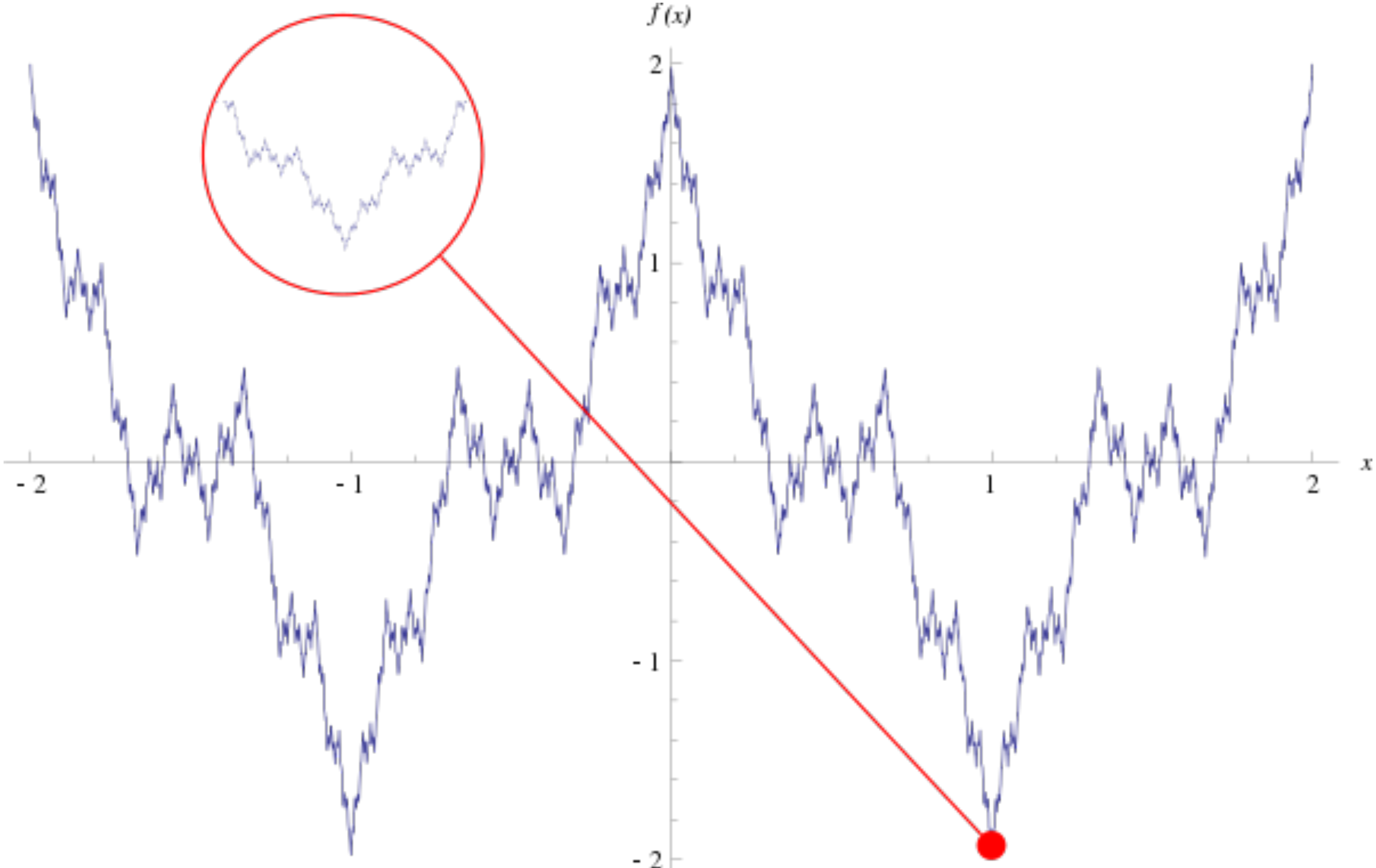
The straight line l is the tangent to the curve with equation $y = \mathrm{e}^{2x}(x^2 + 1)$ at the point on the curve where $x = 0$

(b) Find an equation for l in the form $y = mx + c$

(3)

Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$



Integral Calculus

*“Calculus is an **integral** part of my life”*

Riemann Sums

Definition 7: Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$.

Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A **Riemann sum** is defined for $f(x)$ as

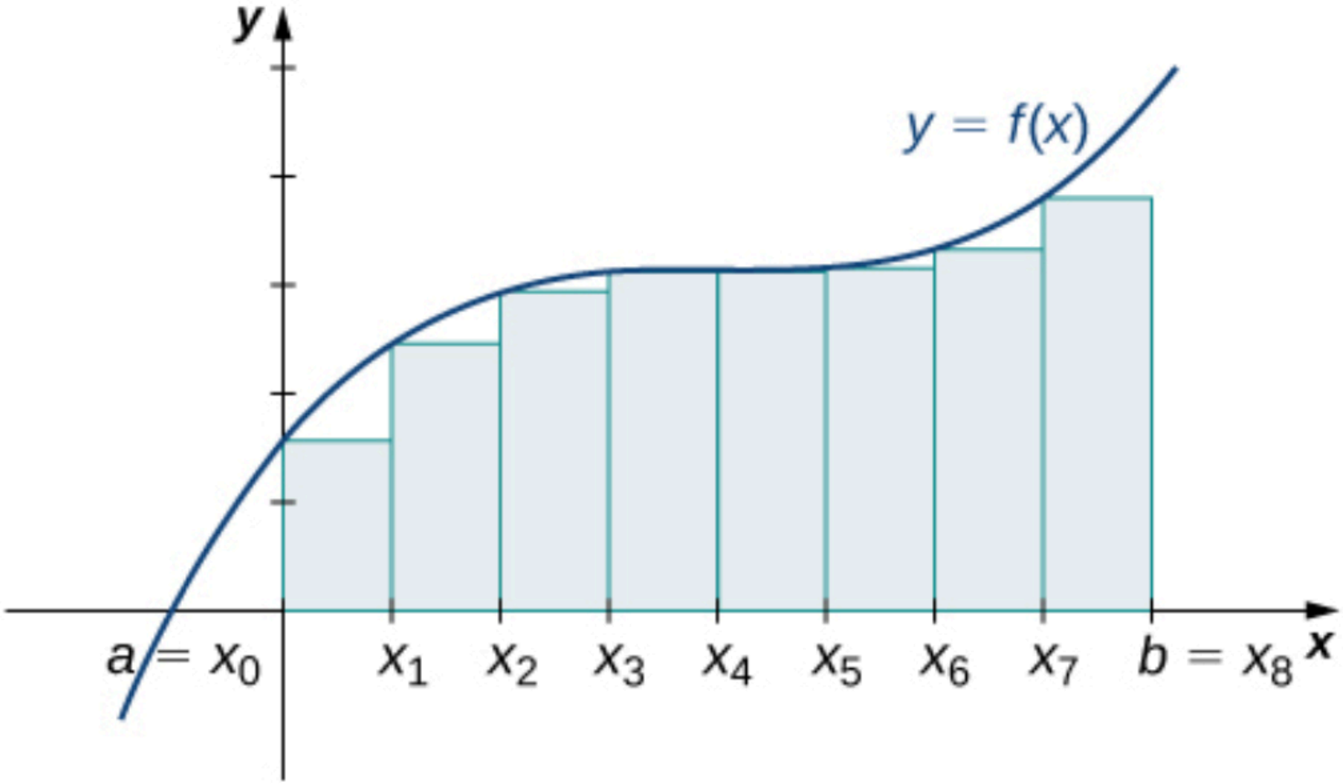
$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Then, the area uder the curve is:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Right and left sided sums

Riemann Sums



Definite Integrals

Definition 8: If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit exists.

If this limit exists, the function $f(x)$ is said to be *integrable* on $[a, b]$.

Theorem 8: If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Fundamental theorem of calculus

Theorem 9:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.

PPQ

Q2.

A curve has equation

$$y = 2x^3 - 4x + 5$$

Find the equation of the tangent to the curve at the point $P(2, 13)$.

Write your answer in the form $y = mx + c$, where m and c are integers to be found.

Solutions relying on calculator technology are not acceptable.

(Total for question = 5 marks)

PPQ

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(Total for question = 5 marks)

$$Answer\colon\ y = 20x - 27$$

PPQ

Q1.

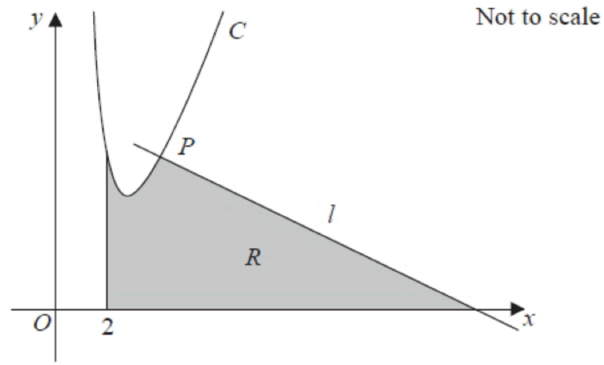


Figure 4

Figure 4 shows a sketch of part of the curve *C* with equation

$$y = \frac{32}{x^2} + 3x - 8, \qquad x > 0$$

The point *P* (4, 6) lies on *C*.
The line *l* is the normal to *C* at the point *P*.

The region *R*, shown shaded in Figure 4, is bounded by the line *l*, the curve *C*, the line with equation *x* = 2 and the *x*-axis.

Show that the area of *R* is 46

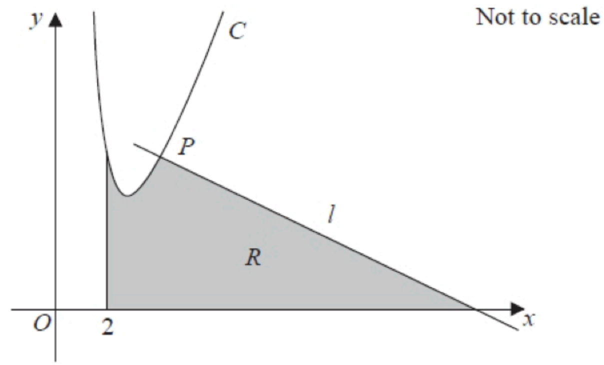
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(10)

(Total for question = 10 marks)

PPQ

Q1.



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Show that the area of *R* is 46

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(10)

(Total for question = 10 marks)

$Answer$: 46

PPQ

12. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Show that

$$\int_1^{e^2} x^3 \ln x \, dx = ae^8 + b$$

where a and b are rational constants to be found.

(5)

4. (a) Express $\lim\limits_{\delta x \to 0} \sum\limits_{x=2.1}^{6.3} \frac{2}{x} \delta x$ as an integral.

(1)

(b) Hence show that

$$\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \ln k$$

where k is a constant to be found.

(2)

PPQ

5.

$$y = \sec^2 x$$

(a) Show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6\sec^4 x - 4\sec^2 x$.

(4)

(b) Find a Taylor series expansion of $\sec^2 x$ in ascending powers of $\left(x - \dfrac{\pi}{4}\right)$, up to and including the term in $\left(x - \dfrac{\pi}{4}\right)^3$.

(6)

3. Find the general solution of the differential equation

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - y\cos x = \sin 2x \sin x,$$

giving your answer in the form $y = \mathrm{f}(x)$.

(8)

PPQ

3. Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+5y=\frac{\ln x}{x}, \qquad x>0$$

giving your answer in the form $y=\mathrm{f}(x)$.

(8)

(next question)

Show that

$$\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x=\frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

PPQ

Q6.

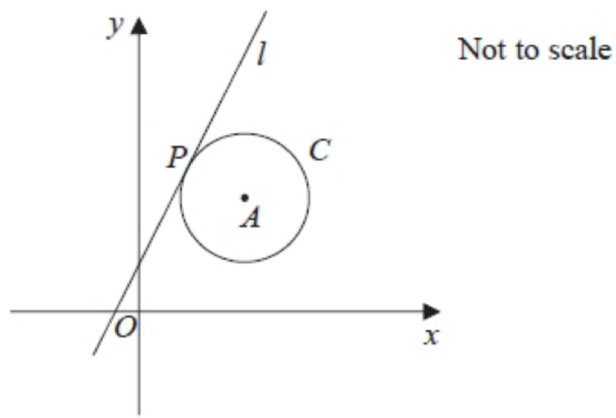


Figure 3

The circle *C* has centre *A* with coordinates (7, 5).

The line *l*, with equation $y = 2x + 1$, is the tangent to *C* at the point *P*, as shown in Figure 3.

(a) Show that an equation of the line *PA* is $2y + x = 17$

(3)

(b) Find an equation for *C*.

(4)

The line with equation $y = 2x + k$, $k \neq 1$ is also a tangent to *C*.

(c) Find the value of the constant *k*.

(3)

PPQ

Q2.

(a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,\mathrm{d}u}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

Integration by parts

Theorem 10 (Integration by parts):

$$\int u dv = uv - \int v du$$

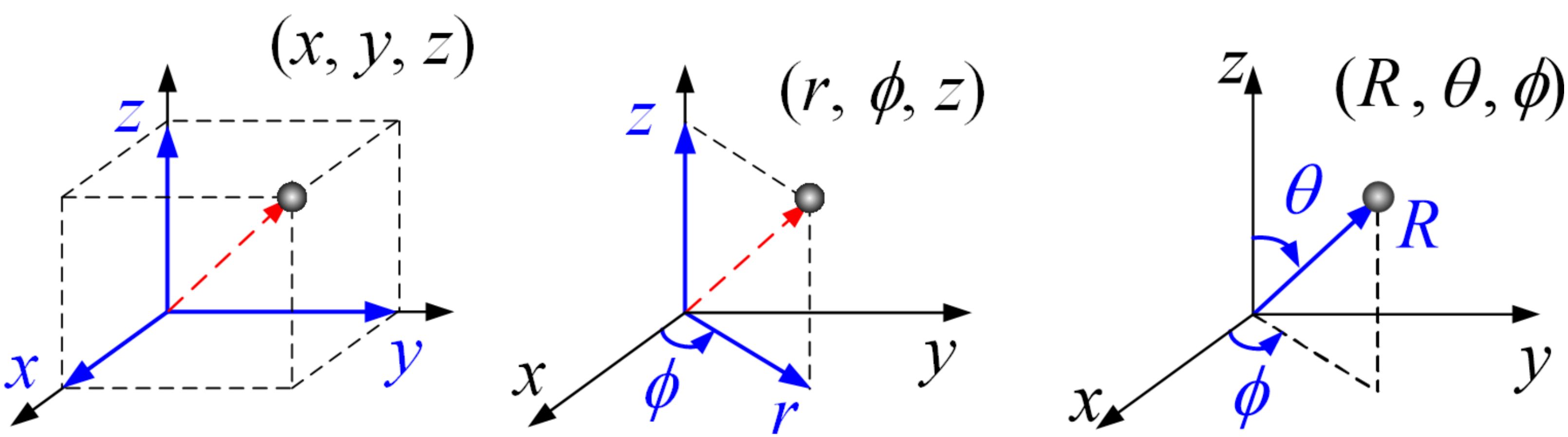
Coordinate systems

2D:

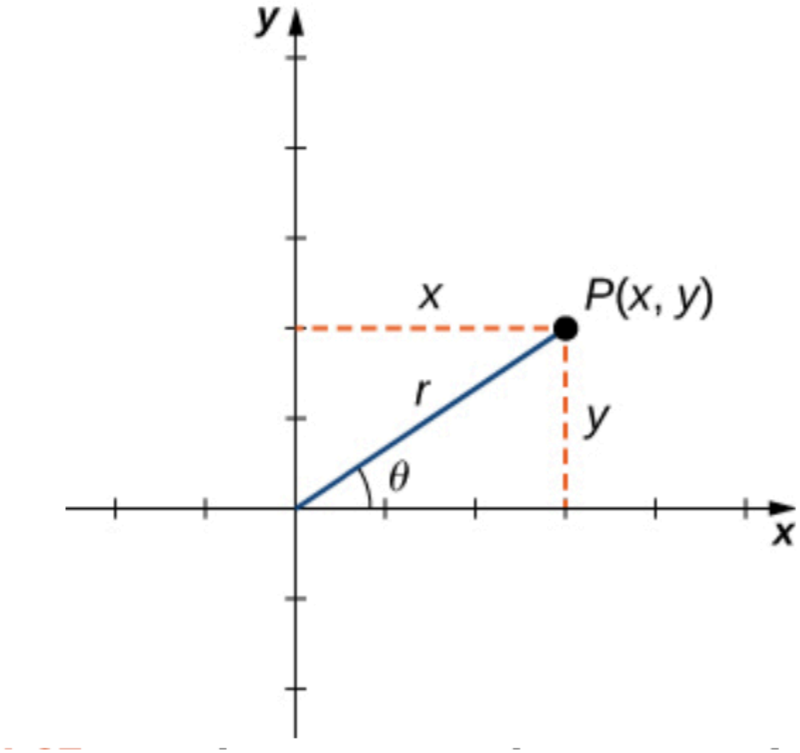
- Cartesian/rectangular coordinates
- Polar coordinatesm

3D:

- Rectangular coordinates
- Spherical coordinates
- Cylindrical coordinates



Polar coordinates



Polar coordinates

Theorem 11 (Polar conversion):

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Polar coordinates

Theorem 12 (Polar conversion):

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and

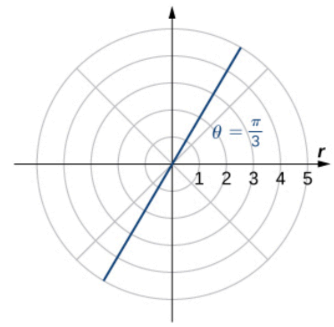
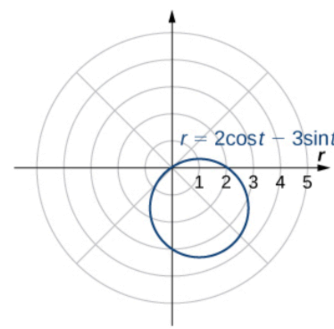
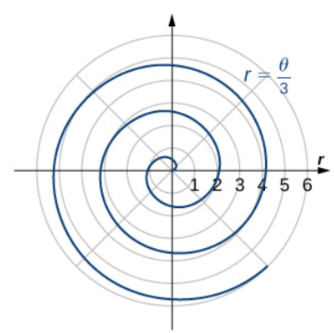
$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Example 18:

- Convert $(-3, 4)$ into polar coordinates
- Convert $\left(3, \frac{\pi}{3}\right)$ into Cartesian coordinates

Polar equations

Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Polar equations

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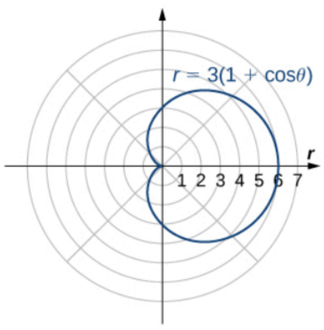
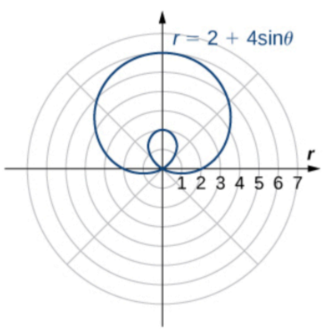
$r = a(1 - \cos\theta)$
$r = a(1 + \sin\theta)$
$r = a(1 - \sin\theta)$</td><td>
</td></tr>
<tr><td>Limaçon</td><td>$r = a\cos\theta + b$
$r = a\sin\theta + b$</td><td>
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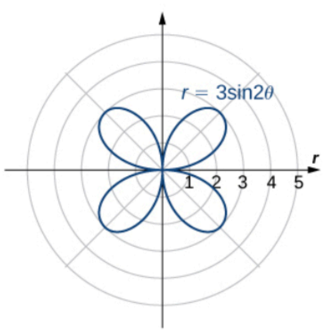
<tr><td>Rose</td><td>$r = a\cos(b\theta)$
$r = a\sin(b\theta)$</td><td>
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Formal definition of limits

Theorem 13 (Epsilon delta definition)**:** $\forall \varepsilon > 0, \exists \delta > 0$ such that:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Epsilon-delta

Example 19:

1. *Prove* that $\lim_{x\to 1}(3x-1)=2$
2. *Prove* that $\lim_{x\to 1}(2x+1)=3$