Calculus Society (Section 1-3)

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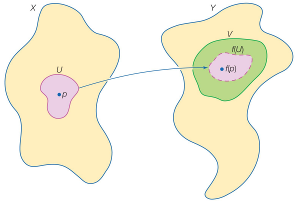
Functions

In order to have a solid understanding of calculus, we must first formally understand what functions are.

Definition 1.1

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function, whereas the set of outputs is called the **range** of the function.

In this way, we can view functions as **maps** between the domain and range. This concept will be important when studying higher mathematics.

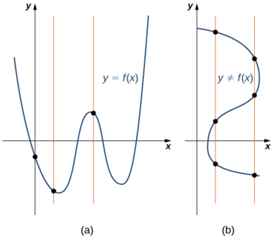


Functions

Since functions have exactly one output for every input, we can use this property to determine if a certain set of points represent a function.

Theorem 1.2 (Vertical line test)

Given a function f, every vertical line that may be drawn will intersect the graph of f **no more than once**. *If any vertical line intersects the graph of f more than once, then the set of points does not represent a function.*



Functions

Now, lets quickly go through how to combine functions, as well as composite functions.

Definition 1.3

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Remark

For the quotient compound function, it is assumed that $g(x) \neq 0$.

Finally, lets look into even and odd graphs.

Definition 1.4

If $f(x) = f(-x)$ for all x in the domain of f, then f is an **even function**. An even function is symmetric about the y-axis. If $f(-x) = -f(x)$ for all x in the domain of f, then f is an **odd function**. An odd function is symmetric about the origin.

Exponential and logarithmic functions?

Limits

We now move on to look at limits.

Definition 1.5

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach L as the values of x approach the number a $(x \neq a)$, then we say that the **limit** of $f(x)$ as x approaches a is L. We write this as:

$$\lim_{x \to a} f(x) = L$$

What is the difference between the limit (say as $x \to \infty$) and the bound of a function? We will come back to this question later on...

Limits

Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number.

Definition 1.6

Left sided limits: If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. We write this as:

$$\lim_{x \to a^-} f(x) = L$$

Right sided limits: If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. We write this as:

$$\lim_{x \to a^+} f(x) = L$$

Limits

Definition 1.7

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Limits

Let $f(x)$ and $g(x)$ be defined $\forall x \neq a$ over an interval containing a. Let c be a constant. Then the following laws hold:

Sum law for limits: $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$

Difference law for limits: $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x\to a} cf(x) = c\cdot \lim_{x\to a} f(x) = cL$

Product law for limits: $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = L\cdot M$

Quotient law for limits: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$ for every positive integer n.

Root law for limits: $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$.

Continuity

Finally, before going into the fundamentals of calculus, lets look into continuity.

Definition 1.8

A function $f(x)$ is **continuous** at a point a if and only if the following three conditions are satisfied:

- $f(a)$ is defined
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

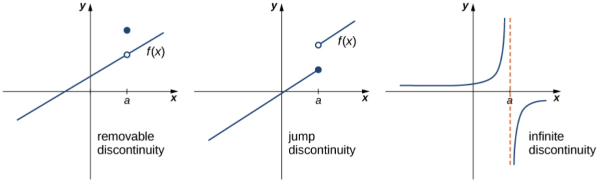
A function is **discontinuous** at a point a if it fails to be continuous at a.

Continuity

Definition 1.9

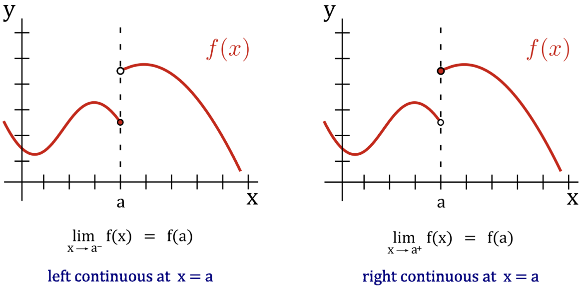
If $f(x)$ is discontinuous at a, then:

- f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists
- f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, but are not equal to each other.
- f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$, but are not equal to each other.



Definition 1.10

A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^-} f(x) = f(a)$. Similarly, A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \to a^+} f(x) = f(a)$.



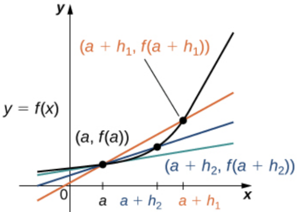
Basic Principles

We can now begin to look at the concept of derivatives. The derivative is defined to be the *rate of change* of functions (i.e, the gradient of its graph at a certain point). Thus we have to determine the gradient of the *tangent line* of a curve at a particular point.

Definition 1.11

Let f (x) be a function defined in an open interval containing a. The tangent line to f (x) at a is the line passing through the point $(a, f(a))$ having gradient:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Basic Principles

Thus, we can define the derivative a function $f(x)$ as follows.

Definition 1.12

Let f be a function. Then, its **derivative** f' is defined for all values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternatively, the notation $\frac{d}{dx}(f(x))$ can be used instead of $f'(x)$.

Also, lets establish the relationship between differentiability and continuity.

Theorem 1.13

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

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Basic rules of differentiation

We now move onto the basic differentiation rules.

Theorem 2.1 (Constant rule)

$$\frac{d}{dx}(c) = 0$$

Theorem 2.2 (Power rule)

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Basic rules of differentiation

Theorem 2.3

The derivative is a **linear operator**, *i.e,*

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

And, for all $k \in \mathbb{R}$

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$$

Now lets try some examples!

Basic rules of differentiation

Example 2.4

Find the derivative of each of the following with respect to x.

- $y = \frac{1}{x^2}$
- $y = \sqrt{x}$
- $f(x) = 1$
- $f(x) = \frac{x^4 - 3x^2 + 4}{x^2}$

Product rule, Quotient rule and Chain rule

Now lets look at the product and quotient rules: Let $u(x)$ and $v(x)$ be differentiable functions.

Theorem 2.5 (Product Rule)

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Theorem 2.6 (Quotient Rule)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 2.7

Find the derivative of

$$f(x) = \frac{2x+5}{3x-4}$$

Example 2.8

Find the derivative of

$$y = (3x+2)\sqrt{4x-1}$$

Answers

- $$-\frac{23}{(3x-4)^2}$$
- $$\frac{18x+1}{\sqrt{4x-1}}$$

We now introduce the chain rule.

Product rule, Quotient rule and Chain rule

Theorem 2.9 (Chain Rule)

If y is a function of u, and u is a function of x, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 2.10

Find the derivative of

$$y = \frac{1}{(x-2)^5}$$

at $x = 3$.

Remark

Note that the derivative of y at $x = a$ can be written as $\frac{dy}{dx}\big|_{x=a}$

Answer

$$-5$$

Higher order derivatives

Definition 2.11

The function $\frac{dy}{dx}$ is the **first derivative** of y with respect to x.
Differentiating $\frac{dy}{dx}$ with respect to x gives the **second derivative**, which can be written as:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Remark

Note that:

$$\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$$

Higher order derivatives

Example 2.12

Given that $y = \frac{2x^2}{x-3}$, find $\frac{d^2y}{dx^2}$.

Higher order derivatives

Answer

$$\frac{36}{(x-3)^3}$$

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Derivative of trigonometric functions

We now begin to look into the derivatives of special functions, namely trigonometric, exponential and logarithmic functions.

Theorem 3.1

$$\frac{d}{dx}(\sin x) = \cos x$$

Theorem 3.2

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivative of trigonometric functions

Example 3.3

Find the derivative of $y = 5x^3 \sin x$ with respect to x.

Example 3.4 (Derivative of $\tan x$)

Find the derivative of $y = \tan x$. (Hint: $\tan x = \frac{\sin x}{\cos x}$)

Derivative of trigonometric functions

Answer

$$\frac{dy}{dx} = 5x^3 \cos x + 15x^2 \sin x$$

Theorem 3.5

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Theorem 3.6

$$\frac{d}{dx}(e^x) = e^x$$

How could this rule be applied to differentiate exponential functions with the general form $f(x) = b^{g(x)}$? We will come back to this.

Derivative of exponential functions

Example 3.7

Find the derivative of $y = e^{x^2+2}$

Example 3.8 (Challenge)

Find the derivative of $y = \frac{e^{\tan 2x}}{x}$

Derivative of exponential functions

Answer

$$\frac{dy}{dx} = 2xe^{x^2+2}$$

Answer

$$\frac{dy}{dx} = \frac{e^{\tan 2x}(2x\sec^2 2x - x)}{x^2}$$

Derivative of logarithmic functions

Theorem 3.9

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Once again, how could this rule be applied to find the derivative of functions of the general form $f(x) = \log_a g(x)$?

Derivative of exp. and log. functions

Theorem 3.10

Let $b > 0$ and $g(x)$ be a differentiable function. Then:

- *If $f(x) = \log_b g(x)$, where $g(x) > 0$,*

$$f'(x) = \frac{g'(x)}{g(x)\ln b}$$

- *If $f(x) = b^{g(x)}$,*

$$f'(x) = b^{g(x)} g'(x) \ln b$$

Derivative of exp. and log. functions

Example 3.11

Find the slope of the line tangent to the graph of $y = \log_2{(3x + 1)}$ at $x = 1$.

Example 3.12

Find the derivative of $y = 3^{x^3}$.

Derivative of exp. and log. functions

Answer

$$m = \frac{3}{\ln 16}$$

Answer

$$\frac{dy}{dx} = 3^{x^3}(3x^2 \ln 3)$$

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Functions

We now briefly look into the idea of increasing and decreasing functions.

Definition 4.1

A function f is **increasing** on an interval I if $\forall x_1, x_2 \in I$ such that $x_1 < x_2$:

$$f(x_1) \leq f(x_2)$$

We say that f is **strictly increasing** on the interval I if if $\forall x_1, x_2 \in I$ such that $x_1 < x_2$:

$$f(x_1) < f(x_2)$$

Similarly, a function f is **decreasing** on an interval I if $\forall x_1, x_2 \in I$ such that $x_1 < x_2$:

$$f(x_1) \geq f(x_2)$$

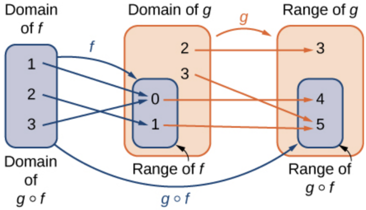
We say that f is **strictly decreasing** on the interval I if $\forall x_1, x_2 \in I$ such that $x_1 < x_2$:

$$f(x_1) > f(x_2)$$

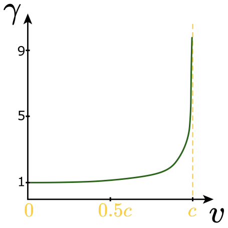
Definition 4.2

Consider the function f with domain A and range B, and the function g with the domain D and range E. If B is a subset of D, then the **composite function** $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x))$$



Lets try an example! This example is an important demonstration of the applications of calculus to various fields, such as special relativity in this case.



Limits (KE e.g.)

Example 4.3 (Speed of light)

The *relativistic kinetic energy* of a particle travelling at a velocity v can be determined the following equation:

$$\mathrm{KE} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 = (\gamma - 1) m_0 c^2$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor and m_0 is the rest mass of the particle (mass of the particle when it is at rest). Use the theory of limits to show that nothing can theoretically exceed the speed of light (Hint: $\lim_{x \to a} \frac{1}{(x-a)^n} = \infty$).

Limits (epsilon delta)

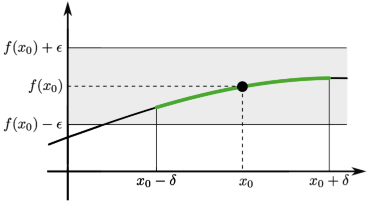
Now, lets look at the *formal/precise definition* of limits.

Definition 4.4

Let $f(x)$ be defined $\forall x \neq a$ over an open interval containing a. Let L be a real number. Then:

$$\lim_{x \to a} f(x) = L$$

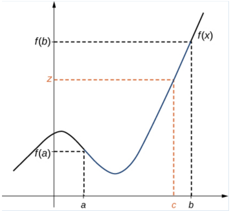
if, $\forall \varepsilon \exists \delta > 0$ such that if $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$. This is known as the **epsilon-delta definition of the limit**.



Continuity (IVT)

Theorem 4.5 (Intermediate Value Theorem)

Let f be a continuous function over a closed, bounded interval $[a, b]$. If z is any real number such that $f(a) < z < f(b)$, then there is a number $c \in [a, b]$ satisfying $f(c) = z$.



Derivative of log functions e.g

Example 4.6

Find the derivative of $y = (2x^4 + 1)^{tanx}$ using logarithmic differentiation.

Answer

$$\frac{dy}{dx} = \left(2x^4 + 1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4 + 1\right) + \frac{8x^3}{2x^4 + 1} \cdot \tan x\right)$$