The Fundamental Group

Big Picture

"Given a top. space, X, the fundamental group of X, $\pi_1(X)$, encodes the 'holes' of X"

Outline

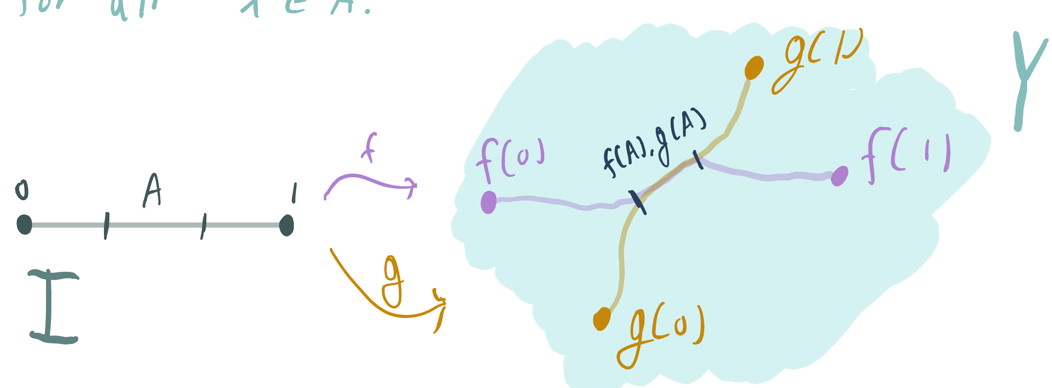
- Paths?
- Def of π_1 (and showing it's a group).
- Generalizations and example

Paths?

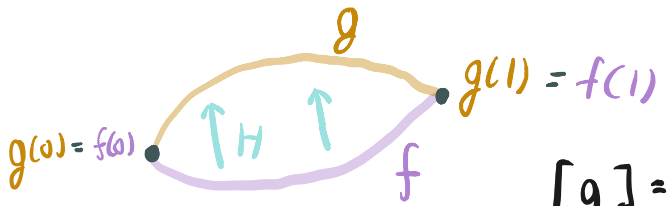
Def: A <u>path</u> is a cont. map, $f: I \to X$. "Given $f(0)=p$, $f(1)=q$, we say f is a path from p to q."

Given $f, g: X \to Y$ cont and $A \subseteq X$ subset, we say $H: f \simeq g$ is "stable on A" if $H(x,t) = f(x)$ $\forall x \in A$, $t \in I$.

Note that $1 \in I$, so $H(x,1) = f(x)$. But by def. of H, $H(x,1) = g(x)$, so $f(x) = g(x)$ for all $x \in A$.



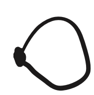
Def: A "path homotopy" from f, g which is stationary on $\{0,1\} \subseteq I$. We write this as $f \simeq g$.



$[g] = [f]$

Prop: For any $p, q \in X$, $\simeq$ is an equiv. relation on all paths from p to q.

Now for a crap-ton of defs:

- $[f]$ is the equiv class of a path f under $\simeq$.
- A "loop" is a path s.t. $f(0) = f(1)$ 
- $\Omega(X, p)$ is the set of all loops at p 
- $c_p \in \Omega(X, p)$ is the map $c_p(s) = p \; \forall s \in I$.

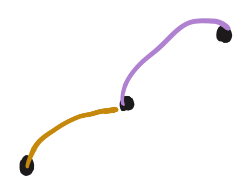
Fundamental Group

Def: We define fund. grp. of X based at p, $\pi_1(X,p)$, to be the path homotopy classes of $\Omega(X,p)$

"How is this a grp?"

· Given $f,g : I \to X$ paths, s.t. $f(1)=g(0)$ let

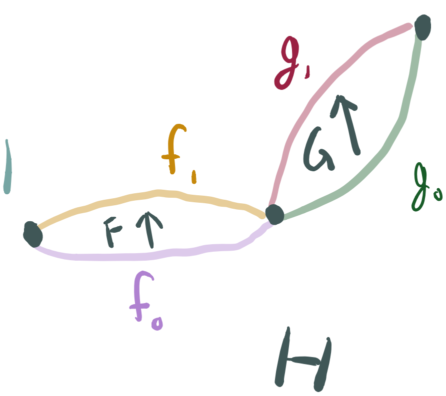
$$f\cdot g\,(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(\underbrace{2s-1}_{2(s-\frac{1}{2})}), & \frac{1}{2} \le s \le 1 \end{cases}.$$



Prop: If $f_0 \sim f_1$ and $g_0 \sim g_1$, then $f_0\cdot g_0 \sim f_1\cdot g_1$. (when $f_0(1)=g_0(0)$ and $f_1(1)=g_1(0)$).

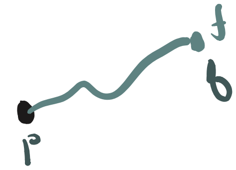
"Pf": Let $F: f_0 \sim f_1$ and $G: g_0 \sim g_1$ then $H: f_0\cdot g_0 \sim f_1\cdot g_1$ is given as

$$H(s,t) = \begin{cases} F(2s, t) : & 0 \le s \le \frac{1}{2} \\ G(2s-1, t) : & \frac{1}{2} \le s \le 1 \end{cases}$$



We can now define $[f]\cdot[g]$ to be $[f\cdot g]$. We'll now state the group prop. without prf.

Let $\bar{f}(s) = f(1-s)$. Then the following hold:



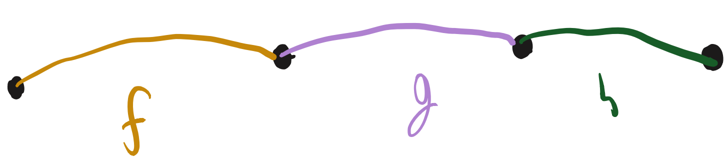
- $[c_p]\cdot[f] = [f]\cdot[c_b] = [f]$
- $[f]\cdot[\bar{f}] = [\bar{f}]\cdot[f] = [c_p]$
- $[f]\cdot([g]\cdot[h]) = ([f]\cdot[g])\cdot[h]$

$\downarrow \uparrow$

$f\cdot(g\cdot h) \sim (f\cdot g)\cdot h$

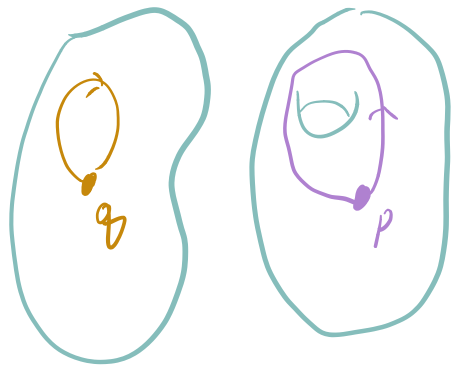
$$\begin{cases} f(2s) : & 0 \le s \le \frac{1}{2} \\ g(4s-2) : & \frac{1}{2} \le s \le 3/4 \\ h(4s-3) : & 3/4 \le s \le 1 \end{cases}$$

$$\begin{cases} f(4s) : & 0 \le s \le \frac{1}{4} \\ g(4s-1) : & \frac{1}{4} \le s \le \frac{1}{2} \\ h(2s-1) : & 1/2 \le s \le 1 \end{cases}$$



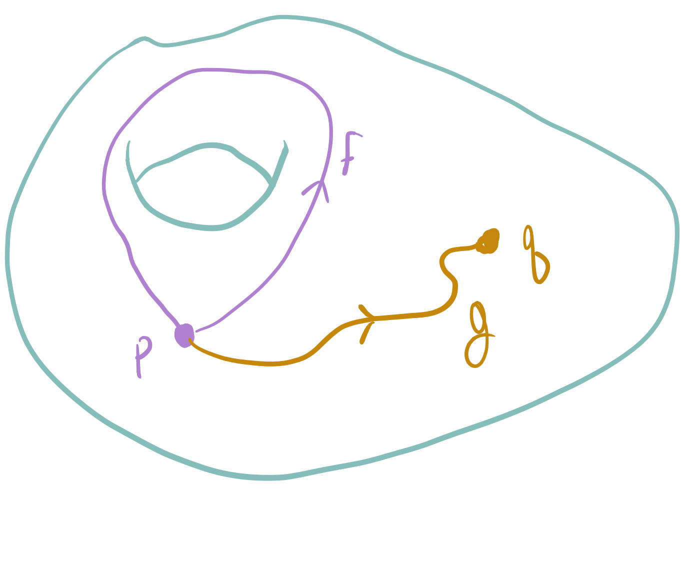
Generalizations + Example

X not path connected
$\Rightarrow$ $\pi_1(X,p)$ and $\pi_2(X,p)$
not really related.



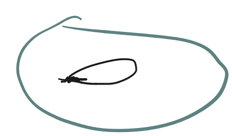
Thrm: <u>Change of base point</u>

Let X be pth connected, $p, q \in X$ and
g is a path from p to q. The map $\phi_g : \pi_1(X,p) \to \pi_1(X,q)$
defined as $\phi_g[f] = [\bar{g}] \cdot [f] \cdot [g]$ is an isomorphism
whose inverse is $\phi_{\bar{g}}$.



Because of this thrm., we can often write $\pi_1(X)$ to represent an arbitrary fund. grp. since they will be isomorphic.

- This is good since we are studying spaces.



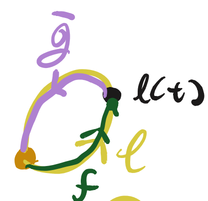
Exc 7.14:

a) $f \sim g$ so $[g]=[f]$ and since $[f][\bar{f}]=[c_p]$, $[g][\bar{f}]=[c_p]$

Similarly, $f\cdot\bar{g} \sim c_p \Rightarrow [f]\cdot[\bar{g}]=[c_p]$ and $[g][\bar{g}]=[c_p]$

since inverses are unique $[g]=[f]$.

b) Let f,g be paths from p to q. If X simply connected, then $f\cdot\bar{g}$ will be a loop and $\bar{f}\cdot g \sim c_p$ so $f \sim g$.



Let ℓ be loop with a base point p and $f,\bar{g}$ be the paths defined by following ℓ and splitting at a point $x=\ell(t)$.

Then by def. $f\cdot\bar{g} \sim \ell$ but by assumption $g \sim f$ so $f\cdot\bar{g} \sim c_p$ and $\ell \sim c_p$.