1.1 Variables

Here are three types of statements:

- **Universal**: A certain property is true **for all** the set.
- **Conditional**: **If** one statement is true, **then** another must also be true.
- **Existential**: **There exists** at least one member of the set where a certain property is true.

Here are three more types of statements combining the previous statements:

- **Universal-Conditional**: First part of statement is universal, next is conditional.
- **Universal-Existential**: First part of statement is universal, next is existential.
- **Existential-Universal**: First part of statement is existential, next is universal.

1.2 The Language of Sets

A **set** is a collection of **elements**, which can include other sets. Only what is in the set matters; order of elements or duplicates do not. This is known as the **Axiom of Extensionality**.

Given a set S, the set of all elements x in S that satisfies a property P can be denoted using **set-builder notation**: $\{x \in S \mid P\}$.

A $\subseteq$ B means that A is a **subset** of B, a.k.a A is contained in B, a.k.a B contains A.
A is a **proper subset** of B if A is a subset of B and the sets are unequal.

Given sets A and B, a **Cartesian product** is the set of all ordered pairs (a, b) where a is in A and b is in B:

$$[A \times B = (a, b) \mid a \in A, b \in B]$$

1.3 The Language of Relations and Functions

A **relation** R from a set A to a set B is a subset of the Cartesian product $A \times B$. Given an ordered pair $(x, y) \in A \times B$, we say x is **related to** y by R, written xRy, if and only if $(x, y) \in R$. The **domain** of R from A to B is the set A. The **codomain** is the set B.

A relation R from A to B is a **function** if it satisfies two properties:

1. For every $x \in A$, there exists $y \in B$ such that xRy. (Every element of A is related to some element of B.)
2. For every $x \in A$ and $y, z \in B$, if xRy and xRz, then $y = z$. (No two distinct elements of B are related to the same element of A.)

If f is a function from A to B, then for every $x \in A$, the unique $y \in B$ such that xfy is denoted $f(x)$.

An **arrow diagram** for a relation depicts elements of the domain and codomain as points, with arrows connecting related elements. Let $R = (2, 4), (3, 4), (1, 5), (-5, 2), (-4, 1)$ be a relation from $A = -5, -4, 1, 2, 3$ to $\mathbb{Z}$. R is a function since each element of A is related to exactly one integer. The circle relation $C \subseteq \mathbb{R} \times \mathbb{R}$ defined by $x^2 + y^2 = 4$ is not a function. For instance, 0 is related to both 2 and -2.

1.4 The Language of Graphs

A **graph** $G = (V, E)$ consists of a non-empty set V of **vertices** and a set E of **edges**. Each edge $e \in E$ connects either one or two vertices called its **endpoints**.

The **edge endpoint function** $\phi : E \to V \times V$ maps each edge to its endpoint(s). An edge with one endpoint is a **loop**. Multiple edges with the same endpoints are **parallel edges**.

Key terminology:

- An edge e is **incident on** its endpoint(s).
- Vertices connected by an edge are **adjacent**.
- A vertex on no edges is **isolated**.
- The **degree** $\deg(v)$ of a vertex v counts incident edges (a loop is counted twice).

The **handshake lemma**: Sum of all vertex degrees $= 2|E|$.

Special types of graphs:

- A **simple graph** has no loops or parallel edges.
- K_n: **Complete graph** on n vertices ($\binom{n}{2}$ edges).
- $K_{m,n}$: **Complete bipartite graph** with partite sets of size m and n.
- A **subgraph** has a vertex subset $V' \subseteq V$ and edge subset $E' \subseteq E$.

In a **directed graph**, edges are ordered pairs of vertices indicating direction. Loops and parallel edges with opposite directions are allowed.