2.5 Number Systems and Circuits for Addition

Decimal notation represents numbers using powers of 10. Each digit represents a coefficient and power of 10.

Binary notation represents numbers using powers of 2. Each digit is 0 or 1, called a **bit**. The bits represent coefficients and powers of 2.

Hexadecimal notation represents 4 bits as a single hexadecimal digit 0-9, A-F. This makes binary more compact.

Two's complement represents negative integers in binary:

This allows easy addition using binary:

- Write the positive binary number
- Flip all the bits
- Add 1
- Add using regular binary rules
- Ignore overflow bits (leading 1)

Two's complement can represent -128 to 127 with 8 bits. Leading 0 = positive, leading 1 = negative.

Logic Circuits

A **half adder** circuit adds two bits p and q, producing a **sum** $s = p \oplus q$ and **carry** $c = pq$ output.

A **full adder** circuit adds three bits p, q, and r, using two half adders. The sum is $s = p \oplus q \oplus r$ and the carry is $c = pq + (p \oplus q)r$.

Chaining together full adders creates a **parallel adder** circuit to add binary numbers of any length. Ripple carry adders reduce hardware but have propagation delay.

3.1 Predicates and Quantified Statements

A **predicate** $P(x_1, x_2, ..., x_n)$ is an expression with n variables that becomes a statement when specific values are substituted for the variables. The set D of allowable values for the variables is called the **domain** of the predicate.

For example, $x^2 > 2x$ is a predicate. When we substitute a value for x, such as $x = 2$, we obtain the statement $2^2 > 2(2)$ which can be determined true or false.

The **truth set** of a predicate $P(x)$, with domain D, is the set $T = x \in D : P(x)$ is true. That is, T contains the elements of D that make $P(x)$ true.

A **universal quantifier** is a phrase such as "for all" that specifies a predicate holds for every element of the domain. A **universal statement** has the form:

$$\forall x \in D, P(x)$$

This is read "for all x in D, $P(x)$" and it is true precisely when $P(x)$ is true for every $x \in D$. Otherwise, it is false.

An **existential quantifier** specifies that a predicate holds for at least one element of the domain. An **existential statement** has the form:

$$\exists x \in D \text{ such that } P(x)$$

This reads "there exists x in D such that $P(x)$" and is true when $P(x)$ holds for at least one $x \in D$. If $P(x)$ is false for all x, the statement is false.

3.2 Predicates and Quantified Statement Part 2

Consider the universal statement about dogs:

$$\forall x \in D, P(x) : x \text{ has black fur}$$

where D represents the set of all dogs.

Negating Quantified Statements
The negation of a universal statement $\forall x \in D, P(x)$ is logically equivalent to the existential statement $\exists x \in D$ such that $\neg P(x)$. Conversely, the negation of an existential statement produces a universal statement with the negation applied to the predicate.

Using symbolic notation:

$\neg[\forall x \in D, P(x)] \equiv \exists x \in D, \neg P(x)$

$\neg[\exists x \in D, P(x)] \equiv \forall x \in D, \neg P(x)$

Now consider a universal conditional statement:

$$\forall x, P(x) \to Q(x)$$

The negation of a universal conditional statement is:

$$\exists x, P(x) \land \neg Q(x)$$

Where the negation applies to the conditional statement $P(x) \to Q(x)$.

3.3 Statements with Multiple Quantifiers

Consider the statement: "You can fool some of the people all of the time."

Two ways to write this formally: $\exists$ person $p, \forall$ time $t,$ you can fool p
$\forall$ persons $p, \exists$ time $t,$ you can fool p

The first statement correctly captures the original meaning. The second statement says "every person can be fooled some of the time", which is different.

The order of quantifiers is crucial and changes the meaning of statements with multiple quantifiers.

As another example, consider:

> Given any positive real number, there is a smaller positive real number.

In formal logic:

$$\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+, y < x$$

This is true because we can always divide a positive number by 2. However, if we replace $\mathbb{R}^+$ with $\mathbb{N}$, the natural numbers, then this becomes false, since 1 has no smaller natural number.

Negating Multiply Quantified Statements

To negate a statement with multiple quantifiers, switch $\forall$ and $\exists$ quantifiers, and negate the predicate:

$\neg[\forall x \in D, \exists y \in E, P(x, y)] \equiv \exists x \in D, \forall y \in E, \neg P(x, y)$

$\neg[\exists x \in D, \forall y \in E, P(x, y)] \equiv \forall x \in D, \exists y \in E, \neg P(x, y)$