10.1 Trails, Paths, and Circuits

An **Eulerian circuit** is a circuit through a graph that uses every edge exactly once.

Euler proved: A graph has an Eulerian circuit *if and only if* the graph is connected and every vertex has even degree. This condition was not satisfied in the Konigsberg graph, so no Eulerian circuit was possible.

Key Terms

- **Graph**: Vertices and edges
- **Vertex**: Node in a graph
- **Edge**: Connection between vertices
- **Eulerian circuit**: Circuit using every edge once
- **Connected graph**: Paths between all vertices
- **Even degree**: Even number of incident edges

Walking in Graphs

A **walk** in a graph is an alternating sequence of vertices and edges that begins at one vertex and ends at another.

- A **trail** does not contain any repeated edges.
- A **path** does not contain any repeated vertices.
- A **trivial walk** does not move from the starting vertex.
- A **closed walk** starts and ends at the same vertex.
- A **circuit** is a closed walk with no repeated edges (also known as a **cycle**).
- A **simple circuit** has no repeated vertices except the starting/ending vertex.

Two vertices v and w in a graph G are **connected** if there is a walk between them.

G is a **connected graph** if every pair of vertices is connected.

10.4 Trees: Examples and Basic Properties

A **tree** is a connected graph that has no circuits. Formally, a graph is a tree if it is connected and has no circuits. A **trivial tree** is a single vertex. A **forest** is a disconnected graph with no circuits - essentially a group of separate trees. The Konigsberg bridges graph is not a tree because it has circuits.

A vertex of degree 1 in a tree is called a **leaf** or **terminal vertex**. A vertex with degree > 1 is an **internal vertex** or **branch vertex**. If the tree only has 1 vertex, that vertex is the leaf or terminal vertex.

Tree Theorems:

- All trees have at least one vertex of degree 1.
- A tree with n vertices has $n-1$ edges.
- Deleting any edge from a circuit in a connected graph leaves a connected graph.
- A connected graph with n vertices and $n-1$ edges is a tree.
- A graph with n vertices and $m \geq n$ edges has a circuit.

10.5 Rooted Trees

A **rooted tree** has one vertex distinguished as the **root**. For any vertex v, there is a unique path from the root to v. The number of edges in this path is the **level** of v.

The **height** of the tree is the length of the longest path from root to leaf.

The **children** of a vertex v are adjacent to v and farther from the root. The **parent** is the opposite relation. **Siblings** have the same parent. **Ancestors** of v are vertices closer to the root. **Descendants** of v are vertices farther from the root.

A **binary tree** has root with at most two children designated left and right. A **full binary tree** has each parent with exactly two children. The **left subtree** and **right subtree** of a vertex are the trees rooted at its children.

Binary Tree Theorems:

- A full binary tree with n internal vertices has $2n+1$ total vertices and $n+1$ leaves.
- A binary tree of height h with t leaves satisfies $t \leq 2^h$ or $\log_2 t \leq h$.