ECON 591: Lecture #4

Topic 4: Multivariable Calculus

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From Last Time...

Functions are rules that assign objects from one space to individual objects in another space.

- Motivating example: Production Function
- Linear
  - Level sets in linear $\rightarrow$ hyperplanes $ax = b$
  - Common for econ: budget sets

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- Motivating example: Production Function
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- Next up... Quadratic forms

Quadratic Forms

$$f(x) = bx^2$$

Definition 4.D

A quadratic form on $\mathbb{R}^k$ *is a real-valued function of the form*

$$Q(x_1, ..., x_k) = \sum_{i,j=1}^{k} a_{ij} x_i x_j.$$

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Level curve: $a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 = b$
$\rightarrow$ ellipse, hyperbola, pair of lines, empty set (i.e. conic sections)
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Matrix Representation ("Expansion around a matrix")

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Quadratic Forms

Theorem 4.3

The general quadratic form $Q(x_1, ..., x_n) = \sum_{i \leq j} a_{ij} x_i x_j$ *can be written as*

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \dots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \dots & \frac{1}{2}a_{2n} \\ \vdots & & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or $x^T Ax$*, where* A *is a unique symmetric matrix. Conversely, if* A *is a symmetric matrix, then* $Q(x) = x^T Ax$ *is a quadratic form.*

Polynomials

Linear and quadratic are special cases of polynomials.

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Definition 4.E

A function $f : \mathbb{R}^k \to \mathbb{R}^1$ is a monomial if it can be written as

$$f(x_1, ..., x_k) = cx_1^{a_1} x_2^{a_2} ... x_k^{a_k}$$

where c is a scalar and the exponents $a_1, ..., a_k$ are nonnegative integers. The sum of exponents is called the degree of the monomial.

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where c *is a scalar and the exponents* $a_1, ..., a_k$ *are nonnegative integers. The sum of exponents is called the* *degree* *of the monomial.*

Definition 4.F

A function $f : \mathbb{R}^k \to \mathbb{R}^1$ *is a* *polynomial* *if* f *is the finite sum of monomials on* $\mathbb{R}^k$*. The highest degree which occurs among the monomials is the degree of the polynomial.*

A function $f : \mathbb{R}^k \to \mathbb{R}^m$ *is called a polynomial if each of its component functions is a real-valued polynomial.*

Continuous Functions

Definition 4.G

Let f be a function from $\mathbb{R}^k$ to $\mathbb{R}^m$. Let x_0 be a vector in $\mathbb{R}^k$, and $y = f(x_0)$ its image. The function f is continuous at x_0 if wherever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^k$ which converges to x_0, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ in $\mathbb{R}^m$ converges to $f(x_0)$.

The function is said to be continuous if it is continuous at every point in its domain.

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Theorem 4.4

Let $f = (f_1, ..., f_m)$ be a function from $\mathbb{R}^k$ to $\mathbb{R}^m$. Then f is continuous at x iff each of its component functions $f_i : \mathbb{R}^k \to \mathbb{R}^1$ is continuous at x.

Continuous Functions

Theorem 4.5

Let f and g be functions from $\mathbb{R}^k$ to $\mathbb{R}^m$. Suppose f and g are continuous at x. Then $f + g, f - g$ and $f \cdot g$ are all continuous at x.

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Proof.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to x.

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Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to x. By continuity, $f(x_n)$ converges to $f(x)$ and $g(x_n)$ converges to $g(x)$.

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$f(x_n) + g(x_n) = (f+g)(x_n) \rightarrow$ this converges to
$f(x) + g(x) = (f+g)(x)$.

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Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to x. By continuity, $f(x_n)$ converges to $f(x)$ and $g(x_n)$ converges to $g(x)$. From earlier, $f(x_n)+g(x_n)=(f+g)(x_n) \to$ this converges to $f(x)+g(x)=(f+g)(x)$. Therefore $f+g$ is continuous at x as well.

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Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to x. By continuity, $f(x_n)$ converges to $f(x)$ and $g(x_n)$ converges to $g(x)$. From earlier, $f(x_n)+g(x_n)=(f+g)(x_n) \rightarrow$ this converges to $f(x)+g(x)=(f+g)(x)$. Therefore $f+g$ is continuous at x as well.

Ditto for $f-g$, $f \cdot g$. $\square$

Continuous Functions

Definition 4.H

A function $f : S \to \mathbb{R}^m$, where S is an open set in $\mathbb{R}^n$, is said to be <u>*differentiable*</u> *at a point $x \in S$ s.t. $\exists$ an $m \times n$ matrix A s.t. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $y \in S$ and $||x - y|| < \delta$*

$$\Rightarrow ||f(x) - f(y) - A(x - y)|| < \varepsilon ||x - y||.$$

Equivalently, f is differentiable at $x \in S$ if

$$\lim_{y \to x} \left(\frac{||f(y) - f(x) - A(y - x)||}{||y - x||} \right) = 0$$

where y represents $\forall$ sequences $y_k \to x$.

The matrix A is called the <u>derivative</u> of f at x and is denoted $Df(x)$.

When $n = m = 1$ (so $S \in \mathbb{R}$ and $f : S \to \mathbb{R}$) denote $Df(x)$ as $f'(x)$.

Continuous Functions

Definition 4.I

When f is differentiable on S, the derivative Df itself forms a function from S to $\mathbb{R}^{m\times n}$. If $Df : S \to \mathbb{R}^{m\times n}$ is a continuous function, then f is said to be continuously differentiable on S, and we say f is C^1.

Differentiable everywhere gives some idea of continuity, but is not the same thing as continuously differentiable.

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Theorem 4.6

If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are both differentiable at point $x \in \mathbb{R}^n$, then so is $(f+g)$; and in fact $D(f+g)(x) = Df(x) + Dg(x)$.

Continuous Functions

Proof.

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Choose some $\varepsilon > 0$. f differentiable $\Rightarrow \exists\, Df(x)$ s.t. $\forall\, \varepsilon$ (particularly $\frac{\varepsilon}{2}$), $\exists\, \delta_f > 0$ s.t. for $y \in S$

$$||x - y|| < \delta_f \Rightarrow ||f(x) - f(y) - Df(x)|| \leq \frac{\varepsilon}{2}||x - y||$$

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g differentiable $\Rightarrow \exists\, Dg(x)$ s.t. for $\frac{\varepsilon}{2}$, $\exists\, \delta_g > 0$ s.t. for $y \in S$

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Let $\delta = \min\{\delta_f, \delta_g\}$. Then $||x - y|| < \delta \Rightarrow$

$$||f(x) - f(y) - Df(x)|| + ||g(x) - g(y) - Dg(x)|| \leq \varepsilon||x - y||$$

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Then $(Df(x) + Dg(x)) = D(f + g)(x)$. $\square$

Vocabulary

Definition 4.J

The <u>image</u> of C under f is: $f(C) \equiv \{b \in B : b = f(a) \,\text{for some}\, a \in C\}$

The <u>preimage</u> is the set of all points in the domain whose image is in V: $f^{-1}(V) \equiv \{a \in A : f(a) \in V\}$

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If $\forall\, b \in B$, $\exists\, a \in A$ s.t. $b = f(a)$, or if the target space of f is the image of f, we say f maps onto B or that f is surjective.

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Definition 4.L

A function $f : A \to B$ is one-to-one (or injective) on a subset C of A iff $\forall\, x, y \in C$, $f(x) = f(y) \Rightarrow x = y$. f is one-to-one on $C \subseteq A$ if each $b \in f(c)$ is the image of precisely one element of C.

Vocabulary

Surjective (onto) & Injective (one-to-one)

Importance for $f(x) = b$

- f is onto if $f(x) = b$ has at least one solution for each b
- f is one-to-one if $f(x) = b$ has at most one solution for each b

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Definition 4.M

A bijective function is a one-to-one and onto mapping of A to B.

A bijection has an inverse for sure, but other functions can as well.

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Definition 4.N

When $f : A \to B$ is one-to-one on a set $C \subseteq A$, there is a natural function that takes $f(C)$ back to C which assigns to each $b \in f(C)$ the unique point in C which is mapped to it. This map is called the inverse of f on C and is written as $f^{-1}(C) : f(C) \to C$.

Composition of Functions

Definition 4.O

Let $f : A \to B$ and $g : C \to D$ be two functions. Suppose that B, the image of A is a subset of C, the domain of g. Then the composition of f with g, $g \circ f : A \to D$ is defined as $(g \circ f)(x) = g(f(x)) \, \forall \, x \in A$.

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Theorem 4.7

Let $f : \mathbb{R}^k \to \mathbb{R}^m$ be a continuous function at $x \in \mathbb{R}^k$. Let $g : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function at $f(x) \in \mathbb{R}^m$. Then the composition $g \circ f : \mathbb{R}^k \to \mathbb{R}^n$ is a continuous function at $x \in \mathbb{R}^k$.

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Theorem 4.8

Let $f : \mathbb{R}^k \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$. Let $x \in \mathbb{R}^k$. If f is differentiable at x, and g is differentiable at $f(x)$, then $g \circ f$ is itself differentiable at x, and its derivative may be obtained through the "chain rule" as

$$D(g \circ f)(x) = Dg(f(x))Df(x).$$

Calculus of Several Variables

$F : \mathbb{R}^n \to \mathbb{R}$ $\qquad\qquad$ $F(x_1, ..., x_n) = y$

We're interested in how changes in x_i affect F.

Partial derivative of F wrt x_i denoted $\frac{\partial F}{\partial x_i}$.

Recall that for one variable function $f(x)$, the derivative was defined as:

$$\frac{df(x_0)}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

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Definition 4.P

Let $f : \mathbb{R}^n \to \mathbb{R}$. Then $\forall\, x_i$ at each point $x^0 = (x_1^0, ..., x_n^0)$ in the domain of f, the partial derivative of f wrt x_i is

$$\frac{\partial f(x_1^0, ..., x_n^0)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1^0, ..., x_i + h, ..., x_n^0) - f(x_1^0, ..., x_i^0, ..., x_n^0)}{h}.$$

EX// Cobb-Douglas Production Function

$$Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$$

where K is capital and L is labor

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At $K = 10,000$ and $L = 625$, $Q = 20,000$

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At $K = 10,000$ and $L = 625$, $Q = 20,000$
Let's take the partial derivatives:

$$\frac{\partial Q}{\partial K}$$

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At $K = 10,000$ and $L = 625, Q = 20,000$
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$$\frac{\partial Q}{\partial K} = \left(4L^{\frac{1}{4}}\right)\left(\frac{3}{4}K^{-\frac{1}{4}}\right)$$

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At $K = 10,000$ and $L = 625$, $Q = 20,000$
Let's take the partial derivatives:

$$\frac{\partial Q}{\partial K} = \left(4L^{\frac{1}{4}}\right)\left(\frac{3}{4}K^{-\frac{1}{4}}\right) = 3\left(\frac{L}{K}\right)^{\frac{1}{4}} \quad (MPK)$$

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At (10,000, 625), $\frac{\partial Q}{\partial K} = 1.5$, $\frac{\partial Q}{\partial L} = 8$.

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  - $\Rightarrow$ derivatives are slopes
  - $\Rightarrow$ changes are vectors
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Put in matrix format:

$$\begin{pmatrix} \frac{\partial F(x_1^*, x_2^*)}{\partial x_1} & \frac{\partial F(x_1^*, x_2^*)}{\partial x_2} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$$

We call this matrix of derivatives the Jacobian (4.Q): $J = \{J_{ij} = \frac{\partial f_i}{\partial x_j}\}$

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$\rightarrow$ here just one F, 2 variables; expand to multivariable:

$$DF_x = \begin{pmatrix} \frac{\partial F(x^*)}{\partial x_1} & \dots & \frac{\partial F(x^*)}{\partial x_n} \end{pmatrix}$$

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Inverse Function Theorem

We can think of the Jacobian as its own function, in some ways.

For $F : \mathbb{R}^n \to \mathbb{R}^m$ which is C^1, where $F(x^*) = b$ and $DF(x^*)$ is the $m \times n$ Jacobian, then if

- $DF(x^*)$ is onto, then F is locally onto.
- $DF(x^*)$ is one-to-one, then F is locally one-to-one.

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Theorem 4.9

(Inverse Function Theorem) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function with $F(x^*) = y^*$. If $DF(x^*)$ is nonsingular, then $\exists$ on an open ball $B_r(x^*)$ about x^* and on open set V about y^* s.t. F is a one-to-one and onto map from $B_r(x^*)$ to V. The natural inverse map $F^{-1} : V \to B_r(x^*)$ is also C^1 and $D(F(x^*)^{-1}) = (DF(x^*))^{-1}$.*

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$\Rightarrow$ follows from the above properties

Parameterized Curves

Definition 4.Q

A curve in $\mathbb{R}^n$ is an n-tuple of continuous functions

$$x(t) = (x_1(t), ..., x_n(t))$$

where each $x_i : \mathbb{R} \to \mathbb{R}$. The functions $x_i(t)$ are called coordinate functions and t is the parameter describing the curve.

EX// if t is elapsed time, then $x(t)$ gives the position in $\mathbb{R}^n$ on its trajectory at time t

$\to$ distance, a common econ problem for traffic

$\to$ any cts line segment

Parameterized Curves

Definition 4.R

The *tangent vector* *at* t *is the component vector of first derivatives of* x*, i.e.* $x'(t) = (x_1'(t), ..., x_n'(t))$

$\Rightarrow$ get us back to the idea of continuously differentiable when these tangent vectors form a continuous function themselves

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Theorem 4.10

[Chain Rule.] *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and let $a : \mathbb{R}^1 \to \mathbb{R}^n$ be C^1 functions. Then, the composible function $g(t) = F(a(t))$ is a C^1 function from $\mathbb{R}^1 \to \mathbb{R}^m$ and*

$$g_i'(t) = \sum_j \frac{\partial f_i}{\partial x_j} (a_1(t), ..., a_n(t))\, a_j'(t) = Df_i(a(t)) \cdot a'(t)$$

This gives the vector equation $g'(t) = D(F \circ a)(t) = DF(a(t)) \cdot a'(t)$.

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Recall C^1, continuously differentiable.

If $f'(x)$ has a continuous derivative, then f is <u>twice-continuously differentiable</u> (4.T), C^2.

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If we have at least C^1, we can calculate some higher order derivatives.

$\frac{\partial f}{\partial x_i}$ is the change in f wrt x_i

$\rightarrow$ what if this is something like MPK, where we might be interested in how the variable changes that?

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$\rightarrow$ put in a matrix called the **Hessian (4.U)** whose (ij)th entry is $\frac{\partial^2 f(x^*)}{\partial x_j \partial x_i}$:

$$H \equiv D^2 f_x = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

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Theorem 4.11

[Young's Theorem.] *Suppose that $y = f(x_1, x_2, ..., x_n)$ is C^2 on an open region J in $\mathbb{R}^n$. Then $\forall\, x \in J$ and for each pair (i, j):*

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EX// Recall Cobb-Douglas: $Q = 4L^{\frac{1}{4}}K^{\frac{3}{4}}$ which had $\frac{\partial MPK}{\partial L} = \frac{3}{4}\frac{1}{L^{\frac{3}{4}}K^{\frac{1}{4}}}$

$$\frac{\partial Q}{\partial L} = L^{-\frac{3}{4}}K^{\frac{3}{4}}$$

$$\frac{\partial}{\partial K}\left(\frac{\partial Q}{\partial L}\right) = L^{-\frac{3}{4}} \cdot \frac{3}{4}K^{-\frac{1}{4}} = \frac{3}{4}\frac{1}{L^{\frac{3}{4}}K^{\frac{1}{4}}}$$

$\Rightarrow$ the Hessian is a symmetric matrix

Higher-Order Derivatives

Theorem 4.11

[Young's Theorem.] *Suppose that $y = f(x_1, x_2, ..., x_n)$ is C^2 on an open region J in $\mathbb{R}^n$. Then $\forall\, x \in J$ and for each pair (i, j):*

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$\Rightarrow$ the Hessian is a symmetric matrix

Higher orders: if a function is C^3, then third order partials exist and are continuous and Young's Theorem holds