Math 360 - Project 1: Symbiosis (Mutualism)

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Problem 1

Honey bees and clover plants have a symbiotic relationship that is mutually beneficial. The assigned problem attempts to model this relationship using a non-linear dynamical system. The model variables are defined in table 1 below.

Variable	Description
n	Number of years since 1990
X_n	Population of bees at year n
Y_n	Population of clover plants at year n
ΔX_n	Net change in bee population during year n
ΔY_n	Net change in clover plant population during year n
a	Proportionality constant for death rate of bees
b	Proportionality constant for birth rate of bees
c	Proportionality constant for death rate of clover plants
d	Proportionality constant for birth rate of bees

Table 1: Definition of all variables used in this paper.

The exact model is essentially the sum of two terms for each population. The first term in both models can be thought of as the death rate function: $-aX_n$ for bees and $-cY_n$ for clovers. More organism will die just by random chance or aging even the population of that organism is higher. The parameters a and c represent proportionality constants for the death function in both populations. And because they decrease the population, the sign of the terms are negative. Because bees and clovers have a mutually beneficial relationship we wouldn't expect there to be any interaction effects here.

The second term is the birth rate function: bX_nY_n for bees and dX_nY_n for clover plants. Here is where the interaction effects become relevant. Bees benefit from having a higher population of clover plants because they can use feed on more of their nectar. Simultaneously, clover plants benefit from having a higher population of bees because there is a greater chance each bee will spread more pollen. b and d represent proportionality constants for the birth function in both populations.

Put together, the model is simply the sum of the death rate function and the birth rate function for each population:

$$\Delta X_n = -aX_n + bX_nY_n$$

$$\Delta Y_n = -cY_n + dX_nY_n$$

Problem 2

For the next section, we're looking at the model when the parameters in table 2 are specified along with the initial points $X_0 = 200$ and $Y_0 = 300$. The results of the model are recorded in table 3. For the year 1998, the population of bees is 7,475 while the population of clovers is 14,360. The population of both organisms quickly grows and becomes too large to calculate using Python by the year 2007! So we are unable to

Parameter	Value
a	0.2
b	0.001
c	0.3
d	0.002

Table 2: Set of values for each parameter

Year	X	Y	ΔX	ΔY
1990	2.000e+02	3.000e+02	2.000e+01	3.000e+01
1991	2.200e+02	3.300e+02	2.860e+01	4.620e+01
1992	2.486e+02	3.762e+02	4.380e+01	7.419e+01
1993	2.924e+02	4.504e+02	7.321e+01	1.283e+02
1994	3.656e+02	5.787e+02	1.384e+02	2.495e+02
1995	5.041e+02	8.282e+02	3.167e+02	5.865e+02
1996	8.207e+02	1.415e+03	9.969e+02	1.898e+03
1997	1.818e+03	3.312e+03	5.657e+03	1.105e+04
1998	7.475e+03	1.436e+04	1.058e+05	2.104e+05
1999	1.133e+05	2.247e+05	2.544e+07	5.086e+07
2000	2.555e+07	5.108e+07	1.305e+12	2.611e+12
2001	1.305e+12	2.611e+12	3.408e+21	6.816e+21
2002	3.408e+21	6.816e+21	2.323e+40	4.646e+40
2003	2.323e+40	4.646e+40	1.079e+78	2.158e+78
2004	1.079e+78	2.158e+78	2.329e+153	4.659e+153
2005	2.329e+153	4.659e+153	1.085e+304	2.170e+304
2006	1.085e+304	2.170e+304	inf	inf
2007	inf	inf	nan	nan
2008	nan	nan	nan	nan

Table 3: The model values calculated using a Python script. An overflow error is thrown by the year 2006 for ΔX and ΔY.

provide an answer for the population in the year 2008 but the long run behavior of the model using this set of parameters is clear. The population becomes infinitely large without reaching an equilibrium.

Problem 3

For problem 3 we use the same parameters in table 2 but with initial values $X_0 = 100$ and $Y_0 = 150$. The results here are much more amenable to a visual representation. Figure 1 depicts the model as a vector field with the specific path taken by the populations when they start at the given initial values.

In 1998, the population of bees declines to 43 and the population of clovers is 42. By 2008 there are only 6 bees and 2 clovers. More specific values for the evolution of the population model are given in table 4. Over time it appears that both organisms go

extinct.

The model does not result in the populations being cut in half from the populations in table 3 from problem 2. This is because the model is defined by non-linear difference equations. To illustrate this, set $X_n = 2X_n$ and $Y_n = 2Y_n$ in the birth functions for bees and clovers:

$$X_{Birth} = b(2X_n)(2Y_n) = 4bX_nY_n$$
$$Y_{Birth} = d(2X_n)(2Y_n) = 4dX_nY_n$$

As you can see, doubling the populations results in the birth rate increasing by a factor of 4 for both populations! That is, the birth function has increasing returns to scale and is non-linear.
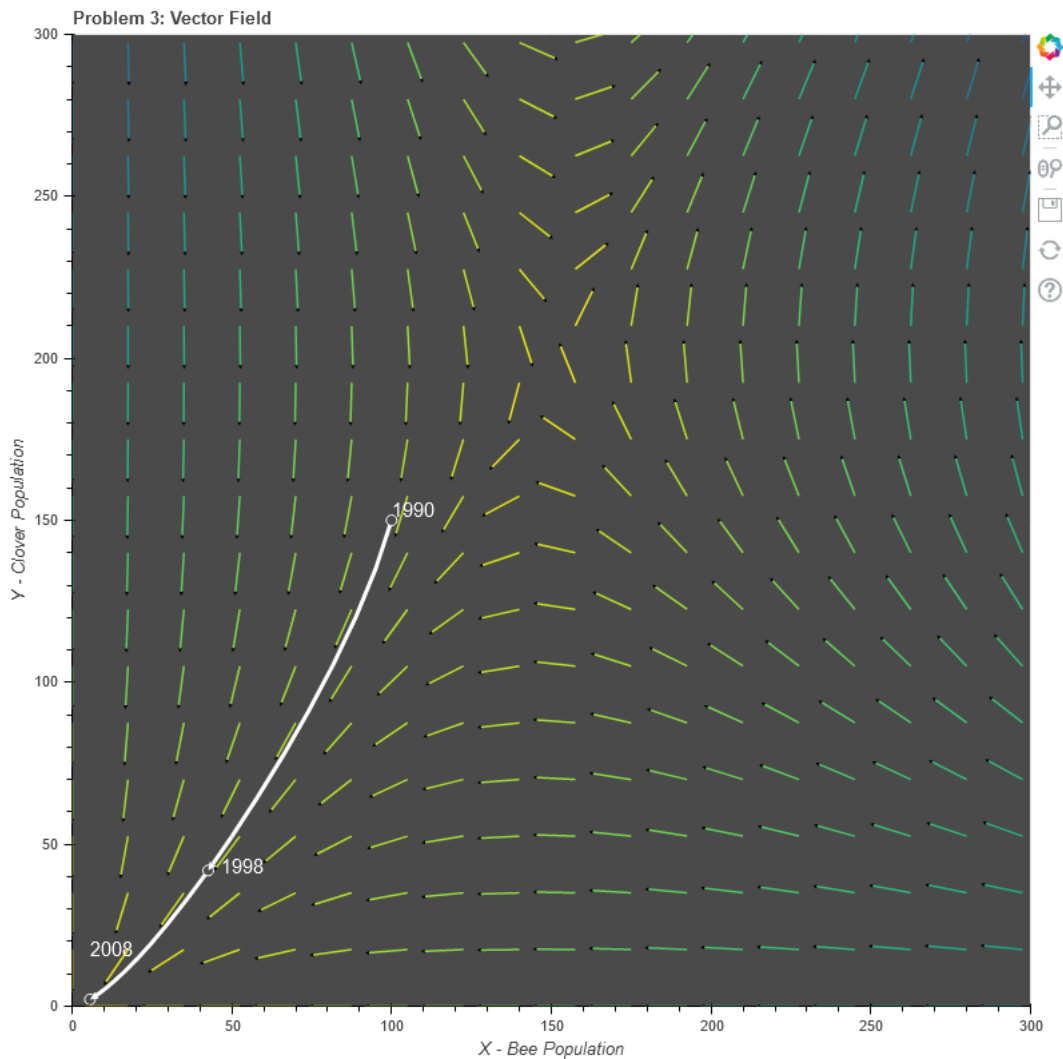

Figure 1: The model is represented as a vector field. The lengths of each vector are kept the same, but the color of each vector is determined by the magnitude of the vector at that point. Light yellow vectors have the lowest magnitude and dark purple vectors have the largest. The white shows the evolution of the populations when the initial conditions are $X_0 = 100$ and $Y_0 = 150$ are chosen.

	X	Y	ΔX	ΔY
Year				
1990	100.00	150.00	-5.00	-15.00
1991	95.00	135.00	-6.18	-14.85
1992	88.83	120.15	-7.09	-14.70
1993	81.73	105.45	-7.73	-14.40
1994	74.00	91.05	-8.06	-13.84
1995	65.94	77.21	-8.10	-12.98
1996	57.85	64.23	-7.85	-11.84
1997	49.99	52.39	-7.38	-10.48
1998	42.61	41.91	-6.74	-9.00
1999	35.88	32.91	-5.99	-7.51
2000	29.88	25.40	-5.22	-6.10
2001	24.66	19.30	-4.46	-4.84
2002	20.21	14.46	-3.75	-3.75
2003	16.46	10.71	-3.12	-2.86
2004	13.34	7.85	-2.56	-2.14
2005	10.78	5.70	-2.09	-1.59
2006	8.68	4.11	-1.70	-1.16
2007	6.98	2.95	-1.38	-0.84
2008	5.61	2.11	-1.11	-0.61

Table 4: The model evolution for the initial conditions $X_0 = 100$ and $Y_0 = 150$ given the parameters in table 2.

At the same time, the death functions have constant returns to scale:

$$X_{Death} = -a(2X_n) = -2aX_n$$

$$Y_{Death} = -c(2Y_n) = -2cY_n$$

As a result, the birth rate tends to increase faster than the death rate as the populations get larger. At the same time however, the death rate will decrease at a slower rate than the birth rate for decreasing populations. If both populations are declining, then generally speaking the death function will stay higher than the birth function in future years. If both populations are increasing, then the birth function will generally be larger than the death function. This can help explain why small changes in the initial condition of the populations can result in very different long term behaviors - the starting conditions basically determine whether the death function will be larger or smaller than the birth function very early on. And consistent with this description, the vector field in 1 indicates that lower starting populations tend to decline more often than higher starting populations.

Problem 4

In this problem we have to impose a population shock on the bees in the year 1996 using the initial conditions $X_0 = 200$ and $Y_0 = 300$. The results of the script are depicted visually in figure 3 and numerically in table 4.
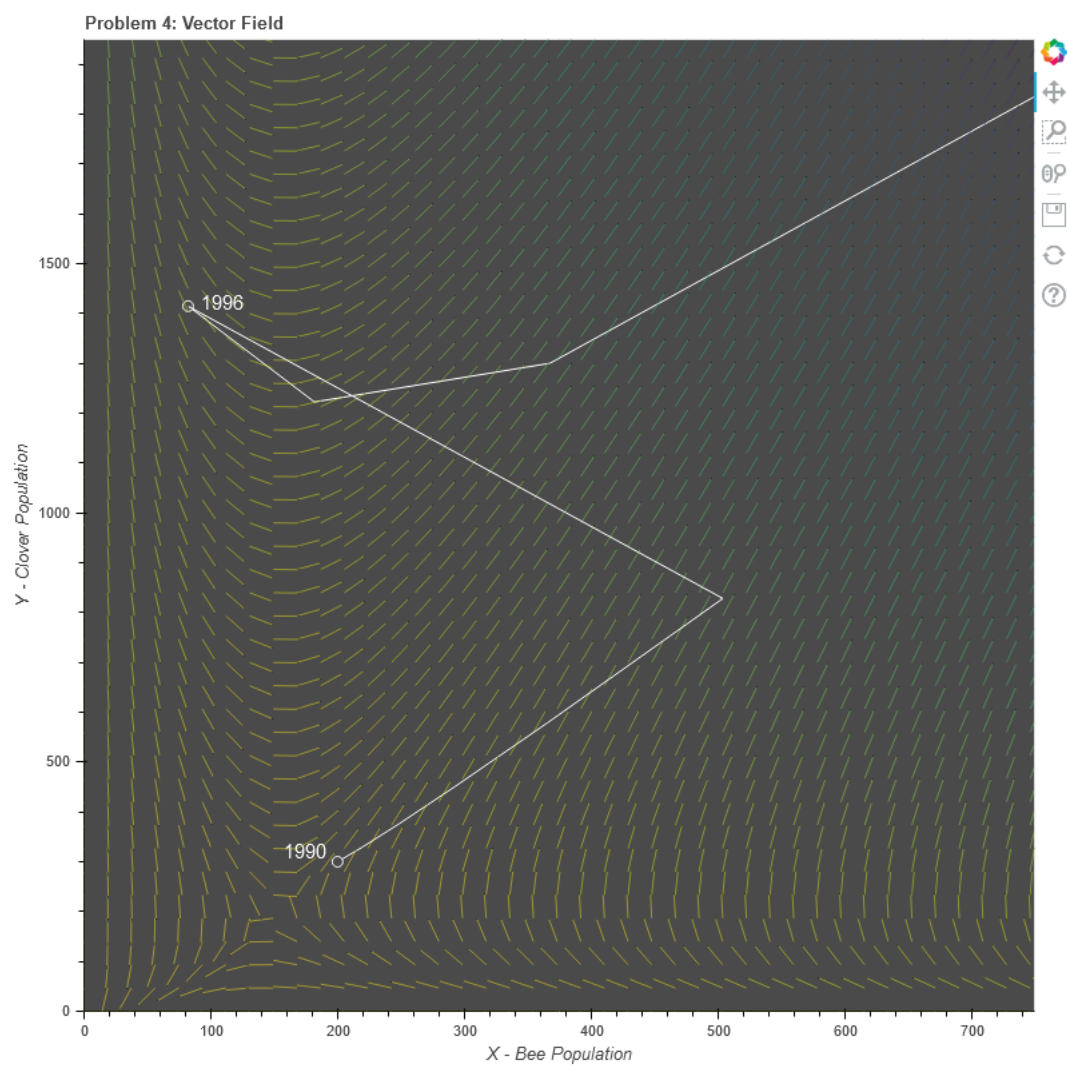
Figure 2: Here the model runs just the same as it did in problem 2 until the year 1996 when the bee population shock is imposed.

	X	Y	ΔX	ΔY
Year				
1990	2.00e+02	3.00e+02	2.00e+01	3.00e+01
1991	2.20e+02	3.30e+02	2.86e+01	4.62e+01
1992	2.49e+02	3.76e+02	4.38e+01	7.42e+01
1993	2.92e+02	4.50e+02	7.32e+01	1.28e+02
1994	3.66e+02	5.79e+02	1.38e+02	2.50e+02
1995	5.04e+02	8.28e+02	3.17e+02	5.86e+02
1996	8.21e+01	1.41e+03	9.97e+01	-1.92e+02
1997	1.82e+02	1.22e+03	1.86e+02	7.77e+01
1998	3.68e+02	1.30e+03	4.04e+02	5.66e+02
1999	7.72e+02	1.87e+03	1.29e+03	2.32e+03
2000	2.06e+03	4.19e+03	8.21e+03	1.60e+04
2001	1.03e+04	2.02e+04	2.05e+05	4.08e+05
2002	2.15e+05	4.28e+05	9.21e+07	1.84e+08
2003	9.23e+07	1.85e+08	1.70e+13	3.41e+13
2004	1.70e+13	3.41e+13	5.81e+23	1.16e+24
2005	5.81e+23	1.16e+24	6.74e+44	1.35e+45
2006	6.74e+44	1.35e+45	9.09e+86	1.82e+87
2007	9.09e+86	1.82e+87	1.65e+171	3.31e+171
2008	1.65e+171	3.31e+171	inf	inf

Table 5: The model evolution for the initial conditions $X_0 = 200$ and $Y_0 = 300$ given the parameters in table 2 with an exogenous shock in to the bee population in the year 1996. The bee population is reduced by a factor of 10 relative to what was predicted in 1996 in the model from problem 1.

In 1998 there are only 368 bees, which is much smaller than the 7,475 bee population from problem 2 at the same year. The clover population is 1,300 which is smaller than the original model prediction in problem 2 by more than a factor of 10. But figure 3 shows that both populations only experience a population decline for a very short period of time. The clover population only declines for one year, and the model never actually gives bees a negative population growth rate outside of the exogenous shock. Both populations quickly recover and by 2008 there are 1.65e+171 bees and 3.31e+171 clover plants.

Problem 5

The most obvious equilibrium that can be seen from the model alone is a trivial equilibrium - when both populations are zero. Nothing changes because both the birth and death terms of the equations are zero. The more interesting question is whether there are non-zero equilibria. By definition, an equilibrium point is when neither population changes. Setting ΔX and ΔY equal to zero and solving for X_n and Y_n) will yield another equilibrium point. First, solving for Y_n:

$$0 = -aX_n + bX_nY_n$$

$$a = bY_n$$

$$Y_n = \frac{a}{b} = \frac{0.2}{0.001} = 200$$

In order to solve for X_n, the clover equation is needed:

$$0 = -cY_n + dX_nY_n$$

$$c = dX_n$$

$$X_n = \frac{c}{d} = \frac{0.3}{0.002} = 150$$

Therefore, 150 bees and 200 clover plants should be an equilibrium point for the system. This can be seen visually by looking at a slightly different the vector field in figure 1. In
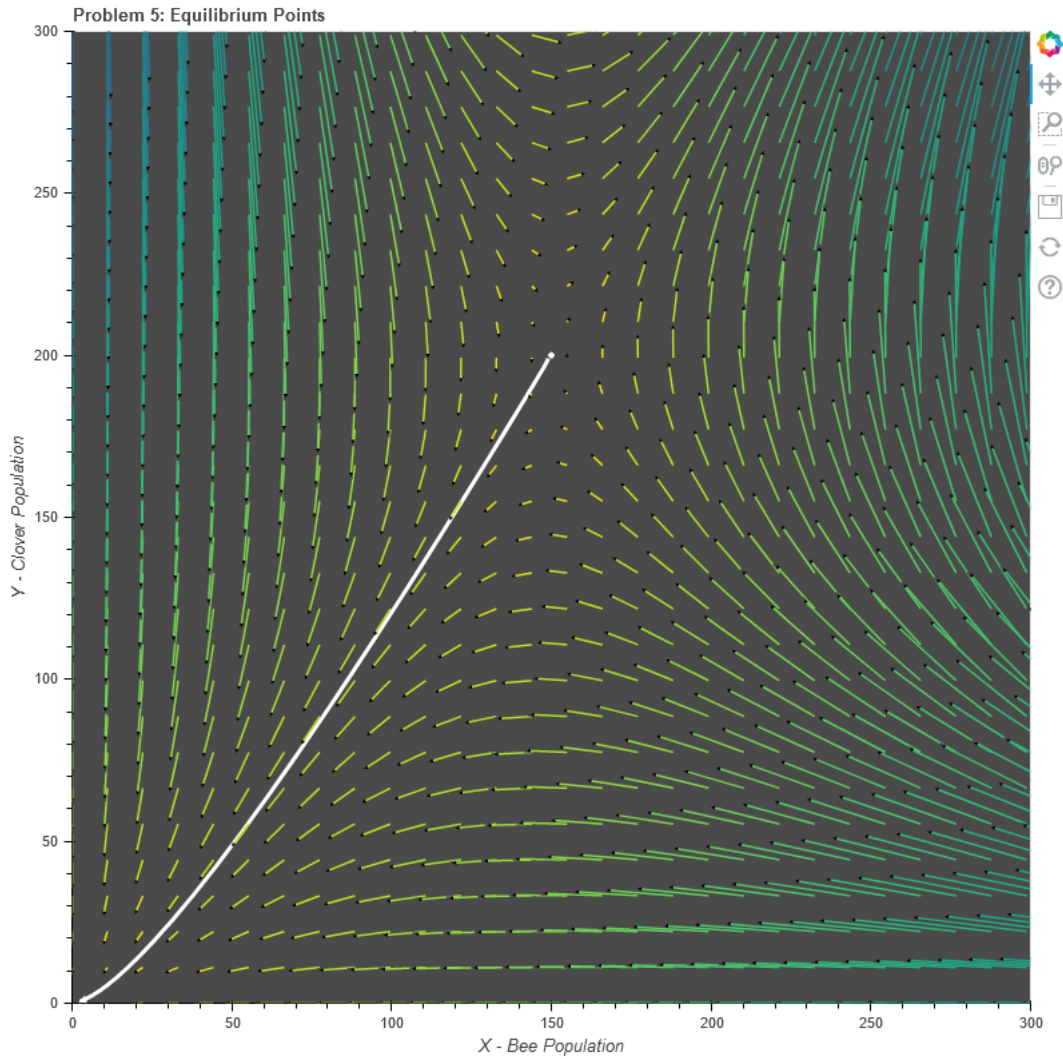

Figure 3: Here the model runs just the same as it did in problem 2 until the year 1996 when the bee population shock is imposed.