Math 360
Project 2 - Symbiosis (Mutualism)

Symbiosis is the interaction of two species whose survival depends upon their mutual cooperation. As an example we might take one species to be the common honey bee and the other to be the clover plant; the bee feeds on the nectar of the plant and simultaneously pollinates the plant. In this project, we will use difference equations to study the change in their populations over time.

We begin our observations by counting the populations of these two species in the Afton prairie preserve in 1990. Let X_n and Y_n denote the populations of the bees and the clover, respectively, after n years.

Problem 1. Assume that the bees have no other source of food besides the clover and that the clover have no other source of pollination besides the bee. Explain why a reasonable model for the changes in population from year to year is

$$\Delta X_n = -aX_n + bX_nY_n$$
$$\Delta Y_n = -cY_n + dX_nY_n$$

for some positive constants a, b, c, d. You should be able to describe in words what the constants a, b, c, d represent.

Problem 2. For this particular prairie field, the constants $a = 0.2, b = 0.001, c = 0.3, d = 0.002$ have been estimated from observations of a few representative plants and bees. It was observed in 1990 that $X_0 = 200$ and $Y_0 = 300$. Use the model to estimate the populations which should have been attained in 1998. Estimate the populations in 2008.

Problem 3. Suppose a bookkeeping error in 1990 has overstated the populations by a factor of 2, that is, suppose it had really been correct that $X_0 = 100$ and $Y_0 = 150$. Use the same model to estimate the populations which would have been attained in 1998. Explain in words why the model does not predict the populations to be half as big as suggested in Problem 2. You should observe that the initial populations can have a dramatic effect on the long-term behavior of the two populations: they can experience either dramatic growth or dramatic decline.

Problem 4. During the winter between 1996 and 1997, the honey bees of north America were largely decimated by an infection by mites. Suppose again that $X_0 = 200$ and $Y_0 = 300$, and that the populations X_n and Y_n change according to our model, except that X_6 is reduced by a factor of 10 from the value predicted by our model. Compute the populations in 1998 and 2008. Are the bees recovering?

Problem 5. Are there populations of the two species which can remain in balance, with neither species' population growing or diminishing? If there is a minor change in the populations from such an equilibrium, would the populations tend to return to the equilibrium or to move further from it?

Problem 6. Try some experiments with other initial values X_0 and Y_0. Can you predict which combinations allow both populations to flourish? Which combinations allow one species to flourish while the other declines? You should be able to identify a pattern of points (X_0, Y_0) in the XY-plane which lead to different long-term behaviors.

Problem 7. Experiments suggest, and some mathematics can prove, that this model allows the populations to grow without bound if X_0 and Y_0 are chosen suitably. This conclusion seems implausible. Suggest improvements to the model chosen to take into account factors which would eventually limit the populations of the two species.