

Problem Set II

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1 Multiplicity of an Einstein Solid

Since the particles are distinguishable the number of unique micro states we can be calculated following formula for computing the multiplicity of an Einstein Solid.

$$\Omega(N, q) = \frac{(q + N - 1)!}{q!(N - 1)!} \quad (1)$$

Then using the sum of the output of the above from 0 to 8 we get

$$\boxed{165} \quad (2)$$

2 Quantum Harmonic Ladder Operators

$$\int \Psi^*(a_+\phi)dx = \int \phi(\hat{a}_-\Phi)^*dx \quad (3)$$

$$\int \Psi^*(a_-\phi)dx = \int \phi(\hat{a}_+\Phi)^*dx \quad (4)$$

$$\hat{a}_\pm \hat{a}_\mp \psi_n = (E_n \mp \frac{\hbar\omega}{2})\psi_n \quad (5)$$

Let $E_n = (n + \frac{1}{2})\hbar\omega$

$$\hat{a}_+ \hat{a}_- \psi_n = (n)\psi_n \quad (6)$$

$$\hat{a}_- \hat{a}_+ \psi_n = (n + 1)\psi_n \quad (7)$$

$$\int_{-\infty}^{\infty} (\hat{a}_- \psi_n)^* (\hat{a}_+ \psi_n) dx = |c_n|^2 \int_{-\infty}^{\infty} |\psi_n| dx = (n + 1) \quad (8)$$

$$\int_{-\infty}^{\infty} (\hat{a}_+ \psi_n)^* (\hat{a}_- \psi_n) dx = |d_n|^2 \int_{-\infty}^{\infty} |\psi_n| dx = (n) \quad (9)$$

$$\hat{a}_+ \psi_n = \sqrt{n+1} \psi_{n+1} \quad (10)$$

$$\hat{a}_- \psi_n = \sqrt{n} \psi_{n-1} \quad (11)$$

$$A_n = \sqrt{n+1} \quad (12)$$

$$B_n = \sqrt{n} \quad (13)$$

Substitue $E_n = (n + \frac{1}{2})\hbar\omega$ back in to get:

$$A_n = \sqrt{\frac{E_n + \frac{1}{2}\hbar\omega}{\hbar\omega} - \frac{1}{2}} \quad (14)$$

$$B_n = \sqrt{\frac{E_n - \frac{1}{2}\hbar\omega}{\hbar\omega} - \frac{1}{2}} \quad (15)$$

Relied heavily on Griffiths Into to QM Pg 44-46.

3 Bound Dirac Delta Well

Assumptions $\lim_{x \rightarrow \pm\infty} \Psi(x) = 0$, $V(x \neq 0) = 0$

Starting from Schrodinger's Equation:

$$\frac{\hbar}{2m} \frac{d^2 \Psi}{dx^2} = E \Psi \quad (16)$$

$$\frac{d^2 \Psi}{dx^2} = k^2 \Psi \quad (17)$$

Where $k = \frac{-2mE}{\hbar}$ since $E < 0$ k will be real in the following general solution.

$$\Psi(x) = A e^{-kx} + B e^{kx} \quad (18)$$

When we apply the Boundary Conditions at $\pm\infty$ we find:

For $x \rightarrow -\infty$

$$\Psi(x) = A e^{-kx} + B e^{kx} \quad (19)$$

$\Rightarrow A = 0$ to keep the equation from blowing up. And for $x \rightarrow \infty$

$$\Psi(x) = A e^{-kx} + B e^{kx} \quad (20)$$

$$\Psi'(0)_{x>0} - \Psi'(0)_{x<0} = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\epsilon} \frac{2m}{\hbar} (V(x) - E) \Psi(x) dx \quad (21)$$

$$Bk e^{\frac{1}{\hbar} x} - Ak e^{-\frac{1}{\hbar} x} = \frac{-2ma}{\hbar^2} \Psi(0) \quad (22)$$

Since $\Psi(x)$ is continuous at $\Psi(0)$:

$$\Psi(0) = A e^{\frac{1}{\hbar} \cdot 0} = B e^{-\frac{1}{\hbar} \cdot 0} \quad (23)$$

$$Bk + Ak = \frac{-2ma}{\hbar^2} B \quad (24)$$

Given that $k = \frac{ma}{\hbar^2} = -\frac{2mE}{\hbar}$

$$\Psi(x) = \begin{cases} B e^{\frac{max}{\hbar^2}} & \text{for } x < 0 \\ B e^{-\frac{max}{\hbar^2}} & \text{for } x < 0 \end{cases} \quad (25)$$

To find B we can leverage the fact the the partical must be between $-\infty$ and ∞ .

$$\int_{-\infty}^{\infty} \Psi^* \Psi = 1 \quad (26)$$

$$\int_{-\infty}^0 B^2 e^{\frac{2max}{\hbar}} + \int_0^{\infty} B^2 e^{-\frac{2max}{\hbar}} = 1 \quad (27)$$

Because the function is symmetric about the y-axis:

$$2 \int_0^{\infty} B^2 e^{-\frac{2max}{\hbar}} dx = 1 \quad (28)$$

$$\int_0^{\infty} B^2 e^{-\frac{2max}{\hbar}} dx = \frac{1}{2} \quad (29)$$

$$\int B^2 e^{-\frac{2max}{\hbar}} dx \Big|_{\infty}^0 - \int B^2 e^{-\frac{2max}{\hbar}} dx \Big|_0 = \frac{1}{2} \quad (30)$$

$$\frac{B^2 \hbar}{2ma} e^{-\frac{2max}{\hbar}} dx \Big|_0 = \frac{1}{2} \quad (31)$$

$$B = \frac{\sqrt{ma}}{\hbar} \quad (32)$$

$$\boxed{\Psi(x) = \begin{cases} \frac{\sqrt{ma}}{\hbar} e^{\frac{max}{\hbar^2}} & \text{for } x < 0 \\ \frac{\sqrt{ma}}{\hbar} & \text{for } x = 0 \\ \frac{\sqrt{ma}}{\hbar} e^{-\frac{max}{\hbar^2}} & \text{for } x < 0 \end{cases}} \quad (33)$$

4 Positive Energy Particle with a $-a\delta$ Well

Given:

$$V_x = -a\delta(x)$$

$$\Psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & \text{for } x < 0 \\ Be^{ikx} & \text{for } x > 0 \end{cases}$$

$$\Psi(0)|_{x<0} = \Psi(0)|_{x>0} \quad (34)$$

$$\cancel{e^{ikx}|_{x=0}}^1 + \cancel{Ae^{-ikx}|_{x=0}}^1 = \cancel{Be^{-ikx}|_{x=0}}^1 \quad (35)$$

$$1 + A = B \quad (36)$$

From P.3 we can use (21) to get:

$$\cancel{ike^{ikx}|_0}^1 + \cancel{-Aike^{-ikx}|_0}^1 - \cancel{Bike^{ikx}|_0}^1 = \frac{2ma}{\hbar^2}\Psi(0) = \cancel{Be^{ikx}|_0}^1 \quad (37)$$

Introduce $\beta = \frac{ma}{\hbar^2 k}$

$$i(1 - A - B) = 2\beta(B) \quad (38)$$

Given that $R = A^*A$, $T = B^*B$ and $R + T = 1$, and $1 + A = B$:

$$\begin{aligned} 1 + A - 2 &= B - 2 \\ A - 1 &= B - 2 \end{aligned} \quad (39)$$

Substituting (39) into (38)

$$i(2 - 2B) = 2\beta B \quad (40)$$

$$B = \frac{1}{1 - i\beta} \quad (41)$$

$$A = \frac{-i\beta}{1 - i\beta} \quad (42)$$

In Terms of Energy:

$$A = \frac{-i\frac{a}{\hbar^2 E}}{1 - i\frac{a}{\hbar^2 E}} \quad (43)$$

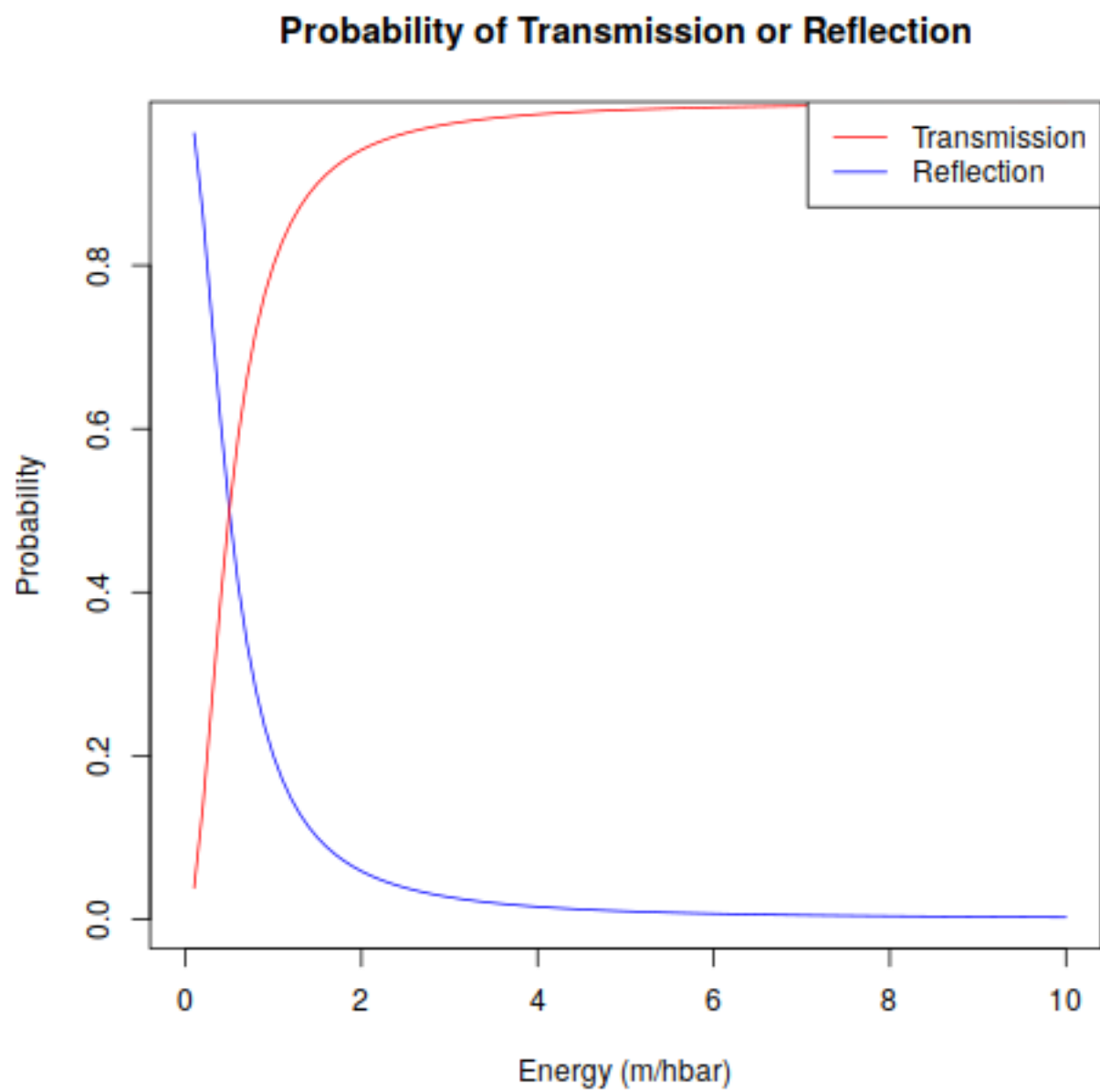


Figure 1: The Probability of a Reflection Decreases as the Energy of the incoming wave goes up. a and $\hbar = 1$

5 Computational

First turn the Schrödinger Equation into a system of first order differential equations:

```
void f( double complex* vect, double complex* res , double x, double energy)
{
    res[1] = k(x, energy)*cexp(I*k(x, energy)*x);
    res[0] = vect[1]; //dx/dt = v_x
}
```

Next this was used in an rk2 integrator function from $x = 1$ to $x = 0$. This should have given me the value of $\Psi(0)$

```
while ( x > 0 )
{
    integrate_rk2(Psi, dx, Psin, x, e);
    x -= dx;
    Psi[0] = Psin[0];
    Psi[1] = Psin[1];
}
```

Then once The value of Ψ was determined I could use that to determine the value of B and then determine the value of the transmission and reflection at various energies. Which could then be plotted. It seems the code I wrote expects the probability of reflection to converge at 50%. That seems none physical.

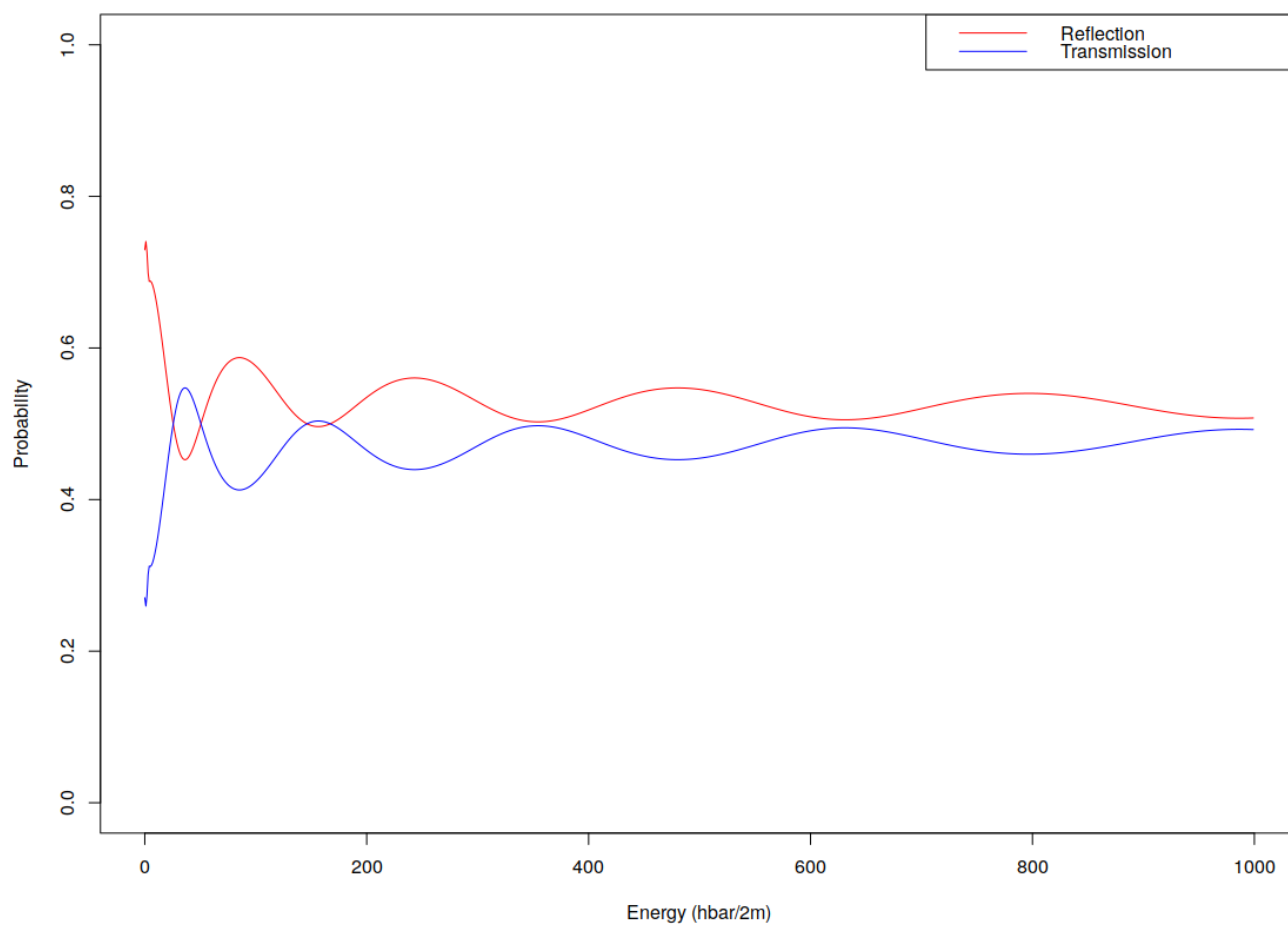


Figure 2: I think I made an Error somewhere as I would expect the Transmission Probability trend up as Energy Increases.