Problem Set I

Ethan Rooney
Department of Physics

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1 Multiplicity of an Einstein Solid

Since the particles are distinguishable the number of unique micro states we can be calculated following formula for computing the multiplicity of an Einstein Solid.

$$\Omega(N,q) = \frac{(q+N-1)!}{q!} \tag{1}$$

$$\Omega(3,8) = \frac{(10)!}{8!} = 90 \tag{2}$$

2 Quantum Harmonic Ladder Operators

$$\int \Psi^*(a_+\phi)dx = \int \phi(\hat{a}_-\Phi)^*dx \tag{3}$$

$$\int \Psi^*(a_-\phi)dx = \int \phi(\hat{a}_+\Phi)^*dx \tag{4}$$

$$\hat{a}_\pm\hat{a}_\mp\psi_n = (E_n \mp \frac{\hbar\omega}{2})\psi_n \tag{5}$$

Let $E_n = (n + \frac{1}{2})\hbar\omega$

$$\hat{a}_+\hat{a}_-\psi_n = (n)\psi_n \tag{6}$$

$$\hat{a}_-\hat{a}_+\psi_n = (n-1)\psi_n \tag{7}$$

$$\int_{-\infty}^{\infty} (\hat{a}_-\psi_n)^*(\hat{a}_+\psi_n)dx = |c_n|^2 \cancelto{1}{\int_{-\infty}^{\infty}|\psi_n|} = (n+1) \tag{8}$$

$$\int_{-\infty}^{\infty} (\hat{a}_+\psi_n)^*(\hat{a}_-\psi_n)dx = |d_n|^2 \cancelto{1}{\int_{-\infty}^{\infty}|\psi_n|} = (n) \tag{9}$$

$$\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1} \tag{10}$$

$$\hat{a}_-\psi_n = \sqrt{n}\psi_{n-1} \tag{11}$$

$$A_n = \sqrt{n+1} \tag{12}$$

$$B_n = \sqrt{n} \tag{13}$$

Substitue $E_n = (n + \frac{1}{2})\hbar\omega$ back in to get:

$$\boxed{A_n = \sqrt{\frac{E_n + 1}{\hbar\omega} - \frac{1}{2}}} \tag{14}$$

$$\boxed{B_n = \sqrt{\frac{E_n}{\hbar\omega} - \frac{1}{2}}} \tag{15}$$

Relied heavily on Griffiths Into to QM Pg 44-46.

3 Bound Dirac Delta Well

Assumptions $lim_{x\to\pm\infty}\Psi(x) = 0$, $V(x \neq 0) = 0$

Starting from Schrodinger's Equation:

$$\frac{\hbar}{2m}\frac{d^2\Psi}{dx^2} = E\Psi \tag{16}$$

$$\frac{d^2\Psi}{dx^2} = k^2\Psi \tag{17}$$

Where $k = \frac{-2mE}{\hbar}$ since $E < 0$ k will be real in the following general solution.

$$\Psi(x) = Ae^{-kx} + Be^{kx} \tag{18}$$

When we apply the Boundary Conditions at $\pm\infty$ we find:
For $x \to -\infty$

$$\Psi(x) = A\cancelto{\infty}{e^{-kx}} + Be^{kx} \tag{19}$$

$\Rightarrow A = 0$ to keep the equation from blowing up. And for $x \to \infty$

$$\Psi(x) = Ae^{-kx} + B\cancelto{\infty}{e^{kx}} \tag{20}$$

$$\Psi'(0)_{x>0} - \Psi'(0)_{x<0} = lim_{\epsilon\to 0}\int_{\epsilon}^{\epsilon}\frac{2m}{\hbar}(V(x) - E)\Psi(x)dx \tag{21}$$

$$Bk\cancelto{1}{e^{kx}} - Ak\cancelto{1}{e^{kx}} = \frac{-2ma}{\hbar^2}\Psi(0) \tag{22}$$

Since $\Psi(x)$ is continuous at $\Psi(0)$:

$$\Psi(0) = A\cancelto{1}{e^{-k0}} = B\cancelto{1}{e^{k0}} \tag{23}$$

$$\cancel{B}k + \cancel{B}k = \frac{-2ma}{\hbar^2}\cancel{B} \tag{24}$$

Given that $k = \frac{ma}{\hbar^2} = -\frac{2mE}{\hbar}$

$$\Psi(x) = \begin{cases} Be^{\frac{max}{\hbar^2}} & \text{for } x < 0 \\ Be^{-\frac{max}{\hbar^2}} & \text{for } x < 0 \end{cases} \tag{25}$$

To find B we can leverage the fact the the partical must be between $-\infty$ and ∞.

$$\int_{-\infty}^{\infty} \Psi^* \Psi = 1 \tag{26}$$

$$\int_{-\infty}^{O} B^2 e^{\frac{2max}{\hbar}} + \int_{0}^{\infty} B^2 e^{\frac{-2max}{\hbar}} = 1 \tag{27}$$

Because the function is symmetric about the y-axis:

$$2\int_{0}^{\infty} B^2 e^{\frac{-2max}{\hbar}} dx = 1 \tag{28}$$

$$\int_{0}^{\infty} B^2 e^{\frac{-2max}{\hbar}} dx = \frac{1}{2} \tag{29}$$

$$\cancelto{0}{\int B^2 e^{\frac{-2max}{\hbar}} dx\Big|_{\infty}} - \int B^2 e^{\frac{-2max}{\hbar}} dx\Big|_{0} = \frac{1}{2} \tag{30}$$

$$\frac{B^2\hbar}{2ma}\cancelto{1}{e^{\frac{-2max}{\hbar}} dx\Big|_{0}} = \frac{1}{2} \tag{31}$$

$$B = \frac{\sqrt{ma}}{\hbar} \tag{32}$$

$$\boxed{\Psi(x) = \begin{cases} \frac{\sqrt{ma}}{\hbar} e^{\frac{max}{\hbar^2}} & \text{for } x < 0 \\ \frac{\sqrt{ma}}{\hbar} & \text{for } x = 0 \\ \frac{\sqrt{ma}}{\hbar} e^{-\frac{max}{\hbar^2}} & \text{for } x < 0 \end{cases}} \tag{33}$$

4 Positive Energy Particle with a $-a\delta$ Well

Given:

$$V_x = -a\delta(x)$$

$$\Psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & \text{for } x < 0 \\ Be^{ikx} & \text{for } x > 0 \end{cases}$$

$$\Psi(0)|_{x<0} = \Psi(0)|_{x>0} \tag{34}$$

$$\cancelto{1}{e^{ikx}|_{x=0}} + A\cancelto{1}{e^{-ikx}|_{x=0}} = B\cancelto{1}{e^{-ikx}|_{x=0}} \tag{35}$$

$$1 + A = B \tag{36}$$

From P.3 we can use (21) to get:

$$ik\cancelto{1}{e^{ikx}|_0} + -Aik\cancelto{1}{e^{-ikx}|_0} - Bik\cancelto{1}{e^{ikx}|_0} = \frac{2ma}{\hbar^2}\Psi(0) = B\cancelto{1}{e^{ikx}|_0} \tag{37}$$

Introduce $\beta = \frac{ma}{\hbar^2 k}$

$$i(1 - A - B) = 2\beta(B) \tag{38}$$

Given that $R = A^*A$, $T = B^*B$ and $R + T = 1$, and $1 + A = B$:

$$\begin{aligned} 1 + A - 2 &= B - 2 \\ A - 1 &= B - 2 \end{aligned} \tag{39}$$

Substituting (39) into (38)

$$i(\cancel{2} - \cancel{2}B) = \cancel{2}\beta B \tag{40}$$

$$B = \frac{1}{1 - i\beta} \tag{41}$$

$$A = \frac{-i\beta}{1 - i\beta} \tag{42}$$

In Terms of Energy:

$$A = \frac{-i\frac{a}{\hbar 2E}}{1 - i\frac{a}{\hbar 2E}} \tag{43}$$

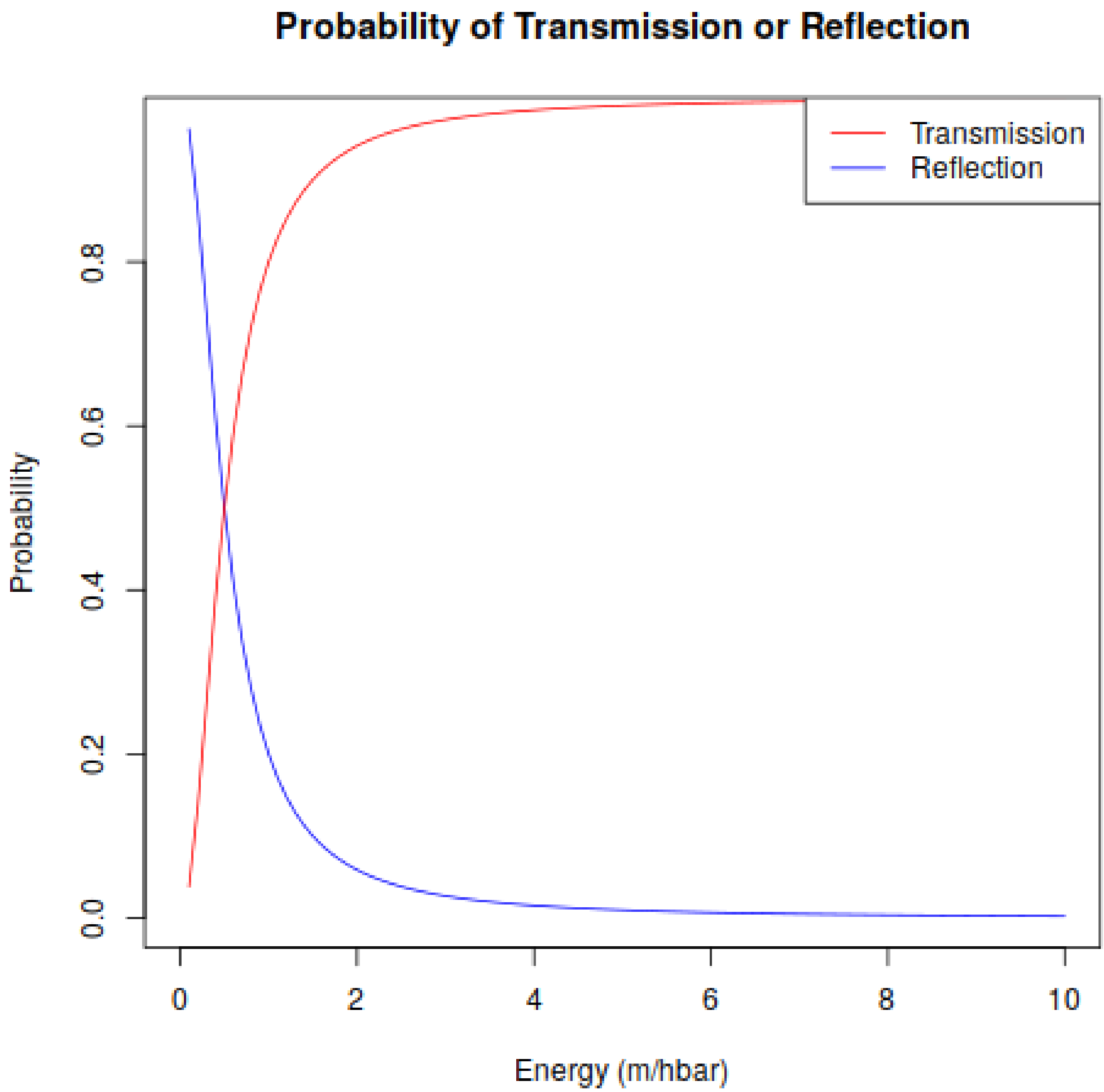

Figure 1: The Probability of a Reflection Decreases as the Energy of the incoming wave goes up. a and $\hbar = 1$

5 Computational

First turn the Schrödinger Equation into a system of first order differential equations:

```
void f( double complex* vect, double complex* res , double x, double energy)
{
  res[1] = k(x, energy)*cexp(I*k(x, energy)*x);
  res[0] = vect[1]; //dx/dt = v_x
}
```

Next this was used in an rk2 integrator function from $x = 1$ to $x = 0$. This should have given me the value of $\Psi(0)$

```
while ( x > 0 )
{
  integrate_rk2(Psi, dx, Psin, x, e);
  x -= dx;
  Psi[0] = Psin[0];
  Psi[1] = Psin[1];
}
```

Then once The value of Ψ was determined I could use that to determine the value of B and then determine the value of the transmission and reflection at various energies. Which could then be plotted. It seems the code I wrote expects the probability of reflection to converge at 50%. That seems none physical.

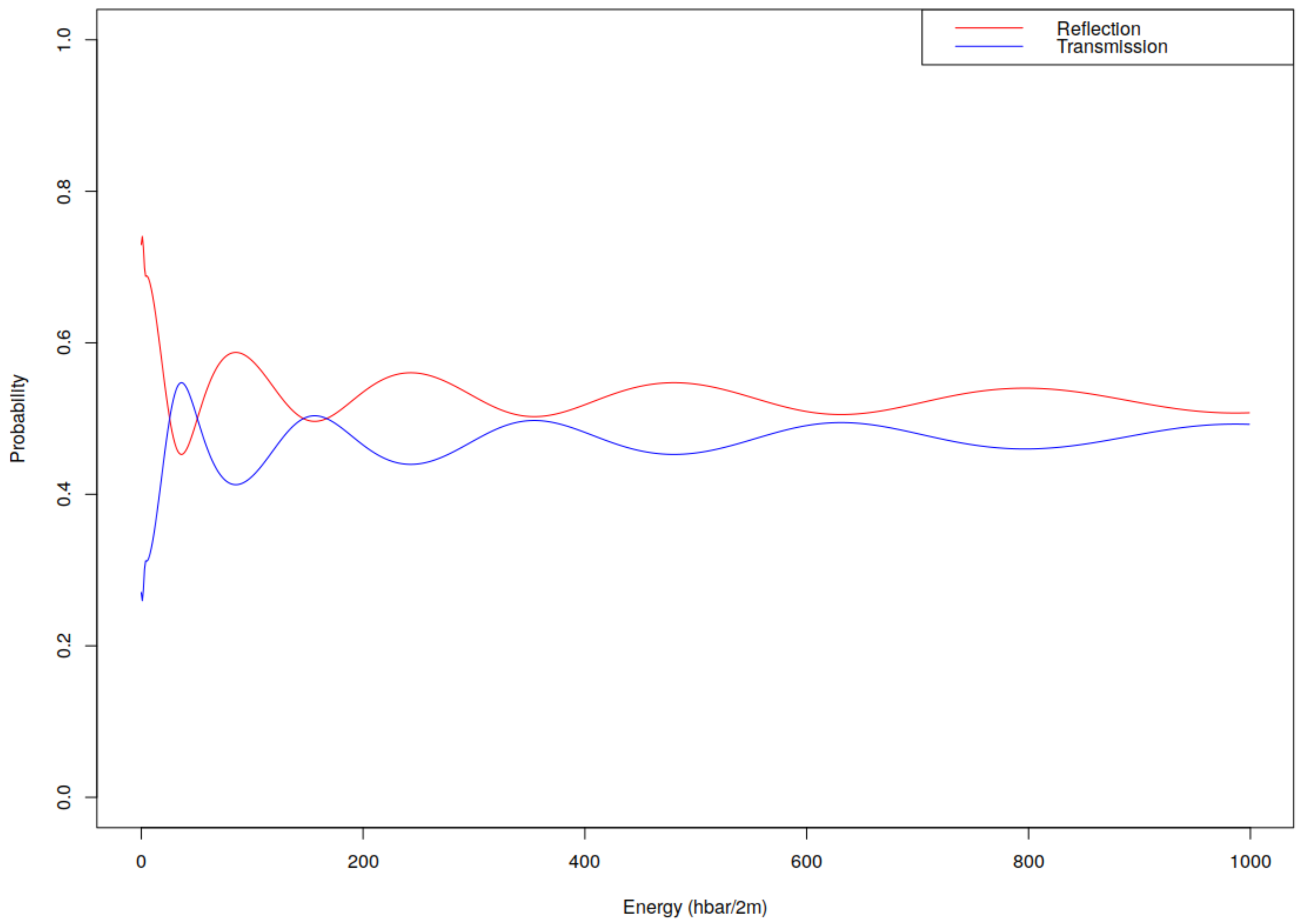

Figure 2: I think I made an Error somewhere as I would expect the Transmission Probability trend up as Energy Increases.