The Scattering of Fast Electrons by Atomic Nuclei

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then the conditions imply that ψ shall vanish unless

$$\frac{\varepsilon^2}{R_{12}} \ll \text{[Kinetic energy of the electrons].}$$

It is clear, therefore, that over all that part of $\mathbf{x}_2$ space which contributes anything to the integral in (2), ψ must satisfy the wave equation for two electrons in free space *with no interaction.* The equation (2) follows in the same way as the equation (1) for one electron.

It appears, therefore, that the interpretation of the two-electron equation is consistent, and further that all results of physical importance can be obtained by using one time only, putting $t_1 = t_2$ in the wave equation. It would, however, be of interest to obtain a solution of the wave equation containing the two times, so that we could see what happens in the case excluded above, when one experiment is in the absolute future of the other, and so may be affected by it.

The Scattering of Fast Electrons by Atomic Nuclei.

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Section 1.—The hypothesis that the electron has a magnetic moment was, as is well known, first introduced to account for the duplexity phenomena of atomic spectra. More recently, however, Dirac has succeeded in accounting for these same phenomena by the introduction of a modified wave equation, which conforms both to the principle of relativity and to the general transformation theory. Formally, at least, on the new theory also, the electron has a magnetic moment of $\varepsilon h/mc$, but when the electron is in an atom we cannot observe this magnetic moment directly; we can only observe the moment of the whole atom, or, of course, the splitting of the spectral lines, which we may say is " caused " by this moment. The question arises, has the *free* electron " really " got a magnetic moment, a magnetic moment that we can by any conceivable experiment observe ? The question is not so simple as it might seem, because a magnetic moment $\varepsilon h/mc$ can never be observed directly, *e.g.*, with a magnetometer ; there is always an uncertainty in the external electromagnetic field, due to the uncertainty in the position and velocity of the electron, and this uncertainty is greater than the effect of the electron magnet which we are trying to observe.* Our only hope of observing the moment of a free electron is to obtain a "polarised" beam, in which all the spin axes are pointing in the same direction, or at any rate more in one direction than another. The obvious method of obtaining such a polarised beam is a Stern-Gerlach experiment, but here again the Uncertainty Principle shows that this is impossible*; in fact, it appears certain that no experiment based on the classical idea of an electron magnet can ever detect the magnetic moment of the electron.

We are, however, unwilling to give up altogether the idea of the direction of the spin axis of the free electron, because of the form that the solution of the wave equation has for this case. Whether we consider an infinite plane wave, or a wave packet, there are, in the solution, two arbitrary constants A, B, which are just enough to determine a "spin" direction. Further, it has been shown by Darwin† that the electromagnetic field due to a wave packet can be separated formally into two parts, the one due to the charge and current, and the other due to the magnetic moment of the electron, which points in a definite direction and is determined by A, B. As we have pointed out, this second part cannot be observed, because it is less than the uncertainty in the first; but nevertheless, we can associate formally a direction of the spin axis with any given solution of the wave equation.

Now, have these constants A, B, this direction of the spin axis, any physical meaning? Suppose, for example, a wave packet were to fall on a nucleus; would the scattered intensity depend on the A and B of the initial wave packet? This can only be decided by a mathematical investigation, to which the greater part of this paper is devoted. If the scattered intensity does not depend on A, B, that would be very satisfactory; we should consider A and B to be constants used in the mathematics, but with no physical meaning, and the spin of a free electron to be something non-existent. However, we shall find that the scattered intensity does depend on A and B, so that the spin direction has some meaning after all. Suppose an electron, about whose spin direction we know nothing, falls on a nucleus and is scattered through a given angle; we now know that its spin axis is more likely to be in one direction than another. Suppose an unpolarised beam, in which the spin axes are pointing in all directions at random, falls on a target and is scattered; the scattered beam is partly polarised; more spin axes point in one direction than another; and this

* These arguments are due to Prof. Niels Bohr, and are discussed further in an appendix.

† C. G. Darwin, 'Roy. Soc. Proc.,' A, vol. 120, p. 631 (1928).

polarisation could be detected by letting the scattered beam fall on a second target. Since the beam is polarised it will not be scattered in the same way as an unpolarised beam; actually we shall find that the scattering is asymmetrical about the direction in which the beam falls on the second target, and this could be detected experimentally.

In this paper we shall investigate the scattering of fast electrons by atomic nuclei, using the wave equation of Dirac. As well as investigating the polarisation, we shall obtain a formula* for the scattering of an unpolarised beam, which is to replace the Rutherford formula for fast electrons. In Section 2 we shall obtain certain general results for scattering by a field of force $V(r)$. In Section 3 we shall investigate the scattering by a Coulombian field of force, and determine the scattering law and the polarisation to be expected. The mathematics can be interpreted without difficulty, since the energy is an integral of the equations of motion, and we are not troubled by transitions to negative energy. We may emphasise once again that we do not want to know how the spin axis is turned when the electron is deflected, so much as how the direction of the spin axis affects the probability of the electron being scattered in a given direction, as it is this last that will be observable experimentally.

Section 2.—We consider the scattering of an infinite plane wave by a centre of force $V(r)$. If we were working with Schrödinger electrons, the wave equation would be

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}(W + V)\,\psi = 0, \tag{1}$$

and we should have to find a solution ψ, such that for large r

$$\psi \sim I + S \,.\, u(\theta\ \phi), \tag{1.1}$$

where I is written for $\exp(2\pi ipz/h)$ and represents the incident wave, and S for $\exp(2\pi ipr/h)/r$ to represent the scattered wave. Then, if a beam of electrons were to fall on a foil, say, of thickness t and containing n nuclei per unit volume, the proportion of the original beam scattered in a given solid angle will be

$$nt\,|u(\theta\ \phi)|^2 \sin\theta\, d\theta\, d\phi.$$

With Dirac electrons, we have, of course, four components of the wave function, ψ_1, ψ_2, ψ_3, ψ_4. The wave equation is the familiar wave equation of Dirac.†

$$[p_0 + V(r)/c - 2\pi ih\,(\sigma, \text{grad}) + \rho_3\, mc]\,\psi = 0, \tag{2}$$

* This formula, of course, includes "Relativity correction" as well as "spin correction," but does not include the effect of radiative force.

† 'Roy. Soc. Proc.,' A, vol. 117, p. 610 (1928).

and, as before, we want a solution representing an incident plane wave falling on the nucleus and a scattered wave—a solution ψ therefore, such that, for large r

$$\psi \sim a_\lambda \mathrm{I} + \mathrm{S}u_\lambda(\theta\phi). \tag{3}$$

The a_λ are constants, but not all arbitrary constants, for if any two are given, the other two are known.† We set

$$a_3 = \mathrm{A}, \qquad a_4 = \mathrm{B},$$

where A and B are arbitrary complex constants. Then we have

$$a_1 = -\mathrm{A}p/(p_0 + mc), \qquad a_2 = \mathrm{B}p/(p_0 + mc).$$

The current represented by the incident wave is equal to $\sum_{\lambda=1}^{4} |a_\lambda|^2$, which is proportional to $\mathrm{AA}^* + \mathrm{BB}^*$. In the same way, the current scattered depends on u_3, u_4 only.

To interpret our formulæ, therefore, we choose A, B in such a way that

$$\mathrm{AA}^* + \mathrm{BB}^* = 1.$$

Then, if $nt\,\mathrm{P}\sin\theta\,d\theta\,d\phi$ is the proportion of the original beam scattered in a given solid angle, we have

$$\mathrm{P} = |u_3(\theta\phi)|^2 + |u_4(\theta\phi)|^2. \tag{4}$$

The constants A, B determine also the polarisation, or direction of the spin axis, of the incident electrons. When we speak of the direction of the spin, we shall mean the direction referred to axes with respect to which the electron is at rest; it is this that will be distributed equally in all directions in an unpolarised beam. If χ, ω are the spherical polar angles of the spin direction, then‡

$$-\frac{\mathrm{B}}{\mathrm{A}} = \cot \tfrac{1}{2}\chi \,.\, e^{i\omega}. \tag{5}$$

In the same way, u_4/u_3 will determine the polarisation of the electrons scattered in any direction. To determine the proportion scattered from an unpolarised beam, we must average P of equation (4) over all values of χ, ω.

If we find ψ_3 and ψ_4 for the two cases $\mathrm{A} = 1$, $\mathrm{B} = 0$ and $\mathrm{A} = 0$, $\mathrm{B} = 1$,

† Darwin, 'Roy. Soc. Proc.,' A, vol. 118, p. 654 (1928).
‡ Darwin, 'Roy. Soc. Proc.,' A, vol. 120, p. 631 (1928).

then we can form the general solution (3) by superposition of these two. We shall show in the next section that these two solutions are of the form

$$\left.\begin{aligned}\psi_3 &\sim \mathrm{I} + \mathrm{S}f(\theta) \\ \psi_4 &\sim \qquad \mathrm{S}g(\theta)\, e^{i\phi}\end{aligned}\right\}, \tag{6.1}$$

and

$$\left.\begin{aligned}\psi_3 &\sim \quad - \mathrm{S}g(\theta)\, e^{-i\phi} \\ \psi_4 &\sim \mathrm{I} + \mathrm{S}f(\theta)\end{aligned}\right\}, \tag{6.2}$$

where $f(\theta)$, $g(\theta)$ are functions of θ (not ϕ) which depend on the form of $\mathrm{V}(r)$.

By superposition of these two, we have at once the general solution of the form (3), with

$$u_3(\theta\phi) = \mathrm{A}f - \mathrm{B}g\, e^{-i\phi},$$
$$u_4(\theta\phi) = \mathrm{B}f + \mathrm{A}g\, e^{i\phi}.$$

Hence we have

$$|u_3|^2 + |u_4|^2 = (|\mathrm{A}|^2 + |\mathrm{B}|^2)(|f|^2 + |g|^2) + (fg^* - f^*g)(-\mathrm{AB}^*e^{i\phi} + \mathrm{A}^*\mathrm{B}e^{-i\phi}), \tag{7}$$

so, if $nt\,\mathrm{P}\sin\theta\, d\theta\, d\phi$ is the proportion of the beam scattered in the solid angle $\sin\theta\, d\theta\, d\phi$, then we see from (4), (5) that

$$\mathrm{P} = |f|^2 + |g|^2 + \mathrm{D}\sin\chi\sin(\omega - \phi), \tag{8}$$

where

$$\mathrm{D}(\theta) = i(fg^* - f^*g)$$

and χ, ω determine the direction of the spin axis of the incident electrons.

To obtain the number $\overline{\mathrm{P}}$ scattered from an unpolarised beam, we must average over all directions of the spin axis; we obtain

$$\overline{\mathrm{P}} = |f|^2 + |g|^2. \tag{9}$$

Unless, however, $\mathrm{D}(\theta) = 0$ for the angle of scattering considered, the function P will depend on the polarisation of the incident beam; and if the incident beam is unpolarised, the scattered beam will not be. We shall be able to detect this polarisation by scattering the beam again by a second nucleus.

Before considering this double scattering in detail, it will be well to point out an obvious trap. On the old Quantum Theory, one used to say that a magnet, such as an electron magnet, must orientate itself either parallel or anti-parallel to a magnetic field. Such an assumption would in our case lead to inconsistent results. For from equation (8) we see that electrons whose spin axes lie parallel and anti-parallel to the direction of motion are scattered in the same way as an unpolarised beam. Suppose then we always had a weak magnetic field in the direction of motion, before and after scattering; then the scattering would always be normal, and the double scattering experiment would give a null result, which is contrary to the result of the following calculation. The fallacy is probably this, that we must not think of the axes of the electron magnets as lying parallel and anti-parallel to the field, but as precessing round it.

We shall now consider the double scattering experiment. A beam of electrons LT_1 falls on a target T_1 and is scattered. A second target is placed at T_2 so that the electrons scattered through an angle θ_1 in the plane of the paper (for which $\phi = 0$) undergo a second scattering. We observe the number of electrons scattered by T_2 at a given angle θ_2. If the beam T_1T_2 is polarised, the second scattering will not be symmetrical about T_1T_2; the number scattered in the directions T_2M, T_2M' will not be the same.

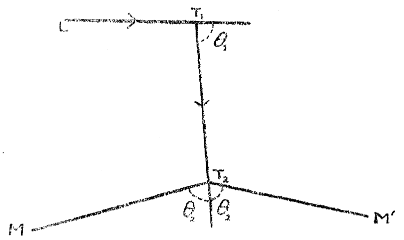

FIG. 1.

Suppose we represent the initial beam LT_1 by

$$\psi_3 = AI \qquad \psi_4 = BI.$$

We shall, of course, have to average over all spin directions later. The direction of the spin axis of the scattered beam T_1T_2 is determined, according to (5), by the ratio of the amplitudes of the two components of the wave function of the scattered beam, namely*

$$Af_1 - Bg_1, \qquad Ag_1 + Bf_1.$$

We now rotate our axes through an angle θ_1 so that T_1T_2 becomes the axis of z and we can represent the beam of electrons T_1T_2 by

$$\psi_3 = A_1I \qquad \psi_4 = B_1I$$

with†

$$A_1 = (Af_1 - Bg_1) \cos \tfrac{1}{2}\theta_1 + (Ag_1 + Bf_1) \sin \tfrac{1}{2}\theta_1$$
$$B_1 = (Ag_1 + Bf_1) \cos \tfrac{1}{2}\theta_1 - (Af_1 - Bg_1) \sin \tfrac{1}{2}\theta_1.$$

We can now obtain the number of electrons scattered by the second target T_2 in a given direction $\theta_2\phi_2$; we must insert these values of A_1, B_1 for A, B,

* f_1 is written for $f(\theta_1)$, etc.

† Darwin, 'Roy. Soc. Proc.,' A, vol. 118, p. 654 (1928).

in (7) and average over all directions of the spin axis of the initial beam LT_1. We are interested primarily in the asymmetry in the scattering about the line T_1T_2. For given θ_1, θ_2, therefore, but variable ϕ_2, a straightforward calculation shows that the number scattered per unit solid angle is proportional to

$$1 - \delta \cos \phi_2, \tag{10}$$

where

$$\delta = 2 \frac{(f_1 g_1^* - f_1^* g_1)(f_2 g_2^* - f_2^* g_2)}{(f_1 f_1^* + g_1 g_1^*)(f_2 f_2^* + g_2 g_2^*)}.$$

The greatest asymmetry, therefore, will be found in the directions TM, TM′, in the plane of the paper. In the plane through T_1T_2 perpendicular to the plane of the paper, the scattering *is* symmetrical about T_1T_2. It was in this plane that asymmetry was looked for by Cox, McIlwraith and Kurrelmeyer,‡ and the asymmetry found by them must be due to some other cause.

We must now show that we can obtain solutions of the wave equation of the form (6.1), (6.2), and obtain expressions for f and g. We shall first consider Schrödinger electrons.§ The general solution of (1) is

$$\Sigma\, a_k \mathrm{P}_k (\cos\theta)\, \mathrm{L}_k(r)$$

where L_k is the bounded solution of

$$\frac{d^2\mathrm{L}}{dr^2} + \frac{2}{r}\frac{d\mathrm{L}}{dr} + \left[\frac{8\pi^2 m}{h^2}(\mathrm{E} + \mathrm{V}) - \frac{k(k+1)}{r^2}\right]\mathrm{L} = 0. \tag{11}$$

For large r, L_k has the form

$$\mathrm{L}_k \sim r^{-1} \cos (2\pi pr/h + \eta_k{}^0).$$

Remembering that

$$e^{ir\cos\theta} = \left(\frac{\pi}{2r}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} (2k+1)\, i^k \mathrm{P}_k(\cos\theta)\, \mathrm{J}_{k+\frac{1}{2}}(r)$$

we see that the solution of (1) of the form (2) is

$$i \sum_{k=0}^{\infty} (2k+1)\, e^{i\eta_k{}^0 + ik\pi}\, \mathrm{P}_k(\cos\theta)\, \mathrm{L}_k(r)$$

with

$$u(\theta\,\phi) = \frac{h}{2\pi p} \sum_{k=0}^{\infty} (k + \tfrac{1}{2}) \left[e^{2i\eta_k{}^0 + \frac{2k+1}{2}\pi i} + 1 \right] \mathrm{P}_k(\cos\theta).$$

The general solution in spherical harmonics of Dirac's wave equation (2)

† 'Proc. Nat. Ac. Sci.,' vol. 14, p. 545 (1928).

‡ *Cf.* Faxen and Holtmark, 'Z. Physik,' vol. 45, p. 307 (1927); Mott, 'Roy. Soc. Proc.,' A, vol. 118, p. 542 (1928); Gordon, 'Z. Physik,' vol. 48, p. 187 (1928).

has been given by Darwin for the case of discrete energy values, and his analysis is immediately applicable to our case. A set of solutions are :—

$$
\begin{array}{lll}
(\alpha) & \psi_3 = (k+1)\,\mathrm{P}_k \mathrm{G}_k & \psi_4 = -\,\mathrm{G}_k \mathrm{P}_k^{\,1} \\
(\beta) & \psi_3 = k\,\mathrm{P}_k \mathrm{G}_{-k-1} & \psi_4 = \mathrm{G}_{-k-1}\,\mathrm{P}_k^{\,1} \\
(\gamma) & \psi_3 = \mathrm{P}_k^{\,1} \mathrm{G}_k & \psi_4 = (k+1)\,\mathrm{G}_k \mathrm{P}_k \\
(\delta) & \psi_3 = -\,\mathrm{G}_{-k-1}\,\mathrm{P}_k^{\,1} & \psi_4 = \mathrm{G}_{-k-1}\,\mathrm{P}_k.
\end{array}
$$

Here P_k is the ordinary Legendre coefficient $\mathrm{P}_k(\cos\theta)$ (not Darwin's notation) and $\mathrm{P}_k^{\,1}$ is $\sin\theta \dfrac{d}{d(\cos\theta)} \mathrm{P}_k(\cos\theta)\, e^{i\phi}$. $\mathrm{G}_k(r)$ is the bounded solution of the pair of equations

$$
\left.
\begin{aligned}
\frac{2\pi}{h}\left(p_0 + \frac{\varepsilon \mathrm{V}}{c} + mc\right)\mathrm{F} + \frac{d\mathrm{G}}{dr} - \frac{k}{r}\,\mathrm{G} &= 0. \\
-\frac{2\pi}{h}\left(p_0 + \frac{\varepsilon \mathrm{V}}{c} - mc\right)\mathrm{G} + \frac{d\mathrm{F}}{dr} + \frac{k+2}{r}\,\mathrm{G} &= 0.
\end{aligned}
\right\} \tag{13}
$$

Now, G_k has the asymptotic form

$$
\mathrm{G}_k \sim r^{-1} \cos(2\pi p r/h + \eta_k) \tag{14}
$$

Hence, in the same way as for Schrödinger electrons, we can form a solution representing an incident wave and a scattered wave. From (α) and (β) we see that a solution with the asymptotic form (6.1) is

$$
\left.
\begin{aligned}
\psi_3 &= i \sum_{k=0}^{\infty} \{(k+1)\, e^{i\eta_k}\, \mathrm{G}_k + k e^{i\eta_{-k-1}}\, \mathrm{G}_{-k-1}\} (-)^k\, \mathrm{P}_k(\cos\theta) \\
\psi_4 &= i \sum_{k=0}^{\infty} \{-\, e^{i\eta_k}\, \mathrm{G}_k + e^{i\eta_{-k-1}}\, \mathrm{G}_{-k-1}\} (-)^k\, \mathrm{P}_k^{\,1}(\cos\theta)\, e^{i\phi},
\end{aligned}
\right\} \tag{15}
$$

and that

$$
\left.
\begin{aligned}
f(\theta) &= \frac{h}{2\pi p} \cdot \tfrac{1}{2} \sum_{0}^{\infty} i\, \{(k+1)(e^{2i\eta_k + ki\pi} + 1) + k\,(e^{2i\eta_{-k-1} + ki\pi} + 1)\}\, \mathrm{P}_k(\cos\theta) \\
g(\theta) &= \frac{h}{2\pi p}\, \tfrac{1}{2} \sum_{0}^{\infty} i\, \{-\,(e^{2i\eta_k + ki\pi} - 1) + (e^{2i\eta_{-k-1} + ki\pi} - 1)\}\, \mathrm{P}_k^{\,1}(\cos\theta)
\end{aligned}
\right\} \tag{16}
$$

In an exactly similar way from (γ) and (δ), we can construct a solution of the form (6.2), with the same f and g.

There is no difficulty in justifying these processes mathematically, provided that the series (12), (16) converge absolutely. And they do converge absolutely, if $\mathrm{V}(r) \to 0$ faster than $1/r^2$, as may be seen by solving equation (11) for very large k, when V may be considered as a perturbation. We shall consider the case of a Coulombian field in the next section.

Section 3.—We shall now consider the case that is of greatest interest, namely, the scattering by an atomic nucleus with inverse square law field, such that

$$V(r) = Z\varepsilon^2/r.$$

We know that, for Schrödinger electrons, with neglect of relativity and spin, the scattering obeys the Rutherford law. It is interesting to compare the second order wave equation for Dirac electrons* with the Schrödinger equation, and to see what are the order of the deviations to be expected from Rutherford scattering. This second order equation is

$$\left[-\left(p_0 + \frac{Z\varepsilon^2}{cr}\right)^2 - 4\pi^2 h^2 \nabla^2 + m^2c^2 + \frac{2\pi i \varepsilon^2 h}{cr^3}\,\rho_1\,(\sigma\, r)\right]\psi = 0,$$

where cp_0 is equal to the energy E of the electron, including the rest mass.

The order of magnitude of the various terms is best seen if we take for our unit of length $1/2\pi$ times the de Broglie wave-length in free space, namely

$$h/2\pi p = h/2\pi\,(p_0{}^2 - m^2c^2)^{\frac{1}{2}} = \frac{h}{2\pi m V}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}.$$

The wave equation then becomes

$$\left[\nabla^2 + 1 + \frac{2\mu\alpha}{r} + \frac{\alpha^2}{r^2} - i\frac{\alpha}{r^3}\,\rho_1\,(\sigma r)\right]\psi = 0, \tag{17}$$

where

$$\alpha = \frac{2\pi Z\varepsilon^2}{hc} = \frac{Z}{137}, \quad \mu = \frac{p_0}{\sqrt{p_0{}^2 - m^2c^2}} = \frac{c}{v}.$$

The last three terms inside the square bracket may be said to "cause" the scattering. For small velocities of the incident electron, it is clear that the term $2\mu\alpha/r$ is much larger than the other terms, and therefore the scattering is approximately inverse square. But for velocities comparable with the velocity of light, μ tend to unity, and so the effect of the "spin" term, $i\alpha/r^3 . \rho_1 (\sigma r)$, will be of the same order as the effect of the inverse square law term. For light nuclei, α is very much smaller than unity, and therefore the "relativity" term α^2/r^2, which is "responsible" for the fine structure of atomic spectral lines, has only a small effect on the scattering.

We can obtain a solution of equation (17) of the form (6.1) by the method of Born† and Wentzel.‡ The method yields a solution of the form

$$\psi^{(0)} + \alpha\psi^{(1)} + \alpha^2\psi^{(2)} + \dots,$$

* Dirac, 'Roy. Soc. Proc.,' vol. 117, p. 610 (1928).

† 'Z. Physik,' vol. 38, p. 803 (1926).

‡ 'Z. Physik,' vol. 40, p. 590 (1927). We should have to use an 'Abschirmungsfeld.'

with $\psi^{(0)}$ representing the incident wave, and the other terms the scattered wave. Such a method is only convenient for the evaluation of $\psi^{(1)}$; the author has actually evaluated $\psi^{(1)}$ by this method, and the calculation provides a useful check upon the subsequent work. For light nuclei, for which α is small, this first approximation would probably be sufficient; but the interest of $\psi^{(2)}$ lies in this, that to the first order of approximation f and g turn out to be real, and there is therefore no polarisation to this order. We shall therefore return to formula (15) and from it evaluate ψ as far as $\psi^{(2)}$.

In our subsequent work we shall take the unit of length to be $h/2\pi p$.

The equations (13) have been solved for F_k and G_k by Darwin, and by Gordon*; for continuous energy values Gordon's solution is more suitable. We introduce the following notation

$$q = \frac{\alpha E/mc^2}{\sqrt{(E/mc^2)^2 - 1}} = \frac{2\pi\varepsilon^2}{hv},$$

$$q' = \frac{\alpha}{\sqrt{(E/mc^2)^2 - 1}} = \frac{2\pi\varepsilon^2}{hv}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$$

$$\rho = \sqrt{k^2 - \alpha^2}, \qquad \alpha = \frac{2\pi Z\varepsilon^2}{hc},$$

v is the classical "velocity" of the particle defined by

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}. \tag{18}$$

With the usual notation for generalised hypergeometric series, we write

$$F(\alpha;\ \beta;\ x) = 1 + \frac{\alpha}{1!\ \beta}x + \frac{\alpha(\alpha+1)}{2!\ \beta(\beta+1)}x^2 + \dots.$$

The asymptotic expansion of this function for large x is well known.† We require the first term only; for pure imaginary x we have

$$\left.\begin{array}{ll} F(\alpha;\ \beta;\ x) \sim \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(-x)^{-\alpha}, & \\ \text{or} & \\ F(\alpha;\ \beta;\ x) \sim \dfrac{\Gamma(\beta)}{\Gamma(\alpha)}e^x x^{\alpha-\beta} & \\ |\arg(-x)| < \pi \quad |\arg x| < \pi & \end{array}\right\}, \tag{19}$$

according as the real part of $-2\alpha + \beta$ is greater or less than zero.

* Darwin, 'Roy. Soc. Proc.,' A, vol. 118, p. 654 (1928); Gordon, 'Z. Physik,' vol. 48, p. 11 (1928).

† *Cf.* for example, Gordon, 'Z. Physik,' vol. 48, p. 187 (1928).

With this notation we have* for G_k,

$$G_{-k-1} = N\left[\frac{e^{-\frac{1}{2}\pi i\rho}\, c_k \zeta_k}{\Gamma(\rho + 1 + iq)} + \frac{e^{+\frac{1}{2}\pi i\rho}\, c_k' \zeta_k'}{\Gamma(\rho + 1 - iq)}\right],$$

where

$$\zeta_k = \frac{\Gamma(\rho + 1 + iq)}{\Gamma(2\rho + 1)}\, e^{+\frac{\pi i\rho}{2} + \frac{q}{2}}\, F(\rho + 1 + iq\,;\; 2\rho + 1\,;\; 2ir)\tfrac{1}{2}(2r)^{\rho}\, e^{-ir}/r,$$

$$\zeta_k' = \frac{\Gamma(\rho + 1 - iq)}{\Gamma(2\rho + 1)}\, e^{-\frac{\pi i\rho}{2} + \frac{q}{2}}\, F(\rho + iq\,;\; 2\rho + 1\,;\; 2ir)\,\tfrac{1}{2}(2r)^{\rho}\, e^{-ir}/r,$$

and

$$c_k/c_k' = -(k - iq')/(\rho - iq)$$

and N is a normalising factor.

From formulæ (19) we have at once

$$\zeta_k \backsim \tfrac{1}{2}(2r)^{iq}\,.\, e^{ir}/r,$$

$$\zeta'_k \backsim \tfrac{1}{2}(2r)^{-iq}\,.\, e^{-ir}/r,$$

and

$$G_{-k-1} \backsim r^{-1}\cos(r + q\log 2r + \eta_{-k-1})$$

with η_k given by

$$e^{2i\eta_{-k-1}} = -\frac{k - iq'}{\rho - iq}\, e^{-\pi i\rho}\, \frac{\Gamma(\rho + 1 - iq)}{\Gamma(\rho + 1 + iq)}$$

$$= B_k \quad \text{say.}$$

The asymptotic expansion of G_k is not quite of the form (14), differing from it by the logarithmic term; as has been pointed out by various authors, for an inverse square law field the incident wave is not quite plane. We can, however, construct the solution of the form (6.1) without difficulty. This solution is

$$\left.\begin{aligned} \psi_3 &= i\sum_{k=0}^{\infty}\left[(2k + 1)\,\zeta_k' + \{kB_k + (k + 1)\,B_{-k-1}\}\,\zeta_k\right](-)^k\, P_k(\cos\theta) \\ \psi_4 &= i\sum_{k=0}^{\infty}\left[B_k - B_{-k-1}\right]\zeta_k\,(-)^k\, P_k^1(\cos\theta)\, e^{i\varphi} \end{aligned}\right\}. \quad (21)$$

These series converge absolutely for given r. A method previously given by the present author† can be used to prove that, for large r

$$i\sum_{k=0}^{\infty}(2k + 1)\,\zeta'_k(r)\,(-)^k\, P_k(\cos\theta) \backsim e^{ir\cos\theta - iq\log r(1 - \cos\theta)}.$$

This represents the incident wave. The remaining terms represent outgoing

* Gordon, *loc. cit.*, p. 13, equation (10). If we put Gordon's j' equal to our k, then his ψ_2 is equal to our $r\,G_{-k-1}$.

† Mott, 'Roy. Soc. Proc.,' A, vol. 118, p. 543.

waves only. We cannot, however, obtain the form of the wave for large r by inserting the asymptotic solution (20) for ζ_k, because the series so obtained do not converge. They can, however, be summed as the limit of a power series on its radius of convergence.* If we express the functions ζ_k as the contour integrals from which the asymptotic expansion is obtained, it is easy to see that these sums do in fact give the asymptotic form of (21). The method is the same as that used in the author's previous paper. We see, therefore, that (21) is a solution of the wave equation with the asymptotic form (6.1) with

$$\mathrm{I} = e^{ir\cos\theta - iq\log r(1-\cos\theta)},$$

$$\mathrm{S} = e^{ir + iq\log 2r}/r,$$

and

$$f(\theta) = \tfrac{1}{2}i\,\Sigma\,[k\mathrm{B}_k + (k+1)\,\mathrm{B}_{-k-1}](-)^k\,\mathrm{P}_k(\cos\theta)$$

$$g(\theta) = \tfrac{1}{2}i\,\Sigma\,[\mathrm{B}_k - \mathrm{B}_{-k-1}]\,(-)^k\,\mathrm{P}_k^1(\cos\theta),$$

the summation of each series being carried out as the limit of a power series on its radius of convergence.

We can express f and g in terms of series which do not contain q'. If we write

$$c_k = -\,e^{-i\pi\rho}\Gamma(\rho - iq)/\Gamma(1+\rho+iq)$$

and

$$\mathrm{F}(\theta) = \tfrac{1}{2}i\sum_0^\infty(-)^k\{k\mathrm{C}_k + (k+1)\,\mathrm{C}_{k+1}\}\,\mathrm{P}_k(\cos\theta),$$

$$\mathrm{G}(\theta) = \tfrac{1}{2}i\sum_0^\infty(-)^k\,\{k^2\mathrm{C}_k - (k+1)^2\,\mathrm{C}_{k+1}\}\,\mathrm{P}_k(\cos\theta),$$

we obtain

$$\left.\begin{aligned} f &= -\,iq'\mathrm{F} + \mathrm{G} \\ g &= [iq'(1+\cos\theta)\,\mathrm{F} + (1-\cos\theta)\,\mathrm{G}]/\sin\theta \end{aligned}\right\}. \tag{23}$$

F and G are functions of θ, α^2 and q. It has not been found possible to sum the series in terms of known functions; we can, however, write $q = \alpha\mu$ and expand F and G as power series in α. We shall obtain the first two terms of the expansion. α has, of course, any value from 1/137 for hydrogen up to about 3/4 for the heavy nuclei; and for fast electrons μ will be about 3, though for slow electrons it will be greater. *Our approximation is best, therefore, for fast electrons and for light nuclei.*

* Whittaker and Watson, 'Modern Analysis,' p. 155.

If we refer to the series (22), and put $q = q'$ and $\alpha^2 = 0$, we obtain

$$\left.\begin{aligned} f(\theta) &= -\tfrac{1}{2}i\,\Sigma\,(2k+1)\,\frac{\Gamma(k+1-iq)}{\Gamma(k+1+iq)}\,\mathrm{P}_k(\cos\theta) \\ g(\theta) &= 0 \end{aligned}\right\}. \tag{24}$$

This corresponds to a neglect of relativity and spin. Since $g = 0$ we see that the direction of the spin axis is unchanged on collision. The series (24) occurs in the investigation of the scattering of Schrödinger electrons; it can be summed by the method of the author's previous paper,* the sum being

$$\mathrm{R}\,\operatorname{cosec}^2\frac{\theta}{2},$$

where

$$\mathrm{R} = \tfrac{1}{2}q\exp\left[2iq\log\sin\frac{\theta}{2} + \frac{\Gamma(1-iq)}{\Gamma(1+iq)} + i\pi\right].$$

The scattering is therefore classical, as it should be.

Now, α occurs in F and G only as α^2, so that as a first approximation we can neglect α altogether. Let F_0, G_0 be the values of F and G to this approximation. Since we know f_0 and g_0 when $q = q'$, and q' does not occur in F and G, we have at once from (23)

$$iq\,\mathrm{F}_0 = -\,\mathrm{R},$$

$$\mathrm{G}_0 = \mathrm{R}\cot^2\frac{\theta}{2}.$$

Hence

$$\left.\begin{aligned} f_0 &= \left(\frac{q'}{q} - 1 + \operatorname{cosec}^2\frac{\theta}{2}\right)\mathrm{R} \\ g_0 &= -\left(\frac{q'}{q} - 1\right)\mathrm{R}\cot\frac{\theta}{2} \end{aligned}\right\}. \tag{24}$$

These are the first order scattering formula that we should obtain by the Born method. The ratio of f_0 and g_0 is real; it follows that to this order there is no polarisation. It is therefore of interest to evaluate f and g to the next power of α, q.

Expanding C_k in powers of α we have

$$\mathrm{C}_k = \frac{(-)^k\,\Gamma(k-iq)}{\Gamma(1+k+iq)} + \frac{\alpha^2}{2k^2}(-)^k\left[i\pi + \frac{1}{k}\right]. \qquad k \neq 0$$

$$+ \text{ terms } \alpha^2 q, \text{ etc.}$$

* Mott, 'Roy. Soc. Proc.,' vol. 118, p. 543 (1928). The result given there can be simplified to that given above. *Cf.* Whittaker and Watson, p. 240.

Neglecting terms of the order $\alpha^2 q$ we obtain†

$$iq\mathrm{F} = iq\mathrm{F}_0,$$
$$\mathrm{G} = \mathrm{G}_0 + \frac{\alpha^2}{4}\left[\pi \operatorname{cosec}^2 \frac{\theta}{2} - i \log \operatorname{cosec}^2 \frac{\theta}{2}\right],$$

whence from (23), (24) we can obtain at once formulæ for f and g.

To this order, we find that f/g is not real; we have therefore some polarisation on a collision ; the constant that determines the polarisation is

$$fg^* - f^*g = \frac{i\alpha^2 q'}{4} \operatorname{cosec} \theta \log \operatorname{cosec} \frac{\theta}{2} + \text{terms of order } \alpha^4.$$

To the same order

$$|f|^2 + |g|^2$$
$$= \tfrac{1}{4}\left[q^2 \operatorname{cosec}^4 \frac{\theta}{2} + \frac{q'^2 - q^2}{4} \operatorname{cosec}^2 \frac{\theta}{2} + \frac{\pi q \alpha^2}{4} \frac{\cos^2 \frac{\theta}{2}}{\sin^3 \frac{\theta}{2}} + \text{terms of order } \alpha^4\right].$$

We shall now return to the ordinary unit of length ; our formulæ are most conveniently expressed in terms of v, the velocity of the electron, defined by (18). We have then

$$|f|^2 + |g|^2 = \frac{Z^2\varepsilon^4}{4m^2v^4}\left(1 - \frac{v^2}{c^2}\right)\left[\operatorname{cosec}^4 \frac{\theta}{2} - \frac{v^2}{c^2} \operatorname{cosec}^2 \frac{\theta}{2} + \frac{v}{c}\pi \frac{2\pi Z\varepsilon^2}{hc} \frac{\cos^2 \frac{\theta}{2}}{\sin^3 \frac{\theta}{2}} + \text{terms of order } \alpha^2\right], \quad (25)$$

and

$$fg^* - f^*g = \frac{Z^2\varepsilon^4}{4m^2v^4}\left(1 - \frac{v^2}{c^2}\right)^{3/2} \frac{v}{c}\, 4i\alpha \operatorname{cosec} \theta \log \operatorname{cosec} \frac{\theta}{2}. \quad (26)$$

The formulæ (25) and (26) determine the total scattering and the polarisation of the scattered beam. They are, of course, calculated with neglect of radiative

† We use the formulæ

$$\sum_0^\infty \frac{\mathrm{P}_k(\cos\theta)}{k+1} = \int_0^1 \frac{dx}{\sqrt{1 - 2x\cos\theta + x^2}} = \log\left(1 + \operatorname{cosec}\frac{\theta}{2}\right)$$

$$\sum_1^\infty \frac{\mathrm{P}_k(\cos\theta)}{k} = \int_0^1 \left(\frac{1}{x\sqrt{1 - 2x\cos\theta + x^2}} - \frac{1}{x}\right) dx = \log \frac{\operatorname{cosec}^2 \frac{\theta}{2}}{1 + \operatorname{cosec}\frac{\theta}{2}}.$$

forces, which, for fast electrons, is a serious matter. An electron deflected through 90° by a nucleus of charge $Z\varepsilon$ would, on the classical theory, lose an amount of energy equal to *

$$\frac{1}{Z}\frac{4}{3}(2\pi + 3)\,\tfrac{1}{2}mv^2\,\frac{v^3}{c^3}.$$

This formula is calculated with neglect of relativity, but it shows that for light nuclei, an electron with a velocity approaching that of light is acted on by forces comparable with the electrostatic field of the nucleus. For heavy nuclei, however, the radiative forces are less important, but for heavy nuclei our approximations are less good—though there would be no difficulty in pushing them to any degree of accuracy required. The author hopes, in a later paper, to consider in greater detail the effect of radiative forces on the scattering.

From (25) and (26) we can see the order of magnitude of the effect that may be expected in the double scattering experiment considered in Section 2. In equation (10), we suppose that both θ_1 and θ_2 are 90°; then we have approximately

$$\delta = 11\cdot 2 \times \frac{(1 - v^2/c^2)\,v^2/c^2}{2 - v^2/c^2}\,\alpha^2,$$

δ has a maximum when $v/c = 0\cdot 764$ so there is an optimum value of the velocity of the incident electrons. With this value of v/c we have

$$\delta = (Z/96)^2,$$

Z being the atomic number of scattering nucleus.

For light elements, therefore, the effect is very small, and, indeed, may not exist, since the radiative forces are so considerable. For heavy elements, however, the effect of the radiative forces falls off inversely as the atomic number, whereas the polarisation effect increases with Z^2, and so it seems certain that the Dirac theory of the electron does predict a polarisation on collision. Whether the effect could be observed experimentally is more doubtful; the K electrons of heavy atoms have themselves velocities of the order of $0\cdot 7\,c$, and would interfere with the nuclear scattering.

The proportion of an unpolarised beam scattered in a given solid angle is given by (9), so that for the scattering of fast electrons (25) is to replace Rutherford's formula $Z^2\varepsilon^4/4m^2v^2 \operatorname{cosec}^4 \frac{\theta}{2}$. Our formula bears no resemblance

* Kramers, 'Phil. Mag.,' vol. 46, p. 845 (1923).

to Darwin's* classical relativity correction; this is not surprising in view of the fact that we are dealing with a case where the wave-length is long compared to the classical distance of closest approach. There is therefore no possibility of forming a wave packet that must follow the classical orbit.

Nothing occurs in the wave mechanics at all analogous to the spiral orbits of the classical theory.

The formula does not agree very well with the available experimental evidence, giving in all cases too little scattering. Chadwick and Mercier, for instance, have investigated the scattering of β particles from Ra C by aluminium. At angles from 10°–20° our formula gives 2/3 of the observed scattering. It is possible that the radiative forces may be sufficient to account for this divergence. Without a fuller investigation nothing can be said on this point.

In conclusion, the author would like to express his thanks to Prof. Niels Bohr for the opportunity to work at his Institute, and for constant help and discussion.

Summary.

The scattering of a beam of fast electrons by an atomic nucleus is investigated, using the wave equation of Dirac. A scattering formula is obtained, and it is found that the scattered beam is polarised. A method by which this polarisation could be detected is discussed.

Appendix.

Suppose we wish to observe the spin of a free electron directly, with a magnetometer. We will suppose the electron to be at a distance R from the magnetometer, so that the order of magnitude of the magnetic field due to the spin is

$$\frac{eh}{mc} \cdot \frac{1}{\mathrm{R}^3}. \tag{1}$$

Now, there may also be a magnetic field due to the motion of the electron; the order of magnitude of this field is

$$\frac{ev}{c} \cdot \frac{1}{\mathrm{R}^2}. \tag{2}$$

Now, by the Uncertainty Principle, R and v cannot both be known at the same time; if $\Delta\mathrm{R}$, Δv are the uncertainties in our knowledge of R and v, then

$$\Delta\mathrm{R} \, . \, \Delta v > h/m. \tag{3}$$

* C. G. Darwin, 'Phil. Mag.,' vol. 25, p. 201 (1925).

Now, in order that (1), the effect of the spin, shall be observable, it must be greater than the uncertainty in (2). That is to say

$$\frac{h}{m} \cdot \frac{1}{\mathrm{R}} > \Delta v.$$

Hence from (3)

$$\Delta \mathrm{R} > \mathrm{R}.$$

The experiment will therefore be impossible, since the uncertainty in the position of the electron would have to be greater than the distance of the electron from the magnetometer; the uncertainty in (1) would be greater than the field (1) that we want to measure.

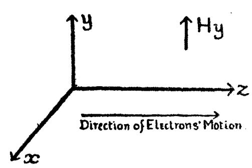

FIG. 2.

Stern Gerlach Experiment.—A beam of electrons travels along the z axis with velocity v_z in an unhomogeneous magnetic field $\mathbf{H}$. We shall suppose that H_z is everywhere zero, and that in the plane Oyx H_x is also zero. The force on the electron magnets tending to split the beam is

$$\frac{eh}{mc} \frac{\partial \mathrm{H}_y}{\partial y},$$

and in the plane Oyz this is the only force in y direction. However, the beam must be of finite breadth, and since

$$\frac{\partial \mathrm{H}_y}{\partial y} = -\frac{\partial \mathrm{H}_x}{\partial x}, \tag{2}$$

it is clear that H_x is only zero in the plane Oyz. In general

$$\mathrm{H}_x = \int_0^x \frac{\partial \mathrm{H}_x}{\partial x} dx$$

$$= \int_0^x \frac{\partial \mathrm{H}_y}{\partial y} dx. \qquad \text{by (2)}$$

Electrons, therefore, travelling at a distance Δx from the plane Oyz will be subject to a force

$$\frac{ev_z \mathrm{H}_x}{c} \tag{3}$$

in the direction Oy due to their motion through the field, and we see that (3) is equal to

$$\frac{ev_z}{c} \frac{\partial \mathrm{H}_y}{\partial y} \Delta x. \tag{4}$$

This force is in different directions according as Δx is positive or negative, and will therefore cause a spreading of the beam, which will mask the Stern-Gerlach splitting, unless (4) is less than (1), *i.e.*, unless

$$mv_z \, \Delta x < h. \tag{5}$$

Now the uncertainty principle states that

$$\Delta v_x \, . \, m \, \Delta x \backsim h. \tag{6}$$

That is to say, that the slit that we use to limit our beam to the dimensions of Δx will introduce an uncertainty in the velocity Δv_x, given by (6). Inequality (5) therefore leads to the inequality

$$\Delta v_x > v_z.$$

That is to say, the slit must be so narrow (of the order of the de Broglie wave-length) that we have not got a beam at all, but a cylindrical wave emerging from it.

Infra-Red Investigations of Molecular Structure.—Part* I. *Apparatus and Technique.*

By C. P. Snow (Keddey Fletcher-Warr Student) and A. M. Taylor (Ramsay Memorial Research Fellow).

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In what is by far the most exhaustive description of practical work in the infra-red, Robertson and Fox† have deplored the lack of detail given in most

* It is the aim of this series of papers to extend knowledge of the structure of molecules by work in the infra-red. There are three obvious lines of development. The first is the study of the spectra of diatomic gases and the deductions which follow according to the classical quantum theory ; this has already been done with success by Imes (Sleator, 'Astrophys. J.,' vol. 48, p. 125 (1918) ; Imes, *ibid.*, vol. 50, p. 251 (1919)) for the hydrogen halides. Comparisons will be made of the molecular constants from the infra-red bands with those obtained from electronic band spectra.

As a second development there is the use of the infra-red results in the light of the newer quantum theory. Dennison's prediction (' Phys. Rev.,' vol. 31, p. 503 (1928)) of the shape of absorption bands needs confirmation by experiment ; and, if absolute intensities of absorption bands can be measured, the wave-mechanics is ready with an interpretation. Thirdly, the problems offered by triatomic molecules will be attempted. —E. K. Rideal, C. P. Snow, F. I. G. Rawlins, A. M. Taylor.

† ' Roy. Soc. Proc.,' A, vol. 120, p. 128 (1928).