I. Introduction

The following report discusses the calculations and considerations needed to measure the vertical depth of Mine #22, as requested by Dr. Clark-Teuben, Director of Mining265 Inc. Dr. Clack-Teuben has proposed measuring the vertical depth of the shaft by dropping a *1-kg* test mass. As such, I have calculated the expected time it will take the test mass to reach the bottom of the shaft, as well as the minimum width required of the mine to ensure the test mass hits the bottom before hitting the wall of the mine. The report also discusses the characteristics of an "infinitely deep" mine, stretching from the North Pole to the South Pole.

The following report utilizes a right-handed coordinate system. Unless otherwise specified, assume that the origin is at the top of the mine, the x-direction points east, the y-direction points away from the Earth's center, and the z-direction points north.

II. Calculation of Fall Time

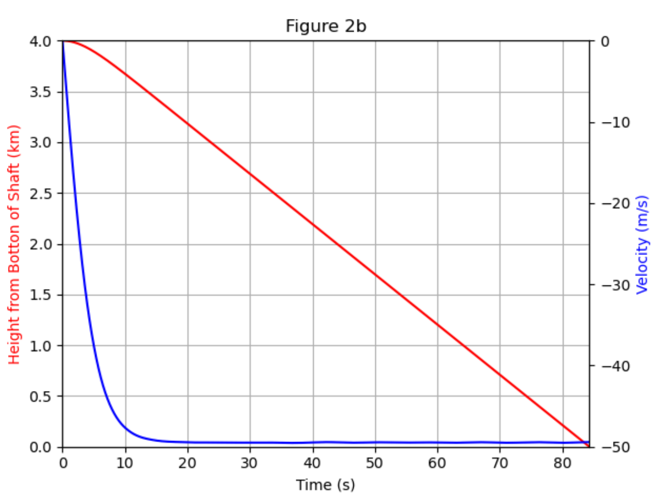
When calculating the time it should take for the mass to reach the bottom of the shaft, there are multiple forces one must take into account. Simply assuming a constant acceleration of $g = 9.8\ m/s^2$ would indicate a fall time of *28.6 s*. However, the gravitational force acting on the mass is not constant due to the inverse square relationship between gravity and the distance to the center of the Earth. Taking gravity's variance into account also yields a fall time of approximately *28.6 s*.

One must also account for the drag force acting on the mass as it falls. Accounting for the variance of gravity and the drag force yields the following differential equation:

$$\frac{d^2y}{dt^2} = \frac{g_0(R_E+y)}{R_E} + \alpha\left(\frac{dy}{dt}\right)^2$$

g_0 is the acceleration due to gravity at the surface of the Earth, R_E is the radius of the Earth, and α is the drag coefficient. The drag force always acts opposite to the velocity, in this case acting up.

The trajectory of the mass is shown in Figure 2b, as calculated using the differential equation above. The mass has a terminal velocity of *50 m/s*, meaning the drag coefficient $\alpha = 4*10^{-3}\ m^{-1}$. Before the mass reaches terminal velocity, the drag force is small. The acceleration is, therefore, almost constant, so the velocity decreases almost linearly and the height decreases almost parabolically. Once the mass reaches terminal velocity, the acceleration is zero. Thus, the velocity is constant, and the height decreases linearly. The mass is expected to hit the bottom of the shaft at time $t = 84.3\ s$.



III. Feasibility of Depth Measurement Approach

Due to the Coriolis force, this method of measuring the depth of the mine is not feasible assuming the mine is only *5 m* wide. The mass has an x-direction velocity associated with the Earth's rotation. Relative to the top of the mine, the mass' x-direction velocity is zero, because the top of the mine has the same velocity associated with the Earth's rotation. The mine's associated velocity decreases, however, as one moves deeper.

As the mass falls, its x-direction velocity decreases relative to the mine's x-direction velocity. This creates an effective "Coriolis force" which follows this equation:

$$\vec{F}_C = -2m(\vec{\Omega} \times \vec{v})$$

Since the mine is at the equator of the Earth, the acceleration due to the Coriolis force $a_{Cx} = -2\Omega v_y$.

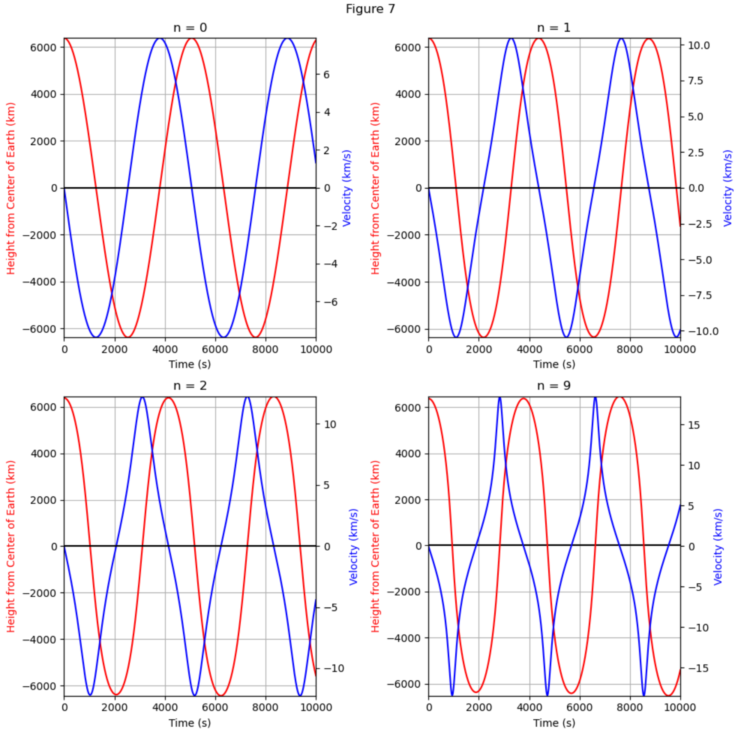
Accounting for the Coriolis force and the drag force yields the following x-direction acceleration:

$$\frac{d^2x}{dt^2} = -2\Omega\frac{dy}{dt} - \alpha\left(\frac{dx}{dt}\right)^2$$

Neglecting drag indicates that if the mass is dropped from the center, the mine would have to be at least *11.1 m* in diameter to ensure the mass does not hit the wall before reaching the bottom. In reality, however, the mine must be over *45.7 m* wide if the mass is to be dropped from the center. The drag force has a negligible effect on the horizontal velocity, but it decreases the average vertical velocity by a factor of almost *3*. Thus, the Coriolis force affects the mass for a longer period of time when taking drag into account, and the mass' horizontal displacement is greater.

If the mass is dropped from the center of a *5 m*-wide mine, the mass will hit the wall after traveling *1.3 km* down. This will take *29.7 s*. If the mass is dropped from the western edge of the mine, the mine must be over *22.8 m* wide to ensure the mass hits the bottom of the shaft before hitting the wall.

Figure 7

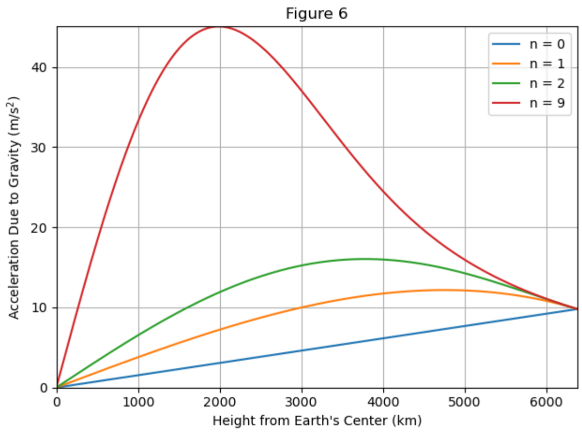


IV. Calculation of Crossing Times for Endless Mine

Let us consider an endless mine that starts on the North Pole, goes straight down through the center of the Earth, and continues straight to the South Pole. Because this mine goes from pole to pole, there is no Coriolis force acting on a mass as it falls through this mine. Let us also assume zero drag force.

If the mass is dropped from one pole, say the North Pole, it will endlessly travel cosinusoidally from pole to pole, as shown in the top left subplot of Figure 7. Gravity always points towards the center of the Earth, resulting in simple harmonic motion. It would take the mass *5,068.2 s* to make it back to the North Pole. This is the same as the time it would theoretically take for an object to orbit around the surface of the Earth, because both objects would be in simple harmonic motion due to the Earth's gravity.

It is important to take into account, however, that the Earth is not of uniform density. The Earth gets increasingly dense as one moves closer to its center. Let us assume that the Earth's density *ρ* follows the following formula as a function of distance *r* from the Earth's center:



$$\rho = \rho_n \left(1 - \frac{r^2}{R^2}\right)^n$$

ρ_n is a normalizing constant, and *n* is a constant. Taking *n = 0* would imply uniform density. Taking a greater *n* would increase the average acceleration due to gravity, as shown in Figure 6, thus increasing the "fall time" it takes for a mass to make it from one pole to the other. Taking *n = 2* is most accurate.

For any given "endless mine" through a planet or moon of a given size, there is an inverse-square root relationship between the crossing time and the density: $T \propto \frac{1}{\sqrt{\rho}}$

V. Discussion and Future Work

The above report details all the calculations and considerations needed in order to drop a *1-kg* test mass as a means of confirming the depth of an approximately *4-km*-deep mine. The report also discusses the characteristics of an endless mine that stretches from the North Pole to the South Pole.

I have calculated that the mass is expected to hit the bottom of the shaft after *84.3 s*. Additionally, if the mass is dropped from the western edge of the mine, the mine must be over *22.8 m* in diameter to ensure the mass hits the bottom before hitting the wall of the mine.

Though helpful, my calculations can be improved upon. My calculation for the fall time of the test mass to the bottom of the shaft can be made more accurate by accounting for the Coriolis force and the varying density of the Earth. My calculation for the width needed to ensure the mass does not hit the wall can be made more accurate by considering more precise aerodynamics. For example, one might consider that the air density toward the bottom of the shaft is not the same as that of the surface of the Earth, meaning the drag force would not be constant. Finally, my calculations regarding a theoretical endless mine can be made more accurate by including the drag force and creating a more accurate model of the density as a function of radius. Due to the Earth's four layers, the relationship between density and distance from the center of the Earth is more complex than described above.