1. Prove $LIST(SEXP_{FG}) \subseteq SEXP_{FG}$

I will be using the following equations from p.116 of the textbook:

(2.1) $LIST(A) = \{'()\} \cup \{(\text{cons } a \text{ } as) \mid a \in A \wedge as \in LIST(A)\}$

(2.3) $SEXP_{FG} = ATOM \cup \{(\text{cons } v_1 \text{ } v_2) \mid v_1 \in SEXP_{FG} \wedge v_2 \in SEXP_{FG}\}$

(2.2) $ATOM = BOOL \cup NUM \cup SYM \cup \{'()\}$

I will be proving that if $X \in LIST(SEXP_{FG})$, then $X \in SEXP_{FG}$

Base case

Let $X = '()$. According to the EMPTYLIST rule: $\dfrac{}{'() \in LIST(SEXP_{FG})}$

Using equations 2.2 and 2.3, we see that '() is an atom, and an atom is a $SEXP_{FG}$. Therefore, we have if $'() \in LIST(SEXP_{FG}) \Rightarrow '() \in SEXP_{FG}$

Inductive case

Let $x = (\text{cons } y \text{ } ys)$. According to the CONSLIST rule: $\dfrac{y \in SEXP_{FG} \quad ys \in LIST(SEXP_{FG})}{(\text{cons } y \text{ } ys) \in LIST(SEXP_{FG})}$

Given equation 2.1, we know that $ys \in SEXP_{FG}$ by the inductive hypothesis. Therefore, using equation 2.3, we see that (cons y ys) is an $SEXP_{FG}$ because $y \in SEXP_{FG}$ and $ys \in SEXP_{FG}$. Therefore, we have if $(\text{cons } y \text{ } ys) \in LIST(SEXP_{FG})$ $\Rightarrow (\text{cons } y \text{ } ys) \in SEXP_{FG}$.

Our proof by induction is now complete after proving the base case and inductive case.

35. Prove: (length (reverse xs)) = (length xs)

I will be utilizing the reverse-simple-reverse law:

(reverse xs) = (simple-reverse xs)

Base case xs = '( )

(length (reverse '()))

= {reverse-empty law}

(length '( )) = (length (reverse '())) ✓

Inductive case xs = (cons y ys)

(length (reverse (cons y ys)))

= {reverse-cons law}

(length (append (reverse ys) (cons y '())))

= {length-append law}

(+ (length (reverse ys) (length (cons y '()))))

= {induction hypothesis}

(+ (length ys) (length (cons y '())))

= {length-cons law}

(+ (length ys) (+1 (length '())))

= {length-null law}

(+ (length ys) (+ 1 0))

= {communative property of addition}

(+ 1 (length ys))

= {length-cons law}

(length (cons y ys)) = (length (reverse (cons y ys))) ✓

A. a.

$$
\text{CDR}\;\dfrac{\langle cdr,\rho,\sigma\rangle \Downarrow \langle \mathrm{PRIM}(cdr),\rho,\sigma\rangle \quad \text{CONS}\;\dfrac{\langle cons,\rho,\sigma\rangle \Downarrow \langle \mathrm{PRIM}(cons),\sigma\rangle \quad \text{VAR}\;\dfrac{x\in \mathrm{dom}\,\rho \quad \rho(x)\in \mathrm{dom}\,\sigma}{\langle \mathrm{VAR}(x),\rho,\sigma\rangle \Downarrow \langle \sigma(\rho(x)),\sigma\rangle} \quad \text{VAR}\;\dfrac{xs\in \mathrm{dom}\,\rho \quad \rho(xs)\in \mathrm{dom}\,\sigma}{\langle \mathrm{VAR}(xs),\rho,\sigma\rangle \Downarrow \langle \sigma(\rho(xs)),\sigma\rangle} \quad \begin{matrix} l_1 \notin \mathrm{dom}\,\sigma \\ l_2 \notin \mathrm{dom}\,\sigma \\ l_1 \neq l_2 \end{matrix}}{\langle \mathrm{APP}(cons, \mathrm{VAR}(x), \mathrm{VAR}(xs)),\rho,\sigma\rangle \Downarrow \langle \mathrm{PAIR}\langle l_1, l_2\rangle, \sigma\{l_1 \mapsto \sigma(\rho(x)), l_2 \mapsto \sigma(\rho(xs))\}\rangle}}{\langle \mathrm{APP}(cdr, \mathrm{APP}(cons, \mathrm{VAR}(x), \mathrm{VAR}(xs))),\rho,\sigma\rangle \Downarrow \langle \sigma(\rho(xs)),\sigma\rangle}
$$

$$
\text{VAR}\;\dfrac{xs\in \mathrm{dom}\,\rho \quad \rho(xs)\in \mathrm{dom}\,\sigma}{\langle \mathrm{VAR}(xs),\rho,\sigma\rangle \Downarrow \langle \sigma(\rho(xs)),\sigma\rangle}
$$

The operational semantics of (cdr (cons x xs)) and xs separately, show that the two expressions evaluate to the same value on the lower-right of the trees: $\sigma(\rho(xs))$.

A.b. I will be proving that (cdr (cons e_1 e_2)) == e_2 is a <u>false</u> conjecture.

Let $e_1 = (\text{set } x\ 3)$ Initial state: $\rho = \{x \mapsto l\}$

$e_2 = (= x\ 3)$ $\sigma = \{l \mapsto 2\}$

$$\frac{}{\langle =, \rho, \sigma\{l \mapsto 3\}\rangle \Downarrow \langle \text{PRIM}(=), \sigma\{l \mapsto 3\}\rangle} \qquad \frac{}{\langle \text{LIT}(3), \rho, \sigma' \rangle \Downarrow \langle 3, \sigma' \rangle}\,\text{LIT}$$

$$\text{VAR}\,\frac{x \in \text{dom}\,\rho \quad \rho(x) \in \text{dom}\,\sigma}{\langle \text{VAR}(x), \rho, \sigma\{l \mapsto 3\}\rangle \Downarrow \langle \sigma(\rho(x)), \sigma\{l \mapsto 3\}\rangle}$$

$$\text{ASSIGN}\,\frac{x \in \text{dom}\,\rho \quad \rho(x) = l \quad \overline{\langle \text{LIT}(3), \rho, \sigma \rangle \Downarrow \langle 3, \sigma \rangle}\,\text{LIT}}{\langle \text{SET}(\text{VAR}(x), \text{LIT}(3)), \rho, \sigma \rangle \Downarrow \langle 3, \sigma\{l \mapsto 3\}\rangle}$$

$$\frac{\langle =, \rho, \sigma\{l \mapsto 3\}\rangle \Downarrow \langle \text{PRIM}(=), \sigma\{l \mapsto 3\}\rangle \quad \langle \text{VAR}(x), \rho, \sigma\{l \mapsto 3\}\rangle \Downarrow \langle \sigma(\rho(x)), \sigma\{l \mapsto 3\}\rangle \quad \langle \text{LIT}(3), \rho, \sigma' \rangle \Downarrow \langle 3, \sigma' \rangle \quad v_1 \equiv v_2}{\langle \text{APP}(=, \text{VAR}(x), \text{LIT}(3)), \rho\ \sigma\{l \mapsto 3\}\rangle \Downarrow \langle \text{BOOLV}(\#t), \sigma\{l \mapsto 3\}\rangle}\,\text{APPEQTRUE}$$

$$\text{CONS}\,\frac{\langle \text{cons}, \rho, \sigma \rangle \Downarrow \langle \text{PRIM}(\text{cons}), \sigma \rangle \quad \langle \text{SET}(\text{VAR}(x), \text{LIT}(3)), \rho, \sigma \rangle \Downarrow \langle 3, \sigma\{l \mapsto 3\} \quad \langle \text{APP}(=, \text{VAR}(x), \text{LIT}(3)), \rho\ \sigma\{l \mapsto 3\}\rangle \Downarrow \langle \text{BOOLV}(\#t), \sigma\{l \mapsto 3\}\rangle}{\langle \text{APP}(\text{cons}, \text{SET}(\text{VAR}(x), \text{LIT}(3)), \text{APP}(=, \text{VAR}(x), \text{LIT}(3)), \rho, \sigma \rangle \Downarrow \langle \text{PAIR}(l_1, l_2), \sigma\{l_1 \mapsto 3, l_2 \mapsto (\text{BOOLV}(\#t))\}\rangle}$$

$$\text{CDR}\,\frac{\langle \text{cdr}, \rho, \sigma \rangle \Downarrow \langle \text{PRIM}(\text{cdr}), \sigma \rangle \quad \langle \text{APP}(\text{cons}, \text{SET}(\text{VAR}(x), \text{LIT}(3)), \text{APP}(=, \text{VAR}(x), \text{LIT}(3)), \rho, \sigma \rangle \Downarrow \langle \text{PAIR}(l_1, l_2), \sigma\{l_1 \mapsto 3, l_2 \mapsto (\text{BOOLV}(\#t))\}\rangle}{\langle \text{APP}(\text{cdr}, \text{APP}(\text{cons}, \text{SET}(\text{VAR}(x), \text{LIT}(3)), \text{APP}(=, \text{VAR}(x), \text{LIT}(3))), \rho, \sigma \rangle \Downarrow \langle \text{BOOLV}(\#t), \sigma\{l_1 \mapsto 3, l_2 \mapsto (\text{BOOLV}(\#t))\}\rangle}$$

$$\text{APPEQFALS}\,\frac{\langle =, \rho, \sigma \rangle \Downarrow \langle \text{PRIM}(=), \sigma \rangle \quad \text{VAR}\,\dfrac{x \in \text{dom}\,\rho \quad \rho(x) \in \text{dom}\,\sigma}{\langle \text{VAR}(x), \rho, \sigma\{l \mapsto 2\} \Downarrow \langle \sigma(\rho(x)), \sigma\{l \mapsto 2\}\rangle} \quad \text{LIT}\,\dfrac{}{\langle \text{LIT}(3), \rho, \sigma\{l \mapsto 2\}\rangle \Downarrow \langle 3, \sigma\{l \mapsto 2\}\rangle} \quad v_1 \not\equiv v_2}{\langle \text{APP}(=, \text{VAR}(x), \text{LIT}(3)), \rho, \sigma\{l \mapsto 2\}\rangle \Downarrow \langle \text{BOOLV}(\#f), \sigma\{\rho(x) \mapsto l, l \mapsto 2\}\rangle}$$

The operational semantics of (cdr(cons e_1 e_2)) and e_2, show that the conjecture (cdr(cons e_1 e_2)) = e_2 is false, because they evaluate to different values on the lower right of the trees.