Concise Review of Multivariable Calculus and Linear Albebra

Multivariable Calculus:

Feedback:(

Things to add/improve upon as of 6/5/2007:

- **Jacobian**
- **Kepler's Laws**
- **A few examples where needed**
-
-
-

Things to explain better as of 6/5/2007:

- Geometric Interpretations
- Analogy between single and multivariable

Linear Albebra:

Part 1: Linear Systems, Matrix Operations
Part 2: Determinants, Eigenvectors, Diagonalization, Orthogonality
Part 3: Vector Spaces, Basis, Linear Transformation, Inner Product Spaces, Least Squares

Feedback:

Things to add as of 6/5/2007:

- **Complex Entries, Numbers, Eigenvalues**
- **Iterative Solution for Lin. Eqs.**
- **A few examples where needed**
- Discrete Dynamical Systems???
- Apps. To Differential Equations
- Final Version of Invertible Matrix
-
-

Things to explain better as of 6/5/2007:

- LU Factorization
-

Overall:

Organization and other improvements:

-
-
-
-

This is a work in progress. More information will be added after an even better understanding of the material has taken place.

$$\begin{pmatrix} Date \\ Qual. \\ Quan. \\ Unders\tan ding \\ Effort \end{pmatrix} \to \begin{pmatrix} 6/5/2007 \\ 85 \\ 84 \\ 80 \\ 89 \end{pmatrix}$$

Vectors:

$\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle$

The angle between the two vectors is θ.

Components: a_1, a_2, a_3 are components of $\mathbf{a}$.

Length: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

Addition: $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

Scalar Multiplication:

$c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$

Dot Product (Scalar Product):

$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$

Ex.: $W = \mathbf{F} \cdot \mathbf{d}$

Cross Product (Vector Product):

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

Direction is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

Ex.: $\tau = \mathbf{r} \times \mathbf{F}$

Projections:

Scalar Projection: $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector Projection: $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|}$

Lines and Planes:

Lines:

Vector Equation: $\mathbf{r} = \mathbf{r}_{t=0} + t\mathbf{v}$

$\mathbf{r}$ is the position vector

$\mathbf{r}_0$ is the initial position vector

t is the parameter

Or for a line segment

$\mathbf{r} = (1-t)\mathbf{r}_0 + t\,\mathbf{r}_1 \quad 0 \le t \le 1$

$\mathbf{r}_1$ is the final position vector

Parametric Equations:

$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$

$\mathbf{v} = \langle a, b, c \rangle$ is a parallel vector

$P_0(x_0, y_0, z_0)$ is an intersection point

General Form:

$ax + by = c$

Planes:

Normal Form:

$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ or $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$

$\mathbf{n} = \langle a, b, c \rangle$ is orthogonal to the plane

Vector Equation: $\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$

$\mathbf{r}$ is the position vector

s, t are the parameters

Parametric Equations:

$x = x_0 + a_1s + a_2t \quad y = y_0 + b_1s + b_2t \quad z = z_0 + c_1s + c_2t$

$\mathbf{v}_1 = \langle a_1, b_1, c_1 \rangle$ are parallel vectors

$\mathbf{v}_2 = \langle a_2, b_2, c_2 \rangle$

$P_0(x_0, y_0, z_0)$ is an intersection point

Scalar Equation through $P_0(x_0, y_0, z_0)$:

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

General Form:

$ax + by + cz = d$

Cylinders and Quadric Surfaces:

Cylinder: A surface consisting of *rulings* parallel to a given line.

Quadric Surface: A second-degree equation in three variables:

$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gy + Hy + Iz + J = 0$

Traces: Intersections between a surface and a coordinate plane.

Ex.: $z = x^2$ This is a parabolic cylinder (Traces on xz-plane are parabolas)

Ex.: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ This is an ellipsoid (Traces are ellipses)

Cylindrical and Spherical Coordinates

-*Cylindrical coordinates* are represented by (r, θ, z) where (r, θ) is the polar coordinates in the xy-plane and z is the distance from the xy-plane.

From cylindrical coordinates (r, θ, z)

To rectangular coordinates (x, y, z):

$\boxed{x = r\cos\theta \quad y = r\cos\theta \quad z = z}$

From rectangular coordinates (r, θ, z)

To cylindrical coordinates (x, y, z):

$\boxed{r^2 = x^2 + y^2 \quad \tan\theta = \dfrac{y}{x} \quad z = z}$

-*Spherical coordinates* are represented by (ρ, θ, ϕ) where ρ is the distance to the origin, ϕ is the angle off of the y-axis, and θ is the angle off of the x-axis.

From spherical coordinates (ρ,θ,ϕ)

To rectangular coordinates (x,y,z):

$$\boxed{x=\rho\sin\phi\cos\theta \quad y=\rho\sin\phi\sin\theta \quad z=\rho\cos\phi}$$

From rectangular coordinates (r,θ,z)

To spherical coordinates (x,y,z):

$$\boxed{\rho^2=x^2+y^2+z^2}$$

Ex.: $r=c$ Cylinder. $z=r$ Cone.
$\rho=c$ Sphere.

Vector Functions

Vector Function:

$$\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle = f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$$

A *vector function* $\mathbf{r}(t)$ with 3 *component functions* $f(t),g(t),h(t)$.

Derivative of a Vector Function:

$$\mathbf{r}'(t)=\langle f'(t),g'(t),h'(t)\rangle = f'(t)\hat{\mathbf{i}}+g'(t)\hat{\mathbf{j}}+h'(t)\hat{\mathbf{k}}$$

Integral of a Vector Function:

$$\int_a^b \mathbf{r}(t)dt=\left(\int_a^b f(t)dt\right)\hat{\mathbf{i}}+\left(\int_a^b g(t)dt\right)\hat{\mathbf{j}}+\left(\int_a^b h(t)dt\right)\hat{\mathbf{k}}$$

Ex.: $\mathbf{r}(t)=\langle t,t^2,t^3\rangle$ twisted cubic function

Then, $\mathbf{r}'(t)=\left\langle \frac{d}{dt}t,\frac{d}{dt}t^2,\frac{d}{dt}t^3\right\rangle$

$$\mathbf{r}'(t)=\langle 1,2t,3t^2\rangle$$

And,

$$\int \mathbf{r}(t)dt=\left(\int t\,dt\right)\hat{\mathbf{i}}+\left(\int t^2dt\right)\hat{\mathbf{j}}+\left(\int t^3dt\right)\hat{\mathbf{k}}$$

$\int \mathbf{r}(t)dt=\frac{t^2}{2}\hat{\mathbf{i}}+\frac{t^3}{3}\hat{\mathbf{j}}+\frac{t^4}{4}\hat{\mathbf{k}}+\mathbf{C}$ where $\mathbf{C}$ is the vector constant of integration.

Arc Length:

$$\mathrm{L}=\int_a^b |\mathbf{r}'(t)|dt$$

And

$$s(t)=\int_a^t |\mathbf{r}'(u)|du=\int_a^t \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}\,du$$

is the ***arc length function***.

Using the fundamental theorem of calculus,

$$\frac{ds}{dt}=|\mathbf{r}'(t)|$$

Unit Tangent Vector:

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Curvature: describes how sharp the curve is or how quickly the curve changes direction at some point t. (Differential change in direction with respect to Differential change in arc length)

$$\kappa=\left|\frac{d\mathbf{T}}{ds}\right|=\frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Unit Normal Vector: the direction in which the curve is turning

$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, $\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$ called the ***binormal vector*** which is perpendicular to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$. It is the axis of rotation.

Ex.:

$\mathbf{r}(t)=\cos t\,\hat{\mathbf{i}}+\sin t\,\hat{\mathbf{j}}+t\,\hat{\mathbf{k}}$ called a "helix"

$$\mathbf{r}'(t)=-\sin t\,\hat{\mathbf{i}}+\cos t\,\hat{\mathbf{j}}+\hat{\mathbf{k}} \quad |\mathbf{r}'(t)|=\sqrt{2}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{1}{\sqrt{2}}\left(-\sin t\,\hat{\mathbf{i}}+\cos t\,\hat{\mathbf{j}}+\hat{\mathbf{k}}\right)$$

$$\mathbf{T}'(t)=\frac{1}{\sqrt{2}}\left(-\cos t\,\hat{\mathbf{i}}-\sin t\,\hat{\mathbf{j}}\right) \quad |\mathbf{T}'(t)|=\frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=-\cos t\,\hat{\mathbf{i}}-\sin t\,\hat{\mathbf{j}}=\langle -\cos t,-\sin t,0\rangle$$

$$\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)=\frac{1}{\sqrt{2}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -\sin t & \cos t & 1\\ -\cos t & -\sin t & 0\end{vmatrix}=\frac{1}{\sqrt{2}}\langle \sin t,-\cos t,1\rangle$$

Motion in Space:

$$\mathbf{a}(t)=\mathbf{v}'(t)=\mathbf{r}''(t)$$

Projectile Motion:

$$\mathbf{a}=-g\,\hat{\mathbf{j}}$$

$$\mathbf{a}'=\mathbf{v}(t)=-gt\,\hat{\mathbf{j}}+(\mathbf{C}=\mathbf{v}_0)$$

$$\mathbf{v}'(t)=\mathbf{r}(t)=-\frac{1}{2}gt^2\,\hat{\mathbf{j}}+t\mathbf{v}_0+(\mathbf{D}=\mathbf{0})$$

T

$$\mathbf{v}_0=v_0\cos\theta\,\hat{\mathbf{i}}+v_0\sin\theta\,\hat{\mathbf{j}}$$

$$\mathbf{r}(t)=(v_0\cos\theta)t\,\hat{\mathbf{i}}+\left[(v_0\sin\theta)t-\frac{1}{2}gt^2\right]\hat{\mathbf{j}}$$

$$x=(v_0\cos\theta)t \quad y=(v_0\sin\theta)t-\frac{1}{2}gt^2$$

he horizontal distance $x=d$ occurs when $y=0$. Setting $y=0$, and solving: $t=(2v_0\sin\theta)/g$. Gives

$$d=x=(v_0\cos\theta)\frac{2v_0\sin\theta}{g}=\frac{2v_0^2\sin 2\theta}{g}$$

Tangential and Normal Components of Acceleration:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

$$\mathbf{v} = v\mathbf{T}$$

$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \quad \text{so} \quad |\mathbf{T}'| = \kappa v$$

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} \Rightarrow \mathbf{T}' = |\mathbf{T}'| = \kappa v \mathbf{N}$$

$$\mathbf{a} = v'\mathbf{T} + v\mathbf{T}' = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$

$a_T = v'$ and $a_N = \kappa v^2$

This is complicated because it is difficult to grasp the concepts of curvature and the multiple differentiations of $\mathbf{r}(t)$.

Just realize that $\mathbf{T}$ and $\mathbf{N}$ are directions and, as in circular motion, the tangential component is the change in velocity and the normal component should increase with velocity (Circular centripetal acceleration: $a_c = \frac{v^2}{r}$, try to prove this.

Hint: you need to find the curvature of a circle which turns out to be 1/radius.)

Kepler's Laws of Planetary Motion:

To be rediscovered by Ethan Suttner, ~~eventually~~ soon.

Partial Derivatives:

Functions of Two Variables:

A rule that assigns a unique real number $f(x, y)$ to every ordered pair (x, y).

The set of ordered pairs (x, y) described by f is the ***domain***.

The set values that f takes on is the ***range***.

The ***graph*** of f in $\mathbb{R}^3$ is $z = f(x, y)$.

Level Curves are curves on f such that $f = k$ (a constant).

Applications: Topographical maps (constant altitudes)
Weather maps (constant pressures, temps.)

Functions of Several Variables:

3 variables: $f(x, y, z)$

n variables: $f(\mathbf{x})$ (difficult to visualize, same basic properties, though)

Limits and Continuity:

The ***limit of*** $f(x, y)$ ***as*** (x, y) ***approaches*** (a, b) is L:

$$\lim_{(x,y)\to(a,b)} f(x, y) = \mathrm{L}$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$|f(x, y) - \mathrm{L}| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

This means that the distance between the numbers $f(x, y)$ and L gets closer and closer to zero as (x, y) gets closer and closer to (a, b).

If $f(x, y) \to \mathrm{L}_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to \mathrm{L}_2$ as $(x, y) \to (a, b)$ along a path C_2, where $\mathrm{L}_1 \neq \mathrm{L}_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

A function f is ***continuous*** if its limit exists everywhere on the domain of f.

Partial Derivatives:

For a function $f : z = f(x, y)$ of two variables, its ***partial derivative with respect to x at*** (a, b) is defined by:

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

Other notations:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$

Rules for finding Partial Derivatives of $z = f(x, y)$

1. *To find* f_x, *regard* y *as a constant.*
2. *To find* f_y, *regard* x *as a constant.*

Ex.: $f(x, y) = x^3 + x^2y^3 - 2y^2$

$$f_x(x, y) = 3x^2 + 2xy^3$$

Then, $f_x(2,1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Since partial derivatives are still functions of two variables, we can take their partial derivatives: ***second partial derivatives*** are intuitively defined:

$(f_x)_x, (f_x)_y, (f_y)_x$, and $(f_y)_y$

Ex.: $f(x, y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3 \qquad f_y = 3x^2y^2 - 4y$

$f_{xx} = 6x + 2y^3 \qquad f_{yx} = 6xy^2$

$f_{xy} = 6xy^2 \qquad f_{yy} = 6x^2y - 4$

Notice $f_{xy} = f_{yx}$; this is ***Clairaut's Theorem***.

This can extend to all continuous functions of several variables.

Tangent Planes:

The equation for the tangent plane at a point $P_0\left(x_0, y_0, z_0\right)$ on the graph of $z = f(x, y)$ is:

$$z - z_0 = f_x(x_0, y_0)\left(x - x_0\right) + f_y(x_0, y_0)\left(y - y_0\right)$$

This is a ***linear function*** of two variables; the tangent plane can be called the ***Linear Approximation*** of f at (a,b):

$$z = \mathrm{L}(x, y) \approx f(a,b) + f_x(a,b)\left(x-a\right) + f_y(a,b)\left(y-b\right)$$

The ***total differential*** is the change the tangent plane (or linear approximation $\mathrm{L}(x, y)$) as $f(x, y)$ changes:

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Setting $dx = x - a$ and $dy = y - b$:

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So, $\mathrm{L}(x, y) \approx f(a,b) + dz$.

Chain Rule:

Case 1:

For a function of two variables $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Case 2:

For $z = f(x, y)$, if both $x = g(s,t)$ and $y = h(s,t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \text{ and } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The General Form:

If u is a function of n variables $x_1, x_2, \ldots, x_n$, and each x is a function of $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

For each $i = 1, 2, \ldots, m$.

Ex.: From the ideal gas equation,

$$PV = 8.31T \Rightarrow P = 8.31\frac{T}{V}$$

If

$$T = 300 \text{ K}, \frac{dT}{dt} = 0.1 \text{ K/s}; V = 100 \text{ L}, \frac{dV}{dt} = 0.2 \text{ L/s}$$

To find how P changes with time:

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) \approx 0.042 \text{ kPa/s}$$

Ex.:

$$w = f(x, y, z, t)$$

$$x = x\left(u, v\right), y = y\left(u, v\right), z = z\left(u, v\right), t = t\left(u, v\right)$$

Using the chain rule,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Implicit Differentiation:

For a function $F(x, y, z) = 0$,

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Directional Derivatives and Gradient:

Directional Derivative:

For a function $f(x, y)$ its ***directional derivative*** in the direction of any **unit** vector $\mathbf{u} = \left\langle a, b \right\rangle$:

$$D_{\mathbf{u}} f(x, y) = f_x\left(x, y\right)a + f_y(x, y)b$$

This can be written as a dot product:

$$f_x\left(x, y\right)a + f_y(x, y)b = \left\langle f_x\left(x, y\right), f_y(x, y)\right\rangle \cdot \mathbf{u}$$

$\left\langle f_x\left(x, y\right), f_y(x, y)\right\rangle$ is called the ***gradient vector***:

$$\nabla f(x, y) = \left\langle f_x\left(x, y\right), f_y(x, y)\right\rangle = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

So, $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$

(which is a maximum when $\nabla f(x, y) \parallel \mathbf{u}$)

The gradient vector should be interpreted as the magnitude of the directional derivative at any point (x, y).

Ex.:

$$f(x, y) = x^2y^3 - 4y$$

$$\nabla f(x, y) = \left\langle f_x, f_y \right\rangle = 2xy^3\hat{\mathbf{i}} + \left(3x^2y^2 - 4\right)\hat{\mathbf{j}}$$

At (2,1),

$$\nabla f(2,1) = -4\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$$

In the direction of $\mathbf{v} = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{29}}\left(2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}\right)$$

$$D_{\mathbf{u}} f(2,1) = \nabla f(2,1) \cdot \mathbf{u} = (-4\hat{\mathbf{i}} + 8\hat{\mathbf{j}}) \cdot \left(\frac{2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}}{\sqrt{29}}\right)$$

$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

The gradient vector is perpendicular to level curves.

Maximum and Minimum Values:

Terminology:

(a,b) is

an ***absolute maximum*** if- $f(x,y) \le f(a,b)$

an ***absolute minimum*** if- $f(x,y) \ge f(a,b)$

for all points (x,y) in the domain of f

a ***local maximum*** if- $f(x,y) \le f(a,b)$

a ***local minimum*** if- $f(x,y) \ge f(a,b)$

for all points (x,y) near (a,b).

First Derivative Test:

If $f_x(a,b)=0$ and $f_y(a,b)=0$,

(a,b) is a ***critical point***. Meaning it either is a local minimum **or** maximum **or** neither.

Second Derivative Test:

Given a critical point (a,b) and D such that

$$D(a,b)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a,b)\,f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$$

a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

c) If $D<0$, then $f(a,b)$ is a neither a local maximum or minimum.

In c) $f(a,b)$ is a ***saddle point***.

If $D=0$, then no conclusion is drawn.

Lagrange Multipliers:

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$$

and $g(x,y,z)=k$

$$f_x=\lambda g_x \quad f_y=\lambda g_y \quad f_z=\lambda g_z \quad g(x,y,z)=k$$

$$\nabla f(x,y)=\lambda\nabla g(x,y)$$

and $g(x,y)=k$

$$f_x=\lambda g_x \quad f_y=\lambda g_y \quad g(x,y)=k$$

Ex.: Find the maximum volume of an open lid box made out of $12\ \text{m}^2$ of cardboard.

$$V=xyz$$

$$2xy+2yz+xy=12$$

$$z=(12-xy)/\left[2(x+y)\right]$$

So, $V=xy\dfrac{(12-xy)}{2(x+y)}=\dfrac{12xy-x^2y^2}{2(x+y)}$

$$\frac{\partial V}{\partial x}=\frac{y^2(12-2xy-x^2)}{2(x+y)^2} \quad \frac{\partial V}{\partial y}=\frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

To maximize, $\partial V/\partial x=\partial V/\partial y=0$

$12-2xy-x^2=0 \quad 12-2xy-y^2=0$

This implies, $x=y\Rightarrow 12-3x^2=0$

$x=2, y=2,\ z=(12-2\cdot 2)/\left[2(2+2)\right]=1$

The box has a base of $2\ \text{m}\times 2\ \text{m}$ and a height of $1\ \text{m}$.

Using Lagrange multipliers,

$$V=xyz$$

$$g(x,y,z)=2xy+2yz+xy=12$$

$$V_x=\lambda g_x \quad V_y=\lambda g_y \quad V_z=\lambda g_z \quad 2xy+2yz+xy=12$$

Solve the system. $\begin{cases} yz=\lambda(2z+y) \\ xz=\lambda(2z+x) \\ xy=\lambda(2x+2y) \\ 2xy+2yz+xy=12 \end{cases}$

(1) $xyz=\lambda(2xz+xy)$

(2) $xyz=\lambda(2yz+yx)\Rightarrow$

(3) $xyz=\lambda(2xz+2yz)$

From (1) and (2):

$2xz+xy=2yz+yx$

$2xz=2yz\Rightarrow x=y$

From (2) and (3):

$2yz+yx=2xz+2yz$

$yx=2xz\Rightarrow y=2z$

$x=y=2z$

$4z^2+4z^2+4z^2=12$

$z=1, x=y=2$

Multiple Integrals:

Double Integrals:

Double Integrals over Rectangles:

If $f(x,y) \geq 0$, then the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

Fubini's Theorem:

$R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

This is the best way to solve ***iterated integrals*** (by evaluating two single integrals)

Average Value over a Region

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

Double Integrals over General Regions:

Type 1 Region:

$D = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Type 2 Region:

$D = \{(x,y) \mid c \leq y \leq d, h_1(x) \leq x \leq h_2(x)\}$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$

Some Important Properties:

1. $V = \iint_D 1\,dA = A(D)$
2. If $m \leq f(x,y) \leq M$ for all (x,y) in D, then $mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$

Double Integrals in Polar Coordinates:

Let R be a ***polar rectangle***.

$R = \{(x,y) \mid 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

$$\iint_R f(x,y)\,dA == \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

Applications of Double Integrals:

Density is mass per unit area:

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A}.$$

So, $m = \iint_D \rho(x,y)dA$

Similar for charge distribution:

$$Q = \iint_D \sigma(x,y)dA$$

Moments and Center of Mass:

The ***moments about the x- and y- axes*** respectively:

$$M_x = \iint_D y\rho(x,y)dA$$

$$M_y = \iint_D x\rho(x,y)dA$$

The ***center of mass*** coordinates of a lamina $(\bar{x}, \bar{y})$ are given by:

$$\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$$

The ***moments of inertia about the x- and y-axes*** and ***about the origin*** (***polar moment***) are given by:

$$I_x = \iint_D y^2\rho(x,y)dA$$

$$I_y = \iint_D x^2\rho(x,y)dA$$

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)dA$$

The ***radius of gyration*** is given by:

$mR^2 = I$

Probability:

The ***joint density function*** of X and Y is a function f such that the probability that (X,Y) lies in a region D:

$$P((X,Y) \in D) = \iint_D f(x,y)dA$$

$$\iint_{\mathbb{R}^2} f(x,y)dA = 1$$

X and Y are ***independent random variables*** if f can be represented by:

$f(x,y) = f_1(x)f_2(y)$

The ***expected values*** of a joint density function are the ***X-mean*** and ***Y-mean***.

$$\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)dA \quad \mu_2 = \iint_{\mathbb{R}^2} yf(x,y)dA$$

A ***single random variable*** is *normally distributed* if its probability density function is of the form:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

Surface Area:

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$$

Derivation:

Let $\mathbf{a}$ and $\mathbf{b}$ be the tangent vectors at point P_{ij}:

$f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ respectively.

So,
$$\mathbf{a} = \Delta x\,\hat{\mathbf{i}} + f_x(x_i, y_j)\Delta x\,\hat{\mathbf{k}}$$
$$\mathbf{b} = \Delta y\,\hat{\mathbf{i}} + f_y(x_i, y_j)\Delta y\,\hat{\mathbf{k}}$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix}$$

$$= -f_x(x_i, y_j)\Delta x\Delta y\,\hat{\mathbf{i}} - f_y(x_i, y_j)\Delta x\Delta y\,\hat{\mathbf{j}} + \Delta x\Delta y\,\hat{\mathbf{k}}$$

Since $\Delta A = \Delta x\Delta y$,

$$\mathbf{a}\times\mathbf{b} = \left[-f_x(x_i, y_j)\,\hat{\mathbf{i}} - f_y(x_i, y_j)\,\hat{\mathbf{j}} + \hat{\mathbf{k}}\right]\Delta A$$

The differential area covered by $\mathbf{a}\times\mathbf{b}$ is

$$\left|\mathbf{a}\times\mathbf{b}\right| = \sqrt{\left[f_x(x_i, y_j)\right]^2 + \left[f_y(x_i, y_j)\right]^2 + 1}\;\Delta A$$

Integrating yields

$$A(S) = \iint_D \sqrt{\left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2 + 1}\;dA\,.$$

Triple Integrals:

Triple Integrals over Boxes:

$$B = [a,b]\times[c,d]\times[s,r]$$

$$\iiint_B f(x,y,z)\,dV = \int_s^r\int_c^d\int_a^b f(x,y,z)\,dxdydz$$

As for all iterated integrals, the integrations can be taken in any order.

Triple Integrals over General Regions:

$$E = \left\{(x,y,z)\,|\,a\le x\le b, g_1(x)\le y\le g_2(x), u_1(x,y)\le z\le u_2(x,y)\right\}$$

$$\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dzdydx$$

Applications of Triple Integrals:

If $f(x,y,z)=1$ for all points in E, then

$$V(E) = \iiint_E dV$$

Ex.: Evaluate the triple integral $\iiint_B xyz^2 dV$ over rectangular box B.

$$B = \left\{(x,y,z)\,|\,0\le x\le 1, -1\le y\le 2, 0\le z\le 3\right\}$$

$$\iint_B xyz^2 dA =$$

$$\int_0^3\int_{-1}^2\int_0^1 xyz^2 dxdydz = \int_0^3\int_{-1}^2\left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dydz$$

$$\int_0^3\int_{-1}^2 \frac{yz^2}{2}dydz = \int_0^3\left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$\int_0^3 \frac{3z^2}{4}dz = \left.\frac{z^3}{4}\right]_0^3 = \frac{27}{4}$$

Ex.: Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, $x = 1$.

$$E = \left\{(x,y,z)\,|\,-1\le y\le 1, y^2\le x\le 1, 0\le z\le x\right\}$$

$$\rho(x,y,z) = \rho$$

$$m = \iiint_E \rho\,dV = \int_{-1}^1\int_{y^2}^1\int_0^x \rho\,dzdxdy$$

$$= \rho\int_{-1}^1\int_{y^2}^1 x\,dxdy$$

$$= \rho\int_{-1}^1\left[\frac{x^2}{2}\right]_{x=y^2}^{x=1} dy = \frac{\rho}{2}\int_{-1}^1\left(1-y^4\right)dy =$$

$$\rho\int_0^1\left(1-y^4\right)dy = \rho\left[y-\frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5}$$

By symmetry, $\overline{y} = 0$

The other moments are

$$M_{yz} = \iiint_E x\rho\,dV = \int_{-1}^1\int_{y^2}^1\int_0^x x\rho\,dzdxdy$$

$$= \rho\int_{-1}^1\int_{y^2}^1 x^2\,dxdy$$

$$= \rho\int_{-1}^1\left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy =$$

$$\frac{2\rho}{3}\int_0^1\left(1-y^6\right)dy = \frac{4\rho}{7}$$

$$M_{xy} = \iiint_E z\rho\,dV = \int_{-1}^1\int_{y^2}^1\int_0^x z\rho\,dzdxdy$$

$$= \rho\int_{-1}^1\int_{y^2}^1\left[\frac{z^2}{2}\right]_{z=0}^{z=x} dxdy = \frac{\rho}{2}\int_{-1}^1\int_{y^2}^1 x^2\,dxdy$$

$$= \frac{\rho}{2}\int_{-1}^1\left[\frac{x^3}{3}\right]_{x=y^2}^{x=1} dy =$$

$$\frac{\rho}{3}\int_0^1\left(1-y^6\right)dy = \frac{2\rho}{7}$$

So the center of mass is

$$\left(\overline{x}, \overline{y}, \overline{z}\right) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\frac{5}{7}, 0, \frac{5}{14}\right)$$

Triple Integrals in Cylindrical and Spherical Coordinates:

The ***triple integral*** of a function f ***in cylindrical coordinates***:

$$\iiint_E f(x,y,z)\,dV =$$

$$\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,rdzdrd\theta$$

The ***triple integral*** of a function f ***in spherical coordinates*** over a spherical wedge E:

$$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

$$\iiint_E f(x,y,z)\,dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\sin\phi)\,\rho^2\sin\phi\; d\rho\, d\theta\, d\phi$$

Vector Calculus:

Vector Field:

A function $\mathbf{F}$ that assigns a vector to each point in a plane $\mathbf{F}(x,y)$, or in space $\mathbf{F}(x,y,z)$.

Gradient Field:

For a **scalar function** $f(x,y)$, the ***gradient vector field*** is defined by:

$$\nabla f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$$

A vector field $\mathbf{F}$ is called a ***conservative vector field*** if it is the gradient of some scalar function: $\mathbf{F} = \nabla f$ (It can be called the ***potential function***).

Line Integrals:

The integral of a function f over a curve $C: \mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$: $\int_C f(x,y)\,ds$

And since the differential arc length $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, the line integral can be written as

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Ex.: Given a function $f(x,y) = 2 + x^2 y$ and a curve $x^2 + y^2 = 1$, Evaluate the function over the *upper half* of the given unit circle given.

$x^2 + y^2 = 1$

$x = \cos t \quad y = \sin t \quad$ from $0 \le t \le \pi$

$$\int_C (2 + x^2 y)\,ds =$$

$$\int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt =$$

$$\int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\;dt =$$

$$\int_0^\pi (2 + \cos^2 t \sin t)\,dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi =$$

$$2\pi + \frac{2}{3}$$

The ***line integrals of f along C with respect to x and y***:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt$$

Line Integrals in Space:

$$\int_C f(\mathbf{r}(t))\left|\mathbf{r}'(t)\right|dt$$

The ***line integral of $\mathbf{F}$ along*** $C: \mathbf{r}(t)$ is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

or

$$\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt =$$

$$\int_a^b (P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}})\cdot\left(x'(t)\,\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}}\right) =$$

$$\int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)dt =$$

$$\int_a^b \left(Pdx + Qdy + Rdz\right) = \int_C \mathbf{F}\cdot d\mathbf{r}$$

Fundamental Theorem for Line Integrals:

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

This means that the line integral for the gradient vector can be determined just by knowing the endpoints of the curve.

Proof: Using the Chain Rule and the Fundamental Theorem of Calculus

$$\int_C \nabla f\cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\right)dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Independence of Path:

For two given ***paths*** C_1 and C_2 with the same initial and terminal points, the line integral is ***independent of path*** iff $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every ***closed*** path C. $\mathbf{F}$ is called ***conservative*** if there exists a function f such that $\nabla f = \mathbf{F}$.

Let $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ be a ***simply-connected*** region D. $\mathbf{F}$ is conservative iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Ex.:
$$\mathbf{F}(x,y)=(x-y)\hat{\mathbf{i}}+(x-2)\hat{\mathbf{j}}$$
$$\frac{\partial P}{\partial y}=-1\neq 1=\frac{\partial Q}{\partial x}$$

$\mathbf{F}$ is not conservative.
$$\mathbf{F}(x,y)=(3x+2xy)\hat{\mathbf{i}}+(x^2-3y^2)\hat{\mathbf{j}}$$
$$\frac{\partial P}{\partial y}=2x=\frac{\partial Q}{\partial x}$$

$\mathbf{F}$ is conservative.

To find a function f such that $\nabla f=\mathbf{F}$, its gradient vector, is:
$$\mathbf{F}(x,y)=(3x+2xy)\hat{\mathbf{i}}+(x^2-3y^2)\hat{\mathbf{j}}$$

Use partial integration:

(1) $f_x(x,y)=3+2xy$

(2) $f_y(x,y)=x^2-3y^2$

Integrating (1) with respect to x,

(3) $f(x,y)=3x+x^2y+g(y)$

Differentiating (3) with respect to y,

(4) $f_y(x,y)=x^2+g'(y)$

Comparing (2) and (4),
$$g'(y)=-3y^2$$

Integrating,
$$g(y)=-y^3+K$$

Substituting into (3),
$$f(x,y)=3x+x^2y+-y^3+K$$

Green's Theorem:
$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_a^b Pdx+Qdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$
$$A(D)=\iint_D 1dA=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$
$$1=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$$

This is true for

$$\begin{array}{lll} P(x,y)=0 & P(x,y)=-y & P(x,y)=-\frac{1}{2}y \\ Q(x,y)=x & Q(x,y)=0 & Q(x,y)=\frac{1}{2}x \end{array}$$

$$A=\oint_C x\,dy=\oint_C -y\,dx=\tfrac{1}{2}\oint_C x\,dy-y\,dx$$

Ex. Area of an Ellipse:
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$x=a\cos t \quad y=b\sin t$

where $0\le t\le 2\pi$
$$A=\tfrac{1}{2}\oint_C x\,dy-y\,dx$$
$$=\tfrac{1}{2}\oint_C a\cos t\,b\cos t\,dt-(b\sin t)(-a\sin t)\,dt$$
$$=\frac{ab}{2}\int_0^{2\pi}dt=\pi ab$$

Curl and Divergence:

The ***curl of*** $\mathbf{F}$ is a vector field of a vector field $\mathbf{F}$:
$$\text{curl}\,\mathbf{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\hat{\mathbf{j}}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

The ***vector differential operator*** is defined as:
$$\nabla=\frac{\partial}{\partial x}\hat{\mathbf{i}}+\frac{\partial}{\partial y}\hat{\mathbf{j}}+\frac{\partial}{\partial z}\hat{\mathbf{k}}$$

∇ acts on a scalar f to produce the gradient vector:
$$\nabla f=\frac{\partial f}{\partial x}\hat{\mathbf{i}}+\frac{\partial f}{\partial y}\hat{\mathbf{j}}+\frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

∇ acts on a vector $\mathbf{F}$ to produce the curl of $\mathbf{F}$:
$$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\text{curl}\,F$$

Proof of: $\text{curl}\,(\nabla f)=\mathbf{0}$
$$\text{curl}\,(\nabla f)=\nabla\times(\nabla f)=$$
$$\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{vmatrix}\text{expanding gives }\mathbf{0}$$

If $\text{curl}\,\mathbf{F}=\mathbf{0}$, then $\mathbf{F}$ is a conservative vector.

Geometric Interpretation: $\text{curl}\,\mathbf{F}(x,y,z)$ represents the rotation about (x,y,z).

Divergence:
$$\text{div}\,\mathbf{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\nabla\cdot\mathbf{F}$$

Geometric Interpretation: $\text{div}\,\mathbf{F}(x,y,z)$ represents the flow away from (x,y,z).

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$$

Parametric Surfaces

$$\mathbf{r}(u,v) = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}} + z(u,v)\hat{\mathbf{k}}$$

Surfaces of Revolution:

$$x = x \quad y = f(x)\cos\theta \quad z = f(x)\sin\theta$$

Tangent Planes:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\hat{\mathbf{i}} + \frac{\partial y}{\partial u}(u_0,v_0)\hat{\mathbf{j}} + \frac{\partial z}{\partial u}(u_0,v_0)\hat{\mathbf{k}}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\hat{\mathbf{i}} + \frac{\partial y}{\partial v}(u_0,v_0)\hat{\mathbf{j}} + \frac{\partial z}{\partial v}(u_0,v_0)\hat{\mathbf{k}}$$

$$\mathbf{n} = \left|\mathbf{r}_u \times \mathbf{r}_v\right|$$

The tangent plane has a normal vector $\mathbf{n}$.

Surface Area:

For a parametric surface $\mathbf{r}(u,v)$,

$$A(S) = \iint_D \left|\mathbf{r}_u \times \mathbf{r}_v\right| dA.$$

Surface Integrals:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

The ***surface integral of*** $\mathbf{F}$ ***over*** S is:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n} \quad \text{(also called \textbf{\textit{flux}})}.$$

Stokes' Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

Divergence Theorem:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}(x,y,z)\,dV$$

Second-Order Differential Equation:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

If $G(x) = 0$, it is called ***homogeneous***.

Solutions are closed under linear combinations.

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions, then $y(x)$ is a general solution.

Differential Equations are not easy to solve, for now- we can do it they have *constant coefficients.*

Differentiating $y = e^{rx}$,

$$ay'' + by' + cy = 0 \Rightarrow$$

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$(ar^2 + br + c)e^{rx} = 0$$

$$ar^2 + br + c = 0$$

This is the ***auxiliary/characteristic equation***.

If:

$$b^2 - 4ac > 0 \;\Rightarrow\; y = c_1e^{r_1x} + c_2e^{r_2x}$$

$$b^2 - 4ac = 0 \;\Rightarrow\; y = c_1e^{rx} + c_2e^{rx}$$

$$b^2 - 4ac < 0 \;\Rightarrow\; y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

where $r_{1,2} = \alpha \pm \beta i$.

Initial Value:

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

Boundary Value

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

Method of Undetermined Coefficients:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \neq 0$$

Ex.:

$$y'' + y' - 2y = x^2$$

$$r = 1, -2$$

$$y_c = c_1e^{x} + c_2e^{-2x}$$

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

$$A = -\frac{1}{2} \quad B = -\frac{1}{2} \quad C = -\frac{3}{4}$$

$$y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

$$y = y_c + y_p = c_1e^{x} + c_2e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$