Physique des marchés
modèles d'agents II

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Variable N games

Rationale:

- Bad player should not play
- Play only if score good enough: compare with fixed reward ϵ.
- Add N_p predictable players with a single strategy ("producers")

$$\Omega^{\mu} = \sum_{j=1}^{N_p} b_j^{\mu} \sim \mathcal{N}(0, N_p) \propto O(\sqrt{N_p})$$

Simplest model

- agent i: 1 strategy a_i^{μ}, 1 score $U_{i,t}$, agent $n_i(t)$

$$U_{i,t+1} = U_{i,t} - a_i^{\mu_t} A(t) - \epsilon$$

$$n_{i,t} = \theta(U_{i,t}) \in \{0,1\}\text{: play or not}$$

$$A(t) = \underbrace{\sum_i a_i^{\mu_t} n_{i,t}}_{} + \underbrace{\Omega^{\mu_t}}_{\text{producers}}$$

Average dynamics

Idea: coarse time by factor P

$$E(U_{i,t+1}|U_{i,t}) = U_{i,t} - E(a_i^{\mu_t} A(t)) - \epsilon E(n_{i,t})$$

Set

$$\phi_i = E(n_{i,t})$$
$$\tau = t/P$$

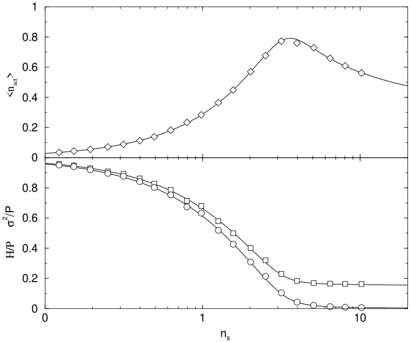
$$\frac{dU_{i,\tau}}{d\tau} \simeq -\sum_\mu a_i^\mu E(A(\tau)|\mu) - \epsilon\phi_i \sim -\frac{\partial H}{\partial \phi_i}$$

where $E(A|\mu) \simeq \sum_j E(n_j)\, a_j^\mu = \sum_j \phi_j\, a^\mu$

if

$$H = \frac{1}{2}\sum_\mu E(A|\mu)^2 + \epsilon \sum_j \phi_j = H_0 + \epsilon N_{act}$$

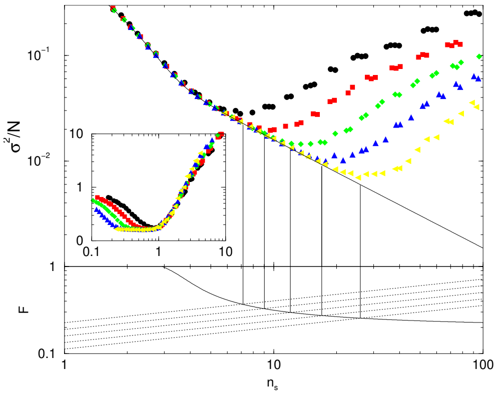
Variable N model: results



small N large N

- $n_s = N/P = 1/\alpha$, $\langle n_{\text{act}} \rangle = \frac{1}{P} \sum_i E(n_i)$
- saturation of the number of active speculators
- $H > 0$ as soon as $\epsilon > 0$ (exact solution)

Variable N model: results for larger N



small N large N

- large fluctuations $\longleftrightarrow$ small signal-to-noise ratio

Suitable mathematical formalism

Is $H_0 = 0$ doable?

$$
\begin{aligned}
E(A|\mu) &= 0 \\
\iff \Omega^\mu + \sum_i a_i^\mu E(n_i) &= 0
\end{aligned}
$$

Set $\phi_i = E(n_i)$.

$$\Omega^\mu + \sum_i a_i^\mu \phi_i = 0 \qquad 0 \leq \phi_i \leq 1$$

P equations, N bounded variables $\implies$ yes for $N = KP$, $K > 1$

MG and frustrated systems

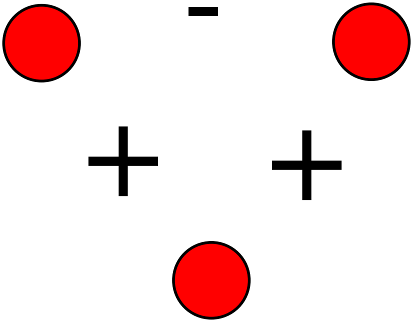
- Predictability $H = \frac{1}{P}\sum_\mu E(A|\mu)^2 + \epsilon \sum_i \phi_i$

$$H = \frac{1}{P}\sum_\mu \left[(\Omega^\mu)^2 + \sum_i \Omega^\mu a_i^\mu \phi_i + \sum_{i,j} a_i^\mu a_j^\mu \phi_i \phi_j \right] + \epsilon \sum_i \phi_i$$
$$= \frac{1}{P}\sum_\mu \left[(\Omega^\mu)^2 \right] + \sum_i (h_i + \epsilon)\phi_i + \sum_{i,j} J_{i,j} \phi_i \phi_j$$

- Random heterogeneity $\Longleftrightarrow$ random h_i and $J_{i,j}$
- Random h_i: cf random field Ising model
- Random $J_{i,j}$: frustrated system, spin-glass

Frustrated systems

- Frustration: friend of friend = friend ?



- Random $J_{i,j}$: if $s_i \in \{-1, +1\}$, 2^N configurations to test
- Special mathematical methods to deal with random $J_{i,j}$.
- Parisi: Nobel prize in Physics 2021 for spin-glass problems

Exact solution of Minority Games

Problem 1: how to compute minimum of H?

- $N \to \infty$ limit
- Predictability H_N minimised: $H_N \equiv$cost function
- Partition function

$$Z(\{a_i, \Omega\}) = \int_0^{+1} \prod_i d\phi_i \, e^{-\beta H(\{a_i,\Omega\},\{\phi_i\})}$$

- Minimisation

$$\min_{m_i} H(\{a_i, \Omega\}, \{\phi_i\}) = \lim_{\beta \to \infty} -\frac{1}{\beta} \log Z(\{a_i, \Omega\})$$
$$= \lim_{\beta \to \infty} -\frac{1}{\beta} \log \left[e^{-\beta \min H} \prod_i d\phi_i \, e^{-\beta(H - \min H)} \right]$$

Exact solution of the MG

Problem 2: average over heterogeneity

- Mathematically

$$H_N(\{a_i\},\{\phi_i\}) = \frac{1}{P}\sum_\mu E(A|\mu)^2 = \frac{1}{P}\sum_\mu(\Omega^\mu + \sum_i a_i^\mu E(\phi_i))^2$$

  $\{a_i, \Omega\}$: random heterogeneity

- Minimum of H_N depends on $\{a_i, \Omega\}$
- Compute $E_{\{a_i,\Omega\}}(\min_{\{m_i\}} H)$: average over heterogeneity

$$\begin{aligned}\tilde{H}_{\min} &= \lim_{N\to\infty}\min_{\phi_i} E_{\{a_i\}} H_N(\{a_i,\Omega\},\{\phi_i\}) \\ &= \lim_{\beta\to\infty} -\frac{1}{\beta}\lim_{N\to\infty} E_{\{a_i,\Omega\}}[\log Z(\{a_i,\Omega\})]\end{aligned}$$

Exact solution of MG: replica trick

- $E(\log Z)$: generally impossible to compute
- Trick:
$$E(\log Z) = \lim_{n \to 0} \frac{E(Z^n) - 1}{n}$$
- What is $E_{\{a_i, \Omega\}}(Z^n)$?
- Z^n : same agents, n duplicates of ϕ_i, $\phi_{i,c}$, $c = 1, \cdots, n$
- Compute now
$$E_{\{a_i, \Omega\}}\left[e^{-\frac{\beta}{P}(\Omega^\mu + \sum_j a_j^\mu \phi_{j,c})^2}\right]$$
- Gaussian integrals, doable.

Exact solution of the standard MG

Eventually (after about 6 A4 pages of calculus)

- Predictability, $\alpha = P/N > \alpha_c = 0.3374\ldots$

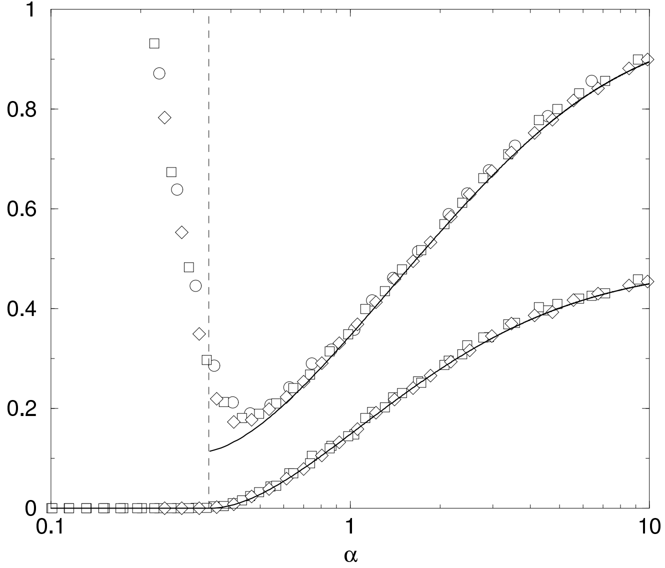
$$\tilde{H}_0 = \frac{1+Q}{2(1+\chi)^2}$$

$$Q = \frac{1}{N}\sum_i \phi_i^2$$

- No predictability $\Longleftrightarrow \chi = \infty$: phase transition.
- $\alpha_c = \operatorname{erf}\left[\sqrt{\left|\log[\sqrt{\pi}(2-\alpha_c)]\right|}\right]$
- Fluctuations

$$\frac{\sigma^2}{N} \to \tilde{H}_0 + \frac{1-Q}{2}$$

Exact solution of Minority Games



Dynamical solutions of interacting agents

- H minimised: stationary state, static approach
- Exact dynamical solutions known: De Dominicis generating functionals
- From N dynamical equations to 1 effective agent equation, with complex time structure
- See Coolen book "Mathematical theory of minority games".

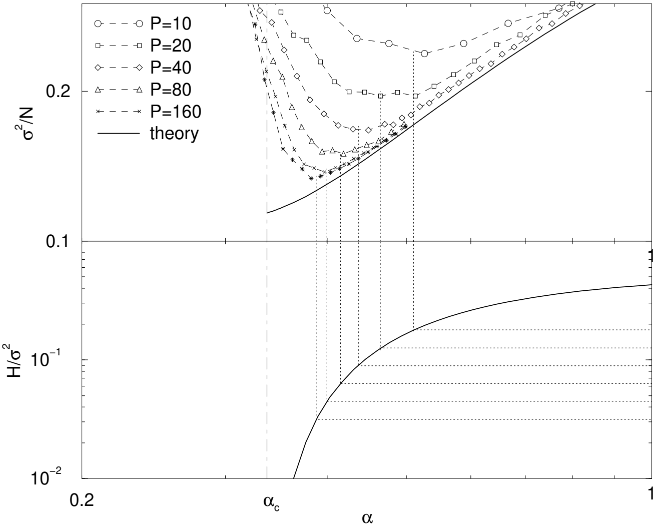
Why solution exact?

- Mathematically complex (non-linear, dynamical, heterogeneity) N-agent model
- Exact solutions generically in 1, 2, ∞ dimensional models
- Payoff

$$-a_i A$$

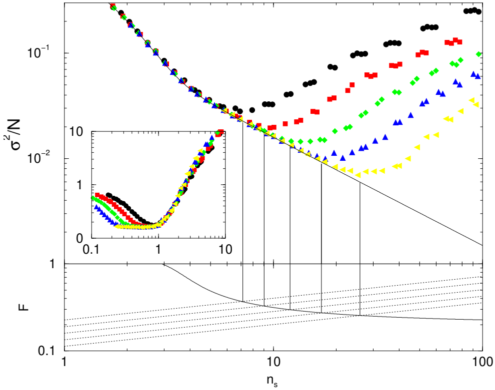
- Everybody interacts with everybody else through aggregate quantity
- Mean-field type of interaction
- When is a modified MG still mean-field?

Signal-to-noise transition: standard MG



Signal-to-noise transition: variable-N MG

$F =$ signal ratio of the strategies of the agents



Summary so far

- Agents minimise predictability by learning
- When *exploitable* predictability is too small $\rightarrow$ explosion of volatility
- Exploitable means

$$\frac{\text{signal}}{\text{noise}} \simeq \frac{H}{\sigma} \text{ large enough}$$

- Signal OF THE STRATEGIES of the agents

Learning $\rightarrow$ instability

Pazelt and Pawelzik: "criticality of adaptive control dynamics" (2011)

- Signal $y_t = \alpha y_{t-1} + \beta_{t-1}$,
  - α unknown
  - $\beta \sim \mathcal{N}(0, \sigma^2)$
- The agents learn α from the last m time steps and try to cancel y_{t+1}
- Example: $m = 2$, minimize $E(y_{t+1}^2 | y_{t,}, y_{t-1}) \rightarrow$

$$\tilde{\alpha}_{t+1} = \frac{y_t}{y_{t-1}} + \tilde{\alpha}_t$$

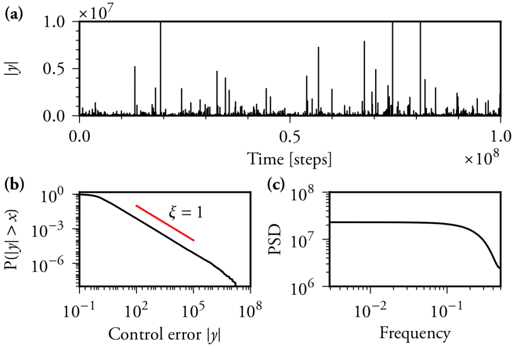
- The signal becomes

$$y_{t+1} = (\alpha - \tilde{\alpha}_{t+1}) y_t + \beta_t = -\frac{y_t}{y_{t-1}} \beta_{t-1} + \beta_t$$

- One shows that

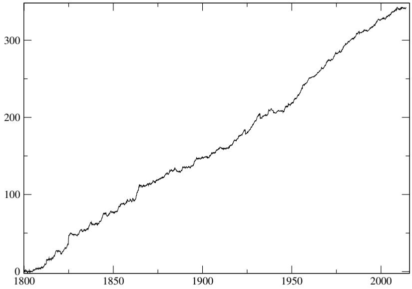
$$P(y > |r|) \propto \frac{1}{|r|^m}$$

Suppression d'un signal



Price predictability

Performance of trend-following
Lempérière *et al.* [link]



- Human bias towards trend following
- Using a trend-following strategy makes the price persistent

How predictability appears: strategy use

Influence of trend-following on random prices

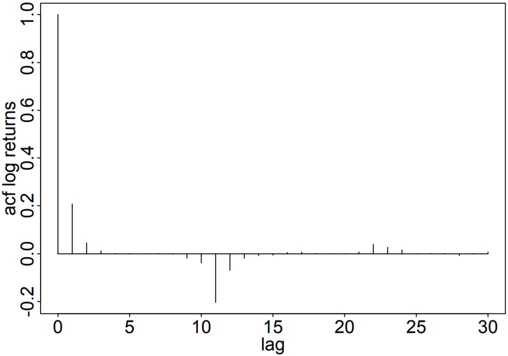


FIGURE 2. The autocorrelation function for Equation 10 with $\alpha = 0.2$ and $\theta = 10$. The positive coefficients for small τ indicate short term trends in prices, and the negative coefficients indicate longer term oscillations.

How predictability appears: constraints

Examples:

- ETFs, no positions overnight, etc
- Control exposure to FX for international companies
- Hedging
- Slow reaction speed (large volume, investment committees)
- Preference for certain strategies
- ...

Measures of price predictability

- Strategies $\longleftrightarrow$ predictability measure

  $$H = |E(\text{gain})| = |E(x_t r_{t+1})|, \quad H > 0 \iff \text{predictability}$$

  $\rightarrow$strategy = tool to find predictability

- Note: use statistical tests on $g_{t+1} = x_t r_{t+1}$ instead

Learning how to remove predictability

1. Strategy $\rightarrow$ detection
2. Use strategy
3. Decrease strategy performance
   $\equiv$ decrease predictability DETECTABLE BY STRATEGY
4. Side-effect: inject predictability for other people
5. Markets are (almost) always predictable with new types of strategies

How to remove predictability

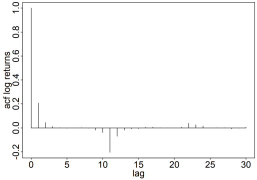


FIGURE 2. The autocorrelation function for Equation 10 with $\alpha = 0.2$ and $\theta = 10$. The positive coefficients for small τ indicate short term trends in prices, and the negative coefficients indicate longer term oscillations.

- ACF induced by strategy use

$$C_r(\tau) \simeq \alpha^{\tau} - \alpha^{(\theta+2-\tau)}, \ \tau \leq \theta + 1$$

- Use anti-strategy $\alpha \to -\alpha$ modifies predictability
- Predictability depends on the fraction of agents using $+\alpha$ and $-\alpha$

Strategy use and Minority Games

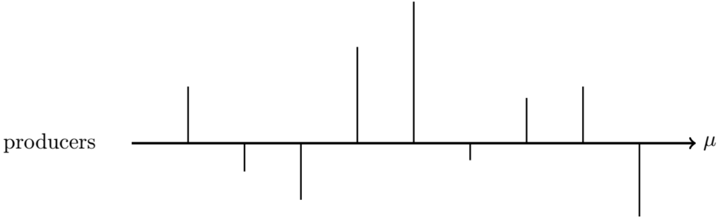
- Two choices $+\alpha$ and $-\alpha$
- If too many people use $+\alpha$, it is best to use $-\alpha$

  $\rightarrow$ Minority Games in the space of strategy use
- More generally, be in minority *in the strategy space*

How to model predictability dynamics

Minority games with P states, denoted by μ

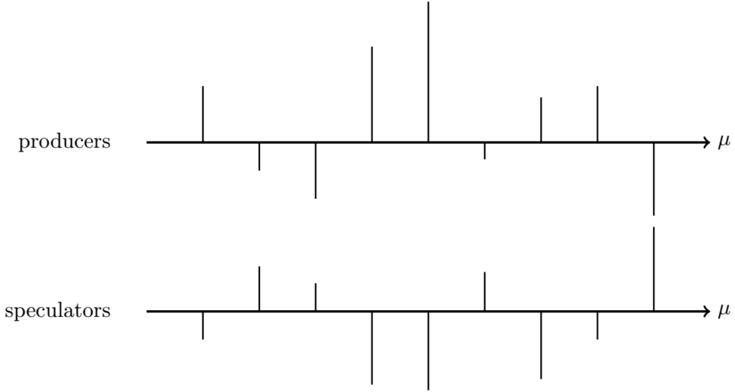
- Producers $\rightarrow \Omega_{\mu}$, fixed, from known distribution.
  Constraint: not adaptive



How to model predictability dynamics

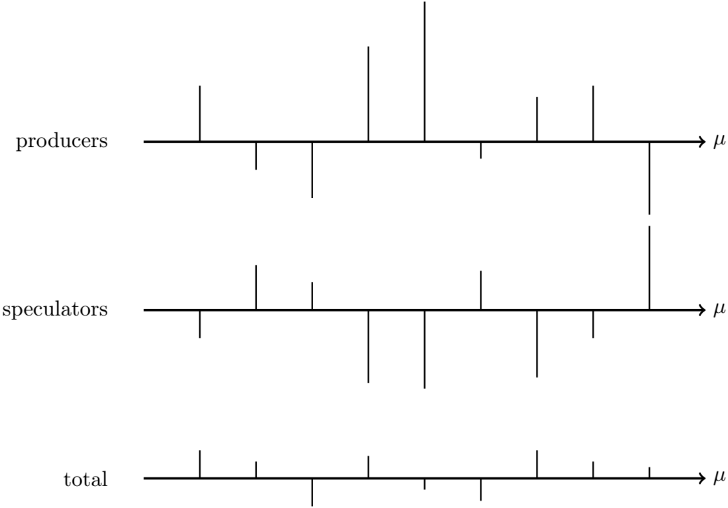
Minority games with P states, denoted by μ

- Producers $\rightarrow \Omega_\mu$
  Not adaptive
- Speculators $\rightarrow A_{spec}$ that try to reduce Ω_μ.
  Play if payoff $> \epsilon$ per time step. Constraint: 1 strategy



How to model predictability dynamics

$$A_{total}(t) = \Omega_{\mu(t)} + A_{spec}(t)$$



Result of learning to use strategies

1. Decrease of predictability

$$H_{total} \leq H_{producers} = E(\Omega^2)$$

2. Contribution of producers almost completely disappears

$$E[\text{sign}(A_{total}\Omega_\mu)|\mu] \to 0$$

3. If H_{total} small, any change may switch the sign of $A_{total}[\mu_t]$ (susceptibility)

Payoffs of producers

- Decomposition $A_{tot}(t) = A_{prod} + A_{spec}$

$$A_{prod}(t) = \Omega_{\mu(t)} \quad A_{spec}(t) = \sum_{i=1}^{N_s} n_i(t) a_{i,\mu(t)}$$

- Total payoff of producers

$$G_{prod}(t) = -\Omega_{\mu(t)} A_{tot}(t)$$

- Average payoff: denote $\frac{1}{P}\sum_{\mu} E(X|\mu) \equiv \overline{X}$,

$$E(G_{prod}) = \overline{G_{prod}} = -\overline{\Omega A_{total}}$$

- Without speculators

$$E(G_{prod}) = -\overline{\Omega^2} < 0$$

- With speculators, if $A_{spec} = -\Omega_{\mu} \ \ \forall \mu$

$$E(G_{prod}) = -\overline{\Omega [A_{spec} + \Omega]} = 0$$

Payoffs of speculators

- Total payoff of speculators

$$G_{spec}(t) = -A_{spec}(t)A_{tot}(t)$$
$$= -A_{spec}(t)\Omega_{\mu(t)} - [A_{spec}(t)]^2$$

- Taking averages

$$E(G_{spec}) = \underbrace{-\overline{A_{spec}\Omega}}_{\lesseqqgtr 0} \underbrace{-E(A^2_{spec})}_{\leq 0}$$

What do agents minimize?

- Each agent has a single, simple strategy, does not care about global behaviour

$$U_i(t+1) = U_i(t) - a_{i,\mu(t)} A_{total}(t) - \epsilon$$

- Continuous time approximation

$$\begin{aligned} \frac{dU_i}{dt} &= -\overline{a_i A_{total}(t)} - \epsilon + \eta_i(t) \\ &= -\frac{1}{2}\frac{\partial}{\partial \phi_i} H_\epsilon(t) + \eta_i(t) \\ \phi_i &= E(n_i) \end{aligned}$$

where

$$\begin{aligned} H_\epsilon &= \overline{(\Omega + a_i \phi_i)^2} + 2\epsilon \sum_i \phi_i \\ &= H_0 + 2\epsilon E(N_{act}) \end{aligned}$$

Left-over predictability from costs and benchmarks

- Predictability in the stationary state
  $\simeq$ minimum performance neeeded $\simeq$ transaction costs

- In the MG with variable number of speculators, the agents minimize
  $$H_\epsilon = H_0 + 2\epsilon E(N_{active})$$
- Assuming no active agents at the beginning, H_0 is minimized first until it is comparable with $2\epsilon E(N_{active})$

Optimal strategy and agent impact

- A trader that observes $E(A_{total}|\mu)$ should play

$$a_\mu^{(opt,naive)} = -\mathrm{sign}[E(A|\mu)]$$

- Naively,

$$E(gain^{(opt,naive)}) = \frac{1}{P}\sum_\mu |E(A|\mu)| \sim H_0$$

- Will play if $H_{total} > \epsilon$

- Practically, *ceteribus paribus*

$$E(gain^{(opt,real)}) = -\frac{1}{P}\sum_\mu a_\mu^{(opt)} E(A + a_\mu^{(opt)}|\mu) \sim H_0 - 1$$

- When $H_0 < 1$, $a_\mu^{(opt,naive)}$ is wrong

The role of self-impact

- Generically, in all MGs

$$gain_i(t) = -a_i(t)A(t) = -a_i(t)A_{-i}(t) - a_i^2(t)$$
$$A_{-i} = \sum_{j \neq i} a_j$$

- In variable-N MGs

$$gain_i(t) = -a_i(t)n_i(t)A(t) = \underbrace{-a_i(t)A_{-i}(t)}_{\text{virtual gain}} - \underbrace{n_i(t)}_{\text{self impact}}$$
$$A_{-i} = \sum_{j \neq i} a_j n_j$$

1. If $n_i(t) = 0$, over-estimation of strategy value
2. When $n_i(t) = 1$, actual gain $=$ virtual gain -1
   - If virtual gain $<$ neglected impact, net loss, agent $\rightarrow$ out

Speculative models and market predictability

- Round trip gain

$$
\begin{aligned}
g &= a_i(t)[p(t'+1) - p(t+1)] \\
&= a_i(t)[p(t') - p(t)] + a_i(t)A(t') - a_i(t)A(t) \\
&= \underbrace{a_i(t)[p(t') - p(t)]}_{\text{gain if no impact}} \underbrace{- a_i(t')A(t')}_{\text{impact}} \underbrace{- a_i(t)A(t)}_{\text{impact}}
\end{aligned}
$$

- Multi-step strategies?

- How to make agents hold positions?

What does trigger agent activity?

Moving averages crossing [source]

- Alternative view point: μ is a signal, e.g., crossing between two moving averages

Agent-based speculation model with holding periods

Challet (2008) [link]

- Collection of possible signals $\boldsymbol{\mu} = \{\mu_1, \cdots, \mu_P\}$

- At each time step, the market is in a unique state μ_t

- Agent i
  1. recognizes two signals $\boldsymbol{\mu}_\mathrm{i} = \{\mu_{i,1}, \mu_{i,2}\} \subset \boldsymbol{\mu}$;
  2. may change her position only when $\mu_t \in \boldsymbol{\mu}_\mathrm{i}$;
  3. computes the performance $\mu_{i,1} \to \mu_{i,2}$ and $\mu_{i,2} \to \mu_{i,1}$
  4. If e.g. $\mu_t = \mu_{i,1}$,
     - open/close position depending on performance $\mu_{i,1} \to \mu_{i,2}$
  5. If $\mu_t \notin \{\mu_{i,1}, \mu_{i,2}\}$, do nothing (hold or wait)

Performance measure and activity criterion

- Same as in variable-N MG:

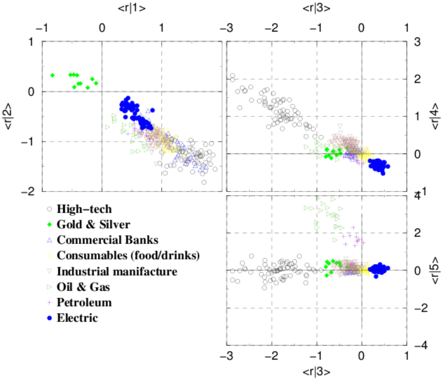
$$|E(\sum A|\mu_1 \to \mu_2)| > \epsilon$$

- Without producers, no active speculators
- Add producers: fixed biases Ω_μ

$$A_{total}(t) = A_{spec}(t) + \Omega_{\mu(t)}$$

Fixed conditional biases Ω_μ

Marsili (2002) [paper]: they exist



Dynamics of market states

- Dynamics for μ: uniform transition probabilities $W_{\mu\to\nu}$ do not produce inter-pattern predictability

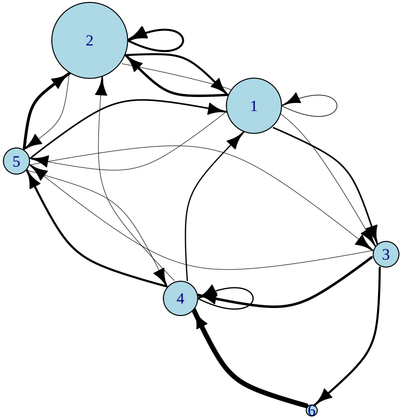
- [Hendricks, Gebbie and Wilcox (2015)], 1h periods

		state$_{t+1}$					
		1	**2**	**3**	**4**	**5**	**6**
	1	0.13	0.49	0.32	0.00	0.06	0.00
	2	0.41	0.41	0.09	0.00	0.09	0.00
state$_t$	**3**	0.00	0.00	0.00	0.52	0.05	0.43
	4	0.25	0.07	0.00	0.25	0.43	0.00
	5	0.32	0.59	0.05	0.05	0.00	0.00
	6	0.00	0.00	0.00	1.00	0.00	0.00

$$\to W_{\mu\to\nu} \neq \frac{1}{P}$$

Dynamics of market states

[Hendricks, Gebbie and Wilcox (2015)], 1h periods



Relevant predictability

- Contemporaneous predictability: not exploitable

$$H = \frac{1}{P} \sum_{\mu} E(A_t | \mu_t)^2$$

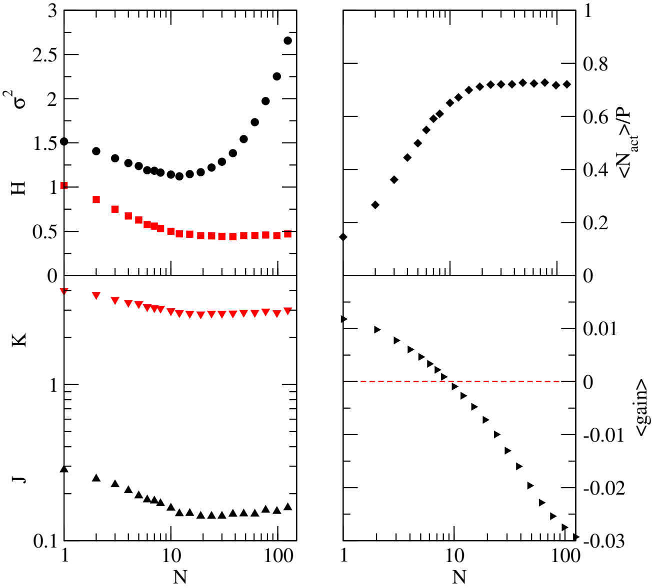
- Predictability associated to strategies

$$J = \frac{2}{P(P-1)} \sum_{\mu,\nu} E(\sum A_{t+1} | \mu_t \to \nu)^2$$

- Naive predictability

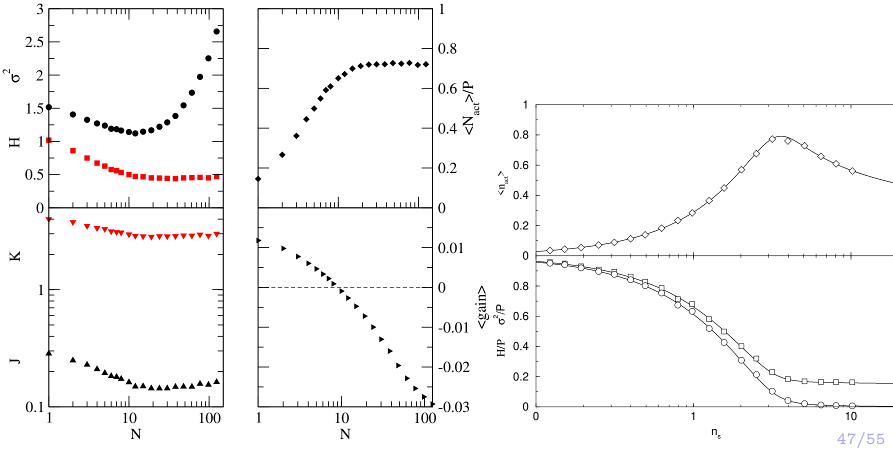
$$K = \frac{2}{P(P-1)} \sum_{\mu,\nu} E(\sum A_t | \mu_t \to \nu)^2$$

Model: results ($\epsilon = 0.05$)



Model: results

- J and K: real and naive predictability between patterns
- Exploitable predictability is decreased by speculation
- H, gain, N_{active} and $\sigma^2 = E(A^2)$: similar to MG



Can inter-pattern predictability disappear?

- Without speculators

$$H = \overline{E(A|\mu)^2} = \overline{\Omega_\mu^2} = \sigma_\Omega^2$$

- Hypothesis: circular history

$$\mu_t = t \, \mathrm{MOD} \, P$$

- Hypothesis: holding period of 1 step

$\rightarrow$ there are P groups of agents
$\rightarrow$ at each time step, one group of agents opens and one closes their position

Predictability in a circular world

- Hypothesis: circular history

$$\mu_t = t \,\mathrm{MOD}\, P$$

- Hypothesis: holding period of 1 step

$\rightarrow$ there are P groups of agents
$\rightarrow$ at each time step, one group of agents opens and one closes their position

- One average, for a given μ

$$\Omega_\mu + A_{Open,\mu} + A_{Close,\mu} = 0$$

P equations, P variables $\rightarrow$ $H = 0$ is possible.

Predictability in a non-deterministic world

- Hypothesis: possibly random history

- Hypothesis: holding period of any number of steps

  $\rightarrow$ there are $P(P-1)/2$ groups of agents
  $\rightarrow$ at each time step, one group of agents opens and one closes their position

- One average, for a given μ

$$\Omega_\mu + A_{Open,\mu} + A_{Close,\mu} = 0$$

P equations, $P(P-1)/2$ variables $\rightarrow$ $H = 0$ is possible.

Relevant predictability

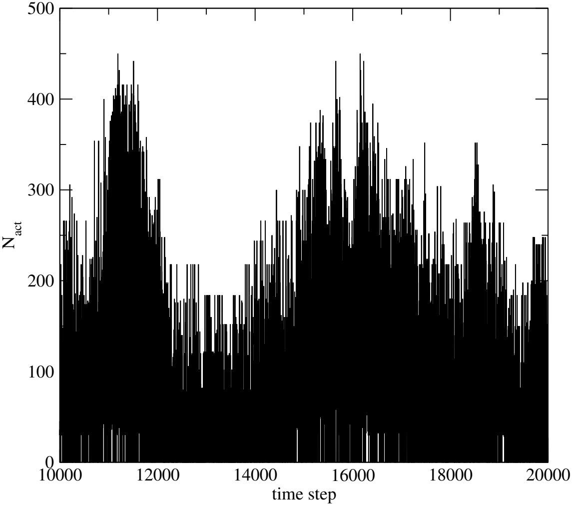
- Hypothesis: deterministic μ
- Unique path between two states
- Given the strategies, the relevant predictability is

$$J = \frac{2}{P(P-1)} \sum_{\mu,\nu} E\left(\sum A | \mu \to \nu\right)^2$$

- $J = 0$: $P(P-1)/2$ equations, $P(P-1)/2$ variables
- $\epsilon > 0 \Rightarrow J \neq 0$

Results: activity memory

Long memory of $N_{active} \propto Volume \propto volatility$ (subordination)



Strategy usage and long memory of activity

Bouchaud *et al.* (2001) On a universal mechanism for long-range volatility correlations [link]

- N agents
- agent i active if $n_i = \theta[U_i(t)]$, where $U_i(t)$ follows a random walk
- Theorem: volume $V(t)$

  $$V(t) = \sum_i n_i$$

  has a long memory
- Proof: n_i has a long memory: see persistence properties of random walks.
- Corollary: activity in any agent-based model in which the volume is mostly modulated by approximately random events has a long memory

Strategy usage and long memory of activity

Variogram of V

$$v(V) = E([V(t) - V(t')]^2)$$

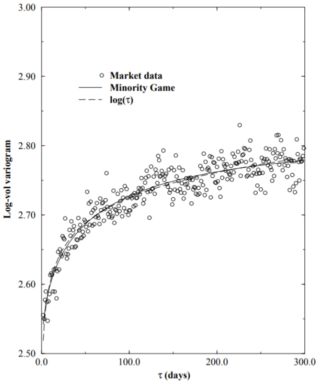


Figure 5: Variogram of the log-volatility, $\langle \log^2(\sigma_t/\sigma_{t+\tau})\rangle$ as a function of τ, averaged over 17 different stock indices (American, European, Asian). The full line is the MG result, with again both axis rescaled and a constant added to account for the presence of 'white noise' trading. The dashed line is the prediction of the multifractal model of [7], and is nearly indistinguishable from the MG result.

Summary

1. Agents use strategies to detect and exploit predictability
2. Predictability decreases w.r.t the strategies they use
   $\rightarrow$ minority game w.r.t. strategies
   $\rightarrow$ may increase the predictability of other strategies
3. Signal-to-noise ratio transitions to herding and large price fluctuations
4. Variable investment:

   long-range memory of activity $\equiv$ volatility