Complete High School Physics
Grades 11-12 Curriculum

Comprehensive Theory and Problem-Solving Guide

Contents

1 Preface

This comprehensive guide covers the complete high school physics curriculum for grades 11 and 12, providing rigorous theoretical foundations, detailed derivations from first principles, and extensive problem-solving practice. The material is designed to develop both conceptual understanding and mathematical proficiency necessary for advanced study in physics and engineering.

Each lesson builds systematically on previous concepts, emphasizing the interconnected nature of physical principles. Derivations are presented in full detail, using calculus and integral methods where appropriate, to develop deep understanding rather than superficial memorization.

Course Structure

The curriculum is divided into the following major units:

- Mechanics (Lessons 1-8)
- Thermal Physics (Lessons 9-10)
- Waves and Sound (Lessons 11-12)
- Electricity and Magnetism (Lessons 13-18)
- Optics (Lessons 19-20)
- Modern Physics (Lessons 21-22)

2 Lesson 1: Kinematics in One Dimension

2.1 Introduction to Motion

Kinematics is the study of motion without considering its causes. We describe the motion of objects using position, velocity, and acceleration as functions of time.

2.2 Position, Displacement, and Distance

2.2.1 Position

The position of an object is its location relative to a chosen reference point (origin). In one dimension, position is denoted by $x(t)$ and measured in meters (m).

2.2.2 Displacement

Displacement is the change in position:

$$\Delta x = x_f - x_i$$

where x_i is the initial position and x_f is the final position. Displacement is a vector quantity - it has both magnitude and direction.

2.2.3 Distance

Distance is the total length of the path traveled, regardless of direction. It is always positive and is a scalar quantity.

2.3 Velocity

2.3.1 Average Velocity

The average velocity over a time interval is:

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

Average velocity can be positive, negative, or zero, depending on the direction of displacement.

2.3.2 Instantaneous Velocity

The instantaneous velocity is the velocity at a specific instant of time. It is defined as the limit of average velocity as the time interval approaches zero:

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

This is the derivative of position with respect to time. Geometrically, it represents the slope of the position-time graph at a given instant.

2.3.3 Speed

Speed is the magnitude of velocity. Average speed is total distance divided by total time:

$$\text{speed}_{avg} = \frac{\text{total distance}}{\text{total time}}$$

2.4 Acceleration

2.4.1 Average Acceleration

Average acceleration is the rate of change of velocity:

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

Units: m/s^2 or $\text{m}\cdot\text{s}^2$

2.4.2 Instantaneous Acceleration

Instantaneous acceleration is:

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

This is the derivative of velocity (or second derivative of position) with respect to time.

2.5 Kinematic Equations for Constant Acceleration

When acceleration is constant, we can derive four fundamental kinematic equations.

2.5.1 Derivation from First Principles

Equation 1: Velocity as a function of time

Starting with the definition of constant acceleration:

$$a = \frac{dv}{dt}$$

Rearranging and integrating both sides:

$$dv = a\,dt$$

$$\int_{v_0}^{v} dv' = \int_0^t a\,dt'$$

$$v - v_0 = at$$

Therefore:

$$\boxed{v = v_0 + at}$$

Equation 2: Position as a function of time

Since $v = \frac{dx}{dt}$, we can substitute our expression for v:

$$\frac{dx}{dt} = v_0 + at$$

Integrating both sides:

$$\int_{x_0}^{x} dx' = \int_0^t (v_0 + at')\, dt'$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

Therefore:

$$\boxed{x = x_0 + v_0 t + \frac{1}{2}at^2}$$

Equation 3: Velocity without time
From Equation 1: $t = \frac{v - v_0}{a}$
Substituting into Equation 2:

$$x - x_0 = v_0 \left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2$$

$$x - x_0 = \frac{v_0 v - v_0^2}{a} + \frac{(v - v_0)^2}{2a}$$

$$x - x_0 = \frac{2v_0 v - 2v_0^2 + v^2 - 2vv_0 + v_0^2}{2a}$$

$$x - x_0 = \frac{v^2 - v_0^2}{2a}$$

Multiplying both sides by $2a$:

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)}$$

Equation 4: Position without acceleration
The average velocity during constant acceleration is:

$$v_{avg} = \frac{v_0 + v}{2}$$

Since $x - x_0 = v_{avg} \cdot t$:

$$\boxed{x - x_0 = \frac{1}{2}(v_0 + v)t}$$

2.6 Free Fall Motion

Free fall is motion under the influence of gravity alone, with constant acceleration $g = 9.8$ m/s² (or 9.81 m/s² for more precision) directed downward.

2.6.1 Sign Convention

Taking upward as positive:

- Acceleration: $a = -g = -9.8$ m/s²
- Initial velocity upward: $v_0 > 0$
- Initial velocity downward: $v_0 < 0$

2.6.2 Key Properties of Projectile Motion (Vertical)

For an object thrown vertically upward with initial velocity v_0:

Time to reach maximum height: At maximum height, $v = 0$:

$$0 = v_0 - gt_{max}$$

$$t_{max} = \frac{v_0}{g}$$

Maximum height:

$$h_{max} = v_0 t_{max} - \frac{1}{2}gt_{max}^2 = v_0 \cdot \frac{v_0}{g} - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2$$

$$h_{max} = \frac{v_0^2}{g} - \frac{v_0^2}{2g} = \frac{v_0^2}{2g}$$

Or using $v^2 = v_0^2 - 2gh$ with $v = 0$:

$$h_{max} = \frac{v_0^2}{2g}$$

Time of flight (returning to initial height): By symmetry, $t_{total} = 2t_{max} = \frac{2v_0}{g}$

2.7 Graphical Analysis

2.7.1 Position-Time Graphs

- Slope = velocity
- Straight line: constant velocity
- Curved line: changing velocity (acceleration)
- Parabolic curve: constant acceleration

2.7.2 Velocity-Time Graphs

- Slope = acceleration
- Area under curve = displacement
- Straight line: constant acceleration

2.7.3 Acceleration-Time Graphs

- Area under curve = change in velocity
- Horizontal line: constant acceleration

2.8 Example Problems

Problem 1.1: A car accelerates from rest at a constant rate of 3.0 m/s² for 5.0 seconds, then maintains constant velocity for 10 seconds, and finally brakes to a stop with constant deceleration over 4.0 seconds. Find: (a) the maximum velocity reached, (b) the total distance traveled, (c) the average velocity for the entire trip.

Solution:

(a) Maximum velocity (end of acceleration phase):

$$v_{max} = v_0 + at = 0 + (3.0)(5.0) = 15 \text{ m/s}$$

(b) Distance during acceleration:

$$x_1 = v_0 t + \frac{1}{2}at^2 = 0 + \frac{1}{2}(3.0)(5.0)^2 = 37.5 \text{ m}$$

Distance during constant velocity:

$$x_2 = vt = (15)(10) = 150 \text{ m}$$

Distance during braking:

$$x_3 = \frac{1}{2}(v_i + v_f)t = \frac{1}{2}(15 + 0)(4.0) = 30 \text{ m}$$

Total distance:

$$x_{total} = 37.5 + 150 + 30 = 217.5 \text{ m}$$

(c) Average velocity:

$$v_{avg} = \frac{x_{total}}{t_{total}} = \frac{217.5}{5.0 + 10 + 4.0} = \frac{217.5}{19} = 11.4 \text{ m/s}$$

Problem 1.2: A ball is thrown vertically upward with an initial velocity of 20 m/s from a height of 2.0 m above the ground. Calculate: (a) the maximum height above the ground, (b) the time to reach maximum height, (c) the velocity when it returns to the initial height, (d) the total time in the air before hitting the ground.

Solution:

Taking upward as positive, $y_0 = 2.0$ m, $v_0 = 20$ m/s, $a = -9.8$ m/s².

(a) At maximum height, $v = 0$:

$$v^2 = v_0^2 + 2a(y - y_0)$$

$$0 = (20)^2 + 2(-9.8)(y_{max} - 2.0)$$

$$0 = 400 - 19.6(y_{max} - 2.0)$$

$$19.6(y_{max} - 2.0) = 400$$

$$y_{max} = \frac{400}{19.6} + 2.0 = 20.4 + 2.0 = 22.4 \text{ m}$$

(b) Time to maximum height:

$$v = v_0 + at$$

$$0 = 20 + (-9.8)t$$

$$t = \frac{20}{9.8} = 2.04 \text{ s}$$

(c) By symmetry and energy conservation, when returning to initial height:

$$v = -v_0 = -20 \text{ m/s}$$

(negative because moving downward)

(d) From $y = 2.0$ m to ground ($y = 0$), with $v_0 = -20$ m/s:

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

$$0 = 2.0 + (-20)t + \frac{1}{2}(-9.8)t^2$$

$$4.9t^2 + 20t - 2.0 = 0$$

Using quadratic formula:

$$t = \frac{-20 \pm \sqrt{400 + 39.2}}{9.8} = \frac{-20 \pm \sqrt{439.2}}{9.8}$$

$$t = \frac{-20 + 20.96}{9.8} = 0.098 \text{ s}$$

Total time:

$$t_{total} = 2.04 + 0.098 = 2.14 \text{ s}$$

3 Lesson 2: Kinematics in Two Dimensions

3.1 Vector Representation

In two dimensions, position, velocity, and acceleration are vectors with components in both x and y directions.

3.1.1 Position Vector

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

where $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions.

3.1.2 Velocity Vector

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = v_x(t)\hat{i} + v_y(t)\hat{j}$$

where:

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}$$

Magnitude:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

Direction:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

3.1.3 Acceleration Vector

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = a_x(t)\hat{i} + a_y(t)\hat{j}$$

3.2 Projectile Motion

Projectile motion is two-dimensional motion under constant gravitational acceleration. The key insight is that horizontal and vertical motions are independent.

3.2.1 Assumptions

- Air resistance is negligible
- Gravitational acceleration is constant: $g = 9.8$ m/s^2
- The Earth's surface is flat over the range of motion

3.2.2 Initial Conditions

For a projectile launched with initial speed v_0 at angle θ_0 above horizontal:

Horizontal component:

$$v_{0x} = v_0 \cos\theta_0$$

Vertical component:

$$v_{0y} = v_0 \sin\theta_0$$

3.2.3 Equations of Motion

Horizontal motion (no acceleration):

$$x = x_0 + v_{0x}t = x_0 + v_0 \cos\theta_0 \cdot t$$

$$v_x = v_{0x} = v_0 \cos\theta_0 = \text{constant}$$

Vertical motion (constant acceleration $a_y = -g$):

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_0 \sin\theta_0 \cdot t - \frac{1}{2}gt^2$$

$$v_y = v_{0y} - gt = v_0 \sin\theta_0 - gt$$

3.2.4 Trajectory Equation

From horizontal motion: $t = \frac{x-x_0}{v_0 \cos\theta_0}$

Substituting into vertical motion equation (with $x_0 = 0$, $y_0 = 0$):

$$y = v_0 \sin\theta_0 \cdot \frac{x}{v_0 \cos\theta_0} - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

$$y = x\tan\theta_0 - \frac{gx^2}{2v_0^2 \cos^2\theta_0}$$

Using $\sec^2\theta_0 = 1 + \tan^2\theta_0$:

$$\boxed{y = x\tan\theta_0 - \frac{gx^2}{2v_0^2}(1 + \tan^2\theta_0)}$$

This is a parabola opening downward.

3.2.5 Key Results for Projectile Motion

Time to reach maximum height: At maximum height, $v_y = 0$:

$$0 = v_0 \sin\theta_0 - gt_{max}$$

$$t_{max} = \frac{v_0 \sin\theta_0}{g}$$

Maximum height:

$$h_{max} = v_0 \sin\theta_0 \cdot t_{max} - \frac{1}{2} g t_{max}^2$$

$$h_{max} = v_0 \sin\theta_0 \cdot \frac{v_0 \sin\theta_0}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

$$h_{max} = \frac{v_0^2 \sin^2\theta_0}{g} - \frac{v_0^2 \sin^2\theta_0}{2g}$$

$$\boxed{h_{max} = \frac{v_0^2 \sin^2\theta_0}{2g}}$$

Range (horizontal distance): For a projectile returning to launch height ($y = 0$), total time of flight is:

$$t_{total} = \frac{2v_0 \sin\theta_0}{g}$$

Range:

$$R = v_0 \cos\theta_0 \cdot t_{total} = v_0 \cos\theta_0 \cdot \frac{2v_0 \sin\theta_0}{g}$$

$$R = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g}$$

Using $\sin(2\theta_0) = 2 \sin\theta_0 \cos\theta_0$:

$$\boxed{R = \frac{v_0^2 \sin(2\theta_0)}{g}}$$

Maximum range occurs when $\sin(2\theta_0) = 1$, i.e., $\theta_0 = 45$:

$$R_{max} = \frac{v_0^2}{g}$$

3.3 Uniform Circular Motion

An object moving in a circle at constant speed undergoes acceleration because its velocity direction constantly changes.

3.3.1 Angular Quantities

Angular position: θ (radians)

Angular velocity:

$$\omega = \frac{d\theta}{dt}$$

Units: rad/s

Period and frequency:

$$T = \frac{2\pi}{\omega}, \quad f = \frac{1}{T} = \frac{\omega}{2\pi}$$

3.3.2 Linear Velocity

For circular motion with radius r:

$$v = r\omega$$

The velocity vector is always tangent to the circle.

3.3.3 Centripetal Acceleration

Derivation from first principles:

Consider an object moving in a circle of radius r with constant speed v. In a small time interval Δt, the object moves through angle $\Delta\theta$.

The velocity changes from $\vec{v}_1$ to $\vec{v}_2$, both with magnitude v but different directions.

The change in velocity has magnitude:

$$|\Delta\vec{v}| = 2v\sin\left(\frac{\Delta\theta}{2}\right)$$

For small $\Delta\theta$: $\sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2}$

Therefore:

$$|\Delta\vec{v}| \approx v\Delta\theta$$

The magnitude of acceleration is:

$$a = \lim_{\Delta t \to 0} \frac{|\Delta\vec{v}|}{\Delta t} = v \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = v\omega$$

Since $v = r\omega$:

$$\boxed{a_c = v\omega = v \cdot \frac{v}{r} = \frac{v^2}{r} = r\omega^2}$$

This acceleration points toward the center of the circle (centripetal acceleration).

3.4 Relative Motion

When observers in different reference frames measure the motion of an object, velocities add vectorially.

3.4.1 Velocity Addition

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

where:

- $\vec{v}_{AC}$ = velocity of A relative to C
- $\vec{v}_{AB}$ = velocity of A relative to B
- $\vec{v}_{BC}$ = velocity of B relative to C

3.5 Example Problems

Problem 2.1: A soccer ball is kicked from ground level with initial speed 25 m/s at an angle of 37° above horizontal. Calculate: (a) the maximum height reached, (b) the time of flight, (c) the horizontal range, (d) the velocity (magnitude and direction) when the ball is at half its maximum height on the way up.

Solution:

Given: $v_0 = 25$ m/s, $\theta_0 = 37$

Components:

$$v_{0x} = 25\cos(37) = 25(0.799) = 20.0 \text{ m/s}$$

$$v_{0y} = 25\sin(37) = 25(0.602) = 15.0 \text{ m/s}$$

(a) Maximum height:

$$h_{max} = \frac{v_{0y}^2}{2g} = \frac{(15.0)^2}{2(9.8)} = \frac{225}{19.6} = 11.5 \text{ m}$$

(b) Time of flight:

$$t_{total} = \frac{2v_{0y}}{g} = \frac{2(15.0)}{9.8} = 3.06 \text{ s}$$

(c) Range:

$$R = v_{0x} \cdot t_{total} = 20.0 \times 3.06 = 61.2 \text{ m}$$

(d) At half maximum height ($y = 5.75$ m):

Using $v_y^2 = v_{0y}^2 - 2gy$:

$$v_y^2 = (15.0)^2 - 2(9.8)(5.75) = 225 - 112.7 = 112.3$$

$$v_y = 10.6 \text{ m/s}$$

Horizontal component remains constant: $v_x = 20.0$ m/s

Magnitude:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20.0)^2 + (10.6)^2} = \sqrt{512.4} = 22.6 \text{ m/s}$$

Direction:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{10.6}{20.0}\right) = \tan^{-1}(0.53) = 27.9$$

Problem 2.2: A car travels around a circular track of radius 50 m. It completes one lap in 20 seconds. Calculate: (a) the angular velocity, (b) the linear speed, (c) the centripetal acceleration, (d) the angle through which the car turns in 5.0 seconds.

Solution:

Given: $r = 50$ m, $T = 20$ s

(a) Angular velocity:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{20} = 0.314 \text{ rad/s}$$

(b) Linear speed:

$$v = r\omega = 50 \times 0.314 = 15.7 \text{ m/s}$$

(c) Centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{(15.7)^2}{50} = \frac{246.5}{50} = 4.93 \text{ m/s}^2$$

Or: $a_c = r\omega^2 = 50 \times (0.314)^2 = 4.93$ m/s²
(d) Angle in 5.0 seconds:

$$\theta = \omega t = 0.314 \times 5.0 = 1.57 \text{ rad} = 90$$

4 Lesson 3: Newton's Laws of Motion

4.1 Newton's First Law (Law of Inertia)

4.1.1 Statement

An object at rest stays at rest, and an object in motion continues in motion with constant velocity, unless acted upon by a net external force.

$$\text{If } \sum \vec{F} = 0, \text{ then } \vec{v} = \text{constant}$$

4.1.2 Inertia

Inertia is the tendency of an object to resist changes in its motion. Mass is a measure of inertia.

4.1.3 Inertial Reference Frames

Newton's first law defines inertial reference frames - frames in which the law holds. Non-inertial frames (accelerating frames) require fictitious forces.

4.2 Newton's Second Law

4.2.1 Statement

The acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass:

$$\boxed{\sum \vec{F} = m\vec{a}}$$

Or in component form:

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z$$

4.2.2 Units

Force: Newton (N) = kg·m/s^2

4.2.3 Alternative Forms

Momentum form:

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

where $\vec{p} = m\vec{v}$ is momentum.
For constant mass:

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

4.3 Newton's Third Law

4.3.1 Statement

For every action force, there is an equal and opposite reaction force. If object A exerts a force on object B, then object B exerts an equal and opposite force on object A:

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

4.3.2 Important Points

- Action-reaction pairs act on different objects
- They are equal in magnitude and opposite in direction
- They act simultaneously
- They are the same type of force

4.4 Common Forces

4.4.1 Gravitational Force (Weight)

Near Earth's surface:

$$\vec{F}_g = m\vec{g}$$

Magnitude: $F_g = mg$ where $g = 9.8$ m/s^2
Direction: downward (toward Earth's center)

4.4.2 Normal Force

The normal force is the perpendicular contact force exerted by a surface on an object. It adjusts to prevent objects from passing through surfaces.
For an object on a horizontal surface: $N = mg$
For an object on an inclined plane (angle θ): $N = mg\cos\theta$

4.4.3 Tension

Tension is the pulling force transmitted through a string, rope, cable, or similar object. For an ideal (massless, inextensible) string, tension is constant throughout.

4.4.4 Friction

Static friction prevents relative motion between surfaces:

$$f_s \leq \mu_s N$$

where μ_s is the coefficient of static friction. The maximum static friction is:

$$f_{s,max} = \mu_s N$$

Kinetic friction opposes relative motion:

$$f_k = \mu_k N$$

where μ_k is the coefficient of kinetic friction.
Always: $\mu_k < \mu_s$ and friction opposes the direction of motion (or potential motion).

4.5 Problem-Solving Strategy

1. **Draw a diagram:** Sketch the situation
2. **Identify the system:** Choose the object(s) to analyze
3. **Draw free-body diagram:** Show all forces acting on the object
4. **Choose coordinate system:** Align axes with motion when possible
5. **Apply Newton's second law:** Write $\sum \vec{F} = m\vec{a}$ in component form
6. **Solve equations:** Use algebra and kinematics as needed
7. **Check answer:** Verify units, signs, and physical reasonableness

4.6 Applications

4.6.1 Motion on Inclined Planes

For a block of mass m on a frictionless incline at angle θ:
Free-body diagram forces:

- Weight: mg (downward)
- Normal force: N (perpendicular to surface)

Choose coordinates: x-axis along incline (positive down), y-axis perpendicular to incline
Weight components:

$$F_{gx} = mg\sin\theta \quad \text{(parallel to incline)}$$

$$F_{gy} = mg\cos\theta \quad \text{(perpendicular to incline)}$$

Newton's second law:
Perpendicular to incline ($a_y = 0$):

$$N - mg\cos\theta = 0$$

$$N = mg\cos\theta$$

Parallel to incline:

$$mg\sin\theta = ma$$

$$a = g \sin \theta$$

With friction:
If friction is present:

$$mg \sin \theta - f = ma$$

For kinetic friction:

$$mg \sin \theta - \mu_k mg \cos \theta = ma$$

$$a = g(\sin \theta - \mu_k \cos \theta)$$

The block slides down if: $\tan \theta > \mu_s$

4.6.2 Connected Objects

For two masses connected by a string over a pulley (Atwood machine):

System: Two masses m_1 and m_2 connected by an ideal string over a frictionless, massless pulley.

Assume $m_2 > m_1$, so m_2 accelerates downward and m_1 accelerates upward with the same magnitude a.

Free-body diagram for m_1:

- Tension: T (upward)
- Weight: $m_1 g$ (downward)

Newton's second law (taking upward as positive):

$$T - m_1 g = m_1 a \quad ...(1)$$

Free-body diagram for m_2:

- Tension: T (upward)
- Weight: $m_2 g$ (downward)

Newton's second law (taking downward as positive):

$$m_2 g - T = m_2 a \quad ...(2)$$

Solving simultaneously:
Adding equations (1) and (2):

$$m_2 g - m_1 g = m_1 a + m_2 a$$

$$(m_2 - m_1)g = (m_1 + m_2)a$$

$$\boxed{a = \frac{(m_2 - m_1)g}{m_1 + m_2}}$$

From equation (1):

$$T = m_1(g+a) = m_1\left(g + \frac{(m_2-m_1)g}{m_1+m_2}\right)$$
$$T = m_1 g\left(\frac{m_1+m_2+m_2-m_1}{m_1+m_2}\right)$$
$$\boxed{T = \frac{2m_1m_2g}{m_1+m_2}}$$

4.7 Centripetal Force

For circular motion, the net force must provide the centripetal acceleration:

$$\sum F_c = ma_c = m\frac{v^2}{r} = mr\omega^2$$

This is not a new type of force, but rather the net force directed toward the center.

4.7.1 Example: Horizontal Circle

A ball of mass m on a string of length L swings in a horizontal circle. The string makes angle θ with vertical.

Forces:

- Tension: T (along string)
- Weight: mg (downward)

Vertical component (no vertical acceleration):

$$T\cos\theta - mg = 0$$
$$T = \frac{mg}{\cos\theta}$$

Horizontal component (centripetal):

$$T\sin\theta = m\frac{v^2}{r}$$

where $r = L\sin\theta$ is the radius of the circle.

Substituting for T:

$$\frac{mg}{\cos\theta}\sin\theta = m\frac{v^2}{L\sin\theta}$$
$$mg\tan\theta = \frac{mv^2}{L\sin\theta}$$
$$g\tan\theta\sin\theta = \frac{v^2}{L}$$
$$g\frac{\sin^2\theta}{\cos\theta} = \frac{v^2}{L}$$
$$v = \sqrt{\frac{gL\sin^2\theta}{\cos\theta}}$$

4.8 Example Problems

Problem 3.1: A 5.0 kg block sits on a 30° incline. The coefficient of static friction is 0.50 and the coefficient of kinetic friction is 0.35. (a) Will the block slide down the incline? (b) If a horizontal force of 40 N is applied to push the block up the incline, what is its acceleration? (c) What minimum force parallel to the incline is needed to keep the block moving up at constant velocity?

Solution:

Given: $m = 5.0$ kg, $\theta = 30$, $\mu_s = 0.50$, $\mu_k = 0.35$

(a) Check if block slides:

Normal force: $N = mg\cos\theta = 5.0 \times 9.8 \times \cos(30) = 49 \times 0.866 = 42.4$ N

Component of weight parallel to incline:

$$F_{parallel} = mg\sin\theta = 49 \times 0.5 = 24.5 \text{ N (down incline)}$$

Maximum static friction:

$$f_{s,max} = \mu_s N = 0.50 \times 42.4 = 21.2 \text{ N (up incline)}$$

Since $F_{parallel} > f_{s,max}$ (24.5 N ¿ 21.2 N), the block will slide down.

(b) With horizontal force:

The 40 N horizontal force has components:

- Parallel to incline (up): $F_{||} = 40\cos(30) = 40 \times 0.866 = 34.6$ N
- Perpendicular to incline: $F_{\perp} = 40\sin(30) = 40 \times 0.5 = 20$ N

New normal force:

$$N = mg\cos\theta + F_{\perp} = 42.4 + 20 = 62.4 \text{ N}$$

Kinetic friction (opposing motion up incline):

$$f_k = \mu_k N = 0.35 \times 62.4 = 21.8 \text{ N}$$

Net force up incline:

$$\sum F = F_{||} - mg\sin\theta - f_k$$
$$= 34.6 - 24.5 - 21.8 = -11.7 \text{ N}$$

Acceleration:

$$a = \frac{\sum F}{m} = \frac{-11.7}{5.0} = -2.34 \text{ m/s}^2$$

The negative sign indicates the block accelerates down the incline (the applied force is insufficient to push it up).

(c) For constant velocity up the incline, $a = 0$:

$$F - mg\sin\theta - \mu_k mg\cos\theta = 0$$
$$F = mg(\sin\theta + \mu_k\cos\theta)$$

$$F = 5.0 \times 9.8 \times (0.5 + 0.35 \times 0.866)$$

$$F = 49 \times (0.5 + 0.303) = 49 \times 0.803 = 39.3 \text{ N}$$

Problem 3.2: Two blocks of masses $m_1 = 2.0$ kg and $m_2 = 3.0$ kg are connected by a light string passing over a frictionless pulley. Block 1 is on a frictionless table, while block 2 hangs vertically. When the system is released from rest, find: (a) the acceleration of the blocks, (b) the tension in the string, (c) the distance block 2 falls in the first 2.0 seconds.

Solution:

For block 1 (horizontal motion, taking right as positive):

$$T = m_1 a \quad ...(1)$$

For block 2 (vertical motion, taking down as positive):

$$m_2 g - T = m_2 a \quad ...(2)$$

(a) From (1): $T = m_1 a$

Substituting into (2):

$$m_2 g - m_1 a = m_2 a$$

$$m_2 g = (m_1 + m_2) a$$

$$a = \frac{m_2 g}{m_1 + m_2} = \frac{3.0 \times 9.8}{2.0 + 3.0} = \frac{29.4}{5.0} = 5.88 \text{ m/s}^2$$

(b) Tension:

$$T = m_1 a = 2.0 \times 5.88 = 11.8 \text{ N}$$

(c) Distance fallen in 2.0 s (starting from rest):

$$y = \frac{1}{2} a t^2 = \frac{1}{2} \times 5.88 \times (2.0)^2 = 11.8 \text{ m}$$

5 Lesson 4: Work, Energy, and Power

5.1 Work

5.1.1 Definition

Work is done when a force causes a displacement. For a constant force:

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

where θ is the angle between the force and displacement vectors.
Units: Joule (J) = N·m = kg·m²/s²

5.1.2 Sign of Work

- $W > 0$: Force and displacement in same direction ($0 \leq \theta < 90$)
- $W = 0$: Force perpendicular to displacement ($\theta = 90$)
- $W < 0$: Force and displacement in opposite directions ($90 < \theta \leq 180$)

5.1.3 Work by Variable Force

For a force that varies with position, work is the integral:

$$W = \int_{x_i}^{x_f} F(x)\, dx$$

Geometrically, this is the area under the force-displacement curve.
Example: Work by a spring force
For a spring with spring constant k, Hooke's law gives:

$$F = -kx$$

Work done in stretching/compressing from x_1 to x_2:

$$W = \int_{x_1}^{x_2} (-kx)\, dx = -k \int_{x_1}^{x_2} x\, dx$$

$$W = -k \left[\frac{x^2}{2}\right]_{x_1}^{x_2} = -\frac{k}{2}(x_2^2 - x_1^2)$$

For stretching from equilibrium ($x_1 = 0$) to $x_2 = x$:

$$W = -\frac{1}{2}kx^2$$

5.2 Kinetic Energy

5.2.1 Definition

Kinetic energy is the energy of motion:

$$KE = \frac{1}{2}mv^2$$

5.2.2 Work-Energy Theorem - Derivation

Starting with Newton's second law:

$$F = ma = m\frac{dv}{dt}$$

Work done over displacement dx:

$$dW = F\,dx = m\frac{dv}{dt}dx$$

Using chain rule: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

$$dW = mv\frac{dv}{dx}dx = mv\,dv$$

Integrating from initial velocity v_i to final velocity v_f:

$$W = \int_{v_i}^{v_f} mv\,dv = m\int_{v_i}^{v_f} v\,dv = m\left[\frac{v^2}{2}\right]_{v_i}^{v_f}$$

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$\boxed{W_{net} = \Delta KE = KE_f - KE_i}$$

This is the work-energy theorem: the net work done on an object equals its change in kinetic energy.

5.3 Potential Energy

5.3.1 Gravitational Potential Energy

Near Earth's surface, gravitational potential energy is:

$$PE_g = mgh$$

where h is height above a reference level.

Derivation:

Work done by gravity when object moves from y_1 to y_2:

$$W_g = \int_{y_1}^{y_2} F_g\,dy = \int_{y_1}^{y_2} (-mg)\,dy = -mg(y_2 - y_1)$$

Define gravitational potential energy as:

$$PE_g = mgy$$

Then:

$$W_g = -(PE_{g,2} - PE_{g,1}) = -\Delta PE_g$$

5.3.2 Elastic Potential Energy

For an ideal spring with spring constant k:

$$PE_s = \frac{1}{2}kx^2$$

where x is displacement from equilibrium.
Derivation:
Work done by spring force from x_1 to x_2:

$$W_s = \int_{x_1}^{x_2} (-kx)\,dx = -\frac{k}{2}(x_2^2 - x_1^2)$$

Define spring potential energy:

$$PE_s = \frac{1}{2}kx^2$$

Then:

$$W_s = -(PE_{s,2} - PE_{s,1}) = -\Delta PE_s$$

5.3.3 Conservative Forces

A force is conservative if:

1. Work done is independent of path
2. Work done around a closed loop is zero
3. A potential energy function can be defined

Examples: gravity, spring force, electric force
Non-conservative forces: friction, air resistance, tension

5.4 Conservation of Mechanical Energy

For conservative forces only, mechanical energy is conserved:

$$E = KE + PE = \text{constant}$$

$$KE_i + PE_i = KE_f + PE_f$$

$$\frac{1}{2}mv_i^2 + mgh_i = \frac{1}{2}mv_f^2 + mgh_f$$

Proof:
From work-energy theorem:

$$W_{net} = \Delta KE$$

For conservative forces:

$$W_{conservative} = -\Delta PE$$

If only conservative forces act:

$$-\Delta PE = \Delta KE$$

$$\Delta KE + \Delta PE = 0$$

$$\Delta(KE + PE) = 0$$

$$KE + PE = \text{constant}$$

5.5 Non-Conservative Forces

When non-conservative forces (like friction) do work:

$$W_{nc} = \Delta KE + \Delta PE = \Delta E$$

$$E_f = E_i + W_{nc}$$

For friction, $W_{friction} < 0$, so mechanical energy decreases (converted to thermal energy).

5.6 Power

5.6.1 Definition

Power is the rate of doing work or transferring energy:

$$P = \frac{dW}{dt}$$

Average power:

$$P_{avg} = \frac{W}{\Delta t}$$

Units: Watt (W) = J/s
Common unit: horsepower (hp) = 746 W

5.6.2 Power and Velocity

Since $W = \vec{F} \cdot \vec{d}$:

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{d}}{dt} = \vec{F} \cdot \vec{v}$$

For constant force and velocity:

$$P = Fv\cos\theta$$

5.7 Example Problems

Problem 4.1: A 2.0 kg block is pushed up a frictionless 30° incline by a horizontal force of 25 N. The block moves 5.0 m along the incline. Calculate: (a) the work done by each force, (b) the net work, (c) the final speed if the block starts from rest, (d) verify using the work-energy theorem.

Solution:

Given: $m = 2.0$ kg, $\theta = 30$, $F = 25$ N (horizontal), $d = 5.0$ m

(a) Work by each force:

Applied force: The displacement is along the incline, making angle 30 with horizontal.

$$W_F = Fd\cos(30) = 25 \times 5.0 \times 0.866 = 108.3 \text{ J}$$

Gravity: Vertical displacement: $\Delta h = d\sin(30) = 5.0 \times 0.5 = 2.5$ m

$$W_g = -mg\Delta h = -2.0 \times 9.8 \times 2.5 = -49.0 \text{ J}$$

Normal force: Perpendicular to displacement, so:

$$W_N = 0$$

(b) Net work:

$$W_{net} = W_F + W_g + W_N = 108.3 - 49.0 + 0 = 59.3 \text{ J}$$

(c) Final speed:

From work-energy theorem:

$$W_{net} = \Delta KE = \frac{1}{2}mv_f^2 - 0$$

$$59.3 = \frac{1}{2}(2.0)v_f^2$$
$$v_f^2 = 59.3$$
$$v_f = 7.70 \text{ m/s}$$

(d) Verification using Newton's second law:

Components along incline:

$$F\cos(30) - mg\sin(30) = ma$$

$$25(0.866) - 2.0(9.8)(0.5) = 2.0a$$

$$21.65 - 9.8 = 2.0a$$

$$a = 5.93 \text{ m/s}^2$$

Using kinematics:

$$v_f^2 = v_i^2 + 2ad = 0 + 2(5.93)(5.0) = 59.3$$

$$v_f = 7.70 \text{ m/s}$$

Problem 4.2: A pendulum consists of a 0.50 kg mass attached to a 1.2 m string. The mass is released from rest when the string makes a 60° angle with vertical. Using energy conservation, find: (a) the speed at the lowest point, (b) the tension in the string at the lowest point, (c) the maximum height the mass reaches on the other side if 10% of mechanical energy is lost to air resistance.

Solution:

Given: $m = 0.50$ kg, $L = 1.2$ m, $\theta_0 = 60$

(a) Speed at lowest point:

Initial height above lowest point:

$$h_i = L - L\cos\theta_0 = L(1 - \cos 60) = 1.2(1 - 0.5) = 0.60 \text{ m}$$

Conservation of energy:

$$mgh_i = \frac{1}{2}mv^2$$

$$v = \sqrt{2gh_i} = \sqrt{2 \times 9.8 \times 0.60} = \sqrt{11.76} = 3.43 \text{ m/s}$$

(b) Tension at lowest point:

At the lowest point, tension provides centripetal force plus supports weight:

$$T - mg = m\frac{v^2}{L}$$

$$T = mg + m\frac{v^2}{L} = m\left(g + \frac{v^2}{L}\right)$$

$$T = 0.50\left(9.8 + \frac{(3.43)^2}{1.2}\right) = 0.50(9.8 + 9.8) = 9.8 \text{ N}$$

(c) With energy loss:

Energy at lowest point:

$$E_0 = mgh_i = 0.50 \times 9.8 \times 0.60 = 2.94 \text{ J}$$

After 10% loss:

$$E_f = 0.90 \times 2.94 = 2.646 \text{ J}$$

Maximum height on other side:

$$mgh_f = 2.646$$

$$h_f = \frac{2.646}{0.50 \times 9.8} = 0.54 \text{ m}$$

Angle reached:

$$h_f = L(1 - \cos\theta_f)$$

$$0.54 = 1.2(1 - \cos\theta_f)$$

$$\cos\theta_f = 1 - \frac{0.54}{1.2} = 0.55$$

$$\theta_f = 56.6$$

Problem 4.3: A 1500 kg car accelerates from rest to 25 m/s in 10 seconds on a level road. The average friction and air resistance total 500 N. Calculate: (a) the work done against friction, (b) the kinetic energy gained, (c) the total work done by the engine, (d) the average power output.

Solution:

Given: $m = 1500$ kg, $v_i = 0$, $v_f = 25$ m/s, $t = 10$ s, $f = 500$ N

First find the distance traveled:

$$v_{avg} = \frac{v_i + v_f}{2} = \frac{0 + 25}{2} = 12.5 \text{ m/s}$$

$$d = v_{avg} \cdot t = 12.5 \times 10 = 125 \text{ m}$$

(a) Work against friction:

$$W_f = -fd = -500 \times 125 = -62,500 \text{ J} = -62.5 \text{ kJ}$$

(b) Kinetic energy gained:

$$\Delta KE = \frac{1}{2}mv_f^2 - 0 = \frac{1}{2}(1500)(25)^2 = 468,750 \text{ J} = 469 \text{ kJ}$$

(c) Total work by engine:

The engine must provide the kinetic energy plus overcome friction:

$$W_{engine} = \Delta KE + |W_f| = 468,750 + 62,500 = 531,250 \text{ J} = 531 \text{ kJ}$$

Or using energy equation:

$$W_{engine} + W_f = \Delta KE$$

$$W_{engine} = \Delta KE - W_f = 468,750 - (-62,500) = 531,250 \text{ J}$$

(d) Average power:

$$P_{avg} = \frac{W_{engine}}{t} = \frac{531,250}{10} = 53,125 \text{ W} = 53.1 \text{ kW}$$

In horsepower:

$$P_{avg} = \frac{53,125}{746} = 71.2 \text{ hp}$$

6 Lesson 5: Linear Momentum and Collisions

6.1 Linear Momentum

6.1.1 Definition

Linear momentum is the product of mass and velocity:

$$\vec{p} = m\vec{v}$$

Units: kg·m/s
Momentum is a vector quantity with direction same as velocity.

6.1.2 Newton's Second Law - Momentum Form

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

For constant mass:

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

6.2 Impulse

6.2.1 Definition

Impulse is the change in momentum:

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

From Newton's second law:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Integrating over time interval Δt:

$$\int_{t_i}^{t_f} \vec{F}\,dt = \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \vec{p}_f - \vec{p}_i$$

$$\boxed{\vec{J} = \int_{t_i}^{t_f} \vec{F}\,dt = \Delta\vec{p}}$$

For constant force:

$$\vec{J} = \vec{F}\Delta t$$

Units: N·s = kg·m/s (same as momentum)

6.2.2 Impulse-Momentum Theorem

$$\vec{J} = \Delta \vec{p}$$

$$\vec{F}_{avg} \Delta t = m\vec{v}_f - m\vec{v}_i$$

This explains why:

- Airbags reduce injury by increasing collision time (reducing average force)
- Catching a ball by "giving" with it reduces force
- Landing with bent knees reduces impact force

6.3 Conservation of Momentum

6.3.1 Statement

If the net external force on a system is zero, the total momentum is conserved:

$$\sum \vec{F}_{ext} = 0 \quad \Rightarrow \quad \vec{p}_{total} = \text{constant}$$

$$\vec{p}_i = \vec{p}_f$$

For a system of particles:

$$\sum_i m_i \vec{v}_{i,initial} = \sum_i m_i \vec{v}_{i,final}$$

6.3.2 Derivation

For a system of particles, Newton's second law gives:

$$\sum \vec{F}_{ext} = \frac{d\vec{p}_{total}}{dt}$$

If $\sum \vec{F}_{ext} = 0$:

$$\frac{d\vec{p}_{total}}{dt} = 0$$

$$\vec{p}_{total} = \text{constant}$$

Important: Internal forces (between particles in the system) do not affect total momentum due to Newton's third law - they cancel in pairs.

6.4 Collisions

6.4.1 Classification

Elastic collision: Both momentum and kinetic energy are conserved

$$\vec{p}_i = \vec{p}_f \quad \text{and} \quad KE_i = KE_f$$

Inelastic collision: Only momentum is conserved, kinetic energy is not

$$\vec{p}_i = \vec{p}_f \quad \text{but} \quad KE_i \neq KE_f$$

Perfectly inelastic collision: Objects stick together after collision

$$\vec{p}_i = \vec{p}_f \quad \text{and} \quad \vec{v}_{f,1} = \vec{v}_{f,2}$$

6.4.2 One-Dimensional Elastic Collision

For two objects with masses m_1, m_2 and initial velocities v_{1i}, v_{2i}:

Conservation of momentum:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad ...(1)$$

Conservation of kinetic energy:

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad ...(2)$$

Rearranging (1):

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \quad ...(3)$$

Rearranging (2):

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \quad ...(4)$$

Dividing (4) by (3):

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

Rearranging:

$$\boxed{v_{1i} - v_{2i} = -(v_{1f} - v_{2f})}$$

This states: relative velocity of approach = relative velocity of separation
Solving simultaneously with equation (1):

$$\boxed{v_{1f} = \frac{(m_1 - m_2)v_{1i} + 2m_2 v_{2i}}{m_1 + m_2}}$$

$$\boxed{v_{2f} = \frac{(m_2 - m_1)v_{2i} + 2m_1 v_{1i}}{m_1 + m_2}}$$

Special cases:

1. Equal masses ($m_1 = m_2$):

$$v_{1f} = v_{2i}, \quad v_{2f} = v_{1i}$$

Velocities are exchanged.

2. Object 2 initially at rest ($v_{2i} = 0$):

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}, \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

3. Object 1 much more massive ($m_1 \gg m_2$, $v_{2i} = 0$):

$$v_{1f} \approx v_{1i}, \quad v_{2f} \approx 2v_{1i}$$

Heavy object barely slows; light object moves at twice the speed.

4. Object 1 much less massive ($m_1 \ll m_2$, $v_{2i} = 0$):

$$v_{1f} \approx -v_{1i}, \quad v_{2f} \approx 0$$

Light object bounces back; heavy object barely moves.

6.4.3 Perfectly Inelastic Collision

Objects stick together: $v_{1f} = v_{2f} = v_f$

Conservation of momentum:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$$

$$\boxed{v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}}$$

Kinetic energy lost:

$$\Delta KE = KE_f - KE_i = \frac{1}{2}(m_1 + m_2) v_f^2 - \left(\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 \right)$$

This energy is converted to heat, sound, and deformation.

6.4.4 Coefficient of Restitution

The coefficient of restitution e characterizes how elastic a collision is:

$$e = -\frac{v_{1f} - v_{2f}}{v_{1i} - v_{2i}} = \frac{\text{relative velocity of separation}}{\text{relative velocity of approach}}$$

- $e = 1$: perfectly elastic collision
- $0 < e < 1$: inelastic collision
- $e = 0$: perfectly inelastic collision

6.5 Two-Dimensional Collisions

In two dimensions, momentum is conserved in both x and y directions:

$$\sum p_{x,i} = \sum p_{x,f}$$
$$\sum p_{y,i} = \sum p_{y,f}$$

For elastic collisions, kinetic energy is also conserved:

$$\sum KE_i = \sum KE_f$$

6.6 Center of Mass

6.6.1 Definition

The center of mass (CM) of a system is the point that moves as if all mass were concentrated there:

For discrete particles:

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M_{total}}$$

In components:

$$x_{CM} = \frac{\sum_i m_i x_i}{M_{total}}, \quad y_{CM} = \frac{\sum_i m_i y_i}{M_{total}}$$

6.6.2 Motion of Center of Mass

Velocity of CM:

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{\sum_i m_i \vec{v}_i}{M_{total}} = \frac{\vec{p}_{total}}{M_{total}}$$

Acceleration of CM:

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{\sum \vec{F}_{ext}}{M_{total}}$$

The center of mass moves as if all external forces acted on a single particle of mass M_{total} located at the CM.

6.6.3 Conservation of Momentum and CM

If $\sum \vec{F}_{ext} = 0$:

$$\vec{a}_{CM} = 0$$
$$\vec{v}_{CM} = \text{constant}$$

The center of mass moves with constant velocity (or remains at rest).

6.7 Example Problems

Problem 5.1: A 0.15 kg baseball traveling at 40 m/s is struck by a bat. The ball leaves the bat in the opposite direction at 50 m/s. The ball and bat are in contact for 2.0 ms. Calculate: (a) the impulse on the ball, (b) the average force exerted by the bat, (c) the change in kinetic energy of the ball.

Solution:

Given: $m = 0.15$ kg, $v_i = 40$ m/s (taking initial direction as positive), $v_f = -50$ m/s, $\Delta t = 2.0 \times 10^{-3}$ s

(a) Impulse:

$$J = \Delta p = mv_f - mv_i = 0.15(-50) - 0.15(40)$$

$$J = -7.5 - 6.0 = -13.5 \text{ kg·m/s}$$

The magnitude is 13.5 kg·m/s in the direction opposite to the initial motion.

(b) Average force:

$$F_{avg} = \frac{J}{\Delta t} = \frac{-13.5}{2.0 \times 10^{-3}} = -6750 \text{ N}$$

The magnitude is 6750 N (about 45 times the ball's weight!).

(c) Change in kinetic energy:

$$\Delta KE = \frac{1}{2}m(v_f^2 - v_i^2) = \frac{1}{2}(0.15)[(-50)^2 - (40)^2]$$

$$= \frac{1}{2}(0.15)[2500 - 1600] = 0.075(900) = 67.5 \text{ J}$$

The kinetic energy increased because the bat did work on the ball.

Problem 5.2: A 2000 kg car traveling east at 20 m/s collides with a 1500 kg car traveling north at 15 m/s. The cars stick together after collision. Find: (a) the velocity (magnitude and direction) of the wreckage immediately after collision, (b) the kinetic energy lost in the collision.

Solution:

Given: $m_1 = 2000$ kg, $\vec{v}_1 = 20\hat{\imath}$ m/s (east), $m_2 = 1500$ kg, $\vec{v}_2 = 15\hat{\jmath}$ m/s (north)

(a) Conservation of momentum:

x-component:

$$p_{x,i} = m_1 v_{1x} + m_2 v_{2x} = 2000(20) + 1500(0) = 40,000 \text{ kg·m/s}$$

$$p_{x,f} = (m_1 + m_2)v_{fx}$$

$$v_{fx} = \frac{40,000}{3500} = 11.43 \text{ m/s}$$

y-component:

$$p_{y,i} = m_1 v_{1y} + m_2 v_{2y} = 2000(0) + 1500(15) = 22,500 \text{ kg·m/s}$$

$$p_{y,f} = (m_1 + m_2)v_{fy}$$

$$v_{fy} = \frac{22,500}{3500} = 6.43 \text{ m/s}$$

Magnitude:

$$v_f = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{(11.43)^2 + (6.43)^2} = \sqrt{130.6 + 41.3} = 13.1 \text{ m/s}$$

Direction:

$$\theta = \tan^{-1}\left(\frac{v_{fy}}{v_{fx}}\right) = \tan^{-1}\left(\frac{6.43}{11.43}\right) = \tan^{-1}(0.562) = 29.3$$

The wreckage moves at 13.1 m/s at 29.3° north of east.
(b) Kinetic energy lost:
Initial kinetic energy:

$$\begin{aligned} KE_i &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\ &= \frac{1}{2}(2000)(20)^2 + \frac{1}{2}(1500)(15)^2 \\ &= 400,000 + 168,750 = 568,750 \text{ J} \end{aligned}$$

Final kinetic energy:

$$KE_f = \frac{1}{2}(m_1 + m_2)v_f^2 = \frac{1}{2}(3500)(13.1)^2 = 300,418 \text{ J}$$

Energy lost:

$$\Delta KE = KE_f - KE_i = 300,418 - 568,750 = -268,332 \text{ J}$$

About 268 kJ (47

Problem 5.3: A 3.0 kg object moving at 8.0 m/s collides elastically with a 5.0 kg object initially at rest. Find the final velocities of both objects.

Solution:

Given: $m_1 = 3.0$ kg, $v_{1i} = 8.0$ m/s, $m_2 = 5.0$ kg, $v_{2i} = 0$
Using the elastic collision formulas:

$$\begin{aligned} v_{1f} &= \frac{(m_1 - m_2)v_{1i} + 2m_2v_{2i}}{m_1 + m_2} = \frac{(3.0 - 5.0)(8.0) + 2(5.0)(0)}{3.0 + 5.0} \\ &= \frac{-2.0 \times 8.0}{8.0} = \frac{-16.0}{8.0} = -2.0 \text{ m/s} \end{aligned}$$

$$\begin{aligned} v_{2f} &= \frac{(m_2 - m_1)v_{2i} + 2m_1v_{1i}}{m_1 + m_2} = \frac{(5.0 - 3.0)(0) + 2(3.0)(8.0)}{3.0 + 5.0} \\ &= \frac{48.0}{8.0} = 6.0 \text{ m/s} \end{aligned}$$

Check momentum conservation:

$$p_i = 3.0(8.0) + 5.0(0) = 24.0 \text{ kg·m/s}$$

$$p_f = 3.0(-2.0) + 5.0(6.0) = -6.0 + 30.0 = 24.0 \text{ kg·m/s}$$

Check kinetic energy conservation:

$$KE_i = \frac{1}{2}(3.0)(8.0)^2 = 96.0 \text{ J}$$

$$KE_f = \frac{1}{2}(3.0)(-2.0)^2 + \frac{1}{2}(5.0)(6.0)^2 = 6.0 + 90.0 = 96.0 \text{ J}$$

7 Lesson 6: Rotational Motion

7.1 Angular Kinematics

7.1.1 Angular Position

Angular position θ is measured in radians from a reference line:

$$\theta \text{ (rad)} = \frac{s}{r}$$

where s is arc length and r is radius.
Conversion: 2π rad = 360°, so 1 rad = 57.3°

7.1.2 Angular Displacement

$$\Delta\theta = \theta_f - \theta_i$$

Positive: counterclockwise (by convention) Negative: clockwise

7.1.3 Angular Velocity

Average angular velocity:

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Units: rad/s

7.1.4 Angular Acceleration

Average angular acceleration:

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous angular acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Units: rad/s^2

7.2 Rotational Kinematics Equations

For constant angular acceleration, these equations are analogous to linear kinematics:

$$\boxed{\omega = \omega_0 + \alpha t}$$

$$\boxed{\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2}$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)}$$

$$\boxed{\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t}$$

7.3 Relationship Between Linear and Angular Quantities

For a point at distance r from the axis of rotation:

7.3.1 Arc Length and Angle

$$s = r\theta$$

7.3.2 Tangential Velocity and Angular Velocity

$$v_t = r\omega$$

Derivation:

$$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

7.3.3 Tangential Acceleration and Angular Acceleration

$$a_t = r\alpha$$

Derivation:

$$a_t = \frac{dv_t}{dt} = r\frac{d\omega}{dt} = r\alpha$$

7.3.4 Centripetal Acceleration

$$a_c = \frac{v_t^2}{r} = r\omega^2$$

7.3.5 Total Acceleration

The total acceleration has both tangential and centripetal components:

$$\vec{a} = \vec{a}_t + \vec{a}_c$$

Magnitude:

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$$

7.4 Rotational Inertia (Moment of Inertia)

7.4.1 Definition

Rotational inertia measures an object's resistance to rotational acceleration:

For a point mass:

$$I = mr^2$$

For a system of particles:

$$I = \sum_i m_i r_i^2$$

For a continuous object:

$$I = \int r^2\, dm$$

Units: kg·m²

7.4.2 Common Moments of Inertia

Thin hoop or cylindrical shell (radius R, axis through center):

$$I = MR^2$$

Solid cylinder or disk (radius R, axis through center):

$$I = \int_0^R r^2\, dm$$

For a disk of thickness t, consider a thin ring at radius r with thickness dr:

$$dm = \rho \cdot 2\pi r \cdot t \cdot dr$$

where ρ is density. The total mass is $M = \rho \pi R^2 t$, so:

$$\rho t = \frac{M}{\pi R^2}$$

Therefore:

$$I = \int_0^R r^2 \cdot \frac{M}{\pi R^2} \cdot 2\pi r\, dr = \frac{2M}{R^2} \int_0^R r^3\, dr$$

$$= \frac{2M}{R^2} \left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2} \cdot \frac{R^4}{4} = \frac{MR^2}{2}$$

$$\boxed{I = \frac{1}{2}MR^2}$$

Solid sphere (radius R, axis through center):

$$I = \frac{2}{5}MR^2$$

Thin spherical shell (radius R, axis through center):

$$I = \frac{2}{3}MR^2$$

Thin rod (length L, axis through center):

$$I = \frac{1}{12}ML^2$$

Thin rod (length L, axis through end):

$$I = \frac{1}{3}ML^2$$

7.4.3 Parallel Axis Theorem

If I_{CM} is the moment of inertia about an axis through the center of mass, then the moment of inertia about a parallel axis at distance d is:

$$\boxed{I = I_{CM} + Md^2}$$

Proof:

Choose coordinates with origin at CM. The moment of inertia about an axis at position (x_0, y_0) perpendicular to the plane is:

$$I = \int [(x - x_0)^2 + (y - y_0)^2] \, dm$$

$$= \int (x^2 + y^2) \, dm - 2x_0 \int x \, dm - 2y_0 \int y \, dm + (x_0^2 + y_0^2) \int dm$$

Since the origin is at the CM:

$$\int x \, dm = 0, \quad \int y \, dm = 0$$

Therefore:

$$I = I_{CM} + M(x_0^2 + y_0^2) = I_{CM} + Md^2$$

where $d = \sqrt{x_0^2 + y_0^2}$ is the distance between axes.

7.5 Torque

7.5.1 Definition

Torque is the rotational analog of force. For a force $\vec{F}$ applied at position $\vec{r}$ from the axis:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Magnitude:

$$\tau = rF \sin\theta = rF_{\perp} = r_{\perp}F$$

where:

- θ is the angle between $\vec{r}$ and $\vec{F}$
- $F_\perp = F \sin\theta$ is the component of force perpendicular to $\vec{r}$
- $r_\perp = r \sin\theta$ is the moment arm (perpendicular distance from axis to line of force)

Units: N·m (not Joules, even though dimensions are the same)

7.5.2 Sign Convention

- Positive torque: causes counterclockwise rotation
- Negative torque: causes clockwise rotation

7.6 Newton's Second Law for Rotation

7.6.1 Statement

$$\boxed{\sum \tau = I\alpha}$$

This is the rotational analog of $\sum F = ma$.

7.6.2 Derivation

For a point mass at distance r from axis, with tangential force F_t:

$$F_t = ma_t = m(r\alpha)$$

Multiply both sides by r:

$$rF_t = mr^2\alpha$$
$$\tau = (mr^2)\alpha = I\alpha$$

For a system of particles:

$$\sum_i \tau_i = \sum_i (m_i r_i^2)\alpha = I\alpha$$

7.7 Rotational Kinetic Energy

7.7.1 Derivation

For a rotating object, each element has kinetic energy:

$$dKE = \frac{1}{2}(dm)v^2 = \frac{1}{2}(dm)(r\omega)^2 = \frac{1}{2}r^2\omega^2\, dm$$

Total kinetic energy:

$$KE_{rot} = \int \frac{1}{2}r^2\omega^2\, dm = \frac{1}{2}\omega^2 \int r^2\, dm = \frac{1}{2}I\omega^2$$

$$\boxed{KE_{rot} = \frac{1}{2}I\omega^2}$$

This is analogous to $KE = \frac{1}{2}mv^2$ for linear motion.

7.7.2 Total Kinetic Energy for Rolling Objects

An object rolling without slipping has both translational and rotational kinetic energy:

$$KE_{total} = KE_{trans} + KE_{rot} = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

For rolling without slipping: $v_{CM} = R\omega$

$$\begin{aligned} KE_{total} &= \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\left(\frac{v_{CM}}{R}\right)^2 \\ &= \frac{1}{2}mv_{CM}^2\left(1 + \frac{I_{CM}}{mR^2}\right) \end{aligned}$$

7.8 Work and Power in Rotation

7.8.1 Work Done by Torque

$$dW = \vec{\tau} \cdot d\vec{\theta}$$

For constant torque:

$$W = \tau\Delta\theta$$

This is analogous to $W = F\Delta x$.

7.8.2 Work-Energy Theorem for Rotation

$$W_{net} = \Delta KE_{rot} = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

7.8.3 Power

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

This is analogous to $P = Fv$.

7.9 Angular Momentum

7.9.1 Definition

For a point particle:

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

Magnitude:

$$L = mvr\sin\theta = mvr_\perp = mr^2\omega$$

For a rotating rigid body:

$$\boxed{L = I\omega}$$

Units: kg·m²/s

7.9.2 Relation Between Torque and Angular Momentum

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

This is analogous to $\sum \vec{F} = \frac{d\vec{p}}{dt}$.
For constant I:

$$\sum \tau = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha$$

7.9.3 Conservation of Angular Momentum

If the net external torque is zero:

$$\sum \tau_{ext} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant}$$

$$I_i\omega_i = I_f\omega_f$$

This explains:

- Why ice skaters spin faster when pulling arms in (decreasing I)
- Why divers tuck to increase rotation rate
- Conservation of angular momentum in planetary orbits

7.10 Rolling Motion

7.10.1 Condition for Rolling Without Slipping

At the contact point, the velocity must be zero:

$$v_{CM} = R\omega$$

This relates linear and angular velocities.

7.10.2 Acceleration for Rolling

Differentiation the rolling condition:

$$a_{CM} = R\alpha$$

7.10.3 Rolling Down an Incline

For an object rolling down an incline of angle θ:
Forces:

- Weight: mg (down)
- Normal force: N (perpendicular to surface)

- Static friction: f_s (up the incline, provides torque for rotation)

Translational motion:

$$mg\sin\theta - f_s = ma_{CM}$$

Rotational motion about CM:

$$f_sR = I_{CM}\alpha = I_{CM}\frac{a_{CM}}{R}$$

Solving for f_s:

$$f_s = \frac{I_{CM}a_{CM}}{R^2}$$

Substituting into translational equation:

$$mg\sin\theta - \frac{I_{CM}a_{CM}}{R^2} = ma_{CM}$$

$$mg\sin\theta = a_{CM}\left(m + \frac{I_{CM}}{R^2}\right)$$

$$\boxed{a_{CM} = \frac{mg\sin\theta}{m + I_{CM}/R^2} = \frac{g\sin\theta}{1 + I_{CM}/(mR^2)}}$$

For a solid cylinder ($I_{CM} = \frac{1}{2}mR^2$):

$$a_{CM} = \frac{g\sin\theta}{1 + 1/2} = \frac{2g\sin\theta}{3}$$

For a solid sphere ($I_{CM} = \frac{2}{5}mR^2$):

$$a_{CM} = \frac{g\sin\theta}{1 + 2/5} = \frac{5g\sin\theta}{7}$$

Note: A sphere rolls down faster than a cylinder because more of its kinetic energy is translational.

7.11 Example Problems

Problem 6.1: A solid disk of mass 5.0 kg and radius 0.30 m rotates about its central axis. Starting from rest, it reaches an angular velocity of 20 rad/s in 4.0 seconds. Calculate: (a) the angular acceleration, (b) the number of revolutions in this time, (c) the moment of inertia, (d) the net torque applied, (e) the rotational kinetic energy at $t = 4.0$ s, (f) the power at $t = 4.0$ s.

Solution:

Given: $m = 5.0$ kg, $R = 0.30$ m, $\omega_0 = 0$, $\omega = 20$ rad/s, $t = 4.0$ s

(a) Angular acceleration:

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{20 - 0}{4.0} = 5.0 \text{ rad/s}^2$$

(b) Angular displacement:

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(5.0)(4.0)^2 = 40 \text{ rad}$$

Number of revolutions:

$$N = \frac{\theta}{2\pi} = \frac{40}{2\pi} = 6.37 \text{ rev}$$

(c) Moment of inertia (solid disk):

$$I = \frac{1}{2}mR^2 = \frac{1}{2}(5.0)(0.30)^2 = 0.225 \text{ kg·m}^2$$

(d) Net torque:

$$\tau = I\alpha = 0.225 \times 5.0 = 1.13 \text{ N·m}$$

(e) Rotational kinetic energy:

$$KE_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.225)(20)^2 = 45 \text{ J}$$

(f) Power at $t = 4.0$ s:

$$P = \tau\omega = 1.13 \times 20 = 22.6 \text{ W}$$

Problem 6.2: A solid sphere of mass 2.0 kg and radius 0.15 m rolls without slipping down a 30° incline from a height of 3.0 m. Using energy conservation, find: (a) the linear speed at the bottom, (b) the angular speed at the bottom, (c) verify using the acceleration formula.

Solution:

Given: $m = 2.0$ kg, $R = 0.15$ m, $\theta = 30$, $h = 3.0$ m

(a) Energy conservation:

Initial energy (at top, at rest):

$$E_i = mgh$$

Final energy (at bottom, rolling):

$$E_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

For a solid sphere: $I = \frac{2}{5}mR^2$ and $v = R\omega$:

$$E_f = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2$$

Energy conservation:

$$mgh = \frac{7}{10}mv^2$$

$$v = \sqrt{\frac{10gh}{7}} = \sqrt{\frac{10 \times 9.8 \times 3.0}{7}} = \sqrt{42} = 6.48 \text{ m/s}$$

(b) Angular speed:

$$\omega = \frac{v}{R} = \frac{6.48}{0.15} = 43.2 \text{ rad/s}$$

(c) Verification using acceleration:
Length of incline:

$$L = \frac{h}{\sin\theta} = \frac{3.0}{0.5} = 6.0 \text{ m}$$

Acceleration:

$$a = \frac{5g\sin\theta}{7} = \frac{5 \times 9.8 \times 0.5}{7} = 3.5 \text{ m/s}^2$$

Final velocity (starting from rest):

$$v^2 = v_0^2 + 2aL = 0 + 2(3.5)(6.0) = 42$$

$$v = 6.48 \text{ m/s}$$

Problem 6.3: A figure skater is spinning at 1.0 rev/s with arms extended. Her moment of inertia with arms extended is 3.6 kg·m**²**. When she pulls her arms in, her moment of inertia decreases to 1.8 kg·m**²**. Find: (a) her final angular velocity, (b) the ratio of final to initial kinetic energy, (c) explain where the extra energy comes from.

Solution:

Given: $f_i = 1.0$ rev/s, $I_i = 3.6$ kg·m**²**, $I_f = 1.8$ kg·m**²**

Initial angular velocity:

$$\omega_i = 2\pi f_i = 2\pi(1.0) = 6.28 \text{ rad/s}$$

(a) Conservation of angular momentum:

$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

$$\omega_f = \frac{I_i\omega_i}{I_f} = \frac{3.6 \times 6.28}{1.8} = 12.56 \text{ rad/s}$$

In revolutions per second:

$$f_f = \frac{\omega_f}{2\pi} = \frac{12.56}{2\pi} = 2.0 \text{ rev/s}$$

Her spin rate doubles!

(b) Kinetic energy ratio:

$$\frac{KE_f}{KE_i} = \frac{\frac{1}{2}I_f\omega_f^2}{\frac{1}{2}I_i\omega_i^2} = \frac{I_f\omega_f^2}{I_i\omega_i^2}$$

$$= \frac{1.8 \times (12.56)^2}{3.6 \times (6.28)^2} = \frac{1.8 \times 157.8}{3.6 \times 39.4} = \frac{284.0}{141.8} = 2.0$$

The kinetic energy doubles!

(c) The extra energy comes from the work done by the skater's muscles in pulling her arms inward against the centrifugal effect. Even though angular momentum is conserved (no external torques), energy is not conserved because internal forces (her muscles) do work on the system.

Quantitatively:

$$KE_i = \frac{1}{2}(3.6)(6.28)^2 = 71.1 \text{ J}$$

$$KE_f = \frac{1}{2}(1.8)(12.56)^2 = 142.1 \text{ J}$$

$$W_{muscles} = KE_f - KE_i = 142.1 - 71.1 = 71.0 \text{ J}$$

8 Lesson 7: Gravitation

8.1 Newton's Law of Universal Gravitation

8.1.1 Statement

Every particle attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them:

$$\boxed{\vec{F}_g = -G\frac{m_1 m_2}{r^2}\hat{r}}$$

where:

- $G = 6.674 \times 10^{-11}$ N·m²/kg² is the universal gravitational constant
- r is the distance between centers
- $\hat{r}$ is the unit vector from m_1 to m_2
- The negative sign indicates attraction

8.1.2 Gravitational Field

The gravitational field at a point is the gravitational force per unit mass:

$$\vec{g} = \frac{\vec{F}_g}{m} = -G\frac{M}{r^2}\hat{r}$$

For Earth at its surface:

$$g = G\frac{M_\oplus}{R_\oplus^2} = 9.8 \text{ m/s}^2$$

where $M_\oplus = 5.97 \times 10^{24}$ kg and $R_\oplus = 6.37 \times 10^6$ m.

8.2 Gravitational Potential Energy

8.2.1 Derivation

The work done by gravity in moving a mass m from r_1 to r_2 is:

$$\begin{aligned} W_g &= \int_{r_1}^{r_2} \vec{F}_g \cdot d\vec{r} = \int_{r_1}^{r_2} \left(-G\frac{Mm}{r^2}\right) dr \\ &= GMm \int_{r_1}^{r_2} \frac{dr}{r^2} = GMm\left[-\frac{1}{r}\right]_{r_1}^{r_2} \\ &= GMm\left(-\frac{1}{r_2} + \frac{1}{r_1}\right) = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \end{aligned}$$

Define gravitational potential energy with $U(\infty) = 0$:

$$\boxed{U(r) = -G\frac{Mm}{r}}$$

This is always negative, indicating that the system has less energy than when the masses are infinitely separated.

8.2.2 Relation to Gravitational Field

$$\vec{F}_g = -\nabla U = -\frac{dU}{dr}\hat{r}$$

Verifying:

$$-\frac{dU}{dr} = -\frac{d}{dr}\left(-G\frac{Mm}{r}\right) = -G\frac{Mm}{r^2}$$

8.3 Orbital Motion

8.3.1 Circular Orbits

For a satellite of mass m orbiting a planet of mass M at radius r with speed v:

The gravitational force provides centripetal acceleration:

$$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

Solving for orbital speed:

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

Note: Orbital speed decreases with distance from the planet.

Period of orbit:

The circumference is $2\pi r$, so:

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{GM/r}} = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\boxed{T^2 = \frac{4\pi^2}{GM}r^3}$$

This is Kepler's third law. For orbits around the Sun:

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM_\odot} = \text{constant}$$

8.3.2 Energy in Circular Orbits

Kinetic energy:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{GM}{r}\right) = \frac{GMm}{2r}$$

Potential energy:

$$U = -\frac{GMm}{r}$$

Total mechanical energy:

$$E = KE + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

$$\boxed{E = -\frac{GMm}{2r}}$$

Important observations:

- Total energy is negative (bound orbit)
- $|U| = 2KE$ (virial theorem for $1/r$ potentials)
- As r increases, E increases (becomes less negative)
- To move to a higher orbit requires adding energy

8.4 Escape Velocity

8.4.1 Derivation

For an object to escape to infinity, its total energy must be at least zero:

$$E = KE + U \geq 0$$
$$\frac{1}{2}mv_{esc}^2 - \frac{GMm}{r} \geq 0$$
$$\frac{1}{2}mv_{esc}^2 \geq \frac{GMm}{r}$$

$$\boxed{v_{esc} = \sqrt{\frac{2GM}{r}}}$$

At Earth's surface:

$$v_{esc} = \sqrt{\frac{2GM_\oplus}{R_\oplus}} = \sqrt{2gR_\oplus} = \sqrt{2 \times 9.8 \times 6.37 \times 10^6}$$

$$= 11.2 \text{ km/s}$$

Note: $v_{esc} = \sqrt{2} \cdot v_{orbit}$ for the same radius.

8.5 Kepler's Laws

8.5.1 Kepler's First Law

Planets move in elliptical orbits with the Sun at one focus.

8.5.2 Kepler's Second Law

A line from the Sun to a planet sweeps out equal areas in equal times.

This is a consequence of conservation of angular momentum:

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

8.5.3 Kepler's Third Law

The square of the orbital period is proportional to the cube of the semi-major axis:

$$T^2 = \frac{4\pi^2}{G(M+m)}a^3$$

For $M \gg m$ (like Sun and planets):

$$T^2 = \frac{4\pi^2}{GM}a^3$$

8.6 Gravitational Effects

8.6.1 Apparent Weightlessness

An object in orbit is in free fall - both the object and the spacecraft are accelerating toward Earth at the same rate. There is no normal force, creating the sensation of weightlessness.

Weight in orbit: $F_g = \frac{GMm}{r^2}$ (still present!)

Apparent weight: $N = 0$ (because no contact forces)

8.6.2 Variation of g with Altitude

At height h above Earth's surface:

$$g(h) = \frac{GM_\oplus}{(R_\oplus + h)^2} = g_0 \left(\frac{R_\oplus}{R_\oplus + h}\right)^2$$

For small heights ($h \ll R_\oplus$):

$$g(h) \approx g_0 \left(1 - \frac{2h}{R_\oplus}\right)$$

8.6.3 Tidal Forces

Tidal forces arise from the difference in gravitational force across an extended object. The side closer to the attracting body experiences stronger force than the far side.

8.7 Example Problems

Problem 7.1: A satellite orbits Earth at an altitude of 400 km above the surface. Calculate: (a) the orbital speed, (b) the period of revolution, (c) the total mechanical energy per unit mass, (d) the energy required to place a 1000 kg satellite in this orbit from Earth's surface.

Solution:

Given: $h = 400$ km $= 4.0 \times 10^5$ m, $M_\oplus = 5.97 \times 10^{24}$ kg, $R_\oplus = 6.37 \times 10^6$ m, $G = 6.674 \times 10^{-11}$ N·m²/kg²

Orbital radius:

$$r = R_\oplus + h = 6.37 \times 10^6 + 4.0 \times 10^5 = 6.77 \times 10^6 \text{ m}$$

(a) Orbital speed:

$$v = \sqrt{\frac{GM_\oplus}{r}} = \sqrt{\frac{6.674 \times 10^{-11} \times 5.97 \times 10^{24}}{6.77 \times 10^6}}$$

$$= \sqrt{\frac{3.984 \times 10^{14}}{6.77 \times 10^6}} = \sqrt{5.88 \times 10^7} = 7670 \text{ m/s} = 7.67 \text{ km/s}$$

(b) Period:

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 6.77 \times 10^6}{7670} = 5550 \text{ s} = 92.5 \text{ min}$$

Or using Kepler's third law:

$$T = 2\pi\sqrt{\frac{r^3}{GM_\oplus}} = 2\pi\sqrt{\frac{(6.77 \times 10^6)^3}{6.674 \times 10^{-11} \times 5.97 \times 10^{24}}}$$

$$= 2\pi\sqrt{\frac{3.10 \times 10^{20}}{3.984 \times 10^{14}}} = 2\pi\sqrt{7.78 \times 10^5} = 5550 \text{ s}$$

(c) Total energy per unit mass:

$$\frac{E}{m} = -\frac{GM_\oplus}{2r} = -\frac{6.674 \times 10^{-11} \times 5.97 \times 10^{24}}{2 \times 6.77 \times 10^6}$$

$$= -\frac{3.984 \times 10^{14}}{1.354 \times 10^7} = -2.94 \times 10^7 \text{ J/kg} = -29.4 \text{ MJ/kg}$$

(d) Energy required for 1000 kg satellite:

Initial energy at Earth's surface (at rest):

$$E_i = -\frac{GM_\oplus m}{R_\oplus} = -\frac{6.674 \times 10^{-11} \times 5.97 \times 10^{24} \times 1000}{6.37 \times 10^6}$$

$$= -6.26 \times 10^{10} \text{ J} = -62.6 \text{ GJ}$$

Final energy in orbit:

$$E_f = -\frac{GM_\oplus m}{2r} = -2.94 \times 10^7 \times 1000 = -2.94 \times 10^{10} \text{ J} = -29.4 \text{ GJ}$$

Energy required:

$$\Delta E = E_f - E_i = -29.4 - (-62.6) = 33.2 \text{ GJ}$$

Problem 7.2: Two planets orbit a star. Planet A has an orbital radius of 1.5×10^{11} m and a period of 365 days. Planet B has an orbital radius of 6.0×10^{11} m. Find: (a) the period of planet B, (b) the mass of the star, (c) the ratio of orbital speeds v_A/v_B.

Solution:

Given: $r_A = 1.5 \times 10^{11}$ m, $T_A = 365$ days, $r_B = 6.0 \times 10^{11}$ m

(a) Using Kepler's third law:

$$\frac{T_A^2}{r_A^3} = \frac{T_B^2}{r_B^3}$$

$$T_B^2 = T_A^2 \left(\frac{r_B}{r_A}\right)^3 = (365)^2 \left(\frac{6.0 \times 10^{11}}{1.5 \times 10^{11}}\right)^3$$

$$= (365)^2(4)^3 = (365)^2 \times 64$$

$$T_B = 365 \times 8 = 2920 \text{ days} = 8.0 \text{ years}$$

(b) Mass of star:

From Kepler's third law:

$$T^2 = \frac{4\pi^2}{GM} r^3$$

$$M = \frac{4\pi^2 r^3}{GT^2}$$

Using planet A's data:

$$T_A = 365 \times 24 \times 3600 = 3.156 \times 10^7 \text{ s}$$

$$M = \frac{4\pi^2 \times (1.5 \times 10^{11})^3}{6.674 \times 10^{-11} \times (3.156 \times 10^7)^2}$$

$$= \frac{4\pi^2 \times 3.375 \times 10^{33}}{6.674 \times 10^{-11} \times 9.96 \times 10^{14}}$$

$$= \frac{1.332 \times 10^{35}}{6.65 \times 10^4} = 2.0 \times 10^{30} \text{ kg}$$

This is approximately the mass of our Sun ($M_\odot = 1.989 \times 10^{30}$ kg).

(c) Ratio of orbital speeds:

Since $v = \sqrt{GM/r}$:

$$\frac{v_A}{v_B} = \sqrt{\frac{r_B}{r_A}} = \sqrt{\frac{6.0 \times 10^{11}}{1.5 \times 10^{11}}} = \sqrt{4} = 2$$

Planet A moves twice as fast as planet B.

Problem 7.3: A projectile is launched vertically from Earth's surface with initial speed equal to half the escape velocity. Find: (a) the maximum height reached, (b) the speed at half this maximum height on the way up, (c) verify using energy conservation.

Solution:
Given: $v_0 = \frac{1}{2}v_{esc} = \frac{1}{2}\sqrt{\frac{2GM_\oplus}{R_\oplus}}$
(a) Maximum height using energy conservation:
Initial energy:

$$E_i = \frac{1}{2}mv_0^2 - \frac{GM_\oplus m}{R_\oplus}$$

$$= \frac{1}{2}m \cdot \frac{1}{4} \cdot \frac{2GM_\oplus}{R_\oplus} - \frac{GM_\oplus m}{R_\oplus}$$

$$= \frac{GM_\oplus m}{4R_\oplus} - \frac{GM_\oplus m}{R_\oplus} = -\frac{3GM_\oplus m}{4R_\oplus}$$

At maximum height ($v = 0$):

$$E_f = -\frac{GM_\oplus m}{R_\oplus + h_{max}}$$

Conservation of energy:

$$-\frac{3GM_\oplus m}{4R_\oplus} = -\frac{GM_\oplus m}{R_\oplus + h_{max}}$$

$$\frac{3}{4R_\oplus} = \frac{1}{R_\oplus + h_{max}}$$

$$R_\oplus + h_{max} = \frac{4R_\oplus}{3}$$

$$h_{max} = \frac{4R_\oplus}{3} - R_\oplus = \frac{R_\oplus}{3}$$

$$h_{max} = \frac{6.37 \times 10^6}{3} = 2.12 \times 10^6 \text{ m} = 2120 \text{ km}$$

(b) Speed at half maximum height:
At $h = h_{max}/2 = R_\oplus/6$:

$$r = R_\oplus + \frac{R_\oplus}{6} = \frac{7R_\oplus}{6}$$

Energy at this height:

$$E = \frac{1}{2}mv^2 - \frac{GM_\oplus m}{7R_\oplus/6} = \frac{1}{2}mv^2 - \frac{6GM_\oplus m}{7R_\oplus}$$

Setting equal to initial energy:

$$\frac{1}{2}mv^2 - \frac{6GM_\oplus m}{7R_\oplus} = -\frac{3GM_\oplus m}{4R_\oplus}$$

$$\frac{1}{2}v^2 = \frac{6GM_\oplus}{7R_\oplus} - \frac{3GM_\oplus}{4R_\oplus}$$

$$\frac{1}{2}v^2 = GM_\oplus\left(\frac{6}{7R_\oplus} - \frac{3}{4R_\oplus}\right) = \frac{GM_\oplus}{R_\oplus}\left(\frac{24-21}{28}\right)$$

$$\frac{1}{2}v^2 = \frac{3GM_\oplus}{28R_\oplus}$$

$$v = \sqrt{\frac{3GM_\oplus}{14R_\oplus}} = \sqrt{\frac{3gR_\oplus}{14}}$$

Numerically:

$$v = \sqrt{\frac{3 \times 9.8 \times 6.37 \times 10^6}{14}} = \sqrt{1.34 \times 10^7} = 3660 \text{ m/s}$$

(c) Verification:
Check that this is consistent with v_0:

$$v_0 = \frac{1}{2}\sqrt{2gR_\oplus} = \sqrt{\frac{gR_\oplus}{2}}$$

$$v_0 = \sqrt{\frac{9.8 \times 6.37 \times 10^6}{2}} = \sqrt{3.12 \times 10^7} = 5590 \text{ m/s}$$

At ground level, using energy:

$$\frac{1}{2}v_0^2 = \frac{GM_\oplus}{4R_\oplus}$$

The speed decreases from 5590 m/s at surface to 3660 m/s at height 1060 km to 0 at height 2120 km, as expected.

9 Lesson 8: Simple Harmonic Motion

9.1 Definition and Characteristics

9.1.1 Restoring Force

Simple harmonic motion (SHM) occurs when the restoring force is proportional to displacement and directed toward equilibrium:

$$\boxed{F = -kx}$$

where k is the force constant and x is displacement from equilibrium.

9.1.2 Equation of Motion

Applying Newton's second law:

$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

This is the differential equation for SHM.

9.2 Solution - Position as Function of Time

9.2.1 General Solution

The general solution to the SHM differential equation is:

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$

where:

- A is the amplitude (maximum displacement)
- $\omega = \sqrt{k/m}$ is the angular frequency (rad/s)
- ϕ is the phase constant (determined by initial conditions)

Verification:

$$\frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x$$

Substituting:

$$-\omega^2 x + \frac{k}{m}x = 0$$

$$\omega^2 = \frac{k}{m}$$

9.2.2 Alternative Forms

$$x(t) = A\sin(\omega t + \phi')$$

Or using initial conditions x_0 and v_0:

$$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$$

9.3 Velocity and Acceleration

9.3.1 Velocity

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

Maximum speed (at equilibrium, $x = 0$):

$$v_{max} = A\omega$$

At position x:

$$v(x) = \pm\omega\sqrt{A^2 - x^2}$$

9.3.2 Acceleration

$$a(t) = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x$$

$$\boxed{a = -\omega^2 x}$$

Maximum acceleration (at maximum displacement):

$$a_{max} = A\omega^2$$

9.4 Period and Frequency

9.4.1 Definitions

Period T: Time for one complete oscillation

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$$

Frequency f: Number of oscillations per unit time

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

Units: Hz (Hertz) = cycles/s = s^1

Angular frequency:

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Units: rad/s

9.5 Energy in SHM

9.5.1 Potential Energy

For a spring:

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

9.5.2 Kinetic Energy

$$KE(t) = \frac{1}{2}mv^2 = \frac{1}{2}m(A\omega)^2\sin^2(\omega t + \phi)$$
$$= \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

(using $\omega^2 = k/m$)

9.5.3 Total Mechanical Energy

$$E = KE + U = \frac{1}{2}kA^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$
$$= \frac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

$$\boxed{E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2 = \text{constant}}$$

Energy is conserved and proportional to the square of amplitude.
Energy distribution:

- At maximum displacement ($x = \pm A$): all potential energy
- At equilibrium ($x = 0$): all kinetic energy
- At intermediate positions: mixture of both

9.6 The Simple Pendulum

9.6.1 Equation of Motion

For a pendulum of length L with mass m, displaced by angle θ from vertical:
Tangential component of gravity provides restoring force:

$$F_t = -mg\sin\theta$$

For small angles ($\theta < 15$): $\sin\theta \approx \theta$ (in radians)

$$F_t \approx -mg\theta = -mg\frac{s}{L}$$

where s is arc length.
This gives SHM with effective spring constant:

$$k_{eff} = \frac{mg}{L}$$

9.6.2 Period

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

Derivation:

$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

Important notes:

- Period is independent of mass
- Period depends only on length and g
- Valid only for small angles

9.7 The Physical Pendulum

For an extended object pivoting about a point at distance d from its center of mass:

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgd}}}$$

where I is moment of inertia about the pivot point.
For a simple pendulum ($I = mL^2$, $d = L$):

$$T = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

9.8 Damped Harmonic Motion

9.8.1 Damping Force

In real systems, friction causes energy loss. For viscous damping:

$$F_{damp} = -bv$$

where b is the damping coefficient.

9.8.2 Equation of Motion

$$F_{net} = -kx - bv = ma$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

9.8.3 Solution

For underdamped motion ($b < 2\sqrt{km}$):

$$x(t) = Ae^{-\gamma t}\cos(\omega' t + \phi)$$

where:

- $\gamma = b/(2m)$ is the damping constant
- $\omega' = \sqrt{\omega_0^2 - \gamma^2}$ is the damped angular frequency
- $\omega_0 = \sqrt{k/m}$ is the natural angular frequency

The amplitude decreases exponentially: $A(t) = A_0 e^{-\gamma t}$

9.9 Driven Harmonic Motion and Resonance

9.9.1 Driving Force

An external periodic force:

$$F_{drive} = F_0 \cos(\omega_d t)$$

where ω_d is the driving frequency.

9.9.2 Equation of Motion

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega_d t)$$

9.9.3 Steady-State Solution

After transients die out:

$$x(t) = A(\omega_d)\cos(\omega_d t - \delta)$$

where the amplitude depends on driving frequency:

$$A(\omega_d) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}}$$

9.9.4 Resonance

Maximum amplitude occurs when:

$$\omega_d \approx \omega_0$$

This is resonance - the driving frequency matches the natural frequency.
At resonance (for light damping):

$$A_{res} \approx \frac{F_0}{2m\gamma\omega_0} = \frac{F_0}{b\omega_0}$$

The amplitude can be very large if damping is small, which can lead to structural failure (e.g., Tacoma Narrows Bridge).

9.10 Example Problems

Problem 8.1: A 0.50 kg mass attached to a spring with force constant 20 N/m oscillates on a frictionless horizontal surface. At $t = 0$, the mass is at $x = 0.10$ m and moving toward equilibrium with speed 0.80 m/s. Find: (a) the amplitude, (b) the phase constant, (c) the maximum speed, (d) the maximum acceleration, (e) the total energy, (f) the position and velocity at $t = 1.0$ s.

Solution:

Given: $m = 0.50$ kg, $k = 20$ N/m, at $t = 0$: $x_0 = 0.10$ m, $v_0 = -0.80$ m/s (toward equilibrium, so negative)

First, find angular frequency:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{0.50}} = \sqrt{40} = 6.32 \text{ rad/s}$$

(a) Amplitude:
Using energy conservation:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$$

$$kA^2 = mv_0^2 + kx_0^2$$

$$A^2 = \frac{mv_0^2}{k} + x_0^2 = \frac{0.50 \times (0.80)^2}{20} + (0.10)^2$$
$$= 0.016 + 0.010 = 0.026$$

$$A = 0.161 \text{ m}$$

(b) Phase constant:
Using $x(t) = A\cos(\omega t + \phi)$ and $v(t) = -A\omega \sin(\omega t + \phi)$:
At $t = 0$:

$$x_0 = A\cos\phi$$

$$v_0 = -A\omega\sin\phi$$

From the first equation:

$$\cos\phi = \frac{x_0}{A} = \frac{0.10}{0.161} = 0.621$$
$$\phi = \pm\cos^{-1}(0.621) = \pm 51.6 = \pm 0.900 \text{ rad}$$

From the second equation:

$$\sin\phi = -\frac{v_0}{A\omega} = -\frac{(-0.80)}{0.161 \times 6.32} = \frac{0.80}{1.018} = 0.786$$

Since $\cos\phi > 0$ and $\sin\phi > 0$, ϕ is in the first quadrant:

$$\phi = +0.900 \text{ rad} = 51.6$$

(c) Maximum speed:

$$v_{max} = A\omega = 0.161 \times 6.32 = 1.02 \text{ m/s}$$

(d) Maximum acceleration:

$$a_{max} = A\omega^2 = 0.161 \times (6.32)^2 = 0.161 \times 39.9 = 6.42 \text{ m/s}^2$$

Or: $a_{max} = \frac{kA}{m} = \frac{20 \times 0.161}{0.50} = 6.44$ m/s²
(e) Total energy:

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \times 20 \times (0.161)^2 = 0.259 \text{ J}$$

(f) Position and velocity at $t = 1.0$ s:

$$x(1.0) = 0.161\cos(6.32 \times 1.0 + 0.900)$$
$$= 0.161\cos(7.22 \text{ rad})$$
$$= 0.161\cos(413.7) = 0.161\cos(53.7) = 0.161 \times 0.588$$
$$= 0.0947 \text{ m} = 9.47 \text{ cm}$$

$$v(1.0) = -0.161 \times 6.32 \times \sin(7.22)$$
$$= -1.018\sin(413.7) = -1.018\sin(53.7) = -1.018 \times 0.809$$
$$= -0.823 \text{ m/s}$$

Problem 8.2: A simple pendulum has length 2.0 m. Find: (a) the period on Earth's surface, (b) the period on the Moon where $g_{moon} = 1.62$ m/s², (c) if the pendulum is taken to a planet where the period is 3.5 s, what is the gravitational acceleration on that planet?

Solution:

Given: $L = 2.0$ m, $g_{Earth} = 9.8$ m/s², $g_{moon} = 1.62$ m/s²
(a) Period on Earth:

$$T_{Earth} = 2\pi\sqrt{\frac{L}{g_{Earth}}} = 2\pi\sqrt{\frac{2.0}{9.8}} = 2\pi\sqrt{0.204} = 2\pi \times 0.452$$
$$= 2.84 \text{ s}$$

(b) Period on Moon:

$$T_{moon} = 2\pi\sqrt{\frac{L}{g_{moon}}} = 2\pi\sqrt{\frac{2.0}{1.62}} = 2\pi\sqrt{1.235} = 2\pi \times 1.111$$
$$= 6.98 \text{ s}$$

The period is much longer on the Moon due to weaker gravity.
(c) Gravitational acceleration:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$T^2 = 4\pi^2\frac{L}{g}$$

$$g = \frac{4\pi^2 L}{T^2} = \frac{4\pi^2 \times 2.0}{(3.5)^2} = \frac{78.96}{12.25} = 6.45 \text{ m/s}^2$$

Problem 8.3: A 2.0 kg block attached to a spring executes SHM with amplitude 0.30 m. At a point 0.20 m from equilibrium, the speed is 3.0 m/s. Find: (a) the spring constant, (b) the total energy, (c) the maximum speed, (d) the maximum acceleration.

Solution:

Given: $m = 2.0$ kg, $A = 0.30$ m, at $x = 0.20$ m: $v = 3.0$ m/s

(a) Spring constant:

Using energy conservation:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

At the given point:

$$kA^2 = mv^2 + kx^2$$

$$k(A^2 - x^2) = mv^2$$

$$k = \frac{mv^2}{A^2 - x^2} = \frac{2.0 \times (3.0)^2}{(0.30)^2 - (0.20)^2}$$

$$= \frac{18}{0.09 - 0.04} = \frac{18}{0.05} = 360 \text{ N/m}$$

(b) Total energy:

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \times 360 \times (0.30)^2 = \frac{1}{2} \times 360 \times 0.09$$

$$= 16.2 \text{ J}$$

(c) Maximum speed:

$$v_{max} = A\omega = A\sqrt{\frac{k}{m}} = 0.30\sqrt{\frac{360}{2.0}}$$

$$= 0.30\sqrt{180} = 0.30 \times 13.4 = 4.02 \text{ m/s}$$

Or using energy:

$$E = \frac{1}{2}mv_{max}^2$$

$$v_{max} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \times 16.2}{2.0}} = \sqrt{16.2} = 4.02 \text{ m/s}$$

(d) Maximum acceleration:

$$a_{max} = \omega^2 A = \frac{k}{m}A = \frac{360 \times 0.30}{2.0} = 54 \text{ m/s}^2$$

10 Lesson 9: Temperature and Heat

10.1 Temperature and Thermal Equilibrium

10.1.1 Temperature

Temperature is a measure of the average kinetic energy of random molecular motion in a substance.

10.1.2 Thermal Equilibrium

Two systems in thermal contact are in thermal equilibrium when they have the same temperature. Heat flows from higher to lower temperature until equilibrium is reached.

10.1.3 Zeroth Law of Thermodynamics

If system A is in thermal equilibrium with system C, and system B is in thermal equilibrium with system C, then A and B are in thermal equilibrium with each other.

This law provides the basis for temperature measurement.

10.2 Temperature Scales

10.2.1 Celsius Scale

Reference points:

- Freezing point of water: 0°C
- Boiling point of water: 100°C (at 1 atm)

10.2.2 Fahrenheit Scale

$$T_F = \frac{9}{5}T_C + 32$$

Reference points:

- Freezing point of water: 32°F
- Boiling point of water: 212°F

10.2.3 Kelvin Scale (Absolute Temperature)

$$T_K = T_C + 273.15$$

The Kelvin scale has its zero at absolute zero, the lowest possible temperature where molecular motion ceases.

Absolute zero: 0 K = −273.15°C = −459.67°F

Important: Temperature differences are the same in Celsius and Kelvin:

$$\Delta T_K = \Delta T_C$$

10.3 Thermal Expansion

10.3.1 Linear Expansion

When temperature changes, length changes:

$$\Delta L = \alpha L_0 \Delta T$$

where α is the coefficient of linear expansion (units: K^{1} or $°C^{1}$).

Final length:

$$L = L_0(1 + \alpha \Delta T)$$

Typical values of α:

- Aluminum: 24×10^{-6} K^{1}
- Steel: 11×10^{-6} K^{1}
- Glass: 9×10^{-6} K^{1}
- Concrete: 12×10^{-6} K^{1}

10.3.2 Area Expansion

$$\Delta A = 2\alpha A_0 \Delta T = \beta A_0 \Delta T$$

where $\beta = 2\alpha$ is the coefficient of area expansion.

10.3.3 Volume Expansion

$$\Delta V = 3\alpha V_0 \Delta T = \gamma V_0 \Delta T$$

where $\gamma = 3\alpha$ is the coefficient of volume expansion.

For liquids, only volume expansion matters (no fixed shape). Water is unusual because it expands when cooled below 4°C.

10.4 Heat and Specific Heat

10.4.1 Heat

Heat is energy transferred between systems due to temperature difference. Heat flows spontaneously from hot to cold.

Units: Joule (J) or calorie (cal), where 1 cal = 4.186 J

10.4.2 Specific Heat Capacity

The specific heat c is the amount of heat needed to raise the temperature of 1 kg of a substance by 1 K:

$$Q = mc\Delta T$$

Units: J/(kg·K) or J/(kg·°C)
Common specific heats:

- Water: $c = 4186$ J/(kg·K) = 1.00 cal/(g·°C)
- Ice: $c = 2090$ J/(kg·K)
- Steam: $c = 2010$ J/(kg·K)
- Aluminum: $c = 900$ J/(kg·K)
- Copper: $c = 387$ J/(kg·K)
- Iron: $c = 450$ J/(kg·K)

Water has an unusually high specific heat, which moderates climate near large bodies of water.

10.4.3 Calorimetry

In an isolated system (no heat exchange with surroundings):

$$\sum Q_i = 0$$

Heat lost by hot objects = Heat gained by cold objects

$$m_1c_1(T_f - T_1) + m_2c_2(T_f - T_2) = 0$$

This principle is used to measure specific heats.

10.5 Phase Changes and Latent Heat

10.5.1 Phase Changes

During a phase change, temperature remains constant even though heat is added or removed. The heat goes into changing the structure (breaking or forming bonds) rather than increasing kinetic energy.

10.5.2 Latent Heat

The latent heat is the heat per unit mass needed for a phase change:

$$Q = mL$$

Latent heat of fusion L_f: Heat to melt/freeze **Latent heat of vaporization** L_v: Heat to vaporize/condense

For water:

- $L_f = 3.33 \times 10^5$ J/kg = 79.7 cal/g
- $L_v = 2.26 \times 10^6$ J/kg = 540 cal/g

Note: $L_v > L_f$ because vaporization requires more bond breaking.

10.6 Heat Transfer Mechanisms

10.6.1 Conduction

Heat transfer through direct contact. The rate of heat flow is given by Fourier's law:

$$\frac{Q}{t} = P = kA\frac{\Delta T}{d}$$

where:

- P is power (heat flow rate)
- k is thermal conductivity (W/(m·K))
- A is cross-sectional area
- ΔT is temperature difference
- d is thickness

Thermal conductivities:

- Copper: $k = 401$ W/(m·K)
- Aluminum: $k = 237$ W/(m·K)
- Steel: $k = 50$ W/(m·K)
- Glass: $k = 0.8$ W/(m·K)
- Water: $k = 0.6$ W/(m·K)
- Air: $k = 0.026$ W/(m·K)

Good conductors (metals) have high k; insulators have low k.

10.6.2 Thermal Resistance (R-value)

$$R = \frac{d}{k}$$

For multiple layers in series:

$$R_{total} = R_1 + R_2 + R_3 + ...$$

Heat flow rate:

$$P = \frac{A\Delta T}{R_{total}}$$

10.6.3 Convection

Heat transfer through fluid motion. Warm fluid rises, cool fluid sinks, creating circulation currents.

Natural convection: Driven by density differences **Forced convection:** Driven by external means (fans, pumps)

Convection is more complex to analyze than conduction.

10.6.4 Radiation

Heat transfer through electromagnetic waves. Does not require a medium.

The Stefan-Boltzmann law gives power radiated:

$$P = e\sigma AT^4$$

where:

- e is emissivity ($0 \leq e \leq 1$)
- $\sigma = 5.67 \times 10^{-8}$ W/(m²·K) is the Stefan-Boltzmann constant
- A is surface area
- T is absolute temperature (Kelvin)

For a perfect blackbody: $e = 1$ For a perfect reflector: $e = 0$

Net radiation: An object at temperature T in surroundings at temperature T_0 has net power:

$$P_{net} = e\sigma A(T^4 - T_0^4)$$

10.7 Example Problems

Problem 9.1: A steel railroad rail is 20.0 m long at 0°C. Find: (a) its length at 40°C, (b) the change in length if the rail is cooled to -30°C from 0°C, (c) if the rail is rigidly fixed at both ends and cannot expand, what thermal stress develops at 40°C? (Young's modulus for steel: $Y = 2.0 \times 10^{11}$ Pa)

Solution:

Given: $L_0 = 20.0$ m, $\alpha_{steel} = 11 \times 10^{-6}$ K^{1}

(a) Length at 40°C:

$$\Delta L = \alpha L_0 \Delta T = 11 \times 10^{-6} \times 20.0 \times 40$$
$$= 8.8 \times 10^{-3} \text{ m} = 8.8 \text{ mm}$$

$$L = L_0 + \Delta L = 20.0 + 0.0088 = 20.0088 \text{ m}$$

(b) Change in length at -30°C:

$$\Delta L = \alpha L_0 \Delta T = 11 \times 10^{-6} \times 20.0 \times (-30)$$
$$= -6.6 \times 10^{-3} \text{ m} = -6.6 \text{ mm}$$

The rail contracts by 6.6 mm.

(c) Thermal stress:

If the rail cannot expand, it experiences compressive stress. The strain that would have occurred:

$$\text{strain} = \frac{\Delta L}{L_0} = \alpha \Delta T = 11 \times 10^{-6} \times 40 = 4.4 \times 10^{-4}$$

Stress from Young's modulus:

$$\sigma = Y \times \text{strain} = 2.0 \times 10^{11} \times 4.4 \times 10^{-4}$$

$$= 8.8 \times 10^{7} \text{ Pa} = 88 \text{ MPa}$$

This is a significant stress that can cause buckling if not accommodated with expansion joints.

Problem 9.2: A 200 g aluminum calorimeter contains 500 g of water at 20°C. A 300 g piece of copper at 95°C is dropped into the calorimeter. Find the final equilibrium temperature. Assume no heat loss to surroundings.

Solution:

Given:

- Aluminum: $m_{Al} = 0.200$ kg, $c_{Al} = 900$ J/(kg·K), $T_{Al,i} = 20$°C
- Water: $m_w = 0.500$ kg, $c_w = 4186$ J/(kg·K), $T_{w,i} = 20$°C
- Copper: $m_{Cu} = 0.300$ kg, $c_{Cu} = 387$ J/(kg·K), $T_{Cu,i} = 95$°C

Heat conservation:

$$Q_{Cu} + Q_{Al} + Q_w = 0$$

$$m_{Cu}c_{Cu}(T_f - T_{Cu,i}) + m_{Al}c_{Al}(T_f - T_{Al,i}) + m_w c_w (T_f - T_{w,i}) = 0$$

$$m_{Cu}c_{Cu}T_f - m_{Cu}c_{Cu}T_{Cu,i} + m_{Al}c_{Al}T_f - m_{Al}c_{Al}T_{Al,i} + m_w c_w T_f - m_w c_w T_{w,i} = 0$$

$$T_f(m_{Cu}c_{Cu} + m_{Al}c_{Al} + m_w c_w) = m_{Cu}c_{Cu}T_{Cu,i} + m_{Al}c_{Al}T_{Al,i} + m_w c_w T_{w,i}$$

Calculate each term:

- $m_{Cu}c_{Cu} = 0.300 \times 387 = 116.1$ J/K
- $m_{Al}c_{Al} = 0.200 \times 900 = 180$ J/K
- $m_w c_w = 0.500 \times 4186 = 2093$ J/K

Sum: $116.1 + 180 + 2093 = 2389.1$ J/K
Right side:

- $116.1 \times 95 = 11,029.5$ J
- $180 \times 20 = 3,600$ J
- $2093 \times 20 = 41,860$ J

Sum: $11,029.5 + 3,600 + 41,860 = 56,489.5$ J

$$T_f = \frac{56,489.5}{2389.1} = 23.6\text{C}$$

Problem 9.3: How much heat is required to convert 50 g of ice at -10°C to steam at 110°C?
Solution:
This process involves five stages:
Stage 1: Heat ice from -10°C to 0°C

$$Q_1 = mc_{ice}\Delta T = 0.050 \times 2090 \times 10 = 1045 \text{ J}$$

Stage 2: Melt ice at 0°C

$$Q_2 = mL_f = 0.050 \times 3.33 \times 10^5 = 16,650 \text{ J}$$

Stage 3: Heat water from 0°C to 100°C

$$Q_3 = mc_{water}\Delta T = 0.050 \times 4186 \times 100 = 20,930 \text{ J}$$

Stage 4: Vaporize water at 100°C

$$Q_4 = mL_v = 0.050 \times 2.26 \times 10^6 = 113,000 \text{ J}$$

Stage 5: Heat steam from 100°C to 110°C

$$Q_5 = mc_{steam}\Delta T = 0.050 \times 2010 \times 10 = 1005 \text{ J}$$

Total heat:

$$\begin{aligned} Q_{total} &= Q_1 + Q_2 + Q_3 + Q_4 + Q_5 \\ &= 1045 + 16,650 + 20,930 + 113,000 + 1005 \\ &= 152,630 \text{ J} = 153 \text{ kJ} \end{aligned}$$

Note: Most of the energy (74

11 Lesson 10: Thermodynamics

11.1 The First Law of Thermodynamics

11.1.1 Statement

The change in internal energy of a system equals the heat added to the system minus the work done by the system:

$$\boxed{\Delta U = Q - W}$$

where:

- ΔU is change in internal energy
- Q is heat added to the system (positive if heat enters)
- W is work done by the system (positive if system does work on surroundings)

This is a statement of energy conservation for thermodynamic systems.

11.1.2 Sign Conventions

- $Q > 0$: Heat added to system
- $Q < 0$: Heat removed from system
- $W > 0$: System does work on surroundings (expansion)
- $W < 0$: Surroundings do work on system (compression)

11.2 Work in Thermodynamic Processes

11.2.1 Work Done by Gas

For a gas expanding or contracting, the work done is:

$$W = \int_{V_i}^{V_f} P\,dV$$

Geometrically, this is the area under the P-V curve.
For constant pressure (isobaric process):

$$\boxed{W = P\Delta V = P(V_f - V_i)}$$

11.2.2 PV Diagrams

Thermodynamic processes are represented on pressure-volume (PV) diagrams:

- Horizontal line: isobaric (constant pressure)
- Vertical line: isochoric (constant volume)
- Hyperbola: isothermal (constant temperature)
- Steeper curve: adiabatic (no heat transfer)

11.3 Ideal Gas Law

11.3.1 Equation of State

$$\boxed{PV = nRT}$$

where:

- P is pressure (Pa)
- V is volume (m^3)
- n is number of moles
- $R = 8.314$ J/(mol·K) is the universal gas constant
- T is absolute temperature (K)

Alternative form:

$$PV = Nk_BT$$

where N is number of molecules and $k_B = 1.381 \times 10^{-23}$ J/K is Boltzmann's constant. Relation: $R = N_Ak_B$ where $N_A = 6.022 \times 10^{23}$ is Avogadro's number.

11.3.2 Internal Energy of Ideal Gas

For a monatomic ideal gas:

$$U = \frac{3}{2}nRT$$

For a diatomic ideal gas:

$$U = \frac{5}{2}nRT$$

The internal energy depends only on temperature, not on pressure or volume.

11.4 Thermodynamic Processes

11.4.1 Isothermal Process ($\Delta T = 0$)

Temperature remains constant.
For an ideal gas: $\Delta U = 0$
From first law: $Q = W$
Work done:

$$W = \int_{V_i}^{V_f} P\, dV = \int_{V_i}^{V_f} \frac{nRT}{V}\, dV = nRT \ln\left(\frac{V_f}{V_i}\right)$$

$$\boxed{W = nRT \ln\left(\frac{V_f}{V_i}\right) = nRT \ln\left(\frac{P_i}{P_f}\right)}$$

11.4.2 Isobaric Process ($\Delta P = 0$)

Pressure remains constant.
Work: $W = P\Delta V = P(V_f - V_i)$
Heat: $Q = nC_P \Delta T$
where C_P is molar heat capacity at constant pressure.

11.4.3 Isochoric Process ($\Delta V = 0$)

Volume remains constant.
Work: $W = 0$ (no volume change)
From first law: $\Delta U = Q$
Heat: $Q = nC_V \Delta T$
where C_V is molar heat capacity at constant volume.

11.4.4 Adiabatic Process ($Q = 0$)

No heat transfer occurs (insulated system or very rapid process).
From first law: $\Delta U = -W$
For an ideal gas, the adiabatic equation is:

$$PV^\gamma = \text{constant}$$

where $\gamma = C_P/C_V$ is the heat capacity ratio.
For monatomic gases: $\gamma = 5/3 = 1.67$ For diatomic gases: $\gamma = 7/5 = 1.40$
Also:

$$TV^{\gamma-1} = \text{constant}$$
$$T^\gamma P^{1-\gamma} = \text{constant}$$

Work in adiabatic process:

$$W = -\Delta U = -nC_V \Delta T = \frac{nR(T_i - T_f)}{\gamma - 1}$$

Or:

$$W = \frac{P_i V_i - P_f V_f}{\gamma - 1}$$

11.5 Heat Capacities

11.5.1 Molar Heat Capacities

At constant volume:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

For monatomic ideal gas: $C_V = \frac{3}{2}R$ For diatomic ideal gas: $C_V = \frac{5}{2}R$
At constant pressure:

$$C_P = C_V + R$$

This is the Mayer relation.
For monatomic ideal gas: $C_P = \frac{5}{2}R$ For diatomic ideal gas: $C_P = \frac{7}{2}R$

11.6 The Second Law of Thermodynamics

11.6.1 Statements

Kelvin-Planck statement: No heat engine can convert heat completely into work in a cyclic process.

Clausius statement: Heat cannot flow spontaneously from cold to hot without external work.

These statements are equivalent and establish the direction of natural processes.

11.6.2 Heat Engines

A heat engine converts heat into work in a cyclic process:

- Absorbs heat Q_H from hot reservoir at T_H
- Performs work W
- Rejects heat Q_C to cold reservoir at T_C

Energy conservation:

$$Q_H = W + Q_C$$

Efficiency:

$$\boxed{e = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}}$$

Efficiency is always less than 100

11.6.3 Carnot Engine

The Carnot engine is a theoretical ideal engine operating between two temperatures. It represents the maximum possible efficiency.

Carnot efficiency:

$$\boxed{e_{Carnot} = 1 - \frac{T_C}{T_H}}$$

where temperatures are in Kelvin.

No real engine can exceed this efficiency when operating between the same two temperatures.

11.6.4 Refrigerators and Heat Pumps

A refrigerator removes heat from a cold space and rejects it to a warm space, requiring work input.

Coefficient of performance (refrigerator):

$$COP_R = \frac{Q_C}{W} = \frac{Q_C}{Q_H - Q_C}$$

For a Carnot refrigerator:

$$COP_{Carnot,R} = \frac{T_C}{T_H - T_C}$$

Coefficient of performance (heat pump):

$$COP_{HP} = \frac{Q_H}{W} = \frac{Q_H}{Q_H - Q_C}$$

For a Carnot heat pump:

$$COP_{Carnot,HP} = \frac{T_H}{T_H - T_C}$$

Note: $COP_{HP} = COP_R + 1$

11.7 Entropy

11.7.1 Definition

Entropy is a measure of disorder or randomness in a system. For a reversible process:

$$dS = \frac{dQ_{rev}}{T}$$

Change in entropy:

$$\boxed{\Delta S = \int_i^f \frac{dQ_{rev}}{T}}$$

Units: J/K

11.7.2 Entropy and the Second Law

For any process in an isolated system:

$$\Delta S \geq 0$$

- $\Delta S > 0$: Irreversible process (natural direction)
- $\Delta S = 0$: Reversible process (idealization)
- $\Delta S < 0$: Impossible in isolated system

The entropy of the universe always increases.

11.7.3 Entropy Changes

For constant temperature:

$$\Delta S = \frac{Q}{T}$$

For temperature change (constant pressure or volume):

$$\Delta S = nC \ln\left(\frac{T_f}{T_i}\right)$$

where C is C_P or C_V as appropriate.
For phase change (constant T):

$$\Delta S = \frac{mL}{T}$$

For free expansion (adiabatic, irreversible):

$$\Delta S = nR \ln\left(\frac{V_f}{V_i}\right)$$

11.8 Example Problems

Problem 10.1: One mole of an ideal monatomic gas undergoes the following cycle:

- A→B: Isobaric expansion at $P = 2.0 \times 10^5$ Pa from $V_A = 0.010$ m^3 to $V_B = 0.030$ m^3
- B→C: Isochoric cooling to $P_C = 0.5 \times 10^5$ Pa
- C→A: Isothermal compression back to initial state

Calculate: (a) the temperature at each state, (b) the work done in each process, (c) the heat transferred in each process, (d) the efficiency of this cycle.

Solution:
Given: $n = 1.0$ mol, $R = 8.314$ J/(mol·K), monatomic: $C_V = \frac{3}{2}R$, $C_P = \frac{5}{2}R$
(a) Temperatures:

State A:

$$T_A = \frac{P_A V_A}{nR} = \frac{2.0 \times 10^5 \times 0.010}{1.0 \times 8.314} = 241 \text{ K}$$

State B: (isobaric, so $P_B = P_A$)

$$T_B = \frac{P_B V_B}{nR} = \frac{2.0 \times 10^5 \times 0.030}{1.0 \times 8.314} = 722 \text{ K}$$

State C: (isochoric, so $V_C = V_B$)

$$T_C = \frac{P_C V_C}{nR} = \frac{0.5 \times 10^5 \times 0.030}{1.0 \times 8.314} = 180 \text{ K}$$

Wait - this doesn't work for a cycle! Let me recalculate. For isothermal C→A, we need $T_C = T_A = 241$ K.

Let me reconsider: If C→A is isothermal, then $P_C V_C = P_A V_A$:

$$P_C = P_A \frac{V_A}{V_C} = 2.0 \times 10^5 \times \frac{0.010}{0.030} = 6.67 \times 10^4 \text{ Pa}$$

Let me use this corrected value: $P_C = 6.67 \times 10^4$ Pa

State C:

$$T_C = T_A = 241 \text{ K}$$

(b) Work in each process:

A→B (isobaric):

$$W_{AB} = P(V_B - V_A) = 2.0 \times 10^5 \times (0.030 - 0.010) = 4000 \text{ J}$$

B→C (isochoric):

$$W_{BC} = 0 \text{ (no volume change)}$$

C→A (isothermal):

$$W_{CA} = nRT_A \ln\left(\frac{V_A}{V_C}\right) = 1.0 \times 8.314 \times 241 \times \ln\left(\frac{0.010}{0.030}\right)$$

$$= 2003.7 \times \ln(1/3) = 2003.7 \times (-1.099) = -2202 \text{ J}$$

Net work per cycle:

$$W_{net} = W_{AB} + W_{BC} + W_{CA} = 4000 + 0 - 2202 = 1798 \text{ J}$$

(c) Heat in each process:

A→B (isobaric):

$$Q_{AB} = nC_P \Delta T = 1.0 \times \frac{5}{2} \times 8.314 \times (722 - 241)$$

$$= 20.785 \times 481 = 10,000 \text{ J}$$

Or: $\Delta U_{AB} = nC_V \Delta T = \frac{3}{2} \times 8.314 \times 481 = 6000$ J

$$Q_{AB} = \Delta U_{AB} + W_{AB} = 6000 + 4000 = 10,000 \text{ J}$$

B→C (isochoric):

$$Q_{BC} = nC_V \Delta T = 1.0 \times \frac{3}{2} \times 8.314 \times (241 - 722)$$

$$= 12.471 \times (-481) = -6000 \text{ J}$$

C→A (isothermal):

$$\Delta U_{CA} = 0 \text{ (isothermal)}$$

$$Q_{CA} = W_{CA} = -2202 \text{ J}$$

Net heat per cycle:

$$Q_{net} = Q_{AB} + Q_{BC} + Q_{CA} = 10,000 - 6000 - 2202 = 1798 \text{ J}$$

Check: $Q_{net} = W_{net}$ (as required for a cycle)
(d) Efficiency:
Heat input (positive Q): $Q_H = Q_{AB} = 10,000$ J Heat output (negative Q): $Q_C = |Q_{BC}| + |Q_{CA}| = 6000 + 2202 = 8202$ J

$$e = \frac{W_{net}}{Q_H} = \frac{1798}{10,000} = 0.180 = 18.0\%$$

Or: $e = 1 - \frac{Q_C}{Q_H} = 1 - \frac{8202}{10,000} = 0.180 = 18.0\%$
Problem 10.2: A Carnot engine operates between a hot reservoir at 500 K and a cold reservoir at 300 K. The engine absorbs 1200 J of heat from the hot reservoir per cycle. Find: (a) the efficiency, (b) the work output per cycle, (c) the heat rejected per cycle, (d) if the engine operates at 60 Hz, what is the power output?
Solution:
Given: $T_H = 500$ K, $T_C = 300$ K, $Q_H = 1200$ J, $f = 60$ Hz
(a) Carnot efficiency:

$$e = 1 - \frac{T_C}{T_H} = 1 - \frac{300}{500} = 1 - 0.6 = 0.4 = 40\%$$

(b) Work output:

$$W = eQ_H = 0.4 \times 1200 = 480 \text{ J}$$

(c) Heat rejected:

$$Q_C = Q_H - W = 1200 - 480 = 720 \text{ J}$$

Or: $Q_C = (1 - e)Q_H = 0.6 \times 1200 = 720$ J
(d) Power output:

$$P = Wf = 480 \times 60 = 28,800 \text{ W} = 28.8 \text{ kW}$$

Problem 10.3: A 2.0 kg block of copper at 100°C is placed in thermal contact with a 4.0 kg block of aluminum at 20°C in an insulated container. Find: (a) the final equilibrium temperature, (b) the entropy change of the copper, (c) the entropy change of the aluminum, (d) the total entropy change of the system.

Solution:

Given: $m_{Cu} = 2.0$ kg, $T_{Cu,i} = 373$ K, $c_{Cu} = 387$ J/(kg·K) $m_{Al} = 4.0$ kg, $T_{Al,i} = 293$ K, $c_{Al} = 900$ J/(kg·K)

(a) Final temperature:

Heat conservation:

$$m_{Cu}c_{Cu}(T_f - T_{Cu,i}) + m_{Al}c_{Al}(T_f - T_{Al,i}) = 0$$

$$T_f(m_{Cu}c_{Cu} + m_{Al}c_{Al}) = m_{Cu}c_{Cu}T_{Cu,i} + m_{Al}c_{Al}T_{Al,i}$$

$$T_f = \frac{2.0 \times 387 \times 373 + 4.0 \times 900 \times 293}{2.0 \times 387 + 4.0 \times 900}$$

$$= \frac{288,702 + 1,054,800}{774 + 3600} = \frac{1,343,502}{4374} = 307 \text{ K} = 34\text{C}$$

(b) Entropy change of copper:

$$\Delta S_{Cu} = m_{Cu}c_{Cu}\ln\left(\frac{T_f}{T_{Cu,i}}\right) = 2.0 \times 387 \times \ln\left(\frac{307}{373}\right)$$
$$= 774 \times \ln(0.823) = 774 \times (-0.195) = -151 \text{ J/K}$$

(c) Entropy change of aluminum:

$$\Delta S_{Al} = m_{Al}c_{Al}\ln\left(\frac{T_f}{T_{Al,i}}\right) = 4.0 \times 900 \times \ln\left(\frac{307}{293}\right)$$
$$= 3600 \times \ln(1.048) = 3600 \times 0.0469 = 169 \text{ J/K}$$

(d) Total entropy change:

$$\Delta S_{total} = \Delta S_{Cu} + \Delta S_{Al} = -151 + 169 = 18 \text{ J/K}$$

The positive total entropy change confirms this is an irreversible process, consistent with the second law. Heat spontaneously flows from hot copper to cold aluminum, increasing the universe's entropy.

12 Lesson 11: Wave Motion

12.1 Introduction to Waves

12.1.1 Definition

A wave is a disturbance that propagates through space and time, transferring energy without transferring matter.

12.1.2 Types of Waves

Mechanical waves: Require a medium (sound, water waves, seismic waves) **Electromagnetic waves:** Do not require a medium (light, radio waves, X-rays)

Transverse waves: Particle motion perpendicular to wave direction (light, waves on string) **Longitudinal waves:** Particle motion parallel to wave direction (sound in air)

12.2 Wave Parameters

12.2.1 Wavelength (λ)

The distance between successive crests (or any two corresponding points). Units: meters

12.2.2 Amplitude (A)

The maximum displacement from equilibrium. Related to wave energy.

12.2.3 Period (T)

Time for one complete oscillation. Units: seconds

12.2.4 Frequency (f)

Number of oscillations per unit time:

$$f = \frac{1}{T}$$

Units: Hz (Hertz) = s^1

12.2.5 Wave Speed (v)

Speed of wave propagation:

$$\boxed{v = f\lambda = \frac{\lambda}{T}}$$

This is the fundamental wave equation.

12.3 Mathematical Description of Waves

12.3.1 Wave Function

For a sinusoidal wave traveling in the +x direction:

$$\boxed{y(x,t) = A\sin(kx - \omega t + \phi)}$$

where:

- $y(x,t)$ is displacement at position x and time t
- A is amplitude
- $k = 2\pi/\lambda$ is wave number (rad/m)
- $\omega = 2\pi f = 2\pi/T$ is angular frequency (rad/s)
- ϕ is phase constant

For wave traveling in -x direction:

$$y(x,t) = A\sin(kx + \omega t + \phi)$$

12.3.2 Wave Speed from Wave Function

From the wave function:

$$v = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda$$

12.3.3 Partial Derivatives

Velocity of particle at position x:

$$\frac{\partial y}{\partial t} = -A\omega\cos(kx - \omega t + \phi)$$

Acceleration of particle:

$$\frac{\partial^2 y}{\partial t^2} = -A\omega^2\sin(kx - \omega t + \phi)$$

Spatial derivative:

$$\frac{\partial y}{\partial x} = Ak\cos(kx - \omega t + \phi)$$

$$\frac{\partial^2 y}{\partial x^2} = -Ak^2\sin(kx - \omega t + \phi)$$

12.4 The Wave Equation

12.4.1 General Wave Equation

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

This partial differential equation describes all waves.
Verification: Substituting $y = A\sin(kx - \omega t)$:

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t) = -k^2 y$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t) = -\omega^2 y$$

Therefore:

$$-k^2 y = \frac{1}{v^2}(-\omega^2 y)$$

$$k^2 = \frac{\omega^2}{v^2}$$

$$v = \frac{\omega}{k} = f\lambda$$

12.5 Wave Speed in Different Media

12.5.1 Waves on a String

For transverse waves on a string under tension T with linear mass density μ (mass per unit length):

$$\boxed{v = \sqrt{\frac{T}{\mu}}}$$

Derivation outline: Consider a small segment of string. The net vertical force from tension provides centripetal acceleration for circular motion of the segment.

12.5.2 Sound Waves in Fluids

$$v = \sqrt{\frac{B}{\rho}}$$

where B is bulk modulus and ρ is density.
For ideal gas:

$$v = \sqrt{\frac{\gamma RT}{M}}$$

where $\gamma = C_P/C_V$, R is gas constant, T is temperature, M is molar mass.
For air at 20°C:

$$v \approx 343 \text{ m/s}$$

Temperature dependence for air:

$$v = (331 + 0.6T) \text{ m/s}$$

where T is in Celsius.

12.6 Energy Transport by Waves

12.6.1 Power

The average power transmitted by a sinusoidal wave on a string:

$$\boxed{P_{avg} = \frac{1}{2}\mu v \omega^2 A^2}$$

where:

- μ is linear mass density
- v is wave speed
- ω is angular frequency
- A is amplitude

Key observation: Power is proportional to A^2 and ω^2.

12.6.2 Intensity

Intensity is power per unit area:

$$I = \frac{P}{A}$$

Units: W/m^2
For a point source radiating uniformly in three dimensions:

$$I = \frac{P}{4\pi r^2}$$

Intensity decreases as $1/r^2$ (inverse square law).

12.7 Superposition and Interference

12.7.1 Principle of Superposition

When two or more waves overlap, the resultant displacement is the sum of individual displacements:

$$y_{total}(x,t) = y_1(x,t) + y_2(x,t) + ...$$

12.7.2 Interference

Constructive interference: Waves add in phase

$$\Delta\phi = 2n\pi, \quad n = 0, 1, 2, ...$$

$$\Delta x = n\lambda$$

Maximum amplitude: $A_{max} = A_1 + A_2$
Destructive interference: Waves add out of phase

$$\Delta\phi = (2n+1)\pi, \quad n = 0, 1, 2, ...$$

$$\Delta x = (n + \frac{1}{2})\lambda$$

Minimum amplitude: $A_{min} = |A_1 - A_2|$

12.7.3 Two-Source Interference

For two coherent sources separated by distance d:
At a distant point P, path difference:

$$\Delta L = d\sin\theta$$

Constructive interference:

$$d\sin\theta = n\lambda, \quad n = 0, \pm 1, \pm 2, ...$$

Destructive interference:

$$d\sin\theta = (n + \frac{1}{2})\lambda, \quad n = 0, \pm 1, \pm 2, ...$$

12.8 Standing Waves

12.8.1 Formation

Standing waves form when two waves of equal amplitude and frequency travel in opposite directions:

$$y_1 = A\sin(kx - \omega t)$$
$$y_2 = A\sin(kx + \omega t)$$

Sum:

$$y = y_1 + y_2 = 2A\sin(kx)\cos(\omega t)$$

This represents a standing wave with:

- Amplitude varies with position: $2A\sin(kx)$
- All points oscillate in phase (or 180° out of phase)
- Nodes: points of zero amplitude at $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, ...$
- Antinodes: points of maximum amplitude at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, ...$

12.8.2 Standing Waves on a String

For a string fixed at both ends (length L):

Boundary conditions: $y = 0$ at $x = 0$ and $x = L$

This requires:

$$L = n\frac{\lambda}{2}, \quad n = 1, 2, 3, ...$$

$$\boxed{\lambda_n = \frac{2L}{n}}$$

Frequencies:

$$\boxed{f_n = \frac{nv}{2L} = nf_1}$$

where $f_1 = \frac{v}{2L}$ is the fundamental frequency.

$n = 1$: fundamental (first harmonic) $n = 2$: second harmonic $n = 3$: third harmonic, etc.

12.9 Example Problems

Problem 11.1: A sinusoidal wave traveling along a string is described by:

$$y(x, t) = 0.15 \sin(0.80x - 50t)$$

where x and y are in meters and t is in seconds. Determine: (a) the amplitude, wavelength, frequency, and speed, (b) the maximum transverse speed of a point on the string, (c) the transverse displacement at $x = 2.3$ m when $t = 0.16$ s.

Solution:

Given: $y = 0.15 \sin(0.80x - 50t)$

Comparing with $y = A \sin(kx - \omega t)$:

(a) Wave parameters:

Amplitude: $A = 0.15$ m

Wave number: $k = 0.80$ rad/m

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.80} = 7.85 \text{ m}$$

Angular frequency: $\omega = 50$ rad/s

$$f = \frac{\omega}{2\pi} = \frac{50}{2\pi} = 7.96 \text{ Hz}$$

Wave speed:

$$v = \frac{\omega}{k} = \frac{50}{0.80} = 62.5 \text{ m/s}$$

Or: $v = f\lambda = 7.96 \times 7.85 = 62.5$ m/s

(b) Maximum transverse speed:

$$v_{y,max} = A\omega = 0.15 \times 50 = 7.5 \text{ m/s}$$

(c) Displacement at $x = 2.3$ m, $t = 0.16$ s:

$$y(2.3, 0.16) = 0.15 \sin(0.80 \times 2.3 - 50 \times 0.16)$$
$$= 0.15 \sin(1.84 - 8.0) = 0.15 \sin(-6.16 \text{ rad})$$
$$= 0.15 \sin(-353.0) = 0.15 \sin(7.0)$$
$$= 0.15 \times 0.122 = 0.0183 \text{ m} = 1.83 \text{ cm}$$

Problem 11.2: A string of length 0.80 m and mass 2.0 g is under tension of 50 N. Find: (a) the wave speed on the string, (b) the fundamental frequency, (c) the frequencies of the first three harmonics, (d) if the string vibrates in the third harmonic with amplitude 3.0 mm, what is the maximum transverse speed of a point at an antinode?

Solution:
Given: $L = 0.80$ m, $m = 2.0 \times 10^{-3}$ kg, $T = 50$ N
Linear mass density:

$$\mu = \frac{m}{L} = \frac{2.0 \times 10^{-3}}{0.80} = 2.5 \times 10^{-3} \text{ kg/m}$$

(a) Wave speed:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{50}{2.5 \times 10^{-3}}} = \sqrt{20,000} = 141 \text{ m/s}$$

(b) Fundamental frequency:

$$f_1 = \frac{v}{2L} = \frac{141}{2 \times 0.80} = \frac{141}{1.6} = 88.1 \text{ Hz}$$

(c) First three harmonics:

$$f_1 = 88.1 \text{ Hz}$$
$$f_2 = 2f_1 = 176.2 \text{ Hz}$$
$$f_3 = 3f_1 = 264.3 \text{ Hz}$$

(d) Maximum transverse speed in third harmonic:

$$A = 3.0 \text{ mm} = 3.0 \times 10^{-3} \text{ m}$$
$$\omega_3 = 2\pi f_3 = 2\pi \times 264.3 = 1660 \text{ rad/s}$$

$$v_{max} = A\omega_3 = 3.0 \times 10^{-3} \times 1660 = 4.98 \text{ m/s}$$

13 Lesson 12: Sound Waves

13.1 Nature of Sound

13.1.1 Sound as a Pressure Wave

Sound is a longitudinal mechanical wave consisting of compressions and rarefactions propagating through a medium.

Compression: Region of higher than average pressure **Rarefaction:** Region of lower than average pressure

13.2 Speed of Sound

13.2.1 In Air

$$v = (331 + 0.6T) \text{ m/s}$$

where T is temperature in Celsius.
At 20°C: $v \approx 343$ m/s

13.2.2 In General

$$v = \sqrt{\frac{B}{\rho}}$$

where B is bulk modulus and ρ is density.
Sound travels faster in:

- Solids ¿ Liquids ¿ Gases (higher B)
- Less dense media of same type (lower ρ)
- Higher temperatures (for gases)

Examples:

- Air (20°C): 343 m/s
- Water: 1480 m/s
- Steel: 5960 m/s
- Aluminum: 6420 m/s

13.3 Intensity and Sound Level

13.3.1 Intensity

$$I = \frac{P}{A}$$

Units: W/m²
For a point source:

$$I = \frac{P}{4\pi r^2}$$

Intensity follows inverse square law: $I \propto 1/r^2$

13.3.2 Sound Level (Decibels)

The decibel scale is logarithmic:

$$\boxed{\beta = 10\log_{10}\left(\frac{I}{I_0}\right)}$$

where:

- β is sound level in decibels (dB)
- I is intensity (W/m²)
- $I_0 = 1.0 \times 10^{-12}$ W/m² is reference intensity (threshold of hearing)

Examples:

- Threshold of hearing: 0 dB
- Whisper: 20 dB
- Normal conversation: 60 dB
- Busy traffic: 80 dB
- Jet engine at 30 m: 140 dB (threshold of pain)

13.3.3 Relationship Between Intensities

If intensity changes from I_1 to I_2:

$$\Delta\beta = \beta_2 - \beta_1 = 10\log_{10}\left(\frac{I_2}{I_1}\right)$$

Important relationships:

- Doubling intensity: $\Delta\beta = 10\log_{10}(2) \approx 3$ dB increase
- Halving intensity: $\Delta\beta \approx -3$ dB
- 10× intensity: $\Delta\beta = 10$ dB increase
- 100× intensity: $\Delta\beta = 20$ dB increase

13.4 Doppler Effect

13.4.1 General Principle

The Doppler effect is the change in observed frequency when source and/or observer are in relative motion.

13.4.2 General Formula

$$\boxed{f' = f\left(\frac{v + v_o}{v - v_s}\right)}$$

where:

- f' is observed frequency
- f is source frequency
- v is speed of sound
- v_o is observer speed (+ if moving toward source, - if away)
- v_s is source speed (+ if moving toward observer, - if away)

13.4.3 Special Cases

Stationary observer, moving source:

$$f' = f\frac{v}{v \mp v_s}$$

(- for source approaching, + for receding)
Moving observer, stationary source:

$$f' = f\left(1 \pm \frac{v_o}{v}\right)$$

(+ for observer approaching, - for receding)

13.5 Resonance in Air Columns

13.5.1 Open Pipe (Both Ends Open)

Antinodes at both ends.
Allowed wavelengths:

$$L = n\frac{\lambda}{2}, \quad n = 1, 2, 3, ...$$

$$\lambda_n = \frac{2L}{n}$$

Frequencies:

$$\boxed{f_n = \frac{nv}{2L} = nf_1}$$

All harmonics present.

13.5.2 Closed Pipe (One End Closed)

Node at closed end, antinode at open end.

Allowed wavelengths:

$$L = (2n - 1)\frac{\lambda}{4}, \quad n = 1, 2, 3, ...$$

$$\lambda_n = \frac{4L}{2n - 1}$$

Frequencies:

$$\boxed{f_n = \frac{(2n - 1)v}{4L} = (2n - 1)f_1}$$

Only odd harmonics present: $f_1, 3f_1, 5f_1, ...$

Fundamental frequency:

$$f_1 = \frac{v}{4L}$$

13.6 Beats

13.6.1 Phenomenon

When two waves of slightly different frequencies interfere, the amplitude varies periodically. This is heard as beats.

For two waves:

$$y_1 = A\cos(\omega_1 t), \quad y_2 = A\cos(\omega_2 t)$$

Resultant:

$$y = y_1 + y_2 = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)$$

This represents a wave at average frequency $\frac{\omega_1+\omega_2}{2}$ with slowly varying amplitude.

13.6.2 Beat Frequency

$$\boxed{f_{beat} = |f_1 - f_2|}$$

This is the rate at which the amplitude oscillates (number of loudness maxima per second).

Beats are used to tune musical instruments - when beat frequency approaches zero, the instruments are in tune.

13.7 Example Problems

Problem 12.1: A sound source emits a 512 Hz tone with power 0.80 mW. Assume it radiates uniformly in all directions. Find: (a) the intensity at distance 2.0 m from the source, (b) the sound level at this distance, (c) the distance at which the sound level is 40 dB.

Solution:

Given: $f = 512$ Hz, $P = 0.80 \times 10^{-3}$ W, $I_0 = 1.0 \times 10^{-12}$ W/m²

(a) Intensity at 2.0 m:
For uniform spherical radiation:

$$I = \frac{P}{4\pi r^2} = \frac{0.80 \times 10^{-3}}{4\pi(2.0)^2} = \frac{0.80 \times 10^{-3}}{50.27}$$

$$= 1.59 \times 10^{-5}\ \text{W/m}^2$$

(b) Sound level:

$$\beta = 10\log_{10}\left(\frac{I}{I_0}\right) = 10\log_{10}\left(\frac{1.59 \times 10^{-5}}{1.0 \times 10^{-12}}\right)$$

$$= 10\log_{10}(1.59 \times 10^{7}) = 10(7.20) = 72.0\ \text{dB}$$

(c) Distance for 40 dB:

$$40 = 10\log_{10}\left(\frac{I}{I_0}\right)$$

$$4 = \log_{10}\left(\frac{I}{I_0}\right)$$

$$\frac{I}{I_0} = 10^4$$

$$I = 10^4 \times 1.0 \times 10^{-12} = 1.0 \times 10^{-8}\ \text{W/m}^2$$

Using $I = \frac{P}{4\pi r^2}$:

$$r = \sqrt{\frac{P}{4\pi I}} = \sqrt{\frac{0.80 \times 10^{-3}}{4\pi \times 1.0 \times 10^{-8}}}$$

$$= \sqrt{\frac{0.80 \times 10^{-3}}{1.257 \times 10^{-7}}} = \sqrt{6361} = 79.8\ \text{m}$$

Problem 12.2: A car horn emits sound at frequency 400 Hz. The car moves toward a stationary observer at 30 m/s. The speed of sound is 340 m/s. Find: (a) the frequency heard by the observer as the car approaches, (b) the frequency heard after the car passes and moves away, (c) the change in wavelength observed.

Solution:

Given: $f = 400$ Hz, $v_s = 30$ m/s, $v = 340$ m/s

(a) Frequency as car approaches:
Observer stationary ($v_o = 0$), source approaching (use $v - v_s$ in denominator):

$$f' = f\frac{v}{v - v_s} = 400 \times \frac{340}{340 - 30} = 400 \times \frac{340}{310}$$

$$= 400 \times 1.097 = 438.7\ \text{Hz}$$

(b) Frequency as car recedes:

Source receding (use $v + v_s$ in denominator):

$$f' = f\frac{v}{v + v_s} = 400 \times \frac{340}{340 + 30} = 400 \times \frac{340}{370}$$

$$= 400 \times 0.919 = 367.6 \text{ Hz}$$

(c) Change in wavelength:
Approaching:

$$\lambda'_{app} = \frac{v}{f'} = \frac{340}{438.7} = 0.775 \text{ m}$$

Receding:

$$\lambda'_{rec} = \frac{v}{f'} = \frac{340}{367.6} = 0.925 \text{ m}$$

Original wavelength:

$$\lambda = \frac{v}{f} = \frac{340}{400} = 0.850 \text{ m}$$

Change when approaching: $\Delta\lambda = 0.775 - 0.850 = -0.075$ m (shorter wavelength) Change when receding: $\Delta\lambda = 0.925 - 0.850 = +0.075$ m (longer wavelength)

Problem 12.3: An organ pipe closed at one end has length 0.68 m. The speed of sound is 340 m/s. Find: (a) the fundamental frequency, (b) the frequencies of the first three harmonics, (c) if the pipe is instead open at both ends, what are the fundamental frequency and first overtone?

Solution:
Given: $L = 0.68$ m, $v = 340$ m/s
Closed pipe:
(a) Fundamental frequency:

$$f_1 = \frac{v}{4L} = \frac{340}{4 \times 0.68} = \frac{340}{2.72} = 125 \text{ Hz}$$

(b) First three harmonics (only odd harmonics for closed pipe):

$$f_1 = 125 \text{ Hz}$$

$$f_3 = 3f_1 = 375 \text{ Hz}$$

$$f_5 = 5f_1 = 625 \text{ Hz}$$

Open pipe:
(c) Fundamental frequency:

$$f_1 = \frac{v}{2L} = \frac{340}{2 \times 0.68} = \frac{340}{1.36} = 250 \text{ Hz}$$

First overtone (second harmonic):

$$f_2 = 2f_1 = 500 \text{ Hz}$$

Note: The open pipe has twice the fundamental frequency of the closed pipe of the same length.

14 Lesson 13: Electric Charge and Electric Field

14.1 Electric Charge

14.1.1 Properties of Charge

- Charge is quantized: $q = ne$ where $e = 1.602 \times 10^{-19}$ C (elementary charge)
- Charge is conserved: total charge in an isolated system remains constant
- Two types: positive and negative
- Like charges repel, unlike charges attract

14.1.2 Units

Coulomb (C) is the SI unit of charge.

14.1.3 Conductors and Insulators

Conductors: Allow charge to flow freely (metals, aqueous solutions) **Insulators:** Resist charge flow (glass, rubber, plastic) **Semiconductors:** Intermediate behavior (silicon, germanium)

14.2 Coulomb's Law

14.2.1 Statement

The electric force between two point charges is:

$$\boxed{\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r}}$$

where:

- $k = 8.99 \times 10^9$ N·m²/C² is Coulomb's constant
- q_1, q_2 are the charges
- r is the distance between charges
- $\hat{r}$ is the unit vector from q_1 to q_2

Often written as:

$$k = \frac{1}{4\pi\epsilon_0}$$

where $\epsilon_0 = 8.85 \times 10^{-12}$ C²/(N·m²) is the permittivity of free space.

14.2.2 Principle of Superposition

For multiple charges, the net force on a charge is the vector sum of forces from all other charges:

$$\vec{F}_{net} = \sum_i \vec{F}_i = \sum_i k\frac{qq_i}{r_i^2}\hat{r}_i$$

14.3 Electric Field

14.3.1 Definition

The electric field at a point is the electric force per unit positive test charge:

$$\boxed{\vec{E} = \frac{\vec{F}}{q_0}}$$

Units: N/C or V/m
The electric field exists independent of any test charge placed in it.

14.3.2 Electric Field of a Point Charge

$$\boxed{\vec{E} = k\frac{q}{r^2}\hat{r}}$$

Direction: Away from positive charge, toward negative charge

14.3.3 Electric Field Lines

Electric field lines visualize the field:

- Lines point in the direction of the electric field
- Density of lines indicates field strength
- Lines start on positive charges and end on negative charges
- Lines never cross
- Field is tangent to the lines at each point

14.4 Electric Field of Continuous Charge Distributions

For continuous distributions, integrate:

$$\vec{E} = \int d\vec{E} = k\int \frac{dq}{r^2}\hat{r}$$

14.4.1 Infinite Line of Charge

For an infinite line with linear charge density λ (C/m):

$$\boxed{E = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{2k\lambda}{r}}$$

perpendicular to the line.

14.4.2 Infinite Sheet of Charge

For an infinite sheet with surface charge density σ (C/m^2):

$$\boxed{E = \frac{\sigma}{2\epsilon_0} = 2\pi k\sigma}$$

perpendicular to the sheet, independent of distance!

14.4.3 Ring of Charge

For a ring of radius R with total charge Q, at distance x along the axis:

$$\boxed{E = \frac{kQx}{(x^2 + R^2)^{3/2}}}$$

along the axis.
Derivation:
Consider an element dq on the ring. The field has magnitude:

$$dE = \frac{kdq}{r^2}$$

where $r = \sqrt{x^2 + R^2}$.
By symmetry, perpendicular components cancel. The axial component is:

$$dE_x = dE\cos\theta = \frac{kdq}{r^2} \cdot \frac{x}{r} = \frac{kxdq}{(x^2 + R^2)^{3/2}}$$

Integrating over the ring:

$$E = \int dE_x = \frac{kx}{(x^2 + R^2)^{3/2}} \int dq = \frac{kQx}{(x^2 + R^2)^{3/2}}$$

Special cases:

- At center ($x = 0$): $E = 0$ (by symmetry)
- Far from ring ($x \gg R$): $E \approx \frac{kQ}{x^2}$ (point charge)

14.5 Motion of Charged Particles in Electric Fields

14.5.1 Force and Acceleration

$$\vec{F} = q\vec{E}$$

$$\vec{a} = \frac{q\vec{E}}{m}$$

For constant field, use kinematic equations with $a = qE/m$.

14.5.2 Work and Energy

Work done by electric field on charge q moving through displacement $\vec{d}$:

$$W = \vec{F} \cdot \vec{d} = q\vec{E} \cdot \vec{d}$$

For constant field:

$$W = qEd\cos\theta$$

From work-energy theorem:

$$\Delta KE = W = qEd$$

14.6 Electric Flux and Gauss's Law

14.6.1 Electric Flux

Electric flux through a surface measures the number of field lines passing through:

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

For uniform field perpendicular to surface:

$$\Phi_E = EA$$

Units: N·m^2/C

14.6.2 Gauss's Law

The net electric flux through any closed surface equals the enclosed charge divided by ϵ_0:

$$\boxed{\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}}$$

This is one of Maxwell's equations.

14.6.3 Applications of Gauss's Law

1) Spherical symmetry (point charge or spherical shell):

Choose Gaussian surface as sphere of radius r.
For $r > R$ (outside):

$$E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$E = \frac{kQ}{r^2}$$

For uniformly charged sphere, field outside is as if all charge were at center.
For $r < R$ (inside hollow sphere):

$$E = 0$$

For $r < R$ (inside solid sphere with uniform volume charge density ρ):

$$E = \frac{kQr}{R^3}$$

where Q is total charge.

2) Cylindrical symmetry (infinite line):

Choose Gaussian surface as cylinder of radius r and length L.

$$E \cdot 2\pi r L = \frac{\lambda L}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

3) Planar symmetry (infinite sheet):

Choose Gaussian surface as cylinder perpendicular to sheet.

$$E \cdot 2A = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

14.7 Example Problems

Problem 13.1: Three point charges are located at the corners of an equilateral triangle with side length 0.50 m. The charges are: $q_1 = +2.0$ C at the top vertex, $q_2 = -4.0$ C at the bottom left, and $q_3 = +3.0$ C at the bottom right. Find: (a) the net electric force on q_1, (b) the electric field at the center of the triangle.

Solution:

Given: $a = 0.50$ m, $q_1 = 2.0\times10^{-6}$ C, $q_2 = -4.0\times10^{-6}$ C, $q_3 = 3.0\times10^{-6}$ C, $k = 8.99\times10^9$ N·m²/C²

(a) Net force on q_1:

Force from q_2:

Magnitude:

$$F_{21} = k\frac{|q_2||q_1|}{a^2} = 8.99 \times 10^9 \times \frac{4.0 \times 10^{-6} \times 2.0 \times 10^{-6}}{(0.50)^2}$$

$$= 8.99 \times 10^9 \times \frac{8.0 \times 10^{-12}}{0.25} = 0.288 \text{ N}$$

Direction: Since q_2 is negative, the force is attractive, pointing from q_1 toward q_2.
In component form (taking positive x to the right, positive y up): - Angle from horizontal: 60° down to the left

$$F_{21x} = -F_{21}\cos(60) = -0.288 \times 0.5 = -0.144 \text{ N}$$

$$F_{21y} = -F_{21}\sin(60) = -0.288 \times 0.866 = -0.249 \text{ N}$$

Force from q_3:
Magnitude:

$$F_{31} = k\frac{|q_3||q_1|}{a^2} = 8.99 \times 10^9 \times \frac{3.0 \times 10^{-6} \times 2.0 \times 10^{-6}}{(0.50)^2}$$

$$= 0.216 \text{ N}$$

Direction: Since both charges are positive, force is repulsive, pointing from q_3 toward q_1. - Angle from horizontal: 60° down to the right

$$F_{31x} = F_{31}\cos(60) = 0.216 \times 0.5 = 0.108 \text{ N}$$

$$F_{31y} = -F_{31}\sin(60) = -0.216 \times 0.866 = -0.187 \text{ N}$$

Net force:

$$F_{net,x} = F_{21x} + F_{31x} = -0.144 + 0.108 = -0.036 \text{ N}$$

$$F_{net,y} = F_{21y} + F_{31y} = -0.249 - 0.187 = -0.436 \text{ N}$$

Magnitude:

$$F_{net} = \sqrt{F_{net,x}^2 + F_{net,y}^2} = \sqrt{(-0.036)^2 + (-0.436)^2}$$

$$= \sqrt{0.00130 + 0.190} = 0.437 \text{ N}$$

Direction:

$$\theta = \tan^{-1}\left(\frac{F_{net,y}}{F_{net,x}}\right) = \tan^{-1}\left(\frac{-0.436}{-0.036}\right) = \tan^{-1}(12.1)$$

$$= 85.3 \text{ below negative x-axis}$$

(b) Electric field at center:
Distance from each vertex to center of equilateral triangle:

$$r = \frac{a}{\sqrt{3}} = \frac{0.50}{\sqrt{3}} = 0.289 \text{ m}$$

Field from q_1:

$$E_1 = k\frac{|q_1|}{r^2} = 8.99 \times 10^9 \times \frac{2.0 \times 10^{-6}}{(0.289)^2} = 2.16 \times 10^5 \text{ N/C}$$

Direction: downward (away from positive charge)

Field from q_2:

$$E_2 = k\frac{|q_2|}{r^2} = 8.99 \times 10^9 \times \frac{4.0 \times 10^{-6}}{(0.289)^2} = 4.31 \times 10^5 \text{ N/C}$$

Direction: toward q_2 (toward negative charge), at 60° above horizontal to the left

Field from q_3:

$$E_3 = k\frac{|q_3|}{r^2} = 8.99 \times 10^9 \times \frac{3.0 \times 10^{-6}}{(0.289)^2} = 3.23 \times 10^5 \text{ N/C}$$

Direction: toward q_3 (away from positive charge), at 60° above horizontal to the right

Components:

$$E_{1x} = 0, \quad E_{1y} = -2.16 \times 10^5$$

$$E_{2x} = -4.31 \times 10^5 \cos(60) = -2.16 \times 10^5$$
$$E_{2y} = 4.31 \times 10^5 \sin(60) = 3.73 \times 10^5$$

$$E_{3x} = 3.23 \times 10^5 \cos(60) = 1.62 \times 10^5$$
$$E_{3y} = 3.23 \times 10^5 \sin(60) = 2.80 \times 10^5$$

Net field:

$$E_{net,x} = 0 - 2.16 + 1.62 = -0.54 \times 10^5 \text{ N/C}$$
$$E_{net,y} = -2.16 + 3.73 + 2.80 = 4.37 \times 10^5 \text{ N/C}$$

Magnitude:

$$E_{net} = \sqrt{(-0.54)^2 + (4.37)^2} \times 10^5 = 4.40 \times 10^5 \text{ N/C}$$

Problem 13.2: An electron is accelerated from rest through a potential difference (which we'll learn about next) creating a uniform electric field of magnitude 5.0×10^4 N/C over a distance of 0.020 m. Find: (a) the acceleration of the electron, (b) the final speed, (c) the time taken.

Solution:

Given: $E = 5.0 \times 10^4$ N/C, $d = 0.020$ m, $e = 1.602 \times 10^{-19}$ C, $m_e = 9.109 \times 10^{-31}$ kg

(a) Acceleration:

$$F = eE = 1.602 \times 10^{-19} \times 5.0 \times 10^4 = 8.01 \times 10^{-15} \text{ N}$$

$$a = \frac{F}{m_e} = \frac{8.01 \times 10^{-15}}{9.109 \times 10^{-31}} = 8.79 \times 10^{15} \text{ m/s}^2$$

(b) Final speed:

$$v^2 = v_0^2 + 2ad = 0 + 2 \times 8.79 \times 10^{15} \times 0.020$$

$$v^2 = 3.52 \times 10^{14}$$

$$v = 1.88 \times 10^7 \text{ m/s}$$

This is about 6
(c) Time:

$$v = v_0 + at$$

$$t = \frac{v}{a} = \frac{1.88 \times 10^7}{8.79 \times 10^{15}} = 2.14 \times 10^{-9} \text{ s} = 2.14 \text{ ns}$$

15 Conclusion and Study Tips

This comprehensive guide has covered the complete high school physics curriculum for grades 11 and 12, from mechanics through thermodynamics to electricity. Each lesson builds systematically on previous concepts, emphasizing both theoretical understanding and problem-solving skills.

15.1 Key Study Strategies

1. **Master the fundamentals:** Ensure you understand basic concepts before moving to advanced topics
2. **Practice derivations:** Working through derivations from first principles builds deep understanding
3. **Solve many problems:** Physics is learned by doing - work through varied problem types
4. **Draw diagrams:** Free-body diagrams, circuit diagrams, ray diagrams are essential tools
5. **Check units:** Dimensional analysis catches many errors
6. **Estimate answers:** Develop physical intuition by estimating before calculating
7. **Connect concepts:** Look for relationships between different topics
8. **Use multiple resources:** Textbooks, online resources, and practice exams all help

15.2 Mathematical Tools

Success in physics requires facility with:

- Algebra and trigonometry
- Vector operations
- Calculus (derivatives and integrals)
- Logarithms and exponentials
- Graphical analysis

15.3 Looking Ahead

The remaining lessons (14-22) will cover:

- Electric potential and capacitance
- Current and resistance
- DC circuits
- Magnetic fields and forces
- Electromagnetic induction
- AC circuits
- Geometric and physical optics
- Atomic and nuclear physics
- Special relativity (introduction)

Master the material in Lessons 1-13 before proceeding, as these foundational concepts underpin all advanced topics.

15.4 Final Thoughts

Physics reveals the fundamental laws governing our universe. The journey from kinematics to quantum mechanics is challenging but immensely rewarding. Persistence, curiosity, and systematic study will lead to mastery.

Remember: every physicist started where you are now. With dedication and practice, you too can develop expertise in this beautiful subject.

16 Lesson 14: Electric Potential and Capacitance

16.1 Electric Potential Energy

16.1.1 Work Done by Electric Force

The electric force is conservative. The work done by the electric force on a charge q moving from point A to point B is:

$$W_{AB} = -\Delta U = U_A - U_B$$

where U is the electric potential energy.

16.1.2 Potential Energy of Point Charges

For two point charges q_1 and q_2 separated by distance r:

$$\boxed{U = k\frac{q_1 q_2}{r}}$$

This is defined with $U(\infty) = 0$.

Derivation:

Work done bringing q_2 from infinity to distance r from q_1:

$$W = \int_{\infty}^{r} \vec{F} \cdot d\vec{r} = \int_{\infty}^{r} k\frac{q_1 q_2}{r'^2} dr'$$

$$= -kq_1 q_2 \left[\frac{1}{r'}\right]_{\infty}^{r} = -k\frac{q_1 q_2}{r} - 0$$

$$U = -W = k\frac{q_1 q_2}{r}$$

For a system of multiple charges:

$$U_{total} = \sum_{i<j} k\frac{q_i q_j}{r_{ij}}$$

16.2 Electric Potential

16.2.1 Definition

Electric potential V is the potential energy per unit charge:

$$\boxed{V = \frac{U}{q}}$$

Units: Volt (V) = J/C

The potential difference between two points is:

$$\Delta V = V_B - V_A = -\int_{A}^{B} \vec{E} \cdot d\vec{s}$$

16.2.2 Potential of a Point Charge

$$\boxed{V = k\frac{q}{r}}$$

For multiple point charges:

$$V = \sum_i k\frac{q_i}{r_i}$$

Note: Potential is a scalar - no vector addition needed!

16.2.3 Relationship Between E and V

The electric field is the negative gradient of potential:

$$\vec{E} = -\nabla V$$

In one dimension:

$$\boxed{E_x = -\frac{dV}{dx}}$$

The field points in the direction of decreasing potential.

16.3 Equipotential Surfaces

16.3.1 Properties

- Points on an equipotential surface have the same potential
- Electric field is perpendicular to equipotential surfaces
- No work is done moving a charge along an equipotential surface
- Field lines are perpendicular to equipotentials
- Equipotentials never cross

16.3.2 Examples

- Point charge: concentric spheres
- Uniform field: equally spaced parallel planes
- Dipole: more complex surfaces

16.4 Potential Energy in Uniform Field

For a uniform electric field E in the x-direction:

$$V = -Ex + C$$

The potential difference over distance d:

$$\Delta V = -Ed$$

Potential energy of charge q:

$$U = qV = -qEx$$

16.5 Capacitance

16.5.1 Definition

A capacitor stores electric charge and energy. Capacitance is the ratio of charge to potential difference:

$$\boxed{C = \frac{Q}{V}}$$

Units: Farad (F) = C/V
Common submultiples:

- Microfarad: μF = 10^{-6} F
- Nanofarad: nF = 10^{-9} F
- Picofarad: pF = 10^{-12} F

16.5.2 Parallel-Plate Capacitor

For two parallel plates with area A separated by distance d:

$$\boxed{C = \epsilon_0 \frac{A}{d}}$$

Derivation:
Surface charge density: $\sigma = Q/A$
Electric field between plates: $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$
Potential difference: $V = Ed = \frac{Qd}{\epsilon_0 A}$
Capacitance: $C = \frac{Q}{V} = \epsilon_0 \frac{A}{d}$

16.5.3 Other Capacitor Geometries

Cylindrical capacitor (coaxial cylinders, length L, radii a and b where $b > a$):

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$$

Spherical capacitor (concentric spheres, radii a and b where $b > a$):

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}$$

Isolated sphere (radius R, taking $V(\infty) = 0$):

$$C = 4\pi\epsilon_0 R$$

16.6 Capacitors in Circuits

16.6.1 Series Combination

For capacitors in series:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ...$$

Key properties:

- Same charge on all capacitors: $Q = Q_1 = Q_2 = Q_3$
- Voltages add: $V = V_1 + V_2 + V_3$
- Equivalent capacitance is less than smallest individual capacitance

16.6.2 Parallel Combination

For capacitors in parallel:

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + ...}$$

Key properties:

- Same voltage across all capacitors: $V = V_1 = V_2 = V_3$
- Charges add: $Q = Q_1 + Q_2 + Q_3$
- Equivalent capacitance is greater than largest individual capacitance

16.7 Energy Stored in Capacitors

16.7.1 Energy in a Capacitor

The energy stored in a charged capacitor is:

$$\boxed{U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}}$$

Derivation:
Work done to charge a capacitor by moving charge dq when voltage is v:

$$dW = v\,dq = \frac{q}{C}dq$$

Total work:

$$W = \int_0^Q \frac{q}{C}dq = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q = \frac{Q^2}{2C}$$

This work is stored as potential energy: $U = W$

16.7.2 Energy Density

For a parallel-plate capacitor, the energy is stored in the electric field. The energy density (energy per unit volume) is:

$$\boxed{u = \frac{1}{2}\epsilon_0 E^2}$$

Derivation:
For parallel plates:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\epsilon_0\frac{A}{d}(Ed)^2 = \frac{1}{2}\epsilon_0 E^2(Ad)$$

Volume between plates: $V_{ol} = Ad$
Energy density: $u = \frac{U}{V_{ol}} = \frac{1}{2}\epsilon_0 E^2$
This result is general for any electric field.

16.8 Dielectrics

16.8.1 Definition

A dielectric is an insulating material placed between capacitor plates. It increases capacitance.

16.8.2 Dielectric Constant

The dielectric constant κ (also called relative permittivity) is the factor by which capacitance increases:

$$C = \kappa C_0$$

where C_0 is the capacitance with vacuum between plates.
For parallel plates with dielectric:

$$C = \kappa \epsilon_0 \frac{A}{d}$$

Common dielectric constants:

- Vacuum: $\kappa = 1$ (by definition)
- Air: $\kappa \approx 1.00054$
- Paper: $\kappa \approx 3.5$
- Glass: $\kappa \approx 4 - 7$
- Water: $\kappa \approx 80$

16.8.3 Effects of Dielectrics

With dielectric inserted at constant charge:

- Capacitance increases: $C = \kappa C_0$
- Electric field decreases: $E = E_0/\kappa$
- Voltage decreases: $V = V_0/\kappa$
- Energy decreases: $U = U_0/\kappa$

With dielectric inserted at constant voltage:

- Capacitance increases: $C = \kappa C_0$
- Charge increases: $Q = \kappa Q_0$
- Electric field remains constant: $E = E_0$
- Energy increases: $U = \kappa U_0$

16.8.4 Dielectric Breakdown

Every dielectric has a maximum electric field it can withstand before breaking down and conducting. This is the dielectric strength.

Dielectric strengths:

- Air: 3×10^6 V/m
- Paper: 16×10^6 V/m
- Teflon: 60×10^6 V/m

16.9 Example Problems

Problem 14.1: Three point charges are arranged as follows: $q_1 = +5.0$ nC at the origin, $q_2 = -3.0$ nC at $(3.0, 0)$ m, and $q_3 = +4.0$ nC at $(0, 4.0)$ m. Find: (a) the electric potential at point P at $(3.0, 4.0)$ m, (b) the work required to bring a charge $q = +2.0$ nC from infinity to point P, (c) the electric field at point P.

Solution:

Given: $q_1 = 5.0 \times 10^{-9}$ C, $q_2 = -3.0 \times 10^{-9}$ C, $q_3 = 4.0 \times 10^{-9}$ C, $k = 8.99 \times 10^9$ N·m²/C²

(a) Potential at P:

Distances:

- $r_1 = \sqrt{(3.0)^2 + (4.0)^2} = 5.0$ m
- $r_2 = \sqrt{(3.0 - 3.0)^2 + (4.0 - 0)^2} = 4.0$ m
- $r_3 = \sqrt{(3.0 - 0)^2 + (4.0 - 4.0)^2} = 3.0$ m

Potential (scalar sum):

$$V_P = k\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3}\right)$$

$$= 8.99 \times 10^9 \left(\frac{5.0 \times 10^{-9}}{5.0} + \frac{-3.0 \times 10^{-9}}{4.0} + \frac{4.0 \times 10^{-9}}{3.0}\right)$$

$$= 8.99 \times 10^9 \times 10^{-9}\left(\frac{5.0}{5.0} - \frac{3.0}{4.0} + \frac{4.0}{3.0}\right)$$

$$= 8.99\,(1.0 - 0.75 + 1.33) = 8.99 \times 1.58 = 14.2 \text{ V}$$

(b) Work to bring charge to P:

$$W = q(V_P - V_\infty) = q \cdot V_P = 2.0 \times 10^{-9} \times 14.2 = 2.84 \times 10^{-8} \text{ J}$$

(c) Electric field at P:

This requires vector addition. Field from each charge:

From q_1 (at origin to P):

$$E_1 = k\frac{|q_1|}{r_1^2} = 8.99 \times 10^9 \times \frac{5.0 \times 10^{-9}}{(5.0)^2} = 1.80 \text{ N/C}$$

Direction: along vector from origin to P: $\hat{r}_1 = (0.6\hat{\imath} + 0.8\hat{\jmath})$

$$\vec{E}_1 = 1.80(0.6\hat{\imath} + 0.8\hat{\jmath}) = 1.08\hat{\imath} + 1.44\hat{\jmath} \text{ N/C}$$

From q_2 (at (3.0, 0) to P):

$$E_2 = k\frac{|q_2|}{r_2^2} = 8.99 \times 10^9 \times \frac{3.0 \times 10^{-9}}{(4.0)^2} = 1.69 \text{ N/C}$$

Direction: toward q_2 (negative charge): $-\hat{\jmath}$

$$\vec{E}_2 = -1.69\hat{\jmath} \text{ N/C}$$

From q_3 (at (0, 4.0) to P):

$$E_3 = k\frac{|q_3|}{r_3^2} = 8.99 \times 10^9 \times \frac{4.0 \times 10^{-9}}{(3.0)^2} = 4.00 \text{ N/C}$$

Direction: away from q_3: $\hat{\imath}$

$$\vec{E}_3 = 4.00\hat{\imath} \text{ N/C}$$

Net field:

$$\begin{aligned}\vec{E}_{net} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 &= (1.08 + 4.00)\hat{\imath} + (1.44 - 1.69)\hat{\jmath} \\ &= 5.08\hat{\imath} - 0.25\hat{\jmath} \text{ N/C}\end{aligned}$$

Magnitude:

$$E_{net} = \sqrt{(5.08)^2 + (-0.25)^2} = 5.09 \text{ N/C}$$

Problem 14.2: A parallel-plate capacitor has plates of area 0.020 m² separated by 2.0 mm of air. It is connected to a 12 V battery. Find: (a) the capacitance, (b) the charge on each plate, (c) the electric field between the plates, (d) the energy stored, (e) if a dielectric with $\kappa = 3.5$ is inserted while the battery remains connected, find the new charge and energy.

Solution:

Given: $A = 0.020$ m², $d = 2.0 \times 10^{-3}$ m, $V = 12$ V, $\epsilon_0 = 8.85 \times 10^{-12}$ C²/(N·m²)

(a) Capacitance:

$$\begin{aligned}C = \epsilon_0\frac{A}{d} &= 8.85 \times 10^{-12} \times \frac{0.020}{2.0 \times 10^{-3}} \\ &= 8.85 \times 10^{-12} \times 10 = 8.85 \times 10^{-11} \text{ F} = 88.5 \text{ pF}\end{aligned}$$

(b) Charge:

$$Q = CV = 8.85 \times 10^{-11} \times 12 = 1.06 \times 10^{-9} \text{ C} = 1.06 \text{ nC}$$

(c) Electric field:

$$E = \frac{V}{d} = \frac{12}{2.0 \times 10^{-3}} = 6000 \text{ V/m} = 6.0 \text{ kV/m}$$

(d) Energy:

$$U = \frac{1}{2}CV^2 = \frac{1}{2} \times 8.85 \times 10^{-11} \times (12)^2 = 6.37 \times 10^{-9} \text{ J}$$

(e) With dielectric at constant voltage:
New capacitance:

$$C' = \kappa C = 3.5 \times 8.85 \times 10^{-11} = 3.10 \times 10^{-10} \text{ F} = 310 \text{ pF}$$

New charge (voltage constant):

$$Q' = C'V = 3.10 \times 10^{-10} \times 12 = 3.72 \times 10^{-9} \text{ C} = 3.72 \text{ nC}$$

New energy:

$$U' = \frac{1}{2}C'V^2 = \frac{1}{2} \times 3.10 \times 10^{-10} \times 144 = 2.23 \times 10^{-8} \text{ J}$$

Or: $U' = \kappa U = 3.5 \times 6.37 \times 10^{-9} = 2.23 \times 10^{-8}$ J
The battery does work to add more charge to the capacitor.

Problem 14.3: Three capacitors ($C_1 = 2.0$ F, $C_2 = 4.0$ F, $C_3 = 6.0$ F) are connected: C_1 and C_2 in series, and this combination is in parallel with C_3. The combination is connected to a 120 V battery. Find: (a) the equivalent capacitance, (b) the charge on each capacitor, (c) the voltage across each capacitor, (d) the total energy stored.

Solution:
Given: $C_1 = 2.0 \times 10^{-6}$ F, $C_2 = 4.0 \times 10^{-6}$ F, $C_3 = 6.0 \times 10^{-6}$ F, $V = 120$ V
(a) Equivalent capacitance:
C_1 and C_2 in series:

$$\frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2.0} + \frac{1}{4.0} = \frac{2+1}{4} = \frac{3}{4}$$

$$C_{12} = \frac{4}{3} = 1.33 \text{ F}$$

C_{12} in parallel with C_3:

$$C_{eq} = C_{12} + C_3 = 1.33 + 6.0 = 7.33 \text{ F}$$

(b) and (c) Charges and voltages:
For C_3 (directly across battery):

$$V_3 = V = 120 \text{ V}$$

$$Q_3 = C_3V_3 = 6.0 \times 10^{-6} \times 120 = 7.2 \times 10^{-4} \text{ C} = 720 \text{ C}$$

For series combination C_1 and C_2:

$$V_{12} = V = 120 \text{ V}$$

$$Q_{12} = C_{12}V_{12} = 1.33 \times 10^{-6} \times 120 = 1.60 \times 10^{-4} \text{ C} = 160 \text{ C}$$

In series, same charge: $Q_1 = Q_2 = Q_{12} = 160$ C
Voltages:

$$V_1 = \frac{Q_1}{C_1} = \frac{160 \times 10^{-6}}{2.0 \times 10^{-6}} = 80 \text{ V}$$

$$V_2 = \frac{Q_2}{C_2} = \frac{160 \times 10^{-6}}{4.0 \times 10^{-6}} = 40 \text{ V}$$

Check: $V_1 + V_2 = 80 + 40 = 120$ V
(d) Total energy:

$$U_{total} = \frac{1}{2}C_{eq}V^2 = \frac{1}{2} \times 7.33 \times 10^{-6} \times (120)^2$$

$$= \frac{1}{2} \times 7.33 \times 10^{-6} \times 14400 = 0.0528 \text{ J} = 52.8 \text{ mJ}$$

Or sum individual energies:

$$U_1 = \frac{Q_1^2}{2C_1} = \frac{(160 \times 10^{-6})^2}{2 \times 2.0 \times 10^{-6}} = 6.4 \times 10^{-3} \text{ J}$$

$$U_2 = \frac{Q_2^2}{2C_2} = \frac{(160 \times 10^{-6})^2}{2 \times 4.0 \times 10^{-6}} = 3.2 \times 10^{-3} \text{ J}$$

$$U_3 = \frac{1}{2}C_3V_3^2 = \frac{1}{2} \times 6.0 \times 10^{-6} \times (120)^2 = 0.0432 \text{ J}$$

$$U_{total} = 6.4 + 3.2 + 43.2 = 52.8 \text{ mJ}$$

17 Lesson 15: Current and Resistance

17.1 Electric Current

17.1.1 Definition

Electric current is the rate of charge flow:

$$\boxed{I = \frac{dQ}{dt}}$$

Units: Ampere (A) = C/s

For steady current:

$$I = \frac{Q}{t}$$

Direction: By convention, current flows in the direction positive charges would move (opposite to electron flow in metals).

17.1.2 Current Density

Current density $\vec{J}$ is current per unit cross-sectional area:

$$I = \int \vec{J} \cdot d\vec{A}$$

For uniform current density perpendicular to area:

$$J = \frac{I}{A}$$

Units: A/m^2

17.2 Resistance and Ohm's Law

17.2.1 Ohm's Law

For many materials (ohmic materials), current is proportional to applied voltage:

$$\boxed{V = IR}$$

where R is resistance.

Units: Ohm (Ω) = V/A

17.2.2 Resistivity

Resistance depends on material and geometry:

$$\boxed{R = \rho\frac{L}{A}}$$

where:

- ρ is resistivity (·m)
- L is length
- A is cross-sectional area

The reciprocal of resistivity is conductivity:

$$\sigma = \frac{1}{\rho}$$

Units: $(\cdot\text{m})^{1}$ = S/m (Siemens per meter)
Resistivities at 20°C:

- Silver: 1.59×10^{-8} ·m
- Copper: 1.68×10^{-8} ·m
- Aluminum: 2.65×10^{-8} ·m
- Iron: 9.71×10^{-8} ·m
- Nichrome: 1.0×10^{-6} ·m
- Silicon: 640 ·m
- Glass: $10^{10} - 10^{14}$ ·m

17.2.3 Temperature Dependence

For metals, resistivity increases with temperature:

$$\rho = \rho_0[1 + \alpha(T - T_0)]$$

where α is the temperature coefficient of resistivity (typical value for metals: $\alpha \approx 4 \times 10^{-3}$ K^{1}).
Similarly for resistance:

$$R = R_0[1 + \alpha(T - T_0)]$$

17.3 Microscopic View of Current

17.3.1 Drift Velocity

In a conductor with current, charge carriers have average drift velocity v_d:

$$I = nqv_dA$$

where:

- n is number density of charge carriers (carriers/m^{3})

- q is charge per carrier
- v_d is drift speed
- A is cross-sectional area

For electrons in copper: $n \approx 8.5 \times 10^{28}$ electrons/m³

Typical drift speeds are very slow (mm/s), but the electric field propagates at nearly the speed of light.

17.3.2 Relationship to Resistivity

$$\vec{J} = nq\vec{v}_d = \sigma\vec{E}$$

This connects current density to electric field through conductivity.

17.4 Electrical Power

17.4.1 Power Dissipation

The power delivered to a circuit element is:

$$\boxed{P = IV}$$

For a resistor (using Ohm's law):

$$\boxed{P = I^2R = \frac{V^2}{R}}$$

Units: Watt (W) = J/s

Derivation:

Power is rate of energy transfer. When charge dQ moves through potential difference V:

$$dU = V\,dQ$$

$$P = \frac{dU}{dt} = V\frac{dQ}{dt} = IV$$

17.4.2 Energy

Energy dissipated in time t:

$$U = Pt = IVt = I^2Rt = \frac{V^2t}{R}$$

17.5 Electromotive Force (EMF)

17.5.1 Definition

EMF ($\mathcal{E}$) is the work done per unit charge by a source (battery, generator, etc.):

$$\mathcal{E} = \frac{W}{Q}$$

Units: Volt (V)
Despite the name, EMF is not a force - it's a potential difference.

17.5.2 Real Batteries

A real battery has internal resistance r. The terminal voltage is:

$$V = \mathcal{E} - Ir$$

where I is the current drawn.
When $I = 0$ (open circuit): $V = \mathcal{E}$ When I is maximum (short circuit): $I_{max} = \mathcal{E}/r$

17.6 Resistors in Circuits

17.6.1 Series Combination

For resistors in series:

$$\boxed{R_{eq} = R_1 + R_2 + R_3 + ...}$$

Properties:

- Same current through all resistors
- Voltages add: $V = V_1 + V_2 + V_3$
- Equivalent resistance is greater than any individual resistance

17.6.2 Parallel Combination

For resistors in parallel:

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...}$$

Properties:

- Same voltage across all resistors
- Currents add: $I = I_1 + I_2 + I_3$
- Equivalent resistance is less than any individual resistance

For two resistors in parallel:

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

17.7 Example Problems

Problem 15.1: A copper wire has diameter 1.0 mm and length 2.5 m. It carries a current of 5.0 A. Find: (a) the resistance of the wire, (b) the voltage across the wire, (c) the power dissipated, (d) the drift velocity of electrons, (e) the electric field in the wire.

Solution:

Given: $d = 1.0 \times 10^{-3}$ m, $L = 2.5$ m, $I = 5.0$ A, $\rho_{Cu} = 1.68 \times 10^{-8}$ ·m, $n = 8.5 \times 10^{28}$ electrons/m^3

Cross-sectional area:

$$A = \pi r^2 = \pi \left(\frac{d}{2}\right)^2 = \pi \left(\frac{0.001}{2}\right)^2 = 7.85 \times 10^{-7} \text{ m}^2$$

(a) Resistance:

$$R = \rho \frac{L}{A} = 1.68 \times 10^{-8} \times \frac{2.5}{7.85 \times 10^{-7}} = 0.0535$$

(b) Voltage:

$$V = IR = 5.0 \times 0.0535 = 0.268 \text{ V}$$

(c) Power:

$$P = I^2 R = (5.0)^2 \times 0.0535 = 1.34 \text{ W}$$

Or: $P = IV = 5.0 \times 0.268 = 1.34$ W

(d) Drift velocity:

From $I = nqv_d A$:

$$v_d = \frac{I}{nqA} = \frac{5.0}{8.5 \times 10^{28} \times 1.602 \times 10^{-19} \times 7.85 \times 10^{-7}}$$

$$= \frac{5.0}{1.069 \times 10^4} = 4.68 \times 10^{-4} \text{ m/s} = 0.468 \text{ mm/s}$$

The drift velocity is very slow - less than 0.5 mm per second!

(e) Electric field:

$$E = \frac{V}{L} = \frac{0.268}{2.5} = 0.107 \text{ V/m}$$

Or using $J = \sigma E$:

$$J = \frac{I}{A} = \frac{5.0}{7.85 \times 10^{-7}} = 6.37 \times 10^6 \text{ A/m}^2$$

$$E = \rho J = 1.68 \times 10^{-8} \times 6.37 \times 10^6 = 0.107 \text{ V/m}$$

Problem 15.2: A battery with EMF 12.0 V and internal resistance 0.50 is connected to a variable external resistor R. Find: (a) the current when $R = 5.0$, (b) the terminal voltage, (c) the power delivered to the external resistor, (d) the power dissipated in the internal resistance, (e) the value of R for maximum power transfer to the external resistor.

Solution:

Given: $\mathcal{E} = 12.0$ V, $r = 0.50$
(a) Current:
Total resistance: $R_{total} = R + r = 5.0 + 0.50 = 5.5$

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.0}{5.5} = 2.18 \text{ A}$$

(b) Terminal voltage:

$$V = \mathcal{E} - Ir = 12.0 - 2.18 \times 0.50 = 12.0 - 1.09 = 10.9 \text{ V}$$

Or: $V = IR = 2.18 \times 5.0 = 10.9$ V
(c) Power to external resistor:

$$P_R = I^2 R = (2.18)^2 \times 5.0 = 23.8 \text{ W}$$

(d) Power in internal resistance:

$$P_r = I^2 r = (2.18)^2 \times 0.50 = 2.38 \text{ W}$$

Total power from battery: $P_{total} = I\mathcal{E} = 2.18 \times 12.0 = 26.2$ W Check: $P_R + P_r = 23.8 + 2.38 = 26.2$ W
(e) Maximum power transfer:
Current as function of R:

$$I = \frac{\mathcal{E}}{R + r}$$

Power to external resistor:

$$P_R = I^2 R = \frac{\mathcal{E}^2 R}{(R + r)^2}$$

To maximize, take derivative and set to zero:

$$\frac{dP_R}{dR} = \mathcal{E}^2 \frac{(R + r)^2 - R \cdot 2(R + r)}{(R + r)^4} = 0$$

$$(R + r)^2 - 2R(R + r) = 0$$

$$(R + r)[(R + r) - 2R] = 0$$

$$r - R = 0$$

$$\boxed{R = r = 0.50}$$

Maximum power transfer occurs when external resistance equals internal resistance.
At this condition:

$$I_{max} = \frac{12.0}{0.50 + 0.50} = 12.0 \text{ A}$$

$$P_{R,max} = (12.0)^2 \times 0.50 = 72.0 \text{ W}$$

Problem 15.3: Four resistors are connected: $R_1 = 10$ and $R_2 = 20$ in parallel, this combination in series with $R_3 = 15$, and all of this in parallel with $R_4 = 30$. The combination is connected across a 60 V source. Find: (a) the equivalent resistance, (b) the total current from the source, (c) the current through each resistor, (d) the power dissipated in each resistor.

Solution:

Given: $R_1 = 10$, $R_2 = 20$, $R_3 = 15$, $R_4 = 30$, $V = 60$ V

(a) Equivalent resistance:

R_1 and R_2 in parallel:

$$\frac{1}{R_{12}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{10} + \frac{1}{20} = \frac{2+1}{20} = \frac{3}{20}$$

$$R_{12} = \frac{20}{3} = 6.67$$

R_{12} in series with R_3:

$$R_{123} = R_{12} + R_3 = 6.67 + 15 = 21.67$$

R_{123} in parallel with R_4:

$$\frac{1}{R_{eq}} = \frac{1}{R_{123}} + \frac{1}{R_4} = \frac{1}{21.67} + \frac{1}{30}$$

$$= 0.0461 + 0.0333 = 0.0794$$

$$R_{eq} = \frac{1}{0.0794} = 12.6$$

(b) Total current:

$$I_{total} = \frac{V}{R_{eq}} = \frac{60}{12.6} = 4.76 \text{ A}$$

(c) Current through each resistor:

Branch with R_{123}:

$$I_{123} = \frac{V}{R_{123}} = \frac{60}{21.67} = 2.77 \text{ A}$$

This current goes through R_3: $I_3 = 2.77$ A

Voltage across R_{12} combination:

$$V_{12} = I_{123} \times R_{12} = 2.77 \times 6.67 = 18.5 \text{ V}$$

Current through R_1:

$$I_1 = \frac{V_{12}}{R_1} = \frac{18.5}{10} = 1.85 \text{ A}$$

Current through R_2:

$$I_2 = \frac{V_{12}}{R_2} = \frac{18.5}{20} = 0.925 \text{ A}$$

Check: $I_1 + I_2 = 1.85 + 0.925 = 2.775 \approx 2.77$ A

Branch with R_4:

$$I_4 = \frac{V}{R_4} = \frac{60}{30} = 2.0 \text{ A}$$

Check total: $I_{123} + I_4 = 2.77 + 2.0 = 4.77 \approx 4.76$ A
(d) Power in each resistor:

$$P_1 = I_1^2 R_1 = (1.85)^2 \times 10 = 34.2 \text{ W}$$
$$P_2 = I_2^2 R_2 = (0.925)^2 \times 20 = 17.1 \text{ W}$$
$$P_3 = I_3^2 R_3 = (2.77)^2 \times 15 = 115.1 \text{ W}$$
$$P_4 = I_4^2 R_4 = (2.0)^2 \times 30 = 120.0 \text{ W}$$

Total power: $P_{total} = 34.2 + 17.1 + 115.1 + 120.0 = 286.4$ W
Check: $P = \frac{V^2}{R_{eq}} = \frac{(60)^2}{12.6} = 286$ W

18 Lesson 16: DC Circuits

18.1 Kirchhoff's Rules

18.1.1 Kirchhoff's Current Law (KCL)

The sum of currents entering any junction equals the sum leaving:

$$\boxed{\sum I_{in} = \sum I_{out}}$$

Or equivalently: $\sum I = 0$ (taking entering as positive, leaving as negative)
This is based on conservation of charge.

18.1.2 Kirchhoff's Voltage Law (KVL)

The sum of potential differences around any closed loop is zero:

$$\boxed{\sum V = 0}$$

This is based on conservation of energy.
Sign conventions:

- Traversing a resistor in direction of current: $-IR$ (voltage drop)
- Traversing a resistor against current: $+IR$ (voltage rise)
- Traversing a battery from to +: $+\mathcal{E}$ (voltage rise)
- Traversing a battery from + to : $-\mathcal{E}$ (voltage drop)

18.2 RC Circuits

18.2.1 Charging a Capacitor

When a capacitor C is charged through resistor R from voltage source $\mathcal{E}$:
Charge on capacitor:

$$\boxed{q(t) = C\mathcal{E}(1 - e^{-t/RC}) = Q_f(1 - e^{-t/\tau})}$$

where $Q_f = C\mathcal{E}$ is the final charge and $\tau = RC$ is the time constant.
Current:

$$I(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC} = I_0 e^{-t/\tau}$$

where $I_0 = \mathcal{E}/R$ is the initial current.
Voltage across capacitor:

$$V_C(t) = \frac{q(t)}{C} = \mathcal{E}(1 - e^{-t/RC})$$

Voltage across resistor:

$$V_R(t) = IR = \mathcal{E}e^{-t/RC}$$

Check: $V_C + V_R = \mathcal{E}$

Derivation:

Applying KVL:

$$\mathcal{E} - IR - \frac{q}{C} = 0$$

$$\mathcal{E} = R\frac{dq}{dt} + \frac{q}{C}$$

This is a first-order linear differential equation. Rearranging:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{\mathcal{E}}{R}$$

Solution with initial condition $q(0) = 0$:

$$q(t) = C\mathcal{E}(1 - e^{-t/RC})$$

18.2.2 Time Constant

$$\tau = RC$$

Physical meaning:

- After time τ: capacitor reaches $(1 - e^{-1}) = 63.2\%$ of final charge
- After time 5τ: capacitor reaches 99.3% of final charge (essentially fully charged)

18.2.3 Discharging a Capacitor

When a charged capacitor (initial charge Q_0) discharges through resistor R:

Charge:

$$\boxed{q(t) = Q_0e^{-t/RC} = Q_0e^{-t/\tau}}$$

Current:

$$I(t) = -\frac{dq}{dt} = \frac{Q_0}{RC}e^{-t/RC} = I$$