Orbital Mechanics
Theory and Applications

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1 Preface

The study of celestial mechanics is cloesly tied to concepts in physics, many of which may already seem familiar. In addition to providing a comprehensive review of the basic knowledge needed to solve problems in celestial mechanics, this document will provide a set of equations, dervivations, and examples relevant to the theoretical portion of the IOAA. The complete syllabus can be found here.

2 Foundational Principles

2.1 Newton's Laws and Conservation Principles

The foundation of celestial mechanics rests upon Newton's laws of motion and gravitation. For a system of particles, we can write the equations of motion as:

$$\frac{d\vec{p}_i}{dt} = \sum_{j\neq i} \vec{F}_{ij}$$

where $\vec{F}_{ij}$ represents the force on particle i due to particle j. For gravitational interactions:

$$\vec{F}_{ij} = -\frac{Gm_i m_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_i - \vec{r}_j)$$

2.1.1 Conservation Laws

Three fundamental conservation laws govern celestial systems:

1) Conservation of Linear Momentum

For an isolated system with no external forces:

$$\vec{P}_{\text{total}} = \sum_i m_i \vec{v}_i = \text{constant}$$

This immediately implies that the center of mass moves with constant velocity:

$$\vec{V}_{\text{CM}} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} = \text{constant}$$

2) Conservation of Angular Momentum
The total angular momentum about any fixed point is conserved:

$$\vec{L}_{\text{total}} = \sum_i \vec{r}_i \times m_i \vec{v}_i = \text{constant}$$

For a two-body system, this reduces to:

$$\vec{L} = \vec{r} \times \mu \vec{v} = \text{constant}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass and $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative position vector.
3) Conservation of Energy
The total mechanical energy is conserved:

$$E_{\text{total}} = T + U = \frac{1}{2} \sum_i m_i v_i^2 - \sum_{i<j} \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|} = \text{constant}$$

2.2 The Two-Body Problem

The two-body problem is exactly solvable and forms the basis for understanding more complex systems. Consider two masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$ from an inertial origin.

2.2.1 Reduction to One-Body Problem

Define the center of mass position:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

and the relative position:

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

The equations of motion separate into:

$$M \ddot{\vec{R}} = 0$$

$$\mu \ddot{\vec{r}} = -\frac{G m_1 m_2}{r^3} \vec{r}$$

where $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$. The first equation tells us the center of mass moves uniformly, while the second describes the relative motion as if a single particle of mass μ orbits a fixed center of force.

2.2.2 Equation of Motion in Polar Coordinates

Converting to polar coordinates (r, θ) in the orbital plane:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

$$r^2 \dot{\theta} = h = \text{constant}$$

where $M = m_1 + m_2$ and $h = L/\mu$ is the specific angular momentum. The second equation is simply conservation of angular momentum.

3 Orbital Mechanics - Extended Theory

3.1 The Orbit Equation - Complete Derivation

We will now derive the orbit equation from first principles using the method of changing variables. Starting from the radial equation of motion:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

Using $h = r^2\dot{\theta}$, we can write:

$$\dot{\theta} = \frac{h}{r^2}$$

Now introduce the substitution $u = 1/r$. Then:

$$r = \frac{1}{u}, \quad \frac{dr}{d\theta} = \frac{dr}{du}\frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

We need $\ddot{r}$:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta} \cdot \frac{h}{r^2} = -h\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}\frac{h}{r^2} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Substituting into the equation of motion:

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -GMu^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

This is a simple harmonic oscillator equation with the general solution:

$$u = \frac{GM}{h^2}(1 + e\cos(\theta - \theta_0))$$

where e is the eccentricity and θ_0 is the orientation of the periapsis. Choosing $\theta_0 = 0$ and converting back to r:

$$r = \frac{h^2/GM}{1 + e\cos\theta} = \frac{p}{1 + e\cos\theta}$$

where $p = h^2/GM$ is the semi-latus rectum. We can express this in terms of the semi-major axis using the relation $p = a(1 - e^2)$:

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

This is the orbit equation, valid for all conic sections.

3.2 Energy and Eccentricity Relationships

The total specific energy (energy per unit reduced mass) is:

$$\mathcal{E} = \frac{v^2}{2} - \frac{GM}{r}$$

At periapsis, where $r = r_p = a(1-e)$ and $v = v_p$:

$$\mathcal{E} = \frac{v_p^2}{2} - \frac{GM}{a(1-e)}$$

Using conservation of angular momentum $v_p r_p = h$ and $h^2 = GMa(1-e^2)$:

$$v_p^2 = \frac{h^2}{r_p^2} = \frac{GMa(1-e^2)}{a^2(1-e)^2} = \frac{GM(1+e)}{a(1-e)}$$

Therefore:

$$\mathcal{E} = \frac{GM(1+e)}{2a(1-e)} - \frac{GM}{a(1-e)} = -\frac{GM}{2a}$$

This fundamental result shows that the specific energy depends only on the semi-major axis, not on the eccentricity.

We can also express the eccentricity in terms of energy and angular momentum:

$$e = \sqrt{1 + \frac{2\mathcal{E}h^2}{(GM)^2}}$$

This relationship categorizes orbits:

- $\mathcal{E} < 0$: $e < 1$ (ellipse)
- $\mathcal{E} = 0$: $e = 1$ (parabola)
- $\mathcal{E} > 0$: $e > 1$ (hyperbola)

3.3 The Vis-Viva Equation - Extended Analysis

The vis-viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

is valid for all Keplerian orbits. For hyperbolic orbits where $a < 0$, we typically write:

$$v^2 = GM\left(\frac{2}{r} + \frac{1}{|a|}\right)$$

For parabolic orbits ($a \to \infty$):

$$v^2 = \frac{2GM}{r}$$

This is the escape velocity at distance r.

3.3.1 Velocity Components

At any point in the orbit, we can decompose the velocity into radial and tangential components:

$$v_r = \dot{r} = \frac{GM}{h} e \sin\theta$$

$$v_\theta = r\dot{\theta} = \frac{h}{r} = \frac{GM}{h}(1 + e\cos\theta)$$

The total velocity is:

$$v^2 = v_r^2 + v_\theta^2 = \frac{(GM)^2}{h^2}(e^2 \sin^2\theta + (1 + e\cos\theta)^2)$$

Expanding and simplifying:

$$v^2 = \frac{(GM)^2}{h^2}(1 + 2e\cos\theta + e^2)$$

Using $h^2 = GMa(1 - e^2)$ and the orbit equation $r = \frac{a(1-e^2)}{1+e\cos\theta}$:

$$v^2 = \frac{GM}{a(1 - e^2)}(1 + 2e\cos\theta + e^2)$$

After algebraic manipulation, this reduces to the vis-viva equation.

3.4 Kepler's Laws - Rigorous Treatment

3.4.1 Kepler's First Law

The orbit of each planet is an ellipse with the Sun at one focus. This follows directly from solving the equation of motion under an inverse-square force law, as we demonstrated above.

3.4.2 Kepler's Second Law

The line joining a planet to the Sun sweeps out equal areas in equal times. Mathematically:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2} = \frac{L}{2\mu} = \text{constant}$$

This is a direct consequence of angular momentum conservation. The total area of an ellipse is $A = \pi ab$, so the period is:

$$T = \frac{2A}{h} = \frac{2\pi ab}{h}$$

Using $h^2 = GMa(1 - e^2) = GMb^2/a$:

$$T = \frac{2\pi ab}{\sqrt{GMb^2/a}} = 2\pi\sqrt{\frac{a^3}{GM}}$$

3.4.3 Kepler's Third Law

The square of the orbital period is proportional to the cube of the semi-major axis:

$$T^2 = \frac{4\pi^2}{GM}a^3$$

For a two-body system where both masses are significant:

$$T^2 = \frac{4\pi^2}{G(m_1 + m_2)}a^3$$

This allows us to determine the total mass of a binary system from observations of the period and separation.

3.5 Anomalies - Complete Theory

3.5.1 True Anomaly

The true anomaly θ (or ν) is the angle from periapsis to the current position. It appears directly in the orbit equation and is the most physically intuitive angle. However, it does not vary linearly with time.

3.5.2 Eccentric Anomaly

The eccentric anomaly E is defined geometrically through an auxiliary circle of radius a. The relationship to the radius is:

$$r = a(1 - e\cos E)$$

This can be derived by considering the ellipse equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with the focus at $x = -ae$. In terms of the eccentric anomaly:

$$x = a\cos E - ae, \quad y = b\sin E$$

The distance from the focus is:

$$r^2 = (a\cos E - ae)^2 + b^2\sin^2 E = a^2(\cos E - e)^2 + a^2(1 - e^2)\sin^2 E$$
$$= a^2(\cos^2 E - 2e\cos E + e^2 + \sin^2 E - e^2\sin^2 E)$$
$$= a^2(1 - 2e\cos E + e^2\cos^2 E) = a^2(1 - e\cos E)^2$$

Therefore: $r = a(1 - e\cos E)$.

The relationship between true and eccentric anomaly can be found by equating the two expressions for r:

$$a(1 - e\cos E) = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

Solving for $\cos\theta$:

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E}$$

The corresponding relation for $\sin\theta$ is:

$$\sin\theta = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}$$

These can be combined into the half-angle formulas:

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

3.5.3 Mean Anomaly - Kepler's Equation

The mean anomaly M is defined to vary linearly with time:

$$M = n(t - t_0)$$

where $n = 2\pi/T = \sqrt{GM/a^3}$ is the mean motion and t_0 is the time of periapsis passage. The relationship between mean and eccentric anomaly is Kepler's equation:

$$M = E - e\sin E$$

Derivation of Kepler's Equation:
We start from Kepler's second law:

$$\frac{dA}{dt} = \frac{h}{2} = \frac{\pi ab}{T}$$

Integrating from periapsis ($t = t_0$):

$$A(t) = \frac{\pi ab}{T}(t - t_0)$$

The area swept from periapsis to eccentric anomaly E can be computed geometrically. Consider the auxiliary circle. The area in the circle from periapsis to angle E is:

$$A_{\text{circle}} = \frac{1}{2}a^2 E$$

The area of the triangle formed by the center, periapsis, and the projection is:

$$A_{\text{triangle}} = \frac{1}{2}a \cdot ae\sin E = \frac{1}{2}a^2 e\sin E$$

The actual area in the ellipse is related to the circular area by the ratio b/a:

$$A = \frac{b}{a}(A_{\text{circle}} - A_{\text{triangle}}) = \frac{b}{a}\frac{a^2}{2}(E - e\sin E) = \frac{ab}{2}(E - e\sin E)$$

Setting this equal to the area from Kepler's second law:

$$\frac{ab}{2}(E - e\sin E) = \frac{\pi ab}{T}(t - t_0)$$

$$E - e\sin E = \frac{2\pi}{T}(t - t_0) = M$$

This is Kepler's equation. Note that it cannot be solved analytically for E given M; numerical methods (such as Newton-Raphson iteration) must be used.

3.5.4 Solving Kepler's Equation

The Newton-Raphson method for solving $E - e \sin E = M$ uses the iteration:

$$E_{n+1} = E_n - \frac{E_n - e \sin E_n - M}{1 - e \cos E_n}$$

with initial guess $E_0 = M + e \sin M$ (for small to moderate e). Typically 3-5 iterations yield sufficient precision.

3.6 Orbital Elements

A Keplerian orbit is completely specified by six orbital elements:

1. **Semi-major axis** a: determines the size and energy
2. **Eccentricity** e: determines the shape
3. **Inclination** i: angle between orbital plane and reference plane
4. **Longitude of ascending node** Ω: orientation of the line of nodes
5. **Argument of periapsis** ω: orientation of the orbit in its plane
6. **Time of periapsis passage** t_0 (or equivalently, mean anomaly at epoch)

For orbits in a single plane (as typically treated in introductory problems), only (a, e, ω, t_0) or equivalently (a, e, θ_0, t_0) are needed.

4 Orbital Maneuvers and Transfers

4.1 Hohmann Transfer

The Hohmann transfer is the most energy-efficient two-impulse transfer between two circular coplanar orbits. Consider transferring from an inner orbit of radius r_1 to an outer orbit of radius r_2.

The transfer ellipse has:

$$a_t = \frac{r_1 + r_2}{2}$$

At r_1 (periapsis of transfer orbit), the velocities are:

$$v_1 = \sqrt{\frac{GM}{r_1}} \quad \text{(circular orbit)}$$

$$v_{t1} = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a_t}\right)} = \sqrt{\frac{GM}{r_1}\frac{2r_2}{r_1 + r_2}} \quad \text{(transfer orbit)}$$

The first impulse is:

$$\Delta v_1 = v_{t1} - v_1 = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right)$$

At r_2 (apoapsis of transfer orbit):

$$v_2 = \sqrt{\frac{GM}{r_2}} \quad \text{(circular orbit)}$$

$$v_{t2} = \sqrt{GM\left(\frac{2}{r_2} - \frac{1}{a_t}\right)} = \sqrt{\frac{GM}{r_2}\frac{2r_1}{r_1 + r_2}} \quad \text{(transfer orbit)}$$

The second impulse is:

$$\Delta v_2 = v_2 - v_{t2} = \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

The total Δv is:

$$\Delta v_{\text{total}} = \Delta v_1 + \Delta v_2$$

The transfer time is half the period of the transfer ellipse:

$$t_{\text{transfer}} = \pi\sqrt{\frac{a_t^3}{GM}} = \pi\sqrt{\frac{(r_1 + r_2)^3}{8GM}}$$

4.2 Bi-elliptic Transfer

For large radius ratios ($r_2/r_1 > 11.94$), a bi-elliptic transfer can be more efficient than a Hohmann transfer. This involves three impulses: transfer to an intermediate radius $r_b > r_2$, then to r_2, then circularization.

4.3 Gravity Assists

When a spacecraft performs a close flyby of a planet, it can gain (or lose) velocity relative to the Sun through the gravitational interaction. In the planet's reference frame, the spacecraft's speed is unchanged (elastic scattering), but the direction changes by angle 2δ, where:

$$\sin\delta = \frac{1}{1 + r_p v_\infty^2/(GM_p)}$$

Here r_p is the closest approach distance, v_∞ is the hyperbolic excess velocity, and M_p is the planet's mass.

5 Perturbations and Three-Body Effects

5.1 Perturbation Theory

Real celestial systems are not pure two-body problems. Planets perturb each other, the Sun is not perfectly spherical, general relativity effects exist, etc. Perturbation theory treats these as small corrections to Keplerian motion.

5.1.1 Gauss's Planetary Equations

The variation of orbital elements due to small perturbing forces can be computed using Gauss's equations. For a perturbing force with components (F_r, F_θ, F_h) in the radial, tangential, and normal directions:

$$\frac{da}{dt} = \frac{2a^2}{h}\left[eF_r \sin\theta + \frac{p}{r}F_\theta\right]$$

$$\frac{de}{dt} = \frac{1}{h}\left[pF_r \sin\theta + ((p + r)\cos\theta + re)F_\theta\right]$$

These equations allow tracking how orbital elements evolve under perturbations.

5.2 The Restricted Three-Body Problem

Consider a small mass m_3 moving under the gravitational influence of two larger masses m_1 and m_2 that orbit their common center of mass. If m_3 is sufficiently small, it doesn't affect m_1 and m_2.

5.2.1 Lagrange Points

There exist five equilibrium points in the rotating frame where the small mass experiences zero net force. Three are collinear (L_1, L_2, L_3) and two are triangular $(L_4, L_5$, forming equilateral triangles with m_1 and $m_2)$.

At L_4 and L_5, the equilibrium is stable for mass ratio $m_2/m_1 < 0.04$ (approximately). This explains the existence of Trojan asteroids sharing Jupiter's orbit.

5.3 Sphere of Influence

The sphere of influence (SOI) is the region around a planet where its gravity dominates over the Sun's. The radius is approximately:

$$r_{\text{SOI}} \approx a_p \left(\frac{m_p}{M_\odot}\right)^{2/5}$$

where a_p is the planet's semi-major axis. Within this sphere, the planet-spacecraft two-body problem is a good approximation.

6 Observational Aspects

6.1 Radial Velocity Method

When a planet orbits a star, the star also orbits the common center of mass. This produces a periodic Doppler shift in the star's spectrum. For a circular orbit:

$$v_r = \frac{2\pi a_* \sin i}{T}$$

where a_* is the star's orbital semi-radius and i is the inclination. Using $m_p a_p = m_* a_*$ and $a = a_p + a_*$:

$$v_r = \frac{2\pi}{T} \frac{m_p \sin i}{m_* + m_p} a$$

For $m_* \gg m_p$:

$$v_r \approx \frac{2\pi}{T} \frac{m_p \sin i}{m_*} a = \frac{m_p \sin i}{m_*} \sqrt{\frac{GM_*}{a}}$$

This allows determining $m_p \sin i$ but not m_p or i separately.

For eccentric orbits, the radial velocity curve is:

$$v_r(t) = K[\cos(\theta(t) + \omega) + e \cos \omega]$$

where $K = \frac{2\pi a \sin i}{T\sqrt{1-e^2}}$.

6.2 Transit Method

When a planet passes in front of its star, the star's brightness decreases by:

$$\frac{\Delta F}{F} = \left(\frac{R_p}{R_*}\right)^2$$

The transit duration for a central transit (impact parameter $b = 0$) is approximately:

$$t_{\text{transit}} = \frac{T}{\pi} \arcsin\left(\frac{R_*}{a}\right) \approx \frac{R_* T}{\pi a}$$

For non-central transits with impact parameter b:

$$t_{\text{transit}} = \frac{T}{\pi} \arcsin\left(\frac{R_* \sqrt{1 + b^2}}{a}\right)$$

6.3 Binary Stars

For a visual binary with separation ρ (in arcseconds) at distance d (in parsecs), the physical separation is:

$$a_{\text{AU}} = \rho_{\text{arcsec}} \cdot d_{\text{pc}}$$

Combined with the period, Kepler's third law gives:

$$M_1 + M_2 = \frac{a^3}{T^2}$$

(in solar masses, AU, and years). If the individual orbits can be resolved:

$$\frac{M_1}{M_2} = \frac{a_2}{a_1}$$

7 Advanced Topics

7.1 Orbital Precession

Several effects cause the periapsis of an orbit to precess (rotate):

7.1.1 Oblateness (J2) Perturbation

A planet's equatorial bulge causes:

$$\dot{\omega} = \frac{3nJ_2R^2}{2a^2(1-e^2)^2}\left(\frac{5\cos^2 i - 1}{2}\right)$$

where J_2 is the quadrupole moment coefficient, R is the planet's radius, and i is the inclination.

7.1.2 General Relativistic Precession

General relativity predicts perihelion advance:

$$\dot{\omega}_{\mathrm{GR}} = \frac{6\pi GM}{c^2a(1-e^2)T}$$

For Mercury, this amounts to 43 arcseconds per century, famously confirming Einstein's theory.

7.2 Tidal Effects

Tidal forces transfer angular momentum between a planet and its satellite, causing orbital evolution. For a circular orbit, the semi-major axis changes as:

$$\frac{da}{dt} = \frac{3k_2Q}{2}\frac{m_s}{m_p}\left(\frac{R_p}{a}\right)^5 na$$

where k_2 is the Love number, Q is the tidal quality factor, and n is the mean motion. This causes the Moon to recede from Earth at ~ 3.8 cm/year.

7.3 Hyperbolic Orbits

For $e > 1$ and $\mathcal{E} > 0$, the orbit is hyperbolic. The asymptotic velocity (at $r \to \infty$) is:

$$v_\infty = \sqrt{\frac{2\mathcal{E}}{1}} = \sqrt{\frac{GM}{|a|}}$$

The scattering angle (angle between asymptotes) is:

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right)$$

For a hyperbola, the analogue of Kepler's equation is:

$$M = e \sinh H - H$$

where H is the hyperbolic anomaly, related to the true anomaly by:

$$\tanh \frac{H}{2} = \sqrt{\frac{e-1}{e+1}} \tan \frac{\theta}{2}$$

8 Practice Problems

1. **Advanced Orbital Transfer Problem**

   A spacecraft is in a circular orbit around Earth at altitude $h_1 = 400$ km. Mission control wants to transfer it to a circular orbit at altitude $h_2 = 35,786$ km (geostationary altitude) using a bi-elliptic transfer with an intermediate apoapsis at $h_b = 100,000$ km.

   (a) Calculate the three Δv burns required for this transfer.

   (b) Calculate the total transfer time.

   (c) Compare the total Δv with that of a Hohmann transfer to the same final orbit. Which is more efficient?

   (d) If the spacecraft has initial mass $m_0 = 2000$ kg and the specific impulse of its engine is $I_{sp} = 300$ s, what is the final mass after completing the transfer?

   Use: $R_\oplus = 6371$ km, $g_0 = 9.81$ m/s^2, $GM_\oplus = 3.986 \times 10^5$ km^3/s^2.

2. **Binary Star System Analysis**

   Two stars with masses $M_1 = 2.5M_\odot$ and $M_2 = 1.8M_\odot$ orbit their common center of mass with period $T = 5.2$ years. The orbit has eccentricity $e = 0.42$.

   (a) Calculate the semi-major axis of the relative orbit and the semi-major axes of each star's orbit around the center of mass.

   (b) At what times during the orbit (expressed as fractions of the period after periapsis passage) are the stars separated by exactly $1.5a$?

   (c) The system is observed from Earth edge-on (inclination $i = 90$). Calculate the maximum radial velocity of each star.

   (d) Sketch the radial velocity curves for both stars over one complete orbit, clearly labeling key features.

   Use: $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $M_\odot = 1.989 \times 10^{30}$ kg, 1 AU $= 1.496 \times 10^{11}$ m.

3. **Exoplanet Transit and Radial Velocity**

   An exoplanet is detected orbiting a Sun-like star ($M_* = M_\odot$, $R_* = R_\odot$) with period $T = 8.4$ days. Transit observations show: Transit depth: $\Delta F/F = 0.0121$ - Transit duration: $t_{transit} = 3.2$ hours - The transit is observed to be symmetric (central transit)

   Radial velocity measurements of the star show a semi-amplitude of $K = 42.3$ m/s with the same period.

(a) Determine the planet's radius R_p in units of Jupiter radii.

(b) Calculate the semi-major axis of the orbit.

(c) From the transit duration and assuming a circular orbit, verify the semi-major axis calculation and determine if the orbit is indeed circular or if there might be eccentricity.

(d) Using the radial velocity amplitude, calculate the planet's mass m_p (assuming edge-on orbit, $i = 90$). Express your answer in Jupiter masses.

(e) Calculate the planet's mean density and compare it to Jupiter's density ($\rho_J = 1326$ kg/m^3). What type of planet is this likely to be?

(f) If the orbital inclination were actually $i = 87$ instead of 90, how would this affect your mass determination? Calculate the corrected mass.

Use: $M_\odot = 1.989 \times 10^{30}$ kg, $R_\odot = 6.96 \times 10^8$ m, $R_J = 7.15 \times 10^7$ m, $M_J = 1.898 \times 10^{27}$ kg, $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

9 Solutions to Practice Problems

Problem 1: Bi-elliptic Transfer Solution

(a) Calculating the three Δv burns:

First, we establish the relevant radii:

$$r_1 = R_\oplus + h_1 = 6371 + 400 = 6771 \text{ km}$$

$$r_2 = R_\oplus + h_2 = 6371 + 35786 = 42157 \text{ km}$$

$$r_b = R_\oplus + h_b = 6371 + 100000 = 106371 \text{ km}$$

First burn (Δv_1): Transfer from circular orbit at r_1 to elliptical orbit with periapsis r_1 and apoapsis r_b.

Initial circular velocity:

$$v_1 = \sqrt{\frac{GM_\oplus}{r_1}} = \sqrt{\frac{3.986 \times 10^5}{6771}} = 7.669 \text{ km/s}$$

Transfer ellipse 1 has semi-major axis:

$$a_{t1} = \frac{r_1 + r_b}{2} = \frac{6771 + 106371}{2} = 56571 \text{ km}$$

Velocity at periapsis of transfer ellipse 1:

$$v_{t1,p} = \sqrt{GM_\oplus \left(\frac{2}{r_1} - \frac{1}{a_{t1}}\right)} = \sqrt{3.986 \times 10^5 \left(\frac{2}{6771} - \frac{1}{56571}\right)}$$

$$= \sqrt{3.986 \times 10^5 \times 0.2777} = 10.53 \text{ km/s}$$

First burn:

$$\Delta v_1 = v_{t1,p} - v_1 = 10.53 - 7.669 = 2.861 \text{ km/s}$$

Second burn (Δv_2): At apoapsis of first transfer ellipse (radius r_b), transfer to second ellipse with apoapsis r_b and periapsis r_2.

Velocity at apoapsis of transfer ellipse 1:

$$v_{t1,a} = \sqrt{GM_\oplus \left(\frac{2}{r_b} - \frac{1}{a_{t1}}\right)} = \sqrt{3.986 \times 10^5 \left(\frac{2}{106371} - \frac{1}{56571}\right)}$$

$$= \sqrt{3.986 \times 10^5 \times 0.01118} = 2.110 \text{ km/s}$$

Transfer ellipse 2 has semi-major axis:

$$a_{t2} = \frac{r_2 + r_b}{2} = \frac{42157 + 106371}{2} = 74264 \text{ km}$$

Velocity at apoapsis of transfer ellipse 2:

$$v_{t2,a} = \sqrt{GM_\oplus \left(\frac{2}{r_b} - \frac{1}{a_{t2}}\right)} = \sqrt{3.986 \times 10^5 \left(\frac{2}{106371} - \frac{1}{74264}\right)}$$

$$= \sqrt{3.986 \times 10^5 \times 0.005336} = 1.458 \text{ km/s}$$

Second burn:

$$\Delta v_2 = |v_{t2,a} - v_{t1,a}| = |1.458 - 2.110| = 0.652 \text{ km/s}$$

Third burn (Δv_3): Circularize at radius r_2.

Final circular velocity:

$$v_2 = \sqrt{\frac{GM_\oplus}{r_2}} = \sqrt{\frac{3.986 \times 10^5}{42157}} = 3.074 \text{ km/s}$$

Velocity at periapsis of transfer ellipse 2:

$$v_{t2,p} = \sqrt{GM_\oplus \left(\frac{2}{r_2} - \frac{1}{a_{t2}}\right)} = \sqrt{3.986 \times 10^5 \left(\frac{2}{42157} - \frac{1}{74264}\right)}$$

$$= \sqrt{3.986 \times 10^5 \times 0.01368} = 2.335 \text{ km/s}$$

Third burn:

$$\Delta v_3 = v_2 - v_{t2,p} = 3.074 - 2.335 = 0.739 \text{ km/s}$$

Total Δv for bi-elliptic transfer:

$$\Delta v_{total,BE} = 2.861 + 0.652 + 0.739 = 4.252 \text{ km/s}$$

(b) Total transfer time:

Time for first transfer ellipse (half period):

$$t_1 = \pi\sqrt{\frac{a_{t1}^3}{GM_\oplus}} = \pi\sqrt{\frac{(56571)^3}{3.986 \times 10^5}} = 21278 \text{ s} = 5.91 \text{ hours}$$

Time for second transfer ellipse (half period):

$$t_2 = \pi\sqrt{\frac{a_{t2}^3}{GM_\oplus}} = \pi\sqrt{\frac{(74264)^3}{3.986 \times 10^5}} = 31783 \text{ s} = 8.83 \text{ hours}$$

Total transfer time:

$$t_{total} = t_1 + t_2 = 53061 \text{ s} = 14.74 \text{ hours}$$

(c) Comparison with Hohmann transfer:
For Hohmann transfer, the semi-major axis is:

$$a_H = \frac{r_1 + r_2}{2} = \frac{6771 + 42157}{2} = 24464 \text{ km}$$

First burn:

$$v_{H,1} = \sqrt{GM_\oplus\left(\frac{2}{r_1} - \frac{1}{a_H}\right)} = \sqrt{3.986 \times 10^5\left(\frac{2}{6771} - \frac{1}{24464}\right)} = 10.15 \text{ km/s}$$

$$\Delta v_{H,1} = 10.15 - 7.669 = 2.481 \text{ km/s}$$

Second burn:

$$v_{H,2} = \sqrt{GM_\oplus\left(\frac{2}{r_2} - \frac{1}{a_H}\right)} = \sqrt{3.986 \times 10^5\left(\frac{2}{42157} - \frac{1}{24464}\right)} = 1.597 \text{ km/s}$$

$$\Delta v_{H,2} = 3.074 - 1.597 = 1.477 \text{ km/s}$$

Total Hohmann Δv:

$$\Delta v_{total,H} = 2.481 + 1.477 = 3.958 \text{ km/s}$$

The Hohmann transfer is more efficient! The bi-elliptic transfer requires $4.252 - 3.958 = 0.294$ km/s more Δv.

This makes sense because the ratio $r_2/r_1 = 42157/6771 = 6.23 < 11.94$, so we're below the threshold where bi-elliptic becomes advantageous.

(d) Final mass after transfer:
Using the Tsiolkovsky rocket equation:

$$\Delta v = I_{sp} g_0 \ln\left(\frac{m_0}{m_f}\right)$$

For the Hohmann transfer (which we'd actually use):

$$3958 = 300 \times 9.81 \times \ln\left(\frac{2000}{m_f}\right)$$

$$\ln\left(\frac{2000}{m_f}\right) = \frac{3958}{2943} = 1.345$$
$$\frac{2000}{m_f} = e^{1.345} = 3.839$$
$$m_f = \frac{2000}{3.839} = 521 \text{ kg}$$

The spacecraft would lose $2000 - 521 = 1479$ kg of propellant, or about 74% of its initial mass.

Problem 2: Binary Star System Solution

(a) Semi-major axes:

Using Kepler's third law for the relative orbit:

$$T^2 = \frac{4\pi^2}{G(M_1 + M_2)} a^3$$

First convert period to seconds: $T = 5.2 \text{ yr} = 5.2 \times 3.156 \times 10^7 = 1.641 \times 10^8$ s

$$a^3 = \frac{G(M_1 + M_2)T^2}{4\pi^2} = \frac{6.674 \times 10^{-11} \times 4.3 \times 1.989 \times 10^{30} \times (1.641 \times 10^8)^2}{4\pi^2}$$
$$= \frac{6.674 \times 10^{-11} \times 8.55 \times 10^{30} \times 2.693 \times 10^{16}}{39.48}$$
$$= 3.894 \times 10^{35} \text{ m}^3$$
$$a = 7.31 \times 10^{11} \text{ m} = 4.88 \text{ AU}$$

The individual semi-major axes are found from:

$$M_1 a_1 = M_2 a_2$$

$$a_1 + a_2 = a$$

Solving:
$$a_1 = \frac{M_2}{M_1 + M_2} a = \frac{1.8}{4.3} \times 4.88 = 2.04 \text{ AU}$$
$$a_2 = \frac{M_1}{M_1 + M_2} a = \frac{2.5}{4.3} \times 4.88 = 2.84 \text{ AU}$$

(b) Times when separation equals $1.5a$:

Using the orbit equation:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

We want $r = 1.5a$:

$$1.5a = \frac{a(1-e^2)}{1+e\cos\theta}$$

$$1.5(1+e\cos\theta) = 1-e^2$$

$$1.5 + 1.5e\cos\theta = 1-e^2$$

$$\cos\theta = \frac{1-e^2-1.5}{1.5e} = \frac{-0.5-e^2}{1.5e}$$

With $e = 0.42$:

$$\cos\theta = \frac{-0.5-0.1764}{1.5\times 0.42} = \frac{-0.6764}{0.63} = -1.074$$

Since $|\cos\theta| > 1$, there is no real solution. The stars are never separated by exactly $1.5a$.

Let's check the range of possible separations: - At periapsis: $r_p = a(1-e) = 4.88\times 0.58 = 2.83$ AU - At apoapsis: $r_a = a(1+e) = 4.88\times 1.42 = 6.93$ AU

Indeed, $1.5a = 7.32$ AU is outside the range $[2.83, 6.93]$ AU, confirming no solution exists.

For a physically meaningful answer, let's instead find when the separation is $1.2a = 5.86$ AU:

$$1.2a = \frac{a(1-e^2)}{1+e\cos\theta}$$

$$\cos\theta = \frac{1-e^2-1.2}{1.2e} = \frac{-0.3764}{0.504} = -0.747$$

$$\theta = \pm 138.3$$

Now convert to time using the eccentric anomaly relation:

$$\cos\theta = \frac{\cos E - e}{1-e\cos E}$$

$$-0.747 = \frac{\cos E - 0.42}{1-0.42\cos E}$$

$$-0.747(1-0.42\cos E) = \cos E - 0.42$$

$$-0.747 + 0.314\cos E = \cos E - 0.42$$

$$0.327 = 0.686\cos E$$

$$\cos E = 0.477$$

$$E = \pm 61.5 = \pm 1.073 \text{ rad}$$

Using Kepler's equation:

$$M = E - e\sin E = 1.073 - 0.42\sin(1.073) = 1.073 - 0.368 = 0.705 \text{ rad}$$

Time after periapsis:

$$t = \frac{MT}{2\pi} = \frac{0.705 \times 5.2}{2\pi} = 0.583 \text{ years} = 0.112T$$

By symmetry, the second occurrence is at $t = T - 0.583 = 4.617$ years $= 0.888T$.

(c) Maximum radial velocities:

For an eccentric orbit viewed edge-on, the radial velocity amplitude is:

$$K = \frac{2\pi a \sin i}{T\sqrt{1-e^2}}$$

For star 1:

$$K_1 = \frac{2\pi a_1}{T\sqrt{1-e^2}} = \frac{2\pi \times 2.04 \times 1.496 \times 10^{11}}{1.641 \times 10^8 \times \sqrt{1-0.42^2}}$$

$$= \frac{1.918 \times 10^{12}}{1.641 \times 10^8 \times 0.907} = \frac{1.918 \times 10^{12}}{1.489 \times 10^8} = 12,880 \text{ m/s} = 12.9 \text{ km/s}$$

For star 2:

$$K_2 = \frac{2\pi a_2}{T\sqrt{1-e^2}} = \frac{2.84}{2.04} \times K_1 = 1.392 \times 12.9 = 17.9 \text{ km/s}$$

The maximum radial velocities occur near periapsis (but not exactly at periapsis for eccentric orbits). The exact maxima are:

$$v_{r,max} = K(1 + e\cos\omega)$$

Assuming $\omega = 0$ (periapsis aligned with line of sight):

$$v_{1,max} = 12.9 \times 1.42 = 18.3 \text{ km/s}$$

$$v_{2,max} = 17.9 \times 1.42 = 25.4 \text{ km/s}$$

(d) Radial velocity curves:

The radial velocity as a function of true anomaly is:

$$v_r(\theta) = K[\cos(\theta + \omega) + e\cos\omega]$$

For $\omega = 0$:

$$v_r(\theta) = K[\cos\theta + e]$$

Key features: - At periapsis ($\theta = 0$): $v_r = K(1+e) =$ maximum - At $\theta = 90$: $v_r = Ke$ - At apoapsis ($\theta = 180$): $v_r = K(e-1) =$ minimum (negative, meaning toward us if periapsis was away) - The curves are phase-shifted by 180° (when one star approaches, the other recedes) - The curves are not sinusoidal due to eccentricity

The amplitude ratio is $K_2/K_1 = M_1/M_2 = 1.39$, reflecting the inverse mass ratio.

Problem 3: Exoplanet Characterization Solution

(a) Planet radius from transit depth:

The transit depth directly gives the radius ratio:

$$\frac{\Delta F}{F} = \left(\frac{R_p}{R_*}\right)^2 = 0.0121$$

$$\frac{R_p}{R_*} = \sqrt{0.0121} = 0.110$$

$$R_p = 0.110 R_\odot = 0.110 \times 6.96 \times 10^8 = 7.656 \times 10^7 \text{ m}$$

In Jupiter radii:

$$R_p = \frac{7.656 \times 10^7}{7.15 \times 10^7} = 1.07 R_J$$

(b) Semi-major axis from period:

Using Kepler's third law:

$$T^2 = \frac{4\pi^2}{GM_*} a^3$$

Convert period: $T = 8.4$ days $= 8.4 \times 86400 = 7.258 \times 10^5$ s

$$\begin{aligned} a^3 = \frac{GM_* T^2}{4\pi^2} &= \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30} \times (7.258 \times 10^5)^2}{4\pi^2} \\ &= \frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30} \times 5.268 \times 10^{11}}{39.48} \\ &= 1.77 \times 10^{30} \text{ m}^3 \end{aligned}$$

$$a = 1.209 \times 10^{10} \text{ m} = 0.0808 \text{ AU}$$

(c) Verification from transit duration:

For a central transit, the duration is approximately:

$$t_{transit} = \frac{T}{\pi} \arcsin\left(\frac{R_*}{a}\right)$$

But for small angles, $\arcsin(x) \approx x$:

$$t_{transit} \approx \frac{TR_*}{\pi a}$$

Rearranging:

$$\begin{aligned} a \approx \frac{TR_*}{\pi t_{transit}} &= \frac{7.258 \times 10^5 \times 6.96 \times 10^8}{\pi \times 3.2 \times 3600} \\ &= \frac{5.052 \times 10^{14}}{3.619 \times 10^4} = 1.396 \times 10^{10} \text{ m} \end{aligned}$$

This gives $a \approx 0.0933$ AU, which is about 15% larger than our Kepler's law calculation. This discrepancy suggests either: 1. The small angle approximation isn't quite valid 2. There may be some eccentricity

Using the exact formula:

$$\arcsin\left(\frac{R_*}{a}\right) = \frac{\pi t_{transit}}{T} = \frac{\pi \times 3.2 \times 3600}{7.258 \times 10^5} = 0.0498 \text{ rad}$$

$$\frac{R_*}{a} = \sin(0.0498) = 0.0497$$

$$a = \frac{R_*}{0.0497} = \frac{6.96 \times 10^8}{0.0497} = 1.40 \times 10^{10} \text{ m}$$

The discrepancy persists. For a more refined analysis, if we assume small eccentricity with periapsis at transit:

$$a_{true} \approx \frac{a_{Kepler} + a_{transit}}{2} \approx 1.30 \times 10^{10} \text{ m} = 0.087 \text{ AU}$$

For simplicity, we'll use the Kepler's law value: $a = 0.0808$ AU $= 1.21 \times 10^{10}$ m.

(d) Planet mass from radial velocity:

For a circular orbit with $i = 90$:

$$K = \frac{2\pi a \sin i}{T} \cdot \frac{m_p}{M_* + m_p} \approx \frac{2\pi a}{T} \cdot \frac{m_p}{M_*}$$

Solving for m_p:

$$m_p = \frac{KTM_*}{2\pi a} = \frac{42.3 \times 7.258 \times 10^5 \times 1.989 \times 10^{30}}{2\pi \times 1.209 \times 10^{10}}$$

$$= \frac{6.104 \times 10^{37}}{7.593 \times 10^{10}} = 8.04 \times 10^{26} \text{ kg}$$

In Jupiter masses:

$$m_p = \frac{8.04 \times 10^{26}}{1.898 \times 10^{27}} = 0.424 M_J$$

(e) Mean density:

Volume of planet:

$$V = \frac{4}{3}\pi R_p^3 = \frac{4}{3}\pi(7.656 \times 10^7)^3 = 1.88 \times 10^{24} \text{ m}^3$$

Density:

$$\rho_p = \frac{m_p}{V} = \frac{8.04 \times 10^{26}}{1.88 \times 10^{24}} = 428 \text{ kg/m}^3$$

Compared to Jupiter: $\rho_p/\rho_J = 428/1326 = 0.32$

This planet has only 32% of Jupiter's density despite having similar size. This is characteristic of a "hot Jupiter" or "puffy planet" - a gas giant that has been heated by its proximity to the star, causing its atmosphere to expand significantly. The low density indicates it's likely mostly hydrogen and helium with very little rocky core.

(f) Corrected mass for $i = 87$:

The radial velocity amplitude includes $\sin i$:

$$K = \frac{2\pi a}{T} \cdot \frac{m_p \sin i}{M_*}$$

Therefore:

$$m_p \sin i = \frac{KTM_*}{2\pi a}$$

For $i = 90$, we found $m_p \sin 90 = 0.424 M_J$.
For $i = 87$:

$$m_p = \frac{0.424}{\sin 87} = \frac{0.424}{0.9986} = 0.425 M_J$$

The difference is minimal (only 0.2% increase) because 87 is very close to edge-on. However, for significantly lower inclinations, the correction becomes substantial. For example, at $i = 60$:

$$m_p = \frac{0.424}{\sin 60} = \frac{0.424}{0.866} = 0.490 M_J$$

This illustrates why radial velocity measurements alone can only determine $m_p \sin i$ (the minimum mass), not the true mass. Transit observations are crucial because they confirm the orbital inclination is close to 90, allowing accurate mass determination.