2. Air Muscle

Bailin Wang | Team Canada

*“Place a **balloon** inside a **cylindrical net** (as is sometimes used to wrap garlic) and **inflate** it. The net will **expand and shorten**. Investigate the **properties** of such a “muscle”.”*

Problem Statement

"Place a balloon inside a cylindrical net (as is sometimes used to wrap garlic) and inflate it. The net will expand and shorten. Investigate the properties of such a "muscle"."

Relevant Parameters

- *Balloon internal pressure*
- *Force exerted by the weight*
- *Length of Netting*
- *Contraction Ratio*
- *Rate of Inflation/Deflation*

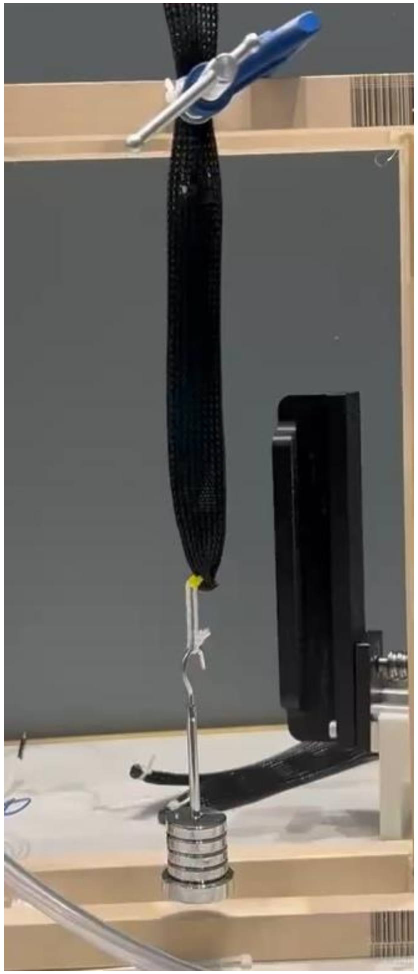
Muscle Properties

- *Force exerted (As well)*
- *Displacement*
- *Hysteresis*

Overview

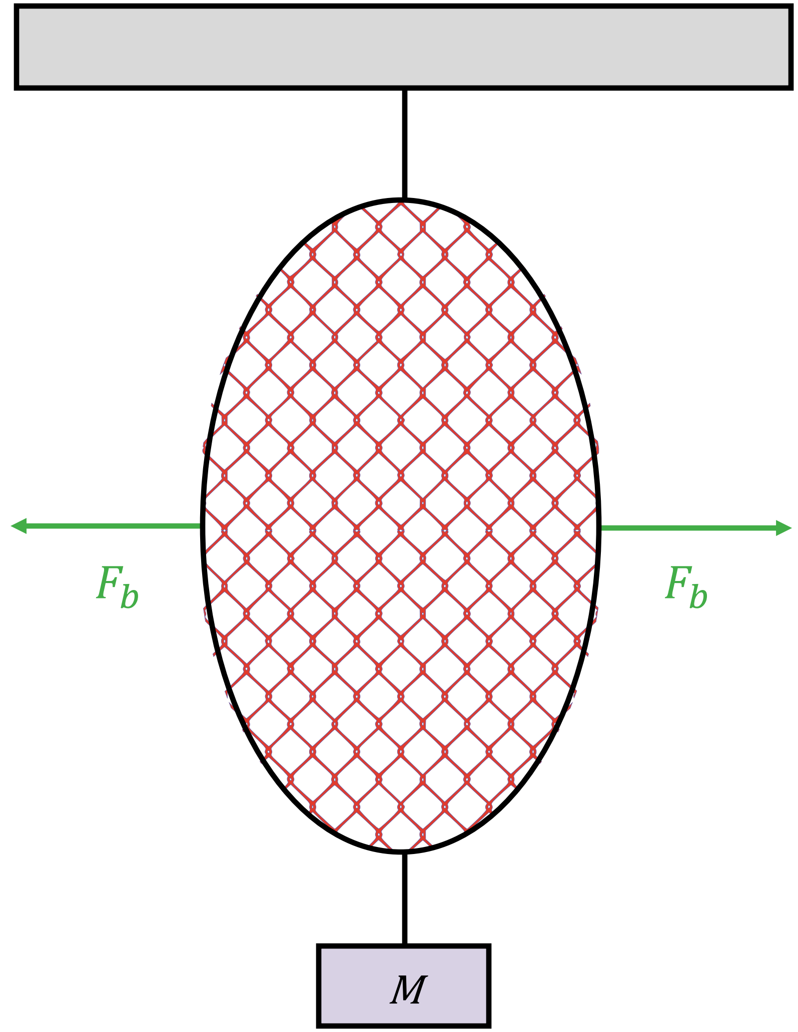
Phenomenon

Phenomenon Demonstration



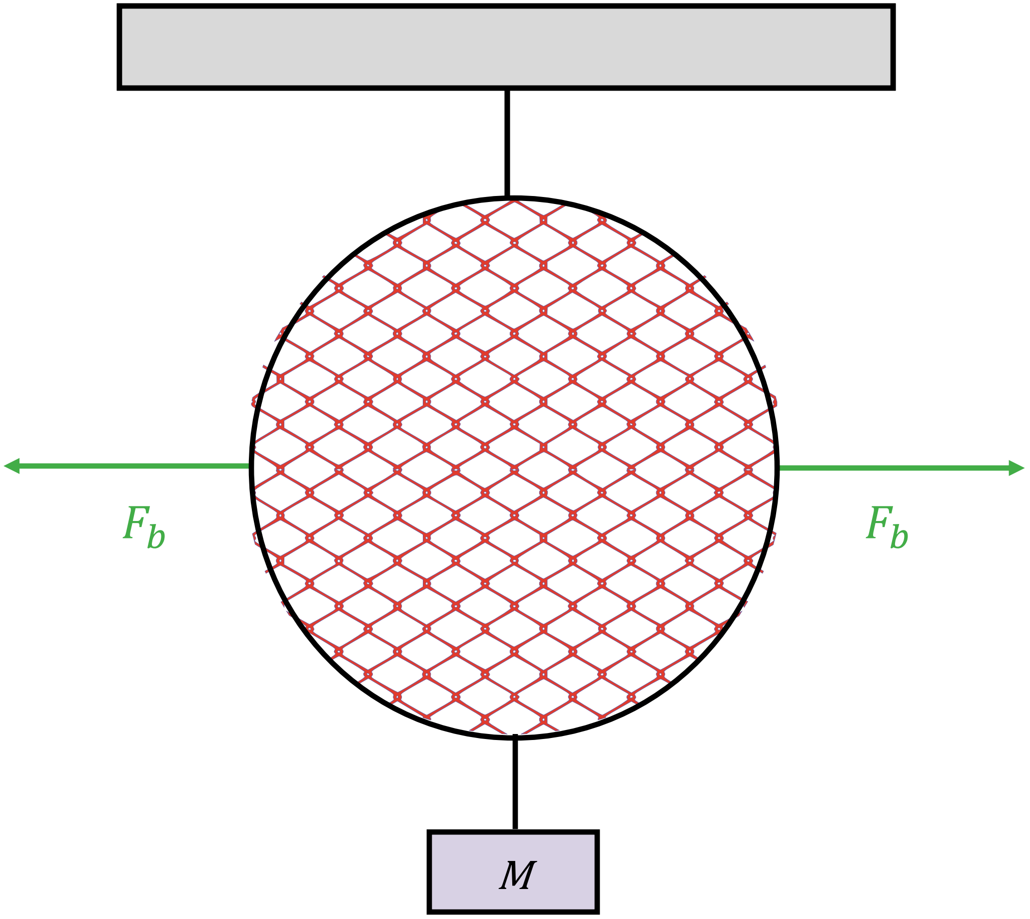
Net Contraction

Pressure from the balloon, can be understood as tension, within the net,



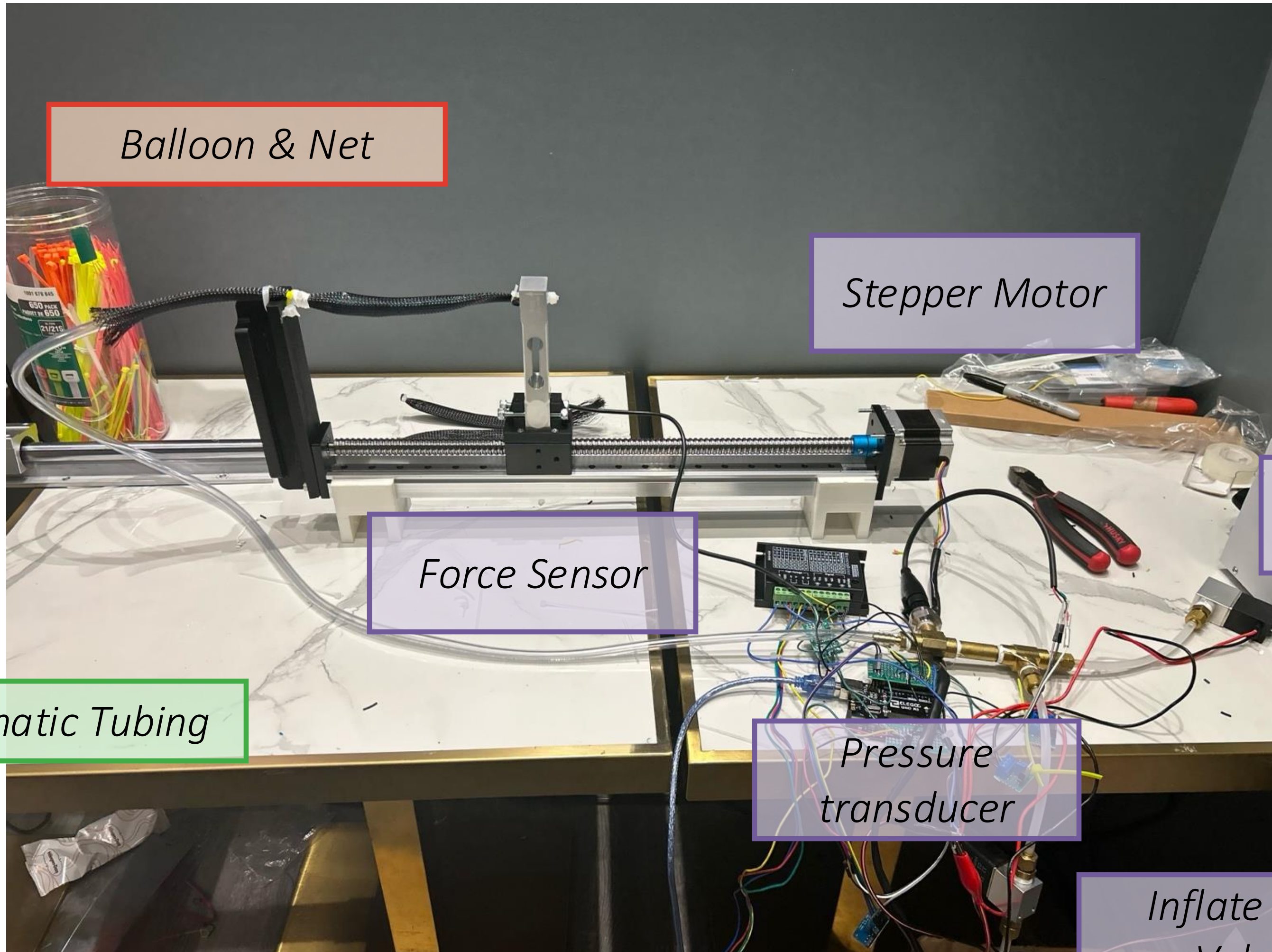
Net Contraction

Mass rises due to geometric constraint of net



Experimental Setup

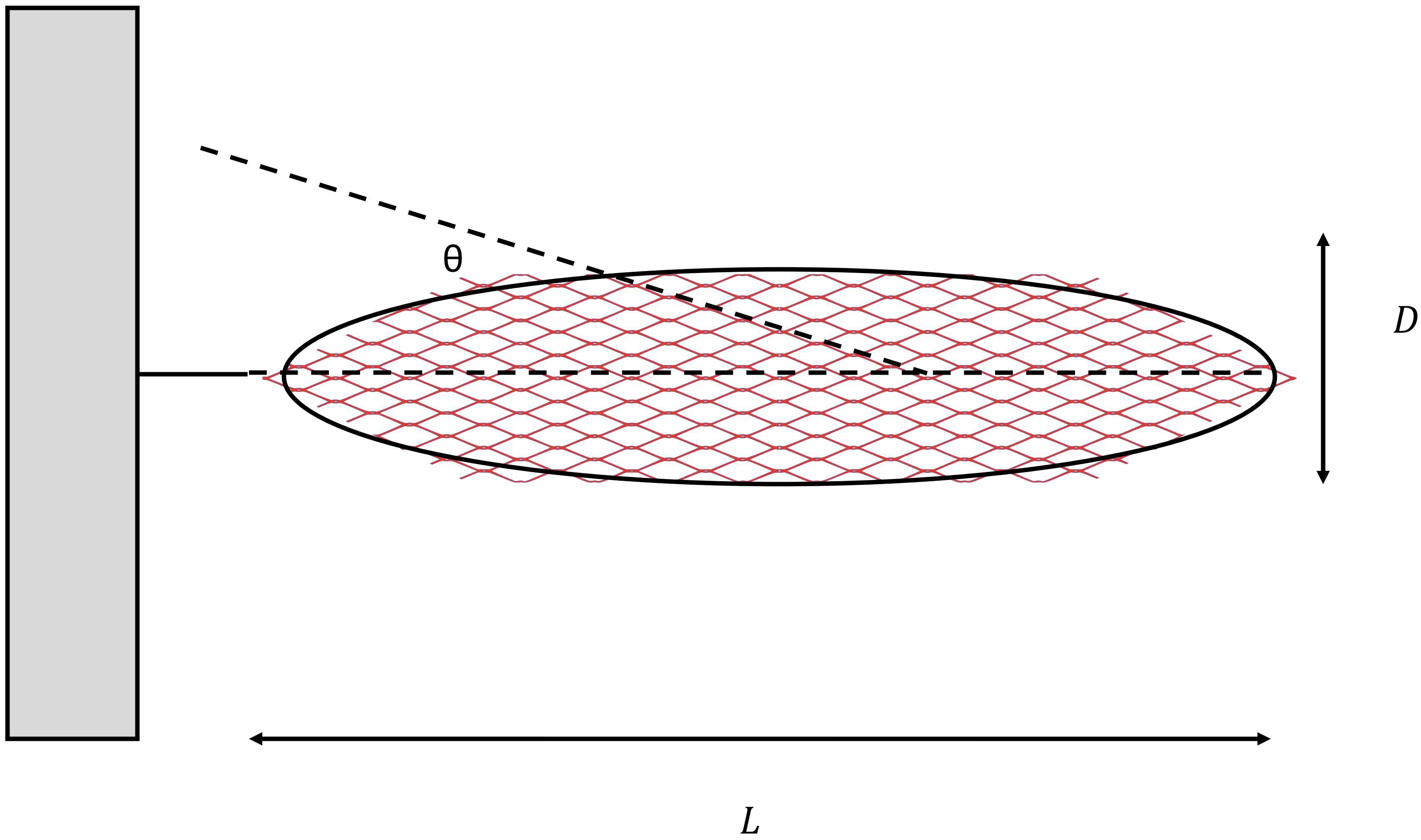
Experimental Setup



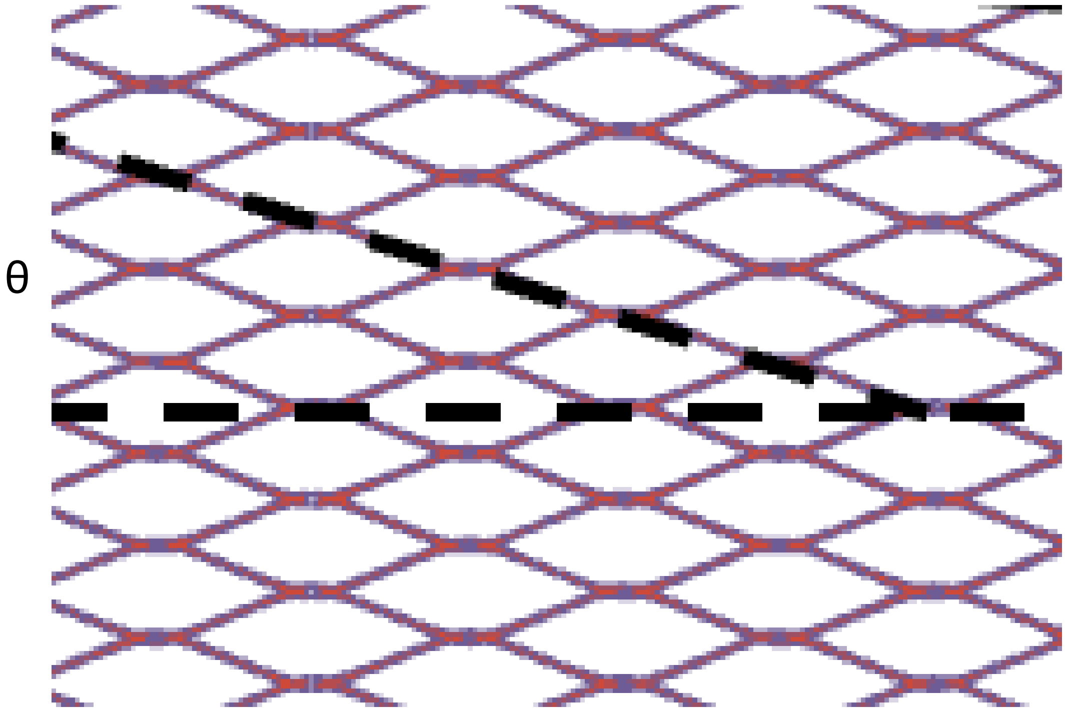
Force sensor Uncertainty:
$\pm 0.001N$

Pressure Transducer:
$\pm 0.001N$

Net Characterization



Net Characterization

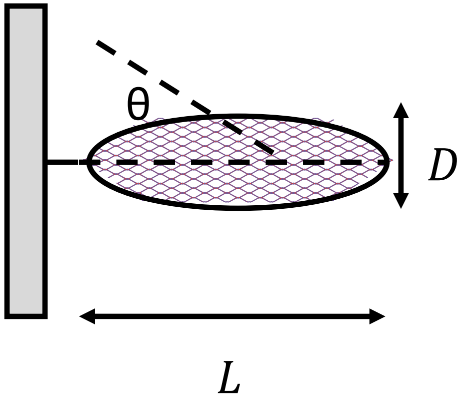


Net Characterization

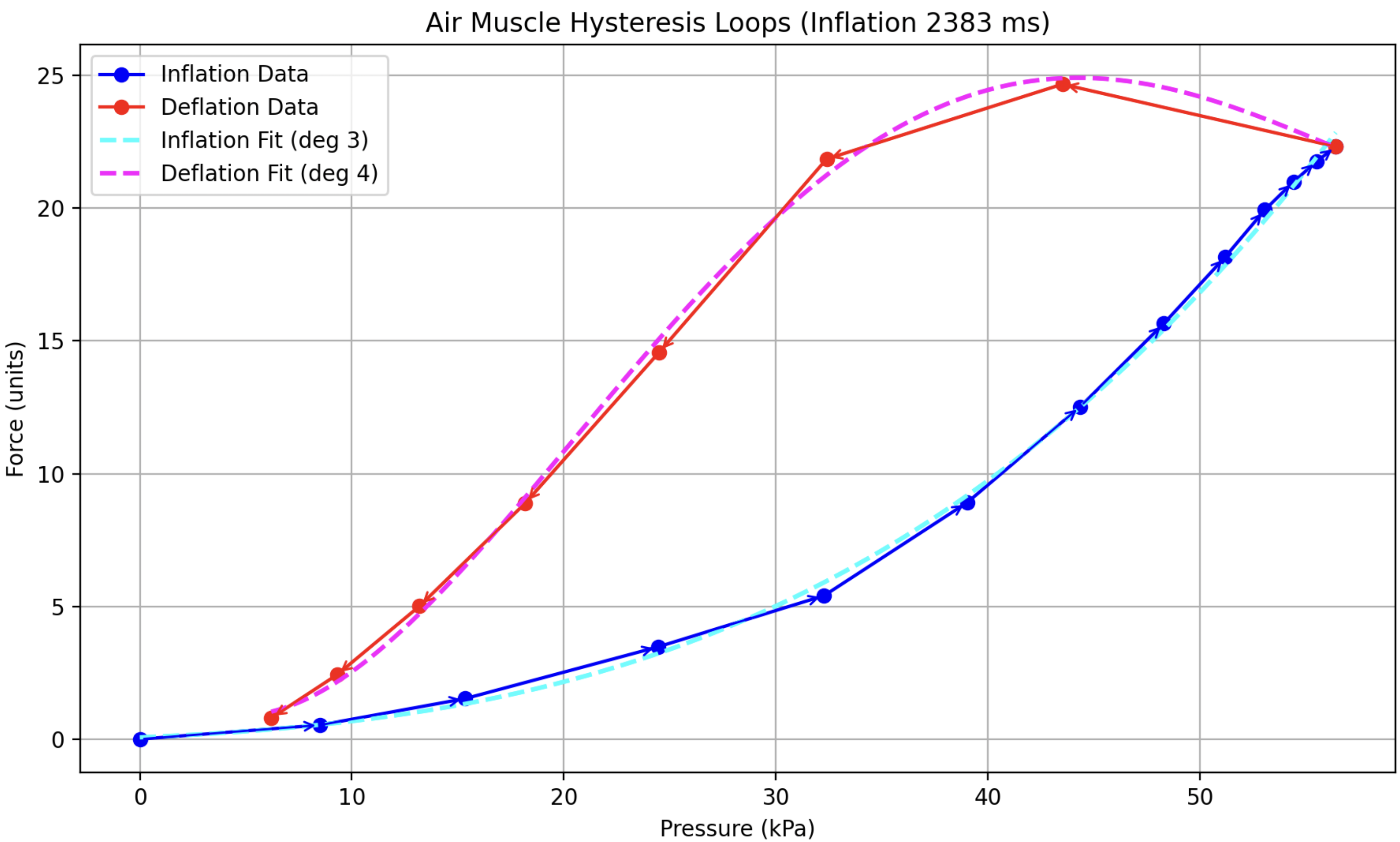
$L_0 = 15.5\ cm, 18\ cm, 20.5\ cm$

$Initial\ Braid\ Angle\text{: } \theta_0 = 20^\circ$

$Number\ of\ turns = L/8cm$



Hysteresis (Force netting)

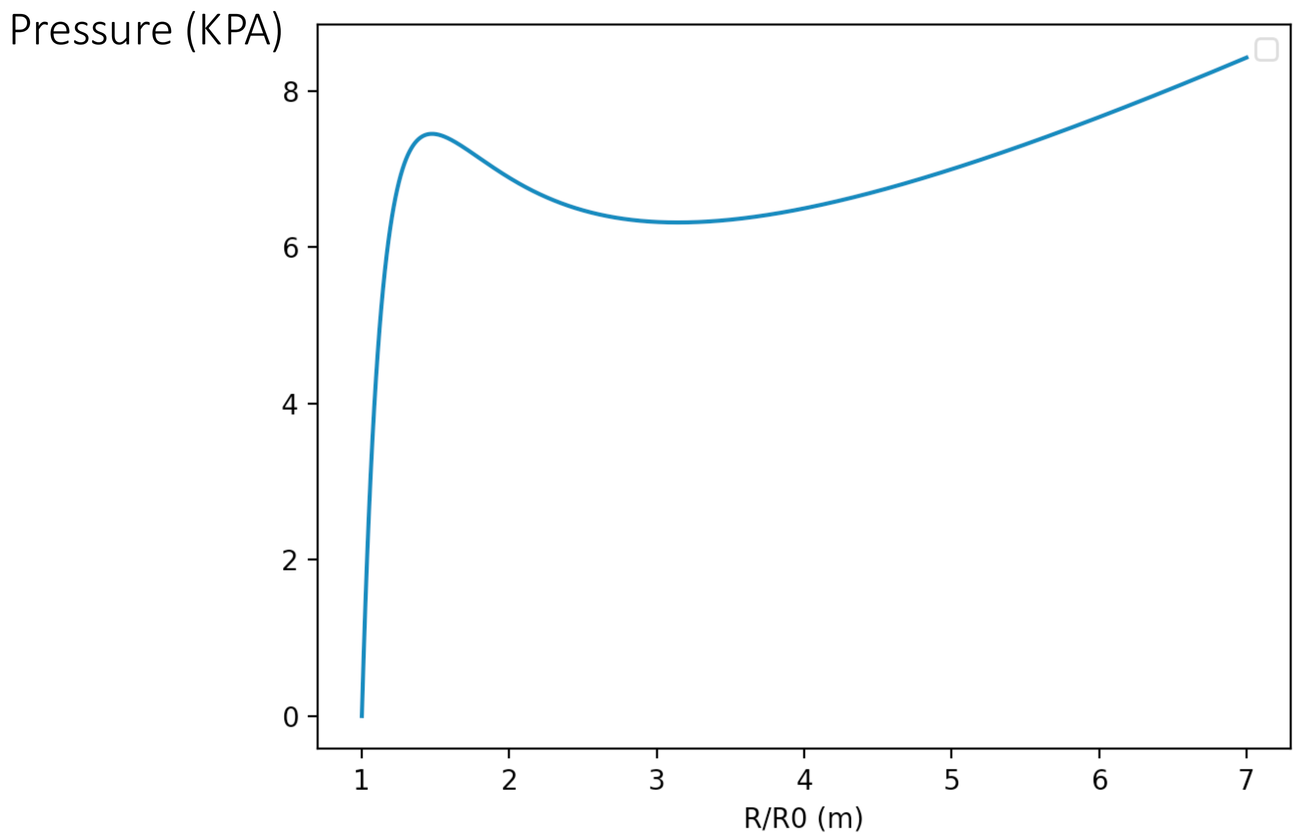


Quantitative

Assumptions

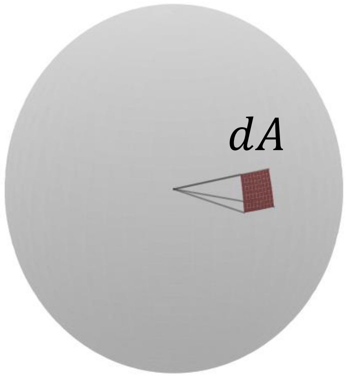
- Surface Energy Density is uniform on the surface of the balloon
- There is no leakage of air from the muscle
- Braids are inextensible

Characterizing Balloon Membrane (First Time Inflate)

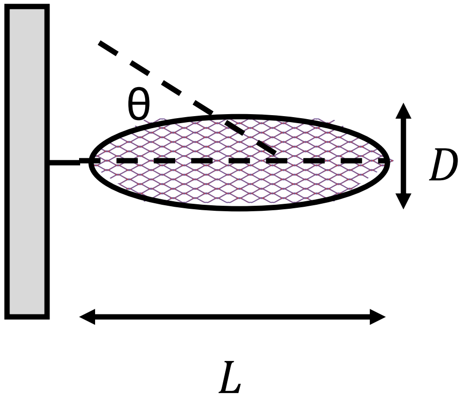


Pressure vs. Radius Curve

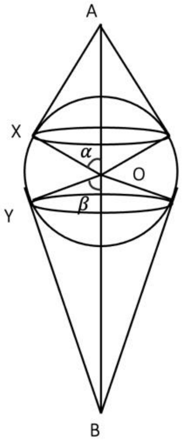
Theoretical Models



Balloon Characterization

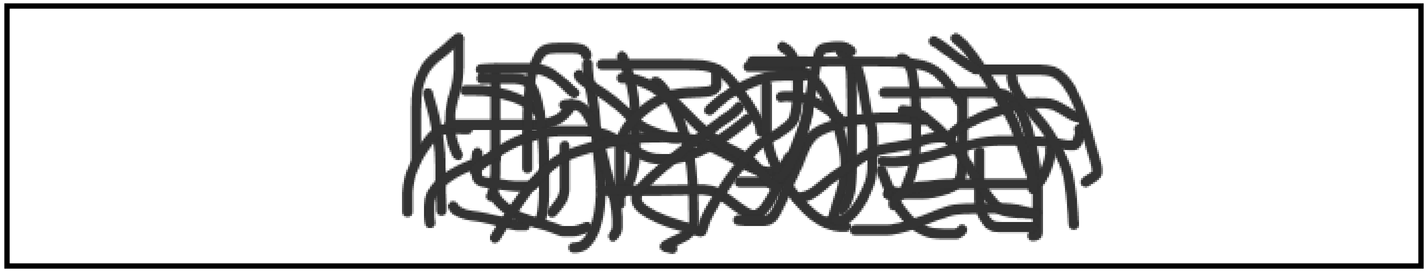


Virtual Work (Force Net)

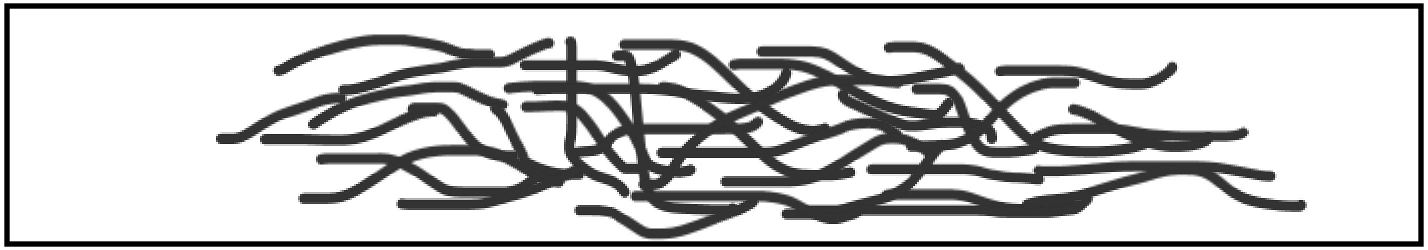


Energy (Contract Net)

Characterizing Balloon Membrane

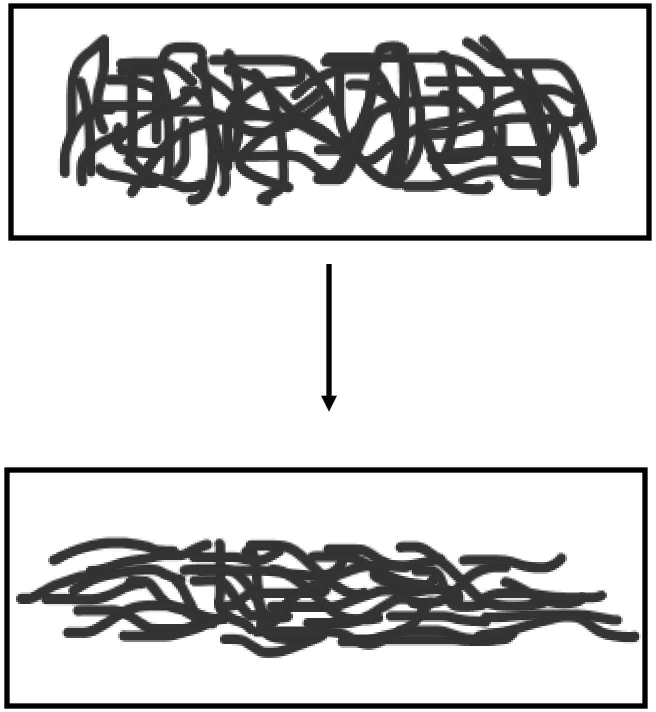


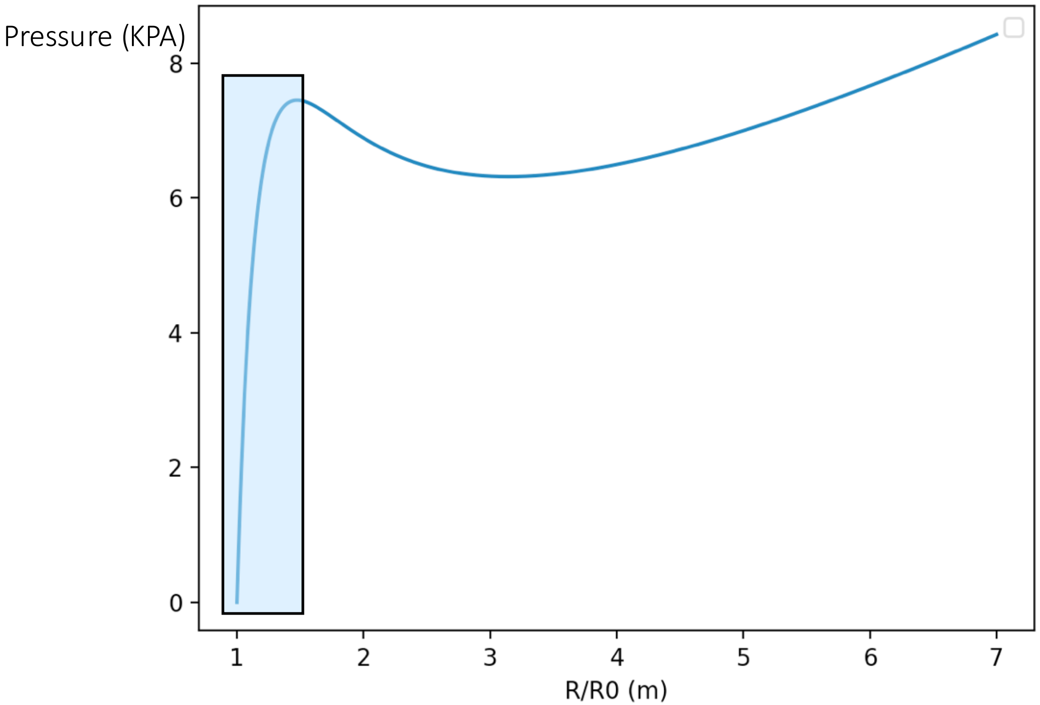
Chains of Polymer inside balloon at different elongations



Moonley-Rivlin Materials

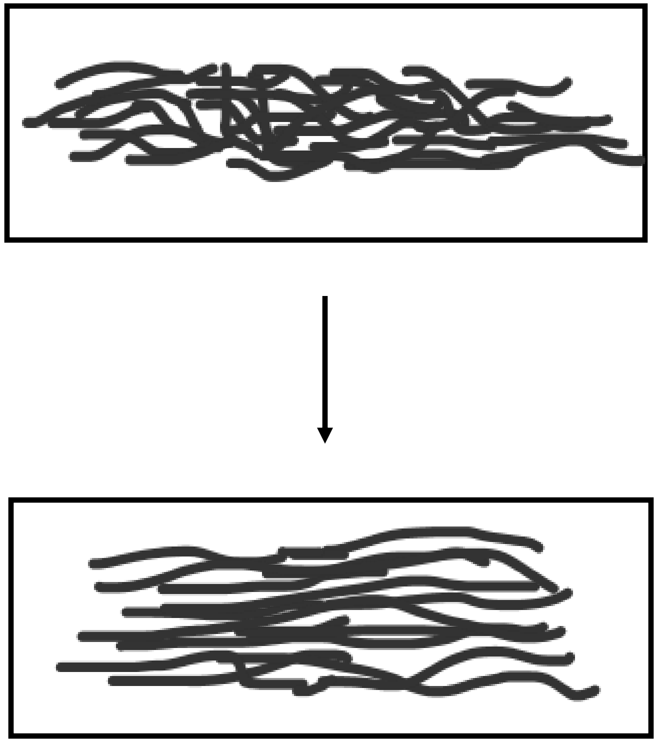
Chains of Polymer inside balloon (Stage 1) [4] for Moonley Rivlin Materials

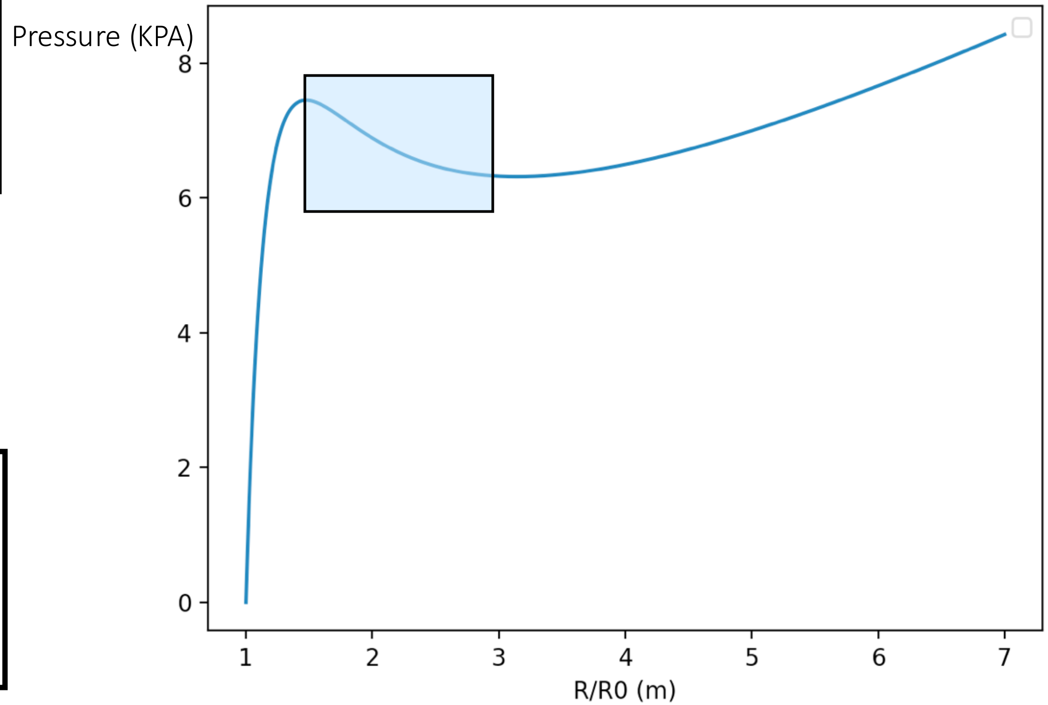




Moonley-Rivlin Materials

Chains of Polymer inside balloon (Stage 2) [4] for Moonley Rivlin Materials

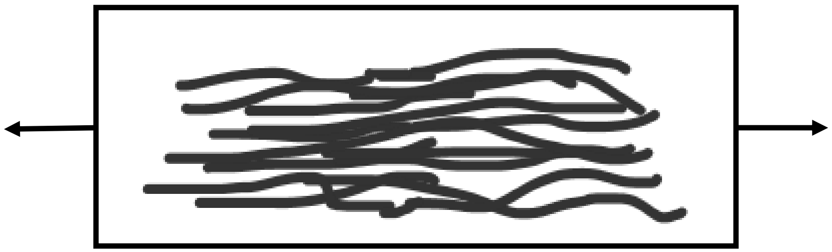


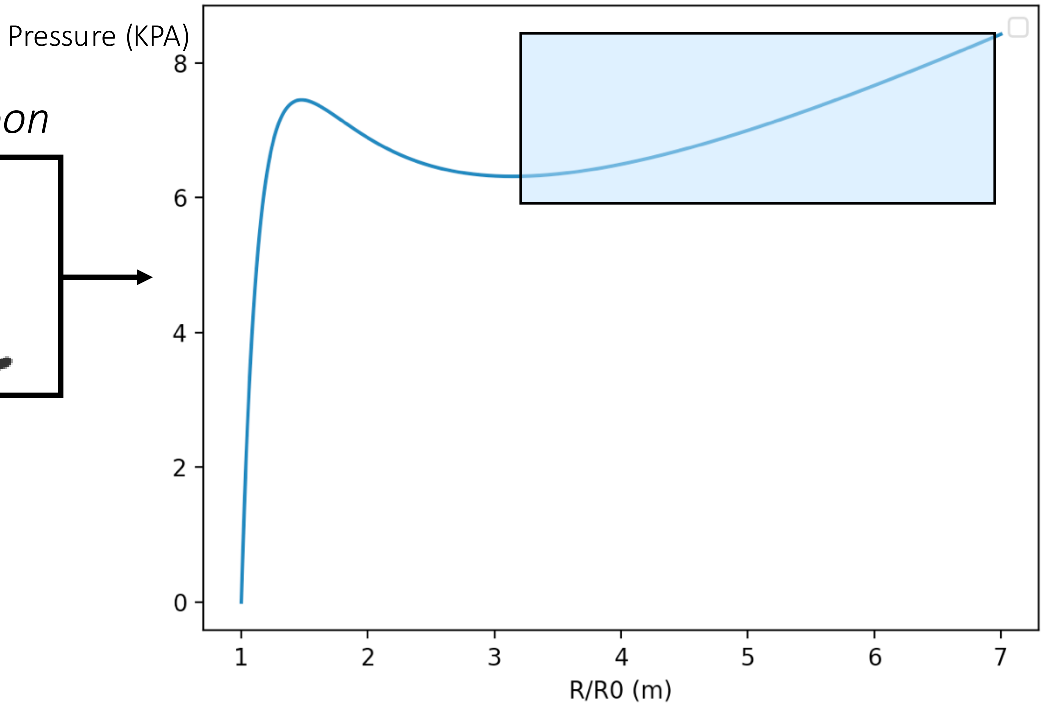


Moonley-Rivlin Materials

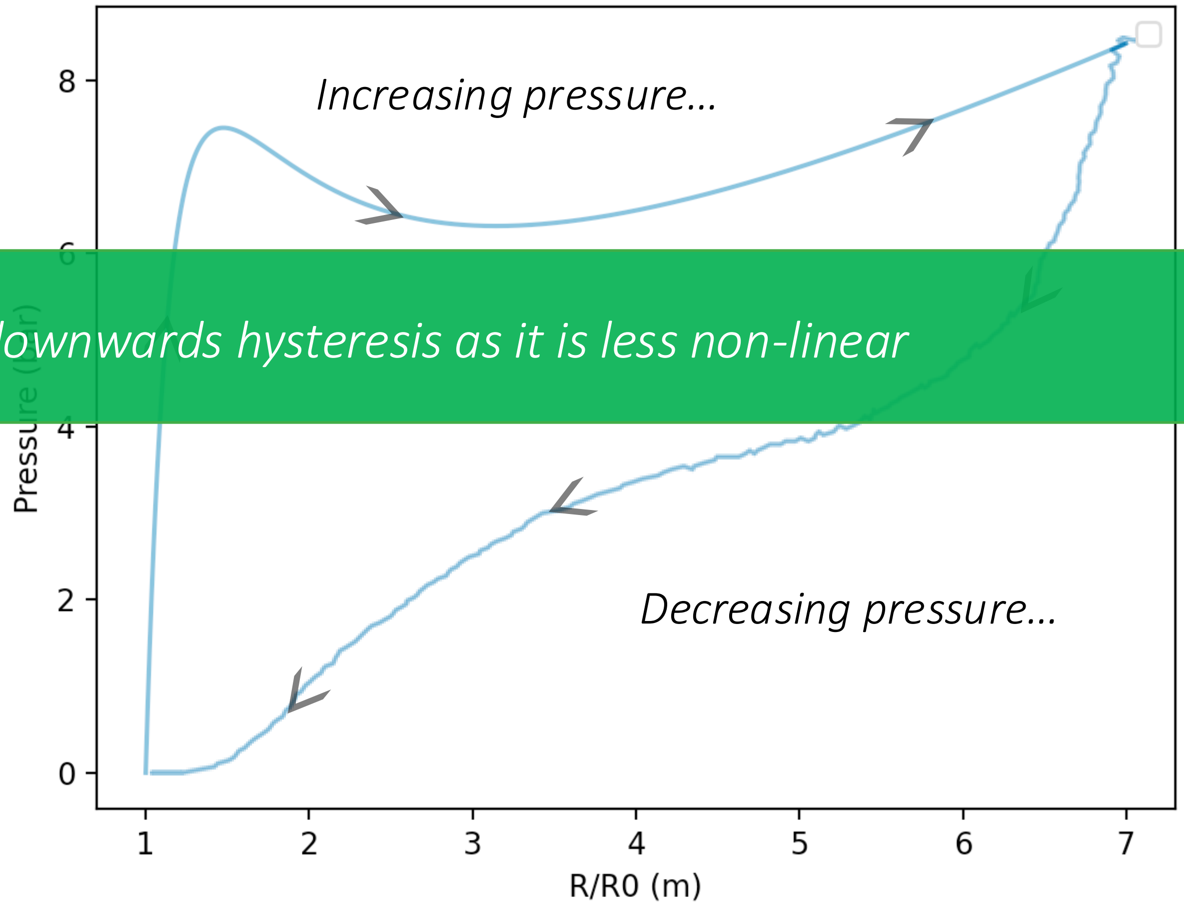
Chains of Polymer inside balloon (Stage 3) [4] for Moonley Rivlin Materials

Chains of Polymer inside balloon



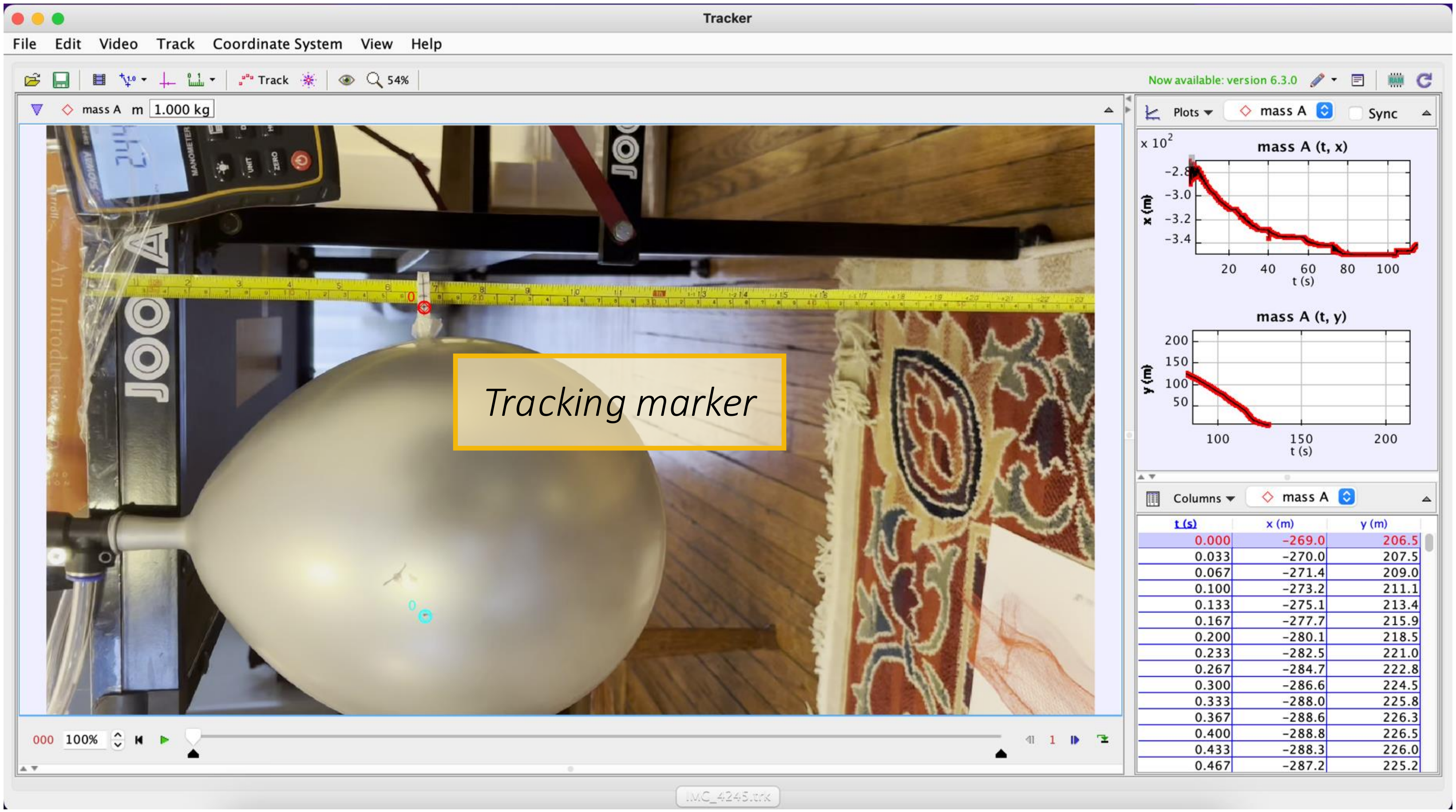


Hysteresis



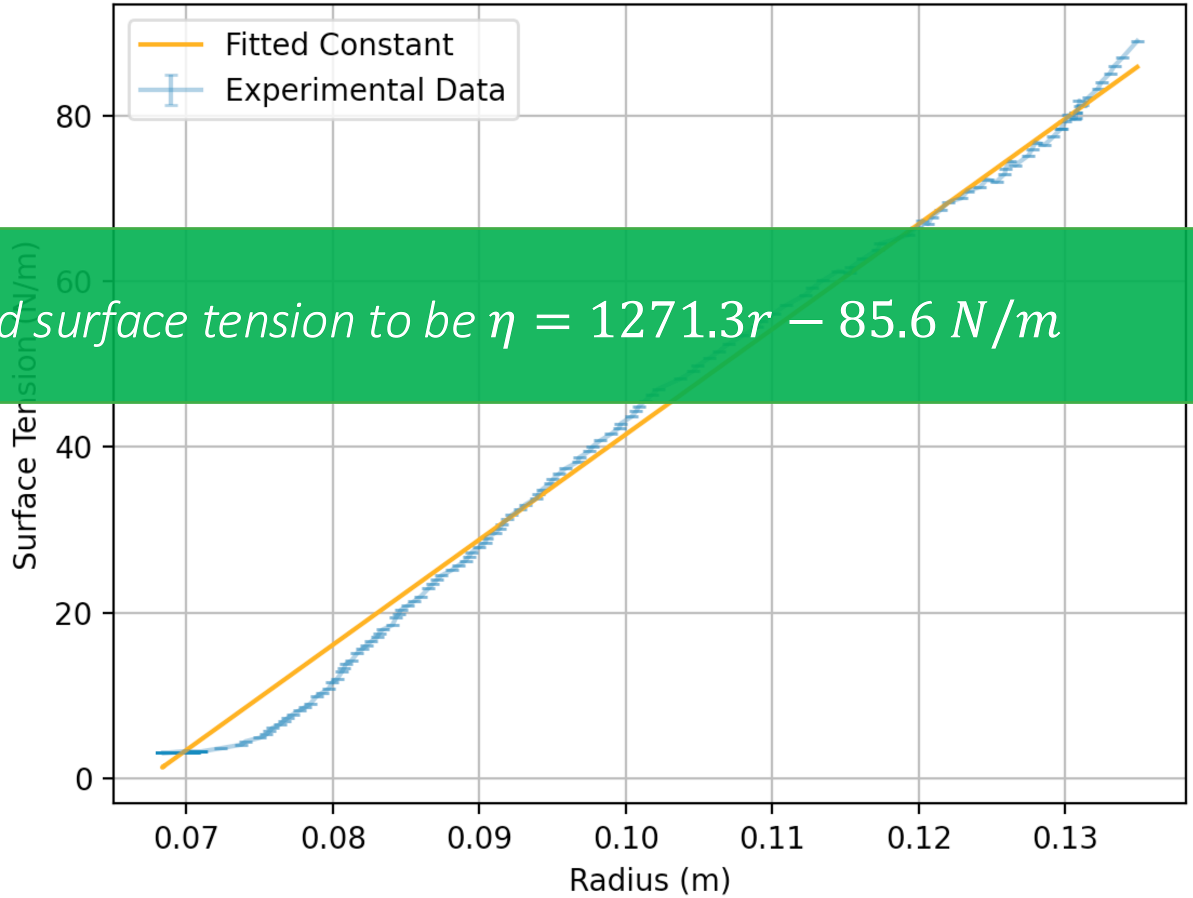
✓ *Use downwards hysteresis as it is less non-linear*

Tracking Balloon



Finding Surface Tension

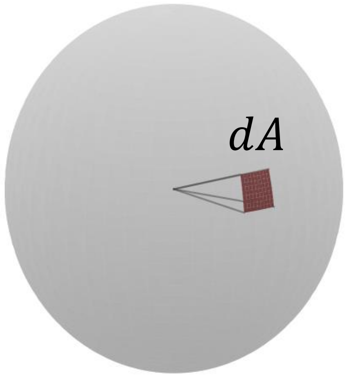
$$\gamma = \frac{PR}{2w}$$ | *Laplace Equation for Spheres*



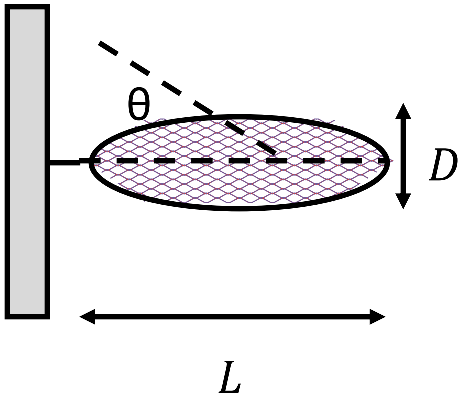
Find surface tension to be $\eta = 1271.3r - 85.6\ N/m$

Where
- γ: Membrane Tension
- P: Pressure
- w: Thickness of net

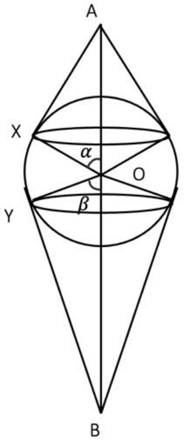
Theoretical Models



Balloon Characterization



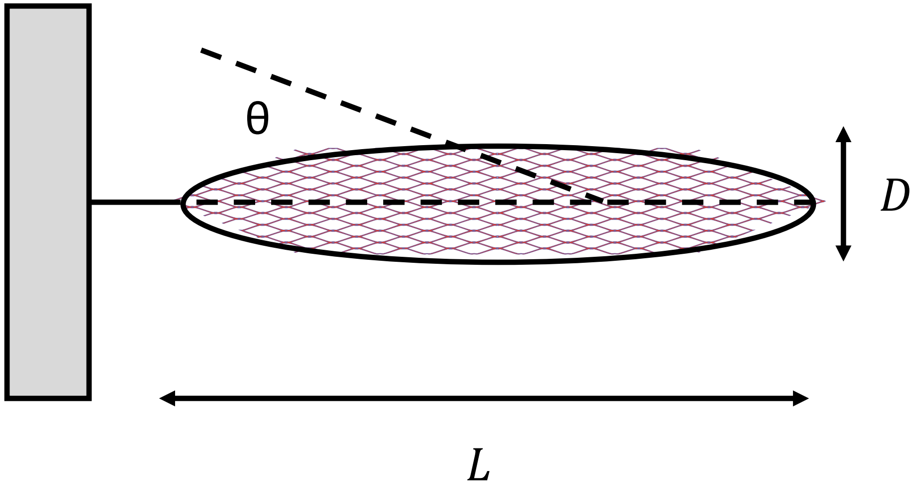
Virtual Work (Force Net)



Energy (Contract Net)

Geometry

Consider a netting, with length L, diameter D, braid angle θ, and number of turns N. The net is composed of inextensible fibers wound helically. Thus, the fiber length B remains constant and obeys:



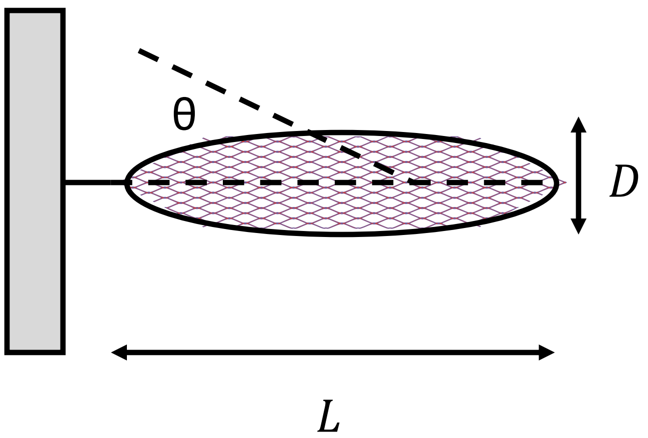
$$\boxed{B^2 = L^2 + (N\pi D)^2}$$

L: Axial length
B: Braid Length (if you unravel one of the braids)
D: Net diameter
– N: Number of complete turns

Geometry

We can define instantaneous “braid angle” θ, by

$$\cos\theta = \frac{L}{B} \qquad \sin\theta = \frac{N\pi D}{B}$$



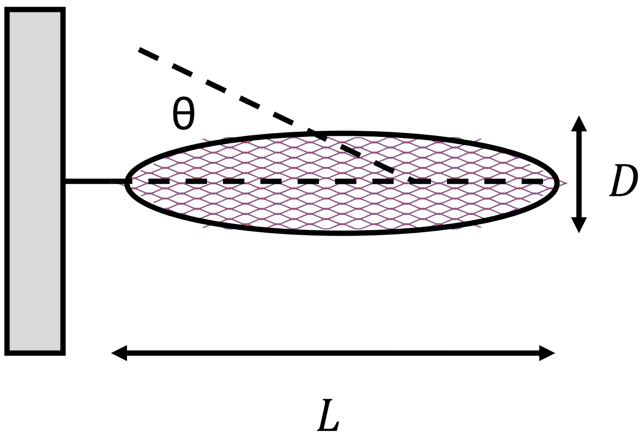
We can then define contraction ratio as:

$$\lambda = \frac{L}{L_0}$$

The relationship can be arranged to express $\cos\theta$ directly in terms of λ and θ_0

$$\cos\theta = \lambda\cos\theta_0$$

Geometry



This relationship can be arranged to express cos θ directly in terms of λ and θ_0:

$$\cos\theta = \lambda \cos\theta_0$$

Since the sine and cosine functions are related by $sin^2\theta + cos^2\theta = 1$, we can write:

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \lambda^2 \cos^2\theta_0}$$

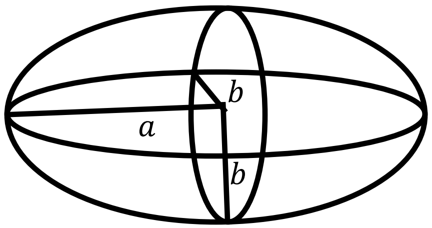
Thus, the complete expressions linking the instantaneous geometry to the contraction ratio are:

$$\boxed{L = B\lambda\cos\theta_0, \quad D = \frac{B}{N\pi}\sqrt{1 - \lambda^2\cos^2\theta_0}}$$

Geometry

The inflated balloon and net may be analyzed as a prolate spheroid, its volume is given by:

$$V = \frac{4}{3}\pi a b^2$$



Substitute the expressions for L and D in terms of θ (or equivalently in terms of λ):

$$V(\theta) = \frac{\pi}{6}(B\cos\ \theta)(\frac{B}{N\pi}\sin\theta)^2 = \frac{B^3}{6N^2\pi}\cos\theta\, sin^2\theta$$

Using algebra, we obtain the dimensionless volume as:

$$\boxed{\bar{V} = \frac{V(\lambda)}{V_0} = \frac{\lambda(1-\lambda^2 cos^2\theta_0)}{sin^2\theta_0}}$$

Virtual Work

By the Work Energy principle :

$$PdV + FdL = 0, F = -P\frac{dV}{dL}$$

By chain rule, $$\frac{dV}{dL} = \frac{dV/d\theta}{dL/d\theta}$$

Taking the derivatives of both in respect to θ, we get

$$\frac{dV}{dL} = \frac{\frac{B^3}{6N^2\pi}\sin\theta(2cos^2\theta - sin^2\theta)}{-Bsin\,\theta}$$

$$= \frac{B^2}{6N^2\pi}(sin^2\theta - 2cos^2\theta)$$

Virtual Work

Through Trig. Ratios,

$$F(\theta) = \frac{PB^2}{6N^2\pi}\frac{2 - tan^2\theta}{1 + tan^2\theta}$$

By Substitution,

$$\boxed{F(\lambda) = \frac{PL_0^2}{6N^2\pi cos^2\theta_0}(3\lambda^2 cos^2\theta_0 - 1)}$$

Where:

P: Pressure in the netting

L_0: Initial Length

N: Number of turns

θ_0: Initial Angle

λ: Contraction Ratio

Balloon Considerations

The membrane tension γ relates to the material properties through:

$$\gamma = \sigma\omega = E\varepsilon\omega$$

Where:

1. E is the elastic modulus of the balloon material
2. ε is the strain in the balloon
3. ω is the wall thickness
4. σ is the tensile stress in the balloon

For the circumferential strain in the balloon:

$$\varepsilon_{hoop} = \frac{D - D_0}{D_0} = \frac{D}{D_0} - 1$$

Balloon Considerations

In the central region where the PAM behaves approximately cylindrically:

- Circumferential radius: $R_1 = \frac{D}{2}$
- Axial radius: $R_2 = \infty$ (no curvature along axis)

The balloon pressure becomes:

$$P_{balloon,cyl} = \frac{\gamma}{R_1} = \frac{2\gamma}{D}$$

At the ellipsoidal ends where both principal radii are approximately equal to $\frac{D}{2}$:

$$P_{balloon,sph} = \gamma(\frac{2}{D}) = \frac{2\gamma}{D}$$

Balloon Considerations

Substituting our geometric relations into our equation:

$$\frac{D}{D_0} = \frac{\frac{B}{N\pi}\sqrt{1-\lambda^2 cos^2\theta_0}}{\frac{B}{N\pi}sin\theta_0} = \frac{\sqrt{1-\lambda^2 cos^2\theta_0}}{sin\theta_0}$$

Therefore:

$$\varepsilon_{hoop} = \frac{\sqrt{1-\lambda^2 cos^2\theta_0}}{sin\theta_0} - 1$$

Where:

ε: is the strain in the balloon

θ_0: Initial Angle

λ: Contraction Ratio

Balloon Considerations

Substituting, D = $\frac{B}{N\pi}\sqrt{1-\lambda^2 cos^2\theta_0}$:

$$P_{balloon} = \frac{2E\omega N\pi}{B\sqrt{1-\lambda^2 cos^2\theta_0}}\left(\frac{\sqrt{1-\lambda^2 cos^2\theta_0}}{sin\theta_0} - 1\right)$$

Simplifying:

$$\boxed{P_{balloon} = \frac{2E\omega N\pi}{B}\left(\frac{1}{sin\theta_0} - \frac{1}{\sqrt{1-\lambda^2 cos^2\theta_0}}\right)}$$

Where:

E: is the Elastic Mod. of Balloon

N: Number of turns

θ_0: Initial Angle

λ: Contraction Ratio

Balloon Considerations

After considering Balloon pressure:

$$F(\lambda) = \frac{(P_{applied} - P_{balloon})L_0^2}{6N^2\pi\, cos^2\theta_0}(3\lambda^2 cos^2\theta_0 - 1),$$

Where: $P_{applied} - P_{balloon} = P_{eff}$

$$\boxed{F(\lambda) = \frac{P_{eff}L_0^2}{6N^2\pi\, cos^2\theta_0}(3\lambda^2 cos^2\theta_0 - 1)}$$

Where:

P_{eff}: Pressure in the netting

L_0: Initial Length

N: Number of turns

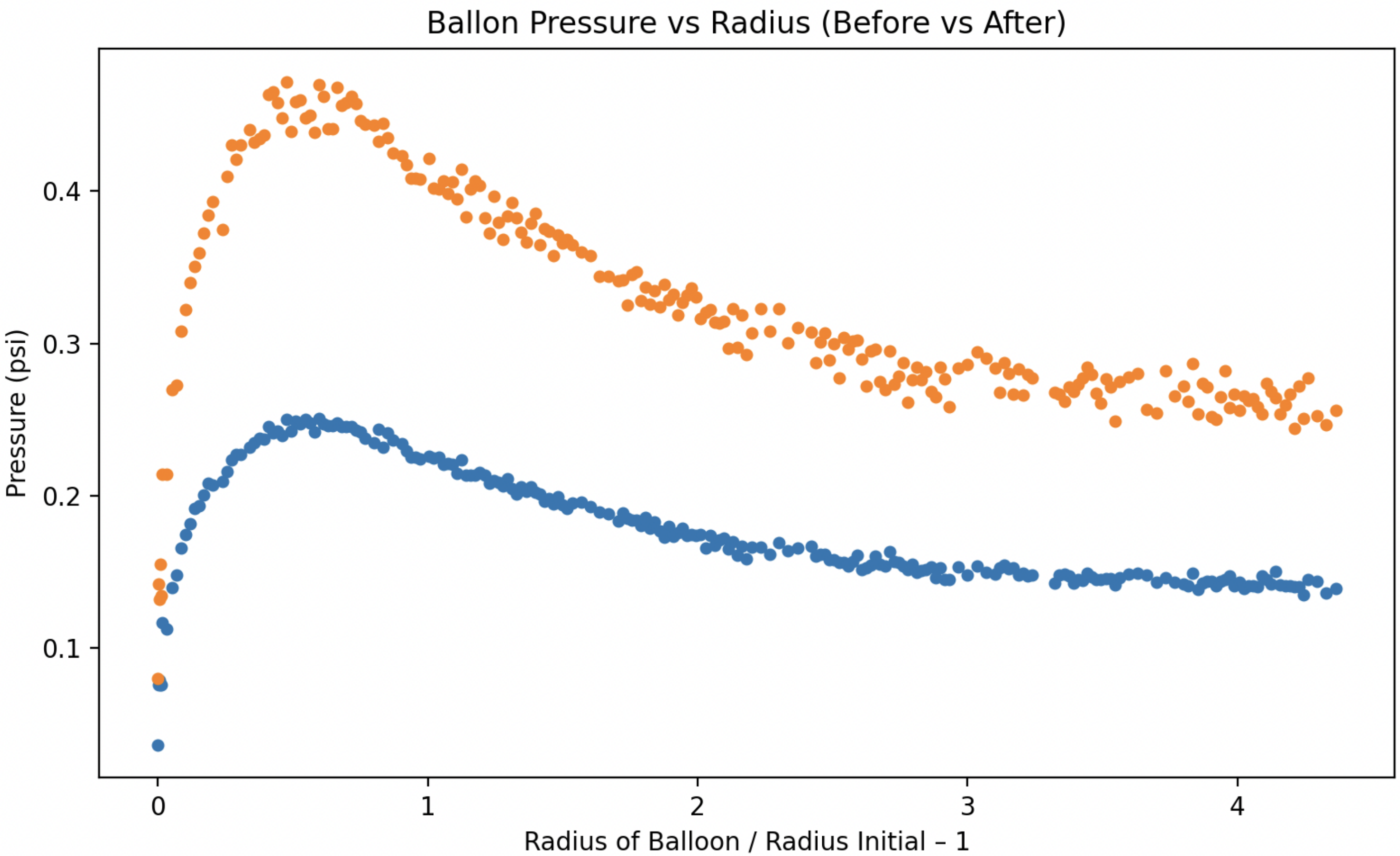
θ_0: Initial Angle

λ: Contraction Ratio

Interesting note, force is negative when $cos^2\theta_0 > \frac{1}{3\lambda^2}$!

Experiments

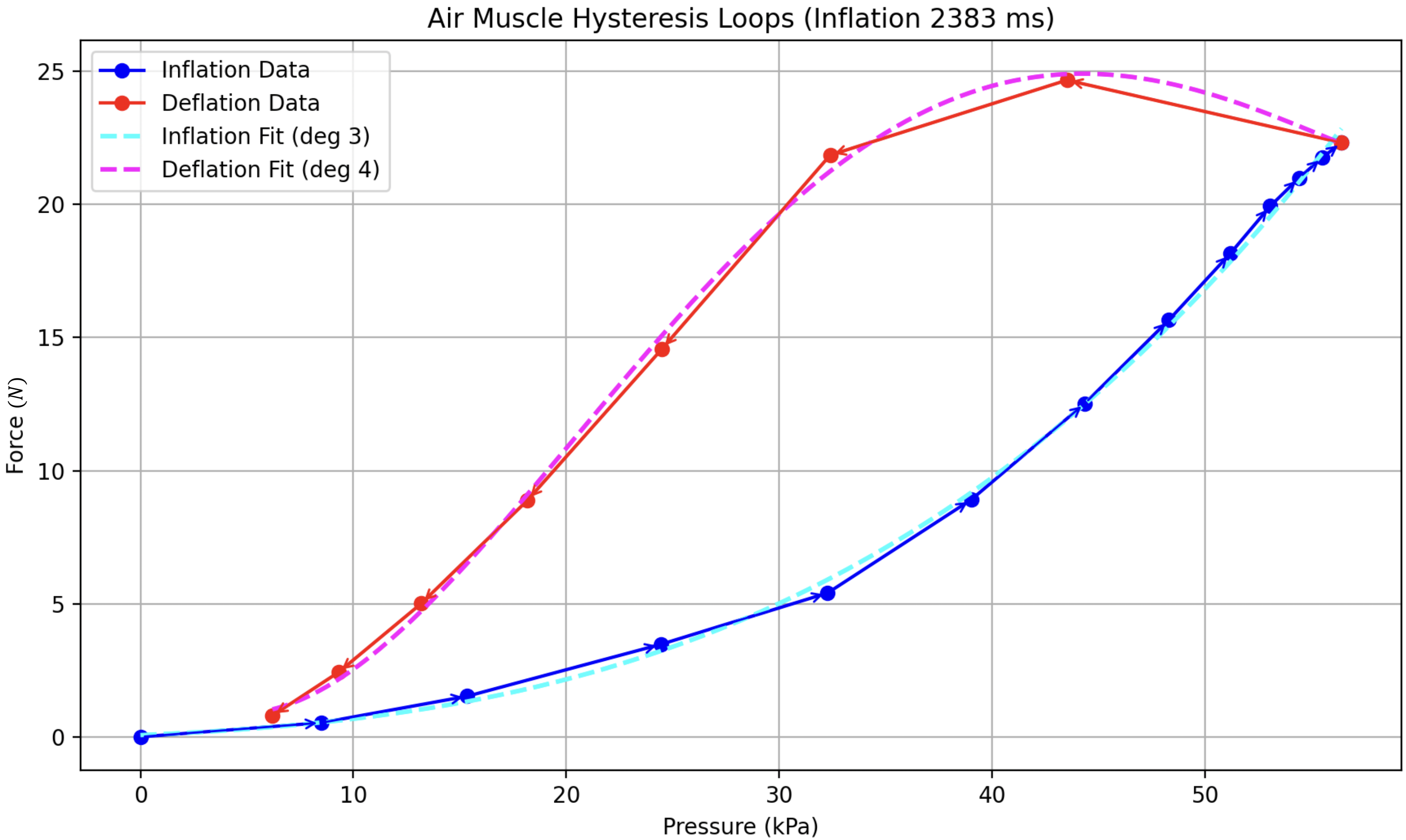
Pressure Profile of Balloon



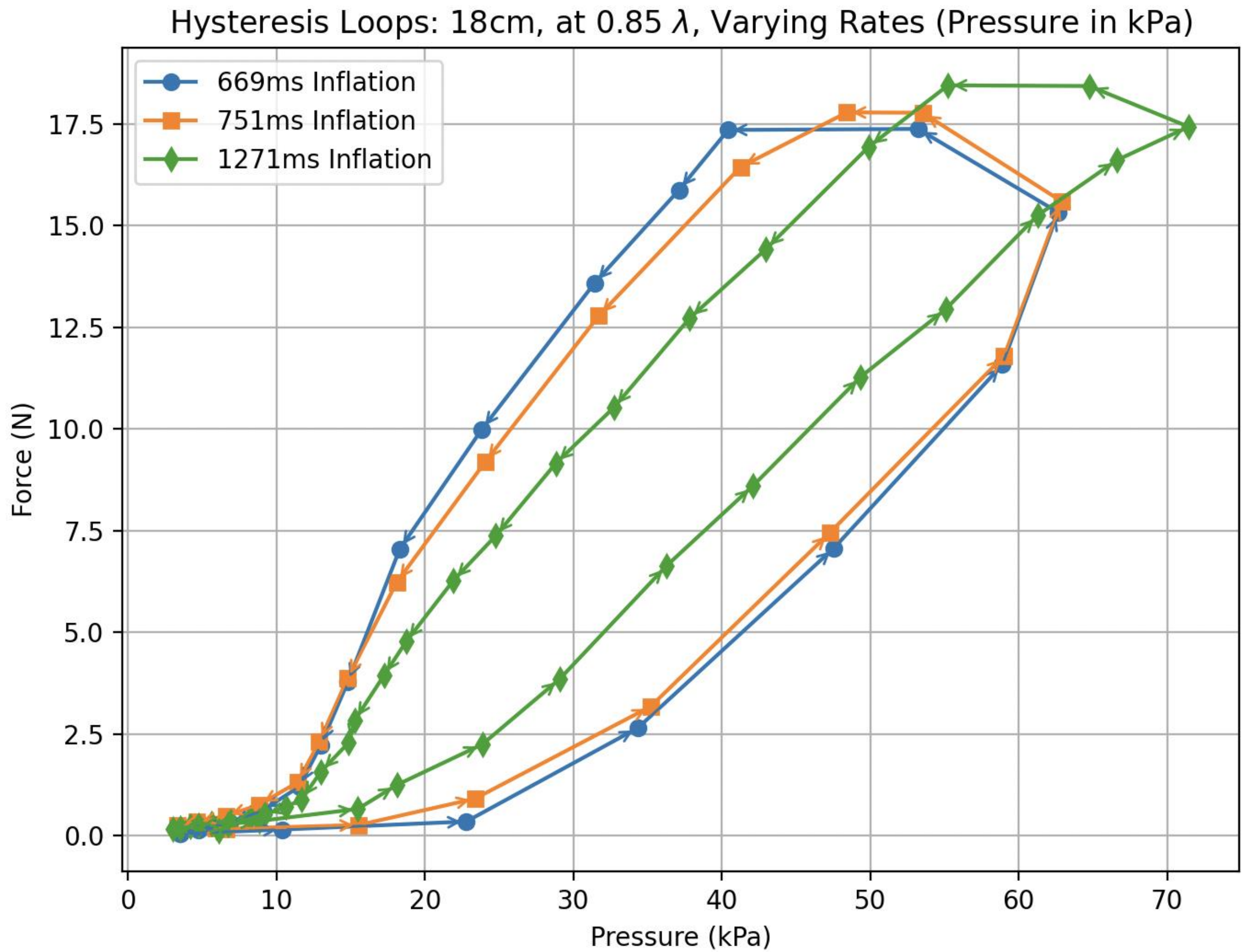
Pressure vs. Radius Curve

Max Inflation pressure decreases by half, after prestressing (3152 times)

Hysteresis (Force netting)

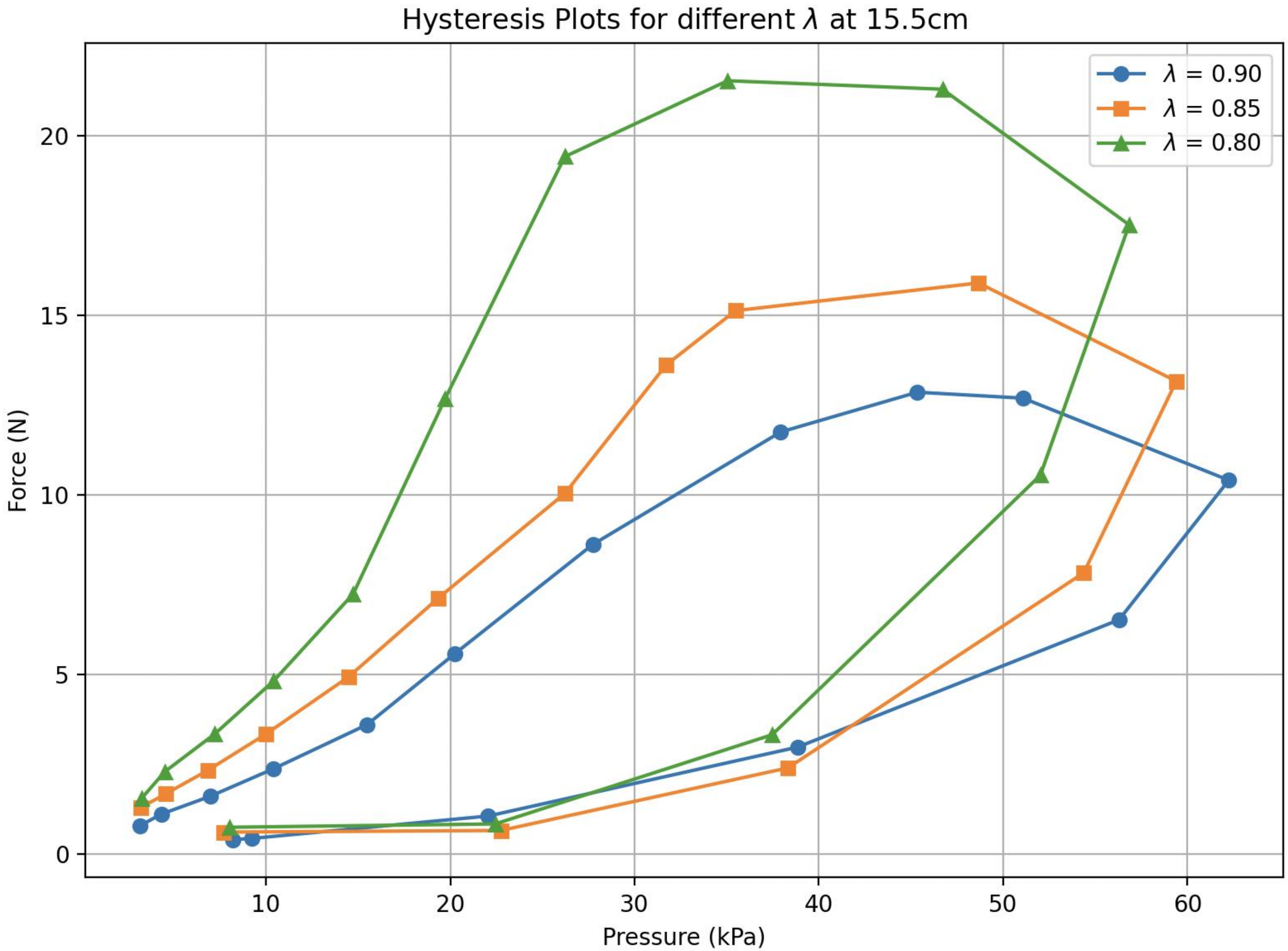


Hysteresis (Force netting)

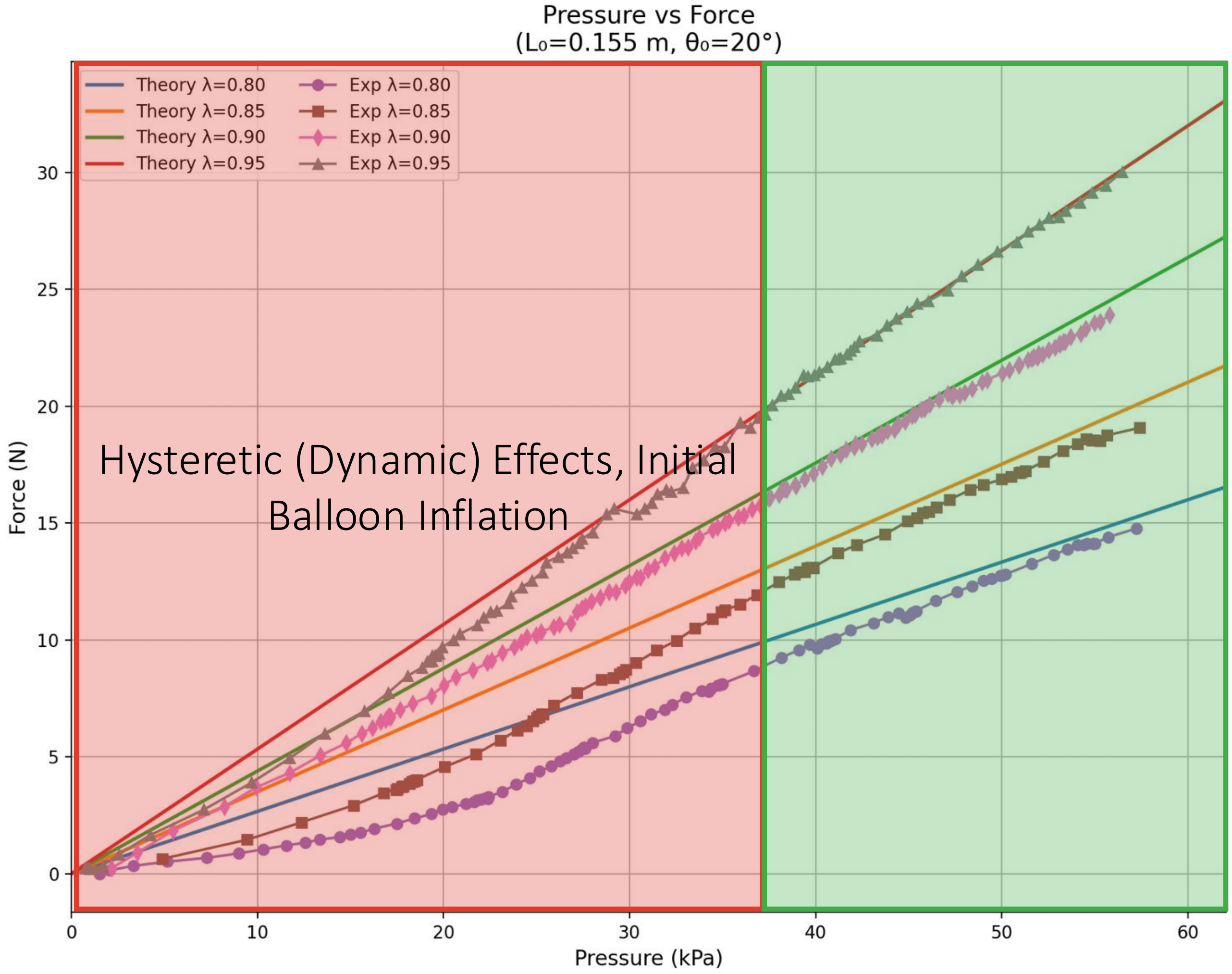


Hysteresis (Force netting)

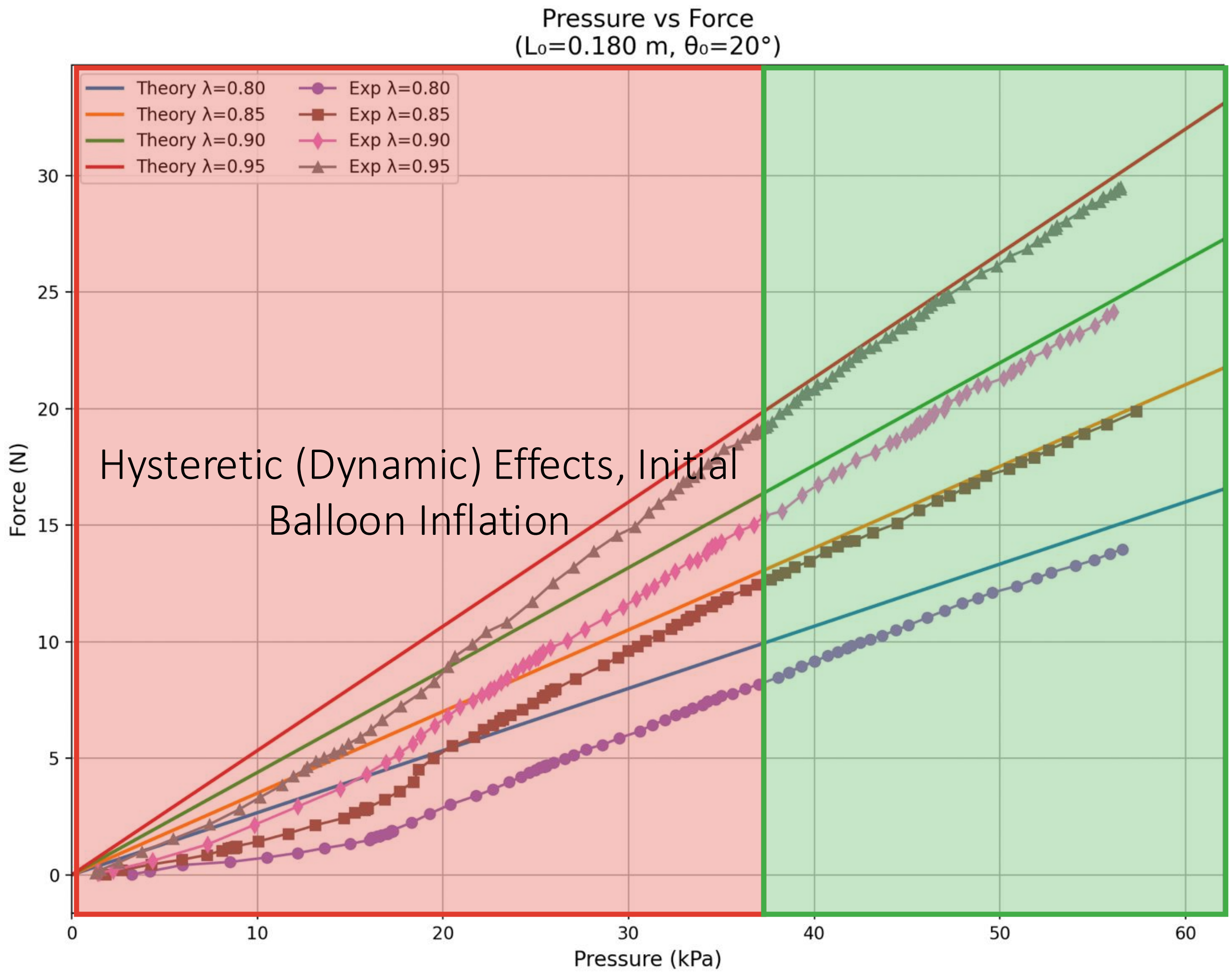
Time frame is 721 ms



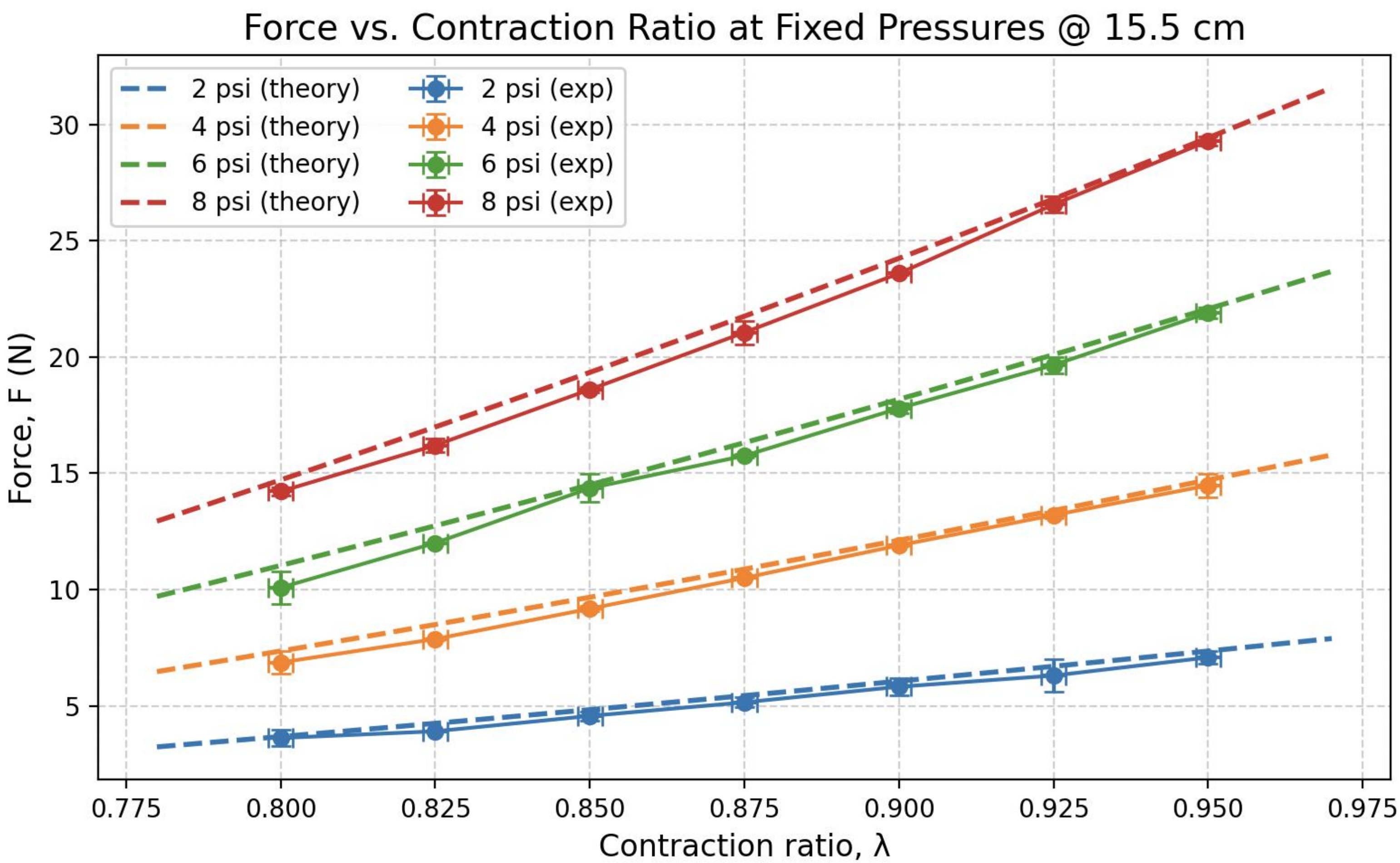
Changing Pressure (Force netting)



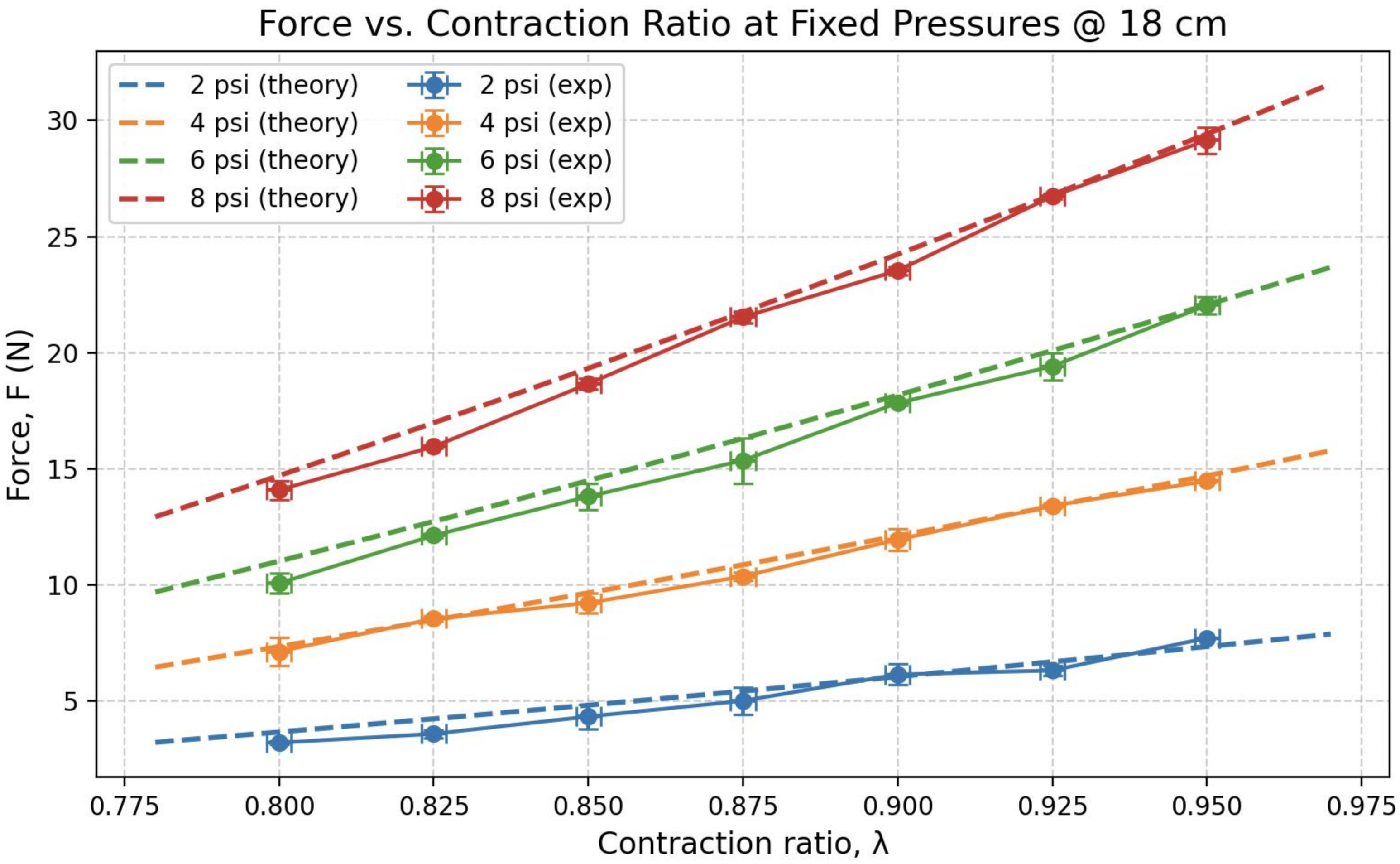
Changing Pressure (Force netting)



Constant Pressure (Force netting)



Constant Pressure (Force netting)



Summary

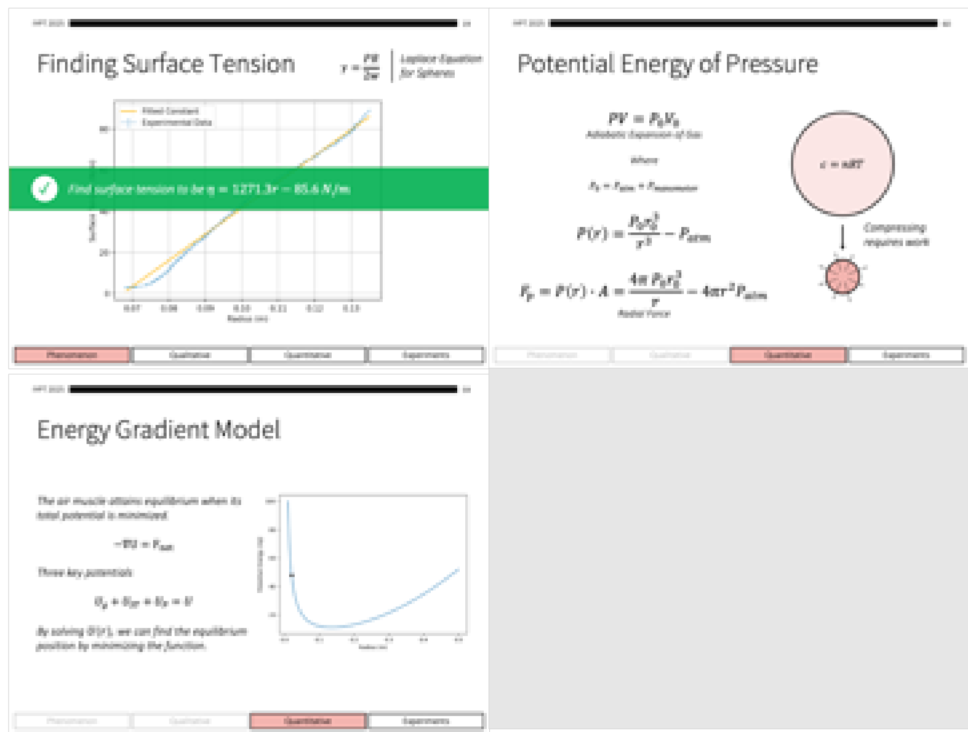
"Place a balloon inside a cylindrical net (as is sometimes used to wrap garlic) and inflate it. The net will expand and shorten. Investigate the properties of such a "muscle"."

Qualitative Account

Qualitatively explained phenomenon

Built experimental setup to test various parameters

Created theory based on volume and virtual work



References

[1] J. D. Hunter, "Matplotlib: A 2D Graphics Environment", Computing in Science & Engineering, vol. 9, no. 3, pp. 90-95, 2007.

[2] Harris, C.R., Millman, K.J., van der Walt, S.J. et al. Array programming with NumPy. Nature 585, 357–362 (2020).

[3] Nave, R. (n.d.). *Wall tension*. Pressure. http://hyperphysics.phy-astr.gsu.edu/hbase/ptens.html

[4] Müller, I., & Struchtrup, H. (2002). Inflating a rubber balloon. *Mathematics and Mechanics of Solids*, *7*(5), 569–577. https://doi.org/10.1177/108128650200700506

[5] Merritt, D. R., & Weinhaus, F. (1978). The pressure curve for a rubber balloon. *American Journal of Physics*, *46*(10), 976–977. https://doi.org/10.1119/1.11486

[6] Surface energy density and surface tension. (n.d.). https://people.bss.phy.cam.ac.uk/~emt1000/scm/old3.pdf

Thanks For Listening!

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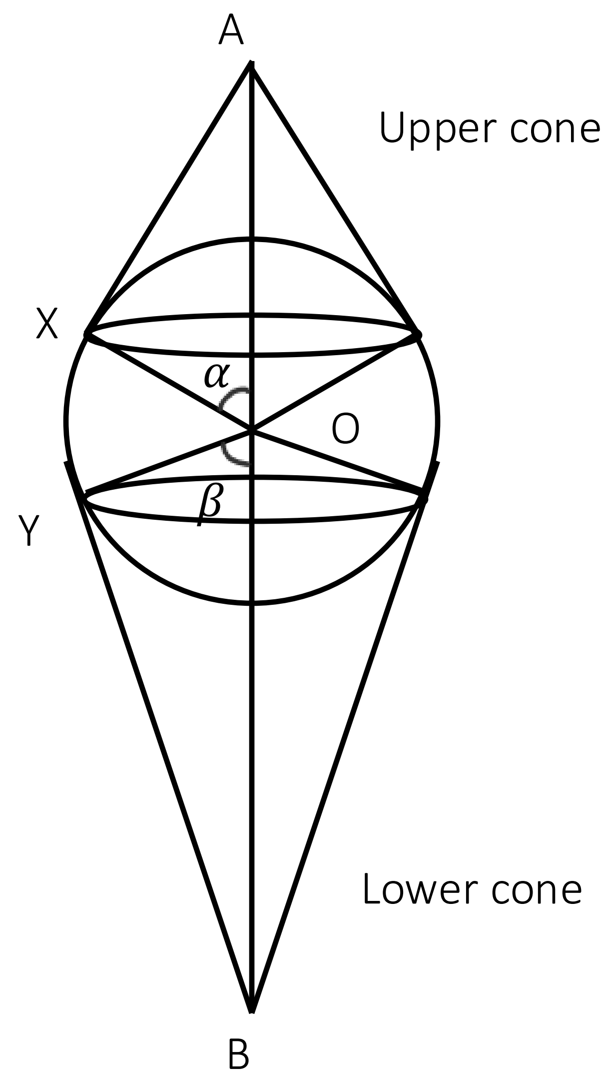
Appendix

Errors / Next Steps

Assumptions:
1. Spherical Balloon Shape/Characterization
2. Cylindrical Geometric Constraint
3. Cosmic Rays

Next Steps:
1. Conic Model for more geometric accuracy
2. Ellipsoid & Other Balloon Shapes
3. Finite-Element Force Model (FEM)
4. Carrying Capacity of Muscle, efficiency of muscle
5. Remove Cosmic Rays



Theory Equations

Cylindrical:

$$K\left[\frac{R_0}{r}-\left(\frac{R_0}{r}\right)^7\right]\left[1+0.1\left(\frac{r}{R_0}\right)^2\right]4\pi r^3=mgh$$

Conic Model:

$$K\left[\frac{R_0}{r}-\left(\frac{R_0}{r}\right)^7\right]\left[1+0.1\left(\frac{r}{R_0}\right)^2\right]2\pi r^2\left(\frac{r}{x}+\frac{r}{L_{net}-x}\right)=mgh$$

Net Characterization – Plastic Net

$L_0 = 15.5\ cm, 18\ cm, 20.5\ cm$

$Initial\ Braid\ Angle\text{:}\ \theta_0 = 20°$

$Number\ of\ turns = L/8cm$

Energy Equations

$$U_{tot} = E_g + E_b$$

Where E_b is the potential energy of the balloon.
At equilibrium, we have:

$$U_{tot} = E_g + E_b = 0$$
$$E_g = mgh$$
$$E_b = F_b \times d = F_b \times (-r) = -p(r) \times SA \times r$$

Where $p(r)$ is the pressure in the balloon given r, r is the radius of the balloon, and SA is the surface area of the balloon in contact with the net.

Balloon Pressure vs Radius

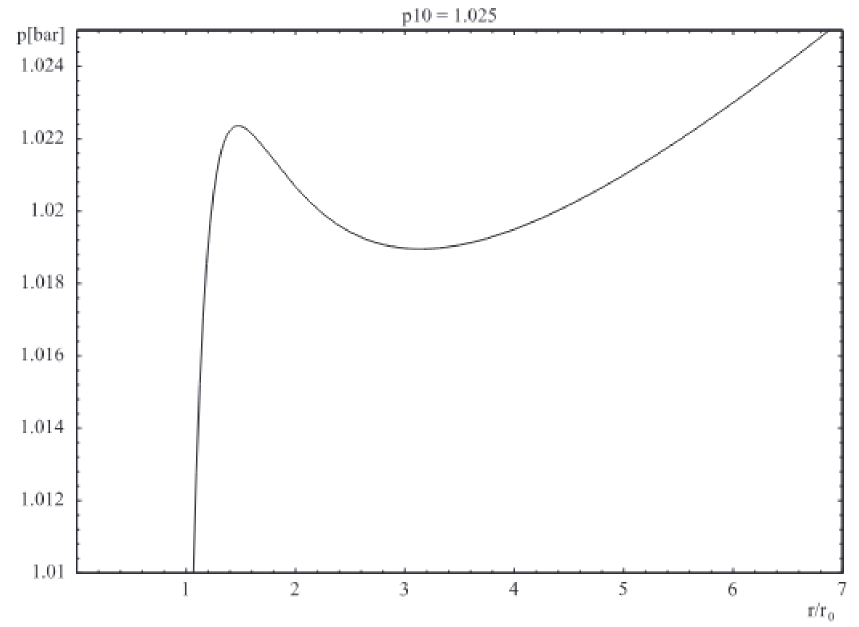


Figure 1. Pressure-radius characteristic.

Balloon Pressure vs Radius

Verron and Marckmann3 proposed Eq. (1) in the case of a Mooney-Rivlin type spherical balloon:

$$P = K\left[\frac{R_0}{R} - \left(\frac{R_0}{R}\right)^7\right]\left[1 + 0.1\left(\frac{R}{R_0}\right)^2\right],$$

Balloon Pressure vs Radius

More specifically, we have

$$K = 2s_1 \frac{d_0}{r_0}$$

where $s_1 = 3bar$ is a constant of a typical Mooney-Rivlin balloon, and d_0 is the thickness of the balloon.

For a typical balloon, we have $K = 10$

More specifically, we have

$$K = 2s_1 \frac{d_0}{r_0}$$

where $s_1 = 3bar$ is a constant of a typical Mooney-Rivlin balloon, and d_0 is the thickness of the balloon.

For a typical balloon, we have $K = 10$

Energy Gradient Model

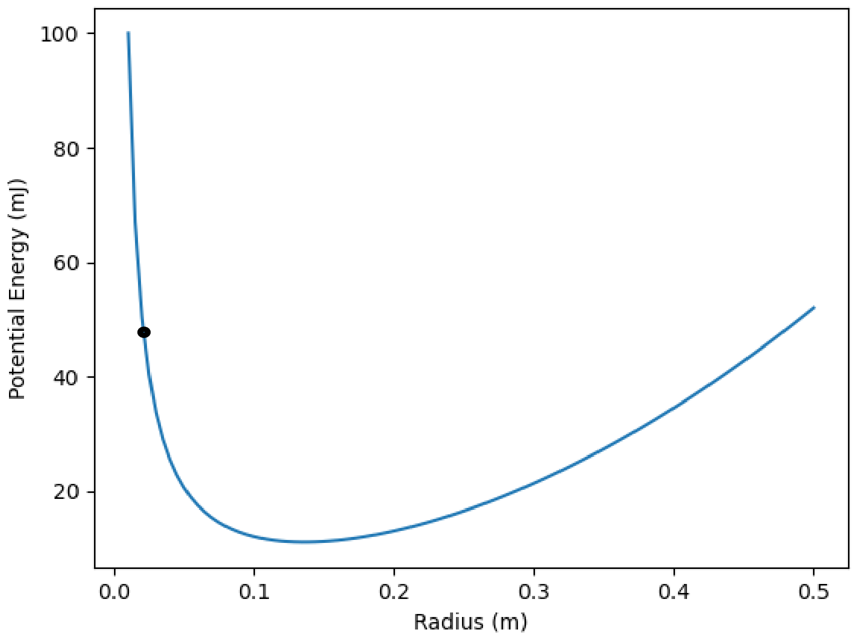
The air muscle attains equilibrium when its total potential is minimized.

$$-\nabla \mathrm{U} = \mathrm{F}_{\mathrm{net}}$$

Three key potentials

$$U_g + U_{ST} + U_P = U$$

By solving $U(r)$, we can find the equilibrium position by minimizing the function.



Potential Energy of Pressure

$$PV = P_0 V_0$$

Adiabatic Expansion of Gas

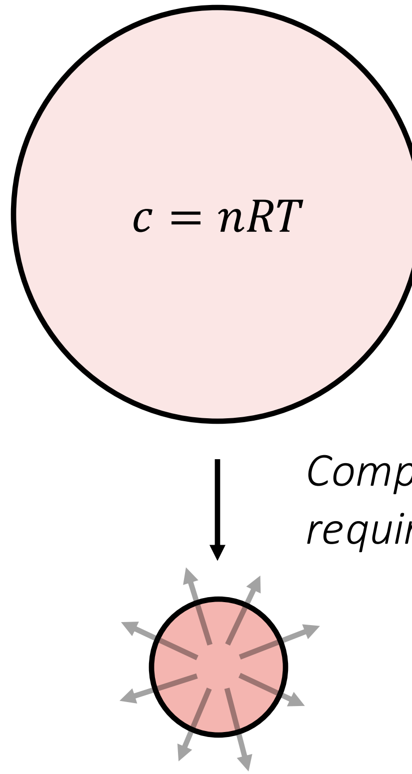
Where

$$P_0 = P_{atm} + P_{manometer}$$

$$P(r) = \frac{P_0 r_0^3}{r^3} - P_{atm}$$

$$F_p = P(r) \cdot A = \frac{4\pi\, P_0 r_0^3}{r} - 4\pi r^2 P_{atm}$$

Radial Force



Compressing requires work

Potential Energy of Pressure

$$W_p = \int_{\infty}^{r} F_p \, dr$$

However, $r = \infty$ is unattainable. Let r_n be max radius before balloon pops.

Pressure potential can be found as $[U_P](r) = -4\pi \, P_0 r_0^3 (\log r - \log r_n) + \frac{4}{3}\pi P_{atm} r^3$

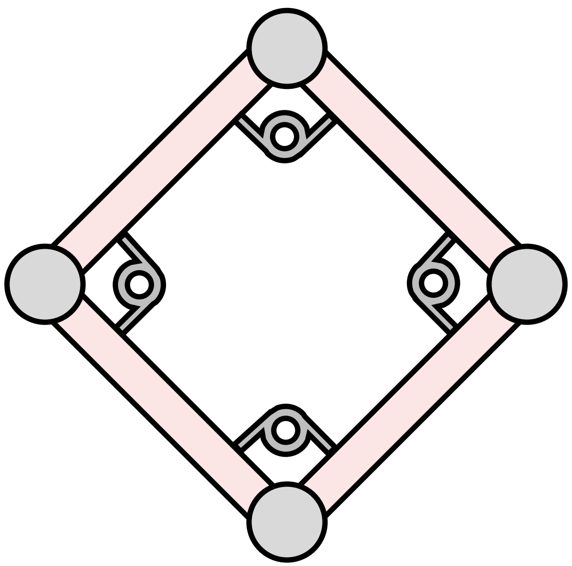
$$W_p = \int_{r_n}^{r} F_p dr$$

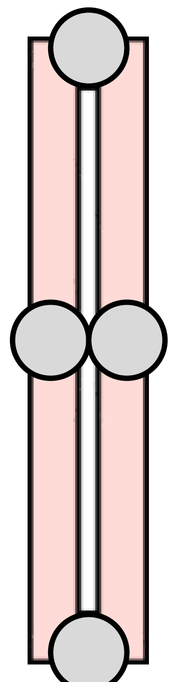
$$= 4\pi \, P_0 r_0^3 (\log r - \log r_n) - \frac{4}{3}\pi P_{atm} r^3$$

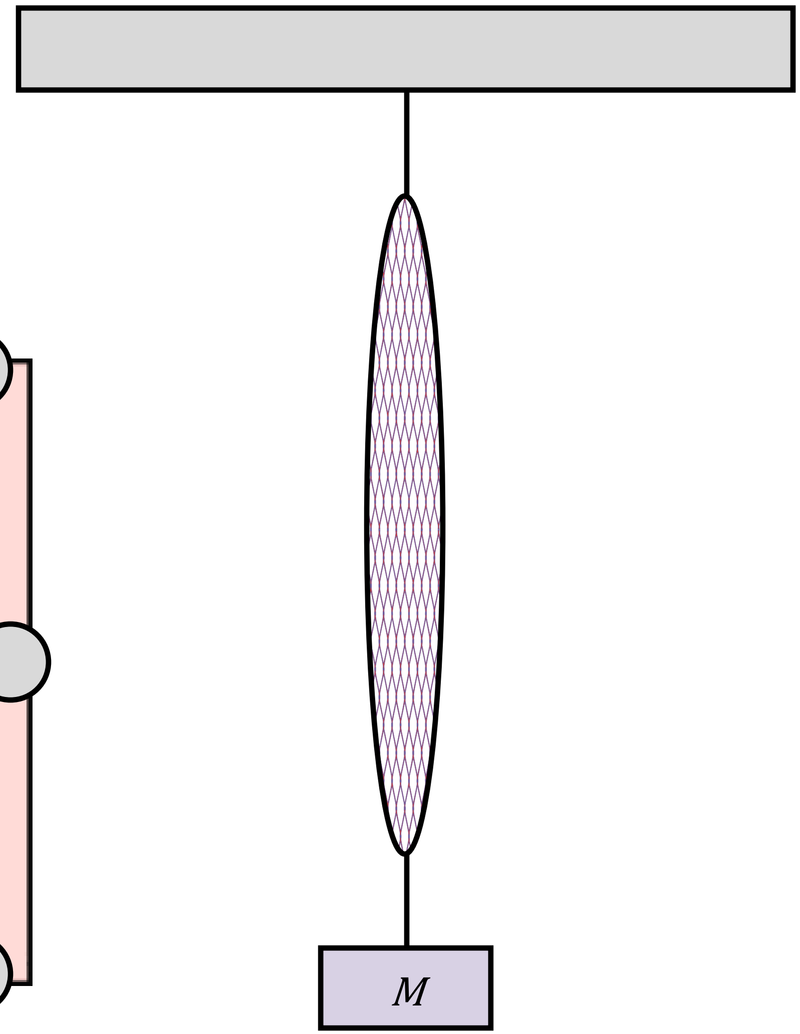
Balloon Contraction

Let the net have n_y vertical subnets, n_x horizontal subnets.

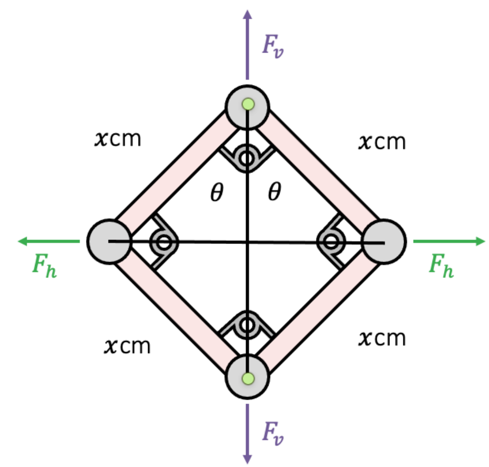
At 0 pressure, $r = 0$ and
$d = d_0 = 2xn_y$







Balloon Contraction

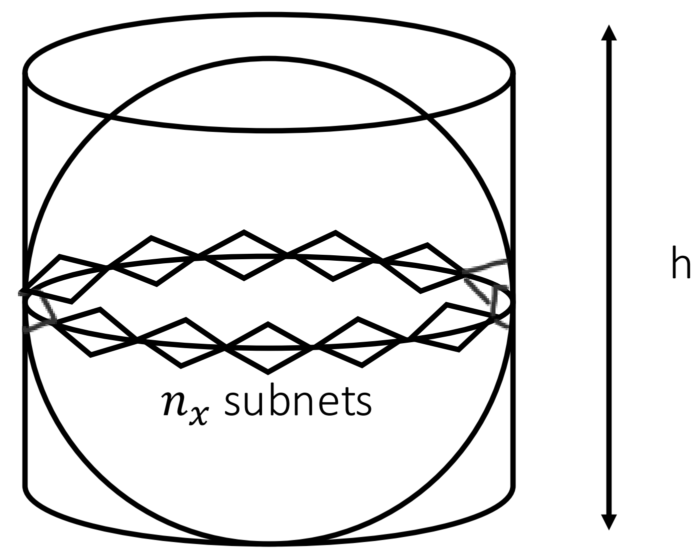


At radius r,

$$2\pi r = 2x \sin\theta\, n_x$$
$$\sin\theta = \frac{r\pi}{x n_x}$$

Therefore the length of net is:

$$d = 2x\cos\theta n_y$$
$$= 2x n_y \sqrt{1 - \left(\frac{r\pi}{x n_x}\right)^2}$$



Balloon Contraction

- Where Δd is the change in length
- x is the length for each cell
- n_y, and n_x are the number of cells
- r is the radius of the net and balloon

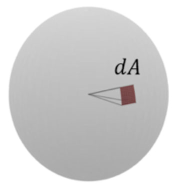
$$\Delta d = 2xn_y - 2xn_y\sqrt{1-\left(\frac{r\pi}{xn_x}\right)^2}$$

$$= 2xn_y\left(1-\sqrt{1-\left(\frac{r\pi}{xn_x}\right)^2}\right)$$

$$U_g = mg\Delta d$$

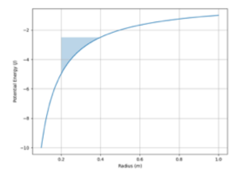
$$= 2mgxn_y\left(1-\sqrt{1-\left(\frac{r\pi}{xn_x}\right)^2}\right)$$

Combined Solution



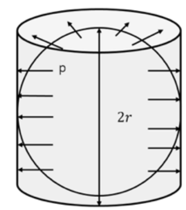
U_{ST}

Express balloon membrane as potential energy



U_P

Solve for potential energy of pressurized air in unconstrained volume



U_g

Solve for gravitational energy using geometric constraint

```
def ST(r):
    return 1271.31272636 * r -85.61180908

def E_gas(r,p):
    return p * 4 * np.pi * r0 * (np.log(r))

def E_b(r):
    return ST(r) * 4 * np.pi * (r ** 2)

def E_grav(r, a):
    return 2 * x * ny * a * g * (1 - np.sqrt(1 - ((np.pi * r)/(x * nx))**2))

def find_h(r):
    return 2 * x * ny * (1 - np.sqrt(1 - ((np.pi * r)/(x * nx))**2))

def find_min(mass=0,pressure=2000):
    rs = np.linspace(0.001, 0.2,1000)
    Es = []

    for R in rs:
        Es.append(E_grav(R, mass) + E_b(R) - E_gas(R,pressure))

    Es = np.array(Es)

    min_en_rad = rs[np.argmin(Es)]

    return find_h(min_en_rad)
```

Numerical Solution

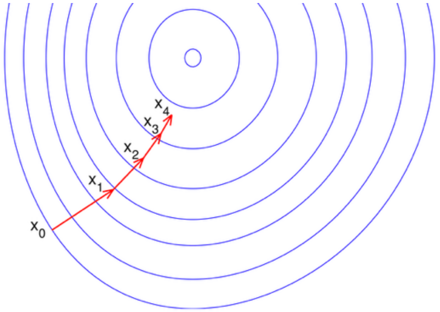
Minimize energy computationally

Discretized Numerical Solution

1. Numerically solve for energy at evenly-spaced R-values

2. Un-discretize using a fit 4th-order polynomial via R^2 *optimization*

3. Minimize energy using Broyden–Fletcher–Goldfarb–Shanno (BFGS) method

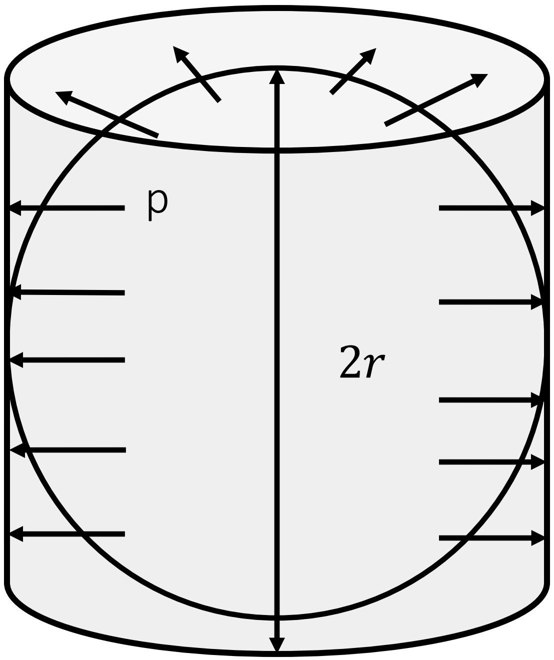


Cylindrical Model

Model the net as a cylinder (without the top and bottom) around the balloon. Assume the pressure due to balloon on the net is constant everywhere on the side of the cylinder.

Thus, the surface area on which the pressure is applied is

$$SA = 2\pi r \times h = 2\pi r \times 2r = 4\pi r^2$$



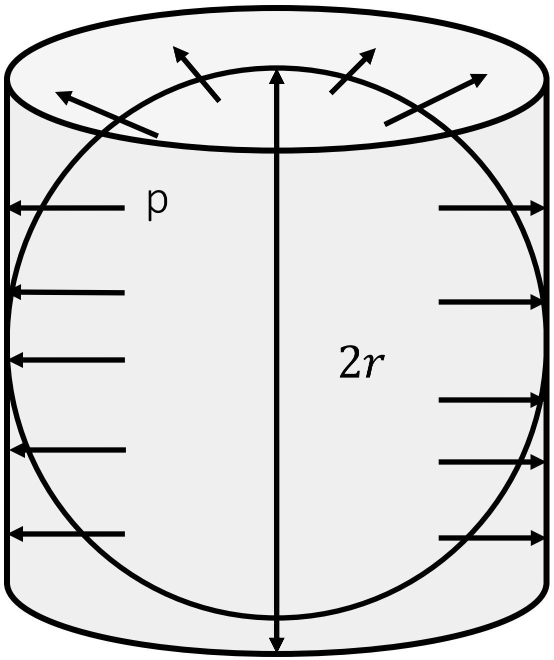
Cylindrical Model

Thus, we would have

$$E_b = -p(r) \times SA \times r = p(r) \times 4\pi r^2 \times r$$
$$E_b = -p(r) \times 4\pi r^3$$

Then, we have

$$E_g + E_b = 0$$
$$r^3 \times p(r) 4\pi = mgh$$



Cylindrical Model cont.

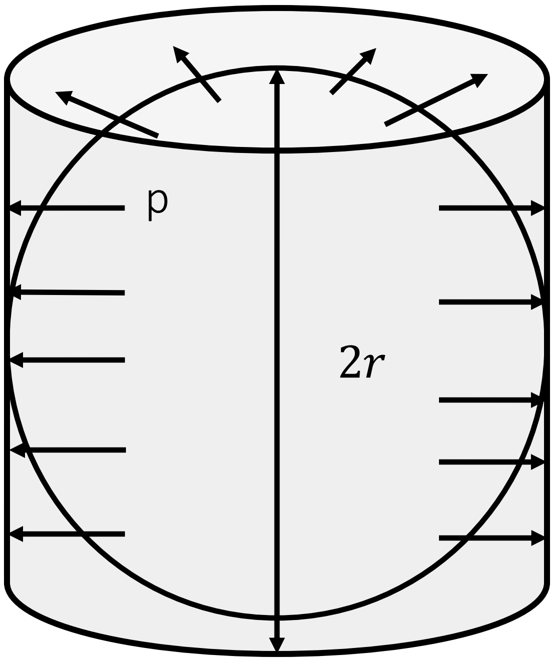
Since we have

$$P(r) = K\left[\frac{R_0}{r} - \left(\frac{R_0}{r}\right)^7\right]\left[1 + 0.1\left(\frac{r}{R_0}\right)^2\right]$$

Plugging this into the previous equation, we have

$$r^3 \times p(r)4\pi = mgh$$

$$K\left[\frac{R_0}{r} - \left(\frac{R_0}{r}\right)^7\right]\left[1 + 0.1\left(\frac{r}{R_0}\right)^2\right]4\pi r^3 = mgh$$

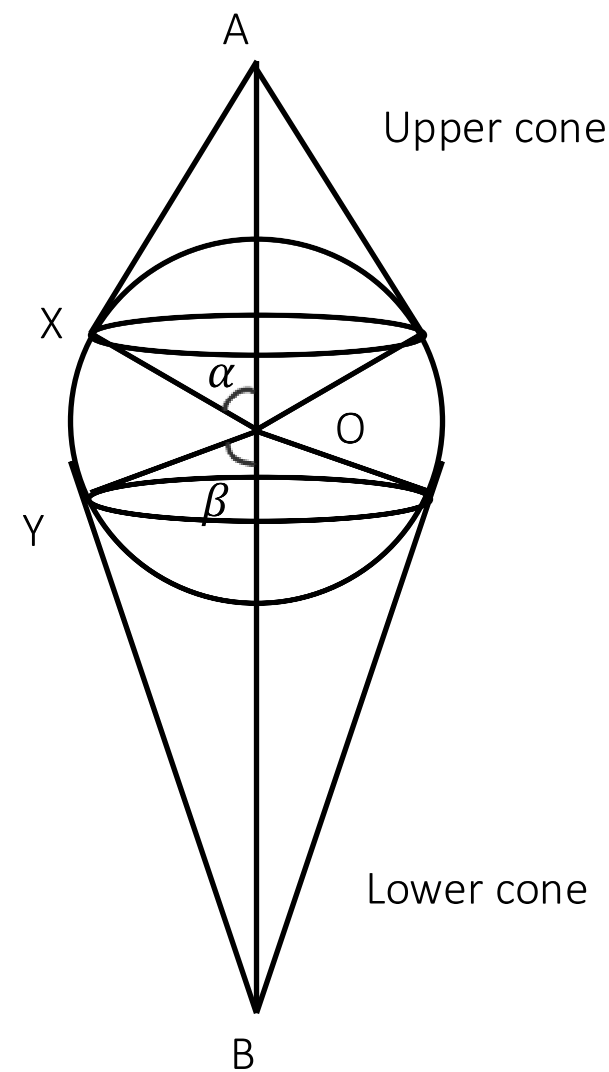


Conic Model

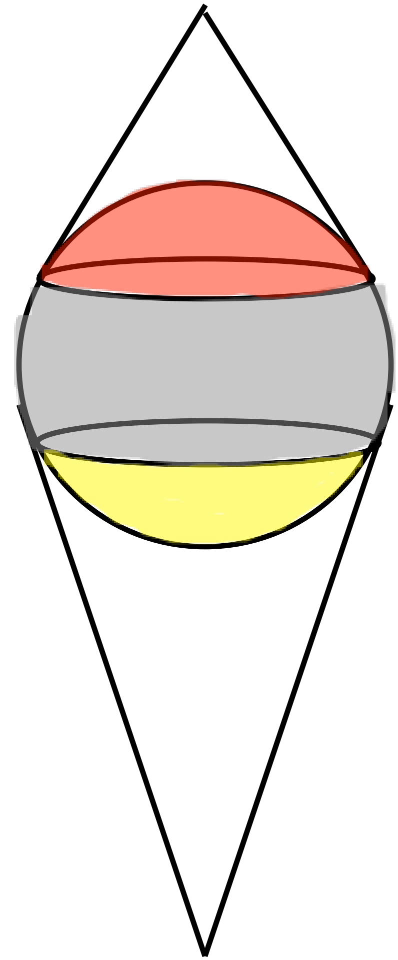
Model the upper and lower part of the net, not touching the balloon as two cones. Thus, the tip of the upper cone is attached to the table/hanger, and the tip of the lower cone is attached to the mass. The bases of the upper and lower cone are tangent to the balloon/sphere.

Let the tip of the upper cone be A, and the tip of the lower cone be B. Let the center of the sphere be O. Let the sphere be tangent to the upper cone at X, and the lower cone at Y.

Let $\angle AOX = \alpha$, and $\angle BOY = \beta$



Conic Model



Thus, the net will only be touching the balloon in the area highlighted in grey. We need to find the surface area of this section of the sphere.

We see that

$$SA_{middle} = SA_{sphere} - SA_{top} - SA_{bottom}$$

Where $\mathbf{SA_{sphere}}$ is the surface area of the entire sphere, SA_{middle} is highlighted in grey, SA_{top} is highlighted in red, and SA_{bottom} is highlighted in yellow.

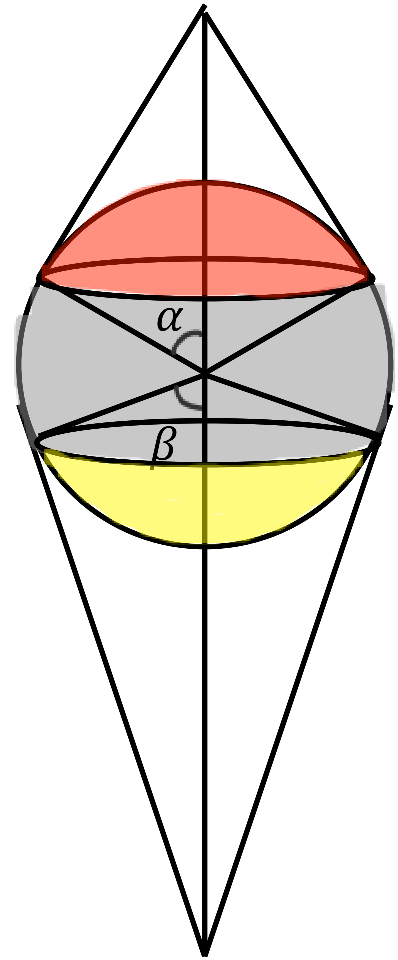
Conic Model (finding SA)

The formula for the Surface Area of a Spherical Cap of a sphere is

$$SA = 2\pi r^2(1 - cos\theta)$$

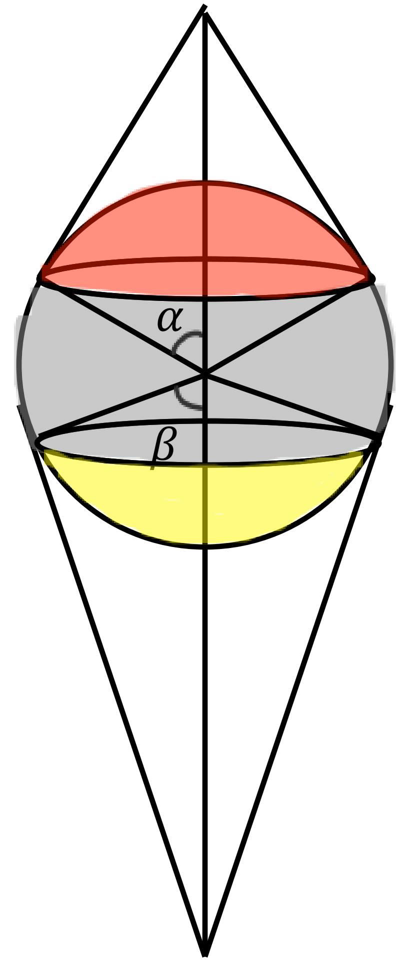
where θ is the angle between the tangent and the vertical.
Thus, we have

$$SA_{top} = 2\pi r^2(1 - cos\alpha)$$
$$SA_{bottom} = 2\pi r^2(1 - cos\beta)$$



Conic Model (finding SA)

$$SA_{middle} = SA_{sphere} - SA_{top} - SA_{bottom}$$
$$= 4\pi r^2 - 2\pi r^2(1 - cos\alpha) - 2\pi r^2(1 - cos\beta)$$
$$= 2\pi r^2 cos\alpha + 2\pi r^2 cos\beta$$
$$= 2\pi r^2(cos\alpha + cos\beta)$$



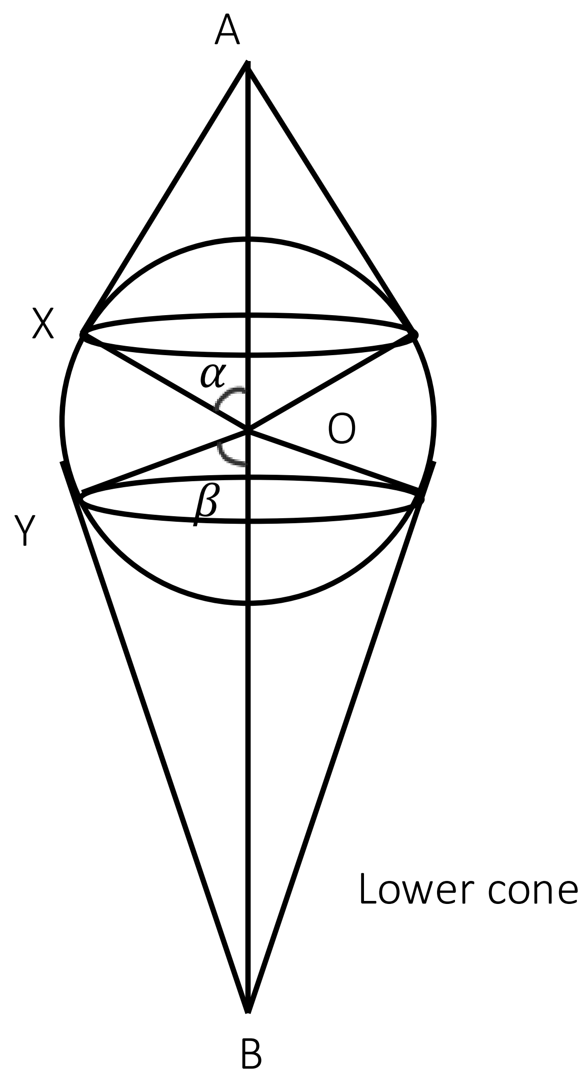
Conic Model (finding α, β)

Assume that as the balloon expands, its center remains the same distance from the table. Thus, assume AO stays constant, $AO = x$m.

Thus, $\cos(\alpha) = \left(\frac{XO}{AO}\right) = \frac{r}{x}$

Let the length of the net be L_{net} (this is constant). Thus, we see that

$$BO = L_{net} - x$$
$$\cos(\beta) = \left(\frac{YO}{BO}\right) = \frac{r}{L_{net} - x}$$



Conic Model (finding α, β)

Thus, $SA_{middle} = 2\pi r^2(cos\alpha + cos\beta)$

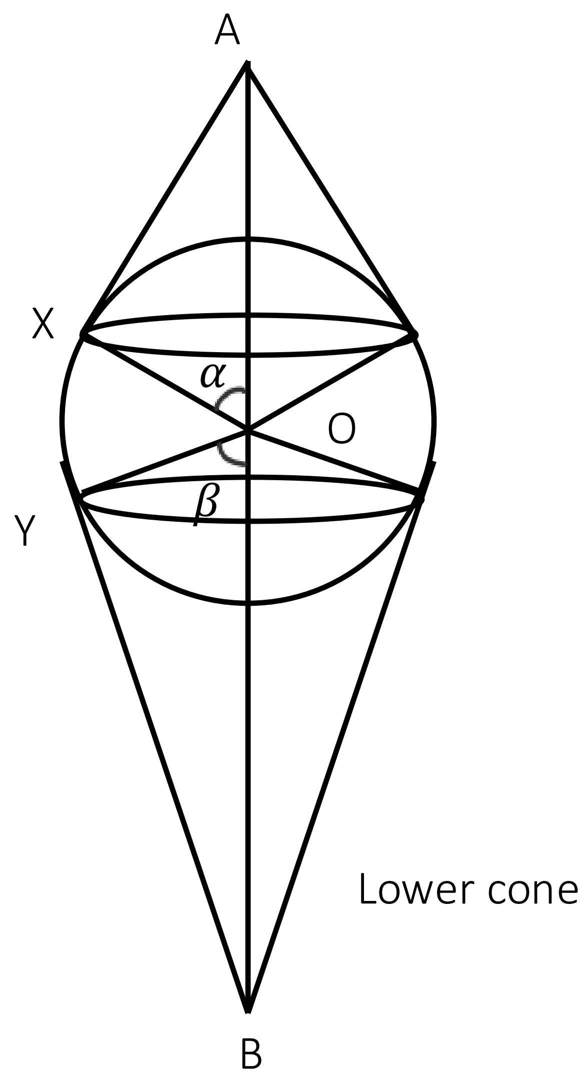
$$= 2\pi r^2(\frac{r}{x} + \frac{r}{L_{net} - x})$$

We have

$$E_b = -p(r) \times SA \times r$$
$$= p(r) \times 2\pi r^2(\frac{r}{x} + \frac{r}{L_{net} - x})$$

Then, we have

$$E_g + E_b = 0$$
$$p(r) \times 2\pi r^2(\frac{r}{x} + \frac{r}{L_{net} - x}) = mgh$$



Conic Model (finding α, β)

Thus, $SA_{middle} = 2\pi r^2(cos\alpha + cos\beta)$

$$= 2\pi r^2(\frac{r}{x} + \frac{r}{L_{net} - x})$$

We have

$$E_b = -p(r) \times SA \times r$$
$$= p(r) \times 2\pi r^2(\frac{r}{x} + \frac{r}{L_{net} - x})$$

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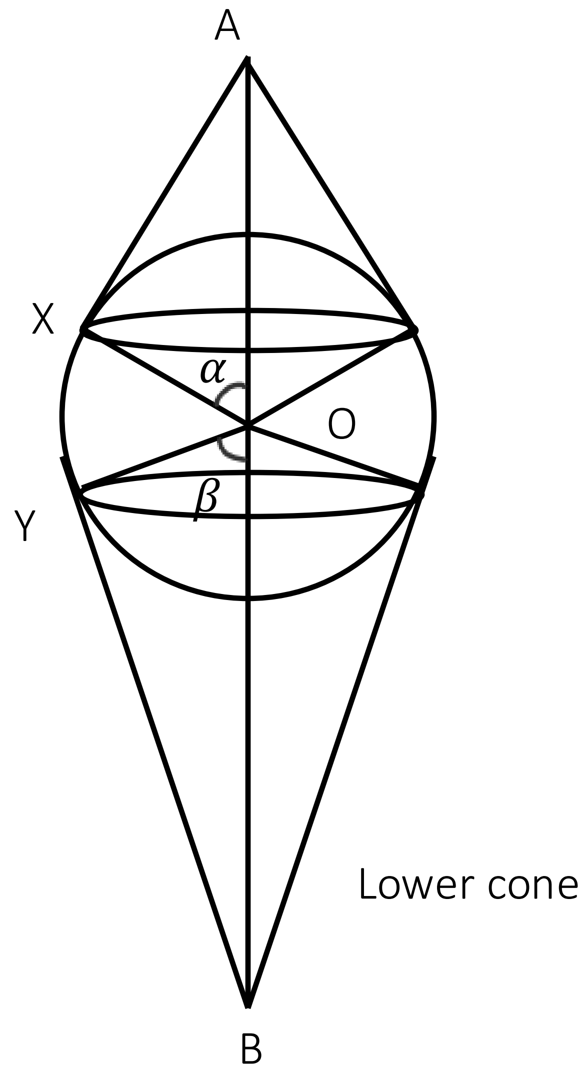

Cylindrical Model cont.

Since we have

$$P(r) = K\left[\frac{R_0}{r} - \left(\frac{R_0}{r}\right)^7\right]\left[1 + 0.1\left(\frac{r}{R_0}\right)^2\right]$$

Plugging this into the previous equation, we have

$$p(r)2\pi r^2\left(\frac{r}{x} + \frac{r}{L_{net} - x}\right) = mgh$$

$$K\left[\frac{R_0}{r} - \left(\frac{R_0}{r}\right)^7\right]\left[1 + 0.1\left(\frac{r}{R_0}\right)^2\right]2\pi r^2\left(\frac{r}{x} + \frac{r}{L_{net} - x}\right) = mgh$$