Cosmology
IOAA Training Camp Material

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1 Preface

If you're reading this, congratulations on making Team Canada for the IOAA! This document presents a cohesive set of theoretical foundations, from core cosmological principles through the Friedmann equations, cosmic expansion, thermal history, and observational cosmology. It's designed for IOAA preparation, with an emphasis on clear derivations, physical intuition, and problem-solving strategies that translate directly to competition settings. I hope it helps!

2 Foundational Concepts

2.1 The Cosmological Principle

Modern cosmology rests on a fundamental assumption: the universe is homogeneous (the same at every point) and isotropic (the same in every direction) on sufficiently large scales (typically > 100 Mpc). This is the cosmological principle.

2.1.1 Observational Evidence

Several observations support this principle:

- The cosmic microwave background (CMB) is isotropic to one part in 10^5 after removing the dipole due to our motion
- Large-scale galaxy surveys show statistical homogeneity above ~ 100 Mpc scales
- The universe appears the same in all directions from our vantage point

2.1.2 Mathematical Consequence

The cosmological principle severely constrains the geometry of spacetime. The most general metric consistent with homogeneity and isotropy is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric.

2.2 The FLRW Metric

In spherical coordinates, the FLRW metric is:

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

where:

- $a(t)$ is the scale factor (dimensionless, normalized so $a(t_0) = 1$ today)
- k is the curvature parameter: $k = +1$ (closed), $k = 0$ (flat), $k = -1$ (open)
- r is the comoving radial coordinate
- c is the speed of light

2.2.1 Physical Interpretation

The scale factor $a(t)$ encodes the expansion of the universe. Physical distances scale as:

$$d_{\text{physical}}(t) = a(t) \cdot d_{\text{comoving}}$$

Comoving coordinates are fixed to the expanding fabric of space - a galaxy at rest in the Hubble flow maintains constant comoving coordinates while its physical distance from us increases.

2.3 Hubble's Law and Expansion

The rate of expansion is characterized by the Hubble parameter:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

where the dot denotes differentiation with respect to time. At the present epoch, $H_0 = H(t_0)$ is the Hubble constant.

2.3.1 Hubble's Law

For nearby galaxies (where $z \ll 1$), the recession velocity is:

$$v = H_0 d$$

where d is the proper distance. This is Hubble's law, discovered observationally in 1929.

2.3.2 Modern Value of H_0

Current measurements give:

$$H_0 = 67.4 \pm 0.5 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad \text{(Planck 2018)}$$

$$H_0 = 73.04 \pm 1.04 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad \text{(SH0ES 2022)}$$

The $\sim 5\sigma$ tension between these values is an active area of research known as the "Hubble tension."

2.3.3 Hubble Time and Distance

The Hubble time provides a characteristic timescale:

$$t_H = \frac{1}{H_0} = 14.4 \text{ Gyr} \quad (\text{for } H_0 = 68 \text{ km s}^{-1} \text{ Mpc}^{-1})$$

The Hubble distance is:

$$d_H = \frac{c}{H_0} = 4.4 \text{ Gpc} = 14.4 \text{ Gly}$$

2.4 Redshift

Due to cosmic expansion, light emitted from distant galaxies is stretched. The redshift is defined as:

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\Delta\lambda}{\lambda}$$

2.4.1 Relation to Scale Factor

The wavelength scales with the universe:

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{emit}})}$$

Since we conventionally set $a(t_0) = 1$ today:

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)}$$

Therefore:

$$a(t) = \frac{1}{1 + z(t)}$$

2.4.2 Redshift Regimes

For small redshifts ($z \ll 1$), we can Taylor expand:

$$v = cz \approx H_0 d$$

This recovers Hubble's law. For larger redshifts, the relationship becomes more complex and depends on the expansion history.

3 The Friedmann Equations

3.1 Derivation from General Relativity

Einstein's field equations relate spacetime curvature to energy-momentum:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

For a perfect fluid with density ρ and pressure p:

$$T_{\mu\nu} = (\rho + p/c^2)u_\mu u_\nu + pg_{\mu\nu}$$

Applying these to the FLRW metric yields the Friedmann equations.

3.2 The First Friedmann Equation

The first Friedmann equation governs the expansion rate:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

This can be rewritten in terms of density parameters:

$$H^2 = H_0^2 \left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda\right]$$

where:

$$\Omega_i = \frac{\rho_i}{\rho_{\text{crit},0}}, \quad \rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G}$$

3.2.1 Critical Density

The critical density today is:

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \approx 1.88 \times 10^{-26} h^2 \text{ kg m}^{-3}$$

where $h = H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$. For $H_0 = 70$:

$$\rho_{\text{crit},0} \approx 9.2 \times 10^{-27} \text{ kg m}^{-3} \approx 5.5 \text{ protons/m}^3$$

3.2.2 Density Parameters

The universe contains several components:

- Matter (baryonic + dark): $\Omega_m \approx 0.31$
- Radiation (photons + neutrinos): $\Omega_r \approx 9 \times 10^{-5}$
- Curvature: $\Omega_k \approx 0$ (observationally)
- Dark energy: $\Omega_\Lambda \approx 0.69$

The closure relation is:

$$\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda = 1$$

3.3 The Second Friedmann Equation

The acceleration equation is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

This can be derived from the first Friedmann equation and the continuity equation.

3.3.1 Equation of State

Each component has an equation of state $w = p/(\rho c^2)$:

- Matter (non-relativistic): $w_m = 0$, $p_m = 0$
- Radiation: $w_r = 1/3$, $p_r = \rho_r c^2/3$
- Dark energy (cosmological constant): $w_\Lambda = -1$, $p_\Lambda = -\rho_\Lambda c^2$

3.3.2 Acceleration Condition

The universe accelerates when:

$$\rho + \frac{3p}{c^2} < 0 \quad \Rightarrow \quad w < -\frac{1}{3}$$

Only dark energy (with $w = -1$) satisfies this in the standard model.

3.4 The Fluid Equation

Energy conservation gives:

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0$$

For a component with constant w:

$$\rho \propto a^{-3(1+w)}$$

Therefore:

- Matter: $\rho_m \propto a^{-3}$ (volume dilution)
- Radiation: $\rho_r \propto a^{-4}$ (volume dilution + wavelength stretching)
- Dark energy: $\rho_\Lambda \propto a^0$ (constant)

4 Solutions to Friedmann Equations

4.1 Single-Component Universes

For a universe dominated by a single component with equation of state w:

$$H^2 = \frac{8\pi G}{3}\rho_0 a^{-3(1+w)}$$

where ρ_0 is the density today. This gives:

$$a(t) \propto t^{2/(3(1+w))}$$

4.1.1 Matter-Dominated Universe

For $w = 0$:

$$a(t) \propto t^{2/3}$$

$$H(t) = \frac{2}{3t}$$

$$\rho(t) = \frac{1}{6\pi G t^2}$$

The age is:

$$t_0 = \frac{2}{3H_0}$$

4.1.2 Radiation-Dominated Universe

For $w = 1/3$:

$$a(t) \propto t^{1/2}$$

$$H(t) = \frac{1}{2t}$$

$$\rho(t) = \frac{3}{32\pi G t^2}$$

4.1.3 Dark Energy Dominated Universe

For $w = -1$:

$$a(t) \propto e^{Ht}$$

This gives exponential (de Sitter) expansion with constant H.

4.2 ΛCDM Model

The standard cosmological model includes matter and dark energy:

$$H^2 = H_0^2[\Omega_m a^{-3} + \Omega_\Lambda]$$

4.2.1 Exact Solution

The scale factor evolves as:

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t\right)$$

4.2.2 Matter-Lambda Equality

The two components were equal when:

$$\Omega_m a_{eq}^{-3} = \Omega_\Lambda$$

$$a_{eq} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \approx 0.76$$

This corresponds to $z_{eq} \approx 0.32$ or about 3.5 Gyr ago. Before this, the universe was decelerating; after, it accelerates.

4.3 Age of the Universe

The age can be computed by integrating:

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{aH(a)}$$

For ΛCDM:

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}}$$

This integral can be evaluated to give:

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \ln\left(\frac{1+\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_m}}\right)$$

For standard values ($\Omega_m = 0.31$, $\Omega_\Lambda = 0.69$, $H_0 = 68$):

$$t_0 \approx 13.8 \text{ Gyr}$$

5 Distances in Cosmology

5.1 Comoving Distance

The comoving distance is the distance in coordinates that expand with the universe:

$$d_c = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = c\int_0^z \frac{dz'}{H(z')}$$

For flat ΛCDM:

$$d_c(z) = \frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}$$

5.1.1 Low Redshift Approximation

For $z \ll 1$:

$$d_c \approx \frac{cz}{H_0}\left[1 - \frac{z}{2}(1+q_0)\right]$$

where q_0 is the deceleration parameter:

$$q_0 = \frac{\Omega_m}{2} - \Omega_\Lambda \approx -0.54$$

5.2 Proper Distance

The proper (physical) distance at time t is:

$$d_p(t) = a(t) \cdot d_c$$

At present ($a = 1$), $d_p = d_c$.

5.3 Luminosity Distance

The luminosity distance is defined so that the inverse square law holds:

$$F = \frac{L}{4\pi d_L^2}$$

where F is the observed flux and L is the intrinsic luminosity.
For a flat universe:

$$d_L = (1+z)d_c = d_c(1+z)$$

The factor of $(1+z)$ accounts for two effects:

- Photons are redshifted: each photon has energy reduced by $(1+z)$
- Arrival rate is reduced: photons arrive at a slower rate by factor $(1+z)$

5.4 Angular Diameter Distance

The angular diameter distance relates physical size to angular size:

$$\theta = \frac{D}{d_A}$$

For a flat universe:

$$d_A = \frac{d_c}{1+z} = \frac{d_L}{(1+z)^2}$$

5.4.1 Distance Duality Relation

The general relation between luminosity and angular diameter distances is:

$$d_L = (1+z)^2 d_A$$

This holds regardless of cosmological model, assuming photon number conservation.

5.5 Distance Modulus

In observational astronomy, distances are often expressed via the distance modulus:

$$\mu = m - M = 5\log_{10}\left(\frac{d_L}{10\ \mathrm{pc}}\right)$$

where m is apparent magnitude and M is absolute magnitude.
For cosmological distances:

$$\mu = 5\log_{10}\left(\frac{d_L(z)}{\mathrm{Mpc}}\right) + 25$$

6 Thermal History of the Universe

6.1 Temperature Evolution

For radiation and matter:

$$T \propto a^{-1} \propto (1+z)$$

The CMB temperature today is $T_0 = 2.7255$ K, so at redshift z:

$$T(z) = T_0(1+z) = 2.7255(1+z)\text{ K}$$

6.2 Planck Era ($t < 10^{-43}$ s)

At times earlier than the Planck time:

$$t_P = \sqrt{\frac{\hbar G}{c^5}} = 5.39 \times 10^{-44}\text{ s}$$

quantum gravity effects dominate and our current theories break down. The corresponding energy scale is:

$$E_P = \sqrt{\frac{\hbar c^5}{G}} = 1.22 \times 10^{19}\text{ GeV}$$

6.3 Grand Unification Era (10^{-43} s $< t < 10^{-36}$ s)

At $t \sim 10^{-36}$ s, $T \sim 10^{27}$ K, the strong nuclear force separates from the electroweak force. The universe undergoes rapid inflation during this epoch.

6.3.1 Inflation

A period of exponential expansion:

$$a(t) \propto e^{Ht}$$

with $H \sim 10^{36}$ s^{-1}. In a tiny fraction of a second ($\Delta t \sim 10^{-32}$ s), the universe expanded by a factor of e^{60} or more. Inflation solves several problems:

Horizon Problem: Causally disconnected regions have the same temperature because they were in causal contact before inflation.

Flatness Problem: Inflation drives Ω very close to 1, explaining why the universe is nearly flat today.

Monopole Problem: Dilutes primordial relics to undetectable levels.

6.4 Quark Era (10^{-12} s $< t < 10^{-6}$ s)

At $T > 10^{12}$ K, quarks and gluons exist freely in a quark-gluon plasma. As the universe cools below $T \sim 10^{13}$ K ($E \sim 1$ GeV), quarks combine into hadrons (protons, neutrons) via confinement. This is the QCD phase transition.

6.5 Lepton Era (10^{-6} s $< t <$ 10 s)

Leptons (electrons, positrons, neutrinos) dominate the energy density. Key events:

6.5.1 Neutrino Decoupling ($t \sim 1$ s, $T \sim 10^{10}$ K)

When T drops below ~ 1 MeV, weak interactions become too slow to maintain equilibrium. Neutrinos decouple and free-stream. Today they form the cosmic neutrino background at $T_\nu \approx 1.95$ K.

6.5.2 Electron-Positron Annihilation ($t \sim 6$ s, $T \sim 5 \times 10^9$ K)

When $kT < m_e c^2 = 0.511$ MeV, e^+e^- pairs annihilate. The energy goes into photons, heating them relative to neutrinos:

$$\frac{T_\gamma}{T_\nu} = \left(\frac{11}{4}\right)^{1/3} = 1.401$$

6.6 Big Bang Nucleosynthesis (BBN) ($t \sim 10$ s to 20 min)

When $T \sim 10^9$ K ($kT \sim 0.1$ MeV), nuclear reactions synthesize light elements.

6.6.1 Deuterium Bottleneck

Deuterium formation ($p + n \to D + \gamma$) is exothermic by 2.22 MeV, but photodisintegration ($\gamma + D \to p + n$) is efficient while $T > 10^9$ K due to the high photon-to-baryon ratio ($\eta \sim 10^{-10}$).

6.6.2 Neutron-Proton Ratio

At $T \gg m_n - m_p = 1.29$ MeV, weak interactions maintain:

$$\frac{n}{p} = e^{-(m_n - m_p)c^2/kT}$$

At freeze-out ($T \sim 0.8$ MeV, $t \sim 1$ s):

$$\frac{n}{p} \approx \frac{1}{6}$$

Free neutrons decay with half-life $\tau_{1/2} = 10.2$ min. By the time BBN begins in earnest ($t \sim 3$ min), this ratio has decreased to about 1/7.

6.6.3 Primordial Abundances

Nearly all free neutrons end up in ^{4}He:

$$Y_P = \frac{2(n/p)}{1 + (n/p)} \approx 0.25$$

where Y_P is the mass fraction of ^{4}He. Other abundances (by number relative to H):

- D/H $\sim 2.5 \times 10^{-5}$
- ^{3}He/H $\sim 10^{-5}$
- ^{7}Li/H $\sim 10^{-10}$

These predictions agree remarkably with observations and provide strong evidence for the Big Bang.

6.7 Matter-Radiation Equality ($t \sim 50,000$ yr)

When $a = a_{eq}$:

$$\rho_m = \rho_r$$

$$\Omega_m a_{eq}^{-3} = \Omega_r a_{eq}^{-4}$$

$$a_{eq} = \frac{\Omega_r}{\Omega_m}$$

For standard values:

$$z_{eq} = \frac{\Omega_m}{\Omega_r} - 1 \approx 3400$$

$$T_{eq} \approx 9300 \text{ K}$$

Before this epoch, the universe was radiation-dominated; after, matter-dominated. This transition is crucial for structure formation.

6.8 Recombination ($t \sim 380,000$ yr)

Recombination occurs when electrons and protons combine to form neutral hydrogen. The relevant reaction is:

$$p + e^- \leftrightarrow H + \gamma$$

6.8.1 Saha Equation

The ionization fraction in equilibrium is given by the Saha equation:

$$\frac{n_e n_p}{n_H} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} e^{-E_I/kT}$$

where $E_I = 13.6$ eV is the ionization energy of hydrogen.

Including the photon-to-baryon ratio $\eta \sim 10^{-9}$, recombination occurs around:

$$z_{rec} \approx 1100, \quad T_{rec} \approx 3000 \text{ K}$$

6.8.2 Photon Decoupling

As the ionization fraction drops, Thomson scattering becomes inefficient and photons decouple from matter. These photons, now redshifted to $T = 2.7$ K, form the cosmic microwave background (CMB).

The surface of last scattering has thickness:

$$\Delta z \sim 80$$

corresponding to the transition from $X_e \approx 0.9$ to $X_e \approx 0.1$.

6.9 Dark Ages ($z \sim 1100$ to $z \sim 20$)

After recombination, the universe was dark - no stars had yet formed. This epoch lasted for ~ 100 million years.

6.10 Reionization ($z \sim 20$ to $z \sim 6$)

The first stars and galaxies formed and their UV radiation reionized the intergalactic medium. Observations of the Gunn-Peterson trough in quasar spectra indicate reionization was complete by $z \sim 6$.

7 Structure Formation

7.1 Jeans Instability

Small density perturbations grow via gravitational instability. In a static medium, the Jeans length is:

$$\lambda_J = c_s \sqrt{\frac{\pi}{G\rho}}$$

where c_s is the sound speed. Perturbations with $\lambda > \lambda_J$ collapse; smaller ones oscillate as sound waves.

The corresponding Jeans mass is:

$$M_J = \frac{4\pi}{3}\rho \left(\frac{\lambda_J}{2}\right)^3 = \frac{\pi^{5/2} c_s^3}{6(G^3\rho)^{1/2}}$$

7.1.1 Matter-Dominated Era

In the matter-dominated era, perturbations in dark matter grow as:

$$\delta \equiv \frac{\delta\rho}{\rho} \propto a \propto t^{2/3}$$

This linear growth continues until $\delta \sim 1$, after which nonlinear collapse occurs.

7.1.2 Radiation-Dominated Era

During radiation domination, perturbations inside the horizon grow only logarithmically:

$$\delta \propto \ln a$$

This slow growth during radiation domination is why structure formation requires dark matter - baryons are coupled to photons and cannot collapse until after recombination.

7.2 Transfer Function and Power Spectrum

The matter power spectrum describes density fluctuations:

$$P(k) = \langle |\delta_k|^2 \rangle$$

where k is the wavenumber. Inflation predicts a nearly scale-invariant primordial spectrum:

$$P(k) \propto k^{n_s}$$

with spectral index $n_s \approx 0.96$ (slightly "red" or tilted toward larger scales).

The transfer function $T(k)$ relates primordial to present-day perturbations:

$$P(k, z) = T^2(k) P_{\text{prim}}(k) D^2(z)$$

where $D(z)$ is the growth function.

7.3 CMB Anisotropies

Temperature fluctuations in the CMB arise from:

- Sachs-Wolfe effect: gravitational potential perturbations
- Acoustic oscillations: sound waves in the baryon-photon fluid
- Doppler shifts: bulk motion of the fluid

7.3.1 Angular Power Spectrum

The CMB temperature is expanded in spherical harmonics:

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

The angular power spectrum is:

$$C_\ell = \langle |a_{\ell m}|^2 \rangle$$

The multipole ℓ corresponds to angular scale $\theta \sim 180/\ell$.

7.3.2 Acoustic Peaks

The CMB power spectrum shows a series of acoustic peaks. The first peak at $\ell \approx 220$ corresponds to the sound horizon at recombination:

$$r_s = \int_0^{z_{rec}} \frac{c_s dz}{H(z)}$$

The angular scale of this feature constrains spatial curvature. Observations show the universe is flat to high precision: $\Omega_k = 0.001 \pm 0.002$.

8 Observational Cosmology

8.1 Standard Candles

Objects with known intrinsic luminosity can measure distances.

8.1.1 Type Ia Supernovae

SNe Ia result from white dwarf detonation at the Chandrasekhar limit ($M \approx 1.4 M_\odot$). Their standardizable peak luminosity makes them excellent distance indicators up to $z \sim 2$.

The Phillips relation correlates peak luminosity with light curve decline rate, allowing $\sim 15\%$ distance precision.

Discovery of Acceleration: In 1998, SNe Ia surveys discovered that distant supernovae were fainter than expected in a decelerating universe, providing the first direct evidence for cosmic acceleration and dark energy.

8.1.2 Cepheid Variables

Cepheids obey a period-luminosity relation:

$$M_V = -2.43(\log P - 1) - 4.05$$

where P is the pulsation period in days. They're crucial for calibrating the distance ladder to ~ 30 Mpc.

8.2 Standard Rulers

Features with known physical size can measure angular diameter distances.

8.2.1 Baryon Acoustic Oscillations (BAO)

Sound waves in the early universe imprint a characteristic scale (~ 150 Mpc today) in galaxy clustering. This "standard ruler" has been measured in galaxy surveys up to $z \sim 2$, providing complementary constraints to SNe Ia.

8.2.2 CMB Acoustic Scale

The angular size of the first acoustic peak in the CMB ($\theta \approx 1$) is a standard ruler at $z = 1100$.

8.3 Gravitational Lensing

Massive objects deflect light. The deflection angle for a point mass M at impact parameter b is:

$$\alpha = \frac{4GM}{c^2 b}$$

8.3.1 Strong Lensing

When a massive galaxy lies along the line of sight to a background source, multiple images can form. The Einstein radius is:

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{d_{LS}}{d_L d_S}}$$

where d_L, d_S, and d_{LS} are angular diameter distances.

Time delays between images depend on both geometry and the Hubble constant: $\Delta t = \frac{1+z_L}{c}\frac{d_L d_S}{d_{LS}}\Delta\phi$

where $\Delta\phi$ is the Fermat potential difference between the images. Measuring time delays provides an independent determination of H_0 (the "time-delay cosmography" method).

8.3.2 Weak Lensing

Weak lensing produces small coherent distortions in background galaxy shapes. The convergence κ and shear γ depend on the projected matter distribution. Weak lensing surveys map dark matter on cosmic scales.

8.4 21-cm Cosmology

Neutral hydrogen emits 21-cm radiation ($\nu = 1420$ MHz) from hyperfine splitting. The brightness temperature relative to the CMB is: $T_b \propto (1+z)x_{HI}\left(1 - \frac{T_{CMB}}{T_S}\right)$

where x_{HI} is the neutral fraction and T_S is the spin temperature.

8.4.1 Cosmic Dawn

The 21-cm signal during the dark ages and reionization encodes information about the first stars and galaxies. Future radio telescopes aim to detect this signal from $z \sim 6$ to $z \sim 30$.

9 Dark Matter

9.1 Evidence for Dark Matter

Multiple independent observations require non-baryonic matter:

9.1.1 Galaxy Rotation Curves

For a visible disk with mass $M(r)$, Kepler's law predicts: $v(r) = \sqrt{\frac{GM(r)}{r}} \propto r^{-1/2}$

Observations show flat rotation curves: $v(r) \approx$ constant at large r. This requires: $M(r) \propto r$

A dark matter halo provides the missing mass.

9.1.2 Galaxy Clusters

The virial theorem relates velocity dispersion to mass: $M = \frac{3\sigma_v^2 R}{G}$

where σ_v is the velocity dispersion. Observed dispersions require $M/L \sim 200 - 300$ (solar units), much larger than visible matter accounts for.

9.1.3 Gravitational Lensing

Weak lensing directly maps the total matter distribution, confirming the presence of dark halos around galaxies and in clusters. The Bullet Cluster (1E 0657-56) shows clear spatial separation between the visible (baryonic) matter and the lensing (total) mass, providing direct evidence that dark matter is collisionless.

9.1.4 CMB and BBN

The CMB acoustic peaks constrain the total matter density: $\Omega_m h^2 \approx 0.14$.

BBN constrains the baryon density: $\Omega_b h^2 \approx 0.022$.

The difference requires $\Omega_{DM} \approx 0.26$, independent of dynamical measurements.

9.2 Dark Matter Candidates

9.2.1 WIMPs (Weakly Interacting Massive Particles)

Particles with mass $\sim 10 - 1000$ GeV that interact via the weak nuclear force. The thermal relic abundance naturally gives $\Omega_{DM} \sim 0.3$ if: $\langle \sigma v \rangle \sim 3 \times 10^{-26}$ cm^{3}s^{-1}

This "WIMP miracle" motivates extensive experimental searches.

9.2.2 Axions

Ultra-light bosons ($m \sim 10^{-5}$ eV) arising from solutions to the strong CP problem in QCD. Axion DM behaves as a cold condensate on galaxy scales.

9.2.3 Primordial Black Holes

Black holes formed in the early universe from density perturbations. Observations constrain various mass ranges, but some windows remain open, particularly around $M \sim 10^{-10} M_\odot$ and $M \sim 10 - 100 M_\odot$.

9.3 Structure of Dark Matter Halos

N-body simulations predict a universal density profile (NFW profile): $\rho(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}$

where ρ_s and r_s are characteristic density and radius. The profile scales as:

- Inner: $\rho \propto r^{-1}$ (cusp)
- Outer: $\rho \propto r^{-3}$

10 Dark Energy and Modified Gravity

10.1 The Cosmological Constant

The simplest dark energy model is Einstein's cosmological constant Λ: $G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

This is equivalent to vacuum energy with $w = -1$ and density: $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$

10.1.1 The Cosmological Constant Problem

Quantum field theory predicts vacuum energy density: $\rho_{vac,QFT} \sim \frac{E^4_{cutoff}}{\hbar^3 c^3}$

Using the Planck scale: $\rho_{vac,QFT} \sim 10^{113}$ J/m^3.

The observed value is: $\rho_\Lambda \sim 10^{-9}$ J/m^3.

This 10^{122} discrepancy is the worst fine-tuning problem in physics.

10.2 Dynamical Dark Energy

Alternative models allow w to vary with time.

10.2.1 Quintessence

A scalar field ϕ with potential $V(\phi)$ can drive acceleration. The equation of state is: $w = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}$

If $V(\phi) > \dot{\phi}^2/2$, then $w < -1/3$ and the universe accelerates.

10.2.2 Parameterized Dark Energy

Observations often parameterize: $w(a) = w_0 + w_a(1-a) = w_0 + w_a \frac{z}{1+z}$

Current constraints: $w_0 = -1.03 \pm 0.03$, $w_a = -0.3 \pm 0.4$ (consistent with Λ).

10.3 Modified Gravity

Rather than dark energy, gravity might deviate from GR on cosmological scales.

10.3.1 $f(R)$ Gravity

Replace the Ricci scalar R in the Einstein-Hilbert action with $f(R)$: $S = \int d^4x \sqrt{-g} \frac{f(R)}{16\pi G}$

For $f(R) = R + \alpha R^2$, this can mimic ΛCDM but with potentially observable deviations.

10.3.2 Tests of GR

The growth rate of structure tests gravity: $f\sigma_8(z) = \sigma_8(z)\frac{d\ln\delta}{d\ln a}$

GR predicts: $f \approx \Omega_m^{0.55}$.

So far, observations are consistent with GR at the $\sim 10\%$ level.

11 Beyond the Standard Model

11.1 Inflationary Perturbations

Quantum fluctuations during inflation seed structure. The power spectrum is: $P_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_0}\right)^{n_s-1}$

where $\mathcal{R}$ is the curvature perturbation. Planck measures: $A_s = 2.1 \times 10^{-9}, \quad n_s = 0.965$

11.1.1 Tensor Modes

Inflation also predicts gravitational waves with amplitude: $r = \frac{P_t}{P_{\mathcal{R}}}$

where P_t is the tensor power spectrum. The tensor-to-scalar ratio r constrains inflation models: $r < 0.036 \quad (95\%\ \mathrm{CL})$

11.2 Primordial Non-Gaussianity

The primordial perturbations are nearly Gaussian, but small deviations encode information about inflation. The bispectrum is parameterized by f_{NL}: $\mathcal{R}(\vec{x}) = \mathcal{R}_G(\vec{x}) + f_{NL}[\mathcal{R}_G^2(\vec{x}) - \langle\mathcal{R}_G^2\rangle]$

Current constraints: $f_{NL}^{local} = 0.8 \pm 5.0$.

11.3 Neutrino Cosmology

Neutrinos are the only Standard Model dark matter component.

11.3.1 Neutrino Mass Constraints

Massive neutrinos suppress small-scale structure. The CMB and large-scale structure constrain: $\sum m_\nu < 0.12\ \mathrm{eV} \quad (95\%\ \mathrm{CL})$

This is remarkable sensitivity to particle physics from cosmology! Oscillation experiments require $\sum m_\nu > 0.06$ eV.

11.3.2 Effective Number of Neutrino Species

The radiation density at BBN and recombination is parameterized by: $\rho_r = \rho_\gamma \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{eff}\right]$

The Standard Model predicts $N_{eff} = 3.046$. Measurements give: $N_{eff} = 2.99 \pm 0.17$ consistent with three neutrino species.

12 Alternative Cosmologies

12.1 Cyclic Models

Some models propose the universe undergoes infinite cycles of expansion and contraction, avoiding the initial singularity. The ekpyrotic scenario involves colliding branes in extra dimensions.

12.2 Steady State Theory

Once a competitor to the Big Bang, steady state theory proposed continuous matter creation to maintain constant density despite expansion. It's now ruled out by:

- The CMB (no natural explanation in steady state)
- Evolution of galaxy populations
- Light element abundances

12.3 Inhomogeneous Models

Most cosmological models assume perfect homogeneity. Void models place us near the center of a large underdensity, potentially explaining acceleration without dark energy. However, they predict strong kinematic dipoles inconsistent with observations.

13 Practice Problems

1. **Multi-Component Universe Evolution**

   Consider a flat universe with matter (Ω_m), radiation (Ω_r), and dark energy with equation of state $w = -0.8$ (not a cosmological constant).

   (a) Write the Friedmann equation $H(a)$ for this universe in terms of H_0, Ω_m, Ω_r, and Ω_w (the dark energy density parameter today).

   (b) For $\Omega_m = 0.30$, $\Omega_r = 9\times10^{-5}$, and $\Omega_w = 0.70$, calculate the scale factor $a_{eq,mr}$ when matter and radiation had equal densities, and $a_{eq,m\Lambda}$ when matter and dark energy had equal densities. What are the corresponding redshifts?

   (c) Determine whether this universe will expand forever or recollapse. Calculate the deceleration parameter q_0 today.

   (d) At what scale factor a_{acc} did the universe transition from deceleration to acceleration? Express your answer in terms of Ω_m and Ω_w only (neglect radiation).

   (e) If $H_0 = 70$ km s^{-1} Mpc^{-1}, numerically compute the age of the universe by evaluating: $t_0 = \frac{1}{H_0}\int_0^1 \frac{da}{a\sqrt{\Omega_r a^{-4}+\Omega_m a^{-3}+\Omega_w a^{-3(1+w)}}}$

   You may use numerical integration or Simpson's rule with at least 10 intervals.

2. **CMB Temperature and Primordial Abundances**

The cosmic microwave background has temperature $T_0 = 2.725$ K today.

(a) Calculate the number density of CMB photons today using: $n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$ where $\zeta(3) = 1.202$. What is the photon-to-baryon ratio $\eta = n_\gamma/n_b$ if the baryon density is $\Omega_b h^2 = 0.022$?

(b) At what redshift was the CMB temperature equal to $T = 3000$ K (approximately when recombination occurred)?

(c) Using the Saha equation, estimate the ionization fraction $X_e = n_e/(n_e + n_H)$ at $z = 1100$ and $T = 3000$ K. The Saha equation is: $\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} e^{-E_I/kT}$ where $E_I = 13.6$ eV. Use $n_b = \eta n_\gamma (1+z)^3$ where n_γ is from part (a).

(d) During Big Bang Nucleosynthesis at $t \sim 3$ minutes, the neutron-to-proton ratio was approximately $n/p = 1/7$. Calculate the mass fraction of ^{4}He formed: $Y_P = \frac{2(n/p)}{1+(n/p)}$

(e) If observations show $Y_P = 0.245$ instead of your calculated value, what would this imply about the expansion rate during BBN (faster or slower)?

3. **Distance Ladder and Cosmological Parameters**

A Type Ia supernova is observed in a distant galaxy.

(a) The supernova has apparent magnitude $m = 24.5$ and absolute magnitude (after standardization) $M = -19.3$. Calculate the luminosity distance in Mpc.

(b) Spectroscopic observations reveal the supernova has redshift $z = 0.85$. For a flat ΛCDM universe with $\Omega_m = 0.30$ and $\Omega_\Lambda = 0.70$, the comoving distance is given by: $d_c = \frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}$

Show that the luminosity distance is $d_L = (1+z)d_c$ and use this to determine H_0 in km s^{-1} Mpc^{-1}. You may evaluate the integral numerically.

(c) The same galaxy hosts Cepheid variables with periods $P_1 = 10$ days and $P_2 = 30$ days. Their apparent magnitudes are $m_1 = 26.8$ and $m_2 = 25.9$. Using the period-luminosity relation: $M_V = -2.43(\log P - 1) - 4.05$

determine if these measurements are consistent with the distance from the supernova.

(d) The angular size of an HII region in the same galaxy is measured to be $\theta = 0.15$ arcseconds, and its physical size is known from spectroscopy to be $D = 100$ pc. Calculate the angular diameter distance: $d_A = \frac{D}{\theta}$

Verify that it satisfies the distance duality relation: $d_L = (1+z)^2 d_A$

(e) Suppose a second Type Ia supernova is discovered at $z = 1.5$ with apparent magnitude $m = 26.2$. Is this supernova brighter or fainter than expected for the cosmological parameters above? Calculate the predicted apparent magnitude and compare.

14 Solutions to Practice Problems

Problem 1: Multi-Component Universe Evolution

(a) Friedmann equation:

For a flat universe ($k = 0$), the first Friedmann equation is:

$$H^2(a) = H_0^2 \left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_w a^{-3(1+w)}\right]$$

For dark energy with $w = -0.8$:

$$a^{-3(1+w)} = a^{-3(1-0.8)} = a^{-0.6}$$

Therefore:

$$H^2(a) = H_0^2 \left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_w a^{-0.6}\right]$$

(b) Equality epochs:

Matter-radiation equality occurs when $\rho_m = \rho_r$:

$$\Omega_m a_{eq,mr}^{-3} = \Omega_r a_{eq,mr}^{-4}$$

$$a_{eq,mr} = \frac{\Omega_r}{\Omega_m} = \frac{9 \times 10^{-5}}{0.30} = 3.0 \times 10^{-4}$$

The corresponding redshift:

$$z_{eq,mr} = \frac{1}{a_{eq,mr}} - 1 = \frac{1}{3.0 \times 10^{-4}} - 1 \approx 3333$$

Matter-dark energy equality occurs when $\rho_m = \rho_w$:

$$\Omega_m a_{eq,m\Lambda}^{-3} = \Omega_w a_{eq,m\Lambda}^{-0.6}$$

$$a_{eq,m\Lambda}^{-2.4} = \frac{\Omega_w}{\Omega_m} = \frac{0.70}{0.30} = 2.333$$

$$a_{eq,m\Lambda} = (2.333)^{-1/2.4} = 0.560$$

The corresponding redshift:

$$z_{eq,m\Lambda} = \frac{1}{0.560} - 1 = 0.786$$

(c) Future evolution and deceleration parameter:

The deceleration parameter is defined as:

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}$$

From the second Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i \rho_i (1 + 3w_i)$$

For our components:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho_r(1 + 3 \cdot 1/3) + \rho_m(1 + 0) + \rho_w(1 + 3(-0.8))]$$

$$= -\frac{4\pi G}{3}[2\rho_r + \rho_m - 1.4\rho_w]$$

Using $H^2 = \frac{8\pi G}{3}\rho_{total}$:

$$q_0 = \frac{1}{2}\sum_i \Omega_i(1 + 3w_i)$$

$$= \frac{1}{2}[\Omega_r(1 + 1) + \Omega_m(1) - \Omega_w(1.4)]$$

$$= \frac{1}{2}[2 \times 9 \times 10^{-5} + 0.30 - 0.70 \times 1.4]$$

$$= \frac{1}{2}[0.00018 + 0.30 - 0.98]$$

$$= -0.340$$

Since $q_0 < 0$, the universe is currently accelerating.

For the long-term future, dark energy with $w = -0.8$ dominates as $a \to \infty$ (since it decays slowest: $\propto a^{-0.6}$). The universe will expand forever, asymptotically approaching:

$$a(t) \propto t^{2/(3(1+w))} = t^{2/0.6} = t^{10/3}$$

This is faster than matter-dominated ($\propto t^{2/3}$) but slower than Λ (exponential).

(d) Acceleration transition:

Neglecting radiation, acceleration begins when $\ddot{a} = 0$:

$$\rho_m(1 + 3 \times 0) + \rho_w(1 + 3(-0.8)) = 0$$

$$\rho_m - 1.4\rho_w = 0$$

In terms of today's values:

$$\Omega_m a_{acc}^{-3} = 1.4\Omega_w a_{acc}^{-0.6}$$

$$a_{acc}^{-2.4} = \frac{1.4\Omega_w}{\Omega_m} = \frac{1.4 \times 0.70}{0.30} = 3.267$$

$$a_{acc} = (3.267)^{-1/2.4} = 0.493$$

This corresponds to $z_{acc} = 1.03$, about 7.7 Gyr ago.

(e) Age of the universe:

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{da}{a\sqrt{\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_w a^{-0.6}}}$$

Let $I = \int_0^1 \frac{da}{a\sqrt{9 \times 10^{-5} a^{-4} + 0.30a^{-3} + 0.70a^{-0.6}}}$

Using Simpson's rule with $n = 10$ intervals, $\Delta a = 0.1$:

$$I \approx \frac{\Delta a}{3}\left[f(a_0) + 4\sum_{odd} f(a_i) + 2\sum_{even} f(a_i) + f(a_{10})\right]$$

where $f(a) = \frac{1}{a\sqrt{9\times10^{-5}a^{-4}+0.30a^{-3}+0.70a^{-0.6}}}$.
Note: $f(0)$ diverges, so we start from $a = 0.01$:
Evaluating numerically (key values):

- $f(0.01) \approx 104.8$
- $f(0.1) \approx 10.95$
- $f(0.5) \approx 1.643$
- $f(1.0) \approx 1.085$

The integral evaluates to approximately $I \approx 0.964$.
Converting to physical time with $H_0 = 70$ km s^{-1} Mpc^{-1} $= 2.27 \times 10^{-18}$ s^{-1}:

$$t_0 = \frac{0.964}{2.27 \times 10^{-18}} = 4.25 \times 10^{17}\ \text{s} = 13.5\ \text{Gyr}$$

This is slightly younger than the standard ΛCDM age (13.8 Gyr) because $w = -0.8$ dark energy provides less early deceleration than $w = -1$.

Problem 2: CMB Temperature and Primordial Abundances

(a) Photon number density and η:

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{kT_0}{\hbar c}\right)^3$$

Substituting values:

- $k = 1.381 \times 10^{-23}$ J/K
- $T_0 = 2.725$ K
- $\hbar = 1.055 \times 10^{-34}$ J·s
- $c = 3 \times 10^8$ m/s
- $\zeta(3) = 1.202$

$$\frac{kT_0}{\hbar c} = \frac{1.381 \times 10^{-23} \times 2.725}{1.055 \times 10^{-34} \times 3 \times 10^8} = 1.189 \times 10^6\ \text{m}^{-1}$$

$$n_\gamma = \frac{2 \times 1.202}{\pi^2}(1.189 \times 10^6)^3 = \frac{2.404}{9.870}(1.681 \times 10^{18})$$

$$= 4.09 \times 10^8 \text{ m}^{-3}$$

For the baryon density, using $\Omega_b h^2 = 0.022$ and $\rho_{\text{crit},0} = 1.88 \times 10^{-26} h^2$ kg/m^3:

$$\rho_b = 0.022 \times 1.88 \times 10^{-26} h^2/h^2 = 4.14 \times 10^{-28} \text{ kg/m}^3$$

$$n_b = \frac{\rho_b}{m_p} = \frac{4.14 \times 10^{-28}}{1.673 \times 10^{-27}} = 0.247 \text{ m}^{-3}$$

The photon-to-baryon ratio:

$$\eta = \frac{n_\gamma}{n_b} = \frac{4.09 \times 10^8}{0.247} = 1.66 \times 10^9$$

Or inversely: $n_b/n_\gamma = 6.0 \times 10^{-10}$ (often quoted this way).

(b) Redshift of recombination:

Since $T \propto (1+z)$:

$$z_{rec} = \frac{T_{rec}}{T_0} - 1 = \frac{3000}{2.725} - 1 = 1101 - 1 = 1100$$

(c) Ionization fraction from Saha equation:

At $z = 1100$, $T = 3000$ K:

First calculate the photon density at that epoch:

$$n_\gamma(z) = n_\gamma(0)(1+z)^3 = 4.09 \times 10^8 \times (1100)^3 = 5.44 \times 10^{17} \text{ m}^{-3}$$

Baryon density:

$$n_b(z) = n_\gamma(z)/\eta = 5.44 \times 10^{17}/(1.66 \times 10^9) = 3.28 \times 10^8 \text{ m}^{-3}$$

The Saha equation right-hand side:

$$RHS = \frac{1}{n_b} \left(\frac{m_e kT}{2\pi\hbar^2} \right)^{3/2} e^{-E_I/kT}$$

Calculate each term:

$$\frac{m_e kT}{2\pi\hbar^2} = \frac{9.109 \times 10^{-31} \times 1.381 \times 10^{-23} \times 3000}{2\pi \times (1.055 \times 10^{-34})^2}$$

$$= \frac{3.774 \times 10^{-50}}{6.991 \times 10^{-68}} = 5.397 \times 10^{17} \text{ m}^{-3}$$

$$(RHS_1)^{3/2} = (5.397 \times 10^{17})^{3/2} = 1.254 \times 10^{27} \text{ m}^{-4.5}$$

The exponential term:

$$\frac{E_I}{kT} = \frac{13.6 \times 1.602 \times 10^{-19}}{1.381 \times 10^{-23} \times 3000} = \frac{2.179 \times 10^{-18}}{4.143 \times 10^{-20}} = 52.59$$

$$e^{-52.59} = 1.30 \times 10^{-23}$$

$$RHS = \frac{1.254 \times 10^{27} \times 1.30 \times 10^{-23}}{3.28 \times 10^8} = \frac{1.630 \times 10^4}{3.28 \times 10^8} = 4.97 \times 10^{-5}$$

From the Saha equation:

$$\frac{X_e^2}{1 - X_e} = 4.97 \times 10^{-5}$$

Assuming $X_e \ll 1$ (which we verify):

$$X_e^2 \approx 4.97 \times 10^{-5}$$

$$X_e \approx 0.0070 = 0.70\%$$

This low ionization fraction confirms that recombination is nearly complete at $z = 1100$.

(d) Helium mass fraction:

With $n/p = 1/7$:

$$Y_P = \frac{2 \times 1/7}{1 + 1/7} = \frac{2/7}{8/7} = \frac{2}{8} = 0.25$$

So 25% of baryonic mass is in ^{4}He.

(e) Interpretation of abundance discrepancy:

The observed $Y_P = 0.245$ is slightly less than the calculated 0.25. This could indicate:

The neutron-to-proton ratio at freeze-out was actually slightly lower than 1/7. This would occur if:

Slower expansion during BBN: If the universe expanded more slowly, weak interactions would stay in equilibrium longer, allowing more neutrons to decay before freeze-out. The equilibrium n/p ratio is:

$$\frac{n}{p} = e^{-\Delta mc^2/kT}$$

which decreases as temperature drops. Slower expansion means reaching freeze-out temperature at a later time when more neutrons have decayed.

Quantitatively, if the actual $n/p = x$ at BBN:

$$Y_P = \frac{2x}{1 + x} = 0.245$$

$$2x = 0.245(1 + x) = 0.245 + 0.245x$$

$$1.755x = 0.245$$

$$x = 0.140 = 1/7.14$$

This is consistent with slightly more neutron decay, indicating the expansion was indeed slightly slower (or equivalently, freeze-out occurred at slightly lower temperature/later time).

Alternatively, this could indicate a different effective number of neutrino species N_{eff}, which affects the expansion rate during BBN.

Problem 3: Distance Ladder and Cosmological Parameters

(a) Luminosity distance from distance modulus:

The distance modulus is:

$$\mu = m - M = 24.5 - (-19.3) = 43.8$$

The luminosity distance in Mpc is:

$$\mu = 5\log_{10}(d_L/\text{Mpc}) + 25$$

$$43.8 = 5\log_{10}(d_L) + 25$$

$$\log_{10}(d_L) = \frac{43.8 - 25}{5} = 3.76$$

$$d_L = 10^{3.76} = 5750 \text{ Mpc} = 5.75 \text{ Gpc}$$

(b) Comoving distance and Hubble constant determination:

The comoving distance integral is:

$$d_c = \frac{c}{H_0}\int_0^{0.85} \frac{dz'}{\sqrt{0.30(1+z')^3 + 0.70}}$$

Let's evaluate this numerically using the trapezoid rule with 100 steps:

$$\Delta z = 0.0085$$

Define $E(z) = \sqrt{0.30(1+z)^3 + 0.70}$:

Key values:

- $E(0) = \sqrt{0.70} = 0.837$
- $E(0.425) = \sqrt{0.30(1.425)^3 + 0.70} = \sqrt{0.868 + 0.70} = 1.252$
- $E(0.85) = \sqrt{0.30(1.85)^3 + 0.70} = \sqrt{1.898 + 0.70} = 1.612$

Using numerical integration:

$$\int_0^{0.85} \frac{dz'}{E(z')} \approx 0.631$$

Therefore:

$$d_c = \frac{c}{H_0} \times 0.631$$

The luminosity distance is:

$$d_L = (1+z)d_c = 1.85 \times \frac{c}{H_0} \times 0.631 = \frac{1.167c}{H_0}$$

We know $d_L = 5750$ Mpc, so:

$$5750 = \frac{1.167c}{H_0}$$

With $c = 3 \times 10^5$ km/s:

$$H_0 = \frac{1.167 \times 3 \times 10^5}{5750} = 60.9 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

This is somewhat lower than typical measurements, suggesting either observational uncertainty or tension with the assumed cosmological parameters.

(c) Cepheid consistency check:

For Cepheid 1 with $P_1 = 10$ days:

$$M_{V,1} = -2.43(\log 10 - 1) - 4.05 = -2.43(1 - 1) - 4.05 = -4.05$$

Distance modulus:

$$\mu_1 = m_1 - M_{V,1} = 26.8 - (-4.05) = 30.85$$

For Cepheid 2 with $P_2 = 30$ days:

$$M_{V,2} = -2.43(\log 30 - 1) - 4.05 = -2.43(1.477 - 1) - 4.05$$

$$= -2.43(0.477) - 4.05 = -1.16 - 4.05 = -5.21$$

Distance modulus:

$$\mu_2 = m_2 - M_{V,2} = 25.9 - (-5.21) = 31.11$$

The two Cepheids give slightly different distance moduli (30.85 vs 31.11), a difference of 0.26 mag. This corresponds to a distance discrepancy of:

$$\frac{d_2}{d_1} = 10^{0.26/5} = 10^{0.052} = 1.127$$

or about 13% difference. This is marginally consistent given typical uncertainties in Cepheid measurements ($\sim 0.1 - 0.2$ mag) and possible differential extinction.

The average distance modulus is $\mu_{avg} = 30.98$, giving:

$$d_{Cep} = 10^{(30.98-25)/5} = 10^{1.196} = 15.7 \text{ Mpc}$$

However, from the supernova we found $d_L = 5750$ Mpc. This enormous discrepancy indicates the Cepheids and supernova cannot be in the same galaxy!

The most likely explanation: the Cepheids are in a foreground galaxy at $d \approx 16$ Mpc, while the supernova is in a background galaxy at $d \approx 5750$ Mpc. At $z = 0.85$, individual Cepheids cannot be resolved - the measurements given must be hypothetical or there's an error in the problem setup.

(d) Angular diameter distance:

Convert angular size: $\theta = 0.15$ arcsec $= 0.15/(206265)$ rad $= 7.27 \times 10^{-7}$ rad

Physical size: $D = 100$ pc $= 100 \times 3.086 \times 10^{16}$ m $= 3.086 \times 10^{18}$ m

$$d_A = \frac{D}{\theta} = \frac{3.086 \times 10^{18}}{7.27 \times 10^{-7}} = 4.24 \times 10^{24} \text{ m}$$

Converting to Mpc:

$$d_A = \frac{4.24 \times 10^{24}}{3.086 \times 10^{22}} = 137 \text{ Mpc}$$

Now check the distance duality relation:

$$d_L = (1+z)^2 d_A = (1.85)^2 \times 137 = 3.42 \times 137 = 469 \text{ Mpc}$$

But we found $d_L = 5750$ Mpc from the supernova! The ratio is:

$$\frac{5750}{469} = 12.3$$

This large discrepancy again suggests different redshifts for different objects, or systematic errors in the measurements. For consistency at $z = 0.85$:

From $d_L = 5750$ Mpc:

$$d_A = \frac{5750}{(1.85)^2} = \frac{5750}{3.42} = 1681 \text{ Mpc}$$

This would require:

$$\theta = \frac{100 \text{ pc}}{1681 \text{ Mpc}} = \frac{100}{1.681 \times 10^6} = 5.95 \times 10^{-5} \text{ rad} = 12.3 \text{ arcsec}$$

So the HII region should subtend 12.3 arcseconds, not 0.15 arcseconds, if it's at the same redshift as the supernova.

(e) Second supernova prediction:

For $z = 1.5$ in our ΛCDM model:

$$d_c = \frac{c}{H_0} \int_0^{1.5} \frac{dz'}{\sqrt{0.30(1+z')^3 + 0.70}}$$

Evaluating numerically:

$$\int_0^{1.5} \frac{dz'}{E(z')} \approx 1.065$$

$$d_c = \frac{c}{H_0} \times 1.065$$

Using $H_0 = 60.9$ km s^{-1} Mpc^{-1} from part (b):

$$d_c = \frac{3 \times 10^5 \times 1.065}{60.9} = 5246 \text{ Mpc}$$

$$d_L = (1+z) d_c = 2.5 \times 5246 = 13,115 \text{ Mpc}$$

The predicted apparent magnitude is:

$$m = M + 5 \log_{10}(d_L/\text{Mpc}) + 25$$

$$= -19.3 + 5 \log_{10}(13115) + 25$$

$$= -19.3 + 5 \times 4.118 + 25$$
$$= -19.3 + 20.59 + 25 = 26.3$$

The observed magnitude is $m_{obs} = 26.2$, which is slightly brighter (lower magnitude) than predicted ($m_{pred} = 26.3$). The difference is:

$$\Delta m = 26.2 - 26.3 = -0.1 \text{ mag}$$

This is barely significant (within typical uncertainty of ~ 0.15 mag for SNe Ia). If real, it could suggest:

- Slightly lower Ω_m or higher Ω_Λ
- Evolution in SNe Ia properties with redshift
- Gravitational lensing magnification
- Systematic errors in standardization

Such small deviations are at the frontier of current cosmological constraints.

15 Problem-Solving Strategies

When approaching cosmology problems:

1. Identify the epoch: Different components dominate at different times. Check which terms in the Friedmann equation matter.

2. Check limiting cases: For $z \ll 1$, linear approximations often work. For $z \gg 1$, certain components dominate.

3. Use conservation: Energy conservation (fluid equation) and the Friedmann equations are related. Sometimes one is easier than the other.

4. Match units carefully: Cosmology mixes units: Mpc, km/s, Gyr, eV. Keep track of conversions.

5. Numerical integration: Many cosmological integrals have no closed form. Be comfortable with numerical methods.

6. Physical intuition: Does the answer make sense? Is $z_{rec} \sim 1100$? Is $t_0 \sim 14$ Gyr? Is $H_0 \sim 70$ km/s/Mpc?

7. Key scales to memorize:

- $t_0 \approx 13.8$ Gyr
- $H_0 \approx 70$ km s^{-1} Mpc^{-1}
- $T_{CMB} = 2.725$ K
- $z_{rec} \approx 1100$
- $z_{eq} \approx 3400$
- $\Omega_m \approx 0.3$, $\Omega_\Lambda \approx 0.7$
- $\Omega_b h^2 \approx 0.022$

16 Mathematical Toolkit

16.1 Useful Integrals

For flat ΛCDM: $\int \frac{dx}{\sqrt{x+\lambda x^{-2}}} = \frac{2}{3\sqrt{\lambda}} \sinh^{-1}\left(\sqrt{\frac{\lambda}{x^3}}\right)$
where $x = a$ and $\lambda = \Omega_\Lambda/\Omega_m$.
For pure matter: $\int \frac{da}{a^{1/2}} = 2a^{1/2}$
For pure radiation: $\int \frac{da}{a} = \ln a$

16.2 Series Expansions

For small z: $E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda} \approx \sqrt{\Omega_m + \Omega_\Lambda}\left(1 + \frac{3\Omega_m z}{2(\Omega_m+\Omega_\Lambda)}\right)$
$d_c(z) \approx \frac{cz}{H_0}\left[1 + \frac{z}{2}(1-q_0) + O(z^2)\right]$

16.3 Dimensionless Hubble Parameter

Often written as: $E(z) = \frac{H(z)}{H_0}$
So the Friedmann equation becomes: $E^2(z) = \Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda$