Confidence Interval

Today's Class

- Point Estimate
- Confidence Interval
  - Distribution: Normal vs. Not normal
  - Variance: known vs. unknown
  - Sample size: small vs. large

Estimating a Population Parameter

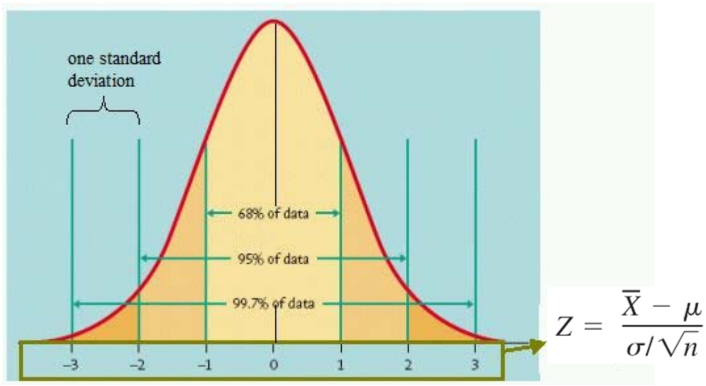
- What is the population mean?
  - Don't know μ? Estimate it.
  - How?
    - Take a sample (n=?)
    - Use $\bar{X}$ to estimate μ
- What is a point estimate?
  - A point estimate is a sample statistic used to estimate the corresponding population parameter

What is Point Estimate of the True Population Mean?

- Use the CLT to know
  - The sample mean ≈ the population mean
  - 68% of all possible sample means drawn from samples you took should be within one standard error of the mean
  - The Standard error = $\frac{\sigma_X}{\sqrt{n}}$
  - So take a large sample and you should have a sample mean very close to μ

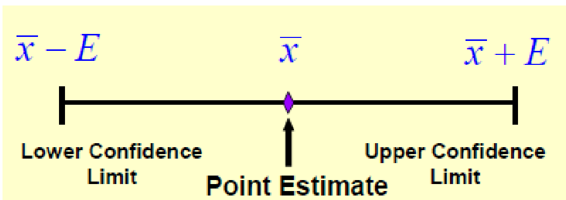
Normal Curve, revisit

- Approximate percentage of area within given standard deviations



Confidence Intervals

- Developed from sample data
- If all possible intervals of a given width were constructed, a percentage of these intervals, known as the ***confidence level***, would include the true population parameter



Confidence Intervals Example

- The internal pressure strength of glass bottles used to package a carbonated beverage is an important quality characteristic
  - Standard deviation: 10 psi
  - Sample mean: 182 psi
  - Sample size: 25
  - Find the 95% two-sided confidence interval

Table 4.1 Standard Normal Percentiles and Critical Values

Percentile	90	95	97.5	99	99.5	99.9	99.95
α (tail area)	.1	.05	.025	.01	.005	.001	.0005
$z_\alpha = 100(1 - \alpha)$th percentile	1.28	1.645	1.96	2.33	2.58	3.08	3.27

Solution

- n = 25
- $\bar{X}$ = 182 psi, σ = 10 psi
- 95% two-side confidence interval:
  - α = 0.05
  - $z_{\alpha/2} = z_{.025} = 1.96$

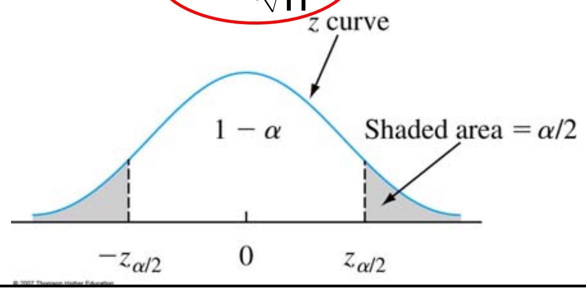
$$\bar{x} \pm 1.96 \cdot \frac{\sigma}{\sqrt{n}} = 182 \pm 1.96 \cdot \frac{10}{\sqrt{25}}$$
$$= 182 \pm 3.92$$
$$= (178.08, 185.92)$$

Confidence Intervals: Normal, STD known

- A 100(1-α)% confidence interval for the mean μ of a normal population when the value of σ is known:

$$\bar{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

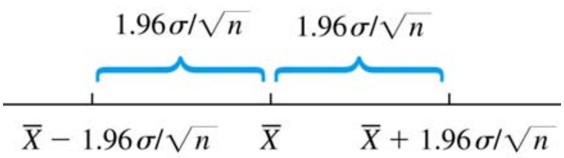
Margin of Error (ME)

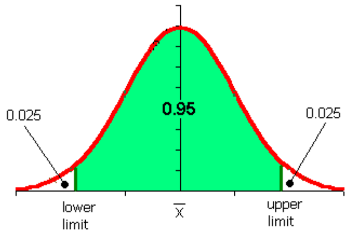


Confidence Intervals: Example

- A 95% confidence interval

$$\bar{x} \pm z_{(1-0.95)/2} \cdot \frac{\sigma}{\sqrt{n}} = \bar{x} \pm z_{0.025} \cdot \frac{\sigma}{\sqrt{n}}$$
$$= \bar{x} \pm 1.96 \cdot \frac{\sigma}{\sqrt{n}}$$

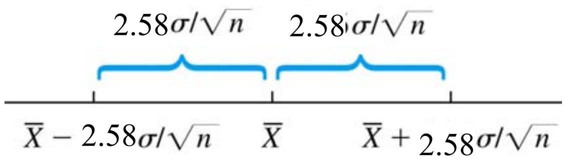


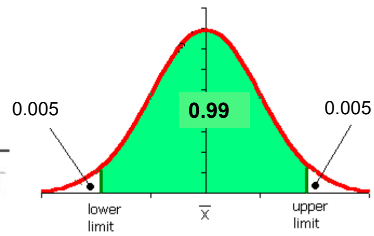


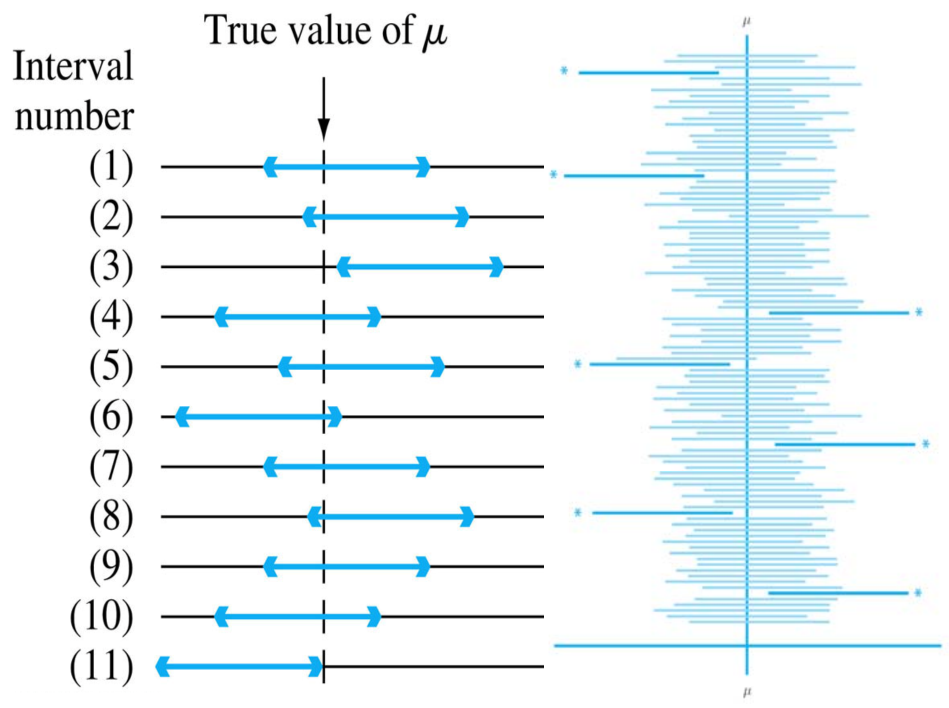
Confidence Intervals: Example

- A 99% confidence interval

$$\bar{x} \pm z_{(1-0.99)/2} \cdot \frac{\sigma}{\sqrt{n}} = \bar{x} \pm z_{0.005} \cdot \frac{\sigma}{\sqrt{n}}$$
$$= \bar{x} \pm 2.58 \cdot \frac{\sigma}{\sqrt{n}}$$







Interpreting a Confidence Interval

- A correct interpretation of “95% confidence” relied on the long-run relative frequency interpretation of probability
- Suppose we obtain another sample and compute another 95% interval, and so on. In the long run 95% of our computed CIs will contain μ.

Confidence Intervals Example

- The internal pressure strength of glass bottles used to package a carbonated beverage is assumed to be normal
  - Sample size: 25
  - Sample mean: 182 psi
  - Sample standard deviation: 10 psi
  - 95% two-sided confidence interval?

Solution

- n = 25
- $\bar{X}$ = 182 psi, s = 10 psi
- 95% two-sided confidence interval:
  - α =0.05
  - $\bar{x} \pm t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}} = 182 \pm t_{0.025,24} \cdot \frac{10}{\sqrt{25}}$

Confidence Interval for Normal, STD Unknown

- A 100(1-a)% two-sided confidence interval for the mean μ of a normal population with $\overline{x}$, the sample mean and s, the sample standard deviation from a random sample of size n:

$$\overline{x} \pm t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}$$

T-distribution

- When $\overline{X}$ is the mean of a random sample of size n from a Normal with mean μ, then

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

where S is sample standard deviation

- A t distribution (Appendix A.5) has one parameter, ν = n-1 degrees of freedom

df/α =	.40	.25	.10	.05	.025	.01	.005	.001	.0005
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	318.309	636.619
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.256	0.685	1.318	1.711	$\boxed{2.064}$	2.492	2.797	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.385	3.646
35	0.255	0.682	1.306	1.690	2.030	2.438	2.724	3.340	3.591
40	0.255	0.681	1.303	1.684	2.021	2.423	2.704	3.307	3.551
50	0.255	0.679	1.299	1.676	2.009	2.403	2.678	3.261	3.496
60	0.254	0.679	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	0.254	0.677	1.289	1.658	1.980	2.358	2.617	3.160	3.373
inf.	0.253	0.674	1.282	1.645	1.960	2.326	2.576	3.090	3.291

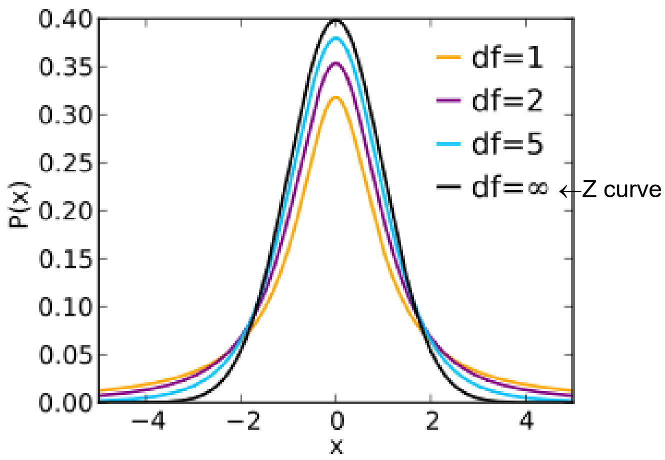
Solution

- n = 25
- $\bar{x}$ = 182 psi, s = 10 psi
- 95% two-sided confidence interval:
  - α =0.05
  - $\bar{x} \pm t_{\alpha/2,n-1} \cdot \dfrac{s}{\sqrt{n}} = 182 \pm t_{0.025,24} \cdot \dfrac{10}{\sqrt{25}}$

$$= 182 \pm 2.064 \cdot \frac{10}{\sqrt{25}}$$

= (177.87, 186.13)

T and Z Curve



Properties of *t* Distributions

- Let t_n denote the *t* distribution with *n* df.
  - Each t_n curve is bell-shaped and centered at 0
  - Each t_n curve is more spread out than the standard normal (z) curve
  - As n increases, the spread of the corresponding t_n curve decreases
  - As $n \to \infty$, the sequence of t_n curves approaches the standard normal curve

    (so the z curve is often called the t curve with df = ∞).

Example

- Consider the following sample of fat content (in percentage) of n = 10 randomly selected hot dogs ("Sensory and Mechanical Assessment of the Quality of Frankfurters," *J. of Texture Studies,* 1990: 395–409):

25.2	21.3	22.8	17.0	29.8	21.0
25.5	16.0	20.9	19.5		

Assuming that these were selected from a normal population distribution, what is a 95% CI for the population mean fat content?

Solution

- n = 10
- $\bar{X}$ = 21.90, s = 4.134
- 95% two-sided confidence interval:
  - α =0.05

$$\bar{x} \pm t_{.025,9} \cdot \frac{s}{\sqrt{n}}$$

$$= 21.90 \pm 2.262 \cdot \frac{4.134}{\sqrt{10}}$$

$$= 21.90 \pm 2.96$$

$$= (18.94, 24.86)$$

df/α =	.40	.25	.10	.05	.025	.01	.005	.001	.0005
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	318.309	636.619
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inf.	0.253	0.674	1.282	1.645	1.960	2.326	2.576	3.090	3.291

Large Sample CI: Not Necessarily Normal

- If n is sufficiently large (the CLT applies) then a large-sample confidence interval for μ with confidence level approximately $100(1-\alpha)\%$ is:

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

  - Generally about 30 observations

Large Sample CI: Not Necessarily Normal, STD Unknown

- If n is sufficiently large (the CLT applies) then a large-sample confidence interval for μ with confidence level approximately $100(1-\alpha)\%$ is:

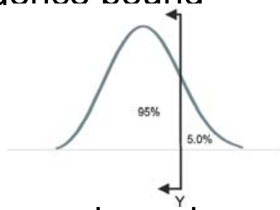
$$\bar{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

  - Note that it uses the sample STD
  - Generally at least 40 observations are needed

One-Sided Confidence Interval

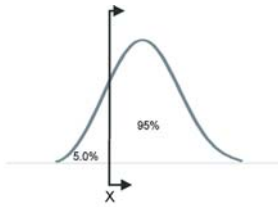
- A large-sample upper confidence bound for μ:

$$\mu < \bar{x} + z_{\alpha} \cdot \frac{s}{\sqrt{n}}$$



- A large-sample lower confidence bound for μ:

$$\mu > \bar{x} - z_{\alpha} \cdot \frac{s}{\sqrt{n}}$$



Example
One-sided CI

- A sample of 48 shear strength observations gave a sample mean strength of 17.17 N/mm^2 and a sample standard deviation of 3.28 N/mm^2. Find a lower confidence bound for true average shear strength μ with confidence level 95%.

Solution

- N= 48
- $\bar{X}$ =17.17 N/mm^2, S = 3.28 N/mm^2
- α = 0.05, $\therefore Z_{0.05}$=1.645

$$\mu > \bar{x} - z_\alpha \cdot \frac{s}{\sqrt{n}}$$

$$\mu > 17.17 - 1.645 \frac{3.28}{\sqrt{48}}$$

$$\mu > 16.39$$

Table 4.1 Standard Normal Percentiles and Critical Values

Percentile	90	95	97.5	99	99.5	99.9	99.95
α (tail area)	.1	.05	.025	.01	.005	.001	.0005
$z_\alpha = 100(1 - \alpha)$th percentile	1.28	1.645	1.96	2.33	2.58	3.08	3.27