## **Negative Binomial Distribution**

**Expected Value** 

$$\begin{split} \mathrm{E}[X] &= \sum_{i} f(x_{i}) \cdot x_{i} = \sum_{x=0}^{\infty} \binom{x+r-1}{r-1} p^{x} (1-p)^{r} \cdot x \\ \mathrm{E}[X] &= \binom{0+r-1}{r-1} p^{0} (1-p)^{r} \cdot 0 + \sum_{x=1}^{\infty} \binom{x+r-1}{r-1} p^{x} (1-p)^{r} \cdot x \\ \mathrm{E}[X] &= 0 + \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(r-1)! x!} p^{x} (1-p)^{r} \cdot x \\ \mathrm{E}[X] &= \frac{rp}{1-p} \sum_{x=1}^{\infty} \frac{(x+r-1)!}{r! (x-1)!} p^{x-1} (1-p)^{r+1} \end{split}$$

Now let s = r+1 and w=x-1 inside the summation.

$$egin{aligned} \mathrm{E}[X] &= rac{rp}{1-p} \sum_{w=0}^{\infty} rac{(w+s-1)!}{(s-1)!w!} p^w (1-p)^s \ \mathrm{E}[X] &= rac{rp}{1-p} \sum_{w=0}^{\infty} inom{w+s-1}{s-1} p^w (1-p)^s \end{aligned}$$

The summation is the sum over the complete pmf of a negative binomial rv, which is 1.

$$\mathrm{E}[X] = rac{rp}{1-p}$$

Variance

$$Var[X] = E[X^2] - (E[X])^2$$

$$egin{aligned} \mathrm{E}[X^2] &= \sum_i f(x_i) \cdot x^2 = \sum_{x=0}^\infty inom{x+r-1}{r-1} p^x (1-p)^r \cdot x^2 \ \mathrm{E}[X^2] &= 0 + \sum_{x=1}^\infty inom{x+r-1}{r-1} p^x (1-p)^r x^2 \ \mathrm{E}[X^2] &= \sum_{x=1}^\infty rac{(x+r-1)!}{(r-1)! x!} p^x (1-p)^r x^2 \ \mathrm{E}[X^2] &= rac{rp}{1-p} \sum_{x=1}^\infty rac{(x+r-1)!}{r! (x-1)!} p^{x-1} (1-p)^{r+1} x \end{aligned}$$

Again, let let s = r+1 and w=x-1.

$$egin{aligned} \mathrm{E}[X^2] &= rac{rp}{1-p} \sum_{w=0}^{\infty} rac{(w+s-1)!}{(s-1)!w!} p^w (1-p)^s (w+1) \ &\mathrm{E}[X^2] &= rac{rp}{1-p} \sum_{w=0}^{\infty} inom{w+s-1}{s-1} p^w (1-p)^s (w+1) \ &\mathrm{E}[X^2] &= rac{rp}{1-p} \left[ \sum_{w=0}^{\infty} inom{w+s-1}{s-1} p^w (1-p)^s w + \sum_{w=0}^{\infty} inom{w+s-1}{s-1} p^w (1-p)^s 
ight] \end{aligned}$$

The first summation is the mean of a negative binomial random variable distributed NB(s.p)

$$egin{aligned} \mathrm{E}[X^2] &= rac{rp}{1-p} \left[rac{sp}{1-p} + 1
ight] \ \mathrm{E}[X^2] &= rac{rp(1+rp)}{(1-p)^2} \end{aligned}$$

We now insert values into the original variance formula.

$$egin{split} \operatorname{Var}[X] &= rac{rp(1+rp)}{(1-p)^2} - \left(rac{rp}{1-p}
ight)^2 \ \operatorname{Var}[X] &= rac{rp}{(1-p)^2} \end{split}$$