Conditional Probabilities

Today's Class

- Conditional Probability
- Independence
- Bayes' Theorem

Conditional Prob. Example

	Cancer	No Cancer	Total
Smoke	18	12	30
No Smoke	22	48	70
Total	40	60	100

- What is probability of cancer?
- What is probability of smoking?
- What is probability of cancer given a person smokes?
- What is probability of smoking given cancer?

Conditional Probability

- Probability depends on another event occurring, P(A|B):

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Multiplication Rule

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Example

- Suppose you roll a die, what is the probability of the die is greater than or equal to 5 given that it is even?

Solution

A={5,6}, B={2,4,6}

$A \cap B$={6}

$$\mathrm{P(A|B)} = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

Example

- Suppose you roll a die,
  - What is the probability of the die is greater than or equal to 5 given that it is even?
  - Are these two events independent?

Solution

A={5,6}, B={2,4,6}

P(A|B)=1/3 (from previous example)

P(A)=2/6=1/3

P(A|B)=P(A)

$\rightarrow$ A and B are independent

Example

- Suppose you roll a die,
  - What is the probability of the die is greater than 5 given that it is even?
  - Are these two events independent?

Solution

A={6}, B={2,4,6}

$A \cap B=\{6\}$

P(A|B)=1/3 (from previous example)

P(A)=1/6

$P(A|B) \neq P(A)$

$\rightarrow$ A and B are NOT independent

Independence

- Two events A and B are independent of each other if the occurrence of one has no influence on the probability of the other

Definition: $P(A | B) = P(A)$

Implication: $P(A \cap B) = P(A) \times P(B)$

Independence of more than two events

- Events A_1, .., A_n are mutually independent if for every k (k=2,3,..,n) and every subset of indices i_1, i_2,…, i_k

$$P(A_1 \cap A_2 \cap ... \cap A_k) = P(A_1) \times P(A_2) \times ... \times P(A_k)$$

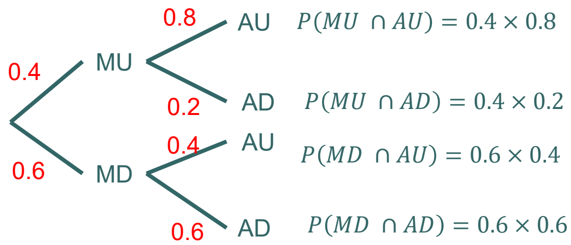
Special Cases

- A and B are mutually exclusive
  - $A \cap B = \varnothing$
  - $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0$
- $A \subset B$
  - $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$
- $B \subset A$
  - $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Law of Total Prob. Example

- We want to know the probability that Apple stock will increase this year.
  - Assume that the probability that the market goes up this year is 40%;
  - The probability that Apple goes up if the market goes up is 80%; and
  - The probability that Apple goes up if the market goes down is 40%.
  - What is the probability that Apple goes up this year?

Solution



$$
\begin{aligned}
P(AU) &= P(AU \cap MU) + P(AU \cap MD) \\
&= P(AU \mid MU)P(MU) + P(AU \mid MD)P(MD) \\
&= 0.4 * 0.8 + 0.6 * 0.4 = 0.56
\end{aligned}
$$

Law of Total Probability

- Let $A_1,\ldots,A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$
\begin{aligned}
P(B) &= P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k) \\
&= \sum_{i=1}^{k} P(B \cap A_i) \\
&= \sum_{i=1}^{k} P(B|A_i)P(A_i)
\end{aligned}
$$

Bayes' Theorem Example

- In the previous example, say that we know that Apple stock in fact goes up this year. Given that, what is the probability that the market goes up?
  - P(MU)=0.4
  - P(AU|MU)=0.8
  - P(AU|MD)=0.4

Solution

- Find P(MU|AU)

  P(MU)=0.4, P(AU|MU)=0.8, P(AU|MD)=0.4, P(AU)=0.56

$$
\begin{aligned}
P(MU|AU) &= \frac{P(AU \cap MU)}{P(AU)} \\
&= \frac{P(AU\,|\,MU) \times P(MU)}{P(AU)} \\
&= \frac{0.8 \times 0.4}{0.56} = 0.57
\end{aligned}
$$

The Reverend Thomas Bayes (1701-1761)

"Probability is that degree of confidence dictated by the evidence through Bayes' Theorem"

- E. T. Jaynes

Bayes' Theorem

$$P(A \cap B) = P(A \mid B) \times P(B) = P(B \mid A) \times P(A)$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Bayes' Theorem, Cont'd

- Let $A_1, \ldots, A_k$ be a collection of mutually exclusive and exhaustive events $P(A_i) > 0$ with for i=1,…,k. then for an event B:

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)\,P(A_j)}{\sum_{i=1}^{k} P(B|A_i)\,P(A_i)}$$

Bayes' Theorem

- Bayes' Theorem indicates how probabilities change in the light of evidence
- It is the most important tool in statistics!

Example

- Approximately 1% of women aged 40-50 have breast cancer. A woman with breast cancer has a 90% chance of a positive test from a mammogram, while a woman without has a 10% chance of a false positive result.
  - What is the probability a woman has breast cancer given that she just had a positive test?

Solution

Let B = "the woman has breast cancer"

A = "a positive test"

P(B|A)?

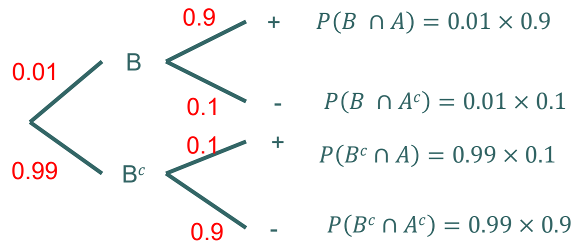
$P(A|B)=0.9$

$P(A|B^C)=0.1$

$P(B)=0.01$

$P(B^C)=1\text{-}P(B)=0.99$

Solution, Cnt'd

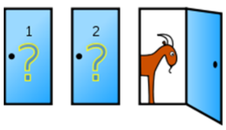


$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B \cap A)}{P(B \cap A) + P(B^C \cap A)}$$

$$= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^C)P(B^C)}$$

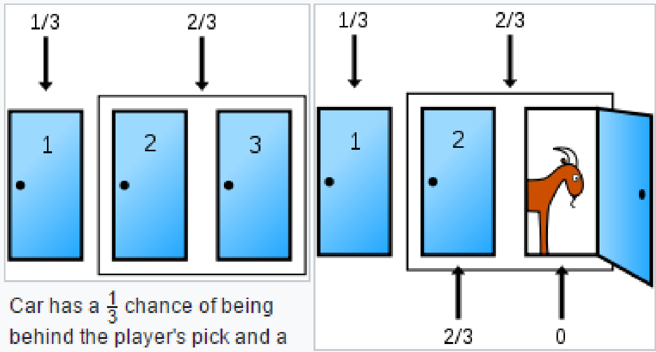
$$= \frac{0.01*0.9}{0.01*0.9+0.99*0.1} = \frac{9}{108} = 0.083$$

Monty Hall Problem



- Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice? Justify this using conditional probability.

Solution I



Car has a $\frac{1}{3}$ chance of being behind the player's pick and a $\frac{2}{3}$ chance of being behind one of the other two doors.

The host opens a door, the prob for the two sets don't change but the prob move to 0 for the open door and $\frac{2}{3}$ for the closed door.

Solution II

- The player picks Door 1,