The Poisson Distribution derived from Binomial Distribution

Assume X ~ B(n,p) , then the pdf of X is,

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ k=0,1,2…}$$

Let λ = np, and also assume that when n approaches infinity and p approaches zero, λ=np stays constant. Thus, p can be written as λ/n. Then we are going to rewrite the pdf,

$$\lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{n^k (n-k)!\left(1-\frac{\lambda}{n}\right)^k} \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\,(n-\lambda)^k} \left(1-\frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\,(n-\lambda)^k} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-k+1)}{(n-\lambda)^k} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \tag{1}$$

Before continuing on, we need to introduce two lemmas.

Lemma 1 From the definition of the number e, we get that $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. Assume that x = -n/λ, we can rewrite the equation as,

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x(-\lambda)} = e^{-\lambda}. \ (2)$$

Lemma 2 The value of $\lim_{n\to\infty}\frac{n(n-1)(n-2)\ldots(n-k+1)}{(n-\lambda)^k} = \lim_{n\to\infty}\frac{n^k}{n^k}$=1. (3)

By plugging (2) and (3) back into (1), we can see that,

$$\lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} * 1 * e^{-\lambda} = \frac{\lambda^k e^{-\lambda}}{k!} \ .$$

Expected value and Variance

The Poisson distribution is deducted from approximating a binomial random variable with parameters (n,p) when n is large and p is small, and λ = np. Since the binomial random variable has an expected value np =λ and variance np(1-p) = λ(1-p) $\to$ λ, when p $\to$ 0, it seems reasonable that a Poisson Variable would have both its expected value and variance equal to λ as well.

To find the expected value, we can compute the following:

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{\infty} \frac{ie^{-\lambda}}{i!}\lambda^{i} \\
&= \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{i}{i!}\lambda^{i-1} = \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!} \quad \text{Letting j = i-1} \\
&= \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^{j}}{j!} = \lambda \quad \text{since} \sum_{j=0}^{\infty}\frac{\lambda^{j}}{j!} = e^{\lambda}
\end{aligned}
$$

Next, we can determine its variance. First, recall that Var[X] = E[X^2] – (E[X])^2. So now we just need to find E[X^2].

$$
\begin{aligned}
E[X^2] &= \sum_{i=0}^{\infty}\frac{i^2e^{-\lambda}}{i!}\lambda^{i} \\
&= \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{i^2}{i!}\lambda^{i-1} = \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{i\lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{(j+1)\lambda^{j}}{j!} \\
&= \lambda\left[e^{-\lambda}\sum_{j=0}^{\infty}\frac{j\lambda^{j}}{j!} + e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^{j}}{j!}\right] \\
&= \lambda(\lambda+1) \\
&= \lambda^2 + \lambda
\end{aligned}
$$

Since we have shown that E[X] = λ, we obtain that

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$