Probability

Today's Class

- Notion of Probability
  - Classical probability
  - Relative frequency
  - Subjective probability
- Probability Axioms
- Properties of Probability

How Do We Assign Probabilities?

- Classical framework
  - Based on gambling ideas, the fundamental assumption is that the game is fair and all elementary outcomes have the same probability
- Frequentist framework: relative frequency
  - When an experiment can be repeated, an event's probability is the proportion of times the event occurs in the long run
- Bayesian framework: subjective probability
  - Degree of belief given evidence
  - Many events in life are not repeatable
  - Bayesians apply formal laws of probability to their own, personal probabilities

Axioms

- For any event A, $0 \le P(A) \le 1$
- $P(S) = 1$
- If $\{A_1, A_2, A_3, \dots, A_k\}$ is a finite collection of mutually exclusive events,

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i)$$

Proposition

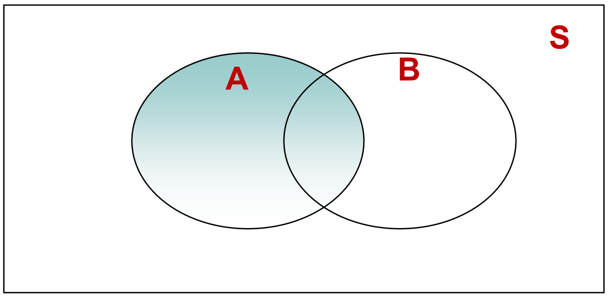
- For any event A,
  - $P(\varnothing)=0$ where $\varnothing$ is the null event
  - $P(A)=1-P(\bar{A})$
  - $P(\bar{A})=1-P(A)$
  - $P(A \cup B)=P(A)+P(B)-P(A \cap B)$

Example

- In LA, 60% of all households get internet service from the local cable company, 80% get TV service from the company, and 50% get both services from the company.
  - If a household is randomly selected, what is the probability that it gets at least one of these two services from the company?
  - What is the probability that it gets exactly one of these services from the company?

Example: Solution

- A = {Internet}, B={TV}
- P(A)=0.6, P(B)=0.8, P(A∩B)=0.5



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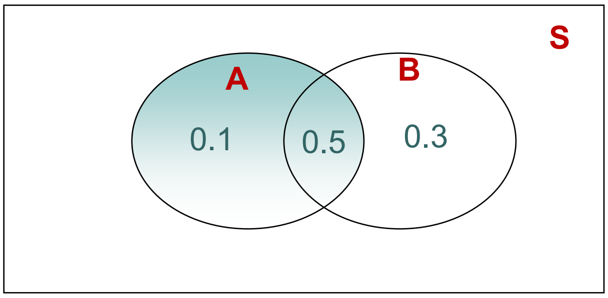


Interrupted Game

Interrupted Game Outcomes

H H H H	H H H T	H H T H	H H T T
H T H H	H T H T	H T T H	H T T T
T H H H	T H H T	T H T H	T H T T
T T H H	T T H T	T T T H	T T T T

Interrupted Game Outcomes

HHHH	HHHT	HHTH	HHTT
HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT
TTHH	TTHT	TTTH	TTTT

Equally likely events

- When the outcome of each event is equally likely to occur

$$p = P(E_i)$$

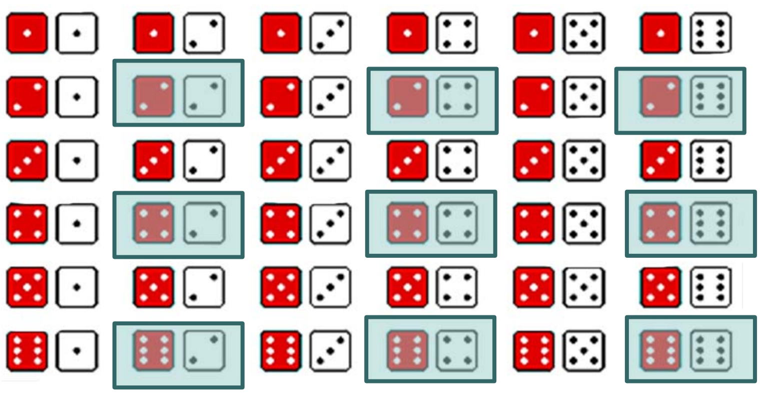
$$1 = \sum_{i=1}^{N} P(E_i) = \sum_{i=1}^{N} p = p \times N$$

$$\therefore P = \frac{1}{N}$$

Probability of Equally Likely Events Example: Dice

- Suppose you have two fair dice. There are N=36 possible outcomes, which are equally likely. With random selection, what is the probability of the event A that both dice are even?

Example, Cont'd



$$P(A) = \frac{N(A)}{N} = \frac{9}{36} = 0.25$$

Equally Likely Events

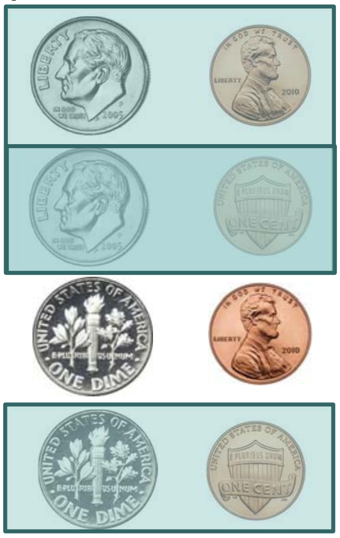
- Consider event A where N(A) is the number of outcomes contained in E. Then,

$$P(A) = \sum_{E_i \text{ in } A} P(E_i)$$
$$= \sum_{E_i \text{ in } A} \frac{1}{N}$$
$$= \frac{N(A)}{N}$$

Example: Coin

- Suppose you flip two coins (a dime and a penny)
  - What is P(DH)? DH: the dime comes up heads
  - What is P(PT)? PT: the penny comes up tails

Example, Cont'd



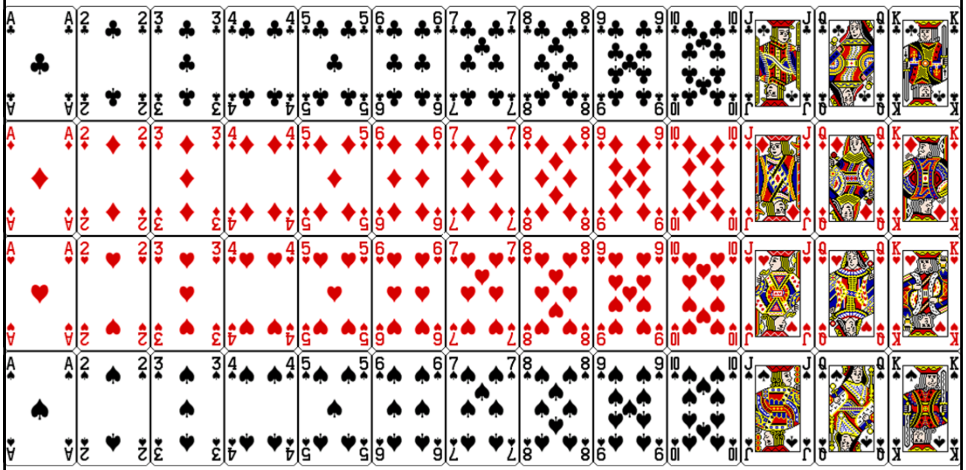
$$P(DH) = \frac{N(DH)}{N} = \frac{2}{4} = 0.5$$

$$P(PT) = \frac{N(PT)}{N} = \frac{2}{4} = 0.5$$

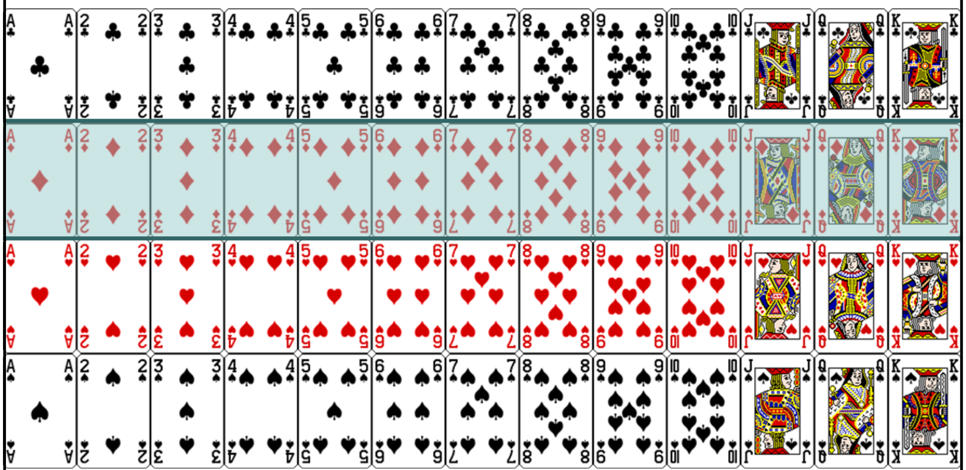
Example: Cards

- Suppose you have a deck of cards. What is the probability of the event that a diamond is selected?

A Deck of Playing Cards



A Deck of Playing Cards



$$P(D) = \frac{n(D)}{n(S)} = \frac{13}{52}$$

Odds

- If a horse has 12 – 1 odds (12 to 1), it means that you will get paid $12 if he wins, and lose $1 if he loses.
- This implies a probability of 1/(12+1) of winning and 12/13 of losing.
- In general if the odds are W to L, the probability of winning is L/(W+L)
- Odds ≠ Probability

Frequentist Framework

- The probability of an event A, denoted P(A), is the proportion of times event A is expected to occur if the random phenomenon is repeated a very large number of times under identical conditions

$$r(A) = \frac{n(A)}{n}$$

$$P(A) = \lim_{n \to \infty} r(A)$$

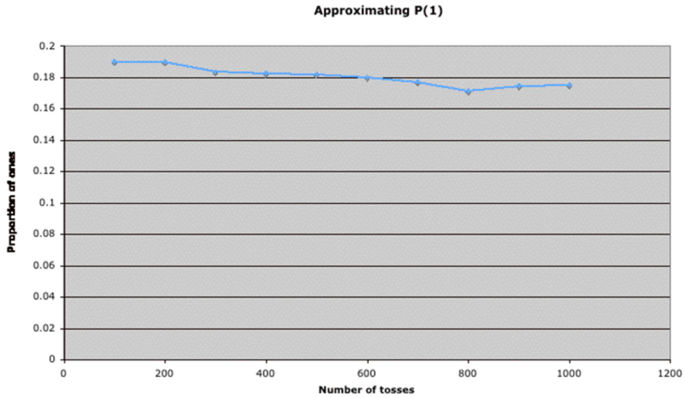
r(A) = relative frequency of A in n trials

n(A) = # of occurrences of A

n= # of repetitions (trials) of the phenomenon

Frequentist Example

- Relative frequencies to the event that the die is 1 for tossing a fair die



Chevalier de Méré Example

- Chevalier de Méré bet on a roll of a die that at least one 6 would appear during a total of four rolls. Later, he bet that he would get a total of 12, or a double 6, on twenty-four rolls of two dice. Soon he realized that his old approach to the game resulted in more money. He asked his friend Blaise Pascal why his new approach was not as profitable.

Solution

- To find the probability of getting at least one 6 in four rolls

$$1 - (5/6)^4$$

or 51.8 %

- To find the probability of getting a total of 12, or a double 6, in 24 rolls of two dice

$$1 - (35/36)^{24}$$

or 49.1 percent

Subjective Probability

- Why Subjective Interpretation
  - Frequentist interpretation is not always appropriate
  - Some events cannot be repeated many times
    - Core meltdown of a nuclear power
    - You being alive at the age of 80
  - The actual event has taken place, but you are uncertain about the result unless you have known the answer
    - The baseball game between Los Angeles Dodgers and the San Francisco Giants

Probability as degree of belief

- A subjective assessment concerning whether the event in question *will occur*, or *has occurred*.
  - What is the probability that the U.S. GDP will grow more than 6.5% this year?
  - What is the probability that the distance from Los Angeles to San Francisco is greater than 300 miles?

Probability as degree of belief

- Consider a "wheel of fortune".

  Would you rather get $100 if:
  - The spinner lands on blue
  - The distance from Los Angeles to San Francisco is greater than 300 miles

Exercise 2.13

- A computer consulting firm presently has bids out on three projects. Let A; = {awarded project i}, for i= 1, 2, 3, and suppose that $P(A_1)=.22$, $P(A_2)=.25$, $P(A_3)=.28$, $P(A_1 \cap A_2) =.11$, $P(A_1 \cap A_3)=.05$, $P(A_2 \cap A_3)= .07$, $P(A_1 \cap A_2 \cap A_3)=.01$
  - $P(A_1 \cup A_2)$?
  - $P(A_1 \cup A_2 \cup A_3)$?

Solution

- $A_1 \cup A_2$

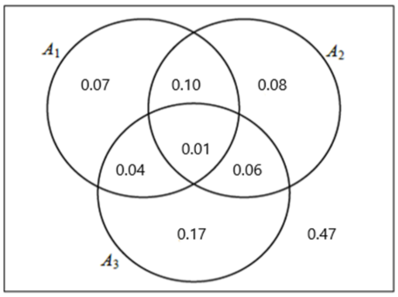
  from the addition rule,

  $$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$
  $$= .22 + .25 - .11 = .36$$

- $A_1 \cup A_2 \cup A_3$

  using the addition rule for 3 events

  $$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2)$$
  $$- P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$
  $$=.22 +.25 + .28 - .11 - .05 - .07 + .01 = .53$$


A_1
A_2
A_3
0.07
0.10
0.08
0.01
0.04
0.06
0.17
0.47