Negative Binomial Distribution

Expected Value

$$\mathrm{E}[X] = \sum_i f(x_i) \cdot x_i = \sum_{x=0}^{\infty} \binom{x+r-1}{r-1} p^x (1-p)^r \cdot x$$

$$\mathrm{E}[X] = \binom{0+r-1}{r-1} p^0 (1-p)^r \cdot 0 + \sum_{x=1}^{\infty} \binom{x+r-1}{r-1} p^x (1-p)^r \cdot x$$

$$\mathrm{E}[X] = 0 + \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(r-1)!x!} p^x (1-p)^r \cdot x$$

$$\mathrm{E}[X] = \frac{rp}{1-p} \sum_{x=1}^{\infty} \frac{(x+r-1)!}{r!(x-1)!} p^{x-1} (1-p)^{r+1}$$

Now let $s = r+1$ and $w = x-1$ inside the summation.

$$\mathrm{E}[X] = \frac{rp}{1-p} \sum_{w=0}^{\infty} \frac{(w+s-1)!}{(s-1)!w!} p^w (1-p)^s$$

$$\mathrm{E}[X] = \frac{rp}{1-p} \sum_{w=0}^{\infty} \binom{w+s-1}{s-1} p^w (1-p)^s$$

The summation is the sum over the complete pmf of a negative binomial rv, which is 1.

$$\mathrm{E}[X] = \frac{rp}{1-p}$$

Variance

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$\mathrm{E}[X^2] = \sum_i f(x_i) \cdot x^2 = \sum_{x=0}^{\infty} \binom{x+r-1}{r-1} p^x (1-p)^r \cdot x^2$$

$$\mathrm{E}[X^2] = 0 + \sum_{x=1}^{\infty} \binom{x+r-1}{r-1} p^x (1-p)^r x^2$$

$$\mathrm{E}[X^2] = \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(r-1)!x!} p^x (1-p)^r x^2$$

$$\mathrm{E}[X^2] = \frac{rp}{1-p} \sum_{x=1}^{\infty} \frac{(x+r-1)!}{r!(x-1)!} p^{x-1} (1-p)^{r+1} x$$

Again, let let $s = r+1$ and $w=x-1$.

$$\mathrm{E}[X^2] = \frac{rp}{1-p} \sum_{w=0}^{\infty} \frac{(w+s-1)!}{(s-1)!w!} p^w (1-p)^s (w+1)$$

$$\mathrm{E}[X^2] = \frac{rp}{1-p} \sum_{w=0}^{\infty} \binom{w+s-1}{s-1} p^w (1-p)^s (w+1)$$

$$\mathrm{E}[X^2] = \frac{rp}{1-p} \left[\sum_{w=0}^{\infty} \binom{w+s-1}{s-1} p^w (1-p)^s w + \sum_{w=0}^{\infty} \binom{w+s-1}{s-1} p^w (1-p)^s \right]$$

The first summation is the mean of a negative binomial random variable distributed NB(s,p)

$$\mathrm{E}[X^2] = \frac{rp}{1-p} \left[\frac{sp}{1-p} + 1 \right]$$

$$\mathrm{E}[X^2] = \frac{rp(1+rp)}{(1-p)^2}$$

We now insert values into the original variance formula.

$$\mathrm{Var}[X] = \frac{rp(1+rp)}{(1-p)^2} - \left(\frac{rp}{1-p} \right)^2$$

$$\mathrm{Var}[X] = \frac{rp}{(1-p)^2}$$