Lognormal distribution

Let X have a lognormal distribution such *that* ln X~N(μ,σ^2). *Then*

$$E(X) = \exp(\mu + \frac{1}{2}\sigma^2),\ V(X) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1) \quad (1)$$

Proof. Probability density function of the variable *X:*

$$\phi(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), -\infty < x < \infty$$

$$\Phi(x,\mu,\sigma) = \int_{-\infty}^{x} \phi(t,\mu,\sigma)dt$$

the probability density function and cumulative distribution function, respectively, for a normal random variable with mean μ and standard deviation σ. In order to derive the probability density function of *X* we start with its cumulative distribution function. Notice that $X > 0$, so for $x > 0$ we have:

$$F(x) = P(X \le x) = P(\ln(X) \le \ln x) = \Phi(\ln x,\mu,\sigma) \quad (2)$$

Now the probability density function of *X* can be found by differentiating in (2):

$$f(x) = \frac{dF(x)}{dx} = \frac{d\Phi(\ln x,\mu,\sigma)}{dx} = \frac{1}{x}\Phi'(\ln x,\mu,\sigma) = \frac{1}{x}\phi(\ln x,\mu,\sigma)$$

$$= \frac{1}{x}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), x > 0 \quad (3)$$

Then

$$E(X) = \int_0^\infty xf(x)dx = \int_0^\infty x\frac{1}{x}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)dx$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right)e^u du$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(u-\mu)^2 - 2u\sigma^2}{2\sigma^2}\right)du = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(u-\mu-\sigma^2)^2 - 2\mu\sigma^2 - \sigma^4}{2\sigma^2}\right)du$$

$$= \exp\left(\frac{2\mu\sigma^2 + \sigma^4}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(u-\mu-\sigma^2)^2}{2\sigma^2}\right)du = \exp\left(\mu + \frac{\sigma^2}{2}\right). (4)$$

The last equality above follows from the fact that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(u-\mu-\sigma^2)^2}{2\sigma^2}\right)du = 1$$

as $\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(u-\mu-\sigma^2)^2}{2\sigma^2}\right)$ is the probability density of a normal variable with mean $\mu + \sigma^2$ and standard deviation σ.

In order to compute the variance we need $E(X^2)$. We have

$$E(X^2) = \int_0^\infty x^2 f(x)dx = \int_0^\infty x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^u \exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right)e^u du$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(u-\mu)^2-4u\sigma^2}{2\sigma^2}\right)du = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(u-\mu-2\sigma^2)^2}{2\sigma^2}+\frac{4\mu\sigma^2+4\sigma^4}{2\sigma^2}\right)du$$

$$= \exp\left(\frac{4\mu\sigma^2+4\sigma^4}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(u-\mu-2\sigma^2)^2}{2\sigma^2}\right)du = \exp(2\mu+2\sigma^2)$$

Then

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \exp(2\mu+\sigma^2)\left(\exp(\sigma^2)-1\right) \quad (5)$$