If X and Y are independent, then ρ=0,  but ρ=0 does not imply independence

Proof

We will show $\mu_{XY} = \mu_X\mu_Y$ from which it will follow that $Cov(X, Y) = \rho_{X,Y} = 0$.

We will assume that X and Y are jointly continuous with jdf f(x,y) and marginal densities $f_x(x)$ and $f_y(y)$.

The key to the proof is the fact that since X and Y are independent, $f(x, y) = f_x(x)f_y(y)$.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_x(x)f_y(y)dxdy$$

$$= \int_{-\infty}^{\infty} xf_x(x)dx \int_{-\infty}^{\infty} yf_y(y)dy$$

$$= E(X)E(Y)$$

Therefore $Cov(X, Y) = \rho_{X,Y} = 0$

The proof in the case that X and Y are jointly discrete is similar, with the integrals replaced by sums.