Random Variables

Today's Class

- Random Variables
- Discrete Random Variables
- Probability Mass Function
- Cumulative Distribution Function
- Expected Values
- Variances

Random Variables

- A **random variable (rv)** associates a number with each outcome in the sample space
  - We denote random variables with upper case letter, X
  - The observed numerical value once the experiment is run is denoted by the corresponding lower case letter, x
- In mathematical terms, a rv is a function whose domain is the sample space and the range is the set or real numbers

Random Variable Example

- A tall antenna is built on a mountain top where an extreme wind event occurs. Either the antenna fails (F) or survives (S)

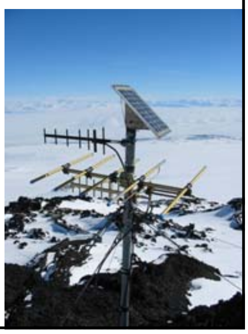
$$s = \{F, S\}$$

  - If the rv X is associated with the outcomes,

$$X(S) = 1,\ X(F) = 0$$

1 indicates that the antenna survived

0 indicates that the antenna failed



Types of Random Variables

- Discrete Random Variable:
  - takes a finite number of values

    e.g. the number of cars lined up at the FasTrak Entrance
- Continuous Random Variable:
  - takes all values in an interval

    e.g. the time each car must wait at FasTrak Entrance

Probability Mass Function Example

- Suppose you flip two coins, a rv, X, is the number of heads in the experiment
  - What is the sample space?
  - What is the probability of each outcome in the sample space?

Example, Cont'd

X	P(X)	
2	1/4	
1	1/4	1/2
1	1/4	
0	1/4	

Probability Mass Function

- A Probability Mass Function (pmf), also called *probability distribution*, is a function p(x) that assigns to each possible value x that the random variable X can take, its probability

$$p(x) = P(X = x)$$
$$= P(\text{all } s \in S : X(s) = x)$$

  - $p(x_i) \geq 0$ for each possible value x_i of X
  - $\sum_{\text{all } x_i} p(x_i) = 1$

Example: pmf

- Consider 10 truckloads of rebar are delivered to the job site. On each of those truckloads, there will be some damaged bars (X).
  - What is the sample space?
  - What is the pmf?

Truckload	1	2	3	4	5	6	7	8	9	10
Number of damaged bars	0	0	1	0	1	0	2	1	0	2

Solution: pmf

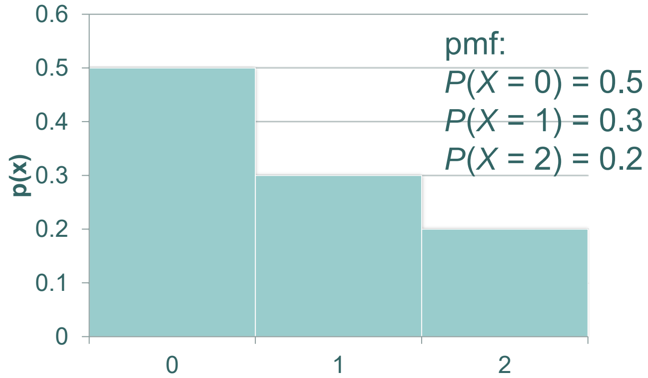
Truckload	1	2	3	4	5	6	7	8	9	10
# of damaged bars	0	0	1	0	1	0	2	1	0	2

$$\begin{aligned} p(0) &= P(X=0) \\ &= P(T_1, T_2, T_4, T_6, T_9) \\ &= \frac{5}{10} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} p(1) &= P(X=1) \\ &= P(T_3, T_5, T_8) \\ &= \frac{3}{10} \\ &= 0.3 \end{aligned}$$

$$\begin{aligned} p(2) &= P(X=2) \\ &= P(T_7, T_{10}) \\ &= \frac{2}{10} \\ &= 0.2 \end{aligned}$$

pmf histogram



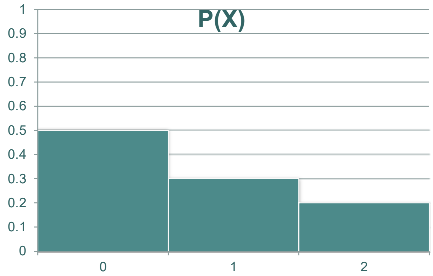
Cumulative Distribution Function Example

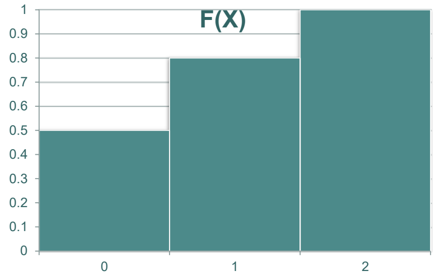
- From the previous example, let F(x) denote the cumulative distribution function (cdf) of the rv X

X	pmf, P(X=x)	cdf, F(X=x)
0	0.5	0.5
1	0.3	0.5+0.3=0.8
2	0.2	0.5+0.3+0.2=1

Cumulative Distribution Function Example

- From the previous example, let F(x) denote the cdf of the rv X





Cumulative Distribution Function

- The cumulative distribution function (cdf) of a r.v. X is

$$F(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i)$$

which gives the sum of the probabilities up to that value x

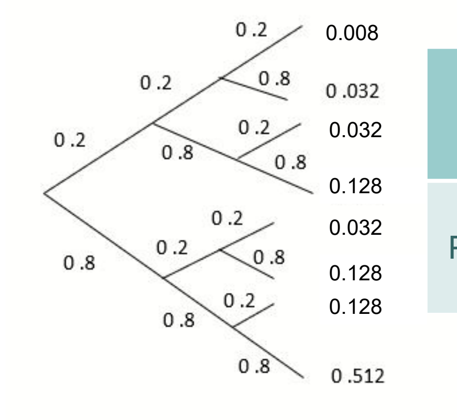
Cumulative Distribution Function

- Any probability distribution must follow the axioms of probability
  - $F(-\infty)=0$; $F(\infty) = 1$
  - $F(x) \geq 0$ and is weakly increasing
  - It is continuous in x
- Any function that satisfies these axioms is a cdf

Example: Expected Value

- Let X = number of working bulldozers after 6 months. Assume the probability that a bulldozer is working after 6 months is 0.8, and there are 3 dozers.
  - Find the pmf

Solution: pmf



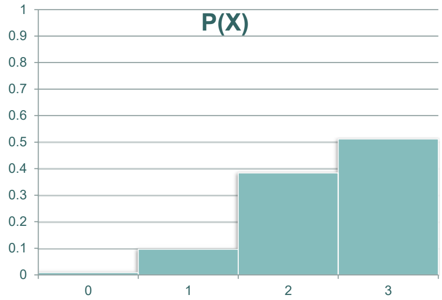
X	0	1	2	3
P(X=x)	.008	.096	.384	.512

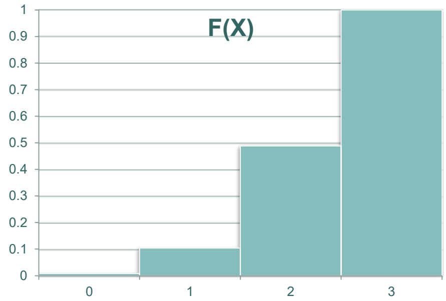
Expected Value Example

- Let X = number of working bulldozers after 6 months. Assume the probability that a bulldozer is working after 6 months is 0.8, and there are 3 dozers.
  - Find the cdf

Solution: cdf

X	0	1	2	3
P(X=x)	0.008	0.096	0.384	0.512
F(X=x)	0.008	0.104	0.488	1.000





Expected Value Example

- Let X = number of working bulldozers after 6 months. Assume the probability that a bulldozer is working after 6 months is 0.8, and there are 3 dozers.
  - What is the expected value of the number of dozers working after 6 months?

Solution: Expected Value

- The expected value of the number of dozers after 6 months is:

X	0	1	2	3
P(X=x)	.008	.096	.384	.512

  - $E[X] = 0 \times .008 + 1 \times .096 + 2 \times .384 + 3 \times .512 = 2.4$

Expected Value

- The *expected value* is the long run expected mean, if you were to see X over and over again

$$E[X] = \mu_x = \sum_{x \in D} x \cdot p(x)$$

Example: Expected Value II

- In previous example, at least 2 dozers are needed to finish a $100K job.

  Every dozer that was brought in after 6 months costs $10K.

  What is your expected profit if you start with 3 dozers?

Solution: Expected Value II

- Let x = # of working dozers at 6 months

X	0	1	2	3
Profit	100 – 20 = 80K	100 –10 = 90K	100K	100K
P(X=x)	0.008	0.096	0.384	0.512

- E[profit] = 0.008×80K + 0.096×90K + 0.384×100K

  +0.512×100K

  = 98.88K

Expected Value of a Function

- Consider that there might be a functional relationship with X with a set of possible values D and pmf p(x) such that we have a probability of h(X)

$$E[h(X)] = \sum_{D} h(x) \cdot p(x)$$

Example: E[X] Properties

- Your net profit is equal to $6000 plus $10 for every dozer that is working at 6 months.

  What is the expected value of your net profit?

Solution: E[X] Properties

- Let x = # of working dozers at 6 months

X	0	1	2	3
net	0	10	20	30
P(X=x)	0.008	0.096	0.384	0.512

- E[net]
  = 6000+(0.008×0 + 0.096×10 + 0.384×20 +0.512×30)
  = 6,024

Solution: E[X] Properties

$$E(aX+b)=aE(X)+b$$

$$\begin{aligned} E(10X+6000) &= 10 \times 2.4 + 6000 \\ &= 24 + 6000 \\ &= 6{,}024 \end{aligned}$$

Properties of Expected Value

$$E(X_1 + X_2) = E[X_1] + E[X_2]$$

$$E(aX + b) = a \times E[X] + b$$

Variance

- The variance is defined as
  $V(X) = \sigma^2 = E[(X-\mu)^2] = \sum_D (x-\mu)^2 \cdot p(x)$
- Using the properties of expected values:

$$\sigma^2 = E[(X-\mu)^2]$$

$$= E[X^2 - 2X\mu + \mu^2]$$

$$= E[X^2] - 2E[X] \cdot \mu + \mu^2$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2 = E[X^2] - (E[X])^2$$

Variance Example

- In the previous example of bull dozer, find variance of the number of dozers working after 6 months?

X	0	1	2	3
P(X=x)	.008	.096	.384	.512

Example Solution

P(x)	.008	.096	.384	.512
x	0	1	2	3
x^2	0	1	4	9

$$\sigma^2 = E[X^2] - \mu^2$$

$E[X^2] = 0 \times .008 + 1 \times .096 + 4 \times .384 + 9 \times .512$

$= 6.24$

$E[X] = \mu = 2.4$

$(E[X])^2 = \mu^2 = 2.4^2 = 5.76$

$\therefore \sigma^2 = 6.24 - 5.76 = 0.48$

Variance

- The variance is defined as

$$
\begin{aligned}
V(X) &= \sigma^2 \\
&= E[(X-\mu)^2] \\
&= \sum_D (x-\mu)^2 \cdot p(x) \\
&= E[X^2] - \mu^2
\end{aligned}
$$

Properties of Variance & Standard Deviation

$$V(aX+b) = a^2 V(X) = a^2 \sigma^2{}_X$$

$$\sigma_{aX+b} = |a|\,\sigma_X$$

Example: Variance Properties

- In previous example, at least 2 dozers are needed to finish a \$100K job. Every dozer that was brought in after 6 months costs \$10K. Suppose you start with 3 dozers.
  - What is the variance of your profit?

Solution

x	0	1	2	3
\$	80K	90K	100K	100K
P(x)	.008	.096	.384	.512

- $V(\$)=E(\$^2)-E(\$)^2$

  $E(\$)=\$98.88K$

  $E(\$^2)= \$80K^2\times 0.008 + \$90K^2\times 0.096 + \$100K^2\times 0.384 + \$100K^2\times 0.512 = \$\ 9{,}788.8M$

  $\therefore\ V(\$)=\$9{,}788.8M-(\$98.88K)^2=\$11.55M$

Solution 2

x	0	1	2	3
Y	2	1	0	0
$	80K	90K	100K	100K
P(x)	.008	.096	.384	.512

- $V(\$)=V(\$100K-\$10K*Y)=(10K)^2*V(Y)$

  $V(Y)=E(Y^2)-E(Y)^2$

  $E(Y)=2\times0.008+1\times0.096+0\times0.384+0\times0.512=0.112$

  $E(Y^2)=2^2\times0.008+1^2\times0.096+0^2\times0.384+0^2\times0.512$

  $=0.128$

  $\therefore V(Y)=0.128-0.112^2=0.1155$

  $V(\$)=(10K)^2\times0.1155=\$11.55M$