# Conditional Probabilities

## Today's Class

- Conditional Probability
- Independence
- Bayes' Theorem

## Conditional Prob. Example

| | Cancer | No Cancer | Total |
|---|---|---|---|
| Smoke | 18 | 12 | 30 |
| No Smoke | 22 | 48 | 70 |
| Total | 40 | 60 | 100 |

- What is probability of cancer?
- What is probability of smoking?
- What is probability of cancer given a person smokes?
- What is probability of smoking given cancer?

## Conditional Probability

- Probability depends on another event occurring, P(A|B):

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Multiplication Rule

$$P(A \cap B) = P(A|B) \cdot P(B)$$

## Example

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## Independence

- Two events A and B are independent of each other if the occurrence of one has no influence on the probability of the other

Definition: $P(A \mid B) = P(A)$

Implication: $P(A \cap B) = P(A) \times P(B)$

## Independence of more than two events

- Events $A_1$, .., $A_n$ are mutually independent if for every k (k=2,3,..,n) and every subset of indices $i_1, i_2, \ldots, i_k$

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1) \times P(A_2) \times \ldots \times P(A_k)$$

## Special Cases

- A and B are mutually exclusive
  - $A \cap B = \emptyset$
  - $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0$
- $A \subset B$
  - $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$
- $B \subset A$
  - $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

## Law of Total Prob. Example

- We want to know the probability that Apple stock will increase this year.
  - Assume that the probability that the market goes up this year is 40%;
  - The probability that Apple goes up if the market goes up is 80%; and
  - The probability that Apple goes up if the market goes down is 40%.
  - What is the probability that Apple goes up this year?

## Law of Total Probability

- Let $A_1,\ldots,A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$
\begin{aligned}
P(B) &= P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k) \\
&= \sum_{i=1}^{k} P(B \cap A_i) \\
&= \sum_{i=1}^{k} P(B|A_i)P(A_i)
\end{aligned}
$$

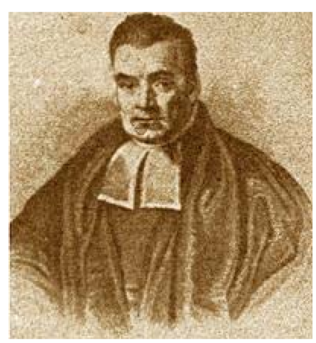
## Bayes' Theorem Example

- In the previous example, say that we know that Apple stock in fact goes up this year. Given that, what is the probability that the market goes up?
  - P(MU)=0.4
  - P(AU|MU)=0.8
  - P(AU|MD)=0.4

## The Reverend Thomas Bayes (1701-1761)

"Probability is that degree of confidence dictated by the evidence through Bayes' Theorem"

- E. T. Jaynes



T. Bayes.

# Bayes' Theorem

$$P(A \cap B) = P(A \mid B) \times P(B) = P(B \mid A) \times P(A)$$

$$P(A|B) = \frac{P(B|A)\, P(A)}{P(B)}$$

# Bayes' Theorem, Cont'd

- Let $A_1$, …, $A_k$ be a collection of mutually exclusive and exhaustive events $P(A_i) > 0$ with for i=1,…,k. then for an event B:

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)\, P(A_j)}{\sum_{i=1}^{k} P(B|A_i)\, P(A_i)}$$

## Bayes' Theorem

- Bayes' Theorem indicates how probabilities change in the light of evidence
- It is the most important tool in statistics!

## Example

- Approximately 1% of women aged 40-50 have breast cancer. A woman with breast cancer has a 90% chance of a positive test from a mammogram, while a woman without has a 10% chance of a false positive result.
  - What is the probability a woman has breast cancer given that she just had a positive test?