CEE110
Homework #4 Solution

1. a. The experiment is binomial with n = 200 and p = 1/88, so
$\mu = np = 2.27$ and $\sigma = \sqrt{npq} = \sqrt{2.247} = 1.50.$
The experiment is Poisson with n = 200 and p = 1/88,
$\lambda = np = 2.27$ and $\sigma = \sqrt{\lambda} = \sqrt{2.247} = 1.50.$

b. X has approximately a Poisson distribution with $\mu = 2.27$,
$P(X \geq 2) = 1 - P(X = 0, 1)$
$\approx 1 - \left[\frac{e^{-2.27} 2.27^0}{0!} + \frac{e^{-2.27} 2.27^1}{1!}\right] = 1 - 0.3378 = 0.6622$
(The exact binomial answer is 0.6645.)

c. Now $\mu = 352\left(\frac{1}{88}\right) = 4$,
$P(X < 5) = P(X \leq 4) \approx F(4; 4) = 0.629.$

2.

a. Let X be the number of servers that fail. Then X is the number of successes in n = 1000 Bernoulli trials, each of which has success probability $p = 0.003$. The mean of X is $np = (1000)(0.003) = 3$. Since n is large and p is small, $X \sim$ Poisson(3) to a very close approximation.

$$P(X = 2) = e^{-3}\frac{3^2}{2!} = 0.2240$$

b. The event that fewer than 998 servers fail is the same as the event that more than 2 servers function, or equivalently, that $Y > 2$.
The Y has the probability of 0.997
Therefore,

$$P(Y > 2) = 1 - P(Y \leq 2) = 1 - P(0) - P(1) - P(2)$$

$$= 1 - \frac{1000!}{0!\,1000!} 0.997^0 0.003^{1000} - - \frac{1000!}{1!\,999!} 0.997^1 0.003^{999} - - \frac{1000!}{2!\,998!} 0.997^2 0.003^{998}$$

≈ 1

c. $\mu_X = 3$
$\sigma_X = \sqrt{3} = 1.732$

3. a. The expected number of bacteria in 1 m^3 of water is 10, so X~Poisson(10).
$P(X \geq 8) = 1 - P(X \leq 7) = 1 - F(7; 10) = 1 - 0.220 = 0.780.$
(From cumulative table)

b. The expected number of bacteria in 1.5 m^3 of water is 10(1.5) = 15, so X~Poisson(15).
Since X is Poisson, $\sigma = \sqrt{\mu} = \sqrt{15} = 3.87.$

c. $P(X > \mu + \sigma) = P(X > 15 + 3.87) = P(X > 18.87) = 1 - P(X \leq 18.87)$
$= 1 - P(X \leq 18) = 1 - F(18; 15) = 1 - 0.8195 = 0.1805.$
(From cumulative table)

4.

a. Find c.

$$1 = \int_0^\infty f(x)dx = \int_0^\infty ce^{-4x}dx = -\frac{c}{4}[-e^{-4x}]_0^\infty = \frac{c}{4}.$$

Thus, c=4

b. To find the CDF of X, we use $F(x) = \int_{-\infty}^{x} f(x)dx$, so for $x < 0$, we obtain $F(x) = 0$. for $x \geq 0$, we have

$$F_X(x) = \int_{-\infty}^{x} 4e^{-4x}dx = -[e^{-4x}]_0^x = 1 - e^{-4x}.$$

Thus,

$$F(x) = \begin{cases} 1 - e^{-4x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

c. We can find P(2<X<5) using either the cdf or the pdf.

$$P(2 < X < 5) = F(5) - F(2) = [1 - e^{-20}] - [1 - e^{-8}] = e^{-8} - e^{-20}$$

Equivalently, we can use the PDF. We have

$$P(2 < X < 5) = \int_2^5 f(x)dx = \int_2^5 4e^{-4x}dx = e^{-8} - e^{-20}$$

d. $E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$=\int_0^\infty x4e^{-4x}dx$

$$= -[xe^{-4x}]_0^\infty + \int_0^\infty e^{-4x}dx$$

$$= 0 + \left[-\frac{1}{4}e^{-4x}\right]_0^\infty$$

$=\frac{1}{4}$

5.

a. P(X<0) = F(0) = 1/2

b. As shown in (a), median = 0

c. $f(x) = F'(x) = \frac{d}{dx}\left(\frac{1}{2} + \frac{3}{32}\left(4x - \frac{x^3}{3}\right)\right) = 0 + \frac{3}{32}\left(4 - \frac{3x^3}{3}\right) = 0.09375(4 - x^2)$

6.

a. As it is uniformly distributed E(X)=$\frac{7.5+20}{2} = 13.75$

$V(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \int_{7.5}^{20} X^2 \frac{1}{20-7.5} dx = \frac{1}{3 * 12.5} X^3\Big]_{7.5}^{20} = \frac{20^3 - 7.5^3}{37.5} = 202.0833$$

V(X)= $202.0833 - 13.75^2 = 13.02$

b.

$F(x) = \int_{7.5}^{x} \frac{1}{20-7.5} dy = \frac{1}{12.5} y]_{7.5}^{x} = \frac{x-7.5}{12.5}, \quad 7.5 < x < 20$

$$F(x) = \begin{cases} 0 & x \le 7.5 \\ \dfrac{x-7.5}{12.5} & 7.5 < x < 20 \\ 1 & 20 \le x \end{cases}$$

c. P(10≤X≤15) = F(15) − F(10) = $\frac{15-7.5}{12.5} - \frac{10-7.5}{12.5} = \frac{5}{12.5} = 0.4$

d. σ = $\sqrt{13.02}$= 3.61, so μ ± σ = (10.14, 17.36).
Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = P(10.14 \le X \le 17.36) = F(17.36) - F(10.14) = .5776$
(From cumulative table)