Joint Probability Distribution

Today's Class

- Joint Probability Mass Function
- Marginal Probability Mass Function
- Joint Probability Density Function
- Marginal Probability Density Function
- Conditional Distribution

Joint Probability Distributions

- We may care about multiple random variables
  - rainfall intensity at a gage, and river runoff
  - hours worked and productivity per hour
  - Lifetime of tarmac and total cost
  - Number of defects detected on different days

Joint pmf

- Let *X* and *Y* be two **discrete rv's** defined on the sample space of an experiment
- The **joint probability mass function** *p*(*x*, *y*):

$$p(x, y) = P(X = x \text{ and } Y = y)$$

  where $p(x, y) \geq 0$ and $\sum_x \sum_y p(x, y) = 1$

- Let *A* be a set consisting of pairs of (x, y) values, then

$$P[X, Y \in A] = \sum\sum_{(x,y) \in A} p(x, y)$$

Joint pmf Example

- Let X and Y denote the percent productivity and the amount of hours of work per day, respectively.

X\Y	6 hrs	8 hrs	10 hrs	12 hrs
50%	.014	.036	.058	.072
70%	.036	.216	.180	.043
90%	.072	.180	.079	.014

  - What is P(h=8)?

Marginal pmf

- The marginal probability mass functions of X and of Y:

$$p_X(x) = P(X = x) = \sum_y p(x, y)$$
$$p_Y(y) = P(Y = y) = \sum_x p(x, y)$$

Marginal pmf Example

- Let X and Y denote the percent productivity and the amount of hours of work per day, respectively.

X\Y	6 hrs	8 hrs	10 hrs	12 hrs	Marg-inal
50%	.014	.036	.058	.072	.180
70%	.036	.216	.180	.043	.475
90%	.072	.180	.079	.014	.345
Marg-inal	.122	.432	.317	.129	1

Exercise 5.1

- A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X and Y denote the number of hoses being used on the self-service and full-service islands at a particular time, respectively.

		y		
P(x,y)		0	1	2
	0	.10	.04	.02
x	1	.08	.20	.06
	2	.06	.14	.30

- What do the numbers inside the table add to?
- What is P(X=1 and Y=1)?
- What is P(X=1) ?
- What is P($X \leq 1$) ?

Solution

P(x,y)		y = 0	y = 1	y = 2
x	0	.10	.04	.02
	1	.08	.20	.06
	2	.06	.14	.30

- Then $P(X=1 \text{ and } Y=1) = = .20$.
- The probability $P(X = 1)$ is computed by summing probabilities of all (x, y) pairs for which $x = 1$:

  $P(X=1) = p(1,0) + p(1,1) + p(1,2) \quad = .34$
- The probability $P(X\leq 1)$ is computed by summing probabilities of all (x, y) pairs for which $x\leq 1$:

  $P(Y\geq 100) = p(0,0) + p(0,1) + p(0,2) + p(1,0) +$

  $p(1,1) + p(1,2) = .5$

Joint pdf Example

- For r.v. X and Y

  $$f(x, y) = x + y \quad 0 \leq x \leq 1,\ 0 \leq y \leq 1$$

  - Find P$(0 \leq x \leq 1/2,\ 0 \leq y \leq 1/2)$

Solution

P$(0 \le x \le 1/2,\ 0 \le y \le 1/2)$

$$= \int_0^{1/2} \int_0^{1/2} (x+y)dxdy$$

$$= \int_0^{1/2} \frac{1}{2}x^2 + yx\Big]_0^{1/2} dy$$

$$= \int_0^{1/2} \left(\frac{1}{8} + \frac{1}{2}y\right) dy$$

$$= \frac{1}{8}y + \frac{1}{4}y^2\Big]_0^{1/2}$$

$$= \frac{1}{16} + \frac{1}{16}$$

$$= \frac{1}{8}$$

Joint pdf Example

- For r.v. X and Y

$$f(x,y) = x + y \quad 0 \le x \le 1,\ 0 \le y \le 1$$

  - Find $f_x(x)(i.e. = \int_{-\infty}^{\infty} f(x,y)dy\ )$
  - Find $f_y(y)(i.e. = \int_{-\infty}^{\infty} f(x,y)dx\ )$

Solution

- Marginal pdf

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 (x+y)dy = xy + \frac{1}{2}y^2\Big]_0^1 = x + \frac{1}{2}$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 (x+y)dx = \frac{1}{2}x^2 + yx\Big]_0^1 = y + \frac{1}{2}$$

Joint pdf

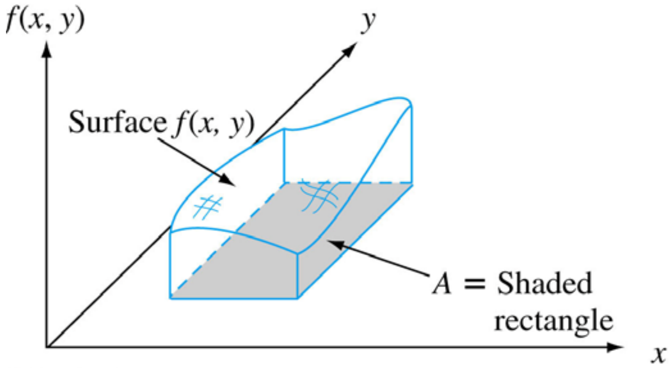
- Let X and Y be **continuous rv's**
- The **joint probability density function** f(x, y):

$$P[X,Y \in A] = \iint_A f(x,y)dxdy$$

- If *A* is the two-dimensional rectangle

  {(x,y): a ≤ x ≤ b, c ≤ y ≤d}

$$P[X,Y \in A] = \int_c^d \int_a^b f(x,y)dxdy$$



P[(X,Y)∈A]

= Volume under density surface above *A*

Exercise 5.9

- Each front tire on a particular vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a r.v.: X for the right tire and Y for the left tire, with joint pdf

$$f(x,y)\begin{cases} K(x^2+y^2) & 20 \le x \le 30, 20 \le y \le 30 \\ 0 & otherwise \end{cases}$$

  - What is the value of K?

Solution

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{20}^{30}\int_{20}^{30} K(x^2+y^2)dxdy$$

$$= K\int_{20}^{30}\int_{20}^{30}(x^2+y^2)dxdy$$

$$= K\int_{20}^{30} \frac{1}{3}x^3 + xy^2\Big]_{20}^{30} dy$$

$$= K\int_{20}^{30}(\frac{19000}{3} + 10y^2)dy$$

$$= K \times (\frac{19000}{3}y + \frac{10}{3}y^3)\Big]_{20}^{30}$$

$$= 10K \times (\frac{19,000}{3} + \frac{19,000}{3})$$

$$\text{Therefore, } K = \frac{3}{380,000}$$

Exercise 5.9

- Each front tire on a particular vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a r.v.: X for the right tire and Y for the left tire, with joint pdf

  $$f(x,y)\begin{cases} K(x^2+y^2) & 20 \le x \le 30, 20 \le y \le 30 \\ 0 & otherwise \end{cases}$$

  - What is the probability that both tires are underfilled?

Solution

$$P(X < 26 \; and \; Y < 26) = \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)dxdy$$

$$= K\int_{20}^{26}\int_{20}^{26}(x^2+y^2)dxdy$$

$$= K\int_{20}^{26} x^2y + \frac{y^3}{3}\Big]_{20}^{26} dx$$

$$= K\int_{20}^{26}(6x^2+3192)dx$$

$$= K \times (\frac{6}{3}x^3 + 3192x)]_{20}^{26}$$

$$= K \times (19{,}152 + 19{,}152)$$

$$= \frac{3}{380{,}000} \times 38{,}304$$

$$= .3024$$

Marginal Probability Density Function

- The marginal probability density functions of *X* and *Y*

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

Example: Marginal pdf

- Let the joint probability density function of rvs X and Y be

  f(x,y)=2 for 0 ≤ x ≤ y ≤ 1

  - Find the marginal probability distributions of random variable X and Y

Solution

- Marginal pdf

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{x}^{1} 2dy = 2y]_x^1 = 2-2x$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{y} 2dx = 2x]_0^y = 2y$$

Independence of rvs

- Two rvs, X and Y, are said to be independent if for every pair of x and y values,

$p(x,y) = p_X(x) \cdot p_Y(y)$ when X and Y are discrete

$f(x,y) = f_X(x) \cdot f_Y(y)$ when X and Y are continuous

Example

- Are X and Y independent?

p(x,y)	y=0	y=5	y=10	y=15	p(x)
x=0	.02	.06	.02	.10	.20
x=5	.04	.15	.20	.10	.49
x=10	.01	.15	.14	.01	.31
p(y)	.07	.36	.36	.21	

Solution

- Are X and Y independent?

p(x,y)	y=0	y=5	y=10	y=15	p(x)
x=0	.02	.06	.02	.10	.20
x=5	.04	.15	.20	.10	.49
x=10	.01	.15	.14	.01	.31
p(y)	.07	.36	.36	.21	

  - p(0,0) = .02 ≠ .2 * .07 = .014 = p(0)*p(0)
  - Answer: No

More Than Two RVs

- If X_1, X_2, X_3, …, X_n are all discrete rvs, the joint pmf of the variables is

$$p(x_1, x_2, x_3, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

- If the variables are continuous, the joint pdf of X_1 to X_n

$$P(a_1 \le X \le b_1, \ldots, a_n \le X \le b_n)$$
$$= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \cdots, x_n) dx_n \cdots dx_1$$

Conditional Distribution

- Let X and Y be rvs, the conditional probability density of Y, given X = x is

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} \quad \text{for discrete}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad \text{for continuous}$$

Example

- Let I and F be rvs of insurance and flood and the joint pmf is as follows:

	Flood	No flood
Has insurance	.04	.36
Doesn't have insurance	.06	.54

  - Find the conditional pmf, $P_{F|I}$(Flood|Has insurance)

Solution

	Flood	No flood	$p_I(i)$
Has insurance	.04	.36	.40
Doesn't have insurance	.06	.54	.60
$p_F(f)$	.10	.90	1.0

$P_{F|I}$(Flod|Has insurance)

P(Flood AND Has insurance) = 0.04

P(Has Insurance)=0.40

$$P_{F|I}(\text{Flood}|\text{Has Insurance}) = \frac{P(\text{Flood AND Has insurance})}{P(\text{Has Insurance})} = \frac{0.04}{0.4} = 0.1$$

Example

- Let I and F be rvs of insurance and flood and the joint pmf is as follows:

	Flood	No flood
Has insurance	.04	.36
Doesn't have insurance	.06	.54

- Find the conditional pmf, $P_{F|I}$(No Flood|Has insurance)

Solution

	Flood	No flood	$p_I(i)$
Has insurance	.04	.36	.40
Doesn't have insurance	.06	.54	.60
$p_F(f)$	.10	.90	1.0

$P_{F|I}$(No Flood|Has insurance)

$$P_{F|I}(\text{No Flood}|\text{Has Insurance})$$

$$= \frac{\text{P(No Flood AND Has insurance)}}{\text{P(Has Insurance)}}$$

$$= \frac{0.36}{0.4}$$

$$= 0.9$$