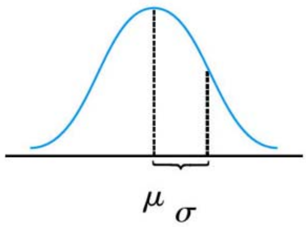
Normal Distribution

Today's Class

- Normal Distribution
- Normal Curve
- Standard Normal Distribution
- Z-score
- Percentiles

Normal Distribution

- Gaussian Distribution
- Bell shaped curve
- Two parameters
  - Mean, μ
  - Standard Deviation, σ



Carl Friedrich Gauss
(1777–1855)

pdf of a Normal rv

- Probability density function:

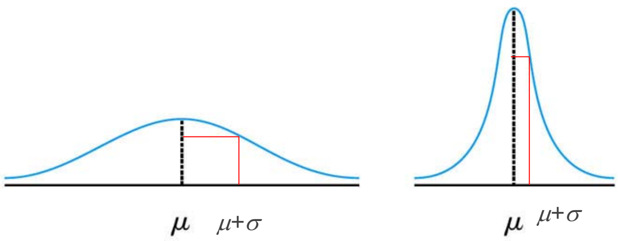
$$f(x) = f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

- If X is a rv whose pdf is normal with mean μ and variance σ^2,

$$X \sim N(\mu, \sigma^2)$$

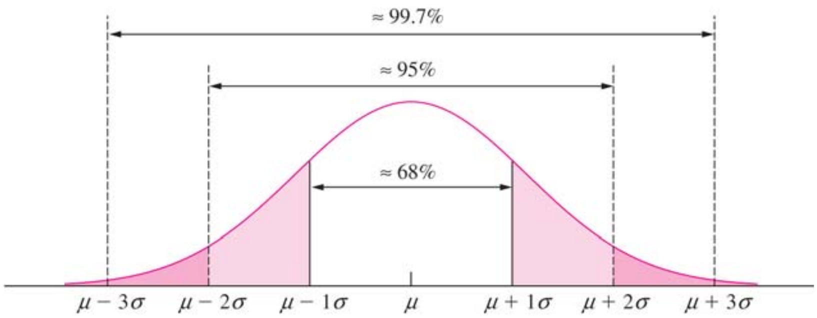
Observe…

- The effect of changes in the variance of X on the shape of the distribution
  - In particular, the larger is the standard deviation σ, the greater is the spread



Normal Curve

- Approximate percentage of area within given standard deviations (empirical rule)



(Navidi, 2009)

Standard Normal Distribution

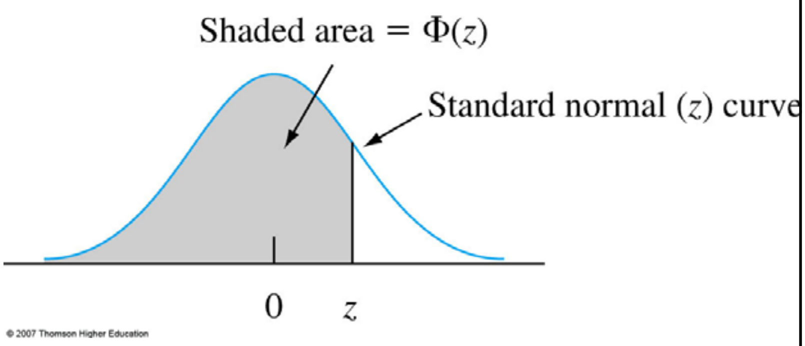
- Normal distribution with parameters:
  - Mean, $\mu = 0$
  - Standard Deviation, $\sigma = 1$
- Z: standard normal r.v. $Z = \frac{X-\mu}{\sigma}$
- The pdf of Z:

$$f(z;0,1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < x < \infty$$

- The cdf of Z, $\Phi(z)$:

$$P(Z \leq z) = \int_{-\infty}^{z} f(y;0,1)dy$$

Standard Normal Cumulative Areas

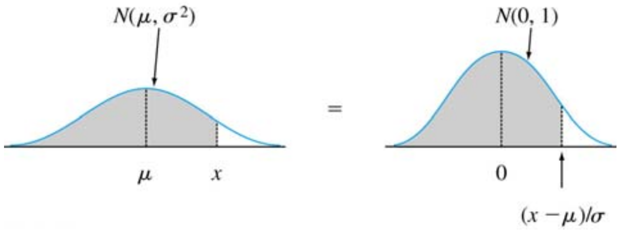


Standardizing Nonstandard Normal Distribution

- If X has a normal distribution with mean μ and standard deviation σ, then

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution.



→ Use Z-table *(Appendix A-3)*

Reading Z-Table

- The values of z are listed
  - down the rows (up to first decimal digit) and
  - across the top of the columns (second decimal digit)

- The probability that $Z \leq z$ is listed within the cell

Standard Normal Curve Areas: Z-Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example

- Let X be a normal random variable with μ=81 and σ=6.
  - Find P(X≤69)

Solution

μ=81 and σ=6

$$P(X \le 69) = P(Z \le \frac{69-81}{6})$$

$$= P(Z \le -2)$$

Check the probability from the z-table

Z-Table

$P(Z \le -2) = ?$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183

Example

- Resistors made by a certain manufacturer have resistances that are normally distributed with a mean of 9.9 ohms and SD of 0.1 ohms. If the specification limits are 10 ± 0.2 ohms, what fraction of the resistors conform to the specification limits?

Solution

- X: resistance of a randomly selected resistor

$$X \sim N(9.9, 0.1^2)$$

- Fraction conforming to specification limits:

$$\begin{aligned} P(9.8 \le X \le 10.2) &= P\left(\frac{9.8-9.9}{0.1} \le Z \le \frac{10.2-9.9}{0.1}\right) \\ &= P(-1 \le Z \le 3) \\ &= \Phi(3) - \Phi(-1) \\ &= 0.9987 - 0.1587 \\ &= 0.84 \end{aligned}$$

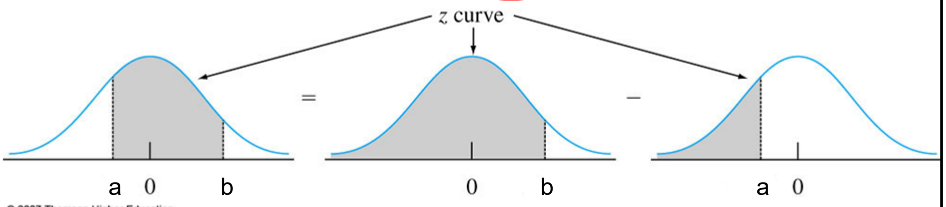
Standardization

- $X \sim N(\mu,\sigma^2) \rightarrow Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$
- Then, we can use the standard tables

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

P(X≤b) P(X≤a)

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$



Example 4.16

- The reaction time for an in-traffic response to a brake signal from standard brake lights can be modeled with a normal distribution having mean value 1.25 sec and standard deviation of .46 sec.
  - What is the probability that reaction time is between 1.00 sec and 1.75 sec?

Solution

- If we let X denote reaction time, then standardizing gives

$$P(1.00 \le X \le 1.75)$$

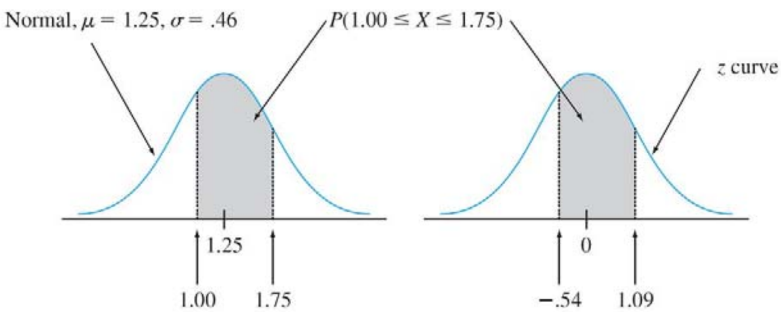
$$P(1.00 \le X \le 1.75) = P\left(\frac{1.00 - 1.25}{.46} \le Z \le \frac{1.75 - 1.25}{.46}\right)$$

$= P(-.54 \le Z \le 1.09) = \Phi(1.09) - \Phi(-.54)$

$= .8621 - .2946$

$= .5675$

Solution, Cont'd



Percentile Example I

- Find the 30th percentile of the standard normal.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.8	0.2119	0.209	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.242	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483

Percentile Solution I

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	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-0.8	0.2119	0.209	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.242	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483

$X_{30\%} \approx -0.52$ from table

Percentiles

- The (100p)th percentile is identified by the row and column in which the entry p appears
- To find the (100p)th percentile, find the value z that has probability of p
- If *p* does not appear, the number closest to it is often used, although linear interpolation gives a little better answer

Percentiles Example II

- What is the 99th percentile of $X \sim N(50, 20^2)$?

Solution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952

Solution

- What is the 99th percentile of X~N(50,20²)?
  - We want the x such that p(X<x) = .99

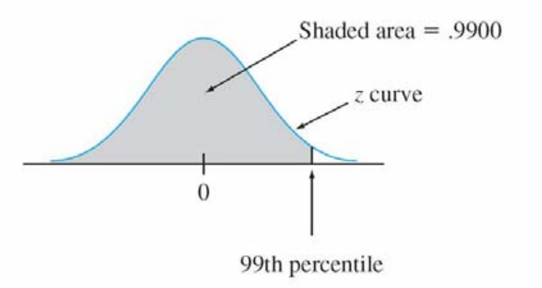
$$P(Z < \frac{x-50}{20}) = 0.99$$

  - Find 99th percentile of Z from table $\rightarrow$ 2.33

$$\frac{x-50}{20} = 2.33$$

x = 2.33 × 20 + 50 = 96.6

Percentiles

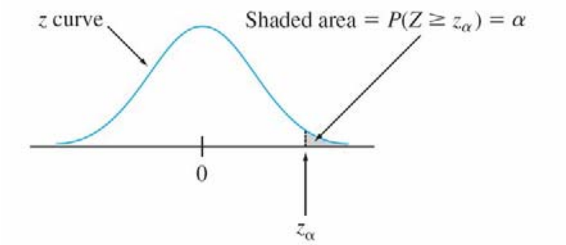


- X(percentile) = z(percentile) * σ + μ

Standard Normal Percentiles

Table 4.1 Standard Normal Percentiles and Critical Values

Percentile	90	95	97.5	99	99.5	99.9	99.95
α (tail area)	.1	.05	.025	.01	.005	.001	.0005
$z_\alpha = 100(1-\alpha)$th percentile	1.28	1.645	1.96	2.33	2.58	3.08	3.27



Percentiles Example III

- Test scores X ~ N(50, 10^2)
  - What does your score have to be to assure that you are among the top 10%?

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$z_\alpha = 100(1-\alpha)$th percentile	1.28	1.645	1.96	2.33	2.58	3.08	3.27

Solution

Let's calculate the corresponding percentiles of the distribution:

90th percentile of standard normal

$$\frac{x-50}{10} = 1.28$$

90th percentile=50+1.280*10=62.8

Thus, you need at least a score of 62.8 to be among the top 10%

Percentiles Example III

- Test scores X ~ N(50, 10^2)
  - What does your score have to be to assure that you are among the top 5%?

Table 4.1 Standard Normal Percentiles and Critical Values

Percentile	90	95	97.5	99	99.5	99.9	99.95
α (tail area)	.1	.05	.025	.01	.005	.001	.0005
$z_\alpha = 100(1 - \alpha)$th percentile	1.28	1.645	1.96	2.33	2.58	3.08	3.27

Solution

Let's calculate the corresponding percentiles of the distribution:

$$\frac{x-50}{10} = 1.645$$

95th percentile of standard normal

95th percentile=50+1.645*10=66.45

Thus, you need at least 66.45 to be on the top 5%

Percentiles Example III

- Test scores $X \sim N(50, 10^2)$
  - How well have you done in relation to the others if your score is 75?

Solution

Now, your score of 75 corresponds to a standard value

$$z = \frac{75 - 50}{10} = 2.5$$

2.5 standard deviations above the mean

Then, $P(X < 2.5) = 0.9938$

You are among the 0.62% top students

Example 4.18

- The amount of distilled water dispensed by a certain machine is normally distributed with mean value 64 oz and standard deviation .78 oz.
  - What container size *c* will ensure that overflow occurs only .5% of the time?

Table 4.1 Standard Normal Percentiles and Critical Values

Percentile	90	95	97.5	99	99.5	99.9	99.95
α (tail area)	.1	.05	.025	.01	.005	.001	.0005
$z_\alpha = 100(1 - \alpha)$th percentile	1.28	1.645	1.96	2.33	2.58	3.08	3.27

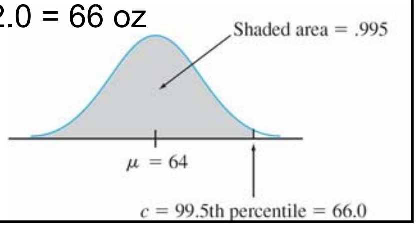
Solution

- If X denotes the amount dispensed, the desired condition is that $P(X \leq c) = .995$.
- The 99.5th percentile of the standard normal distribution is 2.58

$$\frac{c - 64}{0.78} = 2.58$$

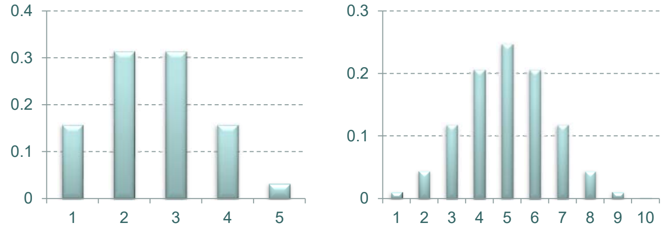
$$c = 64 + (2.58)(.78)$$

$$= 64 + 2.0 = 66 \text{ oz}$$



Normal Distribution

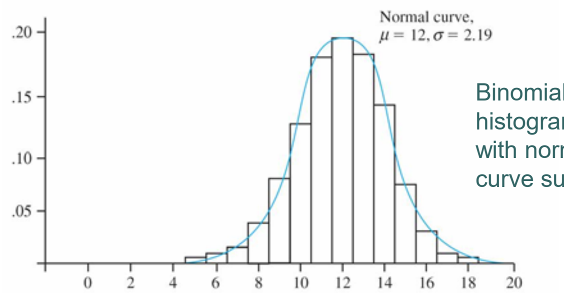
- Probability histograms for binomial for p=0.5, and n=5 and 10



- As n increases → Normal Distribution

Approximating the Binomial Distribution

- Recall binomial distribution
  - Binomial rv X
  - $\mu_X = np$ and $\sigma_X = \sqrt{npq}$



Binomial probability histogram for n = 20, p = .6 with normal approximation curve superimposed

Approximating the Binomial Distribution

- **Proposition**
  Let X be a binomial rv based on n trials with success probability p. Then if the binomial probability histogram is not too skewed, X has approximately a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$. (np ≥ 10 and nq ≥ 10)

$$P(X \leq x) = B(x, n, p) \approx \begin{pmatrix} \text{area under the normal curve} \\ \text{to the left of } x + .5 \end{pmatrix}$$

$$= \Phi\left(\frac{x + .5 - np}{\sqrt{npq}}\right)$$

0.5: continuity correction factor
P(X=n): use P(n – 0.5 < X < n + 0.5)
P(X>n): use P(X > n + 0.5)
P(X≤n): use P(X < n + 0.5)
P (X<n): use P(X < n – 0.5)
P(X ≥ n): use P(X > n – 0.5)

Example 4.20

- Suppose that 25% of all students at a large public university receive financial aid. Let X be the number of students in a random sample of size 50 who receive financial aid.
  - What is the probability that at most 10 students receive aid?

Solution

- N=50, p = .25.
  Then μ= np=12.5 and σ= $\sqrt{npq}$ =3.06.
- Since np = 12.5 ≥ 10 and nq = 37.5 ≥ 10, the approximation can safely be applied.
- The probability that at most 10 students receive aid is

$$P(X \leq 10) = B(10; 50, .25) \approx \Phi\left(\frac{10 + .5 - 12.5}{3.06}\right)$$

$$= \Phi(-0.65) = 0.2578$$