Continuous Random Variables

Today's Class

- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Uniform Distribution
  - Expected Values
  - Variance

Probability Distribution

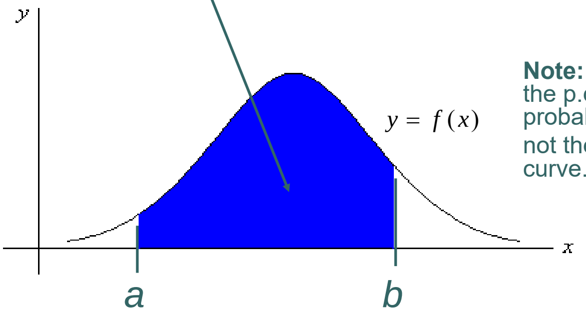
Continuous r.v. and Probability Distribution

- Let X be a continuous r.v.
- Then a probability distribution or probability density function (pdf) of X is a function f(x) such that for any two numbers a and b with a < b,

$$P(a \le X \le b) = \int_a^b f(x)dx$$

Probability Density Function

- $P(a \le X \le b)$ is given by the area of the shaded region.



Note: It is the area under the p.d.f. curve that has a probability interpretation- not the height of the p.d.f. curve.

Its graph, called the density or **p.d.f. curve** shows how the total probability of 1 is spread over the range of X

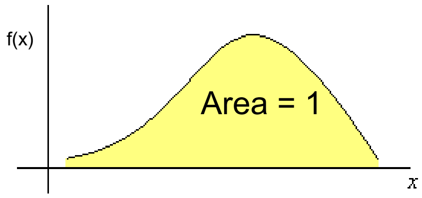
Observe that….

- If X is a continuous r.v., then for any number c, P(X = c) = 0
- Furthermore, for any two numbers a and b with a < b,

$$\begin{aligned} P(a \le X \le b) &= P(a < X \le b) \\ &= P(a \le X < b) \\ &= P(a < X < b) \end{aligned}$$

Properties of pdf

- For a function f(x) to be a legitimate pdf. it must satisfy the following properties:
  - $f(x) \geq 0$ for all x
  - $\int_{-\infty}^{\infty} f(x)dx = 1$



Uniform Distribution Example

- When a motorist stops at a red light at a certain intersection, the waiting time for the light to turn green, in seconds, is uniformly distributed on the interval (0,30).
  - Find the probability that the waiting time is between 10 and 15 seconds.

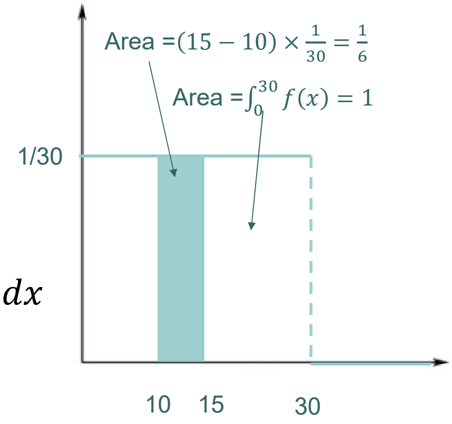
Solution

- X: waiting time
- f(x) $\begin{cases} 1/30, & 0<x<30 \\ 0, & \text{otherwise} \end{cases}$

  because $\int_0^{30} f(x) = 1$
- P(10<x<15)=$\int_{10}^{15} 1/30\, dx$

$$= \frac{15}{30} - \frac{10}{30}$$

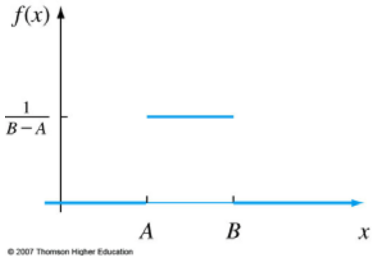
$$= \frac{1}{6}$$

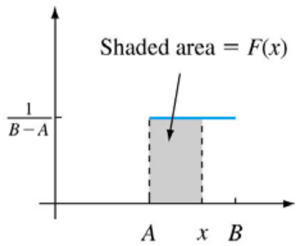


Uniform Distribution

- A continuous rv *X* is said to have a **uniform distribution** on the interval [*A*, *B*] if the pdf of *X* is

$$f(x; A, B) = \begin{cases} \dfrac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$





Example

- Suppose the time to complete a homework is uniformly distributed between 1 and 3 hours.
  - What is the probability that you finish within 2 hours?

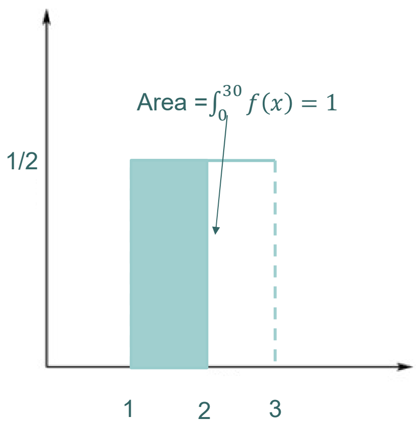
Solution

$$f(x) \begin{cases} 1/(3-1), & 1<x<3 \\ 0, & \text{otherwise} \end{cases}$$

$$P(x \le 2) = \int_1^2 1/2\,dx$$

$$= 1/2 \times (2-1)$$

$$= 0.5$$



Example

- Suppose the time to complete a homework is uniformly distributed between 1 and 3 hours.
  - What is the probability that you take more than 2.5 hours?

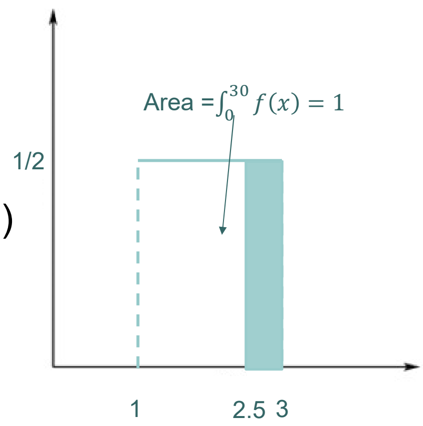
Solution

$$f(x)\begin{cases}1/(3-1), & 1<x<3\\ 0, & \text{otherwise}\end{cases}$$

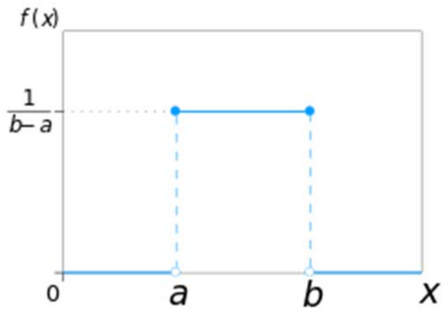
$$P(x>2.5) = \int_{2.5}^{3} 1/2\,dx$$

$$= 1/2\times(3\text{-}2.5)$$

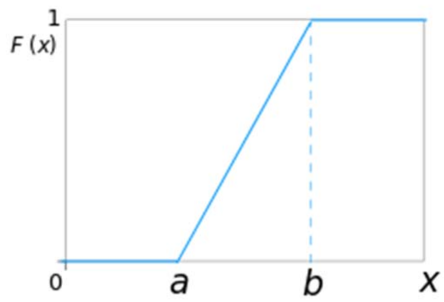
$$= 0.25$$



Cumulative Uniform Distribution



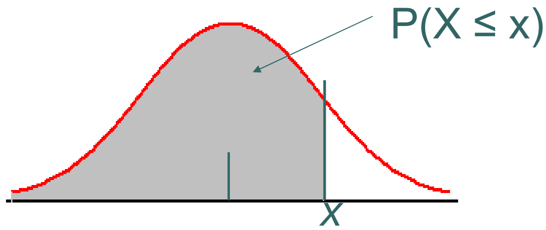
pdf



cdf

Cumulative Distribution Function

- For each x, F(x) is the area under the density curve to the left of x



$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(y)dy$$

- F(x) increases smoothly as x increases

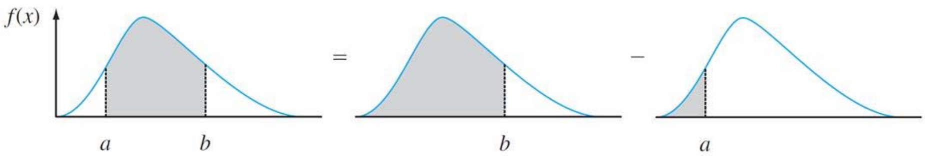
Compute Probabilities

- Let X be a continuous rv with pdf f(x) and cdf F(x). Then for any number a:

$$P(X > a) = 1 - F(a)$$

- For any two numbers a and b with a < b:

$$P(a \le X \le b) = F(b) - F(a)$$



Example, cdf

- Let X be a continuous r.v. and suppose the pdf is

$$f(x) = \begin{cases} Ae^{-x} & x \ge 0 \\ 0 & else \end{cases}$$

  - Find A

Solution

- To find A,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^{\infty} Ae^{-x}dx$$

$$= A[-e^{-\infty} + e^{-0}]$$

$$= A = 1$$

Example, cdf

- Let X be a continuous r.v. and suppose the pdf is

$$f(x) = \begin{cases} Ae^{-x} & x \geq 0 \\ 0 & else \end{cases}$$

  - Find cdf, F(x)

Solution

$$F(X)=\int_{-\infty}^{x} f(x)dx$$

$$= \int_{0}^{x} e^{-x}dx$$

$$= [-e^{-x} + e^{-0}]$$

$$= 1 - e^{-x}$$

$$F(X) = \begin{cases} 1 - e^{-x}, & \text{for } X \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$$

Example, cdf

- Let X be a continuous r.v. and suppose the pdf is

$$f(x) = \begin{cases} Ae^{-x} & x \geq 0 \\ 0 & else \end{cases}$$

  - Find P(1<X<3)

Solution

- $P(1 < X < 3) = F(3) - F(1)$
  $$= (1 - e^{-3}) - (1 - e^{-1})$$
  $$= e^{-1} - e^{-3}$$

Alternatively,

$$\int_1^3 e^{-x}dx = -e^{-3} + e^{-1}$$

$$= 0.318$$

Example, f(x) from F(x)

- Let X be the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf is
  $$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{4} & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$
  - Find the density function f(x)

Solution

- $f(x) = F'(X) = \left(\frac{x^2}{4}\right)' = \frac{x}{2}$ for $0 \le X < 2$

  0 otherwise

Obtaining f(x) from F(x)

- Recall that

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(y)dy$$

- If *X* is a continuous r.v. with pdf *f*(*x*) and cdf *F*(*x*), then at every *x* at which the derivative *F*′(*x*) exists

$$f(x) = F'(x) = \frac{d}{dx}F(x)$$

Example, Percentile

- Suppose the time to complete a homework is uniformly distributed between 1 and 3 hours.
  - What is the 95th percentile? (This means that the probability that you are done before this time is 95%)

Solution

$$F(X) = \int_1^x 1/2\, dx$$

$$= \frac{1}{2}X - \frac{1}{2} = 0.95$$

X = (0.95+0.5) × 2 = 2.9

Example, Percentile

- Suppose the time to complete a homework is uniformly distributed between 1 and 3 hours.
  - You want to be 80% sure to make an important date. What time should you set the date, if you are starting your homework at 1 pm?

Solution

$$F(X) = \int_1^x 1/2\,dx$$

$$= \frac{1}{2}X - \frac{1}{2} = 0.8$$

X = (0.8+0.5) $\times$ 2 = 2.6

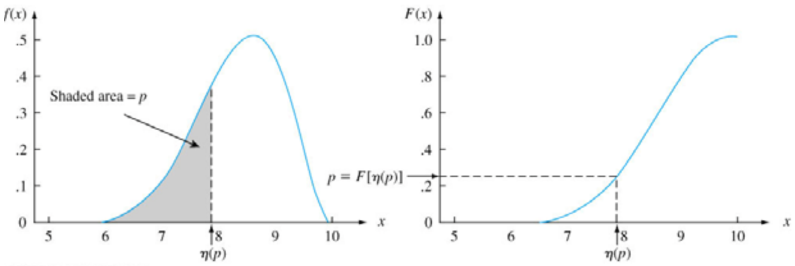
1 pm + 2.6 hours = 3.6

Therefore 3:36 pm

Percentiles Example

- The 25th percentile of the distribution of a continuous r.v. X, denoted by η(.25), is defined by

$$0.25 = F(\eta(.25)) = \int_{-\infty}^{\eta(.25)} f(y)dy$$



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Quartiles

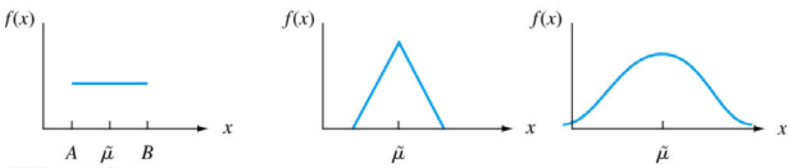
- The values that leave 25%, 50% and 75% of the distribution to the left

  Q_1 = {x s.t. F(x) = .25}

  Q_2 = {x s.t. F(x) = .50}, Median ($\tilde{\mu}$)

  Q_3 = {x s.t. F(x) = .75}

- If $\mu = \tilde{\mu}$ then distribution is symmetric



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Example, E(X) & V(x)

- When a motorist stops at a red light at a certain intersection, the waiting time for the light to turn green, in seconds, is uniformly distributed on the interval (0,30).
  - Find the mean of the waiting time

Solution

Let X represent the waiting time

$$f(x)\begin{cases} 1/30 & 0<x<30 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_x = \int_A^B x \frac{1}{B-A} dx$$

$$= \frac{(B^2 - A^2)}{2(B-A)}$$

$$= \frac{A+B}{2}$$

$$= \frac{0+30}{2} = 15$$

Alternatively, $\int_0^{30} \frac{1}{30} x dx = \frac{1}{60}(30^2 - 0^2)$ =15

Example, E(X) & V(x)

- When a motorist stops at a red light at a certain intersection, the waiting time for the light to turn green, in seconds, is uniformly distributed on the interval (0,30).
  - Find the variance of the waiting time

Solution

Let X represent the waiting time.

$$f(x)\begin{cases} 1/30 & 0<x<30 \\ 0 & \text{otherwise} \end{cases}$$

$V(X) = E(X^2)-(E(X))^2$

$$E(X^2) = \int_A^B (X^2 \times \frac{1}{B-A})dx = \frac{x^3}{3} \times \frac{1}{B-A} \Big|_A^B = \frac{B^2+AB+A^2}{3}$$

$$\therefore V(X) = \frac{B^2+AB+A^2}{3} - (\frac{A+B}{2})^2 = \frac{(B-A)^2}{12}$$

$$= \frac{(30-0)^2}{12} = 75$$

Expected Values

- Expected value or mean of a continuous rv X with pdf f(x):

$$E(X) = \mu_x = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

- If X is a continuous rv with pdf f(x) and h(X) is any function of X:

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x)dx$$

- E(aX+b)=aE(X)+b

Variance

- Variance of a continuous rv

$$\begin{aligned}\sigma_X^2 &= V(X) \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx \\ &= E(X-\mu)^2 \\ \sigma_X &= \sqrt{V(X)}\end{aligned}$$

or

$$V(X) = E(X^2) - [E(X)]^2$$

Example

- An electrician charges \$65 for any customer visit under half an hour. Any time beyond that is charged at \$60/hour. If the time taken to complete the job is estimated to be uniformly distributed on the interval [10, 180] minutes.
  - Calculate the expected charges of a service call to the electrician

Solution

X = time to complete job, X~Uniform [10, 180]

Y= charges associated with job

$$Y = G(X) = \begin{cases} \$65 & \text{if } X<30 \\ \$65+(X-30) & \text{if } X>30 \end{cases}$$

$$E[Y] = E[G(X)] = \int_{10}^{180} G(x) f(x) dx$$

$$E[Y] = \int_{10}^{30} 65 \frac{1}{170} dx + \int_{30}^{180} (65+(x-30)) \frac{1}{170} dx$$

$$E[Y] = \int_{10}^{180} 65 \frac{1}{170} dx + \int_{30}^{180} (x-30) \frac{1}{170} dx = 65 + \frac{1}{170}\left(\frac{x^2}{2} - 30x\right)\Bigg|_{30}^{180}$$

$$E[Y] = 131.18$$