Continuous Random Variables

Today's Class

- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Uniform Distribution
  - Expected Values
  - Variance

Probability Distribution

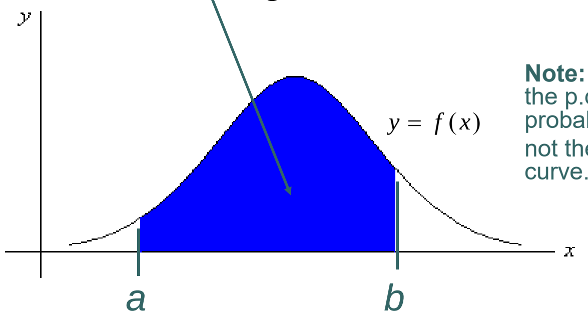
Continuous r.v. and Probability Distribution

- Let X be a continuous r.v.
- Then a probability distribution or probability density function (pdf) of X is a function f(x) such that for any two numbers a and b with a < b,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Probability Density Function

- $P(a \le X \le b)$ is given by the area of the shaded region.



Note: It is the area under the p.d.f. curve that has a probability interpretation- not the height of the p.d.f. curve.

Its graph, called the density or **p.d.f. curve** shows how the total probability of 1 is spread over the range of X

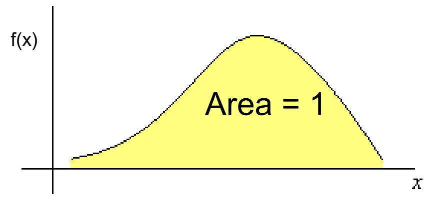
Observe that….

- If X is a continuous r.v., then for any number c, P(X = c) = 0
- Furthermore, for any two numbers a and b with a < b,

$$\begin{aligned} P(a \le X \le b) &= P(a < X \le b) \\ &= P(a \le X < b) \\ &= P(a < X < b) \end{aligned}$$

Properties of pdf

- For a function f(x) to be a legitimate pdf. it must satisfy the following properties:
  - $f(x) \geq 0$ for all x
  - $\int_{-\infty}^{\infty} f(x)dx = 1$



Example

- Is the following f(x) a legitimate pdf?

  $f(x) = 1-x$ for $0 \leq x \leq 1$

  - A: True
  - B: False

Example

- f(x) = 2-2x for $0 \le x \le 1$
  - Is f(x) a legitimate pdf?

    A: True

    B: False

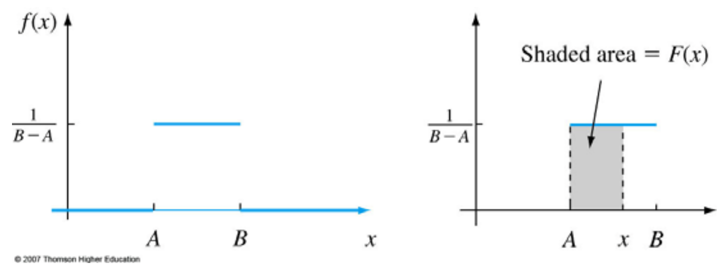
Uniform Distribution Example

- When a motorist stops at a red light at a certain intersection, the waiting time for the light to turn green, in seconds, is uniformly distributed on the interval (0,30).
  - Find the probability that the waiting time is between 10 and 15 seconds.

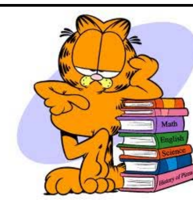
Uniform Distribution

- A continuous rv X is said to have a **uniform distribution** on the interval [A, B] if the pdf of X is

$$f(x; A, B) = \begin{cases} \dfrac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$



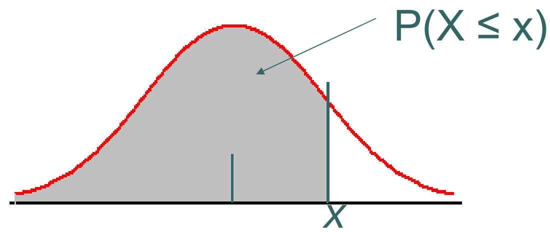
Example



- Suppose the time to complete a homework is uniformly distributed between 1 and 3 hours.
  - What is the probability that you finish within 2 hours?
  - What is the probability that you take more than 2.5 hours?

Cumulative Distribution Function

- For each x, F(x) is the area under the density curve to the left of x



$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(y)dy$$

- F(x) increases smoothly as x increases

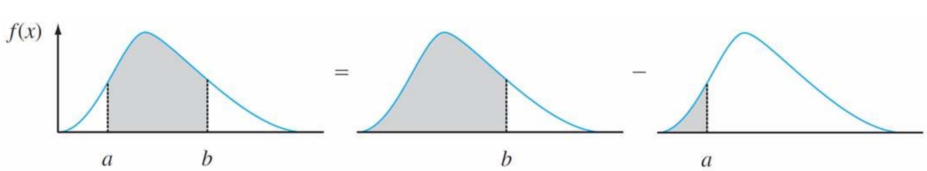
Compute Probabilities

- Let X be a continuous rv with pdf f(x) and cdf F(x). Then for any number a:

$$P(X > a) = 1 - F(a)$$

- For any two numbers a and b with a < b:

$$P(a \le X \le b) = F(b) - F(a)$$



Example, cdf

- Let X be a continuous r.v. and suppose the pdf is

$$f(x) = \begin{cases} Ae^{-x} & x \geq 0 \\ 0 & else \end{cases}$$

  - Find A
  - Find cdf, F(x)
  - Find P(1<X<3)

Example, f(x) from F(x)

- Let X be the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf is

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \leq x < 2 \\ 1 & 2 \leq x \end{cases}$$

  - Find the density function f(x)

Obtaining f(x) from F(x)

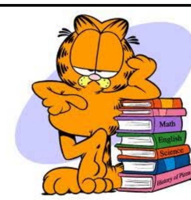
- Recall that

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(y)dy$$

- If *X* is a continuous r.v. with pdf *f*(*x*) and cdf *F*(*x*), then at every *x* at which the derivative *F'*(*x*) exists

$$\begin{aligned} f(x) &= F'(x) \\ &= \frac{d}{dx} F(x) \end{aligned}$$

Example, Percentile

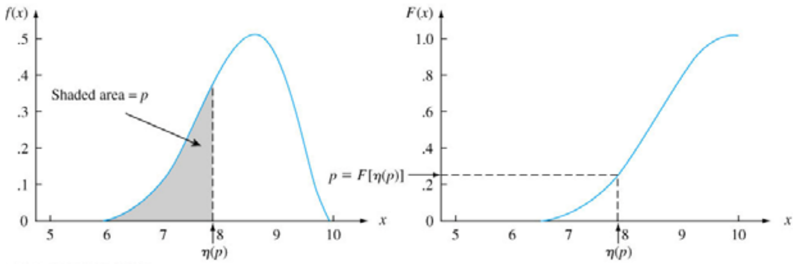


- Suppose the time to complete a homework is uniformly distributed between 1 and 3 hours.
  - What is the 95th percentile? (This means that the probability that you are done before this time is 95%)
  - You want to be 80% sure to make an important date. What time should you set the date, if you are starting your homework at 1 pm?

Percentiles Example

- The 25th percentile of the distribution of a continuous r.v. X, denoted by η(.25), is defined by

$$0.25 = F(\eta(.25)) = \int_{-\infty}^{\eta(.25)} f(y)\,dy$$



Quartiles

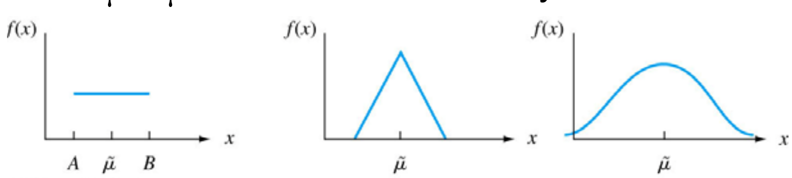
- The values that leave 25%, 50% and 75% of the distribution to the left

  Q_1 = {x s.t. F(x) = .25}

  Q_2 = {x s.t. F(x) = .50}, Median ($\tilde{\mu}$)

  Q_3 = {x s.t. F(x) = .75}

- If $\mu = \tilde{\mu}$ then distribution is symmetric



Example, E(X) & V(x)

- When a motorist stops at a red light at a certain intersection, the waiting time for the light to turn green, in seconds, is uniformly distributed on the interval (0,30).
  - Find the mean of the waiting time
  - Find the variance of the waiting time

Expected Values

- Expected value or mean of a continuous rv X with pdf f(x):

$$E(X) = \mu_x$$
$$= \int_{-\infty}^{\infty} x \cdot f(x)dx$$

- If X is a continuous rv with pdf f(x) and h(X) is any function of X:

$$E[h(X)] = \mu_{h(x)}$$
$$= \int_{-\infty}^{\infty} h(x) \cdot f(x)dx$$

- E(aX+b)=aE(X)+b

Variance

- Variance of a continuous rv

$$\sigma_X^2 = V(X)$$
$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx$$
$$= E(X-\mu)^2$$
$$\sigma_X = \sqrt{V(X)}$$

or

$$V(X) = E(X^2) - [E(X)]^2$$