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CEE 110 HW 6

1. X : # of new batteries Y : # of used batteries

a) Look at table for value where $x=1$, $y=1$ → 0.06

The probability that there's exactly one new and one used battery is 0.06

b) At most one new and one used ($x \le 1$, $y \le 1$)

$X=0, Y=0 \rightarrow 0.125$ $X=0, Y=1 \rightarrow 0.075$

$X=1, Y=0 \rightarrow 0.1$ $X=1, Y=1 \rightarrow 0.06$

$0.125 + 0.075 + 0.1 + 0.06 = 0.36$

The probability that there's at most one new and at most one used battery is 0.36

c) Exactly one new battery ($X=1$)

$X=1, Y=0 \rightarrow 0.1$ $X=1, Y=1 \rightarrow 0.06$ $X=1, Y=2 \rightarrow 0.04$

$0.1 + 0.06 + 0.04 = 0.2$

Exactly one used battery ($y=1$)

$y=1, x=0 \rightarrow 0.075$ $y=1, x=1 \rightarrow 0.06$

$y=1, x=2 \rightarrow 0.075$ $y=1, x=3 \rightarrow 0.09$

$0.075 + 0.06 + 0.075 + 0.09 = 0.3$

The probability of exactly one new battery is 0.2
The probability of exactly one used battery is 0.3

d)

| X↓ Y→ | 0 | 1 | 2 | $P_X(x)$ |
|---|---|---|---|---|
| 0 | 0.125 | 0.075 | 0.05 | 0.25 |
| 1 | 0.1 | 0.06 | 0.04 | 0.2 |
| 2 | 0.125 | 0.075 | 0.05 | 0.25 |
| 3 | 0.15 | 0.09 | 0.06 | 0.3 |
| $P_Y(y)$ | 0.5 | 0.3 | 0.2 | |

X and Y are independent. This is because the probabilities in each of the cells is the product of the marginal probabilities of the row and column.

2.

| gas $X \downarrow$ diesel $Y \rightarrow$ | 0 | 1 | 2 | $P_X(x)$ |
|---|---|---|---|---|
| 0 | 0.09 | 0.08 | 0.03 | 0.2 |
| 1 | 0.07 | 0.18 | 0.05 | 0.3 |
| 2 | 0.05 | 0.15 | 0.3 | 0.5 |
| $P_Y(y)$ | 0.21 | 0.41 | 0.38 | |

a) $P(A|B) = \frac{P(A \cap B)}{P(A)}$ $\quad X=1 \quad P(X=1) = 0.3$

| Y | 0 | 1 | 2 |
|---|---|---|---|
| p(y) | $\frac{0.07}{0.03} = 0.233$ | $\frac{0.18}{0.3} = 0.6$ | $\frac{0.05}{0.03} = 0.167$ |

$\boxed{P(y=0) = 0.233 \quad P(Y=1) = 0.6 \quad P(y=2) = 0.167}$

b) $X=2$ [same process as part a] $P(X=2) = 0.5$

| Y | 0 | 1 | 2 |
|---|---|---|---|
| P(y) | $\frac{0.05}{0.5} = 0.10$ | $\frac{0.15}{0.5} = 0.3$ | $\frac{0.3}{0.5} = 0.6$ |

$\boxed{P(y=0) = 0.10 \quad P(y=1) = 0.3 \quad P(y=2) = 0.6}$

c) $P(Y \leq 1 \mid X=2) = P(Y=0|X=2) + P(Y=1|X=2)$

$= 0.10 + 0.30$

$= 0.40$

$\boxed{P(Y \leq 1 \mid X=2) = 0.40}$

d) $Y=2 \quad P(Y=2) = 0.38$

| X | 0 | 1 | 2 |
|---|---|---|---|
| p(x) | $\frac{0.03}{0.38} = 0.079$ | $\frac{0.05}{0.38} = 0.132$ | $\frac{0.3}{0.38} = 0.789$ |

$\boxed{P(X=0) = 0.079 \quad P(X=1) = 0.132 \quad P(X=2) = 0.0789}$

3. $\lambda = 1$ Exponential Distribution

a) $f(t,\lambda) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & \text{else} \end{cases}$

$f(x,y) = f_X(x) \cdot f_Y(y) = e^{-x} \cdot e^{-y}$

$\boxed{\text{The joint pdf of } X \text{ and } Y \text{ is } e^{-x} \cdot e^{-y}}$

b) $P(X \leq 1, Y \leq 1) = \int_0^1 \int_0^1 f(x,y)\,dx\,dy$

$= \int_0^1 \int_0^1 e^{-x} \cdot e^{-y}\,dx\,dy$

$= \int_0^1 e^{-y}(1 - \frac{1}{e})\,dy$

$= (1 - \frac{1}{e})^2 \longrightarrow 0.3996$

$\boxed{\text{The probability that } X \leq 1 \text{ and } Y \leq 1 \text{ is } 0.3996}$

c) $P(X + Y \leq 2) = \int_0^2 \int_0^{2-y} e^{-x} \cdot e^{-y}\,dx\,dy$

$= \int_0^2 -e^{-y} \cdot (e^{-2+y} - 1)\,dy$

$= -\frac{3}{e^2} + 1 \rightarrow 0.594$

$\boxed{\text{The probability that the total lifetime of the chips is at most 2 is } 0.594}$

d) $P(1 \leq X + Y \leq 2) = P(X + Y \leq 2) - P(X + Y \leq 1)$

$P(X + Y \leq 1) = \int_0^1 \int_0^{1-y} e^{-x} \cdot e^{-y}\,dx\,dy$

$= \int_0^1 -e^{-y} \cdot (e^{-1+y} - 1)\,dy$

$= -\frac{2}{e} + 1 \rightarrow 0.262$

$P(X + Y \leq 2) - P(X + Y \leq 1)$

$0.594 - 0.262 = 0.332$

$\boxed{\text{The probability that the total lifetime is between 1 and 2 is } 0.332}$

4. a) $E(X+Y) = \sum_x \sum_y (x+y)\, p(x,y)$

$[(0+0)\cdot 0.01] + [(0+50)\cdot 0.05] + [(0+75)\cdot 0.03] + [(0+100)\cdot 0.01]$
$+ [(50+0)\cdot 0.02] + [(50+50)\cdot 0.2] + [(50+75)\cdot 0.15] + [(50+100)\cdot 0.1]$
$+ [(100+0)\cdot 0.01] + [(100+50)\cdot 0.14) + [(100+75)\cdot 0.15] + [(100+100)\cdot 0.13]$
$= 134.75$

$\boxed{\text{The expected recorded score } E(X+Y) \text{ is } 134.75}$

b) $E(\max(x,y)) = \sum_x \sum_y \max(x,y)\, p(x,y)$

$[0 \cdot 0.01] + [50 \cdot 0.05] + [75 \cdot 0.03] + [100 \cdot 0.01]$
$+ [50 \cdot 0.02] + [50 \cdot 0.2] + [75 \cdot 0.15] + [100 \cdot 0.1]$
$+ [100 \cdot 0.01] + [100 \cdot 0.14] + [100 \cdot 0.15] + [100 \cdot 0.13]$
$= 81$

$\boxed{\text{In this case, the expected recorded score would be } 81}$

c) $Cov[X,Y] = E[XY] - E[X]E[Y]$

$E[XY] = [2500 \cdot 0.2] + [3750 \cdot 0.15] + [5000 \cdot 0.1] + [5000 \cdot 0.14]$
$+ [7500 \cdot 0.15] + [10000 \cdot 0.13] = 4687.5$

$E[X] = [0 \cdot (0.01 + 0.05 + 0.03 + 0.01)] + [50 \cdot (0.02 + 0.2 + 0.15 + 0.1)]$
$+ [100 \cdot (0.01 + 0.14 + 0.15 + 0.13)] = 66.5$

$E[Y] = [0 \cdot (0.01 + 0.02 + 0.01)] + [50 \cdot (0.05 + 0.2 + 0.14)]$
$+ [75 \cdot (0.03 + 0.15 + 0.15)] + [100 \cdot (0.01 + 0.1 + 0.13)] = 68.25$

$4687.5 - [66.5 \cdot 68.25] = 148.75$

$\boxed{\text{The covariance is } 148.75}$

d) $V(x) = E(x^2) - [E(x)]^2$

$V(x) = [50^2 \cdot (0.02 + 0.2 + 0.15 + 0.1)] + [100^2 \cdot (0.01 + 0.14 + 0.15 + 0.13)] - 66.5^2$

$V(x) = 1052.75 \qquad \sigma_x = \sqrt{1052.75} = 32.446$

$V(y) = [50^2 \cdot (0.05 + 0.2 + 0.14)] + [75^2 \cdot (0.03 + 0.15 + 0.15)]$
$+ [100^2 \cdot (0.01 + 0.1 + 0.13)] - 68.25^2 = 573.1875$

$\sigma_y = \sqrt{573.1875} = 23.94$

$$corr(x,y) = \frac{Cov[x,y]}{\sigma_x \sigma_y} = \frac{148.75}{32.446 \cdot 23.94} = 0.1915$$

$\boxed{\text{The correlation coefficient is } 0.192}$

5. $f(x,y) = \begin{cases} K(x^2+y^2) & 20 \le x \le 30 \quad 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$ X: right Y: left
26 psi

a) First solve for K

$\int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy = 1$

$K = \frac{3}{380000}$ [we did this in lecture]

$f_X(x) = \int_{20}^{30} \frac{3}{380000} \cdot (x^2+y^2)\,dy$

$f_X(x) = \frac{3}{380000} \int_{20}^{30} x^2+y^2\,dy$

$\int_{20}^{30} x^2\,dy \to 10x^2 \qquad \int_{20}^{30} y^2\,dy \to \frac{19000}{3}$

$f_X(x) = \frac{3}{380000} \cdot \left(10x^2 + \frac{19000}{3}\right)$

$f_X(x) = \frac{3x^2+1900}{38000}$

$\boxed{\text{The marginal distribution of the right tire's air pressure is } \frac{3x^2+1900}{38000}}$

b) Use the same method as above to solve for $f_Y(y)$

$f_Y(y) = \frac{3y^2+19000}{38000}$

$f(x,y) = \frac{3}{380000}(x^2+y^2) \neq \frac{3y^2+19000}{38000} \cdot \frac{3x^2+19000}{38000}$

$\boxed{\text{X and Y are not independent because } f(x,y) \neq f_X(x) \cdot f_Y(y)}$

c) $E(XY) = \int_{20}^{30}\int_{20}^{30} xy \cdot \frac{3}{380000}(x^2+y^2)\,dx\,dy$

$\int_{20}^{30} xy \cdot \frac{3}{380000}(x^2+y^2)\,dx$

$= \frac{3y(y^2+650)}{1520}$

$\int_{20}^{30} \frac{3y(y^2+650)}{1520}\,dy$

$= \frac{24375}{38} \to 641.447$

$E(x) = \int_{20}^{30} x \cdot \left(\frac{3x^2+1900}{38000}\right)dx$ $\qquad E(y) = \int_{20}^{30} y \cdot \left(\frac{3y^2+1900}{38000}\right)dy$

$= \frac{1}{38000}\int_{20}^{30} x(3x^2+1900)\,dx$ $\qquad = \frac{1}{38000}\int_{20}^{30} x(3y^2+1900)\,dy$

$= 25.329$ $\qquad = 25.329$

$Cov(x,y) = 641.447 - (25.329 \times 25.329) = -0.11$

$\boxed{\text{The covariance is } -0.11}$

d) $V(x) = E(x^2) - [E(x)]^2$ $\quad$ $V(y) = E(y^2) - [E(y^2)]$

$E(x^2) = \int_{20}^{30} x^2 \cdot \left(\frac{3x^2+1900}{38000}\right) dx$

$= \frac{1}{380000}\left[\int_{20}^{30} 3x^4 dx + \int_{20}^{30} 1900x^2 dx\right]$

$= \frac{1}{380000}\left[12660000 + \frac{36100000}{3}\right]$

$= 649.825$

$E(y^2) = \int_{20}^{30} y^2 \cdot \left(\frac{3y^2+1900}{38000}\right) dy$

$= \frac{1}{380000}\left[\int_{20}^{30} 3y^4 + \int_{20}^{30} 1900y^2 dy\right]$

$= \frac{1}{380000}\left[12660000 + \frac{36100000}{3}\right]$

$= 649.825$

$V(x) = 649.825 - 25.329^2 = 8.267$ $\quad$ $\sigma_x = 2.875$

$V(y) = 649.825 - 25.329^2 = 8.267$ $\quad$ $\sigma_y = 2.875$

$Corr(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y}$

$= \frac{-0.11}{2.875^2}$

$\boxed{\text{The correlation coefficient is } -0.0133}$

6

$$f(x,y) = \begin{cases} k(x+y)^2 & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

25pts

a) $\int_0^1 \int_0^1 k(x+y)^2 \, dx\, dy = 1$

$\int_0^1 k(x+y)^2 dx = k(y^2 + y + \frac{1}{3})$

$\int_0^1 k(y^2 + y + \frac{1}{3}) dy = 1$

$k\int_0^1 (y^2 + y + \frac{1}{3}) dy = 1$

$\frac{7}{6}k = 1 \qquad k = \frac{6}{7}$

$\boxed{\text{k should be } \frac{6}{7} \text{ to make the joint-density function legitimate}}$

b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$= \int_0^1 \frac{6}{7}(x+y)^2 dy$

$= \frac{6}{7} \int_0^1 x^2 + 2xy + y^2 dy$

$= \frac{6}{7}\left[\int_0^1 x^2 dy + \int_0^1 2xy dy + \int_0^1 y^2 dy\right]$

$= \frac{6}{7}(x^2 + x + \frac{1}{3})$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx$

$= \int_0^1 \frac{6}{7}(x+y)^2 dx$

$= \frac{6}{7}\int_0^1 x^2 + 2xy + y^2 dx$

$= \frac{6}{7}\left[\int_0^1 x^2 dx + \int_0^1 2xy dx + \int_0^1 y^2 dx\right]$

$= \frac{6}{7}(y^2 + y + \frac{1}{3})$

$\boxed{f_X(x) = \frac{6}{7}(x^2 + x + \frac{1}{3}) \qquad f_Y(y) = \frac{6}{7}(y^2 + y + \frac{1}{3})}$

c) X and Y are independent if $f(x,y) = f_X(x) \cdot f_Y(y)$

$\frac{6}{7}(x+y)^2 = \left[\frac{6}{7}(x^2 + x + \frac{1}{3})\right] \cdot \left[\frac{6}{7}(y^2 + y + \frac{1}{3})\right]$

$\frac{6}{7}(x+y)^2 = \left[\frac{6}{7}x^2 + \frac{6}{7}y + \frac{2}{7}\right] \cdot \left[\frac{6}{7}y^2 + \frac{6}{7}y + \frac{2}{7}\right]$

$6(x+y)^2 = \left[6x^2 + 6y + 2\right] \cdot \left[6y^2 + 6y + 2\right]$

$6(x+y)^2 = 36y^2x^2 + 36y^2x + 12y^2 + 36yx^2 + 36yx + 12y + 12x^2 + 12x + 4$

These are not equivalent.

$\boxed{\text{X and Y are not independent because } f(x,y) \neq f_X(x) \cdot f_Y(y)}$

d) Correlation coefficient : $\rho(x,y) = Corr(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y}$

$Cov(x,y) = E(XY) - E(X)E(Y)$

$E(XY) = \int_0^1 \int_0^1 xy\left[\frac{6}{7}(x+y)^2\right]dxdy$

$\int_0^1 xy\left[\frac{6}{7}(x+y)^2\right]dx = \frac{1}{14}\left[y(6y^2+8y+3)\right]$

$E(XY) = \int_0^1 \frac{1}{14}\left[6y^3+8y^2+3y)\right]dy$

$E(XY) = \frac{17}{42} \approx 0.405$

$E(x) = \int_0^1 x\left[f_X(x)\right]dx$

$= \int_0^1 x\left[\frac{6}{7}(x^2+x+\frac{1}{3})\right]dx$

$= \frac{6}{7}\int_0^1 x(x^2+x+\frac{1}{3})dx$

$= \frac{6}{7}\left[\int_0^1 x^3dx + \int_0^1 x^2dx + \int_0^1 \frac{1}{3}x\,dx\right]$

$= \frac{6}{7}\left[\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right]$

$= 9/14 \approx 0.643$

$E(y) = \int_0^1 y\left[f_y(y)\right]dx$

$= \int_0^1 y\left[\frac{6}{7}(y^2+y+\frac{1}{3})\right]dx$

$= \frac{6}{7}\int_0^1 y(y^2+y+\frac{1}{3})dy$

$= \frac{6}{7}\left[\int_0^1 y^3dy + \int_0^1 y^2dy + \int_0^1 \frac{1}{3}y\,dy\right]$

$= \frac{6}{7}\left[\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right]$

$= 9/14 \approx 0.643$

$Cov(x,y) = E(XY) - E(X)E(Y)$

$= 17/42 - \left[9/14\right]^2 = -\frac{5}{588} \approx -0.00850$

$V(x) = E(x^2) - \left[E(x)\right]^2$

$E(x^2) = \int_0^1 x^2\left[\frac{6}{7}(x^2+x+\frac{1}{3})\right]dx$

$= \frac{6}{7}\int_0^1 x^4+x^3+\frac{x^2}{3}dx$

$= \frac{6}{7}\left[\int_0^1 x^4dx + \int_0^1 x^3dx + \int_0^1 \frac{x^2}{3}dx\right]$

$= \frac{6}{7}\left[\frac{1}{5}+\frac{1}{4}+\frac{1}{9}\right] = \frac{101}{210} \approx 0.481$

$V(x) = \frac{101}{210} - \left[9/14\right]^2$

$V(x) = \frac{199}{2940} \approx 0.06768$

$\sigma_x = \sqrt{V(x)} = 0.26$

$V(y) = E(y^2) - \left[E(y)\right]^2$

$E(y^2) = \int_0^1 y^2\left[\frac{6}{7}(y^2+y+\frac{1}{3})\right]dy$

$= \frac{6}{7}\int_0^1 y^4+y^3+\frac{y^2}{3}dy$

$= \frac{6}{7}\left[\int_0^1 y^4dy + \int_0^1 y^3dy + \int_0^1 \frac{y^2}{3}dy\right]$

$= \frac{6}{7}\left[\frac{1}{5}+\frac{1}{4}+\frac{1}{9}\right] = \frac{101}{210} \approx 0.481$

$V(y) = \frac{101}{210} - \left[9/14\right]^2$

$V(y) = \frac{199}{2940} \approx 0.06768$

$\sigma_y = \sqrt{V(y)} = 0.26$

$$\rho(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{-0.00850}{0.26^2}$$

$$\rho(x,y) = -0.12573$$

$\boxed{\text{The correlation of x and y is } -0.12573}$