Covariance and Correlation

Covariance is a measure of the extent to which two random variables are associated. If large values (greater than the mean) of one variable tend to occur with large values of the other variable, and small values (smaller than the mean) of the variables tend to occur together, then the covariance is positive. On the other hand, if large values of one variable tend to occur with small values of the second variable, then the covariance is negative.

Let X and Y be random variables with finite means μ_X and μ_Y. The covariance of X and Y is defined to be

$$\mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)],$$

provided that the expectation exists. Exercise 2 proves that if both X and Y have finite variance, then the covariance of X and Y exists.

For all random variables X and Y such that ${\sigma^2}_X < \infty$ and ${\sigma_Y}^2 < \infty$,

$$\mathrm{Cov}(X, Y) = E(XY) - \mu_X \mu_Y.$$

The proof is straightforward:

$$\begin{aligned} E[(X - \mu_X)(Y - \mu_Y)] &= E(XY) - E(X\mu_Y) - E(Y\mu_X) + \mu_X\mu_Y \\ &= E(XY) - \mu_X\mu_Y. \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{Cov}(X, X) &= E(X^2) - E(X)^2 \\ &= \mathrm{Var}(X). \end{aligned}$$

Suppose that the variance of X and variance of Y are finite. Then

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y).$$

The proof expands $\mathrm{Var}(X + Y) = E[(X + Y) - (\mu_X + \mu_Y)^2]$:

$$\begin{aligned} \mathrm{Var}(X + Y) &= E[(X - \mu_X + Y - \mu_Y)^2] \\ &= E[(X - \mu_X)^2 + (Y - \mu_Y)^2 + 2(X - \mu_X)(Y - \mu_Y)] \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y). \end{aligned}$$