Discrete Probability Distribution II

Today's Class

- Poisson Distribution
  - Poisson Process
  - Cumulative Function
  - Expected Value
  - Variance

Poisson as Limiting Distribution of Binomial

- A mass contains 10,000 atoms of a radioactive substance. The probability that a given atom will decay in a one-minute time period is 0.0002. Let X represent the number of atoms that decay in one minute. E(X)? P(X=3)?
- Another mass contains 5,000 atoms of a radioactive substance with the probability of decay is 0.0004. Let Y represent the number of atoms that decay in one minute. E(Y)? P(Y=3)?

Solution

X ~ Bin (10,000, 0.0002)

$$E(X) = np = 10{,}000 \times 0.0002 = 2$$

Y ~ Bin (5,000, 0.0004)

$$E(Y) = np = 5{,}000 \times 0.0004 = 2$$

$$P(X = 3) = \binom{10{,}000}{3} 0.0002^3 0.9998^{9997}$$

$$\cong 0.1804$$

$$P(X = 3) = \binom{5{,}000}{3} 0.0004^3\ 0.9996^{4997}$$

$$\cong 0.1804$$

Poisson as Limiting Distribution of Binomial

- Suppose that in the binomial distribution X~*Bin*(*n*,*p*), we let n→∞ and p→0 in such a way that *np* approaches a value λ (λ=*np*). Then

$$X \sim Bin(n,p) \rightarrow X \sim \text{Poisson}(\lambda)$$

$$\mathrm{P(X = x)} = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\approx e^{-\lambda} \frac{\lambda^x}{x!}$$

Solution, Cont'd

X ~ Bin (10,000, 0.0002)

Y ~ Bin (5,000, 0.0004)

$$E(X) = E(Y) = np = \lambda = 2$$

$$\mathrm{P(X = 3)} = \binom{10{,}000}{3} 0.0002^3 0.9998^{9997}$$

$$\cong e^{-\lambda} \frac{\lambda^x}{x!} = e^{-2} \frac{2^3}{3!} = 0.18$$

$$\mathrm{P(X = 3)} = \binom{5{,}000}{3} 0.0004^3\ 0.9996^{4997}$$

$$\cong e^{-2} \frac{2^3}{3!} = 0.18$$

Poisson Process

- An event can occur at random and at any time (or point in space)
- The occurrence(s) of an event in a fixed time (or space) interval is independent of that prior to this interval
- We can model such occurrences with a Poisson Process

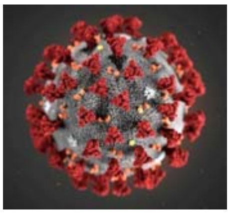
Siméon Denis Poisson
(1781-1840)

Poisson Process

- Often we are interested in events that can occur at any point in time or space
  - Fatigue cracks may occur continuously along a weld
  - Earthquakes may occur anytime and anywhere over a seismically active region
  - Traffic accidents can happen anywhere on a given highway
- We can model such occurrences with a Poisson Process

Poisson Distribution Example

- About 1 in 10,000 people has coronavirus.
  - What is the probability that 5 people out of 500 people would have this disease?



Solution

- Use Poisson approximation

  λ = np =500 × .0001 = .05

  $$P(5) \cong e^{-\lambda}\frac{\lambda^x}{x!} = e^{-.05}\frac{.05^5}{5!} = 2.5 \times 10^{-9}$$

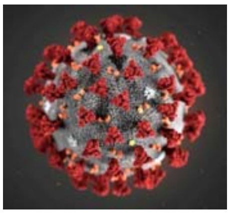
- Compare with Binomial Distribution

  $$P(5) = \binom{500}{5}.0001^5 \times (1 - 0.0001)^{495}$$

  $$= 2.43 \times 10^{-9}$$

Poisson Distribution Example

- About 1 in 10,000 people has coronavirus.
  - What is the probability that one or more out of 500 people would have it?



Solution

- Use Poisson approximation

  λ = np =500 × .0001 = .05

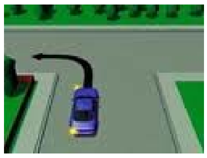
  p(1 or more) = 1-p(0)

  $$= 1 - e^{-0.05}\frac{.05^0}{0!} = .05$$

- Compared to binomial

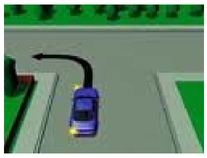
  $$1 - p(0)$$
  $$= 1 - \binom{500}{0}.0001^0 \times (1 - 0.0001)^{500}$$
  $$= 0.05$$

Example



- Suppose that average of 60 cars per hour making left turns at Westwood Plaza.
  - What is the probability that exactly 5 cars make a left turn in a 10 minute interval?

Solution



- If *X* has a Poisson distribution with $\lambda = \alpha t$
- α = 60 per hr and t = 1/6 hr

$$P(X_t = 5) = e^{-60/6} \frac{(60/6)^5}{5!} = e^{-10} \frac{10^5}{5!} = 0.038$$

Or, α = 1 per minute and t = 10 minutes

$$P(X_t = 5) = e^{-1 \times 10} \frac{(1 \times 10)^5}{5!} = e^{-10} \frac{10^5}{5!} = 0.038$$

Example

- Suppose that the average number of having accidents occurring on the I-405 is 3 based on daily traffic volume of 75,000 vehicles
  - Calculate the probability that there will be no accident today

Solution

$$P(x:\lambda) = e^{-\lambda}\frac{\lambda^x}{x!} \qquad \lambda = 3$$

$$P(X = 0) = \frac{e^{-3}3^0}{0!}$$

$$= e^{-3}$$

$$= 0.0498$$

Cumulative Poisson Probability

- See Appendix A.2

$$F(x;\lambda) = \sum_{y=0}^{x} e^{-\lambda} \frac{\lambda^y}{y!}$$

Cumulative Poisson Table

	λ									
x	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0	0.6065	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0111	0.0067
1	0.9098	0.7358	0.5578	0.4060	0.2873	0.1991	0.1359	0.0916	0.0611	0.0404
2	0.9856	0.9197	0.8088	0.6767	0.5438	0.4232	0.3208	0.2381	0.1736	0.1247
3	0.9982	0.9810	0.9344	0.8571	0.7576	0.6472	0.5366	0.4335	0.3423	0.2650
4	0.9998	0.9963	0.9814	0.9473	0.8912	0.8153	0.7254	0.6288	0.5321	0.4405
5	1.0000	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160
6	1.0000	0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622
7	1.0000	1.0000	0.9998	0.9989	0.9958	0.9881	0.9733	0.9489	0.9134	0.8666
8	1.0000	1.0000	1.0000	0.9998	0.9989	0.9962	0.9901	0.9786	0.9597	0.9319
9	1.0000	1.0000	1.0000	1.0000	0.9997	0.9989	0.9967	0.9919	0.9829	0.9682
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990	0.9972	0.9933	0.9863
11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9976	0.9945
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9992	0.9980
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Expected Values and Variance

- If X has a Poisson distribution with a parameter λ, then

$$E(X) = \lambda$$

$$V(X) = \lambda$$

Example

- Assume that the number of hits on a certain website during a fixed time interval follows a Poisson distribution. Assume that the mean rate of hits is 5 per minute. Find the probability that there will be exactly 17 hits in the next three minutes.

Solution

- Let X be the number of hits in three min.
- The mean number of hits in three min is 5×3=15

X ~ Poisson (15)

$$P(X = 17) = e^{-15}\frac{15^{17}}{17!} = .0847$$