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CEE 110 HW 5

1. Normal Distribution $\mu = 0.40$ $\sigma = 0.04$

a) $P(X > 0.51) = 1 - P(X \le 0.51)$

$= 1 - P(Z \le \frac{0.51 - 0.40}{0.04})$

$= 1 - P(Z \le 2.75)$ [look at normal distribution table]

$= 1 - 0.9970 = 0.003$

$\boxed{\text{The probability of the concentration exceeding 0.51 is 0.003}}$

b) $P(X \le 0.33) = P(Z \le \frac{0.33 - 0.40}{0.04})$

$= P(Z \le -1.75)$ [look at normal distribution table]

$= 0.0401$

$\boxed{\text{The probability that the concentration is at most 0.33 is 0.0401}}$

c) largest 10% of all concentration values

↳ 90th percentile

$P(Z \le ?) = 0.9$

$P(Z \le 1.29) = 0.9$

$Z = \frac{X - \mu}{\sigma}$

$1.29 = \frac{X - 0.4}{0.04}$

$X = 0.4516$

$\boxed{\text{The largest 10\% of all concentration values is 0.4516}}$

2. mean: 110 μm std. deviation 12.9 μm

a) "at least 140"

$P(x \geq 140) = 1 - P(x \leq 139)$

$= 1 - P(z > \frac{139-110}{12.9})$

$= 1 - P(z > 2.25)$ [look at Normal Table (WK 6)]

$= 1 - 0.9878 = 0.0122$

The probability of a single droplet of at least 140 μm in size is 0.0122

b) $P(90 < x < 140) \longrightarrow P(89 \leq x \leq 139)$

$= P(\frac{89-110}{12.9} \leq z \leq \frac{139-110}{12.9})$

$= P(-1.63 \leq z \leq 2.25)$

$= P(2.25) - P(-1.63)$

$= 0.9878 - 0.0516 = 0.9362$

The probability of a droplet to be this size is 0.9362.

c) smallest 3% of droplets

$X_{3\%} \approx z = -1.88$ [look at normal table]

$z = -1.88 \rightarrow \frac{x-110}{12.9}$

$x = 85.748 \ \mu m$

The smallest 3% of droplets is 85.748 μm

d) exactly two exceed 140 μm

From part a), we saw that $P(x \geq 140) = 0.0122$

Use Binomial distribution to solve

$Bin(2; 5, 0.0122)$

$\binom{5}{2} 0.0122^2 (1-0.0122)^{5-2} = 0.00143$

There is a 0.00143 probability of exactly two droplets exceeding 140 μm out of five droplets

3. lognormal distribution $\mu = 12933$ $CV = 0.37$ $CV = \frac{\sigma}{\mu}$

a) $E(x) = 12933 = e^{\mu + \frac{\sigma^2}{2}}$

$V(x) = (0.37 \cdot 12{,}933)^2 = e^{2\mu + \sigma^2} \cdot (e^{\sigma^2} - 1)$

$[E(x)]^2 = 12933^2 = e^{2\mu + \sigma^2}$

$$\frac{V(x)}{E(x)} = \frac{(0.37 \cdot 12{,}933)^2}{12933^2} = \frac{e^{2\mu+\sigma^2} \cdot (e^{\sigma^2} - 1)}{e^{2\mu + \sigma^2}}$$

$0.37^2 = e^{\sigma^2} - 1$

$0.128 = \sigma^2$

$E(x) = 12933 = e^{\mu + \frac{0.128}{2}}$

$\mu = 9.404$ $\sigma^2 = 0.128$ $\sigma = 0.358$

$\boxed{\text{The mean is } 9.404 \text{ and the std. deviation is } 0.358}$

b) $P(x \le 12{,}500)$

$P(Z \le \frac{\ln(12500) - 9.404}{0.358})$

$P(Z \le 0.08235)$ [look at normal table]

$= 0.5319$

$\boxed{\text{The probability that X is at most 12,500 kg/day/km is } 0.5319}$

c) $P(x > 12933) = 1 - P(x \le 12933)$

$= 1 - P(z \le \frac{\ln(12933) - 9.404}{0.358})$

$= 1 - P(z \le 0.17747)$

$= 1 - P(z \le 0.18)$

$= 1 - 0.5714 = 0.4286$

The probability that X exceeds its mean value is 0.4286. The probability is not 0.5 due to the fact that it is a lognormal distribution – it is not symmetric around the mean.

d) $Z = \frac{\ln(17000) - 9.404}{0.358} = 0.9413$

$P(z < 0.94) = 0.8264$

$0.8264 \neq 0.95$

$\boxed{\text{17,000 is not the } 95^{th} \text{ percentile of the distribution}}$

4. Exponential Distribution $\lambda = 0.02368$

a) at most 100 m

$P(t \leq 100) = [1-e^{-0.02368(100)}] = 0.90633$

at most 200 m

$P(t \leq 200) = [1-e^{-0.02368(200)}] = 0.99122$

between 100m and 200 m

$P(100 \leq t \leq 200) = F(200) - F(100) = 0.99122 - 0.90633 = 0.08489$

$\boxed{P(\text{at most } 100\text{ m}) = 0.906 \quad P(\text{at most } 200\text{ m}) = 0.991 \quad P(\text{between } 100\text{m and } 200\text{m}) = 0.08489}$

b) distance exceeds mean by more than 1 std. deviation

mean : Expected value

$E = \frac{1}{\lambda} \rightarrow E = \frac{1}{0.02368} \qquad E = 42.23$

Standard deviation = mean = 42.23

$P(t > 84.46) = 1 - P(t \leq 84.46)$

$= 1 - [1-e^{-0.02368(84.46)}]$

$= 0.13533 \approx 0.14$

$\boxed{\text{The probability that distance exceeds the mean by more than 1 standard deviation is 0.14}}$

c) Median distance $\rightarrow$ 50th percentile

$0.5 = 1 - e^{-0.02368(t)}$

$0.5 = e^{-0.02368t}$

$\ln(0.5) = -0.02368t$

$t = \frac{-\ln(0.5)}{0.02368} = 29.27$

$\boxed{\text{The median distance is } 29.27\text{ m}}$

5. mean: 24 months std. deviation: 12 months [gamma distribution]

a) $24 = \alpha\beta$ $144 = \alpha\beta^2$

$\alpha = 4$ $\beta = 6$

$\int_0^{\infty} x^3 e^{-x} dx = 6$

$\int_0^{1.5} \frac{1}{6} y^3 e^{-y} dy = 0.06564$

$P(9 \le x \le 18) = P(x \le 18) - P(x \le 9)$

$= F(\frac{18}{6}; 4) - F(\frac{9}{6}; 4)$

$= F(3; 4) - F(1.5; 4)$ [look at gamma table (wk 7)]

$= 0.353 - 0.06564 = 0.287$

$\boxed{\text{The probability of the battery lasting between 9 and 18 months is 0.287}}$

b) $P(x \le 24) = F(\frac{24}{6}; 4)$

$= F(4; 4) = 0.567$

$\boxed{\text{The probability that the battery lasts at most 24 months is 0.567. The median is less than 24 months because } 0.5 < 0.57}$

c) 99^{th} percentile

$x = 10$ $\alpha = 4$ $\rightarrow 0.990$

$10 = \frac{x}{6}$ $\rightarrow x = 60$

$\boxed{\text{The } 99^{th} \text{ percentile is 60 months}}$

d) terminated after t months

0.1% of batteries work $\rightarrow$ 99.9^{th} percentile

$F(\frac{x}{6}; 4) = 0.999$

When $\alpha = 4$, $x = 13$ $\rightarrow 0.999$

$\frac{x}{6} = 13$

$x = 13 \cdot 6$

$x = 78$

$\boxed{t \text{ has to be 78 months such that only 0.1\% of batteries would still be operating.}}$

6. 3 tsunamis/yr between 1961 - 1990 Poisson Distribution

a) $P(X \geq 1) = 1 - P(X = 0)$

$\lambda = \frac{3 \text{ tsunamis}}{1 \text{ year}} = 3$

$P(x=0) = e^{-\lambda} \frac{\lambda^x}{x!}$

$= e^{-3} \frac{3^0}{0!}$

$= .049$

$1 - 0.049 = 0.950$

$\boxed{\text{The probability that at least one tsunami strikes is } 0.950}$

b) $P(t \geq 1) = 1 - P(t < 1)$

$= 1 - [1 - e^{-\lambda t}]$

$= 1 - [1 - e^{-3(1)}] = 0.04978 \approx 0.05$

$\boxed{\text{The probability that it'll be at least one year until the next tsunami is } 0.05}$

c) Doesn't matter that one year has passed already

$P(t < 0.5) = 1 - e^{-\lambda t}$

$= 1 - e^{-3(0.5)}$

$= 0.77686 \approx 0.777$

$\boxed{\text{The probability of a tsunami striking in the next 6 months is } 0.777}$

d) $\alpha = 5$ $\beta = \frac{1}{3}$ $(\beta = \frac{1}{\lambda})$

$X \sim \Gamma(5, \frac{1}{3})$

$P(1 \leq x \leq 2) = F(\frac{2}{1/3}, 5) - F(\frac{1}{1/3}, 5)$

$F(6, 5) - F(3, 5)$ Look at gamma table

$0.715 - 0.185 = 0.53$

$\boxed{\text{The probability that T is between 1 and 2 years is } 0.53}$

* Use Gamma distribution because finding the probability of something happening between "time X" and "time Y"