Discrete Probability Distribution

Today's Class

- Bernoulli Random Variable
- Binomial Distribution
  - pmf
  - cdf
  - Expected values
  - Variance
- Hypergeometric Distribution
- Negative Binomial Distribution

Bernoulli Random Variable

- Bernoulli RV:
  - Any random variable X whose only possible values are 0 and 1

$$x = \begin{cases} 1 & \text{if success} \\ 0 & \text{if failure} \end{cases}$$

Bernoulli Trial

- Each trial has only two outcomes
  - dichotomous trials
- The trials are independent
  - the outcome on any particular trial does not influence the outcome on any other trial
- The probability of success, p, is constant

Bernoulli Distribution

- Bernoulli Probability Distribution

x	0	1
p(x)	1-p	p

$\mu = E[X] = ?$

$\sigma^2 = Var[X] = ?$

Bernoulli Example

- Suppose that we sample the temperature of a manufacturing process
  - Each of the sample can be categorized as:

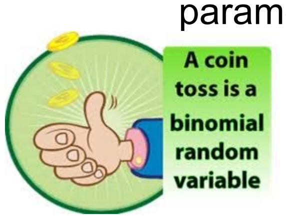
$$x = \begin{cases} 1 & \text{temp} \geq 212° \\ 0 & \text{temp} < 212° \end{cases}$$

  - Note here the sample space is the real numbers and the random variable maps from the real numbers into the integers
  - p = probability that the temp $\geq 212°$

Binomial Distribution

- If a total of n Bernoulli trials are conducted, the trials are independent
- Each trial has the same success probability p,

  X is the number of successes in the n trials
- Then X has the binomial distribution with parameters n and p



$$X \sim \mathrm{Bin}(n, p)$$

More Examples

- # of defects out of 100 inspected parts
- # of people with corona virus out of 1000
- # of working bulldozers after 6 months
- # of years in which river flow is above a flood level

Binomial Distributions Example

- A biased coin has probability 0.6 of coming up heads. The coin is tossed 3 times. Let X be the number of heads. What is the probability of 2 heads in three tosses of a coin?

Binomial Distributions

$$b(x;n,p)=\begin{cases}\binom{n}{x}p^{x}(1-p)^{n-x} & x=0,\,1,\,2,\ldots,n\\ 0 & \text{otherwise}\end{cases}$$

$$\binom{n}{x}=\frac{n!}{x!\,(n-x)!}$$

Binomial Example

- The probability that a river flow exceeds the annual maximum flood of 12 ft in any year is 0.10.
  - What is the probability that the flood level will exceed exactly once in the next 5 years?
  - What is the probability that it will not exceed in next 5 years?

Binomial Example

- Suppose you toss a fair coin 8 times. Find the probability that no more than 2 heads come up.

Binomial Cumulative Distributions

$$B(x:n,p)=\begin{cases}\sum_{i=0}^{x}\binom{n}{i}p^{i}(1-p)^{n-i} & x=0,1,2,\ldots,n\\ 0 & \text{otherwise}\end{cases}$$

$$B(x:n,p)=P[X\le x|n,p]$$

- See Appendix A.1

Binomial Cumulative Table

n=8					p					
x	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	0.663	0.430	0.272	0.168	0.100	0.058	0.032	0.017	0.008	0.004
1	0.943	0.813	0.657	0.503	0.367	0.255	0.169	0.106	0.063	0.035
2	0.994	0.962	0.895	0.797	0.679	0.552	0.428	0.315	0.220	0.145
3	1.000	0.995	0.979	0.944	0.886	0.806	0.706	0.594	0.477	0.363
4	1.000	1.000	0.997	0.990	0.973	0.942	0.894	0.826	0.740	0.637
5	1.000	1.000	1.000	0.999	0.996	0.989	0.975	0.950	0.912	0.855
6	1.000	1.000	1.000	1.000	1.000	0.999	0.996	0.991	0.982	0.965
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.998	0.996
8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Binomial Distributions Expected Values & Variance

- Expected value and Variance of X

$$X \sim \text{Bin}(n,p)$$

$$E[X] = np$$

$$Var[X] = np(1-p)$$

The Hypergeometric Box

- There is a box containing 5 red balls and 5 blue balls.
- Take one ball out of box, record color, and put away.
- If you draw 3 balls and what is the probability distribution of the number of blue balls?

Hypergeometric Experiment

- Similar to binomial, but…
- Choosing objects from a *finite* population (*N*)
- Each trial has only two outcomes
  - S or F: *M* successes in the *N* population
- A sample of size *n* is selected without replacement
- Equal probability to pick any individual outcome
- Similar to binomial, but the population is finite, so the probability of picking a 2nd outcome is different than the first
- *M/N* plays a similar role to *p*

Hypergeometric Distribution

- If *X* is the number of *S*'s in a completely random sample of size *n* drawn from a population consisting of *M S*'s and (*N–M) F*'s,

$$P(X = x) = h(x; n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

Expected Value and Variance

- Expected value of X

$$E(X) = n\frac{M}{N}$$

- Variance of X

$$V(X) = \left(\frac{N-n}{N-1}\right) n \frac{M}{N}\left(1 - \frac{M}{N}\right)$$

The Negative Binomial Box

- Consider a box with red and blue balls:
  - replace each ball after inspecting it, and
  - sample until you got *2* red balls
- Let *X* equal the number of blue balls chosen
- If you repeated this experiment a great many times, the distribution of *X* would be negative binomial

Negative Binomial Experiment

- The experiment consists of a sequence of independent trials
- Each trial has two outcomes: S or F
- The probability of success is constant from trial to trial: P(X=S) = p for i = 1,2,3,…
- The experiment continues until a total of r successes have been observed, where r is a specified positive integer
  - The r.v. of interest is X = the number of failures that precede the rth success
  - X is called a **negative binomial random variable** because, in contrast to the binomial r.v., the number of successes is fixed and the number of trials is random

Negative Binomial Distribution

- The probability that takes X=x failures to get r successes, with probability of success p is:

$$nb(x;r,p)=\binom{x+r-1}{r-1}p^r(1-p)^x,\quad x=0,1,2,\ldots$$

Expected Value and Variance

- Expected value of X

$$E(X) = \frac{r(1-p)}{p}$$

- Variance of X

$$V(X) = \frac{r(1-p)}{p^2}$$