CEE110

Homework #5 Solution

1. $\mu = .40$, $\sigma = .04$

   a. $P(X > .51) = P(Z > 2.75) = 1 - \Phi(2.75) = 1 - .9970 = .0030$.

   b. $P(X \leq .33) = P(Z \leq -1.75) = \Phi(-1.75) = .0401$

   c. We want the 90th percentile, c, of this normal distribution, so that 10% of the values are higher. The 90th percentile of the standard normal distribution satisfies $\Phi(z) = .90$, which from the normal table yields $z = 1.282$.
   So, $c = .40 + (1.282)(.04) = .4513$.
   The largest 10% of all concentration values are above .4513 mg/cm^3.

2.

   a. $P(X < 140) = P(Z < 2.33) = \Phi(2.33) = .9901$

   b. $P(90 < X < 140) = P(-1.55 < Z < 2.33) = \Phi(2.33) - \Phi(-1) = .9901 - .0606 = .9295$

   c. From the table, $\Phi(z) = .03 \rightarrow z = -1.88 \rightarrow x = 110 - 1.88\,(12.9) = 85.748$ μm.
   The smallest 3 % of droplets are those smaller than 85.748 μm in size.

   d. Let Y = the number of droplets, out of 5, that exceed 140 μm. Then Y is binomial, with n =5 and p =.0099 from **a**.

   $$P(Y = 2) = \binom{5}{2}(0.0099)^2(0.9901)^3 \approx 9.51 \times 10^{-4}$$

3. Notice that μ_X and σ_X are the mean and standard deviation of the lognormal variable X in this example; they are not the parameters μ and σ which usually refer to the mean and standard deviation of ln(X). We're given μ_X = 12,933 and σ_X/μ_X = .37, from which σ_X = .37μ_X = 4785.21.

   a. To find the mean and standard deviation of ln(X), set the lognormal mean and variance equal to the appropriate quantities: $12{,}933 = E(X) = e^{\mu+\sigma^2/2}$ and $(4785.21)^2 = V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$.
   Square the first equation: $(12{,}933)^2 = e^{2\mu+\sigma^2}$.
   Now divide the variance by this amount:
   $$\frac{(4785.21)^2}{(12{,}933)^2} = \frac{e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)}{e^{2\mu+\sigma^2}} = e^{\sigma^2} - 1 = (0.37)^2 = 0.1369$$
   $$\sigma = \sqrt{\ln(1.1369)} = 0.3582$$

   That's the standard deviation of ln(X). Use this in the formula for E(X) to solve for μ:
   $$12{,}933 = e^{\mu+\frac{(0.3582)^2}{2}} = e^{\mu+0.06415}$$
   $\mu = 9.4034$. That's E(ln(X)).

   b. $P(X \le 12{,}500) = P\left(Z \le \frac{\ln(12{,}500)-9.4034}{0.3582}\right) = P(Z \le 0.08) = \Phi(0.08) = 0.5319$

   c. $P(X \ge \mu_X) = P(X \ge 12{,}933) = P\left(Z \ge \frac{\ln(12{,}933)-9.4034}{0.3582}\right) = P(Z \ge 0.18)$
   $= 1 - \Phi(0.18) = 0.4286$.
   Even though the normal distribution is symmetric, the lognormal distribution is <u>not</u> a symmetric distribution (See the lognormal graphs in the textbook.) So, the mean and the median of X aren't the same and, in particular, the probability X exceeds its own mean doesn't equal 0.5.

   d. One way to check is to determine whether $P(X < 17{,}000) = .95$; this would mean 17,000 is indeed the 95th percentile.
   However, we find that $P(X < 17{,}000) = \Phi\left(\frac{\ln(17{,}000)-9.4034}{0.3582}\right) = \Phi(0.94) = 0.8264$, so 17,000 is <u>not</u> the 95th percentile of this distribution (it's the 82.64 %ile).

4.

a. $P(X \le 100) = 1 - e^{-(100)(0.02368)} = 1 - e^{-2.368} = 0.9063$
$P(X \le 200) = 1 - e^{-(200)(0.02368)} = 1 - e^{-4.736} = 0.9912$
$P(100 \le X \le 200) = P(X \le 200) - P(X \le 100) = 0.9912 - 0.9063 = 0.0849$

b. First, since X is exponential, $\mu = \frac{1}{\lambda} = \frac{1}{0.02368} = 42.23, \sigma = 42.23$. Then
$P(X > \mu + \sigma) = p(X > 42.23 + 42.23) = P(X > 84.46) = 1 - \left(1 - e^{-0.02368(84.46)}\right)$
$= e^{-1} = 0.1353$

c. The median is the solution to F(x) = 0.5. Use the formula for the exponential cdf and solve for x:
$F(X) = 1 - e^{-0.02368x} = 0.5$
$e^{-0.02368x} = 0.5$
$-0.02368x = \ln(0.5)$
$x = -\frac{\ln(0.5)}{0.02368} = 29.27\ m.$

5. Notice that $\mu = 24$ and $\sigma^2 = 144$.
So $\alpha\beta = 24,\ \alpha\beta^2 = 144 => \ \beta = \frac{144}{24} = 6,\ \ \alpha = \frac{24}{\beta} = 4$

a. $P(9 \le X \le 18) = F(3; 4) - F(1.5;\ 4) = 0.353 - 0.066 = 0.287$

b. $P(X \le 24) = F(4; 4) = 0.567$,
so while the mean is 24, the median is less than 24, since $P(X \le \mu) = 0.5$.
This is a result of the positive skew of the gamma distribution.

c. We want a value for x for $F\left(\frac{x}{\beta}, \alpha\right) = F\left(\frac{x}{6}, 4\right) = 0.99$.
In the gamma distribution table, we see F(10; 4) = 0.990.
So x/6 = 10, and the 99th percentile is 6(10) = 60.

d. We want a value t for which P(X>t) = 0.001, i.e. $P(X \le t) = 0.999$.
The left-hand side is the cdf of X, so we really want $F\left(\frac{t}{6}, 4\right) = 0.999$.
In the gamma distribution table, F(13; 4) = 0.999, so t/6 = 13, and t = 6(13) = 78.
At 78 months, only 0.1% of all transistors would still be operating.

6.

a. α= 3 tsunami/year

$$p(X \geq 1) = 1 - p(X = 0) = 1 - \frac{e^{-3}(3)^0}{0!} = 0.95$$

b. T ~ Exp (λ)

λ=3 occurrences per year × 1 year = 3

$$p(T \geq 1) = 1 - p(T < 1) = 1 - F(1)$$

$$= 1 - (1 - e^{-3\times 1}) = e^{-3} = 0.04979$$

≈ 0.050

c. The probability is $P(T \leq (0.5{+}1)|T > 1) = 1 - P(T > (0.5{+}1)|T > 1)$

By the property of memoryless,

$$p(T > t + t_0|T > t_0) = \frac{p(T > t + t_0 \cap T > t_0)}{p(T > t_0)} = \frac{p(T > t + t_0)}{p(T > t_0)} = \frac{e^{-\lambda(t+t_0)}}{e^{-\lambda t_0}}$$

$$= e^{-\lambda t}$$

$P(T > (0.5{+}1)|T > 1) = P(T > 0.5)$

so $P(T \leq (0.5{+}1)|T > 1) = 1 - P(T > 0.5) = P(T \leq 0.5) = 1 - e^{-3(0.5)} = 0.7769 \approx 0.78$

d. T ~ $\Gamma(\alpha, \beta)$

$\alpha = 5$

$\beta = 1/\lambda = 1/3$

P(1<T<2)

$$= F(\frac{2}{1/3}; 5) - F(\frac{1}{1/3}; 5)$$

$$= F(6; 5) - F(3; 5)$$

= 0.715 – 0. 185

= 0.53