INDEPENDENCE

Independence: For random variables X and Y, the intuitive idea behind "Y is independent of X" is that the distribution of Y shouldn't depend on what X is. This can be expressed in terms of the conditional pdf's to say "f(y|x) doesn't depend on x."

Caution: "Y is not independent of X" means simply that the *distribution* of X may vary as Y varies. It doesn't mean that X depends on Y.

If Y is independent of X, then:

1. μ_x = E(Y|X = x) does not depend on x.

(*Question*: Is the converse true? That is, if E(Y|X = x) does not depend on x, can we conclude that Y is independent of X?)

2. Let h(y) be the common pdf of the conditional distributions Y|X. Then for every x, h(y) = f(y|x) = $\frac{f(x,y)}{f_X(x)}$, where f(x,y) is the joint pdf of X and Y. Therefore

$$f(x,y) = h(y)\, f_X(x)$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \\ &= \int_{-\infty}^{\infty} h(y)\, f_X(x)\,dx \\ &= h(y)\int_{-\infty}^{\infty} f_X(x)\,dx = h(y) = f(y|x) \end{aligned}$$

In other words: *If Y is independent of X, then the conditional distributions of Y given X are the same as the marginal distribution of Y.*

3. Now we have

$$f_Y(y) = f(y|x) = \frac{f(x,y)}{f_X(x)},$$

so

$$f_Y(y)f_X(x) = f(x,y).$$

In other words: *If Y is independent of X, then the joint distribution of X and Y is the product of the marginal distributions of X and Y.*