Continuous Distribution

Today's Class

- Lognormal distribution
- Exponential Distribution
- Gamma Distribution

Lognormal Distribution

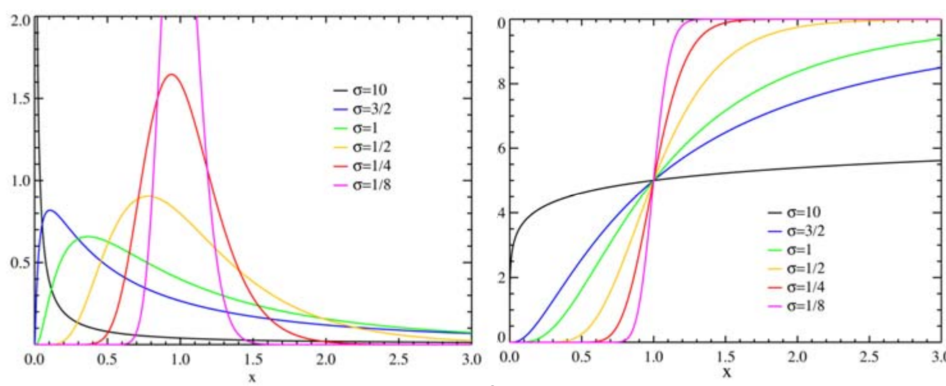
- A nonnegative rv X has a lognormal distribution if Y= ln(X) has a normal distribution with parameters μ and σ

$$f(x;\mu,\sigma)=\begin{cases}\dfrac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

- cdf $F(x;\mu,\sigma)=P(X\leq x)=P[\ln(X)\leq \ln(x)]$

$$P\left(Z\leq \frac{\ln(x)-\mu}{\sigma}\right)=\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)$$

Lognormal Density Curve



μ=0

pdf

cdf

Lognormal Examples

- A theoretical justification based on a certain material failure mechanism underlies the assumption that ductile strength X of a material has a lognormal distribution with the parameters $\mu = 5$ and $\sigma = 0.1$
  - Find the probability that the strength X of a material is greater than 125

Lognormal Distribution

- Expected Values

$$E(X) = e^{\mu+\sigma^2/2}$$

- Variance

$$V(X) = e^{2\mu+\sigma^2} \cdot \left(e^{\sigma^2} - 1\right)$$

Lognormal Examples

- A theoretical justification based on a certain material failure mechanism underlies the assumption that ductile strength X of a material has a lognormal distribution with the parameters $\mu = 5$ and $\sigma = 0.1$
  - Find the expected strength
  - Find the variance of strength

Recall Poisson Process

- Let X_t be the number of occurrences in the time interval t ($\lambda = \alpha t$)

$$p(X_t = x) = \frac{(\lambda)^x}{x!} e^{-\lambda} = \frac{(\alpha t)^x}{x!} e^{-\alpha t}$$

- Let T_1 = time until the first occurrence of an event, which is a continuous rv and t is time interval. If $t<T_1$, no event has occurred before time t

$P(t<T_1) = P(X_t = 0) = e^{-\alpha t}$

$P(t \geq T_1) = 1- P(T_1 > t) = 1 - e^{-\alpha t}$

Exponential Distribution

- The number of events occurring in any time interval of length t has a Poisson distribution
  - α is the expected number of events occurring in 1 unit of time
  - numbers of occurrences in nonoverlapping intervals are independent of one another
- Then the distribution of elapsed time between the occurrence of two successive events is exponential with parameter $\lambda = \alpha$
  - λ is the expected number of events occurring in 1 unit of time

$$P(T<t) = F(t) = 1-e^{-\lambda t}$$

Exponential Distribution

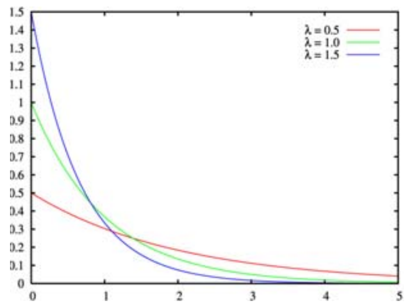
- Exponential Distribution

$$f(t;\lambda) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- cdf

$$F(t;\lambda) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Exponential Density Curve



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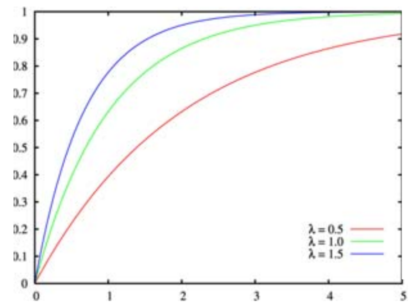

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Expected Values and Variance

$$E(T) = \int_0^{\infty} t\lambda\, e^{-\lambda t} dt = \frac{1}{\lambda}$$

$$V(T) = E(T^2) - \left(E(T)\right)^2 = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

Exponential Example

- In the period (1836 – 1961], 16 high-intensity earthquakes occurred in CA. If we assume these follow a Poisson process,
  - What is the probability that it will be at least 10 years until the next high-intensity earthquake?

Exponential Example

- A light bulb has a lifetime represented by an exponential distribution. On average, the light bulbs last 3000 hours. This light bulb has been burning for 1000 hours. What is the probability that it will last 3000 hours more?

Exponential is Memoryless

- Suppose the lifetime of a component can be represented by an exponential with parameter λ
- What is the probability that the component lasts t more hours, given that it has already lasted t_0 hours?
  - $P[T> t+t_0|T>t_0] =P[T> t]$

$$\frac{p[T > t + t_0 \cap T > t_0]}{p[T > t_0]} = \frac{p[T > t + t_0]}{p[T > t_0]} = \frac{e^{-\lambda(t+t_0)}}{e^{-\lambda t_0}} = e^{-\lambda t}$$

Percentile Example

- The lifetime of lightbulbs is represented by an exponential distribution with mean 3000 hours. How long would a lightbulb have to burn to be in the 80th percentile?

Exponential Distribution: Finding Percentile

- Let x_p represent the (100p)th percentile,

$$p = F(T) = 1 - e^{-\lambda t_p}$$

- Then

$$t_p = \frac{-\ln(1-p)}{\lambda}$$

Gamma Distribution

- If X is a rv whose pdf is gamma with parameters α and β

$$X \sim \Gamma(\alpha, \beta)$$

- Gamma Distribution

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$, $\beta > 0$ ($\beta = 1$, standard gamma distribution)

- cdf (Appendix A.4)

$$P(X \leq x) = F(x;\alpha,\beta) = F\left(\frac{x}{\beta};\alpha\right) = \int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dy$$

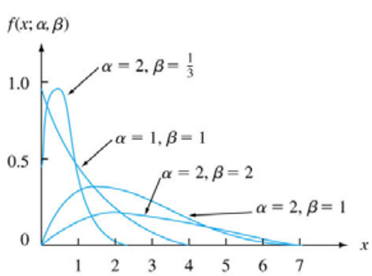
Gamma Function

- Gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \text{for } \alpha > 0$$

- The properties of the gamma function
  - For any $\alpha>1$, $\Gamma(\alpha)=(\alpha-1)\,\Gamma(\alpha-1)$
  - For any positive integer, n, $\Gamma(\alpha)=(\alpha-1)!$
  - $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Gamma Density Curve



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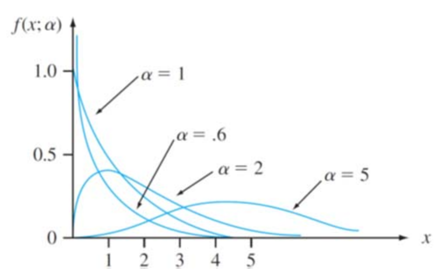

standard gamma pdf

$\beta=1$

Gamma Example

- Suppose the survival time X in weeks of a randomly selected mouse exposed to 240 rads of gamma radiation has a gamma distribution with $\alpha = 8$, $\beta = 15$
  - What is the probability that a mouse survives between 60 and 120 weeks?

Expected Values, Variance

- Expected values

$$E(X) = \mu = \alpha\beta$$

- Variance

$$V(X) = \sigma^2 = \alpha\beta^2$$

Other Continuous Distributions (Not covered)

- Chi-Squared Distribution

Let ν be a positive integer. Then a random variable X is said to have a **chi-squared distribution** with parameter ν if the pdf of X is the gamma density with $\alpha = \nu/2$ and $\beta = 2$. The pdf of a chi-squared rv is thus

$$f(x;\nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{4.10}$$

The parameter ν is called the **number of degrees of freedom** (df) of X. The symbol χ^2 is often used in place of "chi-squared."

Other Continuous Distributions (Not covered)

- Weibull Distribution

A random variable X is said to have a **Weibull distribution** with shape parameter α and scale parameter β ($\alpha > 0, \beta > 0$) if the pdf of X is

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(x/\beta)^{\alpha}} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{4.11}$$

The cdf of a Weibull rv having parameters α and β is

$$F(x;\alpha,\beta) = \begin{cases} 0 & x < 0 \\ 1 - e^{-(x/\beta)^{\alpha}} & x \geq 0 \end{cases} \tag{4.12}$$

Other Continuous Distributions (Not covered)

- Beta Distribution

A random variable X is said to have a **beta distribution** with parameters α, β (both positive), A, and B if the pdf of X is

$$f(x; \alpha, \beta, A, B) = \begin{cases} \dfrac{1}{B-A} \cdot \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \left(\dfrac{x-A}{B-A}\right)^{\alpha-1} \left(\dfrac{B-x}{B-A}\right)^{\beta-1} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

The case $A = 0$, $B = 1$ gives the **standard beta distribution.**