Random Variables

Today's Class

- Random Variables
- Discrete Random Variables
- Probability Mass Function
- Cumulative Distribution Function
- Expected Values
- Variances

Random Variables

- A **random variable (rv)** associates a number with each outcome in the sample space
  - We denote random variables with upper case letter, X
  - The observed numerical value once the experiment is run is denoted by the corresponding lower case letter, x
- In mathematical terms, a rv is a function whose domain is the sample space and the range is the set or real numbers

Random Variable Example

- A tall antenna is built on a mountain top where an extreme wind event occurs. Either the antenna fails (F) or survives (S)

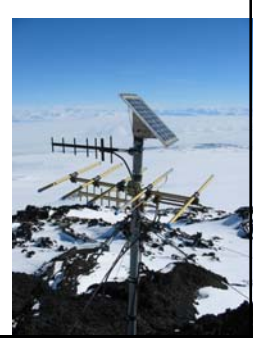
$$s = \{F, S\}$$

  - If the rv X is associated with the outcomes,

$$X(S) = 1, \ X(F) = 0$$

1 indicates that the antenna survived

0 indicates that the antenna failed



Types of Random Variables

- Discrete Random Variable:
  - takes a finite number of values

    e.g. the number of cars lined up at the FasTrak Entrance
- Continuous Random Variable:
  - takes all values in an interval

    e.g. the time each car must wait at FasTrak Entrance

Probability Mass Function Example

- Suppose you flip two coins, a rv, X, is the number of heads in the experiment
  - What is the sample space?
  - What is the probability of each outcome in the sample space?

Probability Mass Function

- A Probability Mass Function (pmf), also called *probability distribution*, is a function p(x) that assigns to each possible value x that the random variable X can take, its probability

$$p(x) = P(X = x)$$
$$= P(\text{all } s \in S : X(s) = x)$$

  - $p(x_i) \geq 0$ for each possible value x_i of X
  - $\sum_{\text{all } x_i} p(x_i) = 1$

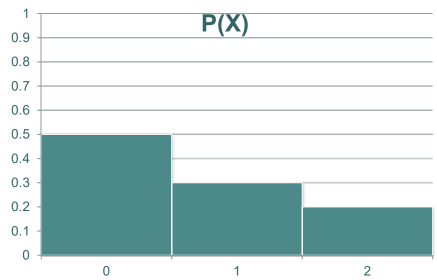
Example: pmf

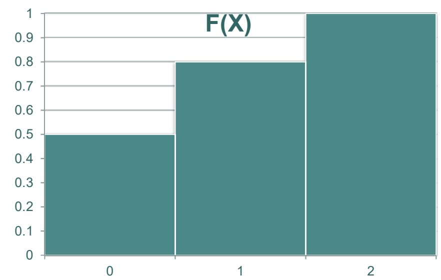
- Consider 10 truckloads of rebar are delivered to the job site. On each of those truckloads, there will be some damaged bars (X).
  - What is the sample space?
  - What is the pmf?

Truckload	1	2	3	4	5	6	7	8	9	10
Number of damaged bars	0	0	1	0	1	0	2	1	0	2

Cumulative Distribution Function Example

- From the previous example, let F(x) denote the cdf of the rv X





Cumulative Distribution Function

- The cumulative distribution function (cdf) of a r.v. X is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i)$$

which gives the sum of the probabilities up to that value x

Cumulative Distribution Function

- Any probability distribution must follow the axioms of probability
  - $F(-\infty)=0$; $F(\infty) = 1$
  - $F(x) \geq 0$ and is weakly increasing
  - It is continuous in x
- Any function that satisfies these axioms is a cdf

Expected Value Example

- Let X = number of working bulldozers after 6 months. Assume the probability that a bulldozer is working after 6 months is 0.8, and there are 3 dozers.
  - Find the pmf
  - Find the cdf
  - What is the expected value of the number of dozers working after 6 months?

Expected Value

- The *expected value* is the long run expected mean, if you were to see X over and over again

$$E[X] = \mu_x = \sum_{x \in D} x \cdot p(x)$$

Example: Expected Values

- In previous example, at least 2 dozers are needed to finish a $100K job. Every dozer that was brought in after 6 months costs $10K. What is your expected profit if you start with 3 dozers?

Expected Value of a Function

- Consider that there might be a functional relationship with X with a set of possible values D and pmf p(x) such that we have a probability of h(X)

$$E[h(X)] = \sum_D h(x) \cdot p(x)$$

Example: E[X] Properties

- Your profit is equal to $6000 plus $10 for every dozer that is working at 6 months. What is the expected value of your profit?

Properties of Expected Value

$$E(X_1 + X_2) = E[X_1] + E[X_2]$$

$$E(aX + b) = a \times E[X] + b$$

Variance Example

- In the previous example of bull dozer, find variance of the number of dozers working after 6 months?

X	0	1	2	3
P(x)	.008	.096	.384	.512

Variance

- The variance is defined as

$$
\begin{aligned}
V(X) &= \sigma^2 \\
&= E[(X-\mu)^2] \\
&= \sum_D (x-\mu)^2 \cdot p(x) \\
&= E[X^2] - \mu^2
\end{aligned}
$$

Properties of Variance & Standard Deviation

$$V(aX+b) = a^2\sigma^2{}_X$$

$$\sigma_{aX+b} = |a|\,\sigma_X$$

Example: Variance Properties

- In previous example, at least 2 dozers are needed to finish a $100K job. Every dozer that was brought in after 6 months costs $10K. Suppose you start with 3 dozers.
  - What is the variance of your profit?