Proposition.

E(X+Y) = E(X) + E(Y)

Proof

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xf(x,y)dxdy \;+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}yf(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x[f(x,y)dy]dx \;+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y[f(x,y)dx]dy$$

$$= \int_{-\infty}^{\infty}xf_x(x)dx \;+ \int_{-\infty}^{\infty}yf_y(y)dy$$

$$= E(X) + \; E(Y)$$

The proof in the case that X and Y are jointly discrete is similar, with the integrals replaced by sums.

Proposition.

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y).$$

Proof

$$\begin{aligned} \mathrm{Var}(X+Y) &= \mathrm{E}([X+Y]^2) - \mathrm{E}(X+Y)^2 \\ &= [\mathrm{E}(X^2) - \mathrm{E}(X)^2] + [\mathrm{E}(Y^2) - \mathrm{E}(Y)^2] + 2[\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)] \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) \end{aligned}$$