CEE 110
Homework #6 Solution

1.

a. $p(1,1) = 0.06$

b. $p(X \leq 1 \; and \; Y \leq 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = 0.36$

c. $p(X = 1) = p(1,0) + p(1,1) + p(1,2) = 0.20$
$p(Y = 1) = p(0,1) + p(1,1) + p(2,1) + p(3,1) = 0.30$

d. The marginal probabilities for X (rows sums from the joint probability table) are $p_x(0) = 0.25$, $p_x(1) = 0.2$, $p_x(2) = 0.25$, $p_x(3) = 0.3$; those for Y (column sums) are $p_y(0) = 0.5$, $p_y(1) = 0.3$, $p_y(2) = 0.2$. It is now easily verified that for every $(x, y), p(x, y) = p_x(x) \times p_y(y)$, so X and Y are independent.

2.

a. $p_{Y|X}(y \mid 1)$ results from dividing each entry in x = 1 row of the joint probability table by $p_x(1) = 0.3$:
$p_{Y|X}(0 \mid 1) = \frac{0.07}{0.30} = 0.2333$ $\quad p_{Y|X}(1 \mid 1) = \frac{0.18}{0.30} = 0.60$ $\quad p_{Y|X}(0 \mid 1) = \frac{0.07}{0.30} = 0.1667$

b. b. $p_{Y|X}(y \mid 2)$ is requested; to obtain this divide each entry in the x = 2 row by $p_x(2) = 0.5$:

y	0	1	2
$p_{Y\|X}(y \mid 2)$	0.1	0.3	0.6

c. $p(Y \leq 1 | X = 2) = p_{Y|X}(0 \mid 2) + p_{Y|X}(1 \mid 2) = 0.1 + 0.3 = 0.4$

d. $p_{X|Y}(x \mid 2)$ results from dividing each entry in the y = 2 column by $p_y(2) = 0.38$

x	0	1	2
$p_{X\|Y}(x \mid 2)$	0.0789	0.1316	0.7895

3.

a. $f(x, y) = f_x(x) \times f_y(y) = \begin{cases} e^{-x-y} & x \geq 0, \; y \geq 0 \\ 0 & otherwise \end{cases}$

b. By independence, $P(X \leq 1 and \; Y \leq 1) = P(X \leq 1) \times P(Y \leq 1) = (1 - e^{-1})(1 - e^{-1}) = 0.400$

c. c.$P(X + Y \; \leq 2) = \int_0^2 \int_0^{2-x} e^{-x-y} dydx = \int_0^2 e^{-x}[1 - e^{-(2-x)}]dx \; = \int_0^2 (e^{-x} - e^{-2})dx = 1 - e^{-2} - \quad 2e^{-2} = 0.594$

d. $P(X + Y \; \leq 1) = \int_0^1 e^{-x}[1 - e^{-(1-x)}]dx = 1 - 2e^{-1} = 0.264$
so $P(1 \leq X + Y \leq 2) = P(X + Y \; \leq 2) - P(X + Y \leq 1) = 0.594 - 0.264 = 0.330$

4.

a. $E(X+Y) = \sum\sum(x+y)p(x,y) = (0+0)(0.01) + (0+50)(0.05) + \cdots + (100+100)(0.13) = 134.75$. Note: It can be shown that E(X+Y) always equals E(X) + E(Y), so in this case we could also work out the means of X and Y from their marginal distributions: E(X) = 66.5, E(Y) = 68.25, so E(X+Y) = 66.5+68.25 = 134.75

b. For each coordinate, we need the maximum: e.g., max(0 ,0) = 0, while max(50, 0) = 50 and max(50, 75) = 75. Then, calculate the sum: $E\big(max(X,\ Y)\big) = \sum\sum max(x+y)p(x,y) = max(0,0)(0.01) + max(0,50)(0.05) + \cdots + max(100,100)(0.13) = 0(0.01) + 50(0.05) + \cdots + 100(0.13) = 81$

c. E(x) = 66.5, E(Y) = 68.25, E(XY) = (0)(0.01)+(0)(0.05)+…+(10000)(0.13) = 4687.5, so COV(X,Y) = 4687.5 – (66.5)(68.25) = 148.875

d. By direct computation, $\sigma_x^2 = 1052.75$ and $\sigma_y^2 = 573.19$, so $\rho_{X,Y} = \frac{148.875}{\sqrt{(1052.75)(573.19)}} = 0.192$

5.

a. $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{20}^{30} K(x^2+y^2)dy = 10Kx^2 + K\frac{y^3}{3}\Big]_{20}^{30} = 10Kx^2 + 0.05, for 20 \leq x \leq 30$

b. $f_Y(y)$ can be obtained by substituting y for x in (d); clearly $f(x,y) \neq f_X(x) \cdot f_Y(y)$, so X and Y are not independent.

c. $E(X) = \int_{20}^{30} x\, f_X(x)dx = \int_{20}^{30} x[10Kx^2 + 0.05]dx = \frac{1925}{76} = 25.329 = E(Y)$,
$E(XY) = \int_{20}^{30}\int_{20}^{30} xy \cdot K(x^2+y^2)\, dxdy = \frac{24375}{38} = 641.447$
$Cov(X,Y) = 641.447 - (25.329)^2 = -0.1082$

d. $E(X^2) = \int_{20}^{30} x^2[10Kx^2 + 0.05]dx = \frac{37040}{57} = 649.8246 = E(Y^2)$
$V(X) = V(Y) = 649.8246 - (25.329)^2 = 8.2664$
$\rho = -\frac{0.1082}{\sqrt{(8.2664)(8.2664)}} = -0.0131$

6.

a. $\int_0^1 \int_0^1 c(x+y)^2 dxdy = 1$

$$\int_0^1 \int_0^1 k(x^2 + 2xy + y^2)dxdy = 1$$

$$\int_0^1 k\left(x^2y + xy^2 + \frac{1}{3}y^3\right)dx = 1$$

$$\int_0^1 k\left(x^2 + x + \frac{1}{3}\right)dx = 1$$

$$k\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x\right) = 1$$

$$k\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3}\right) = 1$$

$k = \frac{6}{7}$ ≈0.86

b. For 1<y<0,

$$\int_0^1 \frac{6}{7}(x+y)^2 dx = \frac{2}{7}x^3 + \frac{6}{7}x^2y + \frac{6}{7}y^2x|_0^1$$

$$f_Y(y) = \frac{2}{7} + \frac{6}{7}y + \frac{6}{7}y^2$$

Otherwise 0

Likewise,

$$f_X(x) = \frac{2}{7} + \frac{6}{7}x + \frac{6}{7}x^2$$

c. No. Because $f(x,y) \neq f_X(x)f_Y(y)$

$$\frac{6}{7}(x+y)^2 \neq (\frac{2}{7} + \frac{6}{7}x + \frac{6}{7}x^2)(\frac{2}{7} + \frac{6}{7}y + \frac{6}{7}y^2)$$

d. Cov(X,Y)=E(XY)-E(X)E(Y)

$E(X) = \int_0^1 x f_X(x)dx = \int_0^1 x\left(\frac{2}{7} + \frac{6}{7}x + \frac{6}{7}x^2\right)dx = \frac{1}{7}x^2 + \frac{2}{7}x^3 + \frac{3}{14}x^4|_0^1 = \frac{9}{14}$ ≈ 0.6429

Likewise,

$$E(Y) = \int_0^1 y f_Y(y)dx = \frac{9}{14}$$

≈ 0.6429

$$E(XY) = \int_0^1 \int_0^1 xy\frac{6}{7}(x+y)^2 dxdy$$
$$= \int_0^1 \int_0^1 \frac{6}{7}x(x^2y + 2xy^2 + y^3)dxdy$$
$$= \int_0^1 \frac{6}{7}\left(\frac{1}{2}x^3y^2 + \frac{2}{3}x^2y^3 + \frac{1}{4}xy^4\right)|_0^1 dx$$
$$= \int_0^1 \frac{6}{7}\left(\frac{1}{2}x^3 + \frac{2}{3}x^2 + \frac{1}{4}x\right)dx$$
$$= \frac{6}{7}\left(\frac{1}{8}x^4 + \frac{2}{9}x^3 + \frac{1}{8}x^2\right)|_0^1$$
$$= \frac{17}{42}$$

≈ 0.4048

Therefore, Cov(X,Y)

$$= \frac{17}{42} - \frac{9}{14}\frac{9}{14} = -\frac{5}{588}$$

≈ -0.0085

Or Cov(X,Y) = 0.4048-0.6429*0.6429 = -0.0085

Or Cov (X,Y)= 0.40-0.64*0.64 = -0.0096

$$Corr(X,Y) = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$$

$V(X)=E(X^2) - (E(X))^2$

$$E(X^2) = \int_0^1 x^2 f_X(x)dx = \int_0^1 x^2 \left(\frac{2}{7} + \frac{6}{7}x + \frac{6}{7}x^2\right) dx = \frac{2}{21}x^3 + \frac{6}{28}x^4 + \frac{6}{35}x^5|_0^1$$
$$= \frac{40 + 90 + 72}{420} = \frac{101}{210} \approx 0.4801\ (or\ 0.48)$$

$V(X) = 0.4801 - 0.6429^2 = 0.0668$ (or 0.48-0.64^2 =0.0704)

$\sigma_X = \sqrt{0.0668} = 0.2585\ (or\ \sqrt{0.0704} = 0.2653)$

Likewise,

$$E(Y^2) = \int_0^1 y^2 f_Y(y)dx = \int_0^1 y^2 \left(\frac{2}{7} + \frac{6}{7}y + \frac{6}{7}y^2\right) dx = \frac{101}{210} \approx 0.4801\ (or\ 0.48)$$

$V(Y) == 0.4801 - 0.6429^2 = 0.0668$ (or 0.48-0.64^2 =0.0704)

$\sigma_Y = \sqrt{0.0668} = 0.2585\ (or\ \sqrt{0.0704} = 0.2653)$

$$Corr(X,Y) = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$$
$$= \frac{-0.0085}{0.2585 \times 0.2585} = -0.1272$$
$$(or = \frac{-0.0085}{0.2653 \times 0.2653} = -0.1208)$$

Therefore, X and Y are not correlated.