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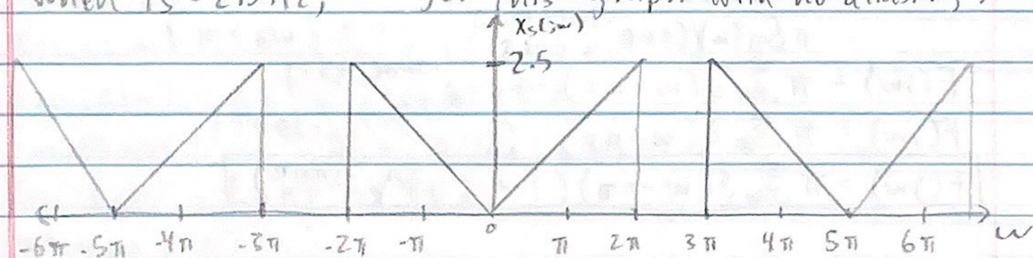
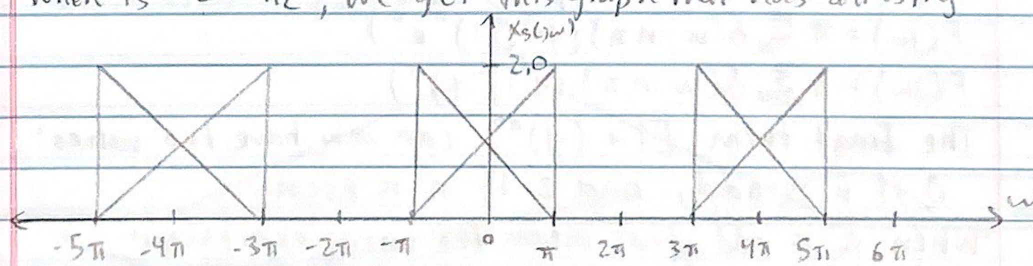
ECE 102 HW 7

1. a) Based on the given graph, the Nyquist rate is:
[$2 \times 5\pi$] = 10 rad/s, (5 Hz)
The cutoff frequency is half of the sampling frequency, so it is $\boxed{2.5 \text{ Hz.}}$

b) The period will be $W_s = 2\pi F_s$
There will be scaling by $F_s = \frac{1}{T_s}$
Using $F_s = 2.5$ Hz is better in this situation as it prevents accidental aliasing, which can ruin the signal.
When $F_s = 2.5$ Hz, we get this graph with no aliasing:




When $F_s = 2.0$ Hz, we get this graph that has aliasing:




Clearly, the $F_s = 2.5$ Hz ($w = 5\pi$ rad/s) graph has better results. A bandpass filter from $3\pi \le |w| \le 5\pi$ would help to recover the signal. The minimum value for $F_s$ would be $5\pi$ rad/s. This is because any lower values would create aliasing with respect to the frequency domain, which should be avoided if possible.

[Sorry these graphs are a bit ugly]

2. a) Looking at the graph, we observe that:

Odd samples (close to -3, -1, 1, 3) - $\delta$ train separated by 2, with a delay of $1+\tau$

Even samples (-2, 0, 2) - $\delta$ train with no delay

Adding the odd and even samples yields:

$$\boxed{\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} \delta(t-(1+\tau+2k)) + \sum_{k=-\infty}^{\infty} \delta(t-2k) \\ f(t) &= \delta_2(t-(1+\tau)) + \delta_2(t) \end{aligned}}$$

b) Find the Fourier Transform

$$F(j\omega) = \pi\delta_{\pi}(\omega)e^{-j\omega(1+\tau)} + \pi\delta_{\omega}(\omega)$$

$$= \pi\delta_{\pi}(\omega)(1+e^{-j\omega(1+\tau)}) \qquad [\omega_0 = \pi]$$

$$F(j\omega) = \pi\sum_{n=-\infty}^{\infty}\delta(\omega-n\pi)\cdot(1+e^{-j\omega\pi(1+\tau)})$$

$$F(j\omega) = \pi\sum_{n=-\infty}^{\infty}\delta(\omega-n\pi)\cdot(1+e^{-j\pi n}e^{-j\pi n\tau})$$

$$\boxed{F(j\omega) = \pi\sum_{n=-\infty}^{\infty}\delta(\omega-n\pi)(1+(-1)^n e^{-j\pi n\tau})}$$

c) $F(j\omega)$ when $\tau=0$

Plug into answer from part b

$$F(j\omega) = \pi\sum_{n=-\infty}^{\infty}\delta(\omega-n\pi)(1+(-1)^n e^0)$$

$$F(j\omega) = \pi\sum_{n=-\infty}^{\infty}\delta(\omega-n\pi)(1+(-1)^n)$$

The final term $[1+(-1)^n]$ can only have two values: 0 if $n$ is odd, and 2 if $n$ is even

When $n$ is odd, that means the entire expression will get zeroed out

When $n$ is even, that means the entire expression will be scaled by a factor of 2

$$F(j\omega) = \pi\sum_{n=-\infty}^{\infty}\delta(\omega-2n\pi)\cdot 2$$

$$F(j\omega) = 2\pi\,\delta_{2\pi}(\omega)$$

This result is actually the Fourier transform for $\delta_1(t)$

This was the result I expected before performing any calculations because when $\tau$ goes to 0, the non-uniform cases will just go to the uniform case.

d) The sampled signal is represented by $f(t)g(t)$.
We must find the Fourier Transform of this.

$G_s(j\omega) = \frac{1}{2\pi} F(j\omega) * G(j\omega)$ [convolution prop.?]

$G_s(j\omega) = \frac{1}{2\pi}\left[\pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)\cdot\left(1+(-1)^n e^{-j\pi n\tau}\right)\right] * \Delta\left(\frac{\omega}{\pi}\right)$

$G_s(j\omega) = 0.5 \sum_{n=-\infty}^{\infty} \Delta\left(\frac{\omega - n\pi}{\pi}\right)\cdot\left(1+(-1)^n e^{-j\pi n\tau}\right)$

Find the baseband replica [$n=0$]

$G_{s,0}(j\omega) = 0.5\,\Delta\left(\frac{\omega}{\pi}\right)(2)$

$G_{s,0}(j\omega) = \Delta\left(\frac{\omega}{\pi}\right)$

When [$n=1$]

$G_{s,1}(j\omega) = 0.5\,\Delta\left(\frac{\omega-\pi}{\pi}\right)\cdot\left(1+(-1)^1 e^{-j\pi\tau}\right)$

$G_{s,1}(j\omega) = \Delta\left(\frac{\omega-\pi}{\pi}\right)\cdot\left(1-e^{-j\pi t}\right)$

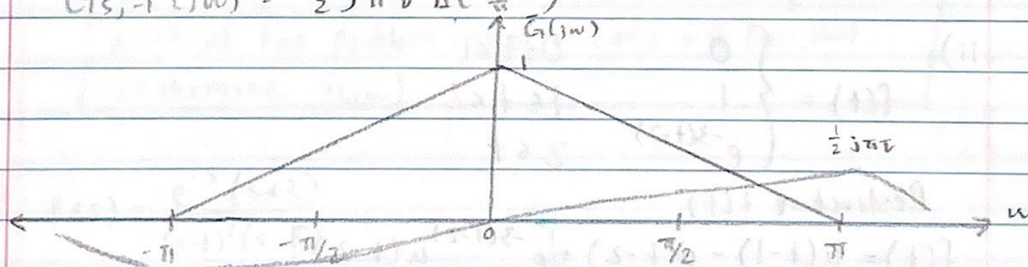
$G_{s,1}(j\omega) = \Delta\left(\frac{\omega-\pi}{\pi}\right)\cdot\left(1-(1-j\pi\tau)\right)$

$G_{s,1}(j\omega) = \frac{1}{2} j\pi\tau\,\Delta\left(\frac{\omega-\pi}{\pi}\right)$

When [$n=-1$]

Same steps as $n=1$, but flip signs

$G_{s,-1}(j\omega) = -\frac{1}{2} j\pi\tau\,\Delta\left(\frac{\omega+\pi}{\pi}\right)$



3 a) i) $f(t) = te^{-at}(\sin w_0 t)^2 u(t)$

Restructure $f(t)$

$$f(t) = te^{-at}\left[\tfrac{1}{2}\left(1-\cos(2w_0 t)\right)\right] \cdot u(t)$$

$$f(t) = \left[\tfrac{1}{2}te^{-at} \cdot u(t)\right] - \left[\tfrac{1}{2}te^{-at}\cos(2w_0 t) \cdot u(t)\right]$$

$$\cos(2w_0 t) \cdot u(t) \rightarrow \frac{s}{(2w_0)^2 + s^2}$$

$$t\cos(2w_0 t) \cdot u(t) \rightarrow -\frac{d}{ds}\frac{s}{(2w_0)^2 + s^2}$$

$$\rightarrow \frac{s^2 - 4w_0^2}{(s^2 + 4w_0^2)^2}$$

$$e^{-at}t\cos(2w_0 t) \cdot u(t) \rightarrow \frac{[s+a]^2 - 4w_0^2}{[4w_0^2 + (s+a)^2]^2}$$

Combine all this together

$$\boxed{F(s) = \left[0.5\frac{1}{(a+s)^2}\right] - \left[0.5\frac{(a+s)^2 - 4w_0^2}{[(a+s)^2 + 4w_0^2]^2}\right]}$$

Region of convergence: $Re\{s\} > -a$

ii)

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ e^{-3(t-2)} & 2 \le t \end{cases}$$

Restructure $f(t)$

$$f(t) = u(t-1) - u(t-2) + \left[e^{-3(t-2)}u(t-2)\right]$$

Pretty straightforward Laplace transform, simple terms

$$\boxed{F(s) = \left(\tfrac{1}{s}e^{-s}\right) - \left(\tfrac{1}{s}e^{-2s}\right) + \left(\tfrac{1}{s+3}e^{-2s}\right)}$$

Region of convergence : $Re\{s\} > -3$

Just look at the last term to determine ROC here, wand -3 so fraction is nonnegative

b) $X(s) = \frac{1}{s^2+2s+5}$   ROC: $Re\{s\} > -1$

i) $F\{x(t)e^{-t}\}$

$y(t) = x(t)e^{-t} \rightarrow Y(s) = X(s+1)$

Determine region of convergence for $Y(s)$

$Re\{s+1\} > -1$

$Re\{s\} > -2$

Therefore, the result is:

$\boxed{Y(j\omega) = [(j\omega+1)^2+5+2(j\omega+1)]^{-1}}$

ii) $F\{x(t)e^{3t}\}$

$y(t) = x(t)e^{3t} \rightarrow Y(s) = X(s-3)$

Determine region of convergence for $Y(s)$

$Re\{s-3\} > -1$

$Re\{s\} > 2$

In this case, the region of convergence does not include the $j\omega$-axis like it did in the previous part of the problem. This means we can not determine $Y(j\omega)$.

4. a) $F(s) = \frac{e^{-s}(s+3)}{(s-1)^2(s-2)}$

Ignore the $e^{-s}$ for now, that's easy and can be handled later

$$\frac{(s+3)}{(s-1)^2(s-2)} = \frac{r_1}{(s-1)^2} + \frac{r_2}{(s-1)} + \frac{r_3}{(s-2)}$$

$r_3 = \frac{(s+3)}{(s-1)^2}$, $s=2 \rightarrow r_3 = 5$

$r_1 = \frac{(s+3)}{(s-2)}$, $s=1 \rightarrow r_1 = -4$

Now: $$\frac{(s+3)}{(s-1)^2(s-2)} = \frac{-4}{(s-1)^2} + \frac{r_2}{s-1} + \frac{5}{s-2}$$

Solve $r_2$, $s=0$

$$-\frac{3}{2} = -\frac{4}{1} - \frac{r_2}{1} - \frac{5}{3} \rightarrow r_2 = -5$$

Add back the $e^{-s}$ now, calculations are done

$$F(s) = e^{-s}\left[-\frac{4}{(s-1)^2} - \frac{5}{s-1} + \frac{5}{s-3}\right]$$

Simplify this into $f(t)$

$$\boxed{f(t) = [u(t-1)] \cdot [5e^{3(t-1)} - 4(t-1)e^{(t-1)} - 5e^{(t-1)}]}$$

b) $F(s) = \frac{s+4}{s^3+4s}$

$$F(s) = \frac{r_1}{s} + \frac{Xs+Y}{s^2+4}$$

$r_1 = \frac{s+4}{(s^2+4)}$, $s=0$ $\qquad r_1 = 1$

Solve for numerator $(Xs+Y)$

$$Xs+Y = \frac{s-s^2}{s} = -s+1$$

$$F(s) = \frac{1}{s} + \frac{(-s+1)}{s^2+4}$$

$$F(s) = \frac{1}{s} + \frac{1}{s^2+4} - \frac{s}{s^2+4}$$

Simplify this into $f(t)$

$$\boxed{f(t) = [u(t)] \cdot [\tfrac{1}{2}\sin(2t) - \cos(2t) + 1]}$$

c) $F(s) = \frac{1}{(s+1)(s^2+2s+2)}$

$$F(s) = \frac{1}{(s+1+j)(s+1-j)(s+1)}$$

$$F(s) = \frac{r_1}{(s+1)} + \frac{r_2}{(s+1+j)} + \frac{r_3}{(s+1-j)}$$

$r_1 = \frac{1}{s^2+2s+2}$, $s=-1$ $\quad \rightarrow r_1 = 1$

$r_2 = \frac{1}{(s+1)(s+1-j)}$, $s = -1-j$ $\quad \rightarrow r_2 = -\frac{1}{2}$

$r_3 = -\frac{1}{2}$ as well

$$f(t) = u(t) \cdot [-\tfrac{1}{2}e^{-(1+j)t} - \tfrac{1}{2}e^{-(1-j)t} + e^{-t}]$$

$$\boxed{f(t) = [u(t)] \cdot [e^{-t}(1-\cos(t))]}$$

5. $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = ax(t)$

$y(0) = 0 \quad y'(0) = 0$

a : constant   input : $e^t$ -> output : $\frac{1}{2}e^t$

a) Take Laplace Transform

$s^2Y(s) + 3sY(s) + 2Y(s) = aX(s)$

$H_1(s) = \frac{Y(s)}{X(s)} \rightarrow \frac{a}{s^2+3s+2}$

$H_1(s) = \frac{a}{(s+2)(s+1)}$

Based on the given input/output pair, we know that we can use the Eigenfunction property

$x(t) = e^{st}$

$y(t) = H(s)e^{st}$

$y(t) = |H(s)| \cdot [e^{st + j(\angle H(s))}]$

$H_1(s)$, $s = 1 \rightarrow \frac{1}{2}$

$\frac{1}{2} = \frac{a}{(s+2)(s+1)}$, $s = 1$

$\frac{1}{2} = \frac{a}{6} \rightarrow a = 3$

$\boxed{H_1(s) = \frac{3}{(s+2)(s+1)}}$

b) $x(t) = u(t)$

Find Laplace Transform of $y(t)$

$Y(s) = H_1(s)X(s)$

$= \frac{3}{s(s+2)(s+1)} \rightarrow \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$

$r_1 = \frac{3}{(s+2)(s+1)}$, $s = 0 \rightarrow r_1 = \frac{3}{2}$

$r_2 = \frac{3}{s(s+2)}$, $s = -1 \rightarrow r_2 = -3$

$r_3 = \frac{3}{s(s+1)}$, $s = -2 \rightarrow r_3 = \frac{3}{2}$

Combine all this together

$\boxed{y(t) = [u(t)] \cdot \left[\frac{3}{2}e^{-2t} - 3e^{-t} + \frac{3}{2}\right]}$

c) $S_2$ input : $u(t) \rightarrow$ output : $r(t)$

$$H_2(s) = \frac{\mathcal{L}(r(t))}{\mathcal{L}(u(t))}$$

$$H_2(s) = \frac{1}{s}$$

$$H(s) = H_1(s) \cdot H_2(s)$$

$$H(s) = \frac{3}{s(s+2)(s+1)}$$

Simplify . [Note, this is same calculation as 5b]

$$h(t) = [u(t)] \cdot \left[\frac{3}{2}e^{-2t} - 3e^{-t} + \frac{3}{2}\right]$$