ECE102, Fall 2019 **Final**

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UCLA True Bruin academic integrity principles apply.
Open: Four cheat sheets allowed.
Closed: Book, computer, internet.
8:00-11:00am.
Wednesday, 11 Dec 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	_____	/ 40
Problem 2	_____	/ 45
Problem 3	_____	/ 40
Problem 4	_____	/ 30
Problem 5	_____	/ 45
BONUS	_____	/ 15 bonus points
Total	_____	/ 200 points + 15 bonus points

1. **Signal and System Basics** (40 points)

   (a) (16 points) For each of the statement below, determine whether it is true or false. You must justify your answer to receive full credit.

      i. (8 points) If $f(t)$ is a real and even signal, and $g(t)$ is a real and odd signal, the convolution of $f(t)$ and $g(t)$ is real and odd.

      Solution: True. $F(j\omega)$ is real and even, $G(j\omega)$ is imaginary and odd. The Fourier transform of convolution $h(t) = f(t) * g(t)$,

$$H(j\omega) = F(j\omega)G(j\omega)$$

      is imaginary and odd, hence h(t) is real and odd.

      ii. (8 points) All LTI systems are stable.

      Solution: False. Consider an integrating system, it's LTI but not stable.

   (b) (12 Points) Suppose we have an unknown system (black box). We input

$$x(t) = \mathrm{sinc}(t)$$

   into the system, and measure that its output is

$$y(t) = e^{-t}u(t).$$

   Can this system be LTI? You must justify your answer to receive full credit.

   Solution:

$$X(j\omega) = \mathrm{rect}(\omega/2\pi)$$
$$Y(j\omega) = \frac{1}{1 + j\omega}$$

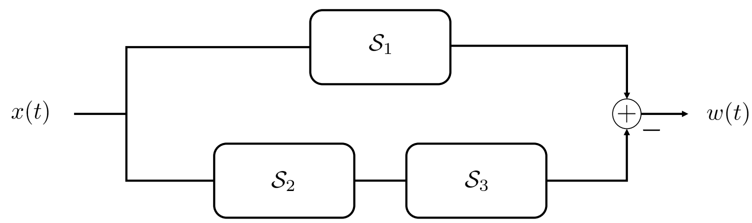
   For an LTI system, we need $Y(j\omega) = H(j\omega)X(j\omega)$. For $|\omega| > 1$, $X(j\omega)$ is zero ,while $Y(j\omega)$ is non-zero. Therefore, no frequency response of a LTI system $H(j\omega)$ could explain this pair.

   (c) (12 Points) Determine whether the following system is (1) causal, and whether it is (2) stable.

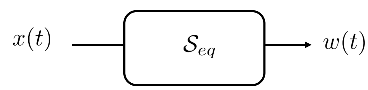
$$y(t) = \int_{-\infty}^{t} (x(\tau) + e^{-\tau})u(\tau + 1)d\tau$$

   Solution: This is causal, and not stable. The signal is two pieces.
   When $t \leq -1$, $y(t) = 0$
   When $t > -1$, $y(t) = \int_{-1}^{t} (x(t) + e^{-t})dt$
   Causality: The system is causal in both cases.
   Stability: The output takes the integral of input x(t). Take a DC signal x(t) for instance. As t approaches ∞, the integral would goes to ∞ and not be bounded. Hence, the system is not stable.

2. **Frequency Response and LTI system** (45 points)
Suppose the three systems are interconnected as shown below.



And we denote the equivalent system as below.



(a) (8 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are all LTI systems. Is the equivalent system $\mathcal{S}_{eq}$ an LTI system? Please justify your answer to receive full credit.

Solution: Yes. A cascade of two LTI systems is still a LTI system. A summation of two LTI system again preserves linearity and time-invariance.

(b) (8 points) Suppose the equivalent system $\mathcal{S}_{eq}$ is an LTI system. Are $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ all necessarily LTI systems? Please justify your answer to receive full credit.

Solution: No. For example, we could construct a system such that

$$y_1(t) = f[x(t)],$$

$$y_2(t) = f[x(t)],$$

$$y_3(t) = y_2(t) - 1,$$

where $f(...)$ is a non-LTI system. Therefore $w(t) = 1$, which represents an LTI system. Meanwhile, $\mathcal{S}_1$ and $\mathcal{S}_2$ are, by design, non-LTI.
The same argument could be made for with other system examples as well.

(c) (15 points) Suppose $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are each characterized by an LTI system,

- The first system $\mathcal{S}_1$ is given by its input-output relationship: $y(t) = x(t-3)$;
- The second system $\mathcal{S}_2$ is given by its impulse response: $h_2(t) = u(t-3)$;
- The third system $\mathcal{S}_3$ is given by its input-output relationship: $y(t) = \frac{d}{dt}x(t) + \frac{d^2}{dt^2}x(t)$.

Dtermine the frequency response $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$ of each system as well as $H_{eq}(j\omega)$ of the equivalent system.

Solution: The frequency response of the three systems:

$$H_1(j\omega) = e^{-3j\omega}$$

$$H_2(j\omega) = (\pi\delta(\omega) + 1/j\omega)e^{-3j\omega}$$

$$H_3(j\omega) = j\omega - \omega^2$$

Hence,

$$\begin{aligned} H_{eq}(j\omega) &= H_1 - H_2 H_3 \\ &= e^{-3j\omega}\left[-j\omega - (j\omega - \omega^2)\pi\delta(\omega)\right] \\ &= e^{-3j\omega}(-j\omega) \end{aligned}$$

(d) (14 points) For the system in part(c), the output $w(t)$ to an input $x(t) = e^{j\pi t/3}$ can be written as:

$$w(t) = Ae^{j\theta}x(t).$$

Determine A and θ.

Solution: Since $x(t) = e^{j\pi t/3}$ is an eigenfunction of the LTI system, we could write the output as,

$$w(t) = |H(j\pi/3)|e^{j\angle H(j\pi/3)}x(t)$$

where

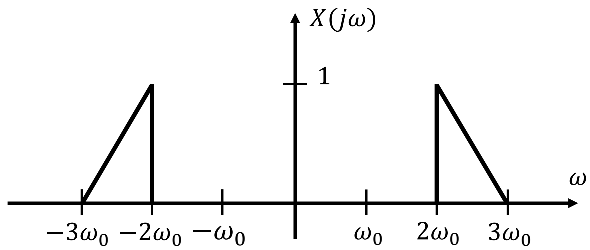
$$H(j\pi/3) = e^{-j\pi}(-j\pi/3) = (-1)(-j\pi/3) = j\pi/3$$

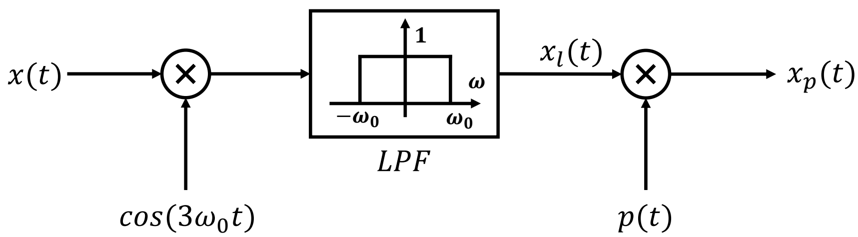
Therefore

$$A = |H(j\pi/3)| = \pi/3,$$

$$\theta = \pi/2$$

3. **Sampling and Modulation** (40 points)
Assume we have a continuous bandpass signal $x(t)$ with frequency spectrum as shown below. We also assume that $x(t)$ is real. The sampling theorem states that, to recover a signal without distortion, a signal must be sampled at a rate greater than twice its bandwidth. However, since $x(t)$ has most of its energy concentrated in a narrow band, it would seem reasonable to expect that a sampling rate lower than the Nyquist rate could be used. Now consider the system shown below where $p(t)$ is the sampling function.





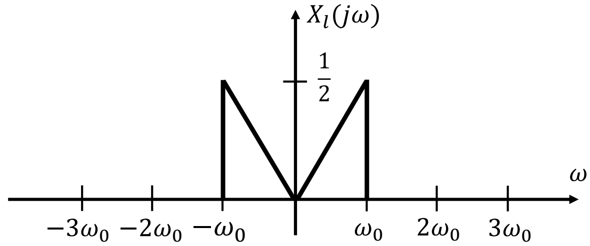
(This question continues on the next page)

(a) (5 points) What is the Nyquist rate of $x(t)$?

Solution: The Nyquist rate is $6\omega_0$ rad/s for $x(t)$, or $3\omega_0/\pi$ Hz.

(b) (5 points) What is the Nyquist rate of $x_l(t)$? Sketch the frequency spectrum after the low pass filter, i.e. $X_l(j\omega)$

Solution: The Nyquist rate of $x_l(t)$ is $2\omega_0$ rad/s, or ω_0/π Hz.

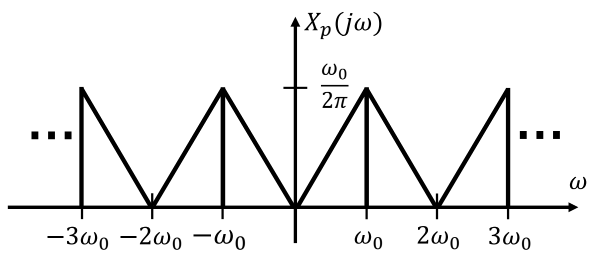


(c) (10 points) If the sampling function is an impulse train

$$p(t) = \sum_{k=-\infty}^{k=+\infty} \delta(t - kT)$$

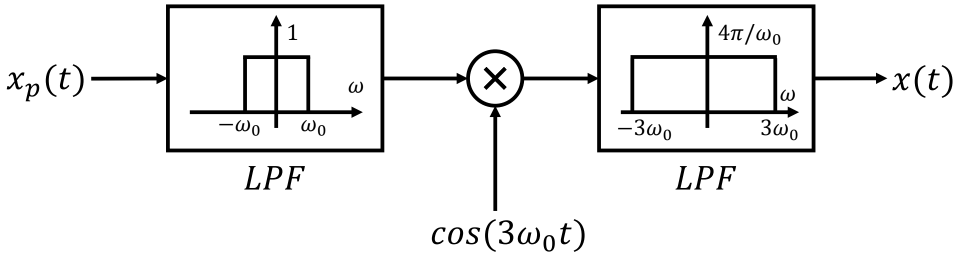
find the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$. Sketch the output frequency spectrum $X_p(jw)$.

Solution: The Nyquist sampling rate of $x_l(t)$ is $2\omega_0$. Therefore, the sampling period T must be at most $\frac{2\pi}{2\omega_0} = \frac{\pi}{\omega_0}$ in order to avoid aliasing.



(d) (20 points) With the $p(t)$ found in part (c), design a system to recover $x(t)$ from $x_p(t)$ without using a bandpass or highpass filter. Note that the recovered signal should have the same amplitude as $x(t)$ in frequency spectrum. Draw a flow diagram of your system and clearly state each component (including cutoff frequencies of any lowpass filter). Write out the explicit mathematical expression of any signal involved.

Solution:



- $x_p(t)$ is first passed through a LPF that cuts off at ω_0 with amplitude equals to 1. This is to get the baseband components.
- Then we modulate the baseband components by multiplying $\cos(3\omega_0 t)$ in time domain. Note that this halves the amplitude in frequency spectrum.
- Finally, we pass the signal through another LPF that cuts off at $3\omega_0$ with amplitude equals to $\frac{4\pi}{\omega_0}$ to get the recovered signal $x(t)$.

(Solutions may not be unique.)

Therefore,

$$y(t) = m_1(t) + jm_2(t).$$

4. **Laplace Transform** (30 points)

A system can be described by the following differential equation:

$$y''(t) + y'(t) - 2y(t) = 6x'(t) - 3x(t)$$

where the initial conditions are all zero, i.e. $y''(0) = 0$, $y'(0) = 0$ and $y(0) = 0$.

(a) (10 points) Find the transfer function $H(s)$.

Solution: Applying the Laplace transform to the differential equation:

$$s^2Y(s) + sY(s) - 2Y(s) = 6sX(s) - 3X(s)$$

Therefore we have,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{6s-3}{s^2+s-2}$$

(b) (20 points) If the input is

$$x(t) = e^{-t}u(t)$$

find the output $y(t)$.

Solution: Since

$$x(t) = e^{-t}u(t) \Longrightarrow X(s) = \frac{1}{s+1}$$

we have,

$$Y(s) = \frac{6s-3}{(s^2+s-2)(s+1)} = \frac{6s-3}{(s-1)(s+2)(s+1)} = \frac{r_1}{s-1} + \frac{r_2}{s+2} + \frac{r_3}{s+1}$$

We can find r_1, r_2 and r_3 by

$$r_1 = \left.\frac{6s-3}{(s+2)(s+1)}\right|_{s=1} = \frac{1}{2}$$

$$r_2 = \left.\frac{6s-3}{(s-1)(s+1)}\right|_{s=-2} = -5$$

$$r_3 = \left.\frac{6s-3}{(s-1)(s+2)}\right|_{s=-1} = \frac{9}{2}$$

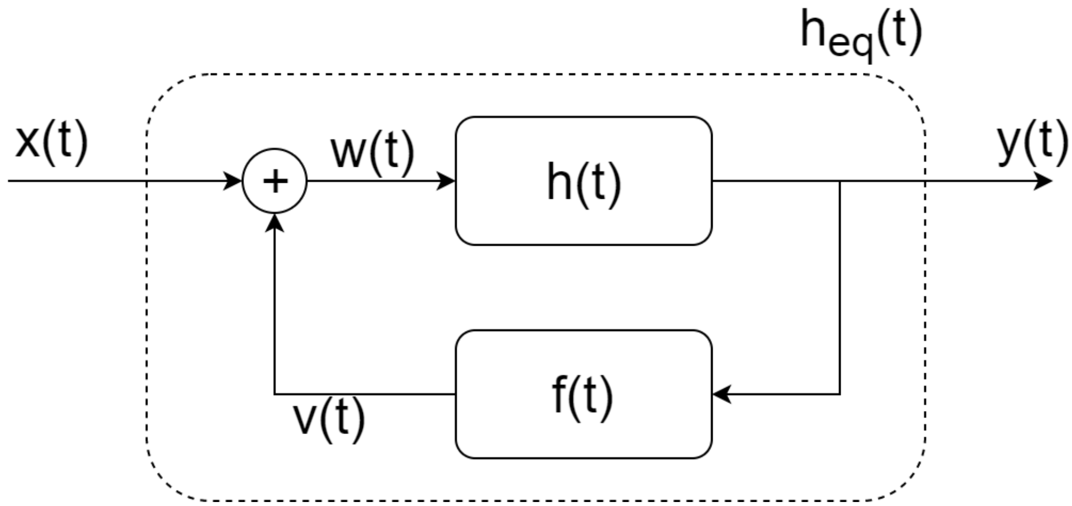
Therefore, we have,

$$Y(s) = \frac{1}{2}\frac{1}{s-1} - 5\frac{1}{s+2} + \frac{9}{2}\frac{1}{s+1}$$

and

$$y(t) = \frac{1}{2}e^{t}u(t) - 5e^{-2t}u(t) + \frac{9}{2}e^{-t}u(t)$$

5. **Feedback System** (45 points)
Consider the feedback system shown below (all components are LTI):



where $h(t) = e^{-2t}u(t)$ and $y(0) = 0$.

(a) (10 points) Show that

$$H_{eq}(s) = \frac{H(s)}{1 - H(s)F(s)}$$

Solution: From the diagram, we get:

$$\begin{aligned} W(s) &= X(s) + V(s) \\ Y(s) &= H(s)W(s) \\ V(s) &= F(s)Y(s) \end{aligned}$$

Then, we solve for $H_{eq}(s) = \frac{Y(s)}{X(s)}$:

$$\begin{aligned} Y(s) &= H(s)(X(s) + V(s)) \\ &= H(s)X(s) + H(s)V(s) \\ &= H(s)X(s) + H(s)F(s)Y(s) \\ Y(s)(1 - H(s)F(s)) &= H(s)X(s) \\ H_{eq}(s) = \frac{Y(s)}{X(s)} &= \frac{H(s)}{1 - H(s)F(s)} \end{aligned}$$

(b) (10 points) Find the Laplace Transform $H(s)$ of $h(t)$. What is the frequency response $H(j\omega)$? Why is this a low-pass filter?

Solution: Applying the Laplace Transform from the table, we have:

$$H(s) = \frac{1}{s+2}$$

Since the ROC of $H(s)$ is $\{s : Re(s) > -2\}$, which contains the $j\omega$ axis, we can directly substitute $s = j\omega$:

$$\begin{aligned} H(j\omega) &= H(s)|_{s=j\omega} \\ &= \frac{1}{j\omega + 2} \end{aligned}$$

This is a low-pass filter because:

$$\lim_{\omega \to 0^+} |H(j\omega)| = \frac{1}{2}$$
$$\lim_{\omega \to \infty} |H(j\omega)| = 0$$

(c) (10 points) $v(t)$ and $y(t)$ satisfy the differential equation

$$v(t) = \frac{d}{dt}y(t) + y(t) - 10\int_0^t y(\tau)d\tau$$

What is $F(s)$?

Solution:

$$\begin{aligned} V(s) &= sY(s) - y(0) + Y(s) - 10\frac{1}{s}Y(s) \\ &= \left(s + 1 - \frac{10}{s}\right)Y(s) \\ F(s) &= \frac{V(s)}{Y(s)} \\ &= s + 1 - \frac{10}{s} \end{aligned}$$

(d) (15 points) Using $F(s)$ found in part c, what is $h_{eq}(t)$? Is this a low-pass, band-pass, or high-pass filter?

Solution:

$$
\begin{aligned}
H_{eq}(s) &= \frac{H(s)}{1 - H(s)F(s)} \\
&= \frac{\frac{1}{s+2}}{1 - \frac{1}{s+2}F(s)} \\
&= \frac{1}{s + 2 - F(s)} \\
&= \frac{1}{s + 2 - (s + 1 - \frac{10}{s})} \\
&= \frac{1}{1 + \frac{10}{s}} \\
&= \frac{s}{s + 10} \\
&= \frac{s + 10}{s + 10} + \frac{-10}{s + 10} \\
&= 1 - 10\frac{1}{s + 10} \\
h_{eq}(t) &= \delta(t) - 10e^{-10t}u(t)
\end{aligned}
$$

Since the ROC of $H_{eq}(s)$, which is $\{s : Re(s) > -10\}$, contains the $j\omega$ axis, we can use

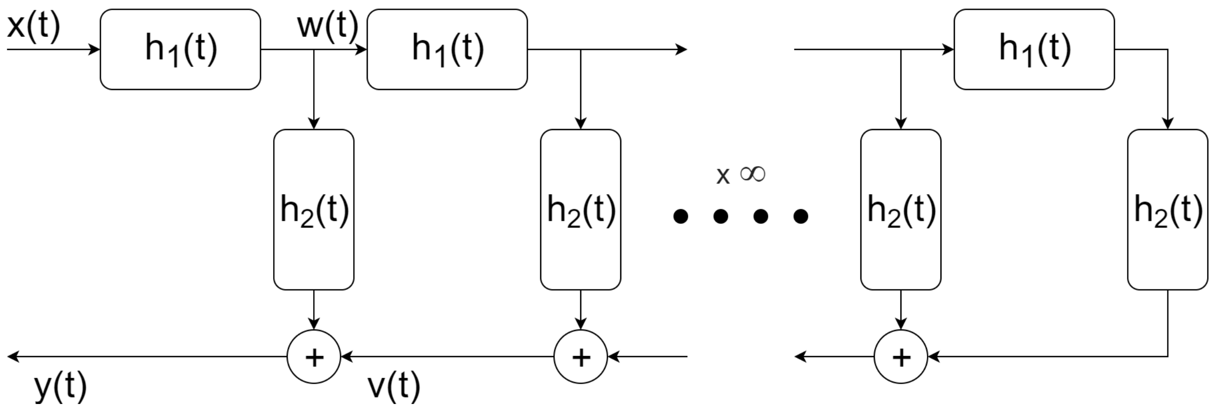
$$
\begin{aligned}
H_{eq}(j\omega) &= H_{eq}(s)|_{s=j\omega} \\
&= \frac{j\omega}{j\omega + 10}
\end{aligned}
$$

Then,

$$
\begin{aligned}
H_{eq}(j\omega)|_{\omega=0} &= 0 \\
\lim_{\omega\to\infty} |H_{eq}(j\omega)| &= 1
\end{aligned}
$$

Thus, this is a high-pass filter.

Bonus (15 points) Consider the LTI system S shown below, which is a system ladder with an infinite number of rungs. Let $y(t) = S[x(t)]$.



(a) (8 points) In terms of $H_1(s)$ and $H_2(s)$, what is the equivalent transfer function $H_{eq}(s)$ between $Y(s)$ and $X(s)$? *Hint: how does* $\frac{V(s)}{W(s)}$ *relate to* $\frac{Y(s)}{X(s)}$*?*

Solution:

$$\begin{aligned}
\frac{Y(s)}{X(s)} &= \frac{V(s)}{W(s)} \\
W(s) &= H_1(s)X(s) \\
V(s) &= \frac{W(s)Y(s)}{X(s)} \\
&= \frac{H_1(s)X(s)Y(s)}{X(s)} = H_1(s)Y(s) \\
Y(s) &= H_2(s)W(s) + V(s) \\
&= H_2(s)H_1(s)X(s) + V(s) \\
&= H_2(s)H_1(s)X(s) + H_1(s)Y(s) \\
Y(s)(1 - H_1(s)) &= H_2(s)H_1(s)X(s) \\
H_{eq}(s) = \frac{Y(s)}{X(s)} &= \frac{H_2(s)H_1(s)}{1 - H_1(s)}
\end{aligned}$$

Alternatively, we can work starting from the right-hand side, arriving at:

$$\begin{aligned}
Y(s) &= \left(H_2(s)H_1(s) \lim_{n\to\infty} \sum_{i=0}^{n} H_1(s)^i \right) X(s) \\
&= H_2(s)H_1(s)\frac{1}{1 - H_1(s)}X(s)
\end{aligned}$$

(b) (7 points) Suppose $h_1(t) = e^{-a_1 t}u(t)$ and $h_2(t) = e^{-a_2 t}u(t)$, where a_1 and a_2 are real and positive. For what values of a_1 is S BIBO stable?

Solution:

$$
\begin{aligned}
H_1(s) &= \frac{1}{s+a_1} \\
H_2(s) &= \frac{1}{s+a_2} \\
H_{eq}(s) &= \frac{H_2(s)H_1(s)}{1-H_1(s)} \\
&= \frac{\frac{1}{s+a_2}\frac{1}{s+a_1}}{1-\frac{1}{s+a_1}} \\
&= \frac{1}{s+a_2}\frac{1}{s+a_1-1} \\
&= \frac{r_2}{s+a_2} + \frac{r_1}{s+a_1-1} \text{ for some } r_1, r_2 \text{ if } a_2 \neq a_1 - 1. \\
h_{eq}(t) &= r_2 e^{-a_2 t} u(t) + r_1 e^{-(a_1-1)t} u(t) \\
\text{if } a_2 &= a_1 - 1, \text{ then} \\
H_{eq}(s) &= \frac{1}{(s+a_2)^2} \\
h_{eq}(t) &= t e^{-a_2 t} u(t)
\end{aligned}
$$

By linearity, we know that the summation of two BIBO stable systems is stable; equivalently, their impulse response has finite energy. The $r_2 e^{-a_2 t} u(t)$ term is guaranteed to have finite energy for any value of r_2, but for the $r_1 e^{-(a_1-1)t} u(t)$ term to have finite energy, we need $a_1 - 1 > 0$, or $a_1 > 1$. This is also equivalent to looking at the poles of $H_{eq}(s)$ and forcing them to be in the left-half-plane of the Laplace plane (the set $\{s : Re(s) < 0\}$).