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EE 102 Midterm

1. a) $x(t) = e^{5+j3t}\left[\cos(t) + j\sin(3t)\right]$ Note: $e^{jt} = \cos(t) + j\sin(t)$ -> Euler's

$= e^{5} \cdot e^{j3t}\left[\cos(t) + j\sin(3t)\right]$

$= e^{5} \cdot \left[\cos(3t) + j\sin(3t)\right] \cdot \left[\cos(t) + j\sin(3t)\right]$

$= e^{5}\left[\cos(t)\cos(3t) - \sin^2(3t)\right] + je^{5}\left[\cos(t)\sin(3t) + \sin(3t)\cos(3t)\right]$

$$\boxed{\begin{array}{l}\text{Real part: } e^{5}\left[\cos(t)\cos(3t) - \sin^2(3t)\right] \\ \text{Imaginary part: } e^{5}\left[\cos(t)\sin(3t) + \sin(3t)\cos(3t)\right]\end{array}}$$

b) i) $x(t) = Ae^{-a|t|}$ $a > 0$

Power: $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |Ae^{-a|t|}|^2 dt$ Undefined?

$\frac{1}{2T}\left(\left(\int_{-\infty}^{0} (-Ae^{-a|t|})^2 dt\right) + \left(\int_{0}^{\infty} (-Ae^{-a|t|})^2 dt\right)\right.$

Undefined! Not a power signal

Energy: $\int_{-\infty}^{\infty} |Ae^{-a|t|}|^2 dt$

$\int_{-\infty}^{0} (-Ae^{-a|t|})^2 dt + \int_{0}^{\infty} (-Ae^{-a|t|})^2 dt$

$\left[A^2 \frac{1}{2a} e^{2at}\right]_{-\infty}^{0} + \left[A^2 e^{-2at} \frac{1}{2a}\right]_{0}^{\infty}$

$\frac{A^2}{2a} + \frac{A^2}{2a} = \frac{A^2}{a}$

$$\boxed{\begin{array}{c} P = \infty \\ \hline E = \frac{A^2}{a}\end{array}}$$

ii) $x(t) = A\sin(\omega t)u(t)$

Power: $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |A\sin(\omega t)u(t)|^2 dt$

$\lim_{T\to\infty} \frac{1}{2T} \cdot \frac{A^2}{2}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_{0}^{\infty}$

$\lim_{T\to\infty} \frac{1}{4}A^2\left[\frac{t}{T} - \frac{\sin(2\omega t)}{2\omega}\right]_{0}^{\infty}$

$\frac{1}{4}A^2\left(\left[\frac{\infty}{\infty} - \frac{\sin(2\omega\infty)}{2\omega}\right] - [0]\right)$

$\frac{1}{4}A^2 \cdot [1] - [0]$ $P = \frac{1}{4}A^2$

Energy: $\int_{-\infty}^{\infty} |A\sin(\omega t)u(t)|^2 dt$

$\int_{0}^{\infty} A^2\sin^2(\omega t)\, dt$

$A^2\int_{0}^{\infty} \frac{1}{2}(1 - \cos(2\omega t)\, dt$

$\frac{1}{2}A^2\left[\frac{1}{2}(1 - \cos 2\omega t\right]_{0}^{\infty}$

Approaches infinity, not energy signal

$$\boxed{\begin{array}{l} P = \frac{1}{4}A^2 \\ E = \infty\end{array}}$$

c) $S(t) = \sum_{k=-\infty}^{\infty} \frac{1}{1+(t+5k)^4}$ Show that $S(t+T) = S(t)$

$S(t+T) = \sum_{k=-\infty}^{\infty} \frac{1}{1+(t+T+5k)^4}$

S is an infinite sum, so when T shifts k, the function will be the same function

$= \sum_{k=-\infty}^{\infty} \frac{1}{1+(t+5(k+\frac{1}{5}T)^4}$

$= \sum_{k=-\infty}^{\infty} \frac{1}{1+(t+5(k+1))^4}$

$= \sum_{k=-\infty}^{\infty} \frac{1}{1+(t+5k)^4} = S(t)$

$S(t+T) = S(t)$ $\boxed{\text{Periodic}}$

d) i) $x(t) = \sin(3t) - \cos^2(2t)$

$x(-t) = \sin(-3t) - \cos^2(-2t)$

$X_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$X_e(t) = \frac{1}{2}[\sin(3t) - \cos^2(2t) + (\sin(-3t) - \cos^2(-2t))]$

$\boxed{X_e(t) = -\cos^2(2t)}$

$X_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$X_o(t) = \frac{1}{2}[\sin(3t) - \cos^2(2t) - (-\cos^2(2t) - \sin(3t))]$

$\boxed{X_o(t) = \sin(3t)}$

ii) $x(t) = t \cdot \sin(2t)$

$x(-t) = -t \cdot \sin(-2t) = t\sin(2t)$

$X_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$X_e(t) = \frac{1}{2}[(t \cdot \sin(2t)) + (t \cdot \sin(2t))]$

$\boxed{X_e(t) = t \cdot \sin(2t)}$

$X_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$\boxed{X_o(t) = 0}$

e) $\Lambda(t) = \begin{cases} 1-|t| & |t|<1 \\ 0 & \text{otherwise} \end{cases}$

The given statement is true. The triangle function will be shifted by T. This can be thought of as periodic extension whenever T is greater than or equal to 2. In the other scenario when T is less than 2, overlap occurs with consecative triangles, which creates periods through the peaks and valleys. All of the possible values of t, the given statement is true.

2. a) $y(t) = \frac{d}{dt}(\sin(t)\,x(t)) + x(3-t)$ linear time-invariant causal stable

- Delay input: $\frac{d}{dt}(\sin(t) \cdot x(t-\tau)) + x(3-t-\tau)$] not equivalent

U'v cos(t) x(t) Delay output: $\frac{d}{dt}(\sin(t-\tau)\,x(t-\tau)) + x(3-(t-\tau))$ Time variant

v'U x'(t) sin(t) • Not causal, For instance, if $t=1$, then $y(t)$ depends on $x(2)$

- $y(t)$ is banded because if the input t is finite, then the output will also be finite $\left|\frac{d}{dt}[\sin(t)\,x(t)] + x(3-t)\right| \leq B_2$ when $B_2 \leq \infty$
- $y(t) = \frac{d}{dt}(\sin(t)\,x(t)) + x(3-t)$

$F(ax(t) + b\bar{x}(t)) = \frac{d}{dt}[\sin(t)(ax(t) + b\bar{x}(t)) + [ax(3-t) + b\bar{x}(3-t)]$

$= \frac{d}{dt}[\sin(t) \cdot ax(t)] + \frac{d}{dt}[\sin(t) \cdot b\bar{x}(t)] + [ax(3-t) + b\bar{x}(3-t)]$

$= a\,F(x(t)) + bF(\bar{x}(t))$

$\boxed{\text{time-varriant, not causal, stable, linear}}$

b) i) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is bounded, the system is stable

- To test this, we can try to plug in a signal to the system and perform convolution.

$|x(t)| < C_1 < \infty$

$|h(t)| < D_1 < \infty$

$|x(t) * h(t)|$

$= |\int_{-\infty}^{\infty} x(\tau)h(t-\tau)| d\tau$

$\leq \int_{-\infty}^{\infty} C_1 D_1 \, d\tau$

Multiplying two constants and integrating from $-\infty$ to ∞ means it diverges.

$\boxed{\text{False}}$

ii) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is even and nonzero, the system is causal

- causal means the system can not depend on future values

"even" means that $h(t) = h(-t)$

causal system: $h(t) = 0$ for $t < 0$

$h(t-\tau) = 0$ if $\tau > t$

Because the system is even, let's take an example:

Say that we take $h(-3)$. For an even function, $h(-3) = h(3)$ and the system relies on a future value. The system is not causal.

$\boxed{\text{FALSE}}$

3. $h_3(t) = \int_{-\infty}^{t-1} \delta(\tau-2)\,d\tau$ $\quad y(t) = 3x(t-1) + \int_{-\infty}^{t+2} x(\tau)\,d\tau$

a) $y(t) = x(t) * h(t)$

Impulse response : $h_{eq}(t) = 3x(t-1) + \int_{-\infty}^{t+\frac{9}{2}} \delta(\tau)\,d\tau$

$h_{eq}(t) = 3\delta(t-1) + u(t-\frac{9}{2})$

Step response : $u_{eq}(t) = 3u(t-1) + \int_{-\infty}^{t+\frac{9}{2}} u(\tau)\,d\tau$

$u_{eq}(t) = 3u(t-1) + r(t-\frac{9}{2})$

$$\boxed{\begin{aligned} h_{eq}(t) &= 3\delta(t-1) + u(t-\tfrac{9}{2}) \\ u_{eq}(t) &= 3u(t-1) + r(t-\tfrac{9}{2}) \end{aligned}}$$

b) $w_1(t) = 3x(t-1)$ $\quad w_3(t) = \int_{-\infty}^{t+\frac{9}{2}} x(\tau)\,d\tau$

$h_1(t) = 3\delta(t-1)$

Find impulses : $h_2(t)$ and $h_3(t)$

$h_2(t) = \int_{-\infty}^{t+9/2} \delta(\tau)\,d\tau$

$= u(t-\frac{9}{2})$

$h_2(t) = \delta(t-\frac{3}{2})$

$$\boxed{\begin{aligned} h_1(t) &= 3\delta(t-1) \\ h_2(t) &= \delta(t-\tfrac{3}{2}) \end{aligned}}$$

c) System 1 - Causality

$w_1(t) = 3x(t-1)$

Causal, as $x(t-1)$ doesn't ever rely on future values - it may only need past values

System 2 and System 3 - Causality

$w_3(t) = \int_{-\infty}^{t+\frac{9}{2}} x(\tau)\,d\tau$

Not causal, because the upper bound is $t+\frac{9}{2}$. This means the integral relies on future values of t.

System 1 - Stability

Say that $|x(t)| < D < \infty$

$|y_1(t)| = |3x(t-1)| \leq 3|x(t-1)| \leq 3D$ stable

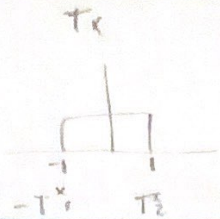
System 2 and System 3 -

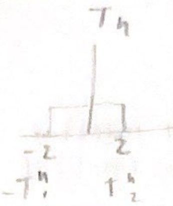
Say that $|x(t)| < D < \infty$

$|y_{23}(t)| = \left|\int_{-\infty}^{t+\frac{9}{2}} x(\tau)\,d\tau\right| = \int_{-\infty}^{t+9/2} |x(\tau)|\,d\tau \leq \int_{-\infty}^{t+9/2} D\,d\tau$ not stable

Seq is not causal - the System 2 and 3 are not causal

Seq is not stable - the systen 2 and 3 are unstable





4. a) $x(t) = 0$ for $t > T_2^x$ and $t < -T_1^x$

$h(t) = 0$ for $t > T_2^h$ and $t < -T_1^h$ $\quad T_1^x, T_2^x, T_1^y, T_2^y \geq 0$

· Think of this in terms of flip-and-drag convolution

There is a nonzero value for the convolution when there is overlap of both functions. (see above pictures)

When doing the dragging, we see that when one side of a function (i.e. rect function) enters the domain of the other, we have a nonzero value.

$\boxed{\text{y(t) is nonzero in the range of } (-T_1^h - T_1^x) \leq t \leq (T_2^x + T_2^h) \text{ because that's when intersection occurs.}}$

b) $x(t) * y(t) = z(t)$

$x(t-T_1) * y(t-T_2) = z(t-(T_1+T_2))$

$[x(t) * \delta(t-T_1)] * [y(t) * \delta(t-T_2)] = z(t-(T_1+T_2))$

$[x(t) * y(t)] * [\delta(t-T_1) * \delta(t-T_2)] = z(t-(T_1+T_2))$

$z(t) * [\delta(t-T_1) * \delta(t-T_2)] = z(t-(T_1+T_2))$

Distributing the convolution on LHS

$z(t-(T_1+T_2)) = z(t-(T_1+T_2))$ ✓

d) $y(t) = x(t) * h(t)$

$y(t) = \int_{-\infty}^{\infty} [e^{-3\tau} u(\tau-3)] [\text{rect}(\tfrac{1}{2}(t-\tau+1))] d\tau$

$y(t) = \int_{3}^{\infty} e^{-3\tau} [\text{rect}(\tfrac{1}{2}(t-\tau+1))]$

$\frac{1}{2} \geq \left|\frac{t-1}{2}\right|$

$2 \geq t$

Because the convolution doesn't produce overlap between $x(t)$ and $h(t)$, $y(t) = 0$.

$\boxed{y(t) = 0}$

$\text{rect}(\frac{t}{2})$

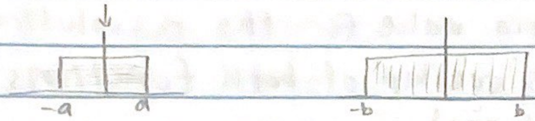
$t = 0 \quad \text{rect}(0) = 1$

$t = -\frac{1}{2}, \frac{1}{2} \quad \text{rect}(\frac{1}{4}) = 1 \quad \text{rect}(\frac{1}{4}) = 1$

$t = -1, 1 \quad \text{rect}(\frac{1}{2}) = 1 \quad \text{rect}(\frac{1}{2}) = 1$

c) $\text{rect}(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$ $\quad a, b > 0$ $\quad b > a$

$y(t) = \text{rect}(\frac{t}{2a}) * \text{rect}(\frac{t}{2b})$

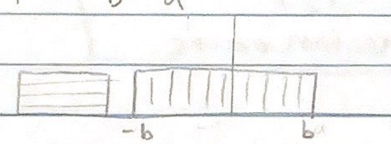


As a increases, gets wider

Use flip-and drag convolution to solve

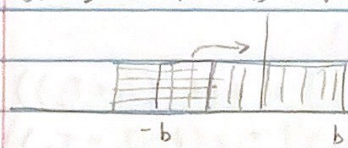
Remember: Convolution of two rect functions with same height but different width leads to convolution in the shape of trapezoid.

$t < -b-a$



$y(t) = 0$

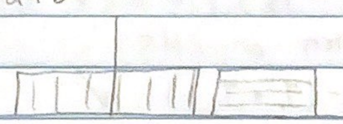
$a-b \le t \le b-a \quad y(t) = 2a$



$b-a \le t < b+a \quad y(t) = 2a - (t-b+a)$

$-(a+b) \le t < -b+a \quad y(t) = t + (b+a)$

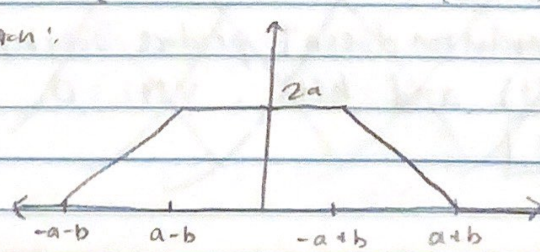
$t > a+b$



$y(t) = 0$

$$y(t) = \begin{cases} 0 & t < (-b-a) \\ t+b+a & -(a+b) \le t < (-b+a) \\ 2a & (a-b) \le t \le (b-a) \\ 2a-(t-b+a) & (b-a) \le t < b+a \\ 0 & t \ge (b+a) \end{cases}$$

Convolution:

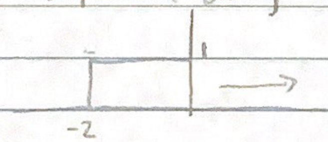


d) $y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} \left[e^{-3\tau} u(\tau - 3)\right] \left[\mathrm{rect}\left(\tfrac{1}{2}(t - \tau + 1)\right)\right] d\tau$$

$$= \int_{3}^{\infty} e^{-3\tau} \cdot \left[\mathrm{rect}\left(\tfrac{1}{2}(t - \tau + 1)\right)\right] d\tau$$

Think of flip and drag and see the overlap



$$\int_{3}^{t} e^{-3\tau} d\tau = -\tfrac{1}{3} e^{-3t} + \tfrac{1}{3} e^{-9}$$

$$\int_{t-2}^{t} e^{-3\tau} d\tau = -\tfrac{1}{3} e^{-3t} + \tfrac{1}{3} e^{-3(t-2)}$$

$$\boxed{y(t) = \begin{cases} 0 & t \le 3 \\ -\tfrac{1}{3} e^{-3t} + \tfrac{1}{3} e^{-9} & 3 < t \le 5 \\ -\tfrac{1}{3} e^{-3t} + \tfrac{1}{3} e^{-3(t-2)} & t > 5 \end{cases}}$$