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ECE 102 HW 6

1. a) $y(t) = (4e^{-t} - 2e^{-2t})u(t)$ $y_1(t) = 2e^{-t}u(t)$

$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 3x(t)$

i) Transform the differential equation

$(j\omega)^2 Y(j\omega) + 5(j\omega)Y(j\omega) + 6Y(j\omega) = 3X(j\omega)$

Isolate $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$

$H(j\omega) = \frac{3}{6 + 5(j\omega) + (j\omega)^2} \to \boxed{H(j\omega) = \frac{3}{(j\omega+3)(j\omega+2)}}$

ii) $X(j\omega) = \frac{Y(j\omega)}{H(j\omega)}$

$y(t) = (4e^{-t} - 2e^{-2t})u(t)$

$Y(j\omega) = -\frac{2}{j\omega+2} + \frac{4}{j\omega+1}$

$Y(j\omega) = \frac{2(j\omega+3)}{(j\omega+2)(j\omega+1)}$

$X(j\omega) = \frac{\frac{2(j\omega+3)}{(j\omega+2)(j\omega+1)}}{\frac{3}{(j\omega+3)(j\omega+2)}} = \frac{2(j\omega+3)^2}{3(j\omega+1)}$

$H_1(j\omega) = \frac{Y_1(j\omega)}{X(j\omega)}$

$y_1(t) = 2e^{-t}u(t)$

$Y_1(j\omega) = \frac{2}{j\omega+1}$

$H_1(j\omega) = \frac{\frac{2}{j\omega+1}}{\frac{2(j\omega+3)^2}{3(j\omega+1)}} = \frac{1}{3(j\omega+3)^2}$

$H(j\omega) = H_1(j\omega)H_2(j\omega)$

$$\boxed{H_2(j\omega) = \frac{H(j\omega)}{H_1(j\omega)} = \frac{j\omega+3}{j\omega+2} \qquad H_1(j\omega) = \frac{1}{3(j\omega+3)^2}}$$

iii) $H_1(j\omega) = \frac{1}{3 \cdot (j\omega+3)^2}$

$h_1(t) = \frac{1}{3}te^{-3t} \cdot u(t)$

$H_2(j\omega) = \frac{j\omega+3}{j\omega+2}$

$h_2(t) = \frac{d}{dt}(e^{-2t}u(t)) + 3e^{-2t}u(t))$

$h_2(t) = \frac{d}{dt}[e^{-2t}u(t)] + e^{-2t}\frac{d}{dt}[3e^{-2t}u(t) + u(t)]$

$h_2(t) = 2e^{-2t}u(t) + 3e^{-2t}u(t) + \delta(t)e^{-2t}$

$H(j\omega) = \frac{3}{(j\omega+3)(j\omega+2)}$

$$\boxed{h(t) = 3[e^{-2t} - e^{-3t}] \cdot u(t) \qquad h_1(t) = \frac{1}{3}te^{-3t} \cdot u(t)}$$

$$\boxed{h_2(t) = -2e^{-2t}u(t) + 3e^{-2t}u(t) + \delta(t)e^{-2t}}$$

b) First case: System is LTI

$y(t) = h(t) * x(t)$ [convolution]

$Y(j\omega) = H(j\omega)X(j\omega)$ [Fourier Transform]

$X(j\omega) = 0$ when $|\omega| > \omega_0$ $\rightarrow$ $Y(j\omega) = 0$ when $|\omega| > \omega_0$

There aren't any new frequency components.

Second case: System is NOT LTI

The signal will expand with regard to frequency domain.

For example, say that $y(t) = x(7t)$

$Y(j\omega) = \frac{1}{7}X(\frac{1}{7}j\omega)$

$Y(j\omega)$ isn't zero when $\omega_0 \le |\omega| \le 7\omega_0$

There is different behavior depending on whether the system is LTI.

c) $h_1(t) = \text{sinc}(\frac{t}{2})\cos(\pi t)$ $h_2(t) = 2\,\text{sinc}(2t)$

$x(t) = \cos(3\pi t)\cos(4\pi t)$

$x(t) = \cos(\omega_0 t) \rightarrow y(t) = |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$

simplify $x(t)$: $x(t) = \frac{1}{2}[\cos(7\pi t) + \cos(\pi t)]$

$x(t) = e^{j\omega_0 t} \rightarrow y(t) = |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$

Solve for frequency response $H_1(j\omega)$

$H_1(j\omega) = \text{rect}(\frac{\omega+\pi}{\pi}) + \text{rect}(\frac{\omega-\pi}{\pi})$

Plug in $(7\pi \rightarrow 0)$, $(\pi \rightarrow 1)$

$y_1(t) = \frac{1}{2}\cos(\pi t)$

Solve for frequency response $H_2(j\omega)$

$H_2(j\omega) = \text{rect}(\frac{1}{4\pi}\omega)$

Plug in $(7\pi \rightarrow 0)$, $(\pi = 1)$

$y_2(t) = \frac{1}{2}\cos(\pi t)$

There were 2 different inputs $x(t) = \cos(\omega_0 t)$ and $x(t) = e^{j\omega_0 t}$, but they both ended up with the same output: $y(t) = \frac{1}{2}\cos(\pi t)$. This shows that you can not always identify an LTI system solely from its input and output because different LTI systems can have the same output with the same input.

2. a) $\alpha : \pm 1$

Impulse Response: $h_{eq}(t) = h_{LP,1}(t) + \alpha\delta(t)$

Transform: $H_{eq}(j\omega) = H_{LP,1}(j\omega) + \alpha$

Try $\alpha = -1$, solve for $H_{eq}(j\omega) = H_{LP,1}(j\omega) - 1$

$$H_{eq}(j\omega) = \begin{cases} -1 & |\omega| \geq 2\pi \\ 0 & \text{else} \end{cases}$$

$\boxed{\text{The new filter has a phase of } \pi \text{ in its frequency response.}}$

b) First of all, ideal filters are not stable. This makes them non-realizable. Secondly, ideal filters non-causal. This also renders them non-realizable. Finally, the fact that a filter impulse response has a neverending duration means the convolution would take an infinitely long time. All three of these facts make ideal filters non-realizable.

c) $H_{LP,2}(j\omega) = \frac{k}{\beta + j\omega}$ cutoff frequency: $\omega_0 = 2\pi$

$H_{LP,2}(0) = 1$ $|H_{LP,2}(j2\pi)|^2 = \frac{1}{2}$

$k = \beta$ $\frac{\beta^2}{\beta^2 + 4\pi^2} = \frac{1}{2}$

Based on these equations, we see that:

$$\beta^2 = 4\pi^2 \rightarrow \frac{4\pi^2}{4\pi^2 + 4\pi^2} = \frac{1}{2}$$

$\beta = \pm 2\pi$

$\beta = 2\pi$, want to ensure that $h_2(t)$ remains causal

$\boxed{k = \beta = 2\pi}$

d) $H_{eq}(j\omega) = \frac{2\pi}{2\pi + j\omega} - 1$

$H_{eq}(j\omega) = \frac{-j\omega}{2\pi + j\omega}$

Find the magnitude: $|H_{eq}(j\omega)| = \sqrt{\frac{\omega^2}{4\pi^2 + \omega^2}}$

The system does actually behave like a high-pass filter. This is because when $\omega \approx 0 \rightarrow |H_{eq}(j\omega)| \approx 0$ and when $\omega \gg 2\pi \rightarrow |H_{eq}(j\omega)| \approx 1$

3. a) First find $h(t)$ based on the given graph

$h(t) = \frac{1}{T}\,\text{rect}\left(\frac{t-\frac{1}{2}T}{T}\right)$

$\frac{1}{T}\,\text{rect}\left(\frac{t}{T}\right) \Longleftrightarrow \text{sinc}\left(\frac{\omega T}{2\pi}\right)$ [look at Fourier Table for pairs]

Apply this pair to $h(t)$

$h(t) = \frac{1}{T}\,\text{rect}\left(\frac{t-\frac{1}{2}T}{T}\right) \Longleftrightarrow \text{sinc}\left(\frac{\omega T}{2\pi}\right)\cdot e^{-j\omega\frac{T}{2}}$

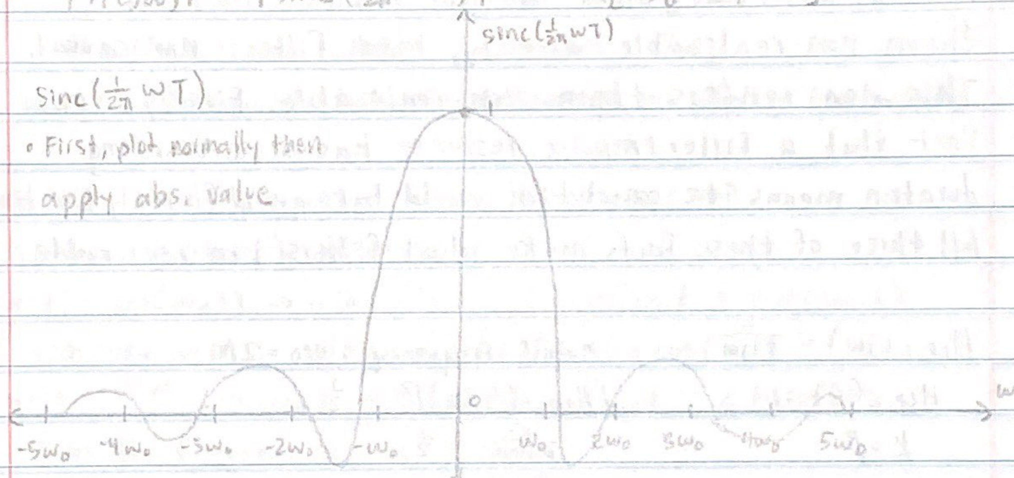
$\boxed{H(j\omega) = \text{sinc}\left(\frac{1}{2\pi}\omega T\right)\cdot e^{-j\omega T/2}}$

b) Amplitude Response $|H(j\omega)|$

$|H(j\omega)| = \left|\text{sinc}\left(\frac{1}{2\pi}\omega T\right)\right|$ [$\omega_0 = \frac{1}{T}2\pi$]

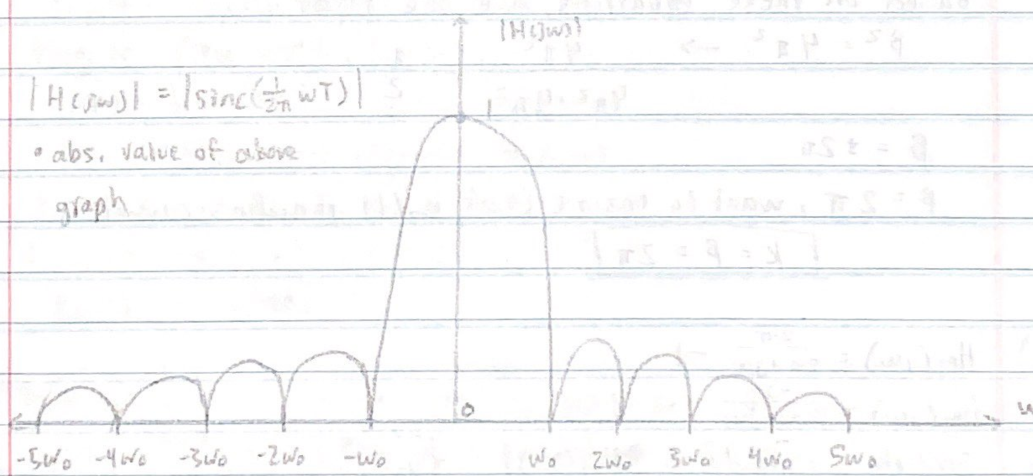
$\text{sinc}\left(\frac{1}{2\pi}\omega T\right)$

• First, plot normally then apply abs. value



$|H(j\omega)| = \left|\text{sinc}\left(\frac{1}{2\pi}\omega T\right)\right|$

• abs. value of above graph



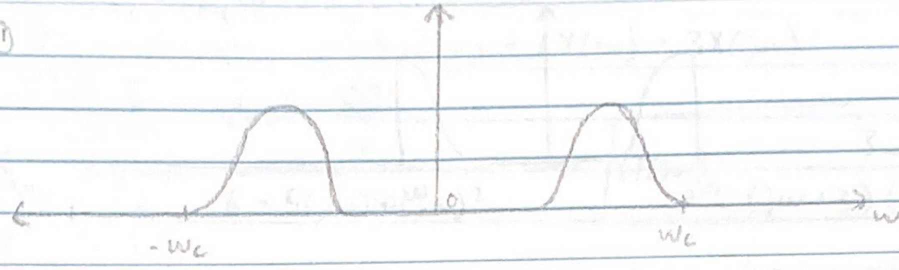
As seen in the bottom graph, the humps become smaller and smaller as the graph gets further from 0.

Therefore, $\lim_{\omega\to\infty} |H(j\omega)| = 0$

4. a) i) the output of the modulator, i.e. $X(j\omega)$

$$X(j\omega) = \frac{M(j(\omega \pm \omega_c))}{2}$$

①

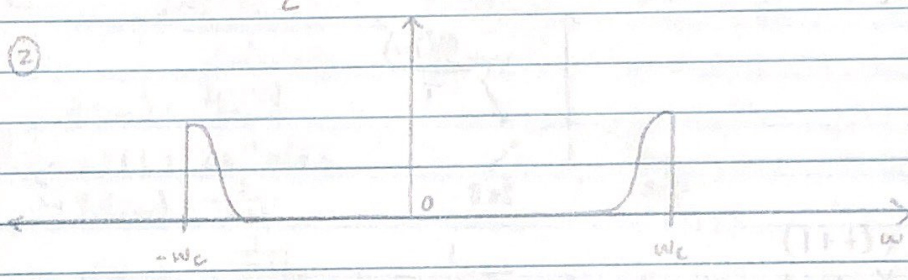


• Two humps that end at $\pm\omega_c$

ii) the output of the ideal bandpass $Y(j\omega)$

$$Y(j\omega) = \frac{M(j(\omega \pm \omega_c))}{2}$$

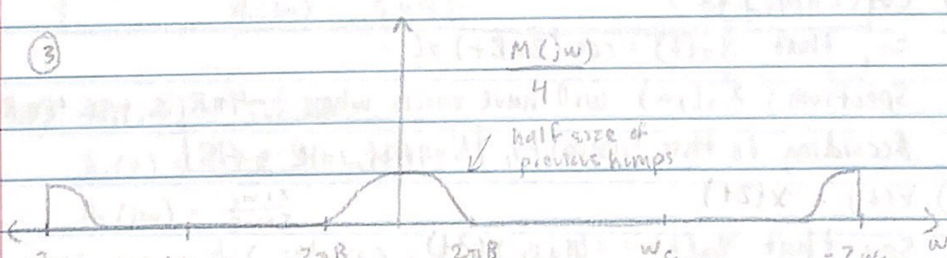
②



• Two half humps that end at $\pm\omega_c$

iii) the output of the demodulator, $Z(j\omega)$

③



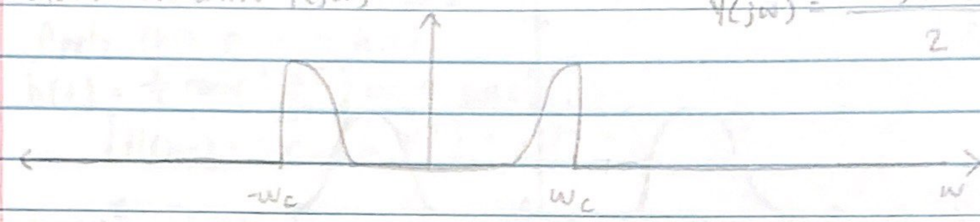
• Two half humps and one full

At the end of all this, $m(t)$ does get recovered but it is scaled by 2 (i.e. factor of two)

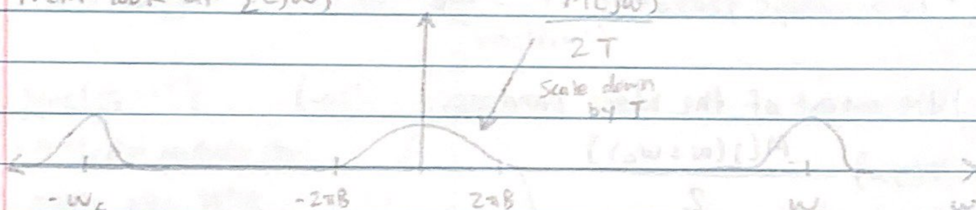
b) Based on the graphs below, we can see that $m(t)$ can be recovered with a factor of T. This could be expected from the problem statement.

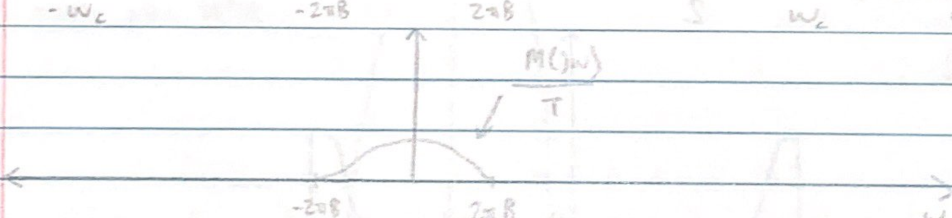
Start off with $Y(j\omega)$:

$$Y(j\omega) = \frac{M(j(\omega \pm \omega_c))}{2}$$



Next look at $Z(j\omega)$





5. a) i) $x(t+1)$

$X_a(j\omega) = X(j\omega) \cdot e^{j\omega}$

Both bandwidths are the same in this instance

$\boxed{\text{Nyquist rate} = 2B}$

ii) $\cos(2\pi Bt)\, x(t)$

Say that $x_a(t) = \cos(2\pi Bt)\, x(t)$

Spectrum: $X_a(j\omega)$ will have values when $-4\pi B \le \omega \le 4\pi B$

According to this inequality, $\boxed{\text{Nyquist rate} = 4B}$

iii) $x(t) + x(2t)$

Say that $x_a(t) = x(t) + x(2t)$

Find the spectrum of $x_a(t)$

$$X_a(j\omega) = \tfrac{1}{2} X(\tfrac{1}{2} j\omega) + X(j\omega)$$

$X(\tfrac{1}{2}j\omega)$: nonzero when $|\omega| \le 4\pi B$ (just think of scaling)

$X(j\omega)$: nonzero when $|\omega| \le 2\pi B$

According to these inequalities, $\boxed{\text{Nyquist rate} = 4B}$