Michael : Neuro 2

Jim: EE2

Justin: EE2

Zeid : CE2

EE102

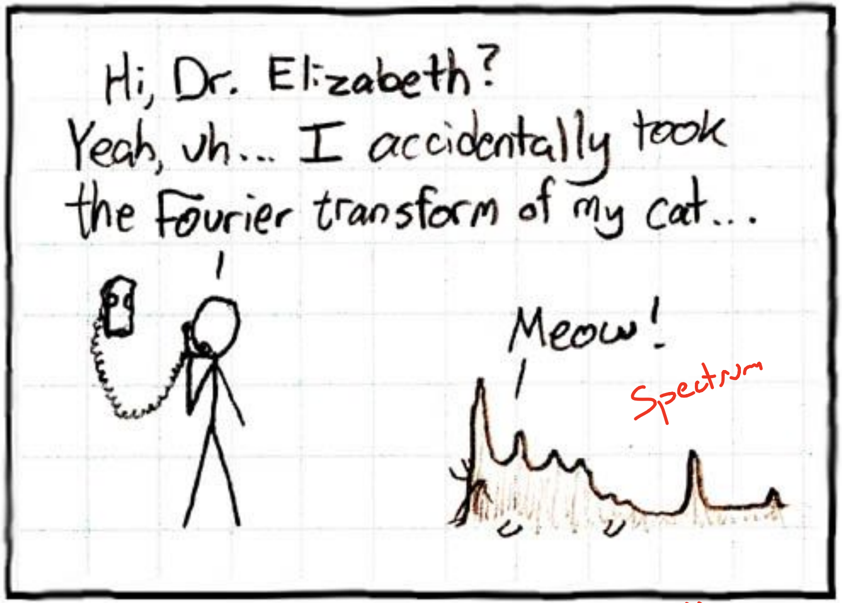
Lecture 9

EE102 Announcements

- **Midterm released ~~Wednesday~~ Thursday 4/29 at 8am and due ~~Thurs~~ Friday 4/30 at 8am**
  - **Open Internet!**
  - **Honor Code: Please do the exam individually**
- **Short video on exam topics released** on Monday
- **This lecture not covered in midterm**
- **No homework this week.**
- Midterm grade will be replaced w/ Final if you do better on the final.

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

This Lecture Breaks a New Seal



XKCD ← Check this comic out.

Joseph Fourier

Fourier was a Super Interesting Guy

- Born, France.
- Son of a Tailor; Orphan at 10 years old.
- Aspires to be Newton, but at age 21 feels like a failure.
- Joins politics and goes to Jail
- Narrowly escapes guillotine in the French Revolution
- Confidante of Napoleon

French Revolution

Intuition: Why Fourier Series?

Why Fourier Series?

Bottom Line of F.S.

Any periodic signal can be expressed as a sum of sinusoids

1) $\underbrace{x(t)}_{\text{some signal}} = \underbrace{x_1(t)}_{\text{sinusoid}} + \underbrace{x_2(t)}_{\text{sinusoid}} + \ldots$

Simplify the analysis of systems

2) A different and often simpler way to think about data/info

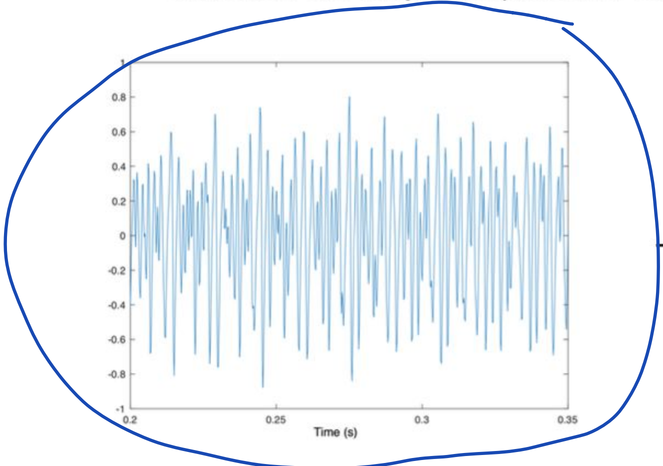
Reveals interesting + unexpected structure.

Fourier Series

Extracts *frequency structure* from a signal.

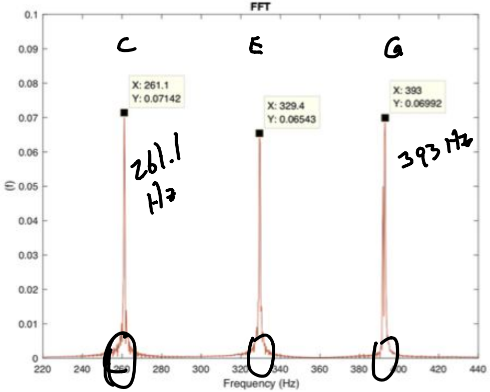
Below: we have a C-major chord. It consists of three notes: C (261.1 Hz), E (329.4 Hz) and G (393 Hz). When we play these three notes at the same time, they create a waveform indicated by the blue line. It's hard to see structure here.

The Fourier series is able to write this as a sum of sinusoids. When we do, we find that there are only 3 frequencies, corresponding to our C-major chord.



Fourier

~ Diracs



C: 261 Hz

Fourier Series - Bottom Line

$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$

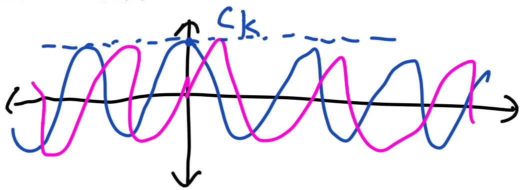
These are the main mathematical results of this lecture, written here for convenience.

$\omega_0 = 2\pi f = \frac{2\pi}{T_0}$

If $f(t)$ is a well-behaved periodic signal with period T_0, then $f(t)$ can be written as a Fourier series

1) Any periodic signal

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$



where $\omega_0 = \frac{2\pi}{T_0}$ and

2) Find the coefficients of each sinusoid

$$\boxed{c_k = \frac{1}{T_0}\int_{\tau}^{\tau+T_0} f(t)e^{-jk\omega_0 t}\,dt}$$

for all integers k. The c_k are called the *Fourier coefficients* of $f(t)$.

Here, $f(t)$ is the *weighted average* of complex exponentials (which are simply complex sines and cosines).

Eigenfunctions

$x(t)$ is an *eigenfunction* of a system if, when inputting $x(t)$ to the system, the output is simply a scaled version of $x(t)$, i.e., $y(t) = ax(t)$ where a is a constant (called an eigenvalue). Note that a may be a complex constant.

$$x(t) \longrightarrow \boxed{H} \longrightarrow ax(t)$$

Eigenfunction

★★★ Complex Exponentials are eigenfunctions of LTI systems

$$e^{(\sigma + j\omega)t} \longrightarrow \boxed{H} \longrightarrow ae^{(\sigma + j\omega)t}$$

Eigenfunction

$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$x \to \boxed{h} \to y$

Eigenfunctions of LTI Systems

Consider an LTI system with impulse response $h(t)$. If the input is a complex exponential, i.e., e^{st} where $s = \sigma + j\omega$, then

$e^{st} \to \boxed{h(t)} \to \cancel{y(t)}\ \hat{H}(s)e^{st}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

algebra

$$= \int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau}\, d\tau$$

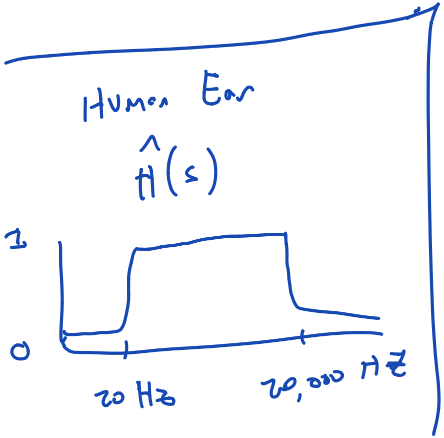
pop out the constant

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau}_{\hat{H}(s)}$$

use transfer fn definition

$$= \hat{H}(s) e^{st}$$

$$\hat{H}(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau \triangleq \text{"Transfer Function"}$$



Eigenfunctions of LTI Systems (Intuition)

This shows that the complex exponential is an eigenfunction of an LTI system.

- If I input a complex exponential into the LTI system, I get the same complex exponential out scaled by $H(s)$.

From here, we can see how Fourier series might help:

- First, I decompose my signal, $x(t)$, into the sum of complex exponentials.
- After this, I put each complex exponential into my LTI system. Since the system is LTI, each complex exponential comes out scaled by $H(s)$.
- Then, I can add my scaled complex exponentials at the output to get the system output.
- Conveniently, since the output is a sum of (scaled) complex exponentials, it is also a Fourier series.

Let's formalize this intuition we've stated here and get to the math.

Eigenfunctions of LTI Systems

Say $x(t)$ is periodic. Then, $x(t) = \sum_{-\infty}^{\infty} c_k e^{jk\omega_0 t}$ (Fourier Series of $x(t)$)

$$x(t) = x_1(t) + x_2(t) + x_3(t) + \dots$$

// Take 1 particular complex exponential

for $k=1 \Longrightarrow x_1(t) = c_1 e^{j\omega_0 t}$

Now, lets apply the system to $x_1(t) \Longrightarrow y_1(t) = \hat{H}(j\omega_0) c_1 e^{j\omega_0 t}$

// Take another complex exponential

for $k=2 \Longrightarrow x_2(t) = c_2 e^{j2\omega_0 t}$

output $\Longrightarrow y_2(t) = \hat{H}(2j\omega_0) c_2 e^{j2\omega_0 t}$

$\tilde{c}_k \triangleq \hat{H}(jk\omega_0)\, c_k$

// Say H is linear, so $H(x_1 + x_2) = y_1 + y_2$

Input $x(t) = \sum_{-\infty}^{\infty} c_k e^{jk\omega_0 t} = x_1(t) + x_2(t) + \dots$

$x \to \boxed{H} \to y$

Output $\Longrightarrow$ $y(t) = \sum_{k=-\infty}^{\infty} \hat{H}(jk\omega_0)\, c_k e^{jk\omega_0 t}$

$= \sum_{-\infty}^{\infty} \tilde{c}_k e^{jk\omega_0 t}$

Review of last couple slides.

Eigenfunctions of LTI Systems

For LTI systems, a simple way to analyze them is to:

- Decompose $x(t)$, the input, into its Fourier series, i.e., a sum of complex exponentials. This represents its decomposition into "fundamental" components, which are sinusoids at different frequencies $k\omega_0$.
- Because the complex exponential is an eigenfunction of an LTI system, if I pass in a complex exponential into my system, I get the same complex exponential at the output, scaled by $\hat{H}(jk\omega_0)$.
- Since LTI systems are distributive, if I pass in a sum of these complex exponentials into my system, I get back an output, $y(t)$, that is a sum of scaled complex exponentials (where the scale term is $\hat{H}(jk\omega_0)$).

(Simple) Fourier Series Example

Motivation for Fourier series

With this motivation, we now need to know how to actually calculate Fourier series, i.e., how do I find the c_k so that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

And further, when is this possible? The rest of this lecture will cover this.

(Simple) Fourier Series Example

Fourier series of a cosine (cont.)

Let's start simple. Consider the sinusoid:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Euler's: $\cos(\omega t) = \frac{1}{2} e^{-j\omega t} + \frac{1}{2} e^{j\omega t}$

A: amplitude

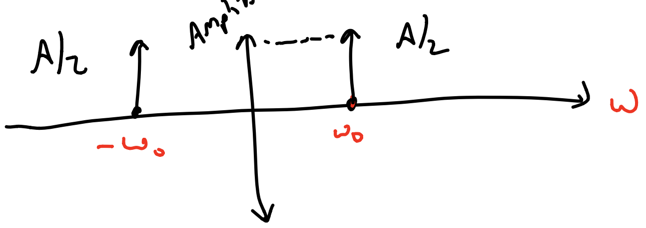
ω_0: frequency

θ: phase

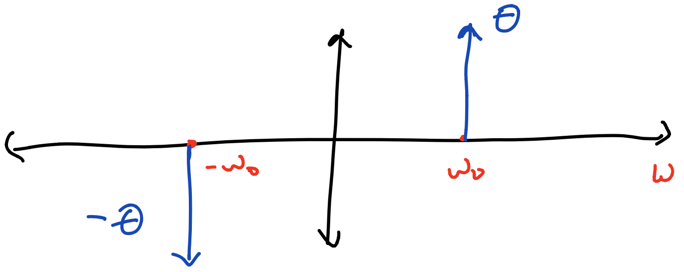
$$f(t) = A\cos(\omega_0 t + \theta)$$

$$= \frac{A}{2} e^{-j(\omega_0 t + \theta)} + \frac{A}{2} e^{j(\omega_0 t + \theta)}$$

SPECTRUM



Amplitude Spectrum



Phase Spectrum

Phase Values $[-\pi, \pi]$ or $[0, 2\pi]$

(Simple) Fourier Series Example

Using spectrum to find structure

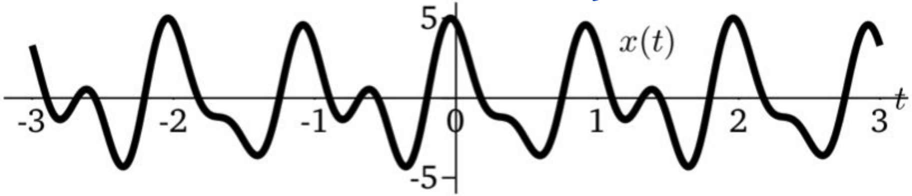
For the cosine example, it doesn't look like we made our lives easier by representing it as a spectrum. But for any more complex signal, it can.

Consider the signal

$$x(t) = 3\cos(2\pi t) + \cos(3\pi t - \pi/4) + 2\cos(4\pi t + \pi/3)$$

This signal is very simple. However, if I gave you a plot of its time domain representation, it'd be hard to recover exactly what $x(t)$ is.

$$x(t) = \frac{3}{2}e^{-j2\pi t} + \frac{3}{2}e^{j2\pi t} + \frac{1}{2}e^{j(3\pi t - \pi/4)} + \frac{1}{2}e^{-j(3\pi t - \pi/4)} + e^{j(4\pi t + \pi/3)} + e^{-j(4\pi t + \pi/3)}$$

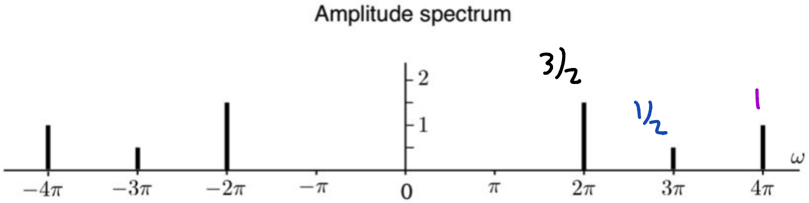


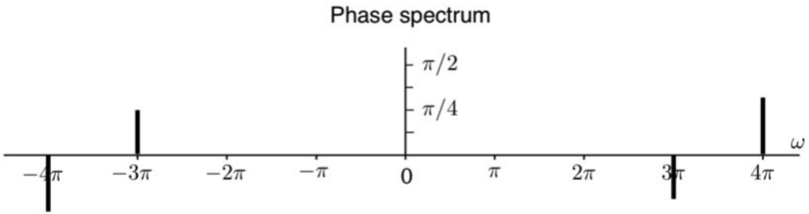
(Simple) Fourier Series Example

Using spectrum to find structure (cont.)

cyu: Given a visual plot of spectrum, how would you write a ground truth $x(t)$?

However, if we plotted the spectrum of this signal, it would look like the following:





From the spectrum, we can read off that this signal is composed of sinusoids at three different frequencies (2π, 3π, and 4π) with amplitudes given by the amplitude spctrum and phases given by the phase spectrum.

Ad-hoc

Deriving the Fourier Series Coefficients

Deriving Fourier series

$$f(t) = \sum_k c_k e^{jk\omega_0 t}$$

How do we find the c_k?

Our derivation is as follows:

- First, we *assume* that the signal $f(t)$ can be written as a sum of complex exponentials that are scaled by coefficients c_k.
- Given this assumption, we find if there are c_k such that we can represent $f(t)$ in this way.

Learning Goals from today:

1. Periodic Signals can be broken down into sum of sinusoids
2. If you know the breakdown, you can plot it on a spectrum
3. Recipe to find c_k's in general

Next Time

Deriving the Fourier Series Coefficients

A preliminary result on integrating complex exponentials

Before proceeding, we're going to introduce a handy trick that will simplify our derivation. Let $T_0 = 2\pi/\omega_0$. Consider the complex exponential

$$e^{jk\omega_0 t}$$

in our Fourier series.

- When $k = 0$, then this complex exponential is equal to 1.
- When $k \neq 0$, then this complex exponential is equal to

$$\cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

Deriving the Fourier Series Coefficients

A preliminary result on integrating complex exponentials (cont.)

If I integrate this expression over a period, I get the following:

$$\int_{t_0}^{t_0+T_0} e^{jk\omega_0 t} dt \quad = \quad \int_{t_0}^{t_0+T_0} e^{j\frac{2\pi k}{T_0}t} dt$$

Deriving the Fourier Series Coefficients

Deriving Fourier series

Let's begin with the derivation then.

Define $\omega_0 \triangleq 2\pi/T_0$, and assume that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

To use our preliminary result, what we'll do is multiply both sides by a complex exponential,

$$e^{-jn\omega_0 t}$$

and then integrate over one period, T_0.

Deriving the Fourier Series Coefficients

Deriving Fourier series (cont.)

$$\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}dt \quad = \quad \int_{t_0}^{t_0+T_0} \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right) e^{-jn\omega_0 t} dt$$

Deriving the Fourier Series Coefficients

$$\boxed{c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt}$$

These are the Fourier coefficients (!) and demonstrate that indeed, a periodic signal (or one defined over a length T_0) can be written as a sum of complex exopnentials.