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ECE 102 HW 4

1. a) i) $f(t) = \cos(3\pi t) + \frac{1}{2}\sin(4\pi t)$

$\cos(3\pi t) \rightarrow T_a = \frac{2}{3}$ $\quad\quad \sin(4\pi t) \rightarrow T_b = \frac{1}{2}$

$(\frac{2}{3}) \div (\frac{1}{2}) = \frac{4}{3}$

$T_0 = 4T_b = 3T_a \rightarrow T_0 = 2$ sec, $\quad w_0 = \pi$ rad/s

Eulers $\quad f(t) = \frac{1}{2}[e^{j3\pi t} + e^{-j3\pi t}] + \frac{1}{4j}[e^{j4\pi t} - e^{-j4\pi t}]$

$f(t) = \frac{1}{2}e^{j3\pi t} + \frac{1}{2}e^{-j3\pi t} - \frac{1}{4}je^{j4\pi t} + \frac{1}{4}je^{-j4\pi t}$

$$\boxed{C_K = \begin{cases} \frac{1}{4}j & K = -4 \\ -\frac{1}{4}j & K = 4 \\ \frac{1}{2} & K = \pm 3 \\ 0 & \text{else} \end{cases}}$$

ii) $T = 1s \quad e^{-2t}$ for $0 < t < 1$

Fourier : $f(t) = \sum_{K=-\infty}^{\infty} C_K e^{jKw_0 t}$ $\quad\quad w_0 = \frac{2\pi}{1} = 2\pi$ rad/s

$C_K = \frac{1}{T}\int_0^T f(t) e^{-jKw_0 t} dt \quad (T=1)$

$C_K = \int_0^1 e^{-2t} \cdot e^{-jKw_0 t} dt$

$$\boxed{\frac{-e^{-2} + 1}{2j\pi K + 2}}$$

iii) $T = 3s$ (based on graph)

Fourier : $f(t) = \sum_{K=-\infty}^{\infty} C_K e^{jKw_0 t}$ $\quad\quad w_0 = \frac{2\pi}{3}$ rad/s

$C_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{3}[\int_1^2 1dt + \int_0^1 2dt] = 1$

$C_K = \frac{1}{T}\int_0^T f(t)e^{-jKw_0 t}dt$

$C_K = \frac{1}{3}[\int_0^1 2e^{-\frac{2\pi}{3}jKt}dt + \int_1^2 e^{-\frac{2\pi}{3}jKt}dt]$

$$C_K = \left[\frac{2 - e^{-\frac{4\pi}{3}jK} - e^{-\frac{2\pi}{3}jK}}{2\pi jK}\right]$$

$$C_K = \frac{2 - 2\cos(\frac{2}{3}\pi K)}{2\pi jK} = \boxed{\frac{1 - \cos(\frac{2}{3}\pi K)}{\pi jK}}$$

b) $x(t) \rightarrow T_1 \rightarrow X_K \quad\quad y(t) \rightarrow T_2 \rightarrow Y_K$

i) $T_1 = T_2 \quad\quad z(t) = x(t) + y(t) \quad\quad X_K, Y_K$

$T_0 = T_1 = T_2$

$x(t) = \sum_{K=-\infty}^{\infty} X_K e^{jKw_0 t} \quad\quad y(t) = \sum_{K=-\infty}^{\infty} Y_K e^{jKw_0 t}$

$$\boxed{\begin{gathered} z(t) = \sum_{K=-\infty}^{\infty}(X_K + Y_K)e^{jKw_0 t} \\ Z_K = X_K + Y_K \end{gathered}}$$

ii) $T_1 = 2T_2$ $\quad w(t) = x(t) + y(t)$

$T_0 = T_1 = 2T_2$ $\quad w_0 = w_1 = \frac{1}{2} w_2$

$x(t) = \sum_{a=-\infty}^{\infty} X_a e^{jaw_0 t}$ $\quad y(t) = \sum_{b=-\infty}^{\infty} Y_b e^{jbw_0 t}$

$w(t) = \sum_{a=-\infty}^{\infty} X_a e^{jaw_0 t} + \sum_{b=-\infty}^{\infty} Y_b e^{jbw_0 t}$

Substitute $a' = 2b$

$w(t) = \sum_{e(a)} X_a e^{jaw_0 t} + \sum_{o(a)} X_a{}^{jaw_0 t} + \sum_{e(a')} Y_{\frac{1}{2}a'} e^{ja'w_0 t}$

$$\boxed{W_K = \begin{cases} X_K + Y_{\frac{1}{2}K} & \text{even } K \\ X_K & \text{odd } K \end{cases}}$$

2. $f(t) = \sum_{k=-\infty}^{\infty} C_K e^{jKw_0 t}$

a) $g(t) = f(t) + 1$

The period is the same, as adding a constant wouldn't affect period.

$g(t) = C_0 + 1 + \sum_{K\neq 0} C_K e^{jKw_0 t}$

$g(t) = \sum_{K=-\infty}^{\infty} C_K' e^{jKw_0 t}$

$$\boxed{C_K' = \begin{cases} 1 + C_0 & K = 0 \\ C_K & K \neq 0 \end{cases}}$$

b) $g(t) = f(-t)$

The period is the same, think of it as a reflection - reflecting something does not change its period, just its orientation.

$g(t) = \sum_{K=-\infty}^{\infty} C_K e^{-jKw_0 t}$

$g(t) = \sum_{K=-\infty}^{\infty} C_{-K} e^{jKw_0 t}$

$g(t) = \sum_{K=-\infty}^{\infty} C_K' e^{jKw_0 t}$ $\quad \boxed{C_{-K} = C_K'}$

c) $g(t) = f(at)$ $\quad a$ is positive, real number

The period does change this time.

$T = \frac{T_0}{a}$ $\quad w = aw_0$

$g(t) = \sum_{K=-\infty}^{\infty} C_K e^{jKw_0 (at)}$

$g(t) = \sum_{K=-\infty}^{\infty} C_K e^{jKw_0' t}$

$\boxed{\text{The Fourier series coefficients do not change.}}$

3. a) $f(t) = \cos(w_0 t)$ not an eigenfunction

$h(t)$: impulse response for system

Eulers : $\cos(w_0 t) = \frac{1}{2}[e^{-jw_0 t} + e^{jw_0 t}]$

$y(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau) d\tau$

$y(t) = \frac{1}{2}[\int_{-\infty}^{\infty} h(\tau) e^{-jw_0(t-\tau)} d\tau + \int_{-\infty}^{\infty} h(\tau) e^{jw_0(t-\tau)} d\tau$

$y(t) = \frac{1}{2} e^{-jw_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{jw_0 \tau} d\tau}_{m} + \frac{1}{2} e^{jw_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-jw_0 \tau} d\tau}_{n}$

We see that the two integrals "m" and "n" present in the final form of $y(t)$ are not identical. This means that the original $f(t)$ can not be an eigenfunction because its output would not be $af(t)$. An impulse responce such as $h(t) = \delta(t-20)$ would yield $y(t) = \cos(w_0(t-20))$, which is not equivalent to $a\cos(w_0 t)$

b) $f(t) = t$ not an eigenfunction

$h(t)$: impulse response for system

$y(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau) d\tau$

$y(t) = \int_{-\infty}^{\infty} h(\tau)(t-\tau) d\tau$

$y(t) = t\int_{-\infty}^{\infty} h(\tau) d\tau - \int_{-\infty}^{\infty} \tau h(\tau) d\tau$

Because there is no "t" coefficient in front of the second integral, the function does not have the form $y(t) = af(t)$. Therefore, this isn't an eigenfunction.

4. a) $y(t) = [(e^t x(t)) * h(t)] e^{-t}$

First part : input $[x(t)]$ multiplied by e^t

$y_a(t) = e^t x(t)$

Second part : $y_b(t)$ is convolution of $h(t)$ and $y_a(t)$

$y_b(t) = (e^t x(t)) * h(t)$

Third part : $y_b(t)$ multiplied by e^{-t}

$y(t) = [(e^t x(t) * h(t)] e^{-t}$ ✓

b) $y(t) = [(e^t x(t)) * h(t)]e^{-t} \rightarrow y(t) = \int_{-\infty}^{\infty} h'(\tau)x(t-\tau)d\tau$

$y(t) = e^{-t}\int_{-\infty}^{\infty} x(t-\tau)h(\tau)e^{t-\tau}d\tau$

$y(t) = e^{-t}\int_{-\infty}^{\infty} x(t-\tau)h(\tau)e^{t}e^{-\tau}d\tau$

$y(t) = e^{t}e^{-t}\int_{-\infty}^{\infty} x(t-\tau)h(\tau)e^{-\tau}d\tau$

$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)e^{-\tau}d\tau$

$h'(\tau) = h(\tau)e^{-\tau}$

$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h'(\tau)d\tau \rightarrow y(t) = \int_{-\infty}^{\infty} h'(\tau)x(t-\tau)d\tau$ ✓

c) Prove LTI, find impulse response $h_{eq}(t)$

Delay input: $y_{t'}(t) = \int_{-\infty}^{\infty} h'(\tau)x(t-\tau-t')d\tau$

Delay output: $y(t-t') = \int_{-\infty}^{\infty} h'(\tau)x(t-t'-\tau)d\tau$ } equivalent $\boxed{\text{Time-invariant!}}$

$y(t) = [(e^t x(t)) * h(t)]e^{-t}$

$y(t) = [(e^t(ax_a(t) + bx_b(t)) * h(t)]e^{-t}$

$y(t) = [[ae^t x_a(t) * h(t)] + [be^t x_b(t) * h(t)]]e^{-t}$

$y(t) = e^{-t}[ae^t x_a(t) * h(t)] + e^{-t}[be^t x_b(t) * h(t)]$

$y(t) = ay_a(t) + by_b(t)$ $\boxed{\text{Linear}}$

The equivalent system is indeed LTI. ✓

$h'(\tau) = h(\tau)e^{-\tau} \rightarrow h(t) = h(t)e^{-t}$

$\boxed{h_{eq}(t) = h(t)e^{-t}}$

d) $s(t) = r(t-1)$

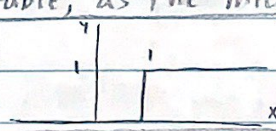
Impulse: $h(t) = \frac{ds(t)}{dt} = \frac{d}{dt}[r(t-1)]$

$\boxed{h(t) = u(t-1)}$

Stable: $\int_{-\infty}^{\infty}|h(t)|dt \rightarrow \int_{-\infty}^{\infty}|u(t-1)|dt = \infty$

$\boxed{\text{Not stable, as the integral will approach infinity}}$

Causal: $u(t-1)$



$h(t) = 0$ when $t < 0$

$\boxed{\text{causal}}$

Equivalent Impulse: $\boxed{h_{eq}(t) = [u(t-1)]e^{-t}}$

Stable: $\int_{-\infty}^{\infty}|h_{eq}(t)|dt \rightarrow \int_{-\infty}^{\infty}[u(t-1)]e^{-t}dt$

$\int_{1}^{\infty} e^{-t}dt = \frac{1}{e} < \infty$ $\boxed{\text{stable}}$

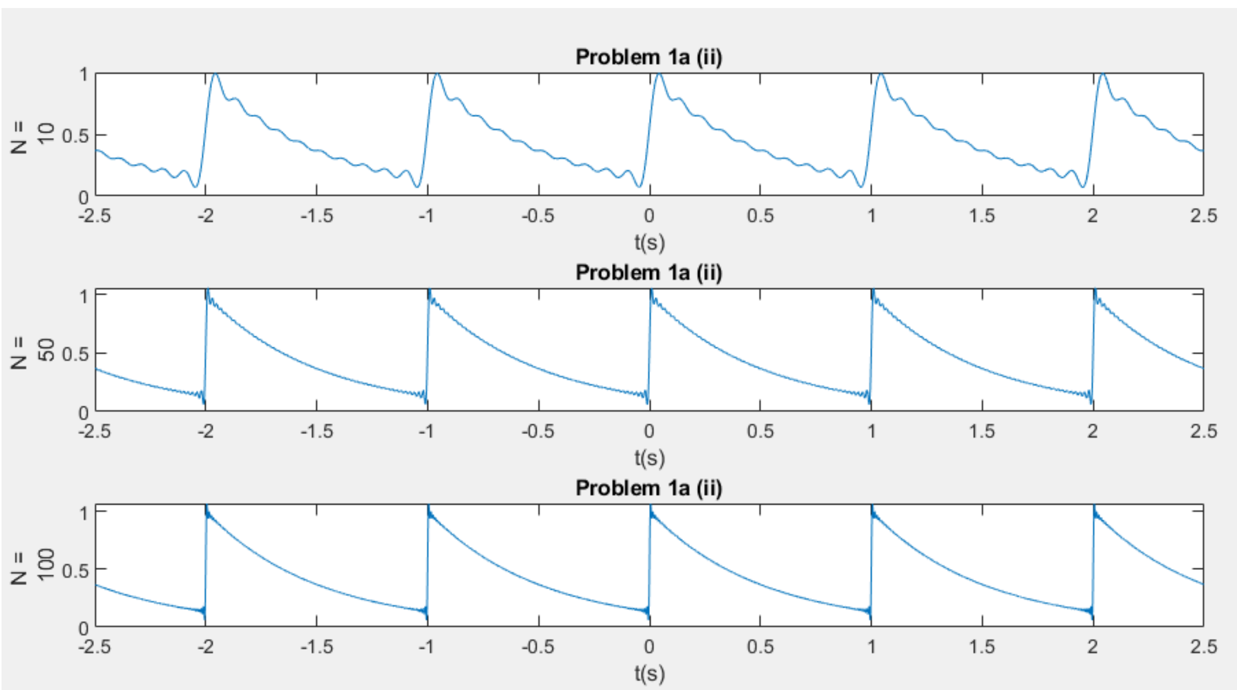
Causal: Based on graph from above, multiplying be e^{-t} doesn't change the fact that $h(t) = 0$ when $t < 0$ $\boxed{\text{causal}}$

5a)

```matlab
% fn = myfs(Dn,omega0,t)
% Evaluates the truncated Fourier Series at times t
% Dn -- vector of Fourier series coefficients
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
% fn -- truncated Fourier series evaluated at t

function fn = myfs(Dn, omega0, t)
fn = zeros(size(t));
N = (1/2) * (length(Dn) - 1);
for i = -N:N
 temp = Dn(N + i + 1);
 fn = fn + (temp * exp(1i * t * i * omega0));
end
```

5b)

```
i = 0;

for N = [10, 50, 100]
omega0 = (2*pi);
n = -N:N;
ck = (1 - exp(-2)) ./ (2+2*1i*n*pi);

t = -2.5:0.001:2.5;
f = myfs(ck, omega0, t);
i = i + 1;
subplot(3, 1, i);
plot(t, f);
title('Problem 1a (ii)'); ylabel(['N = ', string(N)]); xlabel('t(s)');
end
```



5c)

```
i = 0;

for N = [10, 50, 100]
omega = (2*pi)/3;
n1 = -N:1:-1;
k = n1;
neg = (1/3) * (-2+exp(-1i*omega*k) + exp(-1i*2*omega*k)) ./ (-1i*omega*k);

n2 = 1:1:N;
k = n2;
pos = (1/3) * (-2+exp(-1i*omega*k) + exp(-1i*2*omega*k)) ./ (-1i*omega*k);

t = -2.5:0.001:2.5;
f = myfs([neg, 1, pos], omega, t);
i = i + 1;
subplot(3, 1, i);
plot(t, f);
title('Problem 1a (iii)'); ylabel(['N = ', string(N)]); xlabel('t(s)');
end
```