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EE102

Lecture 10

EE102 Announcements

- Syllabus link is tiny.cc/ucla102
- **Fourth Homework - due Friday**
- **Midterm Grading / Solutions** : TA / Grader

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Last Lecture

These are the main mathematical results of this lecture, written here for convenience.

If $f(t)$ is a well-behaved periodic signal with period T_0, then $f(t)$ can be written as a Fourier series

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

where $\omega_0 = \frac{2\pi}{T_0}$ and

Last Lecture

Today: Inverse Problem of finding c_k.

$$\boxed{c_k = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) e^{-jk\omega_0 t} dt}$$

for all integers k. The c_k are called the *Fourier coefficients* of $f(t)$.

Here, $f(t)$ is the *weighted average* of complex exponentials (which are simply complex sines and cosines).

Deriving the Fourier Series Coefficients

Deriving Fourier series

$$f(t) = \sum_k c_k e^{jk\omega_0 t}$$

$$c_0 e^{j0\omega_0 t} = c_0$$

How do we find the c_k?

Our derivation is as follows:

- First, we *assume* that the signal $f(t)$ can be written as a sum of complex exponentials that are scaled by coefficients c_k.
- Given this assumption, we find if there are c_k such that we can represent $f(t)$ in this way.

Deriving the Fourier Series Coefficients

A preliminary result on integrating complex exponentials

Before proceeding, we're going to introduce a handy trick that will simplify our derivation. Let $T_0 = 2\pi/\omega_0$. Consider the complex exponential

$$e^{jk\omega_0 t}$$

in our Fourier series.

- When $k = 0$, then this complex exponential is equal to 1. $e^{j0\omega_0 t} = e^0 = 1$
- When $k \neq 0$, then this complex exponential is equal to

$$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

L.G. Integrating complex exponential over 1 period is 0 unless $k=0$, in which case its T_0.

Deriving the Fourier Series Coefficients

A preliminary result on integrating complex exponentials (cont.)

If I integrate this expression over a period, I get the following: c.y.u: Compute the Integral

$$\int_{t_0}^{t_0+T_0} e^{jk\omega_0 t}dt \quad = \quad \int_{t_0}^{t_0+T_0} e^{j\frac{2\pi k}{T_0}t}dt$$

$$= \int_{t_0}^{t_0+T_0} \cos\left(\frac{2\pi k}{T_0}t\right)dt + j\int_{t_0}^{t_0+T_0} \sin\left(\frac{2\pi k}{T_0}t\right)dt$$

$$= \begin{cases} T_0, & \text{when } k=0 \\ 0, & \text{when } k\neq 0 \end{cases}$$

If $k=0$

$$= \int_{t_0}^{t_0+T_0} 1\,dt + j\int_{t_0}^{t_0+T_0} 0\,dt = T_0$$

If $k \neq 0$

$$\underbrace{\int_{t_0}^{t_0+T_0} \cos\left(\frac{2\pi k}{T_0}t\right)dt}_{0} + j\underbrace{\int_{t_0}^{t_0+T_0} \sin\left(\frac{2\pi kt}{T_0}\right)dt}_{0}$$

Deriving the Fourier Series Coefficients

$f(t)$ is periodic w/ period T_0

Deriving Fourier series

Let's begin with the derivation then.

Define $\omega_0 \triangleq 2\pi/T_0$, and assume that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

To use our preliminary result, what we'll do is multiply both sides by a complex exponential,

$$e^{-jn\omega_0 t}$$

and then integrate over one period, T_0.

$$\int_{t_0}^{t_0+T_0} e^{jk\omega_0 t} e^{-jn\omega_0 t}\, dt = \int_{t_0}^{t_0+T_0} e^{j(k-n)\omega_0 t}\, dt = \begin{cases} T_0 & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$$

Deriving the Fourier Series Coefficients

Deriving Fourier series (cont.)

LHS

$$\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}dt = \int_{t_0}^{t_0+T_0}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)e^{-jn\omega_0 t}dt$$

$$= \int_{t_0}^{t_0+T_0} \sum_k c_k e^{j(k-n)\omega_0 t}\, dt$$

$$= \sum_k c_k \underbrace{\int_{t_0}^{t_0+T_0} e^{j(k-n)\omega_0 t}\, dt}$$

$$= \begin{cases} c_n T_0 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

RHS

$$c_n T_0 = \int_{t_0}^{t_0+T_0} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$\boxed{c_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\, e^{-jn\omega_0 t}\, dt}$$

Deriving the Fourier Series Coefficients

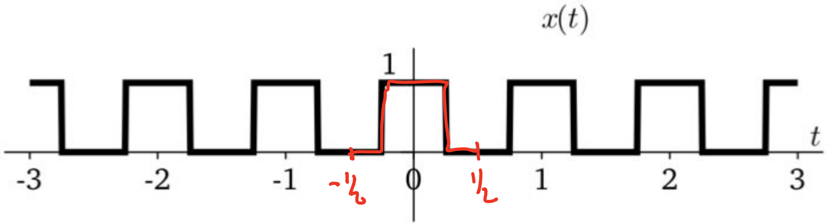
$$\boxed{c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-jk\omega_0 t}dt}$$

These are the Fourier coefficients (!) and demonstrate that indeed, a periodic signal (or one defined over a length T_0) can be written as a sum of complex exopnentials.

Square Wave

Example: square wave

Consider the square wave below.



Let's calculate the Fourier series for this. First, we note that we only need to look at one period, so let's look at from $t = -0.5$ to 0.5, where the square wave starts at 0 and transitions to 1 at $t = -0.25$ and from 1 back to 0 at $t = 0.25$. Let's define

$$s(t) = \begin{cases} 0, & -0.5 \leq t < -0.25 \text{ and } 0.25 \leq t < 0.5 \\ 1, & -0.25 \leq t < 0.25 \end{cases}$$

Square Wave

Example: square wave (cont.)

When we calculate the Fourier series, we should worry about two cases: when $k = 0$ and $k \neq 0$. (Usually, if we just solve for when $k \neq 0$, we'll get an expression that is undefined for $k = 0$. This is why we do both.)

For our square wave, we have that $T_0 = 1$. When $k = 0$,

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt$$

$$\boxed{c_0} = \frac{1}{T_0} \int_{-0.5}^{0.5} s(t) e^{-j0\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{-0.25}^{0.25} 1 \cdot 1 \, dt$$

$$= \frac{1}{2T_0} = \boxed{\frac{1}{2}}$$

Square Wave

$$-\frac{1}{jk2\pi}\left(-2j\sin\left(\frac{k\pi}{2}\right)\right) = \boxed{\frac{\sin(k\pi/2)}{k\pi}}$$

Example: square wave (cont.)

When $k \neq 0$,

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T_0}\int_{-0.5}^{0.5} s(t)e^{-jk\omega_0 t}dt = \int_{-0.25}^{0.25} e^{-jk2\pi t}dt$$

$$= -\frac{1}{jk2\pi} e^{-jk2\pi t}\Big|_{t=-0.25}^{t=0.25}$$

$$= -\frac{1}{jk2\pi}\left(\left[\cancel{\cos\left(\frac{k\pi}{2}\right)} - j\sin\left(\frac{k\pi}{2}\right)\right] - \left[\cancel{\cos\left(-\frac{k\pi}{2}\right)} - j\sin\left(-\frac{k\pi}{2}\right)\right]\right)$$

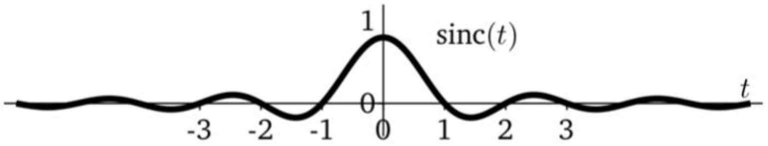
Square Wave

Example: square wave (cont.)

The term $\frac{\sin(\pi t)}{\pi t}$ occurs so frequently in signal processing that it has its own name:

$$\boxed{\operatorname{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}}$$

The sinc function looks like:



(Note that some define $\operatorname{sinc}(t) = \sin(t)/t$. We do NOT use this definition.)
Thus, we have that

$$c_k = \frac{1}{2}\operatorname{sinc}(k/2)$$

Note that $\operatorname{sinc}(0) = 1$.

From last slide

$$c_k = \frac{\sin(\pi k/2)}{\pi k} \cdot \frac{1/2}{1/2}$$

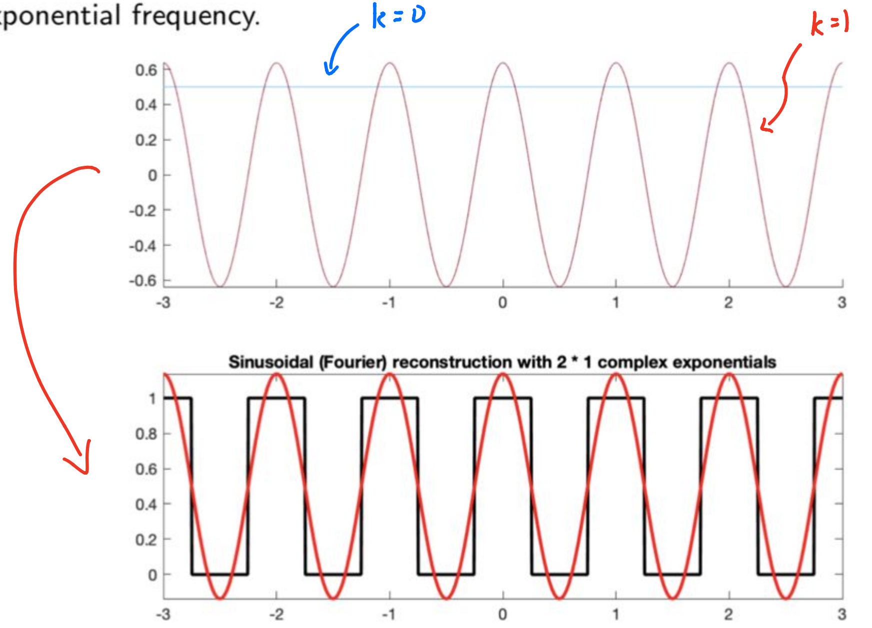
$$= \frac{1}{2}\,\frac{\sin(\pi k/2)}{\pi k/2}$$

$$= \frac{1}{2}\operatorname{sinc}(k/2)$$

Example: square wave (cont.)

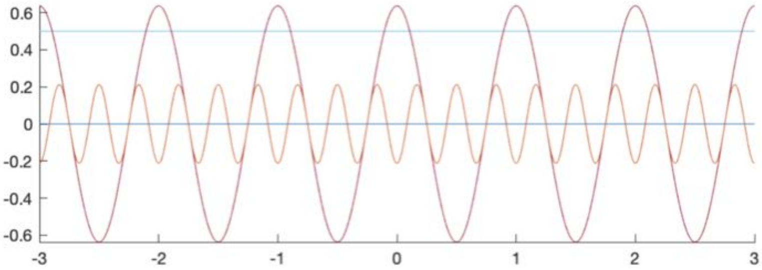
Let's now plot Fourier series fits for our square wave with one complex exponential frequency.

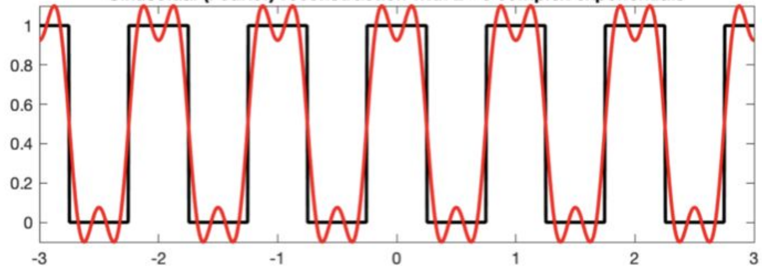
$$c_k = \frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right)$$



Example: square wave (cont.)

Fourier series with 3 complex exponential frequencies...

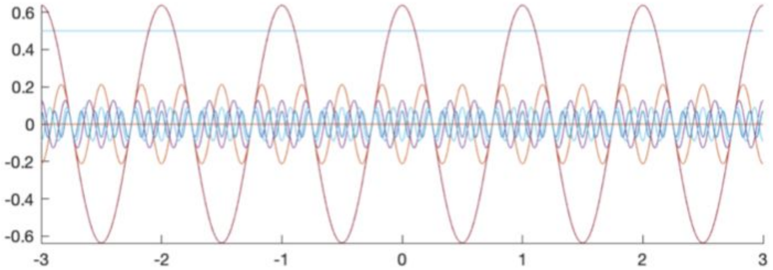


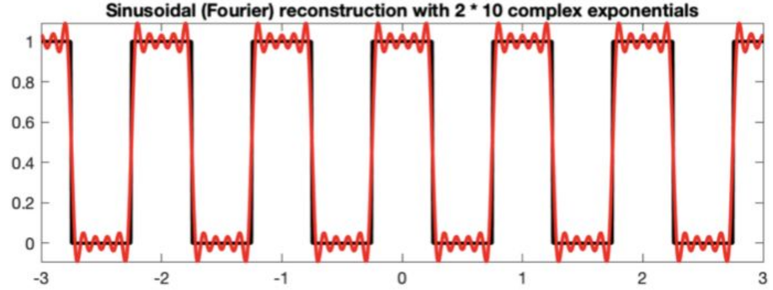


Square wave

Example: square wave (cont.)

Fourier series with 10 complex exponential frequencies...

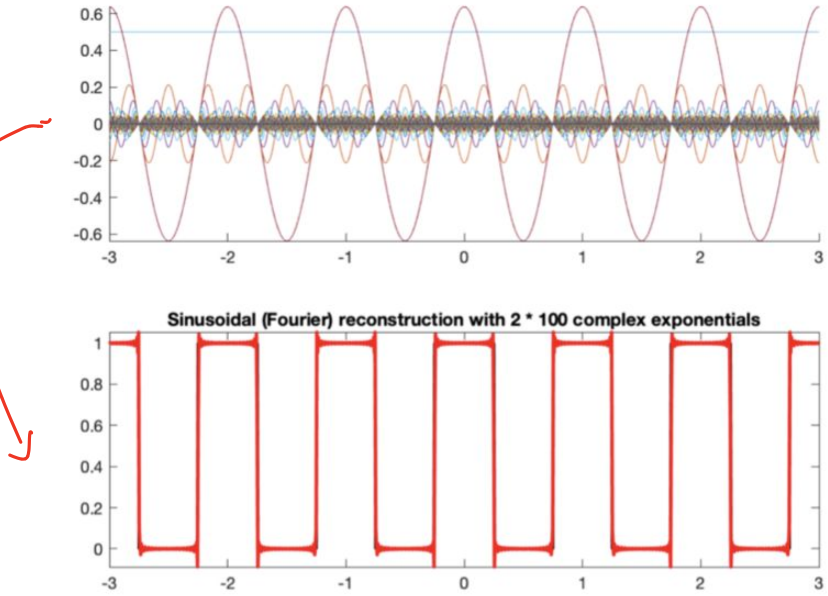




Square wave

Example: square wave (cont.)

Fourier series with 100 complex exponential frequencies...



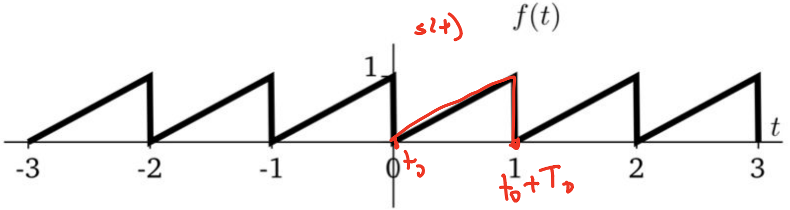
Sawtooth Signal

Example: sawtooth signal

We'll finish this lecture by doing a few more examples to give more familiarity with the Fourier series.

The sawtooth signal is given by $f(t) = t \mod 1$. It is plotted below:

$T_0 = 1$



This signal has a period of $T_0 = 1$. Now, when $k = 0$,

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt$$

$$
\begin{aligned}
c_0 &= \int_0^1 t e^0 dt \\
&= \left. \frac{t^2}{2} \right|_0^1 \\
&= \frac{1}{2}
\end{aligned}
$$

Sawtooth Signal

Now, when $k \neq 0$,

$$c_k = \int_0^1 \overset{u}{t}\,\underbrace{e^{-jk\omega_0 t}dt}_{dv}$$

We use integration by parts, i.e., $\int u dv = uv - \int v du$. We let

$$u = t \qquad dv = e^{-jk\omega_0 t}dt$$

so that

$$du = dt \qquad v = \frac{e^{-jk\omega_0 t}}{-jk\omega_0}$$

This gives that, when $k \neq 0$,

$$\begin{aligned} c_k &= \left.\frac{te^{-jk\omega_0 t}}{-jk\omega_0}\right|_0^1 - \frac{1}{-jk\omega_0}\int_0^1 e^{-jk\omega_0 t}dt \\ &= \frac{je^{-jk\omega_0}}{k\omega_0} + \frac{1}{(k\omega_0)^2}\left. e^{-jk\omega_0 t}\right|_0^1 \\ &= \boxed{\frac{je^{-jk\omega_0}}{k\omega_0} + \frac{e^{-jk\omega_0} - 1}{(k\omega_0)^2}} \implies \text{Simplify to } c_k = \frac{j}{2\pi k} \end{aligned}$$

$c_k = \frac{j}{2\pi k}$ $\qquad$ c_0 + all the c_k's

Now, since $T_0 = 1$, we have that $\omega_0 = 2\pi$. This means $c_k = j/(2\pi k)$ for $k \neq 0$. Therefore, the Fourier series of the sawtooth is:

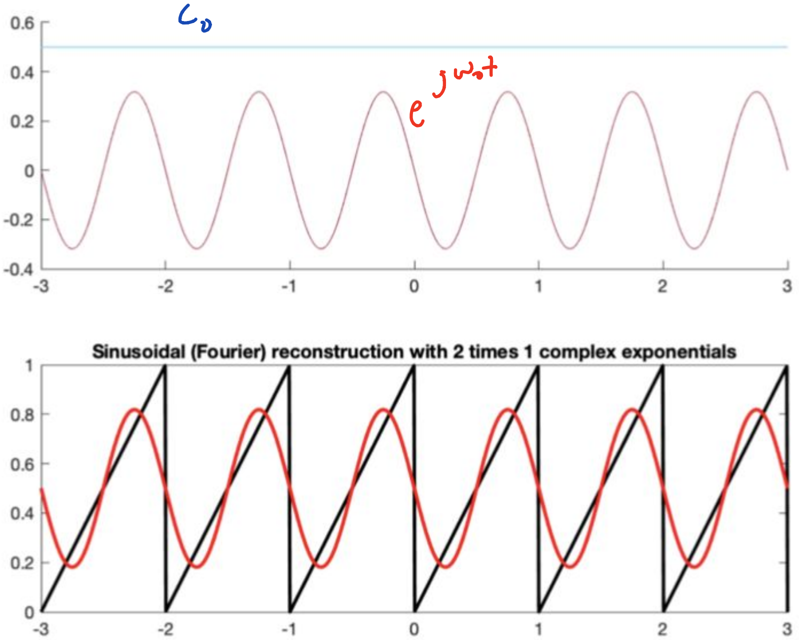
$$t \mod 1 = \underbrace{\frac{1}{2}}_{c_0} + \underbrace{\sum_{k\neq 0}}_{\text{sum over all } k \text{ not equal to zero}} \underbrace{\frac{j}{2\pi k}}_{c_k} \underbrace{e^{jk2\pi t}}_{\text{complex exp}}$$

We show some illustrations below.

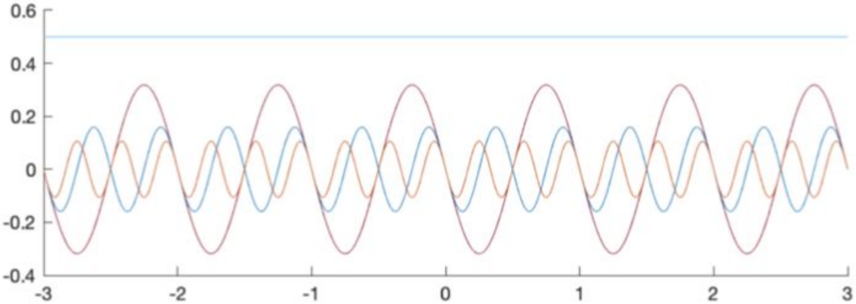
Sawtooth Signal

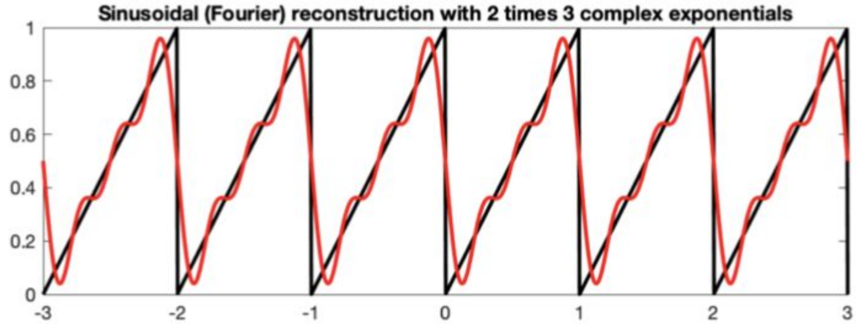
Let's now plot Fourier series fits for our sawtooth signal with one complex exponential frequency.

Legos



Fourier series with 3 complex exponential frequencies...

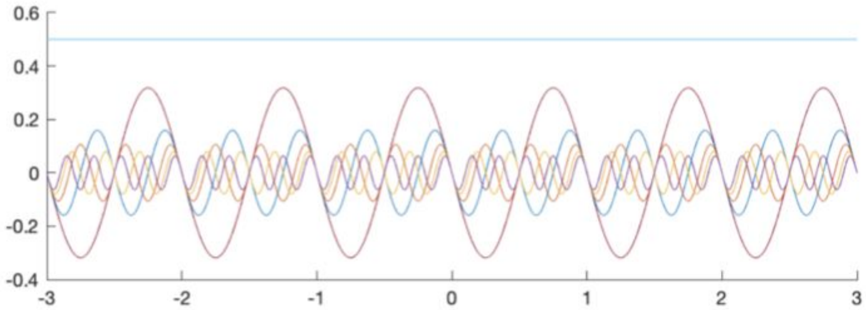


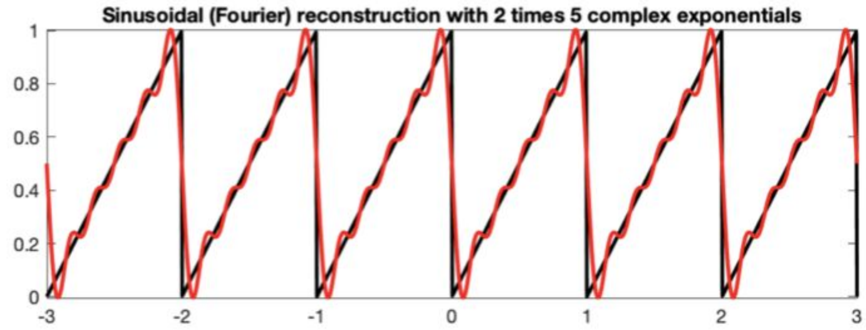


Sawtooth Signal

Fourier series with 5 complex exponential frequencies...

5

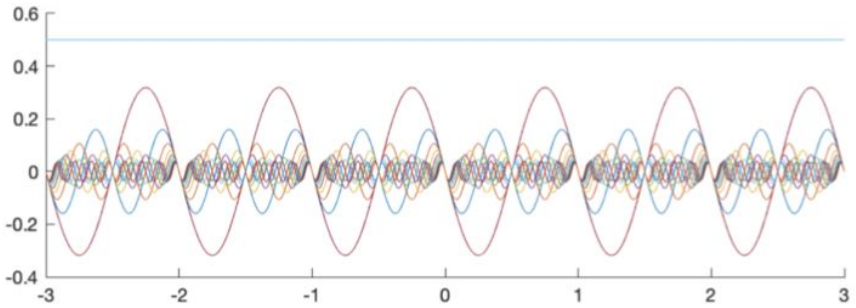


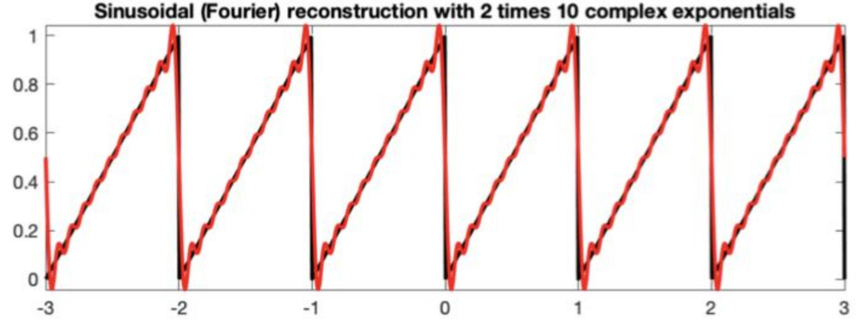


Sawtooth Signal

Fourier series with 10 complex exponential frequencies...

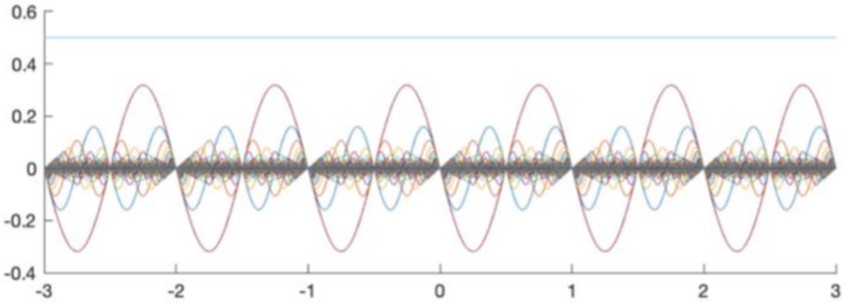
10

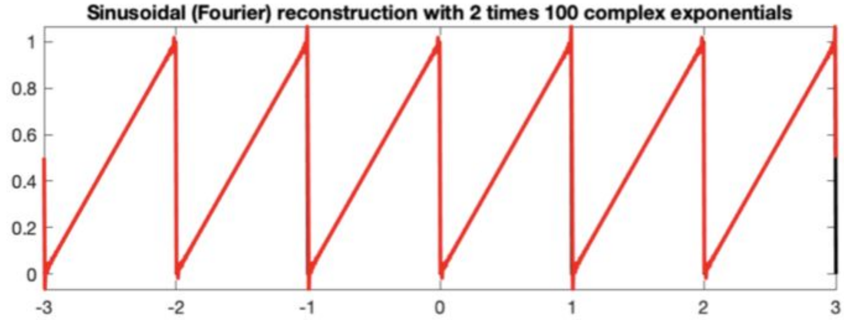




Fourier series with 100 complex exponential frequencies...

100





Cyu:

Equality?

Fourier series convergence?

After our proof, can we definitely say that for

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jk\omega_0 t} dt$$

that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

at every point in time, t?

Equality?

The answer is no. If you notice our derivation, we only showed that this holds when $f(t)$ or its Fourier series representation are in an integral, i.e.,

$$\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}dt \quad = \quad \int_{t_0}^{t_0+T_0}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)e^{-jn\omega_0 t}dt$$

LHS RHS

What this means is that our Fourier series formula really only holds in the sense of the integral average over this period. But it need not be the case that our Fourier series formula holds at exactly every single t. We'll see this in an example.

In plain English: we can establish equality of the integral transform of $f(t)$. This does not imply equality at every time t.

Equality?

Fourier series does not equal $f(t)$ everywhere

As we can see, the Fourier series does not equal $f(t)$ everywhere. However, it does a very reasonable job at fitting the square wave. There is work (we won't cover) on things we observe, like the "ringing" (called Gibbs effect) of the Fourier series at discontinuities. Interestingly, increasing the number of terms compresses the ringing but does not reduce its amplitude. You can see it's still present with $k = 100$ complex exponential frequencies in our square wave example.

There are *Dirichlet conditions* that describe when f can be approximated by a Fourier series. We won't talk about these in depth in class, but essentially, the signal should be "smooth" and "well-behaved" for a Fourier series approximation to be good.

CYU Question

$g(t) = \sum_k c_k' e^{jk\omega_0 t}$

Suppose that $f(t)$ is a periodic signal with period T_0, with the following Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Qst. Suppose $g(t) = f(t) + \varepsilon$.

① Can $g(t)$ be approximated by a F.S.? Yes

② If yes, what is the relation b/w c_k and c_k' where c_k' is the F.S. for $g(t)$?

$c_0' = c_0 + \varepsilon$

$c_k' = c_k$

$$g(t) = \varepsilon + f(t) = \underbrace{\varepsilon + c_0}_{c_0'} + \underbrace{\sum_{k \neq 0} c_k e^{jk\omega_0 t}}_{\text{Not affected}}$$