EE102

Lecture 3

EE102 Announcements

- Syllabus link is tiny.cc/ucla102
- CCLE difficulties, please email help@seas.ucla.edu
- **First Homework due this Friday**
- Pace of course (linear ramp)
- Extra time today after class for homework qst (will be recorded)

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Sinusoids

The most basic signal in this class is the sine or cosine wave. We'll use them *extensively* so it's worth reviewing their properties. By the end of this class, you'll be proficient at manipulating sinusoids.

A cosine is defined by:

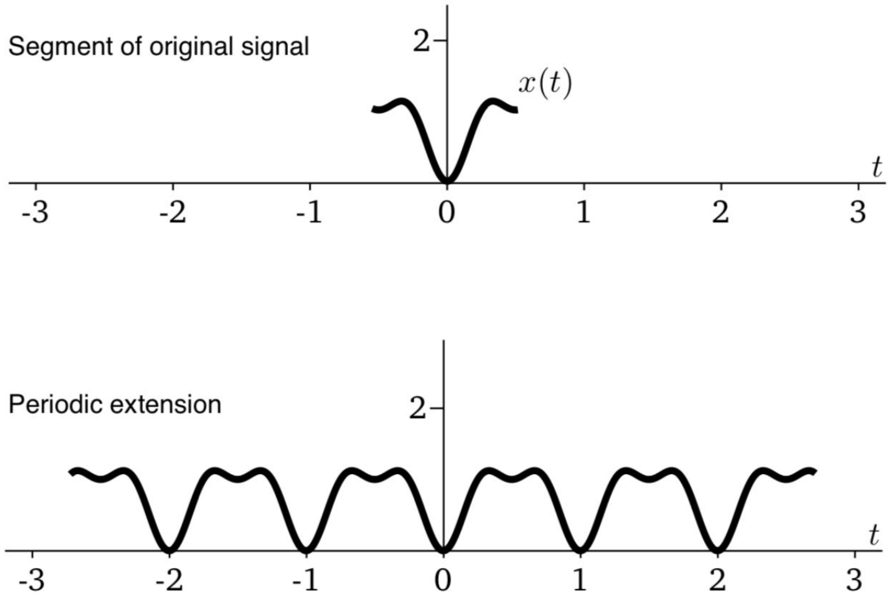
Trigonometric Rules

Some additional properties that you should be familiar with from trigonometry:

- $\sin(\theta) = \cos(\theta - \pi/2)$.
- Are either $\cos(\theta)$ or $\sin(\theta)$ even or odd?
- $\frac{d}{dt}\sin(\theta) = \cos(\theta)$ and $\frac{d}{dt}\cos\theta = -\sin(\theta)$.
- $\sin^2(\theta) + \cos^2(\theta) = 1$.

Periodic Extension

In this class, we will sometimes be interested in taking an aperiodic signal and making its periodic extension. What this mean is that we take some interval on this signal of length T_0 and repeat it, as illustrated below:



CYU Question

Is the sum of the following two signals periodic?

Causality

Complex Numbers Review

So far all signals we've presented are real-valued. But signals can also be complex.

- A complex signal is one that takes the form:

$$z(t) = x(t) + jy(t)$$

where $x(t)$ and $y(t)$ are real-valued signals and $j = \sqrt{-1}$.

Complex Numbers Review

Because complex numbers play a large role in this class, we'll briefly review them.

- A complex number is formed from two real numbers, x and y, via:

$$z = x + jy$$

  with $j = \sqrt{-1}$. Hence, a complex number is simply an ordered pair of real numbers, (x, y).
- $x = \Re(z)$ is called the *real* part of z. (In this class we will also write $x = \mathrm{Re}(z)$.)
- $y = \Im(z)$ is called the *imaginary* part of z. (In this class we will also write $y = \mathrm{Im}(z)$.)

- An aside: why do EE's use j as the imaginary number, while mathematicians and scientists commonly use i?

Complex Numbers Review

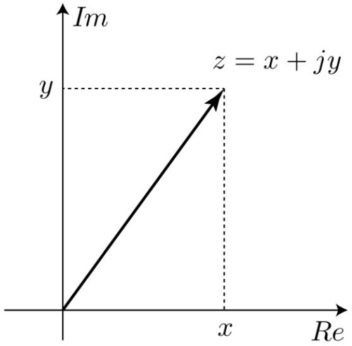
Polar representation of complex numbers

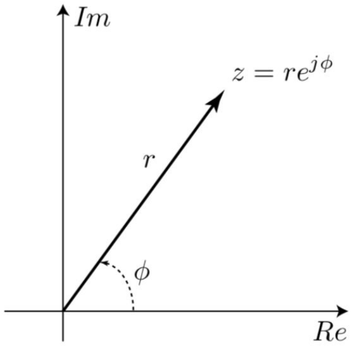
The same complex number can be written in polar form,

$$\begin{aligned} z &= x + jy \\ &= re^{j\phi} \end{aligned}$$

where

- r is the *modulus* or *magnitude* of z.
- ϕ is the *angle* or *phase* of z.
- $e^{j\phi} = \cos(\phi) + j\sin(\phi)$. We will sometimes write this as $\exp(j\phi)$. (More on this below.)





Complex Numbers Review

Cartesian vs polar coordinates

$$
\begin{aligned}
z &= x + jy \\
  &= re^{j\phi}
\end{aligned}
$$

Here, the same intuitions from Cartesian and polar coordinates hold.

- $x = r\cos(\phi)$
- $y = r\sin(\phi)$
- $r = \sqrt{x^2 + y^2}$
- $\phi = \arctan y/x$

Euler's identity

Relating terms in our Cartesian and polar coordinate representation of complex numbers, we arrive at Euler's formula:

$$\begin{aligned} z &= x + jy \\ &= re^{j\phi} \end{aligned}$$

This tells us that, for $r = 1$,

$$\boxed{e^{j\phi} = \cos(\phi) + j\sin(\phi)}$$

Aside: this leads to one of the most elegant equations in mathematics:

$$e^{i\pi} + 1 = 0$$

With five terms, it incorporates Euler's constant (e), pi (π), the imaginary number (i), the multiplicative identity (1) and the additive identity (0).

CYU Question

Complex Conjugate

Some complex relations

Here are a few relations.

- **Complex conjugate**. If $z = x + jy$, then z^*, the complex conjugate of z, is
$$z^* = x - jy$$
- **Modulus and complex conjugate**. The following relation holds:
$$|z|^2 = z^* z = z z^*$$
This is because
$$\begin{aligned} zz^* &= (x+jy)(x-jy) \\ &= x^2 + y^2 \\ &= r^2 \end{aligned}$$
where $r = \sqrt{x^2 + y^2}$ as on the last slide.
- **Inverse of j**. Since $j^2 = -1$, we have that $-j = \frac{1}{j}$.

Signal Energy and Power

Signal Energy and Power

Signal power has units of Watts (Joules per time). Hence, to get the total energy of a signal, $x(t)$, across all time, we integrate the power.

$$\boxed{E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt}$$

(We incorporate the absolute value, $|\cdot|$, in case $x(t)$ is a complex signal, reviewed in the next slides.) Like signal power, signal energy is usually not a *actual* energy.

We can also calculate the *average power* of the signal by calculating:

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}$$

Can we simplify this expression to obtain the power of a periodic signal?

Examples

Finite Energy and Finite Power Signals

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Finite Energy and Finite Power Signals

Causality

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Complex Numbers Review

Phasors

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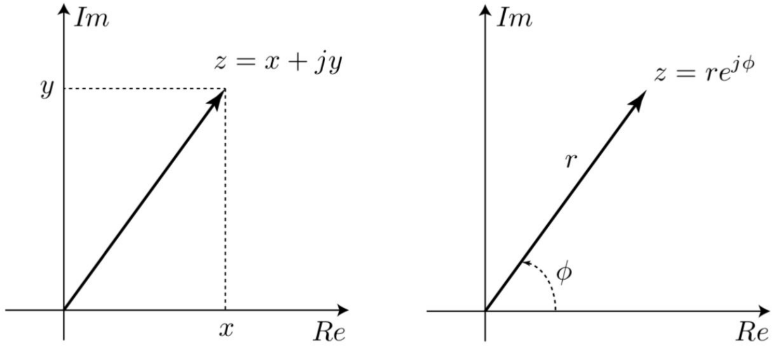

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Check your Understanding (CYU)

Complex Number Addition

Complex Conjugate

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Example using Euler's Formula

Elementary Signal Models

Real sinusoids

We previously discussed the real sinusoid, which we'll recap here for completeness of these notes. A cosine is defined by:

$$
\begin{aligned}
x(t) &= A\cos(\omega t - \theta) \\
&= A\cos(2\pi f t - \theta)
\end{aligned}
$$

with

- A defining the amplitude of the signal (i.e., how large it gets).
- ω defining the *natural* frequency of the signal (in units of radians per second). As ω gets larger, the sinusoid repeats more times in a given time interval.
- The natural frequency is related to the frequency, f, of the signal (in units of Hertz, or s^{-1}) through the relationship: $\omega = 2\pi f$. The frequency, f, is the inverse of the period, i.e.,

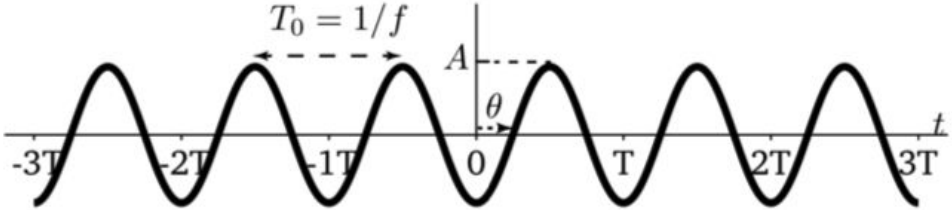
$$T_0 = \frac{1}{f} = \frac{2\pi}{\omega}$$

- θ is the phase of the signal in terms of radians, shifting the sinusoid.

Real sinusoids (cont.)

We illustrate a sinusoid signal below:

$$x(t) = A\cos(\omega t - \theta)$$

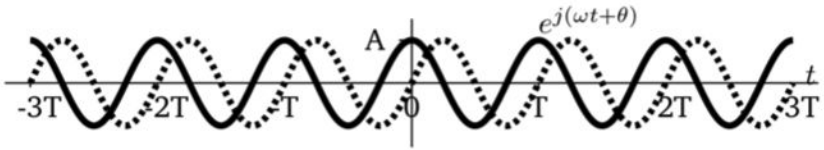


Complex sinusoids

The complex sinusoid is given by:

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + jA\sin(\omega t+\theta)$$

We draw complex signals with dotted lines.



The real part of the complex sinusoid (solid line) is:

$$\Re\left(Ae^{j(\omega t+\theta)}\right) = A\cos(\omega t+\theta)$$

The imaginary part of the complex sinusoid (dotted line) is:

$$\Im\left(Ae^{j(\omega t+\theta)}\right) = A\sin(\omega t+\theta)$$

Exponential

An exponential signal is given by

$$x(t) = e^{\sigma t}$$

- If $\sigma > 0$, this signal grows with increasing t (black signal in plot below). This is called exponential growth.
- If $\sigma < 0$, this signal decays with increasing t (blue signal in plot below). This is called exponential decay.