ECE102, Spring 2020 **Homework #7**

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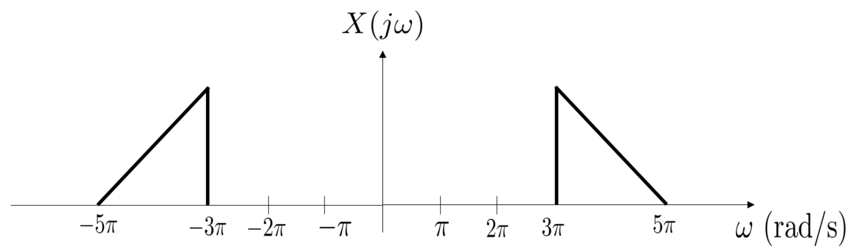
Due Sunday, 31 May 2020, by 11:59pm to CCLE.
100 points total.

1. (16 points) **Bandpass sampling**
The figure below shows the Fourier transform of a real bandpass signal, i.e., a signal whose frequencies are not centered around the origin. We want to sample this signal. Let F_s in Hz



represent the sampling frequency.

(a) (8 points) One option is to sample this signal at the Nyquist rate. Then to recover the signal, we pass its sampled version through a low pass filter. What is the Nyquist rate of this signal? What is the cutoff frequency of the low pass filter?

Solution: The Nyquist rate is $2(5\pi) = 10\pi$ rad/s, or 5 Hz. The cutoff frequency of the filter can be 5π rad/s or 2.5 Hz (In general, the cutoff frequency is chosen to be $F_s/2$ where F_s is the sampling frequency).

(b) (8 points) Since the signal might have high frequency components, Nyquist rate for this signal can be high. In other words, we need to have a lot of samples of the signal, which means that the sampling scheme is costly. It turns out that for this type of signal, we can sample it at a sampling frequency lower that the Nyquist rate and we can still recover the signal, however in this case, we will use a **bandpass** filter. To see this, we have the following two options for the sampling frequency:

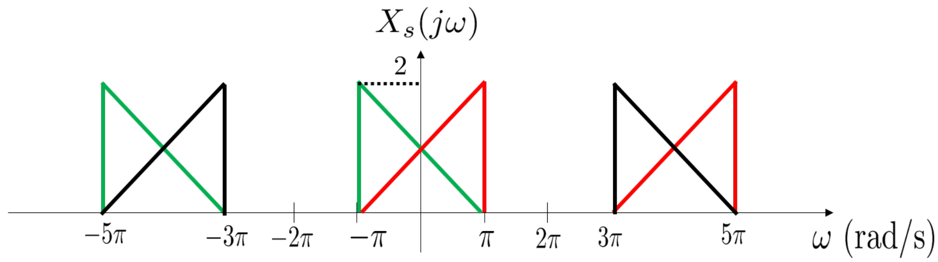
- $F_s = 2$ Hz;
- $F_s = 2.5$ Hz;

For each case, draw the spectrum of the signal after sampling it. To recover the signal, which F_s can we use? How we should choose the frequencies of the bandpass filter? What is the minimum F_s we can use and still recover the signal?

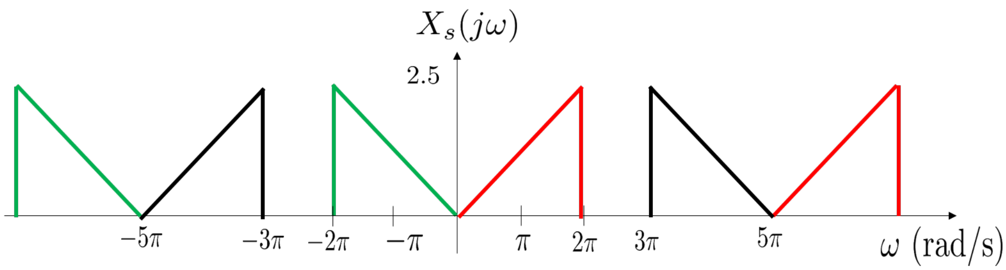
Solution: Sampling a signal at F_s Hz makes its spectrum periodic with period $\omega_s = 2\pi F_s$ rad/s and scales it by $1/T_s = F_s$.

Using $F_s = 2$ Hz creates aliasing in the frequency domain (in the region of interest, for $3\pi \leq |\omega| \leq 5\pi$). On the other hand, using $F_s = 2.5$ Hz, does not create aliasing.

Sampling at $F_s = 2$ Hz (or $\omega_s = 4\pi$ rad/s)

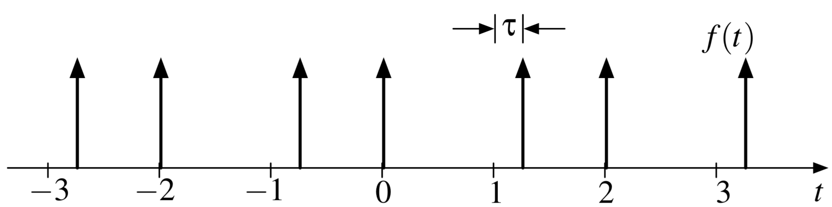


Sampling at $F_s = 2.5$ Hz (or $\omega_s = 5\pi$ rad/s)



To recover the signal, we can use a bandpass filter defined over $3\pi \leq |\omega| \leq 5\pi$. The minimum sampling frequency is $F_s = 2.5$ Hz or $\omega_s = 5\pi$ rad/s. Using any ω_s lower than 5π rad/s, will create aliasing in the frequency domain.

2. (24 points) **Sampling with imperfect sampler**
Imperfections in a sampler cause characteristic artifacts in the sampled signal. In this problem we will look at the case where the sample timing is non-uniform, as shown below: The



sampling function $f(t)$ has its odd samples delayed by a small time τ.

(a) (6 points) Write an expression for $f(t)$ in terms of two uniformly spaced sampling functions.

Solution: The even samples are a δ train separated by 2, with no shift. The odd samples are also a δ train separated by 2, but delayed by $1+\tau$. Adding these together

$$f(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k) + \sum_{k=-\infty}^{\infty} \delta(t-(2k+1+\tau)) = \delta_2(t) + \delta_2(t-(1+\tau))$$

(b) (6 points) Find $F(j\omega)$, the Fourier transform of $f(t)$. Express the impulse trains as sums, and simplify.

Solution: The Fourier transform is:

$$F(j\omega) = \pi\delta_\pi(\omega) + \pi\delta_\pi(\omega)e^{-j\omega(1+\tau)} = \pi\delta_\pi(\omega)(1 + e^{-j\omega(1+\tau)})$$

Where $\omega_0 = \frac{2\pi}{2} = \pi$ for the impulse train. For the impulse train:

$$\begin{aligned}
F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-j\omega(1+\tau)}) \\
&= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-jn\pi(1+\tau)}) \\
&= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-jn\pi}e^{-jn\pi\tau}) \\
&= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau})
\end{aligned}$$

(c) (6 points) Find $F(j\omega)$, for the case where $\tau = 0$, and show that this is what you expect.

Solution: If $\tau = 0$,

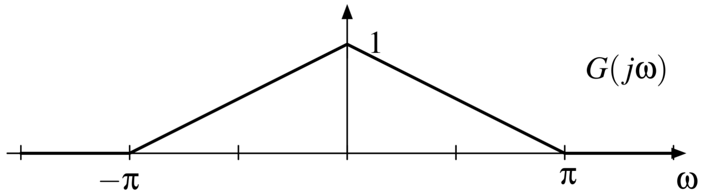
$$\begin{aligned}
F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau}) \\
&= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n)
\end{aligned}$$

The $(1 + (-1)^n)$ term is 2 for n even, and 0 for n odd. Hence, the odd terms drop, and we get a factor of two for the remaining even terms,

$$\begin{aligned}
F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2n\pi)(2) \\
&= 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2n\pi) \\
&= 2\pi\delta_{2\pi}(\omega)
\end{aligned}$$

which is the Fourier transform of $\delta_1(t)$. This is what we expect. As τ goes to zero, we expect that the non-uniform sampling case should go to the uniform case. This will provide a reality check for the next part.

(d) (6 points) Assume the signal we are sampling has a Fourier transform



Sketch the Fourier transform of the sampled signal. Include the baseband replica, and the replicas at $\omega = \pm\pi$. Assume that τ is small, so that $e^{j\omega\tau} \simeq 1 + j\omega\tau$.

Solution: The sampled signal is $f(t)g(t)$, which has a Fourier transform

$$\begin{aligned}
G_s(j\omega) &= \frac{1}{2\pi} F(j\omega) * G(j\omega) \\
&= \frac{1}{2\pi}\left(\pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau})\right) * \Delta(\omega/\pi) \\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty} \Delta\left(\frac{(\omega - n\pi)}{\pi}\right)(1 + (-1)^n e^{-jn\pi\tau})
\end{aligned}$$

We are interested in the baseband replica (n=0) , and the replicas at $\pm\pi (n = \pm 1)$. For $n = 0$,

$$G_{s,0}(j\omega) = \frac{1}{2}\Delta\left(\frac{\omega}{\pi}\right)(1 + 1) = \Delta\left(\frac{\omega}{\pi}\right)$$

which is the same as $G(j\omega)$. For $n = 1$,

$$G_{s,1}(j\omega) = \frac{1}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right)(1 + (-1)^1 e^{-j\pi\tau}) = \Delta\left(\frac{\omega - \pi}{\pi}\right)(1 - e^{-j\pi\tau})$$

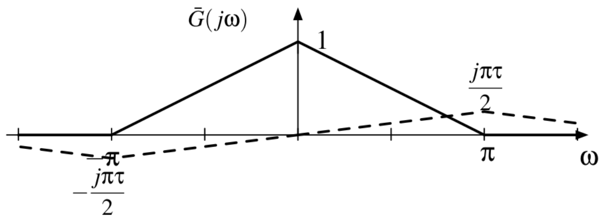
If we approximate $e^{j\omega\tau} \simeq 1 + j\pi\tau$

$$\begin{aligned}
G_{s,1} &= \Delta\left(\frac{\omega - \pi}{\pi}\right)(1 - (1 - j\pi\tau)) \\
&= \frac{j\pi\tau}{2}\Delta\left(\frac{\omega - \pi}{\pi}\right)
\end{aligned}$$

This is a replica of $G(j\omega)$ centered at $\omega = \pi$, multiplied by $j\pi\tau/2$. It is imaginary, and proportional to τ that as τ goes to zero, this replica disappears as we'd expect.
For $n = -1$ we get the same type of term, but with the negative sign,

$$G_{s,-1}(j\omega) = \frac{-j\pi\tau}{2}\Delta\left(\frac{\omega + \pi}{\pi}\right)$$

This is a replica of $G(j\omega)$ centered at $\omega = -\pi$, and scaled by $-j\pi\tau/2$.
If we sketch these three terms, the result is as shown below. .



3. (20 points) **Laplace Transform**

(a) (12 points) Find the Laplace transforms of the following signals and determine their region of convergence.

i. $f(t) = te^{-at}(\sin\omega_0 t)^2 u(t)$

Solution: We can equivalently write $f(t)$ as:

$$f(t) = te^{-at}(\sin\omega_0 t)^2 u(t) = te^{-at}\frac{1-\cos(2\omega_0 t)}{2}u(t) = \frac{1}{2}te^{-at}u(t) - \frac{1}{2}te^{-at}\cos(2\omega_0 t)u(t)$$

We have the following:

$$\begin{aligned}
\cos(2\omega_0 t)u(t) &\to \frac{s}{s^2+(2\omega_0)^2} \\
t\cos(2\omega_0 t)u(t) &\to -\frac{d}{ds}\frac{s}{s^2+(2\omega_0)^2} = -\frac{s^2+4\omega_0^2-2s^2}{(s^2+4\omega_0^2)^2} = \frac{s^2-4\omega_0^2}{(s^2+4\omega_0^2)^2} \\
e^{-at}t\cos(2\omega_0 t)u(t) &\to \frac{(s+a)^2-4\omega_0^2}{((s+a)^2+4\omega_0^2)^2}
\end{aligned}$$

Therefore,

$$F(s) = \frac{1}{2}\frac{1}{(s+a)^2} - \frac{1}{2}\frac{(s+a)^2-4\omega_0^2}{((s+a)^2+4\omega_0^2)^2}, \quad \mathrm{Re}\{s\} > -a$$

ii. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ e^{-3(t-2)} & 2 \le t \end{cases}$

Solution: We can equivalently write $f(t)$ as follows:

$$f(t) = u(t-1) - u(t-2) + e^{-3(t-2)}u(t-2)$$

Therefore,

$$F(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s+3}, \quad \mathrm{Re}\{s\} > -3$$

The ROC of this signal is $\mathrm{Re}\{s\} > -2$, because $f(t)$ can be thought of the sum of a time-limited signal $u(t-1) - u(t-2)$ and a non time-limited signal $e^{-3(t-2)}u(t-2)$. Since the ROC of a time-limited signal is the entire s-plane, the ROC of $f(t)$ is the ROC of $e^{-3(t-2)}u(t-2)$ which is $\mathrm{Re}\{s\} > -3$.

(b) (8 points) The Laplace transform of a causal signal $x(t)$ is given by

$$X(s) = \frac{1}{s^2 + 2s + 5}, \quad \text{ROC: } \text{Re}\{s\} > -1$$

Which of the following Fourier transforms can be obtained from $X(s)$ without actually determining the signal $x(t)$? In each case, either determine the indicated Fourier transform or explain why it cannot be determined.

i. $\mathcal{F}\{x(t)e^{-t}\}$

Solution: Let $y(t) = x(t)e^{-t}$, then $Y(s) = X(s+1)$, and the ROC for $Y(s)$ is:

$$\text{Re}\{s+1\} > -1 \implies \text{Re}\{s\} > -2$$

Since the ROC of $Y(s)$ includes the $j\omega$-axis, we have:

$$Y(j\omega) = Y(s)|_{s=j\omega} = \frac{1}{(j\omega+1)^2 + 2(j\omega+1) + 5}$$

ii. $\mathcal{F}\{x(t)e^{3t}\}$ Let $y(t) = x(t)e^{3t}$, then $Y(s) = X(s-3)$, and the ROC for $Y(s)$ is:

$$\text{Re}\{s-3\} > -1 \implies \text{Re}\{s\} > 2$$

Since the ROC of $Y(s)$ does not include the $j\omega$-axis, we cannot determine $Y(j\omega)$ from $Y(s)$.

4. (20 points) **Inverse Laplace Transform**

Find the inverse Laplace transform $f(t)$ for each of the following $F(s)$: ($f(t)$ is a causal signal)

(a) (6 points) $F(s) = \dfrac{e^{-s}(s+3)}{(s-1)^2(s-2)}$

Solution: Let us first focus on $\frac{(s+3)}{(s-1)^2(s-2)}$. It can be equivalently written as:

$$\frac{s+3}{(s-1)^2(s-2)} = \frac{r_1}{(s-1)^2} + \frac{r_2}{s-1} + \frac{r_3}{s-2}$$

Using the cover-up method,

$$r_1 = \left.\frac{s+3}{s-2}\right|_{s=1} = \frac{4}{-1} = -4$$
$$r_3 = \left.\frac{s+3}{(s-1)^2}\right|_{s=2} = \frac{5}{1} = 5$$

Therefore,

$$\frac{s+3}{(s-1)^2(s-2)} = \frac{-4}{(s-1)^2} + \frac{r_2}{s-1} + \frac{5}{s-2}$$

Now to find r_2, we will evaluate the above equation at $s=0$, we then have:

$$\frac{3}{-2}=\frac{-4}{1}+\frac{r_2}{-1}+\frac{5}{-3}$$

Then

$$r_2=\frac{3}{2}-\frac{8}{2}-\frac{5}{2}=\frac{3-8-5}{2}=-5 \implies r_2=-5$$

Therefore,

$$F(s)=e^{-s}\left(-\frac{4}{(s-1)^2}-\frac{5}{s-1}+\frac{5}{s-3}\right)$$

then

$$f(t)=\left(-4(t-1)e^{(t-1)}-5e^{(t-1)}+5e^{3(t-1)}\right)u(t-1)$$

(b) (6 points) $F\left(s\right)=\dfrac{s+4}{s^3+4s}$

Solution:

$$F(s)=\frac{s+4}{s(s^2+4)}=\frac{r_1}{s}+\frac{r_2}{s+j2}+\frac{r_3}{s-j2}$$

Using the cover-up procedure:

$$r_1=\left.\frac{s+4}{(s^2+4)}\right|_{s=0}=\frac{4}{4}=1$$

and

$$r_2=\left.\frac{s+4}{s(s-j2)}\right|_{s=-j2}=\frac{-2j+4}{(-j2)(-j2-j2)}=\frac{j-2}{4}$$

Then,

$$r_3=r_2^*=\frac{-j-2}{4}$$

We thus have

$$F(s)=\frac{s+4}{s^3+4s}=\frac{1}{s}+\frac{1}{4}\frac{j-2}{s+j2}-\frac{1}{4}\frac{j+2}{s-j2}$$

The inverse Laplace transform is:

$$f(t)=\left(1+\frac{1}{4}(j-2)(e^{-j2t}-\frac{1}{4}(j+2)e^{j2t})\right)u(t)=\left(1+0.5\sin(2t)-\cos(2t)\right)u(t)$$

Alternatively, we can find the inverse as follows:

$$F(s)=\frac{s+4}{s(s^2+4)}=\frac{r_1}{s}+\frac{As+B}{s^2+4}$$

Using the cover-up method, we can determine $r_1=1$ (as previously obtained). Therefore,

$$As+B=\frac{s+4}{s}-\frac{s^2+4}{s}=\frac{s-s^2}{s}=1-s$$

Therefore,

$$F(s) = \frac{1}{s} - \frac{s-1}{s^2+4} = \frac{1}{s} - \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}$$

Therefore,

$$f(t) = \left(1 - \cos(2t) + \frac{1}{2}\sin(2t)\right)u(t)$$

(c) (8 points) $F(s) = \dfrac{1}{(s+1)(s^2+2s+2)}$

Solution:

$$\begin{aligned} F(s) &= \frac{1}{(s+1)((s+1)^2+1)} = \frac{1}{(s+1)(s+1+j)(s+1-j)} \\ &= \frac{r_1}{s+1} + \frac{r_2}{s+1+j} + \frac{r_3}{s+1-j} \end{aligned}$$

Using the cover-up procedure,

$$r_1 = \left.\frac{1}{s^2+2s+2}\right|_{s=-1} = 1$$

and

$$r_2 = \left.\frac{1}{(s+1)(s+1-j)}\right|_{s=-1-j} = \frac{1}{-j(-2j)} = -\frac{1}{2}$$

Thus,

$$r_3 = r_2^* = -\frac{1}{2}$$

Therefore,

$$f(t) = \left(e^{-t} - \frac{1}{2}e^{-(1+j)t} - \frac{1}{2}e^{-(1-j)t}\right)u(t) = e^{-t}\left(1 - \cos(t)\right)u(t)$$

Alternatively, we can find the inverse as follows:

$$F(s) = \frac{1}{(s+1)((s+1)^2+1)} = \frac{r_1}{s+1} + \frac{As+B}{(s+1)^2+1}$$

where $r_1 = 1$ and

$$As + B = \frac{1}{s+1} - \frac{(s+1)^2+1}{s+1} = -(s+1)$$

Therefore,

$$F(s) = \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}$$

Then,

$$f(t) = e^{-t}\left(1 - \cos(t)\right)u(t)$$

5. **LTI system** (20 points)

Assume a causal LTI system $\mathcal{S}_1$ is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = ax(t), \qquad y(0) = 0,\ y'(0) = 0$$

where a is a constant. Moreover, we know that when the input is e^t, the output of the system $\mathcal{S}_1$ is $\frac{1}{2}e^t$.

(a) (6 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

Solution: We know that $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$ and $\mathcal{L}\{y'\} = sY(s) - y(0)$, and here we notice that $y(0) = 0$ and $y'(0) = 0$.
Taking the Laplace transform of above differential equation:

$$s^2Y(s) + 3sY(s) + 2Y(s) = aX(s) \implies H_1(s) = \frac{Y(s)}{X(s)} = \frac{a}{s^2+3s+2} = \frac{a}{(s+1)(s+2)}$$

Eigenfunction property:

$$x(t) = e^{st} \to y(t) = H(s)e^{st} = |H(s)|e^{st+j(\angle H(s))}$$

Since the output to e^t is $\frac{1}{2}e^t$, then using the eigenfunction property we have:

$$H_1(s)|_{s=1} = \frac{1}{2}$$

We thus conclude:

$$\frac{a}{2*3} = \frac{1}{2} \implies a = 3$$

We then conclude:

$$H_1(s) = \frac{3}{(s+1)(s+2)}$$

(b) (8 points) Find the output $y(t)$ when the input is $x(t) = u(t)$.

Solution: The Laplace transform of $y(t)$ is given by

$$Y(s) = H_1(s)X(s) = \frac{3}{(s+1)(s+2)s} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

where

$$r_1 = \left.\frac{3}{(s+1)(s+2)}\right|_{s=0} = \frac{3}{2}$$
$$r_2 = \left.\frac{3}{s(s+2)}\right|_{s=-1} = -3$$
$$r_3 = \left.\frac{3}{s(s+1)}\right|_{s=-2} = \frac{3}{2}$$

Therefore,

$$y(t) = \left(\frac{3}{2} - 3e^{-t} + \frac{3}{2}e^{-2t}\right)u(t)$$

(c) (6 points)The system $\mathcal{S}_1$ is linearly cascaded with another causal LTI system $\mathcal{S}_2$. The system $\mathcal{S}_2$ is given by the following input-output pair:

$$\mathcal{S}_2 \quad \text{input} : u(t) \to \text{output} : r(t)$$

Find the overall impulse response.

Solution: The impulse response of the system $\mathcal{S}_2$ is:

$$H_2(s) = \frac{\mathcal{L}(r(t))}{\mathcal{L}(u(t))} = \frac{\frac{1}{s^2}}{\frac{1}{s}} = \frac{1}{s}$$

Therefore, the overall trasnfer function is given by:

$$\begin{aligned}
H(s) &= H_1(s)H_2(s) \\
&= \left(\frac{3}{2}\frac{1}{s} - 3\frac{1}{s+1} + \frac{3}{2}\frac{1}{s+2}\right)\frac{1}{s} \\
&= \frac{3}{2}\frac{1}{s^2} - \frac{3}{(s+1)s} + \frac{3}{2}\frac{1}{(s+2)s} \\
&= \frac{3}{2}\frac{1}{s^2} - 3\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{3}{4}\left(\frac{1}{s} - \frac{1}{s+2}\right)
\end{aligned}$$

Therefore, $h(t) = \frac{3}{2}t - \frac{9}{4} - 3e^{-t} - \frac{3}{4}e^{-2t}$