ECE102, Spring 2020 | **Homework #1**

Signals & Systems | Prof. A. Kadambi

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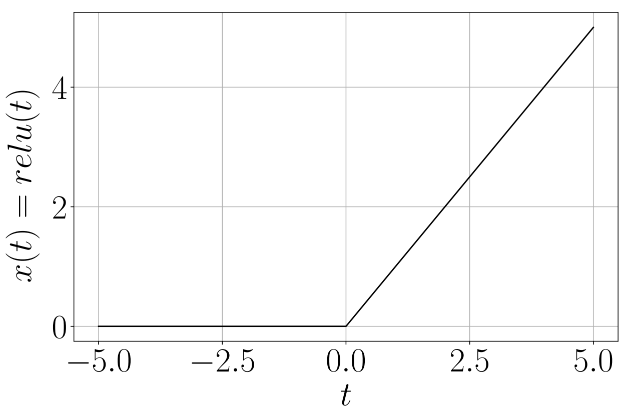
Due Friday, 10 Apr 2020, by 11:59pm to CCLE.
Covers material up to Lecture 3.
100 points total.

1. (10 points) **Even and odd parts.**
   Sketch and write the even and odd components of the following signal:

$$x(t) = relu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

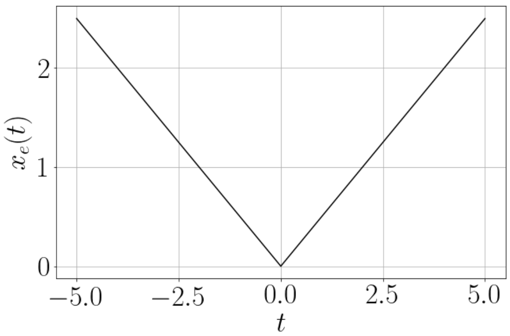


Solutions:
Using the expressions of the even and odd parts,

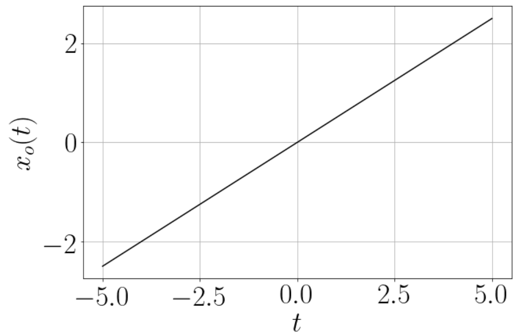
$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$
$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

we can construct the even and odd components of $x(t)$:

$$x_e(t) = \frac{1}{2}|t|$$
$$x_0(t) = \frac{1}{2}t$$



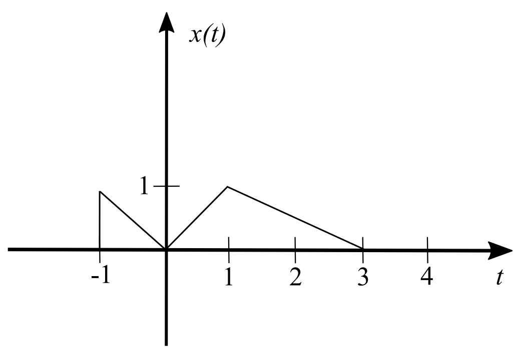
Even component of relu(t)



Odd component of relu(t)

2. (15 points) **Time scaling and shifting.**

   (a) (10 points) Consider the following signal.

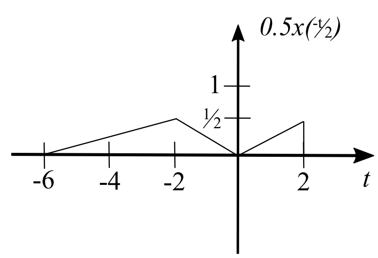


Sketch the following:

i. $\frac{1}{2}x(-t/2)$

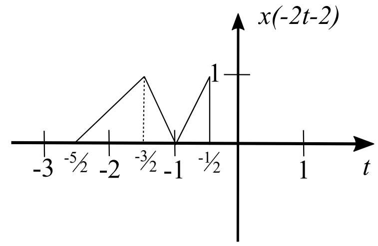
ii. $x(-2t-2)$

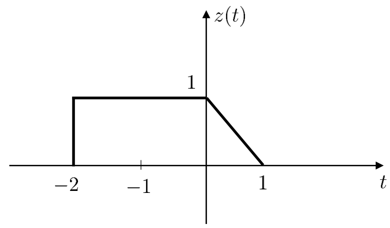
Solutions:

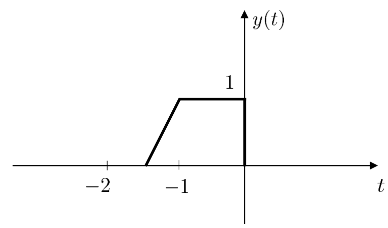


i.

ii.







(b) (5 points) The figure below shows two signals: $z(t)$ and $y(t)$. Express (i) $y(t)$ in terms of $z(t)$, and (ii) $z(t)$ in terms of $y(t)$

Solutions:

$y(t) = z(-2t - 2)$

$z(t) = y(-\frac{1}{2}t - 1)$

3. (20 points) **Periodic signals.**

(a) (15 points) For each of the following signals, determine whether it is periodic or not. If the signal is periodic, determine the fundamental period and frequency.

i. $x_1(t) = \sin(4t/5 + \pi/3)$
ii. $x_2(t) = \cos^2(2\pi t)$
iii. $x_3(t) = x_1(t) + x_2(t)$
iv. $x_4(t) = e^t x_1(t)$
v. $x_5(t) = e^{j(\pi t+1)} x_2(t)$

Solutions:

i. The signal is periodic with period is $2\pi/(4/5) = 5\pi/2$ sec and the frequency is $2/(5\pi)$ Hz.

ii. $x_2(t) = \cos^2(2\pi t) = \frac{1}{2}(1 + \cos(2(2\pi t))) = \frac{1}{2}(1 + \cos(4\pi t))$, therefore the signal is periodic with period $2\pi/(4\pi) = 1/2$ sec, and the frequency is 2 Hz.

iii. $x_3(t) = x_1(t) + x_2(t)$: let T_1 denote the period of $x_1(t)$ and T_2 the period of T_2. If we can find integers m and n such that $mT_1 = nT_2$, $x_3(t)$ will then be periodic with period $T_3 = mT_1 = nT_2$. In other words, the ratio

$$\frac{T_1}{T_2} = \frac{n}{m}$$

need to be rational for $x_3(t)$ to be periodic. However, we have from part (i) $T_1 = \frac{5\pi}{2}$ and from part (ii) $T_2 = 1/2$, so that

$$\frac{T_1}{T_2} = 5\pi$$

The ratio is not rational. Hence, $x_3(t)$ is not periodic.

iv. $x_4(t) = e^t x_1(t)$: this signal is not periodic since its magnitude increases exponentially.

v. $x_5(t) = e^{j(\pi t+1)} x_2(t) = e^{j(\pi t+1)} \times (1+\cos(4\pi t)) = e^{j(\pi t+1)} \times (1 + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right))$. Therefore, $x_5(t)$ can be equivalently written as:

$$x_5(t) = \frac{1}{2} e^j \left(2e^{j\pi t} + e^{j5\pi t} + e^{-j3\pi t}\right)$$

The term $2e^{j\pi t}$ is periodic with period 2 sec. The second term $e^{j5\pi t}$ is periodic with period 2/5. The last term $e^{-j3\pi t}$ is periodic with period 2/3. Since the ratio of any two periods is rational, $x_5(t)$ is periodic with fundamental period of 2 sec, and the frequency is 1/2 Hz.

(b) (5 points) A signal $y(t)$ is periodic with period T_0, and is the sum of two other signals.

$$y(t) = x_1(t) + x_2(t)$$

Must $x_1(t)$ and $x_2(t)$ both be periodic?
What if y(t) = $x_1(t) \times x_2(t)$?

Solutions:
No. For example if $a(t)$ is not periodic and $b(t)$ is periodic, let

$$x_1(t) = a(t) + b(t)$$

$$x_2(t) = b(t) - a(t)$$

Then neither $x_1(t)$ or $x_2(t)$ is periodic, but

$$y(t) = x_1(t) + x_2(t) = (a(t) + b(t)) + (b(t) - a(t)) = 2b(t)$$

is periodic.
Similarly, if $x_1(t) = a(t) \times b(t)$, and $x_2(t) = b(t)/a(t)$, $x_1(t) \times x_2(t) = b^2(t)$, which could be $b(t) = cos(t)$

4. (25 points) **Energy and power signals.**

   (a) (10 points) Determine whether the following signals are energy or power signals. If the signal is an energy signal, determine its energy. If the signal is a power signal, determine its power.

   i. $x(t) = e^{-|t|}$
   ii. $x(t) = 1 + e^{-|t|}$

   Solutions:

   i. $x(t) = e^{-|t|}$

   The energy is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| e^{-|t|} \right|^2 dt = \int_{-\infty}^{\infty} e^{-2|t|} dt = \int_{0}^{\infty} e^{-2t} dt + \int_{-\infty}^{0} e^{2t} dt$$
$$= 2 \int_{0}^{\infty} e^{-2t} dt = -e^{-2t} \Big|_{t=0}^{\infty} = 1$$

   Therefore it's a energy signal. Its power is then 0.

   ii. $x(t) = 1 + e^{-|t|}$

   The energy is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| 1 + e^{-|t|} \right|^2 dt = \int_{-\infty}^{\infty} 1 + 2e^{-|t|} + e^{-2|t|} dt = \infty$$

   Therefore it's not a energy signal.

On the other hand the power is given by:

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (1+e^{-|t|})^2 dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\left(\int_{0}^{T} (1+e^{-t})^2 dt + \int_{-T}^{0} (1+e^{t})^2 dt\right) = 2\lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} (1+e^{-t})^2 dt \\
&= 2\lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} \left(1+2e^{-t}+e^{-2t}\right) dt \\
&= 2\lim_{T\to\infty} \frac{1}{2T}\left(T - 2(e^{-T}+1) - \frac{1}{2}(e^{-2T}-1)\right) \\
&= 2\lim_{T\to\infty} \frac{1}{2T} T = 1
\end{aligned}
$$

(b) (15 points) Show the following two properties:

- If $x(t)$ is an even signal and $y(t)$ is an odd signal, then $x(t)y(t)$ is an odd signal;
- If $z(t)$ is an odd signal, then for any $\tau > 0$ we have:

$$\int_{-\tau}^{\tau} z(t)dt = 0$$

Use these two properties to show that the energy of $x(t)$ is the sum of the energy of its even component $x_e(t)$ and the energy of its odd component $x_o(t)$, i.e.,

$$E_x = E_{x_e} + E_{x_o}$$

Assume $x(t)$ is a real signal.

Solutions:
First property: $x(-t)y(-t) = x(t)(-y(t)) = -x(t)y(t)$, therefore it's odd.
Second property:

$$\int_{-\tau}^{\tau} z(t)dt = \int_{-\tau}^{0} z(t)dt + \int_{0}^{\tau} z(t)dt$$

We apply to the first integral the following variable change: $t = -\lambda$.

$$\int_{-\tau}^{\tau} z(t)dt = -\int_{\tau}^{0} z(-\lambda)d\lambda + \int_{0}^{\tau} z(t)dt$$

We then change the order of the limits of the first integral:

$$\int_{-\tau}^{\tau} z(t)dt = \int_{0}^{\tau} z(-\lambda)d\lambda + \int_{0}^{\tau} z(t)dt$$

Since $z(t)$ is an odd signal, we then have $z(-\lambda) = -z(\lambda)$. Thus,

$$\int_{-\tau}^{\tau} z(t)dt = -\int_{0}^{\tau} z(\lambda)d\lambda + \int_{0}^{\tau} z(t)dt = 0$$

The energy of signal $x(t)$ is given by:

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x_e(t) + x_o(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} \left(x_e^2(t) + x_o^2(t) + 2x_e(t)x_o(t) \right) dt \\
&= \int_{-\infty}^{\infty} x_e^2(t) dt + \int_{-\infty}^{\infty} x_o^2(t) dt = E_e + E_o
\end{aligned}
$$

This is because $2x_e(t)x_o(t)$ is odd, therefore its integral is zero (according to the second property).

5. (16 points) **Euler's identity and complex numbers.**

   (a) (8 points) Use Euler's formula to prove the following identities:

      i. $\cos^2(\theta) + \sin^2(\theta) = 1$

      ii. $\cos(\theta + \psi) = \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)$

   (b) (8 points) $x(t) = (1 + \sqrt{3}j)e^{j(t+2)}$ and $y(t) = \frac{1}{1-j}$.

      i. Compute the real and imaginary parts of $x(t)$ and $y(t)$.

      ii. Compute the magnitude and phase of $x(t)$ and $y(t)$.

Solutions:
(a)

i. $e^{j\theta} = cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = cos(\theta) - j\sin(\theta)$.
Thus, $(cos(\theta) + j\sin(\theta))(cos(\theta) - j\sin(\theta)) = \cos^2(\theta) + \sin^2(\theta) = e^{j\theta} \times e^{-j\theta} = 1$

ii. $\cos(\theta) = (e^{j\theta} + e^{-j\theta})/2$
$\sin(\theta) = (e^{j\theta} - e^{-j\theta})/2j$
$\cos(\theta) \times \cos(\psi) = (e^{j(\theta+\psi)} + e^{-j(\theta+\psi)} + e^{j(\theta-\psi)} + e^{j(\psi-\theta)})/4$
$\sin(\theta) \times \sin(\psi) = (-e^{j(\theta+\psi)} - e^{-j(\theta+\psi)} + e^{j(\theta-\psi)} + e^{j(\psi-\theta)})/(-4)$
$Thus, \cos(\theta) \times \cos(\psi) + \sin(\theta) \times \sin(\psi) = (e^{j(\theta+\psi)} + e^{-j(\theta+\psi)})/2 = \cos(\theta + \psi)$

i. $x(t) = (1+\sqrt{3}j)e^{j(t+2)} = (1+\sqrt{3}j)(\cos(t+2)+j\sin(t+2)) = \cos(t+2)-\sqrt{3}\sin(t+2) + j(\sqrt{3}\cos(t+2) + \sin(t+2))$.
Therefore, the real part is: $\cos(t+2)-\sqrt{3}\sin(t+2)$. The imaginary part is:$\sqrt{3}\cos(t+2) + \sin(t+2)$

$y(t) = 1/(1-j) = \frac{1+j}{(1-j)(1+j)} = 1/2 + 1/2j$, with the real part and imaginary part being 1/2

ii. $x(t) = (1+\sqrt{3}j)e^{j(t+2)} = 2e^{j\pi/3}e^{j(t+2)} = 2e^{j(t+2+\pi/3)}$. Therefore the magnitude is: 2 and the phase is $(t + 2 + \pi/3)$ rad.

$y(t) = (1/\sqrt{2}) \times (\sqrt{2}/2 + \sqrt{2}/2j) = \frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}$, yielding a magnitude of $1/\sqrt{2}$, and a phase of $\pi/4$ rad

6. (14 points) **MATLAB tasks**
For this question, please include all relevant code in text format. For plots, please include axis labels and preferably include a grid.

(a) (5 points) **Task 1**
Plot the waveform

$$x(t) = e^{-t}\cos(2\pi t)$$

for $-10 \le t \le 10$, with a step size of 0.2.

Solutions:
The code is:

```
t=-10:0.2:10;
x=exp(-t).*cos(2*pi*t);
plot(t,x);
grid on;
title('Plot of x(t)=e^{-t}cos(2\pit)'); xlabel('t(sec)');ylabel('x(t)');
```

The code generates the plot shown in Fig. 1.

(b) (4 points) **Task 2**

Create a function `relu(t)` that implements the function from Question 1. You will need to create a file called "relu.m" containing:

```
function out = relu(t)
out = 0; %replace this line with the appropriate implementation of the
%relu function.
end
```

Then plot the function for $-5 \leq t \leq 5$, with a step size of 0.1.

Solutions:

In file relu.m:

```
function out = relu(t)
out = max(0, t);
end
```

Then run:

```
t = -5:0.1:5;
plot(t, relu(t));
xlabel('t');
ylabel('relu(t)')
grid;
```

The code generates the plot shown in Fig. 2.

(c) (5 points) **Task 3**

Create functions `even(t, f)` and `odd(t, f)` that take inputs time `t` and function (handle) `f` that compute the respective even and odd parts of `f(t)` at points t.

For example, the square of a function could be implemented in a file square.m as:

```
function out = square(t, f)
out = f(t).^2;
end
```

and run as:

```
t = -10:0.5:10;
y = square(t, @relu);
```

where `@relu` is called a function handle of the function `relu`, and is necessary for passing a function as input to another function.
Running `plot(t, y); grid;` yields the result:

For this question, plot the even and odd components of $relu(t)$ for $-5 \leq t \leq 5$, with a step size of 0.1 using the functions `even(t, f)` and `odd(t, f)`. Feel free to also define and play around with arbitrary functions to look at their even and odd components.

Solutions:

In file even.m:

```
function out = even(t, f)
out = 0.5*f(t) + 0.5*f(-t);
end
```

In file odd.m:

```
function out = odd(t, f)
out = 0.5*f(t) - 0.5*f(-t);
end
```

Command line code:

```
t = -5:0.1:5;
figure;
plot(t, even(t, @relu));
xlabel('t');
ylabel('even part of relu(t)');

t = -5:0.1:5;
figure;
plot(t, odd(t, @relu));
xlabel('t');
ylabel('odd part of relu(t)');
```

This code generates the plots shown in Fig. 3 and 4.

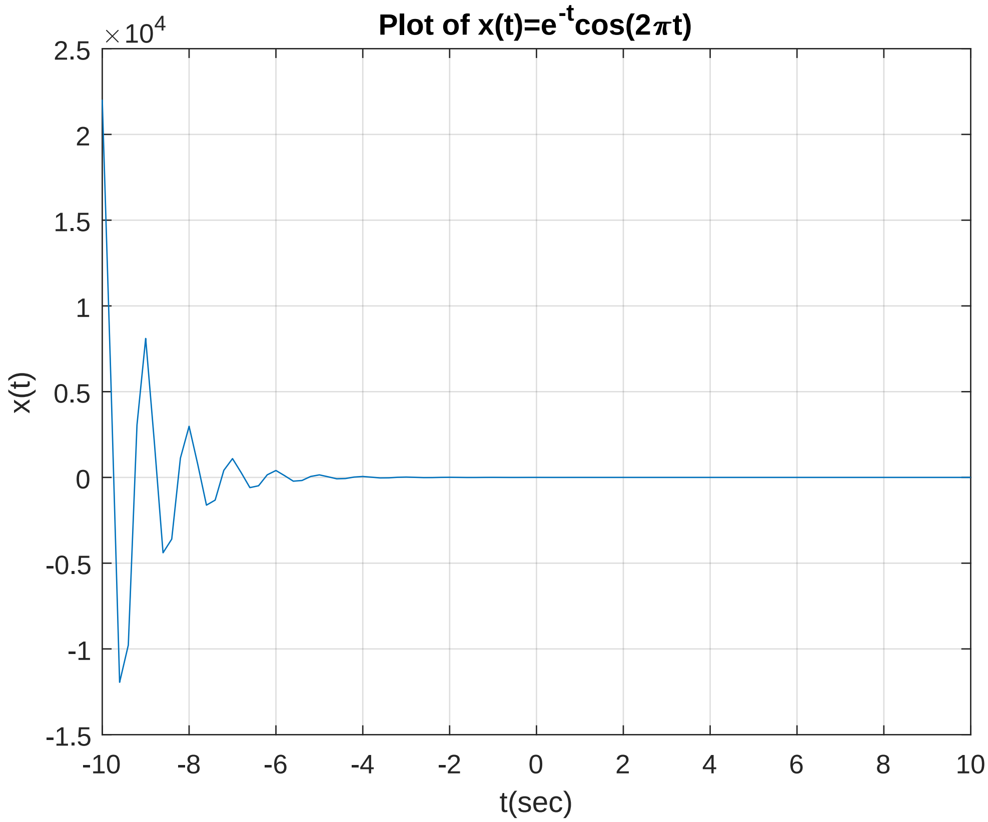
(Note: you can also have a function even_func:

```
function outfunc = even_func(f)
outfunc = @(t)0.5*f(t) + 0.5*f(-t);
end
```

which you can run like so:

```
ef = even_func(@relu)
plot(t, ef(t))
```

which returns the even component of f(t) and you can similarly construct an odd function which return the odd component of f(t)).


Plot of x(t)=e^-t cos(2πt)
×10^4
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
x(t)
-10
-8
-6
-4
-2
0
2
4
6
8
10
t(sec)

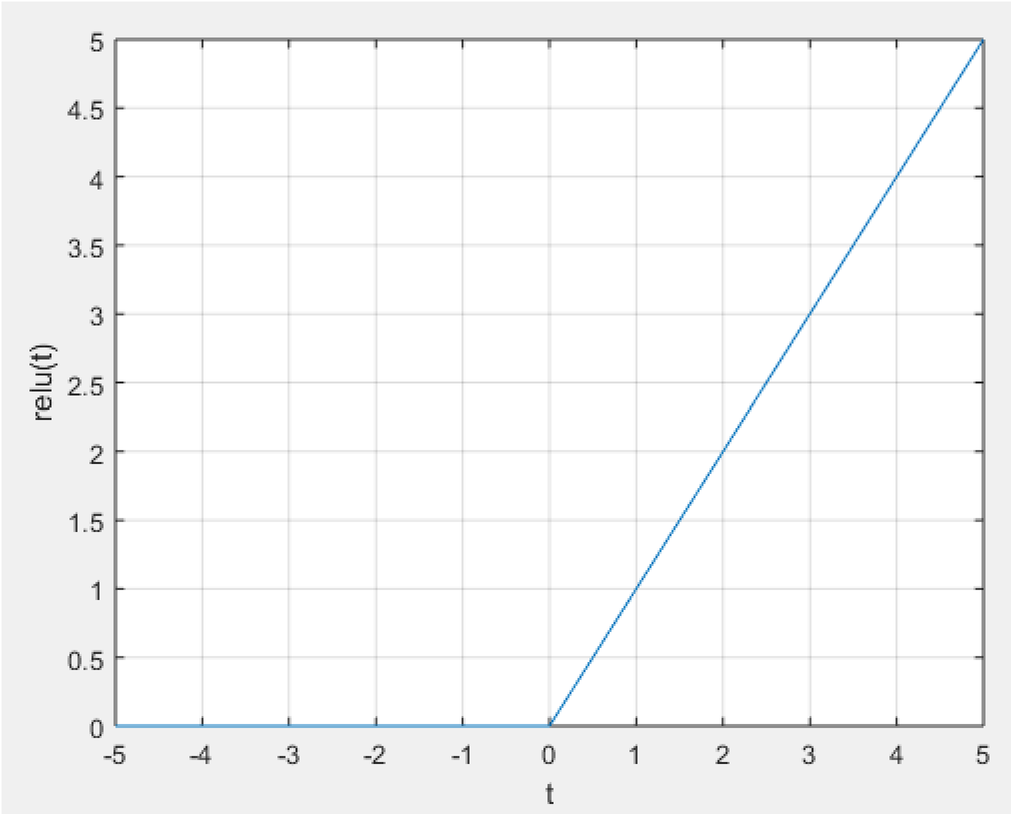
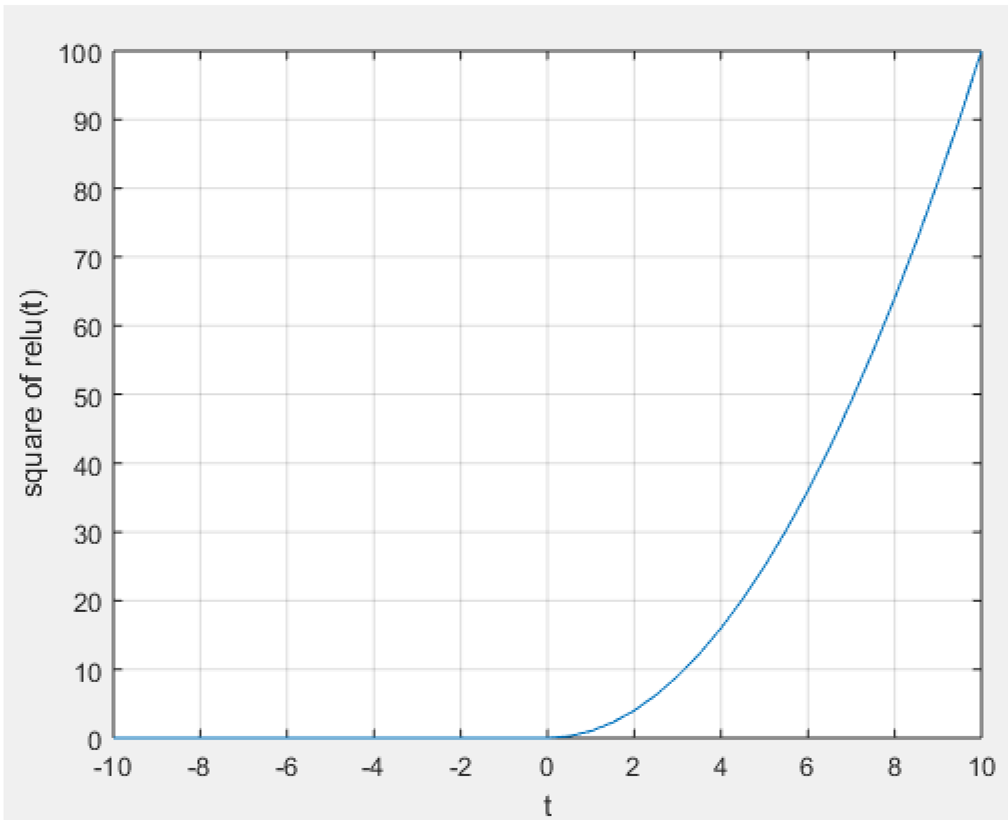


Figure 2: Task 2


100
90
80
70
60
50
40
30
20
10
0
square of relu(t)
-10
-8
-6
-4
-2
0
2
4
6
8
10
t

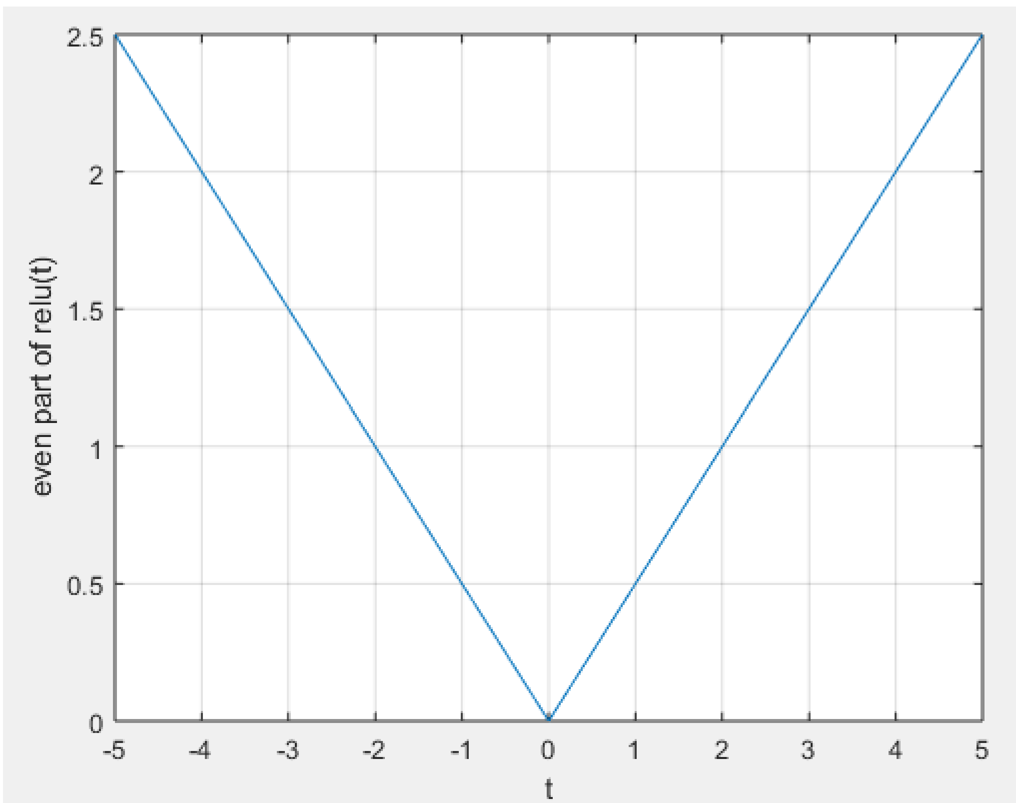


Figure 3: Even component of relu(t)

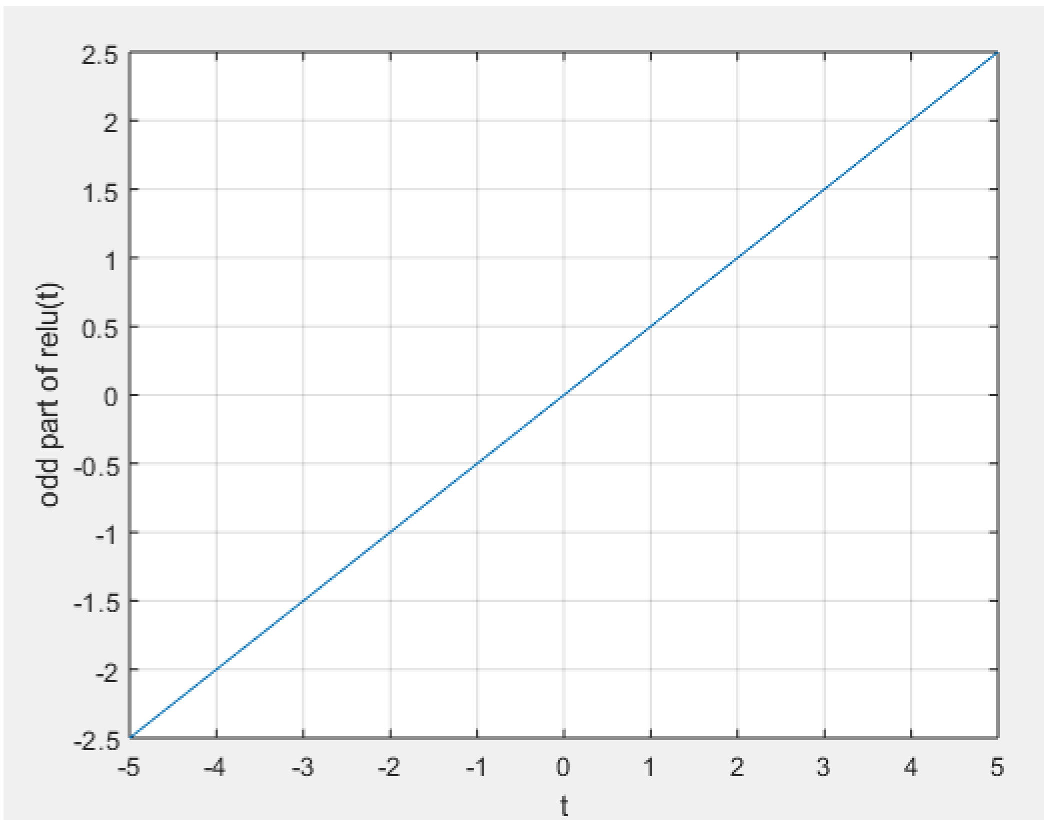


Figure 4: Odd component of relu(t)