EE102

Lecture 15

EE102 Announcements

- Syllabus link is tinyurl.com/ucla102
- **Homeworks due on Tuesday**

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Lots of info .. so summarizing last class

1. (last class) various properties of F.T
2. Applied F.T. to assess RC circuit output.

Today:

3. Analyzing frequency response of RC circuit
4. Sampling a signal and its effects

Example: RC Circuit

Frequency Response

- In addition to *frequency response*, $H(j\omega)$ is sometimes called the *transfer function* of the system.
- The reason its called frequency response is that $H(j\omega)$ describes how the input is changed at every single frequency.
- In particular, the frequency response scales the amplitude response by $|H(j\omega)|$, i.e.,
$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$
- The frequency response shifts the phase response by $\angle H(j\omega)$, i.e.,
$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Frequency Response

To see this, note that if the input to a system is a complex exponential, $e^{j\omega_0 t}$ (recall, these are the eigenfunctions of an LTI system), then

$$\begin{aligned} X(j\omega) &= \mathcal{F}\left[e^{j\omega_0 t}\right] \\ &= 2\pi\delta(\omega - \omega_0) \end{aligned}$$

Therefore, the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)(2\pi\delta(\omega - \omega_0)) \\ &= H(j\omega_0)(2\pi\delta(\omega - \omega_0)) \end{aligned}$$

Frequency Response

This means that

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}[Y(j\omega)] \\ &= \mathcal{F}^{-1}[H(j\omega_0)(2\pi\delta(\omega-\omega_0))] \\ &= H(j\omega_0)e^{j\omega_0 t} \\ &= |H(j\omega_0)|\, e^{j(\omega_0 t+\angle H(j\omega_0))} \end{aligned}$$

To summarize here, we input a sinusoidal input, $x(t) = e^{j\omega_0 t}$ to a LTI system, and saw that the output was

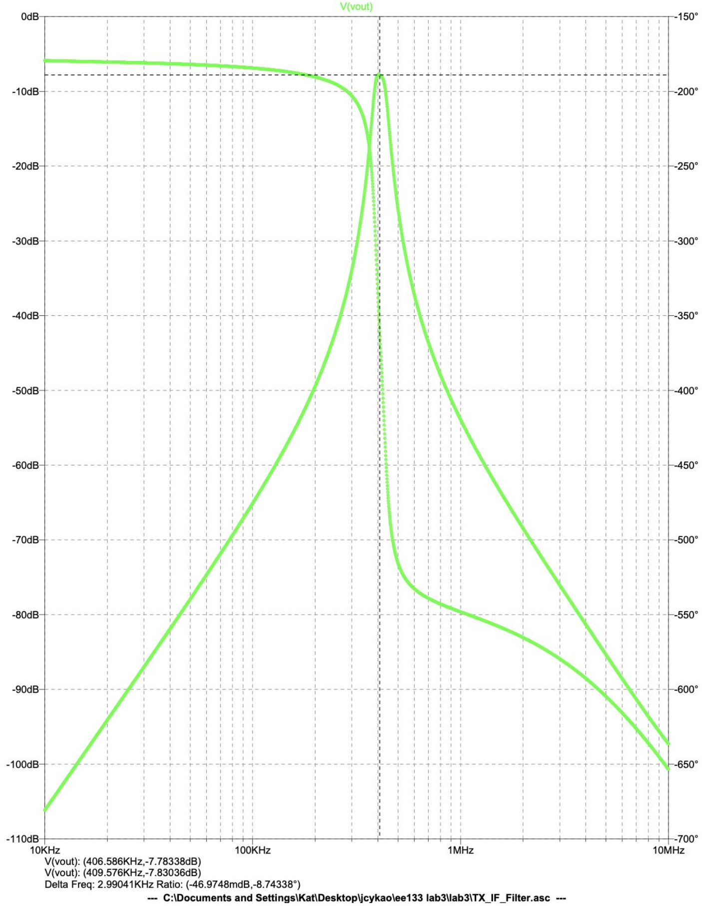
$$y(t) = |H(j\omega_0)|\, e^{j(\omega t+\angle H(j\omega_0))}$$

i.e., inputting a complex sinusoid to an LTI system produces an output that:

- is at the same frequency, ω_0.
- is scaled in amplitude by $|H(j\omega_0)|$.
- is phase shifted by $\angle H(j\omega_0)$.

Application: tuning circuits

Application: tuning circuits



Frequency Response Example

Consider the input:

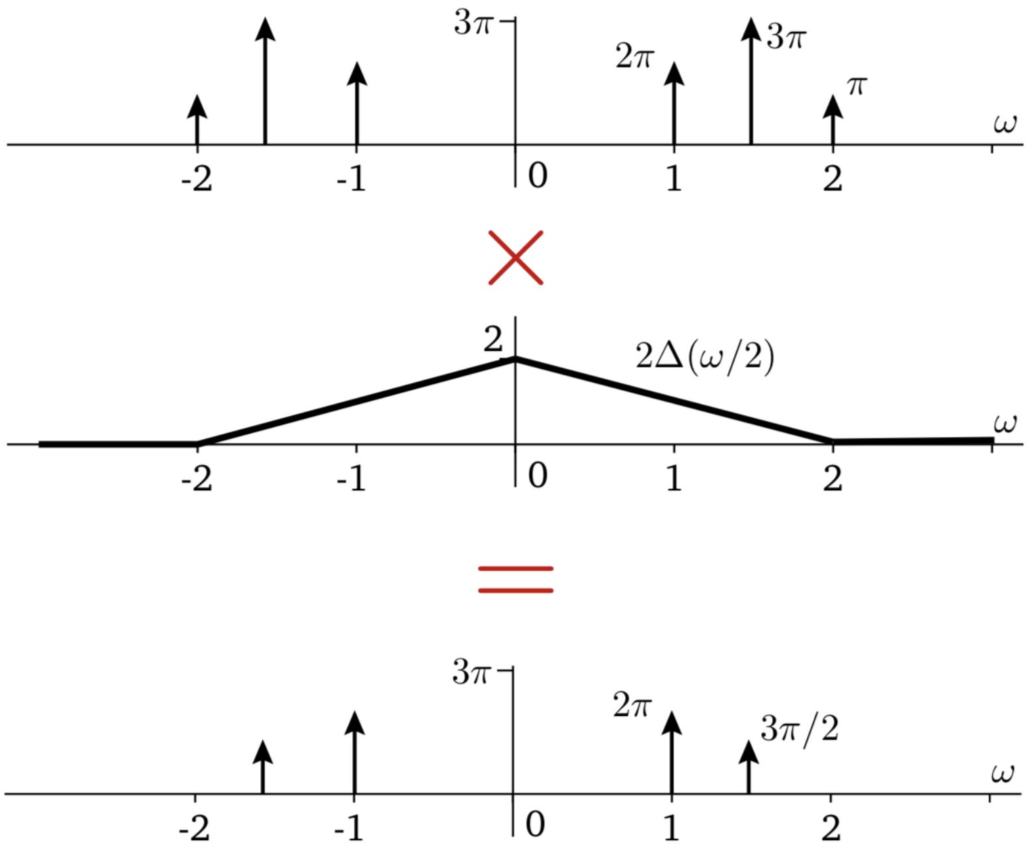
$$x(t) = 2\cos(t) + 3\cos(3t/2) + \cos(2t)$$

and system with impulse response

$$h(t) = \frac{2}{\pi}\,\mathrm{sinc}^2(t/\pi)$$

Find $y(t) = (x * h)(t)$.

Frequency Response Example



Frequency Response Example

This gives that

$$Y(j\omega) = 2\pi\left[\delta(\omega-1)+\delta(\omega+1)\right] + \frac{3\pi}{2}\left[\delta(\omega-3/2)+\delta(\omega+3/2)\right]$$

Taking the inverse Fourier transform, we get that

$$y(t) = 2\cos(t) + \frac{3}{2}\cos(3t/2)$$

Frequency Response Example 2

Let $x(t) = e^{-t}u(t)$. We input this signal into a system with impulse response:

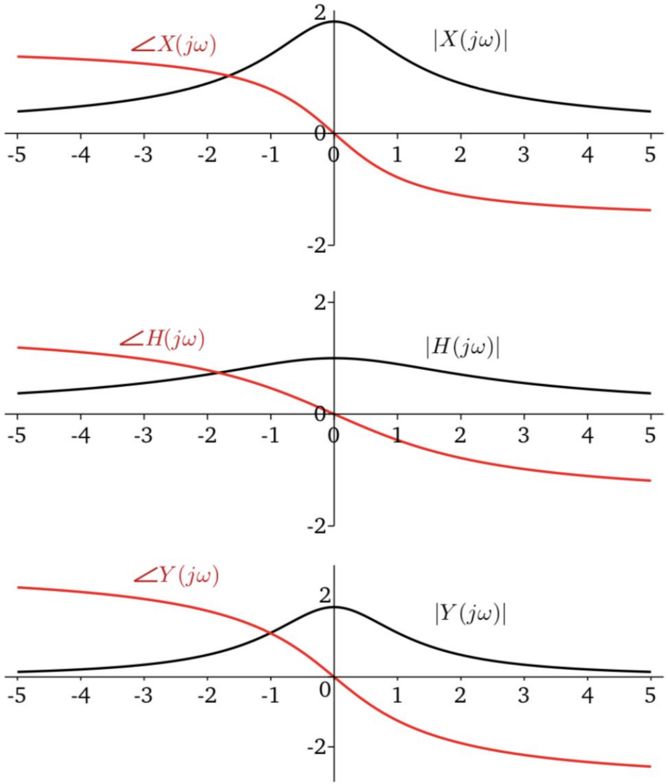
$$h(t) = 2e^{-2t}u(t)$$

What are $Y(j\omega)$ and $y(t)$?

Frequency Reponse Example 2

Frequency Response Example 2

Recall that when we multiply two complex numbers, their magnitudes multiply and their phases add. This is shown below.



Filters

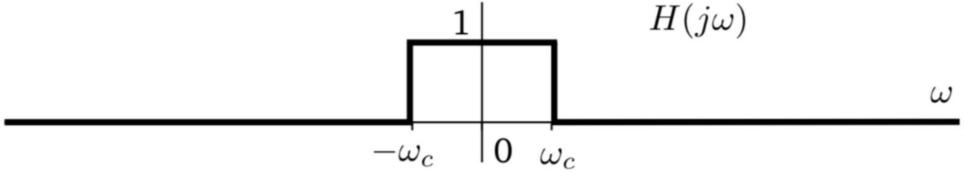
Filters are designed to extract or attenuate certain desired frequencies from a signal. For example, consider a recording of music where the microphones accidentally recorded the sopranos too loudly. It would be possible to rebalance the audio by attenuating higher frequencies in the signal.

We'll first discuss ideal filters, which only pass through certain frequencies. There are three main types of filters we'll discuss:

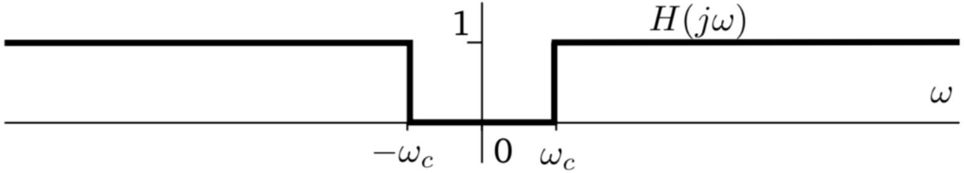
- Low pass filter: suppresses all frequencies that are higher than a specified frequency, ω_c. Its name comes from the fact that it lets frequencies less than ω_c through (i.e., low frequencies).
- High pass filter: supresses all frequencies that are lower than a specified frequency, ω_c.
- Band pass filter: supresses all frequencies outside of a range $\pm\omega_c$ around a chosen frequency ω_0.
- These three filters are illustrated on the next page.

Filter Illustration

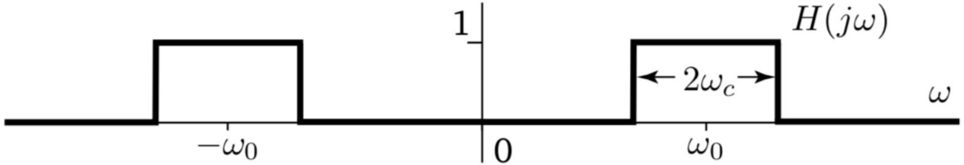
- Low pass filter:



- High pass filter:



- Band pass filter:



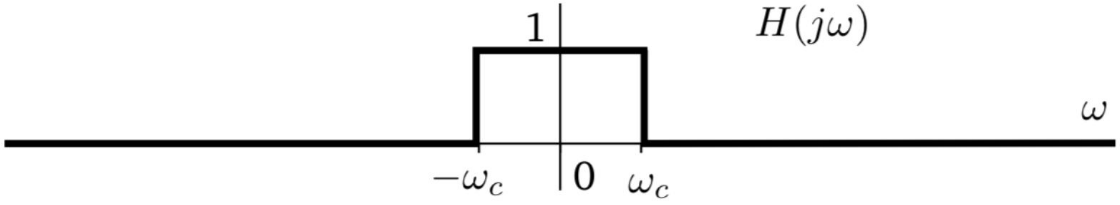
These may alternatively be written as “lowpass” or “low-pass” filter, etc.

CYU on Filters

Imagine I pass the canonical delta function through a low-pass filter. What is the output?

Ideal Low Pass Filter

The ideal low pass filter is



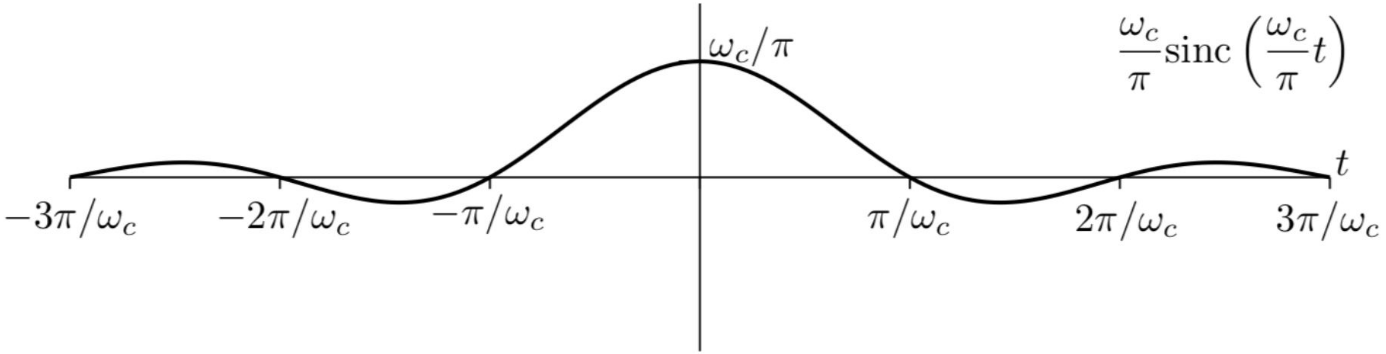
We call the region where frequencies are not suppressed (i.e., up to frequency ω_c for this ideal low pass filter) the "passband." This filter can be represented as

$$H(j\omega) = \mathrm{rect}(\omega/(2\omega_c))$$

Ideal Low Pass Filter

Ideal Low Pass Filter

Thus, the ideal low pass filter's impulse response is:



Note that we've only shown a small interval here, the sinc function is nonzero for t outside of the plotted window.

CYU from HW

The overall LTI system is described by the following differential equation:

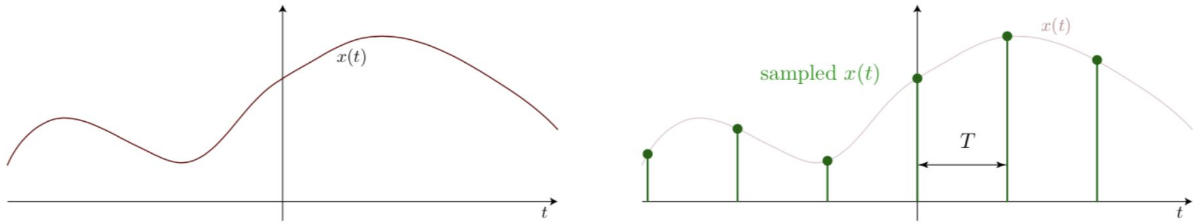
$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 3x(t)$$

i. Find the frequency response, $H(j\omega)$, of the overall system $h(t)$.

Sampling

Motivation

In reality, we could never store a continuous time signal. Instead, as we see in `MATLAB`, we store the signal's value at various times. This is called sampling, as illustrated below.



A key variable of interest is the sampling frequency, i.e., the time in between our samples, denoted T in the above diagram.

This is related to discrete signals, i.e., $x[n] = x(nT)$.

Sampling

How to sample a continuous signal?

How do we sample a continuous signal? You may have several intuitions to do so already using the $\delta(t)$ signal and its property that $f(t)\delta(t) = f(0)\delta(t)$.

- We will arrive at sampling by first studying a related problem: the Fourier transform of periodic signals.
- The reason we approach this is that Fourier series are discrete coefficients, c_k, while the Fourier transform is typically some continuous signal. i.e., it seems like there may be a relationship whereby the Fourier series is like a sampled Fourier transform.
- So we ask: what is the relationship between the Fourier series and the Fourier transform?
- To see this, we can begin by identifying the relationship between the Fourier series and the Fourier transform.

F.T. of Periodic Signal

We cannot directly take the Fourier transform of a periodic signal, since they do not have finite energy. However, we can use a few tricks (like in the Generalized Fourier Transform lecture) to calculate the FT of a periodic signal.

Let $f(t)$ have a Fourier series (with period $T_0 = \omega_0/2\pi$)

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T_0} \int_0^{T_0} f(t) e^{-jk\omega_0 t} dt$$

There's a close relationship between the two, as the Fourier series equation looks like the Fourier transform equation but with a $\sum$ instead of an $\int$.

F.T. of Periodic Signal

Fourier transform of the Fourier series representation

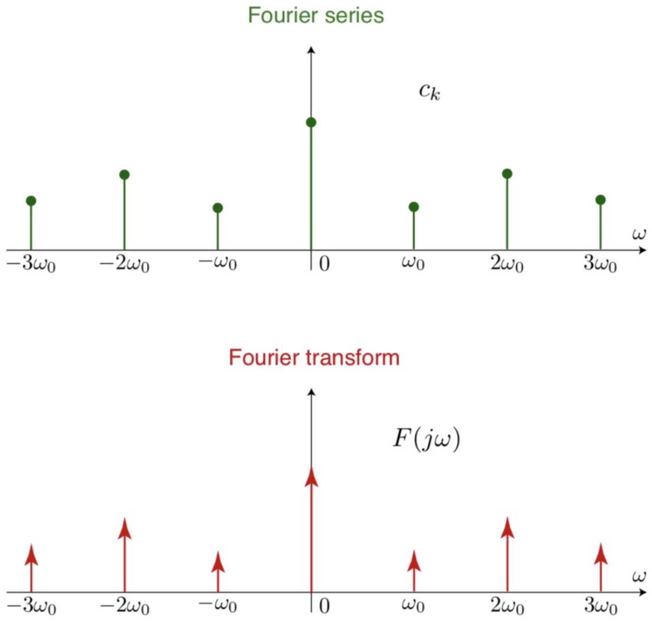
Let's take the Fourier transform of the Fourier series representation.

$$\mathcal{F}[f(t)] \quad = \quad \mathcal{F}\left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]$$

F.T. of periodic signal

Fourier transform of the Fourier series (cont.)

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \iff \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$$



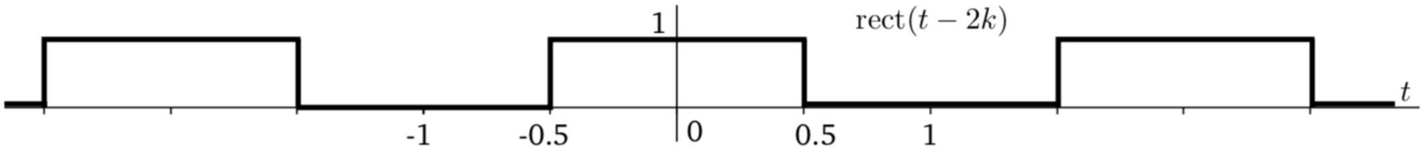
F.T. of periodic signal

Example: square wave

Consider the square wave below:

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t - 2k)$$

This is illustrated below.



F.T. of Periodic Signal

In the Fourier series lecture (slide 8-32), we calculated that the Fourier series of this signal is

$$c_k = \frac{1}{2}\,\mathrm{sinc}(k/2)$$

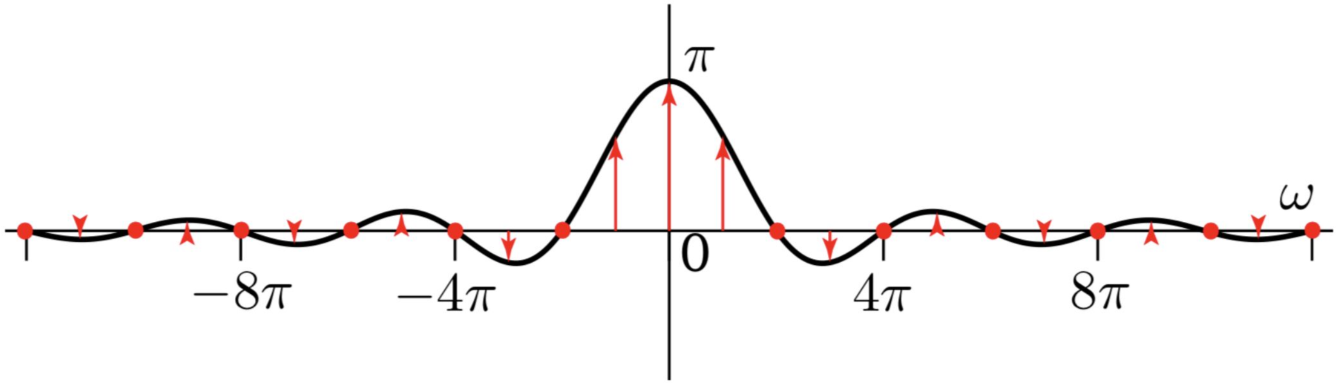
What is its Fourier transform?

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \iff \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0)$$

F.T. of Periodic Signal

Hence, the Fourier transform of the square wave is the Fourier transform of a rect multiplied by evenly spaced δ's, i.e.,

$$F(j\omega) = \pi \sum_{k=-\infty}^{\infty} \operatorname{sinc}(\omega/2\pi)\delta(\omega - k\pi)$$



Impulse Train

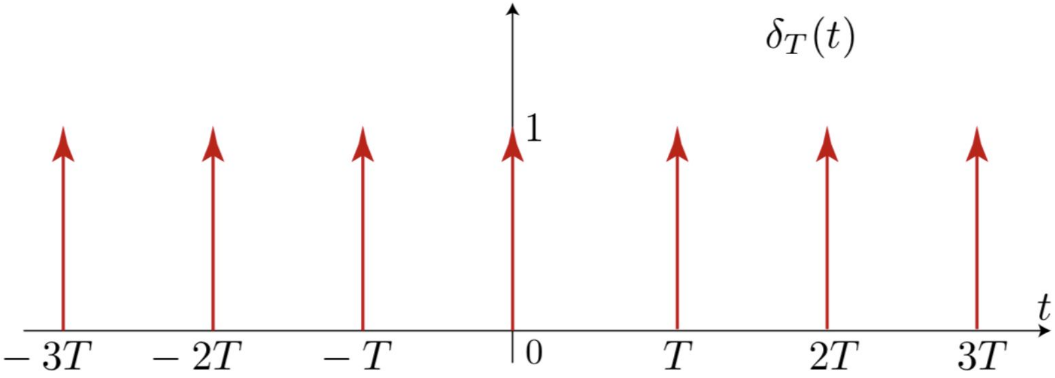
To simplify notation here, we can define an *impulse train*, which ends up being our sampling function. We let $\delta_T(t)$ be a sequence of unit δ functions spaced by T.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Impulse Train

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

This is illustrated below.



With this, we can write the Fourier transform of the square wave as

$$F(j\omega) = \pi\,\mathrm{sinc}(\omega/2\pi)\delta_\pi(\omega)$$

Impulse Train

The impulse train may have been your first thought when thinking of how to sample a signal every T.

Indeed, this signal has very important qualities. Let's start off with a simple question: intuitively, what is the Fourier transform of a impulse train?

Guess CYU: What’s the Fourier Transform of Impulse train

No need to calculate - please think about square wave example and hazard a guess.

Impulse Train F.T.

Let's think through this using our square wave example.

- We know that the Fourier transform of the square wave is a sinc multiplied by $\delta_\pi(\omega)$.
- From the convolution theorem, this means that the inverse Fourier transform (i.e., the square wave) is the inverse Fourier transform of a sinc (i.e., a rect) convolved with the inverse Fourier transform of a impulse train.
- We know that a square wave is simply a rect repeated over and over again, i.e., convolved with a impulse train.
- So intuitively, by duality, the Fourier transform of a impulse train should be a impulse train.

Note, we will sometimes use the term 'delta train' to describe an impulse train.

F.T. of Impulse Train

Let's check our intuition and compute the Fourier transform of an impulse train. To do so, we'll use our trick of finding the Fourier series of the (periodic) impulse train, and then multiplying by $2\pi\delta(\cdot)$.

F.T. of Impulse Train

F.T. of Impulse Train

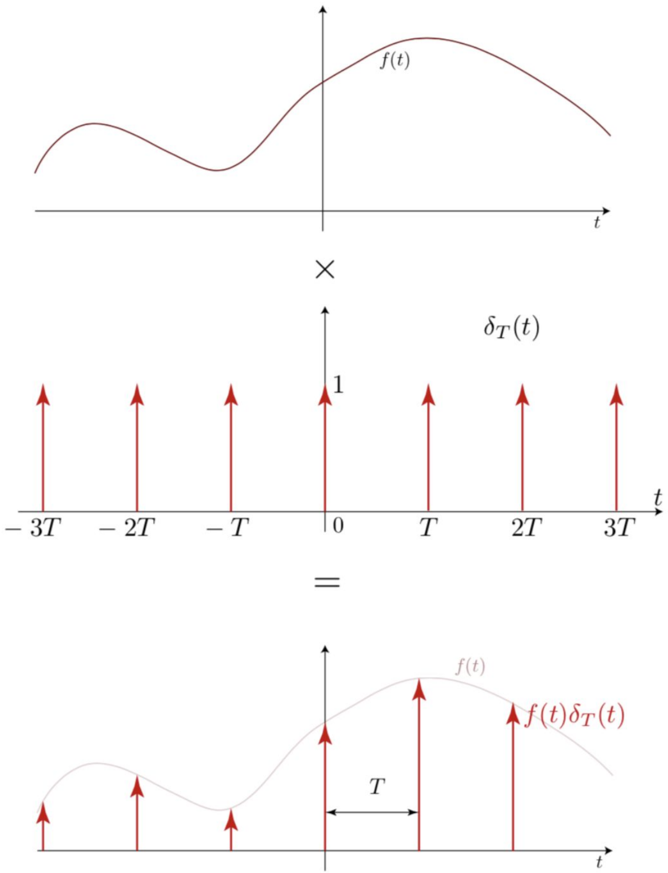
Sampling with an Impulse Train

As we saw earlier, one of the things we will use the impulse train for is to sample signals.

Given a signal $f(t)$,

$$f(t)\delta_T(t) \quad = \quad f(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Sampling with an Impulse Train

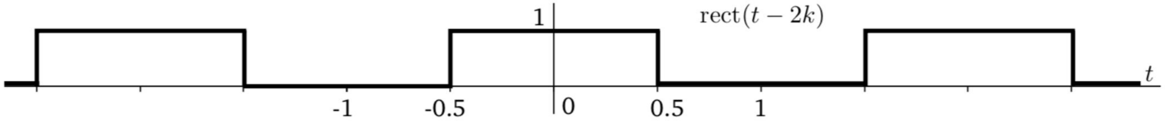


Sampling and Periodicity

Square wave, part 2

Let's revisit our square wave example, where

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t - 2k)$$



Another way to represent this square wave is as follows:

$$f(t) = \mathrm{rect}(t) * \delta_2(t)$$

Hence, we can calculate its Fourier transform by using the convolution theorem. Recall that, for $\omega_0 = 2\pi/T$,

$$\mathrm{rect}(t) \iff \mathrm{sinc}(\omega/2\pi)$$

and

$$\delta_T(t) \iff \omega_0 \delta_{\omega_0}(\omega)$$

Sampling and Periodicity

Square wave, part 2 (cont.)

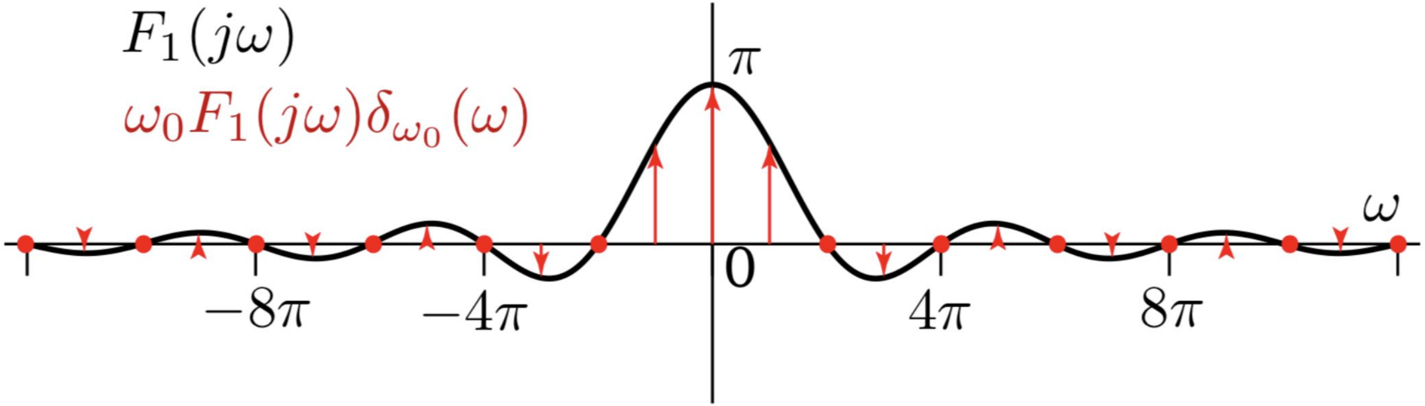
Note that when $T = 2$, then $\omega_0 = \pi$. Then, we have that,

$$\begin{aligned}
\mathcal{F}\left[f(t)\right] &= \mathcal{F}\left[\mathrm{rect}(t) * \delta_2(t)\right] \\
&= \mathcal{F}\left[\mathrm{rect}(t)\right] \mathcal{F}\left[\delta_2(t)\right] \\
&= \mathrm{sinc}(\omega/2\pi)\pi\delta_\pi(\omega)
\end{aligned}$$

This is exactly the same Fourier transform we calculated earlier using the Fourier series of the square wave.

Sampling and Periodicity

Another intuition to remember here is that the Fourier transform of a periodic signal is the Fourier transform of one period of the signal (which we can denote f_1), sampled by an impulse train at multiples of ω_0.



Sampling and Periodicity

Discrete - periodic duality

We can determine the Fourier transform of a signal sampeld in the time-domain. Consider

$$\tilde{f}(t) = f(t)\delta_T(t)$$

Its Fourier transform is

$$\tilde{F}(j\omega) \quad = \quad \mathcal{F}\left[f(t)\delta_T(t)\right]$$

Sampling and Periodicity

Discrete - periodic duality (cont.)

This are merely samples of $F(j\omega)$ repeated every ω_0, since

$$\tilde{F}(j\omega) \quad = \quad \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega)$$

Sampling and Periodicity

Discrete - periodic duality (cont.)

This are merely samples of $F(j\omega)$ repeated every ω_0, since

$$
\begin{aligned}
\tilde{F}(j\omega) &= \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega) \\
&= \frac{1}{T}F(j\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))
\end{aligned}
$$

This leads us to the realization that:

- A signal that is periodic in time is discrete in spectrum.
- A signal that is discrete in time is periodic in spectrum.

There are important consequences from this result when we consider sampling signals in the time domain.

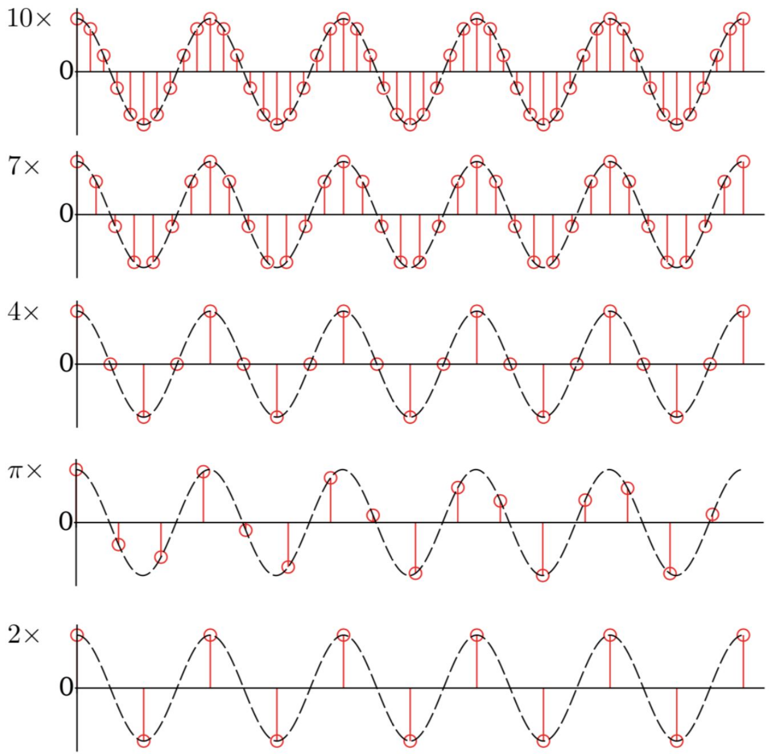
Sampling Theorem Motivation

Consider the following problem. We have a signal $f(t)$, and we need to store it. Our experimental set up is able to sample this signal at an interval T. How do we set T so that we can faithfully store $f(t)$? If T is too large, we sample infrequently and may lose information about $f(t)$. If T is too small, we waste memory and resources to store values we don't need.

The sampling theorem uses the results we've derived to tell us the minimum frequency at which we must sample $f(t)$ to not lose information. It is a very important theorem.

Sampling Theorem

Sampling example: sinusoids (cont.)



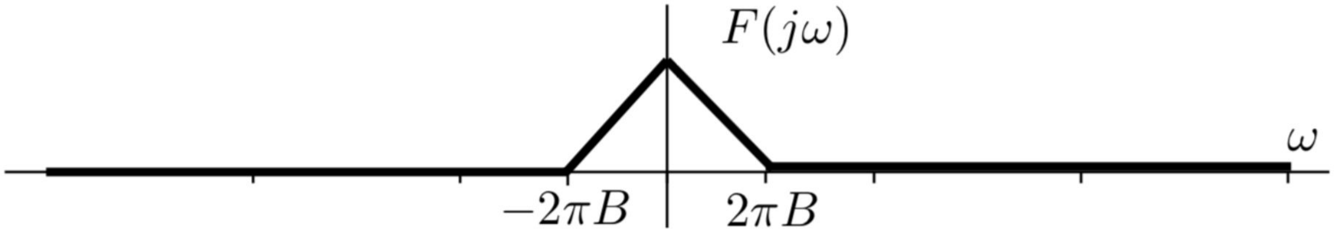
Sampling Theorem

If $\tilde{f}(t) = f(t)\delta_T(t)$, then as shown on the previous slides,

$$\tilde{F}(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))$$

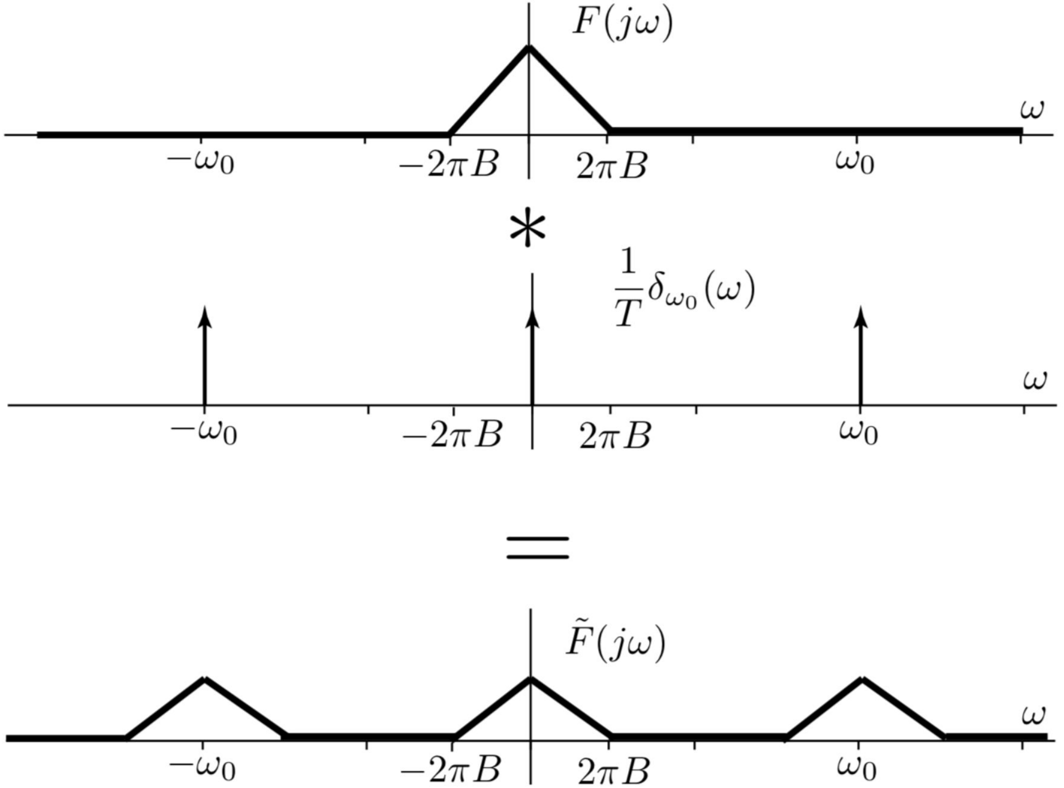
Therefore, the spectrum of $\tilde{f}(t)$ are shifted replicas of the spectrum, $F(j\omega) = \mathcal{F}[f(t)]$ spaced every ω_0 and scaled by $1/T$.

We define the bandwidth of $f(t)$ to be $\pm B$ Hz, e.g.,



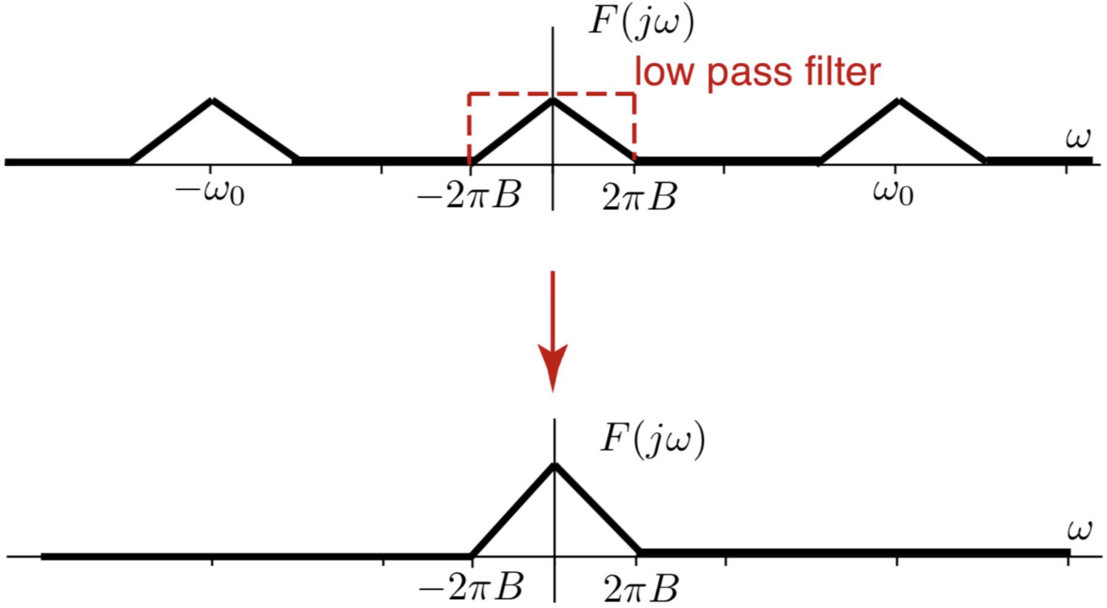
Sampling Theorem

For a particular choice of ω_0, where $\omega_0 \gg 2\pi B$, we see the spectrum of $\tilde{F}(j\omega)$ looks like:



Sampling Theorem

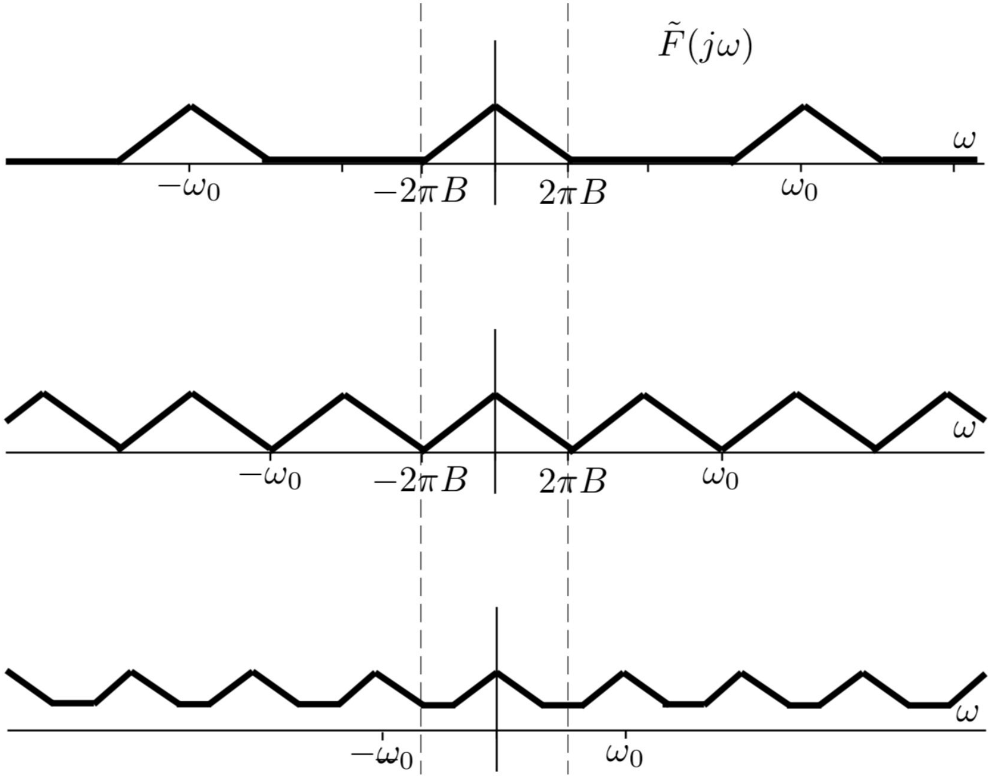
For this choice of ω_0, the original $F(j\omega)$ can be recovered through low pass filtering.



With ideal low pass filtering for the illustrated ω_0, we can *perfectly* recover $f(t)$ after sampling.

Sampling Thm

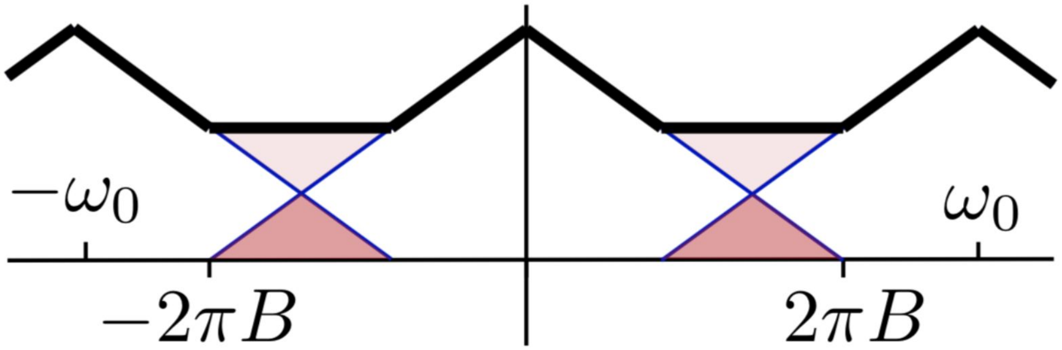
But now, as we increase the time T between samples, which decreases ω_0, the replicas of $F(j\omega)$ get closer and closer together.



Sampling Thm - Aliasing

We see that as ω_0 decreases, the bands start to overlap. When the replicas overlap, even with ideal low pass filtering, we cannot recover the original $F(j\omega)$.

This overlap is called *aliasing* because low frequencies of one spectral replica appear (or alias) as high frequencies in the next spectral replica. The vice versa is true as well; high frequencies of one spectral replica alias as low frequencies in an adjacent spectral replica. The alias'd sections are shown in the darker red.



Sampling Thm - Aliasing

To be able to perfectly recover a signal, we need to sample so as to avoid aliasing. No aliasing happens if $2\pi B < \omega_0/2$. We can simplify this as

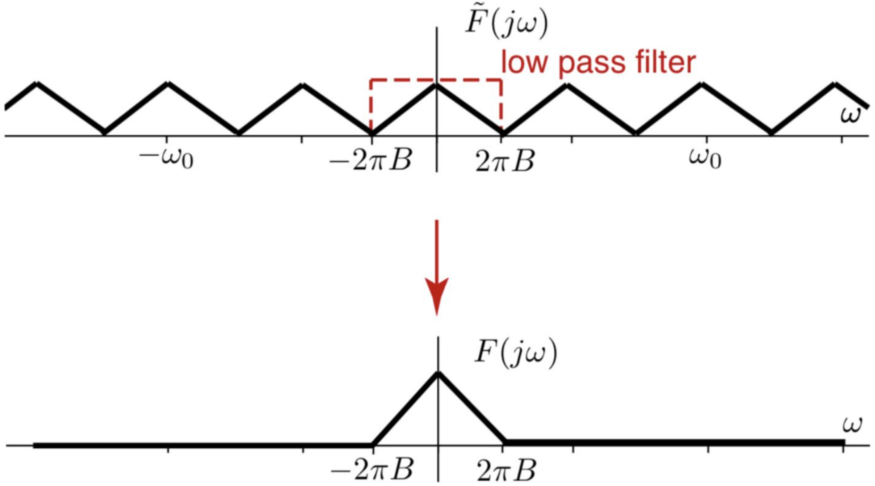
$$\begin{aligned} 2B &< \omega_0/2\pi \\ &= \frac{2\pi}{T}\frac{1}{2\pi} \\ &= \frac{1}{T} \end{aligned}$$

Therefore, the signal can only be recovered exactly if the signal bandwidth $2B$ is less than or equal to the sampling rate $1/T$. Hence, we need to sample at intervals less than or equal to $T = 2B$. This sampling rate, $2B$ is called the *Nyquist rate* for $f(t)$, and it is the lowest rate that we can sample $f(t)$ so that it can be perfectly recovered. T is called the *Nyquist interval*.

Interpolation

Interpolation

With a sampled signal, $\tilde{f}(t)$, as long as we have sampled at a rate $\geq 2B$, we can perfectly recover the original signal through ideal low pass filtering. Let's formalize how this happens, using the particular instantiation that $T = 1/2B$, i.e., we sample at the Nyquist rate.



Our low pass filter has frequency response

$$H(j\omega) = T\mathrm{rect}\left(\frac{\omega}{4\pi B}\right)$$

Interpolation

The inverse Fourier transform of $H(j\omega)$ is

$$h(t) = 2BT\,\mathrm{sinc}(2Bt)$$

Since $T = 1/2B$, we can simplify this expression to

$$h(t) = \mathrm{sinc}(2Bt)$$

Therefore, to reconstruct $f(t)$ from $\tilde{f}(t)$, we calculate:

$$\begin{aligned}
\tilde{f}(t) * h(t) &= \left(\sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)\right) * h(t) \\
&= \sum_{k=-\infty}^{\infty} f(kT)h(t-kT) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2B(t-kT)) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt-k))
\end{aligned}$$

Interpolation

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula.

Intuition?

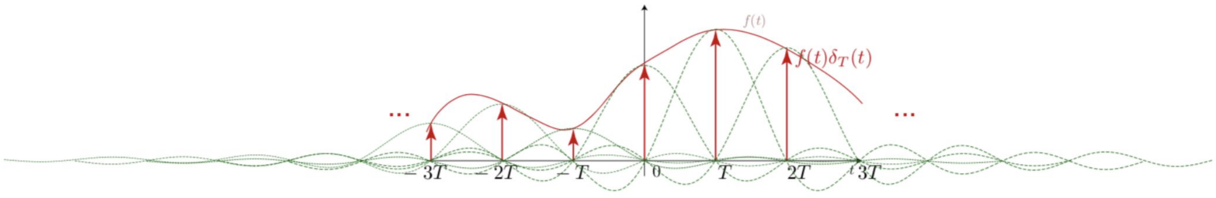
Interpolation

Recovering the original signal through interpolation (cont.)

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula. Intuitively, it does the following:



The sum of the green sinc functions will equal the red function, $f(t)$.

To not mince words, this result, which combines many of the things we've learned thus far, is remarkable. Through this reconstruction, we are able to *perfectly* recover an original signal from samples.

CYU: Homework Qst 5a.i

Assume $x(t)$ a real bandlimited signal where $X(j\omega)$ is non-zero for $|\omega| \leq 2\pi B$ rad/s. If F_s Hz is the Nyquist rate of $x(t)$, determine the Nyquist rate of the following signals in terms of B:

i. $x(t+1)$