EE102

Lecture 11

EE102 Announcements

- Syllabus link is tiny.cc/ucla102
- HW4 due Friday
- Student Feedback (not hitting response rate, but good faith bonus applied). Feedback is extremely helpful.

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Review from Last Time

- **Fourier Series:** Learned that periodic signal => broken down into sines/cosines, or more formally, complex exponentials.
- **Fourier Coefficients:** The coefficients of these sines/cosines can be solved for.
- **Previous Learning Milestones**
  - Given a signal, you should be able to compute its fourier coefficients (we did this for square/sawtooth)
- Today's goals are two-fold
  - Advanced Fourier Properties
  - Does Fourier apply for periodicity alone?

Summary of Fourier Series

These are the main mathematical results of this lecture, written here for convenience.

If $f(t)$ is a well-behaved periodic signal with period T_0, then $f(t)$ can be written as a Fourier series

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

where $\omega_0 = \frac{2\pi}{T_0}$ and

$$\boxed{c_k = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) e^{-jk\omega_0 t} dt}$$

for all integers k. The c_k are called the *Fourier coefficients* of $f(t)$.

Here, $f(t)$ is the *weighted average* of complex exponentials (which are simply complex sines and cosines).

Fourier Series Properties

There are interesting symmetries and properties of the Fourier series that are worth expanding upon.

- c_0 **is the average of the signal**. Note that for $k = 0$, we have that

$$\boxed{c_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t)dt}$$

  Thus, c_0 is exactly the time-averaged mean of the signal and corresponds to a constant value (i.e., it has no sinusoidal component). For this reason, it is sometimes called the "DC component." DC stands for direct current in circuits, and refers to non-alternating (sinusoidal) currents. The DC component is the average value taken on by a signal.

Fourier Series Properties

- **Complex representation**. In general, the c_k may be complex, and so they can be expressed in their real / imaginary form or in magnitude / phase form. i.e.,

$$\begin{aligned} c_k &= \Re(c_k) + j\Im(c_k) \\ &= |c_k|\, e^{j\angle c_k} \end{aligned}$$

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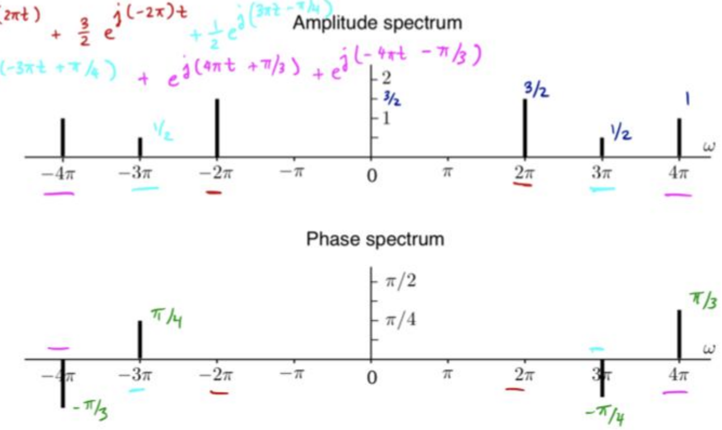
\end{aligned}
$$

$$x(t) = \frac{3}{2} e^{j(2\pi t)} + \frac{3}{2} e^{j(-2\pi)t} + \frac{1}{2} e^{j(3\pi t - \pi/4)} + \frac{1}{2} e^{j(-3\pi t + \pi/4)} + e^{j(4\pi t + \pi/3)} + e^{j(-4\pi t - \pi/3)}$$



Fourier Series Properties

Fourier symmetry

We can apply Euler's formula to re-write the Fourier coefficients, and reveal some symmetries:

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-j\frac{2\pi k t}{T_0}}\,dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right]dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)dt - \frac{j}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)dt
\end{aligned}
$$

If $f(t)$ is real, then so are:

$$
\begin{aligned}
\Re(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos\left(\frac{2\pi k}{T_0}t\right)dt \\
\Im(c_k) &= -\frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin\left(\frac{2\pi k}{T_0}t\right)dt
\end{aligned}
$$

Fourier Series Properties

Fourier symmetry (cont.)

Therefore, for $f(t)$ real, and using the fact that $\cos(k)$ is even and $\sin(k)$ is odd, we have the following symmetries:

$$
\begin{aligned}
\Re(c_k) &= \Re(c_{-k}) \\
\Im(c_k) &= \Im(c_{-k}) \\
c_k^* &= c_{-k} \\
|c_k| &= |c_{-k}| \\
\angle c_k &= -\angle c_k^*
\end{aligned}
$$

CYU

What is the relationship between the Fourier series coefficients c_k and c_{-k} if $x(t)$ is even?

Fourier Series Properties

- Combining facts, we have that if $f(t)$ is even and real, then $c_k = c_{-k}$ and $c_{-k} = c_k^*$, and so $c_k = c_k^*$. This means that the c_k must be real.

$$f(t) \text{ even and real } \implies c_k \text{ real}$$

- If $f(t)$ is odd and real, then $c_k = -c_{-k}$ and because $c_{-k} = c_k^*$, then $c_k = -c_k^*$. This means the c_k must be imaginary.

$$f(t) \text{ odd and real } \implies c_k \text{ imaginary}$$

Parseval’s Theorem

What about non-periodic signals?

This is all well and good, but most signals we care about are not periodic.

Motivation

Last lecture, we learned about the Fourier series, which can model (almost) any **periodic** or **time-limited** function as a sum of complex exponentials.

But the Fourier series is limited because it requires the signals be **periodic** or **time-limited**.

The Fourier transform allows us to calculate the spectrum of aperiodic signals.

How do we go to aperiodic signals?

Intuition of going from Fourier series to Fourier transform

Extending Fourier series to the Fourier transform is fairly intuitive.

The idea is the following.

- We can calculate the Fourier series of a periodic or time-limited signal, over some interval of length T_0.
- A signal that is not periodic can be viewed as a periodic signal, where T_0 is infinite. As T_0 is infinite, it never repeats.
- But the point is that we can replace our Fourier series calculation as, instead of being over a finite period, T_0, being over all time, from $t = -\infty$ to ∞.

Arriving at the Fourier Transform

Intuition (cont.)

Mathematically, we can calculate the Fourier series of $f(t)$ over the interval $[-T/2, T/2)$ via:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0} dt$$

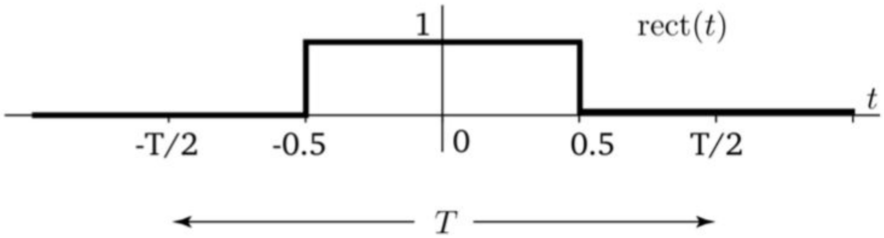
where $\omega_0 = 2\pi/T$.

In the Fourier transform, we're now going to let $T \to \infty$.

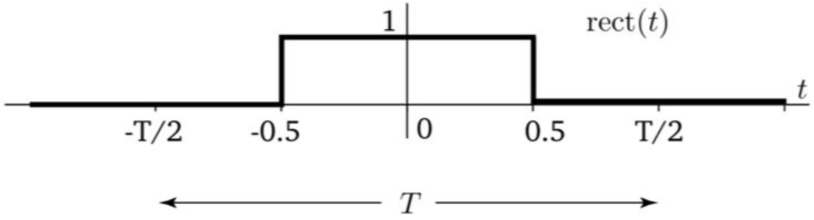
Example: rect

The Fourier series of rect as $T \to \infty$

Consider $x(t) = \mathrm{rect}(t)$. Further, let's define a period T over which, if we made a periodic extension of the rect, it would repeat every T. (Again, we're going to set $T \to \infty$ eventually so that it doesn't repeat.)



Example: rect



The Fourier series of this signal is related to the one we did last lecture, but we'll do it again for the sake of completeness:

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{rect}(t) e^{-jk\frac{2\pi}{T}t} dt \\
&= \frac{1}{T}\int_{-1/2}^{1/2} e^{-jk\frac{2\pi}{T}t} dt \\
&= \frac{1}{T} \left. \frac{Te^{-jk\frac{2\pi}{T}t}}{-jk2\pi} \right|_{-1/2}^{1/2}
\end{aligned}
$$

Example: rect

The Fourier series of rect as $T \to \infty$

Continuing ...

$$
\begin{aligned}
c_k &= \left.\frac{e^{-jk\frac{2\pi}{T}t}}{-jk2\pi}\right|_{-1/2}^{1/2} \\
&= \frac{-j\sin(\pi k/T) - j\sin(\pi k/T)}{-j2\pi k} \\
&= \frac{\sin(\pi k/T)}{\pi k} \\
&= \frac{1}{T}\frac{\sin(\pi k/T)}{\pi k/T} \\
&= \frac{1}{T}\,\mathrm{sinc}\left(\frac{k}{T}\right)
\end{aligned}
$$

Therefore, the Fourier series of $\mathrm{rect}(t)$ with a periodic extension every T is:

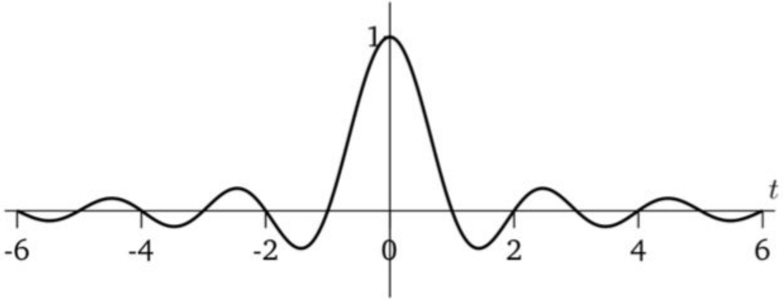
$$
\frac{1}{T}\sum_{k=-\infty} \mathrm{sinc}(k/T)e^{jk\omega_0 t}
$$

for $\omega_0 = 2\pi/T$.

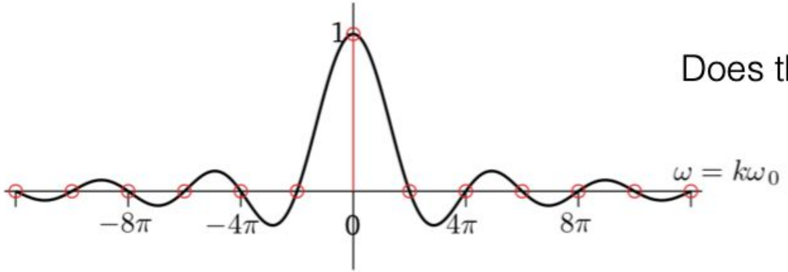
Example: rect

The Fourier series of rect as $T \to \infty$

Let's now look at what the coefficients look like for varying values of T. First, for a refresher, let's recall what the sinc function looks like.



Now, let's set $T = 1$ ($\omega_0 = 2\pi$) and calculate each of the Fourier coefficients.

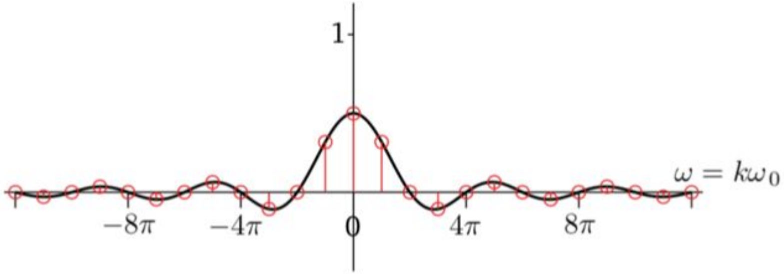


Does this make sense?

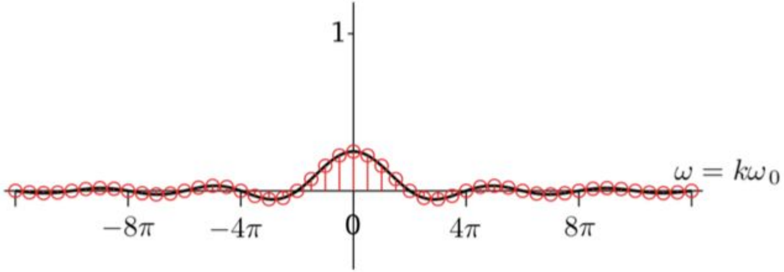
Example: rect

The Fourier series of rect as $T \to \infty$

Set $T = 2$ ($\omega_0 = \pi$) and calculate each of the Fourier coefficients.



Set $T = 4$ ($\omega_0 = \pi/2$) and calculate each of the Fourier coefficients.



What if we replace $k\omega_0$ with ω?

Example: rect

The Fourier series of rect as $T \to \infty$

The trend we see is that as we set T larger, we more densely sample the sinc function. This gives us reason to believe that the spectrum of the rect signal, when $T \to \infty$, is a sinc function.

Let's formalize this intuition with math.

Arriving at the Fourier transform

The Fourier series of $f(t)$ an an interval $[-T/2, T/2)$ is given by:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt$$

Arriving at the Fourier transform

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with

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We define the truncated Fourier transform as:

$$F_T(j\omega) = \int_{-T/2}^{T/2} f(t) e^{-j\omega t} dt$$

so that

$$c_k = \frac{1}{T} F_T(jk\omega_0)$$

Then,

$$f_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(jk\omega_0) e^{jk\omega_0 t}$$

Remember, we're going to replace $k\omega_0$ with ω.

Arriving at the Fourier transform (cont.)

Now, let's set $T \to \infty$. If we do this, then $\omega_0 = 2\pi/T$ will approach 0. So suppose instead that we define a continuous variable,

$$\omega = \frac{2\pi k}{T}$$

which means that k increases with T, so that $\omega = k\omega_0$ is fixed.

The Fourier transform is the limit of the truncated Fourier transform.

$$\begin{aligned} F(j\omega) &= \lim_{T\to\infty} F_T(j\omega) \\ &= \lim_{T\to\infty} \int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \end{aligned}$$

This is the Fourier transform, which takes you from the time domain, $f(t)$, to the frequency domain, $F(j\omega)$.

The Fourier Transform

The Fourier transform is:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

CYU (hw/exam problem difficulty)

Show that $f(t) = \cos(\omega_0 t)$ is not an eigenfunction of an LTI system.