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Ethan Wong

ECE 102 HW 5

1\. a) If f(t) is imaginary: $f(t) = jx(t)$ if $x(t)$ is real

Fourier Series:

$C_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k t}{T_0}\right) - j\sin\left(\frac{2\pi k t}{T_0}\right)\right]dt$

$C_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} jx(t)\left[\cos\left(\frac{2\pi k t}{T_0}\right) - j\sin\left(\frac{2\pi k t}{T_0}\right)\right]dt$

$C_k = \underbrace{j\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos\left(\frac{2\pi k t}{T_0}\right)dt}_{\text{Imaginary}(C_k)} + \underbrace{\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin\left(\frac{2\pi k t}{T_0}\right)dt}_{\text{Real}(C_k)}$

$\text{Real}(C_k) = -\text{Real}(C_{-k})$ $\quad$ $\text{Im}(C_k) = \text{Im}(C_{-k})$

$C_k^* = -C_{-k}$

$|C_k| = |C_{-k}|$

$\angle C_k = -\angle C_k^* \pm \pi$

There is a slight difference, as the version discussed in class was $\angle C_k = -\angle C_k$

b) period = 1s, real, odd $[w_0 = 2\pi]$

one positive frequency component $(C_k, k > 0)$

power of 9

$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jkw_0t}$

$[W_0 = 2\pi]$ $\quad x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk2\pi t}$

Fourier coefficients should be odd and imaginary for x(t)

$x(t) = C_0(e^{j2\pi t} - e^{-j2\pi t})$

C_0 is imaginary, replace it with jx [x is real] and simplify it

$x(t) = -2x\sin(2\pi t)$

The exponent is 9, and according to Parseval's Theorem

$2x^2 = 9 \quad x = \pm\frac{1}{2}3\sqrt{2}$

$\boxed{x(t) = -3\sqrt{2}\sin(2\pi t) \quad x(t) = 3\sqrt{2}\sin(2\pi t)}$

c) $Y(jw) = \int_a^b y(t)e^{-jwt}dt$ $\quad$ Fourier Series y(t)

$Y_k = \frac{1}{T_0}\int_a^b y(t)e^{-j\left(\frac{2\pi k t}{T_0}\right)}dt$ $\quad$ Fourier Series Coefficients $y_T(t)$

Determine relation between these two equations

$\boxed{Y_k = \frac{1}{T_0}Y(jw) \quad \left[w = \frac{2\pi k}{T_0}\right]}$

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2. a) i) $x(t)$ is even
The Fourier transform should be even as well.
$\boxed{A \quad D \quad E}$ are all even based on their graphs

ii) $x(t)$ is odd
The Fourier transform should be odd as well
$\boxed{F}$

iii) $x(t)$ is real
real$[X(j\omega)]$ is even $Im[X(j\omega)]$ is odd
Magnitude is even   phase is odd
$\boxed{C \quad E}$

iv) $x(t)$ is complex (neither real, nor pure imaginary)
$X(j\omega)$ is neither Hermitian, anti-Hermitian
excludes answer from iii)
$\boxed{A \quad B}$ are complex

v) $x(t)$ is real and even
$X(j\omega)$ should be real and even as well
$\boxed{E}$

vi) $x(t)$ is imaginary and odd
$X(j\omega)$ should be real and odd
$\boxed{F}$

vii) $x(t)$ is imaginary and even
$X(j\omega)$ is imaginary and even as well
$\boxed{D}$

viii) There exists a non zero ω_0 such that $e^{j\omega_0 t}x(t)$ is real, even
$X(j(\omega-\omega_0))$ is also real and even
i.e. shifting a signal can create a real, even signal
look for an off center one to start
$\boxed{B}$

b) i) The convolution of a real and even signal and a real and odd signal, is odd

$a(t)$: real and even $\quad b(t)$: real and odd

Convolution : $c(t) = a(t) * b(t)$

Fourier transform : $\underbrace{C(j\omega)}_{\substack{\text{imaginary}\\ \text{odd}}} = \underbrace{A(j\omega)}_{\substack{\text{real}\\ \text{even}}}\underbrace{B(j\omega)}_{\substack{\text{real}\\ \text{odd}}}$

Since the Fourier transform is imaginary and odd, the original function $c(t)$ is real and odd then

$\boxed{\text{TRUE}}$

ii) The convolution of a signal and the same signal reversed is an even signal.

$a(t)$: signal $\quad b(t)$: reversed version of $a(t)$

Convolution : $c(t) = a(t) * b(t)$

Fourier transform : $C(j\omega) = a(j\omega)\,a(-j\omega) = a(j\omega)\,b(j\omega)$

Since the Fourier transform is even, the original function $c(t)$ is even as well.

$\boxed{\text{TRUE}}$

c) i) If $x(t) = x^*(-t)$, then $X(j\omega)$ is real

$X(j\omega)$ will be real if $X^*(j\omega) = X(j\omega)$

Try to check the above statement

$X^*(j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-j\omega t}]^* dt$

$X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t} dt$

Variable substitution $-t = \tau$

$X^*(j\omega) = \int_{-\infty}^{\infty} x^*(-\tau)e^{-j\omega\tau} d\tau$

$X^*(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} d\tau$

$X^*(j\omega) = X(j\omega)$

Therefore, $X(j\omega)$ is real ✓

ii) $x(t)$ is real   Fourier transform $X(j\omega)$

$X_e(j\omega) = Re\{X(j\omega)\}$

$X_o(j\omega) = j\, Im\{X(j\omega)\}$

$X(j\omega) = X_e(j\omega) + X_o(j\omega)$

$X(j\omega) = \int_{-\infty}^{\infty} x_e(t)e^{-j\omega t}dt + \int_{-\infty}^{\infty} x_o(t)e^{-j\omega t}dt$

Simplify with Euler

$X(j\omega) = \underbrace{\int_{-\infty}^{\infty} x_e(t)\cos(\omega t)dt}_{X_e(j\omega)\,:\,\text{real}} - \underbrace{j\int_{-\infty}^{\infty} x_o(t)\sin(\omega t)dt}_{X_o(j\omega)\,:\,\text{imaginary}}$

$Real\{X(j\omega)\} = \int_{-\infty}^{\infty} x_e(t)\cos(\omega t)dt = X_e(j\omega)$

$Im\{X(j\omega)\} = \int_{-\infty}^{\infty} x_o(t)\sin(\omega t)dt = -jX_o(j\omega)$ ✓

3, a) $\int_0^{\infty} X(j\omega)d\omega$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ [Inverse Fourier]

Observe what happens at $t = 0$

$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$

Because $x(t)$ is real and even, the Fourier transform should also be real and even

Therefore: $x(0) = \frac{1}{\pi}\int_0^{\infty} X(j\omega)d\omega$

Reevaluate: $\int_0^{\infty} X(j\omega)d\omega = \pi \cdot x(0) = \pi$

$\boxed{\int_0^{\infty} X(j\omega)d\omega = \pi}$

b) $X(j\omega)|_{\omega=0}$

$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Since $\omega = 0$, the exponential can be ignored

$\int_{-\infty}^{\infty} x(t)dt$

Find the area of the trapezoid in the graph

Rectangle : 4

Each Triangle : $\frac{1}{2}$

$4 + [\frac{1}{2} \times 2] = 5$

$\boxed{X(j\omega)|_{\omega=0} = 5}$

c) $\angle X(j\omega)$

The given signal is even and real.

Therefore, its Fourier Transform is real and even as well.

This means that $\boxed{\begin{cases} \text{phase of } X(j\omega) = 0 & X(j\omega) \geq 0 \\ \text{phase of } X(j\omega) = \pi & X(j\omega) < 0 \end{cases}}$

d) $\int_{-\infty}^{\infty} e^{-j\omega} X(j\omega) d\omega$

Inverse Fourier Transform

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

Say that $t = -1$

$x(-1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega} d\omega$

$\int_{-\infty}^{\infty} X(j\omega) e^{-j\omega} d\omega$

Reevaluate: $2\pi \cdot x(-1) = 2\pi$

$\boxed{\int_{-\infty}^{\infty} e^{-j\omega} X(j\omega) d\omega = 2\pi}$

e) Plot the inverse Fourier transform of $Re\{e^{-3j\omega} X(j\omega)\}$

If $Y(j\omega) = e^{-3j\omega} X(j\omega)$

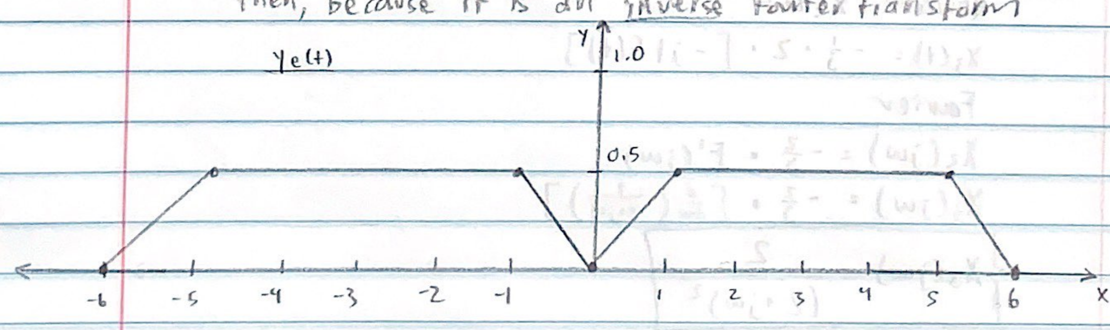
$y(t) = x(t-3)$

Based on this, we can see that $y(t)$ is real

Since $y(t)$ is real it means that the Real component of $Y(j\omega)$ is equivalent to the even component of $Y(j\omega)$.

$Re\{Y(j\omega)\} = Y_e(j\omega) = Y_e(t)$

The graph would be the even component of $y(t)$ then, because it is an inverse Fourier transform



4. a) i) $x_1(t) = \begin{cases} 1 + \cos(\pi t) & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$

$X(j\omega) = \int_{-1}^{1} [1 + \cos(\pi t)] e^{-j\omega t} \, dt$

$= -\frac{1}{j\omega}[e^{-j\omega} - e^{j\omega}] + \frac{1}{2j(\pi-\omega)}[-e^{-j(\pi-\omega)} + e^{j(\pi-\omega)}]$

$- \frac{1}{2j(\pi+\omega)}[e^{-j(\pi+\omega)} - e^{j(\pi+\omega)}]$

$= \frac{2\sin(\omega)}{\omega} + \frac{\sin(\pi-\omega)}{(\pi-\omega)} + \frac{\sin(\pi+\omega)}{\pi+\omega}$

Remember $\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$

Simplify

$$\boxed{X(j\omega) = 2\,\text{sinc}\left(\frac{\pi}{\omega}\right) + \text{sinc}\left(\frac{\omega+\pi}{\pi}\right) + \text{sinc}\left(\frac{\omega-\pi}{\pi}\right)}$$

ii) $x_2(t) = e^{(1+3j)t} u(-t+1)$

Rewrite $x_2(t)$ so it can be split up easily

$x_2(t) = (e^{j3t})(e^{t-1})(e^{1})[u(-t+1)]$

- $e^{t} \cdot u(-t) = \frac{1}{1-j\omega}$
- $e^{(t-1)} \cdot u(-t+1) = \frac{e^{-j\omega}}{1-j\omega}$
- $e^{(t-1)} \cdot u(-t+1) \cdot e^{j3t} = \frac{e^{-j(\omega-3)}}{1-j(\omega-3)}$
- $e^{(t-1)} \cdot u(-t+1) \cdot e^{j3t} \cdot e^{1} = \frac{e^{1-j(\omega-3)}}{1-j(\omega-3)}$

$$\boxed{X_2(j\omega) = \frac{e^{1-j(\omega-3)}}{1-j(\omega-3)}}$$

iii) $x_3(t) = 2te^{-2t} u(t)$

From lecture : $F'(j\omega) = -jt[f(t)]$

Simplify : $e^{-2t}u(t) = \frac{1}{2+j\omega}$

Rewrite $x_3(t)$

$x_3'(t) = -\frac{1}{j} \cdot 2 \cdot [-jtf(t)]$

Fourier

$X_3(j\omega) = -\frac{2}{j} \cdot F'(j\omega)$

$X_3(j\omega) = -\frac{2}{j} \cdot \left[\frac{d}{d\omega}\left(\frac{1}{2+j\omega}\right)\right]$

$$\boxed{X_3(j\omega) = \frac{2}{(2+j\omega)^2}}$$

b) Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)| e^{j\angle X(\omega)} e^{j\omega t} d\omega$$

Set up integrals based on the different sections of $|X(\omega)|$

$$x(t) = \frac{1}{2\pi}\left[\int_{-3}^{-2} e^{-\frac{j}{2}\omega} e^{j\omega t} d\omega + \int_{2}^{3} e^{-\frac{j}{2}\omega} e^{j\omega t} d\omega + \int_{-2}^{2} \frac{1}{2} e^{-\frac{j}{2}\omega} e^{j\omega t} d\omega\right]$$

Simplify

$$x(t) = \frac{1}{2\pi}\left[\int_{-3}^{-2} e^{j(t-\frac{1}{2})\omega} d\omega + \int_{2}^{3} e^{j(t-\frac{1}{2})\omega} d\omega + \int_{-2}^{2} \frac{1}{2} e^{j(t-\frac{1}{2})\omega} d\omega\right]$$

① ② ③

① $\int_{-3}^{-2} e^{j(t-\frac{1}{2})\omega} d\omega = \dfrac{e^{-2j(t-\frac{1}{2})} - e^{-3j(t-\frac{1}{2})}}{j(t-\frac{1}{2})}$

② $\int_{2}^{3} e^{j(t-\frac{1}{2})\omega} d\omega = \dfrac{e^{3j(t-\frac{1}{2})} - e^{2j(t-\frac{1}{2})}}{j(t-\frac{1}{2})}$

③ $\int_{-2}^{2} e^{j(t-\frac{1}{2})\omega} d\omega = \dfrac{e^{2j(t-\frac{1}{2})} - e^{-2j(t-\frac{1}{2})}}{2j(t-\frac{1}{2})}$

$$x(t) = \frac{1}{2\pi}\left[\frac{e^{-2j(t-\frac{1}{2})} - e^{-3j(t-\frac{1}{2})}}{j(t-\frac{1}{2})} + \frac{e^{3j(t-\frac{1}{2})} - e^{2j(t-\frac{1}{2})}}{j(t-\frac{1}{2})} + \frac{e^{2j(t-\frac{1}{2})} - e^{-2j(t-\frac{1}{2})}}{2j(t-\frac{1}{2})}\right]$$

Simplify

$$x(t) = \frac{1}{2\pi}\left[\frac{e^{3j(t-\frac{1}{2})} - e^{-3j(t-\frac{1}{2})}}{j(t-\frac{1}{2})} + \frac{e^{-2j(t-\frac{1}{2})} - e^{2j(t-\frac{1}{2})}}{2j(t-\frac{1}{2})}\right]$$

Simplify (Eulers)

$$\boxed{x(t) = \frac{1}{2\pi}\left[\frac{2\sin(3t-\frac{3}{2})}{(t-\frac{1}{2})} - \frac{\sin(2t-1)}{(t-\frac{1}{2})}\right]}$$

c) $f_1(t) = \text{sinc}(2t)$

$f_2(t) = \text{sinc}(t)\cos(3\pi t)$ $\left[\text{sinc}(t) \stackrel{\Delta}{=} \frac{\sin(\pi t)}{\pi t}\right]$

$f(t) = (f_1 * f_2)(t)$

i) Fourier Transform $F(jw)$

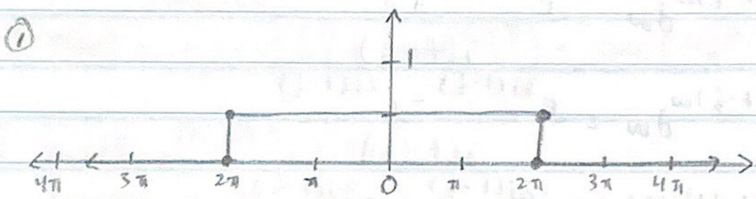
① $F_1(jw) = \frac{1}{2}\,\text{rect}\left(\frac{1}{4\pi}w\right)$

② $F_2(jw) = \frac{1}{2}\left[\text{rect}\left(\frac{1}{2\pi}w - 3\pi\right) + \text{rect}\left(\frac{1}{2\pi}3\pi + w\right)\right]$

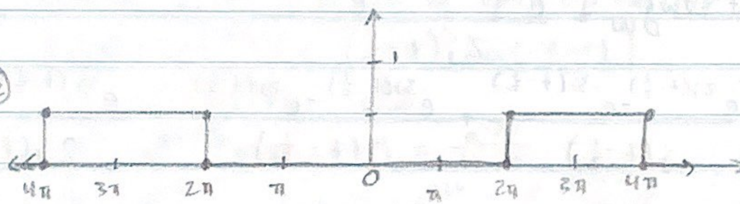
$F_1(jw) \cdot F_2(jw) = F(jw)$

Graph ① and ② to observe how they overlap

①



②



Miraculously, there is no overlap at all.

Therefore, $\boxed{F(jw) = 0}$

ii) $f(t)$

Since the Fourier transform of $f(t)$ was 0, that must mean that $f(t)$ is also 0.

$\boxed{f(t) = 0}$