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EE102

Lecture 13

EE102 Announcements

- Syllabus link is tinyurl.com/ucla102
- **Homework 5 now due Tuesday (Eid Mubarak as well!)**
- **Homework 6 and 7 are extremely difficult**
  - **Extra time: will be due Tuesday (OUT on the same day)**

Midterm Grades

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Last Lecture

The Fourier transform is:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

The inverse Fourier transform is:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega}$$

… and we did a few computations

Summary: Fourier Transforms we Now Know

$$\mathrm{rect}(t/T) \iff T\,\mathrm{sinc}(\omega T/2\pi)$$

$$e^{-at}u(t) \iff \frac{1}{a+j\omega}$$

$\mathrm{rect}(t) \iff$

Fourier transform operator

In this lecture, we'll use the Fourier transform as something that operates on signals. To denote the operation of taking the Fourier transform, we use $\mathcal{F}(\cdot)$ or $\mathcal{F}[\cdot]$. That is, if

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt$$

$$f(t) \iff F(j\omega)$$

$$\mathcal{F}[f(t)]$$

we may alternately write this as

$$F(j\omega) = \mathcal{F}[f(t)]$$

Likewise, the operator $\mathcal{F}^{-1}$ refers to the inverse Fourier transform. Therefore,

$$\mathcal{F}^{-1}[F(j\omega)] = f(t)$$

$$\underset{\text{operator}}{\mathcal{F}} \neq \underset{\text{signal}}{F}$$

This also means that

$$\mathcal{F}^{-1}[\mathcal{F}[f(t)]] = f(t)$$

at all points of continuity in $f(t)$.

Fourier symmetries

Derivations analogous to Fourier series.

Summary of symmetries

- For any $f(t)$, whether it be real, imaginary or complex:
  - $f(t)$ even $\rightarrow F(j\omega)$ even.
  - $f(t)$ odd $\rightarrow F(j\omega)$ odd.
- A *real* signal has a Hermitian Fourier transform:

$$F(-j\omega) = F^*(j\omega)$$

- An *imaginary* signal has an anti-Hermitian Fourier transform:

$$F(-j\omega) = -F^*(j\omega)$$

Please feel free to review

Further, for f(t) real and even, F(jw) is real and even.
For f(t) real and odd, F(jw) is imaginary and odd.
For f(t) imaginary and odd, F(jw) is real and odd.
For f(t) imaginary and even, F(jw) is imaginary and even.

Fourier Transform Properties

1. **Linearity:** $\boxed{\mathcal{F}[af_1(t) + bf_2(t)] = a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]}$
2. **Time scaling:** $\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(j\frac{\omega}{a}\right)}$
3. **Time reversal:** $\boxed{\mathcal{F}[f(-t)] = F(-j\omega)}$
4. **Complex conjugate:** $\boxed{f^*(t) \iff F^*(-j\omega)}$
5. **Duality:** $\boxed{F(t) \iff 2\pi f(-j\omega)}$
6. **Time-shifting:** $\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)}$
7. **Derivative:** $\boxed{\mathcal{F}[f'(t)] = j\omega F(j\omega)}$
8. **Convolution:** $\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$

Fourier Transform Properties

9. Parseval's theorem:

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega}$$

10. Multiplication:

$$\boxed{\mathcal{F}[f_1(t) f_2(t)] = \frac{1}{2\pi} (F_1 * F_2)(j\omega)}$$

11. Modulation:

$$\boxed{\mathcal{F}[f(t) e^{j\omega_0 t}] \quad = \quad F(j(\omega - \omega_0))}$$

Please review the proofs we do not cover in lecture.

Linearity

The Fourier transform is linear.

For two signals, $f_1(t)$ and $f_2(t)$, and two complex numbers a and b,

$$\boxed{\mathcal{F}[af_1(t) + bf_2(t)] = a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]}$$

Another way to write this is

$$af_1(t) + bf_2(t) \iff aF_1(j\omega) + bF_2(j\omega)$$

where $F_1(j\omega) = \mathcal{F}[f_1(t)]$ and $F_2(j\omega) = \mathcal{F}[f_2(t)]$.

Linearity proof

cyu: Please prove that F.T. is linear

$$\mathcal{F}[af_1(t) + bf_2(t)] = a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]$$

$\forall\, a, b, f_1(t), f_2(t)$

LHS

RHS

Pf: $\mathcal{F}\left[af_1(t) + bf_2(t)\right] = \int_{-\infty}^{\infty} \left(af_1(t) + bf_2(t)\right) e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} af_1(t)e^{-j\omega t} + \int_{-\infty}^{\infty} bf_2(t)e^{-j\omega t}\, dt$$

$$= a\int_{-\infty}^{\infty} f_1(t)e^{-j\omega t} + b\int_{-\infty}^{\infty} f_2(t)e^{-j\omega t}\, dt$$

$$= a\mathcal{F}\left[f_1(t)\right] + b\mathcal{F}\left[f_2(t)\right]$$

Extended Linearity

$\mathcal{F}$ has a cost of $\Theta(N \log N)$

This extends to finite combinations, i.e.,

$$\mathcal{F}\left[\sum_{k=1}^{K} a_k f_k(t)\right] = \sum_{k=1}^{K} a_k \mathcal{F}\left[f_k(t)\right]$$

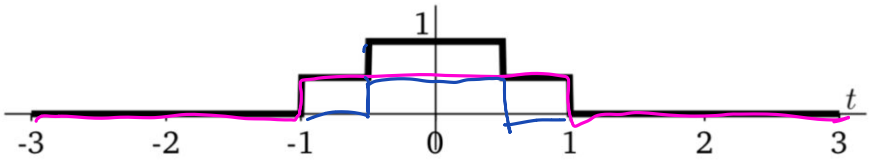
cyu: What's the intuitive implication here?

Linearity Example

Consider the signal:

$$f(t) = \begin{cases} \frac{1}{2}, & \frac{1}{2} \leq |t| \leq 1 \\ 1, & |t| \leq \frac{1}{2} \end{cases}$$

This signal steps up and then steps down, as shown below.



What is its Fourier transform?

$$f(t) = \frac{1}{2} \operatorname{rect}\left(\frac{t}{2}\right) + \frac{1}{2} \operatorname{rect}(t)$$

cyu

Linearity Example

$$f(t) = \underbrace{\frac{1}{2}\,\text{rect}\left(\frac{t}{2}\right)} + \underbrace{\frac{1}{2}\,\text{rect}(t)}$$

$$F(j\omega) = \frac{1}{2}\cdot 2\cdot \text{sinc}\left(\frac{\omega\cdot 2}{2\pi}\right) + \frac{1}{2}\cdot 1\cdot \text{sinc}\left(\frac{\omega\cdot 1}{2\pi}\right)$$

$$= \text{sinc}\left(\frac{\omega}{\pi}\right) + \frac{1}{2}\,\text{sinc}\left(\frac{\omega}{2\pi}\right)$$

Recall:

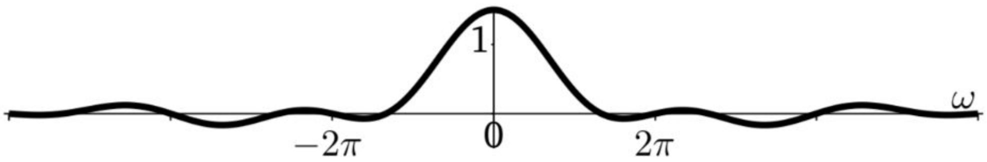
$$\text{rect}\left(t/T\right) \Longleftrightarrow T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right)$$

Linearity Example

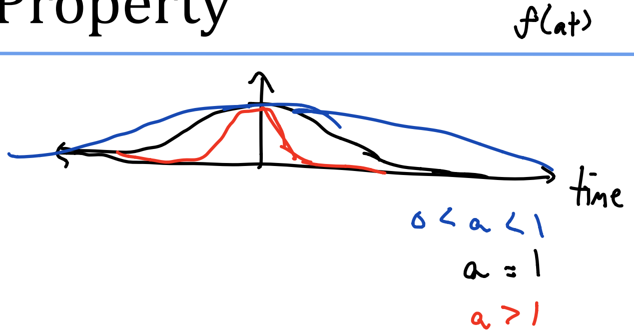
and therefore

$$
\begin{aligned}
F(j\omega) &= \frac{1}{2} 2\,\text{sinc}(2\omega/2\pi) + \frac{1}{2}\,\text{sinc}(\omega/2\pi) \\
&= \text{sinc}(\omega/\pi) + \frac{1}{2}\,\text{sinc}(\omega/2\pi)
\end{aligned}
$$

This is shown below:



Time Scaling Property



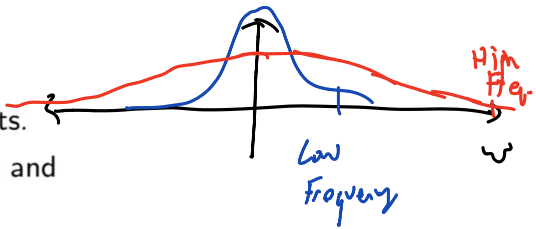
Fourier transform of a time scaled signal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(j\frac{\omega}{a}\right)}$$

Note, for real a:

- If $a > 1$, $f(t)$ contracts, but its Fourier transform expands.
- If $0 < a < 1$, then $f(t)$ expands, but its Fourier transform contracts.
- Thus, stretching a signal in time compresses its Fourier transform, and compacting the signal expands its Fourier transform.
- Does this make intuitive sense?



Time-scaling property

Fourier transform of a time scaled signal (cont.)

To show this, let's consider $a > 0$. (The proof is essentially the same for $a < 0$, which you can do on your own.) We will use a variable change, $\tau = at$, which means that $d\tau = adt$.

$$\mathcal{F}(f(at)) = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt$$

$$\Rightarrow \frac{1}{a}\mathcal{F}\left[j\frac{\omega}{a}\right]$$

$\tau = at$

$d\tau = adt$

$\omega' = \frac{\omega}{a}$

$$= \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\frac{\tau}{a}} \frac{d\tau}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(\tau)e^{-j\frac{\omega}{a}\tau}$$

$$= \frac{1}{a} F(j\omega')$$

$$= \frac{1}{a} F\left(j\frac{\omega}{a}\right)$$

Time-scaling Property

$f(t) \qquad f(at)$

set $a = T$

Example: Knowing that

$$\text{rect}(t/T) \iff T\,\text{sinc}(\omega T/2\pi)$$

we can then determine that the Fourier transform of $\text{rect}(t)$ is $\text{sinc}(\omega/2\pi)$

$\mathcal{F}\left[\text{rect}(t)\right]$, set $a = T$

$$\text{rect}\left(\frac{t}{T}\cdot a\right) = \text{rect}\left(\frac{t}{T}\cdot T\right) \iff \frac{1}{T}\cdot T\cdot\text{sinc}\left(\frac{\omega}{T}\cdot\frac{T}{2\pi}\right)$$

$$= \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

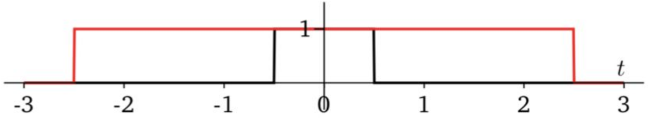
$$\therefore \quad \text{rect}(t) \iff \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

"Rect: Boxcar"

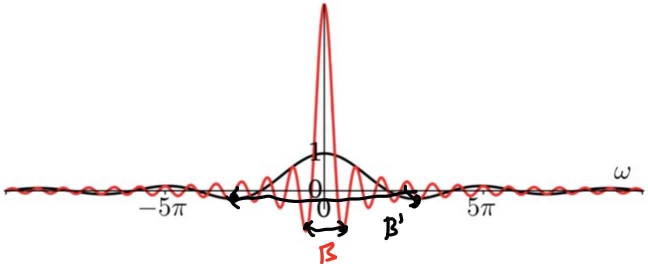
Time-scaling Property

Bandwidth example

Let's nail down our intuition for the time-scaling theorem. Consider two rect pulses, $\mathrm{rect}(t)$ and $\mathrm{rect}(t/5)$.



These are their Fourier transforms.



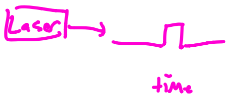
The fatter rect has a narrower spectrum. The width of the spectrum is called bandwidth. So a shorter pulse has a larger bandwidth. What does this mean intuitively?

1 bit/1s = 1 b.p.s.

1 bit/5s = 0.2 b.p.s.

$B' > B$

Laser Communication



Time-shift

Fourier transform of a time shifted signal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau} F(j\omega)}$$

cyu: Tell me why this equation is intuitive?

cyu: Prove this property

$\tau' = t - \tau$

$d\tau' = dt$

$t = \tau' + \tau$

$$\mathcal{F}[f(t-\tau)] = \int_{-\infty}^{\infty} f(t-\tau)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau')\, e^{-j\omega(\tau'+\tau)}\, d\tau'$$

$$= e^{-j\omega\tau} \underbrace{\int_{-\infty}^{\infty} f(\tau')\, e^{-j\omega\tau'}\, d\tau'}_{F(j\omega)}$$

Convolution Theorem

⋆⋆⋆ The Convolution Theorem ⋆⋆⋆

It's probably worth taking up an entire slide here just to say:

This is one of the most important theorems of the class, and is a key reason why a lot of our technology works. (!) This theorem enables us to do convolution, and thus any LTI operation, straightforwardly. With it, we no longer have to do the impulse response integral we saw earlier.

Convolution Theorem

⋆⋆⋆ The Convolution Theorem ⋆⋆⋆

If $f_1(t)$ and $f_2(t)$ are two signals with Fourier transforms $F_1(j\omega)$ and $F_2(j\omega)$, respectively, then

$$\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$$

Stated simply: **convolution in the time domain is multiplication in the frequency domain.**

(And multiplication is easy.)

① Painful

② Impossible to straightforwardly get back x.

$$\begin{array}{l|ccccc}
\text{time} & x(t) & \rightarrow & \boxed{h(t)} & \rightarrow & y(t) = x(t) * h(t) \\
 & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\text{freq} & X(j\omega) & \times & H(j\omega) & = & Y(j\omega)
\end{array}$$

Proof of Convolution Theorem

To show this,

$$\mathcal{F}[(f_1 * f_2)(t)] = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,d\tau\right) e^{-j\omega t}\,dt$$

Time Shifting Property

$$= \int_{-\infty}^{\infty} f_1(\tau) \int_{-\infty}^{\infty} f_2(t-\tau)\, e^{-j\omega t}\,dt \quad d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, e^{-j\omega\tau}\, F_2(j\omega) \quad d\tau$$

$$= F_2(j\omega) \cdot F_1(j\omega)$$

Convolution Example

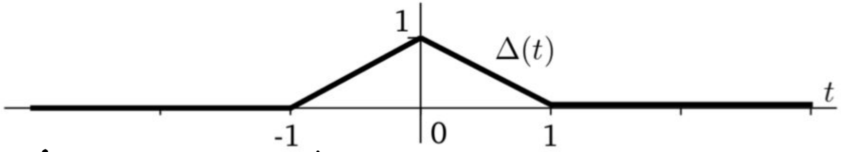
Convolution theorem example

What is the Fourier transform of the unit triangle,

$$\mathrm{rect}(t) \Longleftrightarrow \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$\Delta(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Recall:



$$\Delta(t) = \mathrm{rect}(t) \star \mathrm{rect}(t)$$

$$\mathcal{F}(\Delta(t)) = \mathcal{F}(\mathrm{rect}(t)) \cdot \mathcal{F}(\mathrm{rect}(t)) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) \cdot \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$$

Duality

Duality of the Fourier transform

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{F(t) \iff 2\pi f(-j\omega)}$$

This expression may be opaque at first. What this is saying is that if I take a Fourier transform pair, I can find the dual pair by replacing all the ω's with t's in $F(j\omega)$ and all the t's with $-\omega$'s in $f(t)$. After scaling by 2π, this results in another Fourier transform pair.

Essentially, every Fourier transform pair we derive really gives us two Fourier transform pairs.

Duality

Duality of the Fourier transform (cont.)

To show this, recognize that as

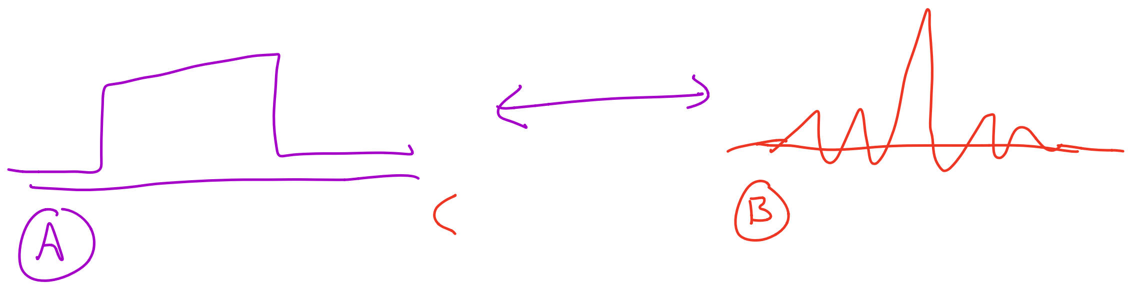
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

then

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega) e^{-j\omega t} d\omega$$

Now, the r.h.s. of this equation is the Fourier transform of $F(j\omega)$ with the roles of ω and t reversed. Hence, $2\pi f(-t)$ is the Fourier transform of $F(j\omega)$ (!) and after we swap the ω and the t's, we arrive at the duality result.

* The Fourier Transform is a math operator.
It doesn't have to specific to time / frequency.



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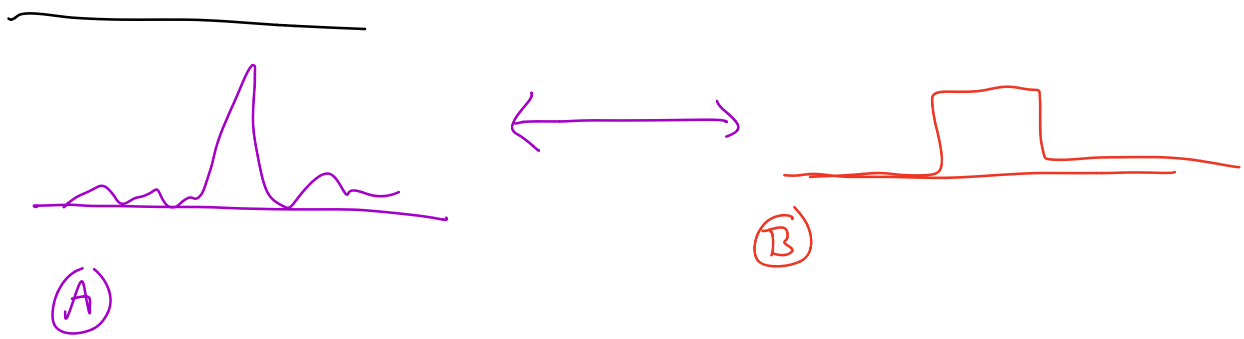


A B

Duality Examples

Duality examples

- Since $\mathrm{rect}(t) \iff \mathrm{sinc}(\omega/2\pi)$, then

$$\begin{aligned} \mathrm{sinc}(t/2\pi) &\iff 2\pi\mathrm{rect}(-\omega) \\ &= 2\pi\mathrm{rect}(\omega) \end{aligned}$$

  Thus, we have that $\mathrm{sinc}(t/2\pi) \iff 2\pi\mathrm{rect}(\omega)$.

- Since

$$e^{-at}u(t) \iff \frac{1}{a+j\omega}$$

  then

$$\frac{1}{a+jt} \iff 2\pi e^{a\omega}u(-\omega)$$

Dual intuition: convolution in time domain is multiplication in frequency domain. Thus, multiplication in the time domain ought be convolution in frequency domain.

Frequency domain convolution

The frequency domain convolution theorem is that for $f_1(t) \iff F_1(j\omega)$ and $f_2(t) \iff F_2(j\omega)$, then

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\nu)F_2(j(\omega-\nu))d\nu}$$

We typically write this as:

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}(F_1 * F_2)(j\omega)}$$

but note that the convolution is w.r.t. ω, not $j\omega$.

This means that multiplication in the time domain is convolution in the frequency domain. This proof is very similar to the time domain proof.

Modulation: duality of time-shifting

Dual intuition: Time shift in the time domain is multiplication by a complex exp. in freq domain
Thus, multiplication by a complex exp. in the freq domain ought be a shift in the freq domain.

Recall that:

$$\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)}$$

Using duality, we arrive at:

$$\mathcal{F}[f(t)e^{j\omega_0 t}] \quad = \quad F(j(\omega-\omega_0))$$

Using linearity, we also see that:

$$\begin{aligned}
\mathcal{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2}\left(F(j(\omega-\omega_0)) + F(j(\omega+\omega_0))\right) \\
\mathcal{F}[f(t)\sin(\omega_0 t)] &= \frac{1}{2j}\left(F(j(\omega-\omega_0)) - F(j(\omega+\omega_0))\right)
\end{aligned}$$

Modulation

Fourier transform of a modulated signal (cont.)

To prove the modulation result, note that if $\mathcal{F}[f(t)] = F(j\omega)$ then

$$\begin{aligned}\mathcal{F}[f(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt \\ &= F(j(\omega-\omega_0))\end{aligned}$$

To get the cosine and sine results, we note that e.g., for cosine,

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

From here, we can use linearity to compute the Fourier transform.

Modulation

Fourier transform of a modulated signal

A major component of communications has to do with *modulation*. For example, AM and FM radio are amplitude modulation and frequency modulation respectively. AM radio involves multiplying $f(t)$, the signal you wish to transmit, with a complex exponential at a carrier frequency, ω_0. This frequency, ω_0, is the frequency you dial in your car to get AM radio.

Here are three ways to modulate a signal: If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\begin{aligned}
\mathcal{F}[f(t)e^{j\omega_0 t}] &= F(j(\omega - \omega_0)) \\
\mathcal{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2}\left(F(j(\omega - \omega_0)) + F(j(\omega + \omega_0))\right) \\
\mathcal{F}[f(t)\sin(\omega_0 t)] &= \frac{1}{2j}\left(F(j(\omega - \omega_0)) - F(j(\omega + \omega_0))\right)
\end{aligned}$$

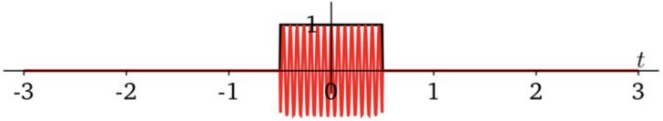
Typically, modulation is done through multiplication by $\cos(\omega_0 t)$. Modulation is dual to the time shift Fourier transform.

What modulation intuitively does is take $F(j\omega)$ and create replicas at $\pm\omega_0$.

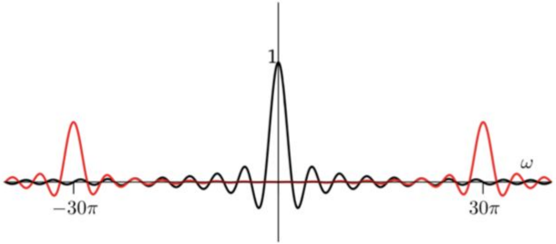
Modulation

Fourier transform of a modulated signal (cont.)

Below, we show what modulation does. We take a signal (here a rect) and multiply it by a cosine with $\omega_0 = 30\pi$. This is denoted in red in the plot below.



The spectrum takes the FT of our signal (i.e., a sinc) and creates replicas at $\pm 30\pi$.



From here, you can gain some intuition for why different radio stations use different frequencies. They're given these frequencies to transmit whatever signals they like; each radio station occupies a different part of the spectrum!

Time-reversal

Fourier transform of a time-reversed signal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(-t)] = F(-j\omega)}$$

To show this, apply the time-scaling result with $a = -1$.

Time-reversal

Time reversal example

Find the Fourier transform of $f(t) = e^{-a|t|}$ (for $a > 0$) without doing integration.

We know that

$$e^{-at}u(t) \iff \frac{1}{a + j\omega}$$

Time-reversal