EE102

Lecture 16

EE102 Announcements

- Syllabus link is tinyurl.com/ucla102
- **Homeworks due on Tuesday**

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

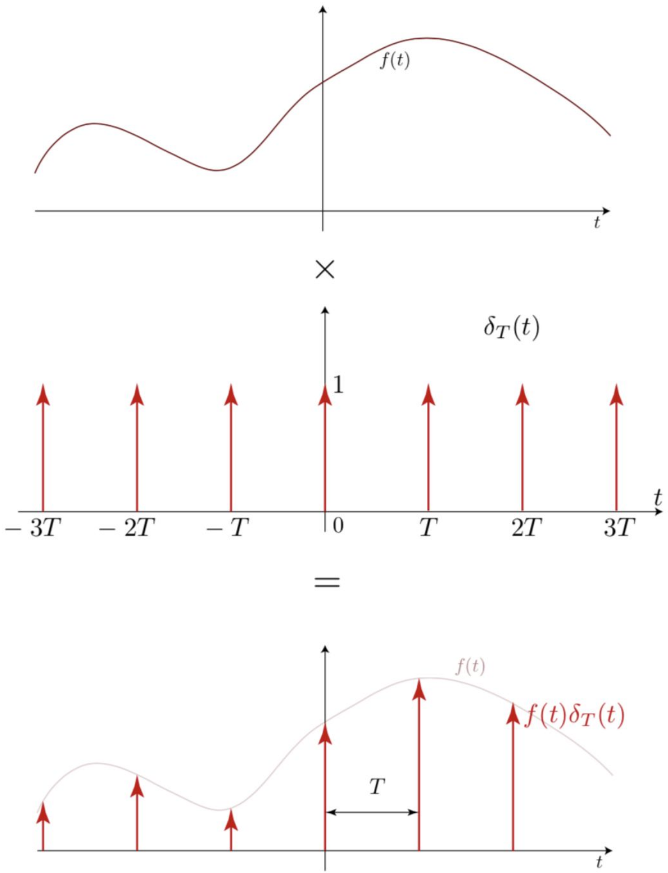
Sampling with an Impulse Train

As we saw earlier, one of the things we will use the impulse train for is to sample signals.

Given a signal $f(t)$,

$$f(t)\delta_T(t) \quad = \quad f(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Sampling with an Impulse Train

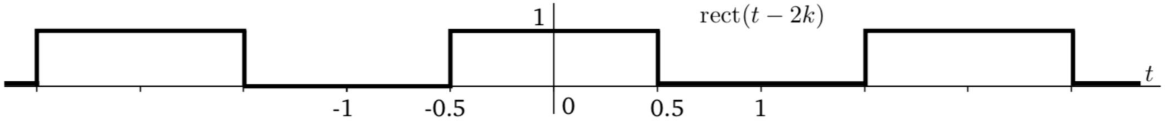


Sampling and Periodicity

Square wave, part 2

Let's revisit our square wave example, where

$$f(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}(t - 2k)$$



Another way to represent this square wave is as follows:

$$f(t) = \operatorname{rect}(t) * \delta_2(t)$$

Hence, we can calculate its Fourier transform by using the convolution theorem. Recall that, for $\omega_0 = 2\pi/T$,

$$\operatorname{rect}(t) \iff \operatorname{sinc}(\omega/2\pi)$$

and

$$\delta_T(t) \iff \omega_0 \delta_{\omega_0}(\omega)$$

Sampling and Periodicity

Square wave, part 2 (cont.)

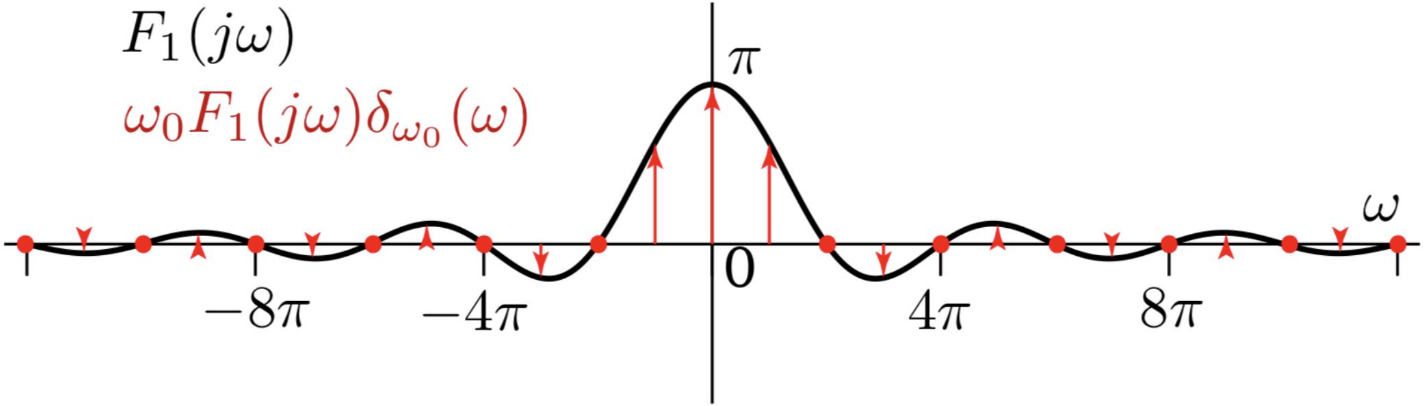
Note that when $T = 2$, then $\omega_0 = \pi$. Then, we have that,

$$
\begin{aligned}
\mathcal{F}\left[f(t)\right] &= \mathcal{F}\left[\text{rect}(t) * \delta_2(t)\right] \\
&= \mathcal{F}\left[\text{rect}(t)\right]\mathcal{F}\left[\delta_2(t)\right] \\
&= \text{sinc}(\omega/2\pi)\pi\delta_\pi(\omega)
\end{aligned}
$$

This is exactly the same Fourier transform we calculated earlier using the Fourier series of the square wave.

Sampling and Periodicity

Another intuition to remember here is that the Fourier transform of a periodic signal is the Fourier transform of one period of the signal (which we can denote f_1), sampled by an impulse train at multiples of ω_0.



Sampling and Periodicity

Discrete - periodic duality

We can determine the Fourier transform of a signal sampeld in the time-domain. Consider

$$\tilde{f}(t) = f(t)\delta_T(t)$$

Its Fourier transform is

$$\tilde{F}(j\omega) \quad = \quad \mathcal{F}\left[f(t)\delta_T(t)\right]$$

Sampling and Periodicity

Discrete - periodic duality (cont.)

This are merely samples of $F(j\omega)$ repeated every ω_0, since

$$\tilde{F}(j\omega) \quad = \quad \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega)$$

Sampling and Periodicity

Discrete - periodic duality (cont.)

This are merely samples of $F(j\omega)$ repeated every ω_0, since

$$\begin{aligned}
\tilde{F}(j\omega) &= \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega) \\
&= \frac{1}{T}F(j\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))
\end{aligned}$$

This leads us to the realization that:

- A signal that is periodic in time is discrete in spectrum.
- A signal that is discrete in time is periodic in spectrum.

There are important consequences from this result when we consider sampling signals in the time domain.

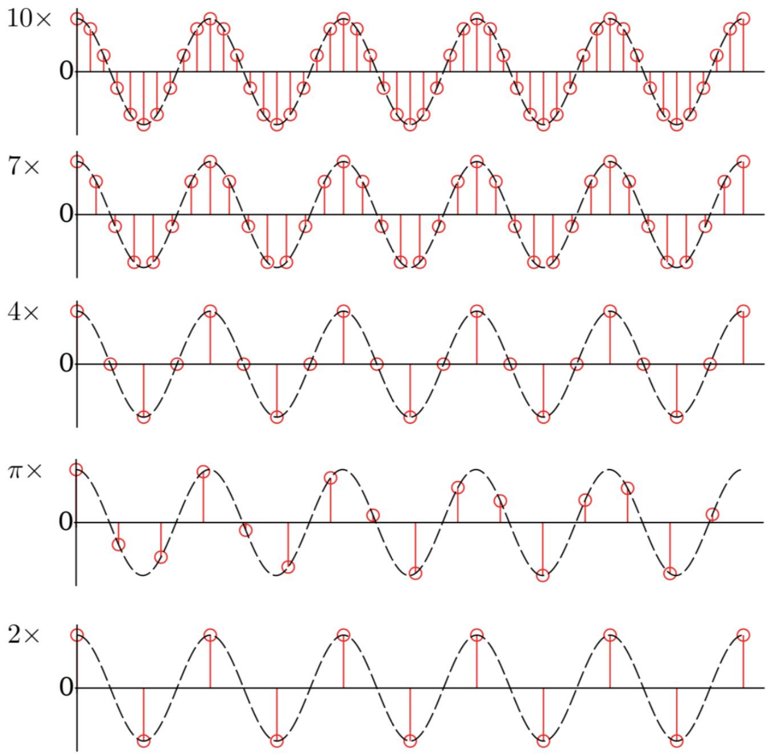
Sampling Theorem Motivation

Consider the following problem. We have a signal $f(t)$, and we need to store it. Our experimental set up is able to sample this signal at an interval T. How do we set T so that we can faithfully store $f(t)$? If T is too large, we sample infrequently and may lose information about $f(t)$. If T is too small, we waste memory and resources to store values we don't need.

The sampling theorem uses the results we've derived to tell us the minimum frequency at which we must sample $f(t)$ to not lose information. It is a very important theorem.

Sampling Theorem

Sampling example: sinusoids (cont.)



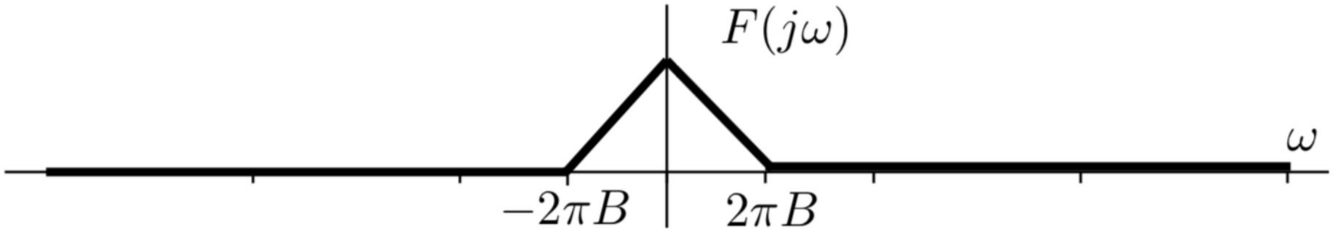
Sampling Theorem

If $\tilde{f}(t) = f(t)\delta_T(t)$, then as shown on the previous slides,

$$\tilde{F}(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))$$

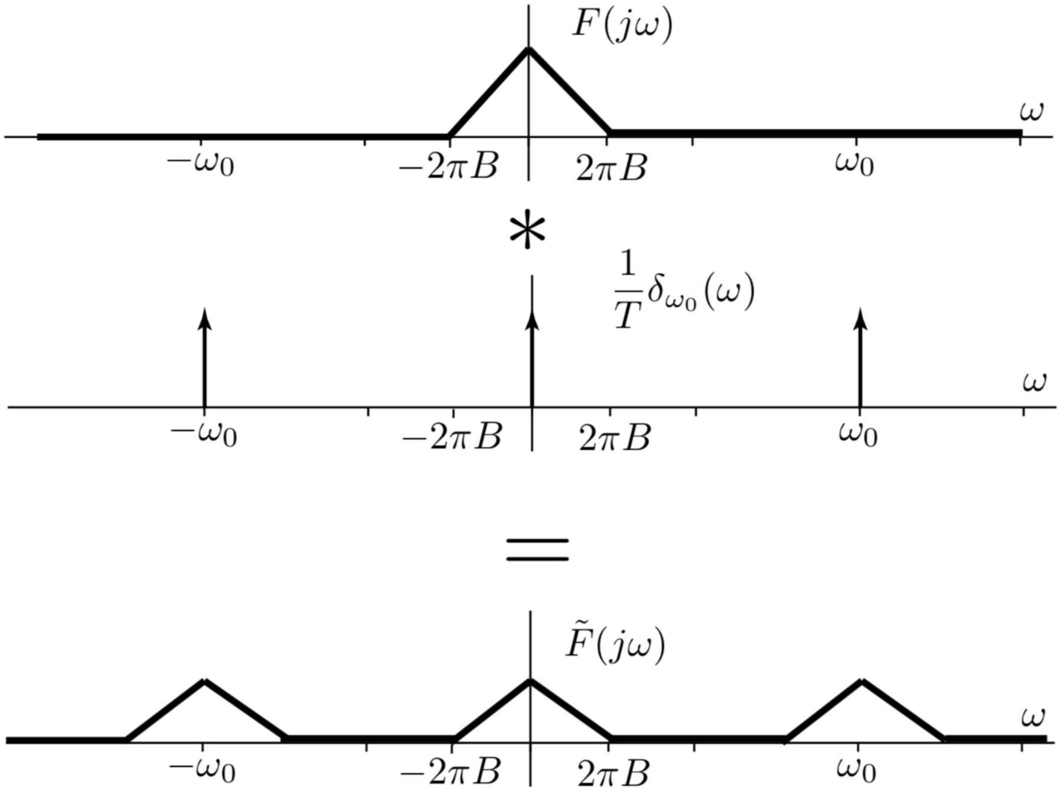
Therefore, the spectrum of $\tilde{f}(t)$ are shifted replicas of the spectrum, $F(j\omega) = \mathcal{F}[f(t)]$ spaced every ω_0 and scaled by $1/T$.

We define the bandwidth of $f(t)$ to be $\pm B$ Hz, e.g.,



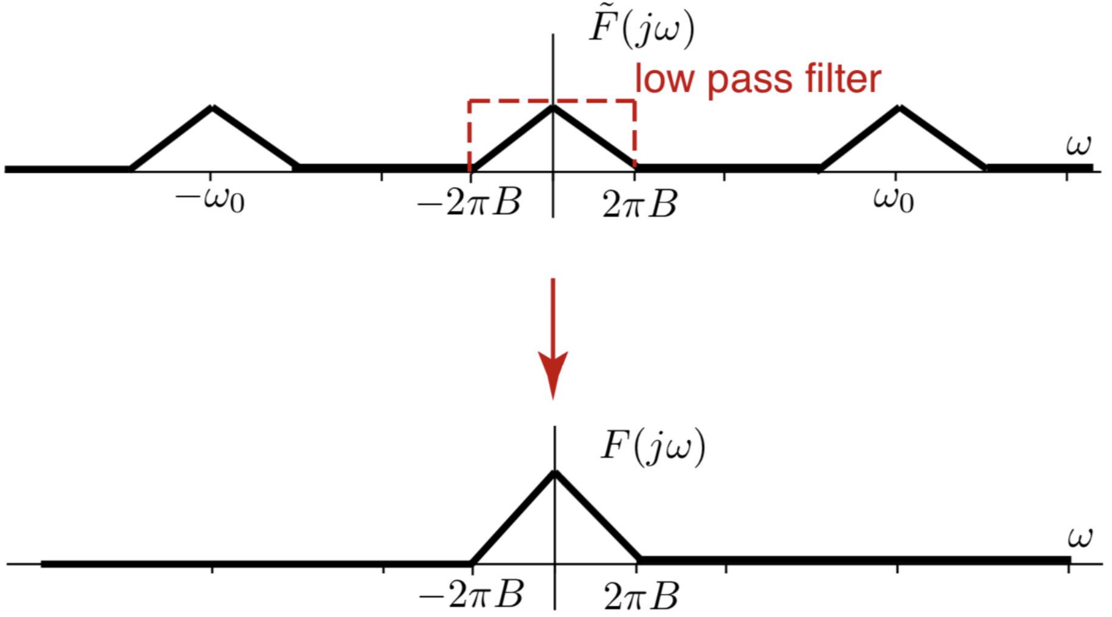
Sampling Theorem

For a particular choice of ω_0, where $\omega_0 \gg 2\pi B$, we see the spectrum of $\tilde{F}(j\omega)$ looks like:



Sampling Theorem

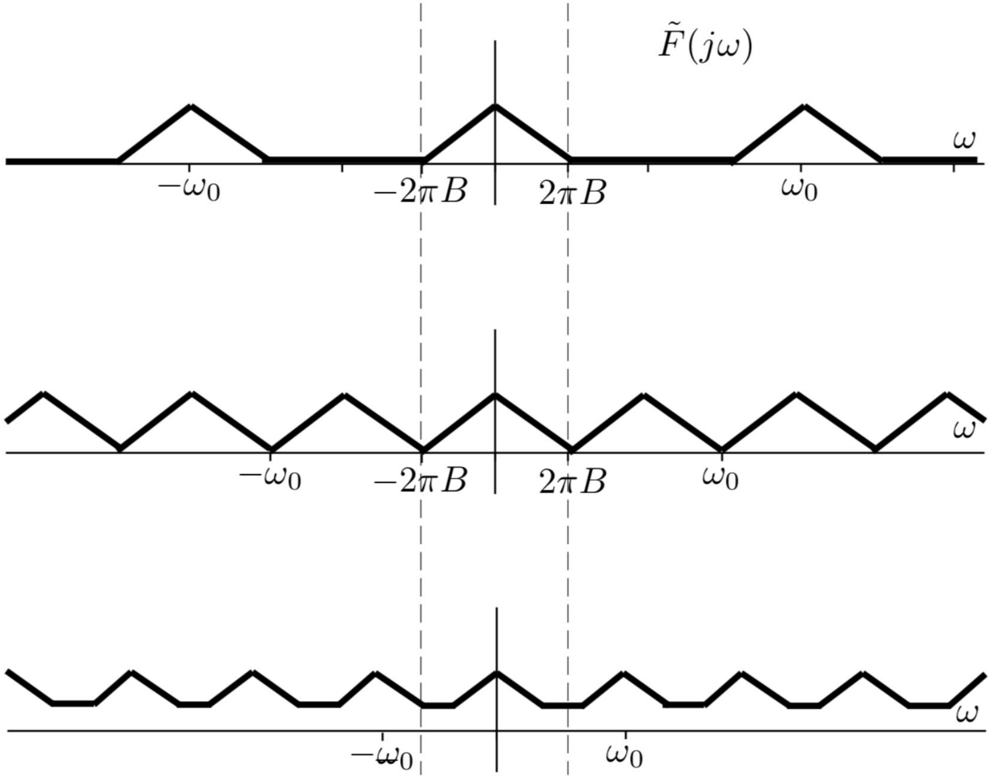
For this choice of ω_0, the original $F(j\omega)$ can be recovered through low pass filtering.



With ideal low pass filtering for the illustrated ω_0, we can *perfectly* recover $f(t)$ after sampling.

Sampling Thm

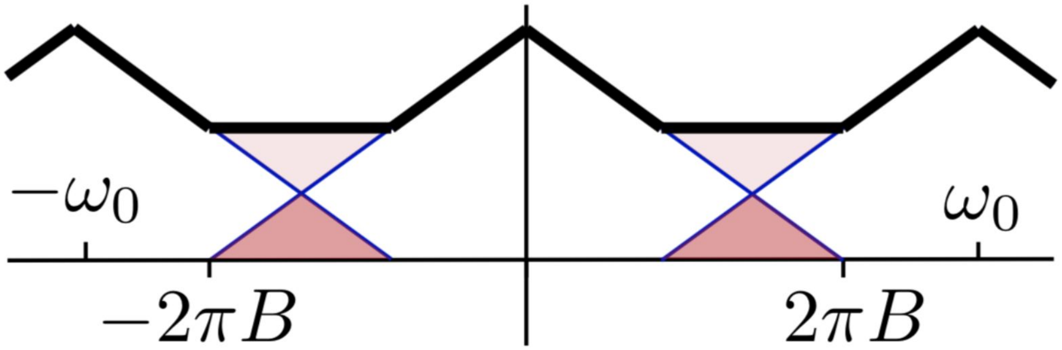
But now, as we increase the time T between samples, which decreases ω_0, the replicas of $F(j\omega)$ get closer and closer together.



Sampling Thm - Aliasing

We see that as ω_0 decreases, the bands start to overlap. When the replicas overlap, even with ideal low pass filtering, we cannot recover the original $F(j\omega)$.

This overlap is called *aliasing* because low frequencies of one spectral replica appear (or alias) as high frequencies in the next spectral replica. The vice versa is true as well; high frequencies of one spectral replica alias as low frequencies in an adjacent spectral replica. The alias'd sections are shown in the darker red.



Sampling Thm - Aliasing

To be able to perfectly recover a signal, we need to sample so as to avoid aliasing. No aliasing happens if $2\pi B < \omega_0/2$. We can simplify this as

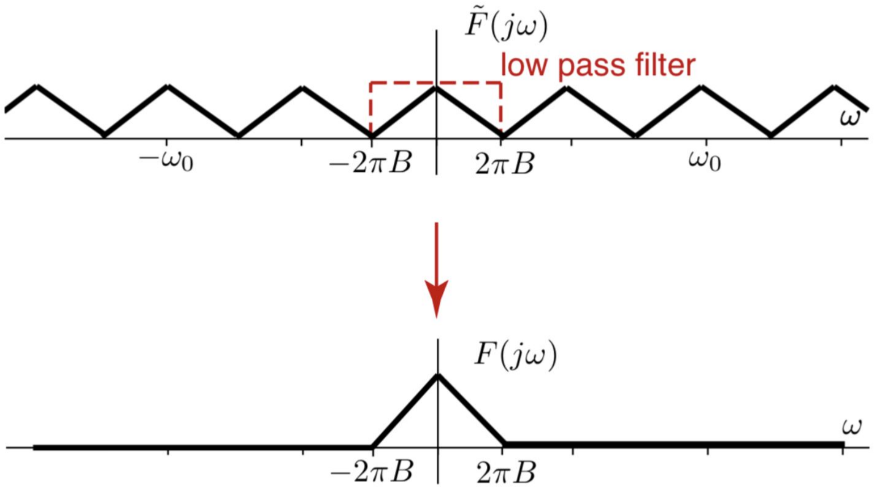
$$\begin{aligned} 2B &< \omega_0/2\pi \\ &= \frac{2\pi}{T}\frac{1}{2\pi} \\ &= \frac{1}{T} \end{aligned}$$

Therefore, the signal can only be recovered exactly if the signal bandwidth $2B$ is less than or equal to the sampling rate $1/T$. Hence, we need to sample at intervals less than or equal to $T = 2B$. This sampling rate, $2B$ is called the *Nyquist rate* for $f(t)$, and it is the lowest rate that we can sample $f(t)$ so that it can be perfectly recovered. T is called the *Nyquist interval*.

Interpolation

Interpolation

With a sampled signal, $\tilde{f}(t)$, as long as we have sampled at a rate $\geq 2B$, we can perfectly recover the original signal through ideal low pass filtering. Let's formalize how this happens, using the particular instantiation that $T = 1/2B$, i.e., we sample at the Nyquist rate.



Our low pass filter has frequency response

$$H(j\omega) = T\text{rect}\left(\frac{\omega}{4\pi B}\right)$$

Interpolation

The inverse Fourier transform of $H(j\omega)$ is

$$h(t) = 2BT\,\mathrm{sinc}(2Bt)$$

Since $T = 1/2B$, we can simplify this expression to

$$h(t) = \mathrm{sinc}(2Bt)$$

Therefore, to reconstruct $f(t)$ from $\tilde{f}(t)$, we calculate:

$$\begin{aligned}
\tilde{f}(t) * h(t) &= \left(\sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)\right) * h(t) \\
&= \sum_{k=-\infty}^{\infty} f(kT)h(t-kT) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2B(t-kT)) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt-k))
\end{aligned}$$

Interpolation

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula.

Intuition?

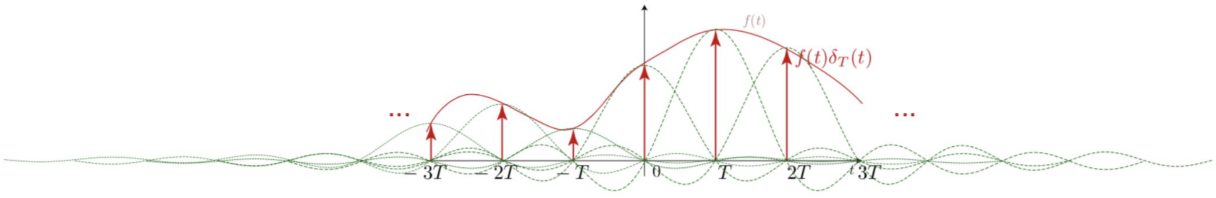
Interpolation

Recovering the original signal through interpolation (cont.)

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula. Intuitively, it does the following:



The sum of the green sinc functions will equal the red function, $f(t)$.

To not mince words, this result, which combines many of the things we've learned thus far, is remarkable. Through this reconstruction, we are able to *perfectly* recover an original signal from samples.

CYU: Homework Qst 5a.i

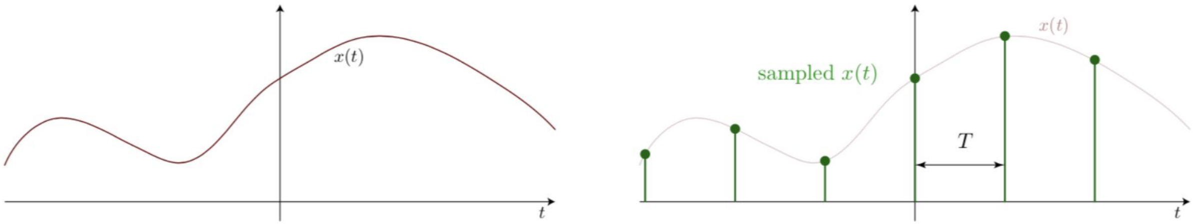
Assume $x(t)$ a real bandlimited signal where $X(j\omega)$ is non-zero for $|\omega| \leq 2\pi B$ rad/s. If F_s Hz is the Nyquist rate of $x(t)$, determine the Nyquist rate of the following signals in terms of B:

i. $x(t+1)$

Review: Sampling

Motivation

In reality, we could never store a continuous time signal. Instead, as we see in `MATLAB`, we store the signal's value at various times. This is called sampling, as illustrated below.



A key variable of interest is the sampling frequency, i.e., the time in between our samples, denoted T in the above diagram.

This is related to discrete signals, i.e., $x[n] = x(nT)$.

Review: Recovering Signal through Interpolation

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

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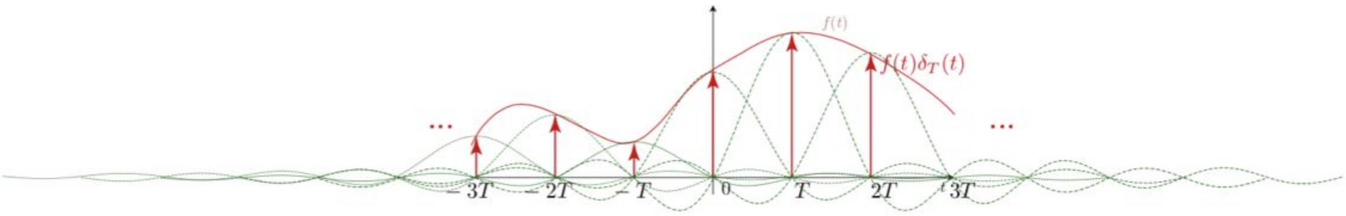
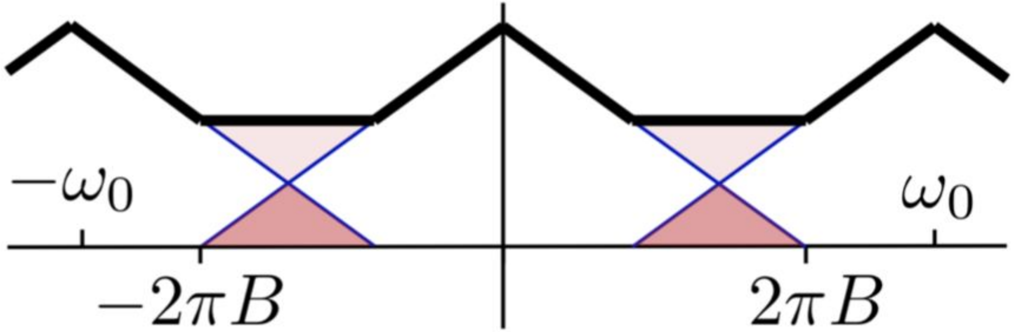


The sum of the green sinc functions will equal the red function, $f(t)$.

Review: Aliasing in Frequency

We see that as ω_0 decreases, the bands start to overlap. When the replicas overlap, even with ideal low pass filtering, we cannot recover the original $F(j\omega)$.

This overlap is called *aliasing* because low frequencies of one spectral replica appear (or alias) as high frequencies in the next spectral replica. The vice versa is true as well; high frequencies of one spectral replica alias as low frequencies in an adjacent spectral replica. The alias'd sections are shown in the darker red.

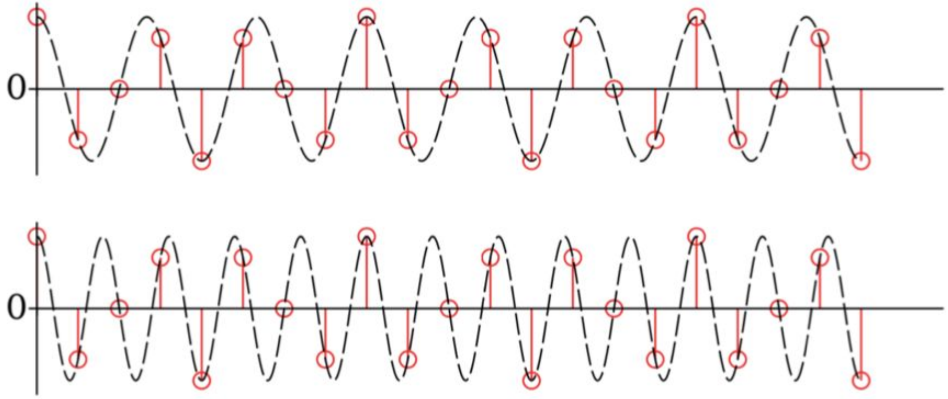


Aliasing (Look at it in Time)

We can intuitively understand aliasing in time domain for sinusoidal signals

Here's an example of aliasing. Below are two sinusoids. The upper one is at a frequency of $f = 0.75$ Hz and the one below is at $f = 1.25$ Hz.

Sampling both signals at $f_s = 2$ Hz,

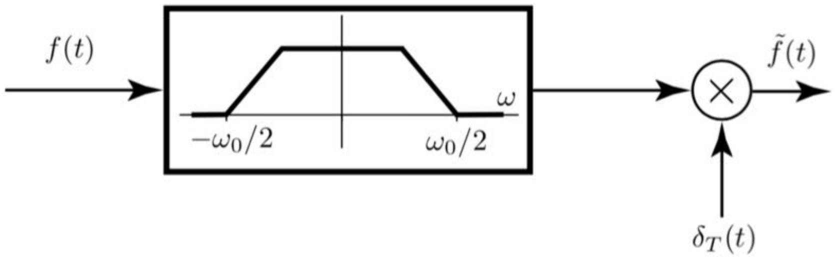


As you can see, the samples are the same for both sinusoids..

How to Ameliorate Aliasing

If you sample below the Nyquist rate, there will be aliasing. Once aliasing happens, there is no way to eliminate aliasing without having additional information about the signal.

One way we can ameliorate aliasing is to first low pass filter the signal, then sample:



- In reality, our low pass filters are not perfect, and so the bandwidth will be larger than $\omega_0/2$, however, we'll attenuate frequencies outside of range.
- Low pass filtering will distort the signal.
- However, the point is that when sampling, frequencies beyond $\omega_0/2$ would cause artifacts. Low pass filtering ameliorates this.
- It also supresses noise outside of $\omega_0/2$.

Laplace Transform

We're in our last major topic of the class: the Laplace transform.

The Laplace transform will extend much of the intuition that you've developed thus far. Informally, this part of the class is more algebraic.

We will see that one major application of the Laplace transform is that it gives us a simple framework to solve differential equations.

Laplace Transform

This lecture introduces the Laplace Transform and its properties. Topics include:

- s spectrum and region of convergence
- Bilateral Laplace transform
- Unilateral Laplace transform
- Relationship between Fourier and Laplace transforms
- Laplace transforms of e^{at}, $u(t)$, t^n, $\delta(t)$, and $\cos(\omega t)$
- Laplace transform properties
- Examples
- Solving differential equations

Motivation for Laplace Transform

The Fourier transform is powerful, but it doesn't exist for some signals and systems. In several applications, including image processing, communications, and circuit design, its sufficient for analysis.

However, some systems are unstable, or are power signals where the Fourier transform can not be straightforwardly generalized. Some examples of this are signals that grow with time, like (ideally) your bank account, or the S&P 500.

How do we analyze these systems in a similar framework to what Fourier analysis enables us to do?

Laplace Transform

Let

$$f(t) = e^{at}u(t)$$

When $a > 1$, this signal does not have a Fourier transform.

Laplace Transform

Let

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When $a > 1$, this signal does not have a Fourier transform.

One approach to arrive at a Fourier transform is to define a new function

$$g(t) = f(t)e^{-\sigma t}$$

If $\sigma > a$, then $g(t)$ is a decreasing exponential, which has a Fourier transform.

Laplace Transform

The function $g(t) = f(t)e^{-\sigma t}$ has a Fourier transform for σ sufficiently large.
The Fourier transform of $g(t)$ comprises how to sum spectral components $e^{j\omega t}$,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)e^{j\omega t} d\omega$$

Laplace Transform

The function $g(t) = f(t)e^{-\sigma t}$ has a Fourier transform for σ sufficiently large. The Fourier transform of $g(t)$ comprises how to sum spectral components $e^{j\omega t}$,

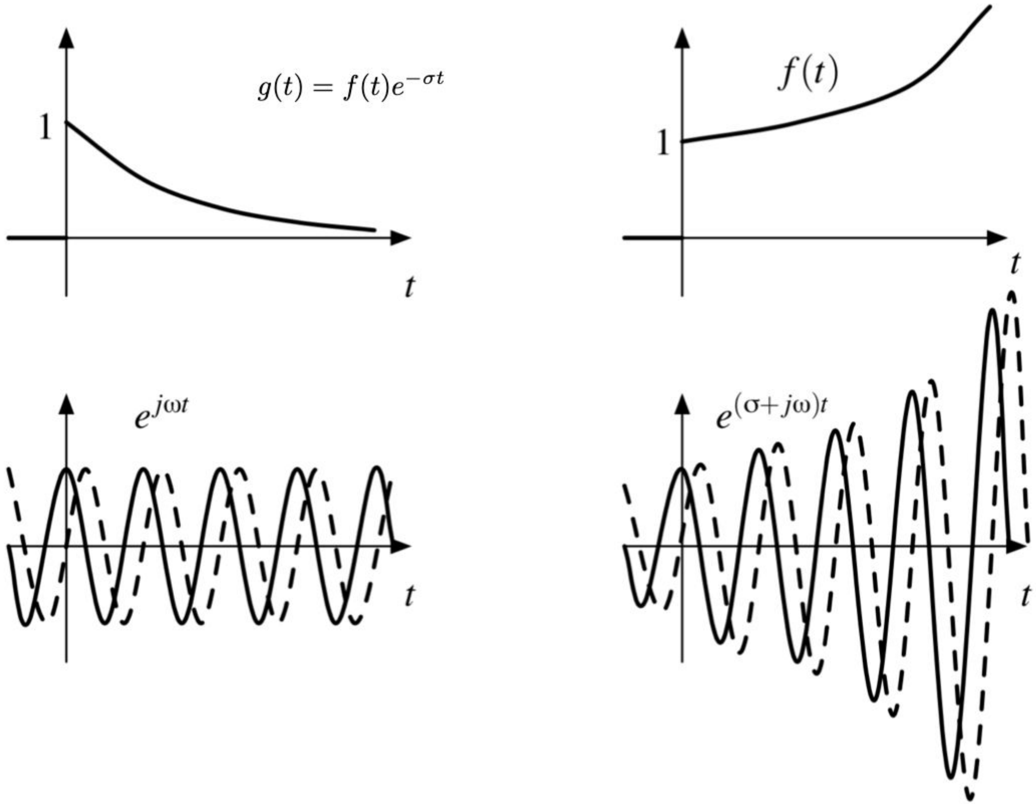
$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

The intuition here is that because $f(t) = g(t)e^{\sigma t}$, $f(t)$ has spectral components

$$e^{\sigma t} e^{j\omega t} = e^{(\sigma + j\omega)t}$$

Hence, the Laplace transform gives us a spectrum of $f(t)$ in terms of a complex exponential with both real and imaginary components (where as the Fourier transform was only with imaginary components).

Laplace Transform



Region of Convergence

When does the s-spectrum exist?

For what values of σ does this work? In the case where $f(t) = e^{at}u(t)$, this is clear, i.e., $\sigma > a$.

In general, there is some σ_0 for which

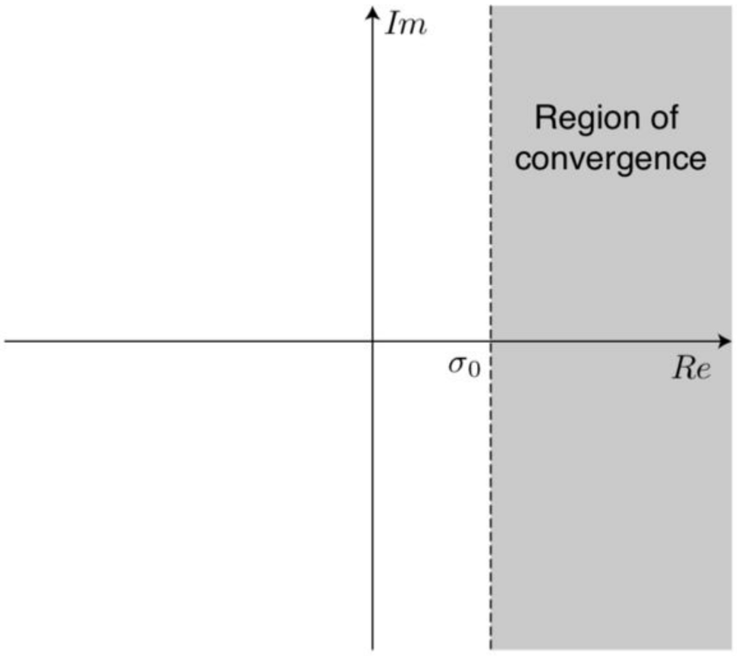
$$f(t)e^{-\sigma_0 t}$$

goes to zero. If it does, then this $f(t)e^{-\sigma_0 t}$ is an energy signal, and its spectrum will exist.

The portion of the complex plane where $\sigma > \sigma_0$ is called the "region of convergence."

Region of Convergence

The region of convergence is illustrated below:



Notation

Laplace transform notation

Our notation for the Laplace transform is very similar to our prior notation. We denote

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)] \\ f(t) &= \mathcal{L}^{-1}[F(s)] \end{aligned}$$

We will also denote this:

$$f(t) \iff F(s)$$

Bilateral Laplace Transform

The Laplace transform incorporates the real exponential. With $s = \sigma + j\omega$, as before,

- $j\omega$ is related to the oscillatory component of the complex exponential
- σ is related to the decay or growth of the complex exponential

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Then, the **bilateral** Laplace transform is:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt}$$

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Then, the **bilateral** Laplace transform is:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt}$$

To invert the bilateral Laplace transform, we calculate:

$$\boxed{f(t) = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds}$$

for $c > \sigma_0$.

Bilateral Laplace Transform

We won't use the bilateral Laplace transform, but it's worth mentioning this for completeness.

Unilateral Laplace Transform

Usually, we are interested in analyzing causal signals. In this case, we can simplify the bilateral Laplace transform. A causal signal can be written as $f(t)u(t)$, and its Laplace transform is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} f(t)e^{st}dt \end{aligned}$$

When we write 0^-, this indicates that impulses at the origin are included (e.g., $\delta(t)$ would have a contribution to this integral).

The Laplace transform is (essentially) unique. From now on, we'll use $\mathcal{L}[f(t)]$ to denote the unilateral Laplace transform of $f(t)$.

Relationship between Fourier and Laplace Transforms

The Fourier transform is a special case of the Laplace transform, i.e.,

$$\boxed{F(j\omega) = F(s)\big|_{s=j\omega}}$$

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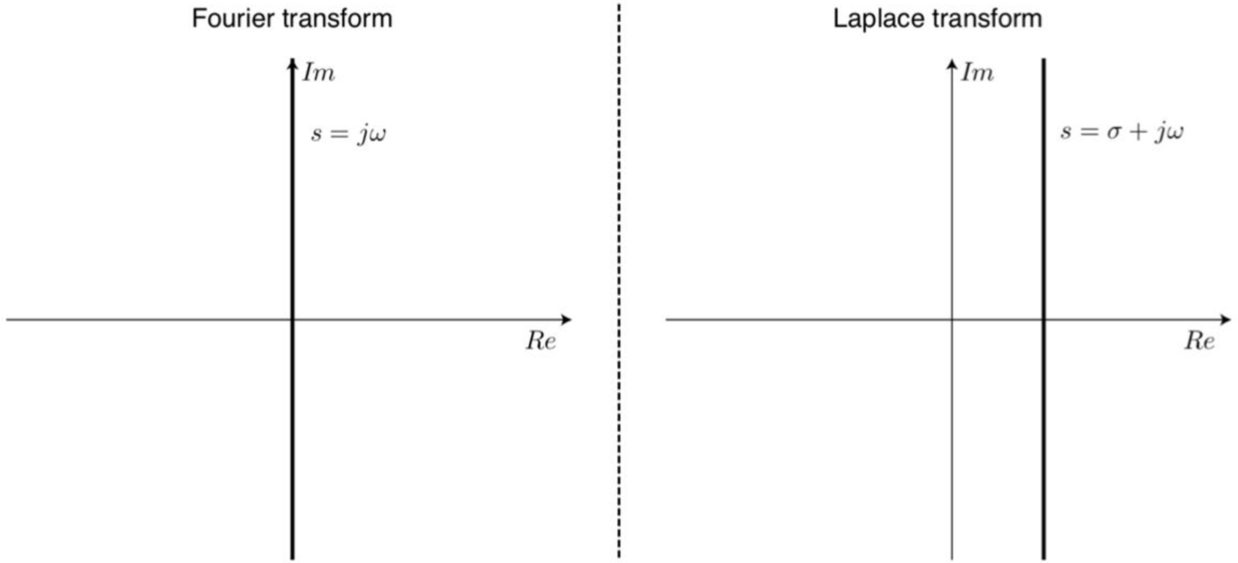
$$\boxed{F(j\omega) = F(s)\big|_{s=j\omega}}$$

The Fourier transform is evaluated at $s = j\omega$ and the Laplace transform is evaluated at a particular $s = \sigma + j\omega$.



Relationship between Fourier and Laplace Transforms

You may imagine that for signals where we know the Fourier transform, the Laplace transform merely replaces $j\omega$ with s. This is sometimes the case. Let's consider $f(t) = e^{-at}u(t)$.

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-at}e^{-st}dt \\
&= \int_0^\infty e^{-(a+s)t}dt \\
&= -\frac{1}{a+s}e^{-(a+s)t}\Big|_0^\infty \\
&= \frac{1}{a+s}
\end{aligned}
$$

as long as $e^{-(a+s)t} \to 0$ as $t \to \infty$. When does this happen?

Relationship b/w Fourier and Laplace (cont'd)

If $e^{-(a+s)t}$ goes to zero, then so does $\left|e^{-(a+s)t}\right|$.

$$\left|e^{-(a+s)t}\right| = \left|e^{-(a+\sigma+j\omega)t}\right|$$

Relationship b/w Fourier and Laplace (cont'd)

Hence, we have that

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

and we know prior, for $a > 0$,

$$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

Here, the Laplace transform is the Fourier transform with $j\omega$ replaced with s.

Relationship b/w Fourier and Laplace (cont'd)

A key thing to note is that with

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

holds for all a, positive or negative, as long as $\sigma > -a$.

This means that, for $a > 0$,

$$\mathcal{L}[e^{at}u(t)] = \frac{1}{s-a}$$

Of course, this signal does not have a Fourier transform.

CYU: Comparing Fourier and Laplace

Let’s take the Laplace transform of the unit step

Relationship between Fourier and Laplace

Recall the Fourier transform of the unit step is:

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

This resembles the Laplace transform with $s = j\omega$, but there is an additional $\pi\delta(\omega)$ term.

Relationship between Fourier and Laplace

We will see this tends to be the case for some of our generalized Fourier transforms. For example, consider the Laplace transform of

$$
\begin{aligned}
f(t) &= \cos(\omega t) \\
&= \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]
\end{aligned}
$$

Example Laplace Transforms

Laplace transform of powers of t

Laplace transforms, given all we've learned thus far, should be fairly straightforward to evaluate. We'll go over a few examples here. Let

$$f(t) = t^n$$

for $n \geq 1$. Then,

$$F(s) = \int_0^{\infty} t^n e^{st} dt$$

Example Laplace Transforms

Example Laplace Transforms

Laplace transform of impulse

Let

$$f(t) = \delta(t)$$

A Trend Emerging ..

Pattern for integration and differentiation?

Notice the following trends:

$$\begin{aligned} \delta(t) &\iff 1 \\ u(t) &\iff \frac{1}{s} \\ tu(t) &\iff \frac{1}{s^2} \\ \frac{1}{2}t^2u(t) &\iff \frac{1}{s^3} \\ \frac{1}{6}t^3u(t) &\iff \frac{1}{s^4} \\ &\vdots \end{aligned}$$

We see a clear pattern: differentiating a signal is equivalent to multiplying the Laplace transform by s while integrating is equivalent to multiplying the Laplace transform by $1/s$.

Laplace Transform Properties

1. **Linearity:** $$\boxed{\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)}$$
2. **Time scaling:** $$\boxed{\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}$$
3. **Time shift:** $$\boxed{\mathcal{L}[f(t-T)] = e^{-sT}F(s)}$$
4. **Frequency shift:** $$\boxed{\mathcal{L}\left[f(t)e^{s_0 t}\right] = F(s - s_0)}$$
5. **Convolution:** $$\boxed{\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)}$$
6. **Integration:** $$\boxed{\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)}$$
7. **Derivative:** $$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$$
8. **Multiplication by *t*:** $$\boxed{\mathcal{L}[tf(t)] = -F'(s)}$$

Differentiation and Integration Property

Key reason to use Laplace Transforms! Turns differential equations into algebraic equations.

$g(t)$	$G(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

CYU: Apply the Property to get L.T. of Step and Ramp

Example: unit step and ramp functions

We were able to calculate the Fourier transform of the unit step; however, this required generalizing the Fourier transform. How can we find the Laplace transforms of the unit step and unit ramp function using the integral properties of the Laplace transform?

If $f(t) = \delta(t)$, then $F(s) = 1$. Then,

Inversion of Laplace Transform

Motivation

The inverse of the Laplace transform is given by

$$f(t) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s)e^{st} ds$$

where σ is large enough that F(s) is defined for $\Re(s) \geq c$.

Catalog of Inverse Transforms to Keep Handy

May seem specific

But come up in Diff Eq!

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

$$\begin{aligned}
\mathcal{L}[\cos(\omega t)] &= \frac{s}{s^2+\omega^2} \\
\mathcal{L}[\sin(\omega t)] &= \frac{\omega}{s^2+\omega^2} \\
\mathcal{L}[e^{-at}\cos(\omega t)] &= \frac{(s+a)}{(s+a)^2+\omega^2} \\
\mathcal{L}[e^{-at}\sin(\omega t)] &= \frac{\omega}{(s+a)^2+\omega^2}
\end{aligned}$$

$$\mathcal{L}^{-1}\left[\frac{r}{(s-\lambda)^k}\right] = \frac{r}{(k-1)!}t^{k-1}e^{\lambda t}$$

Partial Fractions

Partial fraction expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

Partial Fractions

Partial fraction expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

Let's first assume that no poles are repeated and that $m < n$ (i.e., more poles than zeros).

Then, $F(s)$ can be written in its *partial fraction expansion*:

$$F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}$$

where

- $\lambda_1, \ldots, \lambda_n$ are the poles of F.
- The numbers $r_1, \ldots, r_n$ are called residues.
- It turns out when $\lambda_k = \lambda_l^*$, then $r_k = r_l^*$.

Partial Fractions

Inversion of a partial fraction

In partial fraction form, inverting the Laplace transform is easy because

$$\mathcal{L}^{-1}[F(s)] \quad = \quad \mathcal{L}^{-1}\left[\frac{r_1}{s-\lambda_1} + \cdots + \frac{r_n}{s-\lambda_n}\right]$$

Partial Fractions

How to find the partial fraction expansion

To find the partial fraction expansion, we

- Find the poles $\lambda_1, \ldots, \lambda_n$, which means we find the zeros of $a(s)$.
- Find the residues of $r_1, \ldots, r_n$.

3 Main Methods to find Partial Fractions

These are (sometimes painful) exercises in algebra.

Last Homework and Final Exam will use the least painful (cover-up method), which is sufficient to glean the concept. Supplemental lecture 17 may discuss other methods.

Partial Fractions via "Cover-up" Method

Here, we solve for each residual individually in the following way. E.g., to get r_1, we first multiply both sides by $(s - \lambda_1)$.

$$\frac{b_0 + b_1 s + b_2 s^2}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \frac{r_3}{s-\lambda_3}$$

becomes

$$\frac{(s-\lambda_1)(b_0 + b_1 s + b_2 s^2)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = r_1 + \frac{r_2(s-\lambda_1)}{s-\lambda_2} + \frac{r_3(s-\lambda_1)}{s-\lambda_3}$$

Concrete Example of Using the Cover-Up

Let's find the following partial fraction expansion:

$$\frac{s^2 - 2}{s(s+1)(s+2)} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

CYU: Putting it Together

Compute the Inverse Laplace Transform (Hw 4b)

$$F(s) = \frac{s+4}{s^3+4s}$$

CYU: Apply it to Differential Equations

LTI system (20 points)

Assume a causal LTI system $\mathcal{S}_1$ is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = ax(t), \qquad y(0) = 0,\ y'(0) = 0$$

where a is a constant. Moreover, we know that when the input is e^t, the output of the system $\mathcal{S}_1$ is $\frac{1}{2}e^t$.

(a) (6 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

Next Lecture is the Last

Mostly Applications of what we have learned

1. Applying the Laplace Transform to more complicated Diff Eq for those who want to go further into Laplace.
2. Multi-dimensional signals and Applying the Fourier Transform to COVID-19
3. Other Touch-Ups that are good for practicing engineers.

Final Exam is Lectures 1-16, but a bonus qst may be taken from L17.