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ECE 102 Final

1. a) i) $y(t) = x(3t+2)+5$

Not causal - system can depend on future values of the input

Ex: $y(2) = x(8) + 5$

Time variant - $y_\tau(t) \neq y(t-\tau)$

$y_\tau(t) = x(3t+2-\tau)+5$

$y(t-\tau) = x(3(t-\tau)+2)+5$

Stable

If $|x(t)| \leq B_x$ for any t, then

$|y(t)| = |x(3t+2)+5| \leq B_x + 5$

Not linear - $S(ax) \neq aS(x)$

Ex: $x(10(3t+2))+5 \neq 10[x(3t+2)+5]$

$\boxed{\text{Not causal, Time variant, Stable, not linear}}$

ii) $y(t) = \sin\left(\frac{dx(t)}{dt}\right)$

Causal - system never depends on future values of the input

Time invariant

$y_\tau(t) = \sin\left(\frac{dx(t-\tau)}{dt}\right)$

$y(t-\tau) = \sin\left(\frac{dx(t-\tau)}{dt}\right)$

Stable

If $|x(t)| \leq B_x$ for any t, then

$|y(t)| \leq 1$

Not linear - $S(ax) \neq aS(x)$

$$\sin\left(\frac{d\,ax(t)}{dt}\right) \neq a\sin\left(\frac{dx(t)}{dt}\right)$$

$\boxed{\text{Causal, time-invariant, stable, not linear}}$

iii) $y(t) = e^{x^2(t)}$

Causal - system never depends on future values

Time invariant

$y_\tau(t) = e^{x^2(t-\tau)}$

$y(t-\tau) = e^{x^2(t-\tau)}$

Stable

if $|x(t)| \le B_x$ for any t, then

$|y(t)| \le e^{B_x^2}$

The exponent function is finite if the input is finite,

so the output is bounded $\rightarrow$ stable

Not linear

$e^{a x^2(t)} \neq a e^{x^2(t)}$

$\boxed{\text{Causal, time-invariant, stable, not linear}}$

1b) $H(jw) = |H(jw)| e^{j\angle H(jw)}$

$|H(jw)| = 1 \qquad \angle H(jw) = -w \qquad -5 < w < 5$

$X(jw) \cdot H(jw) = Y(jw)$

$X(jw) \cdot e^{-jw} = Y(jw) \qquad -5 < w < 5 \quad , 0 \text{ else}$

$Y(jw) = \Delta(w) \cdot e^{-jw}$

$\Delta(w) \Longleftrightarrow \frac{1}{2\pi} \operatorname{sinc}^2\left(\frac{t}{2\pi}\right)$

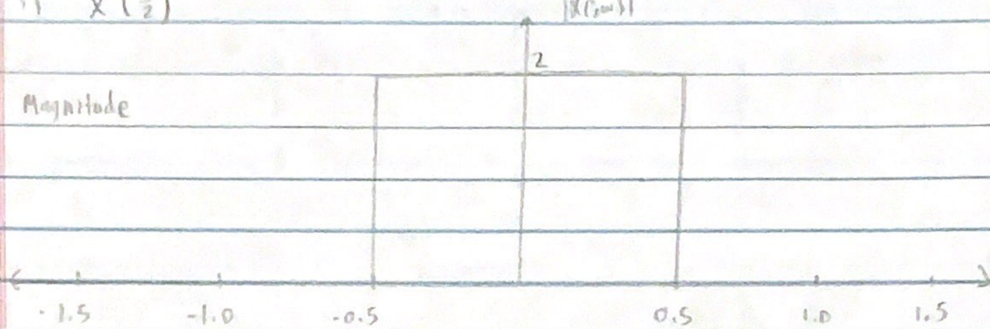
$y(t) = e^{-jw} \cdot \Delta(w)$

$y(t) = e^{-jw} \cdot \Delta(w) \Longleftrightarrow \frac{1}{2\pi} \operatorname{sinc}^2\left(\frac{t-1}{2\pi}\right)$

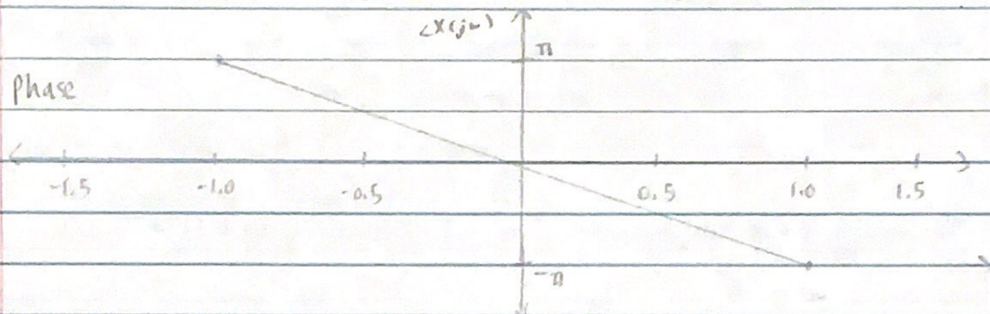
$\boxed{y(t) = \frac{1}{2\pi} \operatorname{sinc}^2\left(\frac{t-1}{2\pi}\right)}$

2a) i) $x\left(\frac{t}{2}\right)$

Magnitude



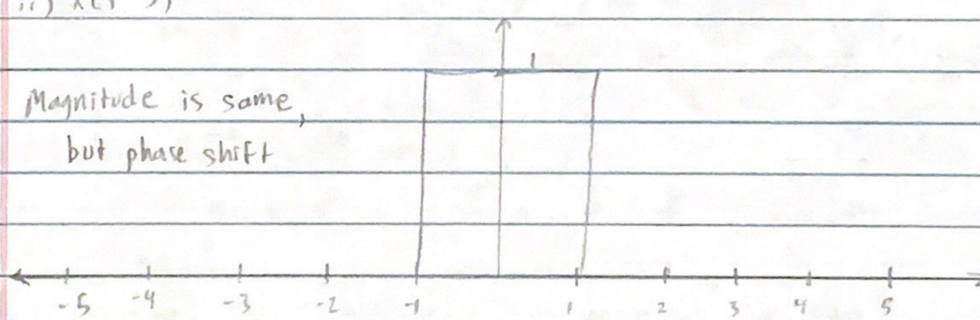
Phase



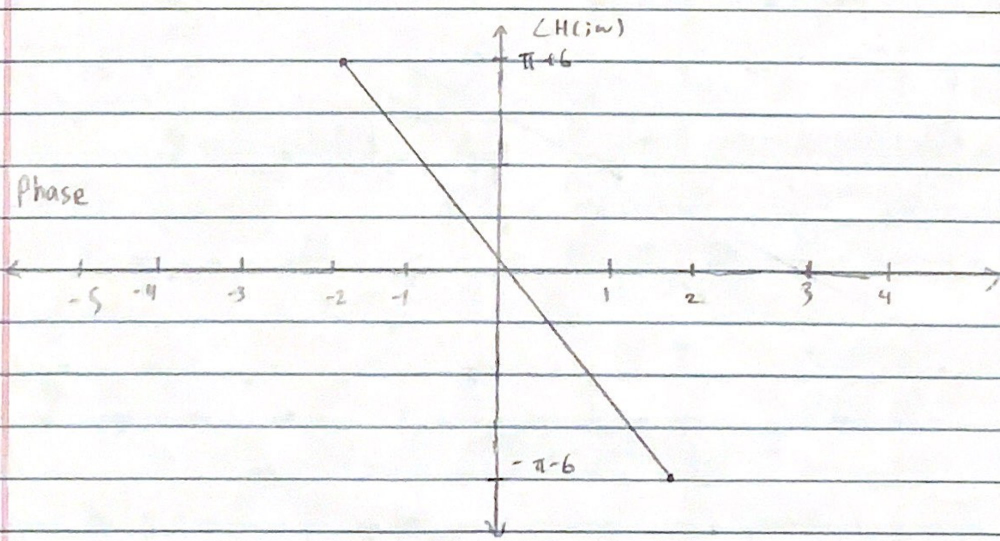
$X(j\omega) = \cos\left(-\frac{\pi\omega}{2}\right) + j\sin\left(-\frac{\pi\omega}{2}\right)$

ii) $x(t-3)$

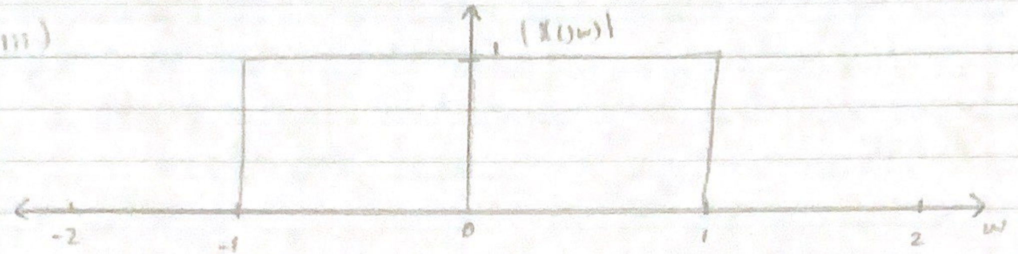
Magnitude is same, but phase shift

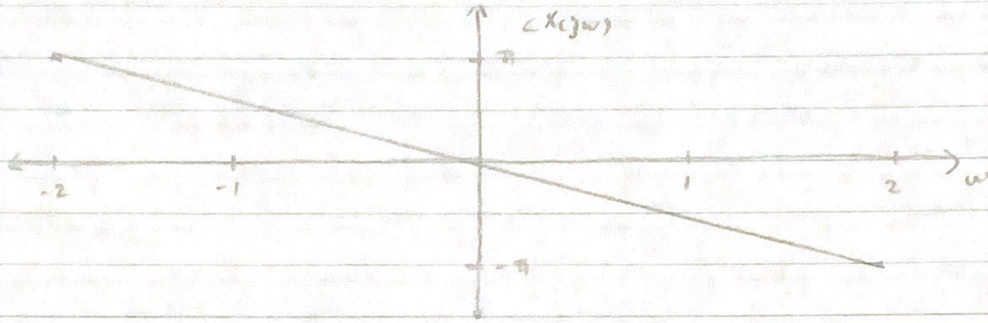


Phase



iii)





Magnitude is the same again as $X(jw)$ is Hermitian

2b) i) $x(t) = e^{-2|t-1|}$

From Fourier Table : $\mathcal{F}(e^{-a|t|}) = \frac{2a}{a^2+w^2}$

$$x(t) = e^{-a|t|} = \frac{1}{a+jw} + \frac{1}{a-jw} = \frac{2a}{a^2+w^2}$$

Plug in a, solve

$$e^{-2|t-1|} \Longleftrightarrow \frac{4}{4+w^2} e^{-jw}$$

$$X(jw) = \frac{4}{4+w^2} e^{-jw}$$

ii) $x(t) = te^{-at}\cos(w_0 t)u(t) \quad a > 0$

- From Fourier Table : $\mathcal{F}(te^{-at}u(t)) = \frac{1}{(a+jw)^2}$
- Linearity, modulation

$$\mathcal{F}[f(t)\cos(w_0 t)] = \frac{1}{2}[F(j(w+w_0)) + F(j(w-w_0))$$

Put these together

$$X(jw) = \frac{1}{2}\left[\frac{1}{(a+jw_0+jw)^2} + \frac{1}{(a+j(w-w_0))^2}\right]$$

2c) i) $X(j\omega) = \begin{cases} 1-|\omega| & |\omega|<1 \\ 0 & \text{otherwise} \end{cases}$ (triangle) $\Lambda(t)$

$\Lambda(t) = \text{rect}(t) * \text{rect}(t)$

$F[\text{rect}(t) * \text{rect}(t)]$

$= F[\text{rect}(t)] \cdot F[\text{rect}(t)]$

$= \text{sinc}^2(\frac{t}{2\pi})$

Need to multiply result by $\frac{1}{2\pi}$

$\boxed{x(t) = \frac{1}{2\pi}\,\text{sinc}^2(\frac{t}{2\pi})}$

ii) $X(j\omega) = \cos(2\omega + \frac{\pi}{6})$

Modify RHS, so it's more workable

$\cos(2\omega + \frac{\pi}{6}) = \frac{1}{2}[e^{2j\omega}e^{\pi/6 j} + e^{-2j\omega}e^{-\pi/6 j}]$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}[e^{2j\omega}e^{\pi/6 j} + e^{-2j\omega}e^{-\pi/6 j}]\,e^{j\omega t}\,d\omega$

$= \frac{1}{4\pi}[e^{\pi/6 j}\int_{-\infty}^{\infty} e^{2j\omega}\cdot e^{j\omega t}\,d\omega + e^{-\pi/6 j}\int_{-\infty}^{\infty} e^{-2j\omega}\cdot e^{j\omega t}\,d\omega]$

Fourier Transform Table: $\delta(t-\tau) \Longleftrightarrow e^{-j\omega\tau}$

Simplify $x(t)$

$\boxed{x(t) = \frac{1}{2}[(e^{\pi/6 j}\cdot\delta(t+2)) + (e^{-\pi/6 j}\cdot\delta(t-2))]}$

3a) $X(j\omega) = \int_{-1}^{1} t e^{-j\omega t}\,dt$

Use substitution

$u = -j\omega t \rightarrow du = -j\omega\,dt$

$X(j\omega) = \int_{j\omega}^{-j\omega} \frac{1}{(j\omega)^2}\,u e^{u}\,du$

$X(j\omega) = \frac{1}{(j\omega)^2}\cdot\int_{j\omega}^{-j\omega} u e^{u}\,du$

$X(j\omega) = [ue^{u} - e^{u}]_{j\omega}^{-j\omega}$

$\boxed{X(j\omega) = \frac{j\omega e^{j\omega} + e^{-j\omega} + j\omega e^{-j\omega} - e^{j\omega}}{\omega^2}}$

b) $X_k = \frac{1}{2} X(j\omega)$

$= \frac{1}{2} X(jk\pi)$

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2} e^{j\pi k t} \cdot \frac{e^{-j\pi k} + j\pi k e^{j\pi k} - e^{j\pi k} + j\pi k e^{-j\pi k}}{(\pi k)^2}$$

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} \frac{e^{j\pi k t}}{\pi k} \cdot \left[(-1)^k \cdot j\right]}$$

c) pass $\tilde{x}(t)$ through system -> filtered out $\pm\pi$, $\pm 2\pi$

-> filter out $k = \pm 1$, ± 2

$k=1$ $\quad f(t) = \frac{e^{j\pi 1 t}}{\pi(1)} \cdot \left[(-1)^1 \cdot j\right] = \frac{-j}{\pi} e^{j\pi t}$

$k=-1$ $\quad f(t) = \frac{e^{j\pi(-1)t}}{\pi(-1)} \cdot \left[(-1)^{-1} \cdot j\right] = \frac{j}{\pi} e^{-j\pi t}$

$k=2$ $\quad f(t) = \frac{e^{j\pi 2 t}}{\pi(2)} \cdot \left[(-1)^2 \cdot j\right] = \frac{j}{2\pi} e^{j2\pi t}$

$k=-2$ $\quad f(t) = \frac{e^{-j\pi 2 t}}{\pi(-2)} \cdot \left[(-1)^{-2} \cdot j\right] = \frac{-j}{2\pi} e^{-j2\pi t}$

Analyze all together

$$f(t) = \left[\frac{j}{\pi} \cdot j \sin(2\pi t)\right] + \left[\frac{j}{\pi} \cdot (-2j) \sin(\pi t)\right]$$

$$\boxed{f(t) = \frac{2\sin(\pi t) - \sin(2\pi t)}{\pi}}$$

4 a) i) False - Band pass signals do not need to be sampled at a frequency greater than Nyquist frequency

ii) False

Say we have two functions:

$a(t) = \cos(4\pi Bt)$

$b(t) = \sin(4\pi Bt)$

$y(t) = a(t)\,b(t) = \frac{1}{2}\sin(16\pi Bt)$

The bandwidth ($8B$) is not the max of the bandwidths of the two original functions,

iii) True

Odd function means $C_k = -C_{-k}$

Say that $f(t)$ is odd

$C_0 = \frac{1}{T}\int_{T}^{T+T} f(t)\,dt = 0$

b) $F(jw) = j2\pi w e^{-2|w|}$

Traits: $F(jw)$ is imaginary, odd

Therefore, $f(t)$ is real and odd

$F(jw) = -\int_{-\infty}^{\infty} f(t) \cdot j\sin(wt)\,dt$

i) Real - proven above

ii) Odd - proven above

iii) $f(0)$

$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(jw)\,dw = 0$

This is because $f(t)$ is odd

This was proven in 4aiii

c) i) $x(t) = \frac{4}{4+t^2}$ Use duality

Treat this as $\frac{2a}{a^2+t^2} \longleftrightarrow 2\pi e^{-a|w|}$ from Fourier Table

$a = 2$ to return this form to $x(t)$

$\boxed{X(jw) = 2\pi e^{-2|w|}}$

ii) Energy of $x(t)$

$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} (2\pi e^{-2|w|})^2 dw$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\pi^2 e^{-4|w|} dw$

Split in two because absolute value

$E = \frac{1}{2\pi}\left[\int_{-\infty}^{0} 4\pi^2 e^{4w} dw + \int_{0}^{\infty} 4\pi^2 e^{-4w} dw\right]$

$E = 2\pi\left[\left[\frac{1}{4}e^{4w}\right]_{-\infty}^{0} + \left[-\frac{1}{4}e^{-4w}\right]_{0}^{\infty}\right]$

$E = \frac{1}{2}(2\pi)$

$\boxed{E = \pi}$

5. a) i) highest frequency : 3000π

$$\frac{3000\pi}{2\pi} \cdot 2 = 3000$$

Nyquist Rate = $\boxed{3000 \text{ Hz}}$

ii) highest frequency : 2000π

$$\frac{2000\pi}{2\pi} \cdot 2 = 2000$$

Nyquist Rate = $\boxed{2000 \text{ Hz}}$

b) i) Squaring a signal doubles its bandwidth

Therefore, the Nyquist Rate is doubled.

$x^2(t) \to \boxed{2w_0}$ is the Nyquist Frequency

ii) $x(t)\cos(w_1 t)$

$$F(x(t)\cos(w_1 t)) = \left[\tfrac{1}{2}\delta(w-w_1) + \tfrac{1}{2}\delta(w+w_1)\right] \cdot \left[X(jw)\right]$$

$$= \tfrac{1}{2}X\left[(jw+jw_1) + (jw-jw_1)\right]$$

maximum frequency $\Rightarrow \frac{1}{2}w_0 + w_1 \Rightarrow \boxed{2w_1 + w_0}$ is the Nyquist frequency

iii) $\frac{d}{dt}x(t)$

The effect of differentiation does not affect the maximum frequency value.

Thus, the Nyquist rate should be unaffected.

$\boxed{w_0}$ is the Nyquist frequency

c) i) $\Delta < \frac{\pi}{W_m}$ $\quad x_p(t)$

$X_p(w) = X(w)P(w)$

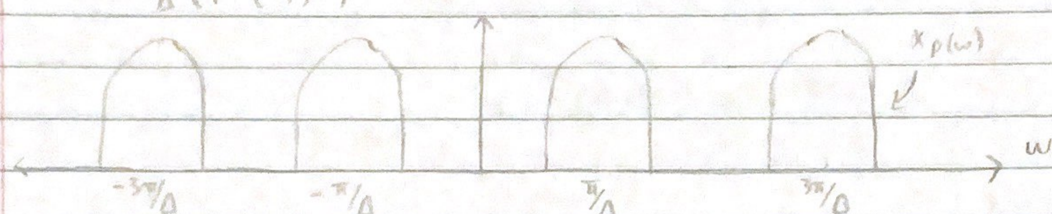
Find Fourier Transform $p(t)$

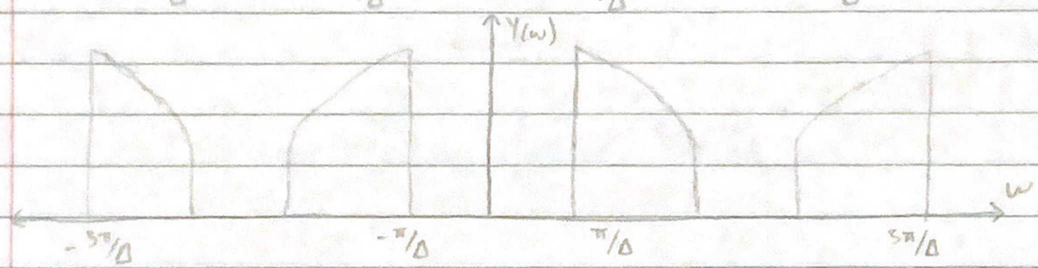
$c_k = \frac{\pi}{\Delta} \int_{2\Delta} p(t)\, e^{j\frac{2\pi k}{2\Delta}t}\, dt$

$= \frac{\pi}{\Delta}\left(1 - e^{j\frac{2\pi k}{2\Delta}\Delta}\right)$

$= \frac{\pi}{\Delta}\left(1 - e^{kj\pi}\right)$

$= \frac{\pi}{\Delta}\left(1 - (-1)^k\right)$





ii) $\Delta < \frac{\pi}{W_m}$

- If you want to recover $x(t)$ from $x_p(t)$ You would want to modulate $x_p(t)$ with $\cos(\frac{\pi}{\Delta}t)$
- The reasoning is that $\cos(\frac{\pi}{\Delta}t)$ has impulses of equivalent strength at $\pm\frac{\pi}{\Delta}$. This meas that the proper low-pass filter will be able to extract the original spectrum $x(t)$.

6. $y''(t) + 4y'(t) + 4y(t) = 4x(t) + 1x''(t)$

$\frac{4}{s+2}$

a) Apply the Laplace Transform

$s^2Y(s) + 4sY(s) + 4Y(s) = 4X(s) + s^2X(s)$

$H(s) = \frac{Y(s)}{X(s)}$

$H(s) = \frac{s^2+4}{s^2+4s+4} \longrightarrow \frac{s^2+4}{(s+2)^2}$

$\boxed{H(s) = \frac{s^2+4}{(s+2)^2}}$

b) Take inverse Laplace Transform of H(s) to get h(t)

$1 - x = \frac{s^2+4}{(s+2)^2}$

$x = -\left(\frac{s^2+4-(s^2+4s+4)}{(s+2)^2}\right)$

$x = \frac{4s}{(s+2)^2}$

$H(t) = 1 - \frac{4s}{(s+2)^2}$

Find values

$\frac{4s(s+2)^2}{(s+2)^2} = \frac{R_1(s+2)^2}{(s+2)^2} + \frac{R_2(s+2)^2}{(s+2)}$

$4s = R_1 + R_2(s+2)$

Find values again

$R_1 = -8 \quad R_2 = 4 \quad s = -2$

Plug in

$H(s) = \frac{8}{(s+2)^2} - \frac{4}{(s+2)} + 1$

Use Fourier Transform Table

$\frac{1}{s^2} \Longleftrightarrow t \qquad \frac{1}{s+a} \Longleftrightarrow e^{-2t}u(t) \qquad 1 \leftrightarrow \partial(t)$

$\frac{1}{(s+2)^2} \Longleftrightarrow 8te^{-2t}$

Put it all together

$\boxed{h(t) = [u(t) \cdot (-4e^{-2t} + 8te^{-2t})] + \delta(t)}$

7. a) The most likely explanation is aliasing or external noise interference. These would both cause some unusual frequencies to occur.