

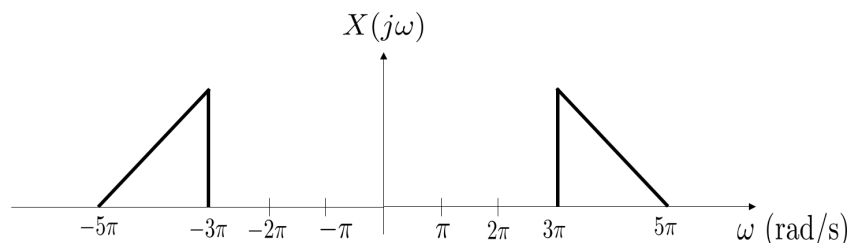
Due Tuesday, 1 June 2021, by 11:59pm to CCLE.

100 points total.

Please follow the UCLA Student Honor Code before and while accessing these homework solutions.

1. (16 points) **Bandpass sampling**

The figure below shows the Fourier transform of a real bandpass signal, i.e., a signal whose frequencies are not centered around the origin. We want to sample this signal. Let F_s in Hz



represent the sampling frequency.

- (a) (8 points) One option is to sample this signal at the Nyquist rate. Then to recover the signal, we pass its sampled version through a low pass filter. What is the Nyquist rate of this signal? What is the cutoff frequency of the low pass filter?

Solution: The Nyquist rate is $2(5\pi) = 10\pi$ rad/s, or 5 Hz. The cutoff frequency of the filter can be 5π rad/s or 2.5 Hz (In general, the cutoff frequency is chosen to be $F_s/2$ where F_s is the sampling frequency).

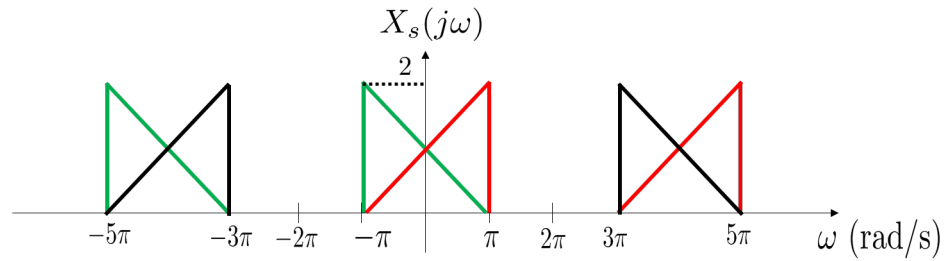
- (b) (8 points) Since the signal might have high frequency components, Nyquist rate for this signal can be high. In other words, we need to have a lot of samples of the signal, which means that the sampling scheme is costly. It turns out that for this type of signal, we can sample it at a sampling frequency lower than the Nyquist rate and we can still recover the signal, however in this case, we will use a **bandpass** filter. To see this, we have the following two options for the sampling frequency:

- $F_s = 2$ Hz;
- $F_s = 2.5$ Hz;

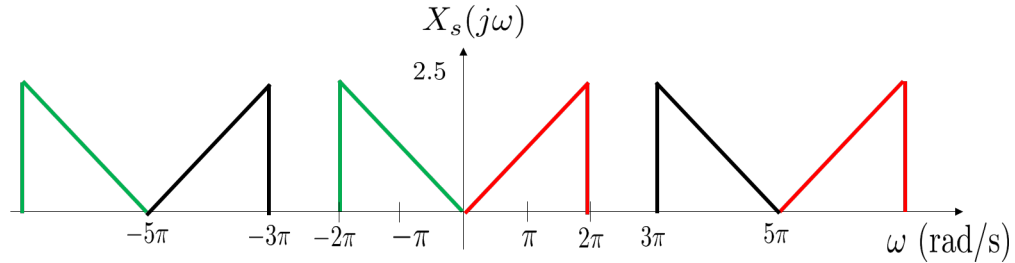
For each case, draw the spectrum of the signal after sampling it. To recover the signal, which F_s can we use? How we should choose the frequencies of the bandpass filter? What is the minimum F_s we can use and still recover the signal?

Solution: Sampling a signal at F_s Hz makes its spectrum periodic with period $\omega_s =$

Sampling at $F_s = 2$ Hz (or $\omega_s = 4\pi$ rad/s)



Sampling at $F_s = 2.5$ Hz (or $\omega_s = 5\pi$ rad/s)

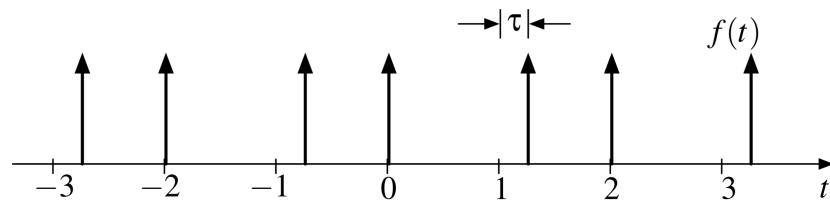


$2\pi F_s$ rad/s and scales it by $1/T_s = F_s$.

Using $F_s = 2$ Hz creates aliasing in the frequency domain (in the region of interest, for $3\pi \leq |\omega| \leq 5\pi$). On the other hand, using $F_s = 2.5$ Hz, does not create aliasing. To recover the signal, we can use a bandpass filter defined over $3\pi \leq |\omega| \leq 5\pi$. The minimum sampling frequency is $F_s = 2.5$ Hz or $\omega_s = 5\pi$ rad/s. Using any ω_s lower than 5π rad/s, will create aliasing in the frequency domain.

2. (24 points) Sampling with imperfect sampler

Imperfections in a sampler cause characteristic artifacts in the sampled signal. In this problem we will look at the case where the sample timing is non-uniform, as shown below: The



sampling function $f(t)$ has its odd samples delayed by a small time τ .

- (a) (6 points) Write an expression for $f(t)$ in terms of two uniformly spaced sampling functions.

Solution: The even samples are a δ train separated by 2, with no shift. The odd

samples are also a δ train separated by 2, but delayed by $1 + \tau$. Adding these together

$$f(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k) + \sum_{k=-\infty}^{\infty} \delta(t - (2k + 1 + \tau)) = \delta_2(t) + \delta_2(t - (1 + \tau))$$

- (b) (6 points) Find $F(j\omega)$, the Fourier transform of $f(t)$. Express the impulse trains as sums, and simplify.

Solution: The Fourier transform is:

$$F(j\omega) = \pi\delta_{\pi}(\omega) + \pi\delta_{\pi}(\omega)e^{-j\omega(1+\tau)} = \pi\delta_{\pi}(\omega)(1 + e^{-j\omega(1+\tau)})$$

Where $\omega_0 = \frac{2\pi}{2} = \pi$ for the impulse train. For the impulse train:

$$\begin{aligned} F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-j\omega(1+\tau)}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-jn\pi(1+\tau)}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + e^{-jn\pi}e^{-jn\pi\tau}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau}) \end{aligned}$$

- (c) (6 points) Find $F(j\omega)$, for the case where $\tau = 0$, and show that this is what you expect.

Solution: If $\tau = 0$,

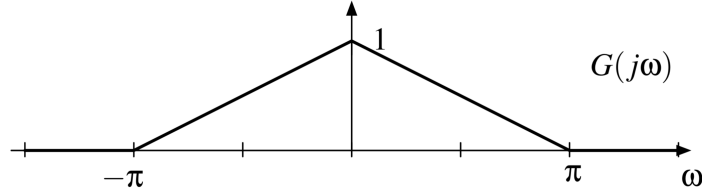
$$\begin{aligned} F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n e^{-jn\pi\tau}) \\ &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi)(1 + (-1)^n) \end{aligned}$$

The $(1 + (-1)^n)$ term is 2 for n even, and 0 for n odd. Hence, the odd terms drop, and we get a factor of two for the remaining even terms,

$$\begin{aligned} F(j\omega) &= \pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2n\pi)(2) \\ &= 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - 2n\pi) \\ &= 2\pi\delta_{2\pi}(\omega) \end{aligned}$$

which is the Fourier transform of $\delta_1(t)$. This is what we expect. As τ goes to zero, we expect that the non-uniform sampling case should go to the uniform case. This will provide a reality check for the next part.

(d) (6 points) Assume the signal we are sampling has a Fourier transform



Sketch the Fourier transform of the sampled signal. Include the baseband replica, and the replicas at $\omega = \pm\pi$. Assume that τ is small, so that $e^{j\omega\tau} \simeq 1 + j\omega\tau$.

Solution: The sampled signal is $f(t)g(t)$, which has a Fourier transform

$$\begin{aligned} G_s(j\omega) &= \frac{1}{2\pi} F(j\omega) * G(j\omega) \\ &= \frac{1}{2\pi} \left(\pi \sum_{n=-\infty}^{\infty} \delta(\omega - n\pi) (1 + (-1)^n e^{-jn\pi\tau}) \right) * \Delta(\omega/\pi) \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \Delta\left(\frac{\omega - n\pi}{\pi}\right) (1 + (-1)^n e^{-jn\pi\tau}) \end{aligned}$$

We are interested in the baseband replica ($n=0$), and the replicas at $\pm\pi$ ($n = \pm 1$). For $n = 0$,

$$G_{s,0}(j\omega) = \frac{1}{2} \Delta\left(\frac{\omega}{\pi}\right) (1 + 1) = \Delta\left(\frac{\omega}{\pi}\right)$$

which is the same as $G(j\omega)$. For $n = 1$,

$$G_{s,1}(j\omega) = \frac{1}{2} \Delta\left(\frac{\omega - \pi}{\pi}\right) (1 + (-1)^1 e^{-j\pi\tau}) = \Delta\left(\frac{\omega - \pi}{\pi}\right) (1 - e^{-j\pi\tau})$$

If we approximate $e^{j\omega\tau} \simeq 1 + j\omega\tau$

$$\begin{aligned} G_{s,1} &= \Delta\left(\frac{\omega - \pi}{\pi}\right) (1 - (1 - j\pi\tau)) \\ &= \frac{j\pi\tau}{2} \Delta\left(\frac{\omega - \pi}{\pi}\right) \end{aligned}$$

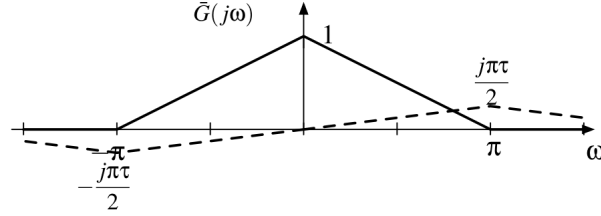
This is a replica of $G(j\omega)$ centered at $\omega = \pi$, multiplied by $j\pi\tau/2$. It is imaginary, and proportional to τ that as τ goes to zero, this replica disappears as we'd expect.

For $n = -1$ we get the same type of term, but with the negative sign,

$$G_{s,-1}(j\omega) = \frac{-j\pi\tau}{2} \Delta\left(\frac{\omega + \pi}{\pi}\right)$$

This is a replica of $G(j\omega)$ centered at $\omega = -\pi$, and scaled by $-j\pi\tau/2$.

If we sketch these three terms, the result is as shown below. .



3. (20 points) Laplace Transform

(a) (12 points) Find the Laplace transforms of the following signals and determine their region of convergence.

i. $f(t) = te^{-at}(\sin \omega_0 t)^2 u(t)$

Solution: We can equivalently write $f(t)$ as:

$$f(t) = te^{-at}(\sin \omega_0 t)^2 u(t) = te^{-at} \frac{1 - \cos(2\omega_0 t)}{2} u(t) = \frac{1}{2} te^{-at} u(t) - \frac{1}{2} te^{-at} \cos(2\omega_0 t) u(t)$$

We have the following:

$$\begin{aligned} \cos(2\omega_0 t) u(t) &\rightarrow \frac{s}{s^2 + (2\omega_0)^2} \\ t \cos(2\omega_0 t) u(t) &\rightarrow -\frac{d}{ds} \frac{s}{s^2 + (2\omega_0)^2} = -\frac{s^2 + 4\omega_0^2 - 2s^2}{(s^2 + 4\omega_0^2)^2} = \frac{s^2 - 4\omega_0^2}{(s^2 + 4\omega_0^2)^2} \\ e^{-at} t \cos(2\omega_0 t) u(t) &\rightarrow \frac{(s+a)^2 - 4\omega_0^2}{((s+a)^2 + 4\omega_0^2)^2} \end{aligned}$$

Therefore,

$$F(s) = \frac{1}{2} \frac{1}{(s+a)^2} - \frac{1}{2} \frac{(s+a)^2 - 4\omega_0^2}{((s+a)^2 + 4\omega_0^2)^2}, \quad \text{Re}\{s\} > -a$$

ii. $f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ e^{-3(t-2)}, & 2 \leq t \end{cases}$

Solution: We can equivalently write $f(t)$ as follows:

$$f(t) = u(t-1) - u(t-2) + e^{-3(t-2)} u(t-2)$$

Therefore,

$$F(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s+3}, \quad \text{Re}\{s\} > -3$$

The ROC of this signal is $\text{Re}\{s\} > -3$, because $f(t)$ can be thought of the sum of a time-limited signal $u(t-1) - u(t-2)$ and a non time-limited signal $e^{-3(t-2)} u(t-2)$. Since the ROC of a time-limited signal is the entire s -plane, the ROC of $f(t)$ is the ROC of $e^{-3(t-2)} u(t-2)$ which is $\text{Re}\{s\} > -3$.

(b) (8 points) The Laplace transform of a causal signal $x(t)$ is given by

$$X(s) = \frac{1}{s^2 + 2s + 5}, \quad \text{ROC: } \text{Re}\{s\} > -1$$

Which of the following Fourier transforms can be obtained from $X(s)$ without actually determining the signal $x(t)$? In each case, either determine the indicated Fourier transform or explain why it cannot be determined.

i. $\mathcal{F}\{x(t)e^{-t}\}$

Solution: Let $y(t) = x(t)e^{-t}$, then $Y(s) = X(s+1)$, and the ROC for $Y(s)$ is:

$$\text{Re}\{s+1\} > -1 \implies \text{Re}\{s\} > -2$$

Since the ROC of $Y(s)$ includes the $j\omega$ -axis, we have:

$$Y(j\omega) = Y(s)|_{s=j\omega} = \frac{1}{(j\omega+1)^2 + 2(j\omega+1) + 5}$$

ii. $\mathcal{F}\{x(t)e^{3t}\}$ Let $y(t) = x(t)e^{3t}$, then $Y(s) = X(s-3)$, and the ROC for $Y(s)$ is:

$$\text{Re}\{s-3\} > -1 \implies \text{Re}\{s\} > 2$$

Since the ROC of $Y(s)$ does not include the $j\omega$ -axis, we cannot determine $Y(j\omega)$ from $Y(s)$.

4. (20 points) **Inverse Laplace Transform**

Find the inverse Laplace transform $f(t)$ for each of the following $F(s)$: ($f(t)$ is a causal signal)

(a) (6 points) $F(s) = \frac{e^{-s}(s+3)}{(s-1)^2(s-2)}$

Solution: Let us first focus on $\frac{(s+3)}{(s-1)^2(s-2)}$. It can be equivalently written as:

$$\frac{s+3}{(s-1)^2(s-2)} = \frac{r_1}{(s-1)^2} + \frac{r_2}{s-1} + \frac{r_3}{s-2}$$

Using the cover-up method,

$$r_1 = \left. \frac{s+3}{s-2} \right|_{s=1} = \frac{4}{-1} = -4$$

$$r_3 = \left. \frac{s+3}{(s-1)^2} \right|_{s=2} = \frac{5}{1} = 5$$

Therefore,

$$\frac{s+3}{(s-1)^2(s-2)} = \frac{-4}{(s-1)^2} + \frac{r_2}{s-1} + \frac{5}{s-2}$$

Now to find r_2 , we will evaluate the above equation at $s = 0$, we then have:

$$\frac{3}{-2} = \frac{-4}{1} + \frac{r_2}{-1} + \frac{5}{-3}$$

Then

$$r_2 = \frac{3}{2} - \frac{8}{2} - \frac{5}{2} = \frac{3-8-5}{2} = -5 \implies r_2 = -5$$

Therefore,

$$F(s) = e^{-s} \left(-\frac{4}{(s-1)^2} - \frac{5}{s-1} + \frac{5}{s-2} \right)$$

then

$$f(t) = \left(-4(t-1)e^{(t-1)} - 5e^{(t-1)} + 5e^{2(t-1)} \right) u(t-1)$$

(b) (6 points) $F(s) = \frac{s+4}{s^3+4s}$

Solution:

$$F(s) = \frac{s+4}{s(s^2+4)} = \frac{r_1}{s} + \frac{r_2}{s+j2} + \frac{r_3}{s-j2}$$

Using the cover-up procedure:

$$r_1 = \left. \frac{s+4}{(s^2+4)} \right|_{s=0} = \frac{4}{4} = 1$$

and

$$r_2 = \left. \frac{s+4}{s(s-j2)} \right|_{s=-j2} = \frac{-2j+4}{(-j2)(-j2-j2)} = \frac{j-2}{4}$$

Then,

$$r_3 = r_2^* = \frac{-j-2}{4}$$

We thus have

$$F(s) = \frac{s+4}{s^3+4s} = \frac{1}{s} + \frac{1}{4} \frac{j-2}{s+j2} - \frac{1}{4} \frac{j+2}{s-j2}$$

The inverse Laplace transform is:

$$f(t) = \left(1 + \frac{1}{4}(j-2)(e^{-j2t} - \frac{1}{4}(j+2)e^{j2t}) \right) u(t) = (1 + 0.5 \sin(2t) - \cos(2t)) u(t)$$

Alternatively, we can find the inverse as follows:

$$F(s) = \frac{s+4}{s(s^2+4)} = \frac{r_1}{s} + \frac{As+B}{s^2+4}$$

Using the cover-up method, we can determine $r_1 = 1$ (as previously obtained). Therefore,

$$As+B = \frac{s+4}{s} - \frac{s^2+4}{s} = \frac{s-s^2}{s} = 1-s$$

Therefore,

$$F(s) = \frac{1}{s} - \frac{s-1}{s^2+4} = \frac{1}{s} - \frac{s}{s^2+4} + \frac{1}{2} \frac{2}{s^2+4}$$

Therefore,

$$f(t) = \left(1 - \cos(2t) + \frac{1}{2} \sin(2t)\right) u(t)$$

(c) (8 points) $F(s) = \frac{1}{(s+1)(s^2+2s+2)}$

Solution:

$$\begin{aligned} F(s) &= \frac{1}{(s+1)((s+1)^2+1)} = \frac{1}{(s+1)(s+1+j)(s+1-j)} \\ &= \frac{r_1}{s+1} + \frac{r_2}{s+1+j} + \frac{r_3}{s+1-j} \end{aligned}$$

Using the cover-up procedure,

$$r_1 = \left. \frac{1}{s^2+2s+2} \right|_{s=-1} = 1$$

and

$$r_2 = \left. \frac{1}{(s+1)(s+1-j)} \right|_{s=-1-j} = \frac{1}{-j(-2j)} = -\frac{1}{2}$$

Thus,

$$r_3 = r_2^* = -\frac{1}{2}$$

Therefore,

$$f(t) = \left(e^{-t} - \frac{1}{2}e^{-(1+j)t} - \frac{1}{2}e^{-(1-j)t}\right) u(t) = e^{-t} (1 - \cos(t)) u(t)$$

Alternatively, we can find the inverse as follows:

$$F(s) = \frac{1}{(s+1)((s+1)^2+1)} = \frac{r_1}{s+1} + \frac{As+B}{(s+1)^2+1}$$

where $r_1 = 1$ and

$$As+B = \frac{1}{s+1} - \frac{(s+1)^2+1}{s+1} = -(s+1)$$

Therefore,

$$F(s) = \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}$$

Then,

$$f(t) = e^{-t} (1 - \cos(t)) u(t)$$

5. **LTI system** (20 points)

Assume a causal LTI system \mathcal{S}_1 is described by the following differential equation:

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = ax(t), \quad y(0) = 0, \quad y'(0) = 0$$

where a is a constant. Moreover, we know that when the input is e^t , the output of the system \mathcal{S}_1 is $\frac{1}{2}e^t$.

- (a) (6 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a , i.e., you should find the value of a).

Solution: We know that $\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$ and $\mathcal{L}\{y'\} = sY(s) - y(0)$, and here we notice that $y(0) = 0$ and $y'(0) = 0$.

Taking the Laplace transform of above differential equation:

$$s^2 Y(s) + 3sY(s) + 2Y(s) = aX(s) \implies H_1(s) = \frac{Y(s)}{X(s)} = \frac{a}{s^2 + 3s + 2} = \frac{a}{(s+1)(s+2)}$$

Eigenfunction property:

$$x(t) = e^{st} \rightarrow y(t) = H(s)e^{st} = |H(s)|e^{st+j(\angle H(s))}$$

Since the output to e^t is $\frac{1}{2}e^t$, then using the eigenfunction property we have:

$$H_1(s)|_{s=1} = \frac{1}{2}$$

We thus conclude:

$$\frac{a}{2 \cdot 3} = \frac{1}{2} \implies a = 3$$

We then conclude:

$$H_1(s) = \frac{3}{(s+1)(s+2)}$$

- (b) (8 points) Find the output $y(t)$ when the input is $x(t) = u(t)$.

Solution: The Laplace transform of $y(t)$ is given by

$$Y(s) = H_1(s)X(s) = \frac{3}{(s+1)(s+2)s} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

where

$$\begin{aligned} r_1 &= \left. \frac{3}{(s+1)(s+2)} \right|_{s=0} = \frac{3}{2} \\ r_2 &= \left. \frac{3}{s(s+2)} \right|_{s=-1} = -3 \\ r_3 &= \left. \frac{3}{s(s+1)} \right|_{s=-2} = \frac{3}{2} \end{aligned}$$

Therefore,

$$y(t) = \left(\frac{3}{2} - 3e^{-t} + \frac{3}{2}e^{-2t} \right) u(t)$$

- (c) (6 points) The system \mathcal{S}_1 is linearly cascaded with another causal LTI system \mathcal{S}_2 . The system \mathcal{S}_2 is given by the following input-output pair:

$$\mathcal{S}_2 \quad \text{input : } u(t) \rightarrow \text{output : } r(t)$$

Find the overall impulse response.

Solution: The impulse response of the system \mathcal{S}_2 is:

$$H_2(s) = \frac{\mathcal{L}(r(t))}{\mathcal{L}(u(t))} = \frac{\frac{1}{s^2}}{\frac{1}{s}} = \frac{1}{s}$$

Therefore, the overall transfer function is given by:

$$\begin{aligned} H(s) &= H_1(s)H_2(s) \\ &= \left(\frac{3}{2} \frac{1}{s} - 3 \frac{1}{s+1} + \frac{3}{2} \frac{1}{s+2}\right) \end{aligned}$$

We solved this expression in the previous problem, so we know that

$$h(t) = \left(\frac{3}{2} - 3e^{-t} + \frac{3}{2}e^{-2t}\right) u(t)$$