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EE102

Lecture 12

CYU (Student Feedback)

Show that $f(t) = \cos(\omega_0 t)$ is not an eigenfunction of an LTI system.

Suppose if true: $\underset{x}{\cos(\omega_0 t)} \longrightarrow \underset{h(t)}{\boxed{LTI}} \longrightarrow \underset{y}{a\cos(\omega_0 t)}$

Counterexample

$h(t) = \delta(t - z)$

$y(t) = \cos(\omega_0 (t - z))$

$\therefore$ Not an eigenfunction

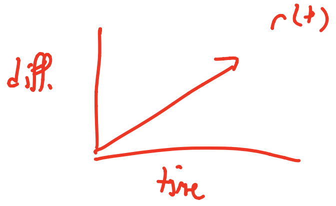
$$y(t) = \int_{-\infty}^{\infty} f(t-\tau) h(\tau)\, d\tau$$

$$= \frac{1}{2} e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0 \tau} h(\tau)\, d\tau}_{a_1} + \frac{1}{2} e^{-j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} e^{j\omega_0 \tau} h(\tau)\, d\tau}_{a_2}$$

$a_1 \neq a_2$

EE102 Announcements

- Syllabus link is tinyurl.com/ucla102
- **Homework 5 due Friday**
- Today's Lecture is probably not difficult



Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Previous Lecture

We discussed F.S. properties and offered some proofs.

CYU (Intuition): In your best opinion, what's the key limitation with Fourier Series?

What about non-periodic signals?

This is all well and good, but most signals we care about are not periodic.

Motivation

Last lecture, we learned about the Fourier series, which can model (almost) any **periodic** or **time-limited** function as a sum of complex exponentials.

But the Fourier series is limited because it requires the signals be **periodic** or **time-limited**.

The Fourier transform allows us to calculate the spectrum of aperiodic signals.

How do we go to aperiodic signals? (F.T. intuition)

Intuition of going from Fourier series to Fourier transform

Extending Fourier series to the Fourier transform is fairly intuitive.

The idea is the following.

- We can calculate the Fourier series of a periodic or time-limited signal, over some interval of length T_0.
- A signal that is not periodic can be viewed as a periodic signal, where T_0 is infinite. As T_0 is infinite, it never repeats.
- But the point is that we can replace our Fourier series calculation as, instead of being over a finite period, T_0, being over all time, from $t = -\infty$ to ∞.

Aperiodic Signal $\longrightarrow$ Periodic Signal with $T = \infty$

Arriving at the Fourier Transform

Intuition (cont.)

Mathematically, we can calculate the Fourier series of $f(t)$ over the interval $[-T/2, T/2)$ via:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0} dt$$

where $\omega_0 = 2\pi/T$.

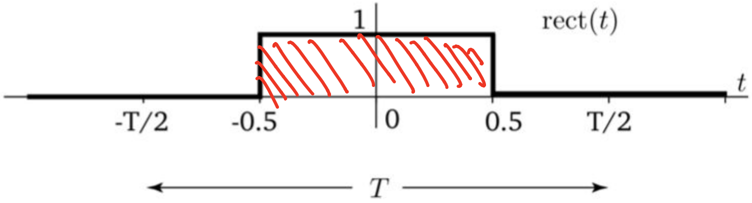
In the Fourier transform, we're now going to let $T \to \infty$.

Example: rect

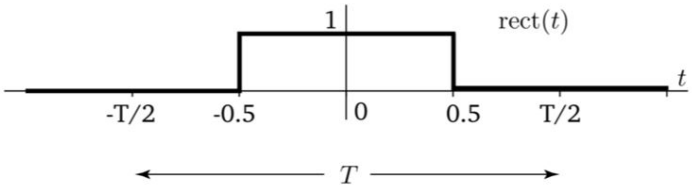
The Fourier series of rect as $T \to \infty$

Consider $x(t) = \mathrm{rect}(t)$. Further, let's define a period T over which, if we made a periodic extension of the rect, it would repeat every T. (Again, we're going to set $T \to \infty$ eventually so that it doesn't repeat.)

Support $\triangleq$ Part of the domain of a fn, where the fn $\neq 0$.



Example: rect



The Fourier series of this signal is related to the one we did last lecture, but we'll do it again for the sake of completeness:

Integrating over support

$$
\begin{aligned}
c_k &= \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{rect}(t) e^{-jk\frac{2\pi}{T}t}\,dt \\
&= \frac{1}{T}\int_{-1/2}^{1/2} e^{-jk\frac{2\pi}{T}t}\,dt \\
&= \frac{1}{T}\left.\frac{Te^{-jk\frac{2\pi}{T}t}}{-jk2\pi}\right|_{-1/2}^{1/2}
\end{aligned}
$$

Example: rect

The Fourier series of rect as $T \to \infty$

Continuing ...

aljebre

$$
\begin{aligned}
c_k &= \left. \frac{e^{-jk\frac{2\pi}{T}t}}{-jk2\pi} \right|_{-1/2}^{1/2} \\
&= \frac{-j\sin(\pi k/T) - j\sin(\pi k/T)}{-j2\pi k} \\
&= \frac{\sin(\pi k/T)}{\pi k} \\
&= \frac{1}{T}\frac{\sin(\pi k/T)}{\pi k/T} \\
&= \boxed{\frac{1}{T}\operatorname{sinc}\left(\frac{k}{T}\right)}
\end{aligned}
$$

$$f(t) = \sum_k c_k e^{jk\omega_0 t}$$

Therefore, the Fourier series of $\mathrm{rect}(t)$ with a periodic extension every T is:

$$\frac{1}{T}\sum_{k=-\infty} \operatorname{sinc}(k/T)e^{jk\omega_0 t}$$

for $\omega_0 = 2\pi/T$.

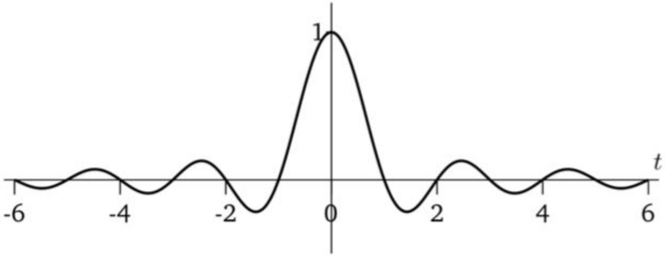
Example: rect

The Fourier series of rect as $T \to \infty$

Let's now look at what the coefficients look like for varying values of T. First, for a refresher, let's recall what the sinc function looks like.

Generic sinc fn

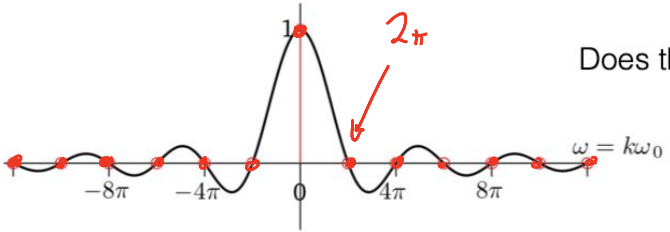
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



F.S. coeff.

$$\frac{1}{T}\,\mathrm{sinc}\left(\frac{k}{T}\right)$$

Now, let's set $T = 1$ ($\omega_0 = 2\pi$) and calculate each of the Fourier coefficients.



2π

Does this make sense?

$$\omega_0 = \frac{2\pi}{T}$$

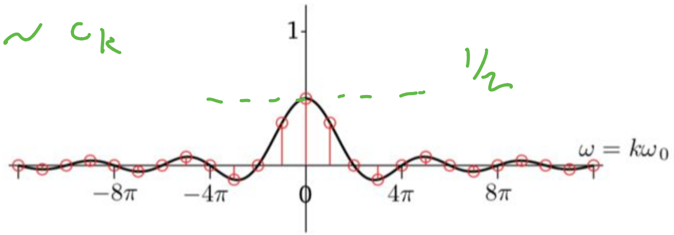
Example: rect

The Fourier series of rect as $T \to \infty$

$\omega_0 = \frac{2\pi}{T} = \pi$

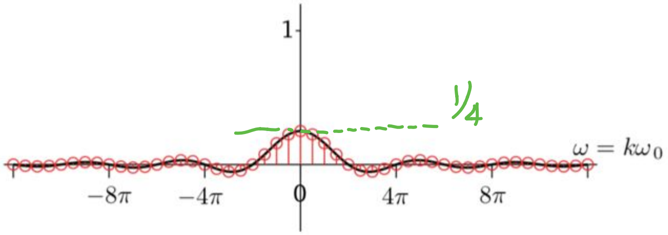
Set $T = 2$ ($\omega_0 = \pi$) and calculate each of the Fourier coefficients.

$\frac{1}{T} \operatorname{sinc}\left(\frac{k}{T}\right) \sim c_k$



Zero crossings of sinc occur every 2π

Set $T = 4$ ($\omega_0 = \pi/2$) and calculate each of the Fourier coefficients.



$\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$

What if we replace $k\omega_0$ with ω?

Example: rect

The Fourier series of rect as $T \to \infty$

The trend we see is that as we set T larger, we more densely sample the sinc function. This gives us reason to believe that the spectrum of the rect signal, when $T \to \infty$, is a sinc function.

Let's formalize this intuition with math.

Learning Goal: Fourier Transform of rect

Support: $-\frac{1}{2}$ to $\frac{1}{2}$

Arriving at the Fourier transform

The Fourier series of $f(t)$ an an interval $[-T/2, T/2)$ is given by:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt$$

Arriving at the Fourier Transform

The first goal is to study F.S. The F.S. of $f(t)$ over $\left[-\frac{T}{2}, \frac{T}{2}\right]$ is:

(1) $f(t) = \sum_k c_k e^{jk\omega_0 t}$ $\quad c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t}\, dt$

cyu: how do these relate?

The second goal is to define a truncated Fourier Transform

(2) $F_T(j\omega) = \int_{-T/2}^{T/2} f(t) e^{-j\omega t}\, dt$

The third goal is to relate this with the Fourier Series

(3) $c_k = \frac{1}{T} F_T(jk\omega_0)$

The fourth goal is to reconstruct the original signal

$$f_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(jk\omega_0) e^{jk\omega_0 t}$$

Arriving at the Fourier Transform (cont'd)

Insight: the Fourier Transform is the limit of truncated F.T.

$$F(j\omega) = \lim_{T\to\infty} F_T(j\omega)$$

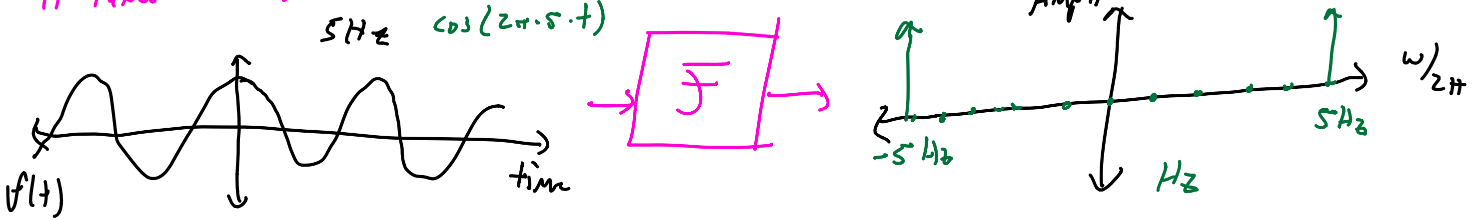
$$= \lim_{T\to\infty} \int_{-T/2}^{+T/2} f(t)\, e^{-j\omega t}\, dt$$ ← Integral over 1 infinitely long period

$$= \boxed{\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt}$$ ← Integral over infinity.

$F(j\omega)$

This is the famous F.T.

It takes a signal $f(t)$ in time domain and maps it to another signal in frequency domain.



The Fourier Transform

The Fourier transform is:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Inverting the Fourier Transform

How do we go from $F(j\omega)$ back to $f(t)$? Let's begin by writing the Fourier series.

$\omega = k\Delta\omega$

$\Delta\omega \triangleq \frac{2\pi}{T}$

$$
\begin{aligned}
f(t) &= \lim_{T\to\infty} f_T(t) \\
&= \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(jk\omega_0) e^{jk\omega_0 t}
\end{aligned}
$$

$$f(t) = \lim_{\Delta\omega \to 0} \sum_{k} F_T(jk\Delta\omega) e^{jk\Delta\omega t} \frac{\Delta\omega}{2\pi}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} \, d\omega$$

Inverting the Fourier transform

Same as last slide

How do we go from $F(j\omega)$ back to $f(t)$? Let's begin by writing the Fourier series.

$$\begin{aligned} f(t) &= \lim_{T\to\infty} f_T(t) \\ &= \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(jk\omega_0) e^{jk\omega_0 t} \end{aligned}$$

Now as $T \to \infty$, what we see is that this approaches an integral. This is an infinite sum, where the integration "widths" are the infinitesimal $1/T$ and the "heights" are $F_T(jk\omega_0)e^{jk\omega_0 t}$. To make this more clear, we denote $\Delta\omega = 2\pi/T$, and note that $\omega = k\Delta\omega$. Then, this sum becomes

$$\begin{aligned} f(t) &= \lim_{\Delta\omega\to 0} \sum_{k=-\infty}^{\infty} F_T(jk\Delta\omega) e^{jk\Delta\omega t} \frac{\Delta\omega}{2\pi} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \end{aligned}$$

This is the inverse Fourier transform, which takes you from the frequency domain, $F(j\omega)$ to the time domain, $f(t)$.

Summarizing the Fourier Transform

The Fourier transform is:

$T \to \infty$

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

The inverse Fourier transform is:

$T \to \infty$

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega}$$

$$\mathcal{F}^{-1}\left[\mathcal{F}\left[f(t)\right]\right] = f(t)$$

The Fourier transform

The Fourier transform is:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

The inverse Fourier transform is:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega}$$

A few notes:

- Like in Fourier series, the inversion formula for $f(t)$ is accurate when $f(t)$ is continuous, but produces the midpoint when $f(t)$ has jumps.
- These two are almost identical in form, except for the sign of the complex exponential and the factor $1/2\pi$.
- Check your intuition when you look at these formulas: to go from the time domain to frequency domain (Fourier transform) you should integrate away time (giving a function of frequency). Likewise, to go from the frequency domain to the time domain (inverse Fourier transform) you should integrate away frequency (giving a function of time).

Fourier Transform Exists for Absolutely Integrable Signals

CYU: Prove that if f(t) is absolutely integrable, that the Fourier Transform is Bounded

Summarizing the F.T. Once Again

The Fourier transform is:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$$

The inverse Fourier transform is:

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega}$$

Now we are going to compute it!

Example: Fourier Transform of Rect

Let's find the Fourier transform of $f(t) = \mathrm{rect}(t/T)$ and see how it relates to the Fourier coefficients we derived earlier.

$$F(j\omega) \quad = \quad \int_{-\infty}^{\infty} \mathrm{rect}(t/T) e^{-j\omega t} dt$$

Example: Fourier Transform of Rect

Note here, we went through some extra algebra to get things into $\mathrm{sinc}(\cdot)$ form. This is out of convenience. Thus, we have that

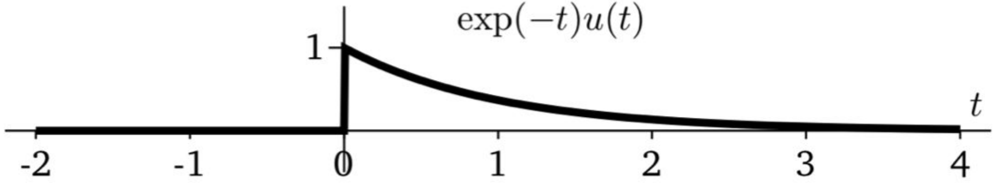
$$\mathrm{rect}(t/T) \iff T\,\mathrm{sinc}(\omega T/2\pi)$$

Fourier Transform Example: Causal Exponential

Let's find the Fourier transform of

$$\begin{aligned} f(t) &= \begin{cases} e^{-at}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \\ &= e^{-at}u(t) \end{aligned}$$

for $a > 0$.



Fourier transform example: causal exponential (cont.)

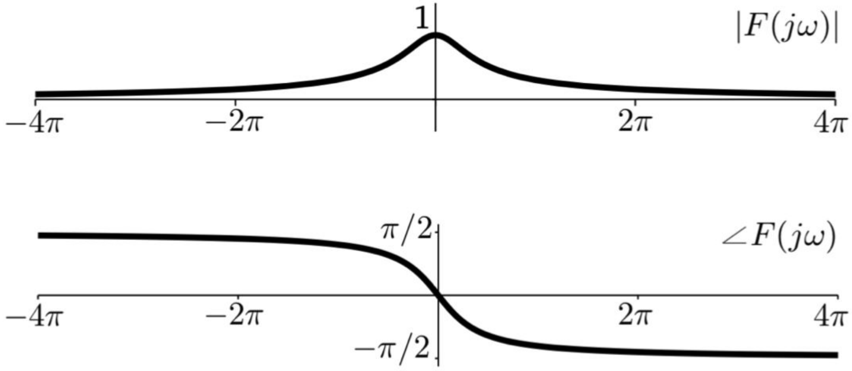
Its Fourier transform is

$$F(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

Fourier Transforms

Fourier transform example: causal exponential

Below is the spectrum of the causal exponential for $a = 1$.



Does the amplitude spectrum make intuitive sense?

Summary: Fourier Transforms we Now Know

$$\mathrm{rect}(t/T) \iff T\,\mathrm{sinc}(\omega T/2\pi)$$

$$e^{-at}u(t) \iff \frac{1}{a + j\omega}$$

HW CYU (qst 1C)

Consider the aperiodic signal shown below. How can the Fourier Series coefficients of the periodic extension be obtained from the Fourier Transform ?