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EE102

Lecture 17

EE102 Announcements

- Syllabus link is tinyurl.com/ucla102
- **Homeworks due on Tuesday**

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Laplace Transform

We're in our last major topic of the class: the Laplace transform.

The Laplace transform will extend much of the intuition that you've developed thus far. Informally, this part of the class is more algebraic.

We will see that one major application of the Laplace transform is that it gives us a simple framework to solve differential equations.

Laplace Transform

This lecture introduces the Laplace Transform and its properties. Topics include:

- s spectrum and region of convergence
- Bilateral Laplace transform
- Unilateral Laplace transform
- Relationship between Fourier and Laplace transforms
- Laplace transforms of e^{at}, $u(t)$, t^n, $\delta(t)$, and $\cos(\omega t)$
- Laplace transform properties
- Examples
- Solving differential equations

Motivation for Laplace Transform

The Fourier transform is powerful, but it doesn't exist for some signals and systems. In several applications, including image processing, communications, and circuit design, its sufficient for analysis.

However, some systems are unstable, or are power signals where the Fourier transform can not be straightforwardly generalized. Some examples of this are signals that grow with time, like (ideally) your bank account, or the S&P 500.

How do we analyze these systems in a similar framework to what Fourier analysis enables us to do?

Laplace Transform

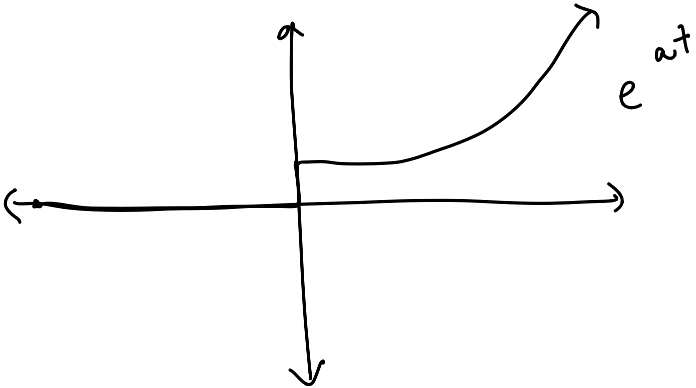
Let

$$f(t) = e^{at}u(t)$$

When $a > 0$, this signal does not have a Fourier transform.

$t \to \infty$ $\quad F(j\omega) < B$

NOT BOUNDED BY B.

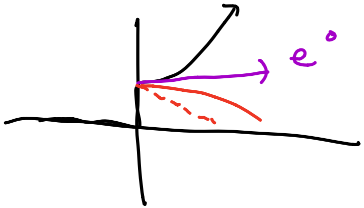


Laplace Transform

Let

$$f(t) = e^{at}u(t)$$

When $a > \cancel{1}\,0$, this signal does not have a Fourier transform.



One approach to arrive at a Fourier transform is to define a new function

$$g(t) = f(t)e^{-\sigma t} = e^{at}u(t)e^{-\sigma t} = e^{(a-\sigma)t}u(t)$$

If $\sigma > a$, then $g(t)$ is a decreasing exponential, which has a Fourier transform.

Laplace Transform

The function $g(t) = f(t)e^{-\sigma t}$ has a Fourier transform for σ sufficiently large.
The Fourier transform of $g(t)$ comprises how to sum spectral components $e^{j\omega t}$,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

Laplace Transform $g(t) = f(t)e^{-\sigma t}$

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$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega$$

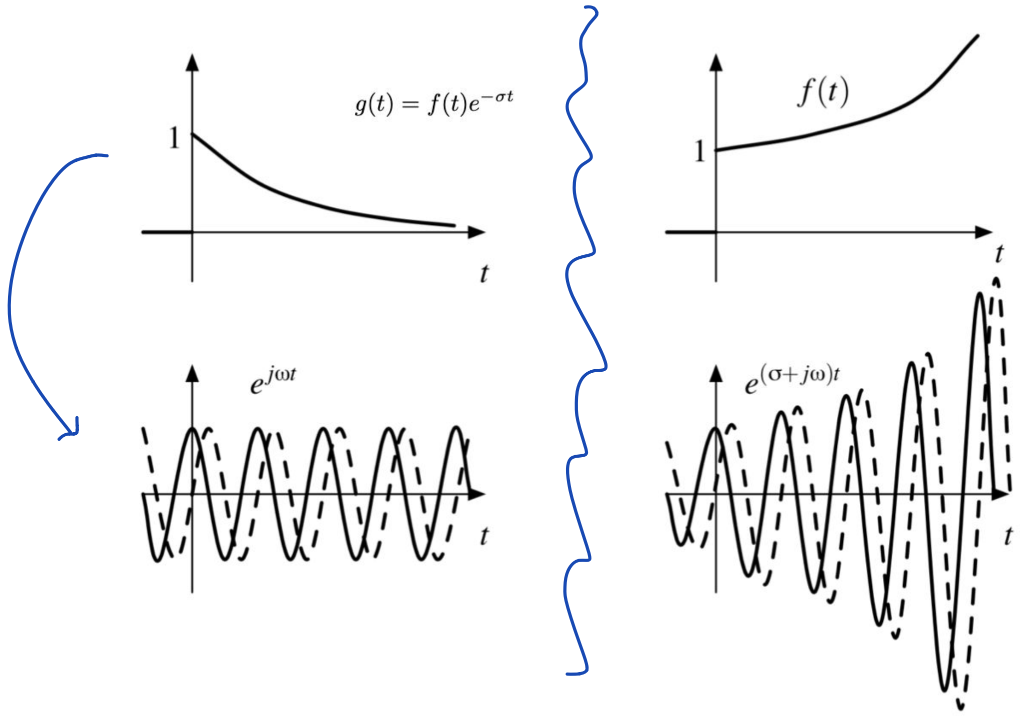
The intuition here is that because $f(t) = g(t)e^{\sigma t}$, $f(t)$ has spectral components

$$e^{\sigma t}e^{j\omega t} = e^{(\sigma + j\omega)t}$$

Hence, the Laplace transform gives us a spectrum of $f(t)$ in terms of a complex exponential with both real and imaginary components (where as the Fourier transform was only with imaginary components).

Laplace Transform

$\mathcal{F}$



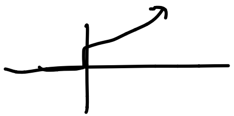
Region of Convergence

$\| \; e^{\underbrace{(\sigma + j\omega)}_{s} t} \rightarrow e^{st}$

$s = \sigma + j\omega$

When does the s-spectrum exist?

For what values of σ does this work? In the case where $f(t) = e^{at}u(t)$, this is clear, i.e., $\sigma > a$.



In general, there is some σ_0 for which

$$f(t)e^{-\sigma_0 t}$$

goes to zero. If it does, then this $f(t)e^{-\sigma_0 t}$ is an energy signal, and its spectrum will exist.

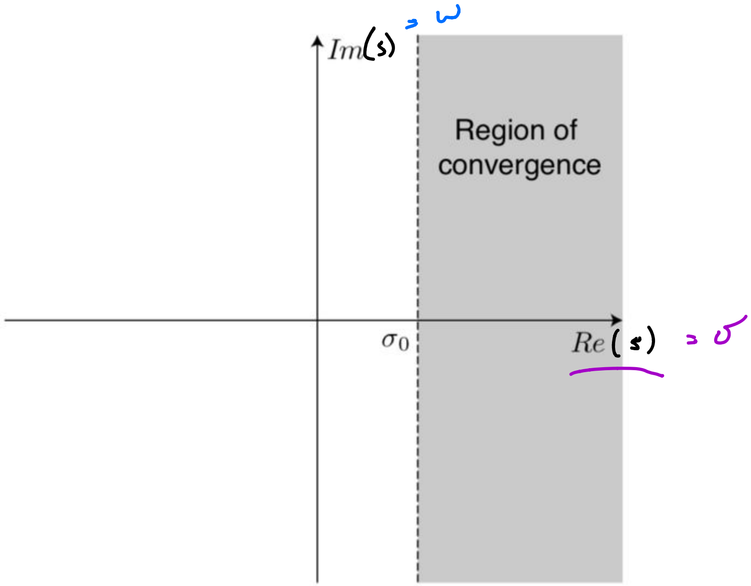
The portion of the complex plane where $\sigma > \sigma_0$ is called the "region of convergence."

Region of Convergence

The region of convergence is illustrated below:

$$e^{(\sigma + j\omega)t} \longrightarrow e^{st}$$

$$s = \underbrace{\sigma}_{\text{Real Part}} + \underbrace{j\omega}_{\text{Imaginary Part}}$$



Notation $\mathcal{L}\{f(t)\} \Rightarrow F(s)$

Laplace transform notation

Our notation for the Laplace transform is very similar to our prior notation. We denote

$s = \sigma + j\omega$

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)] \\ f(t) &= \mathcal{L}^{-1}[F(s)] \end{aligned}$$

We will also denote this:

$$f(t) \Longleftrightarrow F(s) \quad \text{Laplace}$$

$$f(t) \Longleftrightarrow F(j\omega) \quad \text{Fourier}$$

Fourier: $e^{j\omega t}$

Laplace: $e^{j\omega t} e^{\sigma t} = e^{st}$

Bilateral Laplace Transform

The Laplace transform incorporates the real exponential. With $s = \sigma + j\omega$, as before,

- $j\omega$ is related to the oscillatory component of the complex exponential
- σ is related to the decay or growth of the complex exponential

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- σ is related to the decay or growth of the complex exponential

Then, the **bilateral** Laplace transform is:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt} = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt$$

Bilateral Laplace Transform

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- $j\omega$ is related to the oscillatory component of the complex exponential
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Then, the **bilateral** Laplace transform is:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt}$$

We will not use this definition

To invert the bilateral Laplace transform, we calculate:

$$\boxed{f(t) = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds}$$

for $c > \sigma_0$.

Bilateral Laplace Transform

We won't use the bilateral Laplace transform, but it's worth mentioning this for completeness.

Unilateral Laplace Transform

Usually, we are interested in analyzing causal signals. In this case, we can simplify the bilateral Laplace transform. A causal signal can be written as $f(t)u(t)$, and its Laplace transform is

$$
\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt && \text{// Bilateral} \\
&= \int_{0^-}^{\infty} f(t)e^{-st}dt && \text{// Unilateral}
\end{aligned}
$$

When we write 0^-, this indicates that impulses at the origin are included (e.g., $\delta(t)$ would have a contribution to this integral).

The Laplace transform is (essentially) unique. From now on, we'll use $\mathcal{L}[f(t)]$ to denote the unilateral Laplace transform of $f(t)$.

Relationship between Fourier and Laplace Transforms

The Fourier transform is a special case of the Laplace transform, i.e.,

$$\boxed{F(j\omega) = F(s)|_{s=j\omega}}$$

$s = \sigma + j\omega$

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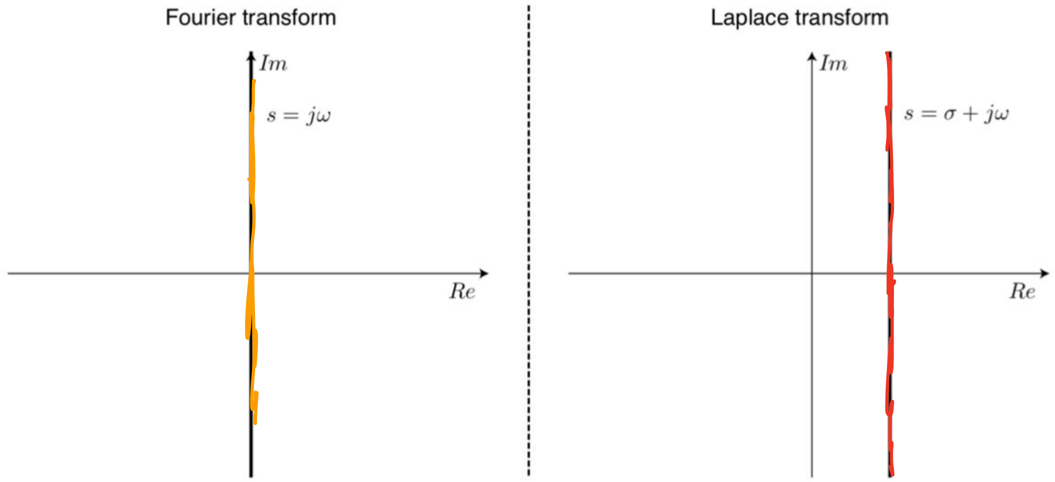
The Fourier transform is a special case of the Laplace transform, i.e.,

$$\boxed{F(j\omega) = F(s)|_{s=j\omega}}$$

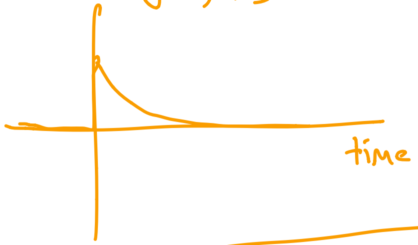
The Fourier transform is evaluated at $s = j\omega$ and the Laplace transform is evaluated at a particular $s = \sigma + j\omega$.



Relationship between Fourier and Laplace Transforms

You may imagine that for signals where we know the Fourier transform, the Laplace transform merely replaces $j\omega$ with s. This is sometimes the case. Let's consider $f(t) = e^{-at}u(t)$.

$f(t) \Leftrightarrow F(j\omega) = \frac{1}{a+j\omega}$



$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-at}e^{-st}dt \\
&= \int_0^\infty e^{-(a+s)t}dt \\
&= -\frac{1}{a+s}e^{-(a+s)t}\Big|_0^\infty \\
&= \frac{1}{a+s}
\end{aligned}
$$

as long as $e^{-(a+s)t} \to 0$ as $t \to \infty$. When does this happen?

Relationship b/w Fourier and Laplace (cont'd)

If $e^{-(a+s)t}$ goes to zero, then so does $\left|e^{-(a+s)t}\right|$.

$$\left|e^{-(a+s)t}\right| = \left|e^{-(a+\sigma+j\omega)t}\right|$$

$$= \left|e^{-(a+\sigma)t} \cdot e^{-j\omega t}\right|$$

$$= \left|e^{-(a+\sigma)t}\right| \cdot \underbrace{\left|e^{-j\omega t}\right|}_{\to 1}$$

$$= e^{-(a+\sigma)t}$$

$e^{-(a+s)t} \to 0$, when

$e^{-(a+\sigma)t} \to 0$ as $t \to \infty$

If $\sigma > -a$, then Laplace Transform Exists

$$\boxed{\text{R.O.C. } \quad Re\{s\} > -a}$$

Relationship b/w Fourier and Laplace (cont'd)

Hence, we have that

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

for R.O.C. Re{s} > -a

and we know prior, for $a > 0$,

$$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

Here, the Laplace transform is the Fourier transform with $j\omega$ replaced with s.

Relationship b/w Fourier and Laplace (cont'd)

$a = 2: \quad \sigma > -2$

$a = -1: \quad \sigma > 1$

A key thing to note is that with

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

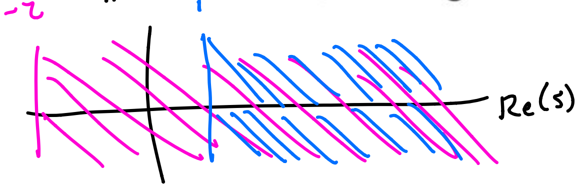
holds for all a, positive or negative, as long as $\sigma > -a$.

$$\mathcal{F}[e^{t}u(t)] \neq \frac{1}{j\omega - a}$$

This means that, for $a > 0$,

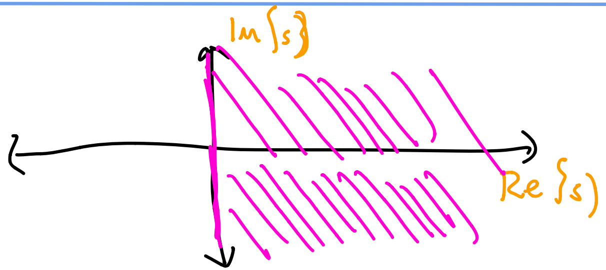
$$\mathcal{L}[e^{at}u(t)] = \frac{1}{s-a}$$

Of course, this signal does not have a Fourier transform.



CYU: Comparing Fourier and Laplace

Let's take the Laplace transform of the unit step



$$f(t) = u(t)$$

$$F(s) = \int_0^{\infty} u(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^{\infty}$$

$$= \frac{1}{s}$$

$$\text{F.T}\left[u(t)\right] = \frac{1}{j\omega} + \pi\delta(\omega)$$

// Find the ROC As long as $e^{-st} \rightarrow 0$ for $t \rightarrow \infty$ ... $\text{Re}\{s\} > 0$

Relationship between Fourier and Laplace

Recall the Fourier transform of the unit step is:

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

This resembles the Laplace transform with $s = j\omega$, but there is an additional $\pi\delta(\omega)$ term.

Relationship between Fourier and Laplace

We will see this tends to be the case for some of our generalized Fourier transforms. For example, consider the Laplace transform of

$$
\begin{aligned}
f(t) &= \cos(\omega t) \\
&= \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]
\end{aligned}
$$

$$F(s) \quad \frac{1}{2}\left[\mathcal{L}\left[e^{j\omega t}\right] + \mathcal{L}\left[e^{-j\omega t}\right]\right]$$

$$= \frac{s}{s^2 + \omega^2}$$

$$\text{R.O.C.} = \mathrm{Re}\{s\} > 0$$

Example Laplace Transforms

Laplace transform of powers of t

Laplace transforms, given all we've learned thus far, should be fairly straightforward to evaluate. We'll go over a few examples here. Let

$$f(t) = t^n u(t)$$

$$u = t^n \qquad dv = e^{-st}\,dt$$

$$du = n t^{n-1}\,dt$$

$$v = -\frac{1}{s} e^{-st}$$

for $n \geq 1$. Then,

$$F(s) = \int_0^\infty t^n e^{-st}\,dt$$

Int. by parts: $\int u\,dv = uv - \int v\,du$

$$\mathcal{L}\left[t^n\right] = F(s) = -\frac{t^n e^{-st}}{s}\Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} n t^{n-1}\,dt$$

$$= 0 - 0 + \frac{n}{s}\underbrace{\int_0^\infty t^{n-1} e^{-st}\,dt} = \frac{n}{s}\mathcal{L}\left[t^{n-1}\right]$$

Example Laplace Transforms

Power Rule

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}]$$

Suppose $n=1 \implies t^0 = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} = u(t)$

$u(t) \iff \frac{1}{s} \quad \therefore \mathcal{L}[t^1] = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$

General

$$\mathcal{L}[t^0] = \frac{1}{s}$$

$$\mathcal{L}[t^1] = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

...

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

cya @ home

Example Laplace Transforms

Laplace transform of impulse

CYU Break : 3 minutes

Let

$$f(t) = \delta(t)$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty \delta(t) e^{-st}\, dt$$

$$= \int_0^\infty \delta(t) e^{0}\, dt$$

$$= \int_0^\infty \delta(t)$$

$$= \boxed{1}$$

$$\therefore \mathcal{L}[\delta(t)] = 1$$

A Trend Emerging ..

Pattern for integration and differentiation?

Notice the following trends:

$$\delta(t) \iff 1$$

$$u(t) \iff \frac{1}{s}$$

$$tu(t) \iff \frac{1}{s^2}$$

$$\frac{1}{2}t^2u(t) \iff \frac{1}{s^3}$$

$$\frac{1}{6}t^3u(t) \iff \frac{1}{s^4}$$

$$\vdots$$

Key: Laplace Transform takes a diff. eq. and converts to an algebra eqn.

$$x'''(t) + 5x'(t) + x(t) = 0$$

$$x(t) \iff X(s)$$

$$s^3X(s) + 5sX(s) + X(s) = 0$$

We see a clear pattern: differentiating a signal is equivalent to multiplying the Laplace transform by s while integrating is equivalent to multiplying the Laplace transform by $1/s$.

Laplace Transform Properties

1. **Linearity:** $$\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)$$

2. **Time scaling:** $$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

3. **Time shift:** $$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$$

4. **Frequency shift:** $$\mathcal{L}\left[f(t)e^{s_0 t}\right] = F(s-s_0)$$

5. **Convolution:** $$\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

6. **Integration:** $$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

7. **Derivative:** $$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

8. **Multiplication by *t*:** $$\mathcal{L}[tf(t)] = -F'(s)$$

$x \to \boxed{h} \to y$

$y = x * h$

$\mathcal{L}(y) = \mathcal{L}(x) \cdot \mathcal{L}(h)$

Differentiation and Integration Property

Key reason to use Laplace Transforms! Turns differential equations into algebraic equations.

$g(t)$	$G(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

CYU: Apply the Property to get L.T. of Step and Ramp

Example: unit step and ramp functions

We were able to calculate the Fourier transform of the unit step; however, this required generalizing the Fourier transform. How can we find the Laplace transforms of the unit step and unit ramp function using the integral properties of the Laplace transform?

If $f(t) = \delta(t)$, then $F(s) = 1$. Then,

$$\mathcal{L}(\delta(t)) = 1$$

$$\mathcal{L}[u(t)] = \mathcal{L}\left[\int_0^t \delta(\tau)\,d\tau\right] = \frac{1}{s}\mathcal{L}(\delta(t)] = \frac{1}{s}$$

$$\mathcal{L}[r(t)] = \frac{\mathcal{L}[u(t)]}{s} = \frac{1}{s^2}$$

Inversion of Laplace Transform

Motivation

The inverse of the Laplace transform is given by

$$f(t) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds$$

where σ is large enough that F(s) is defined for $\Re(s) \geq c$.

Catalog of Inverse Transforms to Keep Handy

May seem specific

But come up in Diff Eq!

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

$$\begin{aligned}
\mathcal{L}[\cos(\omega t)] &= \frac{s}{s^2+\omega^2} \\
\mathcal{L}[\sin(\omega t)] &= \frac{\omega}{s^2+\omega^2} \\
\mathcal{L}[e^{-at}\cos(\omega t)] &= \frac{(s+a)}{(s+a)^2+\omega^2} \\
\mathcal{L}[e^{-at}\sin(\omega t)] &= \frac{\omega}{(s+a)^2+\omega^2}
\end{aligned}$$

$$\mathcal{L}^{-1}\left[\frac{r}{(s-\lambda)^k}\right] = \frac{r}{(k-1)!}t^{k-1}e^{\lambda t}$$

Partial Fractions

Partial fraction expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

Partial Fractions

Partial fraction expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

Let's first assume that no poles are repeated and that $m < n$ (i.e., more poles than zeros).

Then, $F(s)$ can be written in its *partial fraction expansion*:

$$F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}$$

where

- $\lambda_1, \ldots, \lambda_n$ are the poles of F.
- The numbers $r_1, \ldots, r_n$ are called residues.
- It turns out when $\lambda_k = \lambda_l^*$, then $r_k = r_l^*$.

Partial Fractions

Inversion of a partial fraction

In partial fraction form, inverting the Laplace transform is easy because

$$\mathcal{L}^{-1}[F(s)] \quad = \quad \mathcal{L}^{-1}\left[\frac{r_1}{s-\lambda_1} + \cdots + \frac{r_n}{s-\lambda_n}\right]$$

Partial Fractions

How to find the partial fraction expansion

To find the partial fraction expansion, we

- Find the poles $\lambda_1, \ldots, \lambda_n$, which means we find the zeros of $a(s)$.
- Find the residues of $r_1, \ldots, r_n$.

3 Main Methods to find Partial Fractions

These are (sometimes painful) exercises in algebra.

Last Homework and Final Exam will use the least painful (cover-up method), which is sufficient to glean the concept. Supplemental lecture 17 may discuss other methods.

Partial Fractions via "Cover-up" Method

Here, we solve for each residual individually in the following way. E.g., to get r_1, we first multiply both sides by $(s - \lambda_1)$.

$$\frac{b_0 + b_1 s + b_2 s^2}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \frac{r_3}{s-\lambda_3}$$

becomes

$$\frac{(s-\lambda_1)(b_0 + b_1 s + b_2 s^2)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = r_1 + \frac{r_2(s-\lambda_1)}{s-\lambda_2} + \frac{r_3(s-\lambda_1)}{s-\lambda_3}$$

Concrete Example of Using the Cover-Up

Let's find the following partial fraction expansion:

$$\frac{s^2 - 2}{s(s+1)(s+2)} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

CYU: Putting it Together

Compute the Inverse Laplace Transform (Hw 4b)

$$F\left(s\right)=\frac{s+4}{s^3+4s}$$

CYU: Apply it to Differential Equations

LTI system (20 points)

Assume a causal LTI system $\mathcal{S}_1$ is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = ax(t), \qquad y(0) = 0,\ y'(0) = 0$$

where a is a constant. Moreover, we know that when the input is e^t, the output of the system $\mathcal{S}_1$ is $\frac{1}{2}e^t$.

(a) (6 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

Next Lecture is the Last

Mostly Applications of what we have learned

1. Applying the Laplace Transform to more complicated Diff Eq for those who want to go further into Laplace.
2. Multi-dimensional signals and Applying the Fourier Transform to COVID-19
3. Other Touch-Ups that are good for practicing engineers.

New Topics from Next Lecture not on Final! But the applications of Laplace are implicitly helpful.