**ECE102, Spring 2021** **Final Exam**

Signals & Systems

University of California, Los Angeles; Department of ECE

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UCLA True Bruin academic integrity principles apply.

This exam is open book, open note and open internet. Collaboration is not allowed.

8:00 am Thursday, 3 June 2021
- 8:00 am Friday, 4 June 2021.

**Instructions for submission**

State your assumptions and reasoning for all the questions.
No credit without reasoning.

Name: ____________________

Signature: ____________________

ID#: ____________________

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|---|---|---|
| Problem 1 | _____ | / 21 |
| Problem 2 | _____ | / 29 |
| Problem 3 | _____ | / 15 |
| Problem 4 | _____ | / 20 |
| Problem 5 | _____ | / 15 |
| Problem 6 (Bonus) | _____ | / 5 bonus points |
| Problem 7 (Bonus) | _____ | / 1.5 bonus points |
| Total | _____ | / 100 points + 6.5 bonus points |

1. **Signal and Systems Basics** (21 points)

(a) (12 points) **System properties.** For each of the following systems, determine (with reasoning) if they are linear, time invariant, causal and stable.

i. (4 points) $y(t) = x(3t + 2) + 5$

**Solution:** Non-linear, not time invariant, not causal, stable

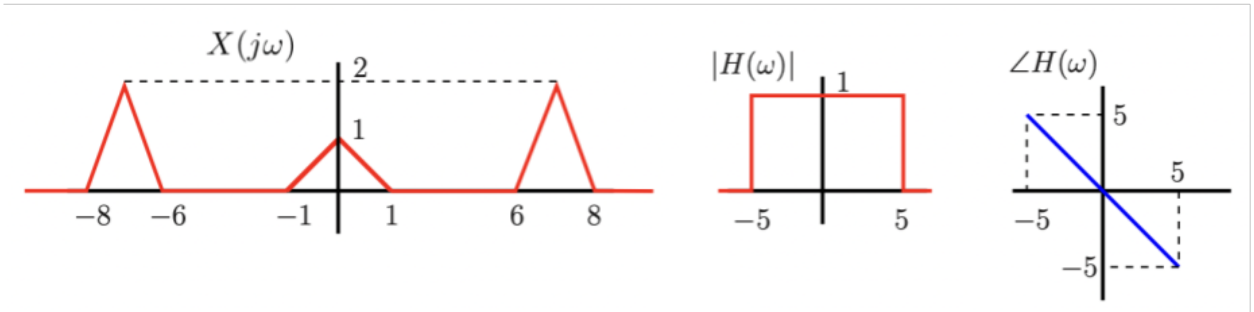
ii. (4 points) $y(t) = sin(\frac{dx(t)}{dt})$

**Solution:** Non-linear, time invariant, causal, stable

iii. (4 points) $y(t) = y(t) = e^{x^2(t)}$

**Solution:** Non-linear, time invariant, causal, stable

(b) (9 points) **LTI System Analysis.** Consider an LTI system with input $x(t)$, output $y(t)$ and impulse response $h(t)$. The Fourier transforms $X(j\omega)$ and $H(j\omega)$ are as shown below.

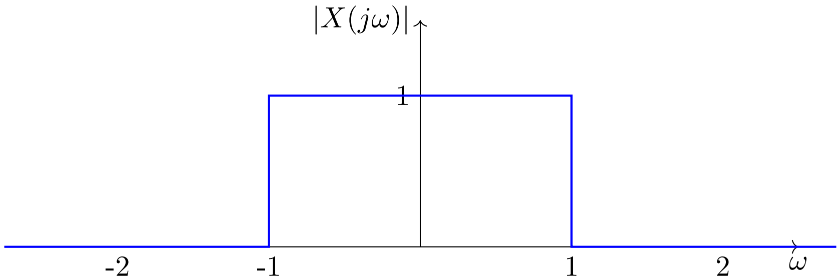


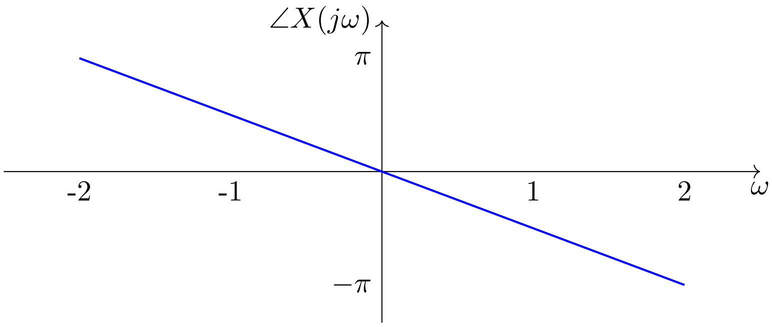

Evaluate $y(t)$.

**Solution:** $\frac{1}{2\pi} sinc^2\left(\frac{t-1}{2\pi}\right)$.

2. **Fourier transform** (29 points)

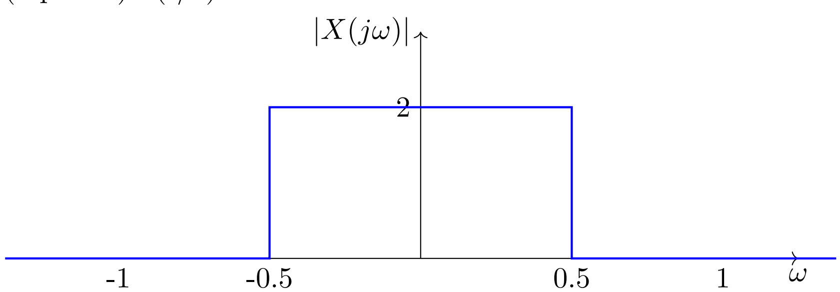
(a) (12 points) A signal x(t) has has the following the Fourier Transform.

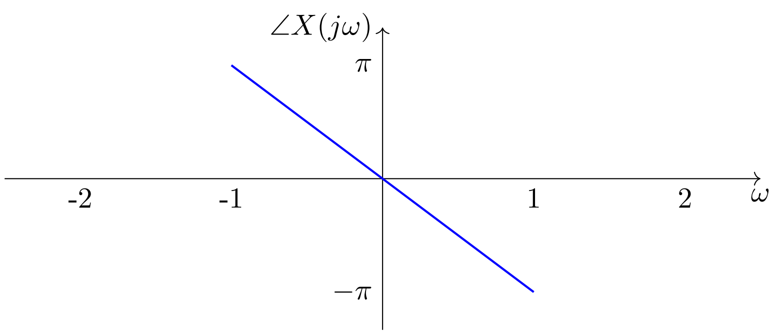


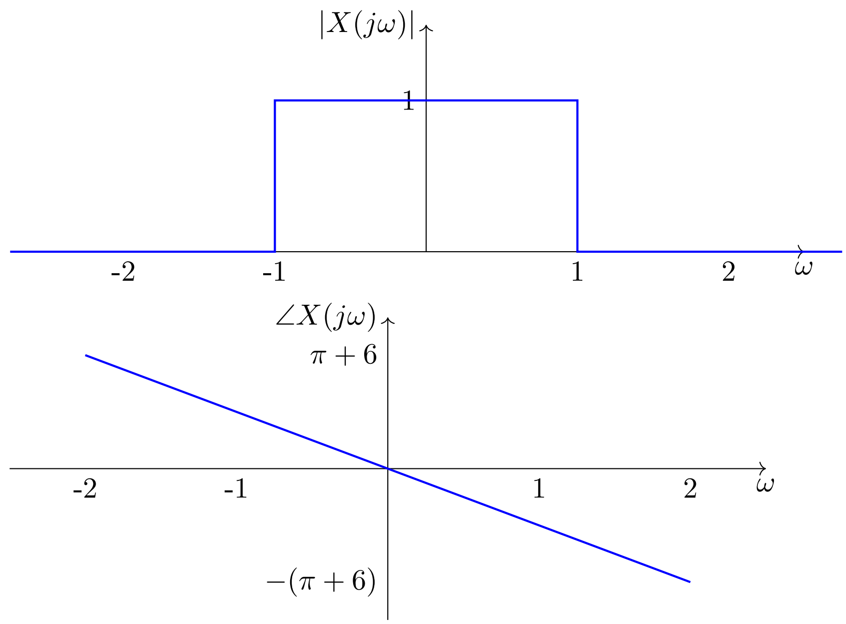

Plot the magnitude and phase plots for the Fourier Transform of the following signals:

i. (4 points) $x(t/2)$

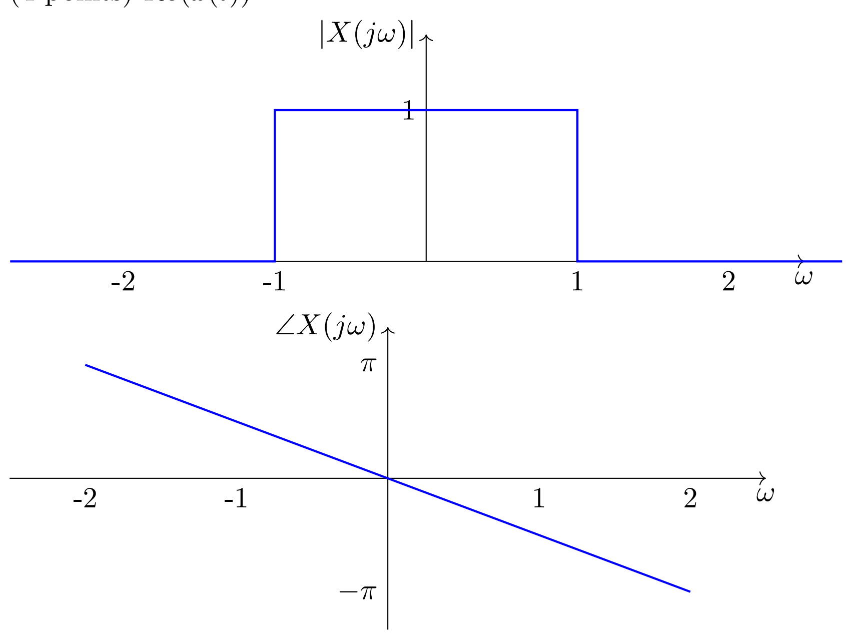



ii. (4 points) $x(t-3)$




iii. (4 points) $Re(x(t))$




(b) (9 points) Evaluate the Fourier Transforms of the following signals:

i. (4 points) $x(t) = e^{-2|t-1|}$

**Solution:** $\frac{4e^{-j\omega}}{4+\omega^2}$.

ii. (5 points) $x(t) = te^{-at}cos(\omega_0 t)u(t)$, $a > 0$

**Solution:** $\frac{(a+j\omega)^2-\omega_0^2}{((a+j\omega)^2+\omega_0^2)^2}$.

(c) (8 points) Evaluate the the time domain signals corresponding to the following Fourier transforms:

i. (4 points)

$$X(j\omega) = \begin{cases} 1 - |\omega|, & |\omega| < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

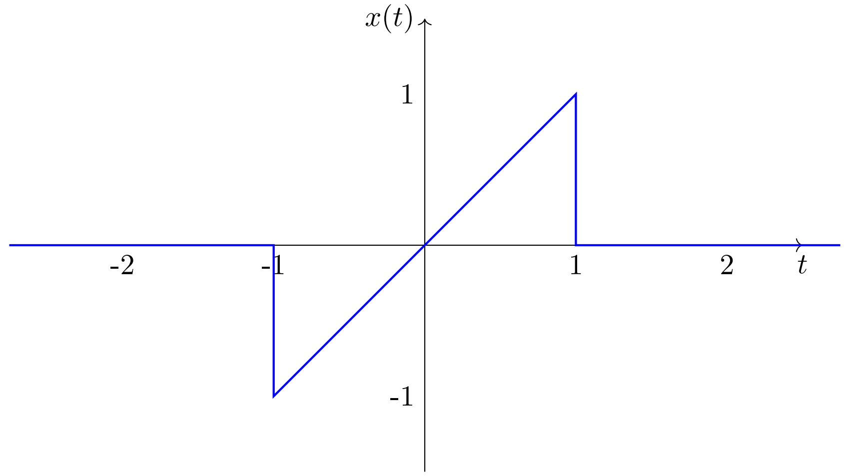
**Solution:** $\frac{1}{2\pi} sinc^2\left(\frac{t}{2\pi}\right)$.

ii. (4 points) $X(j\omega) = cos(2\omega + \frac{\pi}{6})$

**Solution:** $\frac{1}{2}e^{-j\frac{\pi}{12}t}(\delta(t-2) + \delta(t+2))$ OR $\frac{1}{2}(e^{-j\frac{\pi}{6}}\delta(t-2) + e^{j\frac{\pi}{6}}\delta(t+2))$.

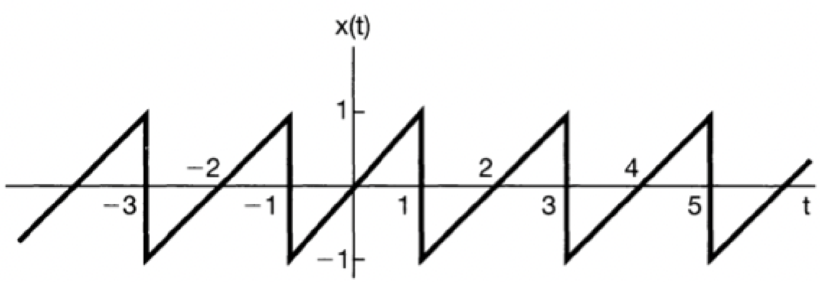
3. **Fourier Series** (15 points)

(a) (5 points) Evaluate the Fourier Transform of the following signal $x(t)$.




**Solution:** $\frac{2j}{\omega}(cos(\omega) - \frac{sin(\omega)}{\omega})$.

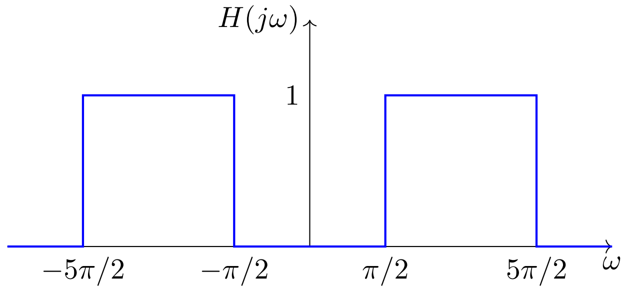
(b) (5 points) Using your solution from part (a), evaluate the fourier series of the following signal $\tilde{x}(t)$.




**Solution:**

$$c_k = \begin{cases} 0, & k = 0 \\ \frac{j}{k\pi}(-1)^k & \text{otherwise.} \end{cases} \tag{2}$$

(c) (5 points) Consider a system whose frequency response $H(j\omega)$ as follows:




What is the output when $\tilde{x}(t)$ is passed through this system?

**Solution:** $\frac{2}{\pi}sin(\pi t) - \frac{1}{\pi}sin(2\pi t)$.

4. **Frequency domain understanding** (20 points)

(a) Identify if the following statements are 'True' or 'False' with appropriately detailed reasoning.

i. (1 points) Sampling at a frequency greater than Nyquist frequency is a necessary condition for perfect reconstruction, for every signal.

**Solution:** False.

ii. (2 points) If we have two bandlimited signals, $x_1(t)$ with a bandwidth $B_1$ and $x_2(t)$ with a bandwidth $B_2$, the signal $y(t) = x_1(t)x_2(t)$ has a bandwidth $max\{B_1, B_2\}$. ($max\{a, b\}$ is equal to the maximum value among $a$ and $b$)

**Solution:** False.

iii. (2 points) Consider a periodic function $x(t)$ with a fundamental period $T$. If $x(t)$ is an odd function, the sum of all its Fourier series coefficients ($\Sigma_{k=-\infty}^{\infty} c_k$) is zero for any odd $x(t)$.

**Solution:** True, if exponential Fourier series used, false if sinusoid Fourier series used.

(b) Let $F(j\omega) = j2\pi\omega e^{-2|\omega|}$. Without computing $f(t)$ answer the following questions with appropriate reasoning.

i. (2 points) Is $f(t)$ real/imaginary/complex?

**Solution:** Real.

ii. (2 points) Is $f(t)$ odd/even/neither?

**Solution:** Odd.

iii. (1 points) What is $f(0)$?

**Solution:** $f(0) = 0$.

(c) Evaluate the following.

i. (7 points) Let $x(t) = \frac{4}{4+t^2}$. Evaluate the Fourier transform $X(j\omega)$. (Hint: use the duality property)

**Solution:** $2\pi e^{-2|\omega|}$.

ii. (3 points) Using the Fourier transform from the previous part, evaluate the energy of $x(t)$.

**Solution:** Energy$=\pi$.

5. **Sampling** (15 points)

(a) (4 points) The sampling theorem says that for a bandlimited signal, a signal must be sampled at a frequency greater than its Nyquist frequency to guarantee perfect reconstruction. Identify the Nyquist frequency for the following signals:

i. (2 points) $x(t) = cos(3000\pi t) - sin(2000\pi t)$

**Solution:** 3000 Hz.

ii. (2 points) $x(t) = \frac{sin(2000\pi t)}{\pi t}$

**Solution:** 2000 Hz.

(b) (6 points) Consider a signal $x(t)$ with a Nyquist frequency $\omega_0$. Determine the Nyquist frequency for the following signals:

i. (2 points) $x^2(t)$

**Solution:** $2\omega_0$.

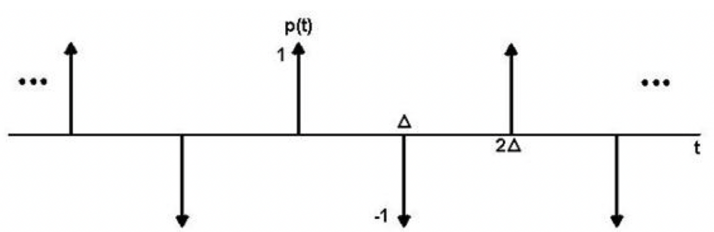
ii. (2 points) $x(t)cos(\omega_1 t)$

**Solution:** $2\omega_1 + \omega_0$.

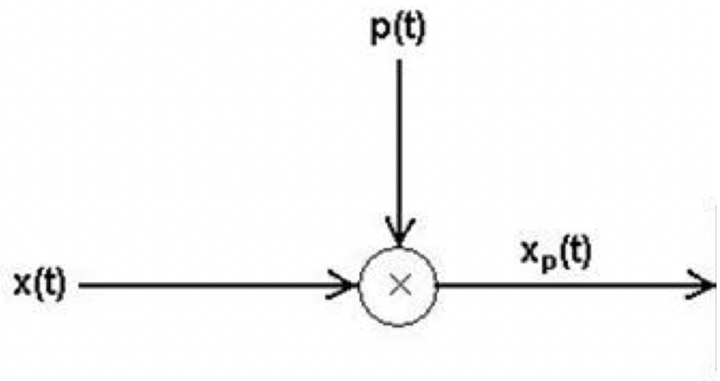
iii. (2 points) $\frac{dx(t)}{dt}$

**Solution:** $\omega_0$.

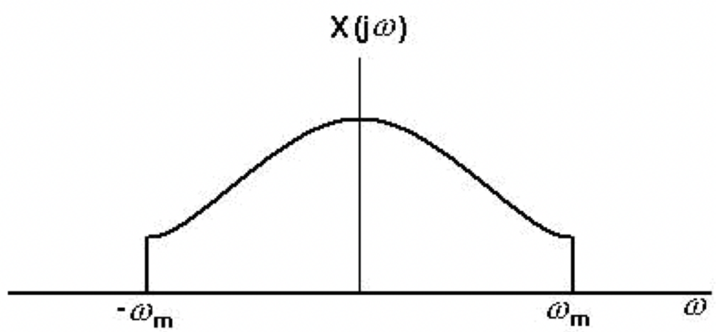
(c) (5 points) We know that ideal sampling is carried out by multiplying the time domain analog signal with an impulse train. Consider a modified sampling regime, where we multiply with the following signal $p(t)$:




The sampling process for a signal $x(t)$ is shown in the following figure:

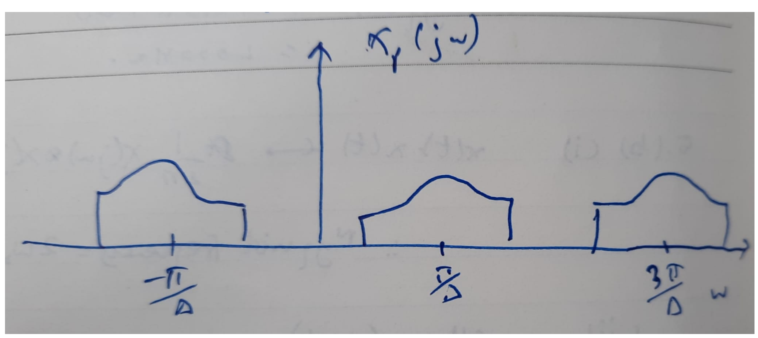



Let $x(t)$ be bandlimited with a one-sided bandwidth of $\omega_m$, with the following fourier transform:




i. (2 points) If $\Delta < \frac{\pi}{\omega_m}$, draw the Fourier transform of $x_p(t)$.

**Solution:**




ii. (3 points) If $\Delta < \frac{\pi}{\omega_m}$, determine a system to recover $x(t)$ from $x_p(t)$.

**Solution:** Multiply by $cos\left(\frac{\pi}{\Delta}t\right)$, and then apply an appropriate low pass filter.

6. **Laplace transform (Bonus)** (5 points)

A casual LTI system can be described by the following differential equation:

$$y''(t) + 4y'(t) + 4y(t) = 4x(t) + 1x''(t).$$

You may assume resting initial conditions (y(0)= 0, y'(0) =0, y"(0) = 0)

(a) (2.5 points) Find the transfer function $H(s)$.

**Solution:**

$$H(s) = \frac{4 + s^2}{s^2 + 4s + 4}.$$

(b) (2.5 points) What is the impulse response $h(t)$ of this system?

**Solution:** $h(t) = \delta(t) - 4e^{-2t}u(t) + 8te^{-2t}u(t)$.

7. **Industry Insight (Bonus)** (1.5 points)

(a) (0.5 points) (sound) You work at a company that has flown in a famous pianist by the name of Bruinhoven who will play a high-pitched version of the UCLA Fight song. The lowest note that the score asks for is 1000 Hz. In the digital recording the song sounds very low pitched, and you hear off-key notes that are in the hundreds of Hertz - much less than the lowest note in the score. Your boss believes Bruinhoven just had a bad day and played poorly. However, when you were listening in the room to Bruinhoven, the song sounded perfect. When you analyze the recording you also find something curious - these off-key notes are at unusual frequencies that do not map to the specific keys of the piano. What do you think is the most likely explanation, and what is your evidence?

**Solution:**
Aliasing occurs when a signal contains frequencies which are higher than half of the sampling rate. When we sample these higher frequencies are interpreted as lower frequencies that are not actually present in the signal (the higher frequencies wrap around the sampling rate). This could result in notes which are not actually present in the signal. Evidence would be that you heard the signal properly, but it sounds differently after digital recording. One way to fix this would be to use an ADC with a higher sampling rate.

(b) (0.5 points) (light) Can you rediscover the intuition behind a Nobel Prize in Physics? Lasers are like superpowered lightbulbs and tools for humanity. We can shoot lasers far into space or use powerful lasers to perform eye surgery. Unfortunately, lasers do not come in many colors. In the early days of its invention, lasers used to be available at infrared frequencies (topping out at 250 THz). This was not as useful since humans can only perceive frequencies of light from 400 THz to 700 THz. Look up the frequency range of what we perceive as "green". Given the infrared laser tech you have on hand and the tools from this class, how might you generate a green laser?

**Solution:**
Green light is the 540-580 THz region. We know that a linear method will not be sufficient for generating a frequency higher than one that is present in the input signal. We can accomplish this with a non-linear interaction between the light and a crystal, which combines the energy of two or more photons into one photon with a harmonic frequency. In this case we can use a Third Harmonic Generating crystal like BBO to generate Green Light within our frequency range.

(c) (0.5 points) (light) In this question, you will play with a simple while very effective trick (or philosophy) in signal and systems. The application scenario of this trick is photoplethysmogram (PPG). A PPG is an optically obtained plethysmogram that can be used to detect blood volume changes in the microvascular bed of tissue. A PPG is often obtained by using a pulse oximeter which illuminates visible light onto the skin and measures changes in light absorption using a camera. The blood volume changes can be further utilized for determining the vital signals like heartbeat rate. In industry, some companies try to use the blinking LED instead of steady-state LED as illumination source for PPG. How do you think it might help improve the performance of PPG (i.e., what is the trick behind it)?

**Solution:**
As we see the keyword 'visible light' in the question, we would have an intuition that the ambient light would be a issue. The ambient light mixes with the PPG signal and strong ambient inevitably deteriorate the accuracy of heartbeat rate measurement.

The blinking LED is a way to enhance the accuracy. We can control the 'on and off' of the LED, when the LED is on, the signal we get from the camera (sensor) is the combination of ambient light and the PPG signal. When the LED is off, we can have only the ambient light. The subtraction of these two enable you to get purified PPG signal.