ECE102, Spring 2020 **Homework #5**

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Due Sunday, 17 May 2020, by 11:59pm to CCLE.
100 points total.
This homework covers questions relate to Fourier series and Fourier transform.

1. (18 points) **Fourier Series**

   (a) (7 points) When the periodic signal $f(t)$ is real, you have seen in class some properties of symmetry for the Fourier series coefficients of $f(t)$ (see the Lecture 11 slide titled: Fourier Series Properties: Fourier Symmetry (cont.)). How do these properties of symmetry change when $f(t)$ is imaginary (with no real component)?

   Solution: Since $f(t)$ is pure imaginary, it can equivalently written as $f(t) = jg(t)$, where $g(t)$ is real.

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right] dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} jg(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right] dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} jg(t)\cos\left(\frac{2\pi k}{T_0}t\right) + g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt + j\frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\cos\left(\frac{2\pi k}{T_0}t\right) dt
\end{aligned}$$

   Now, because $g(t)$ is real:

$$\begin{aligned}
Re(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt \\
Im(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\cos\left(\frac{2\pi k}{T_0}t\right) dt
\end{aligned}$$

   Therefore,

$$\begin{aligned}
Re(c_k) &= -Re(c_{-k}) \\
Im(c_k) &= Im(c_{-k}) \\
c_k^* &= -c_{-k} \\
|c_k| &= |c_{-k}| \\
\angle c_k &= -\angle c_k^* \pm \pi
\end{aligned}$$

   (b) (7 points) A *real* and *odd* signal $x(t)$ has the following properties:

   - it is a periodic signal with period 1 s;

- it has one positive frequency component (positive frequency component meaning c_k with k > 0);
- it has a power of 9 (hint: consider Parseval's relation. The power of the signal in the time domain is the same as the sum of the powers of its frequency components).

What is $x(t)$?

Solution:

The signal has a fundamental period of 1 s, and its frequency is: $\omega_0 = 2\pi$. Therefore, it can be written as:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} c_k e^{j2\pi kt}$$

Since $x(t)$ is real and odd, the coefficients $c_k's$ are imaginary and odd. Since $x(t)$ has one positive frequency component, it can be reduced to the following:

$$x(t) = c_{-1}e^{-j2\pi t} + c_1 e^{j2\pi t} = c_1\left(e^{j2\pi t} - e^{-j2\pi t}\right) = ...$$

We know c_1 to be purely imaginary, so let $c_1 = jb$, where b is a real coefficient.

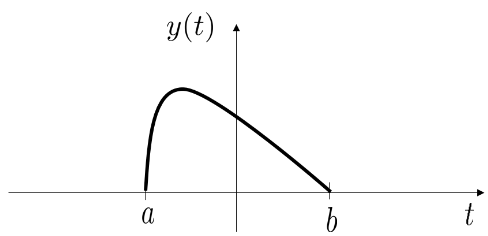
$$... = -\frac{2b}{2j}\left(e^{j2\pi t} - e^{-j2\pi t}\right) = -2b\sin(2\pi t)$$

Using Parseval's relation, we have:

$$2b^2 = 9 \implies b = \pm 3\sqrt{2}/2$$

Therefore, $x(t) = 3\sqrt{2}\sin(2\pi t)$ or $x(t) = -3\sqrt{2}\sin(2\pi t)$

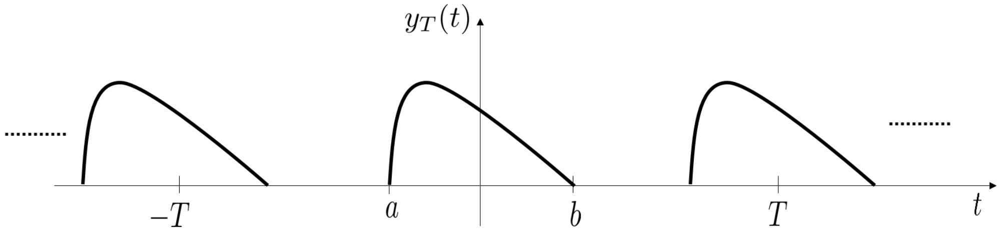
(c) (4 points) Consider the signal $y(t)$ shown below and let $Y(j\omega)$ denote its Fourier transform.



Let $y_T(t)$ denote its periodic extension:

How can the Fourier series coefficients of $y_T(t)$ can be obtained from the Fourier transform $Y(j\omega)$ of $y(t)$? (Note that the figures given in this problem are for illustrative purposes, the question is for any arbitrary $y(t)$).

Solution:



The Fourier transform of $y(t)$ is given by:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_a^b y(t)e^{-j\omega t}dt \tag{1}$$

The coefficients of the Fourier series for $y_T(t)$ are given by:

$$Y_k = \frac{1}{T}\int_a^b y(t)e^{-j(2\pi k/T)t}\,dt \tag{2}$$

for any integer k. Therefore, by comparing (2) to (1), we conclude:

$$Y_k = \frac{1}{T}Y(j\omega)\big|_{\omega=\frac{2\pi k}{T}}$$

2. (32 points) **Symmetry properties of Fourier transform**

(a) (16 points) Determine which of the signals, whose Fourier transforms are depicted in Fig. 1, satisfy each of the following:

i. $x(t)$ is even

Solution: If $x(t)$ is even, then its Fourier transform should be even. Since $X(j\omega)$ in (a), (d) and (e) are even, signals (a), (d) and (e) are all even in the time domain.

ii. $x(t)$ is odd

Solution: If $x(t)$ is odd, then its Fourier transform should be odd. Since $X(j\omega)$ in (f) is odd, signal in (f) is odd in the time domain.

iii. $x(t)$ is real

Solution: If $x(t)$ is real, then $X(j\omega)$ is Hermitian, i.e., $X(-j\omega) = X^*(j\omega)$. This means the real part of $X(j\omega)$ is even and the imaginary part of $X(j\omega)$ is odd. It also means that the magnitude of $X(j\omega)$ is even and the phase of $X(j\omega)$ is odd. Since $X(j\omega)$ in (c) and (e) are both Hermitian, signals (c) and (e) are real in the time domain.

iv. $x(t)$ is complex (neither real, nor pure imaginary)

Solution: For $x(t)$ to be complex (not real neither pure imaginary), $X(j\omega)$ should not be Hermitian or anti-Hermitian. We know from the previous part that $X(j\omega)$ in (c) and (e) are Hermitian. Signals in (d) and (f) are anti-Hermitian. Therefore, signals in (a) and (b) are both complex in the time domain.

v. $x(t)$ is real and even

Solution: If $x(t)$ is real and even, then $X(j\omega)$ is real and even. Therefore, it is (e).

vi. $x(t)$ is imaginary and odd

Solution: If $x(t)$ is imaginary and odd, then $X(j\omega)$ is real and odd. Therefore, it is (f).

vii. $x(t)$ is imaginary and even

Solution: If $x(t)$ is imaginary and even, then $X(j\omega)$ is imaginary and even. Therefore, it is (d).

viii. There exists a non-zero ω_0 such that $e^{j\omega_0 t}x(t)$ is real and even

Solution: If $e^{j\omega_0 t}x(t)$ is real and even, then $X(j(\omega-\omega_0))$ is real and even. In (b), it is symmetric if we shift $X(j\omega)$ to the left by 2 (i.e., $\omega_0 = -2$), giving a real and even Fourier signal. Elsewhere it will not be even if we shift the Fourier signal by a non-zero increment.

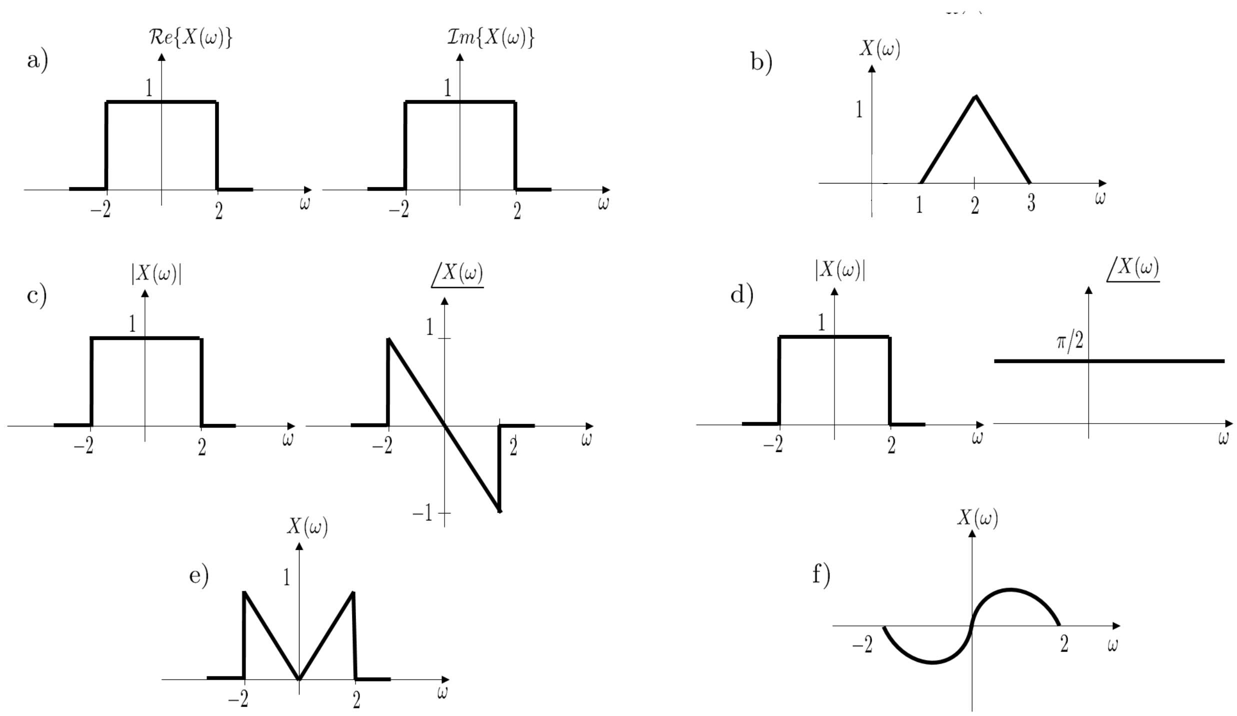


Figure 1: P2.a

(b) (8 points) Using the properties of the Fourier transform, determine whether the assertions are true or false.

i. The convolution of a real and even signal and a real and odd signal, is odd.

Solution: Let $f(t)$ be a real and even signal, and $g(t)$ be a real and odd signal. Then $F(j\omega)$ is real and even, and $G(j\omega)$ is imaginary and odd. The convolution $h(t) = (f * g)(t)$ has the Fourier transform

$$H(j\omega) = F(j\omega)G(j\omega)$$

If $F(j\omega)$ is real and even, and $G(j\omega)$ is imaginary and odd, then $H(j\omega)$ is imaginary and odd, and $h(t)$ is real and odd. The assertion is true.

ii. The convolution of a signal and the same signal reversed is an even signal.

Solution: Let $f(t)$ be a signal, and $f_R(t) = f(-t)$. Let $h(t) = (f * f_R)(t)$. Then

$$H(j\omega) = F(j\omega)F_R(j\omega) = F(j\omega)F(-j\omega)$$

which is even (replacing ω by $-\omega$ results in the same expression). As the Fourier spectrum is even, by symmetry the output of the convolutions is even. The assertion is true.

(c) (8 points) Show the following statements:

i. If $x(t) = x^*(-t)$, then $X(j\omega)$ is real.

Solution: When we reverse this,

$$\begin{aligned}
X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^* \\
&= \int_{-\infty}^{+\infty} \left[x(t)e^{-j\omega t}\right]^*\,dt \\
&= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt \\
&= \int_{-\infty}^{+\infty} x^*(-\tau)e^{-j\omega \tau}\,d\tau, \text{ here we did the variable change } \tau = -t \\
&= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega \tau}\,d\tau, \text{ here we used the fact that } x(\tau) = x^*(-\tau) \\
&= X(j\omega),
\end{aligned}$$

Since $X^*(j\omega) = X(j\omega)$, we conclude that $X(j\omega)$ is real.

ii. If $x(t)$ is a real signal with $X(j\omega)$ its Fourier transform, then the Fourier transforms $X_e(j\omega)$ and $X_o(j\omega)$ of the even and odd components of $x(t)$ satisfy the following:

$$X_e(j\omega) = Re\{X(j\omega)\}$$

and

$$X_o(j\omega) = jIm\{X(j\omega)\}$$

Solution:

Since $x(t) = x_e(t) + x_o(t)$, the Fourier transform of $x(t)$ is given by:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \underbrace{\int_{-\infty}^{\infty} x_e(t)e^{-j\omega t}dt}_{X_e(j\omega)} + \underbrace{\int_{-\infty}^{\infty} x_o(t)e^{-j\omega t}dt}_{X_o(j\omega)}$$

Now using Euler,

$$X_e(j\omega) = \int_{-\infty}^{+\infty} x_e(t)(\cos(\omega t) - j\sin(\omega t))dt = \int_{-\infty}^{+\infty} x_e(t)\cos(\omega t)dt$$

$$X_o(j\omega) = \int_{-\infty}^{+\infty} x_o(t)(\cos(\omega t) - j\sin(\omega t))dt = -j\int_{-\infty}^{+\infty} x_o(t)\sin(\omega t)dt$$

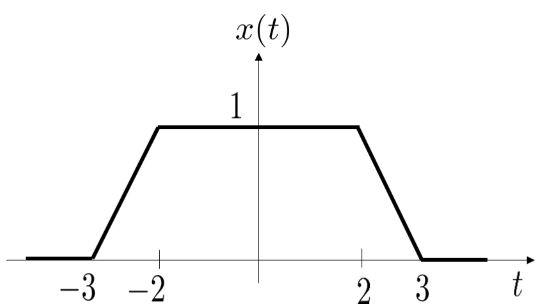
Since $x(t)$ is real, $x_e(t)$ and $x_o(t)$ are both real. Therefore, $X_e(j\omega)$ is real and $X_o(j\omega)$ is pure imaginary. Therefore,

$$\mathcal{R}e\{X(j\omega)\} = \int_{-\infty}^{+\infty} x_e(t)\cos(\omega t)dt = X_e(j\omega)$$

$$\mathcal{I}m\{X(j\omega)\} = -\int_{-\infty}^{+\infty} x_o(t)\sin(\omega t)dt = -jX_o(j\omega)$$

3. (15 points) **Fourier transform properties**

Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ sketched below:



Evaluate the following quantities without explicitly finding $X(j\omega)$:

(a) $\int_0^{\infty} X(j\omega)d\omega$
Hint: Consider the properties of $x(t)$.

Solution:
Intuitively, when we see the integer on $X(jw)$, we will recall the inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\, d\omega$$

Now that we do not have the exponential term in the integral, it comes to mind that when we let $t = 0$, the exponential term becomes a constant value 1:

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, d\omega$$

Also, when we observe the signal $x(t)$, we can find that $x(t)$ is even and real. According to what we have learned in lectures, $X(jw)$ should also be even and real.
Using this property, we can reformulate the previous equality to:

$$x(0) = \frac{1}{\pi} \int_{0}^{+\infty} X(j\omega)\, d\omega$$

Therefore, we find that the integral in the question can calculated as:

$$\int_{0}^{\infty} X(j\omega) d\omega = x(0)\pi = \pi$$

(b) $X(j\omega)|_{\omega=0}$

Solution:
Since

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt$$

we then have:

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt|_{\omega=0} = \int_{-\infty}^{+\infty} x(t)\, dt = \frac{1}{2} + 4 + \frac{1}{2} = 5$$

(c) $\angle X(j\omega)$

Solution: The Fourier transform of a real and even function is real and even. Therefore the phase of $X(j\omega)$ is either 0 or π. It is zero when $X(j\omega) \geq 0$ and it is π when $X(j\omega) < 0$.

(d) $\int_{-\infty}^{\infty} e^{-j\omega} X(j\omega) d\omega$

Solution:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t}\, d\omega \\
x(t)|_{t=-1} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t}\, d\omega|_{t=-1} \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega}\, d\omega
\end{aligned}$$

Therefore,

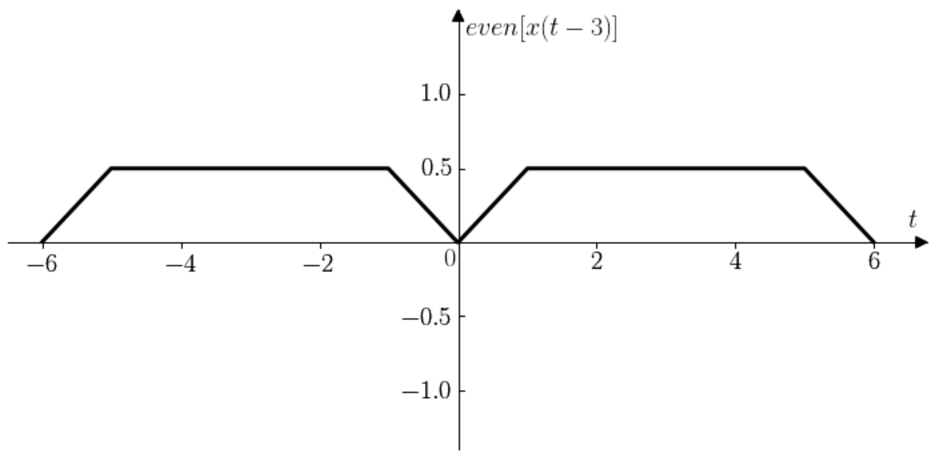
$$\int_{-\infty}^{+\infty} X(j\omega) e^{-j\omega}\, d\omega = 2\pi x(-1) = 2\pi$$

(e) Plot the inverse Fourier transform of $\mathcal{R}e\{e^{-3j\omega}X(j\omega)\}$
Hint: Consider the 'even and odd' properties of the Fourier transform

Solution: Let $Y(j\omega) = e^{-3j\omega}X(j\omega)$, then $y(t) = x(t-3)$. Since $y(t)$ is real,

$$\mathcal{R}e\{e^{-3j\omega}X(j\omega)\} = \mathcal{R}e\{Y(j\omega)\} = Y_e(j\omega)$$

where $Y_e(j\omega)$ is the Fourier transform of the even component of $y(t)$. Therefore, the inverse Fourier transform of $\mathcal{R}e\{e^{-3j\omega}X(j\omega)\}$ is the even component of $x(t-3)$.



4. (35 points) **Fourier transform and its inverse**

(a) (18 points) Find the Fourier transform of each of the signals given below:
Hint: you may use Fourier Transforms derived in class.

i. $x_1(t) = \begin{cases} 1+\cos(\pi t), & |t|<1 \\ 0, & \text{otherwise} \end{cases}$

Solution:
We can compute $X_3(j\omega)$ by applying the definition of Fourier transform:

$$\begin{aligned} X_1(j\omega) &= \int_{-1}^{1}[1+\cos(\pi t)]e^{-j\omega t}dt \\ &= -\frac{1}{j\omega}[e^{-j\omega}-e^{j\omega}] + \frac{1}{j2(\pi-\omega)}[e^{j(\pi-\omega)}-e^{-j(\pi-\omega)}] - \frac{1}{j2(\pi+\omega)}[e^{-j(\pi+\omega)}-e^{j(\pi+\omega)}] \\ &= \frac{2\sin(\omega)}{\omega} + \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega} = 2\text{sinc}\left(\frac{\omega}{\pi}\right) + \text{sinc}\left(\frac{\omega-\pi}{\pi}\right) + \text{sinc}\left(\frac{\omega+\pi}{\pi}\right) \end{aligned}$$

Or we can see that:

$$x_2(t) = \text{rect}\left(\frac{t}{2}\right) + \cos(\pi t)\text{rect}\left(\frac{t}{2}\right)$$

so that,

$$X_2(j\omega) = 2\text{sinc}\left(\frac{\omega}{\pi}\right) + \text{sinc}\left(\frac{\omega-\pi}{\pi}\right) + \text{sinc}\left(\frac{\omega+\pi}{\pi}\right)$$

ii. $x_2(t) = e^{(1+3j)t}u(-t+1)$

Solution:
We can write $x_2(t)$ as follows:

$$x_2(t) = e^{j3t}e^t u(-t+1) = e^{j3t}e^1 e^{t-1}u(-(t-1))$$

We know:

$$\begin{aligned}
e^{-t}u(t) &\longleftrightarrow \frac{1}{1+j\omega} \\
e^{t}u(-t) &\longleftrightarrow \frac{1}{1-j\omega} \\
e^{(t-1)}u(-(t-1)) &\longleftrightarrow \frac{e^{-j\omega}}{1-j\omega} \\
e^{j3t}e^{(t-1)}u(-(t-1)) &\longleftrightarrow \frac{e^{-j(\omega-3)}}{1-j(\omega-3)}
\end{aligned}$$

Therefore,

$$X_2(j\omega) = \frac{e^{1-j(\omega-3)}}{1-j(\omega-3)}$$

iii. $x_3(t) = 2te^{-2t}u(t)$
Hint: You can consider Fourier transform of the derivative and its dual.

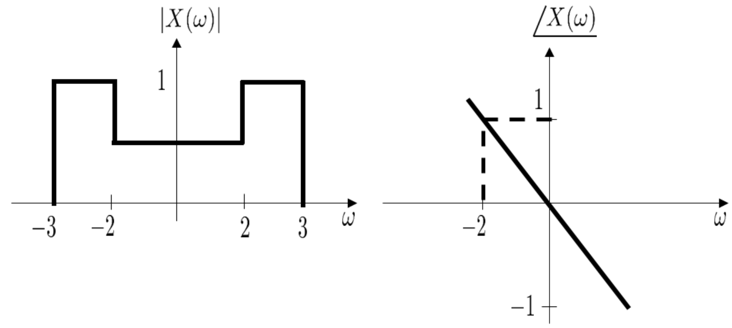
Solution:
We know the following mapping regarding Fourier transform:

$$\begin{aligned}
-jtf(t) &\longleftrightarrow F'(j\omega) \\
e^{-2t}u(t) &\longleftrightarrow \frac{1}{2+j\omega}
\end{aligned}$$

Therefore, we can let $f(t) = e^{-2t}u(t)$. Then $x_3(t) = -\frac{2}{j}(-jtf(t))$. Correspondingly,

$$X_3(j\omega) = -\frac{2}{j}F'(jw) = -\frac{2}{j}\left(\frac{d}{d\omega}\frac{1}{2+j\omega}\right) = \frac{2}{(2+j\omega)^2}.$$

(b) (7 points) Find the inverse Fourier transform of the signal shown below:



Solution: We have:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|e^{j\angle X(\omega)}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-3}^{-2} e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega + \int_{-2}^{2}\frac{1}{2}e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega + \int_{2}^{3} e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{-3}^{-2} e^{j(t-\frac{1}{2})\omega}d\omega + \int_{-2}^{2}\frac{1}{2}e^{j(t-\frac{1}{2})\omega}d\omega + \int_{2}^{3} e^{j(t-\frac{1}{2})\omega}d\omega\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{-j2(t-\frac{1}{2})} - e^{-j3(t-\frac{1}{2})}}{j(t-\frac{1}{2})} + \frac{e^{j2(t-\frac{1}{2})} - e^{-j2(t-\frac{1}{2})}}{j2(t-\frac{1}{2})} + \frac{e^{j3(t-\frac{1}{2})} - e^{j2(t-\frac{1}{2})}}{j(t-\frac{1}{2})}\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{-2j(t-\frac{1}{2})} - e^{2j(t-\frac{1}{2})}}{j2(t-\frac{1}{2})} + \frac{e^{j3(t-\frac{1}{2})} - e^{-j3(t-\frac{1}{2})}}{j(t-\frac{1}{2})}\right) \\
&= \frac{1}{2\pi}\left(-\frac{\sin\left(2(t-\frac{1}{2})\right)}{(t-\frac{1}{2})} + \frac{2\sin\left(3(t-\frac{1}{2})\right)}{(t-\frac{1}{2})}\right)
\end{aligned}
$$

(c) (10 points) Two signals $f_1(t)$ and $f_2(t)$ are defined as

$$
\begin{aligned}
f_1(t) &= \mathrm{sinc}(2t) \\
f_2(t) &= \mathrm{sinc}(t)\cos(3\pi t)
\end{aligned}
$$

Let the convolution of the two signals be

$$f(t) = (f_1 * f_2)(t)$$

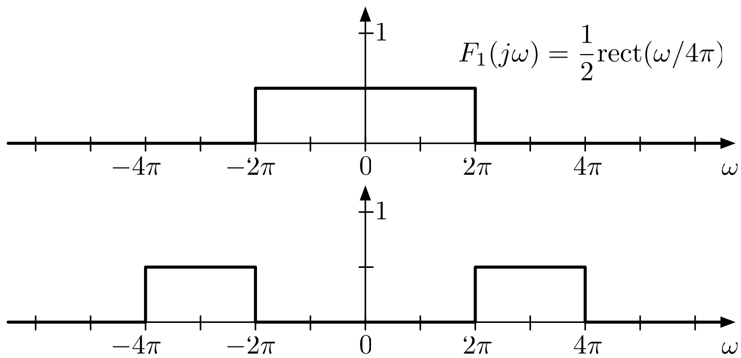
i. Find $F(j\omega)$, the Fourier transform of $f(t)$.

Solution: We know that:

$$
\begin{aligned}
f_1(t) = \mathrm{sinc}(2t) &\longleftrightarrow F_1(j\omega) = \frac{1}{2}\mathrm{rect}\left(\frac{\omega}{4\pi}\right) \\
f_2(t) = \mathrm{sinc}(t)\cos(3\pi t) &\longleftrightarrow F_2(j\omega) = \frac{1}{2}\left(\mathrm{rect}\left(\frac{\omega-3\pi}{2\pi}\right) + \mathrm{rect}\left(\frac{\omega+3\pi}{2\pi}\right)\right)
\end{aligned}
$$

We then have:

$$f(t) = (f_1 * f_2)(t) \longleftrightarrow F(j\omega) = F_1(j\omega)F_2(j\omega)$$



$$F_2(j\omega) = \frac{1}{2}\left[\text{rect}\left(\frac{\omega - 3\pi}{2\pi}\right) + \text{rect}\left(\frac{\omega + 3\pi}{2\pi}\right)\right]$$

To see what the multiplication of $F_1(j\omega)$ and $F_2(j\omega)$ gives us, let us first plot them. We clear see that $F_1(j\omega)$ and $F_2(j\omega)$ do not overlap, therefore $F(j\omega) = 0$.

ii. Find $f(t)$.

Solution:
Since $F(j\omega) = 0$, $f(t)$ is then 0