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DIS 2E

Math 61 HW #1

1. $(1+x)^n \geq 1+nx$ for all $x \geq -1$, and $\underbrace{\text{for all } n \in N}_{n>0}$

Base case: $n=1$

$(1+x)^1 \geq 1+x$

$1+x \geq 1+x$ ✓

Base case holds ✓

Inductive Step: Now suppose $(1+x)^n \geq 1+nx$ for some $n \in N$

Show that: $(1+x)^{n+1} \geq 1+(n+1)x$

$(1+x)^{n+1} = (1+x)^n(1+x)$ $\quad (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+nx+x+nx^2$

$1+nx+x+nx^2 \geq 1+(n+1)x \qquad (nx^2 \geq 0)$

$(1+x)^{n+1} \geq 1+(n+1)x$ ✓

This statement will always be true because after proving that our base case holds, x will always be nonnegative and n will always be positive. By the inductive hypothesis (and double-checking), all nonnegative values of x and all positive values of n make this final statement true.

2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in N$

Base case: $n=1$

$1^2 = 1 = \frac{(1)(2)(3)}{6} = 1$ ✓ Base case holds.

Inductive Step: Assume that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ is true for some $n>1$, show that

$$1^2+2^2+\ldots n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \text{Inductive Hypothesis}$$

Simplify $$1^2+2^2+\ldots n^2+(n+1)^2 = \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad ✓$$

This proves that since the formula holds for n, it holds for n+1. By induction, this identity holds true for all natural numbers

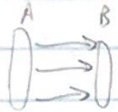
☆ 3. • Prove that $(a,b) = (a',b')$ if and only if $a=a'$ and $b=b'$

- Definition of ordered pairs: $(a,b) = \{\{a\},\{a,b\}\}$
- Assume for now that $(a,b) = (a',b')$
- Thus: $\{\{a\},\{a,b\}\} = \{\{a'\},\{a',b'\}\}$
- These ordered pairs are equal if $\{a\} = \{a'\}$ and $\{a,b\} = \{a',b'\}$
- Since it was just stated that $\{a\} = \{a'\}$, this can only happen if the elements in each of these sets are equal
  - Therefore, $a = a'$, which implies $\{a',b'\} = \{a,b'\}$
- Now, $\{a,b\} = \{a',b'\} = \{a,b'\}$
  - From this it can be observed that $b = b'$
- Now it has been shown that $a=a'$ and $b=b'$
- With this knowledge, finish the proof with def. of ordered pairs:

$$(a,b) = \{\{a\},\{a,b\}\}$$
$$= \{\{a'\},\{a',b\}\}$$
$$= \{\{a'\},\{a',b'\}\} = (a',b')$$

This proves that $(a,b) = (c,d)$. ✓

4. $f: A \to B$ is a bijection



Show: $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$

$id_A : A \to A$ $\quad id_B : B \to B$ $\quad f^{-1} : B \to A$ $\quad f : A \to B$

According to the info in the line above, this is simple.

① $f^{-1} \circ f : (B \to A) \circ (A \to B) \to A \to A = id_A$ ✓

② $f \circ f^{-1} : (A \to B) \circ (B \to A) \to B \to B = id\,B$ ✓

This logic is supported by the fact that the composition in ① simply swaps the "B" in "f^{-1}" into "A" and in ②, the "A" in "f" becomes a "B"

5. ① $n+1 \to$ This is injective because each element in the domain points to one element in the codomain. It's not surjective because the codomain $\neq$ the range.

② $2n \to$ Bijective: Injective because each element in the domain points to one element in the codomain. Surjective because every value of "n" will be onto one of the values of $2n$. Inverse: $f^{-1}(n) = \frac{1}{2}n$.

③ $\lfloor \frac{n+1}{2} \rfloor \to$ Not injective $\to$ multiple values of n map to the same output (eg. $n=1,2$)
It is surjective because each element in the codomain has at least one arrow pointing to it.