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Math 61 HW #2

* 1 When $n \geq 1$, For all $a, b, c \in \mathbb{Z}$

i) $a \equiv a \pmod{n}$

ii) if $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$

iii) if $a \equiv b \pmod{n}$ then $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$

In this case, $n = 5$. $5 \geq 1$ ✓

This means that congruence modulo 5 is an equivalence relation on $\mathbb{Z}$ because it is reflexive, symmetric, and transitive.

There will be 5 different equivalence classes:

$[0]: \{\ldots, -5, 0, 5, 10, 15, \ldots\}$

$[1]: \{\ldots, -4, 1, 6, 11, 16, \ldots\}$

$[2]: \{\ldots, -3, 2, 7, 12, 17, \ldots\}$

$[3]: \{\ldots, -2, 3, 8, 13, 18, \ldots\}$

$[4]: \{\ldots, -1, 4, 9, 14, 19, \ldots\}$

2. X is a Set P: collection of subsets of X

- This is basically what the definition of "partition of a set" entails. In class we found that:

① $\bigcup B = X \quad B \in P$

② $B \cap B' = \emptyset$ for any $B, B' \in P'$ while $B \neq B'$

These conditions must hold for P to be a partition because

① Blocks have to be within P, which are in turn within X

② There shouldn't be equivalent blocks within a partition anyway. The intersection of two completely different blocks would always be an empty set because they share nothing in common.

* 3. R is an equivalence relation on set X.

Define $[x] = \{y \in X : y R x\}$ -> This is the formal definition of an equivalence class when R is an equivalence relation on set X.

$a \to x$

. $P = \{[x] : x \in X\}$ is a partition of X because the distinct set of equivalence classes of R forms a partition of X.

Must show that every element in X belongs to exactly one member of P

Let $x \in X$. Since $x R x$, $x \in [x]$. Thus every element in X belongs to at least one member of S. Now must show that every element in X belongs to exactly one member of S.

eg. if $x \in X$ and $x \in [a] \cap [b]$, then $[a] = [b]$

Show that for all $c, d \in X$ if $c R d$ then $[c] = [d]$. Suppose cRd. Let $x \in [c]$. Then $x R c$. Because $c R d$ and R is transitive, $x R d$. This means $x \in [d]$ and $[c] \subseteq [d]$. Using the same logic, the claim $[d] \subset [c]$ is true as well (just interchange roles of c and d).

Now assume that $x \in X$ and $x \in [a] \cap [b]$. Then xRa and xRb. All of this proof and statements combined show that $[x] = [a]$ and $[x] = [b]$. Thus, $[a] = [b]$ and P is a partition of X.

* This theorem and its inverse are known as the Fundamental theorem on equivalence relations.

4. This is the inverse in #3.

X is a set with partition $P = \{X_1, X_2, X_3 \ldots\}$; R is the relation induced by P

R is reflexive:

Let $x \in X$. Because the union of the sets in the partition $P = X$, x must belong to one of the sets in P. Since $x \in X_i \wedge x \in X_i$, xRx by def. of relation induced by partitions ✓

R is symmetric

Let xRy. $\exists i (x \in X_i \wedge y \in X_i)$ by def. of relation induced by a partition. Because $y \in X_i \wedge x \in X_i$, yRx. ✓

R is Transitive

Suppose $xRy \wedge yRz$.

$\exists i (x \in X_i \wedge y \in X_i)$ and $\exists j (y \in X_j \wedge z \in X_j)$

The sets are pairwise disjoint, either $X_i = X_j$ or $X_i \cap X_j = \emptyset$

y is in both these sets, $X_i \cap X_j \neq \emptyset$, so $X_i = X_j$

x and z are both in the same set, so xRz ✓

These statements showing that R is reflexive, symmetric, and transitive prove that R is an equivalence relation on X.

5. R be a relation on a set X

$[x] = \{ y \in X \mid y R x \}$

Arbitrary relations (eg. nonequivalence relations) will not yield a partition of X because of the Fundamental Theorem on Equivalence Relations. This theorem states that for equivalence relations, the sets [x] would give a partition of X.

In this case of arbitrary relations, we can assume the relations are not reflexive, symmetric, and transitive. This disqualifies them from being equivalence relations and then the theorem on equivalence relations won't apply to them. Therefore, the sets do not necessarily give a partition of X.