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Math 61
11/20/2020

Math 61 Midterm #2

1 It is possible for a graph to have exactly five vertices of degree one.

For a vertex to have degree 1, that means it only has a single neighbor. This would mean the vertex only has a single edge incident to it.

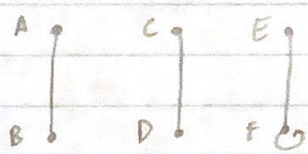
Note: A single vertex looping to itself counts as degree 2, not degree 1. This renders loops invalid.

Example 1 G:

D, E, F, X, Y each have 1 neighbor and are incident to a single edge

For instance, in this graph G shown above, we see that vertices D, E, F, X, and Y all have degree 1. This graph has exactly five vertices of degree 1.

Example 2 T:



A, B, C, D, E each have 1 neighbor and are incident to a single edge.

In this graph T we see that the vertices A, B, C, D, E all have degree 1. This graph has exactly five vertices of degree 1. (F contains a loop to itself, which already guarantees a degree greater than one)

It is possible for a graph to have exactly five vertices of degree 1.

2. $x, y, w, z \in \mathbb{N}$ $x \geq 0,\ y \geq 1,\ w > 0,\ z \geq 3$

$x + y + w + z = 20$

$n = 4$ $k = 20$ $k + n - 1 = 23$

23 total slots: 20 stars, 3 bars

$$\begin{array}{r} y: 1 \\ w: 1 \\ + \ z: 3 \\ \hline \end{array}$$
5 slots excluded

Y, W, and Z have enforced minimums - this will reduce the amount of available slots by 5. (now 18 slots - 15 stars, 3 bars)

So now it would be

x | y | w | z

$x + y + w + z = 15$ (15 because we decreased by 5 due to constraints/enforced minimums)

$x, y, w, z \geq 0$ now

We can fill the above stars and bars diagram however we desire as long as $x + y + w + z = 15$ and $x, y, w, z \geq 0$

This means we have 15 spaces available for all our stars, along with the 3 bars. To calculate this we perform

$$\binom{k+n-1-5}{k-5}$$

$$\binom{18}{15} \rightarrow \frac{18!}{(3!\ 15!)} = \boxed{816 \text{ solutions}}$$

Sample of stars and bars method:

x	y	w	z
☆☆☆☆	☆☆☆☆	☆☆☆	☆☆☆☆

$x = 4$ $y = 4$ $w = 3$ $z = 4$

$4 + 4 + 3 + 4 = 15$ ✓

Basically:

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

There are 18 slots to fill. Just add the 3 bars anywhere and fill the rest with stars to find values for x, y, w, z.

3. $X = \{0, 1, 2\}$ $G = (V, E)$ bipartite graph $V = V_1 \cup V_2$

$V_1 = X$ $V_2 = P(x)$ $v_1 \in V_1$, $v_2 \in V_2$ edge $\{v_1, v_2\} \in E$

iff v_1 is an element of v_2

$V_1 = \{0, 1, 2\}$

$V_2 = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

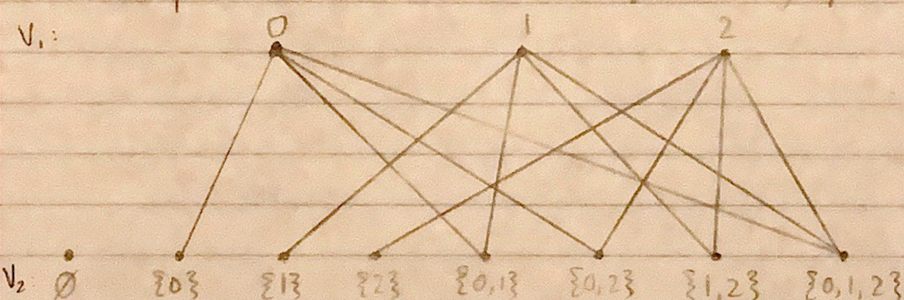
List of all possible edges: $\{v_1, v_2\} \mid v_1 \in v_2$

0: $\{0, \{0\}\}$, $\{0, \{0,1\}\}$, $\{0, \{0, 2\}\}$, $\{0, \{0,1,2\}\}$

1: $\{1, \{1\}\}$, $\{1, \{0,1\}\}$, $\{1, \{1,2\}\}$, $\{1, \{0,1,2\}\}$

2: $\{2, \{2\}\}$, $\{2, \{0,2\}\}$, $\{2, \{1,2\}\}$, $\{2, \{0,1,2\}\}$

Visual representation of these edges in graph G.



1. As shown above, the graph G is not connected. We know this because a graph G is said to be connected when "given any vertices v_1, v_2 in G, there is a path from v_1 to v_2". We clearly see that $\emptyset \in V_2$, so it is a part of the graph. However, none of the elements of V_1 are adjacent with $\emptyset$ of V_2. $\emptyset$ won't ever be incident to an edge – no path will ever reach it in the graph G.
Based on these observations, we can conclude that G is not connected due to $\emptyset$ in V_2.

2. The degree of a vertex v_1 is defined as the total number of its neighbors. We have a power set for a set containing three variables here. This means that the power set will consist of: the null set,

three sets with cardinality 1, three sets with cardinality 2, and one set with cardinality 3.

For any vertex $v_1 \in V_1$, v_1 will be adjacent to one of the sets with cardinality 1, two of the sets with cardinality 2, and to the set with cardinality 3. In total, this would mean any vertex v_1 in V_1 would have a degree of 4 because it would be an element to 4 distinct sets of the powerset.

$\boxed{\deg(v_1) = 4}$ → This is also demonstrated by my graph from the page before.

③ The degree of a vertex v_2 is defined as the total number of its neighbors. V_2 is a powerset that was created from the elements of V_1. In order for an element $v_1 \in V_1$ to be adjacent to an element $v_2 \in V_2$, the elements in v_2 must exist within V_1. This means that the degree of v_2 is dictated by its cardinality, or how many elements it contains because all the elements of V_1 are just a single number. For instance, if $v_2 = \{1\}$, it would only be adjacent to 1 in V_1. If $v_2 = \{1, 2\}$, it would be adjacent to 1 and 2 in V_1. If $v_2 = \{0, 1, 2\}$, it would be adjacent to 1, 2, and 3 in V_1. Based on these observations we can conclude that the degree of v_2 is simply its cardinality.

$\deg(v_2) = |v_2|$ → This is also demonstrated by my graph from the page before.

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4. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4}$ for $n \geq 4$

$x^4 - c_1 x^3 - c_2 x^2 - c_3 x - c_4$

$a_n = r^n$ is a solution

First: Assume r^n is a solution to the recurrence relation $\rightarrow$ plug it in

Original $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4}$

Substitute $r^n = c_1 r^{n-1} + c_2 r^{n-2} + c_3 r^{n-3} + c_4 r^{n-4}$

Terms on one side $r^n - c_1 r^{n-1} - c_2 r^{n-2} - c_3 r^{n-3} - c_4 r^{n-4} = 0$

Divide by r^{n-4} $\frac{r^n}{r^{n-4}} - \frac{c_1 r^{n-1}}{r^{n-4}} - \frac{c_2 r^{n-2}}{r^{n-4}} - \frac{c_3 r^{n-3}}{r^{n-4}} - \frac{c_4 r^{n-4}}{r^{n-4}} = 0$

$r^4 - c_1 r^3 - c_2 r^2 - c_3 r - c_4 = 0$

We have assumed that r^n is a solution. For r to be a root of the polynomial, r has to be a root of the general equation above. Because the equation equates to 0, we know that r is a root.

Our general equation $r^4 - c_1 r^3 - c_2 r^2 - c_3 r - c_4$ is basically the same as $x^4 - c_1 x^3 - c_2 x^2 - c_3 x - c_4$, but just with different variables. Based on the work above, we see that if r is a root of $x^4 - c_1 x^3 - c_2 x^2 - c_3 x - c_4$, then $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4}$, $n \geq 4$. We can conclude this because we have shown that r has to be a root of the polynomial in order for r^n to be a solution to the recurrence relation.