Ethan Wong
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Midterm #1

1. $n \geq 2$ $\quad A_1, \ldots A_n$ and C - arbitrary sets $\quad n$ is a natural number

$$\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$$

<u>Basis step</u> Have $n = 2$

Claim: $\left(\bigcup_{i=1}^{2} A_i\right) \times C = \bigcup_{i=1}^{2} (A_i \times C)$

$\rightarrow (A_1 \cup A_2) \times C = (A_1 \times C) \cup (A_2 \times C)$

let $(a, b) \in (A_1 \cup A_2) \times C = \left(\bigcup_{i=1}^{2} A_i\right) \times C$

$a \in (A_1 \cup A_2)$, $b \in C$

$(a \in A_1$ or $a \in A_2)$, $b \in C$

$(a \in A_1, b \in C)$ or $(a \in A_2, b \in C)$

$(a,b) \in A_1 \times C$ or $(a,b) \in A_2 \times C$

$(a,b) \in (A_1 \times C) \cup (A_2 \times C) = \bigcup_{i=1}^{2} (A_i \times C)$

Therefore: $\left(\bigcup_{i=1}^{2} A_i\right) \times C \subseteq \bigcup_{i=1}^{2} (A_i \times C)$. AND

$\bigcup_{i=1}^{2} (A_i \times C) \subseteq \left(\bigcup_{i=1}^{2} A_i\right) \times C$

Based on the fact that $A = B$ iff $A \subseteq B$ and $B \subseteq A$:

We can conclude that $\left(\bigcup_{i=1}^{2} A_i\right) \times C = \bigcup_{i=1}^{2} (A_i \times C)$ ✓

<u>Inductive step</u> - Here we assume that the result $(P(n))$ is true for $n \geq 2$. (Also assume $n \geq 2$)

$$\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$$

Inductive Hypothesis

Prove for $n+1$

$\left(\bigcup_{i=1}^{n+1} A_i\right) \times C = \left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right) \times C$

$= \left(\left(\bigcup_{i=1}^{n} A_i\right) \times C\right) \cup (A_{n+1} \times C)$

By inductive hypothesis: $= \left(\bigcup_{i=1}^{n} (A_i \times C)\right) \cup (A_{n+1} \times C)$

$= \bigcup_{i=1}^{n+1} (A_i \times C)$

This results in $\rightarrow \left(\bigcup_{i=1}^{n+1} A_i\right) \times C = \bigcup_{i=1}^{n+1} (A_i \times C)$.

By the induction principle, for $n \geq 2$ and when n is a natural number $\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C)$

2. Let R be a relation a set X

a) A relation R on a set X would be antisymmetric if and only if for every pair $x, y \in X$: if $x R y$ then $y \not R x$

A relation R on a set X would be symmetric if and only if for every pair $x, y \in X$: if $x R y$ then $y R x$

The reason why "R is anti-symmetric" isn't the negation of "R is symmetric" because antisymmetric means the only possible way $x R y$ and $y R x$ would be if $x = y$. Not symmetric means that the relation R would never have $x R y$ and $y R x$ even if $x = y$. The negation of symmetric would mean that "for not all pairs $x, y \in X$: if $x R y$ then $y R x$". This shows that the negation of "R is symmetric" isn't synonymas with "R is anti-symmetric"

EX: $A = \{1, 2, 3\}$

$R = \{(1,2), (2,1) (1,3)\}$

R is not symmetric because $(1,3) \in R$ but $(3,1) \notin R$

R is not antisymmetric because $(2,1), (1,2) \in R$ BUT $1 \neq 2$

This example proves "R is anti-symmetric" is not the negation of "R is symmetric"

b) A relation R on a set X would be anti-reflexive if for every $x \in X$: $x \not R x$.

A relation R on a set X would be reflexive if for every $x \in X$: $x R x$.

The negation of "R is reflexive" would be "for not all $x \in X$: $x R x$". This is logically equivalent to saying that "for some $x \in X$: $x \not R x$". This is not equivalent to saying "R is anti-reflexive" because anti-reflexive means "all $x \in X$: $x \not R x$"

Ex: $A = \{4, 5, 6\}$

$R = \{(4,5), (5,6), (4,4)\}$

R is not Reflexive because $(5,5) \notin R$

R is not anti-reflexive because $(1,1) \in R$

This shows that "R is anti-reflexive" is not the negation of "R is reflexive"

3. $X = \{i \in N : 1 \leq i \leq n\}$

$P(x)$: power set of X $\quad P^*(X) := P(X) \setminus \{0\}$

$$f : P^*(x) \to X$$

Injective: The function isn't injective if $n > 1$

For example, $x = \{1, 2\}$

$P(x) = \{\{\emptyset\}, \{1\}, \{2\}, \{1,2\}\}$

$P^*(x) = \{\{1\}, \{2\}, \{1,2\}\}$

$P^*(x)$		X
$\{1\}$	$\to$	1
$\{2\}$	$\to$	2
$\{1,2\}$	$\to$	1

Two elements in $P^*(x)$ mapped to 1, which means when $n > 1$ it is not injective. Similar occurences happen when $n=3$, etc.

The function will only be injective if $n = 1$.

$P(X) = \{\{\emptyset\}, \{1\}\}$

$P^*(x) = \{\{1\}\}$

$P^*(x)$ $\quad$ X

$\{1\} \longrightarrow 1$ $\quad$ This is one-to-one, so it is injective. ✓

Surjective: f will be surjective if the range(f) is equal to the codomain. Every element of the codomain X will have a unique preimage. This means f will be surjective for all elements in $P^*(x)$

Bijective: For f to be bijective, it has to be injective and surjective for all elements of $P^*(x)$. The only time f is injective is when $n = 1$. f is surjective for all elements of $P^*(x)$. This means the only time f can be bijective is when $n = 1$.

4. The result is obtained by adding the number of ways where Averie is first in line with the number of ways such that Charlie is in the line. Then subtract from this number the number of ways such that Averie is first or Charlie is last in line.

# of ways Averie is first = 10!
# of ways Charlie is last = 10!
# of ways Averie is first or Charlie is last = 9!

$$\frac{10! + 10!}{9!}$$

$\boxed{6,894,720 \text{ ways}}$