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Homework 5

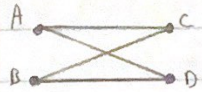
1. The logic behind this statement is simple. If you double the number of edges, you get the sum of all the degrees. The sum of degrees is even. The sum of the degrees of even-degree vertices is also even. Because an even number subtracted from an even number yields another even number, we can conclude that the sum of the degrees of odd-degree vertices must also be even. Because the sum of the degrees of odd-degree vertices is even, we can further conclude that there will be an even amount of these vertices in the graph. □

4. There is no Hamiltonian cycle in this graph. The fact that a and b are the only entry points to the inner part of the graph makes it impossible for a Hamiltonian cycle to be found. No matter which vertex you start from, it is inevitable that a and/or b will show up multiple times when attempting to create a Hamiltonian cycle. Maybe if there were more entry points to the inner part of the graph there would be a Hamiltonian cycle.

5. For the complete bipartite graph $K_{m,n}$ to contain an Eulerian cycle, m and n must both be even. This is because an Eulerian cycle only forms when each of the vertices has an even degree.

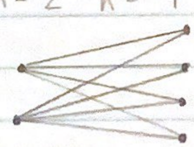
For example:

$m = n = 2$



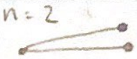
Each vertex has an even degree (deg(2))
Cycle goes from $A \to D \to B \to C \to A$ ✓

$m = 2$ $n = 4$



Each vertex has an even degree of either deg(4) or deg(2), so there's an Eulerian cycle. ✓

$m = 1$ $n = 2$



Each vertex does not have an even degree so there's no Eulerian cycle. X

2. $G = (V, E)$ a graph

$v \sim w$ if and only if there's a path from v to w

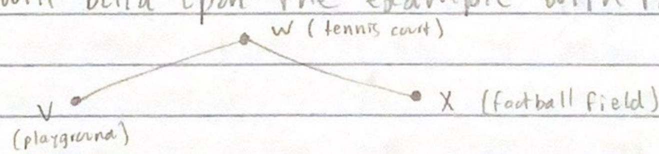
Show this relation is an equivalence relation

equivalence relation: a reflexive, symmetric, and transitive relation

① Reflexive: This relation $\sim$ is reflexive because if we let $v \in V$, there will be a path $v - v$. This is essentially an empty path, or a path from a vertex to itself. This means the relation $\sim$ is reflexive.

② Symmetric: This relation $\sim$ is symmetric because if $v, w \in V$ and there is a path $v - w$, there will also be a path $w - v$. A real life example would be this: if you walked a path in the park that started at the playground (v) and followed the path to the tennis court (w), you can follow the path from the tennis court back to the playground. This shows that the relation $\sim$ is symmetric.

③ Transitive: This relation $\sim$ is transitive because of a similar logic to how the relation is symmetric. I will build upon the example with the park.



In this diagram there is a path from v to w and another path from w to x. Because v is connected to w and w is connected to x, this shows that v is connected (or related) to x. This shows the relation $\sim$ is transitive.

Because the relation $\sim$ is reflexive, symmetric, and transitive, we can conclude the relation is indeed an equivalence relation.

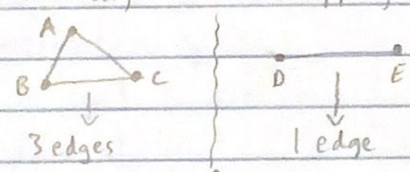
3. $V_1, \ldots V_m$ the blocks in the partition of V given by equivalence relation in Problem #2

$G_i = (V_i, E_i)$ induced subgraph of G on V_i for $i = 1, \ldots, m$

$\bigcup_{i=1}^{m} E_i = E$ $\longrightarrow$ G_i consists of all vertices in V_i and the edges of G that exist within paths of the vertices in V_i

This problem is saying to prove the sum of the edges of all the induced subgraphs (which are formed by the blocks of the partition) is equivalent to the number of edges E from the complete graph. We know the graph is symmetric, reflexive, transitive. A partition will form in a graph when there are vertices that aren't connected, or non-overlapping (because of eq. relation) For example,

EX Graph G:
(4 total edges)



partition because neither D or E are connected to/neighbors with A, B, or C.

In the above example, we see that there is one partition in graph G, whic creates 2 blocks. The first block has 3 edges and the second block has 1 edge. When adding them together, $3 + 1 = 4$ = total number of edges of G. This is one example that satisfies the statement $\bigcup_{i=1}^{m} E_i = E$. ✓

Every vertex in a block V_i is connected to each other, so every vertex in V_i is only incident on edges that're incident on other vertices in V_i. All of these edges combine to form E_i. This means E_i consists of the $e \in E$ when the vertex $x \in V_i$, and x is incident on the edge e.

Every edge $e \in E$ is made of 1 or 2 vertices in V_i. These two vertices are adjacent since they reside in the same block. If an edge only has 1 vertex, that just means the block only has one vertex in it.

Every edge $e \in E$ belongs in an E_i, the same E_i that corresponds with V_i where its incident vertices reside.

Every edge $e \in E$ belongs to a set E_i of $E_1, \ldots, E_m$. This means $E_1, \ldots E_m$ must be a partition of E, proving that $\bigcup_{i=1}^{m} E_i = E$. $\square$