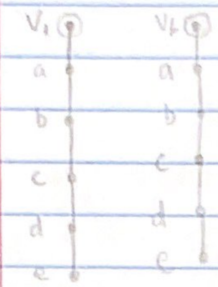
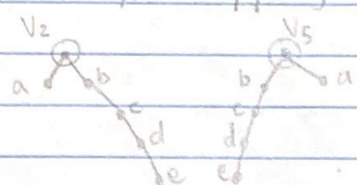
Ethan Wong

Math 61 Final

1.



Note: V_1 and V_6 are the roots, a-e are just arbitrary names for the remaining vertices.
These two trees are isomorphic because they have the same number of levels* (6 total levels) and the same number of vertices per level (1 per level). Both trees also have the same overall degree spectrum. The root vertices and both 'e' vertices have degree 1, while the rest have degree 2. Simply put, the right tree can be obtained by the left tree by swapping the children.



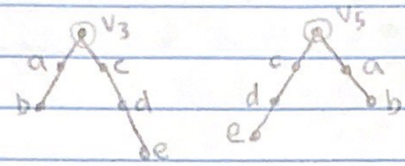
Note: V_2, V_5 are the roots, a-e are just arbitrary names for the remaining vertices.

These two trees are isomorphic because they have the same number of levels (5 total levels) and the same number of vertices per level (1 per level, except 2 in the second level). Both trees also have the same overall degree spectrum. The roots have degree 2. One vertex in level 2 has degree 1 while the other has degree 2 in both trees. The vertices in levels 3 and 4 also have degree 2. Finally, the vertices in both of the level 5's of the trees have degree 1. Basically, the right tree can be obtained from the left tree by switching the halves of the tree.

* By levels, I mean the depth of the tree. The first pair of trees has a level of 6 because it goes 6 vertices deep when including the root vertex.

②

(cont.) 1.

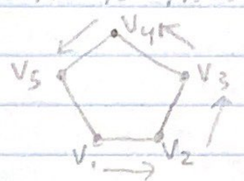


Note: v_3, v_5 are the roots, a - e are just arbitrary names for the remaining vertices.

These two trees are isomorphic because they have the same number of levels (4 total levels) and the same number of vertices per level (1 vertex for the first and last level, 2 vertices for level 2 and level 3). Both trees also have the same overall degree spectrum. The roots have degree 2, the second level vertices have degree 2. In the third level of both graphs, there is one vertex of degree 1 and one vertex of degree 2. Finally in the fourth level, the lone vertex in both graphs has degree 1. Basically, the right tree can be obtained from the left tree by switching the halves of the tree.

2. The spanning trees for breadth-first-search (BFS) and depth-first-search (DFS) will be different for a cycle graph as the algorithms behave differently. Suppose we had a cycle graph with an arbitrary number of vertices (we'll call this number n). We'll assume that the total order $v_1 < \ldots < v_n$.

DFS: For DFS, the core idea of the algorithm is to fully investigate one path and then backtrack to fully investigate the next path until all paths have been examined. For cycle graphs, there isn't really any need to backtrack as there will only be one path to investigate. For example, if $n = 5$:

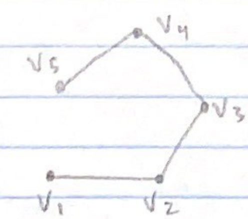


order: v_1, v_2, v_3, v_4, v_5

We would start at v_1 according to our order. Then we see the children of v_1 are v_2 and v_5. According to our order, we would choose v_2. The only child of v_2 not in the graph yet is v_3, so we add that to the tree. The only child of v_3 not in the graph yet is v_4, so we add that to the tree. The only child of v_4 not yet in the tree is v_5, so we add that to the tree. v_5's children are already in the tree. Now that every vertex in the graph has been added to the tree, our spanning tree is complete.

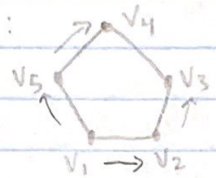
$v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \rightarrow v_5$

Spanning Tree



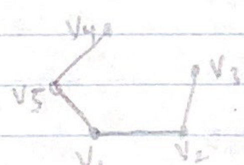
(cont) 2. For BFS, the core idea of the algorithm is to investigate all the paths one step at a time until all paths have been fully investigated.

For example, If $n=5$: order: V_1, V_2, V_3, V_4, V_5



We would start at V_1 and add the edges that V_1 is incident to. So our tree would consist of $V_1 \to V_2$ and $V_1 \to V_5$. Next, we want to investigate edges that V_2 and V_5 are incident that aren't currently in the spanning tree. According to our order, we would look at V_2. The only edge V_2 is incident to that isn't in the tree yet is $V_2 \to V_3$, so we add that to the tree. The only edge V_5 is incident to that isn't in the tree yet is $V_5 \to V_4$. Now we investigate the edges V_3 and V_4 are incident to. The only edge in the graph that we haven't added yet is $V_3 \to V_4$. We shouldn't add this edge because all of the vertices have already been added to the tree, and it would turn our spanning tree into a cycle, thereby negating its status as a tree.

Spanning Tree



Of course, these same ideas for BFS and DFS can be applied similary for any value of n where $n \geq 3$. I only demonstrated on $n=5$, but the concept holds for all cycle graphs. As we can see, the spanning trees are slightly different.

3. Define the following relation R on $\mathbb{Z}$:

$$aRb \longleftrightarrow a = b + i \cdot 3 \quad \text{for some } i \in \mathbb{N}$$

A relation is a partial order when it's reflexive, anti-symmetric, and transitive. We must check if the given relation for these properties to determine if it is a partial order.

Reflexive: For every $x \in \mathbb{Z}$, xRx

To check xRx we see that

$$x = x + i \cdot 3$$
$$0 = i \cdot 3$$
$$0 = i$$

This is valid and true because $i = 0$ is contained within $\mathbb{Z}$. The relation is reflexive.

Transitive: For every $x, y, z \in \mathbb{Z}$: if xRy and yRz then xRz

xRy $\quad$ yRz

$x = y + i \cdot 3$ $\quad$ $y = z + j \cdot 3$ $\quad$ Note: $i, j \in \mathbb{N}$

$$x = z + j \cdot 3 + i \cdot 3$$
$$x = z + 3(i + j)$$

Because i and j are Natural numbers, the sum $i + j$ is guaranteed to be nonnegative. Because of this, we see that the above equation $x = z + 3(i + j)$ is the same as xRz since $i + j$ is a natural number. Therefore, we can conclude that xRz and R must be transitive.

Antisymmetric: For every pair $x, y \in \mathbb{Z}$: if xRy then $y \not R x$

Alternatively: For all $x, y \in \mathbb{Z}$: if xRy and yRx, then $x = y$

xRy $\quad$ yRx

$x = y + i \cdot 3$ $\quad$ $y = x + j \cdot 3$ $\quad$ Note: $i, j \in \mathbb{N}$

$$x = x + j \cdot 3 + i \cdot 3$$
$$0 = j \cdot 3 + i \cdot 3$$
$$0 = j + i$$
$$-j = i$$

For this to hold true, the only possible values for j and i must be 0 in the context of this problem.

$i = 0$
$j = 0$

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Therefore we see that

$$x = y + 0 \cdot 3 \longrightarrow x = y \checkmark$$
$$y = x + 0 \cdot 3 \longrightarrow y = x \checkmark$$

The relation must be antisymmetric.

In conclusion, we know that R must be a partial order because it is reflexive, antisymmmetric, and transitive.

4. $n \in N$ be a positive natural number

$X_n = \{ i \in N \mid 1 \le i \le n \}$

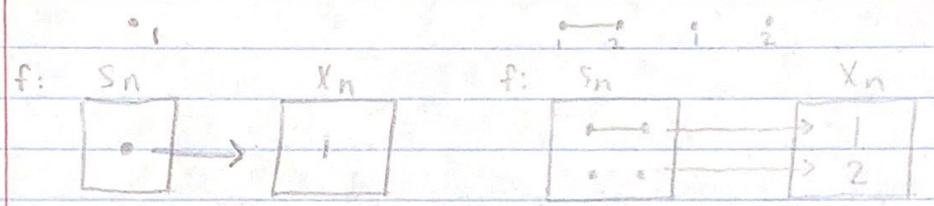
S_n : set whose elements are all graphs with X_n as vertices

$f : S_n \to X_n$

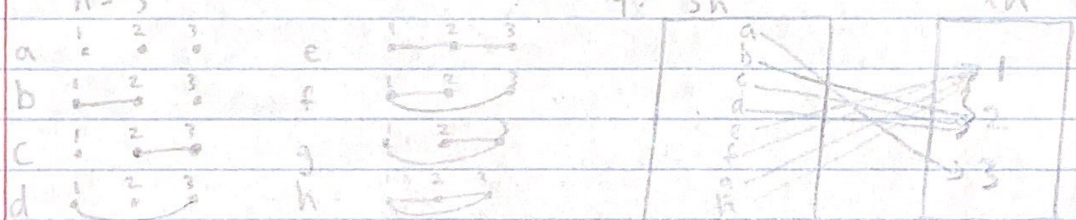
that sends a graph to its number of connected components

Samples for different values of n:

$n = 1$ $\qquad$ $n = 2$



$n = 3$



Note : I am defining a component as a subgraph G' of G consisting of all edges and vertices in G that are contained in some path beginning at a vertex V within G. I am also considering a lone vertex with no incident edges as a component. Finally, a connected graph has exactly one component – the entire graph. This definition would be used to define f, which sends a graph to # of connected components.

Surjective : We can tell that the functions for $n = 1$, $n = 2$, and $n = 3$ are surjective because every value in X_n has a value in S_n that points to it. We can expand this idea to every value of n. We know the value n in X_n will always have something pointed to it as that is the graph of n vertices with no edges. The value 1 in X_n will always have something pointed at it because that is the completely

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connected graph. The values in between 1 and n will also always have arrows pointing at them because edges can be removed from the complete graph to create new graphs that will correspond to these values. Therefore, the function f is surjective for all values of n.

Injective: We can tell that when $n=1$ and $n=2$, the function is injective because it is one-to-one. Each element of S_n corresponds exactly to one element of X_n. However, once $n \geq 3$, the function is no longer injective. In the case of $n=3$, we see there are multiple elements from S_n that are pointing to "1" and "2" in X_n. The function is not one-to-one here and therefore is not injective. In a more general case (when $n > 3$), we will see a similar ocurrence. The number of elements in S_n increases exponentially while the number of elements in X_n increases linearly. By the pigeonhole principle, we see there are not enough holes (elements of X_n) to hold all the pigeons (elements of S_n). Therefore, we observe for when $n \geq 3$ the function f is no longer injective. f is only injective when $n=1$ and $n=2$.

Bijective: For f to be bijective, it must simultaneously be surjective and injective. Since f is surjective for all n, the values where f is bijective depends on the values where f is injective. Since f is injective only when $n=1$ and $n=2$, f is only bijective when $n=1$ and $n=2$.

5. $n \in N \quad n \geq 2$

$A_1, \dots, A_n$ are sets such that $A_{i_1} \cap \dots \cap A_{i_k} \neq 0$

for all $1 \leq i_1 < \dots < i_k \leq n$ and all $1 \leq k < n$

- n describes the number of sets A_1 to A_n
- k is not allowed to equal n, which means the set i cannot include $k = n$. The set i would just be $\{1, \dots, n-1\}$ as there cannot be "n" many elements in this set i.
- There are no constraints on the size of these sets, it will always be possible to select elements of the sets in a manner such that intersections of fewer than n sets are non-empty and that the intersection of all n sets is the empty set.
- For instance, we could use $n = 3$ and provide these sets:

  $A_1 = \{10, 20, 30\}$ $A_2 = \{15, 25, 30\}$

  $A_3 = \{10, 25, 100\}$

  Because $n = 3$, our possible values for k are $k=1$, $k=2$

  When $k = 1$: $k = 2$:

  $A_1 \neq \emptyset$ $A2 \cap A3 = \{25\} \neq \emptyset$

  $A_2 \neq \emptyset$ $A1 \cap A3 = \{10\} \neq \emptyset$

  $A_3 \neq \emptyset$ $A1 \cap A2 = \{30\} \neq \emptyset$

Note: when $k = 1$, this is esentially intersecting a set against itself. The result of this operation is the set itself, which we are assuming is non-empty.

We see from the above example that for $k=1$ and $k=2$, none of the intersections yield an empty set. However, we see that $A1 \cap A2 \cap A3 = \emptyset$ because there is no single element that we see in all three sets. This example that I created above is a valid counterexample to $A_1 \cap \dots \cap A_n \neq 0$ in the prompt.

(10)

Ethan Wong
12/17/20

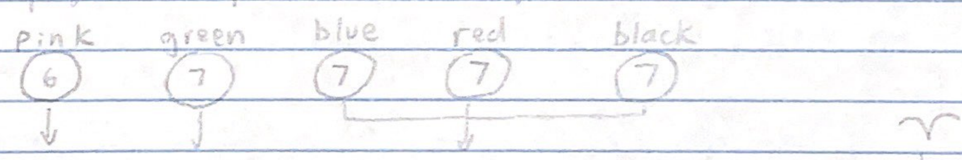
Math 61 Final

(35) 6. 6 pink 7 green 10 blue 12 red 15 black

The minimum number of pens you would have to choose from the drawer to get at least 8 pens of the same color would be 35. If you ere to choose 35 pens from the drawer at random, it's guaranteed at least 8 pens have the same color.

In the absolute worst case scenario, someone cald choose all 6 pink pens, all 7 green pens, 7 blue pens, 7 red pens, and 7 black pens. In this scenario 34 pens will have been chosen already without 8 pens of the same color. Since all of the pink and green have already been chosen, the next selected pen will either be blue, red, or black. This will ensure that the 35^{th} pen will guarantee that there will be 8 pens of the same color. Therefore, the minimum number of pens to choose from the drawer to guarantee there are 8 pens of the same color would be 35.

Visual representation: I will use logic similar to the pigeonhole principle to explain



Full hole. No more pigeons fit.

Full hole. No more pigeons fit.

One more pigeon can fit into these holes to satisfy the problem.

This last pigeon can go into blue, red, or black hole

In this visual representation we see there are 34 pigeons (pens) nestled into 5 holes (the different colors). The 35^{th} pigeon can only fit into the blue, red, or black hole so that there will be at least 8 pens of the same color.

(11)

$n=4 \quad K=7, 12$

chocolate cream plain jam

15

7. Emerson - chocolate Austin - cream + plain
Drew - nothing 7 others - any single croissant

We need to buy at least 10 croissants so that everyone is satisfied. Three of these croissants have been set in stone by Emerson and Austin already. Given that we need to buy at least 10 croissants but can have at most 15 croissants, we can use multiple iterations of the combination equation (n choose k) that is derived from the stars and bars method.

$$\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$$

$n = 4$ (different types of croissants)

$K = 7$ (10 total croissants), 8 (11 total croissants)
9 (12 total croissants), 10 (13 total croissants)
11 (14 total croissants), 12 (15 total croissants)

$K=7$ $\binom{7+4-1}{4-1} \to \binom{10}{3} = 120$ $K=8$ $\binom{8+4-1}{4-1} \to \binom{11}{3} = 165$

$K=9$ $\binom{9+4-1}{4-1} \to \binom{12}{3} = 220$ $K=10$ $\binom{10+4-1}{4-1} \to \binom{13}{3} = 286$

$K=11$ $\binom{11+4-1}{4-1} \to \binom{14}{3} = 364$ $K=12$ $\binom{12+4-1}{4-1} \to \binom{15}{3} = 455$

Once we have obtained all these numbers based on the different values of K, we add them up to get the number of different ways to choose at most 15 croissants.

$120 + 165 + 220 + 286 + 364 + 455 = 1610$

There are 1610 different ways to choose at most 15 croissants so everyone is satisfied.

8. $n! > 2^n$ for all $n \geq 4$. Prove by induction

Basis step: Let's have $n = 4$

$n! > 2^n \rightarrow 4! > 2^4 \rightarrow 24 > 16$

The basis step holds true because $24 > 16$. ✓

Inductive step: Let's say $n = x$ and $x \geq 4$ is given. Suppose $x! > 2^n$ is true for $n = x$,

This would lead us to:

$(x+1)! = x!(x+1)$

$(x+1)! > 2^x(x+1)$ → by inductive hypothesis, know $x! > 2^x$

$(x+1)! \geq 2^x \cdot 2$ → $x+1 \geq 2$ because $x \geq 4$

$(x+1)! = 2^{x+1}$

These procedures showed that $n! > 2^n$ is indeed still true for $n = x+1$, which concludes the induction step's proof.

By the induction principle, we know that $n! > 2^n$ for all $n \geq 4$.