FULL SEMICLASSICAL ASYMPTOTICS NEAR TRANSITION POINTS

ETHAN SUSSMAN

ABSTRACT. We construct complete asymptotic expansions of solutions of the 1D semiclassical Schrödinger equation near transition points. Besides the method, there are three main novelties: (1) transition points of order $\kappa \geq 2$ (i.e. trapped points) are handled, (2) various terms in the operator are allowed to have controlled singularities of a form compatible with the geometric structure of the problem (some applications are given below), and (3) the term-by-term differentiability of the expansions with respect to the semiclassical parameter is included. We prove that any solution to the semiclassical ODE with initial data of exponential type is of exponential-polyhomogeneous type on a suitable manifold-with-corners compactifying the $h \to 0^+$ regime. Consequently, such a solution has an atlas of full asymptotic expansions in terms of elementary functions. This significantly generalizes (and strengthens) the Airy function patching argument found in standard treatments of the JWKB expansion.

CONTENTS

1. INTRODUCTION

In this note, we revisit the old problem of producing asymptotic expansions of solutions of semiclassical ODEs near transition points, where the classical Liouville–Green theory breaks down. Consider the one-dimensional semiclassical Schrödinger operator

$$P = \{P(h)\}_{h>0} = -h^2 \frac{\partial^2}{\partial z^2} + \varsigma z^\kappa W(z) + h^2 \psi(z,h) \tag{1}$$

on the interval $[0, Z]_z$, where $\varsigma \in \{-1, +1\}$ is a sign, $\kappa \in \{-1\} \cup \mathbb{N}$, $W \in C^\infty([0, Z]_z; \mathbb{R}^+)$, and $\psi \in C^\infty((0, Z]_z \times [0, \infty)_{h^2}; \mathbb{C})$ is drawn from some suitable class of admissible functions of the independent variable z and the semiclassical parameter h. We will be more precise later on about

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the meaning of “admissible” in the previous sentence; for now, it suffices to note that any function $\psi : (0, Z]_z \times [0,\infty)_h \to \mathbb{C}$ of the form

$$\psi = \frac{\nu}{z^2} + \frac{\varphi(h)}{z} + G(z, h) \tag{2}$$

for $\nu \in \mathbb{C}$, $\varphi(h) \in C^\infty([0,\infty)_{h^2}; \mathbb{C})$, and $G \in C^\infty([0, Z]_z \times [0,\infty)_{h^2}; \mathbb{C})$ is allowed, but this does not exhaust the admissible ψ. The reader is invited to take $\psi = 0$ if simplification is desired, though some of the intended applications, not to mention the proof itself, require greater generality.

Restated, the problem is producing $h \to 0^+$ asymptotics of solutions $u = \{u(-; h)\}_{h>0}$ to $Pu = 0$. The key structural feature of P is the transition point at $z = 0$, at which the potential $V(z) = \varsigma z^\kappa W(z)$ may have a zero or singularity. The Liouville–Green theory certainly suffices away from the transition point. Moreover, for $h > 0$, the operator $P(h) \in \mathrm{Diff}^2(0, Z)$ is (at worst) a regular singular ordinary differential operator depending smoothly on h, so the behavior of solutions follows from general theory. It is only the behavior near the corner $\{z = 0, h = 0\} \in [0, Z]_z \times [0,\infty)_h$ that needs to be further understood.

The use of the term “corner” is the first instance of our emphasis on geometric structures. Let M denote the manifold-with-corners (mwc, which we use in the sense of Melrose [Mel92], though the precise definition is not important here), depending on κ (and, at the level of sets, on Z), constructed by performing a quasihomogeneous blowup of the corner $\{z = 0, h = 0\}$ of the rectangle $[0, Z]_z \times [0,\infty)_{h^2}$ so as to separate the family $\{\Gamma_\lambda\}_{\lambda>0}$ of curves

$$\Gamma_\lambda = \{z = \lambda h^{2/(2+\kappa)}\}, \tag{3}$$

with the smooth structure at the front edge fe modified in a manner described below. This blowup resolves the ratio $\lambda = z/h^{2/(\kappa+2)}$, which becomes a smooth coordinate $\lambda \in C^\infty(\mathrm{fe}^\circ)$ parametrizing fe° and extending smoothly down to one boundary point. Besides fe, the other edges of M are $\mathrm{ze} = \mathrm{cl}_M\{h = 0, z > 0\}$ (the “zero h edge”), $\mathrm{be} = \mathrm{cl}_M\{h > 0, z = 0\}$ (the “boundary edge”), and $\mathrm{ie} = \{z = Z\}$ (the “initial edge”). See Figure 1 for a depiction of M. The passage to the blowup is a specific instance of a general strategy in geometric singular analysis, that of trading analytic complexity for geometric complexity.

We say that a function $u : M^\circ \cup \mathrm{ie}^\circ \to \mathbb{C}$ with C^1 slices $u|_{h=h_0} \in C^1(0, Z]$ has initial data of exponential-polyhomogeneous type if the restrictions

$$\begin{aligned} u|_{\mathrm{ie}} &: \mathrm{ie}^\circ = \{z = Z \text{ and } h > 0\} \to \mathbb{C}, \\ u'|_{\mathrm{ie}} &: \mathrm{ie}^\circ \to \mathbb{C} \end{aligned} \tag{4}$$

are of exponential-polyhomogeneous type on ie. This restricts their $h \to 0^+$ behavior while saying nothing about the irrelevant $h \to \infty$ regime. Then, our main theorem is a constructive version of:

Theorem A. *If $Pu = 0$ and u has initial data of exponential-polyhomogeneous type, then u is of exponential-polyhomogeneous type on M.*

Roughly, the theorem states the existence of full asymptotic expansions of solutions in suitable asymptotic regimes which suffice to cover all possible ways of following u along some smooth graph $(\Gamma(h), h) : [0,\infty)_h \to [0, Z]_z \times [0,\infty)_h$ as $h \to 0^+$. The key point is that the asymptotic expansions in powers of the boundary-defining-functions of the edges of M do not depend on the angle or other aspects of the manner via which Γ approaches ∂M. Only the endpoint $\lim_{h\to 0^+}(\Gamma(h), h) \in \partial M$ in M matters. This sort of behavior can be contrasted with the behavior of the polar angle

$$\theta = \arctan(y/x) : ([0,\infty)_x \times [0,\infty)_y)\backslash\{0\} \to [0, \pi/2], \tag{5}$$

the limiting value of which when followed along a curve ending at the origin depends on the angle of approach. The difference is that θ is not polyhomogeneous on the punctured quadrant but rather on the blowup $[[0,\infty)^2_{x,y}; (0, 0)]$.

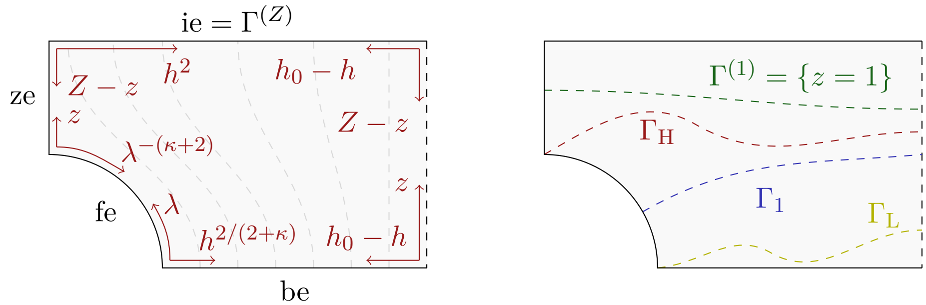

FIGURE 1. *Left*: the manifold-with-corners M (the portion with $h < h_0$, for $h_0 > 0$ arbitrary), with some level sets of h in dashed gray. Some local coordinate charts are denoted in red. *Right*: some of the curves Γ_λ, defined by eq. (3), and $\Gamma^{(z_0)} = \{z = z_0\}$ in M. A curve Γ_{H} probing the "intermediate" regime $\mathrm{ze} \cap \mathrm{fe}$ is in red, and one Γ_{L} probing the other such regime $\mathrm{fe} \cap \mathrm{be}$ is in yellow.

A more explicit and precise version of Theorem A appears below. The proof is constructive, in the sense that it provides an algorithm for computing all asymptotic expansions, as well as joint asymptotic expansions at the corners.

Given how well-trodden this subject is, it may be surprising that there is something left to say. Theorem A improves on the existing literature in three ways:

1. (1) we handle the case $\kappa \geq 3$, for which no full expansions had previously been known (see the remark of Olver quoted below), and the expansion in the $\kappa = 2$ case has been given a proof. (The statement in the $\kappa = 2$ case does not appear explicitly in the existent literature.)
2. (2) Poincaré expandability is improved to polyhomogeneity. This implies control of all derivatives in the semiclassical parameter, filling an apparent hole in the earlier literature even in the otherwise well-understood cases.
3. (3) The potential is allowed to have a natural sort of singularity at the transition point, as arises in several applications discussed below.

Each of these points will be elaborated upon later in the introduction.

1.1. **Regarding generality.** Any second-order semiclassical ordinary differential operator

$$a(z,h)h^2\frac{\partial^2}{\partial z^2} + b(z,h)h\frac{\partial}{\partial z} + c(z,h) \tag{6}$$

on the real line (with coefficients which are smooth save for isolated poles at locations not depending on h) is, assuming that the leading coefficient a is nonvanishing and nonsingular, equivalent modulo conjugation, in the sense of possessing identical kernel, to an operator of the form

$$P = -h^2\frac{\partial^2}{\partial z^2} + V(z) + hQ(z,h). \tag{7}$$

Generically, we should expect that, insofar as the potential V vanishes, it vanishes only at isolated points, at each of which the function vanishes to some finite order. This is the case when V is meromorphic and not identically zero. A very common situation is when $Q = h\psi$ for ψ a smooth function of h^2.

If one is concerned with the local properties of solutions, then it suffices to restrict attention to closed interval $I \subset \mathbb{R}$ of z's containing at most a single zero or singularity of V or singularity of ψ. Without loss of generality, it may be assumed that the interval I is given by $I = [0, Z]$ for some $Z > 0$, with the zero or singularity (if it exists at all) at the endpoint $z = 0$.

Then, we can write the potential as $V(z) = \varsigma z^\kappa W(z)$ for $\varsigma \in \{-1, +1\}$, W as above, and $\kappa \in \mathbb{N}$. We do not consider $\kappa \leq -2$, as then either P has a regular singularity with variable indicial roots (if $\kappa = -2$) or an irregular singularity (if $\kappa \leq -3$). These cases are of a rather different character than those handled here.

1.2. **Brief history.** The $\kappa = 1$ case goes back, of course, to Jeffreys, Wentzel, Kramers, and Brillouin, after whom the whole industry of semiclassical expansions has come to be referred. The definitive treatment of Poincaré expandability is due to Langer–Olver and the better part of a century old. It is quite satisfactory, except in that it stops at Poincaré expandibility. The $\kappa = -1$ case has been brought to a similarly refined level by the same authors. The whole range $\kappa \in (-2, 2)$ can be dispatched via similar techniques, as originally indicated by Langer in his initial works on the subject.

When $\kappa \geq 2$, the potential vanishes degenerately at $z = 0$. One might call $\{z = 0\}$ a *trapped point*, since, in addition to being a zero of the potential, it is a point of equilibrium of the associated classical dynamics, at which the force $-V'$ on a Newtonian particle vanishes. Varied terminology has appeared in the literature. Unlike in the $\kappa \in (-2, 2)$ case, only very incomplete results can be found. Olver describes the situation like so:

> *In a region containing a turning point of [integer] multiplicity κ, uniform asymptotic approximations to the solutions can be constructed in terms of* [Bessel functions of order $1/(\kappa + 2)$]*... When $\kappa > 1$, however, there is no straightforward extension from asymptotic approximations to asymptotic expansions* [Olv75b, §4.3].

Here, "asymptotic approximation" refers to a statement such as $u = (1 + O(h))u_0$ for explicit u_0 (in this case written in terms of Bessel functions), and "asymptotic expansion" refers to the production of a full series in powers of h. For the $\kappa = 2$ case, a different ansatz was tried in [Olv75a], though the idea predates this work. This ansatz uses parabolic cylinder functions rather than Bessel functions, the idea being that a better ansatz allows one to control the error using the same method that works in the $\kappa \in (-2, 2)$ case. Though not stated outright, it is implied in [Olv75a, §11] that the resultant asymptotic approximation can be promoted to a full expansion as long as uniformity is not demanded as $z \to \infty$. The argument does not appear to have been written down.

For $\kappa \geq 3$, in which the trapping is degenerate, it seems that little is known beyond what Olver already states in [Olv54; Olv75b; Olv97]. In particular, I am unable to locate any asymptotic expansions including lower order terms, though it is hard to rule out the existence of folk theorems known to experts on the subject.

For a pedagogical account of the aforementioned developments, see Olver's expository works. An encyclopedic treatment appears in [Fed93, Chp. 4]. An extensive summary of the research literature, including the citations omitted above, can be found in [Was87, §31.2]. Wasow's account does not cover the last few decades, but it still seems current as far as our particular problem is concerned.

1.3. **Primer on exponential-polyhomogeneity.** We refer the reader to the geometric singular analysis literature – of which [Mel92; Mel93; Gri01; She22] is a relevant sample – for the precise definition of polyhomogeneity on manifolds-with-corners (mwc), in addition to the definitions of associated function spaces. The particular notation used here will be outlined as needed.

For the sake of this introduction, it suffices to note that polyhomogeneity is a precisification of the notion of having full asymptotic expansions in terms of powers and logarithms (the particular combinations of which are allowed to appear being specified by an "index set," denoted $\mathcal{E}, \mathcal{F}$, etc. below), with

- each boundary hypersurface (a.k.a. facet) of our mwc M corresponding to one asymptotic regime, and
- the corners corresponding to "intermediate" asymptotic regimes.

These expansions are differentiable term-by-term — this is certainly useful, and the definition guarantees it. So, one can think of polyhomogeneity as being a slight weakening of smoothness, namely smoothness up to the fact that one's "Taylor series" have logarithms. Polyhomogeneity at the corners ensures that the asymptotic expansions at the various adjacent facets match up, and consequently one has well-defined *joint* asymptotic expansions, a point stressed in [She22]. For an application, see [She22, §2.4]. The coefficients in the (joint) expansion at a face f (not necessarily a facet) are polyhomogeneous functions on f, and the expansion of the derivative of the given function along f is the derivative of the expansion. This is part of what it means for the expansion to be differentiable term-by-term, but the latter is stronger because it applies to normal derivatives as well.

A function u on a mwc M is said to be of *exponential-polyhomogeneous type* if it can be written as

$$u = \sum_{n=1}^{N} u_n e^{\theta_n} \tag{8}$$

for some $N \in \mathbb{N}$ and polyhomogeneous $\theta_n, u_n : M \to \mathbb{C}$. Thus, the notion of exponential-polyhomogeneous type is one particular precisification of the notion of having full, well-behaved asymptotic expansions in terms of elementary functions, i.e. polynomials, logarithms, and exponentials. This amounts to exponential-polyhomogeneity being the gold standard for asymptotic expandability, stronger than Poincaré-type expandability.

One interesting aspect of the theorem, Theorem A, above is the possible presence of logarithmic terms, in contrast to the situation, described in appendix §C, when $\kappa \in (-2,2)$ and $\psi \in C^\infty([0,Z]_z \times [0,\infty)_{h^2};\mathbb{C})$. An example, with $\kappa = 2$, in which such logarithmic terms appear is presented in appendix §D.

One advantage of working on the blowup M is the ability to avoid the use of the theory of special functions entirely. Rather than work with parabolic cylinder functions (or some alternative such as Weber functions), as Olver does in [Olv75a] to handle the $\kappa = 2$ case, it suffices here to work entirely in terms of elementary functions. After all, it is only in terms of elementary functions that the notion of exponential-polyhomogeneity is phrased. Insofar as special functions appear, they appear as the solutions to some special differential equations considered — see the examples in §3 — and all of the properties required of them are proven directly from the differential equations they satisfy. No integral representations are used.

One disadvantage, for instance in the $\kappa = 2$ case, is that the possible logarithmic terms, which in the usual treatment are hidden in asymptotics of the parabolic cylinder functions (see appendix §D), become explicit and worse organized. This is especially inconvenient when trying to relate the expansions in the classically allowed and classically forbidden regions. Without care, the result is proving polyhomogeneity statements with unnecessarily large index sets. A similar statement should apply when $\kappa \geq 3$.

1.4. **Smooth structure of M.** Returning to the smooth structure at M, the smooth structure at the front face fe of the blowup chosen such that $\varrho_{\mathrm{fe}} = z + h^{2/(\kappa+2)}$ becomes a defining function (bdf) of it. Defining functions of the faces ze, be, ie are given by $\varrho_{\mathrm{ze}} = h^2/\varrho_{\mathrm{fe}}^{\kappa+2}$, $\varrho_{\mathrm{be}} = z\varrho_{\mathrm{fe}}^{-1}$, and $\varrho_{\mathrm{ie}} = Z - z$. Other choices are possible.

When restricting attention to local coordinate charts, it is often possible to work instead with local bdfs. Outside of any neighborhood of ze, the ratio $z/h^{2/(\kappa+2)}$ is a bdf for be and $h^{2/(\kappa+2)}$ is a bdf for fe, and outside of any neighborhood of be, $h^2/z^{\kappa+2}$ is a local bdf for ze and z is a bdf for fe.

1.5. **Allowed singularities in the operator coefficients.** We describe now the allowed singularities of the term ψ, in full generality. The assumptions to be placed on ψ are

- $\psi \in \varrho_{\mathrm{be}}^{-2}\varrho_{\mathrm{fe}}^{-2}C^\infty(M) = z^{-2}C^\infty(M)$, and
- $z^2\psi|_{\mathrm{be}} : \mathrm{be} \to \mathbb{C}$ is constant.

Allowing the singularity at be is mostly useful when considering the semiclassical ODE arising from partial wave analysis for semiclassical Schrödinger operators with spherical symmetry. The second of the two requirements is placed to avoid having to deal with regular singular differential equations with variable indicial roots.

More interesting is the singularity at fe. Methods such as Olver–Langer's do not seem able to handle this sort of singularity. In fact, Olver–Langer's does not even seem to be able to handle $\psi \in z^{-2}C^\infty([0,Z]_z \times [0,\infty)_{h^2};\mathbb{C})$. Contrastingly, the singular geometric methods below are sufficiently robust.

We now define some notation used below. Given any function ψ satisfying the conditions above, choose $\alpha \in \mathbb{C}$ with $\Re\alpha \geq 0$ such that $z^2\psi|_{\mathrm{be}} = \alpha^2 - 1/4$, and let $\Psi = z^2\psi|_{\mathrm{fe}} - \alpha^2 + 1/4$. We assume for the remainder of the paper that $\Re\alpha > 0$. The arguments below all go through if $\Re\alpha = 0$, with minor modifications and some additional casework.

Notice that $\Psi \in \varrho_{\mathrm{be}}C^\infty(\mathrm{fe})$. Identifying fe\ze with $[0,\infty)_\lambda$ for $\lambda = \zeta/h^{2/(\kappa+2)}$, the previous sentence says, more prosaically, that $\Psi(\lambda) \in \lambda C^\infty[0,\infty)_\lambda$ and satisfies $\Psi(\rho^{-2/(\kappa+2)}) \in C^\infty[0,\infty)_{\rho^2}$. Throughout this paper, we will identify fe\ze with $[0,\infty)_\lambda$. Now define E by

$$\psi(z,h) = \frac{1}{z^2}\Big(\alpha^2 - \frac{1}{4}\Big) + \frac{1}{z^2}\Psi\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + E(z,h) \tag{9}$$

and $\phi = \psi - E$. Since z^2E vanishes at $\mathrm{be} \cup \mathrm{fe}$, it must lie in $\varrho_{\mathrm{be}}\varrho_{\mathrm{fe}}C^\infty(M)$. That is, $E \in \varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{fe}}^{-1}C^\infty(M) = z^{-1}C^\infty(M)$. As it turns out, the term $h^2\phi$ in P, though subleading at ze, is of comparable order to the main terms at fe and must therefore be taken into account in order to understand the $h \to 0^+$ limit.

Let

$$N(P) = -h^2\frac{\partial^2}{\partial z^2} + \varsigma z^\kappa W(z) + \frac{h^2}{z^2}\Big(\alpha^2 - \frac{1}{4}\Big) + \frac{h^2}{z^2}\Psi\Big(\frac{z}{h^{2/(\kappa+2)}}\Big). \tag{10}$$

Thus, $P - N(P) \in (h^2/z)C^\infty(M)$, so $N(P)$ arises from omitting from P terms of order $O(h^2/z)$ and better. Thus, solutions Q to $N(P)Q = 0$ an be considered as $O(h^2/z)$-quasimodes for the original semiclassical ODE $Pu = 0$, hence the use of the notation 'Q' rather than 'u.' These quasimodes will be studied later.

1.6. **Applications with singular coefficients.** Some interesting examples to which we would like to apply the theory come from families $\{P(r;\sigma)\}_{\sigma>0}$ of ODEs on the whole real line $\mathbb{R}_r$ depending non-semiclassically on a parameter σ, in which the semiclassical regime arises as an artificial transitional regime in the joint large r, low σ limit. The coordinate r is to be related to z in the following way: the blown up face $\mathrm{fe} \subset M$ corresponds to the original face $[0,\infty]_r \times \{0\}_\sigma \subset \mathbb{R}_r \times \{0\}_\sigma$. This is the reason why ψ may be singular before passing to M.

Two examples of families $\{P(r;\sigma)\}_{\sigma>0}$ with the structure described are

(1) 1D Schrödinger operators with Coulomb-like potentials [Sus22], in which $\pm\sigma^2$ is the energy and $\kappa = -1$, reflecting the scaling under dilations of the exact Coulomb potential $\pm 1/r$ at $r = 0$, and
(2) the anharmonic oscillator in the limit of small anharmonicity, as analyzed using semiclassical techniques, in which case $\kappa = 4$. To the best of our knowledge, this sort of analysis first appeared in [BW69].

A different situation in which singular a ψ appears is the Regge–Wheeler equation, which describes linear perturbations of the metric of the exterior of the Schwarzschild spacetime. The angular momentum ℓ – i.e. the the azimuthal quantum number – enters as a parameter in the equation. One "semiclassical" regime is the limit of large ℓ. After a conjugation in order to remove the first order term when written with respect to the Eddington–Finkelstein radial coordinate, the differential operator in question can be written as

$$-\Big(1 - \frac{r_{\mathrm{H}}}{r}\Big)^2\frac{\partial^2}{\partial r^2} - \sigma^2 + \Big(1 - \frac{r_{\mathrm{H}}}{r}\Big)\frac{\ell(\ell+1)}{r^2} + V_{\mathrm{RW}}(r), \tag{11}$$

where $r_{\mathrm{H}} > 0$ is the location of the black hole horizon, σ is the perturbation's temporal frequency, and $V_{\mathrm{RW}} \in (r-r_H)C^\infty[r_{\mathrm{H}},\infty)_r$ is some "effective potential." To write this operator in the form considered above, define $z = r-r_{\mathrm{H}}$ and $h^{-2} = \ell(\ell+1)$. Multiplying through by $h^2(1-r_{\mathrm{H}}/r)^{-2} = h^2(z+r_{\mathrm{H}})^2/z^2$, the result is

$$-h^2\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{1}{(z+r_{\mathrm{H}})} + \frac{h^2}{z^2}(z+r_{\mathrm{H}})^2\Big[-\sigma^2 + V_{\mathrm{RW}}(z+r_{\mathrm{H}})\Big]. \tag{12}$$

This has the form studied above, with $\kappa = -1$. Not only is the potential singular, but $\psi = z^{-2}(z+r_{\mathrm{H}})^2(-\sigma^2+V_{\mathrm{RW}}(z+r_{\mathrm{H}}))$ is singular as well. As a corollary of the theorems in this paper, full asymptotics in the joint small z- large ℓ limit follow, refining the analysis of this regime in [Cos+12].

1.7. **More precise theorem.** We now state the precise version of the theorem above, after introducing some notation. Let

$$\zeta(z) = \Big(\frac{\kappa+2}{2}\int_0^z \omega^{\kappa/2}\sqrt{W(\omega)}\,\mathrm{d}\omega\Big)^{2/(\kappa+2)}. \tag{13}$$

This lies in $zC^\infty([0,Z]_z;\mathbb{R}^+)$. Let $\xi \in C^\infty([0,Z]_z;\mathbb{R}^+)$ be defined by $\zeta = z\xi$. For each $j\in\mathbb{C}$, we use $(j,0)$ as an abbreviation for the index set $\{(j+n,0) : n\in\mathbb{N}\}$. In particular, $(0,0)$, which we also abbreviate "$\mathbb{N}$," is the index set for which polyhomogeneity means smoothness. Recall also that "∞" denotes the empty index set.

Letting $\mathcal{E}_{\mathrm{ie}},\mathcal{E}_{\mathrm{ze}},\mathcal{E}_{\mathrm{fe}},\mathcal{E}_{\mathrm{be}}$ denote index sets, we use $\mathcal{A}^{\mathcal{E}_{\mathrm{ie}},\mathcal{E}_{\mathrm{ze}},\mathcal{E}_{\mathrm{fe}},\mathcal{E}_{\mathrm{be}}}(M) \subset C^\infty(M^\circ;\mathbb{C})$ to denote the set of (complex-valued) polyhomogeneous functions on M° with index set $\mathcal{E}_{\mathrm{e}}$ at each edge $\mathrm{e}\in\{\mathrm{ie},\mathrm{ze},\mathrm{fe},\mathrm{be}\}$. To avoid having to write too many index sets, we use $\mathcal{A}^{\mathcal{E},\mathcal{F}}(M)$ as an abbreviation for this set when $\mathcal{E}_{\mathrm{ie}},\mathcal{E}_{\mathrm{ze}} = (0,0)$, with $\mathcal{E} = \mathcal{E}_{\mathrm{fe}}$ and $\mathcal{F} = \mathcal{E}_{\mathrm{be}}$. When working with functions supported away from be, or where the behavior there is unimportant, we will sometimes omit the '$\mathcal{F}$' from "$\mathcal{A}^{\mathcal{E},\mathcal{F}}(M)$."

Below, we use the index set

$$\mathcal{E}_0 = \begin{cases} \bigcup_{j=0}^\infty((\kappa+2)j,j) & (\kappa\notin 2\mathbb{N}), \\ \bigcup_{j=0}^\infty((\kappa+2)j,2j)\cup((\kappa+2)j+\kappa/2,2j+1) & (\kappa\in 2\mathbb{N}). \end{cases} \tag{14}$$

Let $\mathcal{Q}\subset C^\infty(\mathbb{R}^+_\lambda)$ denote

$$\mathcal{Q} = \Big\{v(\lambda)\in C^\infty(\mathbb{R}^+_\lambda) : \frac{\partial^2 v}{\partial\lambda^2} = \Big[\varsigma\lambda^\kappa + \frac{1}{\lambda^2}\Big(\alpha^2-\frac{1}{4}\Big) + \frac{1}{\lambda^2}\Psi\Big(\frac{\lambda}{\sqrt[\kappa+2]{W(0)}}\Big)\Big]v(\lambda)\Big\}. \tag{15}$$

There exists a one-dimensional subspace $\operatorname{span}_{\mathbb{C}} Q_0 \subset \mathcal{Q}$ of solutions that are recessive (see §3) at the origin.

Theorem B. *For any $Q\in\mathcal{Q}$, there exists an index set $\mathcal{G}$ and $\beta,\gamma\in\mathcal{A}^{\mathcal{E}_0}(M)$ with $\operatorname{supp}\beta,\operatorname{supp}\gamma$ disjoint from* be *and $\delta\in\mathcal{A}^{\mathcal{E}_0,\mathcal{G}}(M)$ with $\operatorname{supp}\delta\cap(\mathrm{ie}\cup\mathrm{ze}) = \varnothing$ such that the function u defined by*

$$u = \sqrt[4]{\frac{\xi^\kappa}{W}}\,\Big[(1+\varrho_{\mathrm{ze}}\varrho_{\mathrm{fe}}\beta)Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + \varrho_{\mathrm{ze}}^{(\kappa+1)/(\kappa+2)}\varrho_{\mathrm{fe}}\gamma Q'\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big)\Big] + \varrho_{\mathrm{be}}^{1/2-\alpha}\varrho_{\mathrm{fe}}\delta \tag{16}$$

solves $Pu=0$ in $\{h<h_0\}$ for some $h_0>0$.

The index set $\mathcal{G}$ can be extracted from the argument below, but we will not be explicit.

Thus, using Proposition 3.3 to express Q,Q' in exponential-polyhomogeneous form,

$$\begin{aligned} u - \sqrt[4]{\frac{\xi^\kappa}{W}}Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) &\in \exp\Big(-\frac{2\sqrt{\varsigma}\chi}{\kappa+2}\frac{\zeta^{(\kappa+2)/2}}{h}\Big)\sqrt{\varrho_{\mathrm{ze}}}\varrho_{\mathrm{fe}}\varrho_{\mathrm{be}}^{1/2-\alpha}\mathcal{A}^{\mathbb{N},2^{-1}\mathbb{N},\mathcal{E}_0,\infty}(M) \\ &+ \exp\Big(\frac{2\sqrt{\varsigma}\chi}{\kappa+2}\frac{\zeta^{(\kappa+2)/2}}{h}\Big)\sqrt{\varrho_{\mathrm{ze}}}\varrho_{\mathrm{fe}}\varrho_{\mathrm{be}}^{1/2-\alpha}\mathcal{A}^{\mathbb{N},2^{-1}\mathbb{N},\mathcal{E}_0,\infty}(M) + \varrho_{\mathrm{fe}}\varrho_{\mathrm{be}}^{1/2-\alpha}\mathcal{A}^{\infty,\infty,\mathcal{E}_0,\mathcal{G}}(M) \end{aligned} \tag{17}$$

for $\chi \in C^\infty(M;[0,1])$ identically 1 near $\mathrm{ie} \cup \mathrm{ze}$ and identically vanishing near be.

This version of the theorem says that, at least for h sufficiently small, there exists a solution u to the semiclassical ODE $Pu = 0$ such that

$$u \approx \sqrt[4]{\frac{\xi(z)}{W(z)}} Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big). \tag{18}$$

This is just Langer's approximation. If $|z| \gg h^{2/(\kappa+2)}$, then, as $h \to 0^+$, the large-argument asymptotics of the $Q \in \mathcal{Q}$ allow Q to be approximated by a linear combination of exponentials, in which case the approximation above becomes an instance of the Liouville–Green ansatz. If instead $|z| = O(h^{2/(\kappa+2)})$, then $\zeta \approx W(0)^{1/(\kappa+2)} z$, and so

$$u \approx W(0)^{-(\kappa+1)/(4\kappa+8)} Q(z/h^{2/(\kappa+2)}), \tag{19}$$

which is an ansatz generalizing that appearing in the JWKB connection analysis. (Since Q can be replaced by cQ for any $c \neq 0$, the multiplier out front is not important.) This sort of analysis goes back at least to Langer, but Theorem B provides a natural refinement of it.

If $Q_1, Q_2 \in \mathcal{Q}$ are linearly independent modes, then the functions $u[Q_1](-,h)$, $u[Q_2](-,h)$ produced by the previous theorem are linearly independent for h sufficiently small, so any solution u to $Pu = 0$ can be written as $u = c_1 u[Q_1] + c_2 u[Q_2]$ for h sufficiently small for some functions $c_1, c_2 : (0,\infty)_h \to \mathbb{C}$. Given the values of $u|_{\mathrm{ie}}$ and the derivative $u'|_{\mathrm{ie}}$, the $h \to 0^+$ asymptotics of the coefficients c_1, c_2 can be computed straightforwardly from the asymptotics of the $Q \in \mathcal{Q}$. The preceding theorem therefore contains within it the means of producing semiclassical expansions for *any* such u with prescribed initial data. Thus, Theorem A follows from Theorem B. For completeness, the deduction is included in the appendices. See §B.

We now sketch the proof of Theorem B, which consists of four steps. All steps are carried out in the $W = 1$ case, a simplification which, as originally observed by Langer and discussed in §2, suffices. As preparation for the later sections, some properties of the quasimodes $Q \in \mathcal{Q}$ are proven in §3.

(1) The first step of the main argument, carried out in §4, is to produce a solution to the ODE of the desired form modulo an error of the form $fQ + gQ'$ for

$$f, g \in \varrho_{\mathrm{ze}}^\infty \varrho_{\mathrm{fe}} \mathcal{A}^{\mathcal{E}_0,\infty}(M), \tag{20}$$

i.e. accurate to infinitely many orders at ze, and supported away from be. The proof involves re-interpreting Langer–Olver's asymptotic series in [Olv54; Olv97], which fails to define a uniform expansion down to be°, as an asymptotic expansion at ze, which can then be asymptotically summed in suitable function spaces.

(2) The next step is to solve away the error from the previous step near $\mathrm{ze} \cap \mathrm{fe}$ modulo $\varrho_{\mathrm{ze}}^\infty \varrho_{\mathrm{fe}}^\infty C^\infty(M) = h^\infty C^\infty([0,Z]_z \times [0,\infty)_h; \mathbb{C})$ remainders supported away from be. The argument involves the inversion of the *normal operator* $\hat{N}(P) \in \mathrm{Diff}^2(\mathrm{fe}^\circ)$ in order to produce the approximate solution term-by-term. This step is in §5.

(3) In §6, the remaining $h^\infty C^\infty$ error near ze is solved away using the standard method of variation of parameters [Erd60][Olv97, Chp. 6][Sim15, §15.5]. This is the most technical part of the argument, but the error is as amenable as possible — as long as we stay away from be, the $O(h^\infty)$ suppression is sufficient to kill any large negative powers of h, ζ that arise in the computations. It is necessary to keep careful track of the exponential weights arising in the classically forbidden case, but this is the only delicate point. Estimating derivatives complicates the formulae but is totally straightforward.

(4) For various reasons, the previous steps allow an error supported away from ze. The final step of the argument is to solve away this error completely in $\{h < h_0\}$. Because P is a nondegenerating, smooth family of regular singular differential operator on $\mathrm{fe}\backslash\mathrm{ze}$, this follows from the general theory of regular singular ODE and requires only a short argument. See §7.

The various parts of the argument are combined in §8.

1.8. **Concluding remarks.** This work began with the goal of providing a proof of the main result of [She22] using only the structural features of Bessel's ODE. The results here are not sufficient for this purpose. The missing piece is an analysis of the $\kappa = -2$ case, in which the ODE possesses a regular singularity. However, the clause of Sher's result stating the exponential-polyhomogeneity of the Bessel functions in a neighborhood of the locus of their Airy function asymptotics is a special case of Theorem C, with $\kappa = 1$.

If the coefficients W, E depend on some parameters $p \in \mathbb{R}^d$ (or more generally p valued in some manifold-with-corners), then e.g. smoothness in p implies smoothness of the solutions constructed in Theorem B in p in a suitable topology. Similar results hold for other L^∞-based notions of regularity. The family $\mathcal{Q} = \ker N(P)$ of quasimodes depends smoothly on the coefficients of P with respect to some function space topologies. Then, it is possible to articulate a theorem to the effect that, given a smooth section's worth of chosen quasimodes, the function u constructed in the theorem is also smoothly varying in some suitable sense.

We end this introduction with a few remarks regarding potential extensions:

- For simplicity, we have restricted attention to integral κ, but the arguments below work for all real $\kappa > -2$, except the specific index sets appearing may differ.
- One can handle ψ polyhomogeneous on M with appropriate index sets at the various faces. If ψ has only partial regularity (partially polyhomogeneous or something similar), then partial asymptotic expansions can still be constructed, with the error controlled in a corresponding conormal space. In the argument, the final step, in which the remaining error is solved away via the method of variation of parameters, becomes more delicate, as the forcing is no longer $O(h^\infty)$ but instead only a decaying conormal function. However, it still seems possible to make do.
- The previous point also applies if W is polyhomogeneous or partially polyhomogeneous at 0, except now the Langer transformation is no longer a diffeomorphism. This issue is not critical, as our use of the Langer transformation is only to simplify the algebra. Besides this one complication, the argument works as expected to produce the desired expansions.
- One particularly natural extension is to allow ψ to have a pole of the form $\sim 1/h$, in which case the term $h^2\psi$ in P has an $O(h)$ contribution. Unfortunately, this can dominate the potential at fe. For instance, at fe, the potential is essentially $z^\kappa \sim h^{2\kappa/(\kappa+2)}$, and the right-hand side is smaller than h if $h < 1$ and $\kappa \geq 3$.

  For $\kappa \in (-2, 2)$ at least, where such issues do not occur, this problem has been considered by Langer and others, and there are standard tools which apply. Thus, one can expect results in the vein of the theorems above.
- Complex h can be considered essentially without modification if ψ has appropriate analyticity. In fact, considering $\Im h = p$ as a parameter, this is a special case of smooth parameter dependence. It can be shown that the constructed solutions depend analytically on h in some neighborhood $U \subset \mathbb{C}_h$ of $(0, \infty)_h \subset \mathbb{C}_h$. Of course, U cannot contain the origin, because the Liouville–Green ansatzes have essential singularities there.

  If P has appropriate analytic structure in z, then one can allow complex z as well, but this requires a bit more care, and one has to restrict attention to z in a sector depending on the order of vanishing of the potential.
- One common occurrence when considering parameters is that the transition point disappears or changes character when a parameter is varied. Consider for instance $V(r; \theta) = 1 - re^{i\theta}$, where r is the independent variable and $\theta \in \mathbb{R}$ is a parameter. The semiclassical ODE $Pu = 0$ for $P = -h^2\partial_r^2 + V$ has a turning point at $r = 1$ if $\theta \in 2\pi\mathbb{Z}$ but not otherwise. A more interesting example is that of coalescing turning points, where the potential $V(z; a)$ depends on a parameter $a \geq 0$ and has $N \in \mathbb{N}^{\geq 2}$ turning points for each $a > 0$, with the turning points coalescing in some way as $a \to 0^+$. The $N = 2$ case is the setting of [Olv75a].

  These sorts of problems cannot be readily addressed with the techniques below.

It is an open problem to carry out, in the PDE setting, an analysis similar to that here. In this vein we remark on work by Buchal–Keller [BK60], Ludwig [Lud66], and Guillemin–Schaeffer [GS73], all of which address the case of a simple turning point. This is an incomplete sample of the relevant literature. The cited works essentially solve the problem, giving full asymptotics in the classically allowed region, albeit only rapid decay in the classically forbidden region. The key idea is the representation of solutions of the equation as oscillatory integrals modeled on the integral formula for the Airy functions. The cited works prove Poincaré-type asymptotic expansions, but it is surely the case that e.g. [GS73, Eq. E] can be sharpened to include control on all derivatives.

Whether similar ideas apply in the $\kappa \geq 2$ case remains to be seen, one difficulty being that, in this case, the semiclassical characteristic set is not a smooth submanifold of the semiclassical cotangent bundle. Thus, this falls under the header of semiclassical asymptotics for singular Lagrangians. Some of the relevant theory is developed in [CDV03; Ver03].

The $\kappa = -1$ case gives a prototype for Lagrangian distributions for Lagrangian submanifolds which hit fiber infinity. This situation has been encountered in microlocal studies of the Klein–Gordon [Sus23] and time-dependent Schrödinger equations [GRHG23].

The $\kappa = 2$ case is one of the simplest examples of a semiclassical differential operator whose associated Hamiltonian flow displays normally hyperbolic trapping, the origin of the cotangent bundle over the transition point being the trapped set. In the PDE setting, microlocal estimates near trapped sets have been a popular theme in recent years. We will not attempt to summarize the literature. The ODE case has also been of interest in the study of wave propagation on spherically symmetric spacetimes — see for instance [CPS18] and the references therein.

We mention finally the recent thesis of Sobotta [Sob], who, in ongoing work with Grieser, has constructed $O(h^\infty)$-quasimodes for a wide class of semiclassical ODEs, including many of those considered here. While of a similar spirit to ours, their methods are significantly more general and not restricted to second-order ODEs. They include the use of iterated blowups organized via Newton polygons. When applied to the second-order case, only a single blowup is required, as is true here. It is expected that the constructed quasimodes can be upgraded to full solutions, but this has yet to be done.

2. The Langer diffeomorphism

As a first step in the proof of the main theorem, we follow Langer and Olver in the use of a variant of the "Langer diffeomorphism" — see [Lan31][Olv97, Chp. 12- §14] — to reduce to $W = 1$ case. This serves to simplify the computations. While not strictly necessary in the singular geometric approach, the analysis is somewhat shortened.

Let $M[Z]$ denote the mwc constructed in the introduction and depicted in Figure 1, where we are now making the dependence on Z explicit. For any $Z_0 > 0$, $M[Z] \cong M[Z_0]$, so this is only a distinction at the level of sets. This construction was coordinate invariant in the following sense:

Lemma 2.1. *If $\zeta : [0, Z] \to [0, \zeta(Z)]$ is any diffeomorphism, then the diffeomorphism $[0, Z]_z \times [0, \infty)_{h^{2/(\kappa+2)}} \to [0, \zeta(Z)]_\zeta \times [0, \infty)_{h^{2/(\kappa+2)}}$ given by $(z, h) \mapsto (\zeta(h), h)$ lifts to a diffeomorphism $M[Z] \to M[\zeta(Z)]$.* ■

Proof. The polar blowup of a corner is a coordinate invariant notion, so $\iota : (z, h) \mapsto (\zeta(h), h)$ lifts to a diffeomorphism

$$[[0, Z]_z \times [0, \infty)_{h^{2/(\kappa+2)}}; \{0\}_z \times \{0\}_h] \to [[0, \zeta(Z)]_\zeta \times [0, \infty)_{h^{2/(\kappa+2)}}; \{0\}_\zeta \times \{0\}_h]. \tag{21}$$

This is the desired diffeomorphism $M[Z] \to M[\zeta(Z)]$, but we do not yet know that it is a diffeomorphism, because, although the domain and codomain in eq. (21) agree with $M[Z], M[\zeta(Z)]$ at the level of sets, they differ in terms of smooth structure at ze.

It therefore suffices to verify that the map remains a diffeomorphism after the changes of smooth structures at ze. We only check smoothness, as smoothness of the inverse map is proven analogously.

Concretely, it suffices to check that the functions

$$\begin{aligned}\varrho_{\mathrm{ze}}(\zeta(z),h) &= h^2(\zeta(z)+h^{2/(\kappa+2)})^{-(\kappa+2)},\\ \varrho_{\mathrm{fe}}(\zeta(z),h) &= \zeta(z)+h^{2/(\kappa+2)}\end{aligned} \tag{22}$$

are smooth functions on $M[Z]$. Indeed, writing $\zeta(z) = z\xi(z)$, we have $\xi \in C^\infty([0,Z]_z;\mathbb{R}^+)$, and then the identities

$$\begin{aligned}\varrho_{\mathrm{ze}}(\zeta(z),h) &= \varrho_{\mathrm{ze}}(z,h)(\varrho_{\mathrm{be}}(z,h)(\xi(z,h)-1)+1)^{-(\kappa+2)},\\ \varrho_{\mathrm{fe}}(\zeta(z),h) &= \varrho_{\mathrm{fe}}(z,h)\varrho_{\mathrm{be}}(z,h)(\xi(z)-1)+\varrho_{\mathrm{fe}}(z,h)\end{aligned} \tag{23}$$

hold. The second of these is manifestly smooth on $M[Z]$. The first is a smooth function of $\varrho_{\mathrm{ze}}, \varrho_{\mathrm{be}}, z$ away from the set $\{\varrho_{\mathrm{be}}(\xi-1)\neq -1\}$, but this set is avoided on $M[Z]$, because $\xi(z,h)>0$ and $\varrho_{\mathrm{be}}(z,h)\in[0,1]$. □

We apply this lemma to the map defined by eq. (13). That this is a diffeomorphism follows from

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \xi^{-\kappa/2}\sqrt{W(z)} \in C^\infty([0,Z];\mathbb{R}^+). \tag{24}$$

Thus, the map $(z,h)\mapsto(\zeta(z),h)$ lifts to a diffeomorphism $M[Z]\to M[\zeta(Z)]$. This lift is what we refer to as the *Langer diffeomorphism*. It will be used implicitly below.

Let $P(h)$ be as in the introduction; that is, $P(h) = -h^2\partial_z^2+\varsigma z^\kappa W(z)+h^2\psi(z,h)$ for $\varsigma\in\{-1,+1\}$, $\kappa\in(-2,\infty)$, $W\in C^\infty([0,Z]_z;\mathbb{R}^+)$, and $\psi\in\varrho_{\mathrm{be}}^{-2}\varrho_{\mathrm{fe}}^{-2}C^\infty(M[Z])$ of the form specified in eq. (9). In terms of ζ, the operator $P(h)$ can be written as

$$\begin{aligned}\frac{\xi^\kappa}{W}P(h) &= -h^2\frac{\partial^2}{\partial\zeta^2}+\frac{h^2}{2}\Big(\frac{\kappa}{\xi}\frac{\partial\xi}{\partial\zeta}-\frac{1}{W}\frac{\partial W}{\partial\zeta}\Big)\frac{\partial}{\partial\zeta}+\varsigma\zeta^\kappa+\frac{h^2}{\zeta^2}\frac{\xi^{\kappa+2}}{W}\Big[\Big(\alpha^2-\frac{1}{4}\Big)+\Psi\Big(\frac{\zeta}{\xi h^{2/(\kappa+2)}}\Big)\Big]+h^2\tilde{E}\\ &\overset{\mathrm{def}}{=} P_0(h),\end{aligned} \tag{25}$$

where α,Ψ are as in eq. (9) and $\tilde{E} = \xi^\kappa E/W\in\varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{fe}}^{-1}C^\infty(M[\zeta(Z)])$. Viewing $P_0(h)$ as a differential operator on $[0,Z]_z$, $Pu=0$ is equivalent to $P_0u=0$.

In order to simplify the expression, and to facilitate comparison of P_0 with P, we can rearrange some terms; defining

$$\begin{aligned}E_0 = \tilde{E}+\frac{1}{\zeta^2}\Big(\frac{\xi^{\kappa+2}}{W}-1\Big)\Big[\Big(\alpha^2-\frac{1}{4}\Big)+\Psi\Big(\frac{\zeta}{\xi h^{2/(\kappa+2)}}\Big)\Big]&\\ +\frac{h^2}{\zeta^2}\Big[\Psi\Big(\frac{\zeta}{\xi h^{2/(\kappa+2)}}\Big)-\Psi\Big(\frac{\zeta}{\sqrt[\kappa+2]{W(0)h^2}}\Big)\Big],&\end{aligned} \tag{26}$$

we have, owing to the observation that $\xi(0)^{\kappa+2} = W(0)$, that $E_0\in\varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{fe}}^{-1}C^\infty(M[\zeta(Z)])$. Indeed,

$$(\xi^{\kappa+2}W^{-1}-1)\in zC^\infty([0,Z]_z;\mathbb{C})\subset\varrho_{\mathrm{be}}\varrho_{\mathrm{fe}}C^\infty(M[\zeta(Z)]). \tag{27}$$

Since $\zeta^{-2}\in\varrho_{\mathrm{be}}^{-2}\varrho_{\mathrm{fe}}^{-2}C^\infty(M[\zeta(Z)])$ and

$$\Psi(\zeta\xi^{-1}h^{-2/(\kappa+2)})\in\varrho_{\mathrm{be}}C^\infty(M[\zeta(Z)]), \tag{28}$$

overall the first line on the right-hand side of eq. (26) is in $\varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{fe}}^{-1}C^\infty(M[\zeta(Z)])$. Similarly,

$$\Psi\Big(\frac{\zeta}{\xi h^{2/(\kappa+2)}}\Big)-\Psi\Big(\frac{\zeta}{\sqrt[\kappa+2]{W(0)h^2}}\Big)\in\varrho_{\mathrm{be}}\varrho_{\mathrm{fe}}C^\infty(M[\zeta(Z)]), \tag{29}$$

so the second line is in $\varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{fe}}^{-1}C^\infty(M[\zeta(Z)])$ as well.

In terms of E_0, the operator P_0 can be written

$$P_0 = -h^2\frac{\partial^2}{\partial\zeta^2}+\frac{h^2}{2}\Big(\frac{\kappa}{\xi}\frac{\partial\xi}{\partial\zeta}-\frac{1}{W}\frac{\partial W}{\partial\zeta}\Big)\frac{\partial}{\partial\zeta}+\varsigma\zeta^\kappa+\frac{h^2}{\zeta^2}\Big[\Big(\alpha^2-\frac{1}{4}\Big)+\Psi\Big(\frac{\zeta}{\sqrt[\kappa+2]{W(0)h^2}}\Big)\Big]+h^2E_0. \tag{30}$$

As P_0 has a first order term, unlike P, it is useful to consider the conjugation

$$P_1 = M_{W^{1/4}\xi^{-\kappa/4}} P_0 M_{W^{-1/4}\xi^{\kappa/4}}, \tag{31}$$

where $M_\bullet$ denotes the multiplication operator $u \mapsto \bullet u$. A computation yields

$$P_1 = -h^2\frac{\partial^2}{\partial\zeta^2} + \varsigma\zeta^\kappa + \frac{h^2}{\zeta^2}\Big(\alpha^2 - \frac{1}{4}\Big) + \frac{h^2}{\zeta^2}\Psi\Big(\frac{\zeta}{\sqrt[\kappa+2]{W(0)h^2}}\Big) + h^2 E_1, \tag{32}$$

where

$$E_1 = E_0 + \frac{1}{4}\frac{\partial^2}{\partial\zeta^2}\log\Big(\frac{W}{\xi^\kappa}\Big) + \frac{1}{16}\Big(\frac{\partial}{\partial\zeta}\log\Big(\frac{W}{\xi^\kappa}\Big)\Big)^2 \in \varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{fe}}^{-1}C^\infty(M[\zeta(Z)];\mathbb{C}). \tag{33}$$

The operator P_1 is therefore of the same form as P, in that it also satisfies the hypotheses of Theorem B.

So, if we know the result in the $W=1$ case, then we can apply it to P_1. Let $\mathcal{Q}$ be defined by eq. (15). Given any $Q \in \mathcal{Q}$, Theorem B applied to P_1 gives a solution u_1 to $P_1 u_1 = 0$ of the form

$$u_1(\zeta,h) = (1+\varrho_{\mathrm{ze}}\varrho_{\mathrm{fe}}\beta)Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + \varrho_{\mathrm{ze}}^{(\kappa+1)/(\kappa+2)}\varrho_{\mathrm{fe}}\gamma Q'\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + \varrho_{\mathrm{be}}^{1/2-\alpha}\varrho_{\mathrm{fe}}\delta \tag{34}$$

for β,γ,δ as in the theorem. Define $u(z,h)$ by $u(z,h) = W^{-1/4}\xi^{\kappa/4}u_1(\zeta(z),h)$. This satisfies $0 = P_1 u_1 = W^{1/4}\xi^{-\kappa/4}P_0 u$, so $P_0 u = 0$, and therefore $Pu = 0$. The form of u specified in Theorem B follows from the form of u_1. Thus, if we know the result in the $W=1$ case, then we can deduce the result in general.

Below, we restrict attention to the case $W=1$, in which case $\zeta = z$, and we will mostly write ζ in place of z.

3. $O(h^2/\zeta)$-quasimodes and their properties

Consider

$$P = -h^2\frac{\partial^2}{\partial\zeta^2} + \varsigma\zeta^\kappa + h^2\psi \tag{35}$$

for ψ as in eq. (9), i.e. $\psi(\zeta,h) = \zeta^{-2}(\alpha^2 - 1/4) + \zeta^{-2}\Psi(\zeta/h^{2/(\kappa+2)}) + E$ for $\alpha\in\mathbb{C}$ with $\Re\alpha > 0$, $\Psi \in \varrho_{\mathrm{be}}|_{\mathrm{fe}}C^\infty(\mathrm{fe})$, and $E \in z^{-1}C^\infty(M)$. Now, eq. (10) reads

$$N(P) = -h^2\frac{\partial^2}{\partial\zeta^2} + \varsigma\zeta^\kappa + \frac{h^2}{\zeta^2}\Big(\alpha^2-\frac{1}{4}\Big) + \frac{h^2}{\zeta^2}\Psi\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) = P - h^2E. \tag{36}$$

As discussed in the introduction, solutions Q to $N(P)Q = 0$ can be considered as $O(h^2/\zeta)$-quasimodes for the original semiclassical ODE $Pu = 0$. This section is devoted to studying the properties of these quasimodes.

Conversely, given $\alpha \in \mathbb{C}$ with $\Re\alpha \geq 0$ and $\Psi \in \varrho_{\mathrm{be}}|_{\mathrm{fe}}C^\infty(\mathrm{fe})$, we can consider the semiclassical ordinary differential operator defined by

$$N_0 = -h^2\frac{\partial^2}{\partial\zeta^2} + \varsigma\zeta^\kappa + \frac{h^2}{\zeta^2}\Big(\alpha^2-\frac{1}{4}\Big) + \frac{h^2}{\zeta^2}\Psi\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big). \tag{37}$$

Setting $P = N_0$, the operator P satisfies the hypotheses of our setup, and $N(P) = N$. So, the set of all N_0 of the stipulated form is the set of all $N(P)$ arising as above. Forgetting P, the ODE under investigation is therefore $N_0 Q = 0$. Per the preceding discussion, this can be thought of either as a first step in studying the original semiclassical problem or as a special case of the general problem.

So, though this section is required preparation for the proofs in later sections, it can also be read as providing examples.

The key observation in studying N_0 is that it can be considered as a homogeneous family of operators on fe. Changing coordinates from ζ to $\lambda = \zeta/h^{2/(\kappa+2)}$, we have $N_0 \propto N$ for N the ordinary differential operator defined by

$$N = -\frac{\partial^2}{\partial\lambda^2} + \varsigma\lambda^\kappa + \frac{1}{\lambda^2}\Big(\alpha^2 - \frac{1}{4}\Big) + \frac{\Psi(\lambda)}{\lambda^2}, \tag{38}$$

where '$\propto$' means proportionality up to a nonvanishing function of h. This computation shows that N_0, considered as a family of ordinary differential operators on the positive real axis, has kernel $\ker N_0 = \ker N$ independent of h.

Let $\mathcal{Q} = \{Q \in C^\infty(\mathbb{R}^+_\lambda) : NQ = 0\}$ denote the kernel of N, thought of as a subset of $C^\infty(\mathbb{R}^+_\lambda)$ via h-independence. Cf. eq. (15).

3.1. Asymptotics of quasimodes.

Proposition 3.1. *If $Q \in \mathcal{Q}$, then $Q(\lambda) \in \lambda^{1/2-\alpha}C^\infty[0,\infty)_\lambda + \lambda^{1/2+\alpha}C^\infty[0,\infty)_\lambda$ if $\alpha \notin 2^{-1}\mathbb{Z}$. Otherwise, $Q(\lambda) \in \lambda^{1/2-\alpha}C^\infty[0,\infty)_\lambda + \lambda^{1/2+\alpha}\log(\lambda)C^\infty[0,\infty)_\lambda$. In either case, there exists a nonzero*

$$Q_0 \in \mathcal{Q} \cap \lambda^{1/2+\alpha}C^\infty[0,\infty)_\lambda, \tag{39}$$

unique up to multiplicative constants. If $Q \in \mathcal{Q}$ is such that $\lim_{\lambda\to0^+}\lambda^{-1/2+\alpha}Q = 0$, then $Q \in \operatorname{span}_\mathbb{C} Q_0$, and $Q = (\lim_{\lambda\to0^+}\lambda^{-1/2-\alpha}Q(\lambda))Q_0$. ■

Proof. Recall that a second-order regular singular ordinary differential operator $(\lambda\partial_\lambda)^2 + a(\lambda)\lambda\partial_\lambda + b(\lambda)$ has kernel contained in

$$\begin{cases} \lambda^{\gamma_-}C^\infty[0,\infty)_\lambda + \lambda^{\gamma_+}C^\infty[0,\infty)_\lambda = \mathcal{A}^{(\gamma_-,0)\cup(\gamma_+,0)}[0,\infty)_\lambda & (\gamma_+ - \gamma_- \notin \mathbb{Z}), \\ \lambda^{\gamma_-}C^\infty[0,\infty)_\lambda + \lambda^{\gamma_+}\log(\lambda)C^\infty[0,\infty)_\lambda = \mathcal{A}^{(\gamma_-,0)\cup(\gamma_+,1)}[0,\infty)_\lambda & (\text{otherwise}), \end{cases} \tag{40}$$

where $\gamma_\pm$ are the indicial roots of the equation, i.e. the roots of the polynomial $\gamma^2 + a(0)\gamma + b(0)$, and we have chosen them such that $\Re\gamma_- \le \Re\gamma_+$. Moreover, one of the members of the basis can be chosen to be in $\lambda^{\gamma_+}C^\infty[0,\infty)_\lambda$.

The operator N defined by eq. (38) is regular singular at $\lambda = 0$, which becomes clearer upon writing

$$\lambda^2 N = -\Big(\lambda\frac{\partial}{\partial\lambda}\Big)^2 + \lambda\frac{\partial}{\partial\lambda} + \varsigma\lambda^{\kappa+2} + \alpha^2 - \frac{1}{4} + \Psi(\lambda). \tag{41}$$

Since $\Psi(0) = 0$, the indicial polynomial is $\gamma^2 - \gamma - \alpha^2 + 1/4$, which has roots $\gamma_\pm = \pm\alpha + 1/2$. The difference $\gamma_+ - \gamma_- = 2\alpha$ is integral precisely when $\alpha \in 2^{-1}\mathbb{Z}$. Thus, the proposition is a corollary of this general theory. □

So, every $Q(\lambda) \in \mathcal{Q}$ is polyhomogeneous at $\lambda = 0$, with index set $\mathcal{F}(\alpha) = (1/2-\alpha, 0) \cup (1/2+\alpha, 0)$ if $\alpha \notin 2^{-1}\mathbb{Z}$ and $\mathcal{F}(\alpha) = (1/2-\alpha,0) \cup (1/2+\alpha,1)$ otherwise.

Any nonzero element of $\operatorname{span}_\mathbb{C}\{Q_0\}$ is called the *recessive* solution of the ODE.

Remark 3.2. As seen in the examples below, if $\Psi(\lambda) \in C^\infty[0,\infty)_{\lambda^{\kappa+2}}$, then the argument above can be sharpened, via the coordinate change $\Lambda = \lambda^{\kappa+2}$, to yield the absence of log terms as long as $\alpha \notin 2^{-1}(\kappa+2)\mathbb{Z}$.

Proposition 3.3. *If $Q \in \mathcal{Q}$ and $\varsigma > 0$, then $Q(\rho^{-2/(\kappa+2)}) \in \exp(2(\kappa+2)^{-1}\rho^{-1})\rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho$, and there exists a nonzero $Q_\infty \in \mathcal{Q}$ such that*

$$Q_\infty(\rho^{-2/(\kappa+2)}) \in \exp(-2(\kappa+2)^{-1}\rho^{-1})\rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho. \tag{42}$$

If $\varsigma < 0$, then there exist some $Q_\pm(\rho^{-1/(\kappa+2)}) \in \exp(\pm2i(\kappa+2)^{-1}\rho^{-1})\rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho$ such that $\mathcal{Q} = \operatorname{span}_\mathbb{C}\{Q_-, Q_+\}$. In all cases the leading order terms in the $C^\infty[0,\infty)_\rho$ factors at $\rho = 0$ are all nonzero. ■

Proof. Rewriting N in terms of $\rho = 1/\lambda^{(\kappa+2)/2}$, the result, which is most easily derived from substituting $\lambda\partial_\lambda = -2^{-1}(\kappa+2)\rho\partial_\rho$ into eq. (41), is

$$-\frac{4\lambda^2 N}{(\kappa+2)^2} = \Big(\rho\frac{\partial}{\partial\rho}\Big)^2 + \frac{2\rho}{\kappa+2}\frac{\partial}{\partial\rho} - \frac{4}{(\kappa+2)^2}\Big[\frac{\varsigma}{\rho^2} + \alpha^2 - \frac{1}{4} + \Phi(\rho^2)\Big], \tag{43}$$

where $\Phi(\varrho) = \Psi(\varrho^{-1/(\kappa+2)}) \in C^\infty[0,\infty)_\varrho$. Note that this is *not* a regular singular ODE at $\rho = 0$, because of the ς/ρ^2 term in the brackets. Removing this term, the remainder of the operator is regular singular. Consequently, we can appeal to Liouville–Green theory in the form it is presented in [Olv97, Chp. 7] to conclude the proposition.

To wit, to convert the operator above into the form considered by Olver, let $\Lambda = 1/\rho = \lambda^{(\kappa+2)/2}$, in terms of which

$$-\frac{4\lambda^2 N}{\Lambda^2(\kappa+2)^2} = \frac{\partial^2}{\partial\Lambda^2} + \frac{\kappa}{\kappa+2}\frac{1}{\Lambda}\frac{\partial}{\partial\Lambda} - \frac{4}{(\kappa+2)^2}\Big[\varsigma + \frac{1}{\Lambda^2}\Big(\alpha^2 - \frac{1}{4} + \Phi\Big(\frac{1}{\Lambda^2}\Big)\Big)\Big], \tag{44}$$

and the kernel of this operator is also $\mathcal{Q}$. So, the Liouville–Green expansion applies. Note that the coefficient of the first order term $\kappa(\kappa+2)^{-1}\Lambda^{-1}\partial_\Lambda$ is $O(1/\Lambda)$ and so does not contribute to the phase or decay rate. The phase φ (chosen to be positive in the $\varsigma > 0$ case) is therefore $\varphi = 2(\kappa+2)^{-1}\Lambda$ (up to an arbitrary additive constant) and the decay rate is $\sim \Lambda^{-\nu} = \rho^\nu$ with $\nu = \kappa/(2\kappa+4)$. The conclusion of [Olv97, Chp. 7- Thm. 2.1] applies with these parameters, and the statement of the proposition may be read off of it. A minor bibliographic note is that, in [Olv97, Chp. 7- Thm. 2.1], Olver assumes what in our context is the analyticity of Φ, whereas we only assume smoothness. This assumption is absent in [Olv97, Chp. 7- §1] so is unnecessary for producing solutions to the ODE modulo Schwartz errors (relative to the desired exponential growth or decay in the $\varsigma > 0$ case), and such errors may be solved away via the method of variation of parameters. A more involved variant of the same standard argument appears below, so we do not belabor the details. □

If the coefficients of N are real, then, in the $\varsigma < 0$ case, $Q_0 \neq Q_\pm$. However, in the $\varsigma > 0$ case, $Q_0 = Q_\infty$ is possible.

Combining the preceding two propositions, and identifying $\mathrm{fe} = [0,\infty)_\lambda \cup [0,\infty)_{1/\lambda^{(\kappa+2)/2}}$:

Corollary. *Let $Q \in \mathcal{Q}$. Letting $\chi \in C_c^\infty(\mathbb{R};\mathbb{R})$ be identically 1 near the origin,*

- *if $\varsigma > 0$, then $Q(\lambda) \in \exp(2(\kappa+2)^{-1}\chi(1/\lambda)\lambda^{(\kappa+2)/2})\mathcal{A}^{(-\kappa/(2\kappa+4),0),\mathcal{F}(\alpha)}(\mathrm{fe})$, where $(-\kappa/(2\kappa+4),0)$ is the index set at $\lambda = \infty$ and $\mathcal{F}(\alpha)$ is the index set at $\lambda = 0$. Also,*
$$Q_\infty(\lambda) \in \exp(-2(\kappa+2)^{-1}\chi(1/\lambda)\lambda^{(\kappa+2)/2})\mathcal{A}^{(-\kappa/(2\kappa+4),0),\mathcal{F}(\alpha)}(\mathrm{fe}). \tag{45}$$
- *If $\varsigma < 0$, then instead $Q_\pm(\lambda) \in \exp(\pm 2i(\kappa+2)^{-1}\chi(1/\lambda)\lambda^{(\kappa+2)/2})\mathcal{A}^{(-\kappa/(2\kappa+4),0),\mathcal{F}(\alpha)}(\mathrm{fe})$.*

So, Q is of exponential-polyhomogeneous type on fe. ■□

As a reminder, $[0,\infty]_\lambda = [0,\infty)_\lambda \cup (0,\infty]_{1/\lambda}$.

Proposition 3.4. *Suppose that $f \in \mathcal{A}^{\mathcal{E},\mathcal{F}}[0,\infty]_\lambda$, where $\mathcal{E}$ is the index set at $\lambda = \infty$ and $\mathcal{F}$ is the index set at $\lambda = 0$. Then, $f(\zeta/h^{2/(\kappa+2)}) \in \mathcal{A}^{(0,0),(\kappa+2)^{-1}\mathcal{E},(0,0),\mathcal{F}}(M)$.* ■

Here, $(\kappa+2)^{-1}\mathcal{E}$ denotes the smallest index set containing $\{((\kappa+2)^{-1}j,k) : (j,k) \in \mathcal{E}\}$. If κ is an integer, then this is $\{((\kappa+2)^{-1}j,k) : (j,k) \in \mathcal{E}\}$ itself.

Proof. In terms of $\varrho_{\mathrm{ze}} = h^2/(\zeta + h^{2/(\kappa+2)})^{\kappa+2}$, $\varrho_{\mathrm{fe}} = \zeta + h^{2/(\kappa+2)}$, and $\varrho_{\mathrm{be}} = \zeta/(\zeta + h^{2/(\kappa+2)})$, we have $\zeta/h^{2/(\kappa+2)} = \varrho_{\mathrm{be}}\varrho_{\mathrm{ze}}^{-1/(\kappa+2)}$, so

$$f(\zeta/h^{2/(\kappa+2)}) = f(\varrho_{\mathrm{be}}\varrho_{\mathrm{ze}}^{-1/(\kappa+2)}). \tag{46}$$

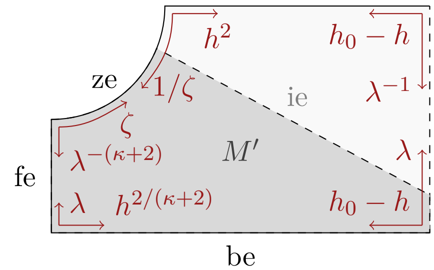

FIGURE 2. The mwc M_2, with the subset $M \cong M' \subset M_2$ labeled, with the edges labeled by the corresponding edges in M. A smooth atlas is depicted.

As ϱ_{ze} is nonvanishing near be, this suffices to imply that f is polyhomogeneous with the claimed index sets $\mathcal{F}$ at be and $(0,0)$ at fe away from ie ∪ ze. To study the situation near ie ∪ ze, consider $g(t) = f(1/t)$, which lies in $\mathcal{A}^{\mathcal{E}}[0,\infty)_t$. Thus,

$$f(\zeta/h^{2/(\kappa+2)}) = g(\varrho_{\mathrm{ze}}^{1/(\kappa+2)}\varrho_{\mathrm{be}}^{-1}). \tag{47}$$

Since ϱ_{be} is nonvanishing near ze, this suffices to imply that f is polyhomogeneous with the claimed index sets $(\kappa+2)^{-1}\mathcal{E}$ at ze and $(0,0)$ at ie and fe away from be. □

There is a geometric interpretation of the previous proposition. We start with $M_1 = [0,\infty]_\lambda \times [0,\infty)_{h^{2/(\kappa+2)}}$ on which f, viewed as a function independent of h, satisfies $f \in \mathcal{A}^{\mathcal{E},(0,0),\mathcal{F}}(M_1)$, where the middle index set is that at $\{h = 0\}$. Performing a polar blowup of the upper corner $\{\lambda = \infty, h = 0\}$, we denote the result, after modifying the smooth structure at the front face of the blowup so that h^2 becomes a boundary-defining-function of its interior,

$$M_2 = [M_1; \{\lambda = \infty, h = 0\}; \kappa + 2]. \tag{48}$$

We have $f \in \mathcal{A}^{\mathcal{E},(\kappa+2)^{-1}\mathcal{E},(0,0),\mathcal{F}}(M_2)$, where the index sets are at $\mathrm{cl}_{M_2}\{\lambda = \infty, h > 0\}$, then the front face of the blowup, $\mathrm{cl}_{M_2}\{\lambda < \infty, h = 0\}$, and $\{\lambda = 0\}$, respectively. Let

$$M' = \mathrm{cl}_{M_2}\{\lambda h^{2/(\kappa+2)} \le Z\} \subseteq M_2. \tag{49}$$

From $f \in \mathcal{A}^{\mathcal{E},(\kappa+2)^{-1}\mathcal{E},(0,0),\mathcal{F}}(M_2)$ follows immediately $f \in \mathcal{A}^{(0,0),(\kappa+2)^{-1}\mathcal{E},(0,0),\mathcal{F}}(M')$, where the first index set is at the curve $\mathrm{cl}_{M_2}\{\lambda h^{2/(\kappa+2)} = Z\}$. The map

$$(0, Z)_\zeta \times (0,\infty)_h \ni (\zeta, h) \mapsto (\zeta/h^{2/(\kappa+2)}, h) \tag{50}$$

extends to a diffeomorphism $M \to M'$. This diffeomorphism identifies $\{\lambda = 0\} \subset M'$ with be, $\mathrm{cl}_{M'}\{\lambda < \infty, h = 0\} = \mathrm{cl}_{M_2}\{\lambda < \infty, h = 0\}$ with fe, the front face of the blowup with ze, and the curve

$$\mathrm{cl}_{M_2}\{\lambda h^{2/(\kappa+2)} = Z\} \tag{51}$$

with ie. See Figure 2. Thus, the already deduced $f \in \mathcal{A}^{(0,0),(\kappa+2)^{-1}\mathcal{E},(0,0),\mathcal{F}}(M')$ is equivalent to the desired result.

Combining the propositions above:

Corollary. *If $Q \in \mathcal{Q}$, then $Q(\zeta/h^{2/(\kappa+2)})$ is of exponential-polyhomogeneous type on M.* ■□

3.2. **Examples.**

Example (Liouville–Green). If $\kappa = 0$, $\alpha = 1/2$, and $\Psi = 0$, then $N = -\partial_\lambda^2 + \varsigma$, so $\mathcal{Q} = \mathrm{span}_{\mathbb{C}}\{\mathcal{Q}_- = e^{-i\lambda}, \mathcal{Q}_+ = e^{+i\lambda}\}$ if $\varsigma < 0$ and $\mathcal{Q} = \mathrm{span}_{\mathbb{C}}\{\mathcal{Q}_\infty = e^{-\lambda}, e^{\lambda}\}$ if $\varsigma > 0$.

The next most classical case is:

Example (JWKB). If $\kappa = 1$, $\alpha = 1/2$, and $\Psi = 0$, then $N = -\partial_\lambda^2 + \varsigma\lambda$, so $NQ = 0$ is Airy's ODE (or its reflection across the origin, depending on the value of ς). So, if $\varsigma > 0$, then

$$\mathcal{Q} = \{c_1 \operatorname{Ai}(\lambda) + c_2 \operatorname{Bi}(\lambda) : c_1, c_2 \in \mathbb{C}\} \tag{52}$$

is precisely the set of Airy functions on the positive real axis. If $\varsigma < 0$, then similarly $\mathcal{Q} = \{c_1 \operatorname{Ai}(-\lambda) + c_2 \operatorname{Bi}(-\lambda) : c_1, c_2 \in \mathbb{C}\}$. Proposition 3.3, applied to these cases, states the qualitative form of the large-argument asymptotic expansions [OMe, §9.7] of the Airy functions and their derivatives. In particular, the large-argument expansions of $\operatorname{Ai}(\lambda), \operatorname{Bi}(\lambda)$ are in integral powers of $\rho = 1/\lambda^{3/2}$.

Generalizing the previous two examples:

Example (Langer). If $\Psi = 0$, then

$$N = -\frac{\partial^2}{\partial\lambda^2} + \varsigma\lambda^\kappa + \frac{1}{\lambda^2}\Big(\alpha^2 - \frac{1}{4}\Big). \tag{53}$$

As observed by Langer, the elements of $\mathcal{Q} = \ker N$ can be written in terms of Bessel functions. (For comparison with the previous two examples, recall that, up to a polynomial weight, the trigonometric functions are Bessel functions of order $1/2$, and the Airy functions are Bessel functions of order $1/3$.) More precisely,

$$\mathcal{Q} = \Big\{\lambda^{1/2} I\Big(\frac{2\lambda^{(\kappa+2)/2}}{\kappa+2}\Big) : I(t) \text{ a solution to } t^2\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + t\frac{\mathrm{d}I}{\mathrm{d}t} - (\varsigma t^2 + \nu^2)I(t) = 0\Big\}. \tag{54}$$

The ODE satisfied by $I(t)$ is Bessel's ODE of order $\nu = 2\alpha/(\kappa+2)$, so I is a (modified, if $\varsigma > 0$, and unmodified otherwise) Bessel function. For this special the case, the conclusions of the previous propositions can be checked explicitly using the small- or large- argument expansions of the Bessel functions. The small argument expansions are in [OMe, §10.8]. The large argument expansions are *Hankel's expansions* [OMe, §10.17(i)][OMe, §10.40(i)]. The expansions given in [OMe] are Poincaré-type expansions of the Bessel functions and their first derivatives. Taken together, these suffice to imply smoothness in terms of $\rho = 1/\lambda^{(\kappa+2)/2}$ at $\rho = 0$, with control of higher derivatives coming from the ODE. Besides the argument used above, Hankel's expansions can be proven in many ways, e.g. extracted from the integral representations of the Bessel functions via the method of stationary phase.

3.3. **The inhomogeneous model problem.** We now study the forced ODE $Nu = fQ + gQ'$ for $f, g \in \mathcal{S}(\mathbb{R}) \cap C_{\mathrm{c}}^\infty(0,\infty]$. That is, f, g are Schwartz functions vanishing near the origin.

The solutions can be produced using the standard Schwartz kernel construction, which we now recall. Let $Q_1, Q_2 \in \mathcal{Q}$ denote linearly independent elements of $\mathcal{Q}$, in which case their Wronskian $\mathfrak{W} \in \mathbb{C}$, which we define with the sign convention

$$\mathfrak{W} = Q_1' Q_2 - Q_2 Q_1', \tag{55}$$

is nonzero. Then, for each $\lambda' > 0$, the function $K(\lambda, \lambda') = K[Q_1, Q_2](\lambda, \lambda') \in C^\infty(\mathbb{R}^+_\lambda \setminus \{\lambda'\})$ defined by

$$K(\lambda, \lambda') = \frac{1}{\mathfrak{W}}\begin{cases} Q_1(\lambda)Q_2(\lambda') & (\lambda > \lambda'), \\ Q_2(\lambda)Q_1(\lambda') & (\lambda < \lambda') \end{cases} \tag{56}$$

solves $NK(\lambda, \lambda') = \delta(\lambda - \lambda')$, where $\delta \in \mathcal{D}'(\mathbb{R})$ denotes a Dirac δ-function. Thus, for nice functions $F \in C^\infty[0,\infty)_\lambda$, including all $F \in C_{\mathrm{c}}^\infty(0,\infty)_\lambda$, the function $N^{-1}[Q_1, Q_2]F$ defined by

$$N^{-1}[Q_1, Q_2]F = K(\lambda, F) = \int_0^\infty K(\lambda, \lambda') F(\lambda') \,\mathrm{d}\lambda' \tag{57}$$

solves $N(N^{-1}[Q_1, Q_2]F) = F$. In this way, each choice of Q_1, Q_2 leads to a right inverse $N^{-1}[Q_1, Q_2] : C_{\mathrm{c}}^\infty(0,\infty)_\lambda \to C^\infty(0,\infty)_\lambda$ of N.

Note that $N^{-1}[Q_1, Q_2]$ depends on Q_1, Q_2 only through $\mathbb{C}^\times Q_1, \mathbb{C}^\times Q_2$.

Proposition 3.5. *Fix independent $Q, \tilde{Q} \in \mathcal{Q}$, and, if $\varsigma > 0$, then, unless $Q = Q_\infty$ is the exponentially decaying mode, let $\tilde{Q} = \mathcal{Q}_\infty$ be the exponentially decaying mode. Then, $N^{-1}[\tilde{Q}, Q]$ extends to a map*

$$\{fQ + gQ' + h : f, g, h \in \mathcal{S}(\mathbb{R}) \cap C^\infty_{\mathrm{c}}(0,\infty]_\lambda\} \to C^\infty(0,\infty)_\lambda \tag{58}$$

such that eq. (57) holds for all F in the codomain and such that the extension is a right inverse of N. ■

Proof. Given the hypotheses, the function $K(\lambda, F)$ given by

$$K(\lambda, F) = \frac{\tilde{Q}(\lambda)}{\mathfrak{W}} \int_0^\lambda Q(s)F(s)\,\mathrm{d}s + \frac{Q(\lambda)}{\mathfrak{W}} \int_\lambda^\infty \tilde{Q}(s)F(s)\,\mathrm{d}s \tag{59}$$

is well-defined for $F = fQ + gQ' + h$ with f, g, h as above, the integrals on the right-hand side being absolutely convergent. Let $N^{-1}[\tilde{Q}, Q]F(\lambda) = K(\lambda, F)$. This is a linear extension of $N^{-1}[\tilde{Q}, Q]$ to $\{fQ + gQ' + h : f, g, h \in \mathcal{S}(\mathbb{R}) \cap C^\infty_{\mathrm{c}}(0,\infty]_\lambda\}$. Differentiating $K(\lambda, F)$ in λ works as before, so $NK(\lambda, F) = F(\lambda)$. □

Denote the extension by $N^{-1}[\tilde{Q}, Q]$ as well.

One annoyance is that, even if $F \in C^\infty_{\mathrm{c}}(0,\infty)_\lambda$, then $v = N^{-1}[\tilde{Q}, Q]F$ need not be $O(\langle\lambda\rangle^{-\infty} Q)$ as $\lambda \to \infty$, as

$$\lim_{\lambda\to\infty} Q(\lambda)^{-1} v(\lambda) = \frac{1}{\mathfrak{W}} \int_0^\infty \tilde{Q}(\lambda)F(\lambda)\,\mathrm{d}\lambda, \tag{60}$$

the right-hand side having no good reason to vanish in general. But this is the only obstruction, so:

Proposition 3.6. *Fix $\Lambda_0 > \lambda_0 > 0$. Given F of the form $F = fQ + gQ' + h$ for $f, g \in \mathcal{S}(\mathbb{R}) \cap C^\infty_{\mathrm{c}}(\lambda_0,\infty]$, there exist $\beta, \gamma \in \mathcal{S}(\mathbb{R}) \cap C^\infty_{\mathrm{c}}(0,\infty]$ such that the function $v \in C^\infty(0,\infty)_\lambda$ defined by*

$$v(\lambda) = \beta(\lambda)Q(\lambda) + \gamma(\lambda)Q'(\lambda) \tag{61}$$

solves $Nv = F + R$ for some $R \in C^\infty_{\mathrm{c}}(0,\infty)_\lambda$ with $\operatorname{supp} R \subseteq [\lambda_0, \Lambda_0]$. ■

Proof. Since $Q, \tilde{Q}$ are linearly independent (which implies linear independence when restricted to $[\lambda_0, \Lambda_0]$) there exists a $R \in C^\infty_{\mathrm{c}}(0,\infty)_\lambda$ with $\operatorname{supp} R \subseteq [\lambda_0, \Lambda_0]$ such that

$$\begin{aligned} \int_0^\infty \tilde{Q}(\lambda)R(\lambda)\,\mathrm{d}\lambda &= -\int_0^\infty \tilde{Q}(\lambda)F(\lambda)\,\mathrm{d}\lambda, \\ \int_0^\infty Q(\lambda)R(\lambda)\,\mathrm{d}\lambda &= -\int_0^\infty Q(\lambda)F(\lambda)\,\mathrm{d}\lambda. \end{aligned} \tag{62}$$

Now define $v = N^{-1}[\tilde{Q}, Q](F + R)$. Then, $Nv = F + R$. Observe that v vanishes identically near $\lambda = 0$. Indeed, for $\lambda < \lambda_0$ such that $\operatorname{supp}(F + R) \cap [0, \lambda] = \varnothing$, we have

$$v(\lambda) = \frac{Q(\lambda)}{\mathfrak{W}} \int_0^\infty \tilde{Q}(s)(F(s) + R(s))\,\mathrm{d}s = 0. \tag{63}$$

By Proposition A.2,

$$\Big\{\tilde{Q}(\lambda) \int_0^\lambda Q(s)(F(s) + R(s))\,\mathrm{d}s,\ Q(\lambda) \int_\lambda^\infty \tilde{Q}(s)(F(s) + R(s))\,\mathrm{d}s\Big\} \subset (|Q(\lambda)|^2 + |Q'(\lambda)|^2)^{1/2}\mathcal{S}(\mathbb{R}). \tag{64}$$

Consequently, the functions $\beta = vQ^*(\lambda)(|Q(\lambda)|^2 + |Q'(\lambda)|^2)$ and $\gamma = v(Q'(\lambda))^*(|Q(\lambda)|^2 + |Q'(\lambda)|^2)$ lie in $\mathcal{S}(\mathbb{R}) \cap C^\infty_{\mathrm{c}}(0,\infty]$. These are defined such that $v = \beta Q + \gamma Q'$. □

4. Quasimodes at ze

The basic strategy of Langer–Olver is to look for a solution u to the ODE

$$Pu = -h^2 \frac{\partial^2 u}{\partial \zeta^2} + \varsigma \zeta^\kappa u + h^2 \psi u = 0, \tag{65}$$

of the form

$$u = \beta Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + h^{\kappa/(\kappa+2)} \gamma Q'\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) \tag{66}$$

for $\beta, \gamma \in C^\infty((0,Z]_\zeta \times [0,\infty)_{h^2}; \mathbb{C})$ that are well-behaved as $\zeta \to 0^+$, where Q is an arbitrary member of $\mathcal{Q}$. In Langer's original work (which allowed a small error on the right-hand side of the ODE), well-behaved meant smooth down to $\zeta = 0$, but something weaker (which ends up being polyhomogeneity on M) is required here.

The power of h in front of the $\gamma Q'$ term in eq. (66) has been chosen so that, for each fixed $\zeta > 0$, both terms on the right-hand side have the same order of magnitude as $h \to 0^+$. This makes the semiclassical structure of the argument below more apparent.

In the rest of the body of the paper, we will abbreviate $Q = Q(\zeta/h^{2/(\kappa+2)})$ and $Q' = Q'(\zeta/h^{2/(\kappa+2)})$, as the $\zeta/h^{2/(\kappa+2)}$ argument will be clear from context.

Applying P to u of the form above, the result is again a function of a similar form:

$$Pu = \begin{bmatrix} Q \\ h^{\kappa/(\kappa+2)} Q' \end{bmatrix}^\intercal \Big(-h^2 \frac{\partial^2}{\partial \zeta^2} - 2 \begin{bmatrix} 0 & \varsigma\zeta^\kappa \\ 1 & 0 \end{bmatrix} h \frac{\partial}{\partial \zeta} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Big(2\phi h^3 \frac{\partial}{\partial \zeta} + h \varsigma \kappa \zeta^{\kappa-1} + h^3 \phi' \Big) + h^2 E \Big) \begin{bmatrix} \beta \\ \gamma \end{bmatrix}, \tag{67}$$

where, as in the introduction, $\phi = \psi - E$. Because $\phi \in \varrho_{\mathrm{be}}^{-2} \varrho_{\mathrm{fe}}^{-2} C^\infty(M)$, we have $\phi' \in \varrho_{\mathrm{be}}^{-3} \varrho_{\mathrm{fe}}^{-3} C^\infty(M)$.

This suggests attempting to choose β and γ so as to satisfy the system of ODEs $LU = 0$, where $U = (\beta, \gamma)$ and $L \in \mathrm{Diff}_\hbar^2((0,Z]_\zeta; \mathbb{C}^2)$ is the matrix-valued semiclassical operator

$$L = -h^2 \frac{\partial^2}{\partial \zeta^2} - 2 \begin{bmatrix} 0 & \varsigma\zeta^\kappa \\ 1 & 0 \end{bmatrix} h \frac{\partial}{\partial \zeta} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Big(2\phi h^3 \frac{\partial}{\partial \zeta} + h \varsigma \kappa \zeta^{\kappa-1} + h^3 \phi' \Big) + h^2 E \tag{68}$$

appearing in eq. (67). It is not true that $Pu = 0$ is equivalent to $LU = 0$, at least without specifying more about the structure of U.

For each $h > 0$, the kernel of $L(h)$ is 4-dimensional. The map $U \mapsto u$, where u is defined in terms of β, γ by eq. (66), sends $\ker L(h)$ to $\ker P$, and it is easily seen that this map has full rank. So, for every individual $h > 0$, any $u(-,h) \in \ker P(h)$ can be decomposed as in eq. (66) for some functions $\beta(-,h), \gamma(-,h) \in C^\infty(0,Z]$ such that the function $U(-,h)$ defined by $U(-,h) = \beta(-,h), \gamma(-,h))$ satisfies $L(h)U(-,h) = 0$. Thus, passage to the vector-valued ODE $LU = 0$ is without loss of generality as far as constructing elements of $\ker P$ is concerned.

Curiously, L is independent of Q; it is only the map $U \mapsto u$ that is Q-dependent. This is related to the semiclassical structure of L, which we examine in §4.1. This subsection can be skipped on first reading.

If U satisfies the ODE $LU = 0$ on the nose, then the function u defined by eq. (66) satisfies $Pu = 0$ on the nose, but this is too much to ask for right away, so instead we try to solve the equation up to small errors. We might like an $O(h^\infty)$ error, which means that the error should be Schwartz at both edges ze and fe comprising the $h \to 0^+$ regime, in which case it is simply Schwartz at $\{h = 0\}$ in $[0,Z]_\zeta \times [0,\infty)_{h^2}$, but this is still too much to ask. In this section, we aim only for an $O(\varrho_{\mathrm{ze}}^\infty)$ error, meaning in

$$\varrho_{\mathrm{ze}}^\infty \varrho_{\mathrm{fe}}^\kappa \mathcal{A}^{\mathcal{E}}(M) \oplus \varrho_{\mathrm{ze}}^\infty \varrho_{\mathrm{fe}}^{\kappa/2} \mathcal{A}^{\mathcal{F}}(M) = \bigcap_{k \in \mathbb{N}} (\varrho_{\mathrm{ze}}^k \varrho_{\mathrm{fe}}^\kappa \mathcal{A}^{\mathcal{E}}(M) \oplus \varrho_{\mathrm{ze}}^k \varrho_{\mathrm{fe}}^{\kappa/2} \mathcal{A}^{\mathcal{F}}(M)) \tag{69}$$

for some to-be-determined index sets $\mathcal{E}, \mathcal{F}$. (The powers of ϱ_{fe} in eq. (69) can be absorbed into a redefinition of $\mathcal{E}, \mathcal{F}$. The convention above is used to match later notation.) That is, the error in our quasimode construction will be Schwartz only at ze, but uniformly so all the way up to fe, in a precise sense.

First, we work on the ray ze\fe, meaning that we will not worry about uniformity up to fe (but uniformity at the other endpoint ze $\cap$ ie is handled). This part of the argument is essentially found in [Olv97, Chp. 12- §14], but we present a full exposition here, in the relevant level of generality. The formal part is contained in §4.2. The situation at the other endpoint, ze $\cap$ fe, will be analyzed second, with uniformity up to fe a consequence. The argument is outlined in §4.3, and the details are in §4.4.

4.1. **Semiclassical structure.** We refer to [Zwo12] for unfamiliar terminology.

Though $L(h)$ is certainly elliptic for each individual $h>0$, L is not elliptic *as a semiclassical operator*. This is true even in the "classically forbidden case" $\varsigma > 0$, in which case P itself *is* semiclassically elliptic. The semiclassically principal part L_0 of L consists of the first two terms in eq. (68),

$$L_0 = -h^2 \frac{\partial^2}{\partial \zeta^2} - 2 \begin{bmatrix} 0 & \varsigma \zeta^\kappa \\ 1 & 0 \end{bmatrix} h \frac{\partial}{\partial \zeta} \tag{70}$$

(Here we are ignoring behavior as $\zeta \to 0^+$. If $\phi = 0$, then $L \in S\,\mathrm{Diff}^2_\hbar([0,Z]_\zeta;\mathbb{C}^2)$, and the principal part is principal in this stronger sense.) Let ${}^\hbar T^*[0,Z]$ denote the semiclassical cotangent bundle over $[0,Z]$, which we parametrize by the map $(h,\zeta,\mu) \mapsto h^{-1}\mu\,\mathrm{d}\zeta$, so μ is the frequency coordinate dual to the sole variable ζ. Then,

$${}^\hbar T^*[0,Z] \cong [0,\infty)_h \times [0,Z]_\zeta \times \mathbb{R}_\mu. \tag{71}$$

In terms of these coordinates, the semiclassical principal symbol $\sigma^2_\hbar(L) : {}^\hbar T^*[0,Z] \to \mathbb{C}$ is

$$\sigma^2_\hbar(L) = \mu^2 \,\mathrm{Id} - 2i\mu \begin{bmatrix} 0 & \varsigma\zeta^\kappa \\ 1 & 0 \end{bmatrix}, \tag{72}$$

gotten by replacing $h\partial_\zeta$ with $i\mu$ in L_0. The semiclassical characteristic set $\mathrm{char}^2_\hbar(L) \subset {}^\hbar T^*[0,Z]$ is by definition the set of points $(h=0,\zeta,\mu)$ at which $\sigma^2_\hbar(L)(0,\zeta,\mu) \in \mathbb{C}^{2\times 2}$ fails to be invertible. This occurs when the determinant

$$\det \sigma^2_\hbar(L) = \mu^4 + 4\varsigma\mu^2\zeta^\kappa = \mu^2(\mu^2 + 4\varsigma\zeta^\kappa) \tag{73}$$

vanishes. This includes the zero section $\{\mu = 0\}$, and, if $\varsigma < 0$, then it also includes the sets $\{\mu = \pm 2\zeta^{\kappa/2}\}$. The interpretation of $\sigma^2_\hbar(L)$ is that it describes the possible oscillations $\exp(ih^{-1}\varphi(\zeta))$ present in the $h \to 0^+$ asymptotics of solutions to $LU = 0$, with the allowed φ being such that $(h=0,\zeta,\varphi'(\zeta)) \in \sigma^2_\hbar(L)$ for each ζ. The key feature of L that allows it to have polyhomogeneous solutions is therefore that its characteristic set includes the zero section $\{\mu = 0\}$, since this corresponds to non-oscillatory asymptotics.

See Figure 3 for an illustration of the characteristic set in the $\kappa = 1,3$ cases, and see Figure 4 for the $\kappa = -1,2$ cases.

The additional characteristic set in the classically allowed case $\varsigma < 0$ signals the existence of highly oscillatory solutions to $LU = 0$ with initial data that is non-oscillatory as $h \to 0$. Consider solutions $u[Q_\pm]$ to $Pu[Q_\pm] = 0$ with the form eq. (66), for $Q_\pm$ in place of Q. The surjectivity of the map $\ker L \to \ker P$ means that we can write $u[Q_\mp]$ in the form eq. (66) for $Q = Q_\pm$ (not $Q = Q_\mp$!) and nonzero $U_\mp \in \ker L$. By assumption, $u[Q_\mp]$ is highly oscillatory as $h \to 0^+$ with the same phase as $Q_\mp$ and therefore with the *opposite* phase as $Q_\pm$, it follows that $U_\mp$ must be oscillating with twice the phase with which $Q_\mp$ is oscillating. Thus, there exist highly oscillatory solutions U to $LU = 0$, but these are not the solutions constructed via asymptotic series below, which are smooth.

The situation is similar to the *conjugated* perspective of Vasy [Vas21a; Vas21b] in microlocal scattering theory (where the setting was the Parenti–Shubin–Melrose sc-calculus rather than the semiclassical calculus, but these are similar in many respects), but there the characteristic set has

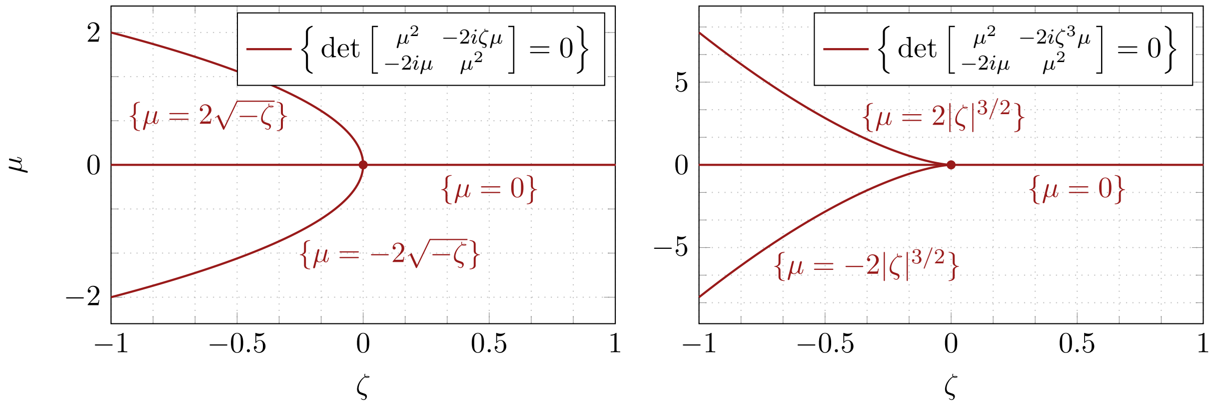

FIGURE 3. The semiclassical characteristic set $\mathrm{char}^2_\hbar(L) = \{\mu = 0 \text{ or } \mu^2 + 4\varsigma\zeta^\kappa = 0\} \subset \{h = 0\} \cap {}^\hbar T^*(-1, +1)$ of L, in the $\kappa = 1, 3$ cases, with $\varsigma = 1$; we have extended L to $[-Z, +Z]$ so as to include both cases of ς in one figure. Various subsets of the characteristic set have been labeled. When $\kappa \geq 3$, the characteristic set has a cusp singularity. When $\kappa = 1$, it is a fold singularity (i.e. *caustic*) with respect to the projection onto the base. When $\kappa = -1$ (not pictured), the characteristic set is unbounded near $\zeta = 0$.

only one nonzero "branch" present, whereas $\sigma^2_\hbar(L)$ has both branches $\{\mu = \pm 2|\zeta|^{\kappa/2}\}$ present, as depicted in the figures. This difference is unsurprising, as L (unlike Vasy's conjugated operator) does not depend on the choice of $Q \in \mathcal{Q}$, so no branch can be singled out.

4.2. **Construction away from** fe**: formalities.** Since ϕ is smooth at ze, and since h^2 serves as a defining function for ze$\backslash$fe in $M\backslash$fe, we can Taylor expand

$$\phi \sim \sum_{k=0}^\infty h^{2k}\phi_k(\zeta) \in C^\infty(\mathrm{ze}\backslash\mathrm{fe})[[h^2]] \tag{74}$$

at ze$\backslash$fe. For later use, not that, because $\phi \in \varrho_{\mathrm{be}}^{-2}\varrho_{\mathrm{fe}}^{-2}C^\infty(M)$, we have $\zeta^{2+k(\kappa+2)}\phi_k \in C^\infty(\mathrm{ze})$. Since E is smooth at ze, we can similarly expand

$$E \sim \sum_{k=0}^\infty h^{2k}E_k(\zeta) \in C^\infty(\mathrm{ze}\backslash\mathrm{fe})[[h^2]]. \tag{75}$$

Since $E \in \varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{fe}}^{-1}C^\infty(M)$, we have $\zeta^{1+k(\kappa+2)}E_k \in C^\infty(\mathrm{ze})$.

Consider the formal version of L,

$$\mathsf{L} = -h^2\frac{\partial^2}{\partial\zeta^2} - 2\begin{bmatrix} 0 & \varsigma\zeta^\kappa \\ 1 & 0 \end{bmatrix} h\frac{\partial}{\partial\zeta} - h\begin{bmatrix} 0 & \varsigma\kappa\zeta^{\kappa-1} \\ 0 & 0 \end{bmatrix} - h^3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\sum_{k=0}^\infty h^{2k}\Big(2\phi_k\frac{\partial}{\partial\zeta} + \phi_k'\Big) + h^2\sum_{k=0}^\infty h^{2k}E_k(\zeta). \tag{76}$$

This is an element of $\mathrm{Diff}^2(\mathrm{ze}\backslash\mathrm{fe};\mathbb{C}^2)[[h]]$. The task before us is to construct $\mathsf{U} \in C^\infty(\mathrm{ze}\backslash\mathrm{fe};\mathbb{C}^2)[[h]]$ satisfying $\mathsf{LU} = 0$.

Consider the following ansatz for U,:

$$\mathsf{U} = \sum_{k=0}^\infty h^{2k}\begin{bmatrix} \beta_k \\ 0 \end{bmatrix} + \sum_{k=0}^\infty h^{2k+1}\begin{bmatrix} 0 \\ \gamma_k \end{bmatrix}, \qquad \beta_k, \gamma_k \in C^\infty(\mathrm{ze}^\circ). \tag{77}$$

This is the Langer–Olver ansatz.

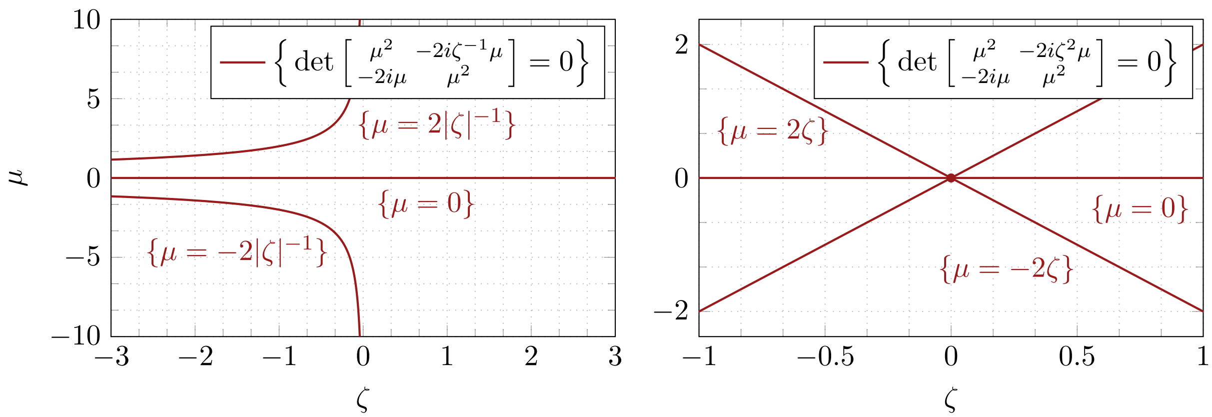

FIGURE 4. The semiclassical characteristic set $\mathrm{char}^2_h(L) = \{\mu = 0 \text{ or } \mu^2 - 4\varsigma\zeta^\kappa = 0\} \subset {}^\hbar T^*(-1,+1)$ of L, in the $\kappa = -1$ and $\kappa = 2$ cases, the latter with $\varsigma < 0$. In the former, the characteristic set hits fiber infinity at $\zeta = 0$.

Given U of this form, the formal ODE LU is equivalent to the conjunction of

$$\frac{\mathrm{d}\beta_0}{\mathrm{d}\zeta} = 0 \tag{78}$$

and

$$2\frac{\mathrm{d}\beta_{k+1}}{\mathrm{d}\zeta} = -\frac{\mathrm{d}^2\gamma_k}{\mathrm{d}\zeta^2} + \sum_{j=0}^k E_j\gamma_{k-j} \tag{79}$$

and

$$2\varsigma\zeta^\kappa\frac{\mathrm{d}\gamma_k}{\mathrm{d}\zeta} + \varsigma\kappa\zeta^{\kappa-1}\gamma_k = -\frac{\mathrm{d}^2\beta_k}{\mathrm{d}\zeta^2} + \sum_{j=0}^k E_j\beta_{k-j} - \sum_{j=0}^{k-1}\Big(2\phi_j\frac{\mathrm{d}\gamma_{k-j-1}}{\mathrm{d}\zeta} + \phi_j'\gamma_{k-j-1}\Big) \tag{80}$$

holding for all $k \in \mathbb{N}$. Integrating these two relations yields the equivalent integral formulas $\beta_0 = 1 + c_0$,

$$\beta_{k+1}(\zeta) = \frac{1}{2}\int_\zeta^Z \Big(\frac{\mathrm{d}^2\gamma_k(\omega)}{\mathrm{d}\omega^2} - \sum_{j=0}^k E_j(\omega)\gamma_{k-j}(\omega)\Big)\,\mathrm{d}\omega + c_k, \tag{81}$$

$$\gamma_k(\zeta) = \frac{\varsigma}{2\zeta^{\kappa/2}}\Big[\int_\zeta^Z \Big(\frac{\mathrm{d}^2\beta_k(\omega)}{\mathrm{d}\omega^2} - \sum_{j=0}^k E_j(\omega)\beta_{k-j}(\omega) + \sum_{j=0}^{k-1}\Big(2\phi_j\frac{\mathrm{d}\gamma_{k-j-1}}{\mathrm{d}\zeta} + \phi_j'\gamma_{k-j-1}\Big)\Big)\frac{\mathrm{d}\omega}{\omega^{\kappa/2}} + C_k\Big], \tag{82}$$

where $c_k, C_k \in \mathbb{C}$ are arbitrary constants of integration. These constants of integration have to do with the fact that if $U = \{U(-,h)\}_{h>0}$ solves the semiclassical ODE $LU = 0$, then so does $(1 + c(h))U$ for any function $c : [0,\infty)_h \to \mathbb{C}$. Analogously, if U solves $\mathsf{LU} = 0$, then so does cU for any $\mathsf{c} \in \mathbb{C}[[h^2]]$. If U is constructed as above, then it can be checked that cU arises from a different choice of c_k. Below, we take $c_k = 0$ for all k.

The interpretation of C_k is similar but more complicated. We will choose C_k so as to minimize the index sets appearing in the $\zeta \to 0^+$ expansions of the $\gamma_k(\zeta)$. In other words, C is to be chosen such that

$$\int_\zeta^Z \Big(\frac{\mathrm{d}^2\beta_k(\omega)}{\mathrm{d}\omega^2} - \sum_{j=0}^k E_j(\omega)\beta_{k-j}(\omega) + \sum_{j=0}^{k-1}\Big(2\phi_j\frac{\mathrm{d}\gamma_{k-j-1}}{\mathrm{d}\zeta} + \phi_j'\gamma_{k-j-1}\Big)\Big)\frac{\mathrm{d}\omega}{\omega^{\kappa/2}} \tag{83}$$

has no constant term in its polyhomogeneous expansion in ζ at $\zeta = 0$.

The equations above give a recursive definition for $\mathsf{U} \in C^\infty(\mathrm{ze}^\circ)[[h]]$. By construction, $\mathsf{LU} = 0$.

4.3. **Behavior at** fe**: sketch.** We now analyze the behavior of the coefficients $\beta_k(\zeta), \gamma_k(\zeta)$ as $\zeta \to 0^+$. Since these Taylor coefficients are thought of as functions on ze, the $\zeta \to 0^+$ limit corresponds to the corner ze $\cap$ fe. Before addressing the situation in full generality, let us suppose that $\phi = 0$ and

$$E \in C^\infty([0,Z]_z \times [0,\infty)_{h^2}; \mathbb{C}). \tag{84}$$

The obstruction encountered by Olver in [Olv97, Chp. 12- §14] already comes into view, namely the fact that, if $\kappa \geq 2$, the right-hand side of eq. (82) will typically be singular as $\zeta \to 0^+$ (as long as E is not decaying too rapidly at fe), and this cannot be fixed by judiciously choosing the C_k's (unlike in the $\kappa < 2$ case). Since $\beta_0 = 1$, γ_0 is already singular, with

$$\gamma_0(\zeta) = \frac{\varsigma E_0(0)(1+o(1))}{(\kappa-2)\zeta^{\kappa-2}} \tag{85}$$

if $\kappa > 2$.

At first glance, the existence of such a singularity may seem paradoxical — since the ODE is regular singular at $\zeta = 0$ with indicial roots $0, 1$, solutions cannot be singular at $\{\zeta = 0, h > 0\}$ except for possible logarithmic terms. So, given that they are singular, how can β_k, γ_k be regarded as Taylor coefficients in h? With the appropriate goemetric perspective, there is no problem; the formal series U should only be taken seriously as a possible asymptotic expansion near ze, which is disjoint from the closure be $= \mathrm{cl}_M\{\zeta = 0, h > 0\}$ of $\{\zeta = 0, h > 0\} = \mathrm{be}^\circ$ in M. In short, irregularity at the edge fe is consistent with regularity at the edge be.

But this is merely a justification for performing *some* blowup of the relevant corner of $[0,Z]_z \times [0,\infty)_{h^2}$. Why is the blowup defining M the correct one for the problem at hand? Certainly, it is natural to hope that the blowup is the same one on which the quasimodes

$$Q(\zeta/h^{2/(\kappa+2)}), Q'(\zeta/h^{2/(\kappa+2)}) \tag{86}$$

are exponential-polyhomogeneous, but it is not obvious that this should be the case. These are quasimodes for P, and here we are constructing quasimodes for L. So, this fails to answer the question.

In order to answer it, we need to work in local coordinates. Let $\rho = h/\zeta^{(\kappa+2)/2}$ and $x = \zeta$. Then (x, ρ^2) is a valid coordinate chart near ze. (We give ζ a new name to avoid conflating the partial derivative $\partial/\partial\zeta$ in the original local coordinate system (ζ, h^2) with the partial derivative $\partial/\partial x$ in the new one.) More precisely, the map

$$(x, \rho^2) \mapsto (\zeta, h^2) = (x, \rho^2 x^{\kappa+2}) \tag{87}$$

defines a diffeomorphism $[0,Z]_x \times [0,\infty)_{\rho^2} \to M \backslash \mathrm{be} \supset \mathrm{ze}$. In terms of these new coordinates, the vector field $\partial/\partial\zeta$ can be written

$$\frac{\partial}{\partial\zeta} = \frac{\partial}{\partial x} - \frac{\kappa+2}{2}\frac{\rho}{x}\frac{\partial}{\partial\rho}. \tag{88}$$

The map

$$C^\infty(0,Z)_\zeta[[h]] \ni \mathsf{s} = \sum_{k=0}^\infty h^k s_k(\zeta) \mapsto \sum_{k=0}^\infty \rho^k x^{k(\kappa+2)/2} s_k(x) \overset{\mathrm{def}}{=} \{\mathsf{s}\} \tag{89}$$

defines an isomorphism

$$\{\cdots\} : C^\infty(0,Z]_\zeta[[h]] \to C^\infty(0,Z]_x[[\rho]] \subset C^\infty(\mathrm{ze}^\circ)[[\rho]] \tag{90}$$

of $\mathrm{Diff}[0,Z]$-modules. The result of applying the right-hand side of eq. (88) to $\rho x^{(\kappa+2)/2}$ gives 0. (This is just the statement that $\partial h/\partial\zeta = 0$, written in the new coordinate system.) Thus,

$$\frac{\partial \mathsf{s}}{\partial\zeta} = \sum_{k=0}^\infty \rho^k x^{k(\kappa+2)/2}\frac{\partial s_k(x)}{\partial x} = \Big\{\sum_{k=0}^\infty h^k \frac{\partial s_k(\zeta)}{\partial\zeta}\Big\}. \tag{91}$$

So, $\{\cdots\}$ commutes with differential operators, replacing derivatives in ζ with derivatives in x. We may therefore drop the "$\{\cdots\}$" in eq. (91) without risk of confusion, identifying elements of $C^\infty(0,Z)_\zeta[[h]]$ with the corresponding elements of $C^\infty(\mathrm{ze}^\circ)[[\rho]]$.

At a heuristic level, we can answer the question asked above by examining the orders of the singularities of $\beta_k(\zeta),\gamma_k(\zeta)$ at $\zeta=0$. We continue to assume $\phi=0$ and eq. (84) holds. Since differentiation in ζ can worsen the singularity by at most one order, and since integration mollifies the singularity by at least one order, the recurrence relations eq. (81), eq. (82) tell us that, when comparing β_{k+1},γ_{k+1} to β_k,γ_k, the order of the singularity has increased by no more than $\kappa+2$ orders. But, the former each appear in eq. (77) with an extra power of $h^2=\rho^2x^{\kappa+2}$ relative to the previous. So, in the formal series U, the worsening singularity of the coefficients is exactly canceled by the extra factors of x present in the multiplier h^2. This is what allows it to be interpreted as an asymptotic expansion at $\mathrm{ze}=\{\rho=0\}$, including the endpoint $\mathrm{ze}\cap\mathrm{fe}$, in powers of ρ. The fortuitous numerology needed to make this cancellation happen would not work with a different blowup.

One notable exception to the above heuristic is in going from $k=0$ to $k=1$, because

$$\frac{\partial^2\beta_0}{\partial^2\zeta}=0. \tag{92}$$

This means that going from β_0 to β_1 increases the order of the singularity by no more than $\kappa+1$ orders, which is an improvement of one order relative to the other k. (Depending on the behavior of E at $\zeta=0$, there may be similar improvements for more k.) Consequently, every subleading term in the formal series for β in ρ is $O(\zeta)$ relative to the leading term as $\zeta\to 0^+$, and so are subleading in that sense too. This does not imply that the expansion is a *joint* expansion, which would require that subsequent terms have further increasing decay at $\zeta=0$; the improvement noted typically stops at $O(\zeta)$. Nevertheless, even this improvement aids in understanding leading behavior at fe.

4.4. **Behavior at fe: details.** A precise version of the argument in the previous section requires keeping track of logarithmic terms (which we avoided in eq. (85), in the main term, by assuming $\kappa\geq 3$), but these complicate matters only slightly. Allowing general $\phi\neq 0$ and E does not modify the conclusions, but only because the assumptions placed on these functions in the introduction were precisely those needed for the argument here to go through.

We lay out some notation. If $\mathcal{F}\subset\mathbb{C}\times\mathbb{N}$ is an index set, then let $\partial\mathcal{F}$, $\mathcal{F}_+$ denote the index sets

$$\partial\mathcal{F}=\{(j-1+n,k):(j,k)\in\mathcal{F}\text{ and }j\neq 0,n\in\mathbb{N}\}\cup\{(j-1,k-1):(j,k)\in\mathcal{F}\text{ and }k\geq 1\} \tag{93}$$

and

$$\mathcal{F}_+=\{(j+1,k)\in\mathbb{C}\times\mathbb{N}:(j,k)\in\mathcal{F}\}\cup\{(n,k+1)\in\mathbb{N}\times\mathbb{N}:(-1,k)\in\mathcal{F}\}. \tag{94}$$

Let $\mathcal{F}_{+\mathrm{c}}=\{(n,0):n\in\mathbb{N}\}\cup\mathcal{F}_+$, which is also an index set. For any $F\in\mathcal{A}^{\mathcal{F},(0,0)}[0,Z]_\zeta$, where $\mathcal{F}$ denotes the index set at $\zeta=0$, we have $\partial F\in\mathcal{A}^{\partial\mathcal{F},(0,0)}[0,Z]_\zeta$, hence the notation. Also,

$$\int_\zeta^Z F(\omega)\,\mathrm{d}\omega\in\mathcal{A}^{\mathcal{F}_{+\mathrm{c}},(1,0)}[0,Z]_\zeta, \tag{95}$$

and, for some choice of constant of integration C,

$$C+\int_\zeta^Z F(\omega)\,\mathrm{d}\omega\in\mathcal{A}^{\mathcal{F}_+,(0,0)}[0,Z]_\zeta. \tag{96}$$

These claims can be verified by expanding $F(\zeta)$ around $\zeta=0$, differentiating or integrating each term in the expansion by hand, and then locating the error in the appropriate conormal space. Also, recall, for each $\lambda\in\mathbb{C}$, the notation $\mathcal{F}+\lambda=\{(j+\lambda,k):(j,k)\in\mathcal{F}\}=\lambda+\mathcal{F}$.

Proposition 4.1. *Except for β_0, the functions $\beta_k,\gamma_k\in C^\infty(\mathrm{ze}^\circ)$ recursively defined above satisfy $x^{k(\kappa+2)}\beta_k\in\mathcal{A}^{\mathcal{E},(0,0)}(\mathrm{ze})$ and $x^{k(\kappa+2)}\gamma_k\in\mathcal{A}^{\mathcal{F},(0,0)}(\mathrm{ze})$ for index sets $\mathcal{E}\subseteq\mathbb{N}\times\mathbb{N}$ and $\mathcal{F}\subseteq\mathbb{Z}\times\mathbb{N}$*

defined by

$$\mathcal{E} = 1 + \begin{cases} \bigcup_{j=0}^\infty ((\kappa+2)j, j) & (\kappa \notin 2\mathbb{N}), \\ \bigcup_{j=0}^\infty ((\kappa+2)j, 2j) \cup ((\kappa+2)j + \kappa/2, 2j+1) & (\kappa \in 2\mathbb{N}), \end{cases} \tag{97}$$

that is by eq. (14), and $\mathcal{F} = \mathcal{E} - \kappa - 1$. ■

As above, in $\mathcal{A}^{\mathcal{I},(0,0)}(\mathrm{ze})$, the index set at $\mathrm{ze} \cap \mathrm{fe}$ is $\mathcal{I}$ and the index set at $\mathrm{ze} \cap \mathrm{ie}$ is $(0,0)$.

Proof. Via a straightforward inductive argument using eq. (81), eq. (82), we have, for all $k \in \mathbb{N}$, $x^{k(\kappa+2)}\beta_k \in \mathcal{A}^{\mathcal{E}_k,(0,0)}(\mathrm{ze})$ and $x^{k(\kappa+2),(0,0)}\gamma_k \in \mathcal{A}^{\mathcal{F}_k}(\mathrm{ze})$ for index sets $\mathcal{E}_k, \mathcal{F}_k$ defined recursively as follows. The index sets $\mathcal{E}_0, \mathcal{F}_0$, are defined by $\mathcal{E}_0 = (0,0)$ and $\mathcal{F}_0 = (\mathcal{E}_0 - 1 - \kappa/2)_+ - \kappa/2$. That is,

$$\mathcal{F}_0 = \begin{cases} (-\kappa, 0) & (\kappa \notin 2\mathbb{N}), \\ (-\kappa, 0) \cup (-\kappa/2, 1) & (\kappa \in 2\mathbb{N}). \end{cases} \tag{98}$$

Then, $\mathcal{E}_1$ is defined by

$$\mathcal{E}_1 = ((\partial^2 \mathcal{F}_0) \cup (\mathcal{F}_0 - 1))_{+\mathrm{c}} + \kappa + 2 = \begin{cases} (1,0) & (\kappa = -1), \\ (1,0) \cup (\kappa+2, 1) & (\kappa \notin 2\mathbb{N}), \\ (1,0) \cup (\kappa/2 + 1, 1) \cup (\kappa + 2, 2) & (\kappa \in 2\mathbb{N}). \end{cases} \tag{99}$$

In order to state the definitions of $\mathcal{F}_1$ and $\mathcal{E}_k, \mathcal{F}_k$ for $k \in \mathbb{N}^{\geq 2}$, let $\mathcal{E}'_k = \mathcal{E}_k - k(\kappa+2)$ and $\mathcal{F}'_k = \mathcal{F}_k - k(\kappa+2)$. Then, for $k \in \mathbb{N}^{\geq 2}$,

$$\mathcal{E}_k = (((\partial^2 \mathcal{F}'_{k-1}) \cup (\mathcal{F}'_{k-1} - 1))_{+\mathrm{c}} + k(\kappa+2)), \tag{100}$$

and, for all $k \in \mathbb{N}^+$,

$$\mathcal{F}_k = (((\partial^2 \mathcal{E}'_k) \cup (\mathcal{E}'_k - 1) \cup (\mathcal{E}_0 - k(\kappa+2)) \cup (\mathcal{F}'_{k-1} - 3) - \kappa/2)_+ - \kappa/2 + k(\kappa+2)). \tag{101}$$

If $\kappa \notin 2\mathbb{N}$, let, for each $K \in \mathbb{N}$, $\bar{\mathcal{E}}_K = \{(j,k) \in \mathcal{E} : k \leq K\} \subseteq \mathcal{E}$ and $\bar{\mathcal{F}}_K = \{(j,k) \in \mathcal{F} : k \leq K\} \subseteq \mathcal{F}$. Otherwise, if $\kappa \in 2\mathbb{N}$, let $\bar{\mathcal{E}}_K = \{(j,k) \in \mathcal{E} : k \leq 2K\} \subseteq \mathcal{E}$ and $\bar{\mathcal{F}}_K = \{(j,k) \in \mathcal{F} : k \leq 2K+1\} \subseteq \mathcal{F}$.

We now check via induction that $\mathcal{E}_k \subseteq \bar{\mathcal{E}}_k$ for all $k \in \mathbb{N}^+$ and $\mathcal{F}_k \subseteq \bar{\mathcal{F}}_k$ for all $k \in \mathbb{N}$. Once demonstrated, this proves the proposition.

As the base case, we first observe that $\mathcal{F}_0 = \bar{\mathcal{F}}_0$, which follows immediately from the definition.

For the inductive step, let $k \in \mathbb{N}^+$, and suppose that $\mathcal{E}_j \subseteq \bar{\mathcal{E}}_{k-1}$ and $\mathcal{F}_j \subseteq \bar{\mathcal{F}}_{k-1}$ for all $j \leq k-1$. Then the recursive formula for $\mathcal{E}_k$ yields

$$\mathcal{E}_k \subseteq (\bar{\mathcal{F}}_{k-1} - (k-1)(\kappa+2) - 2)_{+\mathrm{c}} + k(\kappa+2). \tag{102}$$

The set $(\bar{\mathcal{F}}_{k-1} - (k-1)(\kappa+2) - 2)_{+\mathrm{c}} + k(\kappa+2)$ is, by definition, a union $\mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ of three terms, $\mathcal{I} = (0,0) + k(\kappa+2) = (k(\kappa+2), 0) \subseteq \bar{\mathcal{E}}_0 \subseteq \bar{\mathcal{E}}_k$,

$$\mathcal{J} = (\bar{\mathcal{F}}_{k-1} - (k-1)(\kappa+2) - 2) + 1 + k(\kappa+2) = \bar{\mathcal{F}}_{k-1} + \kappa + 1 \subseteq \bar{\mathcal{E}}_k, \tag{103}$$

and

$$\begin{aligned} \mathcal{K} &= \{(k(\kappa+2) + n, j+1) : n \in \mathbb{N}, (-1, j) \in \bar{\mathcal{F}}_{k-1} - (k-1)(\kappa+2) - 2\} \\ &= \{(k(\kappa+2) + n, j+1) : n \in \mathbb{N}, (k(\kappa+2), j) - \kappa - 1 \in \bar{\mathcal{F}}_{k-1}\}. \end{aligned} \tag{104}$$

By the definition of $\bar{\mathcal{F}}_{k-1}$, the condition $(k(\kappa+2), j) - \kappa - 1 \in \bar{\mathcal{F}}_{k-1}$ holds if and only if $j \leq k$ if $\kappa \notin 2\mathbb{N}$ and $j \leq 2k$ if $\kappa \in 2\mathbb{N}$. So,

$$\mathcal{K} = \begin{cases} (k(\kappa+2), k) & (\kappa \notin 2\mathbb{N}), \\ (k(\kappa+2), 2k) & (k \in 2\mathbb{N}). \end{cases} \tag{105}$$

So, $\mathcal{K} \subseteq \bar{\mathcal{E}}_k$. So, altogether, $\mathcal{E}_k \subseteq \bar{\mathcal{E}}_k$.

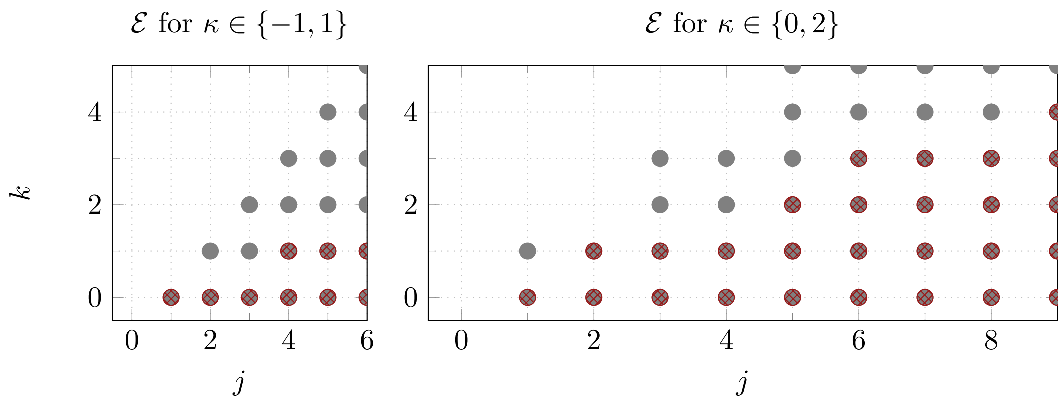

FIGURE 5. The first few points (j,k) in $\mathcal{E}$, for $\kappa = -1,0,1,2$. The cases $\kappa = -1,0$ are in dark gray, and $\kappa = 1,2$ in hatched red.

For the remainder of the inductive step, suppose that $k \in \mathbb{N}^+$, and suppose that $\mathcal{E}_k \subseteq \bar{\mathcal{E}}_k$ and $\mathcal{F}_{k-1} \subseteq \bar{\mathcal{F}}_{k-1}$. Then, the recursive formula for $\mathcal{F}_k$ yields

$$\mathcal{F}_k \subseteq ((\bar{\mathcal{E}}_k - k(\kappa+2) - 2) \cup (\bar{\mathcal{F}}_{k-1} - 3 - (k-1)(\kappa+2)) - \kappa/2)_+ - \kappa/2 + k(\kappa+2). \tag{106}$$

The right-hand side is the union of $(\bar{\mathcal{E}}_k - k(\kappa+2) - 2 - \kappa/2)_+ - \kappa/2 + k(\kappa+2)$ and $(\bar{\mathcal{F}}_{k-1} - 3 - (k-1)(\kappa+2) - \kappa/2)_+ - \kappa/2 + k(\kappa+2)$. Because $\bar{\mathcal{F}}_{k-1} \subseteq \bar{\mathcal{E}}_k - \kappa - 1$, the second of these satisfies

$$(\bar{\mathcal{F}}_{k-1} - 3 - (k-1)(\kappa+2) - \kappa/2)_+ - \kappa/2 + k(\kappa+2) \subseteq (\bar{\mathcal{E}}_k - k(\kappa+2) - 2 - \kappa/2)_+ - \kappa/2 + k(\kappa+2), \tag{107}$$

which is the other. So, in order to show that $\mathcal{F}_k \subseteq \bar{\mathcal{F}}_k$, it suffices to show that $(\bar{\mathcal{E}}_k - k(\kappa+2) - 2 - \kappa/2)_+ - \kappa/2 + k(\kappa+2) \subseteq \bar{\mathcal{F}}_k$. This set is a union $\mathcal{I} \cup \mathcal{J}$ of two terms,

$$\mathcal{I} = (\bar{\mathcal{E}}_k - k(\kappa+2) - 2 - \kappa/2 + 1) - \kappa/2 + k(\kappa+2) = \bar{\mathcal{E}}_k - \kappa - 1 \subset \bar{\mathcal{F}}_k \tag{108}$$

and

$$\begin{aligned} \mathcal{J} &= \{(-\kappa/2 + k(\kappa+2) + n, j+1) : n \in \mathbb{N}, (-1, j) \in \bar{\mathcal{E}}_k - 2 - k(\kappa+2) - \kappa/2\} \\ &= \{(-\kappa/2 + k(\kappa+2) + n, j+1) : n \in \mathbb{N}, (1 + \kappa/2 + k(\kappa+2), j) \in \bar{\mathcal{E}}_k\}. \end{aligned} \tag{109}$$

If $\kappa \notin 2\mathbb{N}$, then $\mathcal{J} = \varnothing$, so $\mathcal{J} \subseteq \bar{\mathcal{F}}_k$ trivially. Otherwise, the condition $(1 + \kappa/2 + k(\kappa+2), j) \in \bar{\mathcal{E}}_k$ is satisfied if and only if $j \leq 2k+1$, so $\mathcal{J} \subseteq \bar{\mathcal{F}}_k$. □

Via the standard asymptotic summation construction, there exist functions β, γ such that, for any $K \in \mathbb{N}$,

$$\beta - 1 - \sum_{k=1}^{K-1} h^{2k} \beta_k \in \varrho_{\mathrm{ze}}^K \mathcal{A}^{\mathcal{E}}(M), \qquad \gamma - \sum_{k=0}^{K-1} h^{2k+1} \gamma_k \in \varrho_{\mathrm{ze}}^{K+1/2} \varrho_{\mathrm{fe}}^{1+\kappa/2} \mathcal{A}^{\mathcal{F}}(M). \tag{110}$$

We may choose β, γ such that $\beta - 1, \gamma$ vanish identically near be.

Let $U = (\beta, \gamma)$. This might not solve $LU = 0$, but:

Proposition 4.2. *$LU \in \varrho_{\mathrm{be}}^{-1} \varrho_{\mathrm{ze}}^{\infty} \varrho_{\mathrm{fe}}^{\kappa} \mathcal{A}^{\mathcal{E}}(M) \oplus \varrho_{\mathrm{be}}^{\infty} \varrho_{\mathrm{ze}}^{\infty} \varrho_{\mathrm{fe}}^{\kappa/2} \mathcal{A}^{\mathcal{E}}(M)$, with the second component supported away from* be. ■

Proof. Since $\beta - 1, \gamma$ are supported away from be,

$$LU = L \begin{bmatrix} 1 \\ 0 \end{bmatrix} = h^2 E \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{111}$$

near be. The right-hand side is in $\varrho_{\mathrm{be}}^{-1} \varrho_{\mathrm{fe}}^{\kappa+1} C^\infty(M) \oplus \{0\} \subset \varrho_{\mathrm{be}}^{-1} \varrho_{\mathrm{fe}}^{\kappa} \mathcal{A}^{\mathcal{E}}(M) \oplus \{0\}$ near be.

It remains to understand the situation away from be, for which we can work in the local coordinate system (x,ρ^2) discussed in eq. (87).

Consider the components of L as weighted elements of $\mathrm{Diff}_{\mathrm{b}} = \mathrm{Diff}_{\mathrm{b}}([0,Z]_x\times[0,\infty)_{\rho^2})$, i.e. the unital algebra of differential operators generated over $C^\infty([0,Z]_x\times[0,\infty)_{\rho^2};\mathbb{C})$ by the vector fields $\rho^2\partial_{\rho^2}\propto\rho\partial_\rho$ and $x\partial_x$. Let $\mathrm{Diff}^k_{\mathrm{b}}$ denote the subset of kth order elements of $\mathrm{Diff}_{\mathrm{b}}$. Equation (88) shows that

$$h\frac{\partial}{\partial\zeta}\in\rho x^{\kappa/2}\,\mathrm{Diff}^1_{\mathrm{b}}\,. \tag{112}$$

Consequently,

$$L\in\begin{pmatrix}\rho^2x^\kappa\,\mathrm{Diff}^2_{\mathrm{b}} & \rho x^{3\kappa/2}\,\mathrm{Diff}^1_{\mathrm{b}}\\ \rho x^{\kappa/2}\,\mathrm{Diff}^1_{\mathrm{b}} & \rho^2x^\kappa\,\mathrm{Diff}^2_{\mathrm{b}}\end{pmatrix}, \tag{113}$$

meaning that the entries L_{00}, L_{01}, L_{10}, L_{11} of L are in the corresponding sets on the right-hand side.

It follows that, since $U\in(1,0)+(\varrho_{\mathrm{ze}}\mathcal{A}^{\mathcal{E}}(M))\oplus\varrho_{\mathrm{ze}}^{1/2}\varrho_{\mathrm{fe}}^{(\kappa+2)/2}\mathcal{A}^{\mathcal{F}}(M)$, LU is polyhomogeneous on M away from be, and more specifically

$$LU\in\varrho_{\mathrm{ze}}\varrho_{\mathrm{fe}}^{\kappa}\mathcal{A}^{\mathcal{E}}(M\backslash\mathrm{be})\oplus\varrho_{\mathrm{ze}}^{3/2}\varrho_{\mathrm{fe}}^{\kappa/2}\mathcal{A}^{\mathcal{E}}(M\backslash\mathrm{be}). \tag{114}$$

Thus, in order to conclude the proposition, it suffices to show that the terms in the Taylor expansion of LU at ze vanish identically. The coefficients of this asymptotic expansion are polyhomogeneous functions on ze and can therefore be identified with their restrictions to ze$^\circ$. These are just the functions β_k,γ_k above.

Consequently, it suffices to check that, for each $\zeta>0$, the Taylor expansion of $U(\zeta,h)$ at $h=0$ vanishes. By construction, this expansion is precisely $\mathsf{LU}=0$. □

Finally, we have the main result of this section:

Proposition 4.3. *There exist functions $\beta_0,\gamma_0\in\mathcal{A}^{\mathcal{E}-1}(M)$ supported away from* be *such that, defining u by*

$$u=(1+\varrho_{\mathrm{ze}}\varrho_{\mathrm{fe}}\beta_0)Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big)+\varrho_{\mathrm{ze}}^{(\kappa+1)/(\kappa+2)}\varrho_{\mathrm{fe}}\gamma_0Q'\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big). \tag{115}$$

we have $Pu=fQ+gQ'$ for $f,g\in\varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{ze}}^{\infty}\varrho_{\mathrm{fe}}^{\kappa}\mathcal{A}^{\mathcal{E}}(M)$ and g supported away from be. ■

Proof. Consider the function u defined by eq. (66), $u=\beta Q+h^{\kappa/(\kappa+2)}\gamma Q'$, with β,γ as above. By Proposition 4.2, this satisfies

$$Pu=\begin{bmatrix}Q\\ h^{\kappa/(\kappa+2)}Q'\end{bmatrix}^{\intercal}LU\in\varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{ze}}^{\infty}\varrho_{\mathrm{fe}}^{\kappa/2}\begin{bmatrix}Q\\ h^{\kappa/(\kappa+2)}Q'\end{bmatrix}^{\intercal}\begin{bmatrix}\varrho_{\mathrm{fe}}^{\kappa/2}\mathcal{A}^{\mathcal{E}}(M)\\ \mathcal{A}^{\mathcal{E}}(M)\end{bmatrix}. \tag{116}$$

Writing out the inner product,

$$\begin{bmatrix}Q\\ h^{\kappa/(\kappa+2)}Q'\end{bmatrix}^{\intercal}\begin{bmatrix}\varrho_{\mathrm{fe}}^{\kappa/2}\mathcal{A}^{\mathcal{E}}(M)\\ \mathcal{A}^{\mathcal{E}}(M)\end{bmatrix}=Q\varrho_{\mathrm{fe}}^{\kappa/2}\mathcal{A}^{\mathcal{E}}(M)+h^{\kappa/(\kappa+2)}Q'\mathcal{A}^{\mathcal{E}}(M), \tag{117}$$

and $h^{\kappa/(\kappa+2)}\mathcal{A}^{\mathcal{E}}(M)=\varrho_{\mathrm{ze}}^{\kappa/(2\kappa+4)}\varrho_{\mathrm{fe}}^{\kappa/2}\mathcal{A}^{\mathcal{E}}(M)$. So $Pu=fQ+gQ'$ for some $f,g\in\varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{ze}}^{\infty}\varrho_{\mathrm{fe}}^{\kappa}\mathcal{A}^{\mathcal{E}}(M)$.

We just need to verify that β,γ can be written in terms of β_0,γ_0 of the desired form. Indeed, the function β_0 defined by $\beta_0=\varrho_{\mathrm{ze}}^{-1}\varrho_{\mathrm{fe}}^{-1}(\beta-1)$ is in $\mathcal{A}^{\mathcal{E}-1}(M)$ by construction, and a short calculation gives

$$h^{\kappa/(\kappa+2)}\gamma\in h^{(2\kappa+2)/(\kappa+2)}\mathcal{A}^{\mathcal{F}}(M)=\varrho_{\mathrm{ze}}^{(\kappa+1)/(\kappa+2)}\varrho_{\mathrm{fe}}\mathcal{A}^{\mathcal{E}-1}(M). \tag{118}$$

Since Proposition 4.2 says that the second component of LU is supported away from be, the same holds for g. Also, since $\beta-1$ and γ are supported away from be, the same applies to β_0,γ_0. □

5. Analysis at the high corner

We now turn to solving, modulo errors which are small relative to Q, the forced ODE $Pu = fQ + gQ'$ near fe. Our assumption on f, g, coming from the previous section, is that

$$f, g \in \varrho_{\mathrm{ze}}^{\infty}\varrho_{\mathrm{fe}}^{\kappa}\mathcal{A}^{\mathcal{E}}(M) = \bigcap_{k=0}^{\infty} \varrho_{\mathrm{ze}}^{k}\varrho_{\mathrm{fe}}^{\kappa}\mathcal{A}^{\mathcal{E}}(M). \tag{119}$$

The key idea is that, as discussed earlier, P is well approximated by $N(P)$ near fe, so an approximate solution can be constructed by asymptotically summing a formal polyhomogeneous series in the bdf of fe. This yields $O(h^\infty)$ errors relative to Q.

It will be most convenient to use the coordinates $\lambda = \zeta/h^{2/(\kappa+2)}$ and $\varrho = h^{2/(\kappa+2)}$. For any $h_0 > 0, \lambda_0 > 0$, let

$$V_{h_0,\lambda_0} = \{(\zeta, h) \in (0, Z) \times (0, h_0) : \zeta > \lambda_0 h^{2/(\kappa+2)}\}, \tag{120}$$

$$U_{h_0,\lambda_0} = \{(\zeta, h) \in (0, Z) \times (0, h_0) : \zeta < \lambda_0 h^{2/(\kappa+2)}\}. \tag{121}$$

These are open subsets of $[0, Z]_\zeta \times [0, \infty)_h$. The subsets $\bar{V}_{h_0,\lambda_0} = (\mathrm{cl}_M V_{h_0,\lambda_0})^\circ \subset M$ and $\bar{U}_{h_0,\lambda_0} = (\mathrm{cl}_M U_{h_0,\lambda_0})^\circ \subset M$ are relatively open subsets of M containing some points of the boundary. (Here, $\circ$ denotes the topological interior as taken in M.)

The set $\bar{U}_{h_0,\lambda_0}$ is a neighborhood of the "low corner" $\mathrm{fe} \cap \mathrm{be}$, and $\bar{V}_{h_0,\lambda_0}$ is a neighborhood of the "high corner" $\mathrm{ze} \cap \mathrm{fe}$. If $\lambda_1 > \lambda_0$, then $V_{h_0,\lambda_0} \cup U_{h_0,\lambda_1} \supset \mathrm{fe}$.

In this section our concern is the situation in V_{h_0,λ_0}, and U_{h_0,λ_0} is considered later.

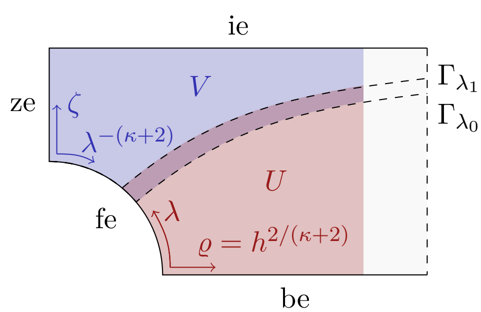

FIGURE 6. The neighborhoods $U = U_{h_0,\lambda_1}$ and $V = V_{h_0,\lambda_0}$ for some $h_0 > 0$ and $\lambda_1 > \lambda_0 > 0$. Note that U, V contain points of ∂M.

Recall that, in the coordinate system (λ, ϱ), P can be considered as a family of ordinary differential operators on $\mathrm{fe}^\circ = \mathbb{R}^+_\lambda$, and $\varrho^{-\kappa}P, \varrho^{-\kappa}N(P)$ have smooth coefficients all the way down to $\varrho = 0$. Let

$$\hat{N}(P) = \varrho^{-\kappa}N(P) \in \mathrm{Diff}^2(\mathrm{fe}^\circ). \tag{122}$$

To leading order at fe, P is indeed given by $N(P)$, as $\varrho^{-\kappa}P - \hat{N}(P) = \varrho^2 E$ is lower order, both with respect to the usual differential sense and with respect to decay as $\varrho \to 0^+$. The latter of these is not immediately obvious, as $E \in \zeta^{-1}C^\infty(M; \mathbb{C})$ can be singular at fe and be, but this just means that $E = \varrho^{-1}\hat{E}$ for some

$$\hat{E} \in \lambda^{-1}C^\infty([0, \infty)_\lambda \times [0, \infty)_\varrho; \mathbb{C}). \tag{123}$$

So, $\varrho^2 E = \varrho\hat{E} = O(\varrho)$ is still decaying as $\varrho \to 0^+$, at least for $0 \ll \lambda \ll \infty$.

Consider the Taylor series

$$E \sim \sum_{k=-1}^{\infty} \varrho^k E_k(\lambda) = \sum_{k=-1}^{\infty} \zeta^k \lambda^{-k} E_k(\lambda) \tag{124}$$

of E at fe. Accordingly, we have the following formal version of $\varrho^{-\kappa}P$:

$$\mathsf{P} = \hat{N}(P) + \sum_{k=-1}^{\infty} \varrho^{k+2} E_k(\lambda) \in \mathrm{Diff}^2(\mathrm{fe}^\circ)[[\varrho]], \tag{125}$$

the coefficients of which are elements of $\mathrm{Diff}^2(\mathrm{fe}^\circ)$.

From $E \in \zeta^{-1}C^\infty(M;\mathbb{C})$ and the fact that the second series in eq. (124) is the Taylor series of E at fe near $\mathrm{ze}\cap\mathrm{fe}$ when ζ is taken as a local bdf of fe near $\mathrm{ze}\cap\mathrm{fe}$, it follows that $E_k(\rho^{-2/(\kappa+2)}) \in \rho^{-2k/(\kappa+2)}C^\infty[0,\infty)_{\rho^2}$.

The main result of this subsection is:

Proposition 5.1. *Let $\mathcal{E}$ denote an index set, and let $h_0,\lambda_0>0$, thus supported away from* be*, $\lambda_1>\lambda_0$, and $\Lambda>\lambda_1$. Given $Q\in\mathcal{Q}$ and $f,g\in\varrho_{\mathrm{ze}}^\infty\varrho_{\mathrm{fe}}^\kappa\mathcal{A}^{\mathcal{E}}(M)$ satisfying* $\operatorname{supp} f, \operatorname{supp} g \Subset \bar{V}_{h_0,\lambda_1}$*, there exist*

- *$\beta,\gamma\in\varrho_{\mathrm{ze}}^\infty\mathcal{A}^{\mathcal{E}}(M)$ satisfying* $\operatorname{supp}\beta,\operatorname{supp}\gamma\Subset\bar{V}_{h_0,\lambda_0}\backslash\mathrm{ie}$*, thus supported away from* $\mathrm{ie}\cup\mathrm{be}$*,*
- *$R\in\varrho_{\mathrm{fe}}^\kappa\mathcal{A}^{\mathcal{E}}(M)$ satisfying*
$$\operatorname{supp} R\Subset\bar{V}_{h_0,\lambda_0}\cap\bar{U}_{h_0,\Lambda}, \tag{126}$$
thus supported away from $\mathrm{ie}\cup\mathrm{ze}\cup\mathrm{be}$*, and*
- *$f_0,g_0\in\varrho_{\mathrm{ze}}^\infty\varrho_{\mathrm{fe}}^\infty C^\infty(M;\mathbb{C})=h^\infty C^\infty(M;\mathbb{C})$ satisfying* $\operatorname{supp} f_0,\operatorname{supp} g_0\Subset\bar{V}_{h_0,\lambda_1}$*, thus supported away from* be*,*

such that the function $u=\beta Q+\gamma Q'$ solves $Pu=(f+f_0)Q+(g+g_0)Q'+R$. ■

Thus, given $O(\varrho_{\mathrm{ze}}^\infty)$ forcing relative to Q, we can solve the ODE modulo $O(h^\infty)$ errors (relative to Q) near $\mathrm{ze}\cap\mathrm{fe}$.

Proof. If $Q=0$, then the claim holds trivially, so assume otherwise. We construct u order-by-order. By linearity, we may assume without loss of generality that one of f,g is identically 0. Both cases are similar, so we focus on the case where $g=0$.

Expand f at fe as

$$\varrho^{-\kappa}f\sim\sum_{(j,k)\in\mathcal{E}}^{\infty} f_{j,k}\varrho^j\log^k\varrho \tag{127}$$

where $f_{j,k}\in\varrho_{\mathrm{ze}}^\infty C^\infty(\mathrm{fe}\backslash\mathrm{be})\subset\mathcal{S}(\mathbb{R}_\lambda)$. Necessarily, $\operatorname{supp} f_{j,k}\cap[0,\lambda_0]=\varnothing$. Let f denote the formal series on the right-hand side of eq. (127).

Fix $\lambda_{1/2}\in(\lambda_0,\lambda_1)$, $\lambda_2\in(\lambda_1,\Lambda)$, and $\lambda_3\in(\lambda_2,\Lambda)$. Suppose that we have formal polyhomogeneous series

$$\mathsf{b}=\sum_{(j,k)\in\mathcal{E}} b_{j,k}\varrho^j\log^k\varrho,\qquad \mathsf{c}=\sum_{(j,k)\in\mathcal{E}} c_{j,k}\varrho^j\log^k\varrho \tag{128}$$

with coefficients $b_{j,k},c_{j,k}\in\mathcal{S}(\mathbb{R}_\lambda)\cap C_{\mathrm{c}}^\infty(\lambda_2,\infty]$ such that, setting $\mathsf{u}=\mathsf{b}Q(\lambda)+\mathsf{c}Q'(\lambda)$, there exists some formal series

$$\mathsf{R}=\sum_{(j,k)\in\mathcal{E}} R_{j,k}(\lambda)\varrho^j\log^k\varrho \tag{129}$$

with coefficients $R_{j,k}\in C_{\mathrm{c}}^\infty(\lambda_{1/2},\lambda_3)$, such that

$$\mathsf{P}(\mathsf{b}Q(\lambda)+\mathsf{c}Q'(\lambda))=\mathsf{R}+\mathsf{f}Q(\lambda) \tag{130}$$

holds as an identity of formal polyhomogeneous series. Then, b,c may be asymptotically summed to yield $\beta\in\varrho_{\mathrm{ze}}^\infty\mathcal{A}^{\mathcal{E}}(M)$ and $\gamma\in\varrho_{\mathrm{ze}}^\infty\mathcal{A}^{\mathcal{E}}(M)$ whose polyhomogeneous expansions at fe are given by b,c respectively, and these can be chosen to be supported in $\bar{V}_{h_0,\lambda_0}\backslash\mathrm{ie}$. Indeed, we necessarily have that β,γ are supported there modulo an $O(h^\infty)$ error, which can be subtracted off. Likewise, R may be asymptotically summed to yield $R\in\varrho_{\mathrm{fe}}^\kappa\mathcal{A}^{\mathcal{E}}(M)$ whose expansion at fe is given by $\varrho^\kappa\mathsf{R}$, and R can be chosen so as to satisfy the support condition eq. (126).

Given β,γ as in the previous paragraph, let $u=\beta Q+\gamma Q'$. By construction, the polyhomogeneous expansion of $\varrho^{-\kappa}Pu$ at fe° is given by $\mathsf{R}+\mathsf{f}Q(\lambda)$, and the error $R_0=Pu-fQ-R$ satisfies

$$R_0\in\varrho_{\mathrm{ze}}^\infty\varrho_{\mathrm{fe}}^\infty\operatorname{Mod}_Q^{1/2}C_{\mathrm{c}}^\infty(\bar V_{h_0,\lambda_0};\mathbb{C})\subset\varrho_{\mathrm{ze}}^\infty\varrho_{\mathrm{fe}}^\infty\varrho_{\mathrm{be}}^\infty\operatorname{Mod}_Q^{1/2}C^\infty(M;\mathbb{C}), \tag{131}$$

where $\operatorname{Mod}_Q(\lambda)$ is the *modulus function* defined in the appendix, by eq. (224).

Letting $f_0=R_0Q^*/(|Q|^2+|Q'|^2)$ and $g_0=R_0(Q')^*/(|Q|^2+|Q'|^2)$, $f_0,g_0\in\varrho_{\mathrm{ze}}^\infty\varrho_{\mathrm{fe}}^\infty C_{\mathrm{c}}^\infty(\bar V_{h_0,\lambda_0};\mathbb{C})$ holds, as does the desired identity $Pu=(f+f_0)Q+(g+g_0)Q'+R$.

To summarize, we have shown that it suffices to construct formal series $\mathsf{b},\mathsf{c},\mathsf{R}$ with the desired properties. Such formal series can be constructed recursively:

$$\begin{aligned}\mathsf{P}(\mathsf{b}Q(\lambda)+\mathsf{c}Q'(\lambda))=\sum_{(j,k)\in\mathcal{E}}\varrho^j\log\varrho^k\Big[&\hat N(P)(b_{j,k}Q+c_{j,k}Q')\\&+\sum_{(j',k)\in\mathcal{E}\text{ and }j'<j}E_{j-j'-2}(\lambda)(b_{j',k}Q+c_{j',k}Q')\Big].\end{aligned} \tag{132}$$

Set

$$v_{j,k}=\hat N(P)^{-1}[Q,Q_0]\Big[f_{j,k}Q-\sum_{(j',k)\in\mathcal{E}\text{ and }j'<j}E_{j-j'-2}(\lambda)(b_{j',k}Q+c_{j',k}Q')\Big], \tag{133}$$

where, for arbitrary $Q_0\in\mathcal{Q}$ linearly independent with Q, $\hat N(P)^{-1}[Q,Q_0]$ is the right-inverse to $\hat N(P)$ constructed in §3.3. By Proposition 3.6, $v_{j,k}$ can be modified in a compact subset so that it can be written as $b_{j,k}Q+c_{j,k}Q'$ for $b_{j,k},c_{j,k}\in\mathcal{S}(\mathbb{R}_\lambda)\cap C_{\mathrm{c}}^\infty(\lambda_2,\infty]$ satisfying

$$\hat N(P)v_{j,k}=R_{j,k}+f_{j,k}Q-\sum_{(j',k)\in\mathcal{E}\text{ and }j'<j}E_{j-j'-2}(\lambda)(b_{j',k}Q+c_{j',k}Q') \tag{134}$$

for $f_{j,k}\in C_{\mathrm{c}}^\infty(\lambda_{1/2},\lambda_2)$ and $R_{j,k}\in C_{\mathrm{c}}^\infty(\lambda_{1/2},\lambda_3)$. By construction, $\mathsf{P}(\mathsf{b}Q(\lambda)+\mathsf{c}Q'(\lambda))=\mathsf{R}+\mathsf{f}Q(\lambda)$ is satisfied. □

6. $O(h^\infty)$ ERROR ANALYSIS

In this section, we study, away from be, the forced ODE $Pu=fQ+gQ'$ for $f,g\in\varrho_{\mathrm{fe}}^\infty\varrho_{\mathrm{ze}}^\infty C^\infty(M)=h^\infty C^\infty([0,Z]_\zeta\times[0,\infty)_h;\mathbb{C})$ supported away from be. The goal is to construct an *exact* solution which is $O(h^\infty)$ relative to the (possibly exponentially growing) quasimode Q. Since we are concerned with the situation away from be, it is acceptable to produce errors away from $\mathrm{ie}\cup\mathrm{ze}$. We restrict attention to $h\in[0,h_0)$ for some $h_0>0$ which can be taken as small as desired.

If $\varsigma<0$, i.e. we are in the classically allowed case, then the forcing $F=fQ+gQ'$ satisfies $F\in h^\infty C^\infty([0,Z]_\zeta\times[0,\infty)_h;\mathbb{C})$ as well. However, when $\varsigma>0$, i.e. we are in the classically forbidden case, then it is critical to keep track of the exponential weights at ze present in the quasimodes and their derivatives, as these can outweigh any $O(h^\infty)$ multiplier. One way of accomplishing this is to work with the modulus function

$$\operatorname{Mod}_Q(\zeta,h)=\operatorname{Mod}_Q(\zeta/h^{2/(\kappa+2)}), \tag{135}$$

where $\operatorname{Mod}_Q(\lambda)$ is defined by eq. (224), except, in order to avoid conflicting notation, replace χ in eq. (224) by χ_0, where $\chi_0\in C_{\mathrm{c}}^\infty(\mathbb{R};[0,1])$ is identically 1 in some neighborhood of the origin. We utilize the weighted Banach spaces

$$\operatorname{Mod}_Q^\nu L^\infty=\operatorname{Mod}_Q(\zeta,h)^\nu L^\infty(V_{h_0,\lambda_0};\mathbb{C}). \tag{136}$$

In addition, let

$$h^\infty\operatorname{Mod}_Q^\nu L^\infty=\bigcap_{k\in\mathbb{N}}h^k\operatorname{Mod}_Q^\nu L^\infty. \tag{137}$$

When $\nu=0$, the "Mod_Q^0" is omitted from all notation. The two main properties of Mod_Q used (mostly without further comment) here are Proposition A.2 and Proposition A.3.

The main result of this section is the following:

Proposition 6.1. *Fix $Q \in \mathcal{Q}$. Given $f, g \in h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h;\mathbb{C})$ satisfying $(\operatorname{supp} f \cup \operatorname{supp} g) \cap \mathrm{be} = \varnothing$, there exist β, γ with the following four properties:*

(1) $\beta, \gamma \in h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h;\mathbb{C})$
(2) $\operatorname{supp}_M \beta, \operatorname{supp}_M \gamma$ are disjoint from be, and
(3) the function $u = \beta Q + \gamma Q'$ solves $Pu = fQ + gQ' + w$ for $w \in h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h;\mathbb{C})$ such that $\operatorname{supp}_M w$ is disjoint from $\mathrm{ze} \cup \mathrm{be}$.

Here, $\operatorname{supp}_M F$ is the closure in M of $\{p \in M^\circ : F(p) \neq 0\}$. ■

Proof. The claim is trivially true if $Q = 0$, so assume otherwise.

We choose a basis A, B for $\mathcal{Q}$. If $\varsigma > 0$, choose $A = Q_\infty$ to be the exponentially decaying mode and B to be any other element of $\mathcal{Q}$ such that $\operatorname{span}_\mathbb{C}\{A, B\} = \mathcal{Q}$. If $\varsigma < 0$, choose instead $A = Q_-$ and $B = Q_+$. Fix $\chi \in C_\mathrm{c}^\infty(M;[0,1])$ identically 1 near ze and vanishing near be, and suppose that $\bar\chi \in C_\mathrm{c}^\infty(M;[0,1])$ is identically 1 near ze and identically vanishing on $\operatorname{supp}(1-\chi)$.

As an ansatz for u, take

$$u = \aleph \bar\chi A\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + \beth \bar\chi B\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) \tag{138}$$

for to-be-constructed $\aleph, \beth \in C^\infty((0,Z]_\zeta \times (0,\infty)_h;\mathbb{C})$. Below, we will abbreviate $A(\zeta/h^{2/(\kappa+2)})$ and $B(\zeta/h^{2/(\kappa+2)})$ as A, B.

We first address sufficient conditions on $(\aleph, \beth)$ for such a u to be expressable as $\beta Q + \gamma Q'$ for β, γ satisfying properties (1) and (2). Define

$$\beta = \operatorname{Mod}_Q(\zeta, h)^{-1} Q(\zeta/h^{2/(\kappa+2)})^* u, \tag{139}$$

$$\gamma = \operatorname{Mod}_Q(\zeta,h)^{-1} \zeta^2 h^{-(\kappa+2)} \langle h^{-1}\chi_0(h^{2/(\kappa+2)}/\zeta)\zeta^{(\kappa+2)/2}\rangle^{-2} Q'(\zeta/h^{2/(\kappa+2)})^* u. \tag{140}$$

Then, $\beta, \gamma \in C^\infty((0,Z)_\zeta \times (0,\infty)_h;\mathbb{C})$ and $u = \beta Q + \gamma Q'$. The presence of the cutoff $\bar\chi$ in u means that property (2) is satisfied automatically.

In order for property (1) to be satisfied, the $\aleph, \beth$ are to be constructed satisfying the following estimates:

- if $\varsigma < 0$ or $Q \notin \operatorname{span}_\mathbb{C} Q_\infty$, then, for every $j, k \in \mathbb{N}$,

$$\frac{\partial^j \partial^k (\chi\aleph)}{\partial h^j \partial \zeta^k} \in h^\infty \operatorname{Mod}_B L^\infty, \qquad \frac{\partial^j \partial^k (\chi\beth)}{\partial h^j \partial \zeta^k} \in h^\infty L^\infty. \tag{141}$$

- If $\varsigma > 0$ and $Q \in \operatorname{span}_\mathbb{C} Q_\infty \backslash \{0\}$, then

$$\frac{\partial^j \partial^k (\chi\aleph)}{\partial h^j \partial \zeta^k} \in h^\infty L^\infty, \qquad \frac{\partial^j \partial^k (\chi\beth)}{\partial h^j \partial \zeta^k} \in h^\infty \operatorname{Mod}_A L^\infty. \tag{142}$$

We now check that these suffice to conclude property (1). If $\varsigma < 0$ or $Q \notin \operatorname{span}_\mathbb{C} Q_\infty$, then this computation is recorded in Lemma 6.2 below. Indeed, β, γ are a linear combination of functions of the form discussed in the lemma with polyhomogeneous coefficients on M that are smooth at ie. Since the product of a polyhomogeneous function on M smooth at ie with an element of

$$h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h;\mathbb{C}) = \varrho_{\mathrm{fe}}^\infty \varrho_{\mathrm{ze}}^\infty C^\infty(M) \tag{143}$$

supported away from be results in an element of $h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h;\mathbb{C})$, it can be concluded that (1) holds. If $\varsigma > 0$ and $Q \in \operatorname{span}_\mathbb{C} Q_\infty$, then we instead appeal to Lemma 6.3, and the desired conclusion follows exactly as before.

In order to satisfy property (3), it suffices to arrange that the function $u_0 = \aleph A + \beth B$ satisfies $Pu = fQ + gQ'$. The property then holds with

$$w = -(1-\bar\chi)(fQ + gQ') + [P, \bar\chi]u_0. \tag{144}$$

As follows from writing P in the coordinate system (λ, ϱ), the result of applying $[P, \bar{\chi}]$ to to u_0 lies in $h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h; \mathbb{C})$, and since $[P,\bar{\chi}]$ is vanishing near $\mathrm{ze} \cup \mathrm{be}$, the support of w is disjoint from $\mathrm{ze}\cup\mathrm{be}$ as well.

To summarize, in order to conclude the proposition it suffices to construct $\aleph, \beth \in C^\infty((0,Z]_\zeta \times (0,\infty)_h; \mathbb{C})$ with the following two properties:

- $\aleph, \beth$ satisfy the estimates above and
- $u_0 = \aleph A + \beth B$ satisfies $Pu = fQ + gQ'$.

The $\aleph, \beth$ are constructed by Proposition 6.4 below. The proposition states that the second item holds. So, we only need to confirm that $\aleph, \beth$ satisfy the desired estimates. This is given by the conjunction of the lemmas below. Indeed, Proposition 6.5 shows that it suffices to prove the $k = 0$ cases of the desired estimates (with a slightly enlarged cutoff), deducing the others. The propositions Proposition 6.8 and Proposition 6.10 provide the required $k = 0$ result, at least if χ is supported in $\mathrm{cl}_M\{\zeta \geq h^{2/(\kappa+2)}\lambda_0\}$ for sufficiently large λ_0, which can be arranged. □

We now fill in the required lemmas.

Lemma 6.2. *Suppose that $\aleph, \beth \in C^\infty((0,Z)_\zeta \times (0,\infty)_h; \mathbb{C})$ satisfy eq. (141) for each $j, k \in \mathbb{N}$. Then, unless $\varsigma > 0$ and $Q \in \operatorname{span}_{\mathbb{C}} Q_\infty$, each*

$$\mathcal{R} \in \Big\{ \frac{\aleph \bar{\chi} Q^* A}{\mathrm{Mod}_Q(\zeta,h)}, \frac{\beth \bar{\chi} Q^* B}{\mathrm{Mod}_Q(\zeta,h)}, \frac{\aleph \bar{\chi} (Q')^* A}{\mathrm{Mod}_Q(\zeta,h)}, \frac{\beth \bar{\chi} (Q')^* B}{\mathrm{Mod}_Q(\zeta,h)} \Big\} \tag{145}$$

satisfies $\mathcal{R} \in h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h; \mathbb{C})$. ■

Proof. We first demonstrate the proof for $\mathcal{R} = \mathrm{Mod}_Q(\zeta,h)^{-1} \aleph \bar{\chi} Q^* A$. For any $j,k \in \mathbb{N}$, $\partial_h^j \partial_\zeta^k \mathcal{R}$ is a linear combination of

$$\frac{\partial^{j_0}\partial^{k_0}(\chi\aleph)}{\partial h^{j_0}\partial\zeta^{k_0}} \frac{\partial^{j_1}\partial^{k_1}\bar{\chi}}{\partial h^{j_1}\partial\zeta^{k_1}} \frac{\partial^{j_2}\partial^{k_2}Q^*}{\partial h^{j_2}\partial\zeta^{k_2}} \frac{\partial^{j_3}\partial^{k_3}A}{\partial h^{j_3}\partial\zeta^{k_3}} \frac{\partial^{j_4}\partial^{k_4}}{\partial h^{j_4}\partial\zeta^{k_4}} \frac{1}{\mathrm{Mod}_Q} \tag{146}$$

for $j_0, j_1, j_2, j_3, j_4, k_0, k_1, k_2, k_3, k_4 \in \mathbb{N}$ with $j_0 + j_1 + j_2 + j_3 + j_4 = j$ and $k_0 + k_1 + k_2 + k_3 + k_4 = k$. By assumption, the first term in eq. (146) is in $h^\infty \mathrm{Mod}_B L^\infty$. Multiplying the second term by some large power of h, the result is in $\varrho_{\mathrm{be}}^\infty L^\infty$. (The reason is that $\partial_h, \partial_\zeta$ are powers of boundary-defining-functions times smooth vector fields on M.) Multiplying the third term by some large power of $\varrho_{\mathrm{be}} h$, the third term is in $\mathrm{Mod}_B^{1/2} L^\infty$, per Proposition A.2. Likewise, multiplying the fourth term by some large power of $\varrho_{\mathrm{be}} h$, it is in $\mathrm{Mod}_A^{1/2} L^\infty$. Finally, using Proposition A.2 again, computing the higher derivatives of $\mathrm{Mod}_Q(\lambda)^{-1}$ yields

$$\varrho_{\mathrm{be}}^K h^K \frac{\partial^{j_4}\partial^{k_4}}{\partial h^{j_4}\partial\zeta^{k_4}} \frac{1}{\mathrm{Mod}_Q} \in \mathrm{Mod}_B^{-1} L^\infty \tag{147}$$

for some $K \in \mathbb{N}$. So, all in all, $\partial_h^j \partial_\zeta^k \mathcal{R} \in h^\infty L^\infty$. We can then conclude $\mathcal{R} \in h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h; \mathbb{C})$.

Consider now the case $\mathcal{R} = \mathrm{Mod}_Q(\zeta,h)^{-1} \beth \bar{\chi} Q^* B$. For any $j,k \in \mathbb{N}$, $\partial_h^j \partial_\zeta^k \mathcal{R}$ is a linear combination of

$$\frac{\partial^{j_0}\partial^{k_0}(\chi\beth)}{\partial h^{j_0}\partial\zeta^{k_0}} \frac{\partial^{j_1}\partial^{k_1}\bar{\chi}}{\partial h^{j_1}\partial\zeta^{k_1}} \frac{\partial^{j_2}\partial^{k_2}Q^*}{\partial h^{j_2}\partial\zeta^{k_2}} \frac{\partial^{j_3}\partial^{k_3}B}{\partial h^{j_3}\partial\zeta^{k_3}} \frac{\partial^{j_4}\partial^{k_4}}{\partial h^{j_4}\partial\zeta^{k_4}} \frac{1}{\mathrm{Mod}_Q} \tag{148}$$

for $j_0, j_1, j_2, j_3, j_4, k_0, k_1, k_2, k_3, k_4 \in \mathbb{N}$ as above. By assumption, the first term in eq. (148) is in $h^\infty L^\infty$. The second and third were understood above. Multiplying the fourth term by some large power of $\varrho_{\mathrm{be}} h$, the fourth term is in $\mathrm{Mod}_B^{1/2} L^\infty$. The fifth term was also understood above. So, all in all, $\partial_h^j \partial_\zeta^k \mathcal{R} \in h^\infty L^\infty$. We can then conclude $\mathcal{R} \in h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h; \mathbb{C})$.

Each of the remaining two cases are similar to one of the previous two. □

Lemma 6.3. *Suppose that $\varsigma > 0$ and $Q \in \operatorname{span}_{\mathbb{C}} Q_\infty \backslash \{0\}$, and suppose that $\aleph, \beth \in C^\infty((0,Z)_\zeta \times (0,\infty)_h; \mathbb{C})$ satisfy eq. (142) for each $j,k \in \mathbb{N}$. Then, the $\mathcal{R}$ as in eq. (145) all satisfy $\mathcal{R} \in h^\infty C^\infty([0,Z]_\zeta \times [0,\infty)_h; \mathbb{C})$.* ■□

The proof is completely analogous to that of Lemma 6.2, so it is omitted.

Proposition 6.4 (Variation of parameters)**.** *Let $F \in C^\infty((0,Z]_\zeta \times (0,\infty)_h; \mathbb{C})$, $\zeta_0 \in C^\infty(\mathbb{R}^+_h; (0,Z])$. There exist unique $\aleph, \beth \in C^\infty((0,Z]_\zeta \times (0,\infty)_h; \mathbb{C})$ vanishing at the graph $\Gamma(\zeta_0) = \{(\zeta_0(h), h) : h \in \mathbb{R}^+\} \subset (0,Z]_\zeta \times (0,\infty)_h$ such that, firstly,*

$$\frac{\partial \aleph}{\partial \zeta} A + \frac{\partial \beth}{\partial \zeta} B = 0 \tag{149}$$

and, secondly, the function u defined by $u = \aleph A + \beth B$ satisfies the forced ODE $Pu = h^2 F$. These $\aleph, \beth$ solve

$$\frac{\partial}{\partial \zeta} \begin{bmatrix} \aleph \\ \beth \end{bmatrix} = \frac{h^{2/(\kappa+2)}}{\mathfrak{W}} \Big(E \begin{bmatrix} -AB & -B^2 \\ A^2 & AB \end{bmatrix} \begin{bmatrix} \aleph \\ \beth \end{bmatrix} + F \begin{bmatrix} B \\ -A \end{bmatrix} \Big), \tag{150}$$

where $\mathfrak{W}$ is the Wronskian $\mathfrak{W} = A(\lambda)B'(\lambda) - A'(\lambda)B(\lambda)$, which is independent of λ and therefore just a constant. Moreover, $\aleph, \beth$ depend linearly on F. ■

Proof. Assuming that $\aleph, \beth \in C^\infty((0,Z]_\zeta \times (0,\infty)_h; \mathbb{C})$ satisfy eq. (149), the function u defined by $u = \aleph A + \beth B$ satisfies $Pu = h^2 F$ if and only if

$$\frac{1}{h^{2/(\kappa+2)}} \Big(\frac{\partial \aleph}{\partial \zeta} A' + \frac{\partial \beth}{\partial \zeta} B' \Big) = (\aleph A + \beth B) E - F. \tag{151}$$

Combining eq. (149), eq. (151) into a single system of ODEs, the result is

$$\begin{bmatrix} A & B \\ A' & B' \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} \aleph \\ \beth \end{bmatrix} = h^{2/(\kappa+2)} \Big(E \begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix} \begin{bmatrix} \aleph \\ \beth \end{bmatrix} - F \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big). \tag{152}$$

Inverting the matrix on the left-hand side,

$$\begin{bmatrix} A & B \\ A' & B' \end{bmatrix}^{-1} = \frac{1}{\mathfrak{W}} \begin{bmatrix} B' & -B \\ -A' & A \end{bmatrix}. \tag{153}$$

Thus, eq. (152) is equivalent to eq. (150). For each $h > 0$, this forced ODE has a unique solution $(\aleph(-,h), \beth(-,h)) \in C^\infty((0,Z]_\zeta; \mathbb{C}^2)$ vanishing at $\zeta_0(h)$, as follows from the theory of linear ODE with smooth dependence on parameters. This solution depends linearly on F. □

Proposition 6.5. *Suppose that $\aleph, \beth \in C^\infty((0,Z]_\zeta \times (0,\infty)_h; \mathbb{C})$ satisfy eq. (150). Let $\chi_0 \in C^\infty(M)$ be identically 1 on $\operatorname{supp} \chi$ and vanishing near* be. *If $\varsigma < 0$ or $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$ and*

$$\frac{\partial^j (\chi_0 \beth)}{\partial h^j} \in h^\infty L^\infty, \qquad \frac{\partial^j (\chi_0 \aleph)}{\partial h^j} \in h^\infty \operatorname{Mod}_Q L^\infty. \tag{154}$$

holds for all $j \in \mathbb{N}$, then eq. (141) holds for each $j,k \in \mathbb{N}$. Similarly, if $\varsigma > 0$ and $Q \in \operatorname{span}_{\mathbb{C}} Q_\infty \backslash \{0\}$ and

$$\frac{\partial^j (\chi_0 \beth)}{\partial h^j} \in h^\infty \operatorname{Mod}_{Q_\infty} L^\infty, \qquad \frac{\partial^j (\chi_0 \aleph)}{\partial h^j} \in h^\infty L^\infty. \tag{155}$$

holds for all $j \in \mathbb{N}$, then eq. (142) holds for each $j,k \in \mathbb{N}$. ■

Proof. Using the ODE, one concludes that, for each $k \in \mathbb{N}$,

$$\begin{aligned}\frac{\partial^j}{\partial h^j}\frac{\partial^k}{\partial \zeta^k}\begin{bmatrix}\chi\aleph\\ \chi\beth\end{bmatrix} &= \frac{1}{\mathfrak{W}}\sum_{j_0=0}^{j}\binom{j}{j_0}\Bigg[\frac{\partial^{j-j_0}\partial^k\chi}{\partial h^{j-j_0}\partial\zeta^k}\frac{\partial^{j_0}}{\partial h^{j_0}}\begin{bmatrix}\chi_0\aleph\\ \chi_0\beth\end{bmatrix} + \sum_{k_0=1}^{k}\frac{\partial^{j-j_0}\partial^{k-k_0}\chi}{\partial h^{j-j_0}\partial\zeta^{k-k_0}} \\ &\times \sum_{j_1=0}^{j_0}\binom{j_0}{j_1}\Big(\frac{\partial^{j_0-j_1}h^{2/(\kappa+2)}}{\partial h^{j_0-j_1}}\Big)\Bigg[\sum_{j_2=0}^{j_1}\sum_{k_1=0}^{k_0-1}\frac{\partial^{j_1-j_2}\partial^{k_0-1-k_1}F}{\partial h^{k_0-1-k_2}\partial\zeta^{j-j_1}}\frac{\partial^{j_2}\partial^{k_1}}{\partial h^{j_2}\partial\zeta^{k_1}}\begin{bmatrix}B\\ -A\end{bmatrix} \\ &+ \sum_{j_2=0}^{j_1}\sum_{k_1=0}^{k_0}\sum_{j_3=0}^{j_2}\sum_{k_2=0}^{k_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\binom{k_0}{k_1}\binom{k_1}{k_2}\Big(\frac{\partial^{j_1-j_2}\partial^{k_0-k_1}E}{\partial h^{j_1-j_2}\partial\zeta^{k_0-k_1}}\Big)\mathcal{M}_{j_3,k_2}\frac{\partial^{j_2-j_3}\partial^{k_1-k_2}}{\partial h^{j_2-j_3}\partial\zeta^{k_1-k_2}}\begin{bmatrix}\chi_0\aleph\\ \chi_0\beth\end{bmatrix}\Bigg]\Bigg]\end{aligned} \tag{156}$$

for any $j \in \mathbb{N}$, where

$$\mathcal{M}_{j,k} = \frac{\partial^j\partial^k}{\partial h^j\partial\zeta^k}\begin{bmatrix}-AB & -B^2\\ A^2 & AB\end{bmatrix}. \tag{157}$$

(The factors of χ_0 in eq. (156) are not required, but we include them for use below.)

Each derivative of χ is (one-step) polyhomogeneous on M and supported away from be, so, using the assumptions,

$$\frac{\partial^{j-j_0}\partial^k\chi}{\partial h^{j-j_0}\partial\zeta^k}\frac{\partial^{j_0}}{\partial h^{j_0}}\begin{bmatrix}\chi_0\aleph\\ \chi_0\beth\end{bmatrix} \in \begin{cases}h^\infty L^\infty \oplus h^\infty \operatorname{Mod}_Q L^\infty & (Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty)\\ h^\infty \operatorname{Mod}_{Q_\infty} L^\infty \oplus h^\infty L^\infty & (\text{otherwise}),\end{cases} \tag{158}$$

so this contribution lies in the desired space. (Here, by "$Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$" we mean that $\varsigma < 0$ or $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$.) If $k = 0$, then this term is the only term in eq. (156), so the claim has been proven in this case.

Applying the ODE $NQ = 0$ to rewrite second and higher derivatives of Q in terms of zeroth and first derivatives, the terms

$$\frac{\partial^{j-j_0}\partial^{k-k_0}\chi}{\partial h^{j-j_0}\partial\zeta^{k-k_0}}\Big(\frac{\partial^{j_0-j_1}h^{2/(\kappa+2)}}{\partial h^{j_0-j_1}}\Big)\frac{\partial^{j_1-j_2}\partial^{k_0-1-k_1}F}{\partial h^{k_0-1-k_2}\partial\zeta^{j-j_1}}\frac{\partial^{j_2}\partial^{k_1}}{\partial h^{j_2}\partial\zeta^{k_1}}\begin{bmatrix}B\\ -A\end{bmatrix} \tag{159}$$

in eq. (156) lie in the same space. If $\varsigma < 0$, this is just a matter of multiplying elements of $h^\infty L^\infty$. If $\varsigma > 0$, then we need to keep track of the exponential factors, with the result being that, if $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$, then the entries lie in the entries of

$$h^\infty \operatorname{Mod}_Q^{1/2}\begin{bmatrix}h^\infty BL^\infty + h^\infty B'L^\infty\\ h^\infty AL^\infty + h^\infty A'L^\infty\end{bmatrix} \subseteq \begin{cases}(h^\infty \operatorname{Mod}_Q L^\infty) \oplus h^\infty L^\infty & (Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty),\\ h^\infty L^\infty \oplus h^\infty \operatorname{Mod}_{Q_\infty} L^\infty & (\text{otherwise}).\end{cases} \tag{160}$$

The remaining terms in eq. (156) only involve fewer ∂_ζ falling on $\aleph, \beth$ than the left-hand side and so can be controlled inductively. Computing $\mathcal{M}_{j,k}$, the result is that

$$(\zeta^\infty h^\infty L^\infty)\mathcal{M}_{j,k} \subseteq \zeta^\infty h^\infty \begin{bmatrix}L^\infty & \operatorname{Mod}_B L^\infty\\ \operatorname{Mod}_A L^\infty & L^\infty\end{bmatrix}. \tag{161}$$

So, if $\varsigma < 0$ or $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$, the corresponding terms in eq. (156) above are, once the result has been proven for all smaller k, known to be in

$$\zeta^\infty h^\infty \begin{bmatrix}L^\infty & \operatorname{Mod}_B L^\infty\\ \operatorname{Mod}_A L^\infty & L^\infty\end{bmatrix}\begin{bmatrix}\operatorname{Mod}_Q L^\infty\\ L^\infty\end{bmatrix} \subseteq \zeta^\infty h^\infty \begin{bmatrix}\operatorname{Mod}_Q L^\infty\\ L^\infty\end{bmatrix}. \tag{162}$$

If $Q \in \operatorname{span}_{\mathbb{C}} Q_\infty \backslash \{0\}$, then the computation is instead

$$\zeta^\infty h^\infty \begin{bmatrix}L^\infty & \operatorname{Mod}_B L^\infty\\ \operatorname{Mod}_A L^\infty & L^\infty\end{bmatrix}\begin{bmatrix}L^\infty\\ \operatorname{Mod}_{Q_\infty} L^\infty\end{bmatrix} \subseteq \zeta^\infty h^\infty \begin{bmatrix}L^\infty\\ \operatorname{Mod}_{Q_\infty} L^\infty\end{bmatrix}. \tag{163}$$

$\square$

The ODE, together with vanishing initial conditions along the graph of a curve $\zeta_0 : \mathbb{R}^+_h \to [0, Z]$, can be combined into an integral equation. Let $G = -h^{2/(\kappa+2)}\mathfrak{W}^{-1}F$ and $\hat{E} = -\mathfrak{W}^{-1}\zeta E \in C^\infty(M)$. Integrating eq. (150), the solution of the initial value problem is

$$\begin{bmatrix} \aleph(\zeta,h) \\ \daleth(\zeta,h) \end{bmatrix} = \int_\zeta^{\zeta_0} \Big(\frac{\hat{E}(\omega,h)}{\lambda(\omega,h)} \begin{bmatrix} -A(\omega,h)B(\omega,h) & -B(\omega,h)^2 \\ A(\omega,h)^2 & A(\omega,h)B(\omega,h) \end{bmatrix} \begin{bmatrix} \aleph(\omega,h) \\ \daleth(\omega,h) \end{bmatrix} + G(\omega,h) \begin{bmatrix} B(\omega,h) \\ -A(\omega,h) \end{bmatrix} \Big) \,\mathrm{d}\omega, \tag{164}$$

where $\lambda(\zeta,h) = \zeta/h^{2/(\kappa+2)}$ as usual. Thus, $(\aleph, \daleth)$ is a fixed point of an affine map whose linear part is given by

$$\begin{bmatrix} \gimel(\zeta,h) \\ \urcorner(\zeta,h) \end{bmatrix} \mapsto \int_\zeta^{\zeta_0} \frac{\hat{E}(\omega,h)}{\lambda(\omega,h)} \begin{bmatrix} -A(\omega,h)B(\omega,h) & -B(\omega,h)^2 \\ A(\omega,h)^2 & A(\omega,h)B(\omega,h) \end{bmatrix} \begin{bmatrix} \gimel(\omega,h) \\ \urcorner(\omega,h) \end{bmatrix} \mathrm{d}\omega. \tag{165}$$

We call this linear map Φ if $\zeta_0 = Z$ and Ξ if $\zeta_0(h) = \lambda_0 h^{2/(\kappa+2)}$. We can consider Φ, Ξ as functions $L^\infty_{\mathrm{loc}}(V_{h_0,\lambda_0};\mathbb{C})^2 \to L^\infty_{\mathrm{loc}}(V_{h_0,\lambda_0};\mathbb{C})^2$.

In the classically forbidden case, $\varsigma > 0$, in order to get a solution with the desired properties, the initial data cannot be prescribed anywhere. At the level of technical details, the difficulty is that, for each $\zeta_0 \in [0, Z]\backslash\{\zeta\}$, the bound

$$\Big| \int_{\zeta_0}^{\zeta} \exp\Big(\pm \frac{4}{\kappa+2} \frac{\omega^{(\kappa+2)/2}}{h} \Big) F(\omega,h) \,\mathrm{d}\omega \Big| \leq |\zeta - \zeta_0| \exp\Big(\pm \frac{4}{\kappa+2} \frac{\zeta^{(\kappa+2)/2}}{h} \Big) \|F\|_{L^\infty} \tag{166}$$

only holds for one choice of sign $\pm$, namely that matching the sign of $\zeta - \zeta_0$. It is only in this case that the exponential has the correct monotone behavior. Estimates of this form are key in the analysis below.

The difficulty is not an artifact of the method. The fundamental issue is that, according to the Liouville–Green expansion, if one specifies initial conditions somewhere and supplies a forcing F with $\operatorname{supp} F \Subset (0, Z)$, then it should be expected that a solution to $Pu = F$ will, for some $\zeta \in (0, Z)$, grow exponentially fast as $h \to 0^+$ relative to F. If the forcing F is $F = fQ + gQ'$ for $O(h^\infty)$ coefficients f, g, then u is growing exponentially faster than Q, Q'. Since the overshoot is exponentially bad, even the $O(h^\infty)$ terms in f, g are not sufficient to restore decay. This is why, unless $Q \in \operatorname{span}_\mathbb{C} Q_\infty$, we instead supply initial conditions for *small* ζ, say at $\zeta = 0$, or, in what ends up being necessary if $\alpha \neq 1/2$, along $\Gamma_{\lambda_0} = \{\zeta = \lambda_0 h^{(\kappa+2)/2}\}$ for some $\lambda_0 > 0$. However, if $Q \in \operatorname{span}_\mathbb{C} Q_\infty\backslash\{0\}$, then instead one has to supply the initial conditions at $\zeta = Z$. In the classically allowed case, all choices work equally well.

See [Sim15, §15.5] for an exposition of the general method in a simpler setting.

Let $\Theta = (1+\varsigma)/2$. Let $\mathcal{X}_{h_0,\lambda_0} \subset L^\infty_{\mathrm{loc}}(V_{h_0,\lambda_0};\mathbb{C})^2$ denote the Banach space

$$\mathcal{X}_{h_0,\lambda_0} = \begin{cases} (\exp(4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h) L^\infty(V_{h_0,\lambda_0};\mathbb{C})) \oplus L^\infty(V_{h_0,\lambda_0};\mathbb{C}) & (Q \notin \operatorname{span}_\mathbb{C} Q_\infty), \\ L^\infty(V_{h_0,\lambda_0};\mathbb{C})) \oplus (\exp(-4(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h) L^\infty(V_{h_0,\lambda_0};\mathbb{C})) & (\text{otherwise}). \end{cases} \tag{167}$$

Lemma 6.6. *Suppose that either $\varsigma < 0$ or that $Q \in \operatorname{span}_\mathbb{C} \backslash\{0\}$, so that the first case of eq. (167) holds. For each $h_0, \lambda_0 > 0$, Φ is a bounded endomorphism of $\mathcal{X}_{h_0,\lambda_0}$. Moreover, for fixed $h_0 > 0$, the operator norm $\|\Phi\|_{\mathcal{X}_{h_0,\lambda_0}\to\mathcal{X}_{h_0,\lambda_0}}$ satisfies*

$$\|\Phi\|_{\mathcal{X}_{h_0,\lambda_0}\to\mathcal{X}_{h_0,\lambda_0}} = O(\lambda_0^{-(\kappa+2)/2}) \tag{168}$$

as $\lambda_0 \to \infty$ (meaning that nothing is implied as $\lambda_0 \to 0^+$). ■

Proof. Let $\|-\|_{L^\infty}$ stand for the $L^\infty(V_{h_0,\lambda_0};\mathbb{C})$ norm. Then,

$$\|\hat{E}\|_{L^\infty} = \|\hat{E}\|_{C^0(\mathrm{cl}_M V_{h_0,\lambda_0})} \tag{169}$$

is finite, since $\mathrm{cl}_M V_{h_0,\lambda_0}$ is a compact subset of M, on which $\hat{E}$ is continuous.

Furthermore,

$$\|\lambda^{-1}AB\|_{L^\infty} = \|\lambda^{-1}A(\lambda)B(\lambda)\|_{L^\infty[\lambda_0,\infty)_\lambda} = O(\lambda_0^{-(\kappa+2)/2}), \tag{170}$$

and, letting $\|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty} = \|\exp(-4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)}/h)\lambda(\omega,h)^{-1}B(\zeta/h^{2/(\kappa+2)})^2\|_{L^\infty}$,

$$\|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty} = \|\exp(-4\Theta(\kappa+2)^{-1}\lambda^{(\kappa+2)/2})\lambda^{-1}B(\lambda)^2\|_{L^\infty[\lambda_0,\infty)_\lambda} = O(\lambda_0^{-(\kappa+2)/2}) \tag{171}$$

as $\lambda_0 \to \infty$. These estimates are immediate corollaries of Proposition 3.3.

Similarly, the quantity

$$\|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty} = \|\exp(4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)}/h)\lambda(\omega,h)^{-1}A(\omega/h^{2/(\kappa+2)})^2\|_{L^\infty} \tag{172}$$

satisfies

$$\|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty} = \|\exp(4\Theta(\kappa+2)^{-1}\lambda^{(\kappa+2)/2})\lambda^{-1}A(\lambda)^2\|_{L^\infty[\lambda_0,\infty)_\lambda} = O(\lambda_0^{-(\kappa+2)/2}). \tag{173}$$

So, in order to prove the proposition, it suffices to bound $\|\Phi(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}}$ by a product of $\|\hat{E}\|_{L^\infty}$, $\|(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}}$, and some linear combination of the three norms

$$\|\lambda^{-1}AB\|_{L^\infty}, \|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty}, \|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty} = O(\lambda_0^{(\kappa+2)/2}). \tag{174}$$

Let $\beth\Phi$ denote the first component of $\Phi(\beth,\daleth)$, and let $\daleth\Phi$ denote the second. We want to bound

$$\|\Phi(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}} = \|\beth\Phi(\zeta,h)\|_{L^\infty} + \|\exp(4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)/2}/h)\daleth\Phi(\omega,h)\|_{L^\infty} \tag{175}$$

for $\beth \in L^\infty(V_{h_0,\lambda_0};\mathbb{C})$ and $\daleth \in \exp(-4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)L^\infty(V_{h_0,\lambda_0};\mathbb{C})$. The first term can be bounded as follows:

$$\begin{aligned}|\beth\Phi(\zeta,h)| \leq Z\|\hat{E}\|_{L^\infty}(&\|\lambda^{-1}AB\|_{L^\infty}\|\beth\|_{L^\infty} \\ &+ \|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty}\|\exp(4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)/2}/h)\daleth(\omega,h)\|_{L^\infty}).\end{aligned} \tag{176}$$

In order to bound $\daleth\Phi$, the inequality

$$\begin{aligned}|\daleth\Phi(\zeta,h)| \leq Z\|\hat{E}\|_{L^\infty}(&\sup\{\lambda^{-1}|A(\omega/h^{(\kappa+2)/2})|^2 : \omega \geq \zeta\}\|\beth\|_{L^\infty} \\ &+ \|\lambda^{-1}AB\|_{L^\infty}\sup\{|\daleth(\omega,h)| : \omega\geq\zeta\})\end{aligned} \tag{177}$$

can be used. Because $\exp(-4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)$ is decreasing on $(0,Z)$ for each individual h,

$$\sup\{\lambda^{-1}|A(\omega/h^{2/(\kappa+2)})|^2 : \omega\geq\zeta\} \leq \exp(-4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)\|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty}, \tag{178}$$

and

$$\sup\{|\daleth(\omega,h)| : \omega\geq\zeta\} \leq \exp(-4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)\|\exp(4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)/2}/h)\daleth(\omega,h)\|_{L^\infty}. \tag{179}$$

So, eq. (177) yields

$$\begin{aligned}&\|\exp(4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)\daleth\Phi(\zeta,h)\|_{L^\infty} \\ &\qquad\leq Z\|\hat{E}\|_{L^\infty}(\|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty} + \|\lambda^{-1}AB\|_{L^\infty})\|(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}}.\end{aligned} \tag{180}$$

Altogether, $\|\Phi(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}} \leq \Sigma Z\|\hat{E}\|_{L^\infty}\|(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}}$ for $\Sigma = \|\lambda^{-1}AB\|_{L^\infty} + \|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty} + \|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty}$.

□

Lemma 6.7. *If $\varsigma < 0$ or $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$, then, for each $h_0,\lambda_0 > 0$, Ξ is a bounded endomorphism of $\mathcal{X}_{h_0,\lambda_0}$. Moreover, for fixed $h_0 > 0$, the operator norm $\|\Xi\|_{\mathcal{X}_{h_0,\lambda_0}\to\mathcal{X}_{h_0,\lambda_0}}$ satisfies*

$$\|\Xi\|_{\mathcal{X}_{h_0,\lambda_0}\to\mathcal{X}_{h_0,\lambda_0}} = O(\lambda_0^{-(\kappa+2)/2}) \tag{181}$$

as $\lambda_0 \to \infty$. ■

Proof. Let $\beth\Xi$ denote the first component of Ξ, and let $\daleth\Xi$ denote the second. We want to bound

$$\|\Xi(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}} = \|\exp(-4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)/2}/h)\beth\Xi(\omega,h)\|_{L^\infty} + \|\daleth\Xi\|_{L^\infty} \tag{182}$$

for $\beth \in \exp(4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)}/h)L^\infty(V_{h_0,\lambda_0};\mathbb{C})$ and $\daleth \in L^\infty(V_{h_0,\lambda_0};\mathbb{C})$. The final term above can be bounded by

$$|\daleth\Xi(\zeta,h)| \leq Z\|\hat{E}\|_{L^\infty}(\|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty}\|\exp(-4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)/2}/h)\beth(\omega,h)\|_{L^\infty} \\ + \|\lambda^{-1}AB\|_{L^\infty}\|\daleth\|_{L^\infty}). \tag{183}$$

In order to bound the first term in eq. (182), the inequality

$$|\beth\Xi(\zeta,h)| \leq Z\|\hat{E}\|_{L^\infty}(\|\lambda^{-1}AB\|_{L^\infty}\sup\{|\beth(\omega,h)| : \omega\leq\zeta\} \\ + \sup\{\lambda^{-1}|B(\omega/h^{2/(\kappa+2)})|^2 : \omega\leq\zeta\}\|\daleth\|_{L^\infty}) \tag{184}$$

can be used. Because $\exp(4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)$ is increasing on $(0,Z)$ for each individual h,

$$\sup\{|\beth(\omega,h)| : \omega\leq\zeta\} \leq \exp(4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)\|\exp(-4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)/2}/h)\beth(\omega,h)\|_{L^\infty}, \tag{185}$$

and

$$\sup\{\lambda^{-1}|B(\omega/h^{2/(\kappa+2)})|^2 : \omega\leq\zeta\} \leq \exp(4\Theta(\kappa+2)^{-1}\zeta^{(\kappa+2)/2}/h)\|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty}. \tag{186}$$

So,

$$\|\exp(-4\Theta(\kappa+2)^{-1}\omega^{(\kappa+2)/2}/h)\beth\Xi(\omega,h)\|_{L^\infty} \\ \leq Z\|\hat{E}\|_{L^\infty}(\|\lambda^{-1}AB\|_{L^\infty} + \|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty})\|(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}}. \tag{187}$$

Altogether, $\|\Xi(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}} \leq \Sigma Z\|\hat{E}\|_{L^\infty}\|(\beth,\daleth)\|_{\mathcal{X}_{h_0,\lambda_0}}$ for $\Sigma = \|\lambda^{-1}AB\|_{L^\infty} + \|\exp(\cdots)\lambda^{-1}B^2\|_{L^\infty} + \|\exp(\cdots)\lambda^{-1}A^2\|_{L^\infty}$. □

Proposition 6.8. *There exists some $\lambda_{00}>0$ such that, if $\lambda_0\geq\lambda_{00}$, then for any $\aleph,\text{ב}\in C^\infty((0,Z]_\zeta\times(0,\infty)_h;\mathbb{C})$ satisfying eq. (164) with $\zeta_0=Z$, then, if $\varsigma<0$ or $A\in\operatorname{span}_{\mathbb{C}}Q_\infty\backslash\{0\}$, then*

$$\frac{\partial^j\aleph}{\partial h^j}\in h^\infty L^\infty(V_{h_0,\lambda_0};\mathbb{C}), \qquad \frac{\partial^j\text{ב}}{\partial h^j}\in h^\infty\operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C}) \tag{188}$$

hold for all $j\in\mathbb{N}$. ■

Proof. Since $\aleph,\text{ב}$ depend linearly on F, which is $O(h^\infty)$ relative to Q, it actually suffices to prove only that $\partial_h^j\aleph\in L^\infty(V_{h_0,\lambda_0};\mathbb{C})$ and

$$\partial_h^j\text{ב}\in\operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C}). \tag{189}$$

Indeed, once this is known, the result can be applied with h^kF in place of F to conclude that $\partial_h^j\aleph\in h^kL^\infty(V_{h_0,\lambda_0};\mathbb{C})$ and $\partial_h^j\text{ב}\in h^k\operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C})$.

Let

$$\mathcal{I} = \int_\zeta^Z \left(\frac{\hat{E}(\omega,h)}{\lambda(\omega,h)}\begin{bmatrix} -A(\omega,h)B(\omega,h) & -B(\omega,h)^2 \\ A(\omega,h)^2 & A(\omega,h)B(\omega,h)\end{bmatrix}\begin{bmatrix}\aleph(\omega,h)\\ \text{ב}(\omega,h)\end{bmatrix}\,\mathrm{d}\omega, \tag{190}$$

$$\mathcal{C} = \int_\zeta^Z G(\omega,h)\begin{bmatrix}B(\omega,h)\\ -A(\omega,h)\end{bmatrix}\,\mathrm{d}\omega. \tag{191}$$

Then, $(\aleph,\text{ב})^\intercal = \mathcal{I}+\mathcal{C}$. Differentiating,

$$\frac{\partial^j}{\partial h^j}\begin{bmatrix}\aleph\\ \text{ב}\end{bmatrix} = \frac{\partial^j\mathcal{I}}{\partial h^j} + \frac{\partial^j\mathcal{C}}{\partial h^j}. \tag{192}$$

First consider

$$\frac{\partial^k \mathcal{C}}{\partial h^k} = \sum_{k_1=0}^{k} \binom{k}{k_1} \int_\zeta^Z \frac{\partial^{k-k_1} G}{\partial h^{k-k_1}} \frac{\partial^{k_1}}{\partial h^{k_1}} \begin{bmatrix} B(\omega,h) \\ -A(\omega,h) \end{bmatrix} \,\mathrm{d}\omega. \tag{193}$$

Each of the integrands above is in $h^\infty \zeta^{-K} \operatorname{Mod}_Q^{1/2}((\operatorname{Mod}_B^{1/2} L^\infty) \oplus (\operatorname{Mod}_A^{1/2} L^\infty)) = h^\infty \zeta^{-K}(L^\infty \oplus (\operatorname{Mod}_A L^\infty))$ for some large $K \in \mathbb{N}$. Thus, by the to-be-proven Lemma 6.9, each term in the sum in eq. (193) is in $h^\infty(L^\infty \oplus \operatorname{Mod}_A L^\infty)$.

On the other hand, consider

$$\frac{\partial^k \mathcal{I}}{\partial h^k} = \sum_{k_1=0}^{k} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \binom{k}{k_1, k_2, k_3} \int_\zeta^Z \frac{\partial^{k_0-k_1} \hat{E}(\omega,h)}{\partial h^{k_0-k_1}} \Big(\frac{\partial^{k_1-k_2}}{\partial h^{k_1-k_2}} \frac{1}{\lambda(\omega,h)} \Big) \\ \times \mathcal{M}_{k_2-k_3,0}(\omega,h) \frac{\partial^{k_3}}{\partial h^{k_3}} \begin{bmatrix} \aleph(\omega,h) \\ \beth(\omega,h) \end{bmatrix} \,\mathrm{d}\omega. \tag{194}$$

Inserting a cutoff $\chi_0 \in C^\infty(M)$ which is identically 1 on V_{h_0,λ_0} and therefore on $\mathrm{cl}_M V_{h_0,\lambda_0}$ and identically vanishing near be, the inductive hypothesis shows, unless $k_3 = k$, that the integrand satisfies

$$\frac{\partial^{k_0-k_1} \hat{E}(\omega,h)}{\partial h^{k_0-k_1}} \Big(\frac{\partial^{k_1-k_2}}{\partial h^{k_1-k_2}} \frac{1}{\lambda(\omega,h)} \Big) \mathcal{M}_{k_2-k_3,0}(\omega,h) \frac{\partial^{k_3}}{\partial h^{k_3}} \begin{bmatrix} \aleph(\omega,h) \\ \beth(\omega,h) \end{bmatrix} \in h^\infty \begin{bmatrix} L^\infty \\ \operatorname{Mod}_A L^\infty \end{bmatrix} \tag{195}$$

on $\bar{V}_{h_0,\lambda_0}$. Thus, by Lemma 6.9, each term in the sum in eq. (194), except possibly the single term with $k_3 = k$, lies in $h^\infty(L^\infty \oplus \operatorname{Mod}_A L^\infty)$.

To summarize, for some $\mathcal{C}_j \in h^\infty(L^\infty(V_{h_0,\lambda_0};\mathbb{C}) \oplus \operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C}))$,

$$\frac{\partial^j}{\partial h^j} \begin{bmatrix} \aleph \\ \beth \end{bmatrix} = \Phi \frac{\partial^j}{\partial h^j} \begin{bmatrix} \aleph \\ \beth \end{bmatrix} + \mathcal{C}_j. \tag{196}$$

Consider the map $L^\infty(V_{h_0,\lambda_0};\mathbb{C}) \oplus \operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C}) \to L^\infty(V_{h_0,\lambda_0};\mathbb{C}) \oplus \operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C})$ given by

$$\begin{bmatrix} \gimel \\ \daleth \end{bmatrix} \mapsto \Phi \begin{bmatrix} \gimel \\ \daleth \end{bmatrix} + \mathcal{C}_j. \tag{197}$$

So, $(\partial_h^j \aleph, \partial_h^j \beth)^\intercal$ is a fixed point of the affine map eq. (197).

By Lemma 6.6, if λ_0 is sufficiently large, then Φ is a contraction map. Thus, the map eq. (197) has a unique fixed point $(\gimel_j, \daleth_j)$, and it is given by a convergent series $\sum_{n=0}^\infty \Phi^n \mathcal{C}_j$. For any $h_1 > 0$, the map eq. (197) is also a contraction map on $L^\infty(V_{h_0,\lambda_0} \cap \{h \geq h_1\};\mathbb{C}) \oplus \operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0} \cap \{h \geq h_1\};\mathbb{C})$, and $(\gimel_j|_{\{h\geq h_1\}}, \daleth_j|_{\{h \geq h_1\}})$ is a fixed point for it. But, so is

$$\partial_h^j(\aleph, \beth)|_{V_{h_0,\lambda_0} \cap \{h \geq h_1\}} \in C^\infty(V_{h_0,\lambda_0} \cap \{h \geq h_1\})^2. \tag{198}$$

Thus, by the uniqueness of the fixed point, $(\gimel_j, \daleth_j)$ agrees with $\partial_h^j(\aleph, \beth)$ in $\{h \geq h_1\}$. Since h_1 can be taken arbitrarily small, the agreement holds everywhere in V_{h_0,λ_0}. We can therefore conclude that

$$\partial_h^j(\aleph, \beth) \in L^\infty(V_{h_0,\lambda_0};\mathbb{C}) \oplus \operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C}). \tag{199}$$

□

Lemma 6.9. *If $H \in h^\infty \operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C})$, then $\int_\zeta^Z H(\omega,h)\,\mathrm{d}\omega \in h^\infty \operatorname{Mod}_A L^\infty(V_{h_0,\lambda_0};\mathbb{C})$.* ■

Proof. In the $\varsigma < 0$ case, the claim is trivially true, so assume $\varsigma > 0$, in which case $A \in \operatorname{span}_{\mathbb{C}} Q_\infty \backslash \{0\}$. We can write $H(\zeta,h) = h^k H_k(\zeta,h) \operatorname{Mod}_A(\zeta,h)$ for some $H_k \in h^k L^\infty(V_{h_0,\lambda_0};\mathbb{C})$. Then,

$$\Big| \int_\zeta^Z H(\omega,h)\,\mathrm{d}\omega \Big| \leq h^k Z \|H_k\|_{L^\infty} \sup\{\operatorname{Mod}_A(\omega,h) : \omega \in [\zeta, Z]\}. \tag{200}$$

The supremum on the right-hand side is finite and can be bounded as

$$\sup\{\mathrm{Mod}_A(\omega,h):\omega\in[\zeta,Z]\}\le\sup\{\mathrm{Mod}_A(\lambda):\lambda\ge\lambda(\zeta,h)\}<\infty \tag{201}$$

on V_{h_0,λ_0}. By Lemma A.4, there exists some $\lambda_1>\lambda_0$ such that for all $\lambda,\lambda'\ge\lambda_1$ with $\lambda>\lambda'$, $\mathrm{Mod}_A(\lambda)\le 2\,\mathrm{Mod}_A(\lambda')$. Consequently:

$$\sup\{\mathrm{Mod}_A(\lambda):\lambda\ge\lambda(\zeta,h)\}\le\begin{cases}2\,\mathrm{Mod}_A(\lambda(\zeta,h)) & (\lambda(\zeta,h)\ge\lambda_1),\\ \max\{2\,\mathrm{Mod}_A(\lambda_1),\sup_{\lambda\in[\lambda(\zeta,h),\lambda_1]}\mathrm{Mod}_A(\lambda)\} & (\text{otherwise})\end{cases} \tag{202}$$

If the second case holds, then $\lambda(\zeta,h)\in[\lambda_0,\lambda_1]$. Because $\mathrm{Mod}_A(\lambda)$ is nonvanishing, there exists some $C>1$ such that $\mathrm{Mod}_A(\lambda')\le C\,\mathrm{Mod}_A(\lambda)$ for all $\lambda,\lambda'\in[\lambda_0,\lambda_1]$. The inequality above therefore yields

$$\sup\{\mathrm{Mod}_A(\lambda):\lambda\ge\lambda(\zeta,h)\}\le 2C\,\mathrm{Mod}_A(\lambda(\zeta,h))=2C\,\mathrm{Mod}_A(\zeta,h). \tag{203}$$

So, all in all,

$$\Big|\int_\zeta^Z H(\omega,h)\,\mathrm{d}\omega\Big|\le 2Ch^kZ\|H_k\|_{L^\infty}\,\mathrm{Mod}_A(\zeta,h)\in h^k\,\mathrm{Mod}_A\,L^\infty. \tag{204}$$

Since k was arbitrary, this completes the proof. □

Proposition 6.10. *There exists some $\lambda_{00}>0$ such that, if $\lambda_0\ge\lambda_{00}$, then for any $\aleph,\beth\in C^\infty((0,Z]_\zeta\times(0,\infty)_h;\mathbb{C})$ satisfying eq. (164) with $\zeta_0(h)=\lambda_0h^{2/(\kappa+2)}$, then, if $\varsigma<0$ or $A\notin\operatorname{span}_{\mathbb{C}}Q_\infty$, then*

$$\frac{\partial^j\aleph}{\partial h^j}\in h^\infty\,\mathrm{Mod}_B\,L^\infty(V_{h_0,\lambda_0};\mathbb{C}),\qquad\frac{\partial^j\beth}{\partial h^j}\in h^\infty L^\infty(V_{h_0,\lambda_0};\mathbb{C}) \tag{205}$$

hold for all $j\in\mathbb{N}$. ■□

The proof is analogous to the previous, for instance using Lemma 6.7 in place of Lemma 6.6, and using Lemma 6.11 in place of Lemma 6.9, so we omit the details.

Lemma 6.11. *If $H\in h^\infty\,\mathrm{Mod}_B\,L^\infty(V_{h_0,\lambda_0};\mathbb{C})$, then, letting $\zeta_0=\lambda_0h^{2/(\kappa+2)}$, $\int_{\zeta_0}^\zeta H(\omega,h)\,\mathrm{d}\omega\in h^\infty\,\mathrm{Mod}_B\,L^\infty(V_{h_0,\lambda_0};\mathbb{C})$.* ■

Proof. In the $\varsigma<0$ case, the claim is trivially true, so assume $\varsigma>0$. We can write $H(\zeta,h)=h^kH_k(\zeta,h)\,\mathrm{Mod}_B(\zeta,h)$ for some $H_k\in L^\infty(V_{h_0,\lambda_0};\mathbb{C})$. Then,

$$\Big|\int_{\zeta_0}^\zeta H(\omega,h)\,\mathrm{d}\omega\Big|\le h^kZ\|H_k\|_{L^\infty}\sup\{\mathrm{Mod}_B(\omega,h):\omega\in[\zeta_0,\zeta]\}. \tag{206}$$

The supremum on the right-hand side is finite and can be written as

$$\sup\{\mathrm{Mod}_B(\omega,h):\omega\in[\zeta_0,\zeta]\}\le\sup\{\mathrm{Mod}_B(\lambda):\lambda_0\le\lambda\le\lambda(\zeta,h)\}<\infty. \tag{207}$$

By Lemma A.4, there exists some $\lambda_1>0$ such that for all $\lambda,\lambda'\ge\lambda_1$ with $\lambda>\lambda'$, $2\,\mathrm{Mod}_B(\lambda)\ge\mathrm{Mod}_B(\lambda')$. Consequently:

$$\sup\{\mathrm{Mod}_B(\lambda):\lambda_0\le\lambda\le\lambda(\zeta,h)\}\le\max\{2\,\mathrm{Mod}_B(\lambda(\zeta,h)),\sup_{\lambda\in[\lambda_0,\lambda_1]}\mathrm{Mod}_B(\lambda)\}. \tag{208}$$

In $\lambda>\lambda_0$, there exists some $C>2$ such that $\mathrm{Mod}_B(\lambda')\le C\,\mathrm{Mod}_B(\lambda)$ whenever $\lambda'\in[\lambda_0,\lambda_1]$. Thus, $\sup\{\mathrm{Mod}_B(\lambda):\lambda_0\le\lambda\le\lambda(\zeta,h)\}\le C\,\mathrm{Mod}_B(\lambda(\zeta,h))$. So,

$$\Big|\int_{\zeta_0}^\zeta H(\omega,h)\,\mathrm{d}\omega\Big|\le Ch^kZ\|H_k\|_{L^\infty}\,\mathrm{Mod}_B(\zeta,h)\in h^k\,\mathrm{Mod}_B\,L^\infty. \tag{209}$$

Since k was arbitrary, this completes the proof. □

7. Analysis at the low corner

We now analyze the forced ODE $Pu = F$ near $\mathrm{fe} \cap \mathrm{be}$. The forcing $F \in C^\infty(M^\circ)$ is now assumed to satisfy

$$\operatorname{supp} F \Subset \bar{U}_{h_0,\Lambda} = \{0 \leq \lambda < \Lambda \text{ and } \varrho < \bar{\varrho}\} \subset M \tag{210}$$

for some $\Lambda > 0$ and $\bar{\varrho} \in (0, Z/\Lambda)$, where $h_0 = \bar{\varrho}^{(\kappa+2)/2}$. (Note that $\bar{U}_{h_0,\Lambda} \cap \mathrm{ie} = \varnothing$.) The behavior of F at $\mathrm{be} = \{\lambda = 0\}$ will be specified. We draw attention to the fact that Λ may be arbitrarily large (but still finite). Thus, it is a bit misleading to refer to the analysis below as occurring "near" $\mathrm{fe} \cap \mathrm{be}$, what matters is excluding $\mathrm{ie} \cup \mathrm{ze}$.

We can identify $\bar{U}_{h_0,\Lambda}$ with $[0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho$. In the coordinates (λ, ϱ), P is a family $\{\hat{P}(\varrho)\}_{\varrho \geq 0}$ of operators on $[0,\Lambda]_\lambda$ converging, as $\varrho \to 0^+$, to $N(P)$ in an appropriate sense, e.g. in the topology of $\mathrm{Diff}^2_{\mathrm{b,E}}[0,\Lambda]_\lambda$. The indicial roots, $\gamma_\pm = 1/2 \pm \alpha$, are independent of ϱ, per the assumption that α be a constant. Recall that we are assuming that $\Re\alpha > 0$.

If $\mathcal{E}, \mathcal{F}$ are two index sets, we use

$$\mathcal{A}^{\mathcal{E},\mathcal{F}}_{\mathrm{c}}([0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho) \subseteq \mathcal{A}^{-\infty,-\infty,\mathcal{E},\mathcal{F}}(M) \tag{211}$$

to denote the set of polyhomogeneous functions u with index set $\mathcal{E}$ at $\{\varrho = 0\}$ and $\mathcal{F}$ at $\mathrm{be} = \{\lambda = 0\}$ satisfying $\operatorname{supp} u \Subset \bar{U}_{h_0,\Lambda} = [0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho$, so in particular u is compactly supported.

Proposition 7.1. *Suppose that, for some index sets $\mathcal{E}, \mathcal{F} \subset \mathbb{C} \times \mathbb{N}$ with $\Re j > -3/2 - \Re\alpha$ for every $(j,k) \in \mathcal{F}$, we are given $F \in \mathcal{A}^{\mathcal{E},\mathcal{F}}_{\mathrm{c}}([0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho)$. Then, there exists a solution*

$$u \in \mathcal{A}^{\mathcal{E},\mathcal{G}}_{\mathrm{c}}([0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho) \tag{212}$$

to $Pu = F$ for some index set $\mathcal{G}$. ■

Proof. The map $[0,\bar{\varrho})_\varrho \ni \varrho \mapsto P(\varrho) \in \lambda^{-2}\,\mathrm{Diff}^2_{\mathrm{b,E}}[0,\Lambda]_\lambda$ is smooth. Recall that $\mathrm{Diff}_{\mathrm{b,E}}[0,\Lambda]$ consists of those differential operators in the algebra generated over $C^\infty[0,\Lambda]$ by $\lambda\partial_\lambda$. Moreover, the coefficient of ∂_λ in $P(\varrho)$ is identically zero. From the theory of regular singular ODE, there exist independent solutions

$$\{v_\pm(\lambda,\varrho)\}_{\varrho \in [0,\bar{\varrho})} \subseteq \mathcal{A}^{\mathcal{F}(\alpha),(0,0)}[0,\Lambda]_\lambda \tag{213}$$

to $P(\varrho)v_\pm(\lambda,\varrho) = 0$ depending smoothly on ϱ, all the way down to $\varrho = 0$. Here, $\mathcal{F}(\alpha)$ is the index set at $\lambda = 0$. Choose v_+ to be recessive.

Now let $W(\varrho) = v_-'(\lambda,\varrho)v_+(\lambda,\varrho) - v_+'(\lambda,\varrho)v_-(\lambda,\varrho)$ denote their Wronskian, where the primes denote differentiation in the first slot. The Wronskian is a function of ϱ alone, smooth down to $\varrho = 0$. Since $v_\pm$ are independent, W is nonvanishing. Now let

$$K(\lambda,\lambda',\varrho) = \frac{1}{W(\varrho)}\begin{cases} v_-(\lambda,\varrho)v_+(\lambda',\varrho) & (\lambda > \lambda'), \\ v_+(\lambda,\varrho)v_-(\lambda',\varrho) & (\lambda < \lambda'). \end{cases} \tag{214}$$

Consider the function $v(\lambda,\varrho) = K(\lambda, F(-,\varrho), \varrho)$, i.e.

$$v(\lambda,\varrho) = \int_0^\infty K(\lambda,\lambda',\varrho)F(\lambda',\varrho)\,\mathrm{d}\lambda'. \tag{215}$$

The integral converges because $v_+(\lambda) \in \lambda^{1/2+\Re\alpha}L^\infty_{\mathrm{loc}}[0,\infty)_\lambda$, so

$$\Big|\int_0^\lambda v_+(\lambda',\varrho)F(\lambda',\varrho)\,\mathrm{d}\lambda'\Big| \preceq \int_0^\lambda s^{1/2+\Re\alpha+\min\mathcal{F}}\,\mathrm{d}s \preceq \int_0^\lambda s^{-1+\epsilon}\,\mathrm{d}s < \infty \tag{216}$$

for some $\epsilon > 0$, where $\min\mathcal{F} = \min\{\Re j : (j,k) \in \mathcal{F}\}$.

The function v satisfies $P(\varrho)v = F$, and it is polyhomogeneous: $v \in \mathcal{A}^{\mathcal{E},\mathcal{G}_0}([0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho)$ for some index set $\mathcal{G}_0$, which can be computed in terms of $\mathcal{F}(\alpha)$ and $\mathcal{F}$.

We have $v(\Lambda,\varrho), \partial_\lambda v(\lambda,\varrho)|_{\lambda=\Lambda} \in \mathcal{A}^{\mathcal{E}}[0,\bar{\varrho})_\varrho$.

Note that v may not satisfy $\operatorname{supp} v \Subset [0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho$, so v is not the desired solution to $Pu = F$. The next step is to add to v a linear combination of $v_\pm$ so that the resulting solution is compactly supported in λ.

Let $w(-,\varrho)$ denote the solution to $P(\varrho)w(-,\varrho) = 0$ with initial conditions $w(\Lambda,\varrho) = -v(\Lambda,\varrho)$ and $\partial_\lambda w(\lambda,\varrho)|_{\lambda=\Lambda} = -\partial_\lambda v(\lambda,\varrho)|_{\lambda=\Lambda}$. This can be written in terms of $v_\pm$ as

$$w(\lambda,\varrho) = \frac{1}{W(\varrho)} \begin{bmatrix} v_-(\lambda,\varrho) \\ v_+(\lambda,\varrho) \end{bmatrix}^\intercal \begin{bmatrix} v_+'(\lambda,\varrho) & -v_+(\lambda,\varrho) \\ -v_-'(\lambda,\varrho) & v_-(\lambda,\varrho) \end{bmatrix} \begin{bmatrix} v(\Lambda,\varrho) \\ v'(\Lambda,\varrho) \end{bmatrix} \tag{217}$$

Thus, w is also in $\mathcal{A}^{\mathcal{E},\mathcal{G}_1}([0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho)$, for some index set $\mathcal{G}_1$. Letting $u = v + w$, we have $u \in \mathcal{A}^{\mathcal{E},\mathcal{G}}([0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho)$ for $\mathcal{G}$ the smallest index set containing $\mathcal{G}_0, \mathcal{G}_1$. This solves the desired ODE with vanishing initial data at Λ. Since the forcing satisfies $\operatorname{supp} F \Subset [0,\Lambda)_\lambda \times [0,\bar{\varrho})_\varrho$, this implies the compact support condition in eq. (212). $\square$

8. Main proof

We turn now to the proof of Theorem B in the $W = 1$ case which we observed in §2 suffices. Our goal, given $Q \in \mathcal{Q}$, is to construct

- $\beta, \gamma \in \mathcal{A}^{\mathcal{E}_0,\infty}(M)$ with $\operatorname{supp}\beta, \operatorname{supp}\gamma$ disjoint from be, and
- $\delta \in \mathcal{A}^{\mathcal{E}_0,\mathcal{F}_0}(M)$ with $\operatorname{supp}\delta \cap (\mathrm{ie} \cup \mathrm{ze}) = \varnothing$,

where $\mathcal{E}_0, \mathcal{F}_0$ are the index sets referenced in the theorem, such that the function u defined by

$$u = (1 + \varrho_{\mathrm{ze}}\varrho_{\mathrm{fe}}\beta) Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + h^{(2\kappa+2)/(\kappa+2)} \gamma Q'\Big(\frac{\zeta}{h^{2/(\kappa+2)}}\Big) + \varrho_{\mathrm{be}}^{1/2-\alpha}\varrho_{\mathrm{fe}}\delta \tag{218}$$

solves $Pu = 0$ in $\{h < h_0\}$ for some $h_0 > 0$.

The upshot of §4, as recorded in Proposition 4.3, was that, letting $\mathcal{E}_0 = \mathcal{E} - 1 \subset \mathbb{N} \times \mathbb{N}$, with $\mathcal{E}$ as in the proposition, there exist $\beta_0, \gamma_0 \in \mathcal{A}^{\mathcal{E}_0}(M)$ with support disjoint from be such that the function

$$u_0 = (1 + \varrho_{\mathrm{ze}}\varrho_{\mathrm{fe}}\beta_0) Q + \varrho_{\mathrm{ze}}^{1/2}\varrho_{\mathrm{fe}}\gamma_0 Q' \tag{219}$$

satisfies $Pu_0 = f_0 Q + g_0 Q'$ for $f_0, g_0 \in \varrho_{\mathrm{be}}^{-1}\varrho_{\mathrm{ze}}^\infty \varrho_{\mathrm{fe}}^\kappa \mathcal{A}^{\mathcal{E}}(M)$, with g_0 supported away from be. Write $f_0 = f_{00} + f_{01}$ for

$$f_{00}, f_{01} \in \varrho_{\mathrm{ze}}^\infty \varrho_{\mathrm{fe}}^\kappa \mathcal{A}^{\mathcal{E}}(M) \tag{220}$$

with f_{00} supported away from be and f_{01} supported away from ze. This decomposition can be arranged because $\mathrm{ze} \cap \mathrm{be} = \varnothing$.

We now apply Proposition 5.1 with $f = -f_{00}$ and $g = -g_0$ to produce $\beta_1, \gamma_1 \in \varrho_{\mathrm{ze}}^\infty \mathcal{A}^{\mathcal{E}_0,\infty}(M)$ supported away from $\mathrm{ie} \cup \mathrm{be}$ and $R \in \varrho_{\mathrm{fe}}^\kappa \mathcal{A}^{\mathcal{E},\infty}(M)$ supported away from $\mathrm{ie} \cup \mathrm{ze} \cup \mathrm{be}$ such that the function $u_0 = \varrho_{\mathrm{fe}}(\beta_1 Q + \gamma_1 Q')$ solves $Pu_1 = (-f_{00} + f_1)Q + (-g_0 + g_1)Q' + R$ for some $f_1, g_1 \in h^\infty C^\infty(M;\mathbb{C})$ supported away from be.

Now apply Proposition 6.1 with $f = -f_1$ and $g = -g_1$ to produce $\beta_2, \gamma_2 \in h^\infty C^\infty(M;\mathbb{C})$ supported away from be such that the function $u_2 = \beta_2 Q + \gamma_2 Q'$ solves $Pu_2 = -f_1 Q - g_1 Q' + H$ for some $H \in h^\infty C^\infty(M;\mathbb{C})$ supported away from $\mathrm{ie} \cup \mathrm{ze} \cup \mathrm{be}$. There exist $\Lambda, h_1 > 0$ such that the quantity $\bar{\varrho}$ defined by

$$\bar{\varrho} = h_1^{2/(\kappa+2)} \tag{221}$$

satisfies $\bar{\varrho} < Z/\Lambda$ (so that the set $\bar{U}_{h_1,\Lambda}$ is defined) and such that $(\operatorname{supp} R \cup \operatorname{supp} H) \cap \{h < h_1\} \Subset \bar{U}_{h_1,\Lambda}$. Let $\chi \in C_c^\infty[0,\infty)_h$ be identically 1 near $h = 0$ and supported in $\{h < h_1\}$.

Finally, we apply Proposition 7.1 with $F = -\chi f_{01} Q - \chi R - \chi H$ to give, for some index set $\mathcal{G}$, $\delta_0 \in \mathcal{A}^{\mathcal{E},\mathcal{G}}(M)$ with $\operatorname{supp}\delta_0 \cap (\mathrm{ie} \cup \mathrm{ze}) = \varnothing$ satisfying

$$P\delta_0 = -\chi f_{01} Q - \chi R - \chi H. \tag{222}$$

The hypothesis of that proposition regarding the index set of the forcing is satisfied, because the only contribution to the index set is $f_{01}Q$, which has index set $\mathcal{F}(\alpha) - 1$ at zero. So, $\min\{\mathcal{F}(\alpha) - 1\} = -1/2 - \Re\alpha > -3/2 - \Re\alpha$.

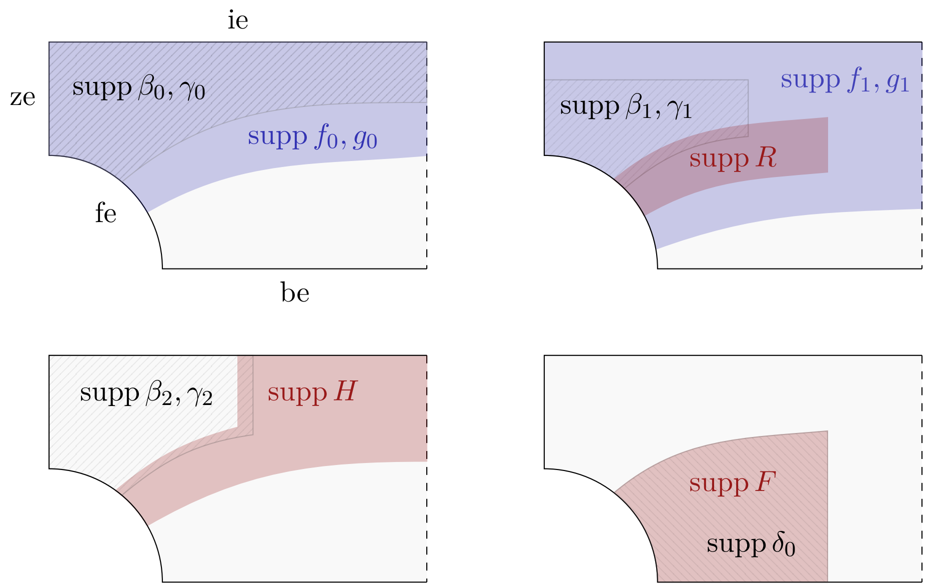

Figure 7. *Top left*: the supports of the functions $\beta_0, \gamma_0, f_0, g_0$ appearing in the first step of the argument, using §4. *Top right*: the supports of the functions $\beta_1, \gamma_1, f_1, g_1, R$ appearing in the second step, using §5. *Bottom left*: the supports of the functions β_2, γ_2, H appearing in the third step, using §6. *Bottom right*: the supports of the functions appearing in the final step of the argument, using §7.

We can write $\delta_0 = \varrho_{\mathrm{be}}^{1/2-\alpha} \varrho_{\mathrm{fe}} \delta$ for $\delta \in \mathcal{A}^{\mathcal{E}_0, \mathcal{G}}$. Set $u = u_0 + u_1 + u_2 + \delta_0$. This solves

$$Pu = (1 - \chi(h))(f_{01} + H + R) \tag{223}$$

and has the form eq. (218) for $\beta = \beta_0 + \varrho_{\mathrm{ze}}^{-1} \beta_1 + \varrho_{\mathrm{ze}}^{-1} \varrho_{\mathrm{fe}}^{-1} \beta_2$ and $\gamma = \gamma_0 + \varrho_{\mathrm{ze}}^{-1/2} \gamma_1 + \varrho_{\mathrm{ze}}^{-1/2} \varrho_{\mathrm{fe}}^{-1} \gamma_2$. By construction, β, γ lie in the desired function spaces, and they have the required support properties.

If a new $h_0 > 0$ is chosen such that $\chi = 1$ identically on $[0, h_0]$, then $Pu = 0$ for $h < h_0$.

Appendix A. Bounds on the modulus function

In the body of the paper, we saw that, when upgrading the quasimode $Q \in \mathcal{Q} \backslash \{0\}$ to a full solution to the ODE, it is convenient to work with the *modulus function* $\mathrm{Mod}_Q(\lambda)$ defined by

$$\mathrm{Mod}_Q(\lambda) = |Q(\lambda)|^2 + \lambda^2 \langle \chi(1/\lambda) \lambda^{(\kappa+2)/2} \rangle^{-2} |Q'(\lambda)|^2, \tag{224}$$

where $\langle \rho \rangle = (1 + \rho^2)^{1/2}$ and $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}; [0, 1])$ is identically 1 in some neighborhood of the origin. In this section, $L^\infty = L^\infty(\mathbb{R}^+_\lambda)$.

Lemma A.1. $\mathrm{Mod}_Q(\lambda) \neq 0$ *for any* $\lambda > 0$. ■

Proof. This is the statement that $Q(\lambda)$ and $Q'(\lambda)$ cannot vanish simultaneously. Because Q satisfies the second order ODE $NQ = 0$, it is the case that, for each $\lambda_0 > 0$, Q is uniquely determined among elements of $\mathcal{Q}$ by the pair $(Q(\lambda_0), Q'(\lambda_0))$. If we were to have $Q(\lambda_0), Q'(\lambda_0) = 0$, then this would force Q to vanish identically, contrary to assumption. □

The key property of the modulus functions is that they control the *other* elements of $\mathcal{Q}$ and their derivatives:

Proposition A.2. *Suppose that either $\varsigma < 0$ or $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$. Let $A \in \mathcal{Q}$. Then, for any $k \in \mathbb{N}$, there exists some $K \in \mathbb{N}$ depending on k such that*

$$\frac{\partial^k A(\lambda)}{\partial \lambda^k} \in \lambda^{-K} \langle \lambda \rangle^{2K} \operatorname{Mod}_Q(\lambda)^{1/2} L^\infty. \tag{225}$$

If $\varsigma > 0$, then $\partial_\lambda^k Q_\infty \in \lambda^{-K}\langle \lambda \rangle^{2K} \operatorname{Mod}_{Q_\infty}(\lambda)^{1/2} L^\infty$ for some $K \in \mathbb{N}$ depending on k. ■

Proof. By Lemma A.1, the claim holds within any bounded interval $I \Subset \mathbb{R}^+_\lambda$ worth of λ.

To prove the claim in an interval $I \Subset [0, \infty)_\lambda$, which may contain 0, first use that, because A is polyhomogeneous at $\lambda = 0$ by Proposition 3.1, it and its derivatives are growing at most polynomially as $\lambda \to 0^+$. Then use that, because Q is nonzero, the functions $\operatorname{Mod}_Q(\lambda), \operatorname{Mod}_{Q_\infty}(\lambda)$ are decaying at most polynomially as $\lambda \to 0^+$. So the proposition also holds here.

It therefore remains only to check that eq. (225) holds in $\{\lambda \geq \lambda_0\}$ for some $\lambda_0 = \lambda_0(k, A, Q) > 0$, and to check the corresponding result for Q_∞.

If $\varsigma < 0$, then the elements of $\mathcal{Q}$ and their derivatives all grow at worst polynomially as $\lambda \to \infty$, as shown by Proposition 3.3. Similarly, Lemma A.5 below shows that Mod_Q obeys a polynomial *lower bound* in the same asymptotic regime. The proposition therefore holds in this case.

If $\varsigma > 0$ and $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$, then, by Proposition 3.3, it is the case that $|A(\lambda)| \leq C|Q(\lambda)|$ for some $C > 0$ if λ is sufficiently large. Thus, since $|Q(\lambda)| \leq \operatorname{Mod}_Q(\lambda)^{1/2}$, the proposition holds in the $k = 0$ case. Consider now the $k = 1$ case. Differentiating the conclusion of Proposition 3.3 once, $A'(\lambda)$ is at worst growing polynomially as $\lambda \to \infty$ relative to Q, so the bound eq. (225) also holds in this case. Having verified the $k = 0, 1$ cases of the proposition, we deduce the remaining cases from the ODE $NA = 0$ which A is defined to satisfy. If $k \in \mathbb{N}^{\geq 2}$, differentiating this ODE $k - 2$ times yields

$$\frac{\partial^k A}{\partial \lambda^k} = \sum_{j=0}^{k-2} \frac{\partial^j A}{\partial \lambda^j} \frac{\partial^{k-2-j}}{\partial \lambda^{k-2-j}} \Big(\varsigma \lambda^\kappa + \frac{1}{\lambda^2}\Big(\alpha^2 - \frac{1}{4}\Big) + \frac{\Psi(\lambda)}{\lambda^2}\Big). \tag{226}$$

Taking $K_0 \in \mathbb{N}$ sufficiently large, each of the derivatives of $\varsigma\lambda^\kappa + \lambda^{-2}(\alpha^2 - 1/4) + \lambda^{-2}\Psi(\lambda)$ lies in $\lambda^{-K_0}\langle\lambda\rangle^{2K_0} L^\infty(\mathbb{R}^+_\lambda)$. We can conclude that

$$\partial_\lambda^k A \in \lambda^{-K-K_0} \langle \lambda \rangle^{2K+2K_0} \operatorname{Mod}_Q(\lambda) L^\infty, \tag{227}$$

assuming we have proven already that $\partial_\lambda^j A \in \lambda^{-K}\langle\lambda\rangle^{2K} \operatorname{Mod}_Q(\lambda)^{1/2} L^\infty$ for each $j \in \{0, \ldots, k-2\}$. This inductive argument completes the proof in the $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$ case.

In the $Q \in \operatorname{span}_{\mathbb{C}} Q_\infty$ case, then the result follows from the same inductive argument, except the base cases $k = 0, 1$ are now trivial because we are only comparing $|Q_\infty|^2, |Q'_\infty|^2$ with $\operatorname{Mod}_{Q_\infty}$. □

Proposition A.3. *Suppose that $\varsigma > 0$ and $Q \notin \operatorname{span}_{\mathbb{C}} Q_\infty$. For some $K \in \mathbb{N}$, we have $\operatorname{Mod}_{Q_\infty} \in \lambda^{-K}\langle\lambda\rangle^{2K} \operatorname{Mod}_Q^{-1} L^\infty$ and $\operatorname{Mod}_{Q_\infty}^{-1} \in \lambda^{-K}\langle\lambda\rangle^{2K} \operatorname{Mod}_Q L^\infty$. Consequently, for any $n, n', m, m' \in \mathbb{Z}$ satisfying $n' - m' = n - m$, it is the case that*

$$\operatorname{Mod}_Q^n \operatorname{Mod}_{Q_\infty}^m L^\infty \subseteq \lambda^{-K}\langle\lambda\rangle^{2K} \operatorname{Mod}_Q^{n'} \operatorname{Mod}_{Q_\infty}^{m'} L^\infty \tag{228}$$

for some $K = K(n, m, n', m') \in \mathbb{N}$. ■

Proof. This is an immediate corollary of Lemma A.4 (and Proposition 3.1). □

Recall that $1/\lambda^{\kappa+2}$ is a boundary-defining-function of $\{\lambda = \infty\} = \mathrm{ze} \cap \mathrm{fe}$ in fe.

Lemma A.4. *If $\varsigma > 0$, then $\operatorname{Mod}_Q(\rho^{-2/(\kappa+2)}) \in \exp(4(\kappa+2)^{-1}\rho^{-1})\rho^{\kappa/(\kappa+2)} C^\infty[0,\infty)_\rho$. Unless $Q \in \operatorname{span}_{\mathbb{C}} Q_\infty$,*

$$\operatorname{Mod}_Q(\rho^{-2/(\kappa+2)})^{-1} \in \exp(-4(\kappa+2)^{-1}\rho^{-1})\rho^{-\kappa/(\kappa+2)} C^\infty[0,\infty)_\rho. \tag{229}$$

Also, $\operatorname{Mod}_{Q_\infty}(\rho^{-2/(\kappa+2)}) \in \exp(-4(\kappa+2)^{-1}\rho^{-1})\rho^{\kappa/(\kappa+2)} C^\infty[0,\infty)_\rho$, and $\operatorname{Mod}_{Q_\infty}(\rho^{-2/(\kappa+2)})^{-1} \in \exp(4(\kappa+2)^{-1}\rho^{-1})\rho^{-\kappa/(\kappa+2)} C^\infty[0,\infty)_\rho$. ■

Proof. Certainly $Q(\rho^{-2/(\kappa+2)}) \in \exp(2(\kappa+2)^{-1}\rho^{-1})\rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho$.

Let $\lambda = \rho^{-2/(\kappa+2)}$. Writing $Q = \exp(2(\kappa+2)^{-1}\rho^{-1})q$ for $q(\rho) \in \rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho$, taking derivatives yields

$$\lambda\partial_\lambda Q(\rho^{-2/(\kappa+2)}) = \exp(2(\kappa+2)^{-1}\rho^{-1})(\rho^{-1}q + \lambda\partial_\lambda q), \tag{230}$$

and $\lambda\partial_\lambda q(\rho^{-2/(\kappa+2)}) \in \rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho$ by the dilation invariance of the operator $\lambda\partial_\lambda$. Equation (230) yields

$$\lambda\langle\lambda^{(\kappa+2)/2}\rangle^{-1}\partial_\lambda Q(\lambda) = \exp(2(\kappa+2)^{-1}\rho^{-1})\rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho. \tag{231}$$

Thus, $\mathrm{Mod}_Q(\lambda)$ lies in the claimed space. The function q has nonvanishing leading order term at $\lambda=\infty$, since otherwise Q would be in $\operatorname{span}_{\mathbb{C}} Q_\infty$ by Proposition 3.3. From this, it follows that the leading order term of $\mathrm{Mod}_Q(\lambda)\exp(-4(\kappa+2)^{-1}\rho^{-1}) \in \rho^{\kappa/(\kappa+2)}C^\infty[0,\infty)_\rho$ is nonvanishing. This implies that $\mathrm{Mod}_Q(\lambda)^{-1}$ lies in the desired space.

The remaining clause of the theorem is proven similarly, switching the signs of the exponentials around. □

If $\varsigma<0$, then, unless $Q \in \operatorname{span}_{\mathbb{C}} Q_\pm$, then the real-valued functions $|Q|^2$ and $\mathrm{Mod}_Q - |Q|^2$ have oscillatory terms as their leading asymptotics in the $\lambda\to\infty$ limit, and thus vanish infinitely often. We already know that the modulus function itself is nonvanishing, but it has to be ruled out that it does not take on any superpolynomially small values. Indeed:

Lemma A.5. *If $\varsigma<0$, then, for some $C=C(Q)>0$, $\mathrm{Mod}_Q(\lambda) - C\lambda^{-\kappa/2} \in \lambda^{-(\kappa+1)}L^\infty_{\mathrm{loc}}(0,\infty]_\lambda$. Thus, $\mathrm{Mod}_Q(\lambda)^{-1} - C^{-1}\lambda^{\kappa/2} \in \lambda^{-1}L^\infty_{\mathrm{loc}}(0,\infty]_\lambda$.* ■

Proof. We write $Q = c_-Q_- + c_+Q_+$ for $c_-,c_+\in\mathbb{C}$ not both zero. As above, we can write $Q_\pm = \exp(\pm 2i(\kappa+2)^{-1}\lambda^{(\kappa+2)/2})q_\pm$ for $q_\pm \in \rho^{\kappa/(2\kappa+4)}C^\infty[0,\infty)_\rho$ having nonvanishing leading parts at $\lambda=\infty$, so that

$$\lim_{\rho\to0^+}\rho^{-\kappa/(2\kappa+4)}q_\pm \neq 0. \tag{232}$$

Thus, for some $C_\pm \in \mathbb{C}\backslash\{0\}$, we have $q_\pm - \rho^{\kappa/(2\kappa+4)}C_\pm \in \rho^{(3\kappa+4)/(2\kappa+4)}C^\infty[0,\infty)_\rho$. Observe that

$$|Q|^2 = |c_-|^2|q_-|^2 + |c_+|^2|q_+|^2 + 2\Re[\exp(4i(\kappa+2)^{-1}\lambda^{(\kappa+2)/2})c_+c_-^*q_+q_-^*] \tag{233}$$

$$= \rho^{\kappa/(\kappa+2)}(|c_-C_-|^2 + |c_+C_+|^2 + 2\Re[\cdots]) + \rho^{(2\kappa+2)/(\kappa+2)}L^\infty, \tag{234}$$

where $\Re[\cdots] = \Re[\exp(4i(\kappa+2)^{-1}\lambda^{(\kappa+2)/2})c_+c_-^*C_+C_-^*]$.

By the same computation as above, $\lambda\langle\lambda^{(\kappa+2)/2}\rangle^{-1}\partial_\lambda Q_\pm(\lambda) = \exp(\pm2i(\kappa+2)^{-1}\lambda^{(\kappa+2)/2})\tilde{q}_\pm$ for $\tilde{q}_\pm$ satisfying

$$\tilde{q}_\pm \mp i\rho^{\kappa/(2\kappa+4)}C_\pm \in \rho^{(3\kappa+4)/(2\kappa+4)}C^\infty[0,\infty)_\rho. \tag{235}$$

Thus,

$$|\lambda\langle\lambda^{(\kappa+2)/2}\rangle^{-1}Q_\pm'|^2 = |c_-|^2|\tilde{q}_-|^2 + |c_+|^2|\tilde{q}_+|^2 + 2\Re[\exp(4i(\kappa+2)^{-1}\lambda^{(\kappa+2)/2})c_+c_-^*\tilde{q}_+\tilde{q}_-^*] \tag{236}$$

$$= \rho^{\kappa/(\kappa+2)}(|c_-C_-|^2 + |c_+C_+|^2 - 2\Re[\cdots]) + \rho^{(2\kappa+2)/(\kappa+2)}L^\infty, \tag{237}$$

So, $\mathrm{Mod}_Q(\rho^{-2/(\kappa+2)}) = \rho^{\kappa/(\kappa+2)}C + \rho^{(2\kappa+2)/(\kappa+2)}L^\infty$ for $C = 2|c_-C_-|^2 + 2|c_+C_+|^2$. Rewriting this in terms of λ, the result is the claim. □

Appendix B. The initial value problem

We now consider the derivation of Theorem A from Theorem B.

Let $Q_1,Q_2 \in \mathcal{Q}$ denote a basis of $\mathcal{Q}$. If $\varsigma<0$, then choose $Q_1 = Q_-$ and $Q_2 = Q_+$. If $\varsigma>0$, choose $Q_1 = Q_\infty$ and Q_2 to be an independent element of $\mathcal{Q}$. Let u_1,u_2 denote the corresponding solutions to $Pu=0$ produced by Theorem B, for $Q=Q_1$ and $Q=Q_2$, respectively.

First, a lemma:

Lemma B.1. *The Wronskian* $W\{u_1,u_2\}(h) = u_1u_2' - u_1'u_2$ *satisfies*

$$W\{u_1,u_2\}(h) - h^{-2/(\kappa+2)}W\{Q_1,Q_2\} \in h^{\kappa/(\kappa+2)}C^\infty[0,\infty)_h, \tag{238}$$

where $W\{u_1,u_2\} = Q_1Q_2' - Q_1'Q_2$. ■

Proof. By the Langer diffeomorphism, it suffices to work in the case where the coefficient W in the PDE is $W = 1$.

Because $u_1u_2' - u_1'u_2$ is only a function of h, we can evaluate it at any ζ. Near ie, we can write $u_1 = (1+ha_1)Q_1(\lambda)$ and $u_2 = (1+ha_2)Q_2(\lambda)$ for $a_1,a_2 \in C^\infty((0,Z]_\zeta \times [0,\infty)_h;\mathbb{C})$ and $\lambda = \zeta/h^{2/(\kappa+2)}$, in part because, for the chosen $Q_\bullet$, the functions $Q_1(\lambda),Q_2(\lambda)$ are not vanishing for λ sufficiently large. (This is just a way of saying that the Liouville–Green ansatz only requires a single term, unlike Langer's ansatz which involves both Q,Q'.)

Then, explicitly,

$$\begin{aligned} W\{u_1,u_2\}(h) = h^{-2/(\kappa+2)}W\{Q_1,Q_2\} &+ hW\{Q_1(\lambda),a_2Q_2(\lambda)\}(h) \\ &+ hW\{a_1Q_1(\lambda),Q_2(\lambda)\}(h) + h^2W\{a_1Q_1(\lambda),a_2Q_2(\lambda)\}(h), \end{aligned} \tag{239}$$

where $W\{Q_1,Q_2\} = Q_1Q_2' - Q_1'Q_2$ is a constant. (Because a_2Q_2, a_1Q_1 typically do not solve the ODE, the "Wronskians" on the right-hand side depend on the ζ at which they are evaluated, but we will not denote this explicitly.)

The particular choice of Q_1,Q_2 means that

$$h^{-\kappa/(\kappa+2)}Q_1(\lambda)Q_2(\lambda), Q_1'(\lambda)Q_2(\lambda), Q_1(\lambda)Q_2'(\lambda) \in C^\infty((0,Z]_\zeta \times [0,\infty)_h;\mathbb{C}), \tag{240}$$

using Proposition 3.3, where λ is still $\zeta/h^{2/(\kappa+2)}$.

Then, we explicitly evaluate

$$\begin{aligned} W\{Q_1(\lambda),a_2Q_2(\lambda)\}(h) = a_2'Q_1(\lambda)Q_2(\lambda) &+ h^{-2/(\kappa+2)}a_2Q_1(\lambda)Q_2'(\lambda) \\ &- h^{-2/(\kappa+2)}a_2Q_1'(\lambda)Q_2(\lambda) \in h^{-2/(\kappa+2)}C^\infty, \end{aligned} \tag{241}$$

$$\begin{aligned} hW\{a_1Q_1(\lambda),Q_2(\lambda)\}(h) = -a_1'Q_2(\lambda)Q_1(\lambda) &- h^{-2/(\kappa+2)}a_1Q_2(\lambda)Q_1'(\lambda) \\ &+ h^{-2/(\kappa+2)}a_1Q_2'(\lambda)Q_1(\lambda) \in h^{-2/(\kappa+2)}C^\infty, \end{aligned} \tag{242}$$

and

$$\begin{aligned} W\{a_1Q_1,a_2Q_2\} = a_1a_2'Q_1Q_2 + h^{2/(\kappa+2)}a_1a_2Q_1Q_2' - a_1'a_2Q_1Q_2 - h^{2/(\kappa+2)}a_1a_2Q_1Q_2' \\ \in h^{-2/(\kappa+2)}C^\infty, \end{aligned} \tag{243}$$

where $C^\infty = C^\infty((0,Z]_\zeta \times [0,\infty)_h;\mathbb{C})$. Adding everything together, the claim follows. □

We return to the proof of Theorem A. It must be the case that $W\{Q_1,Q_2\} \neq 0$. Thus, if h is sufficiently small, Lemma B.1 implies that u_1,u_2 are linearly independent, and a solution u to $Pu = 0$ with prescribed initial data $u|_{\mathrm{ie}}, u'|_{\mathrm{ie}}$ can be written as

$$\begin{aligned} u(\zeta,h) &= \begin{bmatrix} u_1(\zeta,h) \\ u_2(\zeta,h) \end{bmatrix}^\intercal \begin{bmatrix} u_1|_{\mathrm{ie}}(h) & u_2|_{\mathrm{ie}}(h) \\ u_1'|_{\mathrm{ie}}(h) & u_2'|_{\mathrm{ie}}(h) \end{bmatrix}^{-1} \begin{bmatrix} u|_{\mathrm{ie}}(h) \\ u'|_{\mathrm{ie}}(h) \end{bmatrix} \\ &= \frac{1}{W\{u_1,u_2\}(h)} \begin{bmatrix} u_1(\zeta,h) \\ u_2(\zeta,h) \end{bmatrix}^\intercal \begin{bmatrix} u_2'|_{\mathrm{ie}}(h) & -u_2|_{\mathrm{ie}}(h) \\ -u_1'|_{\mathrm{ie}}(h) & u_1|_{\mathrm{ie}}(h) \end{bmatrix} \begin{bmatrix} u|_{\mathrm{ie}}(h) \\ u'|_{\mathrm{ie}}(h) \end{bmatrix}. \end{aligned} \tag{244}$$

Thus, u is a linear combination of products of functions of exponential-polyhomogeneous type on M, so is therefore of exponential-polyhomogeneous type itself.

Appendix C. Collapsing fe

We now consider some simplifications to the main result that apply when $\kappa \in \{-1,0,1\}$, assuming that $\phi = 0$, so that

$$P = -h^2 \frac{\partial^2}{\partial z^2} + \varsigma z^\kappa W(z) + h^2 E, \tag{245}$$

where we assume that $E \in C^\infty([0,Z]_z \times [0,\infty)_{h^2};\mathbb{C})$ if $\kappa \in \{0,1\}$ and $E \in z^{-1}C^\infty([0,Z]_z \times [0,\infty)_{h^2};\mathbb{C})$ if $\kappa = -1$. It is convenient in applications to allow the analysis to extend beyond $z = 0$. Let $Z_- \in (-\infty, 0]$, $W \in C^\infty([Z_-,Z]_z;\mathbb{R}^+)$, and suppose that

$$E \in \begin{cases} C^\infty([Z_-,Z]_z \times [0,\infty)_{h^2};\mathbb{C}) & (\kappa = 0,1), \\ z^{-1}C^\infty([Z_-,Z]_z \times [0,\infty)_{h^2};\mathbb{C}) & (\kappa = -1). \end{cases} \tag{246}$$

With the stated assumptions, we have the following sharpening of the classic result:

Theorem C. *For any $Q \in \mathcal{Q}$, there exist $\beta, \gamma \in C^\infty([0,Z]_z \times [0,\infty)_{h^2};\mathbb{C})$ such that the function u defined by*

$$u = \sqrt[4]{\frac{\xi^\kappa}{W}} \Big[(1 + h^2\beta) Q\Big(\frac{\zeta}{h^{2/(\kappa+2)}} \Big) + h^{(2\kappa+2)/(\kappa+2)} \gamma Q'\Big(\frac{\zeta}{h^{2/(\kappa+2)}} \Big) \Big] \tag{247}$$

solves $Pu = 0$, where

$$\zeta(z) = \operatorname{sign}(z) \Big(\frac{\kappa+2}{2} \int_0^{|z|} \omega^{\kappa/2} \sqrt{W(\operatorname{sign}(z)\omega)} \,\mathrm{d}\omega \Big)^{2/(\kappa+2)} \in C^\infty(\mathbb{R}_z). \tag{248}$$

Under the stated conditions, one can simplify Theorem A in an analogous way.

Proof. By the Langer diffeomorphism, which applies in the present context, it suffices to consider the $W = 1$ case. The proof follows that of Theorem B, with a few simplifications.

The first key simplification is that the coefficients functions $\beta_k, \gamma_k \in C^\infty(0,Z]_\zeta$ defined by eq. (81), eq. (82), with C_k as in the body and an appropriate choice of c_k, are all extendable to elements of the subset $C^\infty[Z_-,Z]_\zeta$. (And if $\kappa = -1$, then the same applies to $\zeta^{-1}\gamma_\bullet$.) If, for some $k \in \mathbb{N}$, we know that $\beta_1(\zeta), \ldots, \beta_k(\zeta) \in C^\infty[Z_-,Z]_\zeta$, then eq. (82), which now reads

$$\gamma_k(\zeta) = -\frac{\varsigma}{2\zeta^{\kappa/2}} \int_0^\zeta \Big(\frac{\mathrm{d}^2 \beta_k(\omega)}{\mathrm{d}\omega^2} - \sum_{j=0}^k E_k(\omega)\beta_{k-j}(\omega) \Big) \frac{\mathrm{d}\omega}{\omega^{\kappa/2}}, \tag{249}$$

tells us that $\gamma_k(\zeta) \in C^\infty[Z_-,Z]_\zeta$. In fact, if $\kappa = -1, 0$, then $\gamma_k(\zeta) \in \zeta C^\infty[Z_-,Z]_\zeta$. Then, eq. (81), for some choice of c_k, reads

$$\beta_{k+1} = -\frac{1}{2} \int_0^\zeta \Big(\frac{\mathrm{d}^2 \gamma_k(\omega)}{\mathrm{d}\omega^2} - \sum_{j=0}^k E_k(\omega)\gamma_{k-j}(\omega) \Big) \,\mathrm{d}\omega, \tag{250}$$

with the key point being that the integral converges because the various terms in the integrand are all smooth. (The only function in the integrand which might not be smooth is E_k when $\kappa = -1$, in which case it can have a simple pole, but $\gamma_{k-j}(\omega)/\omega$ is smooth in this case, so the product $E_k(\omega)\gamma_{k-j}(\omega)$ is still smooth.) So, β_{k+1} is also smooth. Since $\beta_0 = 1$, one proceeds inductively to conclude that all β_k, γ_k are smooth down to $\zeta = 0$ (and through, if $Z_- < 0$). Then, the β_k, γ_k can be asymptotically summed, not in polyhomogeneous spaces on M as in the body of the paper, but in $C^\infty = C^\infty([Z_-,Z]_\zeta \times [0,\infty)_{h^2};\mathbb{C})$. Then, one gets a v of the form eq. (247), for $\beta, \gamma \in C^\infty$ such that $Pv \in h^\infty C^\infty$.

The analysis in §6 applies mutatis mutandis, and there is no longer any need to restrict attention to $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$. Instead, the fixed point argument takes h_0 sufficiently small so as to guarantee the smallness of the operator norms of the maps Φ, Ξ. So the argument in that section directly produces a w of the form $w = \delta Q + \varepsilon Q'$ for $\delta, \varepsilon \in h^\infty C^\infty$ such that $Pw = -Pv$.

Then, $u = w + v$ has the desired form, after absorbing the δ, ε into a redefinition of β, γ, and satisfies $Pu = 0$. $\square$

Appendix D. An example with logarithms

Consider the case $\kappa = 2$, $\varsigma = 1$, $W = 1$, and $E = a$ for constant $a > 0$. Then, $P = -h^2(\partial/\partial z)^2 + z^2 + h^2 a$, and the ODE $Pu = 0$ reads

$$h^2 \frac{\partial^2 u}{\partial z^2} = z^2 u + h^2 a u. \tag{251}$$

The solutions to this can be written

$$u = c_1(h) U\Big(-\frac{ah}{2}, \sqrt{\frac{2}{h}} z \Big) + c_2(h) U\Big(\frac{ah}{2}, i\sqrt{\frac{2}{h}} z \Big), \tag{252}$$

for arbitrary $c_1, c_2 : (0,\infty) \to \mathbb{C}$, where U is the usual parabolic cylinder function [Olv97, Chp. 6-§6]. From eq. (252), it is not apparent where the logarithmic terms at fe discussed above come from. Indeed, if we take $c_1 = 1$ and $c_2 = 0$, or vice versa, then u is simply smooth at fe°, due to the fact that $U(a,z) \in C^\infty(\mathbb{R}^+_a \times K^\circ_z; \mathbb{C})$ for $K \subset \mathbb{R}$ compact.

The paradox is resolved by noting that $u = U(-ah/2, 2^{1/2} z h^{-1/2})$ is not of exponential type at ze° (i.e. smooth up to an exponential prefactor), only exponential-polyhomogeneous type.

In order to see this, we use the coordinate system (x, ρ) for $x = 2^{-1/2} a z^{1/2}$ and $\rho = 2^{-1/2} h z^{-1/2}$. Then, $u = U(-x\rho, 1/\rho)$. The large-argument expansion of the parabolic cylinder function [OMe, §12.9] gives the Poincaré-type expansion

$$u \sim e^{-1/4\rho^2} \rho^{1/2 - x\rho} \sum_{k=0}^\infty (-1)^k \frac{\Gamma(1/2 + 2k - x\rho)}{k!\Gamma(1/2 - x\rho)} \Big(\frac{\rho^2}{2} \Big)^k \tag{253}$$

as $\rho \to 0$. The logarithmic terms are hidden in the $\rho^{-x\rho} = e^{-x\rho \log \rho}$ term. Indeed, we have the following polyhomogeneous expansion:

$$\rho^{-x\rho} \sim \sum_{k=0}^\infty \frac{(-1)^k}{k!} (x\rho \log \rho)^k. \tag{254}$$

So, when organized into the form of an exponential-polyhomogeneous expansion, eq. (253) has logarithmic terms.

If one instead chooses c_1, c_2 so as to make u of exponential-type at ze° (which can be done already by Liouville–Green), then instead logarithmic terms appear at fe.

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Email address: ethanws@stanford.edu

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, CALIFORNIA, USA