

A STRENGTHENED ORLICZ–PETTIS THEOREM VIA ITÔ–NISIO

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ABSTRACT. In this note we deduce a strengthening of the Orlicz–Pettis theorem from the Itô–Nisio theorem. The argument shows that given any series in a Banach space which isn’t summable (or more generally unconditionally summable), we can *construct* a (coarse-grained) subseries with the property that – under some appropriate notion of “almost all” – almost all further subseries thereof fail to be weakly summable. Moreover, a strengthening of the Itô–Nisio theorem by Hoffmann–Jørgensen allows us to replace ‘weakly summable’ with ‘ τ -weakly summable’ for appropriate topologies τ weaker than the weak topology. A treatment of the Itô–Nisio theorem for admissible τ is given.

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1. INTRODUCTION

Let \mathcal{X} denote a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Call a subset $\tau \subseteq 2^{\mathcal{X}}$ an *admissible topology* on \mathcal{X} if

- (1) it is an LCTVS¹-topology on \mathcal{X} identical to or weaker than the norm (a.k.a. strong) topology under which the norm-closed unit ball $\mathbb{B} = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is τ -closed, and
- (2) if \mathcal{X} is *not* separable, then τ is at least as strong as the weak topology.

Cf. [HJ74], from which the separable case of this definition arises. By the Hahn–Banach separation theorem, if τ is an admissible topology then the τ -weak topology (a.k.a. $\sigma(\mathcal{X}, \mathcal{X}_\tau^*)$ -topology) is also admissible (see Lemma A.1).

Besides the norm topology itself, which is trivially admissible (and uninteresting below), the most familiar example of an admissible topology on \mathcal{X} is the weak topology. Many others arise in functional analysis. For example, given a compact Riemannian manifold M , for most function spaces \mathcal{F} it is the case that the $\sigma(\mathcal{F}, C^\infty(M))$ -topology (a.k.a. the topology of distributional convergence) is admissible. An even weaker typically admissible topology is that on \mathcal{F} generated by the functionals $\langle -, \varphi_n \rangle : \mathcal{D}'(M) \rightarrow \mathbb{C}$ for $\varphi_0, \varphi_1, \varphi_2, \dots$ the eigenfunctions of the Laplacian.

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¹By ‘LCTVS’ we mean a *Hausdorff* locally convex topological vector space, so we follow the conventions in [Rud73].

Denote by $\mathcal{X}^{\mathbb{N}}$ the vector space of all \mathcal{X} -valued sequences $\{x_n\}_{n=0}^{\infty} \subseteq \mathcal{X}$. In the usual way, we identify such sequences with \mathcal{X} -valued formal series (and denote accordingly). We say that a formal series $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$ is “ τ -summable” if $\sum_{n=0}^N x_n \in \mathcal{X}$ converges as $N \rightarrow \infty$ in \mathcal{X}_{τ} .

Consider the following (slightly generalized) version of the Orlicz–Pettis theorem [Orl29]:

Theorem 1.1. *Suppose that τ is an admissible topology on \mathcal{X} . If $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$ fails to be unconditionally summable in the norm topology, then*

- *there exist some $\epsilon_0, \epsilon_1, \epsilon_2, \dots \in \{-1, +1\}$ such that the sequence $\Sigma(\{\epsilon_n\}_{n=0}^{\infty}) = \{\Sigma_N\}_{N=0}^{\infty}$ defined by*

$$\Sigma_N = \sum_{n=0}^N \epsilon_n x_n \quad (1)$$

does not τ -converge as $N \rightarrow \infty$ to any element of \mathcal{X} , and

- *there exist some $\chi_0, \chi_1, \chi_2, \dots \in \{0, 1\}$ such that the sequence $S(\{\chi_n\}_{n=0}^{\infty}) = \{S_N\}_{N=0}^{\infty}$ defined by*

$$S_N = \sum_{n=0}^N \chi_n x_n, \quad (2)$$

does not τ -converge as $N \rightarrow \infty$ to any element of \mathcal{X} .

In particular, this applies if $\sum_{n=0}^{\infty} x_n$ is not summable in the norm topology. ■

Remark. From the formulas

$$\Sigma_N(\{\epsilon_n\}_{n=0}^N) = S_N(\{2^{-1}(1 + \epsilon_n)\}_{n=0}^N) - S_N(\{2^{-1}(1 - \epsilon_n)\}_{n=0}^N) \quad (3)$$

$$S_N(\{\chi_n\}_{n=0}^N) = 2^{-1}\Sigma_N(\{1\}_{n=0}^N) + 2^{-1}\Sigma_N(\{2\chi_n - 1\}_{n=0}^N), \quad (4)$$

we deduce that $\Sigma(\{\epsilon_n\}_{n=0}^{\infty})$ is τ -convergent for all $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$ if and only if $S(\{\chi_n\}_{n=0}^{\infty})$ is τ -convergent for all $\{\chi_n\}_{n=0}^{\infty} \in \{0, 1\}^{\mathbb{N}}$. We will phrase the discussion below in terms of whichever of $\Sigma(-)$, $S(-)$ is convenient, but this equivalence should be kept in mind.

See Proposition 2.5 for the probabilistic version of this remark. ■

Example. Let M be a compact Riemannian manifold and $\mathcal{F} \subseteq \mathcal{D}'(M)$ be a function space on M . Let τ denote the topology generated by the functionals $\langle -, \varphi_n \rangle_{L^2(M)}$, where $\varphi_0, \varphi_1, \varphi_2, \dots$ denote the eigenfunctions of the Laplace–Beltrami operator. Suppose that τ is admissible. This holds, for example, if \mathcal{F} is an L^p -based Sobolev space for $p \in [1, \infty)$.

Then, for any $\{x_n\}_{n=0}^{\infty} \subseteq \mathcal{F}$, the formal series $\sum_{n=0}^{\infty} x_n$ is unconditionally summable in \mathcal{F} (in norm) if and only if

$$\sum_{n=0}^{\infty} |\langle x_n, \varphi_m \rangle| < \infty \quad (5)$$

for all $m \in \mathbb{N}$ and, for all $\{\chi_n\}_{n=0}^{\infty} \subseteq \{0, 1\}$, there exists an element $S(\{\chi_n\}_{n=0}^{\infty}) \in \mathcal{F}$ whose m th Fourier coefficient is given by

$$\langle S(\{\chi_n\}_{n=0}^{\infty}), \varphi_m \rangle = \sum_{n=0}^{\infty} \chi_n \langle x_n, \varphi_m \rangle. \quad (6)$$

■

We focus on Banach spaces – as opposed to more general LCTVSs – for simplicity. Most of the considerations below apply equally well to Fréchet spaces. There is a long history of variants of the Orlicz–Pettis theorem for various sorts of TVSs [Die77]. A short proof of the Orlicz–Pettis theorem for Banach spaces can be found in [BP58], and a textbook presentation can be found in [Meg98]. The proof below has much in common with a probabilistic proof [Die84] based on the Bochner integral (due to Kwapien).

The proof below is nonconstructive, in the following sense: upon being given a formal series $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$ which fails to be unconditionally summable, we do not construct any particular sequence $\{\epsilon_n\}_{n=0}^{\infty} \subseteq \{-1, +1\}$ such that $\Sigma(\{\epsilon_n\}_{n=0}^{\infty}) \subseteq \mathcal{X}$ fails to converge in \mathcal{X}_{τ} , or any particular $\{\chi_n\}_{n=0}^{\infty} \subseteq \{0, 1\}$ such that $S(\{\chi_n\}_{n=0}^{\infty}) \subseteq \mathcal{X}$ fails to converge in \mathcal{X}_{τ} . All proofs of the Orlicz–Pettis theorem seem to be nonconstructive in this regard. We do, however, construct a function

$$\mathcal{E} : \{\{x_n\}_{n=0}^{\infty} \in \mathcal{X}^{\mathbb{N}} \text{ not unconditionally summable}\} \rightarrow 2^{\{-1, +1\}^{\mathbb{N}}}, \quad (7)$$

such that, when $\{x_n\}_{n=0}^{\infty}$ is not unconditionally summable, $\Sigma(\{\epsilon_n\}_{n=0}^{\infty})$ and $S(\{2^{-1}(1 - \epsilon_n)\}_{n=0}^{\infty})$ both fail to be τ -summable for $\mathbb{P}_{\text{Coarse}}$ -almost all sequences $\{\epsilon_n\}_{n=0}^{\infty} \in \mathcal{E}$, where

$$\mathbb{P}_{\text{Coarse}} : \text{Borel}(\{-1, +1\}^{\mathbb{N}}) |_{\mathcal{E}(\{x_n\}_{n=0}^{\infty})} \rightarrow [0, 1] \quad (8)$$

is a probability measure on the subspace σ -algebra

$$\text{Borel}(\{-1, +1\}^{\mathbb{N}}) |_{\mathcal{E}(\{x_n\}_{n=0}^{\infty})} = \{S \cap \mathcal{E}(\{x_n\}_{n=0}^{\infty}) : S \in \text{Borel}(\{-1, +1\}^{\mathbb{N}})\}. \quad (9)$$

So, while the proof is nonconstructive, it is only just. Put more colorfully, the proof follows the “hay in a haystack” philosophy familiar from applications of the probabilistic method to combinatorics [AS16]: using an appropriate sampling procedure, we choose a random subseries and show that – with “high probability” (which in this case means probability one) – it has the desired property.

Precisely, letting \mathbb{P}_{Haar} denote the Haar measure on the Cantor group $\{-1, +1\}^{\mathbb{N}} \cong \mathbb{Z}_2^{\mathbb{N}}$ [Die84] (which is a compact topological group under the product topology, by Tychonoff’s theorem):

Theorem 1.2 (Probabilist’s Orlicz–Pettis Theorem). *Suppose that $f : \mathbb{N} \rightarrow \mathbb{N}$ is a function such that $|f^{-1}(\{n\})| < \infty$ for all $n \in \mathbb{N}$. If $\mathcal{T} \subseteq \mathbb{N}$ is infinite and satisfies*

$$\limsup_{\substack{n \rightarrow \infty \\ n \in \mathcal{T}}} \left\| \sum_{n_0 \in f^{-1}(\{n\})} x_{n_0} \right\| > 0, \quad (10)$$

then it is the case that, for \mathbb{P}_{Haar} -almost all $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$, the formal series

$$\sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \epsilon_{f(n)} x_n \in \mathcal{X}^{\mathbb{N}}, \quad \sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \frac{1}{2}(1 - \epsilon_{f(n)}) x_n \in \mathcal{X}^{\mathbb{N}} \quad (11)$$

both fail to be τ -summable. ■

The relation to Orlicz–Pettis is as follows. If $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$ is not unconditionally summable, then we can find some pairwise disjoint, finite subsets $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \dots \subseteq \mathbb{N}$ such that

$$\inf_{N \in \mathbb{N}} \left\| \sum_{n \in \mathcal{N}_N} x_n \right\| > 0. \quad (12)$$

We can then choose some $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n) = f(m)$ if and only if either $n = m$ or $n, m \in \mathcal{N}_N$ for some $N \in \mathbb{N}$. Thus, if we set $\mathcal{T} = \mathbb{N}$, eq. (10) holds. Appealing to Theorem 1.2, we conclude that, for \mathbb{P}_{Haar} -almost all $\{\epsilon_n\}_{n=0}^{\infty}$, the formal series

$$\sum_{n=0}^{\infty} \epsilon_{f(n)} x_n \in \mathcal{X}^{\mathbb{N}}, \quad \sum_{n=0}^{\infty} \frac{1}{2}(1 - \epsilon_{f(n)}) x_n \in \mathcal{X}^{\mathbb{N}} \quad (13)$$

both fail to be τ -summable. Theorem 1.1, therefore, follows from Theorem 1.2. The connection with eq. (7), eq. (8) is that we can choose f such that \mathcal{E} is the set of $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$ such that $\epsilon_n = \epsilon_m$ whenever $f(n) = f(m)$, and $\mathbb{P}_{\text{Coarse}}$ is \mathbb{P}_{Haar} conditioned on the event that $\{\epsilon_n\}_{n=0}^{\infty} \in \mathcal{E}$.

Remark. The Haar measure on the Cantor group is the unique measure on $\text{Borel}(\{-1, +1\}^{\mathbb{N}}) = \sigma(\{\epsilon_n\}_{n=0}^{\infty})$ such that if we define $\epsilon_n : \{-1, +1\}^{\mathbb{N}} \rightarrow \{-1, +1\}$ by $\epsilon_n : \{\epsilon'_m\}_{m=0}^{\infty} \mapsto \epsilon'_n$, the random variables $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ are i.i.d. Rademacher random variables. ■

Remark. It suffices to prove the theorems above when \mathcal{X} is separable. Indeed, if \mathcal{X} is not separable and \mathcal{Y} denotes the norm-closure of the span of $x_0, x_1, x_2, \dots \in \mathcal{X}$, then, for any $\{\lambda_n\}_{n=0}^\infty \subseteq \mathbb{K}$,

$$\tau\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N \lambda_n x_n \quad (14)$$

exists in \mathcal{X} if and only if it exists in \mathcal{Y} . (This is a consequence of the requirement that τ be at least as strong as the weak topology, so the limit in eq. (14) is also a weak limit. Norm-closed convex subsets of \mathcal{X} are weakly closed by Hahn-Banach, so this implies that \mathcal{Y} is τ -closed.)

The subspace topology on $\mathcal{Y} \hookrightarrow \mathcal{X}_\tau$ is admissible, and \mathcal{Y} is separable, so we can deduce Theorem 1.1 and Theorem 1.2 for \mathcal{X} from the same theorems for \mathcal{Y} . ■

Remark. If \mathcal{X} is not separable and τ not at least as strong as the weak topology, then the conclusions of these theorems may fail to hold, even if the norm-closed balls in \mathcal{X} are τ -closed. As a simple counterexample, let $\mathcal{X} = L^\infty[0, 1]$, and let τ be the $\sigma(L^\infty, L^1)$ -topology. This being a weak-* topology, the norm-closed balls are τ -closed (and even τ -compact). Let

$$\Sigma_N(t) = t^N, \quad (15)$$

$x_n(t) = \Sigma_n(t) - \Sigma_{n-1}(t)$ for $n \geq 1$, $x_0(t) = \Sigma_0(t) = 1$. It turns out that the series $\sum_{n=0}^\infty x_n$ is τ -subseries summable. Indeed, if $\{\chi_n\}_{n=0}^\infty \subseteq \{0, 1\}$, then define

$$S(\{\chi_n\}_{n=0}^\infty)(t) = \sum_{n=0}^\infty \chi_n x_n(t) \in \mathbb{R} \quad (16)$$

for each $t \in [0, 1]$. By the monotone convergence theorem, this converges pointwise (so the definition makes sense, and $S(\{\chi_n\}_{n=0}^\infty)$ is a measurable function of t), and satisfies $S(\{\chi_n\}_{n=0}^\infty)(t) \in [0, 1]$, so $S(\{\chi_n\}_{n=0}^\infty) \in L^\infty[0, 1]$. If $f \in L^1[0, 1]$, then

$$\left| \int_0^1 f(t) \sum_{n=N}^\infty \chi_n x_n(t) dt \right| \leq \left| \int_0^{1-1/\sqrt{N}} f(t) \sum_{n=N}^\infty \chi_n x_n(t) dt \right| + \left| \int_{1-1/\sqrt{N}}^1 f(t) \sum_{n=N}^\infty \chi_n x_n(t) dt \right|. \quad (17)$$

For $N \geq 1$, the first term on the right-hand side is bounded above by

$$\|f\|_{L^1} \sup_{t \in [0, 1-1/\sqrt{N}]} \sum_{n=N}^\infty |x_n(t)| = \|f\|_{L^1} \sup_{t \in [0, 1-1/\sqrt{N}]} t^{N-1} = \|f\|_{L^1} \left(1 - \frac{1}{\sqrt{N}}\right)^{N-1}, \quad (18)$$

which converges to 0 as $N \rightarrow \infty$. On the other hand, the second term on the right-hand side of eq. (17) is bounded above by

$$\left(\sup_{t \in [0, 1]} \sum_{n=0}^\infty |x_n(t)| \right) \int_{1-1/\sqrt{N}}^1 |f(t)| dt = 2 \int_{1-1/\sqrt{N}}^1 |f(t)| dt, \quad (19)$$

which converges to 0 as $N \rightarrow \infty$ by the measurability of f . So, we can conclude that the convergence in eq. (16) is in τ .

But, Σ_N does not converge uniformly on $[0, 1]$ as $N \rightarrow \infty$, so $\sum_{n=0}^\infty x_n$ is not strongly summable in $\mathcal{X} = L^\infty[0, 1]$. Thus, the conclusion of Theorem 1.1 does not hold for this space \mathcal{X} and this topology τ . ■

Example. If $\mathcal{X} = C^0[0, 1]$, the set of continuous functions $[0, 1] \rightarrow \mathbb{C}$ with the topology of uniform convergence, and τ is the $\sigma(C^0, L^1)$ -topology, then the hypotheses of the theorems regarding \mathcal{X}, τ are satisfied, since \mathcal{X} is separable and the norm-closed balls in \mathcal{X} are τ -closed. Letting $x_n(t)$ be as in the previous remark, the failure of $\sum_{n=0}^\infty x_n$ to be strongly summable implies (by Theorem 1.2) that $\sum_{n=0}^\infty \chi_n x_n$ cannot be τ -summable in \mathcal{X} for $\mathbb{P}_{\text{Coarse}}$ -almost all $\{\chi_n\}_{n=0}^\infty \subset \{0, 1\}$. But we can define the pointwise limit $S(\{\chi_n\}_{n=0}^\infty)(t)$ as in eq. (16), and we saw convergence in the $\sigma(L^\infty, L^1)$ -topology. Consequently, if there were to exist some

$$\tilde{S}(\{\chi_n\}_{n=0}^\infty)(t) \in C^0[0, 1] \quad (20)$$

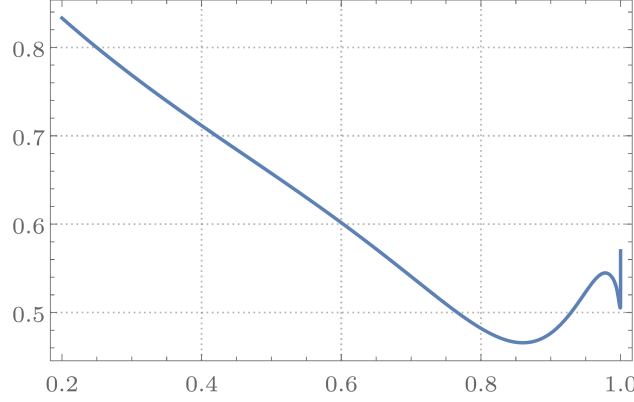


FIGURE 1. A plot of $S_N(t) = 1 + \sum_{n=1}^N \chi_n(t^n - t^{n-1})$ vs. t (horizontal axis) for large N and sampled i.i.d. uniform $\chi_n \in \{0, 1\}$. As N grows, $S_N(t)$ oscillates rapidly as $t \rightarrow 1^-$, much like the topologist's sine curve, in accordance with the prediction (which we know holds for $\mathbb{P}_{\text{Coarse}}$ -almost all $\{\chi_n\}_{n=0}^\infty$) that the full sum $S(t) = \lim_{N \rightarrow \infty} S_N(t)$ does not have a well-defined limit as $t \rightarrow 1^-$.

agreeing with $S(\{\chi_n\}_{n=0}^\infty)(t)$ almost everywhere, then $\sum_{n=0}^\infty \chi_n x_n$ would have to converge to $\tilde{S}(\{\chi_n\}_{n=0}^\infty)$ in $\tau = \sigma(C^0, L^1)$, since this is just the subspace topology of $\sigma(L^\infty, L^1)$. So, it must be the case that, for $\mathbb{P}_{\text{Coarse}}$ -almost all $\{\chi_n\}_{n=0}^\infty$,

$$\tilde{S}(\{\chi_n\}_{n=0}^\infty)(t) \notin C^0[0, 1] \quad (21)$$

if $\tilde{S}(\{\chi_n\}_{n=0}^\infty)$ agrees with $S(\{\chi_n\}_{n=0}^\infty)$ almost everywhere in $[0, 1]_t$. But, the series $\sum_{n=0}^\infty \chi_n x_n$ converges uniformly in $[0, 1 - \delta]$ for every $\delta \in (0, 1)$, so the Weierstrass M-test gives $S(\{\chi_n\}_{n=0}^\infty) \in C^0[0, 1]$. If $\lim_{t \rightarrow 1^-} S(\{\chi_n\}_{n=0}^\infty)(t)$ were to exist, then we could define

$$\tilde{S}(\{\chi_n\}_{n=0}^\infty)(t) = \begin{cases} S(\{\chi_n\}_{n=0}^\infty)(t) & (t < 1) \\ \lim_{s \rightarrow 1^-} S(\{\chi_n\}_{n=0}^\infty)(s) & (t = 1), \end{cases} \quad (22)$$

and this would lie in $C^0[0, 1]$ and agree with $S(\{\chi_n\}_{n=0}^\infty)$ almost everywhere in $[0, 1]_t$. So, it must be the case that $\lim_{t \rightarrow 1^-} S(\{\chi_n\}_{n=0}^\infty)(t)$ fails to exist for $\mathbb{P}_{\text{Coarse}}$ -almost all $\{\chi_n\}_{n=0}^\infty$. See Figure 1. ■

Remark. When \mathcal{X} is separable, it suffices to consider the case when τ is the topology generated by a countable norming set of functionals. Recall that a subset $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ is called *norming* if

$$\|x\| = \sup_{\Lambda \in \mathcal{S}} |\Lambda x| \quad (23)$$

for all $x \in \mathcal{X}$. We can scale the members of a norming subset to get another norming subset whose members Λ satisfy $\|\Lambda\|_{\mathcal{X}^*} = 1$, and this generates the same topology. If τ is admissible, then (by the Hahn-Banach theorem and separability) there exists a countable norming subset $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ (see Lemma A.2). Whenever $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ is a countable norming subset, the $\sigma(\mathcal{X}, \mathcal{S})$ -topology is admissible as well (see Lemma A.3), and identical with or weaker than τ . ■

It is not necessary to consider probability spaces other than

$$(\{-1, +1\}^\mathbb{N}, \text{Borel}(\{-1, +1\}^\mathbb{N}), \mathbb{P}_{\text{Haar}}), \quad (24)$$

but it will be convenient to have a bit more freedom. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space on which i.i.d. Bernoulli random variables

$$\chi_0, \chi_1, \chi_2, \dots : \Omega \rightarrow \{0, 1\} \quad (25)$$

are defined. For example,

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\{-1, +1\}^{\mathbb{N}}, \text{Borel}(\{-1, +1\}^{\mathbb{N}}), \mathbb{P}_{\text{Haar}}), \quad (26)$$

in which case we set $\chi_n = (1/2)(1 - \epsilon_n)$. Given this setup and given a formal series $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$, we can construct a random formal subseries $S : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ by

$$S(\omega) = \sum_{n=0}^{\infty} \chi_n(\omega) x_n. \quad (27)$$

This is a measurable function from Ω to $\mathcal{X}^{\mathbb{N}}$ when \mathcal{X} is separable (see Lemma 2.1)

Suppose that \mathcal{X} is separable. Given any Borel subset $P \subseteq \mathcal{X}^{\mathbb{N}}$ the probability $\mathbb{P}(S^{-1}(P)) \in [0, 1]$ of the “event” $S \in P$ is well-defined. Given some “property” P – which we identify with a not-necessarily-Borel subset $P \subseteq \mathcal{X}^{\mathbb{N}}$ – that a formal series may or may not possess, to say that almost all subseries of $\sum_{n=0}^{\infty} x_n$ have property P means that there exists some $F \in \mathcal{F}$ with

$$\mathbb{P}(F) = 1 \quad (28)$$

and $\omega \in F \Rightarrow S(\omega) \in P$. In this case, we say that S has the property P for \mathbb{P} -almost all ω . (Note that we do not require $S^{-1}(P) \in \mathcal{F}$, although this is automatic if P is Borel, and can be arranged by passing to the completion of \mathbb{P} .) Analogous locutions will be used for random formal series generally. If P is Borel then $S(\omega)$ will have the property P for \mathbb{P} -almost all $\omega \in \Omega$ if and only if $\mathbb{P}(S^{-1}(P)) = 1$.

In order to prove the theorems above, we use the following variant of a theorem of Itô and Nisio [IN68] refined by Hoffmann-Jørgensen [HJ74]:

Theorem 1.3. *Suppose that τ is an admissible topology on \mathcal{X} . Let*

$$\gamma_0, \gamma_1, \gamma_2, \dots : \Omega \rightarrow \{-1, +1\} \quad (29)$$

be independent, symmetric random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. If \mathcal{X} is a Banach space and $\{x_n\}_{n=0}^{\infty} \in \mathcal{X}^{\mathbb{N}}$, the following are equivalent:

- (I) *for \mathbb{P} -almost all $\omega \in \Omega$, $\sum_{n=0}^{\infty} \gamma_n(\omega) x_n$ is summable in \mathcal{X} ,*
- (II) *for \mathbb{P} -almost all $\omega \in \Omega$, $\sum_{n=0}^{\infty} \gamma_n(\omega) x_n$ is τ -summable, i.e. summable in \mathcal{X}_{τ} .*

Moreover, whether or not the conditions above hold depends only on $\{x_n\}_{n=0}^{\infty}$ and the laws of each of $\gamma_0, \gamma_1, \gamma_2, \dots$. ■

This result is essentially contained in [HJ74], but, since our formulation is slightly different, we present a proof in §3 below.

See [Hyt+16] for a modern account of the Itô–Nisio result in the case when τ is the weak topology. Our proof follows theirs.

A special case of this theorem was stated in [Sus22], and the proof was sketched. This paper fills in some details of that sketch.²

Remark. We will refer to Theorem 1.3 as “the Itô–Nisio theorem,” with the following three caveats:

- Unlike in the usual Itô–Nisio theorem, we do not discuss convergence in probability.
- The result is often stated with general Bochner-measurable symmetric and independent random variables $x_n(\omega) : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ in place of $\gamma_n(\omega) x_n$. (A \mathcal{X} -valued random variable X will be called *symmetric* if X and $-X$ are equidistributed, i.e. have the same law.³) In fact, Theorem 1.3 implies the more general version via a rerandomization argument.
- Itô and Nisio only consider the case when τ is the weak topology, the generalization to admissible τ being the result of [HJ74].

²See [Sus22, Thm. 3.11]. The statement there involves convergence in probability, but the proof in §3 below applies.

³Note that, if $\mathbb{K} = \mathbb{C}$, this convention differs from some in the literature, in particular [Hyt+16, Definition 6.1.4]. (We use ‘symmetric’ when they would use ‘real-symmetric.’)

Remark. A strengthening of the Itô–Nisio result in the case when \mathcal{X} does not admit an isometric embedding $c_0 \hookrightarrow \mathcal{X}$ is essentially contained – and explicitly conjectured – in [HJ74]. The proof is due to Kwapien [Kwa74]. If (and only if) \mathcal{X} does not admit an isometric embedding $c_0 \hookrightarrow \mathcal{X}$, then (I), (II) in Theorem 1.3 are equivalent to

$$(III) \text{ for almost all } \omega \in \Omega, \sup_{N \in \mathbb{N}} \|\sum_{n=0}^N \epsilon_n(\omega) x_n\| < \infty.$$

(The event described above, that of “uniform boundedness,” is also measurable. See Lemma 2.2.)

Recall that – by the uniform boundedness principle – the weak convergence of a sequence $\{X_N\}_{N=0}^\infty \subseteq \mathcal{X}$ implies that $\sup_N \|X_N\| < \infty$, so (II) implies (III) when τ is the weak topology. Condition (I) obviously implies (III), so by the Itô–Nisio theorem (once we’ve proven it), (II) implies (III) for any admissible τ . The converse obviously does not hold if \mathcal{X} admits an isometric embedding $c_0 \hookrightarrow \mathcal{X}$. ■

Remark. By Lemma 2.2, the events described in (I), (III) above are measurable, and so, Theorem 1.3 is a statement about their probabilities. If \mathcal{X} is separable and τ is the topology generated by a countable norming collection of functionals, the event in (II) is measurable as well. It is a consequence of Theorem 1.3 that, if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, then (II) is measurable regardless. ■

An outline for the rest of this note is as follows:

- In §2, we fill in some measure-theoretic details related to the main line of argument.
- We prove the Itô–Nisio theorem in §3 using a version of the standard argument based on uniform tightness and Lévy’s maximal inequality.
- Using Theorem 1.3, we prove the probabilist’s Orlicz–Pettis theorem in §4

2. MEASURABILITY

Let \mathcal{X} be an arbitrary separable Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let τ be an admissible topology on it. Below, $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ will be as in Theorem 1.3, i.i.d. Rademacher random variables $\Omega \rightarrow \{-1, +1\}$. Similarly, $\chi_0, \chi_1, \chi_2, \dots$ will be i.i.d. uniformly distributed $\Omega \rightarrow \{0, 1\}$.

Lemma 2.1. *The function $S : \Omega \rightarrow \mathcal{X}^\mathbb{N}$ defined by eq. (27) is measurable with respect to the Borel σ -algebra $\text{Borel}(\mathcal{X}^\mathbb{N})$, so it is a well-defined random formal \mathcal{X} -valued series.* ■

Proof. The Borel σ -algebra of a countable product of separable metric spaces agrees with the product \mathcal{P} of the Borel σ -algebras of the individual factors [Kal02, Lemma 1.2]. So, $\text{Borel}(\mathcal{X}^\mathbb{N}) = \sigma(\text{eval}_n : n \in \mathbb{N}) = \mathcal{P}$, where

$$\text{eval}_n : \mathcal{X}^\mathbb{N} \rightarrow \mathcal{X} \tag{30}$$

is shorthand for the map $\sum_{n=0}^\infty x_n \mapsto x_n$. To deduce that S is Borel measurable, we just observe that it is measurable with respect to the σ -algebra $\sigma(\text{eval}_n : n \in \mathbb{N})$, since $\text{eval}_n \circ S(\omega) = \chi_n(\omega) x_n$. □

Let $P_I, P_{II}, P_{III} \subseteq \mathcal{X}^\mathbb{N}$ denote the sets of (I) strongly summable formal series, (II) τ -summable formal series, and (III) bounded formal series, respectively. In other words,

$$P_I = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \lim_{N \rightarrow \infty} \sum_{n=0}^N x_n \text{ exists in } \mathcal{X}\}, \tag{31}$$

$$P_{II} = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \tau\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n \text{ exists in } \mathcal{X}\}, \tag{32}$$

$$P_{III} = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \sup_{N \in \mathbb{N}} \|\sum_{n=0}^N x_n\| < \infty\}. \tag{33}$$

Likewise, given a countable norming subset $\mathcal{S} \subseteq \mathcal{X}_\tau^*$, let

$$P_{II'} = P_{II'}(\mathcal{S}) = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \mathcal{S}\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n \text{ exists in } \mathcal{X}_{\sigma(\mathcal{X}, \mathcal{S})}\} \tag{34}$$

denote the set of \mathcal{S} -weakly summable formal \mathcal{X} -valued series.

Lemma 2.2. $P_I, P_{II'}, P_{III} \in \text{Borel}(\mathcal{X}^\mathbb{N})$. Consequently, given any random formal series $\Sigma : \Omega \rightarrow \mathcal{X}^\mathbb{N}$, $\Sigma^{-1}(P_i) \in \mathcal{F}$ for each $i \in \{I, II', III\}$. ■

Proof. For each $M, N \in \mathbb{N}$, the function $\mathfrak{N}_{N,M} : \mathcal{X}^{\mathbb{N}} \rightarrow \mathbb{R}$ given by

$$\mathfrak{N}_{N,M}(\{x_n\}_{n=0}^{\infty}) = \left\| \sum_{n=M}^N x_n \right\| \quad (35)$$

satisfies $\mathfrak{N}_{N,M}^{-1}(S) \in \mathcal{P}$ for all $S \in \text{Borel}(\mathbb{R})$. Therefore, $\mathbf{P}_{\text{III}} = \cup_{R \in \mathbb{N}} \cap_{N \in \mathbb{N}} \mathfrak{N}_{N,0}^{-1}([0, R])$ is in \mathcal{P} , as is

$$\mathbf{P}_{\text{I}} = \bigcap_{R \in \mathbb{N}^+} \bigcup_{M \in \mathbb{N}} \bigcap_{N \geq M} \mathfrak{N}_{N,M}^{-1}([0, 1/R]). \quad (36)$$

Let $\mathcal{X}_0 \subseteq \mathcal{X}$ denote a dense countable subset. *Claim:* a sequence $\{X_N\}_{N=0}^{\infty} \subseteq \mathcal{X}$ converges \mathcal{S} -weakly if and only if for each rational $\varepsilon > 0$ there exists $X_{\approx} = X_{\approx}(\varepsilon) \in \mathcal{X}_0$ such that for each $\Lambda \in \mathcal{S}$ there exists a $N_0 = N_0(\varepsilon, \Lambda) \in \mathbb{N}$ such that

$$|\Lambda(X_N - X_{\approx})| < \varepsilon \quad (37)$$

for all $N \geq N_0$.

- Proof of ‘only if:’ if $X_N \rightarrow X$ \mathcal{S} -weakly, then, for each $\varepsilon > 0$, choose $X_{\approx} = X_{\approx}(\varepsilon) \in \mathcal{X}_0$ such that $\|X - X_{\approx}\| < \varepsilon/2$, and for each $\Lambda \in \mathcal{S}$ choose $N_0(\varepsilon, \Lambda)$ such that $|\Lambda(X_N - X)| < \varepsilon/2$ for all $N \geq N_0$.

Since the elements of \mathcal{S} have operator norm at most one, $|\Lambda(X - X_{\approx})| < \varepsilon/2$.

Combining these two inequalities, eq. (37) holds for all $N \geq N_0$.

- Proof of ‘if:’ suppose we are given $X_{\approx}(\varepsilon)$ with the desired property. First, observe that $\{X_{\approx}(1/N)\}_{N=1}^{\infty}$ is Cauchy. Indeed, it follows from the definition of the $X_{\approx}(\varepsilon)$ that $|\Lambda(X_{\approx}(\varepsilon) - X_{\approx}(\varepsilon'))| < \varepsilon + \varepsilon'$ for all $\Lambda \in \mathcal{S}$, which implies (since \mathcal{S} is norming) that $\|X_{\approx}(\varepsilon) - X_{\approx}(\varepsilon')\| \leq \varepsilon + \varepsilon'$. So, by the completeness of \mathcal{X} , there exists some $X \in \mathcal{X}$ such that

$$\lim_{N \rightarrow \infty} X_{\approx}(1/N) = X. \quad (38)$$

We now need to show that, as $N \rightarrow \infty$, $X_N \rightarrow X$ \mathcal{S} -weakly. Indeed, given any $\Lambda \in \mathcal{S}$ and $M \in \mathbb{N}^+$,

$$|\Lambda(X_N - X)| \leq |\Lambda(X_N - X_{\approx}(1/M))| + |\Lambda(X - X_{\approx}(1/M))|. \quad (39)$$

Given any $\varepsilon > 0$, pick M such that $1/M < \varepsilon/2$ and such that $\|X_{\approx}(1/M) - X\| < \varepsilon/2$. Since the elements of \mathcal{S} have operator norm at most one, $|\Lambda(X - X_{\approx}(1/M))| < \varepsilon/2$. By the hypothesis of this direction, we can choose $N_0 = N_0(\varepsilon, \Lambda)$ sufficiently large such that $|\Lambda(X_N - X_{\approx}(1/M))| < 1/M < \varepsilon/2$ for all $N \geq N_0$. Therefore, $|\Lambda(X_N - X)| < \varepsilon$ for all $N \geq N_0$. It follows that $X_N \rightarrow X$ \mathcal{S} -weakly.

We therefore conclude that

$$\mathbf{P}_{\text{II}'} = \bigcap_{\varepsilon > 0, \varepsilon \in \mathbb{Q}} \bigcup_{X_{\approx} \in \mathcal{X}_0} \bigcap_{\Lambda \in \mathcal{S}} \bigcup_{M \in \mathbb{N}} \bigcap_{N \geq M} \{\{x_n\}_{n=0}^{\infty} : |\Lambda(X_N - X_{\approx})| < \varepsilon\} \quad (40)$$

is in \mathcal{P} as well, where $X_N = x_0 + \cdots + x_{N-1}$, which depends measurably on $\{x_n\}_{n=0}^{\infty}$. \square

Remark. We do not address the question of when \mathbf{P}_{II} is Borel. Even when \mathcal{X}_{τ}^* is not second countable, it can be the case that $\mathbf{P}_{\text{II}} \in \mathcal{P}$. For example, if $\mathcal{X} = \ell^1(\mathbb{N})$, then sequential weak convergence is equivalent to sequential strong convergence [Car05, Theorem 6.2], and hence $\mathbf{P}_{\text{I}} = \mathbf{P}_{\text{II}}$. \blacksquare

Let $\pi_N : \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{X}^{\mathbb{N}}$ denote the left-shift map $\sum_{n=0}^{\infty} x_n \mapsto \sum_{n=0}^{\infty} x_{n+N}$. Let $\pi_N^* \mathcal{P} = \{\pi_N^{-1}(S) : S \in \mathcal{P}\}$.

Lemma 2.3. *Let $\mathbf{P}_{\text{I}}, \mathbf{P}_{\text{II}'}, \mathbf{P}_{\text{III}}$ be as above. Then*

$$\mathbf{P}_{\text{I}}, \mathbf{P}_{\text{II}'}, \mathbf{P}_{\text{III}} \in \mathcal{T}, \quad (41)$$

where $\mathcal{T} \subseteq \text{Borel}(\mathcal{X}^{\mathbb{N}})$ is the “tail σ -algebra” $\mathcal{T} = \cap_{N \in \mathbb{N}} \pi_N^* \mathcal{P}$. Consequently, given any \mathbb{K} -valued random variables $\lambda_0, \lambda_1, \lambda_2, \dots : \Omega \rightarrow \mathbb{K}$, the random formal series $\Sigma : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ given by $\Sigma(\omega) = \sum_{n=0}^{\infty} \lambda_n(\omega) x_n$ is such that

$$\Sigma^{-1}(\mathbf{P}_i) \in \cap_{N \in \mathbb{N}} \sigma(\{\lambda_n\}_{n=N}^{\infty}) \quad (42)$$

for each $i \in \{\text{I}, \text{II}', \text{III}\}$. ■

Proof. Clearly, $\pi_N^{-1}(\mathbf{P}_i) = \mathbf{P}_i$ for each $i \in \{\text{I}, \text{II}', \text{III}\}$. By Lemma 2.2, we can therefore conclude that $\mathbf{P}_i \in \mathcal{T}$. If Σ is as above, then $\Sigma^* \circ \pi_N^* \mathcal{P} \subseteq \sigma(\{\lambda_n\}_{n=N}^{\infty})$. Since $\Sigma^{-1}(\mathbf{P}_i)$ is in the left-hand side for each $N \in \mathbb{N}$, eq. (42) follows. □

Proposition 2.4. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfy $|f^{-1}(\{n\})| < \infty$ for all $n \in \mathbb{N}$. Suppose that $\lambda_0, \lambda_1, \lambda_2, \dots : \Omega \rightarrow \mathbb{K}$ are independent random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and consider the random formal series $\Sigma : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ given by*

$$\Sigma(\omega) = \sum_{n=0}^{\infty} \lambda_{f(n)}(\omega) x_n. \quad (43)$$

Then $\mathbb{P}(\Sigma^{-1}(\mathbf{P})) = \mathbb{P}[\Sigma \in \mathbf{P}] \in \{0, 1\}$ for any element $\mathbf{P} \in \mathcal{T}$, and in particular for the sets \mathbf{P}_i for each $i \in \{\text{I}, \text{II}', \text{III}\}$. ■

Proof. Since $\lambda_0, \lambda_1, \lambda_2, \dots$ are now assumed to be independent, that $\mathbb{P}[\Sigma \in \mathbf{P}] \in \{0, 1\}$ follows immediately from the Kolmogorov zero-one law [Dur19, Theorem 2.5.3]. By Lemma 2.3, this applies to $\mathbf{P}_\text{I}, \mathbf{P}_{\text{II}'}, \mathbf{P}_{\text{III}}$. □

Proposition 2.5. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfy $|f^{-1}(\{n\})| < \infty$ for all $n \in \mathbb{N}$. Suppose that $\mathbf{P} \subseteq \mathcal{X}^{\mathbb{N}}$ is a \mathbb{K} -subspace and that $\zeta_0, \zeta_1, \zeta_2, \dots : \Omega \rightarrow \mathbb{K}$ are a collection of symmetric, independent \mathbb{K} -valued random variables.*

Then, letting $\Sigma, S : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ denote the random formal series

$$\Sigma(\omega) = \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \quad \text{and} \quad S(\omega) = \sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n, \quad (44)$$

where $\chi_n = 2^{-1}(1 - \zeta_n)$, the following are equivalent: () $\Sigma \in \mathbf{P}$ for \mathbb{P} -almost all $\omega \in \Omega$ and $\sum_{n=0}^{\infty} x_n \in \mathbf{P}$, (**) $S \in \mathbf{P}$ for \mathbb{P} -almost all $\omega \in \Omega$. Consequently, if $\mathbf{P} \in \mathcal{T}$, by Proposition 2.4 the following are equivalent: (*') $\Sigma \notin \mathbf{P}$ for \mathbb{P} -almost all $\omega \in \Omega$ or $\sum_{n=0}^{\infty} x_n \notin \mathbf{P}$ and (**') $S \notin \mathbf{P}$ for \mathbb{P} -almost all $\omega \in \Omega$.* ■

This is essentially an immediate consequence of eq. (3), eq. (4), *mutatis mutandis*.

Proof. First suppose that (*) holds. In particular, $\sum_{n=0}^{\infty} x_n \in \mathbf{P}$. Then, since \mathbf{P} is a subspace of $\mathcal{X}^{\mathbb{N}}$,

$$\sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n = -\frac{1}{2} \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n + \frac{1}{2} \sum_{n=0}^{\infty} x_n \quad (45)$$

is in \mathbf{P} if $\sum_{n=0}^{\infty} \zeta_n(\omega) x_n$ is. By assumption, this holds for \mathbb{P} -almost all $\omega \in \Omega$, and so we conclude that (**) holds.

Conversely, suppose that (**) holds, so that $S(\omega) \in \mathbf{P}$ for all ω in some subset $F \in \mathcal{F}$ with $\mathbb{P}(F) = 1$. Clearly, the two formal series $S, S' : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$,

$$S(\omega) = \sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n \quad \text{and} \quad S'(\omega) = \sum_{n=0}^{\infty} (1 - \chi_{f(n)}(\omega)) x_n \quad (46)$$

are equidistributed. We deduce that $S'(\omega) \in \mathbf{P}$ for almost all $\omega \in \Omega$, i.e. that there exists some $F' \in \mathcal{F}$ with $\mathbb{P}(F') = 1$ such that $S'(\omega) \in \mathbf{P}$ whenever $\omega \in F'$. This implies, since \mathbf{P} is a subspace of

$\mathcal{X}^{\mathbb{N}}$, that the random formal series

$$S(\omega) + S'(\omega) = \sum_{n=0}^{\infty} x_n \quad (47)$$

$$S(\omega) - S'(\omega) = - \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \quad (48)$$

are both in \mathbf{P} for all $\omega \in F \cap F'$. Since $\mathbb{P}(F \cap F') = 1$, it is the case that $F \cap F' \neq \emptyset$, and so we conclude that $\sum_{n=0}^{\infty} x_n \in \mathbf{P}$. Likewise, $\sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \in \mathbf{P}$ for almost all $\omega \in \Omega$. \square

Proposition 2.5 applies in particular to the sets $\mathbf{P}_I, \mathbf{P}_{IV}, \mathbf{P}_{III}$. We will not discuss \mathbf{P}_{III} further, but the preceding results are useful for the treatment of the Jørgensen–Kwapień and Bessaga–Pełczyński theorems along the lines of §4.

3. PROOF OF ITÔ–NISIO

Let \mathcal{X} be a separable Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We now give a treatment, via the method in [Hyt+16], of the particular variant of the Itô–Nisio theorem stated in Theorem 1.3.

The key result allowing the generalization from the weak topology to all admissible topologies is:

Proposition 3.1. *If τ is an admissible topology on \mathcal{X} , then $\text{Borel}(\mathcal{X}) = \text{Borel}(\mathcal{X}_\tau)$.* \blacksquare

Proof. The inclusion $\text{Borel}(\mathcal{X}) \supseteq \text{Borel}(\mathcal{X}_\tau)$ is an immediate consequence of the assumption that τ is weaker than or identical to the norm topology, so it suffices to prove that $\text{Borel}(\mathcal{X}_\tau)$ contains a collection of sets that generate $\text{Borel}(\mathcal{X})$ as a σ -algebra. Consider the collection

$$\mathcal{B} = \{x + \lambda \mathbb{B} : x \in \mathcal{X}, \lambda \in \mathbb{R}^{\geq 0}\} \subseteq \text{Borel}(\mathcal{X}) \quad (49)$$

of all norm-closed balls in \mathcal{X} . Since \mathcal{X} is separable, the collection of all open balls generates $\text{Borel}(\mathcal{X})$, and each open ball $x + \lambda \mathbb{B}^\circ$, $x \in \mathcal{X}$, $\lambda > 0$, is a countable union

$$x + \lambda \mathbb{B}^\circ = \bigcup_{N \in \mathbb{N}, 1/N < \lambda} (x + (\lambda - 1/N) \mathbb{B}) \quad (50)$$

of closed balls, so the closed balls generate $\text{Borel}(\mathcal{X})$. Since τ is an LCTVS topology, once we know that \mathbb{B} is τ -closed, the same holds for all other norm-closed balls. Because τ is admissible, the elements of \mathcal{B} are τ -closed, so $\mathcal{B} \subseteq \text{Borel}(\mathcal{X}_\tau)$. \square

Suppose now that τ is admissible, and suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which symmetric, independent random variables $\gamma_0, \gamma_1, \gamma_2, \dots : \Omega \rightarrow \mathbb{K}$ are defined.

Proposition 3.2. *Suppose that $\sum_{n=0}^{\infty} \gamma_n(\omega) x_n$ converges in \mathcal{X}_τ for \mathbb{P} -almost all $\omega \in \Omega$, so that we may find some $F \in \mathcal{F}$ with $\mathbb{P}(F) = 1$ such that*

$$\Sigma_\infty(\omega) = \tau\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N \gamma_n(\omega) x_n \quad (51)$$

exists for all $\omega \in F$. Set $\Sigma_\infty(\omega) = 0$ for all $\omega \in \Omega \setminus F$. Then, Σ_∞ is a well-defined \mathcal{X} -valued random variable. \blacksquare

Proof. We want to prove that Σ_∞ is measurable with respect to \mathcal{F} and $\text{Borel}(\mathcal{X})$. By Proposition 3.1 and Lemma A.1, $\text{Borel}(\mathcal{X}) = \text{Borel}(\mathcal{X}_\tau) = \text{Borel}(\sigma(\mathcal{X}, \mathcal{X}_\tau^*)) = \sigma(\mathcal{X}_\tau^*)$, so it suffices to check that $\Lambda \circ \Sigma_\infty$ is a measurable \mathbb{K} -valued function for each $\Lambda \in \mathcal{X}_\tau^*$. Certainly,

$$\Lambda \circ \tilde{\Sigma}_N(\omega) = 1_{\omega \in F} \Lambda \circ \Sigma_N(\omega) = \begin{cases} \Sigma_N(\omega) & (\omega \in F) \\ 0 & (\omega \in \Omega \setminus F) \end{cases} \quad (52)$$

is measurable. Consequently, $\Lambda \circ \Sigma_\infty = \lim_{N \rightarrow \infty} \Lambda \circ \tilde{\Sigma}_N$ is the limit of measurable \mathbb{K} -valued random variables and, therefore, measurable. \square

Proposition 3.3. *Consider the setup of Proposition 3.2. For each $N \in \mathbb{N}$, the \mathcal{X} -valued random variables Σ_∞ and $\Sigma_\infty - 2\Sigma_N$ are equidistributed. ■*

Proof. Denote the laws $\Sigma_\infty, \Sigma_\infty - 2\Sigma_N$ by $\mu, \lambda_N : \text{Borel}(\mathcal{X}) \rightarrow [0, 1]$, respectively. The measures μ, λ_N are uniquely determined by their Fourier transforms $\mathcal{F}\mu, \mathcal{F}\lambda_N : \mathcal{X}_\tau^* \rightarrow \mathbb{C}$,

$$\mathcal{F}\mu(\Lambda) = \int_{\Omega} e^{-i\Lambda\Sigma_\infty(\omega)} d\mathbb{P}(\omega) = \int_{\mathcal{X}} e^{-i\Lambda x} d\mu(x), \quad (53)$$

where $\mathcal{F}\lambda_N$ is defined analogously. For each $\Lambda \in \mathcal{X}_\tau^*$, $\Lambda(\Sigma_\infty - \Sigma_N)$ and $\Lambda(\Sigma_N)$ are clearly independent, and $\Lambda(\Sigma_N)$ is equidistributed with $-\Lambda(\Sigma_N)$, so

$$\begin{aligned} \mathcal{F}\mu(\Lambda) &= \int_{\Omega} e^{-i\Lambda\Sigma_\infty(\omega)} d\mathbb{P}(\omega) = \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} e^{-i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \\ &= \left(\int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} d\mathbb{P}(\omega) \right) \left(\int_{\Omega} e^{-i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \right) \\ &= \left(\int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} d\mathbb{P}(\omega) \right) \left(\int_{\Omega} e^{+i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \right) \quad (54) \\ &= \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} e^{+i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \\ &= \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - 2\Sigma_N(\omega))} d\mathbb{P}(\omega) = \mathcal{F}\lambda_N(\Lambda). \end{aligned}$$

Hence the Fourier transforms of μ, λ_N agree, and we conclude that Σ_∞ and $\Sigma_\infty - 2\Sigma_N$ are equidistributed. □

The proof is identical to the standard one, except we need to know that the law of an \mathcal{X} -valued random variable is uniquely determined by the restriction of its Fourier transform (a.k.a. “characteristic functional”) from \mathcal{X}^* to \mathcal{X}_τ^* , for any admissible τ . The proof of this fact for τ the strong or weak topologies, which is just the proof that a finite Borel measure on \mathcal{X} is uniquely determined by the Fourier transform of its law, is given in [Hyt+16, E.1.16, E.1.17]. The general statement follows from analogous reasoning: the finite-dimensional version (i.e. finite Borel measures on \mathbb{R}^d are identifiable with particular tempered distributions, and are, therefore, uniquely determined by their Fourier transforms), the Dynkin π - λ theorem (which implies that a finite measure is uniquely determined by its restriction to any π -system which generates the σ -algebra on which the measure is defined [Dur19, Theorem A.1.5]), and Proposition 3.1.

Another way to prove the proposition is to show that Σ_∞ agrees, almost everywhere, with the composition of the random formal series $\sum_{n=0}^\infty \gamma_n(-)x_n : \Omega \rightarrow \mathcal{X}^\mathbb{N}$ and $\Sigma_{\infty, \text{Uni}} : \mathcal{X}^\mathbb{N} \rightarrow \mathcal{X}$,

$$\Sigma_{\infty, \text{Uni}}\left(\sum_{n=0}^\infty x_n\right) = \begin{cases} \mathcal{S}\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n & (\sum_{n=0}^\infty x_n \in \mathbf{P}_{\Pi'}), \\ 0 & (\text{otherwise}), \end{cases} \quad (55)$$

where $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ is a countable norming collection of functionals and $\mathbf{P}_{\Pi'}$ is as in §2. By the results in §2, $\Sigma_{\infty, \text{Uni}} : \mathcal{X}^\mathbb{N} \rightarrow \mathcal{X}$ is Borel measurable. Thus, we can form the pushforward under it of the law of the formal series $\sum_{n=0}^\infty \gamma_n(-)x_n$. The initial claim, then, is that the law of Σ_∞ is this pushforwards. Likewise, the pushforwards of the law of the random formal series

$$\omega \mapsto -\sum_{n=0}^N \gamma_n(\omega)x_n + \sum_{n=N+1}^\infty \gamma_n(\omega)x_n \in \mathcal{X}^\mathbb{N} \quad (56)$$

is the law of $\Sigma_\infty - 2\Sigma_N$. Since the random formal series eq. (56) is equidistributed with the original, we deduce that Σ_∞ and $\Sigma_\infty - 2\Sigma_N$ are equidistributed as well.

Recall that an \mathcal{X} -valued random variable $X : \Omega \rightarrow \mathcal{X}$ is called *tight* if for every $\varepsilon > 0$ there exists a norm-compact set $K \subseteq \mathcal{X}$ such that $\mathbb{P}[X \notin K] \leq \varepsilon$. By an elementary argument, every

\mathcal{X} -valued random variable is tight [Hyt+16, Proposition 6.4.5]. A family \mathcal{X} of \mathcal{X} -valued random variables is called *uniformly tight* if we can choose the same $K = K(\varepsilon)$ for every $X \in \mathcal{X}$, i.e. if for each $\varepsilon > 0$ there exists some norm-compact $K \subseteq \mathcal{X}$ such that $\mathbb{P}[X \notin K] \leq \varepsilon$ holds for all $X \in \mathcal{X}$. If \mathcal{X} is uniformly tight, then

$$\mathcal{X} - \mathcal{X} = \{X_1 - X_2 : X_1, X_2 \in \mathcal{X}\} \quad (57)$$

is uniformly tight as well, a fact which is used below. (The map $\Delta : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ given by $(x, y) \mapsto x - y$ is continuous. If $K \subseteq \mathcal{X}$ is compact, then $K \times K$ is a compact subset of $\mathcal{X} \times \mathcal{X}$. Its image $\Delta(K \times K) = K - K$ under Δ is, therefore, also compact. By a union bound,

$$\mathbb{P}[X_1 - X_2 \notin \Delta(K \times K)] \leq \mathbb{P}[X_1 \notin K] + \mathbb{P}[X_2 \notin K]. \quad (58)$$

See [Hyt+16, Lemma 6.4.6].)

To complete the proof of the Itô–Nisio theorem, we use Lévy’s maximal inequality [Hyt+16, Proposition 6.1.12]⁴:

Proposition 3.4 (Lévy’s maximal inequality). *Let \mathcal{X} be a separable Banach space over \mathbb{K} . Let x_0, x_1, x_2, \dots be independent symmetric \mathcal{X} -valued random variables. Then, setting $\Sigma_N = \sum_{n=0}^N x_n$,*

$$\mathbb{P}[(\exists N_0 \in \{0, \dots, N\}) \|\Sigma_{N_0}\| \geq R] \leq 2\mathbb{P}[\|\Sigma_N\| \geq R] \quad (59)$$

for all $N \in \mathbb{N}$ and real $R > 0$. ■□

Proposition 3.5. *Suppose that $\sum_{n=0}^{\infty} \gamma_n(\omega)x_n$ converges in \mathcal{X}_τ for \mathbb{P} -almost all $\omega \in \Omega$, and let Σ_∞ denote the \mathcal{X} -valued random variable constructed in the statement of Proposition 3.2. Then*

$$\Sigma_\infty(\omega) = \lim_{N \rightarrow \infty} \sum_{n=0}^N \gamma_n(\omega)x_n \quad (60)$$

for \mathbb{P} -almost all $\omega \in \Omega$. ■

The limit here is taken in the strong topology.

Proof. The proof is split into three parts. We first show that it suffices to show that $\Sigma_N \rightarrow \Sigma_\infty$ in probability, where $\Sigma_N = \sum_{n=0}^N \gamma_n(\omega)x_n$, i.e. that

$$\lim_{N \rightarrow \infty} \mathbb{P}[\|\Sigma_\infty - \Sigma_N\| > \varepsilon] = 0 \quad (61)$$

for all $\varepsilon > 0$. This part of the argument uses Lévy’s inequality. We then establish (via a standard trick) the uniform tightness of $\{\Sigma_N\}_{N=0}^\infty$. The third step involves showing that, if Σ_N fails to converge to Σ_∞ in probability, then, with positive probability, Σ_N fails to converge to Σ_∞ in \mathcal{X}_τ . Under our assumption to the contrary, we can then conclude that $\Sigma_N \rightarrow \Sigma_\infty$ in probability, which by the first part of the argument completes the proof of the proposition.

- (1) Suppose that $\lim_{N \rightarrow \infty} \mathbb{P}[\|\Sigma_\infty - \Sigma_N\| > \varepsilon] = 0$ for all $\varepsilon > 0$. We want to prove that $\Sigma_N \rightarrow \Sigma_\infty$ \mathbb{P} -almost surely. It suffices to prove that $\{\Sigma_N\}_{N=0}^\infty$ is \mathbb{P} -almost surely Cauchy, since then by the completeness of \mathcal{X} it converges strongly \mathbb{P} -almost surely to some random limit $\Sigma'_\infty : \Omega \rightarrow \mathcal{X}$. Since the τ topology is weaker than (or identical to) the strong topology and Hausdorff, $\Sigma'_\infty = \Sigma_\infty$ \mathbb{P} -almost surely.

By the triangle inequality, for any $M, M', N \in \mathbb{N}$, $\|\Sigma_M - \Sigma_{M'}\| \leq \|\Sigma_M - \Sigma_N\| + \|\Sigma_{M'} - \Sigma_N\|$. Therefore, by a union bound,

$$\mathbb{P}\left[\bigcup_{M, M' \geq N} \|\Sigma_M - \Sigma_{M'}\| \geq \varepsilon\right] \leq 2\mathbb{P}\left[\bigcup_{M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\right]. \quad (62)$$

⁴The statement there uses strict inequalities for the events, but the version for nonstrict inequalities follows by the countable additivity of \mathbb{P} .

By the countable additivity of \mathbb{P} and by Lévy's maximal inequality,

$$2\mathbb{P}\left[\bigcup_{M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\right] = \lim_{N' \rightarrow \infty} 2\mathbb{P}\left[\bigcup_{N' \geq M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\right] \quad (63)$$

$$\leq \lim_{N' \rightarrow \infty} 4\mathbb{P}\left[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2\right]. \quad (64)$$

Consequently,

$$\begin{aligned} \mathbb{P}\left[\bigcup_{\varepsilon > 0} \bigcap_{N=0}^{\infty} \bigcup_{M, M' \geq N} \|\Sigma_M - \Sigma_{M'}\| \geq \varepsilon\right] &= \lim_{\varepsilon \rightarrow 0^+} \lim_{N \rightarrow \infty} \mathbb{P}\left[\bigcup_{M, M' \geq N} \|\Sigma_M - \Sigma_{M'}\| \geq \varepsilon\right] \\ &\leq 4 \lim_{\varepsilon \rightarrow 0^+} \lim_{N \rightarrow \infty} \lim_{N' \rightarrow \infty} \mathbb{P}[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2]. \end{aligned} \quad (65)$$

By the triangle inequality and a union bound,

$$\mathbb{P}[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2] \leq \mathbb{P}[\|\Sigma_{\infty} - \Sigma_N\| \geq \varepsilon/4] + \mathbb{P}[\|\Sigma_{N'} - \Sigma_{\infty}\| \geq \varepsilon/4]. \quad (66)$$

It follows from the assumption that $\Sigma_N \rightarrow \Sigma_{\infty}$ in probability that

$$\lim_{N \rightarrow \infty} \lim_{N' \rightarrow \infty} \mathbb{P}[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2] = 0. \quad (67)$$

Consequently, the right-hand side and thus left-hand side of eq. (65) are zero. The event on the left-hand side of eq. (65) is the event that the sequence $\{\Sigma_N\}_{N=0}^{\infty}$ fails to be Cauchy, so the preceding argument shows that $\{\Sigma_N(\omega)\}_{N=0}^{\infty}$ is Cauchy for \mathbb{P} -almost all $\omega \in \Omega$.

- (2) By Proposition 3.3, Σ_{∞} and $\Sigma_{\infty} - 2\Sigma_N$ are equidistributed, for each $N \in \mathbb{N}$. For any $\varepsilon > 0$, by the (automatic) tightness of Σ_{∞} there is a norm-compact subset $K \subseteq \mathcal{X}$ such that $\mathbb{P}[\Sigma_{\infty} \notin K] < \varepsilon$. Let $L = (1/2)(K - K)$, which is also compact. Then, by a union bound,

$$\mathbb{P}[\Sigma_N \notin L] \leq \mathbb{P}[\Sigma_{\infty} \notin K] + \mathbb{P}[\Sigma_{\infty} - 2\Sigma_N \notin K] = 2\mathbb{P}[\Sigma_{\infty} \notin K] < 2\varepsilon. \quad (68)$$

We conclude that $\{\Sigma_N\}_{N=0}^{\infty}$ is uniformly tight.

Also, since Σ_{∞} is tight, the family $\mathcal{X} = \{\Sigma_N\}_{N=0}^{\infty} \cup \{\Sigma_{\infty}\}$ is uniformly tight, which implies that the family $\{\Sigma_{\infty} - \Sigma_N\}_{N=0}^{\infty} \subseteq \mathcal{X} - \mathcal{X}$ is uniformly tight. Consequently, there exists for each $\varepsilon > 0$ a norm-compact subset $K_0 = K_0(\varepsilon) \subseteq \mathcal{X}$ such that

$$\mathbb{P}[(\Sigma_{\infty} - \Sigma_N) \notin K_0(\varepsilon)] \leq \varepsilon \quad (69)$$

for all $N \in \mathbb{N}$.

- (3) Suppose that Σ_N does not converge to Σ_{∞} in probability, so that there exist some $\varepsilon, \delta > 0$ and some subsequence $\{\Sigma_{N_k}\}_{k=0}^{\infty} \subseteq \{\Sigma_N\}_{N=0}^{\infty}$ such that

$$\mathbb{P}[\|\Sigma_{\infty} - \Sigma_{N_k}\| > \varepsilon] \geq \delta \quad (70)$$

for all $k \in \mathbb{N}$. Consider the set $K_0 = K_0(\delta/2)$ defined in eq. (69), so that $\mathbb{P}[(\Sigma_{\infty} - \Sigma_N) \notin K_0] \leq \delta/2$ for all $N \in \mathbb{N}$. Then, combining this inequality with the inequality eq. (70), $\mathbb{P}[(\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B}] \geq \delta/2$ for all $k \in \mathbb{N}$. It follows that the quantity

$$\mathbb{P}[(\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B} \text{ i.o.}] = \mathbb{P}[\bigcap_{K \in \mathbb{N}} \bigcup_{k \geq K} (\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B}] \quad (71)$$

$$= \lim_{K \rightarrow \infty} \mathbb{P}[\bigcup_{k \geq K} (\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B}] \quad (72)$$

(where “i.o.” means for infinitely many k) is bounded below by $\delta/2$ and is in particular positive. So, for ω in some set of positive probability, there exists an ω -dependent subsequence $\{N'_\kappa(\omega)\}_{\kappa=0}^{\infty} = \{N_{k_\kappa}(\omega)\}_{\kappa=0}^{\infty}$ such that $\Sigma_{\infty}(\omega) - \Sigma_{N'_\kappa}(\omega) \in K_0 \setminus \varepsilon\mathbb{B}$ for all $\kappa \in \mathbb{N}$.

Since K_0 is a compact subset of a metric space, it is sequentially compact, so by passing to a further subsequence we can assume without loss of generality that $\Sigma_{\infty}(\omega) - \Sigma_{N'_\kappa}(\omega)$ converges strongly to some ω -dependent $\Delta(\omega) \in \mathcal{X}$, for ω in some subset of positive

probability. But, for such ω , $\|\Delta(\omega)\| \geq \varepsilon$ necessarily, so $\Delta(\omega) \neq 0$. Since τ is weaker than or identical to the strong topology,

$$(\Sigma_\infty(\omega) - \Sigma_{N'_k}(\omega)) \rightarrow \Delta(\omega) \neq 0 \quad (73)$$

in \mathcal{X}_τ for such ω . Since τ is Hausdorff, $\Sigma_N(\omega)$ does not τ -converge to $\Sigma_\infty(\omega)$ as $N \rightarrow \infty$. We conclude that (60) holds for \mathbb{P} -almost all $\omega \in \Omega$ under the hypotheses of the proposition. \square

It is clear that which of the cases in Theorem 1.3 hold depends only on $\{x_n\}_{n=0}^\infty$ and the laws of the random variables $\gamma_0, \gamma_1, \gamma_2, \dots$.

4. PROOF OF ORLICZ–PETTIS

Let \mathcal{X} be a separable Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let τ be an admissible topology on it.

Proposition 4.1. *Suppose that $\zeta_0, \zeta_1, \zeta_2, \dots : \Omega \rightarrow \mathbb{K}$ are a collection of symmetric, independent \mathbb{K} -valued random variables such that, for some infinite $\mathcal{T} \subseteq \mathbb{N}$,*

$$\mathbb{P}[\exists \varepsilon > 0 \text{ s.t. } |\zeta_n| > \varepsilon \text{ for infinitely many } n \in \mathcal{T}] = 1. \quad (74)$$

Suppose further that $\{X_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N}$ is some sequence satisfying

$$\inf_{n \in \mathcal{T}} \|X_n\| > 0. \quad (75)$$

Then, for any $\mathcal{T}_0 \subseteq \mathbb{N}$ such that $\mathcal{T}_0 \supseteq \mathcal{T}$, it is the case that, for \mathbb{P} -almost all $\omega \in \Omega$, the sequence $\{\Sigma_N(\omega)\}_{N=0}^\infty$ given by

$$\Sigma_N(\omega) = \sum_{n=0, n \in \mathcal{T}_0}^N \zeta_n(\omega) X_n \quad (76)$$

fails to τ -converge as $N \rightarrow \infty$. Therefore, the random formal series $\Sigma : \Omega \rightarrow \mathcal{X}^\mathbb{N}$ defined by $\Sigma(\omega) = \sum_{n=0}^\infty 1_{n \in \mathcal{T}_0} \zeta_n(\omega) X_n$ satisfies $\Sigma(\omega) \notin \mathbf{P}_{\text{II}}$ for \mathbb{P} -almost all $\omega \in \Omega$. \blacksquare

Proof. By Proposition 2.4 and the inclusion $\mathbf{P}_{\text{II}'} \supset \mathbf{P}_{\text{II}}$ (where $\mathbf{P}_{\text{II}'}$ is as in §2), it suffices to prove that it is not the case that $\Sigma(\omega) = \sum_{n=0}^\infty 1_{n \in \mathcal{T}_0} \zeta_n(\omega) X_n$ is \mathbb{P} -almost surely \mathcal{S} -weakly summable, where $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ is a countable collection of norming functionals. Suppose, to the contrary, that Σ were almost surely \mathcal{S} -weakly summable. By the Itô–Nisio theorem, this would imply that $\{\Sigma_N(\omega)\}_{N=0}^\infty$ converges strongly for \mathbb{P} -almost all $\omega \in \Omega$. But, the conjunction of eq. (74) and $\inf_{n \in \mathcal{T}} \|X_n\| > 0$ implies instead that $\{\Sigma_N(\omega)\}_{N=0}^\infty$ almost surely *fails* to converge strongly. \square

Proposition 4.2. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$. If it is the case that*

$$\tau - \lim_{N \rightarrow \infty} \sum_{n=0}^N \epsilon_{f(n)}(\omega) x_n \quad (77)$$

exists for \mathbb{P} -almost all $\omega \in \Omega$, then, for any subset $\mathcal{T} \subseteq \mathbb{N}$,

$$\tau - \lim_{N \rightarrow \infty} \sum_{n=0, f(n) \in \mathcal{T}}^N \epsilon_{f(n)}(\omega) x_n \quad (78)$$

exists for \mathbb{P} -almost all $\omega \in \Omega$. \blacksquare

Proof. Let

$$\epsilon'_n = \begin{cases} \epsilon_n & (n \notin \mathcal{T}) \\ -\epsilon_n & (n \in \mathcal{T}). \end{cases} \quad (79)$$

We can now consider the random formal series

$$\sum_{n=0}^{\infty} (\epsilon'_{f(n)} - \epsilon_{f(n)})x_n = \sum_{n=0}^{\infty} \epsilon'_{f(n)}x_n - \sum_{n=0}^{\infty} \epsilon_{f(n)}x_n \quad (80)$$

$$= 2 \sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \epsilon_{f(n)}x_n. \quad (81)$$

The two random formal series on the right-hand side of eq. (80) are equidistributed, so, under the hypothesis of the proposition, both are τ -summable for \mathbb{P} -almost all $\omega \in \Omega$. Thus, the formal series on the right-hand side of eq. (81) is \mathbb{P} -almost surely τ -summable. \square

We deduce Theorem 1.2 (and thus Theorem 1.1) as a corollary of the previous two propositions. We prove the slightly strengthened claim that, for \mathbb{P}_{Haar} -almost all $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$, the formal series in eq. (11) both fail to even be \mathcal{S} -weakly summable. By Proposition 2.5, we just need to show that it is *not* the case that, for \mathbb{P}_{Haar} -almost all $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$, the formal series

$$\sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \epsilon_{f(n)}x_n \in \mathcal{X}^{\mathbb{N}} \quad (82)$$

is \mathcal{S} -weakly summable. Suppose, to the contrary, that it is \mathcal{S} -weakly summable for \mathbb{P}_{Haar} -almost all $\{\epsilon_n\}_{n=0}^{\infty}$. Owing in part to the assumption that $|f^{-1}(\{n\})| < \infty$ for all $n \in \mathbb{N}$ (along with eq. (10)), there exists a $\mathcal{T}_0 \subseteq \mathcal{T}$ such that

- $f : f^{-1}(\mathcal{T}_0) \rightarrow \mathbb{N}$ is monotone and
- $\inf_{n \in \mathcal{T}_0} \|\sum_{n_0 \in f^{-1}(\{n\})} x_{n_0}\| > 0$.

By the previous proposition, $\sum_{n=0, f(n) \in \mathcal{T}_0}^{\infty} \epsilon_{f(n)}x_n \in \mathcal{X}^{\mathbb{N}}$ is \mathcal{S} -weakly summable \mathbb{P} -almost surely. Since $f|_{f^{-1}(\mathcal{T}_0)}$ is monotone, we deduce that

$$\sum_{n=0, n \in \mathcal{T}_0}^{\infty} \epsilon_n \left[\sum_{n_0 \in f^{-1}(\{n\})} x_{n_0} \right] \in \mathcal{X}^{\mathbb{N}} \quad (83)$$

is \mathcal{S} -weakly summable \mathbb{P} -almost surely. However, this contradicts Proposition 4.1.

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APPENDIX A. ADMISSIBLE TOPOLOGIES

Let \mathcal{X} denote a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let τ be an admissible topology on it.

Lemma A.1. *The τ -weak topology, a.k.a. the $\sigma(\mathcal{X}, \mathcal{X}^*)$ -topology, is admissible. \blacksquare*

Proof.

- (1) The τ -weak topology is an LCTVS-topology on \mathcal{X} [Rud73, §3.10, §3.11] identical to or weaker than the norm topology.

For each $\Lambda \in \mathcal{X}_{\tau}^*$ and closed interval $I \subseteq [-\infty, +\infty]$, let $C_{\Lambda, I}$ denote the τ -weakly closed subset (I) $C_{\Lambda, I} = \Lambda^{-1}(I)$ if $\mathbb{K} = \mathbb{R}$ or (II) $C_{\Lambda, I} = \Lambda^{-1}(\{z \in \mathbb{C} : \Re z \in I\})$ otherwise. By the Hahn-Banach theorem, \mathcal{X}_{τ}^* is not empty — picking any $\Lambda \in \mathcal{X}_{\tau}^* \subseteq \mathcal{X}^*$, there exists some closed interval I such that $C_{\Lambda, I} \supseteq \mathbb{B}$, so we can form the intersection

$$\tilde{\mathbb{B}} = \bigcap_{\substack{\Lambda \in \mathcal{X}_{\tau}^*, I \subseteq [-\infty, +\infty] \\ C_{\Lambda, I} \supseteq \mathbb{B}}} C_{\Lambda, I}. \quad (84)$$

This is a τ -weakly closed set containing \mathbb{B} . If $x \notin \mathbb{B}$, we can apply the Hahn-Banach separation theorem [NB11, Thm. 7.8.6] to the sets $\{x\}$ and \mathbb{B} to get some $\Lambda \in \mathcal{X}_\tau^*$ such that $\Re \Lambda x > 1$ and $\Re \Lambda x_0 < 1$ for all $x_0 \in \mathbb{B}$. Then, since \mathbb{B} is closed under multiplication by -1 , $\Re \Lambda x_0 \in (-1, +1)$ for all $x_0 \in \mathbb{B}$, which means that $C_{\Lambda, [-1, +1]}$ appears on the right-hand side of eq. (84).

Since $x \notin C_{\Lambda, [-1, +1]}$, we get $x \notin \tilde{\mathbb{B}}$. We conclude that $\tilde{\mathbb{B}} = \mathbb{B}$ and, therefore, that the latter is τ -weakly closed.

- (2) If \mathcal{X} is not separable, then τ is at least as strong as the weak topology. Since the weak topology of the weak topology is just the weak topology [Rud73, §3.10, §3.11] – that is, $\sigma(\mathcal{X}, \mathcal{X}_w^*) = \sigma(\mathcal{X}, \mathcal{X}^*)$, where $\mathcal{X}_w = \sigma(\mathcal{X}, \mathcal{X}^*)$ – the τ -weak topology is at least as strong as the weak topology.

Thus, the τ -weak topology is admissible. \square

Lemma A.2. *If \mathcal{X} is separable, there exists a countable norming subset $\mathcal{S} \subseteq \mathcal{X}_\tau^*$.* \blacksquare

Proof. Let $\{x_n\}_{n=0}^\infty$ denote a dense subset of $\mathcal{X} \setminus \{0\}$. By [NB11, Thm. 7.8.6], there exists for each $n \in \mathbb{N}$ and each $R \in (0, \|x_n\|)$ an element $\Lambda_{n,R} \in \mathcal{X}_\tau^*$ such that $\Re \Lambda_{n,R} x_n > 1$ and $\Re \Lambda_{n,R} < 1$ on the closed ball $R\mathbb{B}$ (which is τ -closed by admissibility). Since $R\mathbb{B}$ is closed under multiplication by phases,

$$\|\Lambda_{n,R} x\| < 1 \quad (85)$$

for all $x \in R\mathbb{B}$. Thus, $\|\Lambda_{n,R}\|_{\mathcal{X}^*} \leq 1/R$. It follows that $1 < \Re \Lambda_{n,R} x_n < |\Lambda_{n,R} x_n| \leq \|x_n\|/R$, so $\lim_{R \uparrow \|x_n\|} |\Lambda_{n,R} x_n| = 1$.

Now let \mathcal{S} be the set of all functionals of the form $R\Lambda_{n,R}$ for R of the form $\|x_n\| - 1/m$ for $m \in \mathbb{N}^+$ sufficiently large such that $1/m < \|x_n\|$. Then, it is straightforward to check that \mathcal{S} is a norming subset, and \mathcal{S} is countable. \square

Cf. [Car05, Lemma 6.7].

Lemma A.3. *If \mathcal{X} is separable and $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ is a norming subset, then the $\sigma(\mathcal{X}, \mathcal{S})$ -topology is admissible.* \blacksquare

Proof. We can assume without loss of generality that, if $\mathbb{K} = \mathbb{C}$, $e^{i\theta} \Lambda \in \mathcal{S}$ whenever $\Lambda \in \mathcal{S}$ and $\theta \in \mathbb{R}$. By [Rud73, Thm. 3.10], the $\sigma(\mathcal{X}, \mathcal{S})$ -topology is an LCTVS topology, and it is no stronger than the norm topology. Consider

$$\tilde{\mathbb{B}} = \bigcap_{\substack{\Lambda \in \mathcal{S}, I \subseteq [-\infty, +\infty] \\ C_{\Lambda, I} \supseteq \mathbb{B}}} C_{\Lambda, I}, \quad (86)$$

which is a $\sigma(\mathcal{X}, \mathcal{S})$ -closed set containing \mathbb{B} . If $x \notin \mathbb{B}$, then there exists some $\Lambda \in \mathcal{S}$ such that $|\Re \Lambda x| \in (1, \|x_n\|]$. Since \mathcal{S} is norming, $\|\Lambda\|_{\mathcal{X}^*} \leq 1$, so $C_{\Lambda, [-1, +1]}$ appears on the right-hand side of eq. (86). But,

$$x \notin C_{\Lambda, [-1, +1]}, \quad (87)$$

so $x \notin \tilde{\mathbb{B}}$.

We conclude that $\tilde{\mathbb{B}} = \mathbb{B}$, so \mathbb{B} is $\sigma(\mathcal{X}, \mathcal{S})$ -closed. \square

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