

# A STRENGTHENED ORLICZ–PETTIS THEOREM VIA ITÔ–NISIO

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**ABSTRACT.** In this note we deduce a strengthening of the Orlicz–Pettis theorem from the Itô–Nisio theorem. The argument shows that given any series in a Banach space which isn’t summable (or more generally unconditionally summable), we can *construct* a (coarse-grained) subseries with the property that – under some appropriate notion of “almost all” – almost all further subseries thereof fail to be weakly summable. Moreover, a strengthening of the Itô–Nisio theorem by Hoffmann–Jørgensen allows us to replace ‘weakly summable’ with ‘ $\tau$ -weakly summable’ for appropriate topologies  $\tau$  weaker than the weak topology. A treatment of the Itô–Nisio theorem for admissible  $\tau$  is given.

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## 1. INTRODUCTION

Let  $\mathcal{X}$  denote a Banach space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . Call a subset  $\tau \subseteq 2^{\mathcal{X}}$  an *admissible topology* on  $\mathcal{X}$  if

- (1) it is an LCTVS<sup>1</sup>-topology on  $\mathcal{X}$  identical to or weaker than the norm (a.k.a. strong) topology under which the norm-closed unit ball  $\mathbb{B} = \{x \in \mathcal{X} : \|x\| \leq 1\}$  is  $\tau$ -closed, and
- (2) if  $\mathcal{X}$  is *not* separable, then  $\tau$  is at least as strong as the weak topology.

Cf. [HJ74], from which the separable case of this definition arises. By the Hahn–Banach separation theorem, if  $\tau$  is an admissible topology then the  $\tau$ -weak topology (a.k.a.  $\sigma(\mathcal{X}, \mathcal{X}_\tau^*)$ -topology) is also admissible (see Lemma A.1).

Besides the norm topology itself, which is trivially admissible (and uninteresting below), the most familiar example of an admissible topology on  $\mathcal{X}$  is the weak topology. Many others arise in functional analysis. For example, given a compact Riemannian manifold  $M$ , for most function spaces  $\mathcal{F}$  it is the case that the  $\sigma(\mathcal{F}, C^\infty(M))$ -topology (a.k.a. the topology of distributional convergence) is admissible. An even weaker typically admissible topology is that on  $\mathcal{F}$  generated by the functionals  $\langle -, \varphi_n \rangle : \mathcal{D}'(M) \rightarrow \mathbb{C}$  for  $\varphi_0, \varphi_1, \varphi_2, \dots$  the eigenfunctions of the Laplacian.

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<sup>1</sup>By ‘LCTVS’ we mean a *Hausdorff* locally convex topological vector space, so we follow the conventions in [Rud73].

Denote by  $\mathcal{X}^{\mathbb{N}}$  the vector space of all  $\mathcal{X}$ -valued sequences  $\{x_n\}_{n=0}^{\infty} \subseteq \mathcal{X}$ . In the usual way, we identify such sequences with  $\mathcal{X}$ -valued formal series (and denote accordingly). We say that a formal series  $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$  is “ $\tau$ -summable” if  $\sum_{n=0}^N x_n \in \mathcal{X}$  converges as  $N \rightarrow \infty$  in  $\mathcal{X}_{\tau}$ .

Consider the following (slightly generalized) version of the Orlicz–Pettis theorem [Orl29]:

**Theorem 1.1.** *Suppose that  $\tau$  is an admissible topology on  $\mathcal{X}$ . If  $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$  fails to be unconditionally summable in the norm topology, then*

- *there exist some  $\epsilon_0, \epsilon_1, \epsilon_2, \dots \in \{-1, +1\}$  such that the sequence  $\Sigma(\{\epsilon_n\}_{n=0}^{\infty}) = \{\Sigma_N\}_{N=0}^{\infty}$  defined by*

$$\Sigma_N = \sum_{n=0}^N \epsilon_n x_n \quad (1)$$

*does not  $\tau$ -converge as  $N \rightarrow \infty$  to any element of  $\mathcal{X}$ , and*

- *there exist some  $\chi_0, \chi_1, \chi_2, \dots \in \{0, 1\}$  such that the sequence  $S(\{\chi_n\}_{n=0}^{\infty}) = \{S_N\}_{N=0}^{\infty}$  defined by*

$$S_N = \sum_{n=0}^N \chi_n x_n, \quad (2)$$

*does not  $\tau$ -converge as  $N \rightarrow \infty$  to any element of  $\mathcal{X}$ .*

*In particular, this applies if  $\sum_{n=0}^{\infty} x_n$  is not summable in the norm topology.* ■

*Remark.* From the formulas

$$\Sigma_N(\{\epsilon_n\}_{n=0}^N) = S_N(\{2^{-1}(1 + \epsilon_n)\}_{n=0}^N) - S_N(\{2^{-1}(1 - \epsilon_n)\}_{n=0}^N) \quad (3)$$

$$S_N(\{\chi_n\}_{n=0}^N) = 2^{-1}\Sigma_N(\{1\}_{n=0}^N) + 2^{-1}\Sigma_N(\{2\chi_n - 1\}_{n=0}^N), \quad (4)$$

we deduce that  $\Sigma(\{\epsilon_n\}_{n=0}^{\infty})$  is  $\tau$ -convergent for all  $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$  if and only if  $S(\{\chi_n\}_{n=0}^{\infty})$  is  $\tau$ -convergent for all  $\{\chi_n\}_{n=0}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ . We will phrase the discussion below in terms of whichever of  $\Sigma(-)$ ,  $S(-)$  is convenient, but this equivalence should be kept in mind.

See Proposition 2.5 for the probabilistic version of this remark. ■

*Example.* Let  $M$  be a compact Riemannian manifold and  $\mathcal{F} \subseteq \mathcal{D}'(M)$  be a function space on  $M$ . Let  $\tau$  denote the topology generated by the functionals  $\langle -, \varphi_n \rangle_{L^2(M)}$ , where  $\varphi_0, \varphi_1, \varphi_2, \dots$  denote the eigenfunctions of the Laplace–Beltrami operator. Suppose that  $\tau$  is admissible. This holds, for example, if  $\mathcal{F}$  is an  $L^p$ -based Sobolev space for  $p \in [1, \infty)$ .

Then, for any  $\{x_n\}_{n=0}^{\infty} \subseteq \mathcal{F}$ , the formal series  $\sum_{n=0}^{\infty} x_n$  is unconditionally summable in  $\mathcal{F}$  (in norm) if and only if

$$\sum_{n=0}^{\infty} |\langle x_n, \varphi_m \rangle| < \infty \quad (5)$$

for all  $m \in \mathbb{N}$  and, for all  $\{\chi_n\}_{n=0}^{\infty} \subseteq \{0, 1\}$ , there exists an element  $S(\{\chi_n\}_{n=0}^{\infty}) \in \mathcal{F}$  whose  $m$ th Fourier coefficient is given by

$$\langle S(\{\chi_n\}_{n=0}^{\infty}), \varphi_m \rangle = \sum_{n=0}^{\infty} \chi_n \langle x_n, \varphi_m \rangle. \quad (6)$$

■

We focus on Banach spaces – as opposed to more general LCTVSs – for simplicity. Most of the considerations below apply equally well to Fréchet spaces. There is a long history of variants of the Orlicz–Pettis theorem for various sorts of TVSs [Die77]. A short proof of the Orlicz–Pettis theorem for Banach spaces can be found in [BP58], and a textbook presentation can be found in [Meg98]. The proof below has much in common with a probabilistic proof [Die84] based on the Bochner integral (due to Kwapien).

The proof below is nonconstructive, in the following sense: upon being given a formal series  $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$  which fails to be unconditionally summable, we do not construct any particular sequence  $\{\epsilon_n\}_{n=0}^{\infty} \subseteq \{-1, +1\}$  such that  $\Sigma(\{\epsilon_n\}_{n=0}^{\infty}) \subseteq \mathcal{X}$  fails to converge in  $\mathcal{X}_{\tau}$ , or any particular  $\{\chi_n\}_{n=0}^{\infty} \subseteq \{0, 1\}$  such that  $S(\{\chi_n\}_{n=0}^{\infty}) \subseteq \mathcal{X}$  fails to converge in  $\mathcal{X}_{\tau}$ . All proofs of the Orlicz–Pettis theorem seem to be nonconstructive in this regard. We do, however, construct a function

$$\mathcal{E} : \{\{x_n\}_{n=0}^{\infty} \in \mathcal{X}^{\mathbb{N}} \text{ not unconditionally summable}\} \rightarrow 2^{\{-1, +1\}^{\mathbb{N}}}, \quad (7)$$

such that, when  $\{x_n\}_{n=0}^{\infty}$  is not unconditionally summable,  $\Sigma(\{\epsilon_n\}_{n=0}^{\infty})$  and  $S(\{2^{-1}(1 - \epsilon_n)\}_{n=0}^{\infty})$  both fail to be  $\tau$ -summable for  $\mathbb{P}_{\text{Coarse}}$ -almost all sequences  $\{\epsilon_n\}_{n=0}^{\infty} \in \mathcal{E}$ , where

$$\mathbb{P}_{\text{Coarse}} : \text{Borel}(\{-1, +1\}^{\mathbb{N}}) |_{\mathcal{E}(\{x_n\}_{n=0}^{\infty})} \rightarrow [0, 1] \quad (8)$$

is a probability measure on the subspace  $\sigma$ -algebra

$$\text{Borel}(\{-1, +1\}^{\mathbb{N}}) |_{\mathcal{E}(\{x_n\}_{n=0}^{\infty})} = \{S \cap \mathcal{E}(\{x_n\}_{n=0}^{\infty}) : S \in \text{Borel}(\{-1, +1\}^{\mathbb{N}})\}. \quad (9)$$

So, while the proof is nonconstructive, it is only just. Put more colorfully, the proof follows the “hay in a haystack” philosophy familiar from applications of the probabilistic method to combinatorics [AS16]: using an appropriate sampling procedure, we choose a random subseries and show that – with “high probability” (which in this case means probability one) – it has the desired property.

Precisely, letting  $\mathbb{P}_{\text{Haar}}$  denote the Haar measure on the Cantor group  $\{-1, +1\}^{\mathbb{N}} \cong \mathbb{Z}_2^{\mathbb{N}}$  [Die84] (which is a compact topological group under the product topology, by Tychonoff’s theorem):

**Theorem 1.2** (Probabilist’s Orlicz–Pettis Theorem). *Suppose that  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a function such that  $|f^{-1}(\{n\})| < \infty$  for all  $n \in \mathbb{N}$ . If  $\mathcal{T} \subseteq \mathbb{N}$  is infinite and satisfies*

$$\limsup_{\substack{n \rightarrow \infty \\ n \in \mathcal{T}}} \left\| \sum_{n_0 \in f^{-1}(\{n\})} x_{n_0} \right\| > 0, \quad (10)$$

*then it is the case that, for  $\mathbb{P}_{\text{Haar}}$ -almost all  $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$ , the formal series*

$$\sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \epsilon_{f(n)} x_n \in \mathcal{X}^{\mathbb{N}}, \quad \sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \frac{1}{2} (1 - \epsilon_{f(n)}) x_n \in \mathcal{X}^{\mathbb{N}} \quad (11)$$

*both fail to be  $\tau$ -summable.* ■

The relation to Orlicz–Pettis is as follows. If  $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$  is not unconditionally summable, then we can find some pairwise disjoint, finite subsets  $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \dots \subseteq \mathbb{N}$  such that

$$\inf_{N \in \mathbb{N}} \left\| \sum_{n \in \mathcal{N}_N} x_n \right\| > 0. \quad (12)$$

We can then choose some  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(n) = f(m)$  if and only if either  $n = m$  or  $n, m \in \mathcal{N}_N$  for some  $N \in \mathbb{N}$ . Thus, if we set  $\mathcal{T} = \mathbb{N}$ , eq. (10) holds. Appealing to Theorem 1.2, we conclude that, for  $\mathbb{P}_{\text{Haar}}$ -almost all  $\{\epsilon_n\}_{n=0}^{\infty}$ , the formal series

$$\sum_{n=0}^{\infty} \epsilon_{f(n)} x_n \in \mathcal{X}^{\mathbb{N}}, \quad \sum_{n=0}^{\infty} \frac{1}{2} (1 - \epsilon_{f(n)}) x_n \in \mathcal{X}^{\mathbb{N}} \quad (13)$$

both fail to be  $\tau$ -summable. Theorem 1.1, therefore, follows from Theorem 1.2. The connection with eq. (7), eq. (8) is that we can choose  $f$  such that  $\mathcal{E}$  is the set of  $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$  such that  $\epsilon_n = \epsilon_m$  whenever  $f(n) = f(m)$ , and  $\mathbb{P}_{\text{Coarse}}$  is  $\mathbb{P}_{\text{Haar}}$  conditioned on the event that  $\{\epsilon_n\}_{n=0}^{\infty} \in \mathcal{E}$ .

*Remark.* The Haar measure on the Cantor group is the unique measure on  $\text{Borel}(\{-1, +1\}^{\mathbb{N}}) = \sigma(\{\epsilon_n\}_{n=0}^{\infty})$  such that if we define  $\epsilon_n : \{-1, +1\}^{\mathbb{N}} \rightarrow \{-1, +1\}$  by  $\epsilon_n : \{\epsilon'_m\}_{m=0}^{\infty} \mapsto \epsilon'_n$ , the random variables  $\epsilon_0, \epsilon_1, \epsilon_2, \dots$  are i.i.d. Rademacher random variables. ■

*Remark.* It suffices to prove the theorems above when  $\mathcal{X}$  is separable. Indeed, if  $\mathcal{X}$  is not separable and  $\mathcal{Y}$  denotes the norm-closure of the span of  $x_0, x_1, x_2, \dots \in \mathcal{X}$ , then, for any  $\{\lambda_n\}_{n=0}^\infty \subseteq \mathbb{K}$ ,

$$\tau\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N \lambda_n x_n \quad (14)$$

exists in  $\mathcal{X}$  if and only if it exists in  $\mathcal{Y}$ . (This is a consequence of the requirement that  $\tau$  be at least as strong as the weak topology, so the limit in eq. (14) is also a weak limit. Norm-closed convex subsets of  $\mathcal{X}$  are weakly closed by Hahn-Banach, so this implies that  $\mathcal{Y}$  is  $\tau$ -closed.)

The subspace topology on  $\mathcal{Y} \hookrightarrow \mathcal{X}_\tau$  is admissible, and  $\mathcal{Y}$  is separable, so we can deduce Theorem 1.1 and Theorem 1.2 for  $\mathcal{X}$  from the same theorems for  $\mathcal{Y}$ . ■

*Remark.* If  $\mathcal{X}$  is not separable and  $\tau$  not at least as strong as the weak topology, then the conclusions of these theorems may fail to hold, even if the norm-closed balls in  $\mathcal{X}$  are  $\tau$ -closed. As a simple counterexample, let  $\mathcal{X} = L^\infty[0, 1]$ , and let  $\tau$  be the  $\sigma(L^\infty, L^1)$ -topology. This being a weak-\* topology, the norm-closed balls are  $\tau$ -closed (and even  $\tau$ -compact). Let

$$\Sigma_N(t) = t^N, \quad (15)$$

$x_n(t) = \Sigma_n(t) - \Sigma_{n-1}(t)$  for  $n \geq 1$ ,  $x_0(t) = \Sigma_0(t) = 1$ . It turns out that the series  $\sum_{n=0}^\infty x_n$  is  $\tau$ -subseries summable. Indeed, if  $\{\chi_n\}_{n=0}^\infty \subseteq \{0, 1\}$ , then define

$$S(\{\chi_n\}_{n=0}^\infty)(t) = \sum_{n=0}^\infty \chi_n x_n(t) \in \mathbb{R} \quad (16)$$

for each  $t \in [0, 1]$ . By the monotone convergence theorem, this converges pointwise (so the definition makes sense, and  $S(\{\chi_n\}_{n=0}^\infty)$  is a measurable function of  $t$ ), and satisfies  $S(\{\chi_n\}_{n=0}^\infty)(t) \in [0, 1]$ , so  $S(\{\chi_n\}_{n=0}^\infty) \in L^\infty[0, 1]$ . If  $f \in L^1[0, 1]$ , then

$$\left| \int_0^1 f(t) \sum_{n=N}^\infty \chi_n x_n(t) dt \right| \leq \left| \int_0^{1-1/\sqrt{N}} f(t) \sum_{n=N}^\infty \chi_n x_n(t) dt \right| + \left| \int_{1-1/\sqrt{N}}^1 f(t) \sum_{n=N}^\infty \chi_n x_n(t) dt \right|. \quad (17)$$

For  $N \geq 1$ , the first term on the right-hand side is bounded above by

$$\|f\|_{L^1} \sup_{t \in [0, 1-1/\sqrt{N}]} \sum_{n=N}^\infty |x_n(t)| = \|f\|_{L^1} \sup_{t \in [0, 1-1/\sqrt{N}]} t^{N-1} = \|f\|_{L^1} \left(1 - \frac{1}{\sqrt{N}}\right)^{N-1}, \quad (18)$$

which converges to 0 as  $N \rightarrow \infty$ . On the other hand, the second term on the right-hand side of eq. (17) is bounded above by

$$\left( \sup_{t \in [0, 1]} \sum_{n=0}^\infty |x_n(t)| \right) \int_{1-1/\sqrt{N}}^1 |f(t)| dt = 2 \int_{1-1/\sqrt{N}}^1 |f(t)| dt, \quad (19)$$

which converges to 0 as  $N \rightarrow \infty$  by the measurability of  $f$ . So, we can conclude that the convergence in eq. (16) is in  $\tau$ .

But,  $\Sigma_N$  does not converge uniformly on  $[0, 1]$  as  $N \rightarrow \infty$ , so  $\sum_{n=0}^\infty x_n$  is not strongly summable in  $\mathcal{X} = L^\infty[0, 1]$ . Thus, the conclusion of Theorem 1.1 does not hold for this space  $\mathcal{X}$  and this topology  $\tau$ . ■

*Example.* If  $\mathcal{X} = C^0[0, 1]$ , the set of continuous functions  $[0, 1] \rightarrow \mathbb{C}$  with the topology of uniform convergence, and  $\tau$  is the  $\sigma(C^0, L^1)$ -topology, then the hypotheses of the theorems regarding  $\mathcal{X}, \tau$  are satisfied, since  $\mathcal{X}$  is separable and the norm-closed balls in  $\mathcal{X}$  are  $\tau$ -closed. Letting  $x_n(t)$  be as in the previous remark, the failure of  $\sum_{n=0}^\infty x_n$  to be strongly summable implies (by Theorem 1.2) that  $\sum_{n=0}^\infty \chi_n x_n$  cannot be  $\tau$ -summable in  $\mathcal{X}$  for  $\mathbb{P}_{\text{Coarse}}$ -almost all  $\{\chi_n\}_{n=0}^\infty \subset \{0, 1\}$ . But we can define the pointwise limit  $S(\{\chi_n\}_{n=0}^\infty)(t)$  as in eq. (16), and we saw convergence in the  $\sigma(L^\infty, L^1)$ -topology. Consequently, if there were to exist some

$$\tilde{S}(\{\chi_n\}_{n=0}^\infty)(t) \in C^0[0, 1] \quad (20)$$

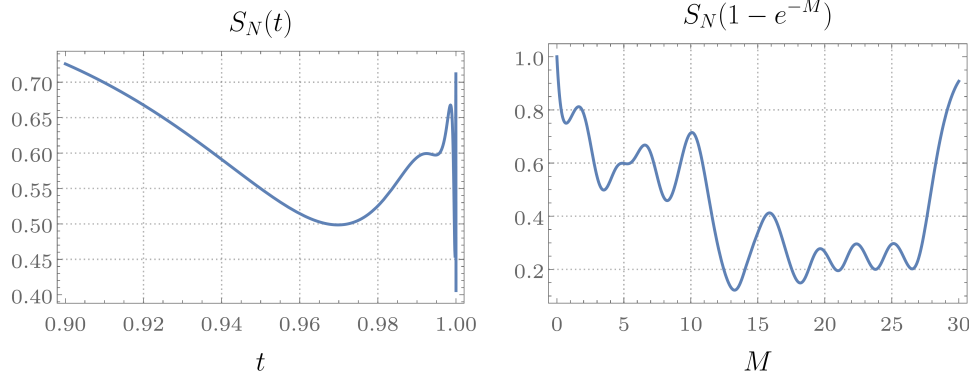


FIGURE 1. (Left:) A plot of  $S_N(t) = 1 + \sum_{n=1}^N \chi_n(t^n - t^{n-1})$  vs.  $t$  for large  $N$  and for  $\{\chi_n\}_{n=0}^\infty$  sampled according to  $\mathbb{P}_{\text{Coarse}}$ . For large  $N$ , the function  $S_N(t)$  oscillates rapidly as  $t \rightarrow 1^-$ , much like the topologist's sine curve, in accordance with the prediction that the full sum  $S(t) = \lim_{N \rightarrow \infty} S_N(t)$  does not have a well-defined limit as  $t \rightarrow 1^-$ . (Right:) The same sum, but plotted as a function of  $M = -\log(1 - t)$ , which allows us to see many more oscillations. (The first plot stops around  $t = 1 - e^{-12}$ , which corresponds to  $M = 12$ .)

agreeing with  $S(\{\chi_n\}_{n=0}^\infty)(t)$  almost everywhere, then  $\sum_{n=0}^\infty \chi_n x_n$  would have to converge to  $\tilde{S}(\{\chi_n\}_{n=0}^\infty)$  in  $\tau = \sigma(C^0, L^1)$ , since this is just the subspace topology of  $\sigma(L^\infty, L^1)$ . So, it must be the case that, for  $\mathbb{P}_{\text{Coarse}}$ -almost all  $\{\chi_n\}_{n=0}^\infty$ ,

$$\tilde{S}(\{\chi_n\}_{n=0}^\infty)(t) \notin C^0[0, 1] \quad (21)$$

if  $\tilde{S}(\{\chi_n\}_{n=0}^\infty)$  agrees with  $S(\{\chi_n\}_{n=0}^\infty)$  almost everywhere in  $[0, 1]_t$ . But, the series  $\sum_{n=0}^\infty \chi_n x_n$  converges uniformly in  $[0, 1 - \delta]$  for every  $\delta \in (0, 1)$ , so  $S(\{\chi_n\}_{n=0}^\infty) \in C^0[0, 1)$ . If  $\lim_{t \rightarrow 1^-} S(\{\chi_n\}_{n=0}^\infty)(t)$  were to exist, then we could define

$$\tilde{S}(\{\chi_n\}_{n=0}^\infty)(t) = \begin{cases} S(\{\chi_n\}_{n=0}^\infty)(t) & (t < 1) \\ \lim_{s \rightarrow 1^-} S(\{\chi_n\}_{n=0}^\infty)(s) & (t = 1), \end{cases} \quad (22)$$

and this would lie in  $C^0[0, 1]$  and agree with  $S(\{\chi_n\}_{n=0}^\infty)$  almost everywhere in  $[0, 1]_t$ . So, it must be the case that  $\lim_{t \rightarrow 1^-} S(\{\chi_n\}_{n=0}^\infty)(t)$  fails to exist for  $\mathbb{P}_{\text{Coarse}}$ -almost all  $\{\chi_n\}_{n=0}^\infty$ . See Figure 1. ■

*Remark.* When  $\mathcal{X}$  is separable, it suffices to consider the case when  $\tau$  is the topology generated by a countable norming set of functionals. Recall that a subset  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$  is called *norming* if

$$\|x\| = \sup_{\Lambda \in \mathcal{S}} |\Lambda x| \quad (23)$$

for all  $x \in \mathcal{X}$ . We can scale the members of a norming subset to get another norming subset whose members  $\Lambda$  satisfy  $\|\Lambda\|_{\mathcal{X}^*} = 1$ , and this generates the same topology. If  $\tau$  is admissible, then (by the Hahn-Banach theorem and separability) there exists a countable norming subset  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$  (see Lemma A.2). Whenever  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$  is a countable norming subset, the  $\sigma(\mathcal{X}, \mathcal{S})$ -topology is admissible as well (see Lemma A.3), and identical with or weaker than  $\tau$ . ■

It is not necessary to consider probability spaces other than

$$(\{-1, +1\}^\mathbb{N}, \text{Borel}(\{-1, +1\}^\mathbb{N}), \mathbb{P}_{\text{Haar}}), \quad (24)$$

but it will be convenient to have a bit more freedom. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  denote a probability space on which i.i.d. Bernoulli random variables

$$\chi_0, \chi_1, \chi_2, \dots : \Omega \rightarrow \{0, 1\} \quad (25)$$

are defined. For example,

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\{-1, +1\}^{\mathbb{N}}, \text{Borel}(\{-1, +1\}^{\mathbb{N}}), \mathbb{P}_{\text{Haar}}), \quad (26)$$

in which case we set  $\chi_n = (1/2)(1 - \epsilon_n)$ . Given this setup and given a formal series  $\sum_{n=0}^{\infty} x_n \in \mathcal{X}^{\mathbb{N}}$ , we can construct a random formal subseries  $S : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$  by

$$S(\omega) = \sum_{n=0}^{\infty} \chi_n(\omega) x_n. \quad (27)$$

This is a measurable function from  $\Omega$  to  $\mathcal{X}^{\mathbb{N}}$  when  $\mathcal{X}$  is separable (see Lemma 2.1)

Suppose that  $\mathcal{X}$  is separable. Given any Borel subset  $P \subseteq \mathcal{X}^{\mathbb{N}}$  the probability  $\mathbb{P}(S^{-1}(P)) \in [0, 1]$  of the “event”  $S \in P$  is well-defined. Given some “property”  $P$  – which we identify with a not-necessarily-Borel subset  $P \subseteq \mathcal{X}^{\mathbb{N}}$  – that a formal series may or may not possess, to say that almost all subseries of  $\sum_{n=0}^{\infty} x_n$  have property  $P$  means that there exists some  $F \in \mathcal{F}$  with

$$\mathbb{P}(F) = 1 \quad (28)$$

and  $\omega \in F \Rightarrow S(\omega) \in P$ . In this case, we say that  $S$  has the property  $P$  for  $\mathbb{P}$ -almost all  $\omega$ . (Note that we do not require  $S^{-1}(P) \in \mathcal{F}$ , although this is automatic if  $P$  is Borel, and can be arranged by passing to the completion of  $\mathbb{P}$ .) Analogous locutions will be used for random formal series generally. If  $P$  is Borel then  $S(\omega)$  will have the property  $P$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$  if and only if  $\mathbb{P}(S^{-1}(P)) = 1$ .

In order to prove the theorems above, we use the following variant of a theorem of Itô and Nisio [IN68] refined by Hoffmann-Jørgensen [HJ74]:

**Theorem 1.3.** *Suppose that  $\tau$  is an admissible topology on  $\mathcal{X}$ . Let*

$$\gamma_0, \gamma_1, \gamma_2, \dots : \Omega \rightarrow \{-1, +1\} \quad (29)$$

*be independent, symmetric random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$ . If  $\mathcal{X}$  is a Banach space and  $\{x_n\}_{n=0}^{\infty} \in \mathcal{X}^{\mathbb{N}}$ , the following are equivalent:*

- (I) *for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ ,  $\sum_{n=0}^{\infty} \gamma_n(\omega) x_n$  is summable in  $\mathcal{X}$ ,*
- (II) *for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ ,  $\sum_{n=0}^{\infty} \gamma_n(\omega) x_n$  is  $\tau$ -summable, i.e. summable in  $\mathcal{X}_{\tau}$ .*

*Moreover, whether or not the conditions above hold depends only on  $\{x_n\}_{n=0}^{\infty}$  and the laws of each of  $\gamma_0, \gamma_1, \gamma_2, \dots$ .* ■

This result is essentially contained in [HJ74], but, since our formulation is slightly different, we present a proof in §3 below.

See [Hyt+16] for a modern account of the Itô–Nisio result in the case when  $\tau$  is the weak topology. Our proof follows theirs.

A special case of this theorem was stated in [Sus22], and the proof was sketched. This paper fills in some details of that sketch.<sup>2</sup>

*Remark.* We will refer to Theorem 1.3 as “the Itô–Nisio theorem,” with the following three caveats:

- Unlike in the usual Itô–Nisio theorem, we do not discuss convergence in probability.
- The result is often stated with general Bochner-measurable symmetric and independent random variables  $x_n(\omega) : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$  in place of  $\gamma_n(\omega) x_n$ . (A  $\mathcal{X}$ -valued random variable  $X$  will be called *symmetric* if  $X$  and  $-X$  are equidistributed, i.e. have the same law.<sup>3</sup>) In fact, Theorem 1.3 implies the more general version via a rerandomization argument.
- Itô and Nisio only consider the case when  $\tau$  is the weak topology, the generalization to admissible  $\tau$  being the result of [HJ74].

<sup>2</sup>See [Sus22, Thm. 3.11]. The statement there involves convergence in probability, but the proof in §3 below applies.

<sup>3</sup>Note that, if  $\mathbb{K} = \mathbb{C}$ , this convention differs from some in the literature, in particular [Hyt+16, Definition 6.1.4]. (We use ‘symmetric’ when they would use ‘real-symmetric.’)

*Remark.* A strengthening of the Itô–Nisio result in the case when  $\mathcal{X}$  does not admit an isometric embedding  $c_0 \hookrightarrow \mathcal{X}$  is essentially contained – and explicitly conjectured – in [HJ74]. The proof is due to Kwapien [Kwa74]. If (and only if)  $\mathcal{X}$  does not admit an isometric embedding  $c_0 \hookrightarrow \mathcal{X}$ , then (I), (II) in Theorem 1.3 are equivalent to

$$(III) \text{ for almost all } \omega \in \Omega, \sup_{N \in \mathbb{N}} \|\sum_{n=0}^N \epsilon_n(\omega) x_n\| < \infty.$$

(The event described above, that of “uniform boundedness,” is also measurable. See Lemma 2.2.)

Recall that – by the uniform boundedness principle – the weak convergence of a sequence  $\{X_N\}_{N=0}^\infty \subseteq \mathcal{X}$  implies that  $\sup_N \|X_N\| < \infty$ , so (II) implies (III) when  $\tau$  is the weak topology. Condition (I) obviously implies (III), so by the Itô–Nisio theorem (once we’ve proven it), (II) implies (III) for any admissible  $\tau$ . The converse obviously does not hold if  $\mathcal{X}$  admits an isometric embedding  $c_0 \hookrightarrow \mathcal{X}$ . ■

*Remark.* By Lemma 2.2, the events described in (I), (III) above are measurable, and so, Theorem 1.3 is a statement about their probabilities. If  $\mathcal{X}$  is separable and  $\tau$  is the topology generated by a countable norming collection of functionals, the event in (II) is measurable as well. It is a consequence of Theorem 1.3 that, if the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is complete, then (II) is measurable regardless. ■

An outline for the rest of this note is as follows:

- In §2, we fill in some measure-theoretic details related to the main line of argument.
- We prove the Itô–Nisio theorem in §3 using a version of the standard argument based on uniform tightness and Lévy’s maximal inequality.
- Using Theorem 1.3, we prove the probabilist’s Orlicz–Pettis theorem in §4

## 2. MEASURABILITY

Let  $\mathcal{X}$  be an arbitrary separable Banach space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , and let  $\tau$  be an admissible topology on it. Below,  $\epsilon_0, \epsilon_1, \epsilon_2, \dots$  will be as in Theorem 1.3, i.i.d. Rademacher random variables  $\Omega \rightarrow \{-1, +1\}$ . Similarly,  $\chi_0, \chi_1, \chi_2, \dots$  will be i.i.d. uniformly distributed  $\Omega \rightarrow \{0, 1\}$ .

**Lemma 2.1.** *The function  $S : \Omega \rightarrow \mathcal{X}^\mathbb{N}$  defined by eq. (27) is measurable with respect to the Borel  $\sigma$ -algebra  $\text{Borel}(\mathcal{X}^\mathbb{N})$ , so it is a well-defined random formal  $\mathcal{X}$ -valued series.* ■

*Proof.* The Borel  $\sigma$ -algebra of a countable product of separable metric spaces agrees with the product  $\mathcal{P}$  of the Borel  $\sigma$ -algebras of the individual factors [Kal02, Lemma 1.2]. So,  $\text{Borel}(\mathcal{X}^\mathbb{N}) = \sigma(\text{eval}_n : n \in \mathbb{N}) = \mathcal{P}$ , where

$$\text{eval}_n : \mathcal{X}^\mathbb{N} \rightarrow \mathcal{X} \tag{30}$$

is shorthand for the map  $\sum_{n=0}^\infty x_n \mapsto x_n$ . To deduce that  $S$  is Borel measurable, we just observe that it is measurable with respect to the  $\sigma$ -algebra  $\sigma(\text{eval}_n : n \in \mathbb{N})$ , since  $\text{eval}_n \circ S(\omega) = \chi_n(\omega) x_n$ . □

Let  $P_I, P_{II}, P_{III} \subseteq \mathcal{X}^\mathbb{N}$  denote the sets of (I) strongly summable formal series, (II)  $\tau$ -summable formal series, and (III) bounded formal series, respectively. In other words,

$$P_I = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \lim_{N \rightarrow \infty} \sum_{n=0}^N x_n \text{ exists in } \mathcal{X}\}, \tag{31}$$

$$P_{II} = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \tau\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n \text{ exists in } \mathcal{X}\}, \tag{32}$$

$$P_{III} = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \sup_{N \in \mathbb{N}} \|\sum_{n=0}^N x_n\| < \infty\}. \tag{33}$$

Likewise, given a countable norming subset  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ , let

$$P_{II'} = P_{II'}(\mathcal{S}) = \{\{x_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N} : \mathcal{S}\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n \text{ exists in } \mathcal{X}_{\sigma(\mathcal{X}, \mathcal{S})}\} \tag{34}$$

denote the set of  $\mathcal{S}$ -weakly summable formal  $\mathcal{X}$ -valued series.

**Lemma 2.2.**  $P_I, P_{II'}, P_{III} \in \text{Borel}(\mathcal{X}^\mathbb{N})$ . Consequently, given any random formal series  $\Sigma : \Omega \rightarrow \mathcal{X}^\mathbb{N}$ ,  $\Sigma^{-1}(P_i) \in \mathcal{F}$  for each  $i \in \{I, II', III\}$ . ■



*Proof.* For each  $M, N \in \mathbb{N}$ , the function  $\mathfrak{N}_{N,M} : \mathcal{X}^{\mathbb{N}} \rightarrow \mathbb{R}$  given by

$$\mathfrak{N}_{N,M}(\{x_n\}_{n=0}^{\infty}) = \left\| \sum_{n=M}^N x_n \right\| \quad (35)$$

satisfies  $\mathfrak{N}_{N,M}^{-1}(S) \in \mathcal{P}$  for all  $S \in \text{Borel}(\mathbb{R})$ . Therefore,  $\mathbf{P}_{\text{III}} = \cup_{R \in \mathbb{N}} \cap_{N \in \mathbb{N}} \mathfrak{N}_{N,0}^{-1}([0, R])$  is in  $\mathcal{P}$ , as is

$$\mathbf{P}_{\text{I}} = \bigcap_{R \in \mathbb{N}^+} \bigcup_{M \in \mathbb{N}} \bigcap_{N \geq M} \mathfrak{N}_{N,M}^{-1}([0, 1/R]). \quad (36)$$

Let  $\mathcal{X}_0 \subseteq \mathcal{X}$  denote a dense countable subset. *Claim:* a sequence  $\{X_N\}_{N=0}^{\infty} \subseteq \mathcal{X}$  converges  $\mathcal{S}$ -weakly if and only if for each rational  $\varepsilon > 0$  there exists  $X_{\approx} = X_{\approx}(\varepsilon) \in \mathcal{X}_0$  such that for each  $\Lambda \in \mathcal{S}$  there exists a  $N_0 = N_0(\varepsilon, \Lambda) \in \mathbb{N}$  such that

$$|\Lambda(X_N - X_{\approx})| < \varepsilon \quad (37)$$

for all  $N \geq N_0$ .

- Proof of ‘only if:’ if  $X_N \rightarrow X$   $\mathcal{S}$ -weakly, then, for each  $\varepsilon > 0$ , choose  $X_{\approx} = X_{\approx}(\varepsilon) \in \mathcal{X}_0$  such that  $\|X - X_{\approx}\| < \varepsilon/2$ , and for each  $\Lambda \in \mathcal{S}$  choose  $N_0(\varepsilon, \Lambda)$  such that  $|\Lambda(X_N - X)| < \varepsilon/2$  for all  $N \geq N_0$ .

Since the elements of  $\mathcal{S}$  have operator norm at most one,  $|\Lambda(X - X_{\approx})| < \varepsilon/2$ .

Combining these two inequalities, eq. (37) holds for all  $N \geq N_0$ .

- Proof of ‘if:’ suppose we are given  $X_{\approx}(\varepsilon)$  with the desired property. First, observe that  $\{X_{\approx}(1/N)\}_{N=1}^{\infty}$  is Cauchy. Indeed, it follows from the definition of the  $X_{\approx}(\varepsilon)$  that  $|\Lambda(X_{\approx}(\varepsilon) - X_{\approx}(\varepsilon'))| < \varepsilon + \varepsilon'$  for all  $\Lambda \in \mathcal{S}$ , which implies (since  $\mathcal{S}$  is norming) that  $\|X_{\approx}(\varepsilon) - X_{\approx}(\varepsilon')\| \leq \varepsilon + \varepsilon'$ . So, by the completeness of  $\mathcal{X}$ , there exists some  $X \in \mathcal{X}$  such that

$$\lim_{N \rightarrow \infty} X_{\approx}(1/N) = X. \quad (38)$$

We now need to show that, as  $N \rightarrow \infty$ ,  $X_N \rightarrow X$   $\mathcal{S}$ -weakly. Indeed, given any  $\Lambda \in \mathcal{S}$  and  $M \in \mathbb{N}^+$ ,

$$|\Lambda(X_N - X)| \leq |\Lambda(X_N - X_{\approx}(1/M))| + |\Lambda(X - X_{\approx}(1/M))|. \quad (39)$$

Given any  $\varepsilon > 0$ , pick  $M$  such that  $1/M < \varepsilon/2$  and such that  $\|X_{\approx}(1/M) - X\| < \varepsilon/2$ . Since the elements of  $\mathcal{S}$  have operator norm at most one,  $|\Lambda(X - X_{\approx}(1/M))| < \varepsilon/2$ . By the hypothesis of this direction, we can choose  $N_0 = N_0(\varepsilon, \Lambda)$  sufficiently large such that  $|\Lambda(X_N - X_{\approx}(1/M))| < 1/M < \varepsilon/2$  for all  $N \geq N_0$ . Therefore,  $|\Lambda(X_N - X)| < \varepsilon$  for all  $N \geq N_0$ . It follows that  $X_N \rightarrow X$   $\mathcal{S}$ -weakly.

We therefore conclude that

$$\mathbf{P}_{\text{II}'} = \bigcap_{\varepsilon > 0, \varepsilon \in \mathbb{Q}} \bigcup_{X_{\approx} \in \mathcal{X}_0} \bigcap_{\Lambda \in \mathcal{S}} \bigcup_{M \in \mathbb{N}} \bigcap_{N \geq M} \{\{x_n\}_{n=0}^{\infty} : |\Lambda(X_N - X_{\approx})| < \varepsilon\} \quad (40)$$

is in  $\mathcal{P}$  as well, where  $X_N = x_0 + \cdots + x_{N-1}$ , which depends measurably on  $\{x_n\}_{n=0}^{\infty}$ .  $\square$

*Remark.* We do not address the question of when  $\mathbf{P}_{\text{II}}$  is Borel. Even when  $\mathcal{X}_{\tau}^*$  is not second countable, it can be the case that  $\mathbf{P}_{\text{II}} \in \mathcal{P}$ . For example, if  $\mathcal{X} = \ell^1(\mathbb{N})$ , then sequential weak convergence is equivalent to sequential strong convergence [Car05, Theorem 6.2], and hence  $\mathbf{P}_{\text{I}} = \mathbf{P}_{\text{II}}$ .  $\blacksquare$

Let  $\pi_N : \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{X}^{\mathbb{N}}$  denote the left-shift map  $\sum_{n=0}^{\infty} x_n \mapsto \sum_{n=0}^{\infty} x_{n+N}$ . Let  $\pi_N^* \mathcal{P} = \{\pi_N^{-1}(S) : S \in \mathcal{P}\}$ .

**Lemma 2.3.** *Let  $\mathbf{P}_{\text{I}}, \mathbf{P}_{\text{II}'}, \mathbf{P}_{\text{III}}$  be as above. Then*

$$\mathbf{P}_{\text{I}}, \mathbf{P}_{\text{II}'}, \mathbf{P}_{\text{III}} \in \mathcal{T}, \quad (41)$$



where  $\mathcal{T} \subseteq \text{Borel}(\mathcal{X}^{\mathbb{N}})$  is the “tail  $\sigma$ -algebra”  $\mathcal{T} = \cap_{N \in \mathbb{N}} \pi_N^* \mathcal{P}$ . Consequently, given any  $\mathbb{K}$ -valued random variables  $\lambda_0, \lambda_1, \lambda_2, \dots : \Omega \rightarrow \mathbb{K}$ , the random formal series  $\Sigma : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$  given by  $\Sigma(\omega) = \sum_{n=0}^{\infty} \lambda_n(\omega) x_n$  is such that

$$\Sigma^{-1}(\mathbf{P}_i) \in \cap_{N \in \mathbb{N}} \sigma(\{\lambda_n\}_{n=N}^{\infty}) \quad (42)$$

for each  $i \in \{\text{I}, \text{II}', \text{III}\}$ . ■

*Proof.* Clearly,  $\pi_N^{-1}(\mathbf{P}_i) = \mathbf{P}_i$  for each  $i \in \{\text{I}, \text{II}', \text{III}\}$ . By Lemma 2.2, we can therefore conclude that  $\mathbf{P}_i \in \mathcal{T}$ . If  $\Sigma$  is as above, then  $\Sigma^* \circ \pi_N^* \mathcal{P} \subseteq \sigma(\{\lambda_n\}_{n=N}^{\infty})$ . Since  $\Sigma^{-1}(\mathbf{P}_i)$  is in the left-hand side for each  $N \in \mathbb{N}$ , eq. (42) follows. □

**Proposition 2.4.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  satisfy  $|f^{-1}(\{n\})| < \infty$  for all  $n \in \mathbb{N}$ . Suppose that  $\lambda_0, \lambda_1, \lambda_2, \dots : \Omega \rightarrow \mathbb{K}$  are independent random variables on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and consider the random formal series  $\Sigma : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$  given by*

$$\Sigma(\omega) = \sum_{n=0}^{\infty} \lambda_{f(n)}(\omega) x_n. \quad (43)$$

*Then  $\mathbb{P}(\Sigma^{-1}(\mathbf{P})) = \mathbb{P}[\Sigma \in \mathbf{P}] \in \{0, 1\}$  for any element  $\mathbf{P} \in \mathcal{T}$ , and in particular for the sets  $\mathbf{P}_i$  for each  $i \in \{\text{I}, \text{II}', \text{III}\}$ .* ■

*Proof.* Since  $\lambda_0, \lambda_1, \lambda_2, \dots$  are now assumed to be independent, that  $\mathbb{P}[\Sigma \in \mathbf{P}] \in \{0, 1\}$  follows immediately from the Kolmogorov zero-one law [Dur19, Theorem 2.5.3]. By Lemma 2.3, this applies to  $\mathbf{P}_\text{I}, \mathbf{P}_{\text{II}'}, \mathbf{P}_{\text{III}}$ . □

**Proposition 2.5.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  satisfy  $|f^{-1}(\{n\})| < \infty$  for all  $n \in \mathbb{N}$ . Suppose that  $\mathbf{P} \subseteq \mathcal{X}^{\mathbb{N}}$  is a  $\mathbb{K}$ -subspace and that  $\zeta_0, \zeta_1, \zeta_2, \dots : \Omega \rightarrow \mathbb{K}$  are a collection of symmetric, independent  $\mathbb{K}$ -valued random variables.*

*Then, letting  $\Sigma, S : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$  denote the random formal series*

$$\Sigma(\omega) = \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \quad \text{and} \quad S(\omega) = \sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n, \quad (44)$$

*where  $\chi_n = 2^{-1}(1 - \zeta_n)$ , the following are equivalent: (\*)  $\Sigma \in \mathbf{P}$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$  and  $\sum_{n=0}^{\infty} x_n \in \mathbf{P}$ , (\*\*)  $S \in \mathbf{P}$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . Consequently, if  $\mathbf{P} \in \mathcal{T}$ , by Proposition 2.4 the following are equivalent: (\*')  $\Sigma \notin \mathbf{P}$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$  or  $\sum_{n=0}^{\infty} x_n \notin \mathbf{P}$  and (\*\*')  $S \notin \mathbf{P}$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ .* ■

This is essentially an immediate consequence of eq. (3), eq. (4), *mutatis mutandis*.

*Proof.* First suppose that (\*) holds. In particular,  $\sum_{n=0}^{\infty} x_n \in \mathbf{P}$ . Then, since  $\mathbf{P}$  is a subspace of  $\mathcal{X}^{\mathbb{N}}$ ,

$$\sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n = -\frac{1}{2} \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n + \frac{1}{2} \sum_{n=0}^{\infty} x_n \quad (45)$$

is in  $\mathbf{P}$  if  $\sum_{n=0}^{\infty} \zeta_n(\omega) x_n$  is. By assumption, this holds for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ , and so we conclude that (\*\*) holds.

Conversely, suppose that (\*\*) holds, so that  $S(\omega) \in \mathbf{P}$  for all  $\omega$  in some subset  $F \in \mathcal{F}$  with  $\mathbb{P}(F) = 1$ . Clearly, the two formal series  $S, S' : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ ,

$$S(\omega) = \sum_{n=0}^{\infty} \chi_{f(n)}(\omega) x_n \quad \text{and} \quad S'(\omega) = \sum_{n=0}^{\infty} (1 - \chi_{f(n)}(\omega)) x_n \quad (46)$$

are equidistributed. We deduce that  $S'(\omega) \in \mathbf{P}$  for almost all  $\omega \in \Omega$ , i.e. that there exists some  $F' \in \mathcal{F}$  with  $\mathbb{P}(F') = 1$  such that  $S'(\omega) \in \mathbf{P}$  whenever  $\omega \in F'$ . This implies, since  $\mathbf{P}$  is a subspace of

$\mathcal{X}^{\mathbb{N}}$ , that the random formal series

$$S(\omega) + S'(\omega) = \sum_{n=0}^{\infty} x_n \quad (47)$$

$$S(\omega) - S'(\omega) = - \sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \quad (48)$$

are both in  $\mathbf{P}$  for all  $\omega \in F \cap F'$ . Since  $\mathbb{P}(F \cap F') = 1$ , it is the case that  $F \cap F' \neq \emptyset$ , and so we conclude that  $\sum_{n=0}^{\infty} x_n \in \mathbf{P}$ . Likewise,  $\sum_{n=0}^{\infty} \zeta_{f(n)}(\omega) x_n \in \mathbf{P}$  for almost all  $\omega \in \Omega$ .  $\square$

Proposition 2.5 applies in particular to the sets  $\mathbf{P}_I, \mathbf{P}_{IV}, \mathbf{P}_{III}$ . We will not discuss  $\mathbf{P}_{III}$  further, but the preceding results are useful for the treatment of the Jørgensen–Kwapień and Bessaga–Pełczyński theorems along the lines of §4.

### 3. PROOF OF ITÔ–NISIO

Let  $\mathcal{X}$  be a separable Banach space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . We now give a treatment, via the method in [Hyt+16], of the particular variant of the Itô–Nisio theorem stated in Theorem 1.3.

The key result allowing the generalization from the weak topology to all admissible topologies is:

**Proposition 3.1.** *If  $\tau$  is an admissible topology on  $\mathcal{X}$ , then  $\text{Borel}(\mathcal{X}) = \text{Borel}(\mathcal{X}_\tau)$ .*  $\blacksquare$

*Proof.* The inclusion  $\text{Borel}(\mathcal{X}) \supseteq \text{Borel}(\mathcal{X}_\tau)$  is an immediate consequence of the assumption that  $\tau$  is weaker than or identical to the norm topology, so it suffices to prove that  $\text{Borel}(\mathcal{X}_\tau)$  contains a collection of sets that generate  $\text{Borel}(\mathcal{X})$  as a  $\sigma$ -algebra. Consider the collection

$$\mathcal{B} = \{x + \lambda \mathbb{B} : x \in \mathcal{X}, \lambda \in \mathbb{R}^{\geq 0}\} \subseteq \text{Borel}(\mathcal{X}) \quad (49)$$

of all norm-closed balls in  $\mathcal{X}$ . Since  $\mathcal{X}$  is separable, the collection of all open balls generates  $\text{Borel}(\mathcal{X})$ , and each open ball  $x + \lambda \mathbb{B}^\circ$ ,  $x \in \mathcal{X}$ ,  $\lambda > 0$ , is a countable union

$$x + \lambda \mathbb{B}^\circ = \bigcup_{N \in \mathbb{N}, 1/N < \lambda} (x + (\lambda - 1/N) \mathbb{B}) \quad (50)$$

of closed balls, so the closed balls generate  $\text{Borel}(\mathcal{X})$ . Since  $\tau$  is an LCTVS topology, once we know that  $\mathbb{B}$  is  $\tau$ -closed, the same holds for all other norm-closed balls. Because  $\tau$  is admissible, the elements of  $\mathcal{B}$  are  $\tau$ -closed, so  $\mathcal{B} \subseteq \text{Borel}(\mathcal{X}_\tau)$ .  $\square$

Suppose now that  $\tau$  is admissible, and suppose that  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space on which symmetric, independent random variables  $\gamma_0, \gamma_1, \gamma_2, \dots : \Omega \rightarrow \mathbb{K}$  are defined.

**Proposition 3.2.** *Suppose that  $\sum_{n=0}^{\infty} \gamma_n(\omega) x_n$  converges in  $\mathcal{X}_\tau$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ , so that we may find some  $F \in \mathcal{F}$  with  $\mathbb{P}(F) = 1$  such that*

$$\Sigma_\infty(\omega) = \tau\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N \gamma_n(\omega) x_n \quad (51)$$

*exists for all  $\omega \in F$ . Set  $\Sigma_\infty(\omega) = 0$  for all  $\omega \in \Omega \setminus F$ . Then,  $\Sigma_\infty$  is a well-defined  $\mathcal{X}$ -valued random variable.*  $\blacksquare$

*Proof.* We want to prove that  $\Sigma_\infty$  is measurable with respect to  $\mathcal{F}$  and  $\text{Borel}(\mathcal{X})$ . By Proposition 3.1 and Lemma A.1,  $\text{Borel}(\mathcal{X}) = \text{Borel}(\mathcal{X}_\tau) = \text{Borel}(\sigma(\mathcal{X}, \mathcal{X}_\tau^*)) = \sigma(\mathcal{X}_\tau^*)$ , so it suffices to check that  $\Lambda \circ \Sigma_\infty$  is a measurable  $\mathbb{K}$ -valued function for each  $\Lambda \in \mathcal{X}_\tau^*$ . Certainly,

$$\Lambda \circ \tilde{\Sigma}_N(\omega) = 1_{\omega \in F} \Lambda \circ \Sigma_N(\omega) = \begin{cases} \Sigma_N(\omega) & (\omega \in F) \\ 0 & (\omega \in \Omega \setminus F) \end{cases} \quad (52)$$

is measurable. Consequently,  $\Lambda \circ \Sigma_\infty = \lim_{N \rightarrow \infty} \Lambda \circ \tilde{\Sigma}_N$  is the limit of measurable  $\mathbb{K}$ -valued random variables and, therefore, measurable.  $\square$

**Proposition 3.3.** *Consider the setup of Proposition 3.2. For each  $N \in \mathbb{N}$ , the  $\mathcal{X}$ -valued random variables  $\Sigma_\infty$  and  $\Sigma_\infty - 2\Sigma_N$  are equidistributed.* ■

*Proof.* Denote the laws  $\Sigma_\infty, \Sigma_\infty - 2\Sigma_N$  by  $\mu, \lambda_N : \text{Borel}(\mathcal{X}) \rightarrow [0, 1]$ , respectively. The measures  $\mu, \lambda_N$  are uniquely determined by their Fourier transforms  $\mathcal{F}\mu, \mathcal{F}\lambda_N : \mathcal{X}_\tau^* \rightarrow \mathbb{C}$ ,

$$\mathcal{F}\mu(\Lambda) = \int_{\Omega} e^{-i\Lambda\Sigma_\infty(\omega)} d\mathbb{P}(\omega) = \int_{\mathcal{X}} e^{-i\Lambda x} d\mu(x), \quad (53)$$

where  $\mathcal{F}\lambda_N$  is defined analogously. For each  $\Lambda \in \mathcal{X}_\tau^*$ ,  $\Lambda(\Sigma_\infty - \Sigma_N)$  and  $\Lambda(\Sigma_N)$  are clearly independent, and  $\Lambda(\Sigma_N)$  is equidistributed with  $-\Lambda(\Sigma_N)$ , so

$$\begin{aligned} \mathcal{F}\mu(\Lambda) &= \int_{\Omega} e^{-i\Lambda\Sigma_\infty(\omega)} d\mathbb{P}(\omega) = \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} e^{-i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \\ &= \left( \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} d\mathbb{P}(\omega) \right) \left( \int_{\Omega} e^{-i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \right) \\ &= \left( \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} d\mathbb{P}(\omega) \right) \left( \int_{\Omega} e^{+i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \right) \quad (54) \\ &= \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - \Sigma_N(\omega))} e^{+i\Lambda\Sigma_N(\omega)} d\mathbb{P}(\omega) \\ &= \int_{\Omega} e^{-i\Lambda(\Sigma_\infty(\omega) - 2\Sigma_N(\omega))} d\mathbb{P}(\omega) = \mathcal{F}\lambda_N(\Lambda). \end{aligned}$$

Hence the Fourier transforms of  $\mu, \lambda_N$  agree, and we conclude that  $\Sigma_\infty$  and  $\Sigma_\infty - 2\Sigma_N$  are equidistributed. □

The proof is identical to the standard one, except we need to know that the law of an  $\mathcal{X}$ -valued random variable is uniquely determined by the restriction of its Fourier transform (a.k.a. “characteristic functional”) from  $\mathcal{X}^*$  to  $\mathcal{X}_\tau^*$ , for any admissible  $\tau$ . The proof of this fact for  $\tau$  the strong or weak topologies, which is just the proof that a finite Borel measure on  $\mathcal{X}$  is uniquely determined by the Fourier transform of its law, is given in [Hyt+16, E.1.16, E.1.17]. The general statement follows from analogous reasoning: the finite-dimensional version (i.e. finite Borel measures on  $\mathbb{R}^d$  are identifiable with particular tempered distributions, and are, therefore, uniquely determined by their Fourier transforms), the Dynkin  $\pi$ - $\lambda$  theorem (which implies that a finite measure is uniquely determined by its restriction to any  $\pi$ -system which generates the  $\sigma$ -algebra on which the measure is defined [Dur19, Theorem A.1.5]), and Proposition 3.1.

Another way to prove the proposition is to show that  $\Sigma_\infty$  agrees, almost everywhere, with the composition of the random formal series  $\sum_{n=0}^\infty \gamma_n(-)x_n : \Omega \rightarrow \mathcal{X}^\mathbb{N}$  and  $\Sigma_{\infty, \text{Uni}} : \mathcal{X}^\mathbb{N} \rightarrow \mathcal{X}$ ,

$$\Sigma_{\infty, \text{Uni}}\left(\sum_{n=0}^\infty x_n\right) = \begin{cases} \mathcal{S}\text{-}\lim_{N \rightarrow \infty} \sum_{n=0}^N x_n & (\sum_{n=0}^\infty x_n \in \mathbf{P}_{\Pi'}), \\ 0 & (\text{otherwise}), \end{cases} \quad (55)$$

where  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$  is a countable norming collection of functionals and  $\mathbf{P}_{\Pi'}$  is as in §2. By the results in §2,  $\Sigma_{\infty, \text{Uni}} : \mathcal{X}^\mathbb{N} \rightarrow \mathcal{X}$  is Borel measurable. Thus, we can form the pushforward under it of the law of the formal series  $\sum_{n=0}^\infty \gamma_n(-)x_n$ . The initial claim, then, is that the law of  $\Sigma_\infty$  is this pushforwards. Likewise, the pushforwards of the law of the random formal series

$$\omega \mapsto -\sum_{n=0}^N \gamma_n(\omega)x_n + \sum_{n=N+1}^\infty \gamma_n(\omega)x_n \in \mathcal{X}^\mathbb{N} \quad (56)$$

is the law of  $\Sigma_\infty - 2\Sigma_N$ . Since the random formal series eq. (56) is equidistributed with the original, we deduce that  $\Sigma_\infty$  and  $\Sigma_\infty - 2\Sigma_N$  are equidistributed as well.

Recall that an  $\mathcal{X}$ -valued random variable  $X : \Omega \rightarrow \mathcal{X}$  is called *tight* if for every  $\varepsilon > 0$  there exists a norm-compact set  $K \subseteq \mathcal{X}$  such that  $\mathbb{P}[X \notin K] \leq \varepsilon$ . By an elementary argument, every

$\mathcal{X}$ -valued random variable is tight [Hyt+16, Proposition 6.4.5]. A family  $\mathcal{X}$  of  $\mathcal{X}$ -valued random variables is called *uniformly tight* if we can choose the same  $K = K(\varepsilon)$  for every  $X \in \mathcal{X}$ , i.e. if for each  $\varepsilon > 0$  there exists some norm-compact  $K \subseteq \mathcal{X}$  such that  $\mathbb{P}[X \notin K] \leq \varepsilon$  holds for all  $X \in \mathcal{X}$ . If  $\mathcal{X}$  is uniformly tight, then

$$\mathcal{X} - \mathcal{X} = \{X_1 - X_2 : X_1, X_2 \in \mathcal{X}\} \quad (57)$$

is uniformly tight as well, a fact which is used below. (The map  $\Delta : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$  given by  $(x, y) \mapsto x - y$  is continuous. If  $K \subseteq \mathcal{X}$  is compact, then  $K \times K$  is a compact subset of  $\mathcal{X} \times \mathcal{X}$ . Its image  $\Delta(K \times K) = K - K$  under  $\Delta$  is, therefore, also compact. By a union bound,

$$\mathbb{P}[X_1 - X_2 \notin \Delta(K \times K)] \leq \mathbb{P}[X_1 \notin K] + \mathbb{P}[X_2 \notin K]. \quad (58)$$

See [Hyt+16, Lemma 6.4.6].)

To complete the proof of the Itô–Nisio theorem, we use Lévy’s maximal inequality [Hyt+16, Proposition 6.1.12]<sup>4</sup>:

**Proposition 3.4** (Lévy’s maximal inequality). *Let  $\mathcal{X}$  be a separable Banach space over  $\mathbb{K}$ . Let  $x_0, x_1, x_2, \dots$  be independent symmetric  $\mathcal{X}$ -valued random variables. Then, setting  $\Sigma_N = \sum_{n=0}^N x_n$ ,*

$$\mathbb{P}[(\exists N_0 \in \{0, \dots, N\}) \|\Sigma_{N_0}\| \geq R] \leq 2\mathbb{P}[\|\Sigma_N\| \geq R] \quad (59)$$

for all  $N \in \mathbb{N}$  and real  $R > 0$ . ■□

**Proposition 3.5.** *Suppose that  $\sum_{n=0}^\infty \gamma_n(\omega)x_n$  converges in  $\mathcal{X}_\tau$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ , and let  $\Sigma_\infty$  denote the  $\mathcal{X}$ -valued random variable constructed in the statement of Proposition 3.2. Then*

$$\Sigma_\infty(\omega) = \lim_{N \rightarrow \infty} \sum_{n=0}^N \gamma_n(\omega)x_n \quad (60)$$

for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . ■

The limit here is taken in the strong topology.

*Proof.* The proof is split into three parts. We first show that it suffices to show that  $\Sigma_N \rightarrow \Sigma_\infty$  in probability, where  $\Sigma_N = \sum_{n=0}^N \gamma_n(\omega)x_n$ , i.e. that

$$\lim_{N \rightarrow \infty} \mathbb{P}[\|\Sigma_\infty - \Sigma_N\| > \varepsilon] = 0 \quad (61)$$

for all  $\varepsilon > 0$ . This part of the argument uses Lévy’s inequality. We then establish (via a standard trick) the uniform tightness of  $\{\Sigma_N\}_{N=0}^\infty$ . The third step involves showing that, if  $\Sigma_N$  fails to converge to  $\Sigma_\infty$  in probability, then, with positive probability,  $\Sigma_N$  fails to converge to  $\Sigma_\infty$  in  $\mathcal{X}_\tau$ . Under our assumption to the contrary, we can then conclude that  $\Sigma_N \rightarrow \Sigma_\infty$  in probability, which by the first part of the argument completes the proof of the proposition.

- (1) Suppose that  $\lim_{N \rightarrow \infty} \mathbb{P}[\|\Sigma_\infty - \Sigma_N\| > \varepsilon] = 0$  for all  $\varepsilon > 0$ . We want to prove that  $\Sigma_N \rightarrow \Sigma_\infty$   $\mathbb{P}$ -almost surely. It suffices to prove that  $\{\Sigma_N\}_{N=0}^\infty$  is  $\mathbb{P}$ -almost surely Cauchy, since then by the completeness of  $\mathcal{X}$  it converges strongly  $\mathbb{P}$ -almost surely to some random limit  $\Sigma'_\infty : \Omega \rightarrow \mathcal{X}$ . Since the  $\tau$  topology is weaker than (or identical to) the strong topology and Hausdorff,  $\Sigma'_\infty = \Sigma_\infty$   $\mathbb{P}$ -almost surely.

By the triangle inequality, for any  $M, M', N \in \mathbb{N}$ ,  $\|\Sigma_M - \Sigma_{M'}\| \leq \|\Sigma_M - \Sigma_N\| + \|\Sigma_{M'} - \Sigma_N\|$ . Therefore, by a union bound,

$$\mathbb{P}\left[\bigcup_{M, M' \geq N} \|\Sigma_M - \Sigma_{M'}\| \geq \varepsilon\right] \leq 2\mathbb{P}\left[\bigcup_{M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\right]. \quad (62)$$

<sup>4</sup>The statement there uses strict inequalities for the events, but the version for nonstrict inequalities follows by the countable additivity of  $\mathbb{P}$ .

By the countable additivity of  $\mathbb{P}$  and by Lévy's maximal inequality,

$$2\mathbb{P}\left[\bigcup_{M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\right] = \lim_{N' \rightarrow \infty} 2\mathbb{P}\left[\bigcup_{N' \geq M \geq N} \|\Sigma_M - \Sigma_N\| \geq \varepsilon/2\right] \quad (63)$$

$$\leq \lim_{N' \rightarrow \infty} 4\mathbb{P}\left[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2\right]. \quad (64)$$

Consequently,

$$\begin{aligned} \mathbb{P}\left[\bigcup_{\varepsilon > 0} \bigcap_{N=0}^{\infty} \bigcup_{M, M' \geq N} \|\Sigma_M - \Sigma_{M'}\| \geq \varepsilon\right] &= \lim_{\varepsilon \rightarrow 0^+} \lim_{N \rightarrow \infty} \mathbb{P}\left[\bigcup_{M, M' \geq N} \|\Sigma_M - \Sigma_{M'}\| \geq \varepsilon\right] \\ &\leq 4 \lim_{\varepsilon \rightarrow 0^+} \lim_{N \rightarrow \infty} \lim_{N' \rightarrow \infty} \mathbb{P}[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2]. \end{aligned} \quad (65)$$

By the triangle inequality and a union bound,

$$\mathbb{P}[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2] \leq \mathbb{P}[\|\Sigma_{\infty} - \Sigma_N\| \geq \varepsilon/4] + \mathbb{P}[\|\Sigma_{N'} - \Sigma_{\infty}\| \geq \varepsilon/4]. \quad (66)$$

It follows from the assumption that  $\Sigma_N \rightarrow \Sigma_{\infty}$  in probability that

$$\lim_{N \rightarrow \infty} \lim_{N' \rightarrow \infty} \mathbb{P}[\|\Sigma_{N'} - \Sigma_N\| \geq \varepsilon/2] = 0. \quad (67)$$

Consequently, the right-hand side and thus left-hand side of eq. (65) are zero. The event on the left-hand side of eq. (65) is the event that the sequence  $\{\Sigma_N\}_{N=0}^{\infty}$  fails to be Cauchy, so the preceding argument shows that  $\{\Sigma_N(\omega)\}_{N=0}^{\infty}$  is Cauchy for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ .

- (2) By Proposition 3.3,  $\Sigma_{\infty}$  and  $\Sigma_{\infty} - 2\Sigma_N$  are equidistributed, for each  $N \in \mathbb{N}$ . For any  $\varepsilon > 0$ , by the (automatic) tightness of  $\Sigma_{\infty}$  there is a norm-compact subset  $K \subseteq \mathcal{X}$  such that  $\mathbb{P}[\Sigma_{\infty} \notin K] < \varepsilon$ . Let  $L = (1/2)(K - K)$ , which is also compact. Then, by a union bound,

$$\mathbb{P}[\Sigma_N \notin L] \leq \mathbb{P}[\Sigma_{\infty} \notin K] + \mathbb{P}[\Sigma_{\infty} - 2\Sigma_N \notin K] = 2\mathbb{P}[\Sigma_{\infty} \notin K] < 2\varepsilon. \quad (68)$$

We conclude that  $\{\Sigma_N\}_{N=0}^{\infty}$  is uniformly tight.

Also, since  $\Sigma_{\infty}$  is tight, the family  $\mathcal{X} = \{\Sigma_N\}_{N=0}^{\infty} \cup \{\Sigma_{\infty}\}$  is uniformly tight, which implies that the family  $\{\Sigma_{\infty} - \Sigma_N\}_{N=0}^{\infty} \subseteq \mathcal{X} - \mathcal{X}$  is uniformly tight. Consequently, there exists for each  $\varepsilon > 0$  a norm-compact subset  $K_0 = K_0(\varepsilon) \subseteq \mathcal{X}$  such that

$$\mathbb{P}[(\Sigma_{\infty} - \Sigma_N) \notin K_0(\varepsilon)] \leq \varepsilon \quad (69)$$

for all  $N \in \mathbb{N}$ .

- (3) Suppose that  $\Sigma_N$  does not converge to  $\Sigma_{\infty}$  in probability, so that there exist some  $\varepsilon, \delta > 0$  and some subsequence  $\{\Sigma_{N_k}\}_{k=0}^{\infty} \subseteq \{\Sigma_N\}_{N=0}^{\infty}$  such that

$$\mathbb{P}[\|\Sigma_{\infty} - \Sigma_{N_k}\| > \varepsilon] \geq \delta \quad (70)$$

for all  $k \in \mathbb{N}$ . Consider the set  $K_0 = K_0(\delta/2)$  defined in eq. (69), so that  $\mathbb{P}[(\Sigma_{\infty} - \Sigma_N) \notin K_0] \leq \delta/2$  for all  $N \in \mathbb{N}$ . Then, combining this inequality with the inequality eq. (70),  $\mathbb{P}[(\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B}] \geq \delta/2$  for all  $k \in \mathbb{N}$ . It follows that the quantity

$$\mathbb{P}[(\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B} \text{ i.o.}] = \mathbb{P}[\bigcap_{K \in \mathbb{N}} \bigcup_{k \geq K} (\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B}] \quad (71)$$

$$= \lim_{K \rightarrow \infty} \mathbb{P}[\bigcup_{k \geq K} (\Sigma_{\infty} - \Sigma_{N_k}) \in K_0 \setminus \varepsilon\mathbb{B}] \quad (72)$$

(where “i.o.” means for infinitely many  $k$ ) is bounded below by  $\delta/2$  and is in particular positive. So, for  $\omega$  in some set of positive probability, there exists an  $\omega$ -dependent subsequence  $\{N'_\kappa(\omega)\}_{\kappa=0}^{\infty} = \{N_{k_\kappa}(\omega)\}_{\kappa=0}^{\infty}$  such that  $\Sigma_{\infty}(\omega) - \Sigma_{N'_\kappa}(\omega) \in K_0 \setminus \varepsilon\mathbb{B}$  for all  $\kappa \in \mathbb{N}$ .

Since  $K_0$  is a compact subset of a metric space, it is sequentially compact, so by passing to a further subsequence we can assume without loss of generality that  $\Sigma_{\infty}(\omega) - \Sigma_{N'_\kappa}(\omega)$  converges strongly to some  $\omega$ -dependent  $\Delta(\omega) \in \mathcal{X}$ , for  $\omega$  in some subset of positive

probability. But, for such  $\omega$ ,  $\|\Delta(\omega)\| \geq \varepsilon$  necessarily, so  $\Delta(\omega) \neq 0$ . Since  $\tau$  is weaker than or identical to the strong topology,

$$(\Sigma_\infty(\omega) - \Sigma_{N'_k}(\omega)) \rightarrow \Delta(\omega) \neq 0 \quad (73)$$

in  $\mathcal{X}_\tau$  for such  $\omega$ . Since  $\tau$  is Hausdorff,  $\Sigma_N(\omega)$  does not  $\tau$ -converge to  $\Sigma_\infty(\omega)$  as  $N \rightarrow \infty$ . We conclude that (60) holds for  $\mathbb{P}$ -almost all  $\omega \in \Omega$  under the hypotheses of the proposition.  $\square$

It is clear that which of the cases in Theorem 1.3 hold depends only on  $\{x_n\}_{n=0}^\infty$  and the laws of the random variables  $\gamma_0, \gamma_1, \gamma_2, \dots$ .

#### 4. PROOF OF ORLICZ–PETTIS

Let  $\mathcal{X}$  be a separable Banach space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , and let  $\tau$  be an admissible topology on it.

**Proposition 4.1.** *Suppose that  $\zeta_0, \zeta_1, \zeta_2, \dots : \Omega \rightarrow \mathbb{K}$  are a collection of symmetric, independent  $\mathbb{K}$ -valued random variables such that, for some infinite  $\mathcal{T} \subseteq \mathbb{N}$ ,*

$$\mathbb{P}[\exists \varepsilon > 0 \text{ s.t. } |\zeta_n| > \varepsilon \text{ for infinitely many } n \in \mathcal{T}] = 1. \quad (74)$$

*Suppose further that  $\{X_n\}_{n=0}^\infty \in \mathcal{X}^\mathbb{N}$  is some sequence satisfying*

$$\inf_{n \in \mathcal{T}} \|X_n\| > 0. \quad (75)$$

*Then, for any  $\mathcal{T}_0 \subseteq \mathbb{N}$  such that  $\mathcal{T}_0 \supseteq \mathcal{T}$ , it is the case that, for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ , the sequence  $\{\Sigma_N(\omega)\}_{N=0}^\infty$  given by*

$$\Sigma_N(\omega) = \sum_{n=0, n \in \mathcal{T}_0}^N \zeta_n(\omega) X_n \quad (76)$$

*fails to  $\tau$ -converge as  $N \rightarrow \infty$ . Therefore, the random formal series  $\Sigma : \Omega \rightarrow \mathcal{X}^\mathbb{N}$  defined by  $\Sigma(\omega) = \sum_{n=0}^\infty 1_{n \in \mathcal{T}_0} \zeta_n(\omega) X_n$  satisfies  $\Sigma(\omega) \notin \mathbf{P}_{\text{II}}$  for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ .  $\blacksquare$*

*Proof.* By Proposition 2.4 and the inclusion  $\mathbf{P}_{\text{II}'} \supset \mathbf{P}_{\text{II}}$  (where  $\mathbf{P}_{\text{II}'}$  is as in §2), it suffices to prove that it is not the case that  $\Sigma(\omega) = \sum_{n=0}^\infty 1_{n \in \mathcal{T}_0} \zeta_n(\omega) X_n$  is  $\mathbb{P}$ -almost surely  $\mathcal{S}$ -weakly summable, where  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$  is a countable collection of norming functionals. Suppose, to the contrary, that  $\Sigma$  were almost surely  $\mathcal{S}$ -weakly summable. By the Itô–Nisio theorem, this would imply that  $\{\Sigma_N(\omega)\}_{N=0}^\infty$  converges strongly for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . But, the conjunction of eq. (74) and  $\inf_{n \in \mathcal{T}} \|X_n\| > 0$  implies instead that  $\{\Sigma_N(\omega)\}_{N=0}^\infty$  almost surely *fails* to converge strongly.  $\square$

**Proposition 4.2.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$ . If it is the case that*

$$\tau - \lim_{N \rightarrow \infty} \sum_{n=0}^N \epsilon_{f(n)}(\omega) x_n \quad (77)$$

*exists for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ , then, for any subset  $\mathcal{T} \subseteq \mathbb{N}$ ,*

$$\tau - \lim_{N \rightarrow \infty} \sum_{n=0, f(n) \in \mathcal{T}}^N \epsilon_{f(n)}(\omega) x_n \quad (78)$$

*exists for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ .  $\blacksquare$*

*Proof.* Let

$$\epsilon'_n = \begin{cases} \epsilon_n & (n \notin \mathcal{T}) \\ -\epsilon_n & (n \in \mathcal{T}). \end{cases} \quad (79)$$

We can now consider the random formal series

$$\sum_{n=0}^{\infty} (\epsilon'_{f(n)} - \epsilon_{f(n)})x_n = \sum_{n=0}^{\infty} \epsilon'_{f(n)}x_n - \sum_{n=0}^{\infty} \epsilon_{f(n)}x_n \quad (80)$$

$$= 2 \sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \epsilon_{f(n)}x_n. \quad (81)$$

The two random formal series on the right-hand side of eq. (80) are equidistributed, so, under the hypothesis of the proposition, both are  $\tau$ -summable for  $\mathbb{P}$ -almost all  $\omega \in \Omega$ . Thus, the formal series on the right-hand side of eq. (81) is  $\mathbb{P}$ -almost surely  $\tau$ -summable.  $\square$

We deduce Theorem 1.2 (and thus Theorem 1.1) as a corollary of the previous two propositions. We prove the slightly strengthened claim that, for  $\mathbb{P}_{\text{Haar}}$ -almost all  $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$ , the formal series in eq. (11) both fail to even be  $\mathcal{S}$ -weakly summable. By Proposition 2.5, we just need to show that it is *not* the case that, for  $\mathbb{P}_{\text{Haar}}$ -almost all  $\{\epsilon_n\}_{n=0}^{\infty} \in \{-1, +1\}^{\mathbb{N}}$ , the formal series

$$\sum_{n=0, f(n) \in \mathcal{T}}^{\infty} \epsilon_{f(n)}x_n \in \mathcal{X}^{\mathbb{N}} \quad (82)$$

is  $\mathcal{S}$ -weakly summable. Suppose, to the contrary, that it is  $\mathcal{S}$ -weakly summable for  $\mathbb{P}_{\text{Haar}}$ -almost all  $\{\epsilon_n\}_{n=0}^{\infty}$ . Owing in part to the assumption that  $|f^{-1}(\{n\})| < \infty$  for all  $n \in \mathbb{N}$  (along with eq. (10)), there exists a  $\mathcal{T}_0 \subseteq \mathcal{T}$  such that

- $f : f^{-1}(\mathcal{T}_0) \rightarrow \mathbb{N}$  is monotone and
- $\inf_{n \in \mathcal{T}_0} \|\sum_{n_0 \in f^{-1}(\{n\})} x_{n_0}\| > 0$ .

By the previous proposition,  $\sum_{n=0, f(n) \in \mathcal{T}_0}^{\infty} \epsilon_{f(n)}x_n \in \mathcal{X}^{\mathbb{N}}$  is  $\mathcal{S}$ -weakly summable  $\mathbb{P}$ -almost surely. Since  $f|_{f^{-1}(\mathcal{T}_0)}$  is monotone, we deduce that

$$\sum_{n=0, n \in \mathcal{T}_0}^{\infty} \epsilon_n \left[ \sum_{n_0 \in f^{-1}(\{n\})} x_{n_0} \right] \in \mathcal{X}^{\mathbb{N}} \quad (83)$$

is  $\mathcal{S}$ -weakly summable  $\mathbb{P}$ -almost surely. However, this contradicts Proposition 4.1.

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#### APPENDIX A. ADMISSIBLE TOPOLOGIES

Let  $\mathcal{X}$  denote a Banach space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ , and let  $\tau$  be an admissible topology on it.

**Lemma A.1.** *The  $\tau$ -weak topology, a.k.a. the  $\sigma(\mathcal{X}, \mathcal{X}^*)$ -topology, is admissible.  $\blacksquare$*

*Proof.*

- (1) The  $\tau$ -weak topology is an LCTVS-topology on  $\mathcal{X}$  [Rud73, §3.10, §3.11] identical to or weaker than the norm topology.

For each  $\Lambda \in \mathcal{X}_{\tau}^*$  and closed interval  $I \subseteq [-\infty, +\infty]$ , let  $C_{\Lambda, I}$  denote the  $\tau$ -weakly closed subset (I)  $C_{\Lambda, I} = \Lambda^{-1}(I)$  if  $\mathbb{K} = \mathbb{R}$  or (II)  $C_{\Lambda, I} = \Lambda^{-1}(\{z \in \mathbb{C} : \Re z \in I\})$  otherwise. By the Hahn-Banach theorem,  $\mathcal{X}_{\tau}^*$  is not empty — picking any  $\Lambda \in \mathcal{X}_{\tau}^* \subseteq \mathcal{X}^*$ , there exists some closed interval  $I$  such that  $C_{\Lambda, I} \supseteq \mathbb{B}$ , so we can form the intersection

$$\tilde{\mathbb{B}} = \bigcap_{\substack{\Lambda \in \mathcal{X}_{\tau}^*, I \subseteq [-\infty, +\infty] \\ C_{\Lambda, I} \supseteq \mathbb{B}}} C_{\Lambda, I}. \quad (84)$$



This is a  $\tau$ -weakly closed set containing  $\mathbb{B}$ . If  $x \notin \mathbb{B}$ , we can apply the Hahn-Banach separation theorem [NB11, Thm. 7.8.6] to the sets  $\{x\}$  and  $\mathbb{B}$  to get some  $\Lambda \in \mathcal{X}_\tau^*$  such that  $\Re \Lambda x > 1$  and  $\Re \Lambda x_0 < 1$  for all  $x_0 \in \mathbb{B}$ . Then, since  $\mathbb{B}$  is closed under multiplication by  $-1$ ,  $\Re \Lambda x_0 \in (-1, +1)$  for all  $x_0 \in \mathbb{B}$ , which means that  $C_{\Lambda, [-1, +1]}$  appears on the right-hand side of eq. (84).

Since  $x \notin C_{\Lambda, [-1, +1]}$ , we get  $x \notin \tilde{\mathbb{B}}$ . We conclude that  $\tilde{\mathbb{B}} = \mathbb{B}$  and, therefore, that the latter is  $\tau$ -weakly closed.

- (2) If  $\mathcal{X}$  is not separable, then  $\tau$  is at least as strong as the weak topology. Since the weak topology of the weak topology is just the weak topology [Rud73, §3.10, §3.11] – that is,  $\sigma(\mathcal{X}, \mathcal{X}_w^*) = \sigma(\mathcal{X}, \mathcal{X}^*)$ , where  $\mathcal{X}_w = \sigma(\mathcal{X}, \mathcal{X}^*)$  – the  $\tau$ -weak topology is at least as strong as the weak topology.

Thus, the  $\tau$ -weak topology is admissible.  $\square$

**Lemma A.2.** *If  $\mathcal{X}$  is separable, there exists a countable norming subset  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$ .*  $\blacksquare$

*Proof.* Let  $\{x_n\}_{n=0}^\infty$  denote a dense subset of  $\mathcal{X} \setminus \{0\}$ . By [NB11, Thm. 7.8.6], there exists for each  $n \in \mathbb{N}$  and each  $R \in (0, \|x_n\|)$  an element  $\Lambda_{n,R} \in \mathcal{X}_\tau^*$  such that  $\Re \Lambda_{n,R} x_n > 1$  and  $\Re \Lambda_{n,R} < 1$  on the closed ball  $R\mathbb{B}$  (which is  $\tau$ -closed by admissibility). Since  $R\mathbb{B}$  is closed under multiplication by phases,

$$\|\Lambda_{n,R} x\| < 1 \quad (85)$$

for all  $x \in R\mathbb{B}$ . Thus,  $\|\Lambda_{n,R}\|_{\mathcal{X}^*} \leq 1/R$ . It follows that  $1 < \Re \Lambda_{n,R} x_n < |\Lambda_{n,R} x_n| \leq \|x_n\|/R$ , so  $\lim_{R \uparrow \|x_n\|} |\Lambda_{n,R} x_n| = 1$ .

Now let  $\mathcal{S}$  be the set of all functionals of the form  $R\Lambda_{n,R}$  for  $R$  of the form  $\|x_n\| - 1/m$  for  $m \in \mathbb{N}^+$  sufficiently large such that  $1/m < \|x_n\|$ . Then, it is straightforward to check that  $\mathcal{S}$  is a norming subset, and  $\mathcal{S}$  is countable.  $\square$

Cf. [Car05, Lemma 6.7].

**Lemma A.3.** *If  $\mathcal{X}$  is separable and  $\mathcal{S} \subseteq \mathcal{X}_\tau^*$  is a norming subset, then the  $\sigma(\mathcal{X}, \mathcal{S})$ -topology is admissible.*  $\blacksquare$

*Proof.* We can assume without loss of generality that, if  $\mathbb{K} = \mathbb{C}$ ,  $e^{i\theta} \Lambda \in \mathcal{S}$  whenever  $\Lambda \in \mathcal{S}$  and  $\theta \in \mathbb{R}$ . By [Rud73, Thm. 3.10], the  $\sigma(\mathcal{X}, \mathcal{S})$ -topology is an LCTVS topology, and it is no stronger than the norm topology. Consider

$$\tilde{\mathbb{B}} = \bigcap_{\substack{\Lambda \in \mathcal{S}, I \subseteq [-\infty, +\infty] \\ C_{\Lambda, I} \supseteq \mathbb{B}}} C_{\Lambda, I}, \quad (86)$$

which is a  $\sigma(\mathcal{X}, \mathcal{S})$ -closed set containing  $\mathbb{B}$ . If  $x \notin \mathbb{B}$ , then there exists some  $\Lambda \in \mathcal{S}$  such that  $|\Re \Lambda x| \in (1, \|x_n\|]$ . Since  $\mathcal{S}$  is norming,  $\|\Lambda\|_{\mathcal{X}^*} \leq 1$ , so  $C_{\Lambda, [-1, +1]}$  appears on the right-hand side of eq. (86). But,

$$x \notin C_{\Lambda, [-1, +1]}, \quad (87)$$

so  $x \notin \tilde{\mathbb{B}}$ .

We conclude that  $\tilde{\mathbb{B}} = \mathbb{B}$ , so  $\mathbb{B}$  is  $\sigma(\mathcal{X}, \mathcal{S})$ -closed.  $\square$

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