

# RADEMACHER SERIES FOR $\eta$ -QUOTIENTS

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ABSTRACT. We apply Rademacher's method in order to compute the Fourier coefficients of a large class of  $\eta$ -quotients.

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## 1. BACKGROUND

A partition of an integer  $n$  is a multiset of positive integers whose sum is  $n$ . Let  $p(n)$  denote the number of partitions of  $n$ . The value of  $p(n)$  may be computed by brute force for sufficiently small  $n$  by simply enumerating all possible partitions and then counting. However,  $p(n)$  grows rapidly and brute force computation rapidly becomes intractible. Another technique, pioneered by Euler [Eul48], is to study the properties of the generating function

$$(1) \quad Z(q) = \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)}.$$

He showed that

$$(2) \quad \frac{1}{Z(q)} = \prod_{n=1}^{\infty} (1 - q^n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(3k-1)/2}.$$

This is a result regarding formal series. However, we can interpret these series as complex valued functions in some appropriate domain. Viewed as complex functions,  $Z(q)$  and  $1/Z(q)$  are both nonvanishing and holomorphic on the open unit disk  $\mathbb{D} \subset \mathbb{C}$  and cannot be analytically continued beyond  $\mathbb{D}$ . One naturally asks (i) what are the analytic properties of these generating functions, and (ii) what properties of  $p(n)$  may be deduced from these analytic properties. With regards to (ii), if we know sufficiently many details regarding the analytic properties of  $Z(q)$ ,  $p(n)$  may simply be extracted by performing a Fourier-Laplace transform:

$$(3) \quad p(n) = \frac{1}{2\pi i} \int_{\gamma} \frac{Z(q)}{q^{n+1}} dq$$

for some suitably chosen contour  $\gamma$ . The difficulty lies in computing this contour integral.

Before continuing, a word on notation: Given a function  $f(q) : \mathbb{D} \rightarrow \mathbb{C}$ , we may pull back  $f(q)$  by the map  $q = e^{2\pi i \tau}$  to get a new function  $f(e^{2\pi i \tau}) : \mathbb{H} \rightarrow \mathbb{C}$ , where  $\mathbb{H}$  is the open upper-half of the

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complex plane. We will often denote  $f(e^{2\pi i\tau})$  as simply  $f(\tau)$  when no confusion should arise. In order to keep track of which variable we are working with, we will often refer to two copies of  $\mathbb{C}$  as the  $q$ -plane and the  $\tau$ -plane.

The first progress with regards to (i) was due to Dedekind. Dedekind considered the eponymous function  $\eta : \mathbb{D} \rightarrow \mathbb{C}$ ,

$$(4) \quad \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where  $q = e^{2\pi i\tau}$  [Apo76]. (This product is convergent.) This is simply  $1/Z(q)$  with a (somewhat mysterious) additional factor of  $q^{1/24}$ . Dedekind showed that  $\eta(\tau)$  is a modular form of weight  $1/2$  (with a nontrivial multiplier system). That is, for any matrix

$$(5) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

where  $\mathrm{SL}_2(\mathbb{Z})$  is the group of integral matrices with determinant  $+1$ ,

$$(6) \quad \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d) \sqrt{c\tau + d} \cdot \eta(\tau),$$

where  $\epsilon(a, b, c, d) = \epsilon(M) : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{S}^1 \subset \mathbb{C}$  is a (nontrivial) homomorphism. (Throughout this paper, when we write square roots or half-integral powers, we mean to refer to the branch of the logarithm with branch cut along the negative real axis.) This phase factor is called a multiplier system. Dedekind computed  $\epsilon(M)$ . It is given by

$$(7) \quad \epsilon(a, b, c, d) = \begin{cases} \exp\left(+\frac{\pi ib}{12}\right) & (c = 0, d = 1), \\ \exp\left(-\frac{\pi ib}{12} + \frac{\pi i}{4}\right) & (c = 0, d = -1), \\ \exp\left(\pi i \left[\frac{a+d}{12c} - s(d, c) - \frac{1}{4}\right]\right) & (c > 0), \\ \exp\left(\pi i \left[\frac{a+d}{12c} - s(-d, -c)\right]\right) & (c < 0). \end{cases}$$

Here,

$$(8) \quad s(h, k) = \sum_{n=1}^{k-1} \frac{n}{k} \left( \frac{hn}{k} - \left\lfloor \frac{hn}{k} \right\rfloor - \frac{1}{2} \right)$$

is known as a Dedekind sum. In retrospect, eq. 6 is rather remarkable and allows us to extract asymptotics of  $\eta(\tau)$  for  $\tau$  near a given rational  $q \in \mathbb{Q} \subset \mathbb{C}$  in terms of the asymptotics of  $\eta(\tau)$  near  $+i\infty$ , asymptotics which are incredibly simple: as  $\tau \rightarrow +i\infty$ , that is as  $q \rightarrow 0$ ,  $\eta(q) \sim q^{1/24}$ . Hardy and Ramanujan [HR18] followed by Rademacher [Rad38a][Rad43] used this in order to explicitly carry out the Fourier transform in eq. 3 and therefore compute  $p(n)$ . This idea is rather general and can be used to compute the Fourier coefficients of a wide variety of automorphic forms [RZ38]. Modifications can be used in order to compute the Fourier coefficients of modular forms which are modular under a congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ . This idea was pioneered by Zuckerman [Zuc39]. In this paper we will use such a modification in order to compute the Fourier coefficients of a finite product of modular forms precomposed with multiplication by different scalar factors  $\mathcal{M} \subset \mathbb{N}$ . These forms are modular forms under a congruence subgroup of the modular group, specifically

$$(9) \quad \Gamma_0(\mathrm{lcm}(\mathcal{M})) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{\mathrm{lcm}(\mathcal{M})} \right\}.$$

So, our result can be obtained using Zuckerman's method applied to  $\Gamma_0(\mathrm{lcm}(\mathcal{M}))$ . We instead modify Rademacher's original method in a slightly different – but ultimately equivalent – way so that the calculation may be done for many  $\eta$ -quotients simultaneously. Our result is also closely

related to a special case of recent work by Bringmann and Ono [BO12], but it is arguably easier to simply derive our expressions from scratch rather than appeal to this more general theory.

Specifically: we consider  $\eta$ -quotients, functions  $Z(q)$  of the form

$$(10) \quad Z(\tau) = \prod_{m=1}^{\infty} \eta(m\tau)^{\delta_m},$$

where  $\{\delta_m\}_{m=1}^{\infty}$  is a sequence of integers of which only finitely many are nonzero. Those with  $\delta_m \geq 0$  for all  $m$  are often called  $\eta$ -products, and we will have little to say about them. These functions,  $\eta$ -quotients, have appeared in several different contexts. One context, into which Rademacher's original paper fits, is being related to the generating functions for the counting function for some partition-like combinatorial quantity, e.g. partitions. Rademacher's method has been applied successfully to a wide variety of counting problems. See the work of Grosswald [Gro58][Gro+60], Haberzette [Hab41], Hagis [Hag62][Hag63][Hag64][Hag65][Hag66], Hua [Hua42], Iseki [Ise60][Ise61], Livingood [Liv45], Niven [Niv40], as well as more recent work by Sills [Sil13][Sil10] and others [KK11][MP12] for examples. Some of the results in these papers are in accord with the main result in this paper, but the proof presented in this paper does not work for many of them; it breaks down when the modular form of interest has zero (or positive) weight, which is the case in some of the applications above. The proof presented here does work for some of them (namely the ones regarding modular forms of negative weight), for example the result of Sills in [Sil13].

A second context is as the partition functions for 1/2-BPS black holes in CHL models of string theory, defined originally in [CHL95], where the frame shape of the  $\eta$ -quotient corresponds to the frame shape of the associated K3 symplectic automorphism [HM14][GK10]. The constants  $d(n)$  then have physical interpretation as the exponential of the entropy of a black hole with a given charge. See [Dab05][Dab+05b][Dab+05a][Sen08] for computations of black hole entropy in non-reduced rank models using Rademacher series. See [GK10][JS06][Gov11] for derivations of CHL 1/2-BPS black hole partition functions. A classification of all CHL models was recently completed [PV15][PVZ17], and the latter work includes a list of all CHL frame shapes. For Type IIB string theory on  $K3 \times T^2$  or dual models, the Rademacher series coefficients may be computed by a gravity path "integral." See [Boe+06][MM10][MP09][DGM15][MR15][Gom17a][Gom17b] for some recent work in this regard.  $\eta$ -products have also been of interest in Mathieu moonshine [HM14][PVZ17]. Just as Rademacher series were used to compute exact black hole entropy from microscopic partition functions for non-CHL models, the original motivation for this work was to compute exact black hole entropy from microscopic partition functions for CHL models.

In order to present our main formula, we need a few preliminary definitions. Let

$$(11) \quad n_0 = -\frac{1}{24} \sum_{m=1}^{\infty} m \cdot \delta_m.$$

The function  $Z(q) \cdot q^{n_0}$  is holomorphic in the open unit disk  $\mathbb{D}$ , and so we may write

$$(12) \quad Z(q) = q^{-n_0} \sum_{n=0}^{\infty} d(n) q^n.$$

for some coefficients  $d(n)$ . From the product formula for  $\eta(q)$  in eq. 4, each  $d(n)$  is an integer. The main result of this paper is an explicit formula for  $d(n)$  for a large class of sequences  $\{\delta_m\}_{m=1}^{\infty}$ . Let

$$(13) \quad c_1 = -\frac{1}{2} \sum_{m=1}^{\infty} \delta_m, \quad c_2(k) = \prod_{m=1}^{\infty} \left[ \frac{\gcd(m, k)}{m} \right]^{\delta_m/2}, \quad c_3(k) = -\sum_{m=1}^{\infty} \delta_m \frac{\gcd(m, k)^2}{m},$$

$$(14) \quad A_k(n) = \sum_{\substack{0 \leq h < k \\ \gcd(h, k)=1}} \exp \left[ -2\pi i \left( \frac{h}{k} \cdot n + \frac{1}{2} \sum_{m=1}^{\infty} \delta_m \cdot s \left( \frac{mh}{\gcd(m, k)}, \frac{k}{\gcd(m, k)} \right) \right) \right].$$

Finally let  $\mathcal{M}$  be the set of  $m$  for which  $\delta_m$  is nonzero. The quantities  $c_1, c_2(k), c_3(k)$  are coefficients which appear in our calculation and formula. The coefficient  $c_1$  is the negative of the weight of  $Z(\tau)$  as a  $\Gamma_0(\text{lcm}(\mathcal{M}))$ -modular form. The sums  $A_k(n)$  closely resemble – and in some cases are – Kloosterman sums. We will call them Kloosterman-like sums. It can be shown that  $A_k(n)$  is real for all  $k$  and  $n$ . With these definitions in hand, we may state:

**Theorem 1.1.** *If  $c_1 > 0$  and the periodic function  $g(k) : \mathbb{N} \rightarrow \mathbb{R}$  given by*

$$(15) \quad g(k) = \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m, k)^2}{m} \right\} - \frac{c_3(k)}{24}$$

*is non-negative, then for  $n \in \{1, 2, \dots\}$  such that  $n > n_0$ ,*

$$(16) \quad d(n) = 2\pi \left( \frac{1}{24(n - n_0)} \right)^{\frac{c_1+1}{2}} \sum_{\substack{k=1 \\ c_3(k) > 0}}^{\infty} c_2(k) c_3(k)^{\frac{c_1+1}{2}} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n - n_0)} \right],$$

*where  $I_{1+c_1}$  is the  $(1 + c_1)$ th modified Bessel function of the first kind. ■*

It is worth spending a moment to comment on the hypotheses of Theorem 1.1. Rademacher’s proof in [RZ38] works only for modular forms of positive “dimension,” that is negative weight. Since  $c_1$ , as defined by eq. 13, is the dimension of our modular form under the congruence subgroup for which it transforms modularly (i.e. (9)), our constraint  $c_1 > 0$  in Theorem 1.1 is analogous to the weight constraint of Rademacher. Rademacher noted that his formula held for other modular forms, including many of weight zero. The various methods of proof for these extreme cases seem to be substantially more delicate, relying on detailed computations involving Kloosterman-like sums. Rademacher’s computation of the Fourier coefficients of  $J(\tau)$  in [Rad38b][Rad39] is an example. The story for  $\eta$ -quotients is expected to be analogous. As a matter of empirical fact, eq. 16 seems to work for many  $\eta$ -quotients with  $c_1 = 0$ , assuming that the second hypothesis regarding  $g(k)$  is satisfied. This is not entirely surprising given recent work by Duncan and others [DF11][CD12][DGO15], and we expect that their results could be used to extend Theorem 1.1 to the  $c_1 = 0$  case.

Rademacher’s formula in [RZ38] includes a sum over the polar part of the relevant modular form. For each  $k$ , each polar term gives rise to a distinct Bessel function in the Rademacher series. We only get a single Bessel function for each  $k$  – as in eq. 16 – when the polar part contains one term. Analogously, the condition that  $g(k) \geq 0$  implies that the polar part of  $Z(\tau)$  at each cusp of  $\mathbb{H}/\Gamma_0(\text{lcm}(\mathcal{M}))$  contains at most one term. (We also allow the polar part to be zero for some  $k$ .) If the polar part of  $Z(\tau)$  at some cusp of  $\mathbb{H}/\Gamma_0(\text{lcm}(\mathcal{M}))$  contains more than one term, the following computation can easily be modified to yield a Rademacher-type formula for the Fourier coefficients of the  $\eta$ -quotient under consideration. The resultant formula will look like eq. 16 but with additional Bessel functions, one for each additional term in the polar part. The existence of only at most a single term in the polar parts entails a simplification of the formulas — since this case seems sufficient for the application to black hole microstate counting (the original motivation for this work), we do not belabor full generality.

An outline of this short paper is as follows: the proof of Theorem 1.1 is contained in §2, we check that eq. 16 has the expected asymptotics in §3, and in §4 we present some numerics serving to illustrate the main result.

## 2. PROOF OF MAIN THEOREM

We will use Rademacher’s modification of the Hardy-Ramanujan-Littlewood circle method to compute  $d(n)$ . We present a couple of lemmas (labeled as such or not) which are contained in the original papers [Rad38a][Rad43] without proof. The reader can find proofs of these lemmas in these papers or in many expositions (such as [Hsu11]). Our notation mostly follows Rademacher, with a few modifications. We will continue working in the setup of the introduction, and unless

stated otherwise, we will be working under the assumption that the hypotheses of the claim, Theorem 1.1, hold.

As in Rademacher's computation of the Fourier coefficients of  $1/\eta(\tau)$ , we extract  $d(n)$  (for integral  $n$ ) by performing a Fourier-Laplace transform:

$$(17) \quad d(n) = \frac{1}{2\pi i} \int_{\gamma} \frac{Z(q) \cdot q^{n_0}}{q^{n+1}} dq,$$

where  $\gamma \subset \mathbb{D}$  is a closed (toy) contour winding once around the origin, contained entirely within the unit disk in the  $q$ -plane. The rest of the proof is simply computing this integral. Following Rademacher, we will define a sequence of suitable contours  $\{\gamma_N\}_{N=1}^{\infty}$ , compute the integral in eq. 17 for  $\gamma = \gamma_N$  up to an error term, take  $N \rightarrow \infty$  and show that the error term converges to zero. As it turns out, it is more convenient to define the contours in the  $\tau$ -plane and then map them into the  $q$ -plane. We will denote a pullback of some contour  $\gamma$  in the  $q$ -plane to the  $\tau$ -plane as  $\tau(\gamma)$ . (Note that  $\tau(\gamma)$  is not uniquely defined, but this is of no consequence below.) Our departure from Rademacher occurs only in the details of the computation of each contour integral.

Some preliminary definitions are in order. By “irreducible fraction  $h/k \in [0, 1]$ ” we mean a pair  $(h, k)$  of coprime integers with  $k > 0$  and  $0 \leq h \leq k$ . (We do not use the parenthetical notation for these ordered pairs as it coincides with our abbreviation for greatest common divisors.) For  $N \in \mathbb{N}^+$ , the  $N$ th Farey sequence  $\mathcal{F}_N \subset \mathbb{Q}$  is the finite sequence containing all irreducible fractions in  $[0, 1]$  of denominator at most  $N$  in increasing order. The Ford circle  $C(h/k)$  associated with an irreducible fraction  $h/k$  is the circle in the (closed upper-half of the)  $\tau$ -plane with center  $h/k + i/2k^2$  and radius  $1/2k^2$ . See Figure 1. We denote by  $q(C(h/k))$  the image of  $C(h/k)$  in the  $q$ -plane,

$$(18) \quad q(C(h/k)) = \{e^{2\pi i \tau} : \tau \in C(h/k)\}.$$

Note that  $q(C(0/1)) = q(C(1/1))$ . It can be shown that the Ford circles corresponding to consecutive fractions  $h_1/k_1$  and  $h_2/k_2$  in some Farey sequence are tangent at the point

$$(19) \quad \tilde{\tau}(h_1/k_1, h_2/k_2) = \frac{h_1 k_1 + h_2 k_2 + i}{k_1^2 + k_2^2}.$$

For irreducible fractions  $h_0/k_0 < h_1/k_1 < h_2/k_2$  with  $C(h_0/k_0)$  and  $C(h_2/k_2)$  tangent to  $C(h_1/k_1)$ , let  $\tau(\gamma_{h_0/k_0, h_1/k_1, h_2/k_2})$  be the arc on  $C(h_1/k_1)$  from the point of tangency with  $C(h_0/k_0)$  to the point of tangency with  $C(h_2/k_2)$  parametrized by arc length. We choose the contour to proceed around the Ford circle clockwise so that the arc does not touch the real line. Likewise, for  $h_2/k_2$  such that  $C(h_2/k_2)$  is tangent to  $C(0/1)$ , let  $\tau(\gamma_{0/1, h_2/k_2})$  be the arc on  $C(0/1)$  from the point  $+i$  to the point of tangency with  $C(h_2/k_2)$  parametrized by arc length. Likewise, for  $h_0/k_0$  such that  $C(h_0/k_0)$  is tangent to  $C(1/1)$ , let  $\tau(\gamma_{h_0/k_0, 1/1})$  be the arc on  $C(0/1)$  from the point of tangency with  $C(h_0/k_0)$  to the point  $1 + i$  parametrized by arc length. If we specify an  $N \in \{1, 2, \dots\}$ , then we may define  $\tau(\gamma_{N, h/k})$  for  $h/k \in \mathcal{F}_N$  to be

$$(20) \quad \tau(\gamma_{N, h/k}) = \begin{cases} \tau(\gamma_{0/1, h_2/k_2}) & (h/k = 0/1), \\ \tau(\gamma_{h_0/k_0, h/k, h_2/k_2}) & (k \neq 1), \\ \tau(\gamma_{h_0/k_0, 1/1}) & (h/k = 1/1), \end{cases}$$

where  $h_0/k_0$  is the element in  $\mathcal{F}_N$  immediately before  $h/k$  if such an element exists and  $h_2/k_2$  is the element in  $\mathcal{F}_N$  immediately after  $h/k$  if such an element exists.

We define  $\tau(\gamma_N)$  to be the concatenation in order of each  $\tau(\gamma_{N, h/k})$  for  $h/k \in \mathcal{F}_N$ . The contour  $\gamma_N$  is then the mapping of  $\tau(\gamma_N)$  into the  $q$ -plane. This is a concatenation of the contours  $\gamma_{N, h/k}$  for  $h/k \in \mathcal{F}_N$ . We redefine  $\gamma_{N, 0/1}$  to be the concatenation of what we used to call  $\gamma_{N, 0/1}$  and  $\gamma_{N, 1/1}$ . These contours meet in the  $q$ -plane at the appropriate endpoints. The full contour  $\gamma_N$  is piecewise smooth and has winding number one about the origin. See Fig. 2 for visualizations of the Rademacher contour  $\gamma_N$  and  $\tau(\gamma_N)$  for various values of  $N$ .

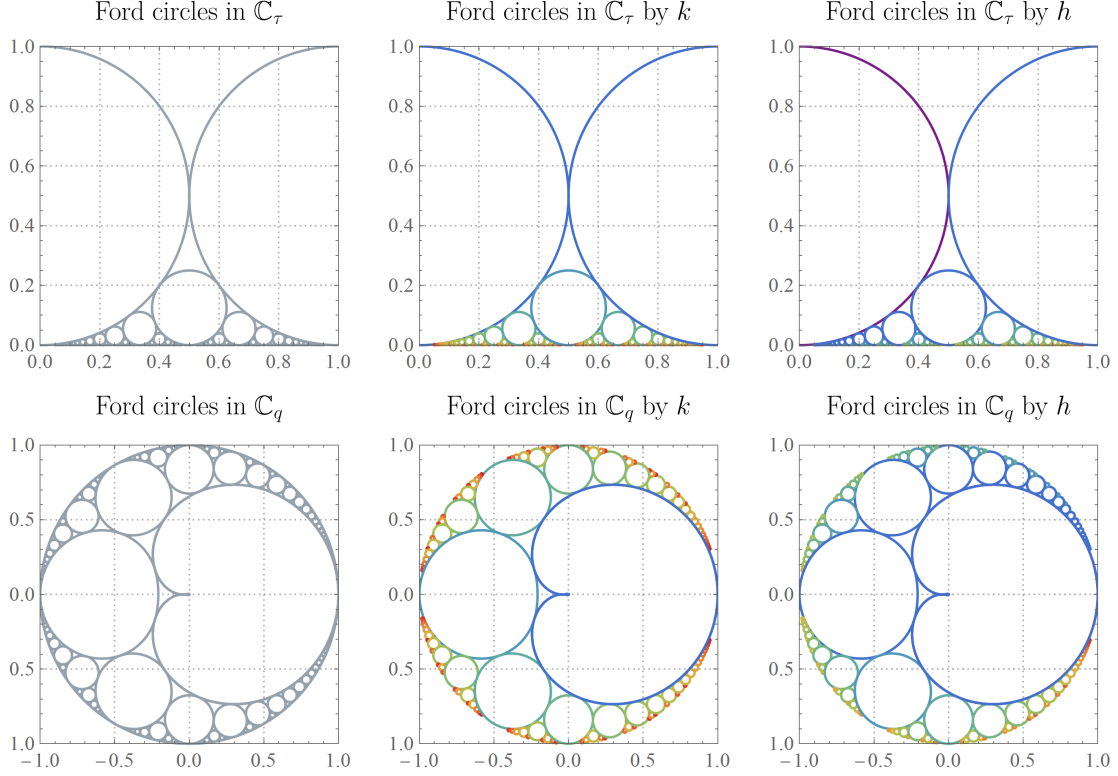


FIGURE 1. The first few Ford circles in the  $\tau$  and  $q$  planes, colored according to various color schemes.

We now split up the contour integral in eq. 17 into a sum of integrals over subcontours:

$$(21) \quad d(n) = \frac{1}{2\pi i} \sum_{k=1}^N \sum_{\substack{1 \leq h < k \\ (h,k)=1}} \int_{\gamma_{N,h/k}} \frac{Z(q)}{q^{(n-n_0)+1}} dq,$$

where  $(h, k)$  is shorthand for  $\gcd(h, k)$ . It is convenient to change coordinates within each subcontour integral in order to write them as integrals over similar contours. Note that, for irreducible  $h/k$ , the coordinate transformation

$$(22) \quad z = -ik^2 \left( \tau - \frac{h}{k} \right) \text{ or equivalently } \tau = i \cdot \frac{z}{k^2} + \frac{h}{k}$$

maps the Ford circle  $C(h/k)$  in the  $\tau$ -plane to the circle  $B_{1/2}(1/2)$  in the  $z$ -plane with center  $1/2$  and radius  $1/2$ . See Fig. 3. The point  $\tilde{\tau}(h/k, h_2/k_2)$  is mapped to the point

$$(23) \quad \tilde{z}_{h/k}(h_2/k_2) = \frac{k^2}{k^2 + k_2^2} + i \left( hk - \frac{k^2}{k^2 + k_2^2} (hk + h_2 k_2) \right).$$

We moved the  $h/k$  into the subscript of  $\tilde{z}$  to emphasize that the coordinate transformation depends on  $h/k$  and that – for this reason – unlike  $\tilde{\tau}$ ,  $\tilde{z}$  is not symmetric under interchanging its arguments. The contour  $\tau(\gamma_{N,h/k}) = \tau(\gamma_{h_0/k_0, h/k, h_2/k_2})$  is mapped to an arc along  $B_{1/2}(1/2)$  from  $\tilde{z}_{h/k}(h_0/k_0)$  to  $\tilde{z}_{h/k}(h_2/k_2)$ , specifically the arc which does not contain the origin. Likewise, the contours  $\tau(\gamma_{0/1, h_2/k_2})$  and  $\tau(\gamma_{h_0/k_0, 1/1})$  are mapped together to an arc along  $B_{1/2}(1/2)$  from  $\tilde{z}_{1/1}(h_0/k_0)$  to  $\tilde{z}_{0/1}(h_2/k_2)$ , also the arc which does not contain the origin. Let

- $\tilde{z}_{1,N,h/k}$  be  $\tilde{z}_{h/k}(h_0/k_0)$ , where  $h_0/k_0$  is the element of  $\mathcal{F}_N$  immediately before  $h/k$  if  $k \neq 1$ , and



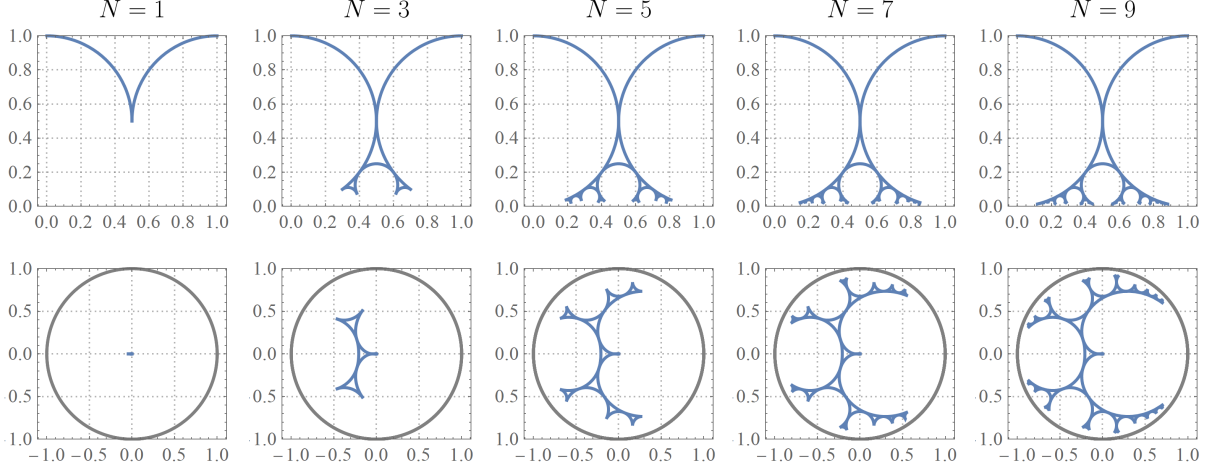


FIGURE 2. The Rademacher contour  $\gamma_N$  for several different values of  $N$ , in the  $\tau$  and  $q$  planes.

- $\tilde{z}_{1/1}(h_0/k_0)$ , where  $h_0/k_0$  is the element of  $\mathcal{F}_N$  immediately before  $1/1$  if  $k = 1$ .

For irreducible  $h/k$  except  $1/1$ , let  $\tilde{z}_{2,N,h/k}$  be  $\tilde{z}_{h/k}(h_2/k_2)$ , where  $h_2/k_2$  is the element of  $\mathcal{F}_N$  immediately after  $h/k$ . It can be checked that

$$(24) \quad \tilde{z}_{1,N,h/k} = \frac{k}{k^2 + k_0^2}(k + ik_0),$$

$$(25) \quad \tilde{z}_{2,N,h/k} = \frac{k}{k^2 + k_2^2}(k - ik_2),$$

where  $h_0/k_0 < h/k < h_2/k_2$  are consecutive fractions in  $\mathcal{F}_N$  or if  $h/k = 0/1$  and  $h_0/k_0$  is immediately preceding  $1/1$  or if  $h/k = 1/1$  and  $h_2/k_2$  is immediately following  $0/1$ . See Fig. 3.

Rewriting eq. 21 in terms of  $z$ ,

$$(26) \quad d(n) = i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \int_{z(\gamma_{N,h/k})} Z \left( \exp \left( 2\pi \left[ i \cdot \frac{h}{k} - \frac{z}{k^2} \right] \right) \right) \exp \left( 2\pi(n - n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right) \right) dz.$$

Here  $z(\gamma_{N,h/k})$  is the mapping of  $\tau(\gamma_{N,h/k})$  into the  $z$ -plane. That is,  $z(\gamma_{N,h/k})$  is the arc of  $B_{1/2}(1/2)$  which avoids the origin and is from  $\tilde{z}_{1,N,h/k}$  to  $\tilde{z}_{2,N,h/k}$ . Tracing through the definitions of  $z$  and  $Z(q)$ , the integrands above are holomorphic in the right-half of the  $z$ -plane. We may therefore deform our subcontours from arcs on  $B_{1/2}(1/2)$  to chords through  $B_{1/2}(1/2)$ . These chords begin at  $\tilde{z}_{1,N,h/k}$  and end at  $\tilde{z}_{2,N,h/k}$ . We denote these chords as  $\overline{z_1 z_2}(N, h/k)$ . See Fig. 3. So,

$$(27) \quad d(n) = i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \int_{\overline{z_1 z_2}(N, h/k)} Z \left( \exp \left( 2\pi \left[ i \cdot \frac{h}{k} - \frac{z}{k^2} \right] \right) \right) \exp \left( 2\pi(n - n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right) \right) dz.$$

Before we proceed, we state two geometric lemmas. The first concerns the properties of the chords  $\overline{z_1 z_2}(N, h/k)$ , and the second concerns the properties of arcs on  $B_{1/2}(1/2)$ . Proofs of these are contained in [Hsu11].

**Lemma 2.1.** *The chord  $\overline{z_1 z_2}(N, h/k)$  has length at most  $2\sqrt{2}k/N$  and on this chord  $|z| \leq \sqrt{2}k/N$ .* ■

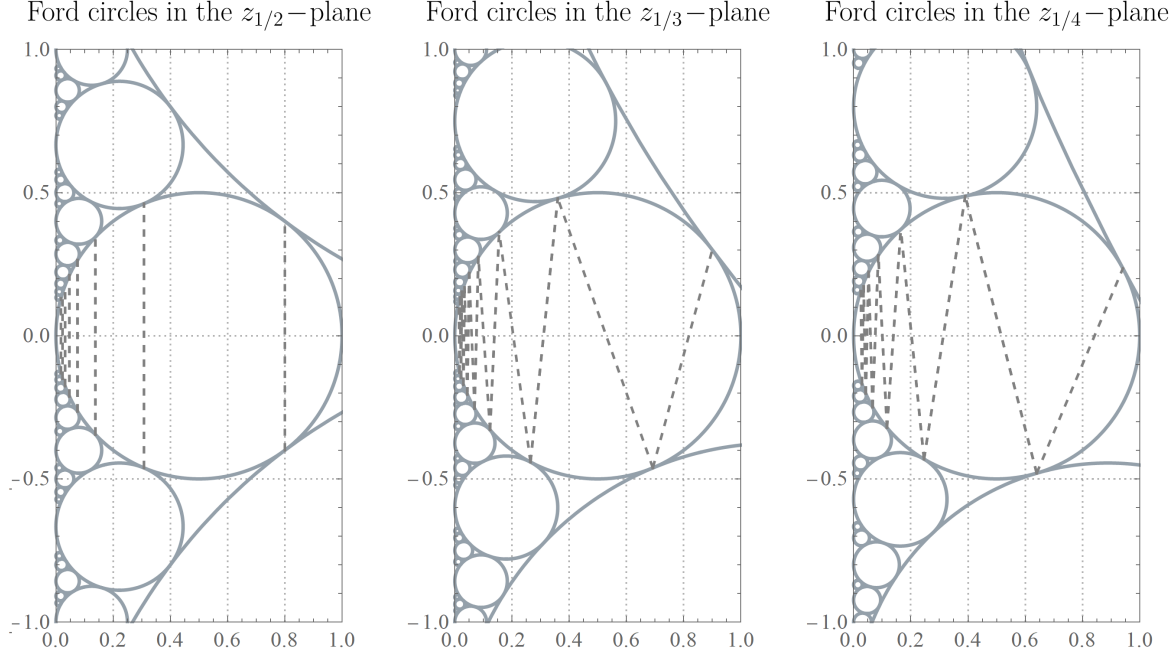


FIGURE 3. The first few Ford circles in the  $z$ -plane using the transformation in eq. 22 for  $h/k = 1/2$  (left),  $h/k = 1/3$  (center), and  $h/k = 1/4$  (right). The chords  $\overline{z_1 z_2}(N, h/k)$  for various  $N$  are denoted by gray dashed lines. (The cases  $h/k = 1/2, 1/4$  look similar but do differ on close examination.)

**Lemma 2.2.** *In the disk bounded by  $B_{1/2}(1/2)$ ,  $\operatorname{Re}(z) \leq 1$  and  $\operatorname{Re}(1/z) \geq 1$ , with  $\operatorname{Re}(1/z) = 1$  on the circle itself.*

*On the arcs from 0 to  $\tilde{z}_{1,N,h/k}$  and  $\tilde{z}_{2,N,h/k}$  to 0,  $|z| \leq \sqrt{2}k/N$ . The length of these arcs is at most  $\pi\sqrt{2}k/N$ . ■*

By the previous two lemmas, if we fix  $h/k$  and send  $N \rightarrow \infty$ , the chords  $\overline{z_1 z_2}(N, h/k)$  get shorter and nearer to the origin. As  $z$  approaches the origin,  $\tau = (h/k) + i(z/k^2)$  approaches  $h/k$ . So, asymptotics for  $\tau \rightarrow h/k + i0$  correspond to asymptotics for  $z \rightarrow 0^+$ , for the  $h/k$ -dependent coordinate transformation above. We can calculate the rather manifest asymptotics of  $\eta(\tau)$  as  $\tau$  approaches  $+i\infty$  from the definition of  $\eta(\tau)$ . We can then calculate the asymptotics of  $Z(\tau)$  near  $h/k$  in terms of the asymptotics of each  $\eta(m\tau)$  near  $+i\infty$  using the modularity properties of  $\eta(\tau)$ . These asymptotics are sufficiently simple to be integrated in terms of some special functions. This is the key insight in the Hardy-Littlewood-Ramanujan circle method, and Rademacher's innovation consists of doing this precisely. We now turn to expressing  $\eta(m\tau)$  for  $m \in \{1, 2, \dots\}$  near  $h/k$  in terms of  $\eta(\tau)$  near  $+i\infty$ .

For an irreducible fraction  $h/k \in [0, 1]$ , we may find by the Euclidean algorithm some integer  $H(m, h, k) = H$  such that

$$(28) \quad mhH \equiv -\gcd(m, k) \pmod{k}.$$

It follows that the matrix  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is in  $\operatorname{SL}_2(\mathbb{Z})$ , where

$$(29) \quad a = H, \quad b = -\frac{1}{k}(mhH + \gcd(m, k)), \quad c = \frac{k}{\gcd(m, k)}, \quad d = -\frac{mh}{\gcd(m, k)}.$$



As a member of the modular group, this matrix defines a linear fractional transformation of the upper-half of the complex plane; this is given by the map  $M(\tau) = \frac{a\tau+b}{c\tau+d}$ ,  $\tau \in \mathbb{H}$ . Under this action, the image of  $m\tau$  is

$$(30) \quad M(m\tau) = M\left(m \left[ \frac{h}{k} + i \cdot \frac{z}{k^2} \right]\right) = \frac{\gcd(m, k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m, k) \right) = \frac{am\tau + b}{cm\tau + d}.$$

As  $z$  approaches 0 – and therefore  $\tau$  approaches  $h/k$  – the right hand side converges (in an appropriate sense) to positive imaginary infinity. Using the modular transformation properties of  $\eta(\tau)$ ,

$$(31) \quad \eta(m\tau) = [\epsilon(a, b, c, d)(cm\tau + d)^{1/2}]^{-1} \eta\left(\frac{am\tau + b}{cm\tau + d}\right).$$

In this case,  $cm\tau + d = imz/k \gcd(m, k)$ , and by eq. 7,

$$(32) \quad \epsilon(a, b, c, d) = \exp\left(\pi i \left( \frac{H}{12k} \gcd(m, k) - \frac{mh}{12k} + s \left( \frac{mh}{\gcd(m, k)}, \frac{k}{\gcd(m, k)} \right) - \frac{1}{4} \right)\right),$$

so that

$$(33) \quad \eta\left(m \left( \frac{h}{k} + i \cdot \frac{z}{k^2} \right)\right) = \exp\left(\pi i \left( -\frac{H}{12k} \gcd(m, k) + \frac{mh}{12k} - s \left( \frac{mh}{\gcd(m, k)}, \frac{k}{\gcd(m, k)} \right) \right)\right) \\ \times \sqrt{\frac{k \gcd(m, k)}{mz}} \eta\left(\frac{\gcd(m, k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m, k) \right)\right).$$

The constant  $H = H(m, h, k)$  depends implicitly on  $m$ ,  $h$ , and  $k$ . Combining eq. 33 for all values of  $m$ :

$$(34) \quad Z\left(\frac{h}{k} + i \cdot \frac{z}{k^2}\right) = \xi(h, k) \omega(h, k) \cdot z^{c_1} \prod_{m=1}^{\infty} \eta\left(\frac{\gcd(m, k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m, k) \right)\right)^{\delta_m},$$

where  $c_1$  was defined in eq. 13 and we defined

$$(35) \quad \xi(h, k) = \prod_{m=1}^{\infty} \left[ \sqrt{\frac{k \gcd(m, k)}{m}} \exp\left(\pi i \left( \frac{mh}{12k} - s \left( \frac{mh}{\gcd(m, k)}, \frac{k}{\gcd(m, k)} \right) \right)\right) \right]^{\delta_m}$$

and

$$(36) \quad \omega(h, k) = \prod_{m=1}^{\infty} \exp\left(-\frac{\pi i}{12k} H \delta_m \gcd(m, k)\right).$$

Plugging in the previous formulas into eq. 27:

$$(37) \quad d(n) = i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ (h, k)=1}} \xi(h, k) \omega(h, k) \int_{\overline{z_1 z_2}(N, h/k)} z^{c_1} \\ \times \left[ \prod_{m=1}^{\infty} \eta\left(\frac{\gcd(m, k)}{k} \left( H + i \cdot \frac{k}{mz} \gcd(m, k) \right)\right)^{\delta_m} \right] \exp\left(2\pi(n - n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right)\right) dz.$$

We expect (say, based on Rademacher's original argument or the rapid convergence of  $e^{-1/z}$  to 0 as  $z \rightarrow 0^+$ .) to be able to replace each  $\eta(q)$  in the integrand with  $q^{1/24}$  and accrue a total  $o(1)$  error as  $N \rightarrow \infty$ . Defining

$$(38) \quad \text{Er}(N) = i \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h, k)=1}} k^{-2} \xi(h, k) \omega(h, k) \int_{\overline{z_1 z_2}(N, h/k)} z^{c_1} \exp\left(2\pi(n - n_0) \left( \frac{z}{k^2} - i \cdot \frac{h}{k} \right)\right) \Delta_{h, k}(z) dz,$$

where  $\Delta_{h,k}(z)$  is the difference between the  $\eta$ -quotient and its leading order asymptotics,

$$(39) \quad \Delta_{h,k}(z) = \left[ \prod_{m=1}^{\infty} \eta \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right] - \left[ \prod_{m=1}^{\infty} \exp \left( \frac{\pi i \gcd(m,k)}{12 k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right],$$

we can write

(40)

$$d(n) = \text{Er}(N) + i \sum_{k=1}^N \frac{1}{k^2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \xi(h,k) e^{-2\pi i(n-n_0) \cdot \frac{h}{k}} \int_{\overline{z_1 z_2}(N, h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz.$$

The second term is the result of replacing each  $\eta$ -function by the appropriate asymptotics, and the first term is the accumulated error from doing so. We wish to show that  $\lim_{N \rightarrow \infty} \text{Er}(N) = 0$ .

Our first task is to bound  $\Delta_{h,k}(z)$ . We will show that  $\Delta_{h,k}(z) = O(1)$  along our contours, where the bound does not depend on  $h, k$ , or  $z$ . That is, there exists some constant  $C = C(\{\delta_m\}_{m=1}^{\infty})$  depending only on  $\{\delta_m\}_{m=1}^{\infty}$  such that  $|\Delta_{h,k}(z)| \leq C$  (for all relevant  $h, k, z$ ). The fact that  $\Delta_{h,k}(z)$  does not blow up at the origin is essentially the assumption that the polar component of  $Z(\tau)$  near  $h/k$  contains at most one term (that which we are approximating  $Z$  by). Let  $\tilde{\eta}(\tau) = \eta(\tau)/q^{1/24}$ , so that

$$(41) \quad \Delta_{h,k}(z) = \left[ \prod_{m=1}^{\infty} \exp \left( \frac{\pi i \gcd(m,k)}{12 k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right] \times \left[ \left[ \prod_{m=1}^{\infty} \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} \right] - 1 \right].$$

Taking the norm,

$$(42) \quad |\Delta_{h,k}(z)| = \exp \left[ \frac{\pi}{12} c_3(k) \text{Re} \left( \frac{1}{z} \right) \right] \times \left| \prod_{m=1}^{\infty} \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} - 1 \right|.$$

If  $c_3(k) > 0$ , the first term on the right hand side can be large for some  $z$  along  $\overline{z_1 z_2}(N, h/k)$ , specifically  $z$  close to the origin. Compensatingly, we expect – based on the  $z \rightarrow 0$  asymptotics of the productand – the second term on the right hand side to be small (hence the claimed uniform bound on  $|\Delta_{h,k}(z)|$ ). This is the content of the following lemma.

**Lemma 2.3.** *For some constant  $D$ , which may depend only on  $\{\delta_m\}_{m=1}^{\infty}$ : for  $z$  in  $\overline{B_{1/2}(1/2)}/\{0\}$ ,*

$$(43) \quad \left| \prod_{m=1}^{\infty} \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{\delta_m} - 1 \right| \leq D \exp \left( -2\pi \text{Re} \left( \frac{1}{z} \right) \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m,k)^2}{m} \right\} \right).$$

Here,  $\overline{B_{1/2}(1/2)}$  denotes the closed disk bounded by  $B_{1/2}(1/2)$ . ■

*Proof of Lemma 2.3.* First consider

$$(44) \quad \tilde{\eta} \left( \frac{\gcd(m,k)}{k} \left( H + i \frac{k}{mz} \gcd(m,k) \right) \right)^{-1},$$

which by definition is

$$(45) \quad \sum_{j=0}^{\infty} p(j) \exp \left( -2\pi j \cdot \frac{\gcd(m,k)^2}{m} \text{Re} \left( \frac{1}{z} \right) \right) \omega(h, m, k, z)^j,$$

where  $p(j)$  is Euler's partition function and  $\omega(h, m, k, z) = \omega$  is a phase factor. Recall the following upper bound for  $p(j)$ , which can be derived by purely classical methods, see e.g. [Apo76]:

$$(46) \quad p(j) = O\left(\exp\left(\pi\sqrt{\frac{2}{3}} \cdot j\right)\right).$$

The tail of the series in eq. 45 is therefore bounded above by a convergent geometric series. To be more precise, consider the sum of all but the first two terms in eq. 45,

$$(47) \quad \exp\left(-2\pi\frac{\gcd(m, k)^2}{m} \operatorname{Re}\left(\frac{1}{z}\right)\right) \omega \sum_{j=2}^{\infty} p(j) \exp\left(-2\pi(j-1)\frac{\gcd(m, k)^2}{m} \operatorname{Re}\left(\frac{1}{z}\right)\right) \omega^{j-1}.$$

Using the triangle inequality and Lemma 2.2,

$$(48) \quad \left| \sum_{j=2}^{\infty} p(j) \exp\left(-2\pi(j-1)\frac{\gcd(m, k)^2}{m} \operatorname{Re}\left(\frac{1}{z}\right)\right) \omega^{j-1} \right| \leq \sum_{j=2}^{\infty} p(j) \exp\left(-2\pi(j-1)\frac{\gcd(m, k)^2}{m}\right),$$

which, using the classical bound on  $p(j)$  in eq. 46, is bounded above by

$$(49) \quad C \sum_{j=2}^{\infty} \exp\left(\pi\sqrt{\frac{2}{3}} \cdot j - 2\pi(j-1) \cdot \frac{\gcd(m, k)^2}{m}\right)$$

for some absolute constant  $C$ . Therefore, since  $\gcd(m, k)/m \geq 1/m$ , the right-hand side above is bounded by

$$(50) \quad C \sum_{j=2}^{\infty} \exp\left(\pi\sqrt{\frac{2}{3}} \cdot j - \frac{2\pi}{m}(j-1)\right).$$

This is bounded above by a convergent geometric series, so that for some constant  $C_m > 0$  which is dependent only on  $m$ ,  $|\sum_{j=2}^{\infty} p(j) \exp(-2\pi(j-1)m^{-1}\gcd(m, k)^2 \operatorname{Re}(1/z)) \omega^{j-1}| \leq C_m$ . It follows that eq. 47 is bounded above in magnitude by  $C_m \exp(-2\pi m^{-1}\gcd(m, k)^2 \operatorname{Re}(1/z))$ . Therefore, replacing the tail in eq. 45 with this bound,

$$(51) \quad \tilde{\eta}\left(\frac{\gcd(m, k)}{k} \left(H + i\frac{k}{mz} \gcd(m, k)\right)\right)^{-1} = 1 + O(1) \exp\left(-2\pi\frac{\gcd(m, k)^2}{m} \operatorname{Re}\left(\frac{1}{z}\right)\right),$$

where the  $O(1)$  term satisfies  $|O(1)| \leq C_m + 1$ . Taking the reciprocal of eq. 51 – and using that  $\operatorname{Re}(1/z) \geq 1$  within  $B_{1/2}(1/2)$  – yields

$$(52) \quad \tilde{\eta}\left(\frac{\gcd(m, k)}{k} \left(H + i\frac{k}{mz} \gcd(m, k)\right)\right) = 1 + O(1) \exp\left(-2\pi\frac{\gcd(m, k)^2}{m} \operatorname{Re}\left(\frac{1}{z}\right)\right)$$

for some other  $O(1)$  term which can be bounded in magnitude depending only on  $m$ .

Taking the appropriate product of eq. 51 and eq. 52 for all  $m$  yields

$$(53) \quad \prod_{m=1}^{\infty} \tilde{\eta}\left(\frac{\gcd(m, k)}{k} \left(H + i\frac{k}{mz} \gcd(m, k)\right)\right)^{\delta_m} = 1 + O(1) \exp\left(-2\pi \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m, k)^2}{m} \right\} \operatorname{Re}\left(\frac{1}{z}\right)\right),$$

where  $\mathcal{M}$  is the (finite) set of all  $m \in \mathbb{N}$  such that  $\delta_m$  is nonzero, and the  $O(1)$  term is bounded in magnitude depending only on  $\{\delta_m\}_{m=1}^{\infty}$ . Consequently, for some constant  $D > 0$  depending only on  $\{\delta_m\}_{m=1}^{\infty}$ ,

$$(54) \quad \left| \prod_m \tilde{\eta}\left(\frac{\gcd(m, k)}{k} \left(H + i\frac{k}{mz} \gcd(m, k)\right)\right)^{\delta_m} - 1 \right| \leq D \exp\left(-2\pi \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m, k)^2}{m} \right\} \operatorname{Re}\left(\frac{1}{z}\right)\right).$$

□

Using Lemma 2.3,  
(55)

$$|\Delta_{h,k}(z)| \leq D \exp \left[ -2\pi \operatorname{Re} \left( \frac{1}{z} \right) \left( \min_{m \in \mathcal{M}} \left\{ \frac{\gcd(m,k)^2}{m} \right\} - \frac{c_3(k)}{24} \right) \right] = D \exp \left[ -2\pi \operatorname{Re} \left( \frac{1}{z} \right) g(k) \right].$$

One of the hypotheses of Theorem 1.1 is that the function  $g(k)$  is non-negative. Then, using Lemma 2.2, we can bound

$$(56) \quad |\Delta_{h,k}(z)| \leq D e^{-2\pi \min\{g(k): k=1, \dots, \operatorname{lcm}(\mathcal{M})\}},$$

and the right hand side depends only on  $\{\delta_m\}_{m=1}^\infty$ . We redefine  $D$  to be this constant (absorbing the exponential). Using eq. 56 and Lemma 2.1, the integral in eq. 38 is bounded above in magnitude:

$$(57) \quad \left| \int_{\overline{z_1 z_2}(N, h/k)} z^{c_1} \exp \left( 2\pi(n - n_0) \left( \frac{z}{k^2} - i \frac{h}{k} \right) \right) \Delta_{h,k}(z) dz \right| \leq 2D \left( \sqrt{2} \cdot \frac{k}{N} \right)^{c_1+1} \exp \left( 2\sqrt{2}\pi(n - n_0) \cdot \frac{k}{N} \right).$$

Substituting this into the definition of  $\operatorname{Er}(N)$  in eq. 38: for some constant  $C$  depending only on  $\{\delta_m\}_{m=1}^\infty$ ,

$$|\operatorname{Er}(N)| \leq C e^{2\sqrt{2}\pi n} N^{-(c_1+1)} \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} k^{-1}.$$

Since there are at most  $k$  terms in the inner sum and  $N$  terms in the outer sum, we can bound this as follows:

$$(58) \quad |\operatorname{Er}(N)| \leq C e^{2\sqrt{2}\pi n} N^{-c_1}.$$

Since  $c_1 > 0$ , this shows that  $\lim_{N \rightarrow \infty} \operatorname{Er}(N) = 0$ , as desired. Referring back to eq. 40, we have shown that

$$(59) \quad d(n) = o(1) + i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \xi(h,k) e^{-2\pi i(n-n_0)h/k} \int_{\overline{z_1 z_2}(N, h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz.$$

We now deform our contours back to arcs along  $B_{1/2}(1/2)$ :

$$(60) \quad d(n) = o(1) + i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \xi(h,k) e^{-2\pi i(n-n_0)h/k} \int_{z(\gamma_{N, h/k})} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz.$$

Our next goal is to show that the main term on the right hand side above,

$$(61) \quad i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \xi(h,k) e^{-2\pi i(n-n_0)h/k} \int_{z(\gamma_{N, h/k})} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz,$$

differs from

$$(62) \quad i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \xi(h,k) e^{-2\pi i(n-n_0)h/k} \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz$$

by an  $o(1)$  term as  $N \rightarrow \infty$  (the rate of convergence depending on  $n$ ). (Lemma 2.2 suffices to show that the integrals in eq. 62 are well-defined, e.g. as improper integrals. Since  $c_1 > 0$ , the integrands are actually continuous at the origin when  $z$  is restricted to  $B_{1/2}(1/2)$ , so we can denote the integrals

as is – that is as a proper Riemann or Lebesgue integral – as long as we interpret the integrands appropriately, namely by removing the removable  $C^0$ -singularity at the origin.) The contour in eq. 62 is traversed clockwise. So, according to the claim: we may replace our integrals over incomplete arcs of  $B_{1/2}(1/2)$  by integrals over the complete circle  $B_{1/2}(1/2)$  and only accrue a total  $o(1)$  error as  $N \rightarrow \infty$ . The former, eq. (61), is the latter, eq. (62), minus  $J_1 + J_2$ , where  $J_1 = J_1(N)$  and  $J_2 = J_2(N)$  are defined by

$$(63) \quad J_1 = i \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} k^{-2} \xi(h, k) e^{-2\pi i(n-n_0)h/k} \int_0^{\tilde{z}_1(N, h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz$$

and

$$(64) \quad J_2 = i \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} k^{-2} \xi(h, k) e^{-2\pi i(n-n_0)h/k} \int_{\tilde{z}_2(N, h/k)}^0 z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz,$$

where the integrals are on arcs of  $B_{1/2}(1/2)$  (hence well-defined). Using the bound  $|\xi(h, k)| \leq k^{-c_1}$ ,

$$(65) \quad |J_1| \leq \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} k^{-(2+c_1)} \left| \int_0^{\tilde{z}_1(N, h/k)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz \right|.$$

Using Lemma 2.2,

$$(66) \quad |J_1| \leq \pi \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} k^{-(2+c_1)} \left( \frac{\sqrt{2}k}{N} \right)^{c_1+1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} + \frac{\pi}{12} c_3(k) \right].$$

So, for some constant  $C$  depending only on  $\{\delta_m\}_{m=1}^\infty$ ,

$$(67) \quad |J_1| \leq C \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} k^{-1} \left( \frac{1}{N} \right)^{c_1+1} e^{2\pi(n-n_0)}.$$

Since the outer sum is over  $N$  terms and the inner sum is over at most  $k$  terms,  $|J_1| \leq CN^{-c_1} e^{2\pi(n-n_0)}$ . Since  $c_1 > 0$ ,  $J_1 = o(1)$  as  $N \rightarrow \infty$  (for fixed  $n$ ), as desired, where the constant depends on  $n$ . An identical argument (*mutatis mutandis*) yields  $J_2 = o(1)$ . Combining all of the previous results, for each  $n > n_0$ ,

$$(68) \quad d(n) = o(1) + i \sum_{k=1}^N k^{-2} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \xi(h, k) e^{-2\pi i(n-n_0) \cdot \frac{h}{k}} \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz$$

as  $N \rightarrow \infty$ , where the rate of convergence can depend on  $n$  (and where the integrand is interpreted as a continuous function on the contour). Taking  $N \rightarrow \infty$ ,

$$(69) \quad d(n) = i \sum_{k=1}^\infty k^{-2} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} \xi(h, k) e^{-2\pi i(n-n_0) \cdot \frac{h}{k}} \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz$$

for each  $n > n_0$ . Referring to the definition of  $\xi(h, k)$  in eq. 35 and of the Kloosterman-like sum  $A_k(n)$  in eq. 14, this is exactly

$$(70) \quad d(n) = i \sum_{k=1}^\infty k^{-(2+c_1)} c_2(k) A_k(n) \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n-n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz.$$

Now we just evaluate the integral  $I = I(n, k)$  given by

$$(71) \quad I = \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{2\pi(n - n_0)}{k^2} z + \frac{\pi}{12} \frac{c_3(k)}{z} \right] dz.$$

First note that if  $c_3(k) = 0$  and  $c_1 \in \mathbb{N}$ , then the integrand is entire, so by the Cauchy integral formula we have  $I = 0$ . If  $c_1 \notin \mathbb{N}$ , then (since we are assuming  $c_1 > 0$ )  $c_1 \in 2^{-1}(1 + 2\mathbb{N})$ . In this case, if  $c_3(k) = 0$ , we can deform the contour in the contour integral so as to avoid the origin (this is easily justified), and conclude (as before) that  $I = 0$ . So, as long as  $c_3(k) = 0$  and  $c_1 > 0$ ,  $I = 0$ .

If  $c_3(k) \neq 0$ , we can rewrite  $I$  as

$$(72) \quad I = \int_{B_{1/2}(1/2)} z^{c_1} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_3(k)|}{6}} (n - n_0) \left( \left( \frac{z}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right) \pm \left( \frac{z}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right)^{-1} \right) \right] dz,$$

where the plus-or-minus is the sign of  $c_3(k)$ . We make the substitution

$$(73) \quad w = \left( \frac{z}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right)^{-1}, \quad z = \left( \frac{w}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right)^{-1},$$

$$(74) \quad dz = - \left( \frac{w^2}{k} \sqrt{\frac{24(n - n_0)}{|c_3(k)|}} \right)^{-1} dw.$$

In terms of these new coordinates,

$$(75) \quad I = - \left( k \sqrt{\frac{|c_3(k)|}{24(n - n_0)}} \right)^{c_1+1} \int_{1-i\infty}^{1+i\infty} w^{-(c_1+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_3(k)|}{6}} (n - n_0) (w^{-1} \pm w) \right] dw.$$

(The integral is well-defined as a Lebesgue integral or an improper Riemann integral. The preceding formula can be proven e.g. by interpreting it as an improper Riemann integral and applying standard theorems on changes-of-variables in Riemann integrals.) We now split into two cases depending on the sign of  $c_3(k)$ . If  $c_3(k) < 0$ , i.e. if the  $\pm$  in the formulas above is negative, then the integrand decays sufficiently rapidly in the right-half plane such that  $I$  is given by

$$(76) \quad I = - \left( k \sqrt{\frac{|c_3(k)|}{24(n - n_0)}} \right)^{c_1+1} \lim_{R \rightarrow \infty} \int_{S(R)} w^{-(c_1+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_3(k)|}{6}} (n - n_0) (w^{-1} - w) \right] dw,$$

where  $S(R)$  is the semicircle contour whose curved arc is the right half of the circumference of the circle of radius  $R$  centered at 1 (and traversed clockwise). The integrand is holomorphic in the region of the complex plane bounded by this contour, since it does not contain the origin, so that, by Cauchy's theorem,  $I = 0$ . Otherwise, that is if  $c_3(k) > 0$  (the one remaining case), then the integrand decays sufficiently rapidly in the left-half of the complex plane such that

$$(77) \quad I = - \left( k \sqrt{\frac{|c_3(k)|}{24(n - n_0)}} \right)^{c_1+1} \oint w^{-(c_1+2)} \exp \left[ \frac{\pi}{k} \sqrt{\frac{|c_3(k)|}{6}} (n - n_0) (w^{-1} + w) \right] dw$$



for any positively oriented closed (toy) contour winding once around the origin, assuming that  $c_1$  is integral. (This formula is fine even if  $c_1$  is half-integral, but then we must be careful about the branch cut.) In order to write this integral in a standard form, we rearrange the terms in the integral slightly:

(78)

$$I = -2\pi i \left( k \sqrt{\frac{|c_3(k)|}{24(n-n_0)}} \right)^{c_1+1} \frac{1}{2\pi i} \oint w^{-(c_1+1)-1} \exp \left[ \frac{1}{2} \frac{\pi}{k} \sqrt{\frac{2}{3} |c_3(k)|(n-n_0)} (w^{-1} + w) \right] dw.$$

This integral (with the factor of  $1/2\pi i$  out front) is a standard form of the modified Bessel function of the first kind, which we cite from e.g. [Boa06][Wei06]. So,

(79)

$$I = -2\pi i \left( k \sqrt{\frac{|c_3(k)|}{24(n-n_0)}} \right)^{c_1+1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} |c_3(k)|(n-n_0)} \right],$$

where (as usual)  $I_{c_1+1}$  is the modified Bessel function of the first kind of “weight”  $c_1$ . In fact, this same formula holds (possibly by definition, depending on choice of definition) for  $c_1$  half-integral, which is a more general integral representation for the modified Bessel function of half-integral order — cf. [Rad38a], which involves the case  $c_1 = 1/2$ . To summarize the casework,

$$(80) \quad I(n, k) = \begin{cases} 0 & (c_3(k) \leq 0), \\ -2\pi i \left( k \sqrt{\frac{c_3(k)}{24(n-n_0)}} \right)^{c_1+1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n-n_0)} \right] & (c_3(k) > 0). \end{cases}$$

Given any  $n > n_0$ : substituting  $I(n, k)$ , as given by eq. 80, into eq. 70 and simplifying, we get

$$(81) \quad d(n) = 2\pi \left( \frac{1}{24(n-n_0)} \right)^{\frac{c_1+1}{2}} \sum_{\substack{k=1 \\ c_3(k) > 0}}^{\infty} c_2(k) c_3(k)^{\frac{c_1+1}{2}} k^{-1} A_k(n) I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n-n_0)} \right].$$

This completes the proof of Theorem 1.1.

### 3. ASYMPTOTICS

We would like to extract useful asymptotics from eq. 81 (“useful” roughly meaning *simple*). These are contained in the following proposition. For this section we (carrying on with the setup in the introduction) assume that the hypotheses of Theorem 1.1 are satisfied, so that eq. 16 applies.

**Proposition 3.1.** *Let  $\mathcal{K} \subset \mathbb{N}$  be the set of  $k$  that maximize  $c_3(k)/k^2$ , and let  $c_3 > 0$  be the maximum value. For any  $\epsilon > 0$ , there exists some constant  $C > 0$  (which may depend only on  $\{\delta_m\}_{m=1}^{\infty}$ ) such that, for all  $n \in \mathbb{N}$  with  $n > n_0$  and*

$$(82) \quad \left| \sum_{k \in \mathcal{K}} c_2(k) k^{c_1} A_k(n) \right| > \epsilon,$$

*it is the case that*

$$(83) \quad d(n) = (1 + O(e^{-C\sqrt{n}})) \cdot 2\pi \left( \frac{c_3}{24(n-n_0)} \right)^{\frac{c_1+1}{2}} I_{1+c_1} \left[ \pi \sqrt{\frac{2}{3} c_3(n-n_0)} \right] \sum_{k \in \mathcal{K}} c_2(k) k^{c_1} A_k(n).$$

(Here  $|O(e^{-C\sqrt{n}})| \leq C_\epsilon e^{-C\sqrt{n}}$  for some constant  $C_\epsilon > 0$  depending on  $\epsilon$  and  $\{\delta_m\}_{m=1}^{\infty}$  only.) ■

The proof of Prop. 3.1 is straightforward and an exercise in using the asymptotics of the modified Bessel functions.

*Proof.* Note that  $c_3(k)$  is periodic with period  $\text{lcm}(\mathcal{M})$ . So,  $\mathcal{K} \subseteq \{1, \dots, \text{lcm}(\mathcal{M})\}$ . We first break up the Rademacher series in eq. 16 into  $\text{lcm}(\mathcal{M})$  sums, one for each possible value of the sum index  $k$  modulo  $\text{lcm}(\mathcal{M})$ . We then show that each of these subsums is “exponentially” dominated by the leading term, in a sense which will be made precise by eq. 93. Collecting the leading terms of the subsums, we have  $\text{lcm}(\mathcal{M})$  different Bessel functions, one for each  $k \in \{1, \dots, \text{lcm}(\mathcal{M})\}$ . We then compare those Bessel functions with  $k \notin \mathcal{K}$  against those with  $k \in \mathcal{K}$ .

So, we first consider (for fixed  $b \in \{1, \dots, \text{lcm}(\mathcal{M})\}$  with  $c_3(b) > 0$ )

$$(84) \quad \sum_{\substack{k \in [b] \\ k > 0}} c_2(k) c_3(k)^{(c_1+1)/2} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n - n_0)} \right],$$

where  $[b]$  is the equivalence class of integers modulo  $\text{lcm}(\mathcal{M})$  containing  $b$ . Because  $c_2(k), c_3(k)$  have period  $\text{lcm}(\mathcal{M})$ , this is

$$(85) \quad c_2(b) c_3(b)^{(c_1+1)/2} \sum_{\substack{k \in [b] \\ k > 0}} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right].$$

Since we wish to show that the sum above is dominated by the first term, consider the rest of the terms,

$$(86) \quad \sum_{\substack{k \in [b] \\ k > \text{lcm}(\mathcal{M})}} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right],$$

which is bounded above in absolute value by

$$(87) \quad \sum_{\substack{k \in [b] \\ k > \text{lcm}(\mathcal{M})}} \left| I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right] \right| = \sum_{\substack{k \in [b] \\ k > \text{lcm}(\mathcal{M})}} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right].$$

(Unlike the unmodified Bessel functions of the first kind, the modified Bessel functions of the first kind are positive on the positive real axis.) Using the series expansion of  $I_{1+c_1}(z)$  — see e.g. [Wei06]

$$(88) \quad I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right] = \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2)j!} \left( \frac{\pi}{2k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right)^{2j+1+c_1},$$

where the sum is convergent. Suppose that  $k_0$  is a real number satisfying  $0 < k_0 \leq k$ . Then,

$$(89) \quad I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right] \leq \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2)j!} \left( \frac{\pi}{2k_0} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right)^{2j+1+c_1} \left( \frac{k_0}{k} \right)^{2j+1+c_1}$$

$$(90) \quad \leq \left( \frac{k_0}{k} \right)^{1+c_1} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + c_1 + 2)j!} \left( \frac{\pi}{2k_0} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right)^{2j+1+c_1}$$

$$(91) \quad = \left( \frac{k_0}{k} \right)^{1+c_1} I_{1+c_1} \left[ \frac{\pi}{k_0} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right].$$

Summing over all relevant  $k > \text{lcm}(\mathcal{M})$  and setting  $k_0 = b + 1/2$ , so that  $k_0 \in (0, k)$ ,

$$(92) \quad \sum_{\substack{k \in [b] \\ k > \text{lcm}(\mathcal{M})}} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right] \leq k_0^{1+c_1} \zeta(1 + c_1) I_{1+c_1} \left[ \frac{\pi}{k_0} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right]$$

$$(93) \quad = O(e^{-C_b \sqrt{n}}) I_{1+c_1} \left[ \frac{\pi}{b} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right],$$

for some constant  $C_b > 0$  depending on  $b$ , where  $\zeta$  denotes the Riemann  $\zeta$ -function. (The constant in the big-O can depend on  $\{\delta_m\}_{m=1}^\infty$ .) Therefore, the expression in eq. 85 is

$$(94) \quad c_2(b) c_3(b)^{(c_1+1)/2} (A_b(n) b^{-1} + O(e^{-C_b \sqrt{n}})) I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right].$$

It follows that

$$(95) \quad \sum_{\substack{k \in [b] \\ k > 0}} c_2(k) c_3(k)^{(c_1+1)/2} A_k(n) k^{-1} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n - n_0)} \right] \\ = c_2(b) c_3(b)^{(c_1+1)/2} (A_b(n) b^{-1} + O(e^{-C_b \sqrt{n}})) I_{1+c_1} \left[ \frac{\pi}{b} \sqrt{\frac{2}{3} c_3(b)(n - n_0)} \right].$$

We can sum this result over  $b \in \{1, \dots, \text{lcm}(\mathcal{M})\}$  with  $c_3(b) > 0$ . After doing so, we can absorb the terms with  $b \notin \mathcal{K}$  into the error term. So, for some constant  $C$  depending only on  $\{\delta_m\}_{m=1}^\infty$ ,

$$(96) \quad d(n) = 2\pi \left( \frac{1}{24(n - n_0)} \right)^{\frac{c_1+1}{2}} \sum_{k \in \mathcal{K}} (A_k(n) k^{-1} + O(e^{-C \sqrt{n}})) c_2(k) c_3(k)^{\frac{c_1+1}{2}} I_{1+c_1} \left[ \frac{\pi}{k} \sqrt{\frac{2}{3} c_3(k)(n - n_0)} \right].$$

Since for  $k \in \mathcal{K}$  it is the case that  $c_3(k) = k^2 c_3$ ,

$$(97) \quad d(n) = 2\pi \left( \frac{c_3}{24(n - n_0)} \right)^{\frac{c_1+1}{2}} I_{1+c_1} \left[ \pi \sqrt{\frac{2}{3} c_3(n - n_0)} \right] \sum_{k \in \mathcal{K}} (A_k(n) k^{-1} + O(e^{-C \sqrt{n}})) c_2(k) k^{1+c_1}.$$

Using the assumption in eq. 82,

$$(98) \quad d(n) = (1 + O(e^{-C \sqrt{n}})) 2\pi \left( \frac{c_3}{24(n - n_0)} \right)^{\frac{c_1+1}{2}} I_{1+c_1} \left[ \pi \sqrt{\frac{2}{3} c_3(n - n_0)} \right] \sum_{k \in \mathcal{K}} A_k(n) k^{c_1} c_2(k),$$

as claimed, where now the  $O(e^{-C \sqrt{n}})$  term is bounded in terms of  $\{\delta_m\}_{m=1}^\infty$  and  $\epsilon$ .  $\square$

#### 4. NUMERICS

In this section we numerically test eq. 16 for several  $\eta$ -quotients  $Z(q)$ .

Here  $d(n, N)$  represents the  $N$ th partial sum of the right hand side of eq. 16 and  $d(n)$  represents the Fourier coefficients of  $Z(q) \cdot q^{n_0}$ . The following  $\eta$ -quotients all satisfy the hypotheses of Theorem 1.1. The absolute value

$$(99) \quad |d(n, N) - d(n)|$$

of the difference between the finite sum  $d(n, N)$  and the exact Fourier coefficient  $d(n)$  – for  $N \in \{1, \dots, 1000\}$  and  $n \in \{1, \dots, 10\}$  with  $n > n_0$  – is plotted in Figure 4, as a log-log plot, for

- $Z(q) = 1/\eta(4\tau)\eta(\tau)^3$  in the upper-left,
- $Z(q) = \eta(4\tau)/\eta(\tau)^3$  in the upper-right,
- $Z(q) = 1/\eta(2\tau)$  in the middle-left,
- $Z(q) = 1/\eta(11\tau)^2\eta(\tau)^2$  in the middle-right,

- $Z(q) = 1/\eta(\tau)\eta(22\tau)$  in the bottom-left,
- $Z(q) = 1/\eta(\tau)\eta(23\tau)$  in the bottom-right.

The convergence of  $d(n, N)$  to  $d(n)$  as  $N \rightarrow \infty$  is clear from the figure, although a few qualitative trends are worth noting. The first is that the convergence is rather haphazard, especially for the last couple of the  $\eta$ -quotients above. (At least, the haphazardness is most obvious for those two.) The second is that the precise rate of convergence of  $d(n, N)$  to  $d(n)$  may depend on  $n$ . Indeed, for the last couple  $\eta$ -quotients above, there are a few  $n$  which seem to lead to significantly delayed decay, although it is not clear whether or not this delay persists as  $N \rightarrow \infty$ . However, since the decay rate appears linear in a  $\log$ - $\log$  plot, with slope independent of  $n$ , the convergence follows an approximate power law (in  $N$ ) whose exponent is independent of  $n$  (depending only on the  $\eta$ -quotient considered).

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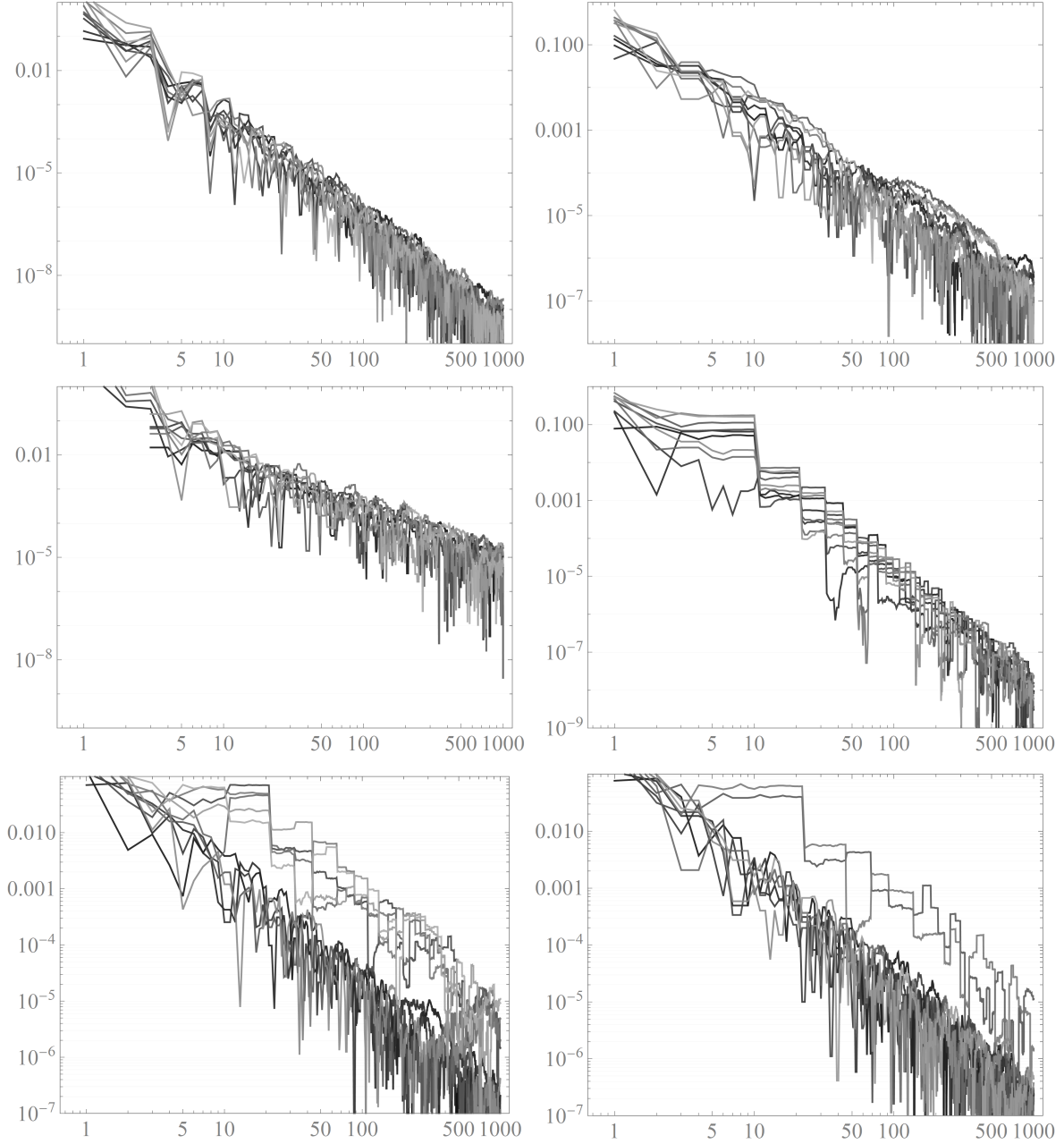


FIGURE 4. Convergence of the  $N$ th partial sums of eq. 16 for the listed  $\eta$ -quotients:  $Z(q) = 1/\eta(4\tau)\eta(\tau)^3$  (upper left),  $Z(q) = \eta(4\tau)/\eta(\tau)^3$  (upper right),  $Z(q) = 1/\eta(2\tau)$  (middle left),  $Z(q) = 1/\eta(11\tau)^2\eta(\tau)^2$  (middle right),  $Z(q) = 1/\eta(\tau)\eta(22\tau)$  (bottom left),  $Z(q) = 1/\eta(\tau)\eta(23\tau)$  (bottom right). The vertical axis is  $|d(n, N) - d(n)|$  and the horizontal axis is  $N$ . Both axes are scaled logarithmically. Each line is a plot of  $d(n, N)$  for fixed  $n$  and variable  $N$ . The lines are different shades of gray to help visually distinguish them.

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