WIGNER'S THEOREM

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Let $\mathcal{H}$ denote a separable Hilbert space over $\mathbb{C}$ of dimension $d \in \mathbb{N}^+ \cup \{\infty\}$ and

$$P\mathcal{H} = \{[\phi] : \phi \in \mathcal{H}\backslash\{0\}\} \tag{1}$$

its projectivization, where $[\phi] = \mathbb{C}^\times \phi$ for $\phi \in \mathcal{H}\backslash\{0\}$. If $d < \infty$, then $P\mathcal{H} = \mathbb{C}P^{d-1}$. Suppose that $\mathcal{H}$ is the Hilbert space used to model some quantum mechanical system. Then, the elements $[\phi] \in P\mathcal{H}$ have the interpretation of being the physical states of the system, possibly modulo unobservable phases whose metaphysical reality is besides the point. Key in the quantum mechanical formalism is the function $(-,-) : P\mathcal{H} \times P\mathcal{H} \to \mathbb{C}$ defined by

$$([\phi],[\psi]) = \frac{|\langle \phi,\psi\rangle|}{\|\phi\|\|\psi\|}, \tag{2}$$

for $\phi,\psi \in \mathcal{H}\backslash\{0\}$. These are called "transition probabilities" because of their central role in Born's rule for predicting the results of measurements.

Unitary maps $U \in \mathrm{U}(\mathcal{H})$ are precisely those bijections $\mathcal{H} \to \mathcal{H}$ preserving the Hilbert space structure. Wigner emphasized that, due to the unobservability of phases, the physically relevant notion of symmetry is not that of a unitary map, but rather the notion that will be called here a "Wigner isomorphism." A *Wigner isomorphism* is a bijection $T : P\mathcal{H} \to P\mathcal{H}$ that preserves transition probabilities. In other words,

$$(T([\phi]), T([\psi])) = ([\phi],[\psi]) \tag{3}$$

for all $\phi,\psi \in \mathcal{H}\backslash\{0\}$. Let $\mathrm{Iso}(P\mathcal{H})$ denote the set of Wigner isomorphisms. How wild can Wigner isomorphisms be? Nothing in the definition seems to require that Wigner isomorphisms be continuous, let alone respect much in the way of the vector space structure of $\mathcal{H}$ (or whatever of that structure is left after projectivization). Wigner's theorem says that, to the contrary, Wigner isomorphisms all lift to real-linear maps.

First, some notation. If $L : \mathcal{H} \to \mathcal{H}$ is injective and either ($\mathbb{C}$-)linear or anti-linear (the latter meaning that $L\lambda\phi = \lambda^* L\phi$ and $L(\phi+\psi) = L\phi + L\psi$ for all $\phi,\psi \in \mathcal{H}$ and $\lambda \in \mathbb{C}$), the map $[L] : P\mathcal{H} \to P\mathcal{H}$ is defined by

$$[L] : [\phi] \mapsto [L\phi], \tag{4}$$

this being well-defined because, for any $\lambda \in \mathbb{C}^\times$, either $[L\lambda\phi] = [\lambda L\phi] = [L\phi]$, if L is linear, or $[L\lambda\phi] = [\lambda^* L\phi] = [L\phi]$, if L is anti-linear.

One obvious source of Wigner isomorphisms is $\mathrm{U}(\mathcal{H})$. If $U \in \mathrm{U}(\mathcal{H})$, then $[U]$ is a Wigner isomorphism, as

$$([U]([\phi]),[U]([\psi])) = ([U\phi],[U\psi]) = \frac{|\langle U\phi,U\psi\rangle|}{\|U\phi\|\|U\psi\|} = \frac{|\langle\phi,\psi\rangle|}{\|\phi\|\|\psi\|} = ([\phi],[\psi]) \tag{5}$$

for any $\phi,\psi \in \mathcal{H}\backslash\{0\}$. A natural guess would be that all Wigner isomorphisms arise in this way, so that the map $\mathrm{U}(\mathcal{H}) \ni U \mapsto [U] \in \mathrm{Iso}(P\mathcal{H})$ is surjective. This is not entirely correct, but it is close:

Theorem 1 (Wigner's theorem)**.** *If $T : P\mathcal{H} \to P\mathcal{H}$ is a Wigner isomorphism, then $T = [U]$ for $U : \mathcal{H} \to \mathcal{H}$ either unitary or anti-unitary.*

Just as a unitary operator on $\mathcal{H}$ is a linear bijection $U : \mathcal{H} \to \mathcal{H}$ such that $\langle U\phi, U\psi\rangle = \langle \phi, \psi\rangle$ for all $\phi, \psi \in \mathcal{H}$, an *anti-unitary* operator on $\mathcal{H}$ is an anti-linear bijection $V : \mathcal{H} \to \mathcal{H}$ such that

$$\langle V\phi, V\psi\rangle = \langle \phi, \psi\rangle^* \tag{6}$$

for all $\phi, \psi \in \mathcal{H}$. Every Hilbert space admits at least one anti-unitary map; if $\mathcal{B} = \{\phi_n\}_{n=1}^{\dim \mathcal{H}}$ is an orthonormal basis of $\mathcal{H}$, then the map $*_{\mathcal{B}} : \mathcal{H} \to \mathcal{H}$ defined by

$$*_{\mathcal{B}} : \sum_{n=1}^{\dim \mathcal{H}} c_n \phi_n \mapsto \sum_{n=1}^{\dim \mathcal{H}} c_n^* \phi_n \tag{7}$$

is anti-unitary. Note that $*_{\mathcal{B}}$ depends on the choice of $\mathcal{B}$. If $d \geq 2$, then, if we are given a map $L : \mathcal{H} \to \mathcal{H}$ which is known to either be linear or be anti-linear, which possibility holds is determined by $[L]$ — see Lemma 4.1. So, $[*_{\mathcal{B}}] \in \mathrm{Iso}(P\mathcal{H})$ is a Wigner isomorphism not coming from any unitary operator.

When $d = 1$, $P\mathcal{H}$ is the singleton $\mathbb{C}P^0$, so the only Wigner isomorphism is the identity $[\mathrm{id}_{\mathbb{C}}] : \mathbb{C}P^0 \to \mathbb{C}P^0$. This edge case is not particularly interesting, so below we assume $d \geq 2$.

Wigner's theorem was first stated by Wigner, who also provided an incomplete argument. The details of that first argument were filled in by Bargmann. Many proofs have appeared in the literature.

The proof of Wigner's theorem is split into several steps.

1. Proof of Wigner's theorem in the $d = 2$ case

As a warmup, and as a lemma for the full result, consider the case $\mathcal{H} = \mathbb{C}^2$, in which case $P\mathcal{H} = \mathbb{C}P^1$. Let $T : \mathbb{C}P^1 \to \mathbb{C}P^1$ denote a Wigner isomorphism. Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. The proof involves constructing an explicit unitary $U \in \mathrm{U}(\mathbb{C}^2)$ such that

$$T \in \{[U], [U*]\}, \tag{8}$$

where $* : (a, b) \mapsto (a^*, b^*)$. In order to understand the argument, note that any element of the group $\mathrm{U}(2)$ of 2×2 unitary matrices, besides scalar multiples of the identity, is uniquely determined by two independent eigenvectors and their eigenvalues. Since $[U]$ does not change if U is multiplied by a phase, this means that T should uniquely determine the eigenvectors and the ratio of the two eigenvalues. Once a suitable U has been found, the Wigner automorphism $[U^{-1}] \circ T$ should be one of $[\mathrm{id}], [*]$.

As a first step, we show that there exists a Wigner automorphism $T_0 : \mathbb{C}P^1 \to \mathbb{C}P^1$ and unitary $U_0 \in \mathrm{U}(2)$ such that $T = [U_0] \circ T_0$ and

($\star$) $T_0([\mathbf{e}_j]) = [\mathbf{e}_j]$ for each $j \in \{1, 2\}$.

Indeed, pick normalized $\mathbf{e}_1' \in T([\mathbf{e}_1])$ and $\mathbf{e}_2' \in T([\mathbf{e}_2])$. Since

$$\begin{aligned} |\langle \mathbf{e}_1', \mathbf{e}_2'\rangle| = ([\mathbf{e}_1'], [\mathbf{e}_2']) &= (T([\mathbf{e}_1]), T([\mathbf{e}_2])) \\ &= ([\mathbf{e}_1], [\mathbf{e}_2]) = |\langle \mathbf{e}_1, \mathbf{e}_2\rangle| = 0, \end{aligned} \tag{9}$$

$\mathbf{e}_1', \mathbf{e}_2'$ are orthogonal. Therefore, there exists a unique unitary map $U_0 \in \mathrm{U}(2)$ such that $\mathbf{e}_1 \mapsto \mathbf{e}_1'$ and $\mathbf{e}_2 \mapsto \mathbf{e}_2'$. Consider $T_0 = [U_0^{-1}] \circ T$, so that $T = [U_0] \circ T_0$. As T_0 is a composition of Wigner automorphisms, it is a Wigner automorphism, and it satisfies ($\star$).

Proposition 1.1. *If $T : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a Wigner automorphism satisfying ($\star$), then, for any $c_1, c_2 \in \mathbb{C}$ not both zero,*

$$T([c_1\mathbf{e}_1 + c_2\mathbf{e}_2]) = [c_1\mathbf{e}_1 + c_2 e^{i\theta}\mathbf{e}_2] \tag{10}$$

for some $\theta = \theta(c_1, c_2) \in \mathbb{R}$. ■

Proof. We can assume without loss of generality that $|c_1|^2 + |c_2|^2 = 1$. Let $\mathbf{e} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$, and choose $\mathbf{f} = d_1\mathbf{e}_1 + d_2\mathbf{e}_2 \in T([\mathbf{e}])$, where $d_1, d_2 \in \mathbb{C}$ are chosen to satisfy $|d_1|^2 + |d_2|^2 = 1$. For each $j \in \{1, 2\}$,

$$\begin{aligned} |d_j| = |\langle \mathbf{f}, \mathbf{e}_j\rangle| = ([\mathbf{f}], [\mathbf{e}_j]) &= (T([\mathbf{e}]), T([\mathbf{e}_j])) \\ &= ([\mathbf{e}], [\mathbf{e}_j]) = |\langle \mathbf{e}, \mathbf{e}_j\rangle| = |c_j|. \end{aligned} \tag{11}$$

Thus, $[\mathbf{f}] = [c_1\mathbf{e}_1 + c_2 e^{i\theta}\mathbf{e}_2]$ for some $\theta \in \mathbb{R}$. □

As a second step, we show that there exists a Wigner automorphism $T_1 : \mathbb{C}P^1 \to \mathbb{C}P^1$ and unitary $U_1 \in \mathrm{U}(2)$ such that $T_0 = [U_1] \circ T_1$ and

($\star\star$) $T_1([\mathbf{e}_1 + \mathbf{e}_2]) = [\mathbf{e}_1 + \mathbf{e}_2]$,

in addition to ($\star$), so that $T_1([\mathbf{e}_j]) = [\mathbf{e}_j]$ for each $j \in \{1, 2\}$. By Proposition 1.1, there exists some $\mathbf{f} = (1, e^{i\theta}) \in T_0([\mathbf{e}_1 + \mathbf{e}_2])$ for $\theta \in \mathbb{R}$. Let $U_1 \in \mathrm{U}(2)$ denote the unitary map

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}, \tag{12}$$

so that $U_1^{-1} : \mathbf{e}_1 \mapsto \mathbf{e}_1$ and $U_1^{-1} : \mathbf{e}_2 \mapsto e^{-i\theta}\mathbf{e}_2$. Consider the Wigner automorphism $T_1 = [U_1^{-1}] \circ T_0$. This satisfies ($\star$), $T_1 : [\mathbf{e}_j] \mapsto [\mathbf{e}_j]$ for each $j \in \{1, 2\}$, and $T_1 : [\mathbf{e}_1 + \mathbf{e}_2] \mapsto [U_1^{-1}\mathbf{f}] = [\mathbf{e}_1 + \mathbf{e}_2]$, so T_1 satisfies ($\star\star$).

Proposition 1.2. *If $T : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a Wigner automorphism satisfying ($\star$) and ($\star\star$), then, the $\theta(1, c)$ in Proposition 1.1 satisfy $|1 + e^{i\theta(1,c)}c| = |1 + c|$, so $e^{i\theta(1,c)}c \in \{c, c^*\}$.* ■

Proof. Letting $C = \sqrt{2}(1 + |c|^2)^{1/2}$,

$$\begin{aligned} |1 + e^{i\theta}c| = C([\mathbf{e}_1 + \mathbf{e}_2], [\mathbf{e}_1 + e^{i\theta}c\mathbf{e}_2]) &= C(T([\mathbf{e}_1 + \mathbf{e}_2]), T([\mathbf{e}_1 + c\mathbf{e}_2])) \\ &= C([\mathbf{e}_1 + \mathbf{e}_2], [\mathbf{e}_1 + c\mathbf{e}_2]) = |1 + c|. \end{aligned} \tag{13}$$

The reason this implies $e^{i\theta}c \in \{c, c^*\}$ is that, for any $r, R > 0$, every circle $1 + r\mathbb{S}^1 = \{1 + z \in \mathbb{C} : |z| = r\}$ centered around 1 intersects the circle $R\mathbb{S}^1$ at no more than two points, which are conjugates. □

Finally, we check that

$$T_1 \in \{\mathrm{id}_{\mathbb{C}P^1}, [*]\}. \tag{14}$$

The key point is to check that which of the two possibilities $e^{i\theta(1,c)}c \in \{c, c^*\}$ in Proposition 1.1 holds does not depend on c, with the possibility $e^{i\theta}c = c$ corresponding to $T_1 = \mathrm{id}_{\mathbb{C}P^1}$ and the possibility $e^{i\theta}c = c^*$ corresponding to $T_1 = [*]$. In order to disambiguate the possibilities, consider $T_1([\mathbf{e}_1 + i\mathbf{e}_2])$. By the previous proposition, this is $[\mathbf{e}_1 + \sigma i\mathbf{e}_2]$ for $\sigma \in \{-1, +1\}$.

- Suppose that $\sigma = +1$. Then, for C as above,

$$\begin{aligned} |1 - ie^{i\theta}c| = C([\mathbf{e}_1 + i\mathbf{e}_2], [\mathbf{e}_1 + e^{i\theta}c\mathbf{e}_2]) &= C(T([\mathbf{e}_1 + i\mathbf{e}_2]), T([\mathbf{e}_1 + c\mathbf{e}_2])) \\ &= C([\mathbf{e}_1 + i\mathbf{e}_2], [\mathbf{e}_1 + c\mathbf{e}_2]) = |1 - ic|. \end{aligned} \tag{15}$$

  But this implies that $-ie^{i\theta}c \in \{-ic, (-ic)^* = ic^*\}$. That is, $e^{i\theta}c \in \{c, -c^*\}$. But Proposition 1.1 says that $e^{i\theta}c \in \{c, c^*\}$. So,

$$e^{i\theta}c \in \{c, c^*\} \cap \{c, -c^*\} = \{c\}. \tag{16}$$

  That is, $e^{i\theta} = 1$.

  To summarize, $T_1([\mathbf{e}_1 + c\mathbf{e}_2]) = [\mathbf{e}_1 + c\mathbf{e}_2]$. Also, $T_1([\mathbf{e}_2]) = [\mathbf{e}_2]$, by ($\star$). Since every element of $\mathbb{C}P^1$ has one of these two forms, we conclude that $T_1 = [\mathrm{id}_{\mathbb{C}^2}] = \mathrm{id}_{\mathbb{C}P^1}$.

- On the other hand, suppose that $\sigma = -1$. Then, for C as above,

$$\begin{aligned} |1+ie^{i\theta}c| = C([\mathbf{e}_1 - i\mathbf{e}_2],[\mathbf{e}_1+e^{i\theta}c\mathbf{e}_2]) &= C(T([\mathbf{e}_1+i\mathbf{e}_2]),T([\mathbf{e}_1+c\mathbf{e}_2])) \\ &= C([\mathbf{e}_1+i\mathbf{e}_2],[\mathbf{e}_1+c\mathbf{e}_2]) = |1-ic|. \end{aligned} \tag{17}$$

But this implies that $ie^{i\theta}c \in \{-ic, (-ic)^* = ic^*\}$. That is, $e^{i\theta}c \in \{-c, c^*\}$. Combining with Proposition 1.1,

$$e^{i\theta}c \in \{c, c^*\} \cap \{-c, c^*\} = \{c^*\}. \tag{18}$$

That is, $e^{i\theta}c = c^*$.

To summarize, $T_1([\mathbf{e}_1+c\mathbf{e}_2]) = [\mathbf{e}_1+c^*\mathbf{e}_2] = [*]([\mathbf{e}_1+c\mathbf{e}_2])$. Also, $T_1([\mathbf{e}_2]) = [\mathbf{e}_2] = [*]([\mathbf{e}_2])$. We conclude that $T_1 = [*]$.

To summarize, $T = [U_0] \circ [U_1] = [U_0 U_1]$ or $T = [U_0] \circ [U_1] \circ [*] = [U_0 U_1 *]$.

2. Proof of Wigner's theorem in the $d \geq 3$ case

Now consider the case $d \geq 3$, and fix normalized $\mathbf{e} \in \mathcal{H}$. Let $T : P\mathcal{H} \to P\mathcal{H}$ denote some Wigner isomorphism. Pick nonzero $\mathbf{f} \in T([\mathbf{e}])$.

Lemma 2.1. *For any $\mathbf{v} \in \mathcal{H}$ independent of $\mathbf{e}$, choosing nonzero $\mathbf{w} \in T([\mathbf{v}])$, $T : P\operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\} \to P\operatorname{span}_{\mathbb{C}}\{\mathbf{f},\mathbf{w}\}$.* ■

Proof. The fact that T preserves orthogonality means that

$$\begin{aligned} T : P\operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\} \to &\bigcap_{\mathbf{x}\in\{\mathbf{e},\mathbf{v}\}^\perp\backslash\{0\}} \{[\mathbf{z}] : \mathbf{z} \perp \mathbf{y} \text{ for all } \mathbf{y} \in T([\mathbf{x}])\} \\ &= P \bigcap_{\mathbf{x}\in\{\mathbf{e},\mathbf{v}\}^\perp\backslash\{0\}} \{\mathbf{z} : \mathbf{z} \perp \mathbf{y} \text{ for all } \mathbf{y} \in T([\mathbf{x}])\}. \end{aligned} \tag{19}$$

The set $\mathcal{S}_{\mathbf{x}} = \{\mathbf{z} : \mathbf{z} \perp \mathbf{y} \text{ for all } \mathbf{y} \in T([\mathbf{x}])\}$ is a subspace of $\mathcal{H}$ containing $\mathbf{f},\mathbf{w}$ and depending on $\mathbf{x}$ only through $\mathbb{C}\mathbf{x}$. The set

$$\mathcal{S} = \bigcap_{\mathbf{x}\in\{\mathbf{e},\mathbf{v}\}^\perp\backslash\{0\}} \mathcal{S}_{\mathbf{x}} \tag{20}$$

is a subspace of $\mathcal{H}$, containing $\mathbf{f},\mathbf{w}$.

In fact, $\mathcal{S} = \operatorname{span}_{\mathbb{C}}\{\mathbf{f},\mathbf{w}\}$. Otherwise, it would contain some $\mathbf{y} \in \mathcal{H}\backslash\{0\}$ orthogonal to both of $\mathbf{f},\mathbf{w}$; because T is a bijection, $\mathbf{y} \in T([\mathbf{x}])$ for some $\mathbf{x} \in \mathcal{H}\backslash\{0\}$, and since the set-theoretic inverse T^{-1} is a Wigner automorphism, $\mathbf{x}$ is orthogonal to both of $\mathbf{e},\mathbf{v}$. But $\mathbf{y} \notin \mathcal{S}_{\mathbf{x}}$, so $\mathbf{y} \notin \mathcal{S}$. So, $\mathcal{S} = \operatorname{span}_{\mathbb{C}}\{\mathbf{f},\mathbf{w}\}$.

Equation (19) says that $T : P\operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\} \to P\mathcal{S}$, and by the previous paragraph $P\mathcal{S} = P\operatorname{span}_{\mathbb{C}}\{\mathbf{f},\mathbf{w}\}$. □

Identifying $P\operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\}$ and $P\operatorname{span}_{\mathbb{C}}\{\mathbf{f},\mathbf{w}\}$ with $\mathbb{C}P^1$, T gives a family $T_{\mathbf{v}}$ of Wigner isomorphisms on $\mathbb{C}P^1$. By the already proven $d = 2$ case of the theorem, there exists a unique unitary or anti-unitary map

$$U_{\mathbf{v}} : \operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\} \to \operatorname{span}_{\mathbb{C}}\{\mathbf{f},\mathbf{w}\} \tag{21}$$

such that

$$[U_{\mathbf{v}}] = T|_{P\operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\}} \tag{22}$$

and $U_{\mathbf{v}}\mathbf{e} = \mathbf{f}$. Define $U : \mathcal{H} \to \mathcal{H}$ by $U\mathbf{v} = U_{\mathbf{v}_0}\mathbf{v}$ for any $\mathbf{v}_0$ such that $\mathbf{v}_0$ and $\mathbf{e}$ are linearly independent and $\mathbf{v} \in \operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}_0\}$. This is well-defined because $U_{\mathbf{v}_0}$ depends on $\mathbf{v}_0$ only through $\operatorname{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}_0\}$, in the sense that if $\mathbf{v}_1$ is a linear combination of $\mathbf{e}$ and $\mathbf{v}_0$ not proportional to $\mathbf{e}$, then $U_{\mathbf{v}_1} = U_{\mathbf{v}_0}$.

The following proposition says that U is either unitary or anti-unitary:

Lemma 2.2. *Suppose that $L : \mathcal{H} \to \mathcal{H}$ is a map of sets that, on every two-dimensional subspace V containing $\mathbf{e}$, the restriction*

$$L|_V : V \to LV \tag{23}$$

is either unitary or anti-unitary, and suppose that $[L]$ preserves transition probabilities. Then L is either unitary or anti-unitary. ■

Proof. First note that either every $L|_V$ for V a two-dimensional subspace containing $\mathbf{e}$ is unitary or every $L|_V$ is anti-unitary. Indeed, the former holds if $L|_{\mathbb{C}\mathbf{e}}$ is linear, and the latter holds otherwise.

In order to prove the lemma, it suffices to consider the case where each $L|_V$ is unitary. This is because, if each is instead, the map $L \circ *_{\mathcal{B}} : \mathbb{C} \to \mathbb{C}$ satisfies the hypotheses of the lemma, and $L \circ *_{\mathcal{B}}|_V$ is unitary for each V as above. Indeed, $*_{\mathcal{B}}$ maps V anti-unitarily onto a two-dimensional subspace $*_{\mathcal{B}}V$ of $\mathcal{H}$, and then L maps $*_{\mathcal{B}}V$ anti-unitarily onto $L \circ *_{\mathcal{B}}V$. If the conclusion of the lemma holds for $L \circ *_{\mathcal{B}}$, then it also holds for L. So, only the unitary case needs to be considered.

Another simplification is that it suffices to consider the case where L maps the elements of $\mathcal{B}$ to themselves: $L\phi = \phi$ for all $\phi \in \mathcal{B}$. Indeed, for general L, the images $L\phi_j$ of the $\phi_j \in \mathcal{B}$ are necessarily orthogonal, by assumption, and they must have norm 1 because, for each j, choosing a two-dimensional subspace V containing $\mathbf{e} = \phi_1$ and ϕ_j,

$$\|L\phi\| = \|L|_V\phi_j\| = \|\phi_j\| = 1, \tag{24}$$

since $L|_V$ is unitary. We can therefore define a unitary map S which maps $L\phi_j \mapsto \phi_j$ for each j. The map SL now fixes the $\phi \in \mathcal{B}$ and satisfies the same hypotheses as L. If SL is known to be unitary then the same applies to L. So, it suffices to consider only the stated case.

Suppose that L fixes the elements of $\mathcal{B}$. It follows that, for each $j \neq 1$, the maps

$$L|_{\mathrm{span}_{\mathbb{C}}\{\mathbf{e},\phi_j\}} : \mathrm{span}_{\mathbb{C}}\{\mathbf{e},\phi_j\} \to \mathrm{span}_{\mathbb{C}}\{\mathbf{e},\phi_j\} \tag{25}$$

are the identity maps on their domains. Given

$$\mathbf{v} = \sum_{j=1}^{\dim\mathcal{H}} v_j\phi_j, \tag{26}$$

since $([\phi_j],[\mathbf{v}]) = ([L\phi_j],[L\mathbf{v}]) = ([\phi_j],[L\mathbf{v}])$, we get $|v_j| = |(L\mathbf{v})_j|$ for each j. So, there exists some $\alpha_j(\mathbf{v}) \in [0, 2\pi)$ such that

$$L\mathbf{v} = \sum_{j=1}^{\dim\mathcal{H}} e^{i\alpha_j(\mathbf{v})}v_j\phi_j. \tag{27}$$

Let $\mathbf{w}_j = v_1^*\phi_j - v_j^*\mathbf{e}$. Then, $\mathbf{w}_j \perp \mathbf{v}$, so $L\mathbf{w}_j = \mathbf{w}_j$ satisfies $\mathbf{w}_j \perp L\mathbf{v}$, unless $v_1 = 0$. This means that the vector formed by the first and jth entries of $L\mathbf{v}$ is proportional to the vector formed by the same entries of $\mathbf{v}$. In other words, $\alpha_1 = \alpha_j \bmod 2\pi\mathbb{Z}$. Thus, $L\mathbf{v} = e^{i\alpha}\mathbf{v}$ for some $\alpha = \alpha(\mathbf{v}) \in \mathbb{R}$. A similar argument using $\mathbf{x}_{j,k} = \mathbf{e} + v_j^*\phi_k - v_k^*\phi_j$ can be used to dispatch the $v_1 = 0$ case. Consequently, $L : \mathrm{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\} \mapsto \mathrm{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\}$. We already knew this for $\mathbf{v} \in \mathcal{B}\backslash\{\mathbf{e}\}$; we now know it for all $\mathbf{v}$. So, L is a linear operator on $\mathrm{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\}$ for which every nonzero vector is an eigenvector. This implies that

$$L|_{\mathrm{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\}} = e^{i\alpha(\mathbf{v})}\mathrm{id}_{\mathrm{span}_{\mathbb{C}}\{\mathbf{e},\mathbf{v}\}}. \tag{28}$$

Since $L\mathbf{e} = \mathbf{e}$, it must be the case that $\alpha = 0$. So, $L\mathbf{v} = \mathbf{v}$, as desired. □

3. Unitarity vs. anti-unitarity

One elegant way of determining whether a Wigner isomorphism $T \in \mathrm{Iso}(P\mathcal{H})$ lifts to a unitary or to an anti-unitary map is to use the following:

Lemma 3.1. *Let $\Delta : (P\mathcal{H})^3 \to \{z \in \mathbb{C} : |z| \leq 1\}$ be defined by $\Delta([\varphi],[\phi],[\psi]) = \langle\varphi,\phi\rangle\langle\phi,\psi\rangle\langle\psi,\varphi\rangle$ when φ,ϕ,ψ are unit vectors. This is well-defined, and, if $\dim\mathcal{H} \geq 2$, then the image of Δ is not a subset of the real line.* ■

Proof. Well-definedness is an immediate consequence of the sesquilinearity of the inner product. In order to prove the other clause, it suffices to prove it in the case $\mathcal{H} = \mathbb{C}^2$. Then, let $\mathbf{e}_1, \mathbf{e}_2$ denote the standard basis vectors, and compute

$$\Delta([\mathbf{e}_1], [a\mathbf{e}_1 + e^{i\alpha}(1-|a|^2)^{1/2}\mathbf{e}_2], [b\mathbf{e}_1 + e^{i\beta}(1-|b|^2)^{1/2}\mathbf{e}_2]) \\ = ab^*(a^*b + e^{-i(\alpha-\beta)}(1-|a|^2)^{1/2}(1-|b|^2)^{1/2}) \quad (29)$$

for $a, b \in \mathbb{C}$ with $|a|, |b| \leq 1$ and $\alpha, \beta \in \mathbb{R}$. Evidently, if $|a|, |b| \in (0,1)$, then this can be real only for one value of $\alpha - \beta$ modulo $\pi\mathbb{Z}$, so Δ is not always real. □

If $L : \mathcal{H} \to \mathcal{H}$ is unitary or anti-unitary, then, for any $\varphi, \phi, \psi \in \mathbb{S}$,

$$\Delta \circ [L]([\varphi], [\phi], [\psi]) = \Delta([L\varphi], [L\phi], [L\psi]) = \begin{cases} \Delta([\varphi], [\phi], [\psi]) & (L \text{ unitary}), \\ \Delta([\varphi], [\phi], [\psi])^* & (L \text{ anti-unitary}). \end{cases} \quad (30)$$

Note that Δ, Δ^* are distinct functions, because they are not real-valued. So, L is unitary if $\Delta \circ [L] = \Delta$, and anti-unitary otherwise.

4. Uniqueness

Frequently, a uniqueness clause is included in Wigner's theorem.

Lemma 4.1. *If $d \geq 2$, and if $L, Q : \mathcal{H} \to \mathcal{H}$ are either linear or anti-linear injective maps such that $[L] = [Q]$, then either L, Q are both linear or they are both anti-linear.* ■

Proof. Suppose, to the contrary, that L is linear and Q anti-linear. Let $\phi, \psi \in \mathcal{H}$ be linearly independent. From $[L]([\phi]) = [Q]([\phi])$ and $[L]([\psi]) = [Q]([\psi])$, it follows that there exist $\lambda, \mu \in \mathbb{C}^\times$ such that $L\phi = \lambda Q\phi$ and $L\psi = \mu Q\psi$.

The requirement that $[L]([\phi+\psi]) = [Q]([\phi+\psi])$ says that $L(\phi+\psi) \in \mathbb{C}^\times Q(\phi+\psi) = \mathbb{C}^\times(Q\phi+Q\psi)$. But,

$$L(\phi + \psi) = L\phi + L\psi = \lambda Q\phi + \mu Q\psi. \quad (31)$$

Because $Q\phi, Q\psi$ are linearly independent – since Q is injective – the only way for $\lambda Q\phi + \mu Q\psi$ to be a multiple of $Q\phi + Q\psi$ is if $\mu = \lambda$.

The requirement that $[L]([\phi + i\psi]) = [Q]([\phi + i\psi])$ says that

$$L(\phi + i\psi) \in \mathbb{C}^\times Q(\phi + i\psi) = \mathbb{C}^\times(Q\phi - iQ\psi). \quad (32)$$

Because L is linear, $L(\phi + i\psi) = L\phi + iL\psi = \lambda(Q\phi + iQ\psi)$. But, since λ is nonzero, $\lambda(Q\phi + iQ\psi) \notin \mathbb{C}^\times(Q\phi - iQ\psi)$, contrading eq. (32). Having reached a contradiction, the assumption that L be linear and Q anti-linear is untenable. □

As a corollary:

Proposition 4.2. *If $d \geq 2$, and if $L, Q : \mathcal{H} \to \mathcal{H}$ are either linear or anti-linear injective maps such that $[L] = [Q]$, then $L = \lambda Q$ for some $\lambda \in \mathbb{C}^\times$.* ■

Proof. By the previous lemma, either L, Q are both linear or both anti-linear. Fix nonzero $\phi \in \mathcal{H}$. From $[L]([\phi]) = [Q]([\phi])$, it follows that $L\phi = \lambda Q\phi$ for some $\lambda \in \mathbb{C}^\times$. If $\psi \in \mathcal{H}$ is linearly independent of ϕ, then, as part of the proof of the previous lemma, it was shown that $L\psi = \lambda Q\psi$.

Given $\varphi \in \mathcal{H}$, write

$$\varphi = \rho\phi + \psi \quad (33)$$

for $\rho \in \mathbb{C}$ and ψ which is either 0 or independent of ϕ; then, if L is linear, $L\varphi = \rho L\phi + L\psi = \lambda(\rho Q\phi + Q\psi) = \lambda Q\varphi$, or, if L is anti-linear, $L\varphi = \rho^* L\phi + L\psi = \lambda(\rho^* Q\phi + Q\psi) = \lambda Q\varphi$.

So, regardless of whether the map L is linear or anti-linear, $L\varphi = \lambda Q\varphi$, and therefore, since φ was arbitrary, $L = \lambda Q$. □

So, given a Wigner isomorphism $T : P\mathcal{H} \to P\mathcal{H}$, while the unitary or anti-unitary operator $U \in \mathrm{U}(\mathcal{H})$ whose existence is guaranteed by Wigner's theorem is not unique, every other unitary or anti-unitary lift of T has the form $e^{i\alpha}U$ for some $\alpha \in \mathbb{R}$.