Contents

CHAPTER 1

Relativistic symmetry

[S]ome good news: quantum field theory is based on the same quantum mechanics that was invented by Schrödinger, Heisenberg, Pauli, Born, and others in 1925–26, and has been used ever since in atomic, molecular, nuclear, and condensed matter physics. – Steven Weinberg in [Wei05, §2.1]

Physicists often describe quantum field theory as the inevitable consequence of reconciling the formalism of quantum mechanics with the strictures of special relativity. This applies most clearly to the description of isolated particles – a kinematical problem whose solution will be covered in later lectures. Electrons, neutrinos, and quarks have many different properties, from charge to color, but they are all "spin-1/2." The force carriers in the standard model of particle physics all are "spin-1." The notion of particle spin, which (despite the suggestive terminology) has no true analogue in classical mechanics, arises naturally from the conjunction of quantum mechanics and a fragment of relativistic covariance.

For our purposes, the conjunction can be encoded in a single definition:

A *relativistic quantum mechanical system* consists of a separable Hilbert space $\mathcal{H}$ together with a strongly-continuous projective unitary representation

$$\rho : \mathrm{P}(1,d) \to \mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})/(\mathrm{U}(1)I) \tag{1.1}$$

of the (restricted) Poincaré group

$$\mathrm{P}(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{SO}(1,d), \tag{1.2}$$

where $d \in \mathbb{N}^+$ is the number of spatial dimensions.

Isolated particles (whether elementary or composite) are relativistic quantum mechanical systems in this sense, as are full-fledged quantum fields.

The goal of this lecture is to unpack the definition above. In particular, the Poincaré group, together with the Lorentz group $\mathrm{O}(1,d)$, is recalled below. We do not assume familiarity with these groups.

1. The Poincaré group

Special relativity is most easily summarized as the requirement that the laws of physics admit as a group of symmetries the (restricted) Poincaré group $\mathrm{P}(1,d)$. From a modern perspective, the central insight contained in Einstein's groundbreaking 1905 paper [Ein05] is that the Poincaré group is among the symmetries of Maxwellian electrodynamics, and, if the same applies to all other fundamental laws of physics, then any observer moving at constant velocity ("inertial frame of reference," see §1.A.1) will observe no departure from Maxwell's theory. In particular, the speed of light

$$c \approx 2.998 \times 10^8 \text{ m/s} \tag{1.3}$$

will appear the same to all. We will work in units where $c = 1$.

The Poincaré group is a particular subgroup of the group

$$\mathrm{Aff}(\mathbb{R}^{1,d}) = \{\text{bijective affine } T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}\} \tag{1.4}$$

of affine transformations of Minkowski spacetime,

$$\mathbb{R}^{1,d} = \mathbb{R}_t \times \mathbb{R}^d_{\mathbf{x}}. \tag{1.5}$$

A spacetime coordinate $x = (t, \mathbf{x}) \in \mathbb{R}^{1,d}$ consists of two components, a "temporal" component $t = x^0$, saying *when* some event occurs, and a "spatial" component $\mathbf{x} \in \mathbb{R}^d$, saying *where*. On Minkowski spacetime is defined the *Minkowski interval*

$$\begin{aligned} d &: (\mathbb{R}^{1,d})^2 \to \mathbb{R} \\ d(x,y) &= (x-y)^2, \end{aligned} \tag{1.6}$$

where $z^2 \overset{\text{def}}{=} -t^2 + \|\mathbf{z}\|^2$ for $z = (t, \mathbf{z}) \in \mathbb{R}^{1,d}$. The Minkowski interval should be compared with the Euclidean interval $(\mathbf{x}, \mathbf{y}) \mapsto \|\mathbf{x} - \mathbf{y}\|^2$. Note the signs.

Just as the Euclidean group is defined to be the group of isometries of Euclidean space, the Poincaré group

$$\mathrm{P}_{\text{full}}(1,d) = \{\text{bijections } T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d} \text{ s.t. } d(T(x), T(y)) = d(x,y)\} \tag{1.7}$$

is defined to be the group of "isometries" of Minkowski spacetime, bijections which preserve the Minkowski interval. Such T are automatically affine, making the Poincaré group a (Lie) subgroup of the affine group $\mathrm{Aff}(\mathbb{R}^{1,d})$. [Problem 1.2]

This group is not connected. The *restricted* Poincaré group $\mathrm{P} = \mathrm{P}(1,d)$ is then defined to be the connected component of $\mathrm{P}_{\text{full}}(1,d)$ containing the identity.

1.1. Basic Poincaré transformations.

Example 1.1 (Spacetime translation). A spacetime translation $T_a : x \mapsto x + a$ is an example of an element of the restricted Poincaré group. This is clear from the translation invariance of the Minkowski interval. The set of translations $T_a, a \in \mathbb{R}^{1,d}$ is a (Lie) subgroup of the restricted Poincaré group, forming a copy of the abelian group $(\mathbb{R}^{1,d}, +)$. Indeed,

$$T_a T_b = T_{a+b}. \tag{1.8}$$

■

Recall the orthogonal group $\mathrm{O}(d) = \{R \in \mathbb{R}^{d\times d} : R^{-1} = R^\intercal\}$. The subgroup $\mathrm{SO}(d) = \{R \in \mathrm{O}(d) : \det R = 1\}$ consists of the orientation-preserving orthogonal transformations, i.e. rotations.

Example 1.2 (Spatial rotations). Let $R \in \mathrm{SO}(d)$. The transformation

$$T_R : \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} t \\ R\mathbf{x} \end{bmatrix} \tag{1.9}$$

which rotates the spatial coordinate $\mathbf{x}$ (but leaves time invariant), is also in the restricted Poincaré group. This is obvious from the rotation invariance of the Minkowski interval. The $T_R, R \in \mathrm{SO}(d)$ form a (Lie) subgroup of the Poincaré group, a copy of $\mathrm{SO}(d)$. Indeed,

$$T_R T_{R'} = T_{RR'}. \tag{1.10}$$

■

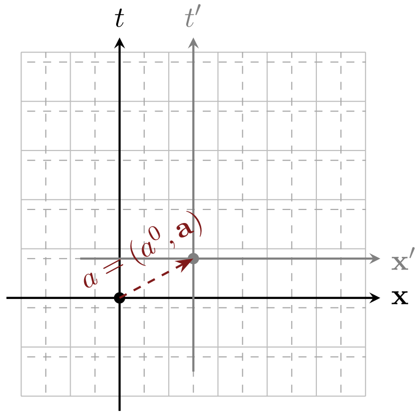
Let $\mathbb{B}^d = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| < 1\}$ denote the open unit ball.

Example 1.3 (Boosts). Let $\mathbf{v} \in \mathbb{R}^d$ with $\|\mathbf{v}\| < 1$. A (Lorentz) *boost* (with velocity $\mathbf{v} \in \mathbb{B}^d$) is a map $T_{\Lambda(\mathbf{v})}$ of the form

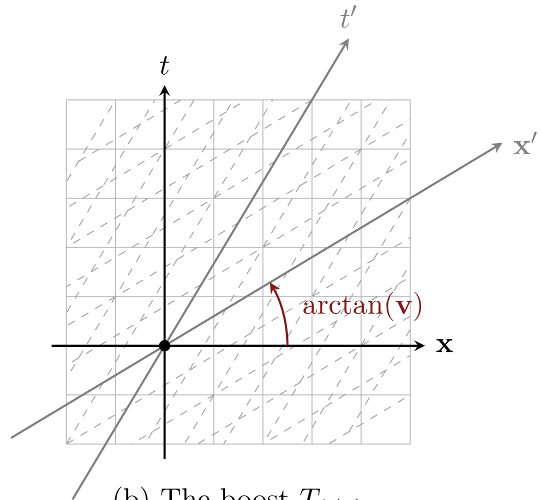
$$T_{\Lambda(\mathbf{v})} : \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \mapsto \Lambda(\mathbf{v}) \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix}, \quad \Lambda(\mathbf{v}) = \frac{1}{\sqrt{1 - \|\mathbf{v}\|^2}} \begin{bmatrix} 1 & -\mathbf{v}^\intercal \\ -\mathbf{v} & \hat{\boldsymbol{\gamma}} \end{bmatrix}, \quad \hat{\boldsymbol{\gamma}} = \sqrt{\frac{1 - \|\mathbf{v}\|^2}{I_d - \mathbf{v}\mathbf{v}^\intercal}}. \tag{1.11}$$

[Exercise 1.2] This lies in the restricted Poincaré group, as a short computation reveals. The factor

$$\gamma = \frac{1}{\sqrt{1 - \|\mathbf{v}\|^2}} \tag{1.12}$$



(a) The translation T_a



(b) The boost $T_{\Lambda(\mathbf{v})}$

FIGURE 1.1

is known as the *Lorentz factor*. The matrix $\hat{\boldsymbol{\gamma}}$ is defined via the functional calculus for symmetric matrices. It acts as the identity on $\operatorname{span}_{\mathbb{R}} \mathbf{v}$ and is multiplication by γ^{-1} on $(\operatorname{span}_{\mathbb{R}} \mathbf{v})^{\perp} \subseteq \mathbb{R}^d$, so its matrix elements are

$$\hat{\gamma}^{ij} = \delta_{ij} + (\gamma - 1)\frac{v_i v_j}{v^2}. \tag{1.13}$$

When $\mathbf{v} = (v, 0, \cdots, 0)$,

$$\Lambda(\mathbf{v}) = \Lambda_{\mathrm{std}}(v) \overset{\mathrm{def}}{=} \begin{bmatrix} \gamma & -\gamma v & 0 \\ -\gamma v & \gamma & 0 \\ 0 & 0 & I_{d-1} \end{bmatrix}. \tag{1.14}$$

That is,

$$\Lambda_{\mathrm{std}}(v) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \ni (t, x, \mathbf{y}) \mapsto \Big(\frac{t - vx}{\sqrt{1 - v^2}}, \frac{x - vt}{\sqrt{1 - v^2}}, \mathbf{y} \Big). \tag{1.15}$$

Restricting attention to $\Lambda_{\mathrm{std}}(v)$ is without essential loss of generality, because any boost $\Lambda(\mathbf{v})$ has the form

$$\Lambda(\mathbf{v}) = T_R T_{\Lambda_{\mathrm{std}}(v)} T_{R^{-1}} \tag{1.16}$$

for some $v \in (-1, 1)$ and $R \in \mathrm{SO}(d)$. If $d \geq 2$, then we can take $v \in [0, 1)$. ■ [Exercise 1.2(b)]

The Lorentz boost $\Lambda_{\mathrm{std}}(v)$ should be contrasted with the Galilei boost:

$$(t, x, \mathbf{y}) \mapsto (t, x - vt, \mathbf{y}). \tag{1.17}$$

This is a symmetry of non-relativistic classical and quantum mechanics. Compared to the Galilei boost, the Lorentz boost has two differences:

(i) the presence of the Lorentz factors $\gamma = 1/\sqrt{1 - v^2}$,
(ii) the correction $-\gamma v x$ to the time coordinate.[1]

The first difference is responsible for length contraction. A moving object will appear to a stationary observer as squashed along its direction of motion, as compared to its shape at rest, by a factor of γ. A conceptual consequence of the second difference is the relativity of simultaneity; events which are simultaneous in one frame need not be simultaneous in other frames. This can be seen by the tilting of the $\mathbf{x}'$-axis in Section 1.1(b).

WARNING: Boosts do not form a subgroup of the Lorentz group (unless $d = 1$). The product of two [Exercise 1.3]

[1] When factors of the speed of light c are restored, this is $-\gamma vx/c^2$, so very small in everyday life.

collinear boosts is a boost, but the same does not apply to non-collinear boosts. The composition of two non-collinear boosts is a boost times a rotation, known as a *Wigner–Thomas rotation.* At the level of the Lie algebra, this can be seen by the commutators of generators of boosts involving the generators of rotations.

[Problem 1.3 and Problem 1.4]

EXAMPLE 1.4 (Time-reversal and parity). Let $\mathscr{R} \in \mathrm{O}(d)$ denote a spatial reflection across an odd number of Cartesian coordinates, say

$$\mathscr{R}(\mathbf{x}) = \begin{cases} (-x^1, x^2, \cdots) & (d \text{ even}), \\ -\mathbf{x} & (d \text{ odd}). \end{cases} \tag{1.18}$$

Consider

$$\begin{aligned} T_{\mathscr{T}} &: (t, \mathbf{x}) \mapsto (-t, \mathbf{x}) \\ T_{\mathscr{P}} &: (t, \mathbf{x}) \mapsto (-t, \mathscr{R}\mathbf{x}). \end{aligned} \tag{1.19}$$

These are elements of the full Poincaré group but not the *restricted* Poincaré group, as we will discuss below. ■

Spacetime translations, spatial rotations, and boosts together generate the restricted Poincaré group — see the problems at the end of this chapter. Any restricted Poincaré transformation T can be written uniquely as

$$T = T_a T_R T_{\Lambda(\mathbf{v})} \tag{1.20}$$

for some $a \in \mathbb{R}^{1,d}$, $R \in \mathrm{SO}(3)$, and $\mathbf{v} \in \mathbb{B}^d$. Moreover, any Poincaré transformation can be written uniquely as

$$T = T_{\mathscr{T}}^{\eta} T_{\mathscr{P}}^{\xi} T_a T_R T_{\Lambda(\mathbf{v})} \tag{1.21}$$

for $\eta, \xi \in \{0, 1\}$, $a \in \mathbb{R}^{1,d}$, $R \in \mathrm{SO}(3)$, and $\mathbf{v} \in \mathbb{B}^d$. So the time-reversal and parity transformations, together with spacetime translations, spatial rotations, and boosts generate the full Poincaré group. The order in which the operators are listed in the expressions above is not important; they do not generally commute, but things can still be rearranged, owing to the following computation:

PROPOSITION 1.5. *Consider the five types of operator above, $T_{\mathscr{T}}$, $T_{\mathscr{P}}$, T_a for $a \in \mathbb{R}^{1,d}$, T_R for $R \in \mathrm{SO}(d)$, and $T_{\Lambda(\mathbf{v})}$ for $\mathbf{v} \in \mathbb{B}^d$. For any A, B of these types, $AB = B'A'$ for A' as the same type as A and B' as the same type of B.* ■

PROOF. Straightforward casework. □

1.2. Lorentz transformations. Any affine transformation $T \in \mathrm{Aff}(\mathbb{R}^{1,d})$ can be written (uniquely) as

$$T = T_a T_\Lambda, \tag{1.22}$$

where $a = T(0)$ is the image of the spacetime origin under T and $T_\Lambda = T_{-a}T$ is some linear transformation. When T is a Poincaré transformation, T_Λ is known as a *Lorentz transformation.* Equation (1.21) shows that any Lorentz transformation has the form

$$T_\Lambda = T_{\mathscr{T}}^{\eta} T_{\mathscr{P}}^{\xi} T_R T_{\Lambda(\mathbf{v})}, \tag{1.23}$$

a product of a pure boost, a spatial rotation, and possibly some reflections. If T_Λ is in the restricted Poincaré group P, then it is a *restricted* Lorentz transformation. This means that $\eta, \xi = 0$ in eq. (1.23):

$$T_\Lambda = T_R T_{\Lambda(\mathbf{v})}. \tag{1.24}$$

REMARK: Physicists often speak of "Lorentz covariance" instead of Poincaré covariance. Insofar as these terms are not used interchangeably, the latter means the former together with translation-invariance.

Being linear, any Lorentz transformation T_Λ has the form $T_\Lambda : \mathbb{R}^{1+d} \ni x \mapsto \Lambda x$ for some (unique) matrix $\Lambda \in \mathbb{R}^{(1+d)\times(1+d)}$. Since $x \mapsto x^2$ is a quadratic form $x^2 = x^\intercal \eta x$ represented by the matrix

$$\eta = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix}, \tag{1.25}$$

the requirement that T_Λ preserves the Minkowski interval can be rewritten

$$\Lambda^\intercal \eta \Lambda = \eta. \tag{1.26}$$

A Lorentz matrix is a matrix satisfying this condition. Spatial rotations, $R \in \mathrm{SO}(d)$,[2] and boosts $\Lambda(\mathbf{v})$, for $\mathbf{v} \in \mathbb{B}^d$, are two sorts of Lorentz matrices. Equation (1.24) says that every restricted Lorentz matrix has the form $\Lambda = R\Lambda(\mathbf{v})$ for unique $R \in \mathrm{SO}(d)$ and $\mathbf{v} \in \mathbb{B}^d$. The parity and time-reflection matrices $\mathscr{P}, \mathscr{T}$ are examples of non-restricted Lorentz transformations.

If Λ is a Lorentz matrix, then taking the determinant of both sides yields $(\det \Lambda)^2 = 1$. So,

$$\det \Lambda = \pm 1. \tag{1.27}$$

In particular, all Lorentz matrices are invertible.

Proposition 1.6. *The Lorentz matrices form a subgroup of the group of invertible* $(1+d)$*-by-*$(1+d)$*-matrices. That is:*

(a) I_{1+d} *is a Lorentz matrix.*
(b) The product of two Lorentz matrices is a Lorentz matrix.
(c) If Λ *is a Lorentz matrix, then* Λ^{-1} *is a Lorentz matrix.*

■

Proof. (a) Obvious.
(b) If Λ, A are both Lorentz, then $(\Lambda A)^\intercal \eta (\Lambda A) = A^\intercal \Lambda^\intercal \eta \Lambda A = A^\intercal \eta A = \eta$.
(c) Multiplying $\eta = \Lambda^\intercal \eta \Lambda$ by Λ^{-1} on the right and $(\Lambda^\intercal)^{-1}$ on the left yields

$$(\Lambda^\intercal)^{-1} \eta \Lambda^{-1} = \eta. \tag{1.28}$$

Since $(\Lambda^{-1})^\intercal = (\Lambda^\intercal)^{-1}$, this says

$$(\Lambda^{-1})^\intercal \eta \Lambda^{-1} = \eta. \tag{1.29}$$

□

The group of Lorentz matrices is denoted

$$\mathrm{O}(1,d) = \{\Lambda \in \mathbb{R}^{(1+d)\times(1+d)} : \Lambda^\intercal \eta \Lambda = \eta\}. \tag{1.30}$$

This is a matrix Lie group, as is easily checked. Since

$$T_\Lambda T_{\Lambda'} = T_{\Lambda\Lambda'} \tag{1.31}$$

for all $\Lambda, \Lambda' \in \mathrm{O}(1,d)$, it is canonically isomorphic to the subgroup of the Poincaré group consisting of Lorentz transformations. Either group is called *the Lorentz group*, and no confusion will arise from conflating the two. We will identify $T_\bullet$ with $\bullet$.

We use $\mathrm{SO}(1,d) \subseteq \mathrm{O}(1,d)$ to denote the subgroup of restricted Lorentz matrices. This is precisely the connected component of the Lorentz group containing the identity matrix I_{1+d}. (This is usually how the restricted Lorentz group is defined. We defined it via connectivity in the Poincaré group P, but this is the same thing — two Lorentz transformations are connected by a path in $\mathrm{O}(1,d)$ if and only if they are connected by a path in the Poincaré group.)

Warning: This notation is not standard; "$\mathrm{O}^+_\uparrow(1,d)$" is more common, but we will use the less-decorated notation.

[2] Strictly speaking, $1 \oplus R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right)$ is the Lorentz matrix describing a spatial rotation, but we will abuse notation and just write this as R.

Proposition 1.7. *If Λ is a Lorentz matrix, then $\Lambda^\intercal$ is a Lorentz matrix.* ■

Proof. Taking the inverse of both sides of $\Lambda^\intercal \eta \Lambda = \eta$ yields $\Lambda^{-1}\eta(\Lambda^\intercal)^{-1} = \eta$, having used $\eta^{-1} = \eta$. Plugging in $\Lambda^{-1} = (\Lambda^\intercal)^{-1\intercal}$, this reads

$$(\Lambda^\intercal)^{-1\intercal}\eta(\Lambda^\intercal)^{-1} = \eta. \tag{1.32}$$

This means that $(\Lambda^\intercal)^{-1}$ is Lorentz; applying Proposition 1.6(c), we deduce that $\Lambda^\intercal$ is Lorentz. □

Multiplying both sides of eq. (1.26) by the invertible matrix η on the left yields $\eta\Lambda^\intercal\eta\Lambda = I_{1+d}$, so

$$\Lambda \in \mathrm{O}(1,d) \iff \Lambda^{-1} = \eta\Lambda^\intercal\eta. \tag{1.33}$$

This is reminiscent of the relationship $R^{-1} = R^\intercal$ that characterizes orthogonal matrices $R \in \mathrm{O}(d)$.

1.3. Semidirect product structure. In summary, the Poincaré group consists of all maps $T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ of the form

$$T : \mathbb{R}^{1,d} \ni x \mapsto a + \Lambda x \tag{1.34}$$

for $a \in \mathbb{R}^{1+d}$ and $\Lambda \in \mathrm{O}(1,d)$ a Lorentz matrix.

Proposition 1.8. *Let $T = T_a T_\Lambda$ and $T' = T_{a'}T_{\Lambda'}$. Then, $TT' = T_{a+\Lambda a'}T_{\Lambda\Lambda'}$.* ■

Proof. $TT'(x) = \Lambda(\Lambda' x + a') + a = \Lambda\Lambda' x + (a + \Lambda a') = T_{a+\Lambda a'}T_{\Lambda\Lambda'}(x)$. □

Proposition 1.9. *$T_\Lambda T_a = T_{\Lambda a}T_\Lambda$.* ■

Proof. $T_\Lambda T_a(x) = \Lambda(x+a) = \Lambda x + a = T_{\Lambda a}T_\Lambda(x)$. □

Proposition 1.10. *The subgroup of $\mathrm{P}(1,d)$ consisting of translations is normal.* ■

Proof. Immediate from above: $T_\Lambda^{-1}T_a T_\Lambda = T_\Lambda^{-1}T_\Lambda T_{\Lambda a} = T_{\Lambda a}$. □

This should be "obvious," because everyone agrees what a translation is — turning your head upside down or taking two steps back does not make a translation look like something else. It may change your description of the direction of the translation, but not whether or not it is a translation. In contrast, two non-collocated observers will not agree about whether an affine transformation of spacetime is linear; being linear means fixing the origin, but there is no objectively correct choice of spacetime origin. Mathematically, this means that the Lorentz subgroup $\mathrm{O}(1,d) \subset \mathrm{P}(1,d)$ is not normal. Indeed, if $a \in \mathbb{R}^{1,d}$ is not fixed by Λ, then

$$T_a T_\Lambda T_{-a}(x) = \Lambda x + (a - \Lambda a) \tag{1.35}$$

is affine, but not linear, so not in the Lorentz group.

Proposition 1.11. *The Poincaré group is an inner semidirect product of the subgroup of translations with the subgroup of Lorentz transformations:*

$$\mathrm{P}_{\mathrm{full}}(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{O}(1,d). \tag{1.36}$$

The multiplication law is $(a,\Lambda)(a',\Lambda') = (a + \Lambda a', \Lambda\Lambda')$. ■

Proof. Immediate from above. □

Consequently, $\mathrm{P}(1,d) = \mathbb{R}^{1,d} \rtimes \mathrm{SO}(1,d)$.

Remark: It is easy to forget which of '$\rtimes$,' '$\ltimes$' is correct. You can figure it out if you remember two things:

- The tip in the triangle points towards the normal subgroup, as it does in "$N \triangleleft G$,"
- The subgroup of translations is the normal one, not the Lorentz group (and see above for the intuition why).

2. Quantum building blocks

Compared to special relativity, quantum mechanics is a bit harder to define. One can easily spend an inordinate amount of time debating what its essentials are — wave/particle duality, superposition, entanglement, or something else? — but that runs orthogonal to our concerns here. The most philosophically vexing features of quantum mechanics, those having to do with the wavefunction's apparent collapse upon measurement by an external observer, are no more vexing when relativity is in play than when it is not. So, we will focus on what quantum mechanics says about the state space. The "process theory" specifying how quantum systems evolve under time-evolution will be ignored.

2.1. The state space. Physical theories typically come with a set whose elements are identified with the states of the system being modeled. This is the state space. In Hamiltonian mechanics, the state space is some finite-dimensional symplectic manifold. One of the defining features of quantum mechanics is that states can be *superposed*. The Schrödinger cat thought experiment involves a superposition of a state in which the cat lives and a state in which the cat dies. As far as we are concerned, quantum mechanics consists of the following principle: that, *in order to accommodate superposition, each quantum mechanical system is associated with a* (separable[3]) *Hilbert space* $\mathcal{H}$*, and the state space is built from it.* More precisely: the distinct (pure) states of the system consist of different *lines*

$$\mathbb{C}\Psi \subseteq \mathcal{H}, \quad \Psi \in \mathcal{H}\backslash\{0\} \tag{1.37}$$

in $\mathcal{H}$ (not unit vectors!). Thus, each pure state $\omega \subseteq \mathcal{H}$ is described – non-uniquely – by a nonzero vector in $\mathcal{H}$, namely a spanning vector $\Psi \in \omega$,

$$\omega = \operatorname{span}_{\mathbb{C}} \Psi \tag{1.38}$$

The ability to add vectors corresponds to the ability to form quantum superpositions. If $\Phi, \Psi \in \mathcal{H}$ are linearly independent vectors, then their linear combination $\Phi + \Psi$ describes the quantum superposition of the two states described by Φ, Ψ, respectively.

The set

$$P\mathcal{H} = \{\mathbb{C}\Psi : \Psi \in \mathcal{H}\backslash\{0\}\} \tag{1.39}$$

of complex lines in $\mathcal{H}$ is known as the *projectivization* of $\mathcal{H}$. Thus, quantum states are identified with points in $P\mathcal{H}$. When $\mathcal{H} = \mathbb{C}^N$ is finite-dimensional, then

$$P\mathcal{H} = \mathbb{C}P^{N-1} \tag{1.40}$$

is a familiar complex manifold, the complex projective space with real dimension $2N-2$. The infinite-dimensional case can be thought of as a "Hilbert manifold," a topological space locally homeomorphic to $\ell^2(\mathbb{N})$, but this is not necessary. We have no need to consider $P\mathcal{H}$ as anything more than a set.

Not all superpositions need be allowed — model builders are allowed to forbid certain superpositions, by fiat. (Whether or not nature adheres to that restriction is another matter.) Then, the state space will be some proper subset of $P\mathcal{H}$. This is the case of *superselection rules*. For now, the reader may assume the absence of superselection rules. Then, the state space is the entirety of $P\mathcal{H}$.

Superselection rules are discussed briefly in §1.D.

2.2. Wigner morphisms. Many different lines of reasoning converge as to what the natural notion of a symmetry of $P\mathcal{H}$ is: a permutation $[U]$ of the state space induced by a unitary operator $U \in \mathrm{U}(\mathcal{H})$. In the absence of superselection rules, this means

$$\begin{aligned} [U] &= U \bmod \mathrm{U}(1)I \\ [U] &\in \mathrm{PU}(\mathcal{H}). \end{aligned} \tag{1.41}$$

[3]In these notes, all Hilbert spaces are separable and, unless stated otherwise, over the complex numbers.

One sometimes begins with a more basic notion of symmetry and then proves that all symmetries are unitarizable in this way (barring special symmetries that invert the arrow of time). This is called *Wigner's theorem* [Wig59]. See [Wei05, §2.A] for an exposition.

The group $\mathrm{PU}(\mathcal{H})$ inherits from $\mathrm{U}(\mathcal{H})$ a topology which makes it into a topological group. Specifically, the topology on the former is the quotient of the strong operator topology on the latter. [Exercise 1.4] The strong and weak operator topologies agree on $\mathrm{U}(\mathcal{H})$. The uniform (a.k.a. norm) topology is [Exercise 1.5] too strong to be useful. Whenever we reference topologies on these groups, we are referring to the strong/weak operator topology or its quotient.

Remark: When $\mathcal{H}$ is finite-dimensional, then $\mathrm{PU}(\mathcal{H}) = \mathrm{PSU}(\mathcal{H})$, where $\mathrm{SU}(\mathcal{H}) = \{U \in \mathrm{U}(\mathcal{H}) : \det U = 1\}$ is the special unitary group. However, "$\det U$" does not make sense when $\mathcal{H}$ is infinite-dimensional, except for special classes of unitary operators. So, we refrain from writing "$\mathrm{PSU}(\mathcal{H})$."

Finally: the natural notion of a (topological) group G of quantum symmetries is a (continuous) homomorphism

$$\rho : G \to \mathrm{PU}(\mathcal{H}). \tag{1.42}$$

Applied to $G = \mathrm{P}(1, d)$, the result is the definition of relativistic quantum mechanical system given at the beginning of the lecture.

There exist two, dual, ways of interpreting the Poincaré action on the state space $P\mathcal{H}$. Let Larry be a scientist working in the laboratory frame, and Moe be a scientist working in some other inertial frame of reference. (**M**oe for **m**oving.) Let $T \in \mathrm{P}(1, d)$ denote the Poincaré transformation such that, if Larry perceives a spacetime event at coordinates x, Moe will perceive the same event at coordinates $T(x)$. Then:

(i) If Larry perceives some quantum system in state $\mathbb{C}\Psi \in P\mathcal{H}$, Moe will perceive the same system in state $\mathbb{C}\rho(T)\Psi$.

The dual way of interpreting the Poincaré-action is this:

(ii) Any state of the system can be translated, rotated, and/or boosted.
- Given some state which Larry labels $\mathbb{C}\Psi$, there exists another state $\mathbb{C}\rho(T_{-a})\Psi$ which Larry sees as a translated version of the original state.
- Similarly, if Larry describes the initial state as possessing energy-momentum $p = (E, \mathbf{p}) \in \mathbb{R}^{1,d}$, there exists another state $\mathbb{C}\rho(T_\Lambda^{-1})\Psi$ which Larry perceives as possessing energy-momentum $\Lambda^{-1}p$.

This distinction is sometimes referred to as that between the *passive* and *active* interpretations of group elements. More colorful phraseology is "alias versus alibi."

2.3. Anti-unitary maps and Wigner representations ($\star$). An anti-unitary operator on $\mathcal{H}$ is a complex anti-linear bijection $V : \mathcal{H} \to \mathcal{H}$ such that

$$\langle \phi, \psi \rangle = \langle V\psi, V\phi \rangle \tag{1.43}$$

for all $\phi, \psi \in \mathcal{H}$. This is equivalent to being of the form $V = C \circ U$ (or equivalently $U \circ C$, for some other U), where $U \in \mathrm{U}(\mathcal{H})$ is unitary and $C : \mathcal{H} \to \mathcal{H}$ is a map that conjugates the coefficients of vectors when expanded in some fixed orthonormal basis.

Together, the unitary and anti-unitary operators form a group $\mathrm{UaU}(\mathcal{H}) \cong C_2 \ltimes \mathrm{U}(\mathcal{H})$.

If V is anti-unitary, then

$$\begin{aligned} [V] : P\mathcal{H} &\to P\mathcal{H} \\ [\Psi] &\mapsto [V\Psi] \end{aligned} \tag{1.44}$$

is a well-defined permutation of $P\mathcal{H}$. Together with the unitarizable automorphisms $[U] \in \mathrm{PU}(\mathcal{H})$, these form a group

$$\mathrm{Aut}(P\mathcal{H}) = [\mathrm{UaU}(\mathcal{H})] \cong C_2 \ltimes \mathrm{PU}(\mathcal{H}). \tag{1.45}$$

Wigner's theorem also allows anti-unitary maps as symmetries. However, these are always associated with "time-reversal symmetry." When considering the symmetries of a quantum system with time-reversal symmetry, one has a group

$$G \cong C_2 \ltimes G_0 \tag{1.46}$$

arising as a semidirect product of $C_2 = \{1, \mathsf{T}\}$ and a group G_0. Then, T is interpreted as time-reversal. Rather than a projective representation of G, the incarnation of G as a group of quantum symmetries takes the form of a (continuous) homomorphism

$$\rho : G \to \mathrm{Aut}(P\mathcal{H}) \tag{1.47}$$

in which the elements of G_0 are mapped to unitarizable symmetries and T is mapped to an anti-unitarizable symmetry, and hence the other elements of $G\backslash G_0$ are as well. We will call these *Wigner representations* of (G, G_0), or just of G for short, leaving the designated subgroup G_0 of unitarizable symmetries implicit.

3. Parity and time-reversal ($\star$)

Above, we were careful to stipulate only that relativistic systems have the restricted Poincaré group $\mathrm{P}(1, d)$ as a symmetry group, not the full Poincaré group

$$\mathrm{P}_{\mathrm{full}}(1, d) \ni T_{\mathscr{T}}, T_{\mathscr{P}} \tag{1.48}$$

for the simple reason that the current reigning theory of particle physics, the standard model, has neither time-reversal nor reflection symmetry, nor a combination thereof.

However, many simplified theories, including QED and QCD, *do* have these fundamental symmetries, in which case the symmetry group is augmented from $\mathrm{P}(1, d)$ to some larger subgroup of $\mathrm{P}_{\mathrm{full}}(1, d)$. This possibility is the topic of this section. The simplest case is where the symmetry group is that generated by $\mathscr{P}$ and P. Such theories have *chiral* symmetry — they look the same as their mirror image, modulo a relabeling of states. A theory with $\mathscr{T}$ symmetry looks the same when run in reverse, modulo a relabeling of states. It has *time-reversal* symmetry. It is also possible for a system to have $\mathscr{P}\mathscr{T}$ symmetry – a "hole-antihole" symmetry – without having $\mathscr{P}$ or $\mathscr{T}$ symmetry individually. Understanding these possibilities, and the interplay between parity and time-reversal symmetry, is surprisingly involved.

3.1. The $\mathscr{P}, \mathscr{T}, \mathscr{C}$ matrices. The parity and time-reversal matrices $\mathscr{P}, \mathscr{T} \in \mathrm{O}(1, d)$ are

$$\mathscr{P} = \begin{bmatrix} 1 & 0 \\ 0 & \mathscr{R} \end{bmatrix} \qquad \mathscr{T} = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix}, \tag{1.49}$$

respectively; $\mathscr{R} \in \mathrm{O}(d)$ is as in eq. (1.18). The product of $\mathscr{P}, \mathscr{T}$ is a third Lorentz matrix, $\mathscr{C} = \mathscr{P}\mathscr{T}$,

$$\mathscr{C} = \begin{bmatrix} -1 & 0 \\ 0 & \mathscr{R} \end{bmatrix} = \begin{cases} \begin{bmatrix} -I_2 & 0 \\ 0 & I_{d-2} \end{bmatrix} & (d \text{ even}), \\ -I_{1+d} & (d \text{ odd}). \end{cases} \tag{1.50}$$

We also use $\mathscr{I} = I_{1+d}$ to denote the identity matrix. The matrices $\mathscr{I}$, $\mathscr{P}$, $\mathscr{T}$, $\mathscr{C}$ constitute a subgroup of $\mathrm{O}(1, d)$ isomorphic to the Klein four-group, $V_4 = C_2 \times C_2$. Together with the identity component, which – as a reminder – we are calling $\mathrm{SO}(1, d)$, they generate the full Lorentz group.

We provide the proof below that $\mathscr{P}, \mathscr{T}, \mathscr{C}$ are not in the identity component of the Lorentz group. From this it follows that the full Lorentz group $\mathrm{O}(1, d)$, as well as the full Poincaré group, have exactly *four* connected components – one component for each

$$\mathscr{A} \in \{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C}\}, \tag{1.51}$$

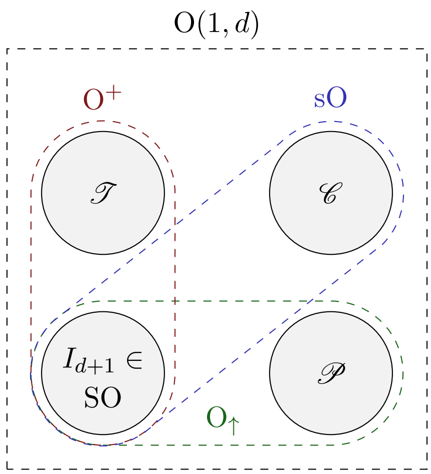


FIGURE 1.2. The various components of the full Lorentz group $\mathrm{O}(1,d)$, and the important subgroups thereof.

namely the connected component containing that $\mathscr{A}$. No two of these components can coincide, since that would imply that one of $\mathscr{P},\mathscr{T},\mathscr{C}$ lies in the identity component. No other components can exist, by the last sentence of the previous paragraph.

It follows from the above that the full Lorentz group $\mathrm{O}(1,d)$ arises as the (inner) semidirect product

$$\mathrm{O}(1,d) = V_4 \ltimes \mathrm{SO}(1,d) \tag{1.52}$$

of its identity component $\mathrm{SO}(1,d)$ and the subgroup $\{\mathscr{I},\mathscr{P},\mathscr{T},\mathscr{C}\} \cong V_4$. Here $\mathrm{SO}(1,d)$ is the normal subgroup. The identity component of a topological group is always normal.

The four components of the Lorentz group can be denoted

$$\mathrm{SO} = \mathrm{O}^+_\uparrow \ni \mathscr{I}, \quad \mathrm{O}^+_\downarrow \ni \mathscr{T}, \quad \mathrm{O}^-_\uparrow \ni \mathscr{P}, \text{ and } \mathrm{O}^-_\downarrow \ni \mathscr{C}. \tag{1.53}$$

From these components, one can form the following three index-two subgroups of $\mathrm{O}(1,d)$:

$$\begin{aligned} \mathrm{O}_\uparrow(1,d) &= \mathrm{SO}(1,d) \sqcup \mathrm{O}^-_\uparrow(1,d), \\ \mathrm{O}^+(1,d) &= \mathrm{SO}(1,d) \sqcup \mathrm{O}^+_\downarrow(1,d), \\ \mathrm{sO}(1,d) &= \mathrm{SO}(1,d) \sqcup \mathrm{O}^-_\downarrow(1,d). \end{aligned} \tag{1.54}$$

This information is summarized in Figure 1.2.

WARNING: "$\mathrm{SO}(1,d)$" is often used to denote $\mathrm{sO}(1,d)$.

Similar notation can be used for the components of $\mathrm{P}(1,d)$, and the corresponding index-two subgroups.

3.2. Relativistic quantum mechanical systems with parity and/or time-reversal. Let G denote one of P, $\mathrm{P}_\uparrow$, P^+, sP, $\mathrm{P}_{\mathrm{full}}$. Each of these has a distinguished subgroup $G_\uparrow = G \cap \mathrm{P}_\uparrow$. A relativistic quantum mechanical system with the reflection symmetries

$$Z = G \cap \{\mathscr{I},\mathscr{P},\mathscr{T},\mathscr{C}\} \tag{1.55}$$

is a Wigner representation of G, i.e. a continuous homomorphism $\rho : G \to \mathrm{Aut}(P\mathcal{H})$ in which the elements of $G_\uparrow$ are mapped to unitarizable symmetries and the elements of $G \backslash G_\uparrow$ are mapped to anti-unitarizable symmetries. The possible pairs $(Z, Z_\uparrow = Z \cap \mathrm{P}_\uparrow)$ are

- Z, Z_0 both trivial,
- $Z = \{\mathscr{I},\mathscr{P}\} \cong \mathbb{Z}_2$, and $Z_0 = Z$,
- $Z = \{\mathscr{I},\mathscr{A}\} \cong \mathbb{Z}_2$ for $\mathscr{A} \in \{\mathscr{T},\mathscr{C}\}$, and Z_0 trivial,

- $Z = \{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C}\} \cong V_4$, and $Z_0 = \{\mathscr{I}, \mathscr{P}\}$.

In the first case, we have neither chiral symmetry nor time-reversal symmetry. In the second and third cases, we have one of $\mathscr{P}, \mathscr{T}, \mathscr{C}$. In the fourth case, we have all three.

For each $\mathscr{A} \in Z$, let

$$\mathsf{A} = \rho(\mathscr{A}). \tag{1.56}$$

The map $\mathscr{A} \mapsto \mathsf{A}$ is a Wigner representation of Z. Since each Poincaré transformation $T \in G$ can be written $T = \mathscr{A} T_0$ for exactly one such $\mathscr{A}$ and restricted Poincaré transformation $T_0 \in \mathrm{P}$, and since

$$\rho(\mathscr{A} T_0) = \mathsf{A}\rho(T_0), \tag{1.57}$$

the full Wigner representation is determined by $\rho|_{\mathrm{P}}$ and the various A.

Conversely, suppose we are given

- a projective unitary representation $\rho : \mathrm{P} \to \mathrm{PU}(\mathcal{H})$,
- together with a Wigner representation $\mathscr{A} \mapsto \mathsf{A} \in \mathrm{Aut}(P\mathcal{H})$ of Z.

We can attempt to extend ρ to all of G by taking eq. (1.57) as a definition for $\mathscr{A} \neq \mathscr{I}$. This is a well-defined continuous function $G \to \mathrm{Aut}(P\mathcal{H})$. It is a homomorphism, and therefore Wigner representation, if and only if

$$\rho(\mathscr{A}_1 T \mathscr{A} T_2) = \mathsf{A}_1 \rho(T) \mathsf{A} \rho(T_2) \tag{1.58}$$

for all $\mathscr{A}_1, \mathscr{A} \in Z$ and $T, T_2 \in \mathrm{P}$. Note that $T\mathscr{A} = \mathscr{A}\tilde{T}_1$, where $\tilde{T} = \mathscr{A} T \mathscr{A} \in \mathrm{P}$. So,

$$\begin{aligned} \rho(\mathscr{A}_1 T \mathscr{A} T_2) = \rho((\mathscr{A}_1 \mathscr{A}) \tilde{T} T_2) &= \mathsf{A}_1 \mathsf{A} \rho(\tilde{T} T_2) \\ &= \mathsf{A}_1 \mathsf{A} \rho(\tilde{T}) \rho(T_2), \end{aligned} \tag{1.59}$$

where the second equality used that $\mathscr{A} \mapsto \mathsf{A}$ is a homomorphism, and the third used that $\rho|_{\mathrm{P}}$ is a homomorphism. So, the desired equality holds if and only if $\rho(T)\mathsf{A} = \mathsf{A}\rho(\tilde{T})$, i.e.

$$\mathsf{A} \rho(T) \mathsf{A} = \rho(\mathscr{A} T \mathscr{A}) \tag{1.60}$$

for all $T, \mathscr{A}$.

To summarize, *a Wigner representation of G is the same thing as a projective unitary representation of the restricted Poincaré group* P *together with a Wigner representation of Z, such that eq.* (1.60) *holds.*

The rest of this section is devoted to the study of the Wigner representations of the group Z. A Wigner representation of the group Z is the same thing as an assignment to each $\mathscr{A} \in Z$ a Wigner automorphism $\mathsf{A} \in \mathrm{Aut}(P\mathcal{H})$ such that

- P is unitarizable and involution (if defined),
- T, C (whichever are defined) are anti-unitarizable and involutive,
- $\mathsf{C} = \mathsf{PT} = \mathsf{TP}$ if all three are defined.

3.3. Wigner representations of $\mathscr{P}, \mathscr{T}, \mathscr{C}$ individually.

Proposition 1.12. *If $\mathsf{P} \in \mathrm{PU}(\mathcal{H})$ is an involution, then there exists a unitary involution $\mathcal{P} \in \mathrm{U}(\mathcal{H})$ such that $\mathsf{P} = [\mathcal{P}]$.* ■

Proof. By the assumption of unitarity, we have $\mathsf{P} = [\mathcal{P}_0]$ for a unitary operator $\mathcal{P}_0$. Since $\mathsf{P}^2 = \mathrm{id}$, this must satisfy $\mathcal{P}_0^2 = cI$ for some $c \in \mathrm{U}(1)$. Let $\mathcal{P} = c^{-1/2}\mathcal{P}_0$. (The branch of the square root does not matter.) Then,

$$\begin{aligned} \mathcal{P}^2 &= c^{-1/2}\mathcal{P}_0 (c^{-1/2}\mathcal{P}_0) \\ &= c^{-1}\mathcal{P}_0^2 = I, \end{aligned} \tag{1.61}$$

which is what it means to be an involution. □

The proof above would not work if $\mathcal{P}$ were *anti*-unitary, because the equality going between the two lines in eq. (1.61) would break. Instead:

PROPOSITION 1.13. *Suppose that $\mathcal{A}$ is an anti-unitary operator such that $\mathsf{A} = [\mathcal{A}]$ satisfies $\mathsf{A}^2 = 1$. Then, $\mathcal{A}^2 = \pm I$.* ■

It is automatic that $\mathcal{A}^2 = \omega\,\mathrm{id}_{\mathcal{H}}$ for some $\omega \in \mathrm{U}(1)$. We cannot get rid of the phase $e^{i\theta}$ by replacing $\mathcal{T}$ by $c\mathcal{T}$ for some $c \in \mathrm{U}(1)$, since

$$(c\mathcal{A})^2 = c\mathcal{A}(c\mathcal{A}) = |c|^2\mathcal{A}^2 = \mathcal{A}^2. \tag{1.62}$$

This actually shows that ω is determined by A.

PROOF. As mentioned above, we have $\mathcal{A}^2 = \omega\,\mathrm{id}_{\mathcal{H}}$ for some $\omega \in \mathrm{U}(1)$. We get a restriction on ω from computing $\mathcal{A}^3$ in two different ways:

$$\begin{aligned} \mathcal{A}^3 &= \mathcal{A}(\mathcal{A}^2) = \mathcal{A}(\omega\,\mathrm{id}_{\mathcal{H}}) = \omega^{-1}\mathcal{A}, \\ \mathcal{A}^3 &= (\mathcal{A}^2)\mathcal{A} = (\omega\,\mathrm{id}_{\mathcal{H}})\mathcal{A} = \omega\mathcal{A}. \end{aligned} \tag{1.63}$$

So, $\omega^{-1} = \omega$, which means that $\omega^2 = 1$, so $\omega \in \{-1, +1\}$. □

If $\mathcal{A}^2 = \mathrm{id}_{\mathcal{H}}$, then $\mathcal{A}$ is a *real structure* on $\mathcal{H}$. If $\mathcal{A}^2 = -\mathrm{id}_{\mathcal{H}}$, it is a *quaternionic structure*. Because the sign of $\mathcal{A}^2$ is an invariant of A, every anti-unitary Wigner transformation comes from a real structure or a quaternionic structure, but not both. Thus, we have two sorts of anti-unitary involutions.

3.4. The tenfold way. Now suppose that our system has *both* mirror and time-reversal symmetry. So, we have Wigner automorphisms P, T, implementing parity and time-reversal, respectively, satisfying the requirements above, including $\mathsf{PT} = \mathsf{TP}$. These constitute a Wigner representation of $(V_4, C_2 \times \{1\})$. By the discussion above, we have $\mathcal{P} \in \mathrm{U}(\mathcal{H})$ and $\mathcal{T} \in \mathrm{aU}(\mathcal{H})$ related to P, T by $\mathsf{P} = [\mathcal{P}]$, $\mathsf{T} = [\mathcal{T}]$ and satisfying

$$\mathcal{P}^2 = I, \text{ and } \mathcal{T}^2 = \varepsilon_{\mathsf{T}} I \tag{1.64}$$

for some $\varepsilon_{\mathsf{T}} \in \{-1, +1\}$.

The condition $\mathsf{PT} = \mathsf{TP}$ is equivalent to the existence of $\theta \in [0, 2\pi)$ such that

$$\mathcal{P}\mathcal{T} = e^{i\theta}\mathcal{T}\mathcal{P}. \tag{1.65}$$

But computing $\mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T}$ in two ways, we get a constraint on θ:

$$\begin{aligned} \mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T} &= e^{i\theta}\mathcal{T}\mathcal{P}^2\mathcal{T} = e^{i\theta}\varepsilon_{\mathsf{T}} I \\ \mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T} &= \mathcal{P}(e^{-i\theta}\mathcal{P}\mathcal{T})\mathcal{T} = e^{-i\theta}\mathcal{P}^2\mathcal{T}^2 = e^{-i\theta}\varepsilon_{\mathsf{T}} I, \end{aligned} \tag{1.66}$$

so that $e^{i\theta} = \pm 1$. Let us call this ϵ_{PT}. To summarize: for some $\varepsilon_{\mathsf{T}}, \epsilon_{\mathsf{PT}} \in \{-1, +1\}$,

$$\boxed{\mathcal{P}^2 = I, \quad \mathcal{T}^2 = \varepsilon_{\mathsf{T}} I, \quad \mathcal{P}\mathcal{T} = \epsilon_{\mathsf{PT}}\mathcal{T}\mathcal{P}.} \tag{1.67}$$

The operator $\mathcal{C} = \mathcal{P}\mathcal{T}$ also has to square to $\pm I$:

$$\mathcal{C}^2 = \mathcal{P}\mathcal{T}\mathcal{P}\mathcal{T} = \epsilon_{\mathsf{PT}}\mathcal{T}\mathcal{P}^2\mathcal{T} = \epsilon_{\mathsf{PT}}\varepsilon_{\mathsf{T}} I. \tag{1.68}$$

So, let

$$\boxed{\varepsilon_{\mathsf{C}} = \epsilon_{\mathsf{PT}}\varepsilon_{\mathsf{T}}} \tag{1.69}$$

denote the sign of $\mathcal{C}^2$. (Note the typographical difference between 'ε' and 'ϵ.') We can equally well use $(\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}})$, instead of $(\varepsilon_{\mathsf{T}}, \epsilon_{\mathsf{PT}})$, to describe the situation. It should be emphasized that $\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}}$ are *invariants* — they depend only on $\mathsf{P}, \mathsf{T} \in \mathrm{Aut}(P\mathcal{H})$ (as the notation indicates) and not on the operators $\mathcal{P}, \mathcal{T}$ used to represent them. Thus, they are invariants of the given Wigner representation, and can be used to classify them.

Each of the four possible cases of $(\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}})$ can be realized, already with $\mathcal{H} = \mathbb{C}^2$.

What eq. (1.67) tells us is that $\mathcal{P}, \mathcal{T}$ constitute a (faithful) representation of some group related to the Klein 4-group. We will call the group generated by $\mathcal{P}, \mathcal{T}$ the "$\mathcal{PT}$-group." Doing the casework, and using conventional labels [nLab25] for the various cases:

- (Class BDI.) If $\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}} = 1$, then $\mathcal{P}, \mathcal{T}$ generate the group $\{I, \mathcal{P}, \mathcal{T}, \mathcal{PT}\}$, which is just a copy of the Klein 4-group.
- (Class DIII.) If $\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}} = -1$, then $\mathcal{P}, \mathcal{T}$ commute but generate a slightly larger group: the $\mathcal{PT}$-group is
$$\{I, \mathcal{T}, -I, -\mathcal{T}, \mathcal{P}, \mathcal{PT}, -\mathcal{P}, -\mathcal{PT}\}. \tag{1.70}$$
This is a copy of $C_4 \times C_2$, with $\mathcal{T}$ generating the $\mathbb{Z}_4$ factor $\{I, \mathcal{T}, -I, -\mathcal{T}\}$ and $\mathcal{P}$ generating the C_2 factor $\{I, \mathcal{P}\}$.
- (Class CII.) If $\varepsilon_{\mathsf{T}} = -1$ but $\varepsilon_{\mathsf{C}} = 1$, then $\mathcal{P}, \mathcal{T}$ no longer commute, but rather anti-commute:
$$\mathcal{PT} = -\mathcal{TP}. \tag{1.71}$$
As a *set*, the $\mathcal{PT}$-group generated is still given by eq. (1.70), but now the group structure is non-abelian. It turns out to be isomorphic to the dihedral group D_8:
$$\begin{aligned} D_8 &= \langle x, a : a^4 = x^2 = 1, xax = a^{-1}\rangle \\ &= \{1, x, a, xa, a^2, xa^2, a^3, xa^3\}. \end{aligned} \tag{1.72}$$
The group D_8 can be interpreted as the group of symmetries of a square; x is a reflection across a median, and a is a 90° rotation. An isomorphism with the $\mathcal{PT}$-group is given by $\mathcal{T} \mapsto a$ and $\mathcal{P} \mapsto x$.
- (Class CI.) If $\varepsilon_{\mathsf{T}} = 1$ and $\varepsilon_{\mathsf{C}} = -1$, then everything in the previous item applies. Just interchange $\mathscr{T}, \mathscr{C}$. The isomorphism between the $\mathcal{PT}$-group and D_8 is now different, because of the interchange (in particular, $\mathcal{T}^2 = I$). An explicit isomorphism is given by $\mathcal{T} \mapsto xa$ and $\mathcal{P} \mapsto x$.

Class	ε_{T}	ε_{C}	ε_{P}	Cover	Example
A	0	0	0	Trivial	Standard model
AI	+1	0	0	C_2	Fermi theory, bosonic sector
AII	-1	0	0	C_4	Fermi theory, fermionic sector
D	0	+1	0	C_2	Yang–Mills θ-term,
C	0	-1	0	C_4	$\bar{\psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu}$-theory
AIII	0	0	1	C_2	$2\Re(e^{i\theta}\phi)$-theory, $\theta \in (0, \pi/2)$
BDI	+1	+1	1	$V_4 = C_2 \times C_2$	Free spin-0 particle
CI	+1	-1	1	D_8	Symplectic boson
CII	-1	+1	1	D_8	Majorana fermion
DIII	-1	-1	1	$C_4 \times C_2$	Electron

TABLE 1.1. The tenfold classification of Wigner representations of the PT-group $(V_4, C_2 \times \{1\})$ and subgroups thereof. For each generator $\mathscr{A}$, $\varepsilon_{\mathsf{A}} = \pm 1$ if $\mathscr{A}$ is present and $\varepsilon_{\mathsf{A}} = 0$ otherwise. For $\mathscr{A} = \mathscr{T}, \mathscr{C}$, the sign of ε_{A}, if nonzero, denotes whether A is implementable by an involution. For $\mathsf{A} = \mathsf{P}$, it always is. The covering group is the group that the Wigner representation lifts to an ordinary unitary/anti-unitary representation of.

Altogether, we have ten possibilities, including those without some of $\mathscr{P}, \mathscr{T}, \mathscr{C}$- symmetry. These possibilities are summarized in Table 1.1. The cases where Z is trivial or the parity group $\{1, \mathsf{P}\}$ each contribute a single possibility. The cases where $Z = \{1, \mathsf{A}\}$ for $\mathsf{A} \in \{\mathsf{T}, \mathsf{C}\}$ each contribute *two* possibilities, one for each possible sign ε_{A}. The case where Z is maximal, $Z = V_4$, contributes four possibilities, one for each possible pair $(\varepsilon_{\mathsf{T}}, \varepsilon_{\mathsf{C}})$. This sort of tenfold classification is known as a *tenfold way* [Bae20]. Recently, condensed matter theorists working on topological superconductivity have popularized closely related tenfold ways. The one here goes back to Wigner [Wig59].

All of the listed classes appear in physically significant examples. For examples, fermions tend to have $\mathcal{T}^2 = -1$, whereas bosons have $\mathcal{T}^2 = 1$. Breaking symmetries is easy — *chiral* terms in the standard model Lagrangian break parity. Some of these, like the axial vector term in the weak sector, $W_\mu \bar{\psi}\gamma^\mu\gamma^5\psi$, break parity without breaking time-reversal. Others, like the θ-term $\theta\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ break both.

Warning: Different conventions exist for defining $\mathcal{P}$, given P. In classes CI, CII, some authors prefer to use $\mathcal{P}_{\text{alt}} = i\mathcal{P}$ instead of $\mathcal{P}$. This only satisfies $\mathcal{P}_{\text{alt}}^4 = I$, not $\mathcal{P}_{\text{alt}}^2 = I$, but it has the advantage that it commutes with time-reversal:

$$\mathcal{T}\mathcal{P}_{\text{alt}} = \mathcal{T}i\mathcal{P} = -i\mathcal{T}\mathcal{P} = i\mathcal{P}\mathcal{T} = \mathcal{P}_{\text{alt}}\mathcal{T}. \tag{1.73}$$

For example, when physicists say that the parity of a Majorana spinor is $\pm i$, this is the convention they are following.

3.5. Classification of Wigner representations within each way. The tenfold way here is *not* a complete classification of the Wigner representations of V_4 or subgroups thereof. A single "way" in the tenfold way can apply to more than one possible representation of Z via Wigner automorphisms. An obvious exception is class A, describing a system with none of the symmetries the extra symmetries $\mathscr{P}, \mathscr{T}, \mathscr{C}$. We will see below that all of the classes except AIII, BDI, and DIII have a unique representation, modulo equivalence, on each $\mathcal{H}$, if a representation exists at all.

Consider class AIII, a system with chiral symmetry but neither $\mathscr{T}$ nor $\mathscr{C}$ symmetry. We saw that any Wigner representation in this class lifts to an ordinary unitary representation of $C_2 = \{I, \mathcal{P}\}$. There are two different irreps of C_2, the one-dimensional representations where $\mathcal{P} = \pm 1$. Call these $\mathbf{1}_\pm$. The two irreps are said to differ in terms of parity; $\mathbf{1}_+$ is even parity, $\mathbf{1}_-$ is odd. The most general finite-dimensional representation, modulo equivalence, is

$$\mathbf{1}_-^{N_-} \oplus \mathbf{1}_+^{N_+}, \tag{1.74}$$

where $N_\pm \in \mathbb{N}$. (The infinite-dimensional cases are analogous.)

Consider now classes AI, AII, D, and C. Regardless of $\dim \mathcal{H}$ (recall this is the complex dimension), then the only representation of classes AI and D modulo equivalence is: $\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^N$, $\mathcal{A} = * \otimes I_N$, where $\mathscr{A} = [\mathcal{A}]$ is whichever of $\mathscr{T}, \mathscr{C}$ is in Z. If $\dim \mathcal{H} = 2N$ is even, then we have the following unique representation in classes AII and C: $\mathcal{H} = \mathbb{H}^N$, $\mathcal{A} = j$ where $\mathbb{H}$ is the quaternions, considered as a complex vector space. Here, $j \in \mathbb{H}$ is a unit quaternion anti-commuting with i and satisfying $j^2 = -1$. This is a complex-antilinear map, acting on $\mathbb{H}^N$, since $jaq = ajq$ and $j(iaq) = -iajq$ for all $q \in \mathbb{R}$ and $q \in \mathbb{H}$. If $\dim \mathcal{H} < \infty$ is odd, then we have no representation of classes AII and C.

Classes BDI and DIII are similarly easy, since $\mathcal{P}$ and $\mathcal{T}$ commute. A representation in these classes is just a combination (really, a tensoring) of the possibilities above. Regardless of $\dim \mathcal{H}$, we have a representation of class BDI: $\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}^{N_-} \oplus \mathbb{R}^{N_+})$, $\mathcal{T} = * \otimes I$, and

$$\mathcal{P} = 1 \otimes \begin{bmatrix} -I_{N_-} & 0 \\ 0 & 1_{N_+} \end{bmatrix}. \tag{1.75}$$

If $\dim \mathcal{H} = 2N$ is even, then we have the following representation of class DIII: $\mathcal{H} = \mathbb{H}^{N_-} \oplus \mathbb{H}^{N_+}$, $\mathcal{T} = j$, and

$$\mathcal{P} = \begin{bmatrix} -\operatorname{id}_{\mathbb{H}^{N_-}} & 0 \\ 0 & \operatorname{id}_{\mathbb{H}^{N_+}} \end{bmatrix}. \tag{1.76}$$

These are the only possibilities, modulo equivalence. The comments above regarding absolute vs. relative parity apply here as well; $\mathcal{P}, -\mathcal{P}$ define the same Wigner representation, so should not be considered inequivalent.

Classes CI, CII are more interesting, since $\mathcal{P}, \mathcal{T}$ now anti-commute. In terms of the $\mathcal{P}$-action, $\mathcal{H} = \mathbf{1}_-^{N_-} \oplus \mathbf{1}_+^{N_+}$. We can identify this with $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}^{N_-} \oplus \mathbb{R}^{N_+})$. Then,

$$\mathcal{P} = 1 \otimes \begin{bmatrix} -I_{N_-} & 0 \\ 0 & I_{N_+} \end{bmatrix}. \tag{1.77}$$

The anti-commutation $\mathcal{T}\mathcal{P} = -\mathcal{P}\mathcal{T}$ tells us that $\mathcal{T} = * \otimes \left(\begin{smallmatrix} 0 & \mathcal{T}_{-+} \\ \mathcal{T}_{+-} & 0 \end{smallmatrix}\right)$, where $\mathcal{T}_{\pm\mp} : \mathbf{1}_\pm^{N_\pm} \to \mathbf{1}_\mp^{N_\mp}$ are two *unitary* maps. This forces $N_- = N_+$. Call their shared value N. Without loss of generality, we can assume that $\mathcal{T}_{-+} = I_N$. Thus,

$$\mathcal{T}^2 = 1 \otimes \begin{bmatrix} \mathcal{T}_{+-} & 0 \\ 0 & \mathcal{T}_{-+} \end{bmatrix}. \tag{1.78}$$

We can therefore read off $\mathcal{T}_{+-}$:

- In class CI, $\mathcal{T}^2 = I$, so $T_{+-} = I_N$, and thus $\mathcal{T} = * \otimes \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$.
- In class CII, $\mathcal{T}^2 = -I$, so $\mathcal{T}_{+-} = -I_N$, and thus $\mathcal{T} = * \otimes \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$.

These indeed define representations in the desired classes, the unique ones of the given dimension, modulo equivalence.

Class	Space hosting irrep	irrep
A	$\mathcal{H} = \mathbb{C}$	Trivial
AI	$\mathcal{H} = \mathbb{C}$	$\mathcal{T} = *$
AII	$\mathcal{H} = \mathbb{H}$	$\mathcal{T} = j$
D	$\mathcal{H} = \mathbb{C}$	$\mathcal{C} = *$
C	$\mathcal{H} = \mathbb{H}$	$\mathcal{C} = j$
AIII	$\mathcal{H} = \mathbb{C}$	$\mathcal{P} = \pm 1$
BDI	$\mathcal{H} = \mathbb{C}$	$\mathcal{T} = *$, $\mathcal{P} = \pm 1$
CI	$\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$	$\mathcal{T} = * \otimes \sigma_1$, $\mathcal{P} = 1 \otimes \sigma_3$
CII	$\mathcal{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$	$\mathcal{T} = * \otimes i\sigma_2$, $\mathcal{P} = 1 \otimes \sigma_3$
DIII	$\mathcal{H} = \mathbb{H}$	$\mathcal{T} = j$, $\mathcal{P} = \pm 1$

TABLE 1.2. The irreps found above. Each Wigner representation of the stated classes is induced (possibly non-uniquely) by a direct sum of irreps of the stated forms. In this table, $\sigma_1, \sigma_2, \sigma_3$ are the three Pauli matrices (eq. (A.1)).

WARNING: Not all of the possibilities above describe distinct Wigner representations. For example, every case with $\dim \mathcal{H} = 1$ is identical, since then $P\mathcal{H}$ is a singleton.

Also, in the classes AIII, BDI, and DIII, replacing $\mathcal{P}$ with $-\mathcal{P}$ leads to the same Wigner representation, since $[\mathcal{P}] = [-\mathcal{P}]$. As a consequence, only *relative* parity is defined. In quantum field theory, everyone agrees that the vacuum state has even parity, so parities of states involving an even number of spinors are defined relative to that. The univalence superselection rule (see §1.D) forbids superposing states with an odd number of fermions with the vacuum, so the parities of spinor states are only defined relative to each other.

1.A. More about special relativity

1.A.1. Inertial frames of reference. Consider an observer moving at some velocity $\mathbf{v} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ relative to some fiducial reference frame (the "laboratory frame"). The term *inertial frame of reference* refers to the standpoint of such an observer.

REMARK 1.14. More accurately, you should imagine a frame of reference as consisting of an army of comoving experimentalists, spread throughout space, each equipped with their own clock (all of which are synchronized from their collective point-of-view) and instructed to record the times of events occurring at their location. Then, for an event to occur at $(t, \mathbf{x}) \in \mathbb{R}^{1,d}$ means that the observer whom they perceive to be at location $\mathbf{x} \in \mathbb{R}^d$ records a hit when their clock reads time $t \in \mathbb{R}$. ■

Let's call the moving observer Moe, and a scientist working in the lab frame Larry. Moe's worldline, the path they trace out in spacetime from the perspective of Larry, is

$$\Gamma = \{(t, t\mathbf{v} + \mathbf{a}) : t \in \mathbb{R}\} \subset \mathbb{R}^{1,d}, \tag{1.79}$$

where $\mathbf{a}$ is their position at time $t = 0$. We are measuring velocity in units relative to the speed of light ("natural units"), so

$$\|\mathbf{v}\| < 1 \tag{1.80}$$

imposes the physical requirement that Moe be moving slower than the speed of light. Another difference between the two observers is that they might not agree to synchronize their clocks. Let $a^0 \in \mathbb{R}$ be the time, according to Larry, when Moe's clock reads $t = 0$. We combine this with $\mathbf{a}$ to get

$$a = (a^0, \mathbf{a}) \in \mathbb{R}^{1,d}. \tag{1.81}$$

This is Larry's description of the point that Moe labels as the spacetime origin. Conceivably, Larry and Moe could be using different Cartesian directions to coordinate space — what one labels as the "x^1-direction" could be labeled by the other as the "x^2-direction" — but this just amounts to a spatial rotation, and we understand these. So, let us assume that any event that Larry perceives at $(0, \mathbf{y})$ for $\mathbf{y} \perp \mathbf{v}$ will also be perceived by Moe at the same coordinates.

In summary: an inertial frame of reference (barring rotations) is specified by two pieces of data, $(\mathbf{v}, a) \in \mathbb{B}^d \times \mathbb{R}^{1,d}$, Moe's velocity and his spacetime origin. **Q.** If Larry perceives an event as occurring at spacetime coordinates $x \in \mathbb{R}^{1,d}$, at what spacetime coordinates will Moe perceive that event?

Let $T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ denote the map such that if Larry perceives an event at x, Moe perceives the same event at $T(x)$. We know the following:

(i) The spacetime point labeled a by Larry will be labeled 0 by Moe, so $T(a) = 0$.
(ii) The worldline Γ will be described as the time axis $\mathcal{T} = \{(t, \mathbf{0} : t \in \mathbb{R}\}$ by Moe, so $T(\Gamma) = \mathcal{T}$,
(iii) If $\mathbf{y} \perp \mathbf{v}$, then $T(0, \mathbf{y}) = (0, \mathbf{y})$.
(iv) T preserves the arrow of time and spatial orientations.

These are the only reasonable requirements, besides that Moe should be using the same laws of physics as Larry. Consider the restricted Poincaré transformation

$$T_0 : (t, \mathbf{x}) \mapsto T_{\Lambda(-\mathbf{v})} T_{-a} \tag{1.82}$$

This obeys (i), (ii), (iii), and (iv). In fact, it is the unique Poincaré transformation with these properties. "Covariance" is a technical term meaning retaining form under a coordinate transformation. A consequence of the Poincaré-covariance of the laws of physics is that Moe can adequately describe the world using the coordinates furnished by T_0. A corollary is that the moving observer Moe cannot tell that it is he that is moving, not Larry. There may exist no objectively correct answer to the question as to which observer is moving.

Special relativity is often expressed as the slightly stronger requirement that Moe will *perceive* at $T_0(x)$ an event that Larry perceives at x. This is a consequence of another assumption: that, besides being Poincaré-covariant, the laws of physics are not covariant under a larger group of spacetime symmetries. The map T_0 is not the only affine transformation satisfying the three properties above — consider, for $Z > 0$, the scaling transformation $T_Z : (t, \mathbf{x}) \mapsto (Zt, Z\mathbf{x})$. Unless $Z = 1$, this is *not* a Poincaré transformation. But, like T_0,

$$T_1 \stackrel{\text{def}}{=} T_Z T_{\Lambda(\mathbf{v})} T_{-a} \tag{1.83}$$

also satisfies properties (i), (ii), and (iii), and therefore serves as a candidate coordinate transformation. Why shouldn't Moe use the coordinate system derived from T_1, instead of T_0, to describe the world?

The answer is: the very fact that we have the ability to measure distances and durations means that nontrivial dilations cannot be symmetries of the laws of physics. This is implicit in the way that we talk about frames of reference. When we ask how an observer perceives the world, we are assuming that there exists an unambiguous answer. Consequently, there exists exactly one adequate T, namely $T = T_0$.

Scaling symmetries are typically symmetries of theories involving only massless particles. A creature made entirely of massless particles would not be able to measure distances or durations. This applies to light. Since light is massless, the symmetry group of classical electrodynamics *without matter* (!) is the full conformal group, including the scaling symmetries T_Z. This means that the behavior of light is not by itself sufficient to derive special relativity. Fortunately, scaling is not a symmetry of theories involving massive particles. The existence of massive charged particles like the electron allows us to design clocks and rulers that, together with the behavior of light, single out Poincaré-covariance.

1.B. Poincaré Lie algebra [*]

1.C. More on the Poincaré group

1.C.1. Dimensions. The dimension of the Lorentz group $\mathrm{O}(1, d)$ is

$$\dim \mathrm{O}(1, d) = \dim \mathrm{P}(1, d) - d - 1 = \frac{d^2}{2} + \frac{d}{2}. \tag{1.84}$$

In the physical case, $d = 3$, this is $\dim \mathrm{O}(1, d) = 6$. Three of these dimensions are from the subgroup of rotations and three from the boosts. Rotations outnumber boosts for $d \geq 4$, and boosts outnumber rotations if $d = 1, 2$.

Since $\mathrm{SO}(d)$ has dimension $d(d-1)/2$, the Poincaré group has dimension

$$\dim \mathrm{P}(1, d) = \frac{d^2}{2} + \frac{3d}{2} + 1. \tag{1.85}$$

In the physical case, $d = 3$, this is $\dim \mathrm{P}(1, d) = 10$. Four of these dimensions are from the subgroup of translations and the remaining six are from the subgroup of Lorentz transformations.

For the Lie algebras, see [Wei05, §2.4].

1.C.2. Topology. A Lorentz matrix Λ is called

(i) *orthochronous* if ${\Lambda^0}_0 > 0$, where ${\Lambda^0}_0$ is the upper-leftmost entry of Λ,
(ii) *orthochorous* if $\det \boldsymbol{\Lambda} > 0$, where $\boldsymbol{\Lambda} \in \mathbb{R}^{d \times d}$ consists of the bottom-right d-by-d submatrix of Λ,
(iii) *special* if $\det \Lambda > 0$, which, by eq. (1.27), means $\det \Lambda = 1$.

An orthochronous Lorentz matrix is one which preserves the arrow of time, and an orthochorous Lorentz matrix is one which preserves spatial orientation. A special Lorentz matrix is one which preserves spacetime orientations.

Among $\mathscr{P}, \mathscr{T}, \mathscr{C}$, the only orthochronous matrix is $\mathscr{P}$, the only orthochorous matrix is $\mathscr{T}$, and the only special matrix is $\mathscr{C}$.

Proposition 1.15. *For any Lorentz matrix Λ, $|\Lambda^0{}_0| \geq 1$.* ■

Proof. Let $N = (1, 0, 0, \cdots)$. This satisfies $N^2 = -1$. Since Λ is a Lorentz matrix, $(\Lambda N)^2 = N^2 = -1$ as well;

$$\Lambda N = (\Lambda^0{}_0, \vec{\Lambda}) \tag{1.86}$$

is the first column of Λ; letting $\vec{\Lambda} \in \mathbb{R}^d$ denote the spatial part, $(\Lambda N)^2 = -|\Lambda^0{}_0|^2 + \|\vec{\Lambda}\|^2$. Setting the left-hand side to -1, we get

$$|\Lambda^0{}_0|^2 = 1 + \|\vec{\Lambda}\|^2 \geq 1. \tag{1.87}$$

□

Second proof. We know that $\Lambda = \mathscr{T}^\eta R\Lambda(\mathbf{v})$ for some $\eta \in \{0, 1\}$, $R \in \mathrm{O}(d)$, and $\mathbf{v} \in \mathbb{B}^d$. Thus,

$$\Lambda^0{}_0 = N^\intercal \Lambda N = \pm N^\intercal \Lambda(\mathbf{v}) N = \pm\gamma, \tag{1.88}$$

where $\gamma = 1/\sqrt{1 - \|\mathbf{v}\|^2} \geq 1$ is the Lorentz factor. □

Proposition 1.16. *For any Lorentz matrix Λ, $|\det \boldsymbol{\Lambda}| \geq 1$.* ■

Proof. Let $\mathbf{x} \in \mathbb{R}^d$ be a unit vector, and $x = (0, \mathbf{x}) \in \mathbb{R}^{1,d}$. Then $x^2 = \|\mathbf{x}\|^2 = 1$. Since Λ is Lorentz, $(\Lambda x)^2 = x^2 = 1$ as well. The spatial component of Λx is $\boldsymbol{\Lambda}\mathbf{x}$, so

$$1 = (\Lambda x)^2 \leq \|\boldsymbol{\Lambda}\mathbf{x}\|^2. \tag{1.89}$$

Since $\mathbf{x}$ was an arbitrary unit vector, this implies that every eigenvalue λ of $\boldsymbol{\Lambda}$ has $|\lambda| \geq 1$. As $\det \boldsymbol{\Lambda}$ is the product of the eigenvalues of $\boldsymbol{\Lambda}$ (with multiplicity), $|\det \boldsymbol{\Lambda}| \geq 1$ follows. □

Proposition 1.17. *Let $\bullet$ stand for "orthochronous," "orthochorous," or "special." Let C be a connected component of* $\mathrm{O}(1, d)$*. If one matrix in C is $\bullet$, then all are.* ■

Proof. The three maps $\Lambda \mapsto \Lambda^0{}_0, \det \boldsymbol{\Lambda}, \det \Lambda$ are all continuous functions $\mathrm{O}(1, d) \to \mathbb{R}$. By the results above, they are non-vanishing, so they cannot swap signs on any connected component of $\mathrm{O}(1, d)$. □

Since the identity matrix $\mathscr{I}$ is orthochronous, orthochorous, and special, we conclude that matrices in the same connected component are as well. This shows that $\mathscr{P}, \mathscr{T}, \mathscr{C}$ are not in the identity component $\mathrm{SO}(d)$.

Proposition 1.18. *(a) A Lorentz matrix lies in the identity component of the Lorentz group if it is orthochronous, orthochorous, and special.*
(b) Any two of these imply the third.

■

Proof. This follows from the representation theorem $\Lambda = \mathscr{T}^\xi \mathscr{P}^\eta R\Lambda(\mathbf{v})$ holding for general Lorentz matrices. Since $\mathrm{SO}(d)$ is connected, $R\Lambda(\mathbf{v})$ lies in the same connected component as the pure boost $\Lambda(\mathbf{v})$, which lies in the same connected component as $\mathscr{I}$ (just take $\mathbf{v} \rightsquigarrow 0$). Thus, $R\Lambda(\mathbf{v})$ is orthochronous, orthochorous, and special. Multiplying this by one of $\mathscr{T}, \mathscr{P}, \mathscr{C}$ on the left ruins two of these. So, the only way Λ can be two or three is if $\Lambda = R\Lambda(\mathbf{v})$. □

Let

$$\begin{aligned}
\mathrm{O}_\uparrow(1, d) &= \{\Lambda \in \mathrm{O}(1, d) : \Lambda^0{}_0 > 0\}, \\
\mathrm{O}^+(1, d) &= \{\Lambda \in \mathrm{O}(1, d) : \det \boldsymbol{\Lambda} > 0\}, \\
\mathrm{sO}(1, d) &= \{\Lambda \in \mathrm{O}(1, d) : \det \Lambda = 1\},
\end{aligned} \tag{1.90}$$

denote the subsets of $\mathrm{O}(1, d)$ consisting of orthochronous, orthochorous, and special Lorentz matrices, respectively. By what we have discussed so far, each of these subsets consists of a disjoint union of components of the Lorentz group. It follows that they are all subgroups, and subgroups of index two. (This can also be checked algebraically.) Each is generated by the restricted Lorentz matrices together with whichever of $\mathscr{A} \in \{\mathscr{P}, \mathscr{T}, \mathscr{C}\}$ it contains.

The decompositions $T = T_a T_\Lambda$ and $T_\Lambda = T_R T_{\Lambda(\mathbf{v})}$ yield:

PROPOSITION 1.19.
- $\mathrm{P}(1,d)$ *is homeomorphic to* $\mathbb{R}^{1+d} \times \mathrm{SO}(1,d)$*,*
- $\mathrm{SO}(1,d)$ *is homeomorphic to* $\mathbb{R}^d \times \mathrm{SO}(d)$*.*

■□

Consequently,

$$\pi_1(\mathrm{P}(1,d)) \cong \pi_1(\mathrm{SO}(d)) \cong \begin{cases} \text{trivial} & (d=1), \\ \mathbb{Z} & (d=2), \\ \mathbb{Z}_2 & (d \geq 3). \end{cases} \tag{1.91}$$

1.C.3. The Haar measure [*].

PROPOSITION 1.20. *The left Haar measure on each of the groups* $G = \mathrm{P}, \mathrm{P}^*, \mathrm{SO}(1,d), \mathrm{Spin}(1,d)$ *is also right-invariant.* ■

1.D. Superselection rules

If we are given two unitary representations $\rho : G \to \mathrm{U}(\mathcal{V})$, $\varrho : G \to \mathrm{U}(\mathcal{W})$ of some group, then we can form their direct sum

$$\rho \oplus \varrho : G \to \mathrm{U}(\mathcal{V} \oplus \mathcal{W}). \tag{1.92}$$

The Hilbert space $\mathcal{V} \oplus \mathcal{W}$ is used to model a system whose state can lie in either summand, $\mathcal{V}$ or $\mathcal{W}$, *or be a quantum superposition thereof*. The matrices in the image of the combined representation $\rho \oplus \varrho$ are block diagonal:

$$(\rho \oplus \varrho)(g) = \begin{bmatrix} \rho(g) & 0 \\ 0 & \varrho(g). \end{bmatrix} \tag{1.93}$$

That is, they lie in the image of the natural embedding $\mathrm{U}(\mathcal{V}) \times \mathrm{U}(\mathcal{W}) \hookrightarrow \mathrm{U}(\mathcal{V} \oplus \mathcal{W})$.

Projective unitary representations, in contrast, cannot generally be summed (unless we are willing to enlarge our group G). The reason is that we lack a natural embedding

$$\mathrm{PU}(\mathcal{V}) \times \mathrm{PU}(\mathcal{W}) \hookrightarrow \mathrm{PU}(\mathcal{V} \oplus \mathcal{W}). \tag{1.94}$$

The natural attempt, $([V],[W]) \mapsto \left[\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \right]$, is not well-defined because

$$\mathrm{id}_\mathcal{V} \oplus e^{i\theta}\, \mathrm{id}_\mathcal{W} = \begin{pmatrix} \mathrm{id}_\mathcal{V} & 0 \\ 0 & e^{i\theta}\, \mathrm{id}_\mathcal{W} \end{pmatrix} \tag{1.95}$$

is not a scalar multiple of $\mathrm{id}_{\mathcal{V}\oplus\mathcal{W}}$ if $e^{i\theta} \neq 1$.

EXAMPLE 1.21. Consider the double cover $\pi : \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$. Because $\ker \pi = \{I_2, -I_2\}$ is in the center of $\mathrm{SU}(2)$ (this is a general property of the kernels of covering maps), Schur's lemma tells us that, in any irreducible unitary representation of $\mathrm{SU}(2)$, both $I_2, -I_2$ are mapped to scalars. Consequently, we get a projective unitary representation of $\mathrm{SO}(3)$.

We have two classes of $\mathrm{SU}(2)$-irreps:

- those, like the trivial representation, where $-I_2$ is mapped to the identity I,
- those, like the fundamental representation, where $-I_2$ is mapped to $-I$.

If we take a direct sum of irreps in *one* of these classes — it can be either class, but all of the irreps have to be from the same class — then $-I_2$ is still mapped to a scalar, and so the $\mathrm{SU}(2)$-rep factors through to a projective unitary representation of $\mathrm{SO}(3)$. *But*, if we take a direct sum of irreps from *both* classes, then $-I_2$ is not mapped to a scalar, and it follows that the resulting $\mathrm{SU}(2)$-rep does not factor through. ■

This is the motivation for superselection rules. It is an empirical fact that spinors exist. One would like to construct a theory in which one-particle spinor states coexist with the vacuum. But the corresponding projective space does not admit a projective representation of the Poincaré group. The problem consists of states involving a quantum superposition of states with an even number of spinors with one with an odd number of spinors. For example, consider the vacuum $|\varnothing\rangle$ and a one-electron state $|\mathrm{e}\rangle$. Under a 360° rotation,

$$\frac{1}{\sqrt{2}}(|\varnothing\rangle + |\mathrm{e}\rangle) \rightsquigarrow \frac{1}{\sqrt{2}}(|\varnothing\rangle - |\mathrm{e}\rangle). \tag{1.96}$$

These vectors are *not* linearly dependent, so they span different states $\in P\mathcal{H}$.

Q. So why does the world appear to be Lorentz invariant? **A.** Because the offending superpositions are not seen in nature.

The interactions present in the standard model of particle physics guarantee that spinors are only created in pairs. This is one way in which Lorentz invariance is preserved. So, we have three choices.

(1) Declare that the actual state space consists *either* of only states in which the parity of the spinor number is even, or states in which the parity is odd.
(2) Accept that Lorentz invariance might hold only at the level of experimental predictions, not the state space.
(3) Cut down the state space by excluding the offending superpositions. This is known as the *univalence* superselection rule [WWW52; SW00].

Until we observe some failure of Lorentz invariance, the choice between these possibilities is an aesthetic one. The first is unappealing because it implies the existence of two different theories, one in which the total number of spinors in the universe is even, and one in which the total number is odd. Good luck figuring out which holds! Regarding the second possibility, manifest Lorentz symmetry is a great simplification arguably not worth abandoning due to the conceivability of quantum superpositions never found in nature. Our preference is (3), but no consensus exists [Ear08]. Physicists seem to like (2), replacing manifest Lorentz covariance with manifest $\mathrm{Spin}(1,3)$ covariance. Nonobservability of problematic superpositions then amounts to a conservation law and concomitant restriction on practical observables.

More generally, a model with superselection rules in force is one in which the state space is defined to be a disjoint union of projectivizations of Hilbert spaces:

$$\text{state space} = P\mathcal{H}[1] \sqcup P\mathcal{H}[2] \sqcup \cdots \tag{1.97}$$

The choice between the different components

$$\mathcal{H}[n] \subsetneq \mathcal{H} \tag{1.98}$$

(*superselection sectors*) is a classical OR, not a quantum OR. The system must lie in some definite superselection sector. No quantum superpositions between the different possibilities are allowed. Thus, the univalence superselection rule accommodates possibility (1) above at the level of the state space. The theory of symmetries of quantum systems with superselection rules is developed in parallel to the version without superselection rules — see [SW00, Thm. 1.1]. The only new complication is that symmetries can permute superselection sectors, but we do not need to consider this possibility.

In the C*-algebra framework for quantum mechanics (which we have not broached), the existence of multiple superselection sectors can become a theorem holding for various models. This applies to QED, where the superselection sectors are labeled by the total charge Q. A rigorous result to this effect is due to Strocchi–Wightman [SW74], but the basic idea goes back to Haag [Haa63]: letting ρ

denote the charge density and $\mathbf{E}$ denote the electric field, we should have

$$Q = \lim_{R\to\infty}\int_{|\mathbf{x}|<R}\rho(t,\mathbf{x})\,\mathrm{d}^3\mathbf{x} = \lim_{R\to\infty}\int_{|\mathbf{x}|<R}\nabla\cdot\mathbf{E}(t,\mathbf{x})\,\mathrm{d}^3\mathbf{x} \\ = \lim_{R\to\infty}\int_{|\mathbf{x}|=R}\mathbf{E}(t,\mathbf{x})\cdot\,\mathrm{d}A(\mathbf{x}), \tag{1.99}$$

using $\nabla\cdot\mathbf{E}=\rho$. This is Gauss's law. Consequently, for any observable O, one expects

$$[Q,O] = \lim_{R\to\infty}\int_{|\mathbf{x}|=R}[\mathbf{E}(t,\mathbf{x}),O]\cdot\,\mathrm{d}A(\mathbf{x}). \tag{1.100}$$

But, if observables are local things, then, if we take R large enough, $[\mathbf{E}(t,\mathbf{x}),O]=0$. So, $[Q,O]=0$. The main task taken up by Strocchi–Wightman is showing that the foregoing reasoning can be rigorized.

In non-abelian gauge theories, topological properties of field configurations are also expected to constitute superselection sectors. An example is the θ-angle in QCD [Col85, §7.3.3].

Exercises and Problems

EXERCISE 1.1: For $v\in(-1,1)$, define the *rapidity* $\beta\in\mathbb{R}$ by $v=\tanh\beta$. Show that the standard boost $\Lambda_{\mathrm{std}}(v)$ defined in eq. (1.14) can be written

$$\Lambda_{\mathrm{std}}(v) = \begin{bmatrix}\cosh\beta & -\sinh\beta & 0\\ -\sinh\beta & \cosh\beta & 0\\ 0 & 0 & I_{d-1}\end{bmatrix}. \tag{1.101}$$

This is a "hyperbolic rotation."

EXERCISE 1.2:
(a) Prove that $\Lambda_{\mathrm{std}}(v)$ preserves the Minkowski interval, hence lies in the Lorentz group.
(b) Let $\mathbf{v}\in\mathbb{R}^d\backslash\{\mathbf{0}\}$. Show that

$$\Lambda(\mathbf{v}) = \begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\Lambda_{\mathrm{std}}(\|\mathbf{v}\|)\begin{bmatrix}1 & 0\\ 0 & R^{-1}\end{bmatrix} \tag{1.102}$$

for $R\in\mathrm{O}(3)$ any orthogonal transformation that takes $\hat{\mathbf{x}}=(1,0,\cdots)$ to $\hat{\mathbf{v}}=\mathbf{v}/\|\mathbf{v}\|$.

So, $\Lambda(\mathbf{v})$ lies in the Lorentz group as well.

EXERCISE 1.3: Let $v,u\in(-1,1)$. Show the following:

- $-1<(v+u)/(1+vu)<1$.
- $$\Lambda_{\mathrm{std}}(v)\Lambda_{\mathrm{std}}(u) = \Lambda\Big(\frac{v+u}{1+vu}\Big). \tag{1.103}$$

  This is the *velocity addition formula* (for collinear boosts).

In particular, $\Lambda_{\mathrm{std}}(v)^{-1}=\Lambda_{\mathrm{std}}(-v)$.

Hint: the use of rapidities is very convenient here.

EXERCISE 1.4: Let $\mathcal{H}$ denote an infinite-dimensional (separable, as always) Hilbert space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$.

(a) Show that the strong and weak operator topologies agree on $\mathrm{U}(\mathcal{H})$.
(b) Show that $\mathrm{U}(\mathcal{H})$ is a topological group under the strong/weak operator topology.
(c) Explain why, when $\mathrm{PU}(\mathcal{H})$ is endowed with the quotient topology, it is also a topological group.

EXERCISE 1.5: For each $a \in \mathbb{R}$, let $T_a \in \mathrm{U}(L^2(\mathbb{R}))$ be the map

$$T_a\psi(x) = \psi(x-a) \tag{1.104}$$

that translates functions to the right by a units. Is the map $a \mapsto T_a$ continuous with respect to the strong operator topology? Uniform operator topology?

EXERCISE 1.6: Let $\mathcal{T}$ denote an anti-unitary operator on $\mathcal{H}$ such that $\mathcal{T}^2 = \varepsilon I$ for $\varepsilon = \pm 1$. If $\varepsilon = +1$, use $\mathcal{T}$ to write $\mathcal{H}$ as the complexification of some real subspace. If $\varepsilon = -1$, use $\mathcal{T}$ to endow $\mathcal{H}$ with the structure of a left $\mathbb{H}$-module.

EXERCISE 1.7: When working with spinors in 1+3 spacetime dimensions, the time-reversal operator $\mathcal{T}$ satisfies $\mathcal{T}^2 = -I$.

(a) Suppose one has a parity operator $\mathcal{P}$ satisfying $\mathcal{P}^2 = I$ and commuting with $\mathcal{T}$. According to Table 1.1, in which tenfold way class do we land?

(b) When working with Majorana spinors, physicists like to define the parity operator $\mathcal{P}$ so that it commutes with $\mathcal{T}$ but satisfies $\mathcal{P}^2 = -I$. Let $\tilde{\mathcal{P}} = i\mathcal{P}$. Show that $\mathcal{T}, \tilde{\mathcal{P}}$ *anti*-commute. Using Table 1.1, determine which tenfold way class this spinor system falls into, when analyzed using the pair $(\mathcal{T}, \tilde{\mathcal{P}})$.

PROBLEM 1.1: Prove rigorously that

- $\mathrm{O}(1,d)$ is a Lie subgroup of $\mathrm{GL}(1+d,\mathbb{R})$
- $\mathrm{P}(1,d)$ is a Lie subgroup of $\mathrm{Aff}(\mathbb{R}^{1,d})$.

While you are at it, prove eq. (1.84), eq. (1.85).

PROBLEM 1.2:

(a) (Optional.) Show that any bijection $T : \mathbb{R}^d \to \mathbb{R}^d$ preserving the Euclidean distance must have the form $\mathbf{x} \mapsto R\mathbf{x} + \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^d$ and $R \in \mathrm{O}(d)$.

Let $T : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ denote a bijection preserving the Minkowski interval $d(x,y) = (x-y)^2$, i.e. a Minkowski isometry. Our goal is to show that T is affine.

(b) Reduce the general case to the case where T fixes the spacetime origin.

Now assume that T fixes the spacetime origin. Our goal is to show that T is *linear*.

(c) Let $\mathbf{e} = (1, \mathbf{0})$. Show that there exists a boost $\Lambda(\mathbf{v})$ such that $\mathscr{T}^j\Lambda(\mathbf{v})\mathbf{e} = T(\mathbf{e})$, for $j = 0, 1$. So, $\Lambda(\mathbf{v})^{-1}T$ is a Minkowski isometry fixing $\mathbf{e}$.

(d) Let $\Sigma = \{(0,\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$. Show that if T is any Minkowski isometry fixing $\mathbf{e}$ and the spacetime origin, then there exists some $R \in \mathrm{O}(d)$ such that $T|_\Sigma = R$.

(e) Then,

$$T_0 \overset{\text{def}}{=} R^{-1}\Lambda(\mathbf{v})^{-1}\mathscr{T}^j T$$

is a Minkowski isometry which fixes the spacetime origin, $\mathbf{e}$, and Σ. Show that T_0 is the identity.

So, $T = \mathscr{T}^j\Lambda(\mathbf{v})R$ is linear. This also shows that the time-reflection and parity operators, spacetime translations, boosts, and rotations together generate the full Poincaré group.

PROBLEM 1.3: Let $\mathbb{B}^d = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| < 1\}$.

(a) (Velocity addition formula.) Show that, for any $\mathbf{v}, \mathbf{u} \in \mathbb{B}^d$ (not necessarily collinear!),

$$\Lambda(\mathbf{v})\Lambda(\mathbf{u}) = W\Lambda(\mathbf{v} \oplus \mathbf{u}) \tag{1.105}$$

for some $W \in \mathrm{SO}(3)$, where

$$\mathbf{v} \oplus \mathbf{u} = \frac{1}{1 + \mathbf{u}\cdot\mathbf{v}}\Big[\Big(1 + \frac{\gamma}{\gamma+1}\mathbf{u}\cdot\mathbf{v}\Big)\mathbf{v} + \frac{\mathbf{u}}{\gamma}\Big] \tag{1.106}$$

and $\gamma = \gamma(\mathbf{v})$ is the Lorentz factor associated to $\mathbf{v}$. Prove that $\mathbf{v} \oplus \mathbf{u} \in \mathbb{B}^d$.

(b) Show that

$$\gamma(\mathbf{u} \oplus \mathbf{v}) = \gamma(\mathbf{u})\gamma(\mathbf{v})(1 + \mathbf{u} \cdot \mathbf{v}). \tag{1.107}$$

(c) Show that $\mathbf{u} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{u}$ in general, but they have the same magnitude.

PROBLEM 1.4: This problem continues Problem 1.3. The rotation W is called a *Thomas–Wigner rotation*. Prove that, if $\mathbf{v}, \mathbf{u}$ are non-collinear, then W is a rotation of the plane in which $\mathbf{u}, \mathbf{v}$ lie by an angle $\epsilon \in (-\pi, \pi)$ satisfying

$$\cos\epsilon = \frac{(1 + \gamma(\mathbf{u}) + \gamma(\mathbf{v}) + \gamma(\mathbf{v} \oplus \mathbf{u}))^2}{(1 + \gamma(\mathbf{v} \oplus \mathbf{u}))(1 + \gamma(\mathbf{u}))(1 + \gamma(\mathbf{v}))} - 1. \tag{1.108}$$

Moreover, the correct sign is the one such that W rotates $\mathbf{u}$ towards $\mathbf{v}$.

PROBLEM 1.5: Consider the matrix exponential

$$e^{\Lambda} = \sum_{j=1} \frac{\Lambda^j}{j!}.$$

Prove that this is a surjective map $\mathfrak{o}(1, 3) \to \mathrm{SO}(1, 3)$. *Hint*: consensus is that no short proof of this exists. You may wish to use the double cover $\mathrm{SL}(2, \mathbb{C}) \twoheadrightarrow \mathrm{SO}(1, 3)$ that we discuss in the next lecture.

CHAPTER 2

Deprojectivization

In the previous lecture, we defined the notion of a relativistic quantum system in $d \in \mathbb{N}^+$ spatial dimensions: a (continuous) projective unitary representation $\rho : \mathrm{P}(1,d) \to \mathrm{PU}(\mathcal{H})$ of the restricted Poincaré group $\mathrm{P}(1,d)$. In this lecture, we present:

THEOREM (Wigner–Bargmann [Wig39; Bar54]). *If $d \geq 2$, then ρ can be lifted to an ordinary unitary representation of the universal cover $\pi : \mathrm{P}^*(1,d) \twoheadrightarrow \mathrm{P}(1,d)$.* ■

Classifying the projective unitary representations of the restricted Poincaré group is thereby reduced to the more amenable problem of classifying *ordinary* unitary representations of a slightly bigger group,

$$\mathrm{P}^*(1,d) = \mathbb{R}^{1,d} \rtimes \widetilde{\mathrm{SO}(1,d)} \tag{2.1}$$

To this latter problem, the ample mathematical tools of linear representation theory apply. This will be the topic of the next chapter. We will call $\mathrm{P}^*(1,d)$, the "universal" Poincaré group, for lack of standard terminology;

$$\widetilde{\mathrm{SO}(1,d)} \cong \begin{cases} \widetilde{\mathrm{SL}(2,\mathbb{R})} & (d=2), \\ \mathrm{Spin}(1,d) & (d \geq 3). \end{cases} \tag{2.2}$$

"Universal" is synonymous with "spinorial" if $d \geq 3$. Then, the relevant covers are all double covers; $\mathrm{Spin}(1,d)$ is a standard semisimple Lie group, the (connected) double cover of the restricted Lorentz group, which exists if $d \geq 2$. As the name indicates, the existence of this cover has to do with spinors.

WARNING: The notation "$\mathrm{Spin}(1,d)$" is also used (by some authors, not us) to refer to a particular double cover of the *full* Lorentz group $\mathrm{O}(1,d)$.

In the physical $d=3$ case, $\mathrm{Spin}(1,3)$ is isomorphic to the group

$$\mathrm{SL}(2,\mathbb{C}) = \{S \in \mathbb{C}^{2\times 2} : \det S = 1\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{C},\ ad - bc = 1 \right\} \tag{2.3}$$

of unimodular two-by-two matrices with complex entries. This isomorphism, which plays the same role that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ plays in non-relativistic quantum mechanics, is the basis for many computations involving Weyl spinors.

1. The Wigner–Bargmann theorem

Let $\mathcal{H}$ denote an arbitrary Hilbert space. Suppose we are given a (continuous, as always) projective unitary representation $\rho : Q \to \mathrm{PU}(\mathcal{H})$ of some topological group Q. **Definition.** Let G denote a set and

$$\pi : G \twoheadrightarrow Q \tag{2.4}$$

a surjection. A *lift* of ρ (to G, via π) is a map of sets $\varrho : G \to \mathrm{U}(\mathcal{H})$ such that

$$[\varrho(g)] = \rho(\pi(g)) \tag{2.5}$$

for all $g \in G$. In other words:

$$\begin{tikzcd} G \arrow[r, "\varrho"] \arrow[d, "\pi"'] & \mathrm{U}(\mathcal{H}) \arrow[d, "U \mapsto [U]"] \\ Q \arrow[r, "\rho"'] & \mathrm{PU}(\mathcal{H}) \end{tikzcd} \tag{2.6}$$

commutes. Note that, here, we are not requiring anything of ϱ other than that it be a map of sets. By the axiom of choice, at least one lift exists. The phase of the operator $\varrho(g)$ is however completely arbitrary:

$$\varrho_{\mathrm{alt}}(g) = e^{i\theta(g)}\varrho(g) \tag{2.7}$$

is also a lift of ρ, for any function $\theta : G \to \mathbb{R}$. Consequently, there is no reason for ϱ, given what we have said so far, to be a homomorphism, or even continuous.

The question of deprojectivization is whether there exists a choice of lift that is a continuous homomorphism. In other words, can we, by choosing θ carefully, arrange that ϱ_{alt} is an ordinary unitary representation of G?

A far-reaching theorem of Bargmann [Bar54] states:

> **Lemma.** (*Bargmann's theorem.*) Any continuous projective unitary representation of a connected Lie group G lifts to a continuous unitary representation of G itself if the following two conditions are satisfied:
> (I) the Lie group is simply connected: the fundamental group $\pi_1(G)$ is trivial,
> (II) a certain cohomology group $\mathrm{H}^2(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}; \mathbb{R})$ (defined below) associated to the Lie algebra $\mathfrak{g}$ of G is trivial.

We will provide an outline of a proof of Bargmann's theorem, omitting the difficult technical bits.

What matters for us is that the hypotheses are satisfied when $G = \mathrm{P}^*(1, d)$ for $d \geq 2$. The topological condition, (I), is automatic: the definition of the universal cover includes that π_1 vanish. For (II): the Lie algebra of $\mathrm{P}^*(1, d)$ is the same as that $\mathfrak{p}$ of $\mathrm{P}(1, d)$, so it suffices to check that

$$\mathrm{H}^2(\mathfrak{p}) = \{0\}. \tag{2.8}$$

This calculation is rather involved, but it is just a calculation. We have included it in §2.A. If the reader is willing to take for granted that Bargmann's theorem applies to $\mathrm{P}^*(1, d)$, then it is not important to understand what $\mathrm{H}^2(\mathfrak{p})$ is.

Proof of the Bargmann–Wigner theorem. Given the above: for any continuous projective representation ρ, let

$$\begin{aligned} \bar{\rho} &: \mathrm{P}^*(1, d) \to \mathrm{PU}(\mathcal{H}) \\ \bar{\rho} &= \rho \circ \pi. \end{aligned} \tag{2.9}$$

This is a continuous projective unitary representation *of* $\mathrm{P}^*(1, d)$. By Bargmann's theorem, there exists a (continuous) unitary representation $\varrho : \mathrm{P}^*(1, d) \to \mathrm{U}(\mathcal{H})$ lifting $\bar{\rho}$. That is, $[\varrho(g)] = \bar{\rho}(g)$. Since $\bar{\rho}(g) = \rho(\pi(g))$, we conclude eq. (2.5). □

The two hypotheses of Bargmann's theorem each reflect an obstruction to lifting a projective representation to an ordinary representation:

(I) The first hypothesis indicates a global topological obstruction to choosing a continuous lift.
(II) The second indicates an "infinitesimal" algebraic obstruction to making the lift a homomorphism.

We discuss these in turn.

1.1. The topological obstruction, $\pi_1(G)$. Topological obstructions show up as a matter of course in lifting problems, so the presence of a topological obstruction here is unsurprising. Some lifting problems are topologically trivial, but this is not one of those: for each $N \in \mathbb{N}^+$, the covering map $\mathrm{U}(N) \twoheadrightarrow \mathrm{PU}(N)$ is a nontrivial U(1)-bundle. It has no continuous sections. So, if a projective unitary representation ρ is to be lifted, this imposes a topological constraint on ρ.

The ur-example, essential to understanding spinors, is:

EXAMPLE 2.1 ($\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$). Famously, the two groups

$$\begin{aligned}\mathrm{SU}(2) &= \{U \in \mathrm{U}(2) : \det U = 1\} \\ \mathrm{SO}(3) &= \{R \in \mathrm{O}(3) : \det R = 1\}\end{aligned} \tag{2.10}$$

are related by a (smooth, homomorphic) double cover $\pi : \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$. Correspondingly, while SU(2) is simply connected, SO(3) is doubly connected:

$$\pi_1(\mathrm{SU}(2)) = \text{trivial}, \qquad \pi_1(\mathrm{SO}(3)) \cong C_2. \tag{2.11}$$

The kernel of π is $\ker \pi = \{I_2, -I_2\}$. Consequently,

$$\mathrm{SO}(3) \cong \mathrm{PU}(2), \tag{2.12}$$

with an explicit isomorphism $\rho : \mathrm{SO}(3) \to \mathrm{PU}(2)$ being $\rho(\pi(U)) = [U]$. The map ρ is a projective unitary representation of SO(3).

Q. Does this lift to a continuous map $\mu : \mathrm{SO}(3) \to \mathrm{SU}(2)$? **A.** No.

Suppose, to the contrary, that there did exist such a μ. This would mean that $[\mu(R)] = \rho(R)$ for all $R \in \mathrm{SO}(3)$. I.e.

$$[\mu(\pi(U))] = [U] \tag{2.13}$$

for all $U \in \mathrm{SU}(2)$. That is, $\mu(\pi(U)), U$ differ by a phase. But $U, \mu(\pi(U))$ are *special* two-by-two unitary matrices, not just general unitary matrices. So differing by a phase implies $\mu(\pi(U)) = \pm U$. That is,

$$U^{-1}\mu(\pi(U)) = \pm I_2. \tag{2.14}$$

A priori, the sign $\pm$ could depend on U — but it is easy to see that it cannot, because the left-hand side depends continuously on U. Since SU(2) is connected, this means that the sign is constant. Without loss of generality, we can assume the $+$ case, which means $\mu(I_3) = I_2$. Then,

$$\mu(\pi(U)) = U. \tag{2.15}$$

But this is impossible, because $\pi(-I_2) = \pi(I_2)$. ■

REMARK 2.2. The reader may have noticed an artificial restriction in the previous example: we required $\mu : \mathrm{SO}(3) \to \mathrm{SU}(2)$. But why not weaken this to land in U(2)? The distinction does not matter: *there does not exist any continuous map* $\mathrm{SO}(3) \to \mathrm{U}(2)$ *lifting* ρ, but proving this requires a bit more topology. ■ [Problem 2.1]

REMARK 2.3. If we demand that $\mu : \mathrm{SO}(3) \to \mathrm{U}(2)$ be a homomorphism, then the lack of a lift follows from representation theoretic facts: the only two-dimensional representation of SO(3) is the trivial one.

One way of proving this is the argument above. Note that $\det \circ \mu : \mathrm{SO}(3) \to \mathrm{U}(1)$ is a one-dimensional representation of SO(3), and the only one-dimensional representation of SO(3) is the trivial one (see Exercise 2.1). Consequently, any two-dimensional unitary representation of SO(3) must land in SU(2), and then the argument above applies. ■

The fundamental group $\pi_1(G)$ is, as a set, the set of homotopy classes of continuous maps $\mathbb{S}^1 \to G$. (Recall that G is connected, so we do not need to worry about the choice of base point.) In the proof of Bargmann's theorem, the vanishing of the fundamental group is used to reduce the problem to one about representations of the Lie *algebra* $\mathfrak{g}$. This is what we discuss next.

1.2. The algebraic obstruction, $\mathrm{H}^2(\mathfrak{g})$**.** Given a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the vector space $\mathrm{H}^2(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}; \mathbb{K})$ is defined as follows. Identify elements of $(\wedge^2 \mathfrak{g})^*$ with anti-symmetric bilinear maps $\mathfrak{g}^2 \to \mathbb{K}$. Then,

$$\mathrm{B}^2(\mathfrak{g}) = \{\omega \in (\wedge^2 \mathfrak{g})^* : \exists \lambda \in \mathfrak{g}^* \text{ s.t. } \omega(X, Y) = \lambda([X, Y])\}, \tag{2.16}$$

and

$$\mathrm{Z}^2(\mathfrak{g}) = \{\omega \in (\wedge^2 \mathfrak{g})^* : \omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0\} \tag{2.17}$$

are the spaces of "coboundaries" and "cocycles" respectively. These are both vector subspaces of $(\wedge^2 \mathfrak{g})^*$. Note that $\mathrm{B}^2(\mathfrak{g}) \subseteq \mathrm{Z}^2(\mathfrak{g})$. Indeed, if $\omega \in \mathrm{B}^2(\mathfrak{g})$,

$$\begin{aligned} &\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) \\ &\qquad = \lambda([[X, Y], Z]) + \lambda([[Y, Z], X]) + \lambda([[Z, X], Y]) = \lambda(J), \end{aligned} \tag{2.18}$$

where $J = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$. But $J = 0$ is the Jacobi identity. So, $\omega \in \mathrm{Z}^2(\mathfrak{g})$. Then,

$$\mathrm{H}^2(\mathfrak{g}; \mathbb{K}) = \mathrm{Z}^2(\mathfrak{g}) / \mathrm{B}^2(\mathfrak{g}) \tag{2.19}$$

is the quotient of the vector space of cocycles by the subspace of coboundaries.

The general theory of Lie algebra cohomology is due to Chevalley–Eilenberg [CE48]. The special case of $\mathrm{H}^\bullet(\mathfrak{g}; \mathbb{K})$ goes further back, to Cartan, in his work on the de Rham cohomology of compact Lie groups.

The relevance of $\mathrm{H}^2(\mathfrak{g})$ to lifting projective representations has to do with its role in classifying central extensions of $\mathfrak{g}$ via $\mathbb{K}$. Recall that a short exact sequence of Lie algebras consists of the data

$$\mathbb{K} \hookrightarrow \mathfrak{e} \overset{\pi}{\twoheadrightarrow} \mathfrak{g}, \tag{2.20}$$

where $\ker \pi$ is exactly the image of $\mathbb{R}$ under the first embedding. If $\ker \pi$ lies in the *center* of $\mathfrak{e}$, then this is called a central extension.

Example 2.4. The *trivial* extension is $\mathfrak{e} = \mathfrak{g} \oplus \mathbb{K}$. More precisely, it is the short exact sequence in which

- the embedding $\mathbb{K} \hookrightarrow \mathfrak{e}$ is $x \mapsto (0, x)$ and
- the projection $\mathfrak{e} \twoheadrightarrow \mathfrak{g}$ is $(X, x) \mapsto X$.

■

Two central extensions $\mathbb{K} \hookrightarrow \mathfrak{e}_\bullet \twoheadrightarrow \mathfrak{g}$ are called *equivalent* if they fit together into a commutative diagram

$$\begin{tikzcd} & \mathfrak{e}_1 \arrow[dd, "\phi"] \arrow[dr] & \\ \mathbb{K} \arrow[ur, hook] \arrow[dr, hook] & & \mathfrak{g} \\ & \mathfrak{e}_1 \arrow[ur, two heads] & \end{tikzcd} \tag{2.21}$$

where ϕ is an isomorphism of Lie algebras. A central extension $\mathfrak{e}$ is also called trivial if it is equivalent to the trivial extension $\mathfrak{g} \oplus \mathbb{K}$. Equivalently, a central extension is trivial if and only if it splits, which means that there exists a map $\mathfrak{g} \ni X \mapsto \bar{X} \in \mathfrak{e}$ of Lie algebras such that $\pi(\bar{X}) = X$.

Given any central extension $\mathbb{K} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{g}$, let $Z \in \mathfrak{e}$ denote a nonzero element in the image of the first embedding. Choose an embedding of vector spaces $\mathfrak{g} \hookrightarrow \mathfrak{e}$, and denote this $X \mapsto \bar{X}$. Note that this will typically not be a map of Lie algebras. Measure its failure to be a map of Lie algebras by defining $\omega : \mathfrak{g}^2 \to \mathbb{K}$ by

$$\omega(X, Y) Z = [\bar{X}, \bar{Y}] - \overline{[X, Y]}, \tag{2.22}$$

where the first Lie bracket is $\mathfrak{e}$'s and the second is $\mathfrak{g}$'s. The right-hand side lies in the kernel of the projection $\mathfrak{e} \twoheadrightarrow \mathfrak{g}$ (this being a map of Lie algebras), so, by exactness, lies in the image of $\mathbb{K} \hookrightarrow \mathfrak{e}$, which is why it has the form $\mathbb{K}Z$.

Proposition 2.5. *(a) The map ω defined above is a cocycle.*
(b) It is a coboundary if and only if the extension is trivial.

■

Proof sketch. The first part of the proposition is the Jacobi identity for ω, as defined by eq. (2.22). This will follow from combining the Jacobi identities for $\mathfrak{e}, \mathfrak{g}$. The second part of the proposition comes from the following observation: an alternative to the lift $X \mapsto \bar{X}$ is $X \mapsto \bar{X} + \lambda(X)Z$, for any $\lambda \in \mathfrak{g}^*$. This alternative has a different ω, one differing from the original by a coboundary. So, we can choose a lift eliminating ω (in which case the lift is a splitting map) if and only if ω is a coboundary. □

Proof. (a) We want to show that ω satisfies the Jacobi identity $\omega([W,X],Y)+\omega([X,Y],W)+\omega([Y,W],X) = 0$. Note that

$$\begin{aligned}\omega([W,X],Y) &= [\overline{[W,X]},\bar{Y}] - \overline{[[W,X],Y]}\\ &= [[\bar{W},\bar{X}] - \omega(W,X)Z,\bar{Y}] - \overline{[[W,X],Y]}\\ &= [[\bar{W},\bar{X}],\bar{Y}] - \overline{[[W,X],Y]}.\end{aligned}\tag{2.23}$$

Likewise,

$$\begin{aligned}\omega([X,Y],W) &= [[\bar{X},\bar{Y}],\bar{W}] - \overline{[[X,Y],W]}\\ \omega([W,X],Y) &= [[\bar{W},\bar{X}],\bar{Y}] - \overline{[[W,X],Y]}.\end{aligned}\tag{2.24}$$

The Jacobi identity for ω therefore follows from three things: the Jacobi identity for $\mathfrak{e}$'s Lie bracket, the Jacobi identity for $\mathfrak{g}$'s Lie bracket, and the linearity of the map $X \mapsto \bar{X}$.

(b)
- ('If.') It suffices to consider the case where $\mathfrak{e} = \mathfrak{g} \oplus \mathbb{K}$ is literally the trivial extension, in which case we can choose $Z = (0,1)$. Because $X \mapsto \bar{X}$ is linear, there must exist some $\lambda \in \mathfrak{g}^*$ such that $\bar{X} = (X, \lambda(X))$. Then,

$$\begin{aligned}\omega(X,Y)Z = [\bar{X},\bar{Y}] - \overline{[X,Y]} &= ([X,Y],0) - ([X,Y],\lambda([X,Y]))\\ &= (0,\lambda([X,Y])).\end{aligned}\tag{2.25}$$

  We conclude that $\omega(X,Y) = \lambda([X,Y])$, which is what it means to be a coboundary.
- ('Only if.') Suppose that $\omega(X,Y) = \lambda([X,Y])$ for some $\lambda \in \mathfrak{g}^*$. Now define $\phi : \mathfrak{g} \oplus \mathbb{K} \to \mathfrak{e}$ by

$$\phi(X,s) = \bar{X} + (s + \lambda(X))Z.\tag{2.26}$$

  This is an isomorphism of vector spaces, fitting into a commutative diagram of vector spaces, as in eq. (2.21). It is also a map of Lie algebras:

$$[\phi(X,s),\phi(Y,t)] = [\bar{X},\bar{Y}]\tag{2.27}$$

$$\begin{aligned}\phi([(X,s),(Y,t)]) = \phi([X,Y],0) &= \overline{[X,Y]} + \lambda([X,Y])Z\\ &= \overline{[X,Y]} + \omega([X,Y])Z = [\bar{X},\bar{Y}].\end{aligned}\tag{2.28}$$

□

It turns out that any continuous projective unitary representation of a Lie group induces an ordinary representation not of $\mathfrak{g}$ but of a one-dimensional central extension thereof. Indeed, consider the topological group

$$E = \{(U,g) \in \mathrm{U}(\mathcal{H}) \times G : [U] = \rho(g)\},\tag{2.29}$$

whose (natural, continuous) unitary representation $\varrho : (U, g) \mapsto U$ lifts ρ. This sits in a central extension

$$\mathrm{U}(\mathcal{H}) \hookrightarrow E \twoheadrightarrow G \tag{2.30}$$

in the category of topological groups. It turns out that E is canonically a Lie group of dimension $\dim G + 1$, and the maps in eq. (2.30) are smooth.[1] Consequently, they can be "differentiated" to yield a central extension $\mathbb{R} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{g}$ of the corresponding Lie algebras. The representation ϱ induces a representation of $\mathfrak{e}$ on $\mathcal{H}$.

If the central extension $\mathbb{R} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{g}$ is trivial, so that $\mathfrak{e} \cong \mathbb{R} \oplus \mathfrak{g}$, then we can ignore the extra $\mathbb{R}$ factor and get a representation on $\mathcal{H}$ of $\mathfrak{g}$ itself. Exponentiating this (using Lie's second theorem) yields an ordinary unitary representation of the universal cover of G. If G is already simply connected, this is G itself. So:

$$\rho \text{ is guaranteed to lift} \iff \text{all 1D central extensions of } \mathfrak{g} \text{ are trivial.}$$

Conversely, nontrivial central extensions of $\mathfrak{g}$ typically yield projective representations of G which only lift to ordinary unitary representations of some central extension of G by $\mathrm{U}(1)$, namely E.

Example 2.6 (The Heisenberg group). In this example, we exhibit a projective representation of the abelian group $(\mathbb{R}^2, +)$ that does not lift to a unitary representation of $(\mathbb{R}^2, +)$ but rather of a three-dimensional Lie group H known as the "Heisenberg group." The existence of such an intrinsically projective representation has to do with the fact that the abelian Lie algebra $\mathfrak{t} = \mathbb{R}^2$ has nontrivial cohomology,

$$\mathrm{H}^2(\mathfrak{t}) \cong \mathbb{R} \tag{2.31}$$

(see Exercise 2.3) and therefore admits a nontrivial central extension, namely the Lie algebra of the Heisenberg group.

The *Heisenberg group* H is the group

$$H = \left\{ \underbrace{\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}}_{M(a,b,c)} : a, b, c \in \mathbb{R} \right\} \tag{2.32}$$

of 3-by-3 matrices $M \in \mathbb{R}^{3\times 3}$ such that $M - I_3$ is upper triangular. The group law is

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a + a' & c + c' + ab' \\ 0 & 1 & b + b' \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.33}$$

So, if we define $\pi : H \twoheadrightarrow \mathbb{R}^2$ by $M(a, b, c) \mapsto (a, b)$, this is a homomorphism onto the abelian group $(\mathbb{R}^2, +)$. This fits into a central extension

$$(\mathbb{R}, +) \hookrightarrow H \overset{\pi}{\twoheadrightarrow} (\mathbb{R}^2, +) \tag{2.34}$$

of groups, where the first map is $c \mapsto M(0, 0, c)$. Differentiating yields a central extension

$$\mathbb{R} \hookrightarrow \mathfrak{h} \twoheadrightarrow \mathfrak{t} = \mathbb{R}^2 \tag{2.35}$$

[1]Establishing this is the main technical step in the proof of Bargmann's theorem. First, one shows straightforwardly that E is a topological manifold [Sim71], being a $\mathrm{U}(1)$-bundle over G. Then, the solution of Hilbert's fifth problem (the Montgomery–Zippin theorem) guarantees that E is canonically a Lie group. The special case relevant here – that of a central extension of a Lie group by another Lie group – is considerably easier than the full problem, and a complete exposition can be found in [Tao14, §2.6].

of Lie algebras. Concretely, $\mathfrak{h} \subset \mathbb{R}^{3\times 3}$ is the Lie algebra of upper-triangular 3-by-3 matrices

$$m(a,b,c) = \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.36}$$

in which the Lie bracket is the usual commutator. The first map in eq. (2.35) is $c \mapsto m(0,0,c)$ and the second map is $m(a,b,c) \mapsto (a,b)$. The central extension $\mathfrak{h}$ is nontrivial, because $\mathfrak{h}$ is not abelian.

The *Schrödinger representation* of the Heisenberg group H is defined on $\mathcal{H} = L^2(\mathbb{R})$. Actually, we have many different Schrödinger representations, one for each $\hbar > 0$. Specifically, let $\varrho_\hbar : H \to \mathrm{U}(\mathcal{H})$ be defined by

$$\varrho_\hbar\left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right)\psi(x) = e^{i\hbar c} e^{ibx}\psi(x+\hbar a) \text{ for all } \psi \in L^2(\mathbb{R}) \tag{2.37}$$

for all $\psi \in L^2(\mathbb{R})$. This is indeed a continuous unitary representation. [Problem 2.3(b)]

Let $\rho_\hbar : \mathbb{R}^2 \to \mathrm{PU}(\mathcal{H})$ be defined by $\rho_\hbar(a,b) = \varrho_\hbar(M(a,b,0)) \bmod \mathrm{U}(1)$. This is a (continuous) projective unitary representation of $(\mathbb{R}^2,+)$. While it lifts to an ordinary representation of H, by construction, it does not lift to an ordinary representation of $\mathbb{R}^2$. The physical interpretation of $\rho_\hbar$ is that it describes the Galilean symmetry of Schrödinger's wave mechanics on the real line. One factor of $\mathbb{R}^2$ is carrying out a translation, and the other is carrying out a Galilean boost. ■ [Problem 2.3(c)]

2. Parity and time-reversal ($\star$)

We now discuss how the considerations above are modified in the presence of parity and/or time-reversal symmetry. In §3, we discussed how these additional symmetries lift to unitary/anti-unitary transformations of the ambient Hilbert space. The possibilities were classified according to a "tenfold way." Excluding class A, a system with no additional symmetries beyond the restricted Poincaré group, there were nine cases: AI, AII, AIII, BDI, C, CI, CII, D, DIII. The goal of this section is to discuss how, in each class, a Wigner representation of the relevant subgroup of the full Poincaré group lifts to an ordinary unitary/anti-unitary representation of some cover thereof.

In each of the ten ways, we have a subgroup $Z \subseteq \{\mathscr{I}, \mathscr{P}, \mathscr{T}, \mathscr{C} = \mathscr{P}\mathscr{T}\}$ telling us which symmetries are present. In Table 1.1 is listed a covering group $c : E \twoheadrightarrow Z$. This is either an isomorphism or a double cover, depending on which way we are in. For each $\mathscr{A} \in Z$, we have an automorphism

$$T \mapsto T_{\mathscr{A}} T T_{\mathscr{A}} \tag{2.38}$$

of $\mathrm{P}(1,d)$. This induces an automorphism of the Lie algebra $\mathfrak{p}(1,d)$, which exponentiates (by Lie's second theorem) to an automorphism $\Phi_{\mathscr{A}} \in \mathrm{Aut}(\mathrm{P}^*(1,d))$ of $\mathrm{P}^*(1,d)$. The map $\mathscr{A} \mapsto \Phi_{\mathscr{A}}$ associating $\mathscr{A}$ with this automorphism is a homomorphism

$$Z \to \mathrm{Aut}(\mathrm{P}^*(1,d)). \tag{2.39}$$

Precomposing with c yields a homomorphism $E \to \mathrm{Aut}(\mathrm{P}^*(1,d))$. Let $E \ltimes \mathrm{P}^*(1,d)$ denote the corresponding semidirect product. This is a double or quadruple cover of the subgroup G of the full Poincaré group containing the identity component and the members of Z.

Recall that we have been assuming $d \geq 2$.

Proposition 2.7. *Every Wigner representation of G lifts to an ordinary unitary/anti-unitary representation of $E \ltimes \mathrm{P}^*(1,d)$.* ■

Proof. By the discussion in the previous lecture, and by the Wigner–Bargmann theorem, we have:

- an ordinary unitary representation $\varrho : \mathrm{P}^*(1,d) \to \mathrm{U}(\mathcal{H})$, lifting the restriction of the given Wigner representation to $\mathrm{P}(1,d)$,

- for each $\mathscr{A} \in Z$, an operator $\mathcal{A}$ (as described in the tenfold way) such that
$$\mathcal{A}\varrho(T)\mathcal{A} = \varepsilon_{\mathsf{A}}\varrho(\Phi_{\mathscr{A}}(T))e^{i\theta(T;\mathscr{A})} \tag{2.40}$$
for each $T \in \mathrm{P}^*(1,d)$, where $\theta(T;\mathscr{A})$ is some phase, and where $\varepsilon_{\mathsf{A}} \in \{-1,+1\}$ is the sign of $\mathcal{A}^2 \propto I$.

The group generated by the $\mathcal{A}$'s is the covering group E.

Let us restrict $\theta(T;\mathscr{A})$. Note that $e^{i\theta(I;\mathscr{A})} = 1$. Compute
$$\begin{aligned}\varepsilon_{\mathsf{A}}e^{i\theta(T_1T_2;\mathscr{A})} = \mathcal{A}\varrho(T_1T_2)\mathcal{A}\varrho(\Phi_{\mathscr{A}}(T_2^{-1}T_1^{-1})) &= \varepsilon_{\mathsf{A}}\mathcal{A}\varrho(T_1)\mathcal{A}\mathcal{A}\varrho(T_2)\mathcal{A}\varrho(\Phi_{\mathscr{A}}(T_2^{-1}))\varrho(\Phi_{\mathscr{A}}(T_1^{-1}))\\ &= \mathcal{A}\varrho(T_1)\mathcal{A}e^{i\theta(T_2;\mathscr{A})}\varrho(\Phi_{\mathscr{A}}(T_1^{-1}))\\ &= e^{i\theta(T_2;\mathscr{A})}\mathcal{A}\varrho(T_1)\mathcal{A}\varrho(\Phi_{\mathscr{A}}(T_1^{-1}))\\ &= \varepsilon_{\mathsf{A}}e^{i\theta(T_1;\mathscr{A})+i\theta(T_2;\mathscr{A})}\end{aligned} \tag{2.41}$$
for all $T_1, T_2 \in \mathrm{P}^*(1,d)$. That is, $T \mapsto e^{i\theta(T;\mathscr{A})}$ is a one-dimensional unitary representation of $\mathrm{P}^*(1,d)$. Every such representation must be trivial (skip ahead to **??** for the proof).

Having now shown that $e^{i\theta(T;\mathscr{A})} = 1$, it suffices to observe that what we have in the $\mathcal{A}$'s, ϱ is a unitary/anti-unitary representation of $E \ltimes \mathrm{P}^*(1,d)$. Explicitly,
$$E \ltimes \mathrm{P}^*(1,d) \ni (\mathcal{A}, \varrho(T)) \mapsto \mathcal{A}\varrho(T) \in \mathrm{UaU}(\mathcal{H}) \tag{2.42}$$
is a unitary/anti-unitary representation lifting the given Wigner representation. □

3. $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$

The trick to connect the Lorentz group with $\mathrm{SL}(2,\mathbb{C})$ is to identify $\mathbb{R}^{1,3}$ with the real vector space $H_2 = \{M \in \mathbb{C}^{2\times 2} : M = M^\dagger\}$ of Hermitian 2-by-2 matrices with complex entries. Let $\boldsymbol{\Sigma} : \mathbb{R}^{1,3} \to H_2$ denote the *Bloch map*,
$$\boldsymbol{\Sigma}(t, x^1, x^2, x^3) = tI_2 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{pmatrix} t + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & t - x^3\end{pmatrix}, \tag{2.43}$$
where $\sigma_\bullet$ are the three Pauli matrices,
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \tag{2.44}$$
which satisfy $\sigma_j\sigma_k = \delta_{jk}I_2 + i\varepsilon_{jk\ell}\sigma_\ell$, where $\varepsilon_{jk\ell} \in \{-1,0,+1\}$ is the Levi-Civita symbol. The Bloch map is a linear isomorphism between $\mathbb{R}^{1,3}$ and H_2. Its inverse is
$$\boldsymbol{\Sigma}^{-1} : \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma\end{pmatrix} \mapsto \Big(\frac{\alpha+\gamma}{2}, \Re\beta, -\Im\beta, \frac{\alpha-\gamma}{2}\Big). \tag{2.45}$$

Let $S \in \mathbb{C}^{2\times 2}$. Whenever $M \in H_2$, then $SMS^\dagger \in H_2$. Consequently, if $x \in \mathbb{R}^{1,3}$, then $\boldsymbol{\Sigma}^{-1}(S\boldsymbol{\Sigma}(x)S^\dagger) \in \mathbb{R}^{1,3}$ is well-defined. The map
$$x \mapsto \boldsymbol{\Sigma}^{-1}(S\boldsymbol{\Sigma}(x)S^\dagger) \tag{2.46}$$
is linear and so can be represented by a matrix, which we denote $\pi(S) \in \mathbb{R}^{4\times 4}$. Evidently, $\pi : \mathbb{C}^{2\times 2} \to \mathbb{R}^{4\times 4}$ is smooth. (An explicit formula will be below.) We can now state:

Proposition 2.8. $\pi : \mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow \mathrm{SO}(1,3)$ *is a surjective 2-to-1 homomorphism of Lie groups.* ■

We will prove this below.

Proposition 2.9. $\pi(S^\dagger) = \pi(S)^\intercal$. ■

PROOF. Note that $\Sigma(x)^{-1}$ exists if $x^2 \neq 0$, and $\Sigma(x)^{-1} = \Sigma(\mathscr{P}x)$. By definition, $\pi(S^{-1}) : x \mapsto \Sigma^{-1}(S^{-1}\Sigma(x)(S^\dagger)^{-1})$, and the right-hand side is, if $x^2 \neq 0$,

$$\Sigma^{-1}((S^\dagger\Sigma(x)^{-1}S)^{-1}) = \mathscr{P}\Sigma^{-1}(S^\dagger\Sigma(\mathscr{P}x)S) = \mathscr{P}\pi(S^\dagger)\mathscr{P}x. \tag{2.47}$$

So, $\pi(S^{-1}) = \mathscr{P}\pi(S^\dagger)\mathscr{P}$. Equivalently, $\pi(S^\dagger) = \mathscr{P}\pi(S^{-1})\mathscr{P}$. The right-hand side is $\mathscr{P}\pi(S)^{-1}\mathscr{P}$ (since π is a homomorphism, which implies $\pi(S^{-1}) = \pi(S)^{-1}$). Now we can use the identity $\Lambda^{-1} = \eta\Lambda^\intercal\eta$ that holds for all Lorentz matrices to get $\pi(S^\dagger) = (-I_4)\pi(S)^\intercal(-I_4) = \pi(S^\intercal)$. □

WARNING: Many facts about the representation theory of $\mathrm{SL}(2,\mathbb{C})$ can be deduced from facts about the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. For example, it is possible to prove Proposition 2.8 by exhibiting an explicit isomorphism

$$\mathfrak{o}(1,3) \cong \mathfrak{sl}(2,\mathbb{C}). \tag{2.48}$$

However, care is required, because the exponential map $\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ is not surjective. For example, $\left(\begin{smallmatrix} -1 & s \\ 0 & -1 \end{smallmatrix}\right)$ is not in the image of the exponential map whenever $s \in \mathbb{C}^\times$. (This does not contradict Problem 1.5.)

3.1. Proof of Proposition 2.8. Observe: if $x \in \mathbb{R}^{1,3}$, then the Minkowski norm $x^2 = -x_0^2 + \|\mathbf{x}\|^2$ is given by $-\det \Sigma(x)$.

LEMMA 2.10. Whenever $S \in \mathrm{SL}(2,\mathbb{C})$, we have $\pi(S) \in \mathrm{SO}(1,3)$. ■

PROOF. For any $x \in \mathbb{R}^{1,3}$, the Minkowski norm of $\pi(S)x = \Sigma^{-1}(S\Sigma(x)S^\dagger)$ is

$$\begin{aligned}\det \Sigma(\Sigma^{-1}(S\Sigma(x)S^\dagger)) &= \det(S\Sigma(x)S^\dagger) \\ &= |\det S|^2 \det(\Sigma(x)) = \det \Sigma(x) = -x^2,\end{aligned} \tag{2.49}$$

the Minkowski norm of x. So, $\pi(S)$ preserves the Minkowski norm. This means that $\pi(S) \in \mathrm{O}(1,3)$ is a Lorentz matrix.

Since $\mathrm{SL}(2,\mathbb{C})$ is connected, the image of π must be a connected subset of $\mathrm{O}(1,3)$, and must therefore lie entirely in one of the four connected components of $\mathrm{O}(1,3)$. Because $\pi(I_2) = I_4$, the relevant component is the one containing the identity I_4, i.e. the restricted Lorentz group $\mathrm{SO}(1,d)$. □

LEMMA 2.11. The map $\pi : \mathrm{SL}(2,\mathbb{C}) \ni S \mapsto \pi(S) \in \mathrm{SO}(1,3)$ is a homomorphism. ■

PROOF. Firstly, $\pi(I_2) = I_4$.

If $S, Q \in \mathrm{SL}(2,\mathbb{C})$, then the Lorentz matrices $\pi(SQ)$, $\pi(S)\pi(Q)$ implement the maps

$$\begin{aligned} x &\mapsto \Sigma^{-1}(SQ\Sigma(x)(QS)^\dagger) = \Sigma^{-1}(SQ\Sigma(x)Q^\dagger S^\dagger), \\ x &\mapsto \Sigma^{-1}(S\Sigma(\Sigma^{-1}(Q\Sigma(x)Q^\dagger))S^\dagger) = \Sigma^{-1}(SQ\Sigma(x)Q^\dagger S^\dagger), \end{aligned} \tag{2.50}$$

which agree. So, $\pi(SQ) = \pi(S)\pi(Q)$. □

LEMMA 2.12. Fix $N \in \mathbb{N}^+$. Let $S \in \mathbb{C}^{N\times N}$, and suppose that $SMS^\dagger = M$ for all $M \in H_N$. Then, $S = e^{i\theta}I_N$ for some $\theta \in \mathbb{R}$. ■

PROOF. Try $M = vv^\intercal$ for $v \in \mathbb{C}^N$. Then, $SMS^\dagger = (Sv)(Sv)^\intercal$. This map, whose range is the span of Sv, is equal to $vv^\intercal$, whose range is the span of v, if and only if

$$Sv = e^{i\theta}v \tag{2.51}$$

for some $\theta \in \mathbb{R}$. So, every nonzero element of $\mathbb{C}^N$ is an eigenvector of S. This implies that S is a scalar multiple of the identity: $S = cI_N$ for some $c \in \mathbb{C}$, which, by eq. (2.51), must be a phase. □

COROLLARY. $\ker \pi = \{-I_2, I_2\}$. ■□

PROOF. Evidently, $\pm I_2 \in \ker \pi$, so the main order of business is the converse.

If $S \in \ker \pi$, then it satisfies the hypotheses of lemma 2.12 with $N = 2$. So $S = e^{i\theta} I$ for some $\theta \in \mathbb{R}$. Then $\det S = e^{2i\theta}$, so S is unimodular only if $S = \pm I_2$. □

LEMMA 2.13. π is onto $\mathrm{SO}(1,3)$. ■

PROOF. This can be shown in several ways. It follows from the fact that $\mathrm{SO}(1,3)$ is connected and $D\pi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{o}(1,3)$ is surjective (which must be true because both Lie algebras have real dimension 6 and $\ker \pi$ is discrete; alternatively, $D\pi$ is computed below). □

This completes the proof of Proposition 2.8.

3.2. $D\pi$**.** The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ can be taken to consist of $X \in \mathbb{C}^{2\times 2}$ such that $e^{sX} \in \mathrm{SL}(2,\mathbb{C})$ for all $s \in \mathbb{R}$. Since

$$e^{(s+\delta s)X} = e^{sX} e^{X\delta s} = e^{sX}(I_2 + X\delta s + O(\delta s^2)), \tag{2.52}$$

$$\begin{aligned} \det e^{(s+\delta s)X} &= (\det e^{sX})(\det(I_2 + X\delta s) + O(\delta s^2)) \\ &= (\det e^{sX})(1 + (\delta s)\operatorname{tr} X + O(\delta s^2)) \end{aligned} \tag{2.53}$$

the derivative

$$\frac{\mathrm{d}}{\mathrm{d}s} \det e^{sX} = (\det e^{sX}) \operatorname{tr} X \tag{2.54}$$

can be computed. So, the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ consists of traceless matrices:

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in \mathbb{C}^{2\times 2} : \operatorname{tr} X = 0\} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a,b,c \in \mathbb{C} \right\}. \tag{2.55}$$

Over the complex numbers, this is spanned by the three Pauli matrices.

Since $e^{sX} = I_2 + sX + O(s^2)$, we can compute $\pi(e^{sX})$ modulo $O(s^2)$ by throwing out all terms in eq. (2.60) that are quadratic in s. This results in

$$\pi(e^{sX}) = I_4 + s \underbrace{\begin{bmatrix} 0 & \Re(b^*+c) & \Im(b^*+c) & 2\Re a \\ \Re(c^*+b) & 0 & 2\Im a & \Re(c^*-b) \\ -\Im(c^*+b) & -2\Im a & 0 & \Im(b-c^*) \\ 2\Re a & \Re(b^*-c) & \Im(b^*-c) & 0 \end{bmatrix}}_{(D\pi)X} + O(s^2). \tag{2.56}$$

The matrix in the previous line is supposed to lie in the Lie algebra

$$\mathfrak{o}(1,3) = \left\{ \begin{bmatrix} 0 & \mathbf{a}^\intercal \\ \mathbf{a} & \mathbf{X} \end{bmatrix} : \mathbf{a} \in \mathbb{R}^3, \mathbf{X} \in \mathbb{R}^{3\times 3}, \mathbf{X} = -\mathbf{X}^\intercal \right\} \tag{2.57}$$

of $\mathrm{O}(1,3)$, and indeed, it does, with

$$\mathbf{a} = \begin{bmatrix} \Re(b^*+c) \\ \Im(b^*+c) \\ 2\Re a \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} 0 & 2\Im a & \Re(c^*-b) \\ -2\Im a & 0 & \Im(b-c^*) \\ -\Re(c^*-b) & -\Im(b-c^*) & 0 \end{bmatrix}. \tag{2.58}$$

In summary:

PROPOSITION 2.14. *The differential* $D\pi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{o}(1,3)$ *is given by*

$$X = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} 0 & \mathbf{a}^\intercal \\ \mathbf{a} & \mathbf{X} \end{bmatrix}, \tag{2.59}$$

where $\mathbf{a}$, $\mathbf{X}$ *are as above. So:*

- *The generators of Lorentz boosts, in* $\mathfrak{sl}(2,\mathbb{C})$*, are those* X *with* $a \in \mathbb{R}$ *and* $b = c^*$*, i.e. satisfying* $X \in \operatorname{span}_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_3\}$*.*
- *The generators of rotations, in* $\mathfrak{sl}(2,\mathbb{C})$*, are those* X *with* $a \in i\mathbb{R}$ *and* $b = -c^*$*, i.e. satisfying* $X \in \operatorname{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2, i\sigma_3\}$*.*

$\blacksquare$

Corollary. The pre-image $\pi^{-1}(\mathrm{SO}(3))$ of the subgroup $\mathrm{SO}(3) \subseteq \mathrm{SO}(1,3)$ of rotations is the subgroup $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$. $\blacksquare\square$

Proof. We know that $\pi^{-1}(\mathrm{SO}(3))$ is a Lie subgroup of $\mathrm{SL}(2,\mathbb{C})$. By the previous proposition, its Lie algebra is $\operatorname{span}_{\mathbb{R}}\{i\sigma_1, i\sigma_2, i\sigma_3\}$, which exponentiates to $\mathrm{SU}(2)$. $\square$

3.3. An explicit formula ($\star$).

Proposition 2.15. *For $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the matrix $\pi(S)$ is*

$$\begin{bmatrix} 2^{-1}(|a|^2+|b|^2+|c|^2+|d|^2) & \Re(ab^*+cd^*) & \Im(ab^*+cd^*) & 2^{-1}(|a|^2-|b|^2+|c|^2-|d|^2) \\ \Re(ac^*+bd^*) & \Re(ad^*+bc^*) & \Im(ad^*-bc^*) & \Re(ac^*-bd^*) \\ -\Im(ac^*+bd^*) & -\Im(ad^*+bc^*) & \Re(ad^*-bc^*) & \Im(bd^*-ac^*) \\ 2^{-1}(|a|^2+|b|^2-|c|^2-|d|^2) & \Re(ab^*-cd^*) & \Im(ab^*-cd^*) & 2^{-1}(|a|^2-|b|^2-|c|^2+|d|^2) \end{bmatrix}. \tag{2.60}$$

$\blacksquare$

Proof. Concretely,

$$S\Sigma(x)S^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t+x^3 & x^1-ix^2 \\ x^1+ix^2 & t-x^3 \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} \tag{2.61}$$

for

$$\begin{aligned} \alpha &= |a|^2(t+x^3) + |b|^2(t-x^3) + 2\Re[(x^1-ix^2)ab^*], \\ \beta &= (t+x^3)ac^* + (t-x^3)bd^* + (x^1-ix^2)ad^* + (x^1+ix^2)bc^*, \\ \gamma &= |c|^2(t+x^3) + |d|^2(t-x^3) + 2\Re[(x^1-ix^2)cd^*], \end{aligned} \tag{2.62}$$

Consequently, $\Sigma^{-1}(S\Sigma(x)S^\dagger) = (t', (x^1)', (x^2)', (x^3)')$ for

$$\begin{aligned} t' &= (|a|^2+|b|^2+|c|^2+|d|^2)\frac{t}{2} + (|a|^2-|b|^2+|c|^2-|d|^2)\frac{x^3}{2} + \Re[(x^1-ix^2)(ab^*+cd^*)], \\ (x^1)'-i(x^2)' &= (t+x^3)ac^* + (t-x^3)bd^* + (x^1-ix^2)ad^* + (x^1+ix^2)bc^*, \\ (x^3)' &= (|a|^2+|b|^2-|c|^2-|d|^2)\frac{t}{2} + (|a|^2-|b|^2-|c|^2+|d|^2)\frac{z}{2} + \Re[(x^1-ix^2)(ab^*-cd^*)]. \end{aligned} \tag{2.63}$$

The matrix implementing the transformation $x \mapsto x'$ is eq. (2.60). $\square$

2.A. Calculation of $\mathrm{H}^2(\mathfrak{p})$

One of the hypotheses of Bargmann's theorem is that the second cohomology group $\mathrm{H}^2(\mathfrak{p}) = \mathrm{H}^2(\mathfrak{p};\mathbb{R})$ of the Lie algebra $\mathfrak{p} = \mathfrak{p}(1,d)$ of $\mathrm{P}(1,d)$ is trivial. It will be, if $d \geq 2$. Carrying out this computation is the purpose of this appendix.

Because $\mathfrak{p}$ is finite-dimensional, $(\wedge^2\mathfrak{p})^*$ is finite-dimensional, so the subspaces $\mathrm{B}^2(\mathfrak{p}), \mathrm{Z}^2(\mathfrak{p})$ of "coboundaries" and "cocycles" are both finite-dimensional. The latter is defined by finitely many linear constraints, and the former is defined as the image of $\mathfrak{p}^*$ under a particular map. So, computing

$$\mathrm{H}^2(\mathfrak{p}) = \mathrm{Z}^2(\mathfrak{p})/\,\mathrm{B}^2(\mathfrak{p}) \tag{2.64}$$

is a matter of finite-dimensional Lie algebra, for each individual d, and could be done with a computer.

Rather than phrasing the computation this way, we revert to the perspective of $\mathrm{H}^2(\mathfrak{p})$ as classifying central extensions (by $\mathbb{R}$). Proving that it is trivial amounts to proving that every central extension is equivalent to the trivial one. Proving that it is *non*-trivial amounts to constructing a central extension that is provably inequivalent to the trivial one. Either way, the computations involved are equivalent to computations done using the cohomological language, the difference being a matter of presentation.

REMARK 2.16 (Complexification). Physicists usually prefer to work with the complexification $\mathfrak{p}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{p}$ of the Lie algebra rather than $\mathfrak{p}$ itself. This does not matter because

$$\mathrm{H}^2(\mathfrak{p}_{\mathbb{C}}; \mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}^2(\mathfrak{p}; \mathbb{R}) \tag{2.65}$$

[Exercise 2.4] naturally. So, the triviality of H^2 does not depend on the choice of base field. Below, we work with $\mathfrak{p}_{\mathbb{C}}$ in place of $\mathfrak{p}$, so as to match physicists' notation (specifically that in [Wei05, §2.7]). Thus, *all Lie algebras in this appendix will be complexified*, and we will not write the '$\mathbb{C}$' subscript. ■

EXAMPLE 2.17 ($d = 1$). If $d = 1$, then, the Poincaré algebra $\mathfrak{p} = \mathfrak{p}(1, 1)$ is three-dimensional. It is spanned over $\mathbb{C}$ by three generators, $\mathsf{P}^0, \mathsf{P}^1, \mathsf{K}$, the two generators of translations and the generator of boosts, respectively, satisfying

$$i[\mathsf{P}^0, \mathsf{K}] = \mathsf{P}^1, \quad i[\mathsf{P}^1, \mathsf{K}] = \mathsf{P}^0 \tag{2.66}$$

and $[\mathsf{P}^0, \mathsf{P}^1] = 0$. Let us try to centrally extend $\mathfrak{p}$. A central extension of $\mathfrak{p}$ consists of a Lie algebra $\mathfrak{e}$ with one more generator than $\mathfrak{p}$ — call it Z (physicists call this a "central charge") — in the center, whose Lie bracket can be written

$$\begin{aligned} i[\mathsf{P}^0, \mathsf{K}] &= \mathsf{P}^1 + D\mathsf{Z}, \quad i[\mathsf{P}^1, \mathsf{K}] = \mathsf{P}^0 + E\mathsf{Z}, \\ i[\mathsf{P}^0, \mathsf{P}^1] &= C\mathsf{Z} \end{aligned} \tag{2.67}$$

for some $C, D, E \in \mathbb{C}$. Usually, the new coefficients would be constrained by the Jacobi identity, but the Jacobi identity is automatically satisfied in this case. So, eq. (2.67) defines a valid Lie algebra.

The extension $\mathfrak{e}$ is trivial if and only if it is possible to replace each of $\mathsf{P}^0, \mathsf{P}^1, \mathsf{K}$ with a linear combination with Z so as to eliminate the C, D, E. This is easy to do for D, E. Indeed, letting

$$\tilde{\mathsf{P}}^1 = \mathsf{P}^1 + D\mathsf{Z}, \qquad \tilde{\mathsf{P}}^0 = \mathsf{P}^0 + E\mathsf{Z}, \tag{2.68}$$

we have

$$\begin{aligned} i[\tilde{\mathsf{P}}^0, \mathsf{K}] &= \tilde{\mathsf{P}}^1, \quad i[\tilde{\mathsf{P}}^1, \mathsf{K}] = \tilde{\mathsf{P}}^0, \\ i[\tilde{\mathsf{P}}^0, \tilde{\mathsf{P}}^1] &= C\mathsf{Z}. \end{aligned} \tag{2.69}$$

Thus, we have eliminated D, E.

But, C cannot be so eliminated — replacing the generators P^j with some linear combinations with Z, we do not change $[\mathsf{P}^0, \mathsf{P}^1]$. So,

$$\mathrm{H}^2(\mathfrak{p}(1, 1)) \cong \mathbb{C} \tag{2.70}$$

is one-dimensional. ■

The fact that $\mathrm{H}^2(\mathfrak{p}(1, 1))$ is non-trivial should make us appreciate:

LEMMA 2.18. For each $d \geq 2$, $\mathrm{H}^2(\mathfrak{p}(1, d)) \cong \{0\}$. ■

PROOF. Let $\mathbb{R} \hookrightarrow \mathfrak{e} \twoheadrightarrow \mathfrak{p}$ denote a central extension of $\mathfrak{p}$. Let Z denote a central charge, and let $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$ denote arbitrarily chosen elements of the preimages of the identically named generators of $\mathfrak{p}$. The Lie bracket of $\mathfrak{e}$ then takes the form

$$\begin{aligned} i[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] &= \eta^{\mu\lambda}\mathsf{J}^{\nu\sigma} - \eta^{\mu\sigma}\mathsf{J}^{\nu\lambda} + \eta^{\nu\sigma}\mathsf{J}^{\mu\lambda} - \eta^{\nu\lambda}\mathsf{J}^{\mu\sigma} + C^{\mu\nu,\sigma\lambda}\mathsf{Z} \\ i[\mathsf{P}^\mu, \mathsf{J}^{\nu\sigma}] &= \eta^{\mu\nu}\mathsf{P}^\sigma - \eta^{\mu\sigma}\mathsf{P}^\nu + C^{\mu,\nu\sigma}\mathsf{Z} \\ i[\mathsf{P}^\mu, \mathsf{P}^\nu] &= C^{\mu,\nu}\mathsf{Z} \end{aligned} \tag{2.71}$$

for some $C^{\mu\nu,\sigma\lambda}, C^{\mu,\nu\sigma}, C^{\mu,\nu} \in \mathbb{C}$. The game is to show that there exist other choices of generators,

$$\tilde{\mathsf{J}}^{\mu\nu} = \mathsf{J}^{\mu\nu} + D^{\mu\nu}\mathsf{Z}, \tilde{\mathsf{P}}^\mu = \mathsf{P}^\mu + D^\mu\mathsf{Z} \in \mathfrak{e}, \quad D^{\mu\nu} = -D^{\nu\mu}, D^\mu \in \mathbb{C}, \tag{2.72}$$

each differing from $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$ by a multiple of the central charge Z, such that, when rewritten in terms of these new generators, eq. (2.71) becomes the usual Poincaré algebra,

$$i[\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{J}}^{\sigma\lambda}] = \eta^{\mu\lambda}\tilde{\mathsf{J}}^{\nu\sigma} - \eta^{\mu\sigma}\tilde{\mathsf{J}}^{\nu\lambda} + \eta^{\nu\sigma}\tilde{\mathsf{J}}^{\mu\lambda} - \eta^{\nu\lambda}\tilde{\mathsf{J}}^{\mu\sigma} \tag{2.73}$$

$$i[\tilde{\mathsf{P}}^{\mu}, \tilde{\mathsf{J}}^{\nu\sigma}] = \eta^{\mu\nu}\tilde{\mathsf{P}}^{\sigma} - \eta^{\mu\sigma}\tilde{\mathsf{P}}^{\nu} \tag{2.74}$$

$$i[\tilde{\mathsf{P}}^{\mu}, \tilde{\mathsf{P}}^{\nu}] = 0. \tag{2.75}$$

But note that the commutators above do not change when we replace $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$ with $\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{P}}^\mu$:

$$[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] = [\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{J}}^{\sigma\lambda}], \quad [\mathsf{P}^{\mu}, \mathsf{J}^{\nu\sigma}] = [\tilde{\mathsf{P}}^{\mu}, \tilde{\mathsf{J}}^{\nu\sigma}], \quad [\mathsf{P}^{\mu}, \mathsf{P}^{\nu}] = [\tilde{\mathsf{P}}^{\mu}, \tilde{\mathsf{P}}^{\nu}]. \tag{2.76}$$

So, what we want to do is choose the coefficients $D^{\mu\nu}, D^\mu$ so that, upon rewriting the right-hand side of eq. (2.71) in terms of $\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{P}}^\mu$, the effect is to absorb the terms involving the central charge Z.

- ($C^{\mu,\nu} = 0$.) The most dangerous terms in eq. (2.71) are the $C^{\mu,\nu}$, because there are no terms on the right-hand side of $i[\mathsf{P}^\mu, \mathsf{P}^\nu] = C^{\mu,\nu}\mathsf{Z}$ besides the central charge. If any of these were nonzero, we would not be able to eliminate them by redefining $\mathsf{J}^{\mu\nu}, \mathsf{P}^\mu$. Fortunately, $C^{\mu,\nu}$ must be zero, so eq. (2.75) holds automatically. In order to prove this, consider the Jacobi identity

$$[\mathsf{J}^{\mu\nu}, [\mathsf{P}^{\sigma}, \mathsf{P}^{\lambda}]] - [\mathsf{P}^{\lambda}, [\mathsf{P}^{\sigma}, \mathsf{J}^{\mu\nu}]] + [\mathsf{P}^{\sigma}, [\mathsf{P}^{\lambda}, \mathsf{J}^{\mu\nu}]] = 0. \tag{2.77}$$

  The terms here, as computed using eq. (2.71), are $[\mathsf{J}^{\mu\nu}, [\mathsf{P}^\sigma, \mathsf{P}^\lambda]] = 0$ and

$$\begin{aligned}
[\mathsf{P}^{\lambda}, [\mathsf{P}^{\sigma}, \mathsf{J}^{\mu\nu}]] &= i[\mathsf{P}^{\lambda}, \eta^{\sigma\nu}\mathsf{P}^{\mu} - \eta^{\sigma\mu}\mathsf{P}^{\nu}] = (C^{\lambda,\mu}\eta^{\sigma\nu} - C^{\lambda,\nu}\eta^{\sigma\mu})\mathsf{Z}, \\
[\mathsf{P}^{\sigma}, [\mathsf{P}^{\lambda}, \mathsf{J}^{\mu\nu}]] &= i[\mathsf{P}^{\sigma}, \eta^{\lambda\nu}\mathsf{P}^{\mu} - \eta^{\lambda\mu}\mathsf{P}^{\nu}] = (C^{\sigma,\mu}\eta^{\lambda\nu} - C^{\sigma,\nu}\eta^{\lambda\mu})\mathsf{Z}.
\end{aligned} \tag{2.78}$$

  So, the Jacobi identity eq. (2.77) says

$$C^{\lambda,\mu}\eta^{\sigma\nu} - C^{\lambda,\nu}\eta^{\sigma\mu} = C^{\sigma,\mu}\eta^{\lambda\nu} - C^{\sigma,\nu}\eta^{\lambda\mu}. \tag{2.79}$$

  Contracting with $\eta_{\mu\sigma}$ yields $(d-1)C^{\lambda,\nu} = 0$, so $C^{\lambda,\nu} = 0$. (This is the only place in the argument where we use that $d \geq 2$.)
- (Eliminating $C^{\mu,\nu\sigma}$.) Next, we check that, by defining $\tilde{\mathsf{P}}^\mu = \mathsf{P}^\mu + D^\mu\mathsf{Z}$ for appropriate D^μ, we can arrange eq. (2.74). However, we only have $1+d$ different D^μ's, and many more $C^{\mu,\nu\sigma}$'s, so it had better be the case that the $C^{\mu,\nu\sigma}$'s are severely restricted by the Jacobi identity. We win if (and only if, as reversing the reasoning shows) the Jacobi identity implies that $C^{\mu,\nu\sigma}$ has the form

$$C^{\mu,\nu\sigma} = \eta^{\mu\nu}\xi^{\sigma} - \eta^{\mu\sigma}\xi^{\nu} \tag{2.80}$$

  for some $\xi^\bullet \in \mathbb{C}$. Indeed, we can then define $D^\mu = \xi^\mu$, and this has the desired effect:

$$\begin{aligned}
\eta^{\mu\nu}\mathsf{P}^{\sigma} - \eta^{\mu\sigma}\mathsf{P}^{\nu} + C^{\mu,\nu\sigma}\mathsf{Z} &= \eta^{\mu\nu}\mathsf{P}^{\sigma} - \eta^{\mu\sigma}\mathsf{P}^{\nu} + (\eta^{\mu\nu}\xi^{\sigma} - \eta^{\mu\sigma}\xi^{\nu})\mathsf{Z} \\
&= \eta^{\mu\nu}\tilde{\mathsf{P}}^{\sigma} - \eta^{\mu\sigma}\tilde{\mathsf{P}}^{\nu}.
\end{aligned} \tag{2.81}$$

  To prove eq. (2.80), consider the Jacobi identity

$$[\mathsf{J}^{\mu\nu}, [\mathsf{P}^{\sigma}, \mathsf{J}^{\lambda\rho}]] + [\mathsf{P}^{\sigma}, [\mathsf{J}^{\lambda\rho}, \mathsf{J}^{\mu\nu}]] - [\mathsf{J}^{\lambda\rho}, [\mathsf{P}^{\sigma}, \mathsf{J}^{\mu\nu}]] = 0. \tag{2.82}$$

The terms here are given by

$$\begin{aligned}
[\mathsf{J}^{\mu\nu},[\mathsf{P}^\sigma,\mathsf{J}^{\lambda\rho}]] &= (\eta^{\sigma\rho}\eta^{\lambda\nu}-\eta^{\sigma\lambda}\eta^{\rho\nu})\mathsf{P}^\mu+(\eta^{\sigma\lambda}\eta^{\rho\mu}-\eta^{\sigma\rho}\eta^{\lambda\mu})\mathsf{P}^\nu \\
&\quad+(\eta^{\sigma\lambda}C^{\rho,\mu\nu}-\eta^{\sigma\rho}C^{\lambda,\mu\nu})\mathsf{Z}, \\
[\mathsf{J}^{\lambda\rho},[\mathsf{P}^\sigma,\mathsf{J}^{\mu\nu}]] &= (\eta^{\sigma\nu}\eta^{\mu\rho}-\eta^{\sigma\mu}\eta^{\nu\rho})\mathsf{P}^\lambda+(\eta^{\sigma\mu}\eta^{\nu\lambda}-\eta^{\sigma\nu}\eta^{\mu\lambda})\mathsf{P}^\rho \\
&\quad+(\eta^{\sigma\mu}C^{\nu,\lambda\rho}-\eta^{\sigma\nu}C^{\mu,\lambda\rho})\mathsf{Z}, \\
[\mathsf{P}^\sigma,[\mathsf{J}^{\lambda\rho},\mathsf{J}^{\mu\nu}]] &= (\eta^{\rho\nu}\eta^{\sigma\lambda}-\eta^{\lambda\nu}\eta^{\sigma\rho})\mathsf{P}^\mu+(\eta^{\lambda\mu}\eta^{\sigma\rho}-\eta^{\rho\mu}\eta^{\sigma\lambda})\mathsf{P}^\nu \\
&\quad+(\eta^{\lambda\nu}\eta^{\sigma\mu}-\eta^{\lambda\mu}\eta^{\sigma\nu})\mathsf{P}^\rho+(\eta^{\rho\mu}\eta^{\sigma\nu}-\eta^{\rho\nu}\eta^{\sigma\mu})\mathsf{P}^\lambda \\
&\quad+(\eta^{\lambda\mu}C^{\sigma,\rho\nu}-\eta^{\lambda\nu}C^{\sigma,\rho\mu}+\eta^{\rho\nu}C^{\sigma,\lambda\mu}-\eta^{\rho\mu}C^{\sigma,\lambda\nu})\mathsf{Z},
\end{aligned} \tag{2.83}$$

so the Jacobi identity eq. (2.82) reads

$$\begin{aligned}
&\eta^{\sigma\lambda}C^{\rho,\mu\nu}-\eta^{\sigma\rho}C^{\lambda,\mu\nu}-\eta^{\sigma\mu}C^{\nu,\lambda\rho}+\eta^{\sigma\nu}C^{\mu,\lambda\rho} \\
&\qquad+\eta^{\lambda\mu}C^{\sigma,\rho\nu}-\eta^{\lambda\nu}C^{\sigma,\rho\mu}+\eta^{\rho\nu}C^{\sigma,\lambda\mu}-\eta^{\rho\mu}C^{\sigma,\lambda\nu}=0.
\end{aligned} \tag{2.84}$$

Contracting with $\eta_{\mu\sigma}$ yields $dC^{\nu,\lambda\rho}-\eta^{\lambda\nu}C^{\sigma,\rho}{}_\sigma+\eta^{\rho\nu}C^{\sigma,\lambda}{}_\sigma=0$. (Here we are using standard conventions regarding raised and lowered indices. A review is in §??.) Rearranging and renaming dummy variables ($\nu\rightsquigarrow\mu$, $\lambda\rightsquigarrow\nu$, $\sigma\rightsquigarrow\lambda$, $\rho\rightsquigarrow\sigma$), this last equation becomes

$$C^{\mu,\nu\sigma}=d^{-1}(\eta^{\mu\nu}C^{\lambda,\sigma}{}_\lambda-\eta^{\mu\sigma}C^{\lambda,\nu}{}_\lambda) \tag{2.85}$$

This says that eq. (2.80) holds for $\xi^\sigma=d^{-1}C^{\lambda,\sigma}{}_\lambda$.

- (Eliminating $C^{\mu\nu,\sigma\lambda}=0$.) This is equivalent to showing that the cohomology $\mathrm{H}^2(\mathfrak{o}(1+d))$ of the Lorentz Lie algebra is trivial. We have separated this as its own lemma, Lemma 2.19.

□

Lemma 2.19. For any $d\geq 1$, $\mathrm{H}^2(\mathfrak{o}(1+d))=\{0\}$. ■

Remark 2.20. *Whitehead's second lemma* [Jac79] says that $\mathrm{H}^2(\mathfrak{g})$ is trivial whenever $\mathfrak{g}$ is a semisimple Lie algebra. Since the Lorentz Lie algebra $\mathfrak{o}(D)$ is semisimple for $D\geq 3$ (in fact simple, with the one exception $D=4$), when $d\geq 2$, Lemma 2.19 is a special case of this. The Lie algebra $\mathfrak{o}(2)$ relevant to the $d=1$ case is abelian and therefore not considered semisimple, but this case is trivial regardless.

Note that $\mathfrak{p}(1,d)$ is *not* semisimple, because of the abelian subalgebra of translations. So, Whitehead's lemma does not apply, and indeed we saw that $\mathfrak{p}(1,1)$ has nontrivial H^2. ■

Proof of Lemma 2.19. The only two-dimensional Lie algebra with nontrivial center is the abelian one, hence a trivial central extension of $\mathfrak{o}(2)\cong\mathbb{R}$. This gives the $d=1$ case of the lemma. For the rest of the proof, suppose $d\geq 2$.

Let $\mathbb{R}\hookrightarrow\mathfrak{e}\twoheadrightarrow\mathfrak{o}(1+d)$ denote a central extension of $\mathfrak{o}(1+d)$, $\mathsf{Z}\in\mathfrak{e}$ denote a central charge, and $\mathsf{J}^{\mu\nu}$ denote arbitrarily chosen elements of the preimages of the identically named generators of $\mathfrak{o}(1+d)$. The Lie bracket of $\mathfrak{e}$ then takes the form

$$i[\mathsf{J}^{\mu\nu},\mathsf{J}^{\sigma\lambda}]=\eta^{\mu\lambda}\mathsf{J}^{\nu\sigma}-\eta^{\mu\sigma}\mathsf{J}^{\nu\lambda}+\eta^{\nu\sigma}\mathsf{J}^{\mu\lambda}-\eta^{\nu\lambda}\mathsf{J}^{\mu\sigma}+C^{\mu\nu,\sigma\lambda}\mathsf{Z}, \tag{2.86}$$

for some structure constants $C^{\mu\nu,\sigma\lambda}$. The game is to show that we can choose $D^{\mu\nu}=D^{-\nu\mu}\in\mathbb{C}$ such that, if we define $\tilde{\mathsf{J}}^{\mu\nu}=\mathsf{J}^{\mu\nu}+D^{\mu\nu}\mathsf{Z}$, then eq. (2.86) can be written

$$i[\tilde{\mathsf{J}}^{\mu\nu},\tilde{\mathsf{J}}^{\sigma\lambda}]=\eta^{\mu\lambda}\tilde{\mathsf{J}}^{\nu\sigma}-\eta^{\mu\sigma}\tilde{\mathsf{J}}^{\nu\lambda}+\eta^{\nu\sigma}\tilde{\mathsf{J}}^{\mu\lambda}-\eta^{\nu\lambda}\tilde{\mathsf{J}}^{\mu\sigma}, \tag{2.87}$$

which says that $\tilde{\mathsf{J}}^{\mu\nu}$ satisfy the usual Lorentz algebra. However, we only have $d(d-1)/2$ different $D^{\mu\nu}$'s, and many more $C^{\mu\nu,\sigma\lambda}$'s that we want to eliminate, so it had better be the case that the latter are severely restricted. By $\mathsf{J}^{\mu\nu}=-\mathsf{J}^{\nu\mu}$ and the anti-symmetry of $\mathfrak{e}$'s Lie bracket, $C^{\mu\nu,\sigma\lambda}$

switches sign under interchanging the two Lorentz indices on either side of the comma, or under interchanging the two sides of the comma.

Owing to the centrality of Z, we have $[\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}] = [\tilde{\mathsf{J}}^{\mu\nu}, \tilde{\mathsf{J}}^{\sigma\lambda}]$, so what we want to do is choose $D^{\mu\nu}$ such that the right-hand sides of eq. (2.86), eq. (2.87) agree. This means

$$C^{\mu\nu,\sigma\lambda} = \eta^{\mu\lambda} D^{\nu\sigma} - \eta^{\mu\sigma} D^{\nu\lambda} + \eta^{\nu\sigma} D^{\mu\lambda} - \eta^{\nu\lambda} D^{\mu\sigma}. \tag{2.88}$$

So, we win if the $C^{\mu\nu,\sigma\lambda}$ can be shown to have this form.

The restriction comes from the Jacobi identity

$$[\mathsf{J}^{\mu\nu}, [\mathsf{J}^{\sigma\lambda}, \mathsf{J}^{\rho\tau}]] + [\mathsf{J}^{\sigma\lambda}, [\mathsf{J}^{\rho\tau}, \mathsf{J}^{\mu\nu}]] + [\mathsf{J}^{\rho\tau}, [\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}]] = 0. \tag{2.89}$$

Note that this is $\mathfrak{e}$'s Lie bracket. Let $\mathscr{J} = \operatorname{span}_{\mathbb{C}}\{\mathsf{J}^{\mu\nu} : 0 \leq \mu < \nu \leq d\} \subseteq \mathfrak{e}$ denote the span of the $\mathsf{J}^{\mu\nu}$'s. Define $J_{\mathrm{I}}, J_{\mathrm{II}}, J_{\mathrm{III}} \in \mathscr{J}$ and $\xi_{\mathrm{I}}, \xi_{\mathrm{II}}, \xi_{\mathrm{III}} \in \mathbb{C}$ by

$$\begin{aligned} [\mathsf{J}^{\mu\nu}, [\mathsf{J}^{\sigma\lambda}, \mathsf{J}^{\rho\tau}]] &= J_{\mathrm{I}} + \xi_{\mathrm{I}}\mathsf{Z} \\ [\mathsf{J}^{\sigma\lambda}, [\mathsf{J}^{\rho\tau}, \mathsf{J}^{\mu\nu}]] &= J_{\mathrm{II}} + \xi_{\mathrm{II}}\mathsf{Z} \\ [\mathsf{J}^{\rho\tau}, [\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}]] &= J_{\mathrm{III}} + \xi_{\mathrm{III}}\mathsf{Z}. \end{aligned} \tag{2.90}$$

The Jacobi identity for $\mathfrak{so}(1+d)$ guarantees that $J_{\mathrm{I}} + J_{\mathrm{II}} + J_{\mathrm{III}} = 0$, so eq. (2.89) (which was the Jacobi identity for $\mathfrak{e}$) says $\xi_{\mathrm{I}} + \xi_{\mathrm{II}} + \xi_{\mathrm{III}} = 0$. Let us compute what these ξ's are:

$$\begin{aligned} [\mathsf{J}^{\mu\nu}, [\mathsf{J}^{\sigma\lambda}, \mathsf{J}^{\rho\tau}]] &= -i[\mathsf{J}^{\mu\nu}, \eta^{\rho\lambda}\mathsf{J}^{\tau\sigma} - \eta^{\rho\sigma}\mathsf{J}^{\tau\lambda} + \eta^{\tau\sigma}\mathsf{J}^{\rho\lambda} - \eta^{\tau\lambda}\mathsf{J}^{\rho\sigma}] \\ &= -(\eta^{\rho\lambda}C^{\mu\nu,\tau\sigma} - \eta^{\rho\sigma}C^{\mu\nu,\tau\lambda} + \eta^{\tau\sigma}C^{\mu\nu,\rho\lambda} - \eta^{\tau\lambda}C^{\mu\nu,\rho\sigma})\mathsf{Z} \bmod \mathscr{J}, \\ [\mathsf{J}^{\sigma\lambda}, [\mathsf{J}^{\rho\tau}, \mathsf{J}^{\mu\nu}]] &= -i[\mathsf{J}^{\sigma\lambda}, \eta^{\mu\tau}\mathsf{J}^{\nu\rho} - \eta^{\mu\rho}\mathsf{J}^{\nu\tau} + \eta^{\nu\rho}\mathsf{J}^{\mu\tau} - \eta^{\nu\tau}\mathsf{J}^{\mu\rho}] \\ &= -(\eta^{\mu\tau}C^{\sigma\lambda,\nu\rho} - \eta^{\mu\rho}C^{\sigma\lambda,\nu\tau} + \eta^{\nu\rho}C^{\sigma\lambda,\mu\tau} - \eta^{\nu\tau}C^{\sigma\lambda,\mu\rho})\mathsf{Z} \bmod \mathscr{J}, \\ [\mathsf{J}^{\rho\tau}, [\mathsf{J}^{\mu\nu}, \mathsf{J}^{\sigma\lambda}]] &= -i[\mathsf{J}^{\rho\tau}, \eta^{\sigma\nu}\mathsf{J}^{\lambda\mu} - \eta^{\sigma\mu}\mathsf{J}^{\lambda\nu} + \eta^{\lambda\mu}\mathsf{J}^{\sigma\nu} - \eta^{\lambda\nu}\mathsf{J}^{\sigma\mu}] \\ &= -(\eta^{\sigma\nu}C^{\rho\tau,\lambda\mu} - \eta^{\sigma\mu}C^{\rho\tau,\lambda\nu} + \eta^{\lambda\mu}C^{\rho\tau,\sigma\nu} - \eta^{\lambda\nu}C^{\rho\tau,\sigma\mu})\mathsf{Z} \bmod \mathscr{J}. \end{aligned} \tag{2.91}$$

That is,

$$\begin{aligned} -\xi_{\mathrm{I}} &= \eta^{\rho\lambda}C^{\mu\nu,\tau\sigma} - \eta^{\rho\sigma}C^{\mu\nu,\tau\lambda} + \eta^{\tau\sigma}C^{\mu\nu,\rho\lambda} - \eta^{\tau\lambda}C^{\mu\nu,\rho\sigma}, \\ -\xi_{\mathrm{II}} &= \eta^{\mu\tau}C^{\sigma\lambda,\nu\rho} - \eta^{\mu\rho}C^{\sigma\lambda,\nu\tau} + \eta^{\nu\rho}C^{\sigma\lambda,\mu\tau} - \eta^{\nu\tau}C^{\sigma\lambda,\mu\rho}, \\ -\xi_{\mathrm{III}} &= \eta^{\sigma\nu}C^{\rho\tau,\lambda\mu} - \eta^{\sigma\mu}C^{\rho\tau,\lambda\nu} + \eta^{\lambda\mu}C^{\rho\tau,\sigma\nu} - \eta^{\lambda\nu}C^{\rho\tau,\sigma\mu}. \end{aligned} \tag{2.92}$$

So, the Jacobi identity $\xi_{\mathrm{I}} + \xi_{\mathrm{II}} + \xi_{\mathrm{III}} = 0$ says

$$\begin{aligned} &\eta^{\rho\lambda}C^{\mu\nu,\tau\sigma} - \eta^{\rho\sigma}C^{\mu\nu,\tau\lambda} + \eta^{\tau\sigma}C^{\mu\nu,\rho\lambda} - \eta^{\tau\lambda}C^{\mu\nu,\rho\sigma} + \eta^{\mu\tau}C^{\sigma\lambda,\nu\rho} - \eta^{\mu\rho}C^{\sigma\lambda,\nu\tau} + \eta^{\nu\rho}C^{\sigma\lambda,\mu\tau} - \eta^{\nu\tau}C^{\sigma\lambda,\mu\rho} \\ &\qquad + \eta^{\sigma\nu}C^{\rho\tau,\lambda\mu} - \eta^{\sigma\mu}C^{\rho\tau,\lambda\nu} + \eta^{\lambda\mu}C^{\rho\tau,\sigma\nu} - \eta^{\lambda\nu}C^{\rho\tau,\sigma\mu} = 0. \end{aligned} \tag{2.93}$$

Contracting with $\eta_{\mu\sigma}$ gives

$$-(d-1)C^{\rho\tau,\lambda\nu} = \eta^{\tau\lambda}C_{\sigma}{}^{\nu,\rho\sigma} + \eta^{\nu\tau}C_{\sigma}{}^{\lambda,\sigma\rho} - \eta^{\rho\lambda}C_{\sigma}{}^{\nu,\tau\sigma} - \eta^{\nu\rho}C_{\sigma}{}^{\lambda,\sigma\tau}, \tag{2.94}$$

having used the various anti-symmetries of $C^{\bullet}$ under interchanging various Lorentz indices. Renaming dummy variables ($\rho \rightsquigarrow \mu$, $\tau \rightsquigarrow \nu$, $\lambda \rightsquigarrow \sigma$, $\nu \rightsquigarrow \lambda$, $\sigma \rightsquigarrow \rho$) and rearranging, we end up with

$$(d-1)C^{\mu\nu,\sigma\lambda} = \eta^{\mu\lambda}C_{\rho}{}^{\sigma,\rho\nu} - \eta^{\mu\sigma}C_{\rho}{}^{\lambda,\rho\nu} + \eta^{\nu\sigma}C_{\rho}{}^{\lambda,\rho\mu} - \eta^{\nu\lambda}C_{\rho}{}^{\sigma,\rho\mu}. \tag{2.95}$$

So, eq. (2.88) holds with $D^{\nu\sigma} = (d-1)^{-1}C_{\rho}{}^{\sigma,\rho\nu} = -(d-1)^{-1}C_{\rho}{}^{\nu,\rho\sigma}$. □

Exercises and problems

Exercise 2.1: Show that the only one-dimensional continuous representation of $\mathrm{SU}(2)$ is the trivial one.

Hint: see the proof of ??.

Exercise 2.2: Show that $\mathrm{SO}(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$.

Exercise 2.3: Let $\mathfrak{t} = \mathbb{R}^2$ denote the abelian two-dimensional Lie algebra. Show that $\mathrm{H}^2(\mathfrak{t};\mathbb{R}) \cong \mathbb{R}$.

Exercise 2.4: Prove eq. (2.65) ($\mathrm{H}^2(\mathfrak{p}_{\mathbb{C}};\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}^2(\mathfrak{p};\mathbb{R})$).

Problem 2.1: Prove the claim in Remark 2.2: there does not exist a continuous map $\mu : \mathrm{SO}(3) \to \mathrm{U}(2)$ such that, for all $U \in \mathrm{SU}(2)$, $\mu(\pi(U))$ differs from U by a phase.

Hint: show that such an existence would imply a homeomorphism $\mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{SO}(3)$. This contradicts

$$\begin{aligned} \pi_1(\mathrm{U}(2)) &\cong \mathbb{Z} \\ \pi_1(\mathrm{U}(1) \times \mathrm{SO}(3)) &\cong \pi_1(\mathrm{U}(1)) \times \pi(\mathrm{SO}(3)) \\ &\cong \mathbb{Z} \times \mathbb{Z}_2, \end{aligned}$$

since $\mathbb{Z}_2 \ncong \mathbb{Z} \times \mathbb{Z}_2$.

Problem 2.2: This problem discusses a few isomorphisms similar to $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C})$ that hold for other numbers of spatial dimensions besides the physical $d = 3$ case.

(a) Show that $\mathrm{Spin}(1,2) \cong \mathrm{SL}(2,\mathbb{R})$.

(b) Show that $\mathrm{Spin}(1,4)$ is isomorphic to the group

$$\mathrm{Sp}(1,1) = \left\{ A \in M_2(\mathbb{H}) : A^\dagger \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{2.96}$$

consisting of 2-by-2 quaternionic "Lorentz matrices."

(c) (Optional.) Show that the group $\mathrm{Spin}(1,5)$ is isomorphic to the group

$$\mathrm{SL}(2,\mathbb{H}) = \{ M \in M_2(\mathbb{H}) : \det\nolimits_{\mathrm{D}}(M) = 1 \} \tag{2.97}$$

of 2-by-2 quaternionic unimodular "matrices." Note: the correct notion of determinant for quaternionic matrices is the Dieudonné determinant, $\det_{\mathrm{D}}(M)$. This is the same thing as $\sqrt{\det(M_{\mathbb{C}})}$, where $M_{\mathbb{C}}$ is the 4-by-4 complex matrix representing M.

(d) (Optional.) Combining the results of (a), (c), the following pattern appears: for the three division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the isomorphism $\mathrm{Spin}(1, 1+\dim\mathbb{K}) \cong \mathrm{SL}(2,\mathbb{K})$ holds, where $\dim\mathbb{K} \in \{1,2,4\}$ is the dimension of $\mathbb{K}$ as a real vector space. What does this suggest about $\mathrm{Spin}(1,9)$?

Problem 2.3: This problem continues Example 2.6.

(a) Consider the following elements of $\mathfrak{h}_{\mathbb{C}}$:

$$\mathsf{x} = -i \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathsf{p} = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathsf{Z} = -i \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.98}$$

Show that these satisfy the "canonical commutation relation"

$$i[\mathsf{x},\mathsf{p}] = \mathsf{Z}. \tag{2.99}$$

(b) Check that the Schrödinger representation is in fact a representation of the Heisenberg group.

(c) Show that in the Schrödinger representation,

$$i[\mathsf{x}, \mathsf{p}] = \hbar. \tag{2.100}$$

Why does this imply that the projective representation $\rho_\hbar : (\mathbb{R}^2, +) \to \mathrm{PU}(\mathcal{H})$ cannot be lifted to a unitary representation of $(\mathbb{R}^2, +)$?

Problem 2.4: (a) Formulate a category relQM of relativistic quantum mechanical systems, and show that, if $d \geq 2$, then this category is equivalent to the category of (continuous) unitary representations of $\mathrm{P}^*(1, d)$.

(b) (Optional.) Formulate and prove a similar statement for $d = 1$.

Problem 2.5: The *celestial sphere* $\mathcal{C}\mathbb{S}^2 = \{\Gamma_{\mathbf{y}} : \mathbf{y} \in \mathbb{S}^2\}$ is the set of null lines $\Gamma_{\mathbf{y}} = \{(t, t\mathbf{y}) : t \in \mathbb{R}\}$ in $\mathbb{R}^{1,3}$. This is naturally given a smooth manifold structure, identifying it with $\mathbb{S}^2$ via the map $\Gamma_{\mathbf{y}} \mapsto \mathbf{y}$.

(1) Show that any restricted Lorentz transformation induces a conformal transformation of the celestial sphere.

Hint: $\mathbb{S}^2 \cong \mathbb{C}P^1$, with conformal transformations being those on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ induced by Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \quad a, b, c, d \in \mathbb{C} \text{ s.t. } ad - bc \neq 0 \tag{2.101}$$

on $\mathbb{C}^2$.

(2) (Optional.) Show that this yields a surjective map $\mathrm{SO}(1, 3) \to \mathrm{PSL}(2, \mathbb{C})$.

APPENDIX A

Notation

Matrices

- σ_1, σ_2, σ_3: the Pauli σ-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

- $\mathscr{P}, \mathscr{T}$: the parity and time-reversal matrices, $\in \mathrm{O}(1,d)$.

$$\mathscr{T} = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix}.$$

- $\mathscr{C} = \mathscr{P}\mathscr{T}$.

Groups.

- $\mathrm{O}(1,d)$: The full Lorentz group

$$\mathrm{O}(1,d) = \left\{ \Lambda \in \mathbb{R}^{(1+d)\times(1+d)} : \Lambda^\intercal \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix} \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & I_d \end{bmatrix} \right\},$$

  including parity and time-reversal.
- $\mathrm{SO}(1,d)$: The identity component of the Lorentz group (restricted Lorentz group).
- $\mathrm{P}(1,d)$: The restricted Poincaré group in $1+d$ spacetime dimensions;

$$\mathrm{P}(1,d) \cong \mathbb{R}^{1,d} \rtimes \mathrm{SO}(1,d)$$

  naturally.
- $\mathrm{P}_{\text{full}}(1,d)$: The full Poincaré group, including parity and time-reversal.

Spacetime and Geometry.

- $x = (t, \mathbf{x})$: A spacetime coordinate in Minkowski spacetime $\mathbb{R}^{1,d}$.
- z^2: the squared Minkowski norm of a vector z (mostly plus convention), $z^2 = -(z^0)^2 + \|\mathbf{z}\|^2$.
- η: The Minkowski metric matrix $\mathrm{diag}(-1, 1, \ldots, 1)$.
- $\gamma = (1 - \|\mathbf{v}\|^2)^{-1/2}$: The Lorentz factor associated with velocity $\mathbf{v}$.

Bibliography

[Bae20] J. C. Baez. "The tenfold way". *Notices Amer. Math. Soc.* **67**.10 (2020), 1599–1601 (cit. on p. 16).

[Bar54] V. Bargmann. "On unitary ray representations of continuous groups". *Ann. of Math. (2)* **59** (1954), 1–46. DOI: 10.2307/1969831 (cit. on pp. 27, 28).

[CE48] C. Chevalley and S. Eilenberg. "Cohomology theory of Lie groups and Lie algebras". *Trans. Amer. Math. Soc.* **63** (1948), 85–124. DOI: 10.2307/1990637 (cit. on p. 30).

[Col85] S. Coleman. "The uses of instantons". *Aspects of Symmetry: Selected Erice Lectures*. Cambridge University Press, 1985, 265–350 (cit. on p. 23).

[Ear08] J. Earman. "Superselection rules for philosophers". *Erkenntnis* **69**.3 (2008), 377–414. DOI: 10.1007/s10670-008-9124-z (cit. on p. 22).

[Ein05] A. Einstein. "On the Electrodynamics of Moving Bodies" (1905) (cit. on p. 3).

[Haa63] R. Haag. "Bemerkungen zum Nahwirkungsprinzip in der Quantenphysik". **466** (1963), 29–34. DOI: 10.1002/andp.19634660107 (cit. on p. 22).

[Jac79] N. Jacobson. *Lie algebras*. Republication of the 1962 original. Dover Publications, Inc., New York, 1979 (cit. on p. 40).

[Sim71] D. J. Simms. "A short proof of Bargmann's criterion for the lifting of projective representations of Lie groups". *Rep. Mathematical Phys.* **2**.4 (1971), 283–287. DOI: 10.1016/0034-4877(71)90011-5 (cit. on p. 32).

[SW00] R. F. Streater and A. S. Wightman. *PCT, spin and statistics, and all that*. Princeton Landmarks in Physics. Corrected third printing of the 1978 edition. 2000 (cit. on p. 22).

[SW74] F. Strocchi and A. S. Wightman. "Proof of the charge superselection rule in local relativistic quantum field theory". *J. Mathematical Phys.* 15 (1974), 2198–2224. DOI: 10.1063/1.1666601 (cit. on p. 22).

[Tao14] T. Tao. *Hilbert's fifth problem and related topics*. Vol. **153**. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2014. DOI: 10.1090/gsm/153 (cit. on p. 32).

[Wei05] S. Weinberg. *The quantum theory of fields. Vol. **I***. Foundations. Cambridge University Press, Cambridge, 2005 (cit. on pp. 3, 10, 19, 38).

[Wig39] E. Wigner. "On unitary representations of the inhomogeneous Lorentz group". *Ann. of Math. (2)* **40**.1 (1939), 149–204. DOI: 10.2307/1968551 (cit. on p. 27).

[Wig59] E. P. Wigner. *Group theory and its application to the quantum mechanics of atomic spectra*. Vol. **5**. Pure and Applied Physics. Expanded and improved ed. Translated from the German by J. J. Griffin. Academic Press, New York-London, 1959 (cit. on pp. 10, 16).

[WWW52] G. C. Wick, A. S. Wightman, and E. P. Wigner. "The intrinsic parity of elementary particles". *Phys. Rev. (2)* **88** (1952), 101–105 (cit. on p. 22).

[nLab25] nLab. *Ten-fold way*. https://ncatlab.org/nlab/show/ten-fold+way. 2025 (cit. on p. 15).