COMPLETE ASYMPTOTIC ANALYSIS OF LOW ENERGY SCATTERING FOR SCHRÖDINGER OPERATORS WITH A SHORT-RANGE POTENTIAL

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ABSTRACT. Recent work by Hintz–Vasy provides a partial asymptotic analysis of the low-energy limit of scattering for Schrödinger operators with a short-range potential. Using a slight refinement of Hintz's algorithm, we complete the asymptotic analysis by providing full asymptotic expansions in every possible asymptotic regime. Moreover, the analysis is done in any dimension $d \geq 3$, for any asymptotically conic manifold, and we keep track of partial multipole expansions.

CONTENTS

1. INTRODUCTION

In this paper, we consider low-energy scattering theory on asymptotically conic manifolds X, including the ur-example of Euclidean space.[1] Since the work of Jensen–Kato [JK79] on the Euclidean case, results in this vein have been used to study wave propagation in both the relativistic (i.e. the wave and Klein–Gordon equations) and non-relativistic (i.e. the Schrödinger equation) settings. See e.g. [GS04][ES04; ES06][GG15] for a small sample of the research literature regarding the Euclidean case. More recently, Hintz [Hin22], building on [Vas21a; Vas21b; Vas21c], has used these methods to prove Price's law without symmetry assumptions. *Price's law* describes the asymptotic tail of radiation on black hole spacetimes and is therefore significant in the mathematical analysis of general relativity. Because the spatial slices of black hole spacetimes are only asymptotically Euclidean and not exactly Euclidean, Hintz's work went beyond the earlier Euclidean analysis.

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[1] We are using the term "asymptotically conic manifold" in the sense of Melrose [Mel95]. For the reader not familiar with such terminological conventions, let us emphasize that X is a compact manifold-*with-boundary* Indeed, any compact manifold-with-boundary can be endowed with a metric that makes it asymptotically conic. When we speak of "Euclidean space" in this paper, what we really mean is the radial compactification $X = \mathbb{R}^d \sqcup \infty\mathbb{S}^{d-1}$ that compactifies $\mathbb{R}^d$ by attaching one point "at infinity" for each ray in $\mathbb{R}^d$ emanating from the origin, in such a way so that $1/\langle r \rangle$ becomes a boundary-defining-function of $\infty\mathbb{S}^{d-1} = \partial X$. This is equivalent to stereographic compactification.

Other works on the asymptotically Euclidean case include [BH10][GH08; GH09; GHS13][Bou11a; Bou11b][VW13][RT15][SW20].

The crux of Hintz's treatment (note that Hintz considers only the case $d = \dim X = 3$, but a similar analysis applies for $d \geq 3$) is the proof that, given Schwartz forcing $f \in \mathcal{S}(X)$, the resolvent output $R(E \pm i0)f$ admits a partial asymptotic expansion on some particular manifold-with-corners X^+_{res} compactifying the low-energy limit:

$$X^+_{\mathrm{res}} \hookleftarrow \mathbb{R}^+_E \times X, \qquad X^+_{\mathrm{res}} \cap \mathrm{cl}_{X^+_{\mathrm{res}}}\{E < E_0\} \Subset X^+_{\mathrm{res}} \text{ for all } E_0 > 0. \tag{1}$$

Specifically, $X^+_{\mathrm{res}} = [[0,\infty)_\sigma \times X; \{0\} \times \partial X]$ is the result of blowing up the corner of $[0,\infty)_\sigma \times X$, where σ is related to the energy E by $E = \sigma^2$. This manifold-with-corners has three faces, bf, tf, and zf — see Figure 1. The fact that Hintz's expansions are only *partial* means that, when using these methods to analyze wave propagation, one only gets partial asymptotic expansions. In fact, Hintz's analysis only gives the leading-order term in the large-time limit. Our goal here is to provide *full* asymptotic expansions, in all possible asymptotic regimes (excluding the "high energy" $E \to \infty$ limit, which is already understood), of the output of the resolvent. We do not provide applications here; the Schrödinger equation in $(1+3)$-dimensions is handled in a companion paper [LS]. The Klein–Gordon and wave equations should be treatable similarly, but we have no intentions of doing so.

Our main theorem is:

Theorem A. *Let X be an asymptotically conic manifold and consider an admissible Schrödinger operator $P(0)$ on it, as described in §2, including the (generically satisfied) assumption that $P(0)$ lack a resonance at zero energy: $\ker_{\mathcal{A}^{d-2}(X)} P(0) = \{0\}$. Let $f \in \mathcal{S}(X)$. Then, the limiting resolvent (see §2) output $R(\sigma^2 \pm i0)f$ has the form*

$$R(\sigma^2 \pm i0)f = e^{\pm i\sigma r}u \tag{2}$$

for some function u which is polyhomogeneous on X^+_{res}, where $1/r$ is a boundary-defining-function of X.[2]

See Theorem C for a more precise version.

Recall that polyhomogeneity is a generalization of smoothness which allows terms of the form $\rho^j(\log\rho)^k$ to appear in the $\rho \to 0^+$ asymptotic expansion, for any $(j,k) \in \mathbb{C} \times \mathbb{N}$, where $\rho = \rho_{\mathrm{f}}$ is a boundary-defining-function for some boundary hypersurface of our manifolds-with-corners. Thus, saying that u is polyhomogeneous on X^+_{res} with index set $\mathcal{E} = \mathcal{E}_{\mathrm{f}} \subset \mathbb{C} \times \mathbb{N}$ is just saying that u can be expanded in generalized Taylor series

$$u \sim \sum_{(j,k)\in\mathcal{E}} u_{j,k}\rho^j(\log\rho)^k, \qquad u_{j,k} = u_{j,k,\mathrm{f}} \in C^\infty(\mathrm{f}^\circ) \tag{3}$$

at each boundary hypersurface $\mathrm{f} \in \{\mathrm{zf}, \mathrm{tf}, \mathrm{bf}\}$ of X^+_{res} (and that the Taylor series can be differentiated term-by-term, and that the different Taylor series at adjacent boundary hypersurfaces are consistent at the corners.) Our proof of Theorem A is constructive, in that it produces the terms in the asymptotic expansions and therefore allows them to be computed, but we have not encapsulated the resultant formulae in a theorem.

One feature of the treatment here is an attention to partial *multipole* expansions (with respect to some fixed boundary collar on X). Thus, when we prove the theorem above, we will actually show that the first few terms in the large radii expansions of u involve only finitely many harmonics of the boundary Laplacian $\triangle_{\partial X} \in \mathrm{Diff}^2(\partial X)$. The quantitative meaning of "few" in the previous sentence depends on the decay rate of any terms in the given Schrödinger operator which break the symmetry associated to preferred boundary collar. Proving this requires a bit of bookkeeping in

[2] In the case where X is Euclidean, r is not the usual Euclidean radial coordinate but its Japanese bracket. This notational elision should not cause confusion.

the analysis of the kernel of the given Schrödinger operator. This bookkeeping can be found in §A, along with some additional motivation. The argument in that section is due to Melrose [Mel93], in much greater generality, but there does not seem to exist in the literature an exposition with the level of detail required here.

In his paper, Hintz uses a recursive algorithm similar to that used here for computing the first few terms in the low-energy asymptotics. Unfortunately, it is not entirely straightforward to extend his algorithm beyond the main order, where "main" means the term contributing to the leading long-time asymptotics for wave propagation. (In Hintz's case, the main order is actually $O(\sigma^{2-})$, despite there being $O(1)$ and $O(\sigma)$ terms in the low-energy asymptotics, because their contributions to the long-time asymptotics end up canceling with those of $R(\sigma^2 \mp i0)f$.) The reason it is not straightforward is, as Hintz explains, that it is necessary to analyze the resolvent output on polyhomogeneous functions

$$f \in \mathcal{A}^{\mathcal{E},\mathcal{F},\mathcal{G}}(X^+_{\mathrm{res}}), \quad \mathcal{E},\mathcal{F},\mathcal{G} \subseteq \mathbb{C}\times\mathbb{N}, \tag{4}$$

where $\mathcal{E},\mathcal{F},\mathcal{G}$ are appropriate index sets and $\mathcal{A}^{\mathcal{E},\mathcal{F},\mathcal{G}}(X^+_{\mathrm{res}})$ is the set of functions admitting full asymptotic expansions on X^+_{res}, with $\mathcal{E},\mathcal{F},\mathcal{G}$ the index sets at bf, tf, and zf respectively; see [Hin22, Def. 2.13].[3] Hintz only analyzes the case where f is partially polyhomogeneous already on $[0,\infty)_\sigma \times X$. Here, we need to allow more singular behavior at the corner $\{0\}\times\partial X \subset [0,\infty)_\sigma \times X$. The algorithm that we employ is actually a small modification of Hintz's. The details are described in §6, but let us emphasize that the key ideas are already contained in [Hin22]. The main difference, besides producing full expansions, is that we use an inductive argument which does not worsen decay at each step, this being necessary to get sharp asymptotics at bf. (In contrast, the argument used by Hintz is simpler, but at the cost of producing a fictitious singularity at bf.) In addition, we work in greater generality.

There are three additional sections of this paper besides those already mentioned: §3, §4, §5. These contain the asymptotic analyses at bf, zf, and tf, respectively.

- In §3, we discuss the limiting absorption principle for σ-dependent forcings with Schwartz behavior as $\sigma \to 0^+$. The low energy limit is singular, but the $O(\sigma^\infty)$ suppression of the forcing is sufficient to push through the asymptotic analysis without trouble.
- §4 involves the analysis at the boundary hypersurface $\mathrm{zf} \subset X^+_{\mathrm{res}}$, that is at *exactly* zero energy. In this section, we solve the PDE modulo terms which are suppressed by one order of the boundary-defining-function of the boundary hypersurface zf.
- Finally, §5, involves the analysis at $\mathrm{tf} \subset X^+_{\mathrm{res}}$. In it, we upgrade the previously constructed quasimode to an $O(\sigma^\varepsilon)$-quasimode for some $\varepsilon > 0$, i.e. an $O(\sigma^{0+})$-quasimode.[4] Such a quasimode solves the PDE modulo terms which are decaying as $\sigma \to 0^+$, *uniformly* in $x \in X$. It is the analysis in this section which forms the heart of the analysis of low energy phenomena.

These results are combined in §6 in order to prove the main theorem, Theorem A. The basic idea is to use §4, §5 to produce a solution w to $(P-\sigma^2)(e^{\pm i\sigma r}w) = f + F$ such that $F = O(\sigma^\infty r^\infty)$, meaning that F is Schwartz at both boundary hypersurfaces of $[0,\infty)_\sigma \times X$. We then appeal to §3

[3] Unfortunately, in our companion paper [LS] our notational conventions included listing index sets in the opposite order. Thus, in that paper, $\mathcal{A}^{\mathcal{G},\mathcal{F},\mathcal{E}}(X^{\mathrm{sp}}_{\mathrm{res}}\backslash\infty\mathrm{f})$ means what we call $\mathcal{A}^{\mathcal{E},\mathcal{F},\mathcal{G}}(X^+_{\mathrm{res}})$ here. We felt the former convention to be more natural when working with the additional face $\infty\mathrm{f} = X^{\mathrm{sp}}_{\mathrm{res}}\backslash X^+_{\mathrm{res}}$ (which corresponds to $E=\infty$), but in the present paper there is no such face (since we do not study the $E\to\infty$ limit). We decided to prioritize matching notational conventions with [Hin22] rather than [LS].

[4] The reader may object that the word "upgrade" is inappropriate given that, if $\varepsilon < 1$, then σ^ε is larger than ρ_{zf} near compact subsets of zf°. However, σ^ε is decaying uniformly as $\sigma \to 0$, unlike ρ_{zf}, which is $\gtrsim 1$ near compact subsets of tf. Worsening the quasimode at zf is a necessary price to pay for bettering the quasimode at tf, this being the origin of Price's law. For understanding the resolvent output $R(\sigma^2 \pm i0)f$, the relevant quasimodes must be uniformly good as $\sigma \to 0$.

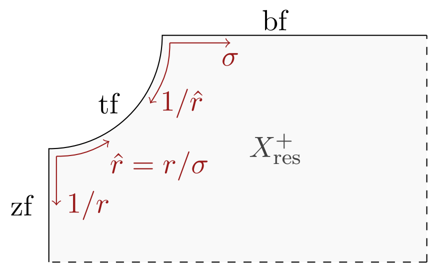

FIGURE 1. The mwc X^+_{res}, with the boundary hypersurfaces $\mathrm{bf} = \mathrm{cl}_{X^+_{\mathrm{res}}}\{(\sigma,\theta) : \sigma > 0, \theta \in \partial X\}$, $\mathrm{zf} = \mathrm{cl}_{X^+_{\mathrm{res}}}\{(0,x) : x \in X^\circ\}$, and tf.

to study $R(\sigma^2 \pm i0)F$. It can then be shown that

$$R(\sigma^2 \pm i0)f = e^{\pm i\sigma r}w - R(\sigma^2 \pm i0)F. \tag{5}$$

So, if w is constructed so as to be polyhomogeneous on X^+_{res}, and $e^{\mp ir}R(\sigma^2 \pm i0)F$ is polyhomogeneous on $[0,\infty)_\sigma \times X$ (in fact, it will be and moreover Schwartz at $\sigma = 0$), then the function u defined by eq. (2) is polyhomogeneous on X^+_{res}. See §6 for the details.

We have five appendices. As already mentioned, the first, §A, is a self-contained exposition of multipole expansions for perturbations of the Laplacian on an exactly conic manifolds, this being the model problem on $\mathrm{zf} \subset X^+_{\mathrm{res}}$. This section was originally intended to be a standalone expository article, so it can actually be read independently of the rest of this paper and is rather more pedagogical. In §B, we provide a similar treatment for a conjugated Schrödinger operator on an exact cone, this being the model problem at the boundary hypersurface $\mathrm{tf} \subset X^+_{\mathrm{res}}$. In §C, we review some elementary facts about the Mellin transform, for reference elsewhere in the paper. This is included just to make the paper a bit more self-contained. Next, §D contains a few straightforward but tedious proofs of propositions stated in §2 that were felt to clutter the exposition.

2. Geometric setup and the admissible operators

Our setup here is a special case of that considered in [Vas21b]. Since we are concerned with asymptotics, we must assume that the coefficients of the PDE under consideration admit at least partial asymptotic expansions themselves, and we assume some additional symmetry of the $O(r^{-1})$ and $O(r^{-2})$ terms, but otherwise the setup is completely analogous. Before proving the main results of this section, we present the details.

First, we discuss notation. Let X denote a connected, smooth manifold-with-boundary equipped with a boundary collar, by which we mean an embedding of the cylinder $\dot{X} = [0,1) \times \partial X$ into X such that such that $(0,\theta) \mapsto \theta$ for all $\theta \in \partial X$. So, we can identify $\dot{X}$ with a subset of X. Moreover, we assume that the coordinate function $(\rho,\theta) \mapsto \rho$ on $\dot{X}$ extends to an element of $C^\infty(X;\mathbb{R}^{\geq 0})$, which we can assume vanishes only at ∂X. Fix such an extension, which we denote for the rest of the paper as ρ. Thus, ρ is a boundary-defining-function for the boundary of X. In order to facilitate the comparison between the asymptotically conic setting and the exact Euclidean setting, define $r = \rho^{-1}$.

The exact Euclidean case is when

$$X = \overline{\mathbb{R}^d} = \mathbb{R}^d \cup \infty\mathbb{S}^{d-1} \tag{6}$$

is the radial compactification of $\mathbb{R}^d$ and when the boundary collar is the map $\dot{X} = [0,1) \times \mathbb{S}^{d-1} \hookrightarrow X$ given by $(\rho,\theta) \mapsto \rho^{-1}\theta$. Inverting, this just means that

$$\rho(\mathbf{x}) = \lVert\mathbf{x}\rVert^{-1}, \qquad \theta = \mathbf{x}/\lVert\mathbf{x}\rVert \tag{7}$$

for all $\mathbf{x} \in \mathbb{R}^d$ with $\|\mathbf{x}\| > 1$. In other words, away from the origin $r = \rho^{-1}$ is the usual Euclidean radial coordinate and $\theta = \hat{\mathbf{x}}$ is the unit vector in the direction of $\mathbf{x}$. We can extend ρ to an element of $C^\infty(X)$. Then, $r = 1/\rho \in C^\infty(\mathbb{R}^d)$ is only given by the usual formula $r(\mathbf{x}) = \|\mathbf{x}\|$ outside of some neighborhood of the origin, but this should not cause any confusion.

Remark. We emphasize the choice of boundary collar (and therefore boundary-defining function) because the forms of the multipole expansions below depend on it. There are two reasons for this:

(I) Consider the fact that in the multipole expansion from Euclidean electrostatics the $1/r^2$ term involves only the spherical harmonics Y^1_m, not $Y^0_0 = 1$ (assuming the charge distribution is sufficiently rapidly decaying at infinity). In other words, the subleading term has no s-wave component. If we were to rewrite the expansion in terms of $r + 1$ instead of r, this would no longer be true. So, the choice of boundary-defining function matters.

(II) More seriously, the presence of logarithmic terms depends on the presence of terms in the PDE that break the conic symmetry of the problem (meaning spherical symmetry in the Euclidean case). This is the reason there are no logarithmic terms in the multipole expansion from Euclidean electrostatics. But to even make sense of what it means to "break the spherical symmetry," we need to fix a boundary collar.

A more thorough discussion of these points may be found in the introduction to §A. If the reader is willing to settle for a theorem involving sub-optimal index sets, then the choice of boundary collar may be forgotten.

2.1. **Some standard notation.** Let $\mathcal{V}_{\mathrm{b}}(X)$ denote the set of smooth vector fields on X (which are smooth all the way up to and including the boundary and) tangent to the boundary. These are the "b-vector fields." Similar notation and terminology is used for tensors, algebras of differential operators, etcetera. For example, for $m \in \mathbb{R}$, $\mathrm{Diff}^m_{\mathrm{b}}(X)$ is the set of linear combinations over $C^\infty(X)$ of compositions of $\leq m$ b-vector fields. Then,

$$\mathrm{Diff}^{m,\ell}_{\mathrm{b}}(X) = \rho^{-\ell}\,\mathrm{Diff}^m_{\mathrm{b}}(X) \tag{8}$$

denotes the set of weighted b-differential operators.

The set $\mathcal{V}_{\mathrm{sc}}(X) = \rho\mathcal{V}_{\mathrm{b}}(X)$ is the set of "sc"-vector fields. These are the sections of a smooth vector bundle over X, which is called ${}^{\mathrm{sc}}TX$. Dualizing, one gets an associated cotangent bundle ${}^{\mathrm{sc}}T^*X$ which is canonically the usual cotangent bundle over the interior. Again, we can form the algebra $\mathrm{Diff}_{\mathrm{sc}}(X)$ of sc-differential operators, and notation similar to that used in the b- case can be employed.

We use $\mathcal{A}^\bullet(M)$ to denote the sets of partially polyhomogeneous functions on a manifold-with-corners M, where the $\bullet$ contains a list of pairs $(\mathcal{E}, \alpha)$, one for each boundary hypersurface f of M, where $\mathcal{E}$ is the index set of the polyhomogeneous expansion at f and $\alpha \in \mathbb{R} \cup \{\infty\}$ is the order of the merely conormal error. When $\alpha = \infty$, in which case one has full polyhomogeneity at f, we simply write $\mathcal{E}$ in place of $(\mathcal{E}, \infty)$. Likewise, if $\mathcal{E} = \varnothing$, we simply write α. For example,

$$\mathcal{A}^{(\mathcal{E},\alpha),\beta,\mathcal{K}}(X^+_{\mathrm{res}}) \tag{9}$$

denotes the set of functions which are partially polyhomogeneous at bf with index set $\mathcal{E}$ and conormal error α, merely conormal at tf with order β, and fully polyhomogeneous at zf with index set $\mathcal{K}$.

Bundles over $\dot{X}$, function spaces on $\dot{X}$, etcetera are denoted similarly to the corresponding objects on X. While we will not state the definitions of such objects explicitly, it should be noted that we never require uniformity as $\rho \to 1^-$. For example,

$$\mathcal{A}^0(\dot{X}) = \{a \in C^\infty(X^\circ) : Qa \in L^\infty_{\mathrm{loc}}(\dot{X}) \text{ for all } Q \in \mathrm{Diff}_{\mathrm{b}}(\dot{X})\}. \tag{10}$$

Thus, for $a \in \mathcal{A}^0(\dot{X})$, $\chi Q a \in L^\infty(\dot{X})$ for all $\chi \in C^\infty_{\mathrm{c}}(\dot{X})$, including those that do not vanish on ∂X. (So, $L^\infty_{\mathrm{loc}}(\dot{X})$ is smaller than $L^\infty_{\mathrm{loc}}(\dot{X}^\circ)$.) We will also add a 'c' subscript to denote compact support, e.g. $\mathcal{A}^0_{\mathrm{c}}(\dot{X})$. Note that this allows functions to be nonvanishing near $\rho = 0$ (because X, with its boundary included, is compact), just not near $\rho = 1$. So, in the Euclidean case, $C_{\mathrm{c}}^\infty(\dot{X})$ consists of smooth functions on $\mathbb{R}^d$ that extend to the radial compactification and that vanish on a neighborhood of the unit ball centered at the origin.

In this section, we will also work with the sc- and b- L^2-based Sobolev spaces $H_{\mathrm{sc}}^{r,\ell}(X), H_{\mathrm{b}}^{r,\ell}(X)$. The parameter $r \in \mathbb{R}$ is the differential order and $\ell \in \mathbb{R}$ is the amount of decay. We refer to [Vas21a] for details when $r \notin \mathbb{N}$. When $r \in \mathbb{N}$,

$$H_{\mathrm{sc}}^{r,\ell}(X) = \{u \in \mathcal{D}'(X^\circ) : \operatorname{Diff}_{\mathrm{sc}}^{r,\ell}(X)u \in L^2(X)\} \tag{11}$$

and similarly for $H_{\mathrm{b}}^{r,\ell}$. In the Euclidean case, these are just the ordinary Sobolev spaces. One convention worth noting is that we follow Vasy in indexing the b-Sobolev spaces so that

$$H_{\mathrm{b}}^{0,0}(X) = H_{\mathrm{sc}}^{0,0}(X) = L^2(X,g), \tag{12}$$

where g is an asymptotically conic metric (see below). This differs from the other commonly used convention where $H_{\mathrm{b}}^{0,0} = L^2(X, g_{\mathrm{b}})$ for g_{b} an asymptotically cylindrical metric (a b-metric). The different conventions just differ by a shift in the decay index.

Remark 2.1. We use the term "pre-index set" to refer to subsets $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}$ such that $\mathcal{E} \cap \{z \in \mathbb{C} : \Re z < \alpha\}$ is finite for each $\alpha \in \mathbb{R}$ and such that $(j, k+1) \in \mathcal{E} \Rightarrow (j,k) \in \mathcal{E}$ for all $j \in \mathbb{C}$ and $k \in \mathbb{N}$. Then, if $\mathcal{E}$ also satisfies $(j,k) \in \mathcal{E} \Rightarrow (j+1,k) \in \mathcal{E}$, then $\mathcal{E}$ is an index set.

In §A, §B we will work with partially polyhomogeneous spaces $\mathcal{A}^{\mathcal{E},\alpha}$ for $\mathcal{E}$ a pre-index set. For example, if α is finite, we say that $f \in \mathcal{A}^{\mathcal{E},\alpha}(X)$ if it is smooth in the interior and

$$\exists f_{j,k} \in C^\infty(\partial X_\theta) \text{ s.t. } f - \sum_{(j,k)\in\mathcal{E}, \Re j \leq \alpha} f_{j,k}(\theta)\rho^j (\log \rho)^k \in \mathcal{A}^\alpha(\dot{X}). \tag{13}$$

Usually, one works with $\mathcal{A}^{\mathcal{E},\alpha}$ for α an index set, which has the advantage that it does not depend on the choice of boundary-defining-function or boundary collar. If $\mathcal{E}$ is only a pre-index set, then $\mathcal{A}^{\mathcal{E},\alpha}$ does depend on these choices. Unfortunately, in order to get sharp statements regarding the index sets involved in multipole expansions, it is necessary to fix a boundary collar and consider pre-index sets.

2.2. **Metric.** Let g denote a smooth "sc-metric" on X, meaning a Riemannian metric on X° of the form $g \in C^\infty(X; \operatorname{Sym}^2 {}^{\mathrm{sc}}T^*X)$, i.e. a smooth section of the symmetric product of two copies of the sc-cotangent bundle. We assume that the boundary collar has been chosen such that g differs from the exactly conic metric $g_0 = \mathrm{d}\rho^2 + \rho^2 h$ on $\dot{X}$ by a well-behaved error:

$$g - g_0 \in \rho C^\infty(\dot{X}; \operatorname{Sym}^2 {}^{\mathrm{sc}}T^*\dot{X}). \tag{14}$$

Thus, $L^2(X, g_0) = L^2(X,g)$ at the level of sets. (One sometimes allows an $\rho^{1+\delta}S^0(\dot{X}; \operatorname{Sym}^2 {}^{\mathrm{sc}}T^*\dot{X})$ term on the right-hand side of eq. (14), for some $\delta > 0$, but we will not do so here.) If $g - g_0$ is decaying faster than eq. (14) requires, then it is worth keeping track of. Moreover, the $\mathrm{d}r^2$ terms in $g - g_0$ have a different effect than the other $d^2 - 1$ terms, so we keep track of these separately. Let $\gimel \in \mathbb{N}$, $\gimel_0 \in \mathbb{N}_+$ be such that

$$g - g_0 \in \rho^{1+\gimel} C^\infty([0,1)_\rho)\mathrm{d}r^2 + \rho^{1+\gimel_0} C^\infty(\dot{X}; \operatorname{Sym}^2 {}^{\mathrm{sc}}T^*\dot{X}) \tag{15}$$

We assume without loss of generality that $\gimel \leq \gimel_0$. We may also allow $\gimel, \gimel_0$ to be infinite, in which case the corresponding terms in $g - g_0$ are Schwartz.

Let $\triangle_g$ denote the positive semidefinite Laplace–Beltrami operator on (X, g). Then:

Proposition 2.2. *Given eq.* (15)*, the Laplacian* $\triangle_g$ *differs from the exactly conic Laplacian*

$$\triangle_{g_0} = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{g_{\partial X}} \tag{16}$$

by an element of $\rho^{3+\gimel} \operatorname{Diff}_{\mathrm{b}}^2([0,1)_\rho) + \rho^{3+\gimel_0} \operatorname{Diff}_{\mathrm{b}}^2(X)$. ■

See §D for the proof.

2.3. **Operators.** Let $P = \{P(\sigma)\}_{\sigma\geq 0}$ denote a family of differential operators $P(\sigma) \in \mathrm{Diff}^2(X^\circ)$ of the form

$$P(\sigma) = \triangle_g + 2i\sigma(1-\chi)\frac{\partial}{\partial r} + \frac{i\sigma(d-1)}{r} + L + \sigma Q + \sigma^2 R, \tag{17}$$

where

- $\chi \in C^\infty_{\mathrm{c}}(X^\circ)$ satisfies $\mathrm{supp}(1-\chi) \Subset \dot{X}$, so that $(1-\chi)\partial_r$ is a well-defined vector field on X,
- $\rho^{-3}L, \rho^{-2}Q \in \mathrm{Diff}^1_{\mathrm{b}}(X)$, and
- $R - \rho\mathfrak{m} \in \rho^2 C^\infty(X)$ for some $\mathfrak{m} \in \mathbb{R}$.

We assume that P is a symmetric operator on $C^\infty_{\mathrm{c}}(X^\circ)$ with respect to the $L^2(X, g)$-inner product. Thus, we can write $P = P^*$.

We can find $\beth, \beth_0, \beth_1, \beth_2, \beth_3, \beth_4 \in \mathbb{N} \cup \{\infty\}$, which may be zero, such that the following conditions hold:

$$\rho^{-3}(\triangle_{g_0} - \triangle_g) \in \rho^{\beth}\ \mathrm{Diff}^2_{\mathrm{b}}([0,1)_\rho) + \rho^{\beth_0}\,\mathrm{Diff}^2_{\mathrm{b}}(\dot{X}), \tag{18}$$

$$\rho^{-3}L \in \rho^{\beth}\ \mathrm{Diff}^1_{\mathrm{b}}([0,1)_\rho) + \rho^{\beth_0}\,\mathrm{Diff}^1_{\mathrm{b}}(\dot{X}), \tag{19}$$

$$\rho^{-2}Q \in \rho^{\beth_1}\mathrm{Diff}^1_{\mathrm{b}}([0,1)_\rho) + \rho^{\beth_2}\,\mathrm{Diff}^1_{\mathrm{b}}(\dot{X}) \tag{20}$$

and

$$R \in \rho^{1+\beth_3}C^\infty([0,1)_\rho) + \rho^{2+\beth_4}C^\infty(X). \tag{21}$$

By Proposition 2.2, the first of these, eq. (18), will be satisfied as long as $\beth \leq \gimel$ and $\beth_0 \leq \gimel_0$. We may assume without loss of generality that $\beth \leq \beth_0$, $\beth_1 \leq \beth_2$, and $\beth_3 \leq \beth_4$.

It will be useful to relate the class of operators considered here to the conjugations of spectral families of Schrödinger operators. Given any family $\{O(\sigma)\}_{\sigma\geq 0} \subset \mathrm{Diff}^2(X^\circ)$, let

$$\hat{O}(\sigma) = e^{i\sigma r}O(\sigma)e^{-i\sigma r}, \qquad \check{O}(\sigma) = e^{-i\sigma r}O(\sigma)e^{i\sigma r}, \tag{22}$$

where by $e^{\pm i\sigma r}$ we mean the multiplication operator $\bullet \mapsto e^{\pm i\sigma r}\bullet$. Then, it is straightforward to prove:

Proposition 2.3. *Let g be as in eq. (15), and let $\beth = \gimel, \beth_0 = \gimel_0$.*

(I) Suppose that $P(\sigma)$ has the form described above. Then, $\check{P}(\sigma)$ has the form

$$\check{P}(\sigma) = P_0 - \sigma^2 + \sigma P_1 + \sigma^2 P_2 \tag{23}$$

for $P_0 = \triangle_g + L$ and P_1, P_2 such that $P_1 \in \rho^{2+\min\{\beth,\beth_1\}}\,\mathrm{Diff}^1_{\mathrm{b}}([0,1)_\rho) + \rho^{2+\min\{\beth_0,\beth_2\}}\,\mathrm{Diff}^1_{\mathrm{b}}(X)$, $P_2 \in \rho^{1+\min\{\beth,\beth_1,\beth_3\}}C^\infty([0,1)_\rho) + \rho^{1+\min\{\beth_0,\beth_2,1+\beth_4\}}C^\infty(X)$.

(II) Conversely, suppose that we are given some Schrödinger operator $P_0 = \triangle_g + L$ for L as in eq. (19). Then, defining $P_0(\sigma) = P_0 - \sigma^2$ and

$$P(\sigma) = \hat{P}_0(\sigma), \tag{24}$$

$P(\sigma)$ has the form described above, with $\beth_1 = \gimel$, $\beth_2 = \gimel_0$, $\beth_4 = \gimel_0 - 1$, and $\beth_3 \geq 0$. If $\gimel \geq 1$, then $\mathfrak{m} = 0$.

■

The proof is contained in §D.

So, the spectral families of Schrödinger operators fall into the framework here. For applications to wave propagation on non-ultrastatic spacetimes, it is important to allow greater generality. For instance, when considering wave propagation on black hole spacetimes, the parameter $\mathfrak{m}$ above is proportional to the mass of the black hole. This is why we do not assume that our operator $P(\sigma)$ arises from Proposition 2.3.(II).

Lemma 2.4. *For each $k \in \mathbb{N}$, $\mathrm{Diff}^k_{\mathrm{b}}(X) \subseteq \mathrm{Diff}^k_{\mathrm{b}}(X^+_{\mathrm{res}})$.* ■

Proof. Any b-vector field on X lifts to a smooth vector field on X^+_{res}, and the lift is tangent to the boundary of X^+_{res}. So, $\mathcal{V}_{\mathrm{b}}([0,\infty)_\sigma \times X) \subseteq \mathcal{V}_{\mathrm{b}}(X^+_{\mathrm{res}})$. Thus,

$$\mathrm{Diff}^k_{\mathrm{b}}(X) \subseteq \mathrm{Diff}^k_{\mathrm{b}}([0,\infty)_\sigma \times X) \subseteq \mathrm{Diff}^k_{\mathrm{b}}(X^+_{\mathrm{res}}) \tag{25}$$

for any $k \in \mathbb{N}$. □

Consequently, $P \in \mathrm{Diff}^{2,-1,-2,0}_{\mathrm{b}}(X^+_{\mathrm{res}})$, where the orders are the differential order, the decay order at bf, the decay order at tf, and the decay order at zf.

2.4. **The Sommerfeld problem.** Assuming that $P(\sigma) = \hat{P}_0(\sigma)$ for $P_0(\sigma) = P_0 - \sigma^2$ the spectral family of a Schrödinger operator of the formed described in Proposition 2.3, let us recall some basic solvability theory for $P(\sigma)$.

First, recall that, since P_0 is essentially self-adjoint on $\mathcal{S}(X)$ with respect to the $L^2(X,g)$ inner product (this follows easily from the theory of deficiency indices, owing to the fact that $\mathrm{ran}(P_0 \pm i)^\perp \subset L^2(X,g)$ is a subset of $\ker_{\mathcal{S}'}(P_0 \mp i)$, but $\ker_{\mathcal{S}'}(P_0 \mp i) = \ker_{\mathcal{S}}(P_0 \mp i)$ consists only of Schwartz functions by ellipticity in the sc-calculus, and $\ker_{\mathcal{S}}(P_0 \mp i)$ is empty by the usual symmetry argument), we have, for all $E \in \mathbb{C}\backslash\mathbb{R}$, a well-defined two-sided inverse

$$R(E) \in \Psi^{-2,0}_{\mathrm{sc}}(X). \tag{26}$$

In other words, for any $f \in \mathcal{S}'(X)$, the unique solution $u \in \mathcal{S}'(X)$ to $P_0 u - Eu = f$ is given by $u = R(E)f$. Every other solution of the PDE fails to be tempered. Moreover, $R(E)$ is a pseudodifferential operator in the Parenti–Shubin–Melrose "sc-calculus" which regularizes by two orders and does not worsen decay, the set of such operators being $\Psi^{-2,0}_{\mathrm{sc}}$.

If $E \in \mathbb{R}$, it may be the case that E lies in the spectrum of P_0, in which case the resolvent $R(E)$ will not be well-defined. In fact, if $E \geq 0$, then it must be in the spectrum. For $E > 0$, in lieu of the resolvent one has the *limiting absorption principle*. One particularly weak form of this principle states that, for any $E > 0$, each of the two limits

$$R(E \pm i0) = \lim_{\epsilon \to 0^+} R(E \pm i\epsilon) : \mathcal{S}(X) \to \mathcal{S}'(X) \tag{27}$$

exist in the strong operator topology. That is, for any $f \in \mathcal{S}(X)$, the Schwartz functions $R(E \pm i\epsilon)f$ converge as $\epsilon \to 0^+$ to some tempered distributions in the topology of $\mathcal{S}'(X)$. Moreover, the "limiting resolvent" $R(E \pm i0)$ is a right inverse to $P_0 - E = P_0(\sigma)$ for $\sigma = E^{1/2}$, so $P_0(\sigma)R(E \pm i0)f = f$ for all $f \in \mathcal{S}(X)$; it produces solutions to the PDE.

The particular solution $u = R(\sigma^2 \pm i0)f$ to $P_0(\sigma)u = f$ produced by the limiting resolvent is distinguished in the set of all solutions by its oscillatory behavior $\sim \exp(\pm i\sigma r)$ at spatial infinity (the "Sommerfeld radiation condition"). Thus, the *Sommerfeld problem*, that of solving $P_0 u - Eu = f$ for u with "outgoing" oscillatory behavior at infinity is well-posed and solved by the limiting resolvent.

A much more precise version of the limiting absorption principle is contained in [Vas21a]. For instance, it, when applied to the case at hand (and stated without assuming that P is the spectral family of a Schrödinger operator), contains:

Proposition 2.5. *Suppose that P satisfies the hypotheses in the previous subsection. Let $\ell, r \in \mathbb{R}$ satisfy the inequalities $\ell < -1/2$ and $r > -1/2 - \ell$. Then, for each $\sigma > 0$,*

$$P(\sigma) : \{u \in H^{r,\ell}_{\mathrm{b}}(X) : P(\sigma)u \in H^{r,\ell+1}_{\mathrm{b}}(X)\} \to H^{r,\ell+1}_{\mathrm{b}}(X) \tag{28}$$

is invertible. Moreover, if $K \Subset \mathbb{R}^+$, the operator norm of $P(\sigma)^{-1} : H^{r,\ell+1}_{\mathrm{b}}(X) \to H^{r,\ell}_{\mathrm{b}}(X)$ is uniformly bounded for $\sigma \in K$. ■

The (almost immediate) proof from [Vas21a, Thm. 1.1] is contained in §D.

Remark 2.6. Part of [Vas21a, Thm. 1.1] is that, when $P(\sigma) = e^{i\sigma r}\check{P}(\sigma)e^{-i\sigma r}$ for $\check{P}(\sigma) = \check{P}(0) - \sigma^2$ the spectral family of a Schrödinger operator, the inverse of eq. (28) is given by

$$P(\sigma)^{-1} = e^{i\sigma r} R(\sigma^2 - i0) e^{-i\sigma r} \tag{29}$$

on $\mathcal{S}(X)$. Thus, Proposition 2.5 directly defines the limiting resolvent (or really its conjugation $e^{i\sigma r}R(\sigma^2 - i0)e^{-i\sigma r}$) via the problem it solves. The connection with the Sommerfeld radiation condition is that, if $u_0 \in H_{\mathrm{b}}^{r,\ell}(X)$ for high r, then u_0 is non-oscillatory at infinity (the b-Sobolev regularity implying that such oscillations are suppressed by at least ℓ powers of ρ), which means that

$$R(\sigma^2 - i0)f = e^{-i\sigma r}P(\sigma)^{-1}(e^{i\sigma r}f) \tag{30}$$

has the correct $e^{-i\sigma r}$ behavior at infinity.

Similar statements apply to $R(\sigma^2 + i0)$.

Here, we prefer to work with the conormal function spaces $\mathcal{A}^\bullet(X)$ instead of the b-Sobolev spaces $H_{\mathrm{b}}^\bullet(X)$. These are related by

$$H_{\mathrm{b}}^{\infty,\ell}(X) \subseteq \mathcal{A}^{\ell+d/2}(X) \subseteq H_{\mathrm{b}}^{\infty,\ell-}(X) = \bigcap_{\varepsilon>0} H_{\mathrm{b}}^{\infty,\ell-\varepsilon}(X), \quad \ell \in \mathbb{R}, \tag{31}$$

which is a consequence of the Sobolev embedding theorems. The shift in the decay order comes from the convention that the indexing of $\mathcal{A}^\bullet$ is based on L^∞, while that of $H_{\mathrm{b}}^\bullet$ is based on L^2. Specifically, $H_{\mathrm{b}}^{0,0}(X) = L^2(X,g)$, so

$$\mathcal{A}^{d/2}(X) \subset \rho^{d/2}L^\infty(X) \subset \rho^{-\varepsilon}L^2(X,g) \subset H_{\mathrm{b}}^{\infty,-\varepsilon}(X). \tag{32}$$

It therefore follows that, for all $\sigma > 0$,

$$P(\sigma)^{-1} : \mathcal{A}^{\alpha+1}(X) \to \mathcal{A}^{\alpha-}(X) \tag{33}$$

for all $\alpha < (d-1)/2$.

We will need to know that $P(\sigma)^{-1}$ varies smoothly in σ as a map $\mathcal{A}^{\alpha+1}(X) \to \mathcal{A}^{\alpha-}(X)$ in the sense that, whenever $f \in \mathcal{A}^{\alpha+1}(X)$, then

$$P(\sigma)^{-1}f \in C^\infty(\mathbb{R}^+_\sigma; \mathcal{A}^{\alpha-}(X)). \tag{34}$$

The argument showing this is standard, so we only illustrate the idea without covering all of the details. The starting point is the usual resolvent identity

$$\begin{aligned} P(\sigma_2)^{-1} - P(\sigma_1)^{-1} &= P(\sigma_1)^{-1}P(\sigma_1)P(\sigma_2)^{-1} - P(\sigma_1)^{-1}P(\sigma_2)P(\sigma_2)^{-1} \\ &= P(\sigma_1)^{-1}(P(\sigma_1) - P(\sigma_2))P(\sigma_2)^{-1}. \end{aligned} \tag{35}$$

(The manipulations are justified because the compositions $P(\sigma_\bullet)^{-1}P(\sigma)P(\sigma_\bullet)^{-1}$ in the middle of eq. (35) are well-defined, mapping $\mathcal{A}^{\alpha+1}(X) \to \mathcal{A}^{\alpha-}(X)$.) We may write $P(\sigma_1) - P(\sigma_2) = (\sigma_1 - \sigma_2)T(\sigma_1,\sigma_2)$ for some smooth two-parameter family of $T \in \mathrm{Diff}_{\mathrm{b}}^{1,-1}(X)$. So, eq. (35) says

$$P(\sigma_2)^{-1} - P(\sigma_1)^{-1} = (\sigma_1 - \sigma_2)P(\sigma_1)^{-1}T(\sigma_1,\sigma_2)P(\sigma_2)^{-1}. \tag{36}$$

The composition $P(\sigma_1)^{-1}T(\sigma_1,\sigma_2)P(\sigma_2)^{-1}$ maps $\mathcal{A}^{\alpha+1}(X) \to \mathcal{A}^{\alpha-}(X)$, and each seminorm of the codomain is uniformly bounded in terms of finitely many seminorms of the domain (by the uniformity clause in Proposition 2.5). Thus, as $\sigma_2 \to \sigma_1$, the right-hand side of eq. (36), when applied to $f \in \mathcal{A}^{\alpha+1}(X)$, goes to 0 in $\mathcal{A}^{\alpha-}(X)$. This shows that $P(\sigma)^{-1}f \in C^0(\mathbb{R}^+_\sigma; \mathcal{A}^{\alpha-}(X))$. So, eq. (36) implies

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}P(\sigma)^{-1}f = -P(\sigma)^{-1}T(\sigma,\sigma)P(\sigma)^{-1}f = -P(\sigma)^{-1}\frac{\mathrm{d}P}{\mathrm{d}\sigma}P(\sigma)^{-1}f. \tag{37}$$

Applying eq. (37) inductively allows upgrading continuity to smoothness.

The limiting absorption principle says nothing about the $\sigma \to 0$ limit; for instance, Proposition 2.5 requires $0 \notin K$. For the analysis of such low-energy phenomena, the key result upon which we will build is the following corollary of [Vas21c]:

Proposition 2.7. *Suppose that $\ker_{\mathcal{A}^{d-2}(X)} P(0) = \{0\}$. Let $\ell < -1/2$, $s > -1/2 - \ell$, $\nu \in (\ell + 2 - d/2, \ell + d/2)$. Then, there exists a $\sigma_0 > 0$ and $C > 0$ such that*

$$\|(\rho + \sigma)^\nu u\|_{H_{\mathrm{b}}^{s,\ell}} \leq C \|(\rho + \sigma)^{\nu - 1} P(\sigma) u\|_{H_{\mathrm{b}}^{s,\ell+1}} \tag{38}$$

holds for all $\sigma \in (0, \sigma_0)$ and $u \in H_{\mathrm{b}}^{s,\ell}(X)$. ■

This is [Hin22, Theorem 2.11] (for real σ), but stated in greater generality. Again, see §D for the details.

Going forwards, we will always assume $\ker_{\mathcal{A}^{d-2}(X)} P(0) = \{0\}$. Thus, when we say that $P(\sigma)$ is as in §2, we are including this assumption.

Let us extract a mapping property of $P(\sigma)$ from the foregoing estimates.

Proposition 2.8. *Let $0 < \alpha < (d-1)/2$. For each $f(\sigma) \in \sigma L^\infty([0,1)_\sigma; \mathcal{A}^{\alpha+1}(X))$, the function $u = P(\sigma)^{-1} f$, which is well-defined by eq. (33), satisfies*

$$u \in L^\infty([0,1)_\sigma; H_{\mathrm{b}}^{s,\ell}(X)) \tag{39}$$

for all $s, \ell \in \mathbb{R}$ such that $\ell < \alpha - d/2$. ■

Proof. We already know, by the discussion in the previous subsection, that $u \in C^\infty((0,1)_\sigma; \mathcal{A}^{\alpha-}(X))$, so it is only the boundedness in the $\sigma \to 0^+$ limit that needs to be understood.

By eq. (31) (and the continuity of the inclusions therein), we have

$$f(\sigma) \in \sigma L^\infty([0,1)_\sigma; H_{\mathrm{b}}^{s,\alpha+1-d/2-}(X)), \tag{40}$$

for any $s \in \mathbb{R}$. Let $\ell < \alpha - d/2 < -1/2$. Then, $f \in \sigma C^0([0,1)_\sigma; H_{\mathrm{b}}^{s,\ell+1}(X))$.

Taking $s > -1/2 - \ell$ (it sufficing to consider this case), and choosing some $\nu \in (\ell+2-d/2, \ell+d/2)$, Proposition 2.7 says that, if $u(-;\sigma) \in H_{\mathrm{b}}^{s,\ell}(X)$ for each σ, then

$$\|(\rho + \sigma)^\nu u\|_{H_{\mathrm{b}}^{s,\ell}} \lesssim \|(\rho + \sigma)^{\nu-1} P(\sigma) u\|_{H_{\mathrm{b}}^{s,\ell+1}} = \|(\rho + \sigma)^{\nu-1} f\|_{H_{\mathrm{b}}^{s,\ell+1}}, \tag{41}$$

at least if we restrict attention to σ is sufficiently small. (Indeed, $u \in C^\infty((0,1)_\sigma; \mathcal{A}^{\alpha-}(X))$ implies

$$u \in C^\infty((0,1)_\sigma; H_{\mathrm{b}}^{s,\alpha-d/2-}(X)) \subset C^\infty((0,1)_\sigma; H_{\mathrm{b}}^{s,\ell}(X)). \tag{42}$$

So, the hypothesis $u(-;\sigma) \in H_{\mathrm{b}}^{s,\ell}(X)$ of Proposition 2.7 is satisfied, and therefore eq. (41) holds.)

Using the lower bound $0 < \alpha$ in the proposition statement, the allowed ℓ includes the whole interval $(-d/2, \alpha - d/2)$. Choosing such ℓ, the interval of allowed ν then includes $\nu = 0$. So, eq. (41) gives

$$\|u\|_{H_{\mathrm{b}}^{s,\ell}} \lesssim \|(\rho+\sigma)^{-1} f\|_{H_{\mathrm{b}}^{s,\ell+1}} = \Big\| \frac{\sigma}{\rho+\sigma} \sigma^{-1} f \Big\|_{H_{\mathrm{b}}^{s,\ell+1}} \lesssim \|\sigma^{-1} f\|_{H_{\mathrm{b}}^{s,\ell+1}}, \tag{43}$$

using the lemma that $\sigma/(\rho+\sigma)$ is a uniformly bounded operator on all b-Sobolev spaces, which follows from the observation that $(\rho\partial_\rho)^k(\sigma/(\rho+\sigma)) \leq 1$ for all $\sigma \geq 0$ and $k \in \mathbb{N}$.

Since $f \in \sigma C^0([0,1)_\sigma; H_{\mathrm{b}}^{s,\ell+1}(X))$, we conclude that $u \in L^\infty([0,1)_\sigma; H_{\mathrm{b}}^{s,\ell}(X))$. □

We can summarize the conclusion of the proposition as stating that $u \in L^\infty([0,1)_\sigma; \mathcal{A}^{\alpha-}(X))$, where the uniform boundedness is understood to mean that each Fréchet seminiorm of u in is uniformly bounded as $\sigma \to 0^+$. So,

$$P^{-1} : \sigma L^\infty([0,1)_\sigma; \mathcal{A}^{\alpha+1}(X)) \to L^\infty([0,1)_\sigma; \mathcal{A}^{\alpha-}(X)). \tag{44}$$

Using the usual resolvent identity eq. (37) and the fact that $\mathrm{d}P/\mathrm{d}\sigma : L^\infty([0,1)_\sigma;\mathcal{A}^\beta(X)) \to L^\infty([0,1)_\sigma;\mathcal{A}^{\beta+1}(X))$ for all $\beta\in\mathbb{R}$, it can be concluded that

$$\frac{\mathrm{d}^k P^{-1}}{\mathrm{d}\sigma^k} : \sigma^{k+1} L^\infty([0,1)_\sigma;\mathcal{A}^{\alpha+1}(X)) \to L^\infty([0,1)_\sigma;\mathcal{A}^{\alpha-}(X)). \tag{45}$$

for all $k\in\mathbb{N}$, α as above. For example, one application of eq. (37) yields

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}P(\sigma)^{-1} = -P(\sigma)^{-1}\frac{\mathrm{d}P}{\mathrm{d}\sigma}P(\sigma)^{-1} : \sigma^2\mathcal{A}^{\alpha+1} \overset{P^{-1}}{\to} \sigma\mathcal{A}^{\alpha-} \overset{P'}{\to} \sigma\mathcal{A}^{1+\alpha-} \overset{P^{-1}}{\to} \mathcal{A}^{\alpha-}, \tag{46}$$

with all the bounds uniform as $\sigma\to 0^+$.

3. Analysis at bf

We now prove the lemma used in the final step of the proof of the main theorem by discussing the asymptotic behavior of the solution of the Sommerfeld problem for σ-dependent forcing $f\in\mathcal{S}([0,1)_\sigma;\mathcal{A}^{\mathcal{E}}(X))$, for any index set $\mathcal{E}\subseteq\mathbb{C}\times\mathbb{N}$. Despite this being the final step of the proof, we discuss it first, the reason being that it is the easiest, owing to the Schwartz behavior of f as $\sigma\to 0^+$.

As a corollary of Proposition 2.7:

Proposition 3.1. *Let $\alpha\in(0,(d-1)/2)$ and $f\in\mathcal{S}([0,1)_\sigma;\mathcal{A}^{1+\alpha}(X))$. Then,*

$$P^{-1}(\sigma)f\in\mathcal{S}([0,1)_\sigma;\mathcal{A}^{\alpha-}(X)), \tag{47}$$

where $P^{-1}(\sigma)f$ is defined by eq. (33). ■

Cf. [Hin22, Theorem 2.14].

Proof. We have, for any $k,\kappa\in\mathbb{N}$,

$$\sigma^{-\kappa}\frac{\partial^k}{\partial\sigma^k}P^{-1}(\sigma)f(-;\sigma) = \sum_{j=0}^k\binom{k}{j}\sigma^{j+1}\frac{\partial^j P^{-1}(\sigma)}{\partial\sigma^j}\Big(\sigma^{-\kappa-j-1}\frac{\partial^{k-j}f}{\partial\sigma^{k-j}}\Big). \tag{48}$$

Via eq. (45), each term on the right-hand side is in $L^\infty([0,1)_\sigma;\mathcal{A}^{\alpha-}(X))$. □

We now upgrade this using a standard "normal operator" argument, varieties of which will be used throughout the other sections of this paper. For each $(j,k)\in\mathbb{C}\times\mathbb{N}$ and pre-index set $\mathcal{E}$, let

$$(j,k)\uplus\mathcal{E} = (j,k)\cup\mathcal{E}\cup\{(j+\delta,k'+\kappa+1) : (j,\kappa)\in\mathcal{E},\delta\in\mathbb{N},k'\in\{0,\dots,k\}\}. \tag{49}$$

Note that this is a pre-index set and is an index set if $\mathcal{E}$ is. It is closely related to the notion of the "extended union" of the given index sets, but it can be smaller. The operation $\uplus$ satisfies the following lemma:

Lemma 3.2. *For any $a\in\mathbb{C}$, and $f\in\mathcal{A}^{((1+a,0)\uplus(1+\mathcal{E}))\cup\mathcal{E}}([0,1)_\rho)$,*

$$\rho^a\int_\rho^1 s^{-a-1}f(s)\,\mathrm{d}s\in\mathcal{A}^{(a,0)\uplus\mathcal{E}}([0,1)_\rho) \tag{50}$$

holds. ■

Proof. Suppose that $f\in\mathcal{A}^{\mathcal{G}}([0,1)_\rho)$ for some index set $\mathcal{G}$. Then, we can write

$$f(\rho) = F + \sum_{(j,k)\in\mathcal{G},\Re j\le a} f_{j,k}\rho^j(\log\rho)^k \tag{51}$$

for some $f_{j,k}\in\mathbb{C}$ and $F\in\mathcal{A}^{a+}\cap\mathcal{A}^{\mathcal{G}}([0,1)_\rho)$. So,

$$\rho^a\int_\rho^1 s^{-a-1}f(s)\,\mathrm{d}s = \rho^a\int_\rho^1 s^{-a-1}F(s)\,\mathrm{d}s + \sum_{(j,k)\in\mathcal{G},\Re j\le a} f_{j,k}\rho^a\int_\rho^1 s^{j-a-1}(\log s)^k\,\mathrm{d}s. \tag{52}$$

The first term on the right-hand side is in $\mathcal{A}^{\mathcal{G}}([0,1)_\rho)$ as well. On the other hand,

$$\rho^a \int_\rho^1 s^{j-a-1}(\log s)^k \,\mathrm{d}s \in \begin{cases} \mathcal{A}^{\mathcal{G}} & (j \neq a) \\ \mathcal{A}^{(a,k+1)} & (j = a). \end{cases} \tag{53}$$

So,

$$\rho^a \int_\rho^1 s^{-a-1} f(s) \,\mathrm{d}s \in \mathcal{A}^{(a,0)\uplus\mathcal{G}}([0,1)_\rho). \tag{54}$$

If $\mathcal{G} = ((1+a,0) \uplus (1+\mathcal{E})) \cup \mathcal{E}$, then $(a,0) \uplus \mathcal{G} = (a,0) \uplus (((1+a,0) \uplus (1+\mathcal{E})) \cup \mathcal{E}) = (a,0) \uplus \mathcal{E}$, so we are done. □

The *normal operator* of P at bf is defined by

$$N_{\mathrm{bf}}(P) = 2i\sigma \frac{\partial}{\partial r} + \frac{i\sigma(d-1)}{r} + \frac{\sigma^2 \mathfrak{m}}{r}. \tag{55}$$

The sense in which this deserves to be called the "normal operator" is that

$$P - N_{\mathrm{bf}}(P) \in \mathrm{Diff}_{\mathrm{b}}^{2,-2,-2,0}(X^+_{\mathrm{res}}). \tag{56}$$

Thus, $P - N_{\mathrm{bf}}(P)$ is faster decaying than $P \in \mathrm{Diff}_{\mathrm{b}}^{2,-1,-2,0}(X^+_{\mathrm{res}})$ (as measured in orders of b-decay) at bf. We will see that the asymptotic behavior of the resolvent output at bf is determined by the "indicial roots" of $N_{\mathrm{bf}}(P)$. These are defined to be the roots of the polynomial in a formed by replacing $r\partial_r$ in $rN_{\mathrm{bf}}(P)$ by $-a$. Concretely, this polynomial is $-2i\sigma a + i\sigma(d-1) + \sigma^2\mathfrak{m}$. So, the only indicial root is

$$a = (d-1)/2 - i\sigma\mathfrak{m}/2. \tag{57}$$

It is worth noting that $\mathfrak{m}$ only affects the imaginary part of a, whereas it is the real part that affects decay rates (e.g. in ρ^a).

Proposition 3.3. *Let $\alpha > 0$ and $\mathcal{E}$ denote an index set such that $\mathcal{E} \subset \mathbb{C}_{>0} \times \mathbb{N}$, where $\mathbb{C}_{>\beta} = \{z \in \mathbb{C} : \Re z > \beta\}$. Suppose that $f \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{1+\mathcal{E},1+\alpha}(X))$, $u \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{0+}(X))$, and also that $Pu = f$. Then,*

$$u \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{(a,0)\uplus\mathcal{E},\alpha}(X)) \tag{58}$$

holds. This applies, in particular, to the solution $u = P^{-1}f$ to the Sommerfeld problem. ■

Proof. It suffices to restrict attention to $\dot{X}^+_{\mathrm{res}} \subset X^+_{\mathrm{res}}$. Let S denote the set of $\beta \in \mathbb{R}$ such that

$$u \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{(a,0)\uplus\mathcal{E},\beta\wedge\alpha}(\dot{X})), \tag{59}$$

where the '$\wedge$' means minimum. Because we are assuming that $u \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{0+}(X))$, we know that $(-\infty, \epsilon] \subset S$ for some $\epsilon > 0$. Our goal is to show that $S = \mathbb{R}$. Suppose that $\beta \in S$.

Writing $P = N_{\mathrm{bf}}(P) + (P - N_{\mathrm{bf}}(P))$, we have

$$N_{\mathrm{bf}}(P)u = Pu - (P - N_{\mathrm{bf}}(P))u = f - (P - N_{\mathrm{bf}}(P))u. \tag{60}$$

Because $P - N_{\mathrm{bf}}(P) \in \rho_{\mathrm{tf}}^2 \rho_{\mathrm{bf}}^2 \mathrm{Diff}_{\mathrm{b}}^2(\dot{X}^+_{\mathrm{res}})$, we have $(P - N_{\mathrm{bf}}(P))u \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{(a+2,0)\uplus(2+\mathcal{E}),2+\beta\wedge\alpha}(\dot{X}))$. So, letting $f_0 = f - (P - N_{\mathrm{bf}}(P))u$, so that eq. (60) reads $N_{\mathrm{bf}}(P) = f_0$,

$$f_0 \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{((2+a,0)\uplus(2+\mathcal{E}))\cup(1+\mathcal{E}),\min\{1+\alpha,2+\beta\}}(X)). \tag{61}$$

Using the explicit formula eq. (55), eq. (60) reads

$$(r\partial_r + a)u = f_1 \tag{62}$$

for $f_1 = irf_0/2\sigma \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{((1+a,0)\uplus(1+\mathcal{E}))\cup\mathcal{E},\min\{\alpha,1+\beta\}}(X))$. This first-order ODE can be solved explicitly via integrating in the radial direction. Rewriting it in terms of $\rho = 1/r$, it becomes $(\rho\partial_\rho - a)u = -f_1$. That is, $\partial_\rho(\rho^{-a}u) = -\rho^{-1-a}f_1$. So,

$$u(\sigma,\rho,\theta) = \rho^a\Big(c(\sigma,\theta) + \int_\rho^{1/2} \frac{f_1(\sigma,s,\theta)}{s^{1+a}} \,\mathrm{d}s\Big) \tag{63}$$

for some $c(\sigma,\theta) : (0,1)_\sigma \times \partial X_\theta \to \mathbb{C}$. This satisfies $c(\sigma,\theta) = 2^a u(\sigma,1/2,\theta) \in \mathcal{S}([0,1)_\sigma; C^\infty(\partial X_\theta))$, so c is Schwartz as $\sigma \to 0^+$. Then, applying Lemma 3.2 gives $u \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{(a,0)\uplus\mathcal{E},\alpha\wedge(1+\beta)}(\dot{X}))$, i.e. that $1+\beta \in S$.

Proceeding inductively, this allows the conclusion that $S = \mathbb{R}$. □

4. Lemma producing $O(\rho_{\mathrm{zf}})$-quasimodes

We discuss in §A the mapping properties of $P(0)^{-1}$ with respect to function spaces

$$\mathcal{A}^\bullet(X;\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}^\bullet_{\mathrm{c}}([0,1))\mathcal{Y}_j, \tag{64}$$

whose definition is in that appendix. In this section, we apply the fruits of that discussion to the construction of $O(\rho_{\mathrm{zf}})$-quasimodes, the first step in our proof of our main theorem. For this section and the ones that follow, let P be as in §2, and fix $\ell \in \mathbb{N}$.

What we want to do is, given forcing f which is polyhomogeneous on X^+_{res}, to find a polyhomogeneous u such that $Pu - f$ is decaying at zf relative to f. This is accomplished by the following lemma. The function spaces used in the statement of the lemma are denoted

$$\mathcal{A}^\bullet(X^+_{\mathrm{res}};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}^\bullet_{\mathrm{c}}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j. \tag{65}$$

These are defined analogously to the spaces in eq. (64). Namely, the first term contains functions on X^+_{res} such that, near the large-r boundary (i.e. on $\dot{X}^+_{\mathrm{res}} = [0,1)^+_{\mathrm{res}} \times \partial X_\theta$), the function is orthogonal to the harmonics in $\mathcal{Y}_0 \cup \cdots \cup \mathcal{Y}_{\ell-1}$. The sum contains functions on $\dot{X}^+_{\mathrm{res}}$ independent of θ times the elements of the $\mathcal{Y}_j$'s.

Lemma 4.1. *Suppose that*

$$f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{F},(0,0)}(X^+_{\mathrm{res}};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_j,2+\mathcal{F}_j,(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j, \tag{66}$$

where the index sets are listed at bf, tf, zf, *respectively. Suppose moreover that* $\mathcal{F}_\bullet \subset \mathbb{C}_{>0} \times \mathbb{N}$. *Then, there exists*

$$u \in \mathcal{A}^{\infty,\mathcal{I},(0,0)}(X^+_{\mathrm{res}};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\infty,\mathcal{I}_j,(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j, \tag{67}$$

where $\mathcal{I},\mathcal{I}_j$ *are as in Theorem D (these satisfying* $\min \Pi\mathcal{I} \geq \min\{\Pi\mathcal{F}_\cup, c_\ell\}$, $\min \Pi\mathcal{I}_j \geq \min\{\Pi\mathcal{F}_\cup, c_j\}$*), such that*

$$Pu - f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{I}^+,(1,0)}(X^+_{\mathrm{res}};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_j,2+\mathcal{I}_j^+,(1,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j, \tag{68}$$

where $\mathcal{I}^+_\bullet$ *are defined by eq. (79).* ■

Here, $\Pi\mathcal{E} = \{j : (j,k) \in \mathcal{E}\}$ is just the set of first components of elements of $\mathcal{E}$, and $\mathbb{C}_{>\alpha} = \{z \in \mathbb{C} : \Re z > \alpha\}$. Thus, u is an $O(\rho_{\mathrm{zf}})$-quasimode (though we may have worsened decay at tf).

Proof. The restriction $f|_{\mathrm{zf}^\circ}$ is well-defined, since f is smooth at zf°. (We will drop the '$\circ$' in $f|_{\mathrm{zf}^\circ}$.) This satisfies

$$f|_{\mathrm{zf}} \in \mathcal{A}^{2+\mathcal{F}}(X;\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{2+\mathcal{F}_j}([0,1)_\rho)\mathcal{Y}_j. \tag{69}$$

Via the solvability theory of $N_{\mathrm{zf}}(P) = \triangle_g + L$ embedded in Theorem D, we have a well-defined element

$$N_{\mathrm{zf}}(P)^{-1} f|_{\mathrm{zf}} \in \mathcal{A}^{\mathcal{I}[\mathcal{F},\ell]}(X;\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{I}_j[\mathcal{F}_j]}([0,1)_\rho)\mathcal{Y}_j, \tag{70}$$

where $\mathcal{I}, \mathcal{I}_j$ are as in Theorem D. In particular, the stated bounds on $\min \Pi \mathcal{I}_\bullet$ hold.

Let $u_0 = \chi_0(r\sigma) N_{\mathrm{zf}}(P)^{-1} f|_{\mathrm{zf}}$ for $\chi_0 \in C_{\mathrm{c}}^\infty(\mathbb{R})$ identically 1 near the origin. Then, $\chi_0(r\sigma)$ is identically 1 near zf and vanishing near bf. So, if we interpret $N_{\mathrm{zf}}(P)^{-1} f|_{\mathrm{zf}}$ as a function on $X^{+\circ}_{\mathrm{res}}$ that is just independent of σ, then

$$u_0 \in \mathcal{A}^{\infty,\mathcal{I}[\mathcal{F}_\bullet,\ell],(0,0)}(X^+_{\mathrm{res}};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\infty,\mathcal{I}_j[\mathcal{F}_\bullet],(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j. \tag{71}$$

This has the form specified in eq. (67).

We now check that eq. (68) holds. Decompose

$$Pu_0 - f = (\chi_0 f|_{\mathrm{zf}} - f) + [N_{\mathrm{zf}}(P), \chi_0] N_{\mathrm{zf}}(P)^{-1} f|_{\mathrm{zf}} + (P - N_{\mathrm{zf}}(P))u_0. \tag{72}$$

Let us analyze each piece of this.

- It immediately follows from the stated regularity of $f|_{\mathrm{zf}}$ that

$$\chi_0 f|_{\mathrm{zf}} \in \mathcal{A}^{\infty,2+\mathcal{F},(0,0)}(X^+_{\mathrm{res}};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\infty,2+\mathcal{F}_j,(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j. \tag{73}$$

  So,

$$\chi_0 f|_{\mathrm{zf}} - f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{F},(1,0)}(X^+_{\mathrm{res}};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E},2+\mathcal{F}_j,(1,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j, \tag{74}$$

  where the improvement at zf of one order comes from the fact that f agrees with $\chi_0 f|_{\mathrm{zf}}$ at zf.
- Because $[N_{\mathrm{zf}}(P), \chi_0]$ is supported away from $\mathrm{zf} \cup \mathrm{bf}$, we have

$$[N_{\mathrm{zf}}(P), \chi_0] \in \rho_{\mathrm{zf}}^\infty \rho_{\mathrm{bf}}^\infty (\rho_{\mathrm{tf}}^2 \operatorname{Diff}_{\mathrm{b}}^1([0,1)^+_{\mathrm{res}}) + \rho^{3+\beth_0} \operatorname{Diff}_{\mathrm{b}}^1(X^+_{\mathrm{res}})). \tag{75}$$

  (The $\rho^{3+\beth_0}$ bound on the non-radial part comes from eq. (18), eq. (19).) Therefore,

$$\begin{aligned} [N_{\mathrm{zf}}(P), \chi_0] N_{\mathrm{zf}}(P)^{-1} f|_{\mathrm{zf}} \in{}& \mathcal{A}^{\infty,(2+\mathcal{I}[\mathcal{F}_\bullet,\ell])\cup(3+\beth_0+\mathcal{I}_\cup),\infty}(X^+_{\mathrm{res}};\ell) \\ &+ \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\infty,(2+\mathcal{I}_j[\mathcal{F}_\bullet])\cup(3+\beth_0+\mathcal{I}_\cup)),\infty}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j, \end{aligned} \tag{76}$$

  where $\mathcal{I}_\cup = \mathcal{I}[\mathcal{F}_\bullet,\ell] \cup \bigcup_{j=0}^{\ell-1} \mathcal{I}_j[\mathcal{F}_\bullet]$.
- Finally, note that

$$\begin{aligned} \sigma^{-1}(P(\sigma) - N_{\mathrm{zf}}(P)) &= 2i(1-\chi)\frac{\partial}{\partial r} + \frac{i(d-1)}{r} + Q + \sigma R \\ &\in \rho_{\mathrm{bf}}\rho_{\mathrm{tf}} \operatorname{Diff}_{\mathrm{b}}^1([0,1)^+_{\mathrm{res}}) + \rho_{\mathrm{tf}}^{(2+\beth_2)\wedge(3+\beth_4)} \rho_{\mathrm{bf}}^{2+\beth_2\wedge\beth_4} \operatorname{Diff}_{\mathrm{b}}^1(X^+_{\mathrm{res}}). \end{aligned} \tag{77}$$

  So,

$$\begin{aligned} \sigma^{-1}(P(\sigma) - N_{\mathrm{zf}}(P))u_0 \in{}& \mathcal{A}^{\infty,(1+\mathcal{I}[\mathcal{F}_\bullet,\ell])\cup(2+\beth_2\wedge(1+\beth_4)+\mathcal{I}_\cup),(0,0)}(X^+_{\mathrm{res}};\ell) \\ &+ \sum_{j=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\infty,(1+\mathcal{I}_j[\mathcal{F}_\bullet])\cup(2+\beth_2\wedge(1+\beth_4)+\mathcal{I}_\cup),(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j. \end{aligned} \tag{78}$$

Combining everything, we get eq. (68), where

$$\begin{aligned}\mathcal{I}^+ &= (2+\mathcal{F})\cup(2+\mathcal{I}[\mathcal{F}_\bullet,\ell])\cup(3+\beth_0+\mathcal{I}_\cup)\cup(2+\beth_2\wedge(1+\beth_4)+\mathcal{I}_\cup),\\ \mathcal{I}_j^+ &= (2+\mathcal{F}_j)\cup(2+\mathcal{I}_j[\mathcal{F}_\bullet])\cup(3+\beth_0+\mathcal{I}_\cup)\cup(2+\beth_2\wedge(1+\beth_4)+\mathcal{I}_\cup).\end{aligned} \tag{79}$$

□

We actually want to apply this with more general index sets at zf:

Lemma 4.2. *Let $\kappa\in\mathbb{N}$. Suppose that $\mathcal{F}_\bullet,\mathcal{J}_\bullet\subset\mathbb{C}_{>0}\times\mathbb{N}$ are some index sets, and that*

$$f\in\mathcal{A}^{1+\mathcal{E},2+\mathcal{F},(0,\kappa)\cup\mathcal{J}}(X^+_{\mathrm{res}};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_j,2+\mathcal{F}_j,(0,\kappa)\cup\mathcal{J}_j}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j. \tag{80}$$

Then, there exists

$$u\in\mathcal{A}^{\infty,\mathcal{I}[\mathcal{F}_\bullet+(0,\kappa)],(0,\kappa)}(X^+_{\mathrm{res}};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{\infty,\mathcal{I}_j[\mathcal{F}_\bullet+(0,\kappa)],(0,\kappa)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j \tag{81}$$

such that $Pu-f$ satisfies eq. (86). ■

Proof. We can find $\tilde{f}_\varkappa$ in the same spaces as f, with index sets $(0,0)$ in place of $\mathcal{J}_\bullet$ for $\tilde{f}_\bullet$, such that $f=F+\sum_{\varkappa=0}^{\kappa}(\log\rho_{\mathrm{zf}})^{\varkappa}\tilde{f}_\varkappa$ for

$$F\in\mathcal{A}^{1+\mathcal{E},2+\mathcal{F},\mathcal{J}}(X^+_{\mathrm{res}};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_j,2+\mathcal{F}_j,\mathcal{J}_j}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j. \tag{82}$$

We can choose the boundary-defining-functions $\rho_\bullet$ such that $\rho_{\mathrm{zf}}=\sigma/\rho_{\mathrm{tf}}$, and then

$$f=F+\sum_{\varkappa=0}^{\kappa}(\log\sigma)^{\varkappa}\sum_{\varkappa'=\varkappa}^{\kappa}\binom{\varkappa'}{\varkappa}(-\log\rho_{\mathrm{tf}})^{\varkappa'-\varkappa}\tilde{f}_{\varkappa'}. \tag{83}$$

Call the $\varkappa$th summand $f_\varkappa$. Then, $f=F+\sum_{\varkappa=0}^{\kappa}(\log\sigma)^{\varkappa}f_\varkappa$, and

$$f_\varkappa\in\mathcal{A}^{1+\mathcal{E},2+\mathcal{F}+(0,\kappa-\varkappa),(0,0)}(X^+_{\mathrm{res}};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_j,2+\mathcal{F}_j+(0,\kappa-\varkappa),(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j. \tag{84}$$

We now apply Lemma 4.1 to get

$$u_\varkappa\in\mathcal{A}^{\infty,\mathcal{I}[\mathcal{F}_\bullet+(0,\kappa-\varkappa)],(0,0)}(X^+_{\mathrm{res}};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{\infty,\mathcal{I}_j[\mathcal{F}_\bullet+(0,\kappa-\varkappa)],(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j \tag{85}$$

as in that lemma. Then, defining $u=\sum_{\varkappa=0}^{\kappa}(\log\sigma)^{\varkappa}u_\varkappa$, we see that u satisfies eq. (81) and

$$\begin{aligned}Pu-f&=-F+\sum_{\varkappa=0}^{\kappa}(Pu_\varkappa-f_\varkappa)\\ &\in\mathcal{A}^{1+\mathcal{E},2+\mathcal{I}[\mathcal{F}_\bullet+(0,\kappa)]^+,(1,\kappa)\cup\mathcal{J}}(X^+_{\mathrm{res}};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_j,2+\mathcal{I}_j[\mathcal{F}_\bullet+(0,\kappa)]^+,(1,\kappa)\cup\mathcal{J}_j}([0,1)^+_{\mathrm{res}})\mathcal{Y}_j.\end{aligned} \tag{86}$$

□

5. Lemma upgrading $O(\rho_{\mathrm{zf}})$-quasimodes to $O(\sigma^{0+})$-quasimodes

Given $f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{F},(0,\kappa)\cup\mathcal{J}}(X^+_{\mathrm{res}})$ and index sets $\mathcal{F},\mathcal{J} \subset \mathbb{C}_{>0}\times\mathbb{N}$, we can now find some $u \in \mathcal{A}^{\mathcal{E},\mathcal{I},(0,\kappa)}(X^+_{\mathrm{res}})$ such that

$$Pu - f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{I}^+,\mathcal{J}^+}(X^+_{\mathrm{res}}), \tag{87}$$

where the index sets $\mathcal{I},\mathcal{I}^+,\mathcal{J}^+ \subset \mathbb{C}_{>0}\times\mathbb{N}$ are defined in the previous section. Thus, $Pu-f$ is suppressed by some (typically one in the Euclidean case) order at zf. In this sense, u is an $O(\sigma^{0+})$-quasimode in zf°. But it is not an $O(\sigma^{0+})$-quasimode at $\mathrm{zf}\cap\mathrm{tf}$, let alone at tf, because the index set $\mathcal{I}^+$ may be much larger than $\mathcal{F}$. For example, even if $\mathcal{F}=\varnothing$, in which case $f|_{\mathrm{zf}}$ is Schwartz, it will typically be the case that the set $\mathcal{I}$ is nonempty (think how Coulomb's law results in solutions of Poisson's equation on $\mathbb{R}^3$ decaying like $1/r$ and therefore not Schwartz), so $Pu-f$ will generally not be Schwartz.

Our next task is to solve away the error at tf to an arbitrarily high degree, though we may find later that it is only useful to solve away a few terms. This process will end up costing us logarithmic losses at zf. Since these logarithms are the ultimate source of Price's law — see the introduction of [Hin22] — this loss is notable.

Thus we want, for $g \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{I}^+,\mathcal{J}^+}(X^+_{\mathrm{res}})$, to find $v \in \mathcal{A}^{\mathcal{E}^+,\mathcal{I}^+,\mathcal{K}}(X^+_{\mathrm{res}})$ such that

$$Pv - g \in \mathcal{A}^{1+\mathcal{E}^+,K,\mathcal{K}^+}(X^+_{\mathrm{res}}), \tag{88}$$

where $\mathcal{K},\mathcal{K}^+$ are some index sets such that $\mathcal{K}\subseteq\mathbb{C}_{>0}\times\mathbb{N}$ and $K\in\mathbb{R}$ is arbitrarily large (though $\mathcal{K},\mathcal{K}^+$ may depend on K). Then, if $g=f-Pu$, the function $w=u+v$ satisfies

$$Pw - f \in \mathcal{A}^{1+\mathcal{E}^+,K,\mathcal{K}^+}(X^+_{\mathrm{res}}). \tag{89}$$

So, w is an honest $O(\sigma^{0+})$-quasimode (at least when $\mathcal{E}^+=\mathcal{E}$, which will be the case in practice).

In order to carry this out, we can use the following lemma:

Lemma 5.1. *Fix $\ell\in\mathbb{N}$ and $K_\bullet\in\mathbb{R}$. Suppose that*

$$f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{I},\mathcal{J}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_l,2+\mathcal{I}_l,\mathcal{J}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \tag{90}$$

for some index sets $\mathcal{J}_\bullet\subset\mathbb{C}_{>0}\times\mathbb{N}$. Then, there exists some

$$u \in \mathcal{A}^{\mathcal{E}^+,\mathcal{I},\mathcal{J}\cup\mathcal{K}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{\mathcal{E}_l^+,\mathcal{I}_l,\mathcal{J}_l\cup\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \tag{91}$$

where $\mathcal{E}^+_\bullet = (2^{-1}(d-1),0)\uplus\mathcal{E}_\bullet$ and $\mathcal{K}_\bullet$ are index sets (depending on $K_\bullet$) such that $\min\Pi\mathcal{K}_\bullet \geq \min\{1, b_\bullet+\Pi\mathcal{I}_\bullet\}$, such that

$$Pu - f \in \mathcal{A}^{1+\mathcal{E}^{++},2+\mathcal{I}^+,\mathcal{J}\cup\mathcal{K}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_l^{++},2+\mathcal{I}_l^+,\mathcal{J}_l\cup\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \tag{92}$$

holds, where

$$\begin{aligned}\mathcal{E}^{++}_\bullet = (1+\mathcal{E}_\bullet)\cup((3+\daleth)\wedge(2+\daleth_1)\wedge(1+\tilde{\daleth}_3)+\mathcal{E}^+_\bullet)\\ \cup((3+\daleth_0)\wedge(2+\daleth_2)\wedge(1+\daleth_4)+\mathcal{E}^+_{\bullet,\cup}),\end{aligned} \tag{93}$$

$\mathcal{I}^+ = \mathcal{I}_{\geq K}\cup(3+\daleth\wedge\daleth_1\wedge\daleth_3+\mathcal{I})\cup(3+\daleth_0\wedge\daleth_2\wedge\daleth_4+\mathcal{I}_\cup)$, and $\mathcal{I}^+_l = \mathcal{I}_{l,\geq K_l}\cup(3+\daleth\wedge\daleth_1\wedge\daleth_3+\mathcal{I}_l)$. Specifically, $\mathcal{K},\mathcal{K}_l$ are given by

$$\begin{aligned}\mathcal{K} &= \bigcup_{(j,k)\in\mathcal{I},\Re j\leq K}(j+\mathcal{B}_{\geq\ell}[(1-j,k_j-k)])\\ \mathcal{K}_l &= \bigcup_{(j,k)\in\mathcal{I}_l,\Re j\leq K_l}(b_l+j,0)\uplus(1,k_{l,j}-k),\end{aligned} \tag{94}$$

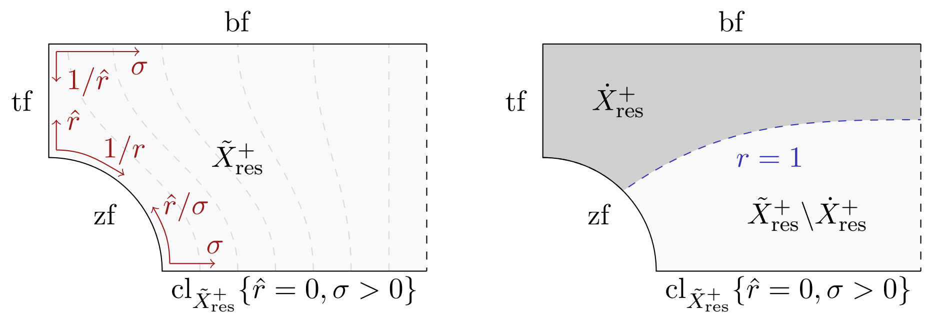

FIGURE 2. The mwc $\tilde{X}^+_{\mathrm{res}}$ (left) and the embedding $\dot{X}^+_{\mathrm{res}} \hookrightarrow \tilde{X}^+_{\mathrm{res}}$ (right). Lines of constant σ are depicted as dashed gray lines in the left figure. Cf. Figure 1.

where $k_{\bullet,j}$ *is the largest* k *such that* $(j,k) \in \mathcal{I}_\bullet$. ■

Remark 5.2. We have stated this theorem with no assumptions on $\mathcal{E}_\bullet, \mathcal{I}_\bullet$, but the result is not useful for the intended applications if $\min \Pi \mathcal{I}_l \leq -b_l$ or $\min \Pi \mathcal{I} \leq -b_\ell$, because then $Pu - f$ might not even be decaying at zf (cf. the discussion above), as then $\mathcal{K}_\bullet$ might have terms (j,k) with $\Re j \leq 0$.

Remark 5.3. Let us highlight that when $\mathcal{I}_\bullet$ is the index set produced by Lemma 4.2, then $\mathcal{I}_l \subset \mathbb{C}_{\geq c_l} \times \mathbb{N}$, and likewise $\mathcal{I} \subset \mathbb{C}_{\geq c_\ell} \times \mathbb{N}$, where $c_j = 2^{-1}(d-2+((d-2)^2+4\lambda_j)^{1/2})$. Recalling that $b_j = -2^{-1}(d-2-((d-2)^2+4\lambda_j)^{1/2})$ it follows that

$$b_j + \min \Pi \mathcal{I}_j \geq \sqrt{(d-2)^2+4\lambda_j} \geq d-2 \geq 1. \tag{95}$$

So, in the application we have in mind, the index sets $\mathcal{K}_\bullet$ do imply decay at zf.

Remark 5.4. When $\mathcal{I}_\bullet \subset \mathbb{C}_{\geq c_\bullet} \times \mathbb{N}$, then, if we choose $K_\bullet$ sufficiently large, $\mathcal{I}^+_\bullet \subset \mathbb{C}_{\geq c_\bullet+3} \times \mathbb{N}$. Thus, if our goal is for $\mathcal{I}_\bullet$ to lie in $\mathbb{C}_{\geq c_\bullet} \times \mathbb{N}$, then $\mathcal{I}^+_\bullet$ is 3 orders better. Combining this observation with the previous remark, we have that, when the assumptions above are satisfied, $Pu - f = O(\sigma^{0+})$ (with respect to the relevant spaces of polyhomogeneous functions), as desired.

Remark 5.5. Note that the index sets defined by eq. (94) depend on K only in $\mathbb{C}_{\geq 1} \times \mathbb{N}$. This will be useful later.

Before proving the theorem, we explain the idea in the case $\mathcal{J}_\bullet = (1,0)$.

Let $\tilde{f} = \tilde{\chi}(\rho) f$, where $\tilde{\chi} \in C^\infty_{\mathrm{c}}((-1,1)_\rho)$ is identically 1 near the origin. Then, we can consider $\tilde{f}$ as a function on $\dot{X}^+_{\mathrm{res}}$, which we embed in

$$\tilde{X}^+_{\mathrm{res}} = [[0,1)_\sigma \times \partial X_\theta \times [0,\infty]_{\hat{r}}; \{\sigma=0,\hat{r}=0\}] \tag{96}$$

via the natural embedding; see Figure 2. Note that closure of $\{\sigma = 0, \hat{r} \in (0,\infty)\} \subset [0,1)_\sigma \times \partial X_\theta \times [0,\infty]_{\hat{r}}$ in $\tilde{X}^+_{\mathrm{res}}$ is identifiable with tf via the embedding $\dot{X}^+_{\mathrm{res}} \hookrightarrow \tilde{X}^+_{\mathrm{res}}$. So, we can write

$$\tilde{f} \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{I},(1,0),\infty}(\tilde{X}^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_l,2+\mathcal{I}_l,(1,0),\infty}(\widetilde{[0,1)}^+_{\mathrm{res}})\mathcal{Y}_l, \tag{97}$$

where the last '∞' in the superscripts denotes Schwartz behavior at $\{\sigma > 0, \hat{r} = 0\} \subset \tilde{X}^+_{\mathrm{res}}$. (Really, $\tilde{f}$ is supported away from this face, so "Schwartz" is an understatement.)

We can consider N_{tf} as a differential operator on $\tilde{X}^{+\circ}_{\mathrm{res}}$. Indeed, since N_{tf}, considered as a differential operator on $X^{+\circ}_{\mathrm{res}}$, commutes with σ, the formula

$$N_{\mathrm{tf}} = \sigma^2 \hat{N}_{\mathrm{tf}}, \quad \hat{N}_{\mathrm{tf}} = -\frac{\partial^2}{\partial \hat{r}^2} - \Big(\frac{d-1}{\hat{r}} + 2i\Big)\frac{\partial}{\partial \hat{r}} + \frac{1}{\hat{r}^2}\triangle_{\partial X} - \frac{i(d-1)}{\hat{r}} \tag{98}$$

can be read as holding on $(\tilde{X}^+_{\mathrm{res}})^\circ$ where the partial derivatives are taken with respect to the coordinate system $(\sigma,\hat{r},\theta)\in\mathbb{R}^+_\sigma\times\mathbb{R}^+_{\hat{r}}\times\partial X_\theta$. Using the canonical diffeomorphism $\mathrm{tf}^\circ=\mathbb{R}^+_{\hat{r}}\times\partial X_\theta$, we are *also* considering $\hat{N}_{\mathrm{tf}}$ as an operator on tf°, given by the same formula eq. (98).

Unfortunately, N_{tf} does not commute with a boundary-defining-function

$$\rho_{\mathrm{tf}}\in C^\infty(\tilde{X}^+_{\mathrm{res}};[0,\infty)). \tag{99}$$

(We previously used "ρ_{tf}" to refer to a boundary-defining-function of tf in X^+_{res}, but since $\tilde{X}^+_{\mathrm{res}}\cong X^+_{\mathrm{res}}$ in a neighborhood of tf, no confusion will arise from changing notation here.) This has the annoying consequence that, in the polyhomogeneous expansion

$$\tilde{f}\sim\sum_{(j,k)\in\mathcal{I}}\rho_{\mathrm{tf}}^{2+j}(\log\rho_{\mathrm{tf}})^k\tilde{f}_{j,k}, \tag{100}$$

though $\tilde{f}_{j,k}=\tilde{f}_{j,k}(\hat{r},\theta)\in C^\infty(\mathrm{tf}^\circ)$ is a function of $\hat{r}$ and θ alone, we should not expect

$$N_{\mathrm{tf}}^{-1}\tilde{f}\sim\sum_{(j,k)\in\mathcal{I}}\rho_{\mathrm{tf}}^{2+j}(\log\rho_{\mathrm{tf}})^kN_{\mathrm{tf}}^{-1}\tilde{f}_{j,k}. \tag{101}$$

Fortunately, we can instead expand in σ:

$$\tilde{f}\sim\sum_{(j,k)\in\mathcal{I}}\sigma^{2+j}(\log\sigma)^k\tilde{f}_{j,k} \tag{102}$$

for some other $\tilde{f}_{j,k}=\tilde{f}_{j,k}(\hat{r},\theta)\in C^\infty(\mathrm{tf}^\circ)$. That is, for any $\beta\in\mathbb{R}$,

$$\tilde{f}-\sum_{(j,k)\in\mathcal{I},\,\Re j\leq\beta}\sigma^{2+j}(\log\sigma)^k\tilde{f}_{j,k}\in\mathcal{A}^{1+\mathcal{E},\beta,(1,0),\infty}(\tilde{X}^+_{\mathrm{res}}). \tag{103}$$

More specifically, $\tilde{f}_{j,k}\in\mathcal{A}^{1+\mathcal{E},(1-j,k_j)}(\mathrm{tf})$; see Lemma 5.6. The cost of expanding in σ rather than in ρ_{tf} is that each term in the expansion is worse than the previous at zf. Fortunately, this does not mean that the *error* that results from truncating the expansion (that is, the function in eq. (103)) needs to be worse at zf. Indeed, in eq. (103), the error is still smooth at zf. Again, see Lemma 5.6 for details.

Having set up these "joint" expansions, we can solve away each $\tilde{f}_{j,k}$, this procedure being facilitated by the fact that

$$[N_{\mathrm{tf}},\sigma^{2+j}(\log\sigma)^k]=0. \tag{104}$$

The expression "N_{tf}^{-1}" means the inverse of N_{tf} on some suitable function spaces in which the $\tilde{f}_{j,k}$ lie, as discussed in §B.

Let us now carry out the details, keeping track of things harmonic-by-harmonic:

Proof of Lemma 5.1. Define $\tilde{f}$ as above, and let $\tilde{f}_l$ denote the component of $\tilde{f}$ in $\mathcal{Y}_l$ and let $\tilde{f}_{\mathrm{rem}}$ denote the component orthogonal to $\mathcal{Y}_0,\ldots,\mathcal{Y}_{\ell-1}$. Letting

$$\tilde{f}_\bullet\sim\sum_{(j,k)\in\mathcal{I}_\bullet}\sigma^{2+j}(\log\sigma)^k\tilde{f}_{\bullet,j,k} \tag{105}$$

denote the joint $\sigma\to0^+$ expansion of $\tilde{f}_\bullet$, we have

$$\tilde{f}_{\mathrm{rem},j,k}\in\mathcal{A}^{1+\mathcal{E},(-1-j,k_j-k)}(\mathrm{tf};\ell),\quad\tilde{f}_{l,j,k}\in\mathcal{A}^{1+\mathcal{E}_l,(-1-j,k_{l,j}-k)}([0,\infty]_{\hat{r}})\mathcal{Y}_l \tag{106}$$

according to Lemma 5.6, where $k_{\bullet,j}$ is the largest k such that (j,k) is in $\mathcal{I}_\bullet$. In the superscripts, the index sets are listed at bf first and zf second. Now let

$$\tilde{v}=\underbrace{\sum_{(j,k)\in\mathcal{I},\Re j\leq K_{\mathrm{rem}}}\sigma^j(\log\sigma)^k\hat{N}_{\mathrm{tf}}^{-1}\tilde{f}_{\mathrm{rem},j,k}}_{\tilde{v}_{\mathrm{rem}}}+\sum_{l=0}^{\ell-1}\underbrace{\sum_{(j,k)\in\mathcal{I}_l,\Re j\leq K_l}\sigma^j(\log\sigma)^k\hat{N}_{\mathrm{tf}}^{-1}\tilde{f}_{l,j,k}}_{\tilde{v}_l}, \tag{107}$$

where $K_\bullet$ is to-be-determined and $\tilde{v}_{\bullet,j,k} = \hat{N}_{\mathrm{tf}}^{-1}\tilde{f}_{\bullet,j,k} \in C^\infty(\mathrm{tf}^\circ)$ solves the PDE $\hat{N}_{\mathrm{tf}}\tilde{v}_{\bullet,j,k} = \tilde{f}_{\bullet,j,k}$, where here N_{tf} is read as an operator on tf. To write this more simply, let

$$f_{\mathrm{approx}} = \underbrace{\sum_{(j,k)\in\mathcal{I},\Re j\leq K_{\mathrm{rem}}} \sigma^{2+j}(\log\sigma)^k \tilde{f}_{\mathrm{rem},j,k}}_{f_{\mathrm{rem,approx}}} + \sum_{l=0}^{\ell-1} \underbrace{\sum_{(j,k)\in\mathcal{I}_l,\Re j\leq K_l} \sigma^{2+j}(\log\sigma)^k \tilde{f}_{l,j,k}}_{f_{l,\mathrm{approx}}} . \tag{108}$$

Then, $\tilde{v} = N_{\mathrm{tf}}^{-1} f_{\mathrm{approx}}$.

By the solvability theory of the model operator $\hat{N}_{\mathrm{tf}}$, specifically Proposition B.2, we can choose

$$\begin{aligned} &\hat{N}_{\mathrm{tf}}^{-1}\tilde{f}_{j,k} \in \mathcal{A}^{(2^{-1}(d-1),0)\uplus\mathcal{E},\mathcal{B}_{\geq\ell}[(1-j,k_j-k)]}(\mathrm{tf};\ell) + \textstyle\sum_{l=0}^{\ell-1}\mathcal{A}^{(2^{-1}(d-1),0),(b_l,0)}([0,\infty]_{\hat{r}})\mathcal{Y}_l, \\ &\hat{N}_{\mathrm{tf}}^{-1}\tilde{f}_{l,j,k} \in \mathcal{A}^{(2^{-1}(d-1),0)\uplus\mathcal{E}_l,(b_l,0)\uplus(1-j,k_{l,j}-k)}([0,\infty]_{\hat{r}})\mathcal{Y}_l, \end{aligned} \tag{109}$$

where we used that $\min\Pi\mathcal{E}_\bullet > 2^{-1}(d-1)$. (And then, $\hat{N}_{\mathrm{tf}}^{-1}\tilde{f}_{\bullet,j,k}$ is unique.) So,

$$\begin{aligned}\tilde{v} \in &\sum_{(j,k)\in\mathcal{I},\Re j\leq K} \sigma^j(\log\sigma)^k \mathcal{A}^{(2^{-1}(d-1),0)\uplus\mathcal{E},\mathcal{B}_{\geq\ell}[(1-j,k_j-k)]}(\mathrm{tf};\ell) \\ &+ \sum_{l=0}^{\ell-1}\sum_{(j,k)\in\mathcal{I}_l,\Re j\leq K_l} \sigma^j(\log\sigma)^k\mathcal{A}^{(2^{-1}(d-1),0)\uplus\mathcal{E}_l,(b_l,0)\uplus(1-j,k_{l,j}-k)}([0,\infty]_{\hat{r}})\mathcal{Y}_l. \end{aligned} \tag{110}$$

Notice that

$$\sigma^j(\log\sigma)^k\mathcal{A}^{(2^{-1}(d-1),0)\cup\mathcal{E},\mathcal{B}_{\geq\ell}[(1-j,k_j-k)]}(\mathrm{tf};\ell) \subseteq \mathcal{A}^{(2^{-1}(d-1),0)\cup\mathcal{E},(j,k),j+\mathcal{B}_{\geq\ell}[(1-j,k_j-k)],-\infty}(\tilde{X}^+_{\mathrm{res}};\ell) \tag{111}$$

and

$$\begin{aligned}&\sigma^j(\log\sigma)^k\mathcal{A}^{(2^{-1}(d-1),0)\cup\mathcal{E}_l,(b_l,0)\uplus(1-j,k_{l,j}-k)}([0,\infty]_{\hat{r}}) \\ &\qquad\subseteq \mathcal{A}^{(2^{-1}(d-1),0)\cup\mathcal{E}_l,(j,k),(b_l+j,0)\uplus(1,k_{l,j}-k),-\infty}(\widetilde{[0,1)}^+_{\mathrm{res}}).\end{aligned} \tag{112}$$

So,

$$\tilde{v} \in \mathcal{A}^{\mathcal{E}^+,\mathcal{I},\mathcal{K},-\infty}(\tilde{X}^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1}\mathcal{A}^{\mathcal{E}^+_l,\mathcal{I}_l,\mathcal{K}_l,-\infty}(\widetilde{[0,1)}^+_{\mathrm{res}})\mathcal{Y}_l \tag{113}$$

for $\mathcal{E}^+_\bullet = (2^{-1}(d-1),0)\uplus\mathcal{E}_\bullet$, $\mathcal{K} = \bigcup_{(j,k)\in\mathcal{I},\Re j\leq K}(j+\mathcal{B}_{\geq\ell}[(1-j,k_j-k)])$, and $\mathcal{K}_l = \bigcup_{(j,k)\in\mathcal{I}_l,\Re j\leq K_l}(b_l+j,0)\uplus(1,k_{l,j}-k)$. Observe that

$$\min\Pi\mathcal{K}_\bullet \geq \min\{1, b_\bullet + \Pi\mathcal{I}_\bullet\}, \tag{114}$$

where $b_{\mathrm{rem}} = b_\ell$.

Now let $v = \chi_1(\sigma/\hat{r})\tilde{v}$ for $\chi_1 \in C_{\mathrm{c}}^\infty((-\infty,1))$ such that $\chi_1 = 1$ identically near the origin. We can interpret v as a function on $\dot{X}^+_{\mathrm{res}}$ and therefore on X^+_{res}. We have

$$v \in \mathcal{A}^{\mathcal{E}^+,\mathcal{I},\mathcal{K}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{\mathcal{E}^+_l,\mathcal{I}_l,\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l. \tag{115}$$

Let v_{rem} denote the first term on the right-hand side and v_l the lth term in the sum. That is, $v_\bullet = \chi_1(\sigma/\hat{r})\tilde{v}_\bullet$.

Let us now examine

$$Pv_\bullet - f_\bullet = (\chi_1 f_{\bullet,\mathrm{approx}} - f_\bullet) + [N_{\mathrm{tf}},\chi_1]\tilde{v}_\bullet + (P - N_{\mathrm{tf}})v_\bullet, \tag{116}$$

piece-by-piece. Here, $f_\bullet$ denote the portions of f in each of the terms on the right-hand side of eq. (90).

- We split $\chi_1 f_{\bullet,\mathrm{approx}} - f_\bullet = \chi_1(f_{\bullet,\mathrm{approx}} - \tilde{f}_\bullet) + (\chi_1 \tilde{f}_\bullet - f_\bullet)$. Beginning with the first term, we have, from eq. (103) (applied with $f_\bullet$ in place of f), that
$$\tilde{f}_\bullet - f_{\bullet,\mathrm{approx}} \in \mathcal{A}^{1+\mathcal{E}_\bullet, K_\bullet, \mathcal{J}_\bullet, -\infty}. \tag{117}$$
(Here, we are omitting the "$(X^+_{\mathrm{res}};\ell)$" or "$([0,1)^+_{\mathrm{res}})\mathcal{Y}_l$" from the notation, as it depends on $\bullet$.) Multiplying by χ_1 then yields
$$\begin{aligned} \chi_1(\tilde{f}_{\mathrm{rem}} - f_{\mathrm{rem,approx}}) &\in \mathcal{A}^{1+\mathcal{E},K_{\mathrm{rem}},\mathcal{J}}(X^+_{\mathrm{res}};\ell), \\ \chi_1(\tilde{f}_l - f_{l,\mathrm{approx}}) &\in \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_l,K_l,\mathcal{J}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l. \end{aligned} \tag{118}$$
The term $\chi_1 \tilde{f}_\bullet - f_\bullet$ vanishes identically near tf, so
$$\chi_1 \tilde{f}_\bullet - f_\bullet \in \mathcal{A}^{1+\mathcal{E}_\bullet,\infty,\mathcal{J}_\bullet}. \tag{119}$$
So, overall, $\chi_1 f_{\bullet,\mathrm{approx}} - f_\bullet \in \mathcal{A}^{1+\mathcal{E}_\bullet,K_\bullet,\mathcal{J}_\bullet}$. Actually, since the left-hand side is polyhomogeneous, we can improve this to $\chi_1 f_{\bullet,\mathrm{approx}} - f_\bullet \in \mathcal{A}^{1+\mathcal{E}_\bullet,\mathcal{I}_{\bullet,\geq K_\bullet},\mathcal{J}_\bullet}$.
- On the other hand, $[N_{\mathrm{tf}},\chi_1] \in \mathrm{Diff}_{\mathrm{b}}^{2,-\infty,-\infty,0}(X^+_{\mathrm{res}})$, where the two $-\infty$ superscripts denote Schwartz coefficients at $\mathrm{tf}\cup\mathrm{bf}$. So, $[N_{\mathrm{tf}},\chi_1]\tilde{v}_\bullet \in \mathcal{A}^{\infty,\infty,\mathcal{K}_\bullet}$.
- Finally,

$$\begin{aligned} P - N_{\mathrm{tf}} &= (\triangle_g - \triangle_{g_0}) + L + \sigma Q + \sigma^2 R \\ &\in \rho_{\mathrm{bf}}^{(3+\beth)\wedge(2+\beth_1)\wedge(1+\beth_3)} \rho_{\mathrm{tf}}^{3+\beth\wedge\beth_1\wedge\beth_3} \mathrm{Diff}^2_{\mathrm{b}}([0,1)^+_{\mathrm{res}}) + \rho_{\mathrm{bf}}^{(3+\beth_0)\wedge(2+\beth_2)\wedge(1+\beth_4)} \rho_{\mathrm{tf}}^{3+\beth_0\wedge\beth_2\wedge\beth_4} \mathrm{Diff}^2_{\mathrm{b}}(\dot{X}), \end{aligned} \tag{120}$$

so $(P - N_{\mathrm{tf}})v_{\mathrm{rem}} \in \mathcal{A}^{\mathcal{E}^{++},\mathcal{I}^+,\mathcal{K}}(X^+_{\mathrm{res}};\ell)$ for
$$\begin{aligned} \mathcal{E}^{++} &= ((3+\beth)\wedge(2+\beth_1)\wedge(1+\beth_3) + \mathcal{E}^+) \cup ((3+\beth_0)\wedge(2+\beth_2)\wedge(1+\beth_4) + \mathcal{E}^+_\cup) \\ \mathcal{I}^+ &= (3+\beth\wedge\beth_1\wedge\beth_3 + \mathcal{I}) \cup (3+\beth_0\wedge\beth_2\wedge\beth_4 + \mathcal{I}_\cup) \end{aligned} \tag{121}$$
for $\mathcal{E}^+_\cup = \cup_\bullet \mathcal{E}^+_\bullet$, and similarly for $\mathcal{I}_\cup$. Likewise, $(P-N_{\mathrm{tf}})v_l \in \mathcal{A}_{\mathrm{c}}^{\mathcal{E}_l^{++},\mathcal{I}_l^+,\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l$ for
$$\begin{aligned} \mathcal{E}_l^{++} &= ((3+\beth)\wedge(2+\beth_1)\wedge(1+\beth_3) + \mathcal{E}_l^+) \\ \mathcal{I}_l^+ &= 3+\beth\wedge\beth_1\wedge\beth_3 + \mathcal{I}_l. \end{aligned} \tag{122}$$

So, all in all,
$$Pv - f \in \mathcal{A}^{(1+\mathcal{E})\cup\mathcal{E}^{++},\mathcal{I}_{\geq K_{\mathrm{rem}}}\cup\mathcal{I}^+,\mathcal{K}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(1+\mathcal{E}_l)\cup\mathcal{E}_l^{++},\mathcal{I}_{l,\geq K_l}\cup\mathcal{I}_l^+,\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l, \tag{123}$$
which has the desired form, though we used different notation in the proof versus in the theorem statement for the various index sets. □

Lemma 5.6. *Suppose that $\mathcal{E},\mathcal{F},\mathcal{G}$ are index sets and $f \in \mathcal{A}^{\mathcal{E},\mathcal{F},\mathcal{G}}(\dot{X}^+_{\mathrm{res}})$. Then, there exist $f_{j,k}(\hat{r},\theta) \in \mathcal{A}^{\mathcal{E},(-j,k_j-k)+\mathcal{G}}([0,\infty]_{\hat{r}} \times \partial X_\theta) = \mathcal{A}^{\mathcal{E},(-j,k_j-k)+\mathcal{G}}(\mathrm{tf})$ such that*
$$f \sim \sum_{(j,k)\in\mathcal{F}} \sigma^j (\log\sigma)^k f_{j,k} \tag{124}$$
in the sense that, for every $\beta \in \mathbb{R}$,
$$f - \sum_{(j,k)\in\mathcal{F},\,\Re j\leq\beta} \sigma^j(\log\sigma)^k f_{j,k} \in \mathcal{A}^{\mathcal{E},\beta,\mathcal{G}}(\dot{X}^+_{\mathrm{res}}). \tag{125}$$
Here, k_j is the largest k such that $(j,k)\in\mathcal{F}$. ■

Remark 5.7. Analogues exist if we restrict attention to functions orthogonal to the first few spherical harmonics near ∂X.

Proof. Let $\tilde{f}_{j,k} \in \mathcal{A}^{\mathcal{E},\mathcal{G}}(\mathrm{tf})$ denote the terms in the polyhomogeneous expansion of f at tf, so

$$f \sim \sum_{(j,k)\in\mathcal{F}} \rho_{\mathrm{tf}}^j (\log \rho_{\mathrm{tf}})^k \tilde{f}_{j,k}, \tag{126}$$

in the usual sense that, for every $\beta \in \mathbb{R}$,

$$f - \sum_{(j,k)\in\mathcal{F},\, \Re j\leq \beta} \rho_{\mathrm{tf}}^j (\log \rho_{\mathrm{tf}})^k \tilde{f}_{j,k} \in \mathcal{A}^{\mathcal{E},\beta,\mathcal{G}}(\dot{X}^+_{\mathrm{res}}) \tag{127}$$

holds. Note that we can consider $\tilde{f}_{j,k}$ as an element of $\mathcal{A}^{\mathcal{E},(0,0),\mathcal{G}}(\dot{X}^+_{\mathrm{res}})$ which only depends on $\hat{r},\theta$.

We have $\sigma \in \rho_{\mathrm{tf}}\rho_{\mathrm{zf}}C^\infty(X^+_{\mathrm{res}};\mathbb{R}^+)$. That is, there exists some positive $\varsigma \in C^\infty(X^+_{\mathrm{res}})$ such that $\sigma\varsigma = \rho_{\mathrm{tf}}\rho_{\mathrm{zf}}$. For the sake of this computation, it is useful to note that we can take ρ_{zf} as a function of $\hat{r}$ alone (though $\hat{r},\theta$ would also suffice) and, simultaneously, that we can take $\varsigma = 1$. Indeed, these conditions are met by choosing $\rho_{\mathrm{tf}} = \sigma+\rho$ and $\rho_{\mathrm{zf}} = \sigma/(\sigma+\rho) = \hat{r}/(1+\hat{r})$. We have

$$\rho_{\mathrm{tf}}^j (\log \rho_{\mathrm{tf}})^k \tilde{f}_{j,k} = \Big(\frac{\sigma}{\rho_{\mathrm{zf}}}\Big)^j \Big(\log\Big(\frac{\sigma}{\rho_{\mathrm{zf}}}\Big)\Big)^k \tilde{f}_{j,k} = \sigma^j \sum_{\kappa=0}^k (\log\sigma)^\kappa \binom{k}{\kappa} (-\log\rho_{\mathrm{zf}})^{k-\kappa}\frac{\tilde{f}_{j,k}}{\rho_{\mathrm{zf}}^j}. \tag{128}$$

Thus, for fixed j,

$$\begin{aligned}\sum_{k \text{ s.t. } (j,k)\in\mathcal{F}} \rho_{\mathrm{tf}}^j (\log \rho_{\mathrm{tf}})^k \tilde{f}_{j,k} &= \sigma^j \sum_{k \text{ s.t. } (j,k)\in\mathcal{F}} \sum_{\kappa=0}^k (\log\sigma)^\kappa \binom{k}{\kappa} (-\log\rho_{\mathrm{zf}})^{k-\kappa}\frac{\tilde{f}_{j,k}}{\rho_{\mathrm{zf}}^j} \\ &= \sigma^j \sum_{\kappa=0}^{k_j} (\log\sigma)^\kappa \sum_{k=\kappa}^{k_j} \binom{k}{\kappa} (-\log\rho_{\mathrm{zf}})^{k-\kappa}\frac{\tilde{f}_{j,k}}{\rho_{\mathrm{zf}}^j} = \sigma^j \sum_{\kappa=0}^{k_j} (\log\sigma)^\kappa f_{j,\kappa}\end{aligned} \tag{129}$$

for

$$f_{j,\kappa} = \sum_{k=\kappa}^{k_j} \binom{k}{\kappa} (-\log\rho_{\mathrm{zf}})^{k-\kappa}\frac{\tilde{f}_{j,k}}{\rho_{\mathrm{zf}}^j} \in \mathcal{A}^{\mathcal{E},(-j,k_j-k)+\mathcal{G}}(\mathrm{tf}). \tag{130}$$

Note that $f_{j,\kappa}$ depends only on the coordinates $\hat{r},\theta$.

□

6. Main proof

This section contains the proof of Theorem A. The first step is the construction of $O(\sigma^{0+})$-quasimodes, which is just a matter of combining the lemmas in the previous two sections. What we want to do is, given $f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{F},\mathcal{J}}(X^+_{\mathrm{res}})$, for appropriate index sets $\mathcal{E},\mathcal{F},\mathcal{J} \subset \mathbb{C}_{\geq 0}\times\mathbb{N}$, solve the quasimode construction problem

$$Pu = f \bmod \sigma^{0+}\mathcal{A}^{1+\mathcal{E}^+,2+\mathcal{F}^+,\mathcal{J}^+}(X^+_{\mathrm{res}}) \tag{131}$$

for some index sets $\mathcal{E}^+,\mathcal{F}^+,\mathcal{J}^+$ which differ from $\mathcal{E},\mathcal{F},\mathcal{J}$ with regards to which subleading indices are present.

Proposition 6.1. *Suppose that $\mathcal{E}_\bullet,\mathcal{F}_\bullet,\mathcal{J}_\bullet$ are index sets such that $\min \Pi\mathcal{E} > (d-1)/2$, $\mathcal{F}_\bullet \subset \mathbb{C}_{>0}\times\mathbb{N}$ and $\mathcal{J}_\bullet \subset \mathbb{C}_{\geq 0}\times\mathbb{N}$. Suppose that*

$$f \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{F},\mathcal{J}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_l,2+\mathcal{F}_l,\mathcal{J}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l. \tag{132}$$

Then, there exists $\varepsilon > 0$ such that there exist index sets $\mathcal{E}^+_\bullet, \mathcal{F}^+_\bullet, \mathcal{J}^+_\bullet$, $\mathcal{E}^{++}_\bullet, \mathcal{F}^{++}_\bullet, \mathcal{J}^{++}_\bullet$ satisfying the same hypotheses as their predecessors such that there exist

$$\begin{aligned} u &\in \mathcal{A}^{\mathcal{E}^+,\mathcal{F}^+,\mathcal{J}^+}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{E}^+_l,\mathcal{F}^+_l,\mathcal{J}^+_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \\ g &\in \mathcal{A}^{1+\mathcal{E}^{++},2+\mathcal{F}^{++},\mathcal{J}^{++}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}^{++}_l,2+\mathcal{F}^{++}_l,\mathcal{J}^{++}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \end{aligned} \tag{133}$$

such that $Pu = f + \sigma^\varepsilon g$. ■

Proof. First, Lemma 4.2 gives $w \in \mathcal{A}^{\infty,\mathcal{I}[\mathcal{F}_\bullet+(0,\kappa)],(0,\kappa)}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\infty,\mathcal{I}_l[\mathcal{F}_\bullet+(0,\kappa)],(0,\kappa)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l$ such that $Pw = f + h$ for

$$h \in \mathcal{A}^{1+\mathcal{E},2+\mathcal{I}[\mathcal{F}_\bullet+(0,\kappa)]^+,(1,\kappa)\cup\mathcal{J}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{1+\mathcal{E}_l,2+\mathcal{I}_l[\mathcal{F}_\bullet+(0,\kappa)]^+,(1,\kappa)\cup\mathcal{J}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l, \tag{134}$$

where $\mathcal{I}, \mathcal{I}_l, \mathcal{I}^+, \mathcal{I}^+_l$ are as in that lemma. Now we apply Lemma 5.1 with h in place of f. In this way, we get

$$v \in \mathcal{A}^{\mathcal{E}^+,\mathcal{I}[\mathcal{F}_\bullet+(0,\kappa)]^+,(1,\kappa)\cup\mathcal{J}\cup\mathcal{K}}(X^+_{\mathrm{res}};\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{E}^+_l,\mathcal{I}_l[\mathcal{F}_\bullet+(0,\kappa)]^+,(1,\kappa)\cup\mathcal{J}_l\cup\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l, \tag{135}$$

where $\mathcal{K}_\bullet \subset \mathbb{C}_{>0} \times \mathbb{N}$ such that $u = w - v$ has the desired properties as long as the parameter K in Lemma 5.1 is sufficiently large. □

We can now iterate this construction. To simplify the induction as much as possible, we assume that $\mathcal{E} = (2^{-1}(d+1), 0)$. Then, eq. (93) gives $\mathcal{E}^{++} = \mathcal{E}$.

Theorem B (Construction of $O(\sigma^\infty)$-quasimodes)**.** *Suppose that*

$$f \in \mathcal{A}^{(2^{-1}(d+1),0),2+\mathcal{F},(0,0)}(X^+_{\mathrm{res}};\ell) + \sum_{l=1}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(2^{-1}(d+1),0),2+\mathcal{F}_l,(0,0)}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l, \tag{136}$$

where $\min \Pi\mathcal{F} > c_\ell$ and $\min \Pi\mathcal{F}_l > c_l$ for $l = 1, \ldots, \ell-1$. Then, there exist index sets $\mathcal{I}_\bullet, \mathcal{K}_\bullet \subset \mathbb{C}_{>0} \times \mathbb{N}$ and an element

$$u \in \mathcal{A}^{(2^{-1}(d-1),0),\mathcal{I},(0,0)\cup\mathcal{K}}(X^+_{\mathrm{res}};\ell) + \sum_{l=1}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(2^{-1}(d-1),0),\mathcal{I}_l,(0,0)\cup\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \tag{137}$$

such that

$$Pu - f \in \mathcal{A}^{(2^{-1}(d-1),0),\infty,\infty}(X^+_{\mathrm{res}}) = \mathcal{S}([0,1)_\sigma; \mathcal{A}^{(2^{-1}(d-1),0)}(X)). \tag{138}$$

Proof. We will recursively define a strictly increasing sequence $\{\alpha_n\}_{n=0}^\infty$ with $\alpha_0 = 0$ and $\lim_{n\to\infty} \alpha_n = \infty$ and a sequence of functions $\{u_n\}_{n=0}^\infty$ such that, in a sense to be made precise,

$$u \sim \sum_{n=0}^\infty \sigma^{\alpha_n} u_n. \tag{139}$$

In the process, we will also define a sequence $\{f_n\}_{n=0}^\infty$ of functions. The first of these is just $f_0 = f$. The recursive step of the algorithm involves defining $\alpha_{n+1}, u_n, f_{n+1}$ in terms of α_n, f_n. At each step in the algorithm, it will be the case that

$$f_n \in \mathcal{A}^{(2^{-1}(d+1),0),2+\mathcal{F}_n,\mathcal{J}_n}(X^+_{\mathrm{res}};\ell) + \sum_{l=1}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(2^{-1}(d+1),0),2+\mathcal{F}_{l,n},\mathcal{J}_{l,n}}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \tag{140}$$

for index sets $\mathcal{F}_{\bullet,n} \subset \mathbb{C}_{>0} \times \mathbb{N}$ and $\mathcal{J}_{\bullet,n} \subseteq \mathbb{C}_{\geq 0} \times \mathbb{N}$. It will be helpful to note that we will have $\mathcal{F}_{l,n} \subset \mathbb{C}_{\geq c_l} \times \mathbb{N}$ and $\mathcal{F}_n \subset \mathbb{C}_{\geq c_\ell} \times \mathbb{N}$ for all n.

Suppose that we have defined α_n, f_n for some $n \in \mathbb{N}$. Note that f_n, by eq. (140), satisfies the hypotheses of Proposition 6.1 (where what we called $\mathcal{J}_\bullet$ in that proposition is the subset of what we are calling $\mathcal{J}_\bullet$ here not containing terms of the form $(0,\kappa)$), so there exists some

$$u_n \in \mathcal{A}^{(2^{-1}(d-1),0),\mathcal{I}_n,\mathcal{K}_n}(X^+_{\mathrm{res}};\ell) + \sum_{l=1}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(2^{-1}(d-1),0),\mathcal{I}_{l,n},\mathcal{K}_{l,n}}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l, \tag{141}$$

where $\mathcal{I}_{\bullet,n} \subset \mathbb{C}_{>0} \times \mathbb{N}$ and $\mathcal{K}_{\bullet,n} \subset \mathbb{C}_{\geq 0} \times \mathbb{N}$, such that

$$Pu_n - f_n \in \mathcal{A}^{(2^{-1}(d+1),0),2+\mathcal{F}^+_n,\mathcal{J}^+_n}(X^+_{\mathrm{res}};\ell) + \sum_{l=1}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(2^{-1}(d+1),0),2+\mathcal{F}^+_{l,n},\mathcal{J}^+_{l,n}}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l, \tag{142}$$

where the $\mathcal{F}^+_\bullet, \mathcal{J}^+_\bullet$ satisfy the same hypotheses as their predecessors. By choosing the parameters K in Proposition 6.1, we can arrange that $\mathcal{F}^+_{\bullet,n} \subset \mathbb{C}_{>K} \times \mathbb{N}$ for arbitrarily large K. Now define $\alpha_{n+1} = \alpha_n + \min\{1, \Pi\mathcal{J}^+_{\bullet,n}\}$. Note that this does not depend on K, by Remark 5.5. Let

$$f_{n+1} = -(Pu_n - f_n)/\sigma^{\alpha_{n+1}-\alpha_n}, \tag{143}$$

which has the required form, eq. (140), for $\mathcal{F}_{\bullet,n+1} = \mathcal{F}^+_{\bullet,n} - (\alpha_{n+1} - \alpha_n)$ and $\mathcal{J}_{\bullet,n+1} = \mathcal{J}^+_{\bullet,n} - (\alpha_{n+1} - \alpha_n)$, at least as long as K is large enough such that

$$\mathcal{F}_{\bullet,n+1} = \mathcal{F}^+_{\bullet,n} - (\alpha_{n+1} - \alpha_n) \subset \mathbb{C}_{>c_\bullet} \times \mathbb{N}. \tag{144}$$

By construction, $\alpha_{n+1} > \alpha_n$. In order to see that $\alpha_n \to \infty$ as $n \to \infty$, use Remark 5.3. Indeed, it must be the case that $\mathcal{J}^+_{\bullet,n}$ differs from $\mathcal{J}_{\bullet,n}$ only in that the former can have more elements in $\mathbb{C}_{\geq 1} \times \mathbb{N}$. It follows that, for some $N = N(n) \in \mathbb{N}$, $\alpha_{n+N} \geq \alpha_n + 1$, which implies that $\alpha_n \to \infty$.

An asymptotic summation argument therefore shows that there exists a polyhomogeneous u whose $\sigma \to 0^+$ asymptotic expansion is given by eq. (139). Specifically, u satisfies eq. (137) for

$$\mathcal{I}_\bullet = \bigcup_{n=0}^{\infty} \Big[(\alpha_n, 0) + \mathcal{I}_{\bullet,n}\Big], \qquad \mathcal{K}_\bullet = \bigcup_{n=0}^{\infty} \Big[(\alpha_n, 0) + \mathcal{K}_{\bullet,n}\Big], \tag{145}$$

which are well-defined index sets cause the real parts of the first components of the elements of the $\mathcal{I}_{\bullet,n}, \mathcal{K}_{\bullet,n}$ are bounded below.

Then,

$$\begin{aligned} Pu - f \sim -f + \sum_{n=0}^{\infty} \sigma^{\alpha_n} Pu_n &= \sum_{n=0}^{\infty} \sigma^{\alpha_n}(Pu_n - f_n + \sigma^{\alpha_{n+1}-\alpha_n} f_{n+1}) \\ &= \sum_{n=0}^{\infty} \sigma^{\alpha_n}(Pu_n - f_n - (Pu_n - f_n)) = 0, \end{aligned} \tag{146}$$

where $f_{-1} = 0$. This means that eq. (138) holds. □

We can now complete the proof of the main theorem.

Theorem C. *Let P be as in §2. Then, letting $P^{-1}(\sigma)$ for $\sigma > 0$ denote the inverse in Proposition 2.5, and supposing that f is as in Theorem B, we have*

$$P^{-1}(\sigma)u \in \mathcal{A}^{(2^{-1}(d-1),0),\mathcal{I},(0,0)\cup\mathcal{K}}(X^+_{\mathrm{res}};\ell) + \sum_{l=1}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(2^{-1}(d-1),0),\mathcal{I}_l,(0,0)\cup\mathcal{K}_l}([0,1)^+_{\mathrm{res}})\mathcal{Y}_l \tag{147}$$

where $\mathcal{I}_\bullet, \mathcal{K}_\bullet$ are as in Theorem B.

Proof. Suppose that $f \in \mathcal{S}(X)$. Then, applying Theorem B, we get v in the set on the right-hand side of eq. (137) such that $g = Pv - f$ is in the set on the right-hand side of eq. (138).

We apply Proposition 3.3 with $-g$ in place of f to get $w \in \mathcal{S}([0,1)_\sigma; \mathcal{A}^{(2^{-1}(d-1),0)}(X))$ such that $Pw = -g$. Then, $u = v + w$ satisfies $Pu = f$, and u is also in the set on the right-hand side of eq. (137). In particular,

$$u(-;\sigma) \in \mathcal{A}^{2^{-1}(d-1)}(X) \tag{148}$$

for each $\sigma > 0$. By the injectivity of P between the spaces in Proposition 2.5, this implies $u = P(\sigma)^{-1} f$. □

Theorem A is an immediate corollary.

Appendix A. Analysis at exactly zero energy

A familiar fact from the theory of electrostatics is that given $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^3)$, say representing the distribution of charge-density in some region of space, then the unique decaying solution

$$\phi(x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{4\pi |x-y|} \,\mathrm{d}^3 y \tag{149}$$

to Poisson's equation $\triangle \phi = \rho$, the voltage generated by that charge-density, satisfies $\phi \in \langle r \rangle^{-1} C^\infty(\overline{\mathbb{R}^3})$, where $\overline{\mathbb{R}^3} = \mathbb{R}^3 \cup \infty\mathbb{S}^2$ is the radial compactification of $\mathbb{R}^3$, r is the spherical radial coordinate, and $\langle r \rangle = (1+r^2)^{1/2}$ is the Japanese bracket. Prosaically, this means that, in the $r \to \infty$ limit, ϕ admits a full asymptotic expansion in terms of negative powers r: there exist $\phi_1(\theta), \phi_2(\theta), \cdots \in C^\infty(\mathbb{S}^2_\theta)$ such that

$$\phi(x) - \sum_{k=1}^{K} \frac{\phi_k(\theta)}{r^k} \in \langle r \rangle^{-K-1} C^\infty(\overline{\mathbb{R}^3} \backslash \{0\}) \subset \langle r \rangle^{-K-1} L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^3} \backslash \{0\}) \tag{150}$$

for all $K \in \mathbb{N}$. It turns out that each coefficient ϕ_k satisfies $\phi_k \in \operatorname{span}_{\mathbb{C}}\{Y_k^{-k}, \cdots, Y_k^0, \cdots, Y_k^k\}$, where, for each $\ell \in \mathbb{N}$ and $m \in \{-\ell, \cdots, \ell\}$, $Y_\ell^m(\theta) \in C^\infty(\mathbb{S}^2_\theta)$ denotes the usual spherical harmonic. This is the *mutlipole expansion*. Explicit formulae can be written down for all the ϕ_k's. For instance, ϕ_0 is proportional to the "monopole moment" of ρ, ϕ_1 to the "dipole moment," and so on; see [Jac75].

The following two generalizations are less well-known:

- For any $\alpha \in \mathbb{N}_{\geq 3}$ and $\rho \in \langle r \rangle^{-\alpha} C^\infty(\overline{\mathbb{R}^3})$, the voltage ϕ, defined by the same formula eq. (149), also admits an asymptotic expansion as $r \to \infty$, but (unless ρ is Schwartz), it is necessary to have logarithmic terms in the expansion,

$$\phi_{j,k}(\theta) r^{-j} \log(r) \tag{151}$$

  for certain $j \in \mathbb{N}_+$. Moreover, rather than each spherical harmonic Y_k^m showing up in the expansion only proportional to $1/r^{k+1}$, they can also show up in later terms, but not in earlier ones.
- If one solves instead an equation such as $\triangle \phi + V\phi = \rho$ for

$$V \in \langle r \rangle^{-3} C^\infty_{\mathrm{c}}[0,1)_{1/r} + \mathcal{S}(\mathbb{R}^3), \tag{152}$$

  i.e. a short-range potential which is spherically symmetric modulo Schwartz terms, then the situation is similar. For $V \in \langle r \rangle^{-3} C^\infty(\overline{\mathbb{R}^3})$ not of the form eq. (152), two new phenomena can occur:
  (i) it may now be possible to find logarithmic terms of the form $\phi_{j,\kappa}(\theta) r^{-j} \log(r)^\kappa$ for $\kappa \geq 2$, not just $\kappa = 1$, and

(ii) more seriously, it may be the case that the spherical harmonics Y_k^m are no longer only found proportional to $r^{-j}(\log r)^\kappa$ for $j \geq k+1$; they may be present in less-decaying terms as well. In fact, the $\phi_{j,\kappa}$ may no longer be linear combinations of *finitely* many spherical harmonics at all.

However, these problems only kick in late in the $r \to \infty$ expansion. Since they are tied to symmetry-breaking terms in V, the more rapidly decaying these terms, the later in the expansion problems arise. It is the essence of the multipole expansion that only finitely many spherical harmonics are present at each individual order. So, in the present situation, one should really speak only of "partial" multipole expansions. One still has an asymptotic expansion in the $r \to \infty$ limit, it just fails to be a multipole expansion past some particular order.

As the presence of logarithmic terms indicates, these results require a different proof than that presented in physics textbooks for the multipole expansion for Laplace's equation, which is based on manipulations of eq. (149). For instance, if the source ρ is not Schwartz, then its high-order multipole moments will not be defined, as the integrals which define them in the Schwartz case fail to be convergent. Moreover, if $V \neq 0$, then eq. (149) no longer applies.

In each of the situations described above, ϕ is polyhomogeneous on $\overline{\mathbb{R}^3}$, with some index set $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}$ keeping track of precisely which pairs (j,κ) are present in the asymptotics. Moreover, for each spherical harmonic Y, one can compute an index set $\mathcal{E}_Y \subsetneq \mathcal{E}$ keeping track in which terms that spherical harmonic shows up. This subset depends, in a rather complicated way, on the decay rate of the source ρ, the decay rate of the potential, and the decay rate of symmetry breaking-terms. Regardless, it can be computed.

Our goal of this section is to carry out this computation in the natural level of generality, that of Schrödinger operators on (compact) manifolds-with-boundary equipped with an exactly-conic structure at their boundary, which is precisely the setup introduced in §2. To keep track of the relevant data, we use the function spaces

$$\mathcal{A}^{\mathcal{E},\alpha}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{E}_l,\alpha_l}([0,1)_\rho)\mathcal{Y}_l, \tag{153}$$

where $\ell \in \mathbb{N}$, $\mathcal{E},\mathcal{E}_l$ are pre-index sets, and $\alpha,\alpha_l \in \mathbb{R}$. Thse are defined as follows. For each $l \in \mathbb{N}$, each $\mathcal{Y}_l$ is a (positive dimensional) subspace of some eigenspace of the boundary Laplacian $\triangle_{\partial X} \in \mathrm{Diff}^2(\partial X)$, and $\mathcal{A}^{\mathcal{E},\alpha}(X;\ell)$ consists of those elements v of $\mathcal{A}^{\mathcal{E},\alpha}(X)$ such that

$$\langle v(\rho,\theta), Y(\theta)\rangle_{L^2(\partial X_\theta)} = 0 \tag{154}$$

for all $\rho \in (0,1/2)$ and $Y \in \mathcal{Y}_l$ for $l \in \{0,\cdots,\ell-1\}$. In particular, $\mathcal{A}^{\mathcal{E},\alpha}(X;0) = \mathcal{A}^{\mathcal{E},\alpha}(X)$. Note that these function spaces depend on the precise choice of exactly-conic structure imposed on X. An f lies in the space eq. (153) if and only if we can write

$$f = F + \sum_{l=0}^{\ell-1} \sum_{Y \in \mathrm{onb}(\mathcal{Y}_l)} f_Y(\rho)Y(\theta) \tag{155}$$

for some $F \in \mathcal{A}^{\mathcal{E},\alpha}(X;\ell)$ and $f_Y \in \mathcal{A}_{\mathrm{c}}^{\mathcal{E}_l,\alpha_l}([0,1)_\rho)$, where $\mathrm{onb}(\mathcal{Y}_l)$ is, for each l, some orthonormal basis of $\mathcal{Y}_l$. We require that $\mathcal{Y}_j \perp \mathcal{Y}_k$ as subspaces of $L^2(\partial X)$ if $j \neq k$. For each $l \in \mathbb{N}$, let λ_l be the eigenvalue under $\triangle_{\partial X}$ of the elements of $\mathcal{Y}_l$. Note that some λ_l's may be repeated. We require that the sequence $\lambda_0,\lambda_1,\lambda_2,\cdots$ be non-decreasing, and we require that each eigenspace of $\triangle_{\partial X}$ be the direct sum of some of the $\mathcal{Y}_l$'s. The simplest two possibilities are when

- $\mathcal{Y}_0,\mathcal{Y}_1,\mathcal{Y}_2,\cdots$ are the eigenspaces of the boundary Laplacian $\triangle_{g_{\partial X}}$, listed in order of increasing eigenvalue, or

- or each $\mathcal{Y}_l$ is given by $\mathcal{Y}_l = \mathbb{C}Y_l$, where Y_l is the lth eigenfunction of the boundary Laplacian, listed in order of increasing eigenvalue (where, if two eigenfunctions share an eigenvalue, their ordering is arbitrary).

All other cases are intermediate to these two.

Theorem D. *Let $P \in \mathrm{Diff}(X^\circ)$ have the form described in §A.2, with $\aleph, \aleph_0 = \infty$. Suppose that $f \in \mathcal{A}^{\mathcal{E}}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{E}_l}([0,1)_\rho)\mathcal{Y}_l$. Then, if $u \in \mathcal{A}^{0+}(X)$ satisfies $Pu = f$,*

$$u \in \mathcal{A}^{\mathcal{I}}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{I}_l}[0,1)\mathcal{Y}_l \tag{156}$$

holds, where $\mathcal{I}, \mathcal{I}_l$ are defined by eq. (196).

Remark. Note that when P differs from $\triangle_{g_0}$ by Schwartz terms, and when f is Schwartz, then this reproduces the usual multipole expansion from electrostatics. So, the index sets are sharp, at least in this case. Though we will not do so, it should be possible to show that $\mathcal{I}, \mathcal{I}_l$ are sharp in general, in the sense that, for each choice of parameters $\mathcal{E}, \alpha, \mathcal{E}_l, \beth, \beth_0$ involved in the assumptions on P, f, there exist some P, f of the hypothesized forms and some u solving $Pu = f$ such that the index sets in eq. (156) are optimal.

Remark. In the asymptotically Euclidean case, then one can shrink $\mathcal{I}, \mathcal{I}_l$ if the terms which break the spherical symmetry only involve finitely many spherical harmonics at each order. That is, if the coefficients of the PDE admit a multipole expansion, then so do the solutions. This improvement is a consequence of the Clebsch–Gordan decomposition and rather special to the asymptotically Euclidean case.

The proof of Theorem D consists of two basic steps:

(I) First, the result is proven for $P = \triangle_{g_0}$, solving the inhomogeneous problem $Pu = f$ explicitly via the Mellin transform. In this case, the result follows from the mapping properties of the Mellin transform. In preparation for the next step, it will be necessary to to handle the case when f is only partially polyhomogeneous. This is not contained in Theorem D, but it is a straightforward extension.

(II) For general P, we write $P = \triangle_{g_0} + L$ for some L which decays faster than $\triangle_{g_0}$ at ∂X (in the sense of regular singular differential operators). Then, when solving $Pu = f$, we rearrange to get

$$\triangle_{g_0} u = f - Lu. \tag{157}$$

Suppose that u is known to be symbolic or partially polyhomogeneous, with the symbolic error kicking in at order $\alpha \in \mathbb{R}$. (We are assuming in Theorem D that $u \in \mathcal{A}^{0+}(X)$, so this assumption holds for some $\alpha > 0$.) Then, Lu is partially polyhomogeneous as well. Applying $\triangle_{g_0}^{-1}$ (making sense of this on appropriate function spaces using the solvability theory of the Laplacian), we write

$$u = \triangle_{g_0}^{-1}(f - Lu) = \triangle_{g_0}^{-1} f - \triangle_{g_0}^{-1} Lu. \tag{158}$$

The term $\triangle_{g_0}^{-1} f$ is understood from step (I) and fully polyhomogeneous. Likewise, from step (I), $w = \triangle_{g_0}^{-1} Lu$ is partially polyhomogeneous, but because L is faster decaying than $\triangle_{g_0}$, the composition w will be faster decaying than u. This applies also to the symbolic errors involved, so, in w, the symbolic error kicks in at some order $\geq \alpha + 1$. But then eq. (158) implies that the symbolic error in u only kicks in at order $\geq \alpha + 1$ as well, at least one order higher than we started out assuming. A straightforward inductive argument then allows us to take $\alpha \to \infty$ and therefore conclude full polyhomogeneity.

Part of Price's law states that gravitational radiation from spherically symmetric black holes radiates away at a rate which depends on the angular pattern according to which the radiation is distributed. Monopole (i.e. spherically symmetric) radiation radiates away more slowly than dipole radiation, dipole radiation radiates away more slowly than quadropole radiation, and so on. As a leading order asymptotic, it is known, from Hintz [Hin22], that Price's law applies to radiation on any asymptotically sub-extremal Kerr background. However, it is now to be expected that e.g. dipole radiation includes a quadropole component which decays more slowly than pure quadropole radiation, and moreover that the index sets describing the long-time asymptotic profile are expected to be more complicated than in the spherically symmetric case. These subleading effects are not studied in [Hin22]. We would like to investigate these delicate effects, which depend on the fact that Kerr is only axially symmetric. This requires the multipole expansions whose systematic discussion we have included here.

A.1. **Some mapping properties of the exactly conic Laplacian.** We next discuss the mapping properties of $\triangle_{g_0}$ vis-a-vis the function spaces in eq. (153). This is step (I) in the outline above. The exactly-conic Laplacian $\triangle_{g_0}$ is given by

$$\triangle_{g_0} = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{g_{\partial X}}. \tag{159}$$

in $\dot{X} = [0,1)_\rho \times \partial X_\theta$.

This operator is an elliptic element of $\rho^2\operatorname{Diff}^2_{\mathrm{b}}(X)$, i.e. is regular singular at ∂X, so its solvability theory is governed by its indicial roots, which are the roots of the polynomial $-c^2+(d-2)c+\lambda \in \mathbb{R}[c]$ for $\lambda \geq 0$ any eigenvalue of the boundary Laplacian $\triangle_{\partial X} = \triangle_{g_{\partial X}}$. If c is a root of this polynomial, then this corresponds (as we will see below) to $O(\rho^{c-})$ bounds for elements of $\ker \triangle_{g_0}$ or $\operatorname{coker} \triangle_{g_0}$. For each value of λ, the two roots of the given quadratic polynomial are

$$c = \frac{1}{2}(d-2 \pm \sqrt{(d-2)^2 + 4\lambda}). \tag{160}$$

Each of the $\mathcal{Y}_l$ in eq. (153) is a set of eigenfunctions of $\triangle_{\partial X}$ with shared eigenvalue λ_l. For each $l \in \mathbb{N}$, let $c_l = 2^{-1}(d-2+((d-2)^2+4\lambda_l)^{1/2})$ denote the positive of these two solutions for $\lambda = \lambda_l$. Also, let

$$-b_l = 2^{-1}(d-2-((d-2)^2+4\lambda_l)^{1/2}) \tag{161}$$

denote the corresponding negative solution. (The sign on the left-hand side in eq. (161) is chosen for later convenience.) In particular, $c_0 = d-2$ and $b_0 = 0$.

Note that $-b_k \leq -b_j < c_j \leq c_k$ whenever $j \leq k$. Also, via Weyl's law, $\lim_{l\to\infty} b_l, \lim_{l\to\infty} c_l = \infty$. So, the set $\cup_{l\in\mathbb{N}}\{-b_l, c_l\}$ is discrete.

Example (Euclidean case). In the Euclidean case, $\partial X = \mathbb{S}^{d-1}$, and $g_{\partial X}$ is the standard metric on the $(d-1)$-sphere. So, assuming that we take $\mathcal{Y}_l$ to denote the lth eigenspace of $\triangle_{\partial X}$, we have $\lambda_l = l(l+d-2)$. Then, one computes that $c_l = d-2+l$ and $b_l = l$.

Lemma A.1. *Let $\ell \in \mathbb{N}$. For $\alpha \in (-b_\ell, c_\ell)$ and $\beta \in (0, d-2)$, we have a well-defined inverse*

$$\triangle_{g_0}^{-1} : \mathcal{A}^{2+\alpha}(X;\ell) + \mathcal{A}^{2+\beta}(X) \to \mathcal{A}^{\alpha-}(X;\ell) + \mathcal{A}^{\beta-}(X). \tag{162}$$

That is, for each $f \in \mathcal{A}^{2+\alpha}(X;\ell) + \mathcal{A}^{2+\beta}(X)$, there exists a unique $u \in \mathcal{A}^{\alpha-}(X;\ell) + \mathcal{A}^{\beta-}(X)$ such that $\triangle_{g_0}u = f$. ■

The key ingredient, beyond elementary aspects of the analysis of the Laplacian, is the Mellin transform $\mathcal{M}$, which we define with the conventions in eq. (229), and some key properties of which we summarize in §C.

Proof. First, one shows that $\triangle_{g_0}$ has trivial kernel acting on $\mathcal{A}^{\alpha-}(X;\ell)+\mathcal{A}^{\beta-}(X)$. Since the operator has real coefficients, its kernel in this space has a basis consisting of real-valued functions. Suppose that $w\in\mathcal{A}^{\alpha-}(X;\ell)+\mathcal{A}^{\beta-}(X)$ is a real-valued function satisfying $\triangle_{g_0}w=0$. Fix $\chi\in C^\infty_{\mathrm{c}}[0,1/2)$ that is identically 1 on a neighborhood of the origin. Let $f=\rho^{-2}\triangle_{g_0}(\chi w)$. Integrating-by-parts, $\mathcal{M}f(c,\theta)=(-c^2+(d-2)c+\triangle_{\partial X})\mathcal{M}(\chi w)(c,\theta)$ as long as c has sufficiently negative real part. Moreover, as long as c does not lie in the discrete set of points $\cup_{l\in\mathbb{N}}\{-b_l,c_l\}$, the resolvent $(-c^2+(d-2)c+\triangle_{\partial X})^{-1}:H^m(\partial X_\theta)\to H^m(\partial X_\theta)$ is well-defined, giving

$$\mathcal{M}(\chi w)(c,\theta)=(-c^2+(d-2)c+\triangle_{\partial X})^{-1}\mathcal{M}f(c,\theta). \tag{163}$$

Since $f=\rho^{-2}\triangle_{g_0}(\chi w)=\rho^{-2}[\triangle_{g_0},\chi]w\in C^\infty_{\mathrm{c}}(\dot{X}^\circ)$, the Mellin transform $\mathcal{M}f(c,\theta)$ is defined for all c. The right-hand side of eq. (163) therefore defines a meromorphic function of c, defined on the entire complex plane $\mathbb{C}_c$, save for the poles. We will call this extension

$$\mathcal{M}(\chi w)_{\mathrm{ext}}:\mathbb{C}_c\backslash\cup_{l\in\mathbb{N}}\{-b_l,c_l\}\to\mathbb{C}. \tag{164}$$

The possible poles, all of which are simple (as follows from the functional calculus applied to $\triangle_{\partial X}$), all lie in $\cup_{l\in\mathbb{N}}\{-b_l,c_l\}$. Any apparent singularities of $\mathcal{M}(\chi w)_{\mathrm{ext}}$ at the $b_0,b_1,b_2,\cdots$ are necessarily removable:

- That this holds for $b_\ell,b_{\ell+1},b_{\ell+2},\cdots$ follows from the fact that $w\in\mathcal{A}^{-b_\ell+\varepsilon}(X)$ for some $\varepsilon>0$ (which follows from the assumptions on α,β) and therefore that the Mellin transform $\mathcal{M}(\chi w)$ is already well-defined and analytic when $\Re c<-b_\ell+\varepsilon$.
- That this holds for the remaining b_l's, $b_0,\ldots,b_{\ell-1}$, follows from a stronger argument. By assumption, $w=u+v$ for $u\in\mathcal{A}^{-b_\ell+}(X;\ell)$ and $v\in\mathcal{A}^{0+}(X)$. The identity

  $$\mathcal{M}(\chi w)(c,\theta)=\mathcal{M}(\chi u)(c,w)+\mathcal{M}(\chi v)(c,\theta) \tag{165}$$

  holds for all c with sufficiently negative real part. We know, a priori, that $\mathcal{M}(\chi v)(c,\theta)$ is analytic in a neighborhood of the left half of the complex plane. On the other hand,

  $$\mathcal{M}(\chi u)(c,w)=(-c^2+(d-2)c+\triangle_{\partial X})^{-1}\mathcal{M}(\rho^{-2}\triangle_{g_0}(\chi u))(c,\theta) \tag{166}$$

  if c has sufficiently negative real part. Because $\triangle_{g_0}w=0$ and therefore $\triangle_{g_0}u=-\triangle_{g_0}v\in\mathcal{A}^{2+}(X)$, we have

  $$\rho^{-2}\triangle_{g_0}(\chi u)=\rho^{-2}[\triangle_{g_0},\chi]u-\rho^{-2}\chi\triangle_{g_0}v\in\mathcal{A}^{0+}(X). \tag{167}$$

  So, $\mathcal{M}(\rho^{-2}\triangle_{g_0}(\chi u))(c,\theta)$ is also analytic in some small neighborhood of the left half of the complex plane. It follows that $(-c^2+(d-2)c+\triangle_{\partial X})^{-1}\mathcal{M}(\rho^{-2}\triangle_{g_0}(\chi u))(c,\theta)$ is meromorphic on some neighborhood of the left half of the complex plane, with possible simple poles at $c\in\cup_{l\in\mathbb{N}}\{-b_l,c_l\}$.

  But, since u is orthogonal to all of the $\mathcal{Y}_0,\cdots,\mathcal{Y}_{\ell-1}$ in $\{\rho<1/2\}$, the function $\rho^{-2}\triangle_{g_0}(\chi u)$ is as well, for all ρ, since $\operatorname{supp}\chi\subset\{\rho<1/2\}$. Therefore

  $$\mathcal{M}(\rho^{-2}\triangle_{g_0}(\chi u))(c)\perp\mathcal{Y}_0,\cdots,\mathcal{Y}_{\ell-1} \tag{168}$$

  for all c. Consequently, the residues of $(-c^2+(d-2)c+\triangle_{\partial X})^{-1}\mathcal{M}(\rho^{-2}\triangle_{g_0}(\chi u))(c,\theta)$ at the various points $b_0,\cdots,b_{\ell-1}$ are all zero. (That is, unless some of these are equal to b_ℓ, which can happen if $\mathcal{Y}_{\ell-1}$ is a proper subspace of some eigenspace of the boundary Laplacian. To avoid having to repeat this caveat, when we say that a pole at b_l or c_l is absent, we mean that it is absent or that $b_{l+1}=b_l$ or $c_{l+1}=c_l$, in which case we can think of the pole as being associated with the index $l+1$ instead of l.)

  So, both terms on the right-hand side of eq. (165) extend analytically to a neighborhood of the left half of the complex plane, and thus the same applies to $\mathcal{M}(\chi w)(c,\theta)$.

If, for $m\in\mathbb{N}$, we define the norm on $H^m(\partial X)$ using $\triangle_{\partial X}$, then we have the operator-norm bound

$$\|(-c^2+(d-2)c+\triangle_{\partial X})^{-1}\|_{H^m(\partial X)\to H^m(\partial X)}\leq d(-c^2+(d-2)c,\sigma(\triangle_{\partial X}))^{-1} \tag{169}$$

for functional analytic reasons. Consequently, if γ_0, γ_1 are real numbers such that $\gamma_0 < \gamma_1$ and $[\gamma_0, \gamma_1]$ is disjoint from $\cup_{l \in \mathbb{N}}\{-b_l, c_l\}$, we have, for all $c \in \mathbb{C}$ satisfying $\Re c \in [\gamma_0, \gamma_1]$, the bound

$$\|\mathcal{M}(\chi w)_{\mathrm{ext}}(c, \theta)\|_{H^m(\partial X_\theta)} \leq C(\gamma_0, \gamma_1) \langle \Im c \rangle^{-2} \|\mathcal{M} f(c, \theta)\|_{H^m(\partial X_\theta)} \tag{170}$$

for some $C(\gamma_0, \gamma_1) > 0$. Since $\mathcal{M}(\chi w)_{\mathrm{ext}}(c, \theta)$ is analytic at each b_l, the same estimate, possibly with a larger value of $C(\gamma_0, \gamma_1)$, holds for any $\gamma_0, \gamma_1 \in \mathbb{R}$ satisfying $\gamma_0 < \gamma_1$ and $\gamma_1 < c_0 = d - 2$.

We apply the Mellin inversion formula eq. (233) to χw, noting that eq. (170) justifies shifting the contour to the right, as long as we replace $\mathcal{M}(\chi w)(c, \theta)$ by the analytic extension $\mathcal{M}(\chi w)_{\mathrm{ext}}(c, \theta)$. So,

$$\chi w(\rho, \theta) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \rho^c \mathcal{M}(\chi w)_{\mathrm{ext}}(c, \theta) \,\mathrm{d}c \tag{171}$$

for any $\gamma < d - 2$. It follows that $\chi w \in \mathcal{A}^{c_0 -}(X) \subseteq \mathcal{A}^{0+}(X)$, as discussed in §C. Therefore, $w \in C^0(X)$.

Since X is compact, continuity implies that, unless $w = 0$, it must be that w (which recall is real-valued) attains a global extremum somewhere in the interior of X. But then the maximum principle for Laplace–Beltrami operators would state that w is constant. Since $w \in \mathcal{A}^{0+}(X)$, w being constant implies that $w = 0$ identically. This completes the proof that the kernel of $\triangle_{g_0}$ acting on $\mathcal{A}^{\alpha-}(X; \ell) + \mathcal{A}^{\beta-}(X)$ is trivial.

In order to show that $\triangle_{g_0} : \mathcal{A}^{\alpha-}(X; \ell) + \mathcal{A}^{\beta-}(X) \twoheadrightarrow \mathcal{A}^{2+\alpha}(X; \ell) + \mathcal{A}^{2+\beta}(X)$, one also uses the Mellin transform. It suffices to consider the cases (i) $f \in \mathcal{A}^{2+\alpha}(X; \ell)$, (ii) $f \in \mathcal{A}^{2+\beta}(X)$ individually.

(i) If $f \in \mathcal{A}^{2+\beta}(X)$, then the existence of $u \in \mathcal{A}^{\beta-}(X)$ satisfying $\triangle_{g_0} u = f$ follows from the standard solvability theory of the Laplacian, e.g. [Hin22, Eq. 2.9] for the $d = 3$, asymptotically Euclidean case, the general case being analogous.

(ii) If $f \in \mathcal{A}^{2+\alpha}(X; \ell)$, then we define a meromorphic function $M(c, \theta)$ by

$$M(c, \theta) = (-c^2 + (d - 2)c + \triangle_{\partial X})^{-1} \mathcal{M}(\rho^{-2} \chi f)(c, \theta), \tag{172}$$

where χ is as above. As $\mathcal{M}(\rho^{-2}\chi f)(c, \theta)$ is well-defined and analytic in c for $\Re c < \alpha$, the function $M(c, \theta)$ has at worst poles there. But actually, because f is orthogonal to each $\mathcal{Y}_0, \ldots, \mathcal{Y}_{\ell-1}$, it follows that the possible poles at $b_0, \ldots, b_{\ell-1}$ (some of which may be $\geq \alpha$) are all absent, as well as those at $c_0, \ldots, c_{\ell-1}$. So, $M(c, \theta)$ is well-defined and analytic for $\Re c < \alpha$ as well. (However, we cannot rule out poles at $b_\ell, b_{\ell+1}, \cdots$.)

Analogous estimates to those above show that

$$u_0 = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \rho^c M(c, \theta) \,\mathrm{d}c \tag{173}$$

is, for each $\gamma \in (-b_\ell, \alpha)$, a well-defined element of $\mathcal{A}^\gamma(\dot{X}; \ell)$ that does not depend on γ. Thus, $u_0 \in \mathcal{A}^{\alpha-}(\dot{X}; \ell)$.

Let $w = \chi u_0 \in \mathcal{A}^{\alpha-}(X; \ell)$. This satisfies $\triangle_{g_0} w = [\triangle_{g_0}, \chi] u + \chi \triangle_{g_0} u_0$. We compute that

$$\begin{aligned} \triangle_{g_0} u_0 &= \frac{\rho^2}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \rho^c (-c^2 + (d - 2)c + \triangle_{\partial X}) M(c, \theta) \,\mathrm{d}c \\ &= \frac{\rho^2}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \rho^c \mathcal{M}(\rho^{-2} \chi f)(c, \theta) \,\mathrm{d}c = \chi f. \end{aligned} \tag{174}$$

So, $\triangle_{g_0} w = f + f_0$ for $f_0 = (\chi^2 - 1) f + [\triangle_{g_0}, \chi] u \in C_{\mathrm{c}}^\infty(\dot{X}^\circ)$.

By the case (i) already discussed, there exists a $v \in \mathcal{A}^{c_0 -}(X) \subset \mathcal{A}^{\beta-}(X)$ such that $\triangle_{g_0} v = -f_0$, so the function $u = w + v$, which lies in the desired function space, satisfies $\triangle_{g_0} u = f$.

□

In order to get more detailed asymptotics, it is necessary to utilize the mapping properties of the Mellin transform on (partially) polyhomogeneous functions, as described in §C. Suppose that f is a meromorphic function on a strip $\{z \in \mathbb{C} : \Re z \in (\gamma_0, \gamma_1)\}$ such that, for each c in the strip, the pole at f at c is at worst of order $k+1$, where k is the largest nonnegative integer such that $(c,k) \in \mathcal{E}$. Then, we say that the poles of f are "given" by $\mathcal{E}$.

Lemma A.2. *Suppose that $u \in \mathcal{A}^{0+}(X)$ satisfies $\triangle_{g_0} u = f$ for some function $f \in \mathcal{A}^{2+\mathcal{E},2+\alpha}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{2+\mathcal{E}_l,2+\alpha_l}([0,1)_\rho)\mathcal{Y}_l$, where $\mathcal{E}, \mathcal{E}_l$ are pre-index sets and $\alpha, \alpha_l \in \mathbb{R}$. Then,*

$$u \in \mathcal{A}^{\mathcal{C}_\ell \uplus \mathcal{E}, \alpha-}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}^{(c_l,0)\uplus \mathcal{E}_l, \alpha_l -}([0,1)_\rho)\mathcal{Y}_l, \tag{175}$$

where $\mathcal{C}_\ell \uplus \mathcal{E} = \bigcup_{l=\ell}^\infty ((c_l,0)\uplus \mathcal{E})$. ■

Proof. Let χ be as in the previous lemma, and let $w = \chi u \in C^\infty(\dot{X}^\circ)$. It suffices to show that $w \in \mathcal{A}^{\mathcal{C}_\ell \uplus \mathcal{E},\alpha-}(\dot{X};\ell) + \sum_{l=0}^{\ell-1}\mathcal{A}^{(c_l,0)\uplus\mathcal{E}_l,\alpha_l-}([0,1)_\rho)\mathcal{Y}_l$.

Taking the Mellin transform, $\mathcal{M}w(c,\theta)$ is defined and analytic in $\Re c < \varepsilon$ for some $\varepsilon > 0$, and the Mellin inversion formula reads

$$w(\rho,\theta) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma+i\infty} \rho^c \mathcal{M}w(c,\theta)\,\mathrm{d}c = \mathcal{M}^{-1}(\mathcal{M}w(-,\theta))(\rho), \tag{176}$$

for any $\gamma < \varepsilon$.

Now let $g = \triangle_{g_0} w = [\triangle_{g_0}, \chi]u + \chi f$. The Mellin transform $\mathcal{M}_{\rho\to c}(\rho^{-2}g(\rho,\theta))(c)$ is defined and analytic if c has sufficiently negative real-part, and, for c with sufficiently negative real part, taking the Mellin transform of the PDE $g = \triangle_{g_0} w$ yields $(-c^2+(d-2)c+\triangle_{\partial X})\mathcal{M}w(c,\theta) = \mathcal{M}_{\rho\to c}(\rho^{-2}g(\rho,\theta))(c)$. Therefore we can write

$$\mathcal{M}w(c,\theta) = \mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}g(\rho,\theta))(c,\theta), \tag{177}$$

where $\mathcal{R}(c) = (-c^2 + (d-2)c + \triangle_{\partial X})^{-1}$, as long as c has sufficiently negative real part and $c \notin \cup_{l\in\mathbb{N}}\{-b_l, c_l\}$.

We can write $g = G + \sum_{l=0}^{\ell-1} g_l$, where $G \in \mathcal{A}_{\mathrm{c}}^{2+\mathcal{E},2+\alpha}(\dot{X};\ell)$, and $g_l \in \mathcal{A}_{\mathrm{c}}^{2+\mathcal{E}_l,2+\alpha_l}[0,1)\mathcal{Y}_l$. Then, eq. (177) becomes

$$\mathcal{M}w(c,\theta) = \mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}G(\rho,\theta))(c) + \sum_{l=0}^{\ell-1}\mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}g_l(\rho,\theta))(c). \tag{178}$$

- First consider the term $\mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}G(\rho,\theta))(c)$. Since $\rho^{-2}G \in \mathcal{A}_{\mathrm{c}}^{\mathcal{E},\alpha}(\dot{X};\ell)$, the mapping properties of the Mellin transform show that $\mathcal{M}_{\rho\to c}(\rho^{-2}G(\rho,\theta))(c)$ is meromorphic in $\{z \in \mathbb{C} : \Re z < \alpha\}$ with poles given by $\mathcal{E}$. Consequently, $\mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}G(\rho,\theta))$ is meromorphic on the same domain, with poles given by $\mathcal{E}\uplus\mathcal{C}$, where $\mathcal{C}$ is the smallest index set containing $(-b_l,0)$ and $(c_l,0)$ for all $l\in\mathbb{N}$.

  However, because $\mathcal{M}_{\rho\to c}(\rho^{-2}G(\rho,\theta))(c)$ is orthogonal to $\mathcal{Y}_0,\ldots,\mathcal{Y}_{\ell-1}$ in $L^2(\partial X_\theta)$, for each c, the poles at $-b_{\ell-1},\ldots,-b_0$ and $c_0,\ldots,c_{\ell-1}$ are absent. Thus, in the strip $\{z\in\mathbb{C} : \Re z \in (0,\alpha)\}$, the poles are given by $\mathcal{E}\uplus\mathcal{C}_\ell$.
- Secondly, consider $\mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}g_l(\rho,\theta))(c)$. Arguing analogously, this is meromorphic in $\{z\in\mathbb{C} : \Re z \in (0,\alpha_l)\}$ with poles given by $\mathcal{E}_l \uplus (c_l,0)$, where the point is because $g_l(\rho,-)\in\mathcal{Y}_l$ for each l, we can apply $\mathcal{R}(c)$ to $\mathcal{M}_{\rho\to c}(\rho^{-2}g_l(\rho,\theta))(c)$ unless $c \in \{-b_l, c_l\}$.

Using eq. (176), we write $w = W + \sum_{l=0}^{\ell-1} w_l$, where

$$W(\rho,\theta) = \mathcal{M}^{-1}_{c\to\rho}(\mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}G(\rho,\theta))(c)) \tag{179}$$

$$w_l(\rho,\theta) = \mathcal{M}^{-1}_{c\to\rho}(\mathcal{R}(c)\mathcal{M}_{\rho\to c}(\rho^{-2}g_l(\rho,\theta))(c)) \tag{180}$$

for each $l \in \{0, \ldots, \ell-1\}$. Each of these inverse Mellin transforms is initially taken along a vertical contour slightly right of the imaginary axis. The mapping properties of the Mellin transform then say that $W \in \mathcal{A}^{\mathcal{C}_\ell \uplus \mathcal{E}, \alpha-}(\dot{X}; \ell)$, $w_l \in \mathcal{A}^{(c_l,0)\uplus\mathcal{E}_l, \alpha_l-}([0,1)_\rho)\mathcal{Y}_l$. □

A.2. **More general operators.** Now suppose that $P \in \mathrm{Diff}^2_{\mathrm{b}}(X)$ has the form $P = \triangle_{g_0} + Q + E$, for some $Q, E \in \mathrm{Diff}^2_{\mathrm{b}}(X)$, which, on $\dot{X} = [0,1)_{1/r} \times \partial X_\theta$, have the form

- $Q \in r^{-3-\daleth}\,\mathrm{Diff}^2_{\mathrm{b}}[0,1)_{1/r} + r^{-3-\aleph} S\,\mathrm{Diff}^2_{\mathrm{b}}[0,1)_{1/r}$, so Q involves only partial derivatives in the radial direction r, and
- $E \in r^{-3-\daleth_0}\,\mathrm{Diff}^2_{\mathrm{b}}(X) + S^{-3-\aleph_0}\,\mathrm{Diff}^2_{\mathrm{b}}(X)$

for some $\aleph, \aleph_0 \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ and nonnegative integers $\daleth, \daleth_0 \in \mathbb{N} \cup \{\infty\}$. Without loss of generality, we may assume that $\aleph, \daleth_0 \in [\daleth, \aleph_0]$. Thus, we keep track of four orders, the order $\daleth$ below subleading at which P differs from $\triangle_{g_0}$, the order $\daleth_0$ below subleading where θ-dependent terms appear, and the analogues $\aleph, \aleph_0$ regarding merely symbolic coefficients.

In this subsection, all pre-index sets will be index sets.

Proposition A.3. *Suppose that $u \in \mathcal{A}^{0+}(X)$ satisfies $Pu = f$ for some $f \in \mathcal{A}^{2+\mathcal{E},2+\alpha}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{2+\mathcal{E}_l,2+\alpha_l}[0,1)\mathcal{Y}_l$. Suppose that $\mathcal{I}, \mathcal{I}_0, \ldots, \mathcal{I}_{\ell-1}$ are index sets such that*

- *$\mathcal{C}_\ell \uplus (\mathcal{E} \cup (1+\daleth_0+\mathcal{I}_\cup) \cup (1+\daleth+\mathcal{I})) \subseteq \mathcal{I}$, where $\mathcal{I}_\cup = \mathcal{I} \cup \mathcal{I}_0 \cup \cdots \cup \mathcal{I}_{\ell-1}$, and*
- *$(c_l, 0) \uplus (\mathcal{E}_l \cup (1+\daleth_0+\mathcal{I}_\cup) \cup (1+\daleth+\mathcal{I}_l))$ for every $l = 0, \ldots, \ell-1$.*

Then,

$$u \in \mathcal{A}^{\mathcal{I},\gamma-}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{I}_l,\gamma_l-}[0,1)\mathcal{Y}_l \tag{181}$$

for any $\gamma \leq \min\{\alpha, 1+\aleph_0+\Pi\mathcal{I}_\cup, 1+\aleph+\Pi\mathcal{I}\}$ and $\gamma_l \leq \{\alpha_l, 1+\aleph_0+\Pi\mathcal{I}_{\cup-}, 1+\aleph+\Pi\mathcal{I}_l\}$ satisfying $\max\{\gamma, \gamma_0, \ldots, \gamma_{\ell-1}\} \leq 1+\daleth_0+\gamma_\wedge$ for $\gamma_\wedge = \min\{\gamma, \gamma_0, \ldots, \gamma_{\ell-1}\}$. ■

Note that such γ, γ_l exist, as one solution is

$$\gamma, \gamma_l = \min\{\alpha, \alpha_l, 1+\aleph_0+\Pi\mathcal{I}_\cup, 1+\aleph+\Pi\mathcal{I}, 1+\aleph+\Pi\mathcal{I}_l : l = 0, \ldots, \ell-1\}. \tag{182}$$

Proof. Suppose that

$$u \in \mathcal{A}^{\mathcal{I},\gamma\wedge\beta-}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{I}_l,\gamma_l\wedge\beta-}([0,1))\mathcal{Y}_l, \tag{183}$$

where $\gamma \wedge \beta = \min\{\gamma, \beta\}$. Because $u \in \mathcal{A}^{0+}(X)$, we know this holds for all nonpositive β, as well as some positive β. Our goal is to show that it holds for all β.

Given eq. (183), $u \in \mathcal{A}^{\mathcal{I}_\cup,\gamma_\wedge\wedge\beta-}(X)$. Since we have the mapping properties

$$\mathrm{Diff}^2_{\mathrm{b}}(X) : \mathcal{A}^{\mathcal{I}_\cup,\gamma_\wedge\wedge\beta-}(X) \to \mathcal{A}^{\mathcal{I}_\cup,\gamma_\wedge\wedge\beta-}(X) \tag{184}$$

$$S\,\mathrm{Diff}^2_{\mathrm{b}}(X) : \mathcal{A}^{\mathcal{I}_\cup,\gamma_\wedge\wedge\beta-}(X) \to \mathcal{A}^{\min\{\gamma_\wedge,\beta,\Pi\mathcal{I}_\cup\}-}(X), \tag{185}$$

where $\Pi\mathcal{I} = \{\Re j : (j,k) \in \mathcal{I}\}$, we have

$$\begin{aligned} Eu \in \mathcal{A}^{3+\daleth_0+\mathcal{I}_\cup,3+\daleth_0+\gamma_\wedge\wedge\beta-}(X) &+ \mathcal{A}^{3+\aleph_0+\min\{\gamma_\wedge,\beta,\Pi\mathcal{I}_\cup\}-}(X) \\ &= \mathcal{A}^{3+\daleth_0+\mathcal{I}_\cup,3+\min\{\daleth_0+\gamma_\wedge\wedge\beta,\aleph_0+\Pi\mathcal{I}_\cup\}-}(X), \end{aligned} \tag{186}$$

using that $\daleth_0 \leq \aleph_0$.

Similarly, we can write $Qu = q + \sum_{l=0}^{\ell-1} q_l$, where

$$\begin{aligned} q &\in \mathcal{A}^{3+\daleth+\mathcal{I},3+\daleth+\gamma\wedge\beta-}(X;\ell) + \mathcal{A}^{3+\aleph+\min\{\gamma_\wedge\beta,\Pi\mathcal{I}\}-}(X;\ell) \\ &= \mathcal{A}^{3+\daleth+\mathcal{I},3+\min\{\daleth+\gamma\wedge\beta,\aleph+\Pi\mathcal{I}\}-}(X;\ell), \end{aligned} \tag{187}$$

$$\begin{aligned} q_l &\in \mathcal{A}_{\mathrm{c}}^{3+\daleth+\mathcal{I}_l,3+\daleth+\gamma_l\wedge\beta-}[0,1)\mathcal{Y}_l + \mathcal{A}_{\mathrm{c}}^{3+\aleph+\min\{\gamma_l\wedge\beta,\Pi\mathcal{I}_l\}-}([0,1))\mathcal{Y}_l \\ &= \mathcal{A}_{\mathrm{c}}^{3+\daleth+\mathcal{I}_l,3+\min\{\daleth+\gamma_l\wedge\beta,\aleph+\Pi\mathcal{I}_l\}-}([0,1))\mathcal{Y}_l \end{aligned} \tag{188}$$

for each $l \in \{0,\dots,\ell-1\}$.

We can rewrite the PDE $Pu = f$ in the form $\triangle_{g_0} u = g$ for $g = f - Qu - Eu$. The computations above show that

$$\begin{aligned} g &\in \mathcal{A}^{(2+\mathcal{E})\cup(3+\daleth_0+\mathcal{I}_\cup)\cup(3+\daleth+\mathcal{I}),\min\{2+\alpha,3+\daleth_0+\gamma_\wedge\wedge\beta-,3+\aleph_0+\Pi\mathcal{I}_\cup-,3+\daleth+\gamma\wedge\beta-,3+\aleph+\Pi\mathcal{I}-\}}(X;\ell) \\ &+\sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(2+\mathcal{E}_l)\cup(3+\daleth_0+\mathcal{I}_\cup)\cup(3+\daleth+\mathcal{I}_l),\min\{2+\alpha_l,3+\daleth_0+\gamma_\wedge\wedge\beta-,3+\aleph_0+\Pi\mathcal{I}_\cup-,3+\daleth+\gamma_l\wedge\beta-,3+\aleph_l+\Pi\mathcal{I}_l-\}}([0,1))\mathcal{Y}_l. \end{aligned} \tag{189}$$

Applying Lemma A.2, we can conclude that

$$u \in \mathcal{A}^{\mathcal{C}_\ell\uplus(\mathcal{E}\cup(1+\daleth_0+\mathcal{I}_\cup)\cup(1+\daleth+\mathcal{I})),\gamma^+-}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{(c_l,0)\uplus(\mathcal{E}_l\cup(1+\daleth_0+\mathcal{I}_\cup)\cup(1+\daleth+\mathcal{I}_l)),\gamma_l^+-}([0,1))\mathcal{Y}_l, \tag{190}$$

where

$$\begin{aligned} \gamma^+ &= \min\{\alpha, 1+\daleth_0+\gamma_\wedge\wedge\beta, 1+\daleth+\gamma\wedge\beta, 1+\aleph_0+\Pi\mathcal{I}_\cup, 1+\aleph+\Pi\mathcal{I}\} \\ &\geq \min\{\gamma, 1+\daleth+\beta, 1+\daleth_0+\gamma_\wedge\} \end{aligned} \tag{191}$$

$$\begin{aligned} \gamma_l^+ &= \{\alpha_l, 1+\daleth_0+\gamma_\wedge\wedge\beta, 1+\aleph_0+\Pi\mathcal{I}_{\cup-}, 1+\daleth+\gamma_l\wedge\beta, 1+\aleph_l+\Pi\mathcal{I}_l\} \\ &\geq \min\{\gamma_l, 1+\daleth+\beta, 1+\daleth_0+\gamma_\wedge\}. \end{aligned} \tag{192}$$

Since we are assuming that $\gamma,\gamma_l \leq 1+\daleth_0+\gamma_\wedge$, we have $\gamma^+ \geq \gamma\wedge\beta^+$ and $\gamma_l^+ \geq \gamma_l\wedge\beta^+$ for $\beta^+ = 1+\daleth+\beta$. Using our assumptions regarding $\mathcal{I},\mathcal{I}_l$, the index sets in eq. (190) are subsets of the respective of $\mathcal{I},\mathcal{I}_0,\dots,\mathcal{I}_{\ell-1}$. For example, $\mathcal{C}_\ell \uplus (\mathcal{E}\cup(1+\daleth_0+\mathcal{I}_\cup)\cup(1+\daleth+\mathcal{I})) \subseteq \mathcal{I}$. So, eq. (190) gives

$$u \in \mathcal{A}^{\mathcal{I},\gamma\wedge\beta^+-}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{I}_l,\gamma_l\wedge\beta^+-}([0,1))\mathcal{Y}_l. \tag{193}$$

Applying this argument inductively, we have $u \in \mathcal{A}^{\mathcal{I},\gamma\wedge\beta^{+\cdots+}-}(X;\ell) + \sum_{l=0}^{\ell-1} \mathcal{A}_{\mathrm{c}}^{\mathcal{I}_l,\gamma_l\wedge\beta^{+\cdots+}-}([0,1))\mathcal{Y}_l$, where $\beta^{+\cdots+} = ((\beta^+)^+)\cdots$. Done enough times, $\beta^{+\cdots+} \geq \max\{\gamma,\gamma_0,\dots,\gamma_{\ell-1}\}$, so we conclude the result. $\square$

Given $\mathcal{E},\mathcal{E}_l$, it is straightforward to construct minimal index sets $\mathcal{I},\mathcal{I}_l$ with closure properties required in the proposition, as we now demonstrate.

Let

$$\mathcal{I}^{(0)} = \mathcal{E}, \qquad \mathcal{I}_l^{(0)} = \mathcal{E}_l, \tag{194}$$

and recursively define index sets

$$\begin{aligned} \mathcal{I}^{(k+1)} &= \mathcal{I}^{(k)} \cup (\mathcal{C}_\ell \uplus (\mathcal{E}\cup(1+\daleth_0+\mathcal{I}_\cup^{(k)})\cup(1+\daleth+\mathcal{I}^{(k)}))), \\ \mathcal{I}_l^{(k+1)} &= \mathcal{I}_l^{(k)} \cup ((c_l,0) \uplus (\mathcal{E}_l\cup(1+\daleth_0+\mathcal{I}_\cup^{(k)})\cup(1+\daleth+\mathcal{I}_l^{(k)}))) \end{aligned} \tag{195}$$

for all $k \in \mathbb{N}$, where $\mathcal{I}_\cup^{(k)} = \mathcal{I}^{(k)} \cup \mathcal{I}_0^{(k)} \cup \cdots \mathcal{I}_{\ell-1}^{(k)}$. Then, one takes

$$\mathcal{I} = \cup_{k\in\mathbb{N}} \mathcal{I}^{(k)}, \qquad \mathcal{I}_l = \cup_{k\in\mathbb{N}} \mathcal{I}_l^{(k)}. \tag{196}$$

In order to show that these are index sets (which is not completely obvious), what one wants to show is that, for any $\alpha \in \mathbb{R}$, the sets $\{(\gamma,k) \in \mathcal{I} : \Re\gamma < \alpha\}$, $\{(\gamma,k) \in \mathcal{I}_l : \Re\gamma < \alpha\}$ are finite. (The other properties required of index sets are preserved in taking the union over k.) This follows from the following claim: letting

$$\mathcal{I}^{(k)}_{<\alpha} = \{(\gamma,k)\in\mathcal{I}^{(k)} : \Re\gamma<\alpha\}, \qquad \mathcal{I}^{(k)}_{l,<\alpha} = \{(\gamma,k)\in\mathcal{I}^{(k)}_l : \Re\gamma<\alpha\}, \tag{197}$$

it is the case that, for each α, there exists some $k(\alpha)$ such that these sets are constant in k for $k\geq k(\alpha)$. In other words, for k large the definitions eq. (195) are only adding to $\mathcal{I}_\bullet$ elements (γ,k) with $\Re\gamma$ large.

Let $S\subset\mathbb{R}$ be the set of α for which this holds. If α is sufficiently negative, then the sets eq. (197) are just empty. So, S is nonempty. Indeed,

$$\begin{aligned}\mathcal{I}^{(k+1)}_{<\alpha} &= \mathcal{I}^{(k)}_{<\alpha}\cup((\mathcal{C}_\ell)_{<\alpha}\uplus(\mathcal{E}_{<\alpha}\cup(1+\beth_0+\mathcal{I}^{(k)}_\cup)_{<\alpha}\cup(1+\beth+\mathcal{I}^{(k)})_{<\alpha}) \\ &= \mathcal{I}^{(k)}_{<\alpha}\cup((\mathcal{C}_\ell)_{<\alpha}\uplus(\mathcal{E}_{<\alpha}\cup(1+\beth_0+(\mathcal{I}^{(k)}_\cup)_{<\alpha-1-\beth_0})\cup(1+\beth+\mathcal{I}^{(k)}_{<\alpha-1-\beth})),\end{aligned}\tag{198}$$

and, likewise,

$$\mathcal{I}^{(k+1)}_{l,<\alpha} = \mathcal{I}^{(k)}_{l,<\alpha}\cup((c_l,0)_{<\alpha}\uplus(\mathcal{E}_{l,<\alpha}\cup(1+\beth_0+(\mathcal{I}^{(k)}_\cup)_{<\alpha-1-\beth_0})\cup(1+\beth+\mathcal{I}^{(k)}_{l,<\alpha-1-\beth})). \tag{199}$$

So, if $\alpha<d-2$, in which case $(\mathcal{C}_\ell)_{<\alpha}=\varnothing$ and $(c_l,0)_{<\alpha}=\varnothing$, and if $\alpha<\min\{\Pi\mathcal{E},\Pi\mathcal{E}_l : l=0,\ldots,\ell-1\}=\min\Pi\mathcal{E}_\cup$, in which case $\mathcal{E}_{<\alpha},\mathcal{E}_{l,<\alpha}=\varnothing$, then $\mathcal{I}_{<\alpha},\mathcal{I}_{l,<\alpha}=\varnothing$. So, $(-\infty,\min\{d-2,\Pi\mathcal{E}_\cup\}]\subseteq S$, and we can take $k(\alpha)=0$ for $\alpha\leq\min\{d-2,\Pi\mathcal{E}_\cup\}$.

More generally, if $\alpha-1-\min\{\beth,\beth_0\}=\alpha-1-\beth\in S$, and if $k\geq k(\alpha-1-\beth)+1$, then we can conclude that

$$\mathcal{I}^{(k+1)}_{<\alpha}=\mathcal{I}^{(k)}_{<\alpha}, \qquad \mathcal{I}^{(k+1)}_{l,<\alpha}=\mathcal{I}^{(k)}_{l,<\alpha}, \tag{200}$$

for all $l\in\{0,\ldots,\ell-1\}$. Thus, $\alpha\in S$, and we can conclude that we may take $k(\alpha)=k(\alpha-1-\beth)+1$. Proceeding inductively, we conclude that $S=\mathbb{R}$, and we may take

$$k(\alpha)=\Big\lceil\frac{\alpha-\min\{d-2,\Pi\mathcal{E}_\cup\}}{1+\beth}\Big\rceil. \tag{201}$$

So, the sets $\mathcal{I},\mathcal{I}_l$ defined above are well-defined index sets, and they satisfy the hypotheses of the proposition. Thus, we get:

Corollary. *Suppose that $u\in\mathcal{A}^{0+}(X)$ satisfies $Pu\in\mathcal{A}^{2+\mathcal{E},2+\alpha}(X;\ell)+\sum_{l=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{2+\mathcal{E}_j,2+\alpha_j}[0,1)\mathcal{Y}_l$. Then,*

$$u\in\mathcal{A}^{\mathcal{I},\gamma-}(X;\ell)+\sum_{l=0}^{\ell-1}\mathcal{A}_{\mathrm{c}}^{\mathcal{I}_l,\gamma-}[0,1)\mathcal{Y}_l, \tag{202}$$

where γ,γ_l are above and $\mathcal{I},\mathcal{I}_l$ are the previously defined index sets. ■□

Example. Consider the case where $\beth_0=\beth$ and $\mathcal{E}_0,\ldots,\mathcal{E}_{\ell-1}=\mathcal{E}$. In other words, the first subleading terms of P are not assumed to preserve the conic symmetry.

Then, eq. (195) yields $\mathcal{I}^{(k+1)}_\cup=\mathcal{I}^{(k)}_\cup\cup(\mathcal{C}_0\uplus\mathcal{E})\cup(\mathcal{C}_0\uplus(1+\beth+\mathcal{I}^{(k)}_\cup))$. This single recurrence relation can be solved to yield

$$\mathcal{I}_\cup=(\mathcal{C}_0\uplus\mathcal{E})\cup(\mathcal{C}_0\uplus(1+\beth+\mathcal{C}_0\uplus\mathcal{E}))\cup(\mathcal{C}_0\uplus(1+\beth+(\mathcal{C}_0\uplus(1+\beth+\mathcal{C}_0\uplus\mathcal{E}))) \cup\cdots. \tag{203}$$

Equation (195) also yields $\mathcal{I}^{(k+1)}=\mathcal{I}^{(k)}\cup(\mathcal{C}_\ell\uplus\mathcal{E})\cup(\mathcal{C}_\ell\uplus(1+\beth+\mathcal{I}^{(k)}_\cup))$, which in turn yields $\mathcal{I}=(\mathcal{C}_\ell\uplus\mathcal{E})\cup(\mathcal{C}_\ell\uplus(1+\beth+\mathcal{I}_\cup))$. Similarly, $\mathcal{I}=((c_l,0)\uplus\mathcal{E})\cup((c_l,0)\uplus(1+\beth+\mathcal{I}_\cup))$.

So, this case is understood explicitly.

Example. Consider now the case where $\beth_0=\infty$, which means that all symmetry-breaking terms in P are Schwartz.

Then, the $1 + \daleth_0 + \mathcal{I}^{(k)}_{\cup}$ terms in eq. (195) are always $\varnothing$, so the recursion relations decouple:

$$\mathcal{I}^{(k+1)} = \mathcal{I}^{(k)} \cup (\mathcal{C}_\ell \uplus \mathcal{E}) \cup (\mathcal{C}_\ell \uplus (1 + \daleth + \mathcal{I}^{(k)})) \tag{204}$$

$$\mathcal{I}^{(k+1)}_l = \mathcal{I}^{(k)}_l \cup ((c_l, 0) \uplus \mathcal{E}_l) \cup ((c_l, 0) \uplus (1 + \daleth + \mathcal{I}^{(k)}_l)). \tag{205}$$

These decoupled recurrence relations are readily solved to yield $\mathcal{I} = (\mathcal{C}_\ell \uplus \mathcal{E}) \cup (\mathcal{C}_\ell \uplus (1 + \daleth + \mathcal{C}_\ell \uplus \mathcal{E})) \cup (\mathcal{C}_\ell \uplus (1 + \daleth + \mathcal{C}_\ell \uplus (1 + \daleth + \mathcal{C}_\ell \uplus \mathcal{E}))) \cup \cdots$, and similarly for $\mathcal{I}_l$.

Example. If, in addition, $\daleth = \infty$ and $\mathcal{E}_\bullet = \varnothing$ (meaning that P differs from $\triangle_{g_0}$ by Schwartz terms, and that f is Schwartz), then $\mathcal{I} = \mathcal{C}_\ell$ and $\mathcal{I}_l = (c_l, 0)$. So, in this case we recover the multipole expansion from classical electrostatics.

Appendix B. Solvability theory of the model problem at tf

We make use of the limiting absorption principle for $\hat{N}_{\mathrm{tf}}$ (the operator defined in eq. (98)), involving the function spaces

$$\mathcal{A}^{\beta,\gamma}(\mathrm{tf}) = \rho_{\mathrm{bf}}^{\beta}\rho_{\mathrm{zf}}^{\gamma}\mathcal{A}^{0,0}(\mathrm{tf}), \qquad \mathcal{A}^{0,0}(\mathrm{tf}) = \{v \in C^\infty(\mathrm{tf}^\circ) : Lv \in L^\infty \text{ for all } L \in \mathrm{Diff}_{\mathrm{b}}(\mathrm{tf})\}, \tag{206}$$

where $\mathrm{Diff}_{\mathrm{b}}(\mathrm{tf}) = \bigcup_{m\in\mathbb{N}} \mathrm{Diff}_{\mathrm{b}}^{m,0,0}(\mathrm{tf})$ is the algebra of differential operators generated over $C^\infty(\mathrm{tf})$ by $\hat{r}\partial_{\hat{r}}$ and the differential operators in the angular directions. In $\mathrm{Diff}_{\mathrm{b}}^{m,\ell,s}(\mathrm{tf})$, the m is the differential order, ℓ is the order at bf, and s is the order at zf.

To begin our discussion, we recall:

Proposition B.1 ([Vas21c, Prop. 5.4], [Hin22, Thm. 2.22])**.** *If $\beta < (d-1)/2$ and $\gamma \in (2-d, 0)$, then $\hat{N}_{\mathrm{tf}} : \mathcal{A}^{\beta,\gamma}(\mathrm{tf}) \to \mathcal{A}^{\beta+1,\gamma-2}(\mathrm{tf})$ is an isomorphism.* ■□

We will not repeat the proof of this result, but it is not difficult to understand the restrictions on β, γ therein:

Proof sketch. Since tf is an exact cone, we can separate variables: for $u(\hat{r}, \theta) = v(\hat{r})Y_j(\theta)$ and $f(\hat{r}, \theta) = g(\hat{r})Y_j(\theta)$, $Y_j \in \mathcal{Y}_j$, u solves $\hat{N}_{\mathrm{tf}}u = f$ if and only if, letting

$$L_j = -\frac{\mathrm{d}^2}{\mathrm{d}\hat{r}^2} - \Big(\frac{d-1}{\hat{r}} + 2i\Big)\frac{\partial}{\partial \hat{r}} + \frac{\lambda_j}{\hat{r}^2} - \frac{i(d-1)}{\hat{r}}, \tag{207}$$

we have $L_j v = g$. Here, as in previous sections, λ_j is the jth eigenvalue of the boundary Laplacian (counted with or without multiplicity, depending on our conventions, so that $\triangle_{\partial X} Y_j = \lambda_j Y_j$). Note that

$$L_j : \mathcal{A}^{\beta,\gamma}([0,\infty]_{\hat{r}}) \to \mathcal{A}^{\beta+1,\gamma-2}([0,\infty]_{\hat{r}}). \tag{208}$$

We will assume, for the sake of this discussion, that the only obstructions to the invertibility of $\hat{N}_{\mathrm{tf}} : \mathcal{A}^{\beta,\gamma}(\mathrm{tf}) \to \mathcal{A}^{\beta+1,\gamma-2}(\mathrm{tf})$ can be seen already as an obstruction to the invertability of L_j for some $j \in \mathbb{N}$. This is in fact true, as can be proven using the Mellin transform. The ordinary differential operator L_j is regular singular at $\hat{r} = 0$ and irregular singular at $\hat{r} = \infty$. It has, up to normalization, a unique element of its kernel which is non-oscillatory at infinity and, up to normalization, a unique element which is recessive (which, we will see, means bounded) at $\hat{r} = 0$. Indeed, L_j is a conjugated form of the differential operator appearing in Bessel's ODE, which yields

$$\ker L_j = e^{-i\hat{r}}\hat{r}^{-(d-2)/2}\,\mathrm{span}_{\mathbb{C}}\,\{J_\nu(\hat{r}), Y_\nu(\hat{r})\}, \tag{209}$$

where $\nu = 2^{-1}((d-2)^2 + 4\lambda_j)^{1/2}$ and J_ν, Y_ν are the usual Bessel functions. We will be able to read the restrictions on β, γ imposed in Proposition B.1 off of the $\hat{r} \to 0$ and $\hat{r} \to \infty$ asymptotics of the elements of $\ker L_j$.

The element of this kernel which is non-oscillatory at infinity is $e^{-i\hat{r}}\hat{r}^{-(d-2)/2}H_\nu^+(\hat{r})$, where $H_\nu^+(\hat{r})$ is the Hankel function with $e^{i\hat{r}}$-type asymptotics. Since conormality excludes oscillatory asymptotics, the only element of $\ker L_j$ that can lie in $\mathcal{A}^{\beta,\gamma}(\mathrm{tf})$ is the non-oscillatory solution $e^{-i\hat{r}}\hat{r}^{-(d-2)/2}H_\nu^+(\hat{r})$

and its scalar multiples, no matter what β,γ are. The small-$\hat{r}$ asymptotics of the Hankel functions show that

$$e^{-i\hat{r}}\hat{r}^{-(d-2)/2}H_\nu^+(\hat{r}) \propto (1+o(1))\hat{r}^{-c_j} \tag{210}$$

as $\hat{r}\to 0^+$, where $c_j = (d-2)/2+\nu$ is as in the previous subsection. Thus, a sufficient condition on β,γ for $\ker \hat{N}_{\mathrm{tf}} \cap \mathcal{A}^{\beta,\gamma}(\mathrm{tf})$ to be zero is that $\gamma > -c_j$. For this to hold for all j means $\gamma > -c_0 = 2-d$.

Solving $L_j v = g$ for v non-oscillatory, even if $g\in C_{\mathrm{c}}^\infty(\mathbb{R}^+_{\hat{r}})$ we cannot expect v to decay more rapidly as $\hat{r}\to\infty$ than $e^{-i\hat{r}}\hat{r}^{-(d-2)/2}H_\nu^+(\hat{r})$. This is especially clear if $g\in C_{\mathrm{c}}^\infty(\mathbb{R}^+_{\hat{r}})$, since then

$$v(\hat{r}) \propto e^{-i\hat{r}}\hat{r}^{-(d-2)/2}H_\nu^+(\hat{r}) \tag{211}$$

for $\hat{r}$ sufficiently large. Since

$$e^{-i\hat{r}}\hat{r}^{-(d-2)/2}H_\nu^+(\hat{r}) \propto (1+o(1))\hat{r}^{-(d-1)/2} \tag{212}$$

as $\hat{r}\to\infty$, this means that for $\hat{N}_{\mathrm{tf}} : \mathcal{A}^{\beta,\gamma}(\mathrm{tf}) \to \mathcal{A}^{\beta+1,\gamma-2}(\mathrm{tf})$ to be surjective it had better be the case that $\beta \le (d-1)/2$. On the other hand, such v will typically have no better decay as $\hat{r}\to 0^+$ than the recessive element of $\ker L_j$, which is $e^{-i\hat{r}}\hat{r}^{-(d-2)/2}J_\nu(\hat{r})$, and this satisfies

$$e^{-i\hat{r}}\hat{r}^{-(d-2)/2}J_\nu(\hat{r}) \propto (1+o(1))\hat{r}^{\nu-(d-2)/2} \tag{213}$$

as $\hat{r}\to 0^+$. So, for $\hat{N}_{\mathrm{tf}} : \mathcal{A}^{\beta,\gamma}(\mathrm{tf}) \to \mathcal{A}^{\beta+1,\gamma-2}(\mathrm{tf})$ to be surjective it had also better be the case that $\gamma \le \nu-(d-2)/2$. In order for this to hold for all j, it just needs to be that $\gamma\le 0$. Conversely, we get surjectivity if both of the conditions $\beta < (d-1)/2$ and $\gamma<0$ are met (excluding the edge cases, for simplicity). One method of proving this is an explicit construction of the Greens function for L_j using H_ν^+, J_ν, and their (nonzero) Wronskian.

So, in summary, if $\gamma\in(2-d,0)$ and $\beta<(d-1)/2$, then $\hat{N}_{\mathrm{tf}} : \mathcal{A}^{\beta,\gamma}(\mathrm{tf}) \to \mathcal{A}^{\beta+1,\gamma-2}(\mathrm{tf})$ is bijective. □

The main proposition of this section is a refinement of the previous result involving the function spaces

$$\mathcal{A}^{(\mathcal{E},\beta),(\mathcal{F},\gamma)}(\mathrm{tf};\ell) + \sum_{j=0}^{\ell-1} \mathcal{A}^{(\mathcal{E}_j,\beta),(\mathcal{F}_j,\gamma_j)}([0,\infty]_{\hat{r}})\mathcal{Y}_j, \tag{214}$$

where $\mathcal{E},\mathcal{E}_j,\mathcal{F},\mathcal{F}_j$ are pre-index sets and $\beta,\gamma,\beta_j,\gamma_j\in\mathbb{R}$. This function space consists of all functions $\mathrm{tf}^\circ\to\mathbb{C}$ of the form

$$u = u_{\mathrm{rem}} + \sum_{j=0}^{\ell-1} u_j(\hat{r}) \tag{215}$$

for $u_{\mathrm{rem}}\in\mathcal{A}^{(\mathcal{E},\beta),(\mathcal{F},\gamma)}(\mathrm{tf};\ell)$ and $u_j\in\mathcal{A}^{(\mathcal{E}_j,\beta_j),(\mathcal{F}_j,\gamma_j)}([0,\infty]_{\hat{r}})\mathcal{Y}_j$. Here, the pre-index sets $\mathcal{E},\mathcal{E}_j$ refer to $\hat{r}\to\infty$ asymptotics and the pre-index sets $\mathcal{F},\mathcal{F}_j$ refer to $\hat{r}\to 0^+$ asymptotics.

Let $b_j = 2^{-1}(2-d+((d-2)^2+4\lambda_j)^{1/2})$, as in the previous section. Let $\mathcal{B}_{\ge\ell}$ denote the smallest index set containing $(b_j,0)$ for all $j\ge\ell$.

Proposition B.2. *Fix $\ell\in\mathbb{N}$. Suppose that $\min\{\Pi\mathcal{F},\Pi\mathcal{F}_j,\gamma,\gamma_j\}>0$. If*

$$f\in\mathcal{A}^{(1+\mathcal{E},1+\beta),(-2+\mathcal{F},-2+\gamma)}(\mathrm{tf};\ell) + \sum_{j=0}^{\ell-1}\mathcal{A}^{(1+\mathcal{E}_j,1+\beta),(-2+\mathcal{F}_j,-2+\gamma_j)}([0,\infty]_{\hat{r}})\mathcal{Y}_j, \tag{216}$$

then $u=\hat{N}_{\mathrm{tf}}^{-1}f$ is well-defined (since $f\in\mathcal{A}^{-\infty,0+}(\mathrm{tf})$), and u satisfies

$$u\in\mathcal{A}^{((2^{-1}(d-1)\uplus\mathcal{E}),\beta-),(\tilde{\mathcal{I}}[\mathcal{F},\ell],\gamma-)}(\mathrm{tf};\ell) + \sum_{j=0}^{\ell-1}\mathcal{A}^{((2^{-1}(d-1)\uplus\mathcal{E}_j),\beta_j-),(\tilde{\mathcal{I}}_j[\mathcal{F}_j],\gamma_j-)}([0,\infty]_{\hat{r}})\mathcal{Y}_j, \tag{217}$$

where $\uplus$ is as in §3 and $\tilde{\mathcal{I}},\tilde{\mathcal{I}}_j$ are defined as follows:

- $\tilde{\mathcal{I}}_j = (b_j,0)\uplus\mathcal{F}_j$,

- *and, similarly,* $\tilde{\mathcal{I}}[\mathcal{F},\ell]$ *is the smallest index set containing* $\mathcal{F}$ *and* $(b_j,0)\uplus(1+\tilde{\mathcal{I}}')$ *whenever* $\tilde{\mathcal{I}}'\subseteq\tilde{\mathcal{I}}$ *and* $j\geq\ell$.

■

We also write $\tilde{\mathcal{I}}[\mathcal{F},\ell]$ as $\mathcal{B}_{\geq\ell}[\mathcal{F}]$. This can be computed explicitly, but we will not do so.

Proof. Since we already know that $u\in C^\infty(\mathrm{tf}^\circ)$, it suffices to check u near $\hat{r}=\infty$ and near $\hat{r}=0$.

The partial polyhomogeneity near $\hat{r}=\infty$ can be proven using the exact same argument as that used to prove Proposition 3.3. Here are the details: the least decaying term in $\hat{N}_{\mathrm{tf}}$ is

$$N_{\mathrm{tf}\cap\mathrm{bf}}=-2i\frac{\partial}{\partial\hat{r}}-\frac{i(d-1)}{\hat{r}}. \tag{218}$$

The sense in which this is least decaying is that $\hat{N}_{\mathrm{tf}}-N_{\mathrm{tf}\cap\mathrm{bf}}\in\hat{r}^{-2}\operatorname{Diff}_{\mathrm{b}}(\mathrm{tf}\backslash\mathrm{zf})$, whereas $N_{\mathrm{tf}\cap\mathrm{bf}}\in\hat{r}^{-1}\operatorname{Diff}_{\mathrm{b}}(\mathrm{tf}\backslash\mathrm{zf})$. So, we rearrange the definition $\hat{N}_{\mathrm{tf}}u=f$ of u to get

$$N_{\mathrm{tf}\cap\mathrm{zf}}u=f-(\hat{N}_{\mathrm{tf}}-N_{\mathrm{tf}\cap\mathrm{bf}})u. \tag{219}$$

A one-sided inverse of $N_{\mathrm{tf}\cap\mathrm{bf}}$ can be constructed just by integrating:

$$N_{\mathrm{tf}\cap\mathrm{bf}}^{-1}=\Big[-2i\frac{\partial}{\partial\hat{r}}-\frac{i(d-1)}{\hat{r}}\Big]^{-1}:f(\hat{r})\mapsto\frac{i}{2\hat{r}^{(d-1)/2}}\int_1^{\hat{r}}f(x)x^{(d-1)/2}\,\mathrm{d}x. \tag{220}$$

Applying this to both sides of $N_{\mathrm{tf}\cap\mathrm{zf}}u=f-(\hat{N}_{\mathrm{tf}}-N_{\mathrm{tf}\cap\mathrm{bf}})u$, we get

$$u=\frac{1}{\hat{r}^{(d-1)/2}}\Big(C(\theta)+\frac{i}{2}\int_1^{\hat{r}}(f(x)-(\hat{N}_{\mathrm{tf}}-N_{\mathrm{tf}\cap\mathrm{bf}})u(x))x^{(d-1)/2}\,\mathrm{d}x\Big), \tag{221}$$

where $C(\theta)$ is an undetermined function of θ (that has to do with $\hat{r}\to0^+$ behavior).

We abbreviate $\mathcal{J}[\mathcal{E}]=2^{-1}(d-1)\uplus\mathcal{E}$. Now, let S denote the set of $B\in\mathbb{R}$ such that $u\in\mathcal{A}^{\mathcal{J}[\mathcal{E}],\beta\wedge B}(\mathrm{tf}\backslash\mathrm{zf};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}^{\mathcal{J}[\mathcal{E}_j],\beta_j\wedge B}((0,\infty]_{\hat{r}})\mathcal{Y}_j$. We already know, from Proposition B.1, that $B\in S$ if it is sufficiently negative, so S is nonempty. Note that

$$(\hat{N}_{\mathrm{tf}}-N_{\mathrm{tf}\cap\mathrm{bf}})u(\hat{r})\in\mathcal{A}^{2+\mathcal{J}[\mathcal{E}],2+\beta\wedge B}(\mathrm{tf}\backslash\mathrm{zf};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}^{2+\mathcal{J}[\mathcal{E}_j],2+\beta_j\wedge B}((0,\infty]_{\hat{r}})\mathcal{Y}_j, \tag{222}$$

so the factor in the integrand $f_1=f-(\hat{N}_{\mathrm{tf}}-N_{\mathrm{tf}\cap\mathrm{bf}})u$ in eq. (221) satisfies

$$f_1\in\mathcal{A}^{(1+\mathcal{E})\cup(2+\mathcal{J}[\mathcal{E}]),(1+\beta)\wedge(2+B)}(\mathrm{tf}\backslash\mathrm{zf};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}^{(1+\mathcal{E}_j)\cup(2+\mathcal{J}[\mathcal{E}_j]),(1+\beta_j)\wedge(2+B)}((0,\infty]_{\hat{r}})\mathcal{Y}_j. \tag{223}$$

Consequently, eq. (221) yields

$$\begin{aligned}u\in\mathcal{A}^{(2^{-1}(d-1),0)\uplus(\mathcal{E}\cup(1+\mathcal{J}[\mathcal{E}])),\beta\wedge(1+B)-}&(\mathrm{tf}\backslash\mathrm{zf};\ell)\\&+\sum_{j=0}^{\ell-1}\mathcal{A}^{(2^{-1}(d-1),0)\uplus(\mathcal{E}\cup(1+\mathcal{J}[\mathcal{E}_j])),\beta_j\wedge(1+B)-}((0,\infty]_{\hat{r}})\mathcal{Y}_j.\end{aligned} \tag{224}$$

By construction, $(2^{-1}(d-1),0)\uplus(\mathcal{E}\cup(1+\mathcal{J}[\mathcal{E}]))\subseteq\mathcal{J}[\mathcal{E}]$, and likewise for the other index sets. Consequently, eq. (224) says that $B+1-\varepsilon\in S$ for any $\varepsilon>0$. So, proceeding inductively, we can conclude that $S=\mathbb{R}$.

Consider now the $\hat{r}\to0^+$ behavior. Let S denote the set of $\Gamma\in\mathbb{R}$ such that

$$u\in\mathcal{A}^{(\tilde{\mathcal{I}},\gamma\wedge\Gamma-)}(\mathrm{tf}\backslash\mathrm{bf};\ell)+\sum_{j=0}^{\ell-1}\mathcal{A}^{\tilde{\mathcal{I}}_j,\gamma_j\wedge\Gamma-}([0,\infty)_{\hat{r}})\mathcal{Y}_j. \tag{225}$$

We want to prove that $S = \mathbb{R}$. We already know from Proposition B.1 that S is nonempty. Given that eq. (225) holds, we show that it holds also for a larger Γ. To do this, we use

$$N_{\mathrm{zf}\cap\mathrm{tf}} = -\frac{\partial^2}{\partial \hat{r}^2} - \frac{d-1}{\hat{r}}\frac{\partial}{\partial \hat{r}} + \frac{1}{\hat{r}^2}\triangle_{\partial X}. \tag{226}$$

Note that $\hat{N}_{\mathrm{tf}} - N_{\mathrm{zf}\cap\mathrm{tf}} \in \mathrm{Diff}_{\mathrm{b}}^{1,-1,1}(\mathrm{tf})$, so $N_{\mathrm{zf}\cap\mathrm{tf}} u = \hat{N}_{\mathrm{tf}} u - (\hat{N}_{\mathrm{tf}} - N_{\mathrm{zf}\cap\mathrm{tf}})u = f - ((\hat{N}_{\mathrm{tf}} - N_{\mathrm{zf}\cap\mathrm{tf}})u$ lies in

$$\begin{aligned}&\mathcal{A}^{(-2+\mathcal{F})\cup(-1+\tilde{\mathcal{I}}),\min\{-2+\gamma,-1+\gamma\wedge\Gamma\}-}(\mathrm{tf}\backslash\mathrm{bf}) \\ &\qquad + \sum_{j=0}^{\ell-1} \mathcal{A}^{(-2+\mathcal{F}_j)\cup(-1+\tilde{\mathcal{I}}_j),\min\{-2+\gamma_j,-1+\gamma_j\wedge\Gamma\}-}([0,\infty)_{\hat{r}})\mathcal{Y}_j.\end{aligned} \tag{227}$$

The indicial roots of $N_{\mathrm{zf}\cap\mathrm{tf}}$ are readily computed to be $b_j, -c_j$. Indeed, in the previous section we saw that $c_j, -b_j$ were the indicial roots of the normal operator of $N_{\mathrm{zf}}(P)$ at $r \to \infty$, which is tf. That is, $N_{\mathrm{tf}}(N_{\mathrm{zf}}(P))$ has indicial roots $c_j, -b_j$. But $N_{\mathrm{tf}}(N_{\mathrm{zf}}(P)) \propto N_{\mathrm{tf}\cap\mathrm{zf}}$, where the proportionality involves a factor of a boundary-defining-function. However, before we were computing indicial roots as $r \to \infty$, whereas now we are computing indicial roots as $\hat{r} \to 0^+$, which accounts for why they are now negative.

So, inverting $N_{\mathrm{zf}\cap\mathrm{tf}}$ and citing the analogue of Lemma A.2 (which just has c_j switched with b_j, but is otherwise identical), we conclude

$$u \in \mathcal{A}^{\tilde{\mathcal{I}}[\mathcal{F}\cup(1+\tilde{\mathcal{I}}[\mathcal{F},\ell]),\ell],\min\{\gamma,1+\Gamma\}-}(\mathrm{tf}\backslash\mathrm{bf}) + \sum_{j=0}^{\ell-1} \mathcal{A}^{(b_j,0)\uplus\mathcal{F}_j,\min\{\gamma_j,1+\Gamma\}-}([0,\infty)_{\hat{r}})\mathcal{Y}_j. \tag{228}$$

Given the definition of $\tilde{\mathcal{I}}$, we have $\tilde{\mathcal{I}}[\mathcal{F}\cup(1+\tilde{\mathcal{I}}[\mathcal{F},\ell]),\ell] \subseteq \tilde{\mathcal{I}}[\mathcal{F},\ell]$. So, we conclude that $1+\Gamma \in S$. Induction then establishes that $S = \mathbb{R}$. □

Appendix C. Some reminders about the Mellin transform

Our conventions on the Mellin transform are as follows: for $\mathcal{X}$ a Banach space and $f \in C^\infty([0,1);\mathcal{X})$, let

$$\mathcal{M}f(c) = \int_0^1 f(\rho)\frac{\mathrm{d}\rho}{\rho^{c+1}} \tag{229}$$

for $c \in \mathbb{C}$ such that the Bochner integral above is absolutely convergent.

If $f \in \mathcal{A}_{\mathrm{c}}^{\gamma}(\dot{X})$ for some $\gamma \in \mathbb{R}$, then we can consider f as being a function of ρ valued in $C^k(\partial X_\theta)$ for every $k \in \mathbb{N}$. Specifically,

$$f(\rho,\theta) \in \rho^\gamma L^\infty_{\mathrm{c}}([0,1)_\rho; C^k(\partial X_\theta)). \tag{230}$$

Then, $\mathcal{M}f(c) \in C^\infty(\partial X)$ exists for every $c \in \mathbb{C}$ with $\Re c < \gamma$, and so we can write $\mathcal{M}f(c,\theta)$ to denote this function of c and $\theta \in \partial X$. Moreover, $\mathcal{M}f(c)$ depends analytically on c in this domain. More precisely, we have uniform bounds in strips:

$$\mathcal{M}f(c,\theta) \in \langle \Im c\rangle^\kappa \mathscr{A}(\{c \in \mathbb{C} : \gamma_1 < \Re c < \gamma_0\}; C^k(\partial X)) \tag{231}$$

for any γ_0, γ_1 satisfying $\gamma_1 < \gamma_0 < \gamma$, $m \in \mathbb{R}$, and $\kappa \in \mathbb{R}$, where $\mathscr{A}(\Omega;\mathcal{X})$ is the set of uniformly bounded analytic functions on $\Omega \subset \mathbb{C}$ valued in the Banach space $\mathcal{X}$. The decay as $\Im c \to \pm\infty$ follows from the definition of the Mellin transform via an integration-by-parts argument: for $c \neq 0$,

$$\begin{aligned}\mathcal{M}f(c) &= \lim_{\epsilon\to 0^+}\int_\epsilon^1 f(\rho)\Big(-\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{1}{\rho^c}\Big)\,\mathrm{d}\rho = \lim_{\epsilon\to 0^+}\Big[\frac{f(\epsilon)}{c\epsilon^c} + \frac{1}{c}\int_\epsilon^1 \rho f'(\rho)\frac{\mathrm{d}\rho}{\rho^{c+1}}\Big] \\ &= \frac{1}{c}\int_0^1 \rho f'(\rho)\frac{\mathrm{d}\rho}{\rho^{c+1}} = \frac{1}{c}\mathcal{M}_{\rho\to c}(\rho f'(\rho)).\end{aligned} \tag{232}$$

Since $\rho f'(\rho)$ is in $\mathcal{A}_{\mathrm{c}}^{\gamma}(\dot{X})$ as well, the Mellin transform $\mathcal{M}_{\rho\to c}(\rho f'(\rho))$ is also defined for $\Re c < \gamma$ and satisfies an L^∞ bound in all vertical strips thereof. This gives eq. (231) for $\kappa = -1$, and an inductive argument extends the conclusion to all κ.

One form of the *Mellin inversion formula* states that any $f \in \mathcal{A}_{\mathrm{c}}^{\gamma}(\dot{X})$ can be recovered from its Mellin transform via the absolutely convergent contour integral

$$f(\rho,\theta) = \mathcal{M}^{-1}\{\mathcal{M}f\} = \frac{1}{2\pi i}\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \rho^c \mathcal{M}f(c)\,\mathrm{d}c, \tag{233}$$

for any $\gamma_0 < \gamma$. Indeed, this can be reduced to the one-dimensional Fourier inversion formula. Letting $x = -\log\rho$ and $\xi = ic$,

$$\mathcal{M}f = \int_{-\infty}^{\infty} e^{-i\xi x} f(e^{-x})\,\mathrm{d}x, \quad (\mathcal{M}^{-1}M)(\rho,\theta) = \frac{1}{2\pi}\int_{i\gamma_0-\infty}^{i\gamma_0+\infty} e^{i\xi x} M\,\mathrm{d}\xi, \tag{234}$$

so the inverse Mellin transform $\mathcal{M}^{-1}$ is just the inverse Fourier transform rewritten using a logarithmic change of variables.

When applying the Mellin inversion formula in order to extract asymptotics, we need the following: suppose that $M \in \langle \Im c\rangle^{\kappa}\mathscr{A}(\{c \in \mathbb{C} : \gamma_1 < \Re c < \gamma_0\}; \mathcal{X})$ for some Banach space $\mathcal{X}$, for every $\kappa \in \mathbb{R}$. Then, the inverse Mellin transform $\mathcal{M}^{-1}M$ satisfies $\mathcal{A}^{\gamma}([0,\infty)_\rho; \mathcal{X})$ for any $\gamma \in (\gamma_1, \gamma_0)$. Indeed, it follows immediately from the assumed estimates that $f \in \rho^{\gamma} L^{\infty}_{\mathrm{loc}}([0,\infty); \mathcal{X})$, and we can differentiate under the integral sign to see that

$$(\rho\partial_\rho)^k \int_{\gamma - i\infty}^{\gamma + i\infty} \rho^c M(c)\,\mathrm{d}c = \int_{\gamma - i\infty}^{\gamma + i\infty} c^k \rho^c M(c)\,\mathrm{d}c \in \rho^{\gamma} L^{\infty}_{\mathrm{loc}}([0,\infty); \mathcal{X}) \tag{235}$$

for every $k \in \mathbb{N}$. Applying this for $\mathcal{X}$ the spaces $C^m(\partial X)$, we get that if $M \in \langle \Im c\rangle^{\kappa}\mathscr{A}(\{c \in \mathbb{C} : \gamma_1 < \Re c < \gamma_0\}; C^m(\partial X))$ for every $\kappa, m \in \mathbb{R}$, then $\mathcal{M}^{-1}M \in \bigcap_{m\in\mathbb{R}} \mathcal{A}^{\gamma}([0,\infty)_\rho; C^m(\partial X)) = \mathcal{A}^{\gamma}(\dot{X})$.

If $\mathcal{E}$ is an index set and $f \in \mathcal{A}_{\mathrm{c}}^{\mathcal{E}}(\dot{X})$, then

- $\mathcal{M}f(c,\theta)$, which is only initially defined if $\Re c < \min \mathcal{E}$, can be extended to a meromorphic function on the whole complex plane, and
- for any $j \in \mathbb{C}$, the pole at j is at worst of order $k+1$, where k is the largest number such that $(j,k) \in \mathcal{E}$.

Indeed, under these assumptions, for any $\alpha \in \mathbb{R}$ there exists some $f_\alpha \in \mathcal{A}_{\mathrm{c}}^{\alpha}(\dot{X})$ such that we can write

$$f(\rho,\theta) = \sum_{(j,k)\in\mathcal{E}, \Re j \le \alpha} f_{j,k}(\theta)\rho^j \log(\rho)^k + f_\alpha \tag{236}$$

for some $f_{j,k}(\theta) \in C^\infty(\partial X_\theta)$. Thus,

$$\mathcal{M}f(c,\theta) = \int_0^1 f(c,\theta)\frac{\mathrm{d}\rho}{\rho^{c+1}} = \mathcal{M}f_\alpha(c,\theta) + \sum_{(j,k)\in\mathcal{E}, \Re j \le \alpha} f_{j,k}(\theta)\int_0^1 \rho^{j-c-1}(\log\rho)^k\,\mathrm{d}\rho. \tag{237}$$

We already know that $\mathcal{M}f_\alpha(c,\theta)$ is well-defined and analytic on $\Re c < \alpha$, and each of the functions in the summand is a meromorphic function of c with poles of the claimed form, as follows from an explicit computation of the integral.

Conversely, suppose that M is a meromorphic $\mathcal{X}$-valued function on $\{c \in \mathbb{C} : \gamma_1 < \Re c < \gamma_0\}$ with finitely many poles $c_1, \ldots, c_N \in \{c \in \mathbb{C} : \gamma_1 < \Re c < \gamma_0\}$, and let $\mathcal{E}$ denote the set of $(j,k) \in \mathbb{C} \times \mathbb{N}$ such that $j \in \{c_1, \ldots, c_N\}$ and the order of the pole of M at j is at least $k+1$. This is a pre-index set in the sense that we are using the term. Suppose moreover that there exists some $R > 0$ such that $M \in \langle \Im c\rangle^{\kappa}\mathscr{A}(\{c \in \mathbb{C} : \gamma_1 < \Re c < \gamma_0, |\Im z| > R\}; \mathcal{X})$ for every $\kappa \in \mathbb{R}$. Then,

$$\mathcal{M}^{-1}M = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \rho^c M(c)\,\mathrm{d}c \tag{238}$$

is well-defined for $\gamma \in (\gamma_1, \min\{\Re c_1, \ldots, \Re c_N\})$, and, as discussed above, $\mathcal{M}^{-1}M \in \mathcal{A}^{\gamma}([0,\infty)_\rho; \mathcal{X})$. The contour can now be shifted to the right, through the possible poles. Specifically,

$$\mathcal{M}^{-1}M = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \rho^c M(c)\,\mathrm{d}c + \sum_{\substack{c \in \{c_1,\ldots,c_N\} \\ \Re c < \gamma}} \frac{1}{2\pi i}\oint_c \rho^\zeta M(\zeta)\,\mathrm{d}\zeta \tag{239}$$

for any $\gamma \in (\gamma_1, \gamma_0)\backslash\{\Re c_1, \ldots, \Re c_N\}$, where $\oint_c$ denotes a clockwise-oriented circular contour containing the pole c but no other members of $\{c_1, \ldots, c_N\}$. The first term on the right-hand side of eq. (239) is in $\mathcal{A}^{\gamma}([0,\infty)_\rho; \mathcal{X})$. On the other hand,

$$\frac{1}{2\pi i}\oint_c \rho^\zeta M(\zeta)\,\mathrm{d}\zeta = \operatorname{Res}_{\zeta = c} \rho^\zeta M(\zeta). \tag{240}$$

Since $\rho^\zeta = \rho^c e^{(\zeta - c)\log\rho} = \sum_{\kappa=0}^\infty \rho^c (\log \rho)^\kappa (\zeta - c)^\kappa/\kappa!$, if $M(\zeta) = \sum_{\kappa = -(k+1)}^\infty M_\kappa (\zeta - c)^\kappa$ denotes the Laurent series of $M(\zeta)$ around $\zeta = c$, then

$$\operatorname{Res}_{\zeta = c} \rho^\zeta M(\zeta) = \sum_{\kappa=0}^{k} \frac{M_{-1-\kappa}\rho^c (\log\rho)^\kappa}{\kappa!} \in \mathcal{A}^{(c,k)}([0,\infty)_\rho; \mathcal{X}). \tag{241}$$

So, $\mathcal{M}^{-1}M \in \mathcal{A}^{(c,k),\gamma}([0,\infty)_\rho; \mathcal{X}) \subseteq \mathcal{A}^{\mathcal{E},\gamma}([0,\infty)_\rho; \mathcal{X})$. Since $\gamma < \gamma_0$ was arbitrary, we get $\mathcal{M}^{-1}M \in \mathcal{A}^{\mathcal{E},\gamma_0 -}([0,\infty)_\rho; \mathcal{X})$. Applied to the case where $\mathcal{X} = C^m(\partial X)$, then, assuming that the above holds for every $m \in \mathbb{R}$, we get $\mathcal{M}^{-1}M \in \mathcal{A}^{\mathcal{E},\gamma_0 -}(\dot{X})$.

Appendix D. Proofs omitted from §2

Proof of Proposition 2.2. Working in some local coordinate chart on ∂X, the metric g can be written as a matrix in block form. Doing so, $g - g_0 = \delta g$ for

$$\delta g = \begin{pmatrix} \rho^{-4}(F + \rho^{1+\beth_0}C^\infty(\dot{X})) & \rho^{-2+\beth_0}C^\infty(\dot{X}) \\ \rho^{-2+\beth_0}C^\infty(\dot{X}) & \rho^{-1+\beth_0}C^\infty(\dot{X}) \end{pmatrix}, \tag{242}$$

for $F \in \rho^{1+\beth}C^\infty([0,1)_\rho)$, near $\dot{X}$, where the bottom-right entries stand for all angular-angular components of the metric and the upper-left entry is the coefficient of $\mathrm{d}\rho^2$. The other entries are the cross terms.

In the chosen local coordinate chart, $\rho \in (0,1), \theta \in \partial X$, the exactly conic metric g_0 has the form

$$g_0 = \begin{pmatrix} \rho^{-4} & 0 \\ 0 & \rho^{-2}g_{\partial X} \end{pmatrix}, \quad \det g_0 = \rho^{-2(d+1)} \det g_{\partial X}, \quad g_0^{-1} = \begin{pmatrix} \rho^4 & 0 \\ 0 & \rho^2 g_{\partial X}^{-1} \end{pmatrix}. \tag{243}$$

Therefore,

$$g_0^{-1}\delta g \in \begin{pmatrix} F + \rho^{1+\beth_0}C^\infty(\dot{X}) & \rho^{2+\beth_0}C^\infty(\dot{X}) \\ \rho^{\beth_0}C^\infty(\dot{X}) & \rho^{1+\beth_0}C^\infty(\dot{X}) \end{pmatrix}. \tag{244}$$

One can then compute that

$$\det(1 + g_0^{-1}\delta g) = 1 + F + \rho^{1+\beth_0}C^\infty(\dot{X}), \quad \det(1 + g_0^{-1}\delta g)^{-1} = (1+F)^{-1} + \rho^{1+\beth_0}C^\infty(\dot{X}). \tag{245}$$

So,

$$\begin{aligned} \det(g) = \det(g_0)\det(1 + g_0^{-1}\delta g) &= \rho^{-2(d+1)}\det(g_{\partial X})\det(1 + g_0^{-1}\delta g) \\ &= \rho^{-2(d+1)}\det(g_{\partial X})(1+F) + \rho^{-2d-1+\beth_0}C^\infty(\dot{X}). \end{aligned} \tag{246}$$

Note that $\det(g) - \det(g_0) \in \rho^{-2(d+1)}F + \rho^{-2d-1+\beth_0}C^\infty(\dot{X})$.

Via Cramer's formula for $(1+g_0^{-1}\delta g)^{-1}$, we have

$$
\begin{aligned}
&(1+g_0^{-1}\delta g)^{-1} \in ((1+F)^{-1}+\rho^{1+\beth_0}C^\infty(\dot X)) \\
&\times \begin{pmatrix} \det(1+\rho^{1+\beth_0}C^\infty(\dot X)) & \rho^{2+\beth_0}C^\infty(\dot X) \\ \rho^{\beth_0}C^\infty(\dot X) & (1+F+\rho^{1+\beth_0}C^\infty(\dot X))\operatorname{Adj}(I_{d-1}+\rho^{1+\beth_0}C^\infty(\dot X))+\rho^{2+2\beth_0}C^\infty(\dot X) \end{pmatrix} \\
&\qquad\subseteq \begin{pmatrix} (1+F)^{-1}+\rho^{1+\beth_0}C^\infty(\dot X) & \rho^{2+\beth_0}C^\infty(\dot X) \\ \rho^{\beth_0}C^\infty(\dot X) & I_{d-1}+\rho^{1+\beth_0}C^\infty(\dot X) \end{pmatrix}. \qquad (247)
\end{aligned}
$$

So,

$$
(1+g_0^{-1}\delta g)^{-1}-I_d \in \begin{pmatrix} F_1+\rho^{1+\beth_0}C^\infty(\dot X) & \rho^{2+\beth_0}C^\infty(\dot X) \\ \rho^{\beth_0}C^\infty(\dot X) & \rho^{1+\beth_0}C^\infty(\dot X) \end{pmatrix} \tag{248}
$$

for $F_1=(1+F)^{-1}-1\in\rho^{1+\beth}C^\infty([0,1)_\rho)$. Since $g^{-1}-g_0^{-1}=((1+g_0^{-1}\delta g)^{-1}-I_d)g_0^{-1}$, it follows that

$$
g^{-1}-g_0^{-1} \in \begin{pmatrix} \rho^4F_1+\rho^{5+\beth_0}C^\infty(\dot X) & \rho^{4+\beth_0}C^\infty(\dot X) \\ \rho^{4+\beth_0}C^\infty(\dot X) & \rho^{3+\beth_0}C^\infty(\dot X) \end{pmatrix}. \tag{249}
$$

Now turning to the formula for the Laplace–Beltrami operator in local coordinates,

$$
\triangle_g = \sum_{i,j=1}^d \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\Big(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x_j}\Big) = \sum_{i,j=1}^d \Big(g^{ij}\frac{\partial^2}{\partial x_i x_j}+\frac{\partial g^{ij}}{\partial x_i}\frac{\partial}{\partial x_j}+\frac{1}{2|g|}\frac{\partial |g|}{\partial x_i}g^{ij}\frac{\partial}{\partial x_j}\Big), \tag{250}
$$

where $|g|=\det(g)$. Here, $x_1=\rho$, and $x_2,\ldots,x_d$ are a local coordinate chart for ∂X, not Cartesian coordinates. So, $\triangle_g$ differs from $\triangle_{g_0}$ by a sum of five terms:

- $$
\begin{aligned}
\sum_{i,j=1}^d (g^{ij}-g_0^{ij})\frac{\partial}{\partial x_i\partial x_j} \in (\rho^4F_1+\rho^{5+\beth_0}C^\infty(\dot X))\frac{\partial^2}{\partial\rho^2}+\rho^{4+\beth_0}C^\infty(\dot X)\frac{\partial}{\partial\rho}\mathcal{V}(\partial X)& \\
+\rho^{3+\beth_0}C^\infty(\dot X)\mathcal{V}(\partial X)^2,& \qquad (251)
\end{aligned}
$$

- $$
\begin{aligned}
\sum_{i,j=1}^d \Big(\frac{\partial}{\partial x_i}(g^{ij}-g_0^{ij})\Big)\frac{\partial}{\partial x_j} &\in \begin{pmatrix} \rho^3F_2+\rho^{4+\beth_0}C^\infty(\dot X) \\ \rho^{3+\beth_0}C^\infty(\dot X) \end{pmatrix}^\intercal \nabla \\
&\subseteq (\rho^3F_2+\rho^{4+\beth_0}C^\infty(\dot X))\frac{\partial}{\partial\rho}+\rho^{3+\beth_0}C^\infty(\dot X)\operatorname{Diff}^1(\partial X) \qquad (252)
\end{aligned}
$$

  for some $F_2\in\rho^{1+\beth}C^\infty([0,1)_\rho)$;

- $$
\begin{aligned}
\sum_{i,j=1}^d \frac{g^{ij}-g_0^{ij}}{2|g|}\frac{\partial|g|}{\partial x_i}\frac{\partial}{\partial x_j} &\in \begin{pmatrix} \rho^{-1}F_3+\rho^{\beth_0}C^\infty(\dot X) \\ C^\infty(\dot X) \end{pmatrix}^\intercal \begin{pmatrix} \rho^4F_1+\rho^{5+\beth_0}C^\infty(\dot X) & \rho^{4+\beth_0}C^\infty(\dot X) \\ \rho^{4+\beth_0}C^\infty(\dot X) & \rho^{3+\beth_0}C^\infty(\dot X) \end{pmatrix}\nabla \\
&\subseteq \begin{pmatrix} \rho^{-1}F_3+\rho^{\beth_0}C^\infty(\dot X) \\ C^\infty(\dot X) \end{pmatrix}^\intercal \begin{pmatrix} (\rho^4F_1+\rho^{5+\beth_0}C^\infty(\dot X))\partial_\rho+\rho^{4+\beth_0}C^\infty(\dot X)\mathcal{V}(\partial X) \\ \rho^{4+\beth_0}C^\infty(\dot X)\partial_\rho+\rho^{3+\beth_0}C^\infty(\dot X)\mathcal{V}(\partial X) \end{pmatrix} \\
&\subseteq (\rho^3F_1F_3+\rho^{4+\beth_0}C^\infty(\dot X))\partial_\rho+\rho^{3+\beth_0}C^\infty(\dot X)\mathcal{V}(\partial X), \qquad (253)
\end{aligned}
$$

  where the inclusion follows from the observation that

  $$
  \frac{1}{|g|}\frac{\partial|g|}{\partial x_i} \in \begin{cases} \rho^{-1}F_3+\rho^{\beth_0}C^\infty(\dot X) & (x_i=\rho), \\ C^\infty(\dot X) & (\text{otherwise}), \end{cases} \tag{254}
  $$

  for some $F_3\in C^\infty([0,1)_\rho)$.

- $$\sum_{i,j=1}^d \frac{g_0^{ij}}{2|g|}\frac{\partial}{\partial x_i}(|g|-|g_0|)\frac{\partial}{\partial x_j} = \frac{g_0^{11}}{2|g|}\frac{\partial}{\partial \rho}(|g|-|g_0|)\frac{\partial}{\partial \rho} + \sum_{i,j=2}^d \frac{g_0^{ij}}{2|g|}\frac{\partial}{\partial \theta_i}(|g|-|g_0|)\frac{\partial}{\partial \theta_j}$$
$$\in \frac{\rho^{2d+6}}{2}\Big(\frac{1}{\det(g_{\partial X})(1+F)} + \rho^{1+\daleth_0}C^\infty(\dot{X})\Big)(\rho^{-2d-3}\det(g_{\partial X})F_4 + \rho^{-2d-2+\daleth_0}C^\infty(\dot{X}))\frac{\partial}{\partial\rho}$$
$$+\frac{1}{2}\sum_{i,j=2}^d g_0^{ij}\Big(\frac{1}{\det(g_{\partial X})(1+F)} + \rho^{1+\daleth_0}C^\infty(\dot{X})\Big)\Big(F\frac{\partial\det(g_{\partial X})}{\partial\theta_i} + \rho^{1+\daleth_0}C^\infty(\dot{X})\Big)\frac{\partial}{\partial\theta_j}$$
$$\subseteq \Big(\frac{\rho^3 F_4}{2(1+F)} + \rho^{4+\daleth_0}C^\infty(\dot{X})\Big)\frac{\partial}{\partial\rho} + \rho^{3+\daleth_0}\mathcal{V}(\partial X) + \mathcal{B}, \quad (255)$$

  where $\theta_j = x_j$ for $j\geq 2$, $F_4 \in \rho^{1+\daleth}C^\infty([0,1)_\rho)$, and
  $$\mathcal{B} = \frac{1}{2}\frac{F}{\det(g_{\partial X})(1+F)}\sum_{i,j=2}^d g_0^{ij}\frac{\partial\det(g_{\partial X})}{\partial\theta_i}\frac{\partial}{\partial\theta_j}. \quad (256)$$

- $$\sum_{i,j=1}^d \frac{g_0^{ij}}{2}\frac{\partial|g_0|}{\partial x_i}\Big(\frac{1}{|g|}-\frac{1}{|g_0|}\Big)\frac{\partial}{\partial x_j} = \Big(\frac{1}{|g|}-\frac{1}{|g_0|}\Big)\Big[\frac{\rho^4}{2}\frac{\partial|g_0|}{\partial\rho}\frac{\partial}{\partial\rho} + \sum_{i,j=2}^d\frac{g_0^{ij}}{2}\frac{\partial|g_0|}{\partial\theta_i}\frac{\partial}{\partial\theta_j}\Big]$$
$$\subseteq \rho^{2d+2}\Big(\frac{F_5}{\det(g_{\partial X})} + \rho^{1+\daleth_0}C^\infty(\dot{X})\Big)\Big[-(d+1)\rho^{-2d+1}\det(g_{\partial X})\frac{\partial}{\partial\rho} + \sum_{i,j=2}^d\frac{g_0^{ij}}{2}\frac{\partial|g_0|}{\partial\theta_i}\frac{\partial}{\partial\theta_j}\Big]$$
$$\subseteq (\rho^3F_5 + \rho^{4+\daleth}C^\infty(\dot{X}))\frac{\partial}{\partial\rho} + \rho^{3+\daleth_0}C^\infty(\dot{X})\mathcal{V}(\partial X) + \mathcal{B}' \quad (257)$$

  for some $F_5 \in \rho^{1+\daleth}C^\infty([0,1)_\rho)$, where
  $$\mathcal{B}' = \frac{1}{2}\frac{F_5}{\det(g_{\partial X})}\sum_{i,j=2}^d g_0^{ij}\frac{\partial\det(g_{\partial X})}{\partial\theta_i}\frac{\partial}{\partial\theta_j}. \quad (258)$$

  Indeed, this follows from
  $$\frac{1}{|g|}-\frac{1}{|g_0|} \in \frac{\rho^{2d+2}}{\det(g_{\partial X})}\Big(\frac{1}{(1+F)}-1\Big) + \rho^{2d+3+\daleth_0}C^\infty(\dot{X}) \quad (259)$$
  and $g_0^{ij}(\partial_{\theta_i}|g_0|)\partial_{\theta_j} \in \rho^{-2d}\mathcal{V}(\partial X)$ for $i,j\geq 2$, from which we also get that $F_5 = -F/(1+F)$, so $\mathcal{B}' = -\mathcal{B}$.

All of the terms on the right-hand sides of eq. (251), eq. (252), eq. (253), eq. (255), eq. (257) lie in $\rho^{3+\daleth}\operatorname{Diff}_{\mathrm{b}}^2([0,1)_\rho) + \rho^{3+\daleth_0}\operatorname{Diff}_{\mathrm{b}}^2(X)$, with the exception of $\mathcal{B},\mathcal{B}'=-\mathcal{B}$. Fortunately, when we sum up these exception contributions to $\triangle_g - \triangle_{g_0}$, they cancel, so we are left only with terms in $\rho^{3+\daleth}\operatorname{Diff}_{\mathrm{b}}^2([0,1)_\rho) + \rho^{3+\daleth_0}\operatorname{Diff}_{\mathrm{b}}^2(X)$. □

Proof of Proposition 2.3. We prove the two parts of the proposition in turn. In both cases, it is immediate that the operators considered have the desired forms in the interior of X. So, we only describe the situation near ∂X.

(I) Suppose that $P(\sigma)$ has the form described in and below eq. (17). We then compute
$$\check{P}(\sigma) = \check{\triangle}_g + 2i\sigma(1-\chi)\frac{\partial}{\partial r} - 2\sigma^2(1-\chi) + \frac{i\sigma(d-1)}{r} + \check{L} + \sigma\check{Q} + \sigma^2 R. \quad (260)$$

(Recall that R is scalar, so $\check{R} = R$.) We evaluate each of the terms on the right-hand side. Starting with the first order terms, note that if $A \in \operatorname{Diff}^1_{\mathrm{b}}(X)$, then $\check{A} - A \in \sigma\rho^{-1}C^\infty(X)$, and if $A \in \operatorname{Diff}^1_{\mathrm{b}}([0,1)_\rho)$, then we can improve this to $\check{A} - A \in \sigma\rho^{-1}C^\infty([0,1)_\rho)$. So,

$$\check{L} = L + \sigma L_1 \quad \text{for } L_1 \in \rho^{2+\beth}C^\infty([0,1)_\rho) + \rho^{2+\beth_0}C^\infty(X), \tag{261}$$

$$\check{Q} = Q + \sigma Q_1 \text{ for } Q_1 \in \rho^{1+\beth_1}C^\infty([0,1)_\rho) + \rho^{1+\beth_2}C^\infty(X). \tag{262}$$

Likewise, writing $\check{\triangle}_g = \check{\triangle}_{g_0} + \check{O}$ for $O = \triangle_g - \triangle_{g_0} \in \rho^{3+\daleth}\operatorname{Diff}^2_{\mathrm{b}}([0,1)_\rho) + \rho^{3+\daleth_0}\operatorname{Diff}^2_{\mathrm{b}}(X)$ (this being Proposition 2.2) and computing

$$\check{\triangle}_{g_0} = \triangle_{g_0} - 2i\sigma\frac{\partial}{\partial r} + \sigma^2 - \frac{i\sigma(d-1)}{r} \tag{263}$$

near ∂X and

$$\begin{aligned}\check{O} - O \in \sigma\rho^{2+\daleth}\operatorname{Diff}^1_{\mathrm{b}}([0,1)_\rho) + \sigma\rho^{2+\daleth_0}\operatorname{Diff}^1_{\mathrm{b}}(X) \\ + \sigma^2\rho^{1+\daleth}C^\infty([0,1)_\rho) + \sigma^2\rho^{1+\daleth_0}C^\infty(X),\end{aligned} \tag{264}$$

we have $\check{\triangle}_g = \triangle_g - 2i\sigma\partial_r + \sigma^2 - i\sigma(d-1)r^{-1} + \sigma O_1 + \sigma^2 O_2$ for

$$O_1 \in \rho^{2+\daleth}\operatorname{Diff}^1_{\mathrm{b}}([0,1)_\rho) + \rho^{2+\daleth_0}\operatorname{Diff}^1_{\mathrm{b}}(X), \tag{265}$$

$$O_2 \in \rho^{1+\daleth}C^\infty([0,1)_\rho) + \rho^{1+\daleth_0}C^\infty(X). \tag{266}$$

Here, we used that, if $O \in \operatorname{Diff}^2_{\mathrm{b}}(X)$, we have $\check{O} - O \in \sigma^2\rho^{-2}C^\infty(X) + \sigma\rho^{-1}\operatorname{Diff}^1_{\mathrm{b}}(X)$. If $O \in \operatorname{Diff}^2_{\mathrm{b}}([0,1)_\rho)$, then we can improve this to $\check{O} - O \in \sigma^2\rho^{-2}C^\infty([0,1)_\rho) + \sigma\rho^{-1}\operatorname{Diff}^1_{\mathrm{b}}([0,1)_\rho)$.

Combining these ingredients, we see that

$$\check{P}(\sigma) = \triangle_{g_0} - \sigma^2 + L + \sigma\underbrace{\Big(-2i\chi\frac{\partial}{\partial r} + L_1 + Q + O_1\Big)}_{P_1} + \sigma^2\underbrace{(2\chi + Q_1 + R + O_2)}_{P_2} \tag{267}$$

near ∂X. So, the stated form eq. (23) of $\check{P}(\sigma)$ holds, with P_1, P_2 as underlined. Thus,

$$P_1 \in \rho^{2+\min\{\daleth,\beth,\beth_1\}}\operatorname{Diff}^1_{\mathrm{b}}([0,1)_\rho) + \rho^{2+\min\{\daleth_0,\beth_0,\beth_2\}}\operatorname{Diff}^1_{\mathrm{b}}(X), \tag{268}$$

$$P_2 \in \rho^{1+\min\{\daleth,\beth_1,\beth_3\}}C^\infty([0,1)_\rho) + \rho^{1+\min\{\daleth_0,\beth_2,1+\beth_4\}}C^\infty(X) \tag{269}$$

which is what was claimed in the proposition (since, in the proposition, $\daleth_\bullet = \beth_\bullet$).

(II) The converse direction is similar. Given $P_0(\sigma) = P_0 - \sigma^2$ for $P_0 = \triangle_g + L$, we have

$$\hat{P}_0(\sigma) = \hat{\triangle}_g + \hat{L} - \sigma^2. \tag{270}$$

As in the previous part, $\hat{L} = L + \sigma L_1$ for $L_1 \in \rho^{2+\beth}C^\infty([0,1)_\rho) + \rho^{2+\beth_0}C^\infty(X)$. We can write

$$\hat{\triangle}_g = \hat{\triangle}_{g_0} + \hat{O} \tag{271}$$

for $O = \triangle_g - \triangle_{g_0}$. The difference $\hat{O} - O$ is in the set on the right-hand side of eq. (264). The conjugate of the computation leading to eq. (263) yields

$$\hat{\triangle}_{g_0} = \triangle_{g_0} + 2i\sigma\frac{\partial}{\partial r} + \sigma^2 + \frac{i\sigma(d-1)}{r}. \tag{272}$$

So, altogether, we have

$$\begin{aligned} P(\sigma) &= \triangle_{g_0} + 2i\sigma\frac{\partial}{\partial r} + \frac{i\sigma(d-1)}{r} + L \\ &+ \sigma\underbrace{\left(\rho^{2+\min\{\gimel,\daleth\}}\operatorname{Diff}^1_{\mathrm{b}}([0,1)_\rho) + \rho^{2+\min\{\gimel_0,\daleth_0\}}\operatorname{Diff}^1_{\mathrm{b}}(X)\right)}_{Q} + \sigma^2\underbrace{\left(\rho^{1+\daleth}C^\infty([0,1)_\rho) + \rho^{1+\daleth_0}C^\infty(X)\right)}_{R}. \end{aligned} \tag{273}$$

The operators Q, R defined here have the desired forms if we take $\daleth, \cdots, \daleth_4$ as in the proposition statement. For example, if we take $\daleth = \gimel$ and $\daleth_0 = \gimel_0$, then $R \in \rho^{1+\daleth}C^\infty([0,1)_\rho) + \rho^{1+\daleth_0}C^\infty(X)$. We can write the $\rho^{1+\daleth}C^\infty([0,1)_\rho)$ term as $\mathfrak{m}\rho + \rho^{\max\{2,1+\daleth\}}C^\infty([0,1)_\rho)$ for $\mathfrak{m} \in \mathbb{R}$ which is equal to 0 if $\daleth > 0$.

□

Proof of Proposition 2.5. Our goal will be to deduce this from [Vas21a, Theorem 1.1]. The only reason why Proposition 2.5 is not a special case of [Vas21a, Theorem 1.1] is that, depending on Q, R, it may not be the case that $\check{P}$ is the spectral family of a Schrödinger operator. Regardless, for each individual σ, we have

$$\check{P}(\sigma) = \check{P}(\sigma;\sigma), \tag{274}$$

where, for each fixed $\omega \in \mathbb{C}$, $\check{P}(\sigma;\omega)$ is the spectral family of a Schrödinger operator $\check{P}(0;\omega)$, which depends on ω. To be concrete, we can take, in terms of the operators in eq. (23),

$$\check{P}(\sigma;\omega) = P_0 - \sigma^2 + \omega P_1 + \omega^2 P_2. \tag{275}$$

We now check that $\check{P}(\sigma;\omega)$ satisfies the hypotheses of [Vas21a, Theorem 1.1]. (Note that what we call P Vasy calls $\hat{P}$, and what we call $\check{P}$ is what Vasy just calls P.) Specifically, we just need to check that, for each $\omega \in \mathbb{R}$, $\check{P}(0;\omega) = \check{P}(0;\omega)^*$, i.e. that $P_0 + \omega P_1 + \omega^2 P_2$ is symmetric on $C^\infty_{\mathrm{c}}(X^\circ)$ with respect to the $L^2(X,g)$ inner product. Indeed, it follows from the symmetry of P that $\check{P}(\sigma) = \check{P}(\sigma)^*$ for all $\sigma \in \mathbb{R}$, since the function r is real-valued. Since $\check{P}(0) = P_0$, we only need to check the symmetry of P_1, P_2. Indeed,

$$P_1 = \frac{\partial \check{P}(\sigma)}{\partial \sigma}\Big|_{\sigma=0} = \frac{\partial \check{P}(\sigma)^*}{\partial\sigma}\Big|_{\sigma=0} = P_1^* \tag{276}$$

and

$$P_2 = 2 + \frac{\partial^2 \check{P}(\sigma)}{\partial\sigma^2}\Big|_{\sigma=0} = 2 + \frac{\partial^2\check{P}(\sigma)^*}{\partial\sigma^2}\Big|_{\sigma=0} = P_2^*. \tag{277}$$

So, $\check{P}(\sigma;\omega)$ satisfies the assumptions of [Vas21a, §3], specifically with $\Im\alpha_\pm = 0$. We may therefore apply [Vas21a, Theorem 1.1] to conclude the proposition. □

Proof of Proposition 2.7. First note that the operator $P(\sigma)$ above falls into the framework discussed in [Vas21c, §2, eq. 2.1]. So, we can appeal to [Vas21c, Thm. 2.5], which, in the case at hand, gives the same conclusion as [Vas21c, Thm. 2.1]. (The latter theorem is stated in a slightly less general case than needed here, but the additional terms in $P(\sigma)$ do not change the statement of [Vas21c, Thm. 2.5].)

In order to apply this theorem, we need that $P(0)$ has trivial kernel acting on $H_{\mathrm{b}}^{\infty,(d-4)/2}(X)$; eq. (31) says that

$$H_{\mathrm{b}}^{\infty,(d-4)/2}(X) \subseteq \mathcal{A}^{d-2}(X), \tag{278}$$

so our assumption that $\ker_{\mathcal{A}^{d-2}(X)} P(0) = \{0\}$ is sufficient to verify this hypothesis.

So, if $r > -1/2$ and ℓ, ν are as in the proposition statement, then [Vas21c, Thm. 2.1] says that, for any $s \in \mathbb{R}$, there exists some $\sigma_0 > 0$ and $C > 0$ such that

$$\|(\rho+\sigma)^\nu u\|_{H^{s,r,\ell}_{\mathrm{sc,b,res}}} \leq C\|(\rho+\sigma)^{\nu-1}P(\sigma)u\|_{H^{s-1,r+1,\ell+1}_{\mathrm{sc,b,res}}} \tag{279}$$

holds for all u,σ such that the left side is finite, where the norms $\|\bullet\|_{H^{s,r,\ell}_{\mathrm{sc,b,res}}}$ are defined in [Vas21a].

In order to put this in a more useful form, we can use the equivalence

$$\|\bullet\|_{H_{\mathrm{b}}^{s,\ell}} \approx \|\bullet\|_{H^{s,s+\ell,\ell}_{\mathrm{sc,b,res}}}, \tag{280}$$

which holds for all $s,\ell\in\mathbb{R}$ (with the σ-*in*dependent constants, notationally suppressed by the "$\approx$," depending s,ℓ); see [Vas21a, Eq. 3.5]. Equation (279) therefore gives, for s,ℓ,ν as in the proposition statement,

$$\begin{aligned}\|(\rho+\sigma)^\nu u\|_{H_{\mathrm{b}}^{s,\ell}} \approx \|(\rho+\sigma)^\nu u\|_{H^{s,s+\ell,\ell}_{\mathrm{sc,b,res}}} &\lesssim \|(\rho+\sigma)^{\nu-1}P(\sigma)u\|_{H^{s-1,s+\ell+1,\ell+1}_{\mathrm{sc,b,res}}} \\ &\lesssim \|(\rho+\sigma)^{\nu-1}P(\sigma)u\|_{H^{s,s+\ell+1,\ell+1}_{\mathrm{sc,b,res}}} \approx \|(\rho+\sigma)^{\nu-1}P(\sigma)u\|_{H_{\mathrm{b}}^{s,\ell+1}}. \end{aligned} \tag{281}$$

□

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