Q1(a) To find c, we integrate over given range which should equal 1.

$$\int_0^\infty \int_0^y ce^{-2y}\,dx\,dy = 1 \Rightarrow \int_0^\infty \int_0^y ce^{-2y}\,dx\,dy = \int_0^\infty [ce^{-2y}x]_0^y\,dy$$

$u = y \quad v' = e^{-2y}$

$u' = 1 \quad v = -\frac{1}{2}e^{-2}$

$$= \int_0^\infty ce^{-2y}y\,dy$$

$$= c\left[[-\tfrac{1}{2}ye^{-2y}]_0^\infty - \int_0^\infty -\tfrac{1}{2}e^{-2y}\,dy\right]$$

$$= c\left[(0-0) + \tfrac{1}{2}\int_0^\infty e^{-2y}\,dy\right]$$

$$= \tfrac{1}{2}c\,[-\tfrac{1}{2}e^{-2y}]_0^\infty$$

$$= \tfrac{1}{2}c\,(0-(-\tfrac{1}{2})) = \tfrac{1}{4}c = 1$$

so $c = 4$.

b) To find $f_X(x)$, we integrate over y for $x \le y \le \infty$

So $\int_x^\infty 4e^{-2y}\,dy = 4[-\frac{1}{2}e^{-2y}]_x^\infty = 4[0 - -\frac{1}{2}e^{-2x}] = 2e^{-2x}$ for $x \geq 0$

To find $f_Y(y)$, we integrate over x for $0 \le x \le y$.

$\int_0^y 4e^{-2y}\,dx = [4e^{-2y}x]_0^y = 4ye^{-2y}$ for $y \geq 0$.

c) We want to find $P(Y \le 2X)$, together with initial bounds, $x \le y \le 2x$.

So integral will be $\int_0^\infty \int_x^{2x} 4e^{-2y}\,dy\,dx = \int_0^\infty [-2e^{-2y}]_x^{2x}\,dx$

$$= \int_0^\infty -2e^{-4x} + 2e^{-2x}\,dx$$

$$= [\tfrac{1}{2}e^{-4x} - e^{-2x}]_0^\infty$$

$$= (0-0) - (\tfrac{1}{2} - 1) = \tfrac{1}{2}$$

So $P(Y \le 2X) = \frac{1}{2}$.

Q2a) Marginal PMF of X is $P(X=2)=\frac{1}{10}+0=\frac{1}{10}$, $P(X=5)=\frac{1}{10}+\frac{3}{10}=\frac{4}{10}$, $P(X=9)=\frac{3}{10}+\frac{2}{10}=$

$$E(X)=\frac{1}{10}\times 2+\frac{4}{10}\times 5+\frac{5}{10}\times 9=\frac{67}{10}=6.7$$

b) Find PMF of $E(X|Y)$. $E(X|Y=0)=\dfrac{\sum x\cdot P(X=x\cap Y=0)}{P(Y=0)}=\dfrac{\frac{5}{10}+\frac{27}{10}}{\frac{4}{10}}=8.$

$$E(X|Y=1)=\frac{\sum x\cdot P(X=x\cap Y=1)}{P(Y=1)}=\frac{\frac{2}{10}+\frac{15}{10}+\frac{18}{10}}{\frac{6}{10}}=\frac{35}{6}.$$

c) $E(E(X|Y))=\sum P(Y=y)\cdot E(X|Y=y)$

$$=8\cdot\frac{4}{10}+\frac{35}{6}\cdot\frac{6}{10}=\frac{32}{10}+\frac{35}{10}=6.7.$$

Q3a) $P(Y=1|X=3)=\frac{1}{3}$, $P(Y=2|X=3)=\frac{1}{3}$, $P(Y=3|X=3)=\frac{1}{3}$

so $E(\cancel{Y}|X=3)=\frac{1}{3}\cdot 1+\frac{1}{3}\cdot 2+\frac{1}{3}\cdot 3=2$.

b) $P(Y=y|X=x)=\frac{1}{x}$ for $1\le y\le x$.

so $E(Y|X=x)=\sum y\cdot P(Y=y|X=x)=\dfrac{1+2+\dots+x}{x}=\dfrac{\frac{x(x+1)}{2}}{x}=\dfrac{x+1}{2}$

c) We first find $P(Y=1)$. $P(Y=1)=\frac{1}{4}\cdot 1+\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}$ (considering all values for X)

$$=\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{16}=\frac{25}{48}$$

$P(X=1|Y=1)=\dfrac{P(X=1\cap Y=1)}{P(Y=1)}=\dfrac{\frac{1}{4}\cdot 1}{\frac{25}{48}}=\dfrac{12}{25}$ $P(X=2|Y=1)=\dfrac{P(X=2\cap Y=1)}{P(Y=1)}=\dfrac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{25}{48}}=\dfrac{6}{25}$

$P(X=3|Y=1)=\dfrac{P(X=3\cap Y=1)}{P(Y=1)}=\dfrac{\frac{1}{4}\cdot\frac{1}{3}}{\frac{25}{48}}=\dfrac{4}{25}$ $P(X=4|Y=1)=\dfrac{P(X=4\cap Y=1)}{P(Y=1)}=\dfrac{\frac{1}{4}\cdot\frac{1}{4}}{\frac{25}{48}}=\dfrac{3}{25}$.

$$E(X|Y=1)=\frac{12}{25}\cdot 1+\frac{6}{25}\cdot 2+\frac{4}{25}\cdot 3+\frac{3}{25}\cdot 4=\frac{48}{25}=1.92.$$

Q4a) First find f_X, so integrate wrt y. Note bounds $y>0$ and $y \le 1-x$.

so $f_X(x) = \int_0^{1-x} 2 \, dy = 2(1-x)$ for $0<x<1$.

For f_Y, $x>0$ $x \le 1-y$

$f_Y(y) = \int_0^{1-y} 2 \, dx = 2(1-y)$ for $0<y<1$.

b) Want to find $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}$ for $0<x<1-y$.

c) $E(X|Y) = \int_0^{1-y} x \cdot f_{x|y}(x|y) \, dx.$

$= \int_0^{1-y} x \cdot \frac{1}{1-y} \, dx$

$= \left[\dfrac{x^2}{2(1-y)} \right]_0^{1-y} = \dfrac{1-y}{2}$

Q5a) There will be 2 cases: $X \le \frac{2}{3} \le Y$ and $Y \le \frac{2}{3} \le X$. Both X & Y are i.i.d so these 2 cases are equally likely to occur.

For $X \le \frac{2}{3} \le Y$, there is $\frac{2}{3}$ chance for $X \le \frac{2}{3}$ & $\frac{1}{3}$ chance for $\frac{2}{3} \le Y$.

so $P(X \le \frac{2}{3} \le Y) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ More formally,

$P(X \le \frac{2}{3} \le Y) = \int_0^{\frac{2}{3}} P(Y \ge \frac{2}{3}) \, dx = \int_0^{\frac{2}{3}} (1-\frac{2}{3}) dx = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$.

so $P(\frac{2}{3}$ between X & $Y) = \frac{2}{9} \cdot 2 = \frac{4}{9}$.

b) Similar to part (a). $P(a$ between X & $Y) = 2 \cdot \int_0^a P(Y \ge a) \, dx = 2\int_0^a (1-a) \, dx = 2a(1-a)$.

c) $P(X \le Z \le Y) = \int_0^1 \int_x^1 P(X \le Z \le y \mid X=x, Y=y) \, f_{XY}(x,y) \, dy \, dx$

$= \int_0^1 \int_x^1 y-x \, dy \, dx = \int_0^1 \left[\frac{y^2}{2} - xy \right]_x^1 dx = \int_0^1 (\frac{1}{2} - x) - (\frac{x^2}{2} - x^2) \, dx$

$= \int_0^1 \frac{x^2}{2} - x + \frac{1}{2} \, dx = \left[\frac{x^3}{6} - \frac{x^2}{2} + \frac{1}{2}x \right]_0^1 = (\frac{1}{6} - \frac{1}{2} + \frac{1}{2}) = \frac{1}{6}$.

$P(Z$ between X & $Y) = 2 \times P(X \le Z \le Y)$ (by symmetry)

$= 2 \times \frac{1}{6} = \frac{1}{3}$