Q1) Want to show $P(X>m+n \mid X>m) = P(X>n)$ for $X \sim Geom(p)$.

For X, $P(X=k) = (1-p)^{k-1} p$.

① Find $P(X>n)$. $P(X>n) = 1 - P(X \leq n)$

Geometric CDF: $P(X \leq n) = 1 - (1-p)^n$ so $P(X>n) = 1-(1-(1-p)^n)$
$= (1-p)^n$.

This makes sense as for # of trials to be greater than n, first n trials has to fail (ie $(1-p)^n$).

② Find $P(X>m+n \mid X>m)$.

$$P(X>m+n \mid X>m) = \frac{P(X>m+n \cap X>m)}{P(X>m)}$$

$X>m+n$ implies $X>m$, so $P(X>m+n \cap X>m) = P(X>m+n)$.

Using result from ①, $P(X>m+n) = (1-p)^{m+n}$ and $P(X>m) = (1-p)^m$

so, we get: $P(X>m+n \mid X>m) = \dfrac{P(X>m+n)}{P(X>m)} = \dfrac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n$.

∴ we have shown ① & ②, and $P(X>m+n \mid X>m) = P(X>n) = (1-p)^n$.

Q2) For first prize drawn, will definitely be new type, so $E_1 = 1$

For second prize, there is $\frac{3}{4}$ chance to get new type, so expected # of boxes to get new type $E_2 = \frac{1}{\frac{3}{4}} = \frac{4}{3}$.

For third prize, There is $\frac{1}{2}$ chance to get new type, so $E_3 = \frac{1}{\frac{1}{2}} = 2$.

For fourth prize, $\frac{1}{4}$ chance to get final new type, so $E_4 = \frac{1}{\frac{1}{4}} = 4$.

So total # expected # boxes to get all 4 types is $E(X) = E_1 + E_2 + E_3 + E_4$
$= 1 + \frac{4}{3} + 2 + 4 = \frac{25}{3}$.

Generalize to n collectibles.

E 1st type $= 1$

E 2nd type $= \dfrac{n}{n-1}$

⋮

E kth type $= \dfrac{n}{n-(k-1)}$

so given $E(X) = E_1 + \ldots + E_k = \sum_{k=1}^{n} \dfrac{n}{n-(k-1)} = n\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right)$

Q3) a) Given parameter λ for Poisson distribution is the average rate of tornadoes in a specified time interval, ie a year, then $\lambda = 0.03 \times 365 = 10.95$.

b) For $X \sim Pois(\lambda)$, $P(X=k) = \frac{\lambda^x e^{-\lambda}}{x!}$ where x is the # of events observed. so.

For $X \sim Pois(10.95)$, $P(X=12) = \frac{10.95^{12} e^{-10.95}}{12!} \approx 0.109$.

c) $P(X=11) = \frac{10.95^{11} e^{-10.95}}{11!} \approx 0.119$ $P(X=10) = \frac{10.95^{10} e^{-10.95}}{10!} \approx 0.120$.

so more likely to observe exactly 10 tornadoes in 365 days.

Q4a) For Binomial Distribution, $X \sim Binom(100, 0.08)$ & $Y \sim Binom(100, 0.15)$

b) We can approximate the Binomial Distribution with Poisson for large n & small p values where $\lambda = np$. so, $\lambda_x = 100 \times 0.08 = 8$, $\lambda_y = 100 \times 0.15 = 15$, thus

$\lambda_z = \lambda_x + \lambda_y = 8 + 15 = 23$.

c) To find $P(Z=20)$ for $\lambda_z = 23$, $P(Z=20) = \frac{23^{20} e^{-23}}{20!} \approx 0.724$.

d) For Binomial Distribution, $Var(X) = np(1-p)$

so $Var(X) = 100 \times 0.08 \times (1-0.08)$ $Var(Y) = 100 \times 0.15 \times (1-0.15)$

$Var(X) + Var(Y) = 20.11$

For $X \sim Pois(\lambda)$, $Var(X) = \lambda$, so $Var(Z) = 23$.

Variance using Poisson distribution is larger than the sum of variances of X & Y.

Intuitive Explanation:

- Poisson is just an approximation when n is large & p is small. It does give good approximation but variance of combined variable $Z = X+Y$ assumes all variance is due to Poisson occurrence itself, without considering seperate X & Y variances.
- ie, Poisson approximation to binomial tends to slightly overestimate variance when aggregating multiple binomial variables because it captures overall event rate (λ) without distinguishing between sources of variance.

Q5)

Find probability B wins. B wins if B gets 3 ~~heads~~ tails before A gets 6 heads.

ie B has to see 3 tails within next 8 flips. If Player B wins on the kth flip,

	k ~~# of flips~~	Probability B wins on the k^{th} flip	
	0	0	
	1	0	
	2	0	
	3	$(\frac{1}{2})^3$	$= (\frac{1}{2})^3$
1H3T	4	$\binom{3}{2}\times\underset{T}{(\frac{1}{2})^2}\times\underset{H}{(\frac{1}{2})}\times(\frac{1}{2})$	$= 3\times(\frac{1}{2})^4$
2H3T	5	$\binom{4}{2}\times\underset{T}{(\frac{1}{2})^2}\times\underset{H}{(\frac{1}{2})^2}\times\underset{T}{(\frac{1}{2})}$	$= 6\times(\frac{1}{2})^5$
3H3T	6	$\binom{5}{3}\times\underset{H}{(\frac{1}{2})^3}\times\underset{T}{(\frac{1}{2})^2}\times\underset{T}{(\frac{1}{2})}$	$= 10\times(\frac{1}{2})^6$
4H3T	7	$\binom{6}{4}\times\underset{H}{(\frac{1}{2})^4}\times\underset{T}{(\frac{1}{2})^2}\times(\frac{1}{2})$	$= 15\times(\frac{1}{2})^7$
5H3T	8	$\binom{7}{5}\times(\frac{1}{2})^5\times(\frac{1}{2})^2\times(\frac{1}{2})$.	$= 21\times(\frac{1}{2})^8$

∴ Probability of B winning is the sum of Pr(B wins on kth flip) from $0 \leq k \leq 8$.

This sum equates to $\left(1+\frac{3}{2}+\frac{6}{4}+\frac{10}{8}+\frac{15}{16}+\frac{21}{32}\right)\times(\frac{1}{2})^3 \approxeq 0.855$

so $P(B\text{ win}) = 0.855$ and thus $P(A\text{ win}) = 1-0.855 = 0.145$

Thus, Player B should get $0.855\times\$100 = \$85.5$ and

Player A should get $0.145\times\$100 = \$14.5$

for the split.