Q1a)

Since B & B^c form partition of Ω, By Law of Total Probability, and $P(B^c) = 1 - P(B) = 1 - \frac{3}{8} = \frac{5}{8}$

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

$$= \frac{7}{18} \times \frac{3}{18} + \frac{11}{15} \times \frac{5}{8} = \frac{29}{48}$$

b) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ so $\Rightarrow$ $P(A \cap B) = P(A|B)P(B) = \frac{7}{18} \times \frac{3}{8} = \frac{7}{48}$

c) $P(C|A \cap B) = \dfrac{P(A \cap B \cap C)}{P(A \cap B)}$ so $\Rightarrow$ $P(A \cap B \cap C) = P(C|A \cap B)P(A \cap B)$

$$= \frac{2}{7} \times \frac{7}{48} = \frac{1}{24}$$

Q2a) $P(\text{Disease}) = 0.004$, $P(\neg\text{Disease}) = 1 - P(\text{Disease}) = 1 - 0.004 = 0.996$.

$P(\text{Positive}|\text{Disease}) = 0.92$, $P(\text{Negative}|\text{Disease}) = 1-0.92 = 0.08$, $P(\text{Positive}|\neg\text{Disease}) = 1 - 0.97 = 0.03$, $P(\text{Negative}|\neg\text{Disease}) = 0.9$

$$P(\text{Disease}|\text{Positive}) = \frac{P(\text{Disease} \cap \text{Positive})}{P(\text{Positive})}$$

$$P(\text{Positive}) = P(\text{Positive}|\text{Disease}) \times P(\text{Disease}) + P(\text{Positive}|\neg\text{Disease}) \times P(\neg\text{Disease})$$

$$= 0.92 \times 0.004 + 0.03 \times 0.996.$$

$$P(\text{Disease} \wedge \text{Positive}) = P(\text{Positive}|\text{Disease}) \times P(\text{Disease}) = 0.92 \times 0.004.$$

$$\text{so, } P(\text{Disease}|\text{Positive}) = \frac{0.92 \times 0.004}{0.92 \times 0.004 + 0.03 \times 0.996} \approx 0.1097$$

b) Let true negative rate be y in this case. We wish to find value of y s.t. $P(\text{Disease}|\text{Positive}) = \frac{1}{2}$.

$$\text{so } P(\text{Disease}|\text{Positive}) = \frac{0.92 \times 0.004}{0.92 \times 0.004 + (1-y) \times 0.996} = \frac{1}{2}$$

$$2 \times 0.92 \times 0.004 = 0.92 \times 0.004 + 0.996 - 0.996y.$$

$$0.996y = 0.996 - 0.92 \times 0.004$$

$$\text{so } y = 1 - \frac{0.92 \times 0.004}{0.996}$$

c) i) Given $P(\text{Disease})$ is now 0.04 and $P(\neg\text{Disease}) = 0.96$,

$$P(\text{Disease}|\text{Positive}) = \frac{0.92 \times 0.04}{0.92 \times 0.04 + 0.03 \times 0.96} \approx 0.5610$$

ii) To get $P(\text{Disease}|\text{Positive}) = \frac{1}{2}$, the true negative rate, y, has to be.

$$y = 1 - \frac{0.92 \times 0.04}{0.96}.$$

Q3) $P(A\cup C) = P(A) + P(C) - P(A\cap C)$

since A & C are independent, $P(A\cap C) = P(A)P(C)$

so $P(A)+P(C)-P(A)P(C) = 0.8$.

Similarly, $P(B\cup C) = P(B) + P(C) - P(B\cap C) = P(B)+P(C)-P(B)P(C) = 0.6$.

$P(A\cup B\cup C) = P(A)+P(B)+P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C) = 0.95$.

since A&B are disjoint, $P(A\cap B) = P(A\cap B\cap C) = 0$.

so we have $P(A)+P(C)-P(A)P(C) = 0.8$ —— ①

$P(B)+P(C)-P(B)P(C) = 0.6$. —— ②

$P(A)+P(B)+P(C)-P(B)P(C)-P(A)P(C) = 0.95$ —— ③.

Taking ①+②−③: we get $P(C) = 0.8+0.6-0.95 = 0.45$.

Then $P(A) = \dfrac{0.8-0.45}{1-0.45}$ and $P(B) = \dfrac{0.6-0.45}{1-0.45}$

$= \dfrac{0.35}{0.55}$ $= \dfrac{0.15}{0.55}$

Q4a) For pairwise independence, it is when for any 2 events A & B, $P(A\cap B) = P(A)P(B)$.

Given there are equal number of people with green & brown eyes, $P(B_1) = P(B_2) = 0.5$.

so $P(B_1\cap B_2) = P(B_1)P(B_2) = 0.5\times 0.5 = 0.25$, since choice of first person does not affect the choice of second person due to independence.

For D, $P(D) = P(B_1\cap \neg B_2) + P(\neg B_1 \cap B_2)$

$= 0.5\times 0.5 + 0.5\times 0.5 = 0.5$.

Checking pairwise independence for B_1 & D, $P(B_1\cap D) = P(B_1\cap\neg B_2) = 0.5\times 0.5 = 0.25$.

Since $P(B_1)P(D) = 0.5\times 0.5 = 0.25$, B_1 & D are pairwise independent.

Without loss of generality, a similar approach shows B_2 & D are pairwise independent too.

since $P(B_1\cap B_2\cap D) \neq P(B_1)P(B_2)P(D)$, B_1, B_2 & D are not mutually independent. This makes sense as the first person & second person cannot have brown eyes $(B_1\cap B_2)$ and have both 2 people have different eye colors as well (D), so $P(B_1\cap B_2\cap D) = 0$, while $P(B_1)P(B_2)P(D) = 0.5^3$.

b) Proportion of individuals with brown eyes is now p, and ~~that~~ proportion of green eyes is $1-p$.

For D, the probability of having different eye colors $P(D) = P(B_1\cap\neg B_2) + P(\neg B_1\cap B_2) = p(1-p) + (1-p)p$
$= 2p(1-p)$.

For B_1 & D to be independent, we must have $P(B_1\cap D) = P(B_1)P(D)$, but $P(B_1\cap D) = p(1-p)$ (the probability that first person has brown eyes & second person green)

This yields $p(1-p) \neq p\times 2p(1-p)$, so $P(B_1\cap D) \neq P(B_1)P(D)$ if $p\neq 0.5$

Therefore, this shows that the only value for p that satisfies the equation and give pairwise independence is $p=0.5$.

Q5) want to show $P(A^c \cap B) = P(A^c)P(B)$

Since A & B are independent, $P(A \cap B) = P(A)P(B)$.

we know $P(A) + P(A^c) = 1$. and that for any events X & Y, $P(Y) = P(X \cap Y) + P(X^c \cap Y)$

(Y can be partitioned into 2 parts, 1 that intersects X & another intersecting X^c)

so $P(B) = P(A \cap B) + P(A^c \cap B)$

$= P(A)P(B) + P(A^c \cap B)$

$P(A^c \cap B) = P(B) - P(A)P(B)$

$= P(B)(1 - P(A))$

$= P(B)(1 - (1 - P(A^c)))$ (as $P(A) + P(A^c) = 1$).

$= P(B)P(A^c)$.

Thus, demonstrating $P(A^c \cap B) = P(B)P(A^c)$, showing A^c & B are independent.