Q1 a) The distribution I would use is hypergeometric STAB11 HW06

- because it describes the probability of a given number of successes (tagged birds), in a sample drawn without replacement from a finite population that contains a fixed number of successes.

$N = 155$, $K = 25$, $n = 40$, X = # of tagged birds in sample

b) $E(X) = n\left(\frac{K}{N}\right) = 40\left(\frac{25}{155}\right) \approx 6.45$

c) Let X be hypergeometric & let estimate for $E(X)$ be what was observed, so $E(X) = \frac{5}{40}$ instead.

Then, to find N, we have $E(X) = 40\left(\frac{25}{N}\right) = 5$, thus, $N = 200$

The scientists would estimate $N = 200$ for the bird population, which is an overestimate.

Q2 a) To find c, we know area under pdf is equal to 1 for normalization.

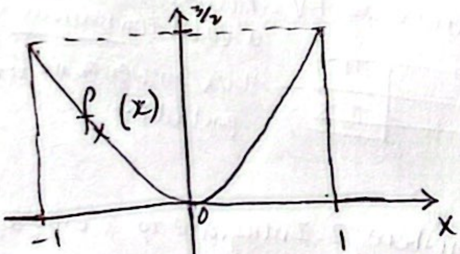
so, $\int_{-1}^{1} cx^2\,dx = 1$, therefore $\left[c\,\frac{1}{3}x^3\right]_{-1}^{1} = 1$

$\frac{c}{3} - -\frac{c}{3} = 1 \Rightarrow c = \frac{3}{2}$.

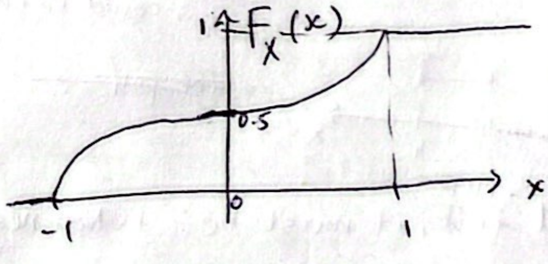
b) The CDF $F_X(x)$ would be $F_X(x) = \int_{-\infty}^{x} f(t)\,dt$ for $x \in [-1, 1]$.

$\int_{-1}^{x} \frac{3}{2} t^2\,dt = \left[\frac{1}{2}t^3\right]_{-1}^{x} = \frac{1}{2}x^3 + \frac{1}{2}$, so $F_X(x) = \frac{1}{2}x^3 + \frac{1}{2}$

c) PDF of X



CDF of X



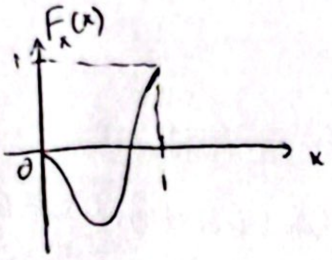
d) Given the symmetry of the PDF about the y-axis, $E(X) = 0$.

Symmetric distribution centered around zero, the expected value should be at the center of symmetry, reflecting the balance of the distribution on either side.

Scanned with CamScanner

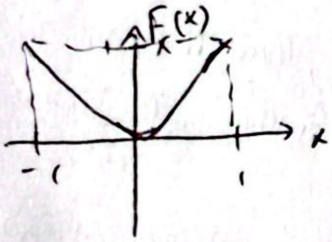
Q3) For valid CDF, it must satisfy the following properties.
1. Non-decreasing 2. approach 0 as $x \to -\infty$ 3. right continuous.
approach 1 as $x \to +\infty$

a)



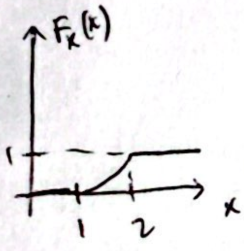
As this function decreases before increasing, it is not a valid CDF.
can plug in $x = 0.5$, we get $F_X(x) < 0$ thus not valid CDF

b)



This function evidently decreases from -1 to 0
$\Rightarrow$ thus it is not a valid CDF.

c)



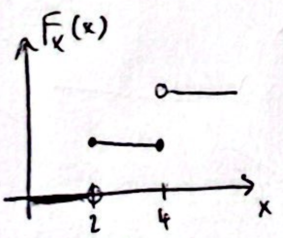
This function is a valid CDF. It satisfies;
1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$
2. non-decreasing
3. right continuous

The pdf is found by differentiation the CDF over the support.

$$\frac{d}{dx}\left[\frac{x^2}{3} - \frac{1}{3}\right] = \frac{2}{3}x.$$

Thus, the corresponding pdf is $f(x) = \begin{cases} \frac{2}{3}x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

d)



This is a valid CDF because it satisfies the 3 conditions. This could be seen as a discrete RV. However, it is not a valid ~~continuous~~ continuous CDF, thus answer is no for part (d).

~~The pmf:~~

~~x~~	~~2~~	~~4~~
~~P(X=x)~~	~~0.5~~	~~0.5~~

Q4) For a valid ~~PMF~~ PDF, it must be: 1. Non-negative everywhere, 2. Integrate to 1 over support.
Considering PDFs $f(x)$ & $g(x)$, & constans a & b, we have new function $h(x) = af(x) + bg(x)$.
1. $h(x)$ must be non-negative everywhere.
- Given we do not know details of $f(x)$ & $g(x)$, a & b must thus be ≥ 0 to ensure $h(x)$ is not negative, so $a, b \geq 0$.

2. Integral of $h(x)$ over space must equal 1, so $\int_{-\infty}^{\infty} h(x)\,dx = 1$.
- Since $f(x)$ & $g(x)$ are PDFs, we know $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx = 1$.
- So $\int_{-\infty}^{\infty} h(x)\,dx = \int_{-\infty}^{\infty} af(x) + bg(x)\,dx = a\int_{-\infty}^{\infty} f(x)\,dx + b\int_{-\infty}^{\infty} g(x)\,dx = 1 \Rightarrow$ Thus, $a+b=1$

3. a & b must be real constants as well.

Thus, $h(x)$ is a valid PDF iff $a \geq 0$, $b \geq 0$, and $a+b=1$, and a & b are real constants.

Scanned with CamScanner

Q5a) For $f(x)$ to be a valid pdf, it must satisfy: 1. Non-negative everywhere
2. Integrate to 1 over support.

1. Given $x \in (-\infty, \infty)$, we know $x^2 \geq 0$. Thus, $1+x^2 \geqslant 1$.

Therefore $\pi(1+x^2) \geqslant \pi$ and $\frac{1}{\pi(1+x^2)} \geq 0$ given $\pi(1+x^2) \geqslant \pi$

more specifically $\frac{1}{\pi(1+x^2)} \to 0$ as $x \to +\infty$ or $x \to -\infty$

and $0 < \frac{1}{\pi(1+x^2)} \leq \frac{1}{\pi}$

thus satisfying the first condition.

2. Check if integrates to 1 over support.

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - -\frac{\pi}{2}\right)$$
$$= 1$$

Therefore, this confirms $f(x)$ is a valid pdf.

b) For $f(x)$, $E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx$

Using the hint. $E(x) = \int_{-\infty}^{c} \frac{x}{\pi(1+x^2)} dx + \int_{c}^{\infty} \frac{x}{\pi(1+x^2)} dx$

$$= \frac{1}{\pi}\left[\frac{1}{2}\ln(1+x^2)\right]_{-\infty}^{c} + \frac{1}{\pi}\left[\frac{1}{2}\ln(1+x^2)\right]_{c}^{\infty}$$

Note that both expressions ~~ln~~ has $\ln(1+x^2)$ and thus, would not converge as $x \to -\infty$ or $x \to +\infty$, they actually diverge.

Therefore, the integral $\int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx$ does not converge as the function does not approach a limit as x approaches $\pm\infty$

Since the function does not diminish fast enough at the tails to produce a finite value, $E(x)$ does not exist.

Scanned with CamScanner