Assignment 1 – Discrete Review

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Related Readings: http://pages.cs.wisc.edu/~hasti/cs240/readings/

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Logic

1. Using a truth table, show the equivalence of the following statements.

(a) $P \vee (\neg P \wedge Q) \equiv P \vee Q$

Solution:

P	Q	$\neg P \wedge Q$	$P \vee (\neg P \wedge Q)$	$P \vee Q$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	F	F	F

Rows match, thus $P \vee (\neg P \wedge Q) \equiv P \vee Q$ holds

(b) $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

Solution:

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Rows math, thus $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$ holds.

(c) $\neg P \vee P \equiv$ true

Solution:

P	$\neg P$	$\neg P \vee P$	true
T	F	T	T
F	T	T	T

Rows match, thus $\neg P \vee P \equiv$ true holds.

(d) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Solution:

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Rows match, thus ~~$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$~~ $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ holds.

Sets

2. Based on the definitions of the sets A and B, calculate the following: $|A|$, $|B|$, $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

(a) $A = \{1, 2, 6, 10\}$ and $B = \{2, 4, 9, 10\}$

Solution:
$|A| = 4$ $|B| = 4$
$A \cup B = \{1, 2, 4, 6, 9, 10\}$
$A \cap B = \{2, 10\}$
$A \setminus B = \{1, 6\}$
$B \setminus A = \{4, 9\}$

(b) $A = \{x \mid x \in \mathbb{N}\}$ and $B = \{x \in \mathbb{N} \mid x \text{ is even}\}$

Solution:
$|A| = |B| = \infty$
$A \cup B = A$
$A \cap B = B$
$A \setminus B = \{x \in \mathbb{N} \mid x \text{ is odd}\}$
$B \setminus A = \phi$

Relations and Functions

3. For each of the following relations, indicate if it is reflexive, antireflexive, symmetric, antisymmetric, or transitive.

(a) $\{(x, y) : x \leq y\}$

Solution: reflexive, transitive, antisymmetric

(b) $\{(x, y) : x > y\}$

Solution: antireflexive, transitive

(c) $\{(x, y) : x < y\}$

Solution: antireflexive, transitive

(d) $\{(x, y) : x = y\}$

Solution: reflexive, symmetric, transitive, antisymmetric

4. For each of the following functions (assume that they are all $f : \mathbb{Z} \to \mathbb{Z}$), indicate if it is surjective (onto), injective (one-to-one), or bijective.

(a) $f(x) = x$

Solution: bijective, surjective & injective

(b) $f(x) = 2x - 3$

Solution: Injective, surjective, bijective

(c) $f(x) = x^2$

Solution: ~~injective~~ none, x & ~~x~~ -x maps to same.

5. Show that $h(x) = g(f(x))$ is a bijection if $g(x)$ and $f(x)$ are bijections.

Solution:
we first prove it is injective by contradiction.
If injective, then ~~for~~ $\forall a, b \in h$, $h(a) \neq h(b)$
Assume $h(a) = h(b)$,
then $g(f(a)) = g(f(b))$
Since $g(x)$ is a bijection, then
$f(a) = f(b)$.
Since f is also a bijection, then
$a = b$.
Thus, from these, we know that
$\forall a, b \in h$, $h(a) = h(b) \Rightarrow g(f(a)) = g(f(b))$
$\Rightarrow f(a) = f(b) = a = b$, so $h(a) = h(b) \Rightarrow a = b$, so h is injective.

To show surjectivity, want to show $\forall y$ in codomain of h, there exist an x in domain of h, ie $h(x) = y$.
Let y be an element in codomain of h, as $h(x) = g(f(x))$, and since $g(x)$ ~~are~~ is a bijection, there is a x' in the domain of g such that $g(x') = y$. Given $f(x)$ is also a bijection, then there is a x'' such that $f(x'') = x'$
now consider the element $x = x''$ in h,
$h(x) = g(f(x)) = g(f(x'')) = g(x') = y$.
Thus, there exist an element x in ~~th~~ domain of h for an arbitrary y in codomain of h, thus h is surjective.
As h is both injective & surjective, h is also a bijection.

Induction

6. Prove the following by induction.

(a) $\sum_{i=1}^{n} i = n(n+1)/2$

Solution:

Show base case holds.

For $P(n): \sum_{i=1}^{n} i = n(n+1)/2$

Base case $P(1)$:

LHS: $\sum_{i=1}^{1} i = 1$

RHS: $1(1+1)/2 = 2/2 = 1$

Thus, base case holds.

Induction step.

Assume $P(k)$ holds, then show $P(k+1)$ also holds.

$P(k): \sum_{i=1}^{k} i = k(k+1)/2$ —— ①

$P(k+1): \sum_{i=1}^{k+1} i = (k+1)(k+2)/2$.

$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k+1 = \frac{k(k+1)}{2} + k+1$ (substituting $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ from ①)

$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

Thus, $\sum_{i=1}^{k+1} = \frac{(k+1)(k+2)}{2}$ holds, completing proof of inductive step.

∴ It follows by induction that $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i = n(n+1)/2$.

(b) $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

Solution:

Let $P(n): \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

Base case $P(1)$:

LHS: $\sum_{i=1}^{1} i^2 = 1^2 = 1$.

RHS: $1(1+1)(2(1)+1)/6 = (2)(3)/6 = 1$

Inductive step.

Assume $P(k)$ holds, show $P(k+1)$ holds.

$P(k): \sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$. —— ①

$P(k+1): \sum_{i=1}^{k+1} i^2 = (k+1)(k+2)(2(k+1)+1)/6$.

$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$

$= k(k+1)(2k+1)/6 + (k+1)^2$ (substitute from ①)

$= \frac{(k+1)\,(k(2k+1)+6(k+1))}{6}$

$= \frac{(k+1)\,(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$

Thus $P(k+1)$ holds, completing proof of inductive step.

∴ It follows by induction that $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

(c) $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$

Solution:

Let $P(n): \sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$

Base case $P(1)$:

LHS: $\sum_{i=1}^{1} i^3 = 1^3 = 1$

RHS: $1^2(1+1)^2/4 = 2^2/4 = 4/4 = 1$

Inductive step. Assume $P(k)$ holds, show $P(k+1)$ also holds.

$P(k): \sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$. —— ①.

$P(k+1): \sum_{i=1}^{k+1} i^3 = (k+1)^2(k+2)^2/4$.

$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$

$= k^2(k+1)^2/4 + (k+1)^3$ (substitute from ①)

$= \frac{(k+1)^2\,(k^2+4(k+1))}{4} = \frac{(k+1)^2(k+2)^2}{4}$

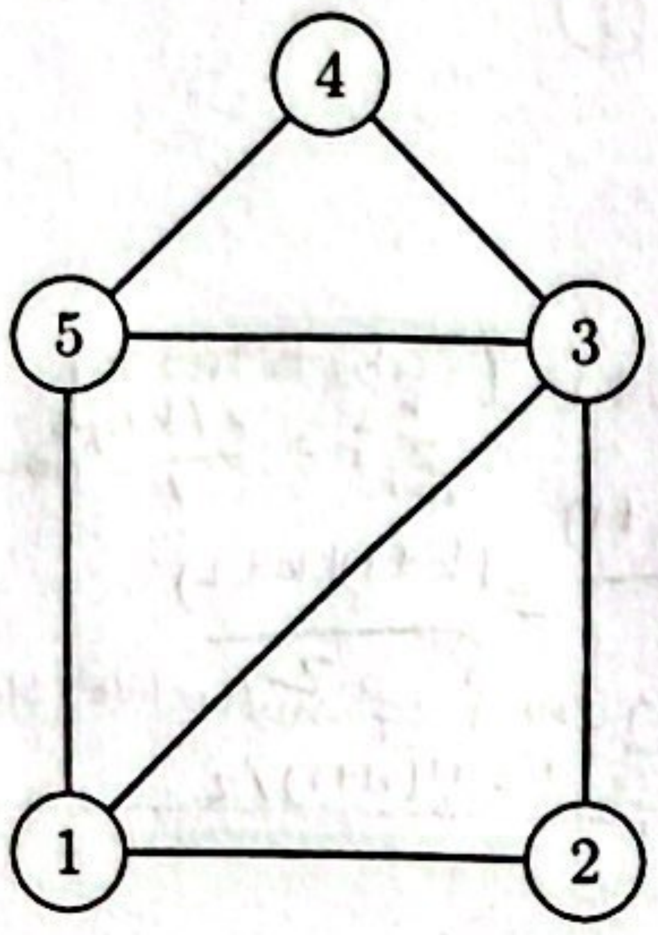
Thus $P(k+1)$ holds, completing proof of inductive step.

It follows by induction that

$\forall n \in \mathbb{N}, \sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$.

Graphs and Trees

7. Give the adjacency matrix, adjacency list, edge list, and incidence matrix for the following graph.



Solution:

adjacency matrix = $\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$

adjacency list:
1 : 2, 3, 5
2 : 1, 3
3 : 1, 2, 4, 5
4 : 3, 5
5 : 1, 3, 4.

edge list: [(1,2), (1,3), (1,5), (2,3), (3,4), (3,5), (4,5)]

incidence matrix: $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$

8. How many edges are there is a complete graph of size n? Prove by induction.

Solution:

edges in complete graph of size $n = n(n-1)/2$

Let $P(n)$ = # of edges in complete graph of size n $= n(n-1)/2$

Base case $P(1)$:

0 vertex (1 vertex graph has no edges)

ie $(\frac{1}{2}) = 0$.

Inductive step:
Assume $P(k)$ holds.
show $P(k+1)$ also holds.
$P(k) = k(k-1)/2$

Show $P(k+1) = \frac{(k+1)(k)}{2}$.
For a graph of size $k+1$,
First consider fully connected graph of size k.
To add n $(k+1)^{st}$ vertex & ensure graph is still fully connected, $(k+1)^{st}$ vertex has to have an edge to every vertex in earlier mentioned fully connected graph of size k.
Thus, $(k+1)^{st}$ vertex has to form k more edges with k vertices.
Thus, total # edges for complete graph of size $k+1$

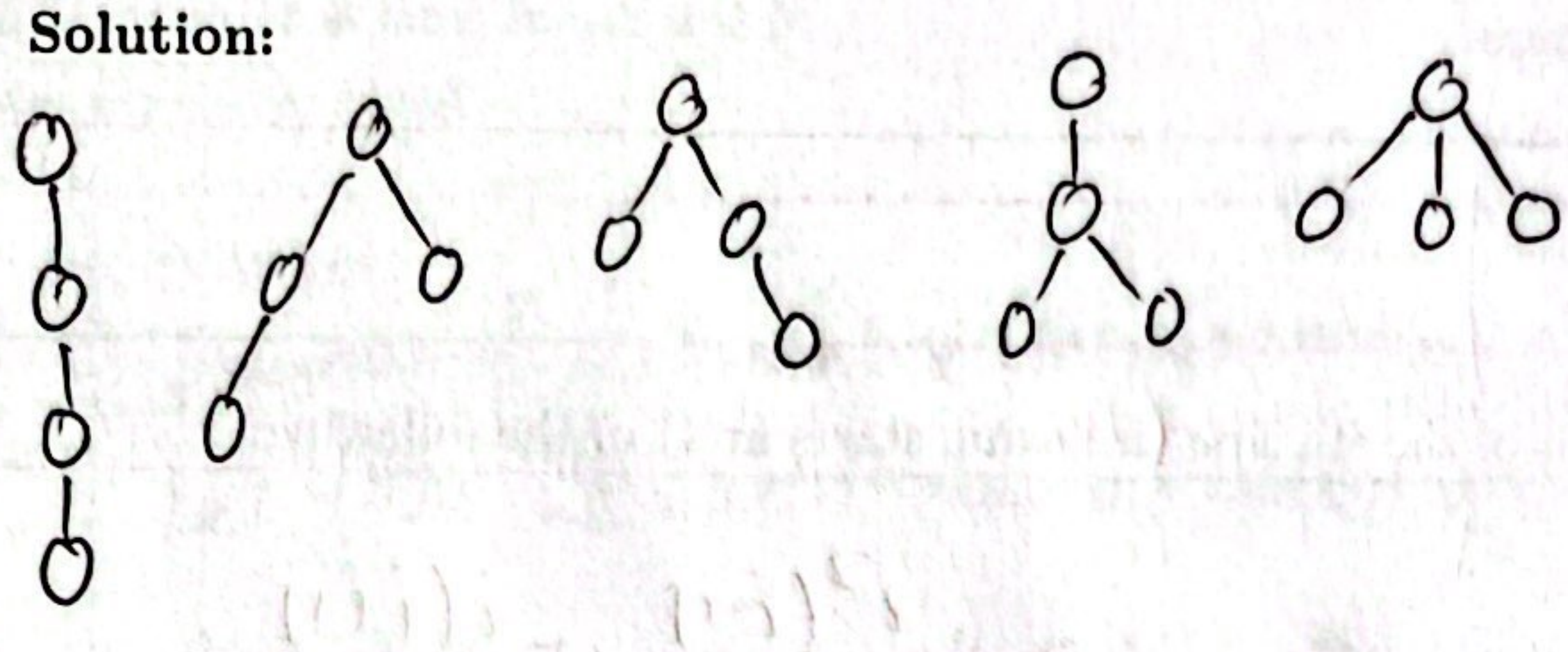
$= \frac{k(k-1)}{2} + k = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$.

Thus $P(k+1) = \frac{k(k+1)}{2}$ holds, completing proof of induction step.

It follows by induction that $\forall n \in \mathbb{N}$, $P(n) = \frac{n(n-1)}{2}$.

9. Draw all possible (unlabelled) trees with 4 nodes.

Solution:



10. Show by induction that, for all trees, $|E| = |V| - 1$.

Solution:

Base case: Consider tree with 1 vertex. There will be no edges, only possible edge is self-loop which is a simple cycle, not allowed in trees. Thus, in base case, $|V|=1$, $|E|=0$, so $|E|=|V|-1$ holds.

Let $P(n)$: For all trees T (V,E) with $|V|=n$, $|E|=n-1$

Given base case $P(1)$ holds,

Inductive step: Assume $P(b)$ holds for all $1 \le b \le k$ (induction hypothesis)

Consider $P(k+1)$, for tree T with $k+1$ nodes, since it has ≥ 2 nodes, there is ≥ 1 edge to be connected. Choose any edge in T & delete it. We now have 2 nonempty trees T_1 & T_2. Every edge in T is part in T_1, part in T_2, or is the deleted edge. Let r be # ~~nodes~~ vertizes in T_1. Since every vertex in T either is in T_1 or T_2, we know T_2 has $k+1-r$ vertizes. Both T_1 & T_2 have between 1 & k vertizes. Apply induction hypothesis. Thus T_1 has $r-1$ edges & T_2 has $k-r$ edges. Including intial deleted edge, we have $(r-1)+(k-r)+1 = k$ edges in T as required. Thus $P(k+1)$ holds, completing the induction.

Counting

11. How many 3 digit pin codes are there?

Solution: $10^3 = 1000$

12. What is the expression for the sum of the ith line (indexing starts at 1) of the following:

1
2 3
4 5 6
7 8 9 10
⋮

Solution:

$\frac{i(i-1)}{2} \times i + (1+2+\ldots+i) = \frac{i^2(i-1)}{2} + \frac{i(i+1)}{2}$

$= \frac{i}{2}(i^2 - i + i + 1) = \frac{i}{2}(i^2+1)$

13. A standard deck of 52 cards has 4 suits, and each suit has card number 1 (ace) to 10, a jack, a queen, and a king. A standard poker hand has 5 cards. For the following, how many ways can the described hand be drawn from a standard deck.

(a) A royal flush: all 5 cards have the same suit and are 10, jack, queen, king, ace.

Solution: 4

(b) A straight flush: all 5 cards have the same suit and are in sequence, but not a royal flush.

Solution: 40

(c) A flush: all 5 cards have the same suit, but not a royal or straight flush.

Solution: $\binom{13}{5} \times 4 = 5148$

(d) Only one pair (2 of the 5 cards have the same number/rank, while the remaining 3 cards all have different numbers/ranks):

Solution: $\binom{13}{1} \times \binom{4}{2} \times \binom{12}{3} \times 4^3$

Proofs

14. Show that $2x$ is even for all $x \in \mathbb{N}$.

(a) By direct proof.

Solution:

Definition of even number: For even number a, $\exists b \in \mathbb{N}$, s.t. $a = 2b$.

Let $y = 2x$, we want to show by direct proof y is even.

Given that $\exists x \in \mathbb{N}$ for all $y \in \mathbb{N}$, then by the definition of even, y is also even. Thus, we have proved by direct proof $2x$ is even for all $x \in \mathbb{N}$.

(b) By contradiction.

Solution:

To prove by contradiction, we first assume the statement is false, that is, ~~[illegible]~~ $2x$ is not even for some $x \in \mathbb{N}$, so $\exists x \in \mathbb{N}$, s.t. $2x$ is odd.

An odd number y is an integer that can be written as $y = 2k+1$, $\exists k \in \mathbb{N}$.

Assuming $\exists x \in \mathbb{N}$, s.t. $2x$ is odd, then $\exists k \in \mathbb{N}$, s.t., $2x = 2k+1 \Rightarrow x = k + \frac{1}{2}$.

Since $x \in \mathbb{N}$, then k cannot be a natural number given it is added with $\frac{1}{2}$ to give x. Thus, there does not exist a $k \in \mathbb{N}$ s.t. $2x = 2k+1$.

Thus, we have proved by contradiction that $2x$ is even $\forall x \in \mathbb{N}$.

15. For all $x, y \in \mathbb{R}$, show that $|x + y| \le |x| + |y|$. (Hint: use proof by cases.)

Solution:

Case 1: $x \geq 0, y \geq 0$,

then $|x+y| = x+y$, $|x| = x$, $|y| = y$

so $|x+y| = x+y = |x|+|y| = x+y$

and $|x+y| \le |x|+|y|$ holds.

Case 2: $x < 0, y < 0$.

then $|x+y| = -x-y$, $|x| = -x$, $|y| = -y$

so $|x+y| = -x-y = |x|+|y| = -x-y$

and $|x+y| \le |x|+|y|$ holds

Case 3: a) $x \geq 0$, $y < 0$ and $|y| \le x$

or b) $y \geq 0$, $x < 0$ and $|x| \le y$.

a) Then $|x+y| = x+y$ and $|x| = x$, $|y| = -y$

and since $y < 0$, then $|x+y| = x - |y| < \underbrace{x + |y|}_{=|x|+|y|}$

and $|x+y| \le |x|+|y|$ holds.

WLOG, case 3b is the same.

Case 4: a) $x \geq 0$, $y < 0$ and $|y| > x$

or b) $y \geq 0$, $x < 0$ and $|x| > y$.

a) Then $|x+y| = |y| - x < |x|+|y| = |y| + x$.

so $|x+y| \le |x|+|y|$ holds.

WLOG, case 4b is the same.

We have proven by cases $\forall x, y \in \mathbb{R}$, $|x+y| \le |x|+|y|$

Program Correctness (and Invariants)

16. For the following algorithms, describe the loop invariant(s) and prove that they are sound and complete.

Algorithm 1: findMin

(a)

Input: a: A non-empty array of integers (indexed starting at 1)
Output: The smallest element in the array

```
begin
    min ← ∞
    for i ← 1 to len(a) do
        if a[i] < min then
            min ← a[i]
        end
    end
    return min
end
```

Solution:

First prove partial correctness:

Invariant: After n iterations of the for loop, min = smallest element of first n elements in the array.

Let min_n be the min element after n iterations of the for loop, $\forall n \in \mathbb{N}$.

We prove by induction $\forall n\, P(n)$, $P(n)$ holds:

Base case: $P(1)$, loop runs 1 time and min would be assigned with this new value since any integer is less than ∞.

Inductive step
Assume $P(k)$ holds, show $P(k+1)$ holds as well.
Since $P(k)$ holds, then min currently, after k iterations of the loop stores min of the first k elements. ∠case 1: $(k+1)^{st}$ element is min of whole array.
For $(k+1)^{st}$ time loop runs, $a[k+1] < min$, min assigned $a[k+1]$, for loop ends and $(k+1)^{st}$ element is returned.
Case 2: $(k+1)^{st}$ element is not min of whole array.
For $(k+1)^{st}$ time loop runs, $a[k+1] \geq min$, so min is not reassigned new value. for loop ends and min element of first k elements is returned, this is definitely min as $P(k)$ holds.

This shows how invariant holds $\forall n \in \mathbb{N}$, completing the proof & program is partially correct.

Termination: Given for loop increments i by 1 every loop, show $len(a)$ is finite and ≥ 1 so program eventually terminates.
$\Rightarrow$ stated in input that a is a non-empty array of integers.
$\Rightarrow$ $len(a) \geq 1$ and $len(a)$ is finite, ie ~~fou~~ num of elts in array a.

Given partial correctness & termination is both proven, we have shown Algorithm 1's correctness.

(b)

Algorithm 2: InsertionSort

Input: a: A non-empty array of integers (indexed starting at 1)
Output: a sorted from largest to smallest

```
begin
    for i ← 2 to len(a) do
        val ← a[i]
        for j ← 1 to i − 1 do
            if val > a[j] then
                shift a[j..i − 1] to a[j + 1..i]
                a[j] ← val
                break
            end
        end
    end
    return a
end
```

Solution: Partial correctness

Invariant: After n iterations of the ~~for~~ loop in line 1, output a is sorted from larger to smallest

~~Base case~~ Let a_n be the array after n iterations of first for loop, $\forall n \in \mathbb{N}$, we prove by induction $\forall n\ P(n)$, $P(n)$ holds.

Base case: $P(1)$, for loop does not execute, algorithm ends, returned array already sorted as there is only 1 element.

$P(2)$, outer for loop runs once, compares 1st to second element, swap if 1st smaller than 2nd return sorted array.

Inductive step

Assume $P(k)$ holds, show $P(k+1)$ holds as well.

Since $P(k)$ holds, this means first k elements in a is already sorted.

For $(k+1)^{st}$ loop, $a[k+2]$ is assigned to val, the inner for loop then compares val to first k elements, starting with the first element. Once it finds first element in first k elements $a[k+2]$ is greater than, it inserts it to the left of this element by shifting every other element to the right. This ensures $a[k+2]$ is inserted in the right position by order.

If it is the smallest element in first $k+2$ elements, it gets compared to every element and stays put.

Termination: Given that for loop increments i & j by 1 each loop, and a is a finite non-empty array of integers, then $len(a) = k$ for some $k \in \mathbb{N}$. Thus, outer for loop terminates. As i increments by 1 each loop and is an integer, inner for loop terminates as well.

Given partial correctness & termination is both proven, we have shown Algorithm 2's correctness.

Recurrences

17. Solve the following recurrences.

(a) $c_0 = 1; c_n = c_{n-1} + 4$

Solution:

n	0	1	2	3	4	5
c_n	1	5	9	13	17	21

$c_n = 4n + 1$.

(b) $d_0 = 4; d_n = 3 \cdot d_{n-1}$

Solution:

$d_n = 3^n \cdot 4$

(c) $T(1) = 1; T(n) = 2T(n/2) + n$ (An upper bound is sufficient.)

Solution: $T(n) = 2T(n/2) + n = 4T(n/4) + 2n = 8T(n/8) + 3n = 2^k T(\frac{n}{2^k}) + kn$

$T(n/2) = 2T(n/4) + \frac{n}{2}$

$T(n/4) = 2T(n/8) + \frac{n}{4}$

Guess: $T(n) = O(n \lg n)$.

Proof by Induction: Base case $T(1) = 1$ (stated.

Inductive step: Assume $T(m) \le cm \lg m$ for all $m < n$, prove $T(n) \le cn \lg n$

$T(n) = 2T(\frac{n}{2}) + n \le 2c(\frac{n}{2}) \lg(\frac{n}{2}) + n = cn \lg n - cn \lg 2 + n$

$= cn \lg n - cn + n \le cn \lg n$

when $c \ge 1$.

$\therefore T(n) \le cn \lg n$.

so $T(n) = O(n \lg n)$.

(d) $f(1) = 1; f(n) = \sum_1^{n-1} (i \cdot f(i))$
(Hint: compute $f(n+1) - f(n)$ for $n > 1$)

Solution:

$f(n+1) - f(n) = \sum_1^{n} i \cdot f(i) - \sum_1^{n-1} i \cdot f(i)$

$= n \cdot f(n)$

so $f(n+1) = (n+1) f(n)$, substituting ~~n~~ with $n-1$

Guess: $f(n) = n f(n-1) = n!$

Check:

n	1	2	3	4	5
$f(n)$	1	1	3	12	60
$n!$	1	2	6	24	120

so, we observe that $f(n) = \frac{n!}{2}$ for $n > 1$ and $f(1) = 1$.