Q1a) Given CDF is $F_X(x) = \left(\frac{x}{b}\right)^n = \frac{x^n}{b^n}$, then PDF is

$$f_X(x) = \frac{d}{dx}\left(\frac{x^n}{b^n}\right) = \frac{nx^{n-1}}{b^n}$$

. To find $E(x) = \int_0^b x \cdot f_X(x)\, dx$

$$= \int_0^b \frac{nx(x^{n-1})}{b^n} dx$$

$$= \frac{n}{b^n}\int_0^b x^n\, dx = \frac{n}{b^n}\left[\frac{x^{n+1}}{n+1}\right]_0^b$$

$$= \frac{n}{b^n}\left[\frac{b^{n+1}}{n+1}\right]$$

$$= \frac{bn}{n+1}.$$

Bias of X would be

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

where $\hat{\theta} = X$ and $\theta = b$, so $\text{Bias}(x) = E(x) - b = \frac{b}{n+1} - b = b\left(\frac{1}{n+1} - 1\right)$

$$= -\frac{1}{n+1} b$$

$E(x) = \frac{n}{n+1} b$ is less than b & Bias(x) is negative and thus is a biased estimator. This means X underestimates b.

b) Given we know $E(X) = \frac{n}{n+1} b$. To have unbiased estimator $g(x)$, we need $E(g(x)) = b$.

We can scale X by $\frac{n+1}{n}$, so $g(x) = \frac{n+1}{n} X = \frac{n+1}{n} \max(U_1, U_2, \dots, U_n)$.

For any scalar a, and RV X, $E(aX) = aE(X)$. so we can check $g(x)$.

$$E(g(x)) = E\left(\frac{n+1}{n} X\right) = \frac{n+1}{n} E(X) = \frac{n+1}{n} \cdot \frac{n}{n+1} b = b.$$

This gives us $\text{Bias}(g(x)) = E(g(x)) - b = b - b = 0$, so $g(x)$ is an unbiased estimator of b.

Q2) For χ^2_v, we know the PDF is $f_X(x) = \dfrac{x^{v/2 - 1} e^{-\frac{x}{2}}}{2^{v/2}\Gamma(v/2)}$

So density would be $\displaystyle\int_0^\infty \frac{x^{\frac{v}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} dx$. want to show it equals 1.

Let $u = \frac{x}{2}$. Then $dx = 2\,du$, so we get $\displaystyle\int_0^\infty \frac{x^{\frac{v}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} dx = \int_0^\infty \frac{x^{\frac{v}{2}-1} e^{-u}}{2^{\frac{v}{2}}\Gamma(\frac{v}{2})} 2\,du$

$$= \int_0^\infty \left(\frac{x}{2}\right)^{\frac{v}{2}-1} \frac{e^{-u}}{\Gamma(\frac{v}{2})} du = \int_0^\infty u^{\frac{v}{2}-1} e^{-u} \frac{1}{\Gamma(\frac{v}{2})} du$$

$$= \frac{\Gamma(\frac{v}{2})}{\Gamma(\frac{v}{2})}$$

(Since $\int_0^\infty u^{\frac{v}{2}-1} e^{-u} du = \Gamma(\frac{v}{2})$. Integral of Gamma func where $x = u$ and $\alpha = \frac{v}{2}$).

$$= 1.$$

Q3) By Markov's Inequality, $P(Y \geq a) \leq \frac{E(Y)}{a}$

$$= P((X-\mu)^2 \geq (k\sigma)^2) \leq \frac{E((X-\mu)^2)}{(k\sigma)^2} \quad \text{(Substituting } Y \text{ and } a\text{)}$$

$$= P((X-\mu)^2 \geq (k\sigma)^2) \leq \frac{\sigma^2}{k^2\sigma^2} \quad (\text{since } E((X-\mu)^2) = V(X) = \sigma^2)$$

$$= P((X-\mu)^2 \geq (k\sigma)^2) \leq \frac{1}{k^2}$$

$P((X-\mu)^2 \geq (k\sigma)^2)$ is the same as $P(|X-\mu| \geq k\sigma)$ as comparing squared real numbers and their absolute values before squaring.

So, $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$. (Chebyshev's Inequality).

Q4a) For each iid Poisson RV, its mean = variance = $\lambda = 1$.

So $E(T_n) = E(X_1 + X_2 + \ldots + X_n) = E(X_1) + \ldots + E(X_n) = n \times 1 = n$.

$V(T_n) = V(X_1 + \ldots + X_n) = V(X_1) + \ldots + V(X_n) = n \times 1 = n$ (since iid)

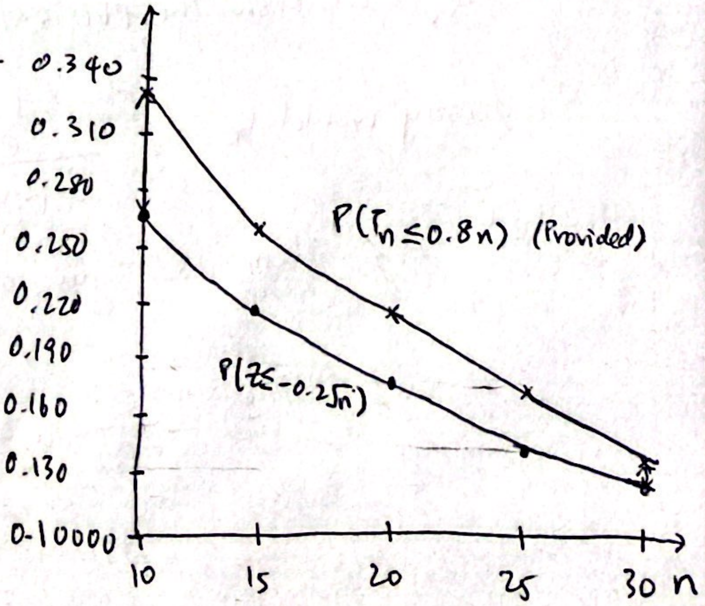
$SD(T_n) = \sqrt{V(T_n)} = \sqrt{n}$.

b) By CLT, as n becomes large, T_n can be approximated by a normal distribution $T_n \sim N(n, n)$.

Normalize T_n: $Z = \frac{T_n - n}{\sqrt{n}} \sim N(0,1)$ so to find $P(T_n \leq 0.8n)$,

$$P\left(\frac{T_n - n}{\sqrt{n}} \leq \frac{0.8n - n}{\sqrt{n}}\right) = P\left(Z \leq \frac{-0.2n}{\sqrt{n}}\right) = P(Z \leq -0.2\sqrt{n}).$$

n	10	15	20	25	30
$P(Z \leq -0.2\sqrt{n})$	.26435	.22065	.18673	.15866	.13786



As n increases, the values for the approximated probabilities by CLT approaches the actual probabilities. (Difference gets smaller).

Q5a) $X_1, X_2, \ldots, X_n \overset{iid}{\sim} Bern(p)$. each has mean p, var $p(1-p)$

$$E(\bar{X}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}(np) = p.$$

$$V(\bar{X}_n) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}.$$

$$SD(\bar{X}_n) = \sqrt{V(\bar{X}_n)} = \sqrt{\frac{p(1-p)}{n}}.$$

b) Chebyshev's Inequality: $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$.

so to find when $k=2$ and applying to $\bar{X}_n$,

$$P\left(|\bar{X}_n - p| \geq 2\sqrt{\frac{p(1-p)}{n}}\right) \leq \frac{1}{4}.$$ $\frac{1}{4}$ is the upper bound.

c) If n is large, we can use CLT to approximate $\bar{X}_n$.

$$\bar{X}_n \,\dot{\sim}\, N\left(p, \frac{p(1-p)}{n}\right).$$

$$Z = \frac{\bar{X}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \,\dot{\sim}\, N(0,1)$$

$$P\left(|\bar{X}_n - p| \geq 2\sqrt{\frac{p(1-p)}{n}}\right) = P\left(\frac{|\bar{X}_n - p|}{\sqrt{\frac{p(1-p)}{n}}} \geq 2\right) = P(|Z| \geq 2).$$

$$P(|Z| \geq 2) = 2\cdot(1-\Phi(2)) = 2\cdot\Phi(-2) = 2\cdot 0.2275 = 0.455$$

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