Q1a) $P(X=1) = \frac{5}{10} = 0.5$ STAT311 HW10

$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = 0.5$.

$P(Y=1) = \frac{2}{10} = 0.2$

$E(Y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1) = 0.2$.

	1	2	3	4	5	6	7	8	9	10
X	0	0	0	0	0	1	1	1	1	1
Y	0	0	0	0	0	0	0	0	1	1
XY	0	0	0	0	0	0	0	0	1	1

b)

		X = 0	X = 1
Y	0	5/10	3/10
	1	0	2/10

c) $E(XY) = \frac{2}{10} = 0.2$

so $Cov(X,Y) = E(XY) - E(X)E(Y) = 0.2 - 0.5 \times 0.2 = 0.1$,,

d) $n=6$. $n=6$ maximizes $Cov(X,Y)$ as it aligns the outcomes of both X & Y most effectively where both X & Y are 1 for $n \geq 6$ and 0 otherwise. This maximizes the dependence, ie the overlap, between X & Y. Since covariance is a term for how much X & Y vary together, the greater the overlap, the more often X & Y vary together. Given X is 1 for $x \geq 6$, it makes sense for $n=6$ so $Y=1$ for $y \geq 6$ as well.

Q2a) Let $f_X(x) = \mu e^{-\mu x}$ and $f_Y(y) = \lambda e^{-\lambda y}$ for $x \geq 0$ and $y \geq 0$ respectively.

Given X & Y are independent, then $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = (\mu e^{-\mu x})(\lambda e^{-\lambda y})$
$= \mu \lambda e^{-\mu x - \lambda y}$ for $x, y \geq 0$.

b) For a passenger to make it on the plane, $X + Y + 30 \geq 0$.

so we wish to find $P(X + Y + 30 \geq 0) = P(X \leq Y + 30)$.

$$P(X \leq Y+30) = \int_0^\infty \int_0^{y+30} \mu \lambda e^{-\mu x - \lambda y} \, dx \, dy = \int_0^\infty \left[-\lambda e^{-\mu x - \lambda y} \right]_0^{y+30} dy$$

$$= \int_0^\infty (-\lambda e^{-\mu(y+30) - \lambda y}) - (-\lambda e^{-\lambda y}) \, dy = \lambda \int_0^\infty e^{-\lambda y} dy - \int_0^\infty e^{-\mu y - 30\mu - \lambda y} dy$$

$$= \lambda \left[\left[-\frac{1}{\lambda} e^{-\lambda y} \right]_0^\infty - \frac{1}{-\mu-\lambda} \left[e^{-\mu y - 30\mu - \lambda y} \right]_0^\infty \right]$$

$$= \lambda \left[\left(0 - -\frac{1}{\lambda}\right) + \frac{1}{\mu+\lambda} \left(0 - e^{-30\mu}\right) \right] = \lambda \left(\frac{1}{\lambda} - \frac{1}{\mu+\lambda} e^{-30\mu} \right) = 1 - \frac{\lambda}{\mu+\lambda} e^{-30\mu}$$

Q3a) Given $X \sim Unif(-1,1)$, then PDF of X is $f_X(x) = \frac{1}{2}$ for $x \in [-1,1]$.

Then $E(Y) = \int_{-1}^{1} x^2 \cdot f_X(x)\,dx = \int_{-1}^{1} x^2 \cdot \frac{1}{2}\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} - -\frac{1}{3}\right)$

$= \frac{1}{3}$.

$V(Y) = E(Y^2) - [E(Y)]^2 \quad E(Y^2) = \int_{-1}^{1} x^4 \cdot f_X(x)\,dx = \frac{1}{2}\int_{-1}^{1} x^4\,dx = \frac{1}{2}\left[\frac{x^5}{5}\right]_{-1}^{1}$

$= \frac{1}{5}$

so $V(Y) = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$ //

b) $Cov(X,Y) = E(XY) - E(X)E(Y)$

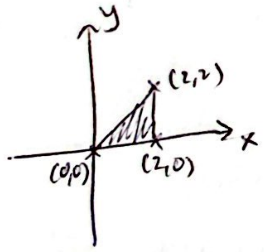
$= E(x^3) - E(x)E(x^2)$

$= \int_{-1}^{1} x^3 \cdot \frac{1}{2}\,dx - \frac{1}{3} \cdot \int_{-1}^{1} x \cdot \frac{1}{2}\,dx$

$= \frac{1}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} - \frac{1}{3} \cdot \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4}\right) - \frac{1}{6}\left(\frac{1}{2} - \frac{1}{2}\right) = 0.$

c) If X & Y are independent, then joint probability of X & Y can be factored & calculated as product of 2 single integrals. However, note $Y = X^2$, so Y is completely determined by X, which contradicts the requirement for independence. Thus, X & Y are not independent, since knowing X gives exact information about Y, violating the definition of independence.

Q4a)



X has support of $[0,2]$.

Y is dependent on value of x, having support of $[0,x]$ as it has to be below $y=x$ line.

b) Given $f_{X,Y}(x,y) = \frac{1}{2}$, then $E(X) = \int_0^2 \int_0^x x \cdot \frac{1}{2}\,dy\,dx = \frac{1}{2}\int_0^2 [xy]_0^x\,dx$

$= \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left(\frac{8}{3}\right) = \frac{4}{3}$ //

$E(Y) = \int_0^2 \int_0^x y \cdot \frac{1}{2}\,dy\,dx = \frac{1}{2}\int_0^2 \left[\frac{y^2}{2}\right]_0^x dx = \frac{1}{2}\int_0^2 \frac{x^2}{2}\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{4}\left(\frac{8}{3}\right) = \frac{2}{3}$ //

Q5a) $E(T) = E(R+M+W) = 497+514+589 = 1500$

$V(T) = V(R+M+W) = V(R)+V(M)+V(W)+2\,Cov(R,M)+2\,Cov(R,W)+2\,Cov(M,W)$

~~For~~ $Cov(X,Y) = Cor(X,Y)\sigma_x \sigma_y$ so,

$V(T) = 114^2+117^2+113^2+2(0.72\times114\times117)+2(0.84\times114\times113)+2(0.72\times117\times113)$
$= 99340.72$

$\sigma_T = \sqrt{99340.72} = 315.18$ (2dp)

b) By R-Studio, qnorm(0.8) = 0.842.

so 0.842 is the point on standard normal where 80% of values lie to left of.

Thus we wish to find x in $\Phi\left(\frac{x-1500}{315.18}\right) = 0.8$, $\Phi^{-1}(0.8) = 0.842$.

so $x = 315.18 * \Phi^{-1}(0.8) + 1500$

$= 315.18 * 0.842 + 1500 = 1765.13$ (2dp)

This means 20% of test takers ~~ha~~ score ≥ 1765.13.