STAT311 HW07

Q1a) $P(R<0.25) = \dfrac{\text{Area of circle with } r=0.25}{\text{Area of entire circle}}$

$$= \frac{\pi(0.25)^2}{\pi(1)^2} = \frac{1}{16}$$

b) To find pdf, we consider the probability is uniformly distributed over the area.

The pdf, $f_R(r)$ is s.t. the integral of $f_R(r)$ from 0 to R is the probability that the distance from the center is less than R. so $\int_0^R f_R(r)\,dr = \dfrac{\text{Area of circle with } r=R}{\text{Area of entire circle}}$.

Thus $$f_R(r) = \frac{d}{dR}\left(\frac{\pi r^2}{\pi(1)^2}\right) = 2r.$$

so pdf is $f_R(r) = 2r.$, for $0 \le r \le 1$

c) Mean is integrating wrt r and multiply by pdf from 0 to 1

$$E(r) = \int_0^1 r f_R(r)\,dr = \int_0^1 r(2r)\,dr = \left[\frac{2}{3}r^3\right]_0^1 = \frac{2}{3}$$

$$E(r^2) = \int_0^1 r^2 f_R(r)\,dr = \int_0^1 r^2(2r)\,dr = \left[\frac{1}{2}r^4\right]_0^1 = \frac{1}{2}.$$

$$V(r) = E(r^2) - [E(r)]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{9}{18} - \frac{4}{9} = \frac{1}{18}$$

Q2a) For $X \sim Exp(\lambda)$, the pdf is defined by $f(x) = \lambda e^{-\lambda x}$ and cdf is $F(x) = 1 - e^{-\lambda x}$.

So to find median, we consider $F(x) = 0.5$, so $1 - e^{-\lambda x} = 0.5$

wish to find x: $1 - 0.5 = e^{-\lambda x} \Rightarrow \ln(\frac{1}{2}) = -\lambda x$

$$\text{so } x = \frac{\ln(\frac{1}{2})}{-\lambda} = \frac{-\ln(2)}{-\lambda} = \frac{\ln(2)}{\lambda}$$

b) For $X \sim Exp(\lambda)$, we know the mean, $\mu = \frac{1}{\lambda}$.

so the difference $= \mu - x = \dfrac{1}{\lambda} - \dfrac{\ln(2)}{\lambda} = \dfrac{1-\ln(2)}{\lambda}$.

Since $1 - \ln(2) > 0$ (given $\ln(2) < 1$), this means that the mean is greater than the median.

This is because, exponential distribution is skewed right (ie has long right tail). The mean would be affected more by the presence of very large values due to long tail. The median, being the middle value, is less influenced by extreme values. Skewness results in the mean being pulled to the right of the median, hence

Q3a) To be valid pdf, $f(x) \geq 0$ for all x & $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Given this pdf, $f(x) \geq 0$ for all x is satisfied since $x^4 \geq 0$ for all x.

we wish to find c such that $\int_c^{\infty} f(x)\,dx = 1$. so $\int_c^{\infty} \frac{24}{x^4}\,dx = 1$.

$$\int_c^{k} \frac{24}{x^4}dx = \left[-8x^{-3}\right]_c^k = (-8k^{-3}) - (-8c^{-3}) = 1 \text{ as } k \to \infty.$$

As $k \to \infty$, $(-8k^{-3}) \to 0$, so now we have $\frac{8}{c^3} = 1 \Rightarrow c^3 = 8$ & thus $c = 2$.

So, lower bound that makes $f(x)$ a valid pdf is $c = 2$.

b) Quantile function $Q(p)$ is inverse of CDF, s.t. $F(Q(p)) = p$ for $0<p<1$.

First, find CDF. CDF found by integrating PDF. $F(x) = \int_2^x \frac{24}{t^4}dt = \left[-\frac{8}{t^3}\right]_2^x = -\frac{8}{x^3} + \frac{8}{2^3} = -\frac{8}{x^3} + 1$

To find $Q(p)$, solve for x in $F(x) = p$, so $-\frac{8}{x^3} + 1 = p \Rightarrow x^3 = \frac{8}{1-p} \Rightarrow x = \sqrt[3]{\frac{8}{1-p}} = \frac{2}{(1-p)^{1/3}}$

Thus $Q(p) = \frac{2}{(1-p)^{1/3}}$

Q4) $E(\bar{A}) = E\left(\frac{1}{10}\sum_{i=1}^{10} X_i\right) = E\left(\frac{X_1 + \dots + X_{10}}{10}\right) = \frac{1}{10}\left[E(X_1) + \dots + E(X_{10})\right] = \frac{20}{10} = 2$.

$V(\bar{A}) = V\left(\frac{X_1 + \dots + X_{10}}{10}\right) = \frac{1}{100}\left[V(X_1) + \dots + V(X_{10})\right] = \frac{160}{100} = 1.6$

want to find $P(\bar{A} > 0)$ in a normal distribution with mean $= 2$, sd $= \sqrt{1.6}$

Z-score $= \frac{0-2}{\sqrt{1.6}}$, So $P(\bar{A} > 0) = 1 - P(\bar{A} \leq 0) = 1 - \text{pnorm}(\text{Z-score})$
$= 0.943$ (From R studio).

$E(\bar{B}) = E\left(\frac{1}{10}\sum_{i=1}^{5} 2X_i\right) = E\left(\frac{2X_1 + 2X_2 + \dots + 2X_5}{10}\right) = \frac{1}{10}\left[E(2X_1) + \dots + E(2X_5)\right] = \frac{1}{10}\left[2E(X_1) + \dots + 2E(X_5)\right] = \frac{20}{10} = 2$.

$V(\bar{B}) = V\left(\frac{2X_1 + \dots + 2X_5}{10}\right) = \frac{1}{100}\left[V(2X_1) + \dots + V(2X_5)\right] = \frac{1}{100}\left[4V(X_1) + \dots + 4V(X_5)\right] = \frac{5 \times 64}{100} = 3.2$, sd $= \sqrt{3.2}$

Z-score $= \frac{0-2}{\sqrt{3.2}}$, so $P(\bar{B} > 0) = 1 - P(\bar{B} \leq 0) = 1 - \text{pnorm}(\text{Z-score})$
$= 0.868$ (From R studio).

$E(\bar{C}) = E\left(\frac{1}{10} \cdot 10X_1\right) = E(X_1) = 2$

$V(\bar{C}) = V\left(\frac{1}{10} \cdot 10X_1\right) = V(X_1) = 16$, sd $= 4$

Z-score $= \frac{0-2}{4}$, so $P(\bar{C} > 0) = 1 - P(\bar{C} \leq 0) = 1 - \text{pnorm}(\text{z-score})$
$= 0.691$ (From R studio).

Q5) Since the normal distribution is symmetric about its mean (0 in this qn), $E[|Z|] = \int_{-\infty}^{\infty} |z| f(z)\,dz = 2\int_0^{\infty} z \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} dz$

given $f(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$, with $\sigma^2 = 1$, $\mu = 0$.

Then $E[|Z|] = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}} = \frac{-2}{\sqrt{2\pi}} \int_0^{\infty} -z e^{-\frac{z^2}{2}} = -\sqrt{\frac{2}{\pi}}\left[e^{-\frac{z^2}{2}}\right]_0^{\infty} = -\sqrt{\frac{2}{\pi}}(0-1) = \sqrt{\frac{2}{\pi}}$.

Thus, $E[|Z|] = \sqrt{\frac{2}{\pi}}$