Q1a) Let X be the random variable representing the number of trials needed to open the door.

i) Each key marked if unsuccessful trial.

k	1	2	3	4	5
$P(X=k)$	$\frac{1}{5}$	$\frac{4}{5}\times\frac{1}{4}$ $=\frac{1}{5}$	$\frac{4}{5}\times\frac{3}{4}\times\frac{1}{3}$ $=\frac{1}{5}$	$\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}$ $=\frac{1}{5}$	$\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}$ $=\frac{1}{5}$

so $P(X=k)=\frac{1}{5}$ for $k=1,2,3,4,5$.

ii) Equally likely to choose any key at each trial

- Each trial has equal success of $1/5$, unsuccessful trial probability of $4/5$.

Since keys are not marked, there is "replacement", all keys equally likely to be chosen again.

so, $P(X=k)=\left(\frac{4}{5}\right)^{k-1}\left(\frac{1}{5}\right)$ for $k=1,2,3,\dots$

b) Repeat (a) with extra duplicate key for each 5 doors.

i) Each key marked if unsuccessful.

k	1	2	3	4	5	6	7	8
$P(X=k)$	$\frac{2}{10}$ $=\frac{1}{5}$	$\frac{8}{10}\times\frac{2}{9}$ $=\frac{8}{45}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{2}{8}$ $=\frac{7}{45}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{2}{7}$ $=\frac{2}{15}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{2}{6}$ $=\frac{1}{9}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{2}{5}$ $=\frac{4}{45}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}$ $=\frac{1}{15}$	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}$ $=\frac{2}{45}$

k	9	10
$P(X=k)$	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2}$ $=\frac{1}{45}$	0 (since door will definitely be unlocked by 9th trial, worst case by 9th trial will be left with 2 duplicate keys, both for opening door).

so $$P(X=k)=\frac{(10-k)}{45} \quad \text{for } k=1,2,\dots,10.$$

ii) Equally likely to choose any key at each trial.

- Probability of choosing a correct key at any trial is still $\frac{2}{10}=\frac{1}{5}$, and unsuccessful trial still has a probability of $\frac{8}{10}=\frac{4}{5}$. As keys are not marked and "replaced", it follows the same geometric distribution.

$P(X=k)=\left(\frac{4}{5}\right)^{k-1}\left(\frac{1}{5}\right)$ for $k=1,2,3,\dots$

Q2a) $X \sim Binom(8,p)$ is a binomial distribution with 8 trials, each with success probability p.

so, pmf of X, $P(X=k)$, where k is the number of successes is

$$P(X=k) = \binom{n}{k} (p^k) (1-p)^{n-k}$$ (ie choose # of successes then find probability of successes & unsuccessful trials)

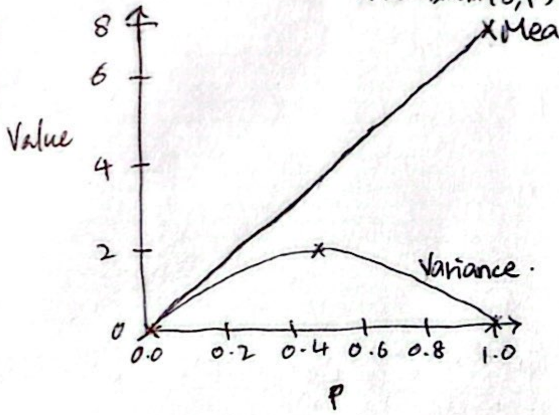
b) For a binomial distribution $X \sim Binom(n,p)$, the mean $\mu = np$, and the variance $\sigma^2 = np(1-p)$

Given $n=8$,

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
μ	0	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2	8.0
σ^2	0	$8\times0.1\times0.9$ $=0.72$	$8\times0.2\times0.8$ $=1.28$	$8\times0.3\times0.7$ $=1.68$	$8\times0.4\times0.6$ $=1.92$	$8\times0.5\times0.5$ $=2.0$	1.92	1.68	1.28	0.72	0

Mean & Variance of $X \sim Binom(8,p)$

c)



d) The mean of X is minimized at $p=0$ and maximized at $p=1$.

The variance of X is minimized at $p=0$ & $p=1$ and maximized at $p=0.5$.

Q3)

R	V	$p(R)$
1	$\frac{4}{3}\pi(1^3) = \frac{4}{3}\pi$	$1/6$
2	$\frac{4}{3}\pi(2^3) = \frac{32}{3}\pi$	$1/6$
3	$\frac{4}{3}\pi(3^3) = 36\pi$	$1/6$
4	$\frac{4}{3}\pi(4^3) = \frac{256}{3}\pi$	$1/6$
5	$\frac{4}{3}\pi(5^3) = \frac{500}{3}\pi$	$1/6$
6	$\frac{4}{3}\pi(6^3) = \frac{864}{3}\pi$	$1/6$

$$E(V) = \sum_{r=1}^{6} \frac{4}{3}\pi r^3 \, p(r)$$

$$= \frac{4}{18}\pi + \frac{32}{18}\pi + 6\pi + \frac{256}{18}\pi + \frac{500}{18}\pi + \frac{864}{18}\pi$$

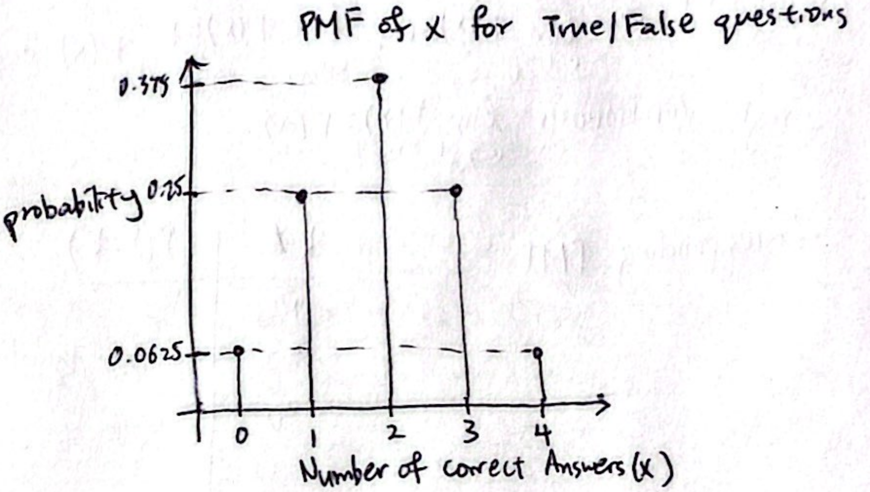
$$= 98\pi$$

Q4a) X follows a Binomial Distribution.

1. Binary outcomes: Only 2 possible outcomes (True/False) for each trial/question
2. Independent trials: Each trial (question) is independent of the others.
3. Number of trials: There is a fixed number of trials/questions, ~~n~~ 4.
4. Success rate: Probability of success (getting a question correct) is constant, for each trial, $p = 0.5$, since there are only True/False options.

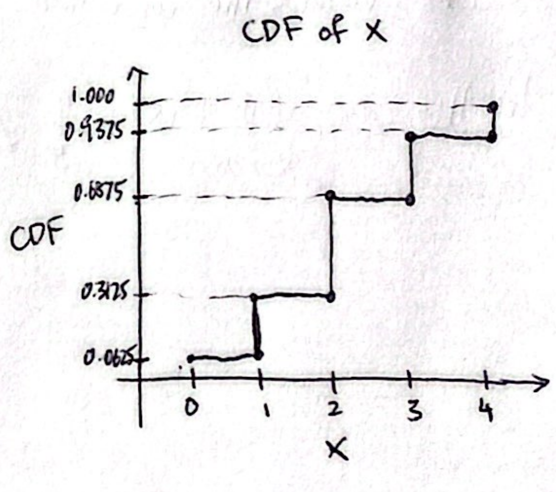
b)

X	$P(X)$
0	$(\frac{1}{2})^4 = \frac{1}{16} = 0.0625$
1	$(\frac{1}{2})(\frac{1}{2})^3 \times \binom{4}{1} = \frac{1}{4} = 0.25$
2	$(\frac{1}{2})^2(\frac{1}{2})^2 \times \binom{4}{2} = \frac{6}{16} = 0.375$
3	$(\frac{1}{2})^3(\frac{1}{2}) \times \binom{4}{3} = \frac{1}{4} = 0.25$
4	$(\frac{1}{2})^4 = \frac{1}{16} = 0.0625$



c)

X	CDF
0	0.0625
1	$0.3125 = 0.0625 + 0.25$
2	$0.6875 = 0.3125 + 0.375$
3	$0.9375 = 0.6875 + 0.25$
4	$1.00 = 0.9375 + 0.0625$



d) $P(X \geq 3) = 1 - P(X \leq 2)$

$= 1 - 0.6875 = 0.3125$

Q5 a) Not a valid CDF. Violates the condition where $\lim_{x \to -\infty} F(x) = 0$.

In this case, ~~F(x)~~ $\lim_{x \to -\infty} F(x) = -\frac{1}{3} \neq 0$.

b) Not a valid CDF. Violates the condition that $F(x)$ is nondecreasing.

In this case, function decreases from $\frac{6}{7}$ to $\frac{4}{7}$ ~~as~~ from $0 \le x < 5$ where $F(x) = \frac{6}{7}$ to $5 \le x < 10$ where $F(x) = \frac{4}{7}$

c) Valid CDF as $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$, $F(x)$ is nondecreasing, and $F(x)$ is right-continuous, $\lim_{x \to a^+} F(x) = F(a)$.

Corresponding PMF:

k	$P(X=k)$
10	$\frac{1}{3}$
20	$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$
30	$1 - \frac{2}{3} = \frac{1}{3}$.

d) Not a valid CDF. Violates the condition $\lim_{x \to \infty} F(x) = 1$.

In this case, $\lim_{x \to \infty} F(x) = \frac{4}{5} \neq 1$