Q1a) $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ (Use integration by parts)

$u = x^{\alpha-1}$ $v' = e^{-x}$

$u' = (\alpha-1)x^{\alpha-2}$ $v = -e^{-x}$

$= \left[-x^{\alpha-1}e^{-x}\right]_0^\infty - \int_0^\infty (\alpha-1)x^{\alpha-2}(-e^{-x})dx$

$= [0-0] + \int_0^\infty (\alpha-1)x^{\alpha-2}e^{-x}dx$

$= (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx$

$= (\alpha-1)\Gamma(\alpha-1)$

~~With integration by parts~~

b) $\Gamma(\frac{1}{2}) = \int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$

Let $y = \sqrt{x}$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,dy$

$\frac{dx}{\sqrt{x}} = 2dy$

$= \int_0^\infty (e^{-y^2})(2)(dy)$

$= 2\int_0^\infty e^{-y^2}dy = \int_{-\infty}^\infty e^{-y^2}dy = \sqrt{\pi}$.

Q2a) This is a gamma distribution with $\alpha=2$ and $\lambda=3$.

Mean of gamma $= \frac{\alpha}{\lambda} = \frac{2}{3}$ Variance of gamma $= \frac{\alpha}{\lambda^2} = \frac{2}{3^2} = \frac{2}{9}$

b) $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i = \frac{1}{n}(X_1+X_2+\dots+X_n)$

Since X_i are iid gamma RVs, $\bar{X}$ will be a gamma distribution as well.

α ~~Mean~~ of $X_1+X_2+\dots+X_n = \alpha_1+\alpha_2+\dots+\alpha_n = n(\alpha) = 2n$.

α ~~Mean~~ of $\frac{1}{n}(X_1+X_2+\dots+X_n) = \alpha$ ~~mean~~ of $X_1+X_2+\dots+X_n = 2n$.

λ ~~Variance~~ of $X_1+X_2+\dots+X_n = \lambda$.

λ ~~Variance~~ of $\frac{1}{n}(X_1+X_2+\dots+X_n)$

$= \frac{\lambda}{\frac{1}{n}} = n\lambda = 3n$

So, $\bar{X} \sim \text{Gamma}(2n, 3n)$

Thus, Mean of $\bar{X} = \frac{2n}{3n} = \frac{2}{3}$

Variance of $\bar{X} = \frac{2n}{(3n)^2} = \frac{2}{9n}$

Q3) $D \sim \text{Exp}(\lambda)$. Given Vol of raindrop $= \frac{4}{3}\pi r^3$ where $r = D$, $V = \frac{4}{3}\pi D^3$. Thus rainfall Intensity, I,

$I = V \times k = \frac{4}{3}\pi D^3 k$. To find $E(I)$, we find $E(\frac{4}{3}\pi D^3 k) = (\frac{4}{3}\pi k) \times (E(D^3))$

$E(D^3) = \int_{-\infty}^\infty d^3 f(d)\,dd = \int_0^\infty d^3\lambda e^{-\lambda d}dd = \frac{1}{\lambda^3}\int_0^\infty t^3 e^{-t}dt$ (with $t = \lambda d$) $= \frac{1}{\lambda^3}\Gamma(3+1) = \frac{3!}{\lambda^3}$.

So, $E(D^3) = \frac{3!}{\lambda^3}$, $E[I] = \frac{4}{3}\pi k \cdot \frac{3!}{\lambda^3} = \frac{8\pi k}{\lambda^3}$.

Q4) I will do this for discrete variables. For RV. Z, $E(Z) = \sum_{z} z \cdot P(Z=z)$.

Now for 2 discrete RVs X & Y, $E(X+Y) = \sum_{x,y} (x+y) \cdot P(X=x, Y=y)$

This is the sum of all possible combinations of x & y values that X & Y can take. The probability $P(X=x, Y=y)$ is the joint probability that $X=x$ & $Y=y$ simultaneously, so $E(X+Y) = \sum_{x}\sum_{y} x \cdot P(X=x, Y=y) + \sum_{x}\sum_{y} y \cdot P(X=x, Y=y)$.

Since $P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$ for independent X & Y, and sum over y for $P(Y=y)$ is 1, vice versa for x, we can simplify the expression.

$$E(X+Y) = \sum_{x} x \cdot P(X=x) \cdot \sum_{y} P(Y=y) + \sum_{Y} y \cdot P(Y=y) \cdot \sum_{x} P(X=x)$$

$$= \sum_{x} x \cdot P(X=x) + \sum_{Y} y \cdot P(Y=y) = E(X) + E(Y).$$

Q5 a) We know integral under curve has to be 1.

so, $\int_0^1 \int_0^1 cxy(1-x)\, dy\, dx = 1$

$$\int_0^1 \left[cx(\tfrac{1}{2}y^2)(1-x)\right]_0^1 dx = \int_0^1 \left(\tfrac{1}{2}cx(1-x)\right) - 0 \; dx$$

$$= \tfrac{1}{2}\int_0^1 cx - cx^2 \; dx$$

$$= \tfrac{1}{2}c\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1$$

$$= \tfrac{1}{2}c\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \frac{c}{12} = 1 \Rightarrow c = 12.$$

b) $f_X(x) = \int_y f_{X,Y}(x,y)\, dx\, dy$

$$= \int_0^1 12xy(1-x)\, dy = \left[12x(\tfrac{1}{2}y^2)(1-x)\right]_0^1 = 6x(1-x) \quad \text{for} \quad 0 \leq x \leq 1.$$

so $E(X) = \int_0^1 x f(x)\, dx = 6\int_0^1 x^2(1-x)\, dx = 6\int_0^1 x^2 - x^3\, dx = 6\left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1$

$$= 6\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \tfrac{1}{2}$$

$E(X^2) = \int_0^1 x^2 f(x)\, dx = 6\int_0^1 x^3(1-x)\, dx = 6\int_0^1 x^3 - x^4\, dx = 6\left[\tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^1 = 6\left(\tfrac{1}{4} - \tfrac{1}{5}\right) = \tfrac{3}{10}$.

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \tfrac{3}{10} - \left(\tfrac{1}{2}\right)^2 = \tfrac{3}{10} - \tfrac{1}{4} = \tfrac{6}{20} - \tfrac{5}{20} = \tfrac{1}{20}$.

c) $E(Z) = \int_0^1\int_0^1 xy \cdot f_{XY}(x,y)\, dy\, dx = \int_0^1\int_0^1 xy \cdot 12xy(1-x)\, dy\, dx = \int_0^1\int_0^1 12x^2y^2 - 12x^3y^2\, dy\, dx$

$$= \int_0^1 \left[4x^2y^3 - 4x^3y^3\right]_0^1 dx$$

$$= \int_0^1 4x^2 - 4x^3\, dx$$

$$= \left[\tfrac{4}{3}x^3 - x^4\right]_0^1 = \tfrac{4}{3} - 1 = \tfrac{1}{3}$$