Probability Fundamentals

The basic tools

Probability is the mathematical study of randomness.

You need \$100, but you only have \$50. Suppose you want to try betting on roulette to double your money. Should you bet it all at once, or place several smaller bets?

You have a box of 24 chocolates, 4 each of 6 different flavors. You offer your brother 3 random chocolates, and he pulls out 3 mint truffles (his favorite). Do you think he was really picking randomly?

Probability is mathematical framework for answering these problems (as well as more serious ones).

Probability makes use of **set theory**.

A **set** is a collection, written as a pair of curly braces with elements separated by commas, e.g.

$$\{\Diamond, \triangle, \Box, \heartsuit\}$$

If an element is a member of a set, the $\in$ symbol is used, e.g.

$$b \in \{a, b, c, d, e\}$$

If all of the members of set A are members of set B, then A is a **subset** of B, denoted $\subseteq$.

$$\{1, 4\} \subseteq \{1, 2, 3, 4\}$$

Any process whose outcome is uncertain (i.e. probabilistic) is called a **random experiment**.

- Rolling a die
- Testing whether a patient has a disease
- Whether the value of a stock increases or decreases
- ...

The result of a random experiment is called an **outcome**. The set of all outcomes form an **outcome space**, denoted Ω.

Toss a coin three times. The outcome space is

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Roll two six-sided dice and add the results. The outcome space is

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

An **event** is a subset of the outcome space, denoted with uppercase letters (A, B,...). For the outcome space of flipping a coin 3 times:

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

One event could be getting at least two heads:

$$A = \{HHH, HHT, HTH, THH\}$$

The empty set $\emptyset$ denotes no possible outcomes.

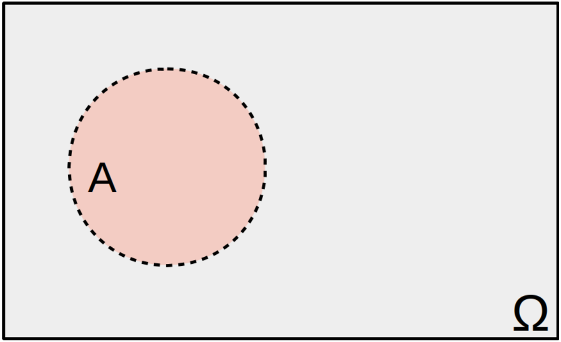
Consider an experiment where three vehicles on a freeway exit ramp either turn right (R) or left (L). Write the following events in set notation:

- *A*: Exactly one of the three vehicles turns right. {RLL, LRL, LLR}
- *B*: At most one of the vehicles turns right. {RLL, LRL, LLR, LLL}
- *C*: All three vehicles turn in the same direction. {LLL, RRR}

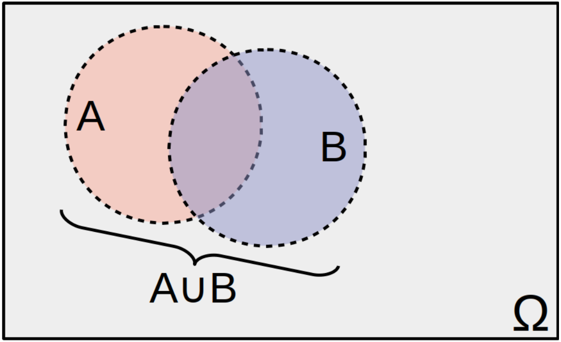
Suppose that when the experiment is performed, the outcome is LLL. Which of the above events have occurred?

A, B

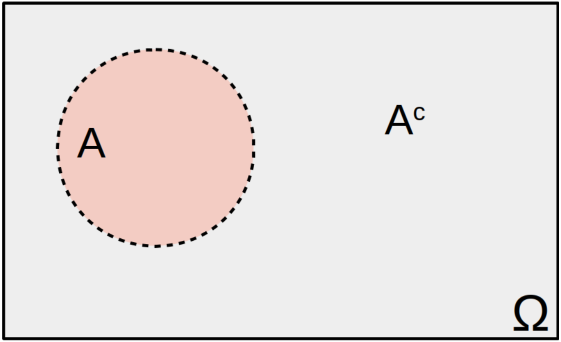
There are more complicated events we can define in terms of other events. Venn diagrams are a useful tool here.



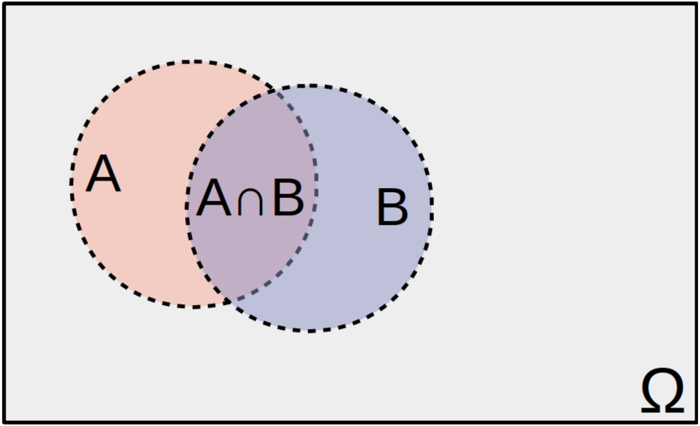
Let A and B be two events. The **union** of A and B, written $A \cup B$, is the set of all outcomes in either A or B (the event that A or B happens, or both).



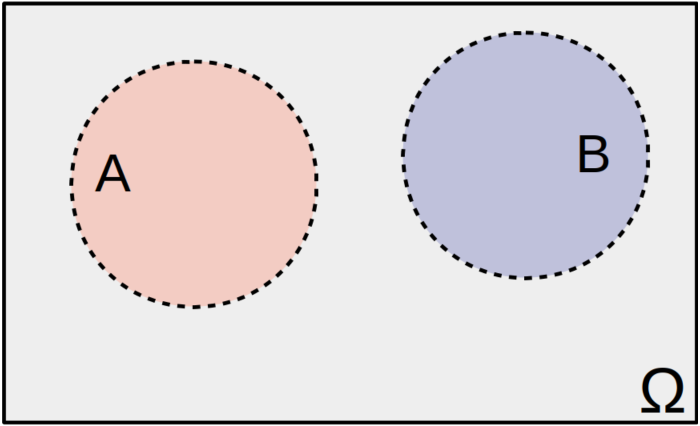
The **complement** of A, written A^c, is the set of all outcomes not in A (the event that A does not happen).



The **intersection** of A and B, written $A \cap B$, is the set of all outcomes in both A and B (the event that both A and B happen).



A and B are said to be **mutually exclusive/disjoint** if they have no outcomes in common. It is impossible for both to happen.



Let's randomly pick a number from 1-10.

$$\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Let

$$A = \{2, 4, 6, 8, 10\} \quad \text{(even)}$$
$$B = \{1, 2, 3, 4, 5\} \quad \leq 5$$
$$C = \{6, 7, 8, 9, 10\} \quad \geq 6$$

- What is A^c? A^c = {1,3,5,7,9) (odd)
- What is $A \cup B$? A U B = {1,2,3,4,5,6,8,10}
- What is $A \cap C$? $(B \cap C)^c$? A n C = {6,8,10} (B n C)^c = omega
- What are the probabilities of these events?

In probability, our goal is to find $\mathbb{P}(A)$, the **probability that event A occurs**. Return to the coin example:

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

What is the probability of $A =$ getting at least two heads?
If the coin is fair, all eight outcomes in Ω are equally likely.

Four of those outcomes include at least two heads, so $\mathbb{P}(A) = \frac{4}{8} = \frac{1}{2}$.

Let Ω be any finite outcome space whose outcomes are equally likely. The probability of A is just

$$\mathbb{P}(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } \Omega}$$

When outcomes are equally likely, calculating probabilities is straightforward. But some outcome spaces are more complicated.

Consider the sum of 2 dice example:

$$\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Some outcome spaces are infinite. For example, flip a coin until you land heads. How many flips does it take?

$$\{1, 2, 3, \ldots\}$$

A **probability distribution** generalizes the intuition of probability to these more difficult cases.

A probability distribution $\mathbb{P}$ maps each event of an outcome space to a probability in $[0,1]$. It assigns each event a probability.

Probability distributions follow three rules:

- **Range:** $0 \leq \mathbb{P}(A) \leq 1$ for all A
- **Total one:** $\mathbb{P}(\Omega) = 1$
- **Additivity:** If $A_1, A_2, \ldots$ are disjoint sets, then

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \mathbb{P}(A_1 \cup A_2 \cup \cdots)$$

We also get $\mathbb{P}(\emptyset) = 0$.

Example: Let $\mathbb{P}$ be the probability distribution for weather.

$$\Omega = \{Sunny, Cloudy, Rainy, Snowy\}$$

$$\mathbb{P}(\{Sunny\}) = 0.3 \quad \mathbb{P}(\{Cloudy\}) = 0.3$$

$$\mathbb{P}(\{Rainy\}) = 0.25 \quad \mathbb{P}(\{Snowy\}) = 0.15$$

Non-example: Let $\mathbb{P}(n) = 0.01$ for all $n \in \mathbb{N}$.
What goes wrong?

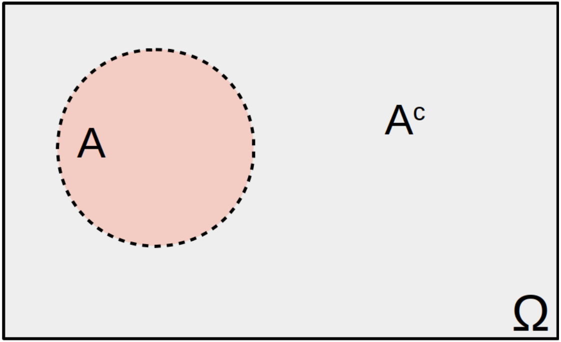
Let $\Omega = \{1, 2, 3, 4\}$. Which of the following are valid probability distributions?

1. $\mathbb{P}(\{x\}) = 0.25$ for all $x \in \Omega$ true
2. $\mathbb{P}(\{1\}) = -0.5$, $\mathbb{P}(\{x\}) = 0.5$ for $x = 2, 3, 4$
3. $\mathbb{P}(\{2\}) = 1, \mathbb{P}(\{x\}) = 0$ for $x = 1$, 3, 4
4. $\mathbb{P}(\{2, 3\}) = \mathbb{P}(\{1, 4\}) = 0.5$

What is one way (4) can be a valid probability distribution? Or a non-valid probability distribution?

Now, let's consider some useful rules of probability distributions.

If we know $\mathbb{P}(A)$, then we should know $\mathbb{P}(A^c)$.



Complement rule: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. Why is this true?

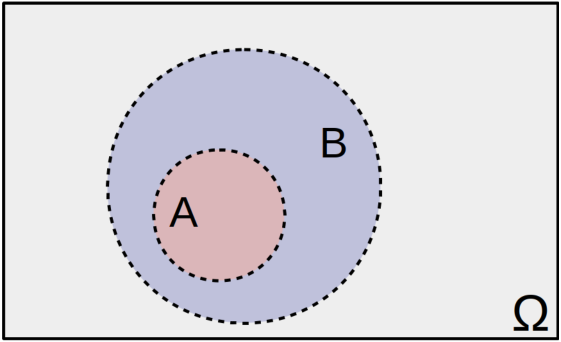
In roulette, the ball can land on either red, black, or green. If $\mathbb{P}(\text{red}) = 0.49$, then $\mathbb{P}(\text{black} \cup \text{green}) = 1 - 0.49 = 0.51$.

In a certain population, 20% of people take both the bus and the train. So, 80% people either don't take the bus, or don't take the train.

Fred has a very analytic mind. He loves patterns and problem-solving, and has a strong grasp of abstract concepts.

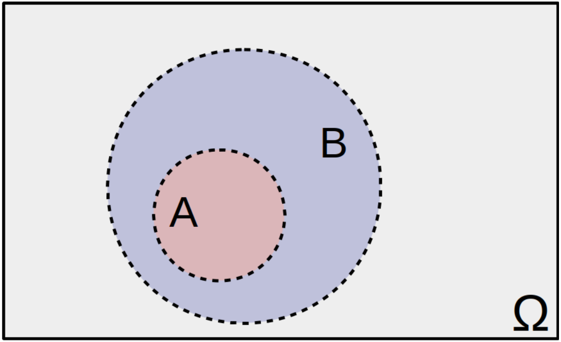
What is more likely to be true?

1. Fred is a student at UC Berkeley.
2. Fred is a math major at UC Berkeley.



Difference rule: if $A \subseteq B$ (A implies B),

- $\mathbb{P}(A) \leq \mathbb{P}(B)$
- $\mathbb{P}(A \cap B) = \mathbb{P}(A)$
- What about $\mathbb{P}(A \cup B)$?



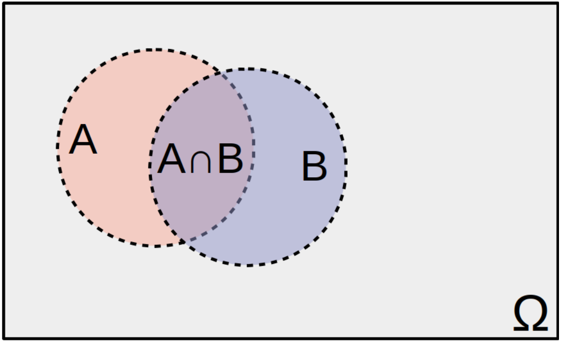
If $A \subseteq B$ we also have:

$$\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A)$$

(B happens but A does not).

One common pitfall is when calculating the probability of the union of events.

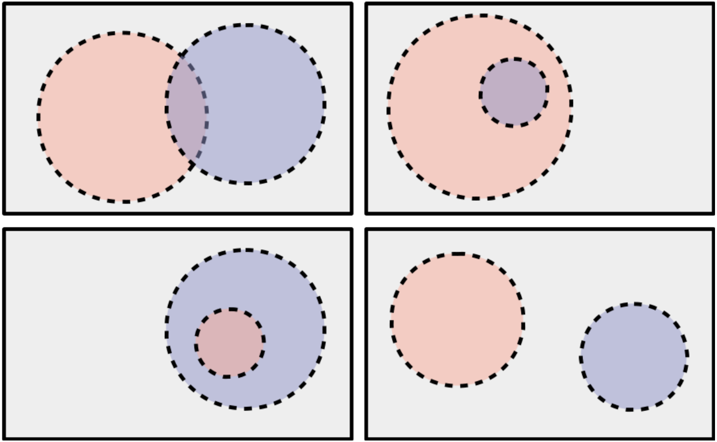
Let $\mathbb{P}(A) = 0.4$ and $\mathbb{P}(B) = 0.3$. What is $\mathbb{P}(A \cup B)$?



For any two events A and B,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

This is the **inclusion-exclusion rule**.

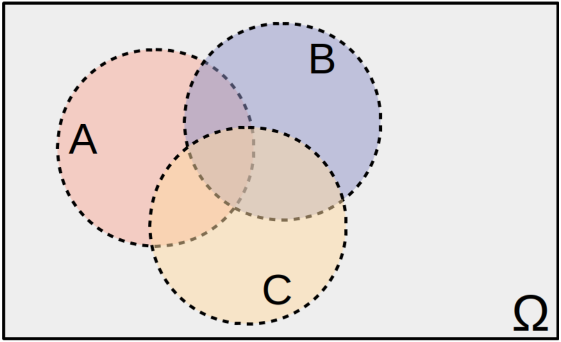


$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

At a diner, 30% of customers like to have cream with their coffee. 50% like to have either cream or sugar with their coffee, and 5% like to have both cream and sugar.

What percent of customers like to have sugar with their coffee?

We know that it works for 2 events, but what about more events?



$$\mathbb{P}(A \cup B \cup C) = ?$$

Consider events A, B, and C with the following probabilities:

$$\mathbb{P}(A) = 0.2 \quad \mathbb{P}(B) = 0.4 \quad \mathbb{P}(C) = 0.1$$

$$\mathbb{P}(A \cap B) = 0.05 \quad \mathbb{P}(C \cap B) = 0.1$$

Find $\mathbb{P}((A \cup B \cup C)^c)$.