1. **Solution 1:** Suppose that we place each square sequentially, so that the bottom left square is placed first. We place $n-1$ additional squares, moving up and right by the same amount at each step. The final square is placed 1 unit up and 1 unit to the right of the initial square, so each new square is placed $\frac{1}{n-1}$ units up and $\frac{1}{n-1}$ units to the right of the previous square.

   Therefore, when we place each new square, the area of overlap with the previous square is a square with side length $2-\frac{1}{n-1}$, so the newly added area is

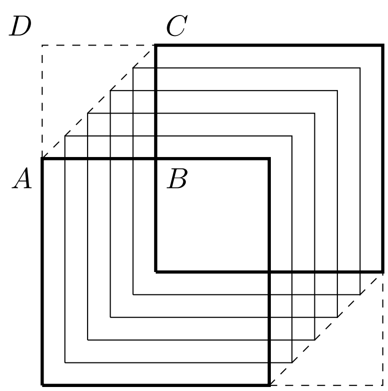
$$2^2-\left(2-\frac{1}{n-1}\right)^2=\left(4-\frac{1}{n-1}\right)\cdot\frac{1}{n-1}.$$

   Since there are $n-1$ new squares added, the total area of the region is

$$2^2+(n-1)\cdot\left(\left(4-\frac{1}{n-1}\right)\cdot\frac{1}{n-1}\right)=8-\frac{1}{n-1}.$$

   Thus we wish to find the smallest positive integer $n\geq 2$ such that $8-\frac{1}{n-1}\geq\sqrt{63}$. This is equivalent to $8-\sqrt{63}\geq\frac{1}{n-1}$, so $n-1\geq\frac{1}{8-\sqrt{63}}$. Multiplying by $\frac{8+\sqrt{63}}{8+\sqrt{63}}$, we find $n-1\geq 8+\sqrt{63}$. Hence $n\geq 9+\sqrt{63}$. The smallest such integer n is $\boxed{17}$.

   Solution 2: First, we extend the sides of the first and last squares, which shows that the region can be contained in a square of side length 3. Let $A=(-1,1)$, $B=(0,1)$, $C=(0,2)$, and $D=(-1,2)$. The top-left vertices of the 2×2 squares will all lie on $\overline{AC}$ as shown below.



   To find the area covered by the figure, we start with the 3×3 square with area $3\cdot 3=9$. From this, we subtract the area of $\triangle ACD$ and its counterpart in the lower right hand corner, obtaining an area of $9-2\left(\frac{1\cdot 1}{2}\right)=8$. In $\triangle ABC$, we must further subtract $n-1$ small congruent isosceles right triangles. Since their leg lengths add up to 1, the leg length of each of these excluded triangles is $\frac{1}{n-1}$. The areas of these small triangles sums to $(n-1)\cdot\frac{(1/(n-1))^2}{2}=\frac{1}{2(n-1)}$. Since we must subtract the same amount in for the region in the lower right hand corner, it follows that the total area covered by the paper is $8-2\cdot\frac{1}{2(n-1)}=8-\frac{1}{n-1}$. As in Solution 1, the smallest integer n for which $8-\frac{1}{n-1}>\sqrt{63}$ is $\boxed{17}$.

2. **Solution 1:** The problem is equivalent to showing there are no primes p dividing both x^2+xy+y^2 and $x^2+3xy+y^2$. If p is such a prime, then p divides their difference, which is $2xy$. Hence either $p=2$, p divides x, or p divides y. Without loss of generality, either $p=2$ or p divides x.

   - If $p=2$, then x^2+xy+y^2 must be even. However, since x and y are coprime, they cannot both be even. If both x and y are odd, then x^2+xy+y^2 is odd, and if one of x, y is even and the other is odd, then x^2+xy+y^2 is also odd. Therefore, it is impossible for x^2+xy+y^2 to be even, contradiction.
   - If p divides x, then since p divides x^2+xy+y^2 and x^2+xy, p divides the difference y^2. Since p is prime, it follows that p divides y. But this contradicts the fact that x and y are coprime.

   Since both cases lead to a contradiction, no such prime p exists.

   Solution 2: We need to show that the greatest common divisor of x^2+xy+y^2 and $x^2+3xy+y^2$ is 1, given that $\gcd(x,y)=1$. To do this, we use the following well-known rules for computing the greatest common divisor:

   (S) Symmetry: $\gcd(a,b)=\gcd(b,a)$
   (E) Euclidean algorithm: $\gcd(a,b)=\gcd(a,b-ka)$ for any $k\in\mathbb{Z}$
   (P) Product rule: $\gcd(a,bc)\mid\gcd(a,b)\cdot\gcd(a,c)$

   Applying these rules, we find:

$$\begin{aligned}
&\gcd(x^2+xy+y^2, x^2+3xy+y^2) \\
&\quad= \gcd(x^2+xy+y^2, 2xy) && \text{by (E)}\\
&\quad\mid\ \gcd(x^2+xy+y^2,2)\cdot\gcd(x^2+xy+y^2,x)\cdot\gcd(x^2+xy+y^2,y) && \text{by (P)}\\
&\quad= \gcd(x^2+xy+y^2,2)\cdot\gcd(x,x^2+xy+y^2)\cdot\gcd(y,x^2+xy+y^2) && \text{by (S)}\\
&\quad= \gcd(x^2+xy+y^2,2)\cdot\gcd(x,y^2)\cdot\gcd(y,x^2) && \text{by (E)}\\
&\quad\mid\ \gcd(x^2+xy+y^2,2)\cdot\gcd(x,y)^2\cdot\gcd(y,x)^2 && \text{by (P)}\\
&\quad= \gcd(x^2+xy+y^2,2)\cdot\gcd(x,y)^4 && \text{by (S)}\\
&\quad= \gcd(x^2+xy+y^2,2) && \text{since } \gcd(x,y)=1\\
&\quad= 1,
\end{aligned}$$

   because x^2+xy+y^2 is always odd for coprime x, y. (This can be seen by cases on whether x and y are odd or even.) Since the gcd of the two polynomials divides 1, it must be 1.

3. Multiplying the second equation by 2 and adding m^2n^2 to both sides, we find

$$2a_{mn}+(mn)^2=(2a_m+m^2)(2a_n+n^2).$$

   Therefore, define $b_n=2a_n+n^2$, so the second equation can be written as $b_{mn}=b_mb_n$. Multiplying the first equation by 2 and adding $(m+n)^2$, we find

$$2a_{m+n}+(m+n)^2=(2a_m+m^2)+(2a_n+n^2).$$

Therefore, $b_{m+n} = b_m + b_n$. Therefore, it suffices to find sequences $b_1, b_2, b_3, \ldots$ that satisfy both $b_{mn} = b_m b_n$ and $b_{m+n} = b_m + b_n$.

We claim that if $b_{m+n} = b_m + b_n$ for all positive integers m, n, then $b_n = nb_1$. Certainly this is true for $n = 1$. If $b_k = kb_1$, then $b_{k+1} = b_k + b_1 = kb_1 + b_1 = (k+1)b_1$, so by induction, it follows that $b_n = nb_1$ for all positive integers n. Additionally, we observe that any sequence of this form will satisfy $b_{m+n} = b_m + b_n$.

Substituting $m = n = 1$ into $b_{mn} = b_m b_n$, we find $b_1 = b_1^2$, so $b_1 = 0$ or $b_1 = 1$. Combining this with the previous paragraph, the only possible sequences are $b_n = 0$ or $b_n = n$. Indeed, both sequences satisfy both $b_{m+n} = b_m + b_n$ and $b_{mn} = b_m b_n$.

Therefore, since $a_n = \frac{b_n - n^2}{2}$, it follows that the two possible sequences are $a_n = -\frac{n^2}{2}$ or $a_n = \frac{n-n^2}{2}$.

4. We claim that Alpha has a winning strategy. To prove this, define a new game where the position denoted (m, n) corresponds to an $(m-1) \times (n-1)$ grid of squares in the original game, so that the starting position is $(11, 101)$. That is, we add one to the two coordinates' names, but the actual moves are the same. Then in the new game, a move $(m, n) \to (m', n)$ is legal if and only if $\frac{m}{2} < m' < m$, and similarly for a move $(m, n) \to (m, n')$. This is because folding the paper either reduces the width by at most half, or the length by at most half, which corresponds to reducing the coordinates in the new game by strictly less than half.

   Now we claim that in the new game, the losing positions (where the second player wins) are exactly the positions of the form $(a, 2^k a)$ or $(2^k a, a)$, for some $a \geq 2$ and $k \geq 0$ – that is, positions where the ratio is a power of two. For example, $(10, 20)$, $(10, 40)$, and $(10, 80)$ are losing, while $(10, 30)$ is winning.

   First, we describe a winning strategy for player 2 in such situations. Given a position $(a, 2^k a)$, there are two cases for the first player's move: either she moves to $(a', 2^k a)$ (where $a/2 < a' < a$), or she moves to (a, j) (where $2^{k-1}a < j < 2^k a$). In the first case, we respond by moving to $(a', 2^k a')$, which is legal because $2^k a' > 2^k (a/2) = (2^k a)/2$, and it preserves the form where one coordinate is a power of two times the other. In the second case, we respond by moving to $(a, 2^{k-1}a)$, which is legal because j is an integer, so $j < 2^k a - 1$, so $j/2 < 2^{k-1}a - 1/2$, so $2^{k-1}a > j/2$. Ultimately, such a strategy is possible as long as $a \geq 2$ and $k \geq 1$, and it will continue until the first player is unable to make a move (position $(2, 2)$), at which point the second player wins.

   Now we argue that there is a winning strategy for player 1 in all other positions. Let the position be (a, b), where without loss of generality $a < b$ (note that $a = b$ is a winning position for player 2). Then let $k \geq 0$ be the unique integer such that $2^k < \frac{b}{a} < 2^{k+1}$—we know that such a k exists because the position is not of the form $a, 2^k a$. Then, player 1's strategy is to move to $(a, 2^k a)$. From here, player 1 can win with player 2's strategy from before.

   To complete the proof, since the starting position $(11, 101)$ has a ratio $\frac{101}{11}$, which is not a power of two (not even an integer), it follows that the position is winning, and Alpha has a winning strategy. In particular, Alpha can win by going to the losing position $(11, 88)$. In the original terms, Alpha can win by folding the 10×100 grid into a 10×87 grid.

5. We claim that the maximum possible value of $D(n)$ is 2. To start, we label the lily pads in clockwise order as pads 0, 1, 2, ..., 2021, such that the frog starts on pad 0. Additionally, if $n > 2021$, and $n \equiv r \pmod{2022}$ for some $0 \leq r < 2022$, we will say that pad n is pad r.

   We prove two lemmas.

   Lemma 1: Suppose that $\gcd(k, 2022) = d$. Then the numbers 0, k, $2k$, ..., $(2022/d - 1)k$ are all distinct modulo 2022.

   Proof of Lemma 1: If $j_1 k \equiv j_2 k \pmod{2022}$, then $k(j_1 - j_2) \equiv 0 \pmod{2022}$, so $2022 \mid k(j_1 - j_2)$. Therefore, $(2022/d) \mid (k/d)(j_1 - j_2)$. But k/d and $2022/d$ are relatively prime (otherwise $\gcd(2022, k) > d$). Therefore, $(2022/d) \mid (j_1 - j_2)$. Since $0 \leq j_1, j_2 \leq 2022/d - 1$, it follows that $j_1 = j_2$. Therefore, it is impossible for the list to contain repeated elements. ■

   Lemma 2: Suppose that $\gcd(k, 2022) = d$, the heights of lily pads 0, k, $2k$, ..., $(2022/d - 1)k$ are $k + 1$, and the heights of the remaining pads are k. Also, suppose that the frog is currently on pad 0. Let $\gcd(k + 1, 2022) = e$. Then after a series of jumps, the heights of lily pads 0, $(k + 1)$, $2(k + 1)$, ..., $(2022/e - 1)(k + 1)$ will all be $k + 2$, while the heights of the remaining pads will be $k + 1$, with the frog on pad 0.

   Proof of Lemma 2: At the start, we note that $0 \equiv k \equiv 2k \equiv \cdots \equiv (2022/d - 1)k \equiv 0 \pmod{d}$, so there are $2022/d$ pads with height $k + 1$, each with pad numbers congruent to $0 \pmod{d}$. Since there are $2022/d$ pads with pad numbers congruent to $0 \pmod{d}$, this means that all of the pads congruent to $0 \pmod{d}$ have height $k + 1$.

   If $d = 1$, this amounts to *all* of the pads having height $k + 1$. Therefore, the frog's next several jumps will be to pads $(k + 1)$, $2(k + 1)$, $3(k + 1)$, ..., until it reaches a pad of height $k + 2$. The pads with height $k + 2$ will be the pads it previously visited (i.e., pads 0, $(k + 1)$, $2(k + 1)$, ...). If $e = \gcd(k + 1, 2022)$, then all of these pads will be $0 \pmod{e}$, and by Lemma 1, we know that 0, $(k + 1)$, $2(k + 1)$, ..., $(2022/e - 1)(k + 1)$ will all be distinct modulo 2022. Since there are $2022/e$ pads that are divisible by e, we have covered each multiple of e exactly once. The next pad visited is $(2022/e)(k + 1) \equiv 0 \pmod{2022}$, so the frog returns to pad 0, having increased the heights of pads 0, $(k + 1)$, $2(k + 1)$, ..., $(2022/e - 1)(k + 1)$ to $k + 2$. This proves the conclusion of the lemma in this case.

   Otherwise, if $d > 1$, then the frog first jumps to pad $k + 1$. This pad is congruent to $1 \pmod{d}$, so it has height k. This increases pad 0 to height $k + 2$. Since this pad (and all of the pads with pad numbers congruent to $1 \pmod{d}$) has height k, its next jump will have length k, and as long as it jumps to distinct pads with pad numbers congruent to $1 \pmod{d}$, it will continue making jumps of length k. If all of its jumps have length k, then it visits the pads $(k + 1)$, $(k + 1) + k$, $(k + 1) + 2k$, ..., $(k + 1) + (2022/d - 1)k$. In particular, we note that these pad numbers are all congruent to $1 \pmod{d}$, and by Lemma 1, they must be distinct modulo 2022. Making one more jump, the frog will visit pad $(k + 1) + (2022/d)k \equiv k + 1 \pmod{2022}$ next, and at this point, all of the pads with numbers congruent to $1 \pmod{d}$ will have height $k + 1$.

   This process continues—the frog jumps next to pad $2(k + 1)$, which is congruent to $2 \pmod{d}$, so it has height k. This increases pad $(k + 1)$ to height $k + 2$. A similar argument to the previous paragraph shows that all of the pads with numbers congruent to $2 \pmod{d}$ will have their heights increased to

$k+1$, and then pad $2(k+1)$ will have its height increased to $k+2$. This moves through all of the residue classes modulo d, and at the end of this process, pads 0, $(k+1)$, $2(k+1)$, $\ldots$, $(d-1)(k+1)$ will all have height $k+2$, and the frog will be at pad $d(k+1)$.

Now since $\gcd(k, k+1) = 1$, where $d \mid k$ and $e \mid k+1$, we know that $\gcd(d, e) = 1$. Therefore, d and e are relatively prime factors of 2022, so $de \leq 2022$. Hence $d \leq \frac{2022}{e}$, so $d - 1 \leq \frac{2022}{e} - 1$. In particular, the frog will keep jumping by $k+1$ until it reaches pad $(2022/e - 1)(k+1)$, at which point it jumps one more time (length $k+1$) to arrive at pad $(2022/e)(k+1) \equiv 0 \pmod{2022}$. This is the desired state, so the lemma is proven. ■

At the start, all of the pads have height 1, so the frog jumps clockwise by 1 step, and it continues to do this until it returns to pad 0. At this point, each pad will have height 2. Note that this satisfies the conditions of Lemma 2 with $k = 1$.

Now if we have an arrangement of heights where the frog is at pad 0 and satisfies the initial conditions of Lemma 2, then the arrangement in the conclusion of Lemma 2 also satisfies the conditions of Lemma 2. Therefore, the frog's movements can be completely described by Lemma 2. In one application of Lemma 2, the shortest lily pad has height k, and the tallest lily pad has height $k+2$, so $D(n)$ is less than or equal to $(k+2) - k = 2$ at every point in time during the steps described by Lemma 2. Therefore, $D(n) \leq 2$. In particular, after 2022 steps, all of the lily pads have height 2. Then after 1011 additional steps, the heights of the lily pads are $3, 2, 3, 2, 3, 2, \ldots, 3, 2$ (where the frog is on pad 0). In the frog's next jump, the height of pad 0 increases to 4, so $D(3034) = 4 - 2 = 2$, hence the maximum value of $D(n)$ is 2.

6. To begin, let x_n be the number of fish-friendly $2 \times n$ grids. Let a_n be the number of fish-friendly $2 \times n$ grids where the two squares in the last column are blue, and let b_n be the number of fish-friendly $2 \times n$ grids where only one square in the last column is blue. Since at least one square in the last column of a fish-friendly grid is blue, we find
$$x_n = a_n + b_n.$$
To compute a_n, note that if a fish-friendly $2 \times n$ has both squares colored blue in the last column, then the prior $2 \times (n-1)$ grid can be any fish-friendly grid, hence
$$a_n = a_{n-1} + b_{n-1}. \tag{1}$$
To compute b_n, consider the second-to-last column. If it has both squares colored blue, then the first $2 \times (n-2)$ grid can be any fish-friendly grid and there are two choices for the last column, so there are $2a_{n-1}$ such grids. If it has only one square colored blue, then the last column is forced to be the same as the second-to-last column, and there are b_{n-1} such grids. Thus,
$$b_n = 2a_{n-1} + b_{n-1}. \tag{2}$$
Adding (1) and (2), we find
$$a_n + b_n = 3a_{n-1} + 2b_{n-1}.$$
Hence $x_n = 2x_{n-1} + a_{n-1}$. Since $a_{n-1} = a_{n-2} + b_{n-2} = x_{n-2}$, we deduce that
$$x_n = 2x_{n-1} + x_{n-2},$$

where $x_1 = 3$ and $x_2 = 7$. This is a homogeneous linear recurrence with characteristic equation $\lambda^2 - 2\lambda - 1 = 0$, which has roots $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$. By the theory of homogeneous linear recurrences, there exist constants c_1 and c_2 such that $x_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$. Using the initial conditions $x_1 = 3$ and $x_2 = 7$, we find $3 = c_1(1+\sqrt{2}) + c_2(1-\sqrt{2})$ and $7 = c_1(1+\sqrt{2})^2 + c_2(1-\sqrt{2})^2$. Multiplying the first equation by $(1-\sqrt{2})$ and subtracting it from the second equation, we find $4 + 3\sqrt{2} = (4+2\sqrt{2}) \cdot c_1$. Multiplying by $2 - \sqrt{2}$, we find $2 + 2\sqrt{2} = 4c_1$, so $c_1 = \frac{1+\sqrt{2}}{2}$. We plug this into the equation to find $c_2 = \frac{1-\sqrt{2}}{2}$. Therefore, the solution to this recurrence is

$$x_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right).$$

It follows that

$$x_{49} = \frac{1}{2}\left((1+\sqrt{2})^{50} + (1-\sqrt{2})^{50}\right).$$

The term $(1-\sqrt{2})^{50}$ is very small, so x_{49} can be approximated by the first term. In particular, $(1-\sqrt{2})^{50} > 0$, so

$$x_{49} > \frac{1}{2} \cdot (1+\sqrt{2})^{50}.$$

Now $(1+\sqrt{2})^2 = 3 + 2\sqrt{2}$, so $(1+\sqrt{2})^4 = 17 + 12\sqrt{2}$, and $(1+\sqrt{2})^5 = 41 + 29\sqrt{2}$. Hence $(1+\sqrt{2})^{10} = 3363 + 2378\sqrt{2}$. Then $3363 + 2378\sqrt{2} > 1.6 \cdot 2^{12}$. Therefore, $(1+\sqrt{2})^{50} > (1.6 \cdot 2^{12})^5 = 1.6^5 \cdot 2^{60}$. Since $1.6^5 = 10.48576 > 2^3$, we find that $(1+\sqrt{2})^{50} > 2^3 \cdot 2^{60} = 2^{63}$. Hence

$$x_{49} > \frac{1}{2} \cdot 2^{63} = 2^{62}.$$

Now suppose that we cover the top two rows of the 42×49 grid with a fish-friendly 2×49 grid, which we can do in at least 2^{62} ways. Then we can color the rest of the grid in $2^{40 \cdot 49} = 2^{1960}$ ways, since each of the remaining squares has two choices of color. Hence there are at least $2^{62+1960} = 2^{2022}$ fish-friendly colorings.

Note: It is possible to improve this count, which would give some leeway on computing the lower bound of x_{49}. We can cut the 42×49 grid into 21 distinct 2×49 grids, denoted $G_1, G_2, \ldots, G_{21}$. let A_i denote the set of all colorings of the 42×49 grid such that G_i is fish-friendly. Observe that the number of fish-friendly colorings is certainly greater than or equal to $|A_1 \cup A_2 \cup \cdots \cup A_{21}|$. We use the fact that

$$|A_1 \cup A_2 \cup \cdots \cup A_{21}| \geq (|A_1| + |A_2| + \cdots + |A_{21}|) - (|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{20} \cap A_{21}|. \quad (3)$$

This is related to the Principle of Inclusion-Exclusion. If a coloring is in exactly k of the sets A_i, then it is counted k times in the first set of parentheses, and it is subtracted $\binom{k}{2}$ times in the second set of parentheses. In particular, if $k = 0$, the coloring is counted 0 times, if $k = 1$, the coloring is counted once, if $k = 2$, the coloring is counted $2 - \binom{2}{2} = 1$ time, and if $k \geq 3$, then $k - \binom{k}{3} \leq 0$. So each coloring in $A_1 \cup A_2 \cup \cdots \cup A_{21}$ is counted at most once by the right side of (3), so the left hand side is greater than or equal to the right hand side.

Using the same methods as above, there are $x_{49} \cdot 2^{40 \cdot 49}$ fish-friendly colorings in A_i. Also, $|A_i \cap A_j| = x_{49}^2 \cdot 2^{38 \cdot 49}$. Since there are $\binom{21}{2}$ intersections, we can use (3) to find that

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_{21}| &\geq 21 \cdot x_{49} \cdot 2^{1960} - \binom{21}{2} \cdot x_{49}^2 \cdot 2^{1862} \\ &= 2^{1862} \cdot 21 \cdot x_{49} \cdot (2^{98} - 10x_{49}). \end{aligned}$$

Also, $x_{49} < \frac{1}{2}((1+\sqrt{2})^{50}+1)$, and $(1+\sqrt{2})^5 = 41 + 29\sqrt{2} < 2^7$, so $x_{49} < \frac{1}{2}(2^{70}+1) < 2^{70}$. Hence $10x_{49} < 16 \cdot 2^{70} < 2^{74}$, so $2^{98} - 10x_{49} > 2^{98} - 2^{74} > 2^{97}$. Hence the number of fish-friendly colorings is at least

$$2^{1862} \cdot 21 \cdot x_{49} \cdot (2^{98} - 10x_{49}) > 2^{1862} \cdot 21 \cdot 2^{62} \cdot 2^{97} > 2^{1862+4+59+97} = 2^{2025}.$$

In fact, using a computer, we can show that the number of fish-friendly colorings is much larger. If the board is split into four 10×49 boards and a 2×49 board, the number colorings that contain a fish-friendly path in one of these five boards that never goes backward is at least $2^{2043.696}$, where to compute this, we used the same Inclusion-Exclusion argument.