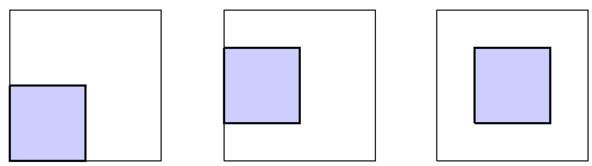
**2019 Utah Math Olympiad Solutions** March 23, 2019

1. We consider the following 3 cases: the largest square is $4 \times 4$; or the largest square is $3 \times 3$; or all squares are $1 \times 1$ or $2 \times 2$. In the last case, there are several subcases.

   - If the largest square is a $4 \times 4$, then there is $\boxed{1}$ way.
   - If the largest square is a $3 \times 3$, then the square has 2 possible $x$-coordinates and 2 possible $y$-coordinates, so there are $2 \cdot 2 = \boxed{4}$ ways.
   - Otherwise, all squares are $2 \times 2$ or $1 \times 1$. We say that each $2 \times 2$ square is either a "corner square" (adjacent to a corner of the original $4 \times 4$ square), an "edge square" (adjacent to an edge of the original $4 \times 4$ square, but not adjacent to a corner), or a "center square" (not adjacent to an edge or a corner of the original $4 \times 4$ square).

     Corner Square Edge Square Center Square

     If there is any center square, then no other $2 \times 2$ squares fit. So there is only $\boxed{1}$ way. If there are no center squares, we consider the number of edge squares, which could be 2, 1, or 0.

     – If there are 2 edge squares, there are no corner squares and only $\boxed{2}$ ways.
     – If there is 1 edge square, there are 4 rotations of the edge square and then two choices for whether to include a corner square in each of the opposite corners, for $4 \times (2 \times 2) = \boxed{16}$ ways.
     – Finally, we arrive at the case where there are no center or edge squares. Then for each corner, we choose whether to put a corner square. So there are $2^4 = \boxed{16}$ ways.

   Adding everything up, we get the answer:

   $$1 + 4 + 1 + 2 + 16 + 16 = \boxed{40 \text{ ways}}.$$ □

2. **Solution 1:** Let $n$ be the degree of the polynomial $p$. Then the degree of $p(x)^2$ is $2n$ and the degree of $p(p(x))$ is $n^2$. Since $p(x)^2 = p(p(x))$, we conclude $2n = n^2$. Therefore, $n = 0$ or $n = 2$.

   - If $n = 0$, then let $p(x) = c$ (a constant). We have $c^2 = c$, so $c = 0$ or $c = 1$.
   - If $n = 2$, say $p(x) = ax^2 + bx + c$, then the leading coefficient of $p(x)^2$ is $a^2x^4$, while the leading coefficient of $p(p(x))$ is $a^3x^4$. Thus $a^2 = a^3$. Since $p(x)$ has degree 2, we know $a \neq 0$, hence $a = 1$. Therefore, we can write

     $$p(p(x)) = p(x)^2 + bp(x) + c.$$

     If this is equal to $p(x)^2$, then $p(x)^2 = p(x)^2 + bp(x) + c$, or $bp(x) + c = 0$. Since $p(x)$ has degree 2, the only way this can be true for all values of $x$ is if $b = 0$, and then $c = 0$. Therefore, $p(x) = x^2$.

Thus all solutions are $\boxed{p(x) = 0, p(x) = 1, \text{ and } p(x) = x^2}$. □

**Solution 2:** If $p(x)$ is constant, then $p(x) = 0$ or $p(x) = 1$. Otherwise, let $u = p(x)$, and the equation becomes $u^2 = p(u)$. Since $p(x)$ is not constant, $u = p(x)$ has infinitely many possible values. This implies that $y^2 = p(y)$ has infinitely many solutions for $y$ (plug in $u = p(x)$ for $y$). But then the polynomial $p(y) - y^2$ has infinitely many roots, so it must be the zero polynomial, so $p(y) = y^2$. Renaming $y$ to $x$, $p(x) = x^2$ for all $x$. □

3. **Solution 1:** For $c \leq 4$, we write $c!$ in binary:

$$
\begin{aligned}
0! = 1! = 1 &= 1_2 \\
2! = 2 &= 10_2 \\
3! = 6 &= 110_2 \\
4! = 24 &= 11000_2
\end{aligned}
$$

Here, $N_2$ denotes that $N$ is the binary representation of an integer. Since binary representation is unique, we see that only 3! and 4! can be written as a sum of two *distinct* powers of two, and each in two different ways (reordering the two powers). Also, only 2! can be written as a sum of two of the *same* power of two, $2! = 2^0 + 2^0$. Therefore, the only possible solutions for these $c$ are:

$$\boxed{(a, b, c) = (0, 0, 2), (1, 2, 3), (2, 1, 3), (4, 3, 4), (3, 4, 4)}.$$

Now we assume that $c \geq 5$. Thus, $c!$ is divisible by both 3 and 5. Also, we may assume without loss of generality that $a \leq b$, and then

$$2^a + 2^b = 2^a(1 + 2^{b-a}).$$

Therefore, if $2^a + 2^b = c!$, then $1 + 2^{b-a}$ must be divisible by 15. But we can list the powers of 2 mod 15, finding

$$1, 2, 4, 8, 1, 2, 4, 8, \ldots.$$

Therefore, powers of 2 are periodic mod 15 with period 4. In particular, $2^n \not\equiv -1 \pmod{15}$ for any $n$. Therefore, $1 + 2^{b-a}$ cannot be divisible by 15, and there are no more solutions. □

**Solution 2:** Modulo 7, for any $k \geq 0$, $2^k \equiv 1, 2,$ or 4. As you cannot add $1, 2,$ or 4 to $1, 2,$ or 4 to get $0 \pmod 7$, it follows that $2^a + 2^b$ is not a multiple of 7 for any $a, b$. However, since $7 \mid c!$ for $c \geq 7$, it follows that $c \leq 6$.

Once we know $c \leq 6$, we proceed as in Solution 1 to write $c!$ in binary. For $c = 0, 1, 2, 3, 4$, we get the same set of solutions as before. For $c = 5$, we get $5! = 120 = 1111000_2$, which contains four 1s, so cannot be a sum of two powers of 2. Finally, for $c = 6$, we get $6! = 720 = 1011010000_2$, which also cannot be a sum of two powers of 2. □

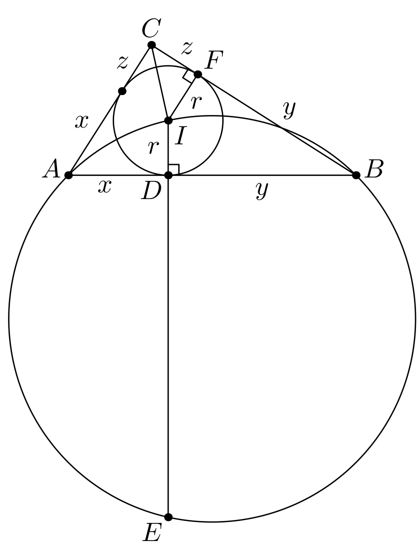
4. We claim that the leader has a winning strategy. That is, it is possible for the leader to give instructions to the players such that the leader always wins, no matter which player is the traitor and no matter how the traitor plays.

   Here is one example of instructions that work. Number the players from $1$ to $10$, and assume that the leader passes first to player $1$. For $i < 10$, let player $i$ pass to player $i + 1$ on the first pass, and to player $10$ on all passes after that. Then let player $10$ pass first to player $9$, then to player $8$, then to player $7$, and so on. These instructions are illustrated below.

| | | | | |
|---|---|---|---|---|
| Player 1 Instructions:<br>1. Pass to Player 2<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 2 Instructions:<br>1. Pass to Player 3<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 3 Instructions:<br>1. Pass to Player 4<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 4 Instructions:<br>1. Pass to Player 5<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 5 Instructions:<br>1. Pass to Player 6<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ |
| Player 6 Instructions:<br>1. Pass to Player 7<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 7 Instructions:<br>1. Pass to Player 8<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 8 Instructions:<br>1. Pass to Player 9<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 9 Instructions:<br>1. Pass to Player 10<br>2. Pass to Player 10<br>3. Pass to Player 10<br>⋮ | Player 10 Instructions:<br>1. Pass to Player 9<br>2. Pass to Player 8<br>3. Pass to Player 7<br>⋮ |

   We claim that with these instructions, the leader wins. If player $10$ is the traitor, the leader wins as soon as the traitor gets the ball for the first time. If player $\ell$ is the traitor, for $\ell < 10$, then players $1$ through $\ell - 1$ will have already gotten the ball when the traitor gets it. No matter who the traitor throws the ball to next, the ball will get to player $10$ before it gets back to player $\ell$. Player $10$ then throws to players $9, 8, \ldots, \ell + 1$, in order, who each throw the ball back to player $10$. Therefore, all of the remaining players will receive the ball before player $\ell$ (the traitor) gets the ball again. $\square$

5. Suppose that $x$, $y$, and $z$ are the lengths of the tangents from $A$, $B$, and $C$, respectively to the incircle.



First we derive a formula for the inradius $r$ of triangle $ABC$ in terms of $x, y$, and $z$, and the semiperimeter $s = \frac{a+b+c}{2}$. Note that $a = y + z$, $b = x + z$, and $c = x + y$. Adding these, we find $a + b + c = 2(x + y + x)$, hence $s = x + y + z$. Subtracting $a = y + z$, $b = x + z$, and $c = y + z$, respectively, from $s = x + y + z$, we find

$$\begin{aligned} s - a &= x \\ s - b &= y \\ s - c &= z. \end{aligned}$$

Therefore, substituting these into Heron's Formula, we find $[ABC] = \sqrt{sxyz}$. Since the area of any triangle is equal to its inradius times its semiperimeter, we also have $[ABC] = rs$ (where $r$ is the inradius). Thus $rs = \sqrt{sxyz}$, and

$$r = \sqrt{\frac{xyz}{s}}. \tag{1}$$

Next, we consider $\angle C$. We know that $I$ lies on the angle bisector of $\angle C$, so if $F$ is the foot of the perpendicular from $I$ to $\overline{BC}$, then $\triangle IFC$ is right with $\angle ICF = \frac{\angle C}{2}$. Therefore,

$$\tan\frac{\angle C}{2} = \tan\angle ICF = \frac{r}{z},$$

and applying the formula for $r$ in (1),

$$\tan\frac{\angle C}{2} = \frac{\sqrt{xyz/s}}{z} = \sqrt{\frac{xy}{sz}}. \tag{2}$$

Finally, we apply Power of a Point to point $D$ and the circumcircle of $\triangle ABI$. We find $BD \cdot AD = ED \cdot DI$. Therefore, since $BD = y$, $AD = x$, $DI = r$, and $ED = IE - DI = b + a - r$, we find

$$xy = (b + a - r) \cdot r. \tag{3}$$

Note that $b + a = (x + z) + (y + z) = s + z$. Therefore, replacing $b + a$ by $s + z$ in (3), and replacing $r$ using equation (1), we find

$$xy = \left((s+z) - \sqrt{\frac{xyz}{s}}\right) \cdot \sqrt{\frac{xyz}{s}}.$$

Hence $xy = (s + z)\sqrt{\frac{xyz}{s}} - \frac{xyz}{s}$, or

$$xy + \frac{xyz}{s} = (s+z)\sqrt{\frac{xyz}{s}}.$$

We can factor this as

$$xy\left(1 + \frac{z}{s}\right) = (s+z)\sqrt{\frac{xyz}{s}}.$$

Therefore,

$$xy\left(\frac{s+z}{s}\right) = (s+z)\sqrt{\frac{xyz}{s}}.$$

Dividing by all the terms on the right hand side, we find

$$\sqrt{\frac{xy}{sz}} = 1.$$

Therefore, by (2), $\tan\frac{\angle C}{2} = 1$. Hence $\frac{\angle C}{2} = 45°$, and $\boxed{\angle C = 90°}$. □

6. **Note:** This turns out to be possible for all $n \geq 0$ with $n \equiv 0$ or $n \equiv 1 \bmod 4$, with only three exceptions: $n = 5$, $n = 8$, and $n = 12$. In particular, as long as $n \geq 13$, it's possible to color the balls such that the probability of two balls being the same is equal to the probability of two balls being different. The problem asks to show this only for $n \geq 200$.

   **Solution 1:** Let the $n$ balls be of $k$ different colors, and let there be $a_i$ balls of each color $i$ for $1 \leq i \leq k$. So $\sum_{i=1}^{k} a_i = n$. Then we would like to have $\sum_{i=1}^{k}\binom{a_i}{2} = \frac{1}{2}\binom{n}{2}$, i.e. the number of ways to pick two balls of the same color is half the total number of ways to pick two balls. However, if any $a_i$ are 1 they contribute nothing to the sum. So it suffices to prove that for any $n \geq 200$, $n \equiv 0$ or 1 mod 4, we can choose nonnegative integers $a_i$ such that

$$\frac{1}{2}\binom{n}{2} = \sum_{i=1}^{k}\binom{a_i}{2} \quad \text{and} \quad \sum_{i=1}^{k} a_i \leq n.$$

   Define a *monochromatic pair* to be a pair of balls with the same color. Thinking in terms of limited resources, the problem is to spend *at most* $n$ balls to create *exactly* $\frac{1}{2}\binom{n}{2}$ *monochromatic pairs*, by distributing the balls into different colors.

   First, pick $a$ maximally so that $\binom{a}{2} \leq \frac{1}{2}\binom{n}{2}$, and color $a$ balls red. We claim that this leaves us with *at least* $\frac{1}{4}n$ balls left to color, and *at most* $\frac{3}{4}n$ monochromatic pairs left to create. To show this, note that $\frac{1}{2}(a-1)^2 \leq \binom{a}{2} \leq \frac{1}{2}\binom{n}{2} \leq \frac{1}{4}n^2$. Rearranging, $a \leq 1 + \frac{1}{\sqrt{2}}n$. This is less than $\frac{3}{4}n$ as long as $n$ is sufficiently large; in particular, as long as $\left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)n > 1$, which is true for $n \geq 20$ since

$$\left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)n > (.75 - .7)\,n = \frac{n}{20} \geq 1.$$

   So we colored at most $a \leq \frac{3}{4}n$ balls. Additionally, since $a$ was maximal, $\binom{a}{2} + a = \binom{a+1}{2} > \frac{1}{2}\binom{n}{2}$. Hence the number of monochromatic pairs left to create, or $\frac{1}{2}\binom{n}{2} - \binom{a}{2}$, is less than $a$, and thus less than $\frac{3}{4}n$.

   Therefore, after coloring these $a$ balls red, we are left with at least $\frac{1}{4}n$ balls to color in order to create at most $\frac{3}{4}n$ monochromatic pairs. At this point, we color groups of 13 balls at a time with their own color. Coloring 13 balls with the same color creates $\binom{13}{2} = 78$ monochromatic pairs. Therefore, if we use $b \leq \left\lfloor \frac{3n/4}{78} \right\rfloor$ different groups of 13 balls, where each group is monochromatic, and where $b$ is chosen maximally, then we have at most 77 monochromatic pairs left to create, and we also colored at most $13 \cdot \frac{3n/4}{78} = \frac{n}{8}$ balls. We are left with at least $\frac{1}{4}n - \frac{1}{8}n = \frac{1}{8}n$ balls, and at most 77 monochromatic pairs to go.

To create the 77 (or fewer) remaining monochromatic pairs, first color the maximum number of balls a single color; this requires coloring at most 12 balls ($\binom{12}{2} = 66$) and leaves us with at most 11 monochromatic pairs to create. To create these at most 11 monochromatic pairs, we color the remaining balls in groups of 3 (3 balls give us 3 monochromatic pairs) until there are less than 3 monochromatic pairs remaining to create, at which point we require at most 4 balls (color 2 and then color 2) to create the remaining 2 monochromatic pairs. Altogether, we can create 77 or fewer monochromatic pairs using at most $12 + 3 + 3 + 3 + 4 = 25$ balls.

Therefore, as long as $\frac{1}{8}n \geq 25$, there are enough balls remaining to be colored to obtain the last 77 pairs. This is equivalent to $n \geq 200$, so we are done. $\square$

**Solution 2:** As in Solution 1, we want to prove that $\frac{1}{2}\binom{n}{2}$ can be written as a summation $\sum \binom{a_i}{2}$ where $\sum a_i \leq n$. We use the following lemma.

*Lemma:* For all positive integers $N$, $N$ can be written as a summation of triangular numbers, $\sum_{i=1}^{k} \binom{a_i}{2}$, such that

$$\sum_{i=1}^{k} a_i \leq 4 + \sqrt{2N} + \sqrt[4]{128N}.$$

*Proof:* To prove the lemma, we write $N = \binom{a}{2} + \binom{b}{2} + c\binom{3}{2} + d\binom{2}{2}$, where $a$ is first chosen maximally, then $b$ is chosen maximally, then $c$ is chosen maximally, and finally $d$ is chosen to pick up everything else. Since $a$ is chosen maximally, $\binom{b}{2} \leq a - 1$. Since $b$ is chosen maximally, $3c + d \leq b - 1$. And finally $d \leq 2$. The sum of all the binomial indices is

$$\sum_{i=1}^{k} a_i = a + b + 3c + 2d \leq a + b + (b-1) + 2 = a + 2b + 1.$$

Now $\binom{a}{2} \leq N$ implies that $\frac{1}{2}(a-1)^2 \leq N$, so $a \leq 1 + \sqrt{2N}$. Similarly, $\binom{b}{2} \leq a - 1 \leq \sqrt{2N}$ implies $b \leq 1 + \sqrt{2\sqrt{2N}}$. So the sum of all the binomial indices is

$$\sum_{i=1}^{k} a_i \leq (1 + \sqrt{2N}) + 2(1 + \sqrt{2\sqrt{2N}}) + 1 = 4 + \sqrt{2N} + \sqrt{128N}. \quad \square$$

Now that the lemma is proven, it only remains to show that for $n \geq 200$, $N = \frac{1}{2}\binom{n}{2}$ satisfies $4 + \sqrt{2N} + \sqrt[4]{128N} \leq n$. Since $N \leq \frac{1}{4}n^2$, we find

$$4 + \sqrt{2N} + \sqrt[4]{128N} - n \leq 4 + \sqrt{n^2/2} + \sqrt[4]{32n^2} - n,$$

so it suffices to show that for $n \geq 200$,

$$4 + \sqrt{n^2/2} + \sqrt[4]{32n^2} - n \leq 0. \tag{1}$$

This function is strictly decreasing, because if $t = \sqrt{n}$, it can be written as $(1/\sqrt{2}-1)t^2 + \sqrt[4]{32}t + 4$, which is a downward opening parabola with a vertex at $t = \frac{\sqrt[4]{32}}{2-\sqrt{2}} < \frac{3}{1/2} = 6$. Hence it is decreasing for $t > 6$, or $n > 36$. So we now plug in $n = 200$, finding

$$\begin{aligned} 4 + 100\sqrt{2} + 40\sqrt[4]{0.5} - 200 &< 4 + 142 + 40 - 200 \\ &< 0. \end{aligned}$$

Thus (1) is true for all $n \geq 200$, which shows that we can complete the process for all $n \geq 200$. In fact, the smallest integer value of $n$ for which (1) is true is 92, so this actually works for $n \geq 92$. $\square$