Applications of the Laplace Transform

1. Differential and Integro-Differential Equations:

The Laplace transform of a differential equation is an algebraic equation that can be readily solved. The method is general and can solve a linear differential equation with constant coefficients of any order.

Differential Equation → Laplace Transform

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Inverse Laplace Transform ← Partial Fractions

This method gives the total response, which includes zero-input and zero-state components. The initial condition terms in the response give rise to the zero-input response and the input terms correspond to the zero-state response.

Q. Using the Laplace transform, solve the following differential equations:

(a). $(D^2 + 3D + 2)\, y(t) = D\, x(t)$

with $y(0^-) = \dot{y}(0^-) = 0$ and $x(t) = u(t)$

(b). $(D^2 + 4D + 4)\, y(t) = (D + 1)\, x(t)$

with $y(0^-) = 2$, $\dot{y}(0^-) = 1$ and $x(t) = e^{-t} u(t)$

A. (a). $(s^2 + 3s + 2)\, Y(s) = s \cdot \frac{1}{s} = 1$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow \quad y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$

(b). $\left(s^2 Y(s) - 2s - 1\right) + 4\left(s\, Y(s) - 2\right) + 4\, Y(s)$

$$= (s+1) \cdot \frac{1}{s+1} = 1$$

$$\Rightarrow \quad Y(s) = \frac{2s + 10}{(s+2)^2} = \frac{2}{s+2} + \frac{6}{(s+2)^2}$$

$$\Rightarrow \quad y(t) = (2 + 6t)\, e^{-2t} u(t)$$

Q. Consider a causal LTIC system described by the DE :

$$2\,\dot{y}(t) + 6\,y(t) = \dot{x}(t) - 4\,x(t)$$

(a). Compute the zero-input response if $y(0^-) = -3$.

(b). Compute the zero-state response if $x(t) = e\,\delta(t-\pi)$.

A. (a). For the zero-input response,

$$2\left[s\,Y(s) - y(0^-)\right] + 6\,Y(s) = 0$$

$$\Rightarrow \quad Y(s) = \frac{-3}{s+3} \quad \text{or} \quad y_{zi}(t) = -3\,e^{-3t}\,u(t)$$

(b). For the zero-state response,

$$2s\,Y(s) + 6\,Y(s) = s\,X(s) - 4\,X(s)$$

$$x(t) = e\,\delta(t-\pi) \Longleftrightarrow e^{-(s\pi-1)}$$

$$\Rightarrow \quad Y(s) = \frac{s-4}{2s+6}\,X(s)$$

$$\Rightarrow \quad y_{zs}(t) = \frac{e}{2}\left[\delta(t-\pi) - 7\,e^{-3(t-\pi)}\,u(t-\pi)\right]$$

Q. Consider a causal LTIC system described by the DE :

$$\dot{y}(t) + 2\,y(t) = \dot{x}(t)$$

(a). Write the system transfer function.

(b). Determine the unit impulse response of this system.

(c). Determine the output $y(t)$ if the input $x(t) = e^{-t}u(t)$ and $y(0^-) = \sqrt{2}$.

A. (a). $$s\,Y(s) + 2\,Y(s) = s\,X(s)$$

Hence, $$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s+2}$$

(b). $$H(s) = \frac{s}{s+2} = 1 - \frac{2}{s+2}$$

$$\Rightarrow \quad h(t) = \delta(t) - 2\,e^{-2t}u(t)$$

(c). $$s\,Y(s) - \sqrt{2} + 2\,Y(s) = s \cdot \frac{1}{s+1}$$

$$\Rightarrow \quad Y(s) = \frac{\sqrt{2}+2}{s+2} - \frac{1}{s+1}$$

$$\Rightarrow \quad y(t) = (\sqrt{2}+2)\,e^{-2t}u(t) - e^{-t}u(t)$$

Q. Using the Laplace transform, solve the following simultaneous differential equations:

$$(D+3)\, y_1(t) - 2\, y_2(t) = x(t)$$

$$-2\, y_1(t) + (2D+4)\, y_2(t) = 0$$

Assume all initial conditions to be zero and the input $x(t) = u(t)$.

A. Taking the Laplace transforms,

$$(s+3)\, Y_1(s) - 2\, Y_2(s) = \frac{1}{s}$$

$$-2\, Y_1(s) + (2s+4)\, Y_2(s) = 0$$

$$\Rightarrow \quad Y_1(s) = \frac{s+2}{s(s^2+5s+4)} = \frac{1}{2s} - \frac{1}{3(s+1)} - \frac{1}{6(s+4)}$$

$$Y_2(s) = \frac{1}{s(s^2+5s+4)} = \frac{1}{4s} - \frac{1}{3(s+1)} + \frac{1}{12(s+4)}$$

$$\Rightarrow \quad y_1(t) = \left(\frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{-4t}\right) u(t)$$

$$y_2(t) = \left(\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\right) u(t)$$

and $\quad H_1(s) = \dfrac{Y_1(s)}{X(s)} = \dfrac{s+2}{s^2+5s+4}, \quad H_2(s) = \dfrac{1}{s^2+5s+4}$