Trignometric Fourier Series:

We saw earlier that a signal can be expressed as a set of orthogonal signals. There exist a large number of orthogonal signal sets which can be used as basis signals for generalized Fourier series.

One such orthogonal set is trignometric (sinusoid) signals.

Consider a signal set:

$$\{1, \sin\omega_0 t, \cos\omega_0 t, \sin 2\omega_0 t, \cos 2\omega_0 t, \dots\dots\}$$

The sinusoid with frequency $n\omega_0$ is called the n^{th} harmonic, ω_0 is called the fundamental frequency, and the constant term 1 is the 0^{th} harmonic.

Then, for any interval $T_0 = \frac{2\pi}{\omega_0}$,

$$\int_{T_0} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = \int_{T_0} \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & n = m \neq 0 \end{cases}$$

and $\int_{T_0} \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0$ for all n and m

Hence, the signal set is orthogonal over T_0.

Therefore, we can express a signal $x(t)$ over T_0 as

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \cdots\cdots$$
$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \cdots\cdots$$
$$(t_1 \leq t \leq t_1 + T_0)$$

$$\Rightarrow \quad x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$
$$(t_1 \leq t \leq t_1 + T_0)$$

where

$$a_n = \frac{\int_{t_1}^{t_1+T_0} x(t) \cos(n\omega_0 t)\, dt}{\int_{t_1}^{t_1+T_0} \cos^2(n\omega_0 t)\, dt} , \quad b_n = \frac{\int_{t_1}^{t_1+T_0} x(t) \sin(n\omega_0 t)\, dt}{\int_{t_1}^{t_1+T_0} \sin^2(n\omega_0 t)\, dt}$$

Simplifying the equations, we get

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} x(t) \cos(n\omega_0 t)\, dt , \quad b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} x(t) \sin(n\omega_0 t)\, dt$$
$$(n = 1, 2, 3, \cdots\cdots)$$

and $$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t)\, dt$$

Note: A trignometric Fourier series is a periodic function of period T_0. That is, if

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

over $[t_1, t_1 + T_0]$

then, $x(t + T_0) = x(t)$

Hence, the signal $x(t)$ and its equivalent Fourier series expansion are both periodic with period T_0.

Dirichlet Conditions

For the existence of Fourier series of $x(t)$, the coefficients a_0, a_n, and b_n must be finite. Dirichlet showed that if $x(t)$ satisfies the following conditions, then its Fourier series will exist:

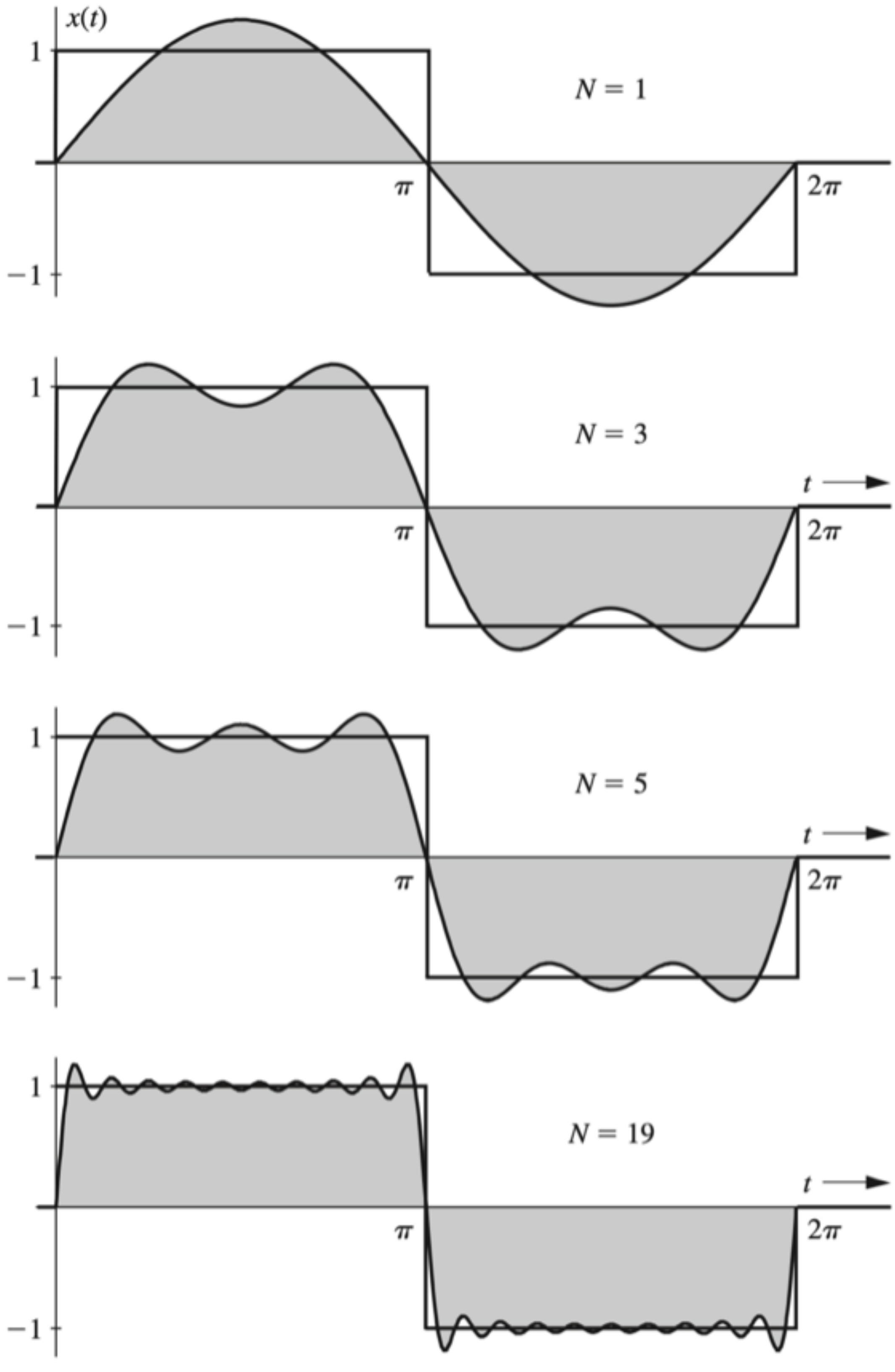
1. $x(t)$ must be absolutely integrable.

   That is, $\int_{T_0} |x(t)|\, dt < \infty$

2. $x(t)$ must have only a finite number of finite discontinuities in one period.
3. $x(t)$ must have only a finite number of maxima and minima in one period.

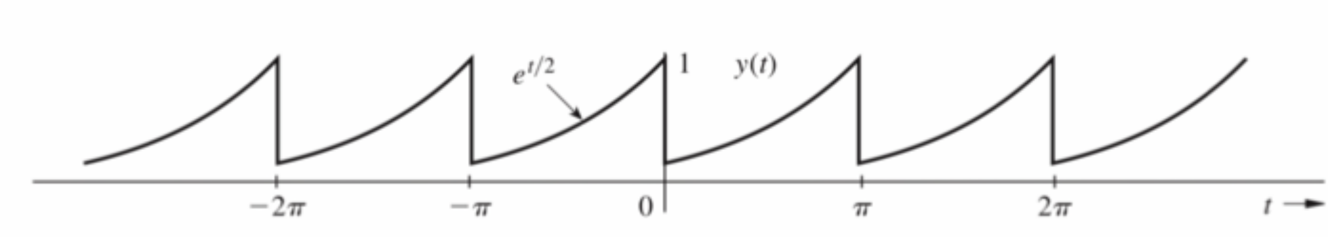
The following figure illustrates approximating a square wave with a set of Harmonic Sinusoids.

Clearly, as N increases, the energy of the error signal decreases improving our approximation.

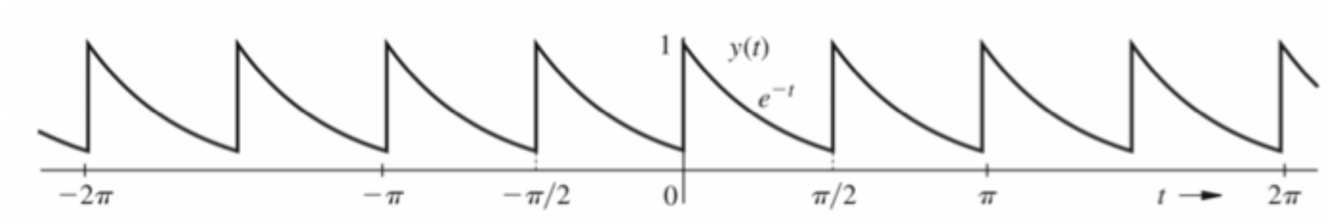


Q. Find the trignometric Fourier series for the signals depicted below.

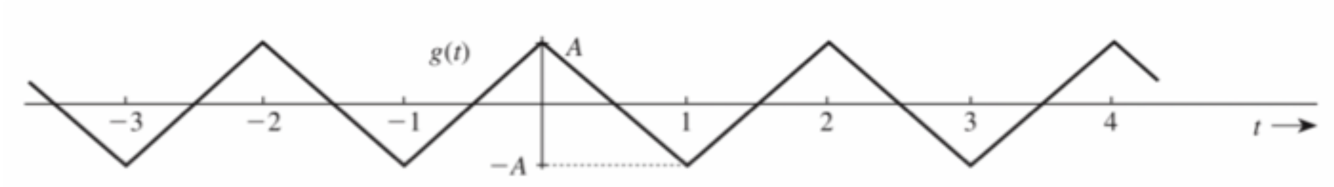
(a).



(b).



(c).



A. (a). $T_0 = \pi \Rightarrow \omega_0 = 2$

Therefore, $y(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{0} e^{t/2} dt = \frac{2(1 - e^{-\pi/2})}{\pi} = 0.504$

$a_n = \frac{2}{\pi} \int_{-\pi}^{0} e^{t/2} \cos(2nt)\, dt = 0.504 \frac{2}{1+16n^2}$

$b_n = \frac{2}{\pi} \int_{-\pi}^{0} e^{t/2} \sin(2nt)\, dt = -0.504 \frac{8n}{1+16n^2}$

(b). $T_0 = \frac{\pi}{2} \Rightarrow \omega_0 = 4$

Therefore, $$y(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(4nt) + b_n \sin(4nt)$$

where

$$a_0 = \frac{1}{\pi/2} \int_0^{\pi/2} e^{-t}\, dt = 0.504$$

$$a_n = \frac{1}{\pi/2} \int_0^{\frac{\pi}{2}} e^{-t} \cos(4nt)\, dt = 0.504 \frac{2}{1+16n^2}$$

$$b_n = \frac{1}{\pi/2} \int_0^{\frac{\pi}{2}} e^{-t} \sin(4nt)\, dt = 0.504 \frac{8n}{1+16n^2}$$

(c). $T_0 = 2 \Rightarrow \omega_0 = \pi$

Therefore, $$y(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$$

where

$$a_0 = \frac{1}{2} \left[\int_{-1}^{0} (A + 2At)\, dt + \int_0^1 (A - 2At)\, dt \right] = 0$$

$$a_n = \frac{1}{2} \left[\int_{-1}^{0} (A + 2At) \cos(n\pi t)\, dt + \int_0^1 (A - 2At) \cos(n\pi t)\, dt \right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{8A}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{2} \left[\int_{-1}^{0} (A + 2At) \sin(n\pi t)\, dt + \int_0^1 (A - 2At) \sin(n\pi t)\, dt \right] = 0$$