Some Properties of LT

If $x(t) \Longleftrightarrow X(s)$, then

1. Linearity:

$$k_1 x_1(t) + k_2 x_2(t) \Longleftrightarrow k_1 X_1(s) + k_2 X_2(s)$$

2. Time Shifting: $x(t-t_0) \Longleftrightarrow e^{-st_0} X(s)$

(for $t_0 \geq 0$)

3. Frequency Shifting: $x(t)\, e^{s_0 t} \Longleftrightarrow X(s-s_0)$

4. Time Differentiation:

$$\frac{d^n x}{dt^n} \Longleftrightarrow s^n X(s) - \sum_{k=1}^{n} s^{n-k} x^{k-1}(0^-)$$

5. Frequency Differentiation: $-t\,x(t) \Longleftrightarrow \dfrac{dX(s)}{ds}$

6. Time Integration: $\displaystyle\int_{0^-}^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{X(s)}{s}$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\, dt$$

7. Frequency Integration: $\frac{x(t)}{t} \Longleftrightarrow \int_s^{\infty} X(z)\,dz$

8. Scaling: $x(at) \Longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$

$(a \geqslant 0)$

9. Convolution: $x_1(t) * x_2(t) \Longleftrightarrow X_1(s) \cdot X_2(s)$

and $x_1(t)\, x_2(t) \Longleftrightarrow \frac{1}{2\pi j} X_1(s) * X_2(s)$

LTIC System Response

If $H(s)$ is the Laplace Transform of an LTIC system with impulse response $h(t)$, then the input $x(t)$ and $y(t)$ are related as:

$$y(t) = x(t) * h(t)$$

Then, using the convolution property,

$$Y(s) = X(s) \cdot H(s)$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)}$$

transfer function

Initial Value Theorem:

If $x(t)$ and $\frac{dx}{dt}$ are both Laplace transformable, then $x(t) \Longleftrightarrow X(s)$ implies

$$\mathcal{L}\left[\frac{dx}{dt}\right] = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st} dt$$

$$= x(t) e^{-st}\Big|_{0^-}^{\infty} + s \int_{0^-}^{\infty} x(t) e^{-st} dt$$

For $X(s)$ to exist, it is necessary that $x(t) e^{-st} \to 0$ as $t \to \infty$ for the values of s in the ROC of $X(s)$.

Hence, $s X(s) - x(0^-) = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st} dt$

$$\Rightarrow s X(s) - x(0^-) = \int_{0^-}^{0^+} \frac{dx}{dt} e^{-st} dt + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st} dt$$

$$\Rightarrow s X(s) = x(0^+) + \int_{0^+}^{\infty} \frac{dx}{dt} e^{-st} dt$$

$$\Rightarrow \lim_{s \to \infty} s X(s) = x(0^+) + \int_{0^+}^{\infty} \frac{dx}{dt} \left(\lim_{s \to \infty} e^{-st}\right) dt$$

Thus, $\lim_{s \to \infty} s X(s) = x(0^+)$

Note: The initial value theorem applies only if $X(s)$ is strictly proper $(M < N)$. If $M \geq N$, $\lim_{s \to \infty} s X(s)$ does not exist, but we still find the answer by using long division to express $X(s)$ as a polynomial in s plus a strictly proper fraction.

Final Value Theorem:

We have
$$s X(s) - x(0^-) = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st} dt$$

$$\Rightarrow \lim_{s \to 0} \left(s X(s) - x(0^-) \right) = \int_{0^-}^{\infty} \frac{dx}{dt} \left(\lim_{s \to 0} e^{-st} \right) dt$$

Thus,
$$\lim_{s \to 0} s X(s) = x(\infty)$$

Note: The final value theorem applies only if the poles of $X(s)$ are in LHP (including $s=0$).
In other cases, $x(\infty)$ either grows exponentially (if pole(s) in RHP) or oscillates indefinitely (if pole(s) on the imaginary axis).

Q. Find the system transfer function for the following systems:

(a). $$\frac{d^2 y}{dt^2} + 11 \frac{dy}{dt} + 24\, y(t) = 5 \frac{dx}{dt} + 3\, x(t)$$

(b). $$\frac{d^4 y}{dt^4} + 4 \frac{dy}{dt} = 3 \frac{dx}{dt} + 2\, x(t)$$

A. (a). $$s^2 Y(s) + 11 s\, Y(s) + 24\, Y(s) = 5 s\, X(s) + 3\, X(s)$$

$$\Rightarrow \left(s^2 + 11 s + 24\right) Y(s) = (5s + 3)\, X(s)$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{5s + 3}{s^2 + 11s + 24}$$

(b). $$s^4 Y(s) + 4 s\, Y(s) = 3 s\, X(s) + 2\, X(s)$$

$$\Rightarrow \left(s^4 + 4s\right) Y(s) = (3s + 2)\, X(s)$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{3s + 2}{s(s^3 + 4)}$$

Q. For the following systems, determine the differential equation relating the output $y(t)$ to the input $x(t)$:

(a). $H(s) = \dfrac{s+5}{s^2+3s+8}$

(b). $H(s) = \dfrac{s^2+3s+5}{s^3+8s^2+5s+7}$

A. (a). $\dfrac{Y(s)}{X(s)} = \dfrac{s+5}{s^2+3s+8}$

$\Rightarrow \quad s^2 Y(s) + 3s\,Y(s) + 8\,Y(s) = s\,X(s) + 5\,X(s)$

$\Rightarrow \quad \dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 8\,y(t) = \dfrac{dx}{dt} + 5\,x(t)$

(b). $\dfrac{Y(s)}{X(s)} = \dfrac{s^2+3s+5}{s^3+8s^2+5s+7}$

$\Rightarrow \quad s^3 Y(s) + 8s^2 Y(s) + 5s\,Y(s) + 7\,Y(s)$

$= s^2 X(s) + 3s\,X(s) + 5\,X(s)$

$\Rightarrow \quad \dfrac{d^3y}{dt^3} + 8\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 7\,y(t) = \dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 5\,x(t)$