Time-Domain Analysis of LTIC Systems

The first step in analyzing any system is to model the system using a mathematical expression or a rule that satisfactorily approximates the dynamical behaviour of the system.

Suppose we have a linear, time-invariant, continuous-time (LTIC) system for which the input $x(t)$ and the output $y(t)$ are related by linear differential equations of the form:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) =$$

$$b_m \frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x(t)$$

where all the coefficients a_i and b_i are constants.

Using a compact notation D for $\frac{d}{dt}$, we can write:

$$\left(D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right) y(t) =$$

$$\left(b_m D^m + b_{m-1} D^{m-1} + \cdots + b_1 D + b_0\right) x(t)$$

substituting $Q(D) = D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$

$P(D) = b_m D^m + b_{m-1} D^{m-1} + \cdots + b_1 D + b_0$

$$\Rightarrow \quad Q(D)\, y(t) = P(D)\, x(t)$$

Note: Noise considerations restrict practical systems to have $m \leq n$.

Since the system is linear, the total response is

$$\text{Total response} = \text{zero-input response} + \text{zero-state response}$$

zero-input response: result of internal system conditions alone (such as energy storages, initial conditions)

zero-state response: result of external input $x(t)$ when the system is in absence of all internal energy storages, initial conditions.

Zero-Input Response:

The zero-input response $y_0(t)$ is the solution to the equation:

$$Q(D)\, y_0(t) = 0$$

or $\left(D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right) y_0(t) = 0$

A linear combination of $y_0(t)$ and its n successive derivatives is zero for all $t \iff y_0(t)$ and all its n successive derivatives are of the same form.

$e^{\lambda t}$ has this property.

Let's assume $y_0(t) = c e^{\lambda t}$

$$\Rightarrow \quad c\left(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\right) e^{\lambda t} = 0$$

$$\Rightarrow \quad \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

$$\Rightarrow \quad Q(\lambda) = 0 = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

Hence, λ has n solutions : $\lambda_1, \lambda_2, \ldots, \lambda_n$

$\Rightarrow Q(D)\, y_0(t) = 0$ has n solutions $c_1 e^{\lambda_1 t}, \ldots, c_n e^{\lambda_n t}$

Hence, a general solution for $y_0(t)$ is :

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}$$

Note: The polynomial $Q(\lambda)$ is called the characteristic polynomial of the system and $Q(\lambda) = 0$ is called the characteristic equation.

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the characteristic roots or eigenvalues. The exponentials $e^{\lambda_i t}$ are called the characteristic modes of the system.

Repeated Roots: If our characteristic equation has repeated roots, then the form of the solution $y_0(t)$ is slightly modified.

For the differential equation $(D-\lambda)^r y_0(t) = 0$, the solution is:

$$y_0(t) = \left(c_1 + c_2 t + \cdots\cdots + c_r t^{r-1}\right) e^{\lambda t}$$

Complex Roots: If our characteristic equation has complex roots, they must occur in conjugate pairs if the coefficients of $Q(\lambda)$ are to be real. Hence, if $\alpha + j\beta$ is a root, then $\alpha - j\beta$ must also be a root.

$$\Rightarrow \quad y_0(t) = c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}$$

For a real system, $y_0(t)$ must also be real,

$\Rightarrow$ c_1 and c_2 must be conjugates

If $c_1 = \frac{c}{2} e^{j\theta} \Rightarrow c_2 = \frac{c}{2} e^{-j\theta}$

$$\Rightarrow \quad y_0(t) = c\, e^{\alpha t} \cos(\beta t + \theta)$$