Q. A 2π-periodic signal $x(t)$ is specified over one period as:

$$x(t) = \begin{cases} \frac{1}{A}t & 0 \le t < A \\ 1 & A \le t < \pi \\ 0 & \pi \le t < 2\pi \end{cases}$$

Compute the exponential Fourier series coefficients D_n for this periodic signal.

A. $$D_n = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

Now, $T_0 = 2\pi$, $\omega_0 = 1$

$$D_0 = \frac{1}{2\pi}\left(\int_0^A \frac{t}{A}\, dt + \int_A^{\pi} dt \right) = \frac{1}{2\pi}\left(\frac{A}{2} + \pi - A \right) = \frac{2\pi - A}{4\pi}$$

For $n \neq 0$,

$$D_n = \frac{1}{2\pi}\left(\int_0^A \frac{t}{A} e^{-jn\omega_0 t}\, dt + \int_A^{\pi} e^{-jn\omega_0 t}\, dt \right)$$

$$= \frac{1}{2\pi}\left(\left. \frac{t e^{-jnt}}{-jAn} \right|_0^A - \int_0^A \frac{e^{-jnt}}{jAn}\, dt + \left. \frac{e^{-jnt}}{-jn} \right|_A^{\pi} \right)$$

$$= \frac{1}{2\pi n}\left(\frac{e^{-jnA} - 1}{An} + j e^{-jn\pi} \right)$$

Hence, $$D_n = \begin{cases} \dfrac{2\pi - A}{4\pi} & n = 0 \\ \dfrac{1}{2\pi n}\left(\dfrac{e^{-jnA} - 1}{An} + j e^{-jn\pi} \right) & \text{otherwise} \end{cases}$$

Q. The signal $x(t) = 1 + 2\cos(5\pi t) + 3\sin(14\pi t)$ is applied to an LTIC system to produce output $y(t)$.

(a). Determine the fundamental radian frequency ω_0 of $x(t)$.

(b). Determine the exponential Fourier Series spectrum of $x(t)$

(c). If the system is an ideal low pass filter with cutoff frequency $f_c = 2\,Hz$, what is the output $y(t)$?

(d). If the system is an ideal high pass filter with cutoff frequency $f_c = 2\,Hz$, what is the output $y(t)$?

(e). If the system is an ideal band pass filter with a $4\,Hz$ passband centered at $4\,Hz$, what is the output $y(t)$?

(f). If the system is an ideal bandstop filter with a $5\,Hz$ stop band centered at $10\,Hz$, what is the output $y(t)$?

A. $x(t) = 1 + 2\cos(5\pi t) + 3\sin(14\pi t)$

(a). The frequencies present in $x(t)$ are $5\pi, 14\pi$.

The fundamental frequency ω_0 is calculated as the greatest common factor of $5\pi, 14\pi = \pi$

Hence, $\omega_0 = \pi$

(b). $x(t) = 1 + 2\cos(5\pi t) + 3\sin(14\pi t)$

$$= e^{0j\pi t} + e^{j5\pi t} + e^{-j5\pi t} + \frac{3}{2j} e^{j14\pi t} - \frac{3}{2j} e^{-j14\pi t}$$

Hence, $D_0 = D_5 = D_{-5} = 1$

$D_{14} = \frac{3}{2j}$, $D_{-14} = \frac{-3}{2j}$, $D_n = 0$ (otherwise)

(c). $f_c = 2\,Hz \Rightarrow \omega_c = 4\pi \Rightarrow$ Both $5\pi, 14\pi$ will be rejected

$\Rightarrow y(t) = 1$

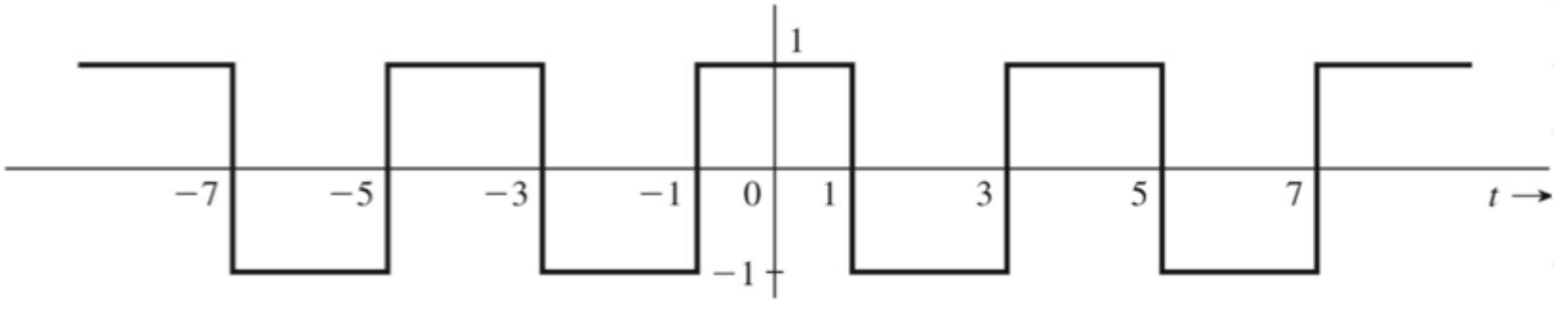
(d). The DC signal will be rejected.

$\Rightarrow y(t) = 2\cos(5\pi t) + 3\sin(14\pi t)$

(e). $y(t) = 2\cos(5\pi t)$

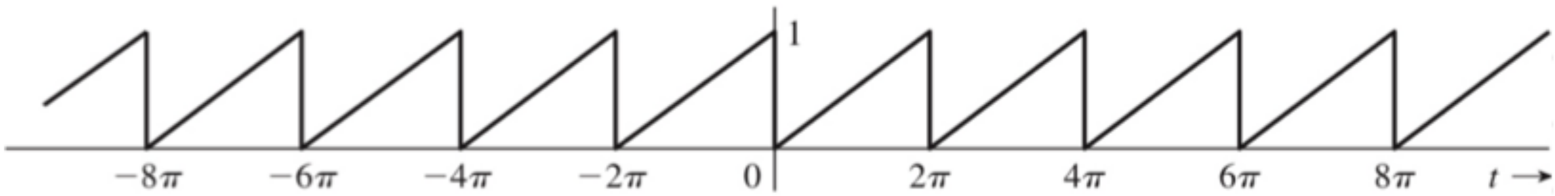
(f). $y(t) = 1 + 2\cos(5\pi t) + 3\sin(14\pi t)$

Q. For each of the following periodic signals, find the exponential Fourier series:

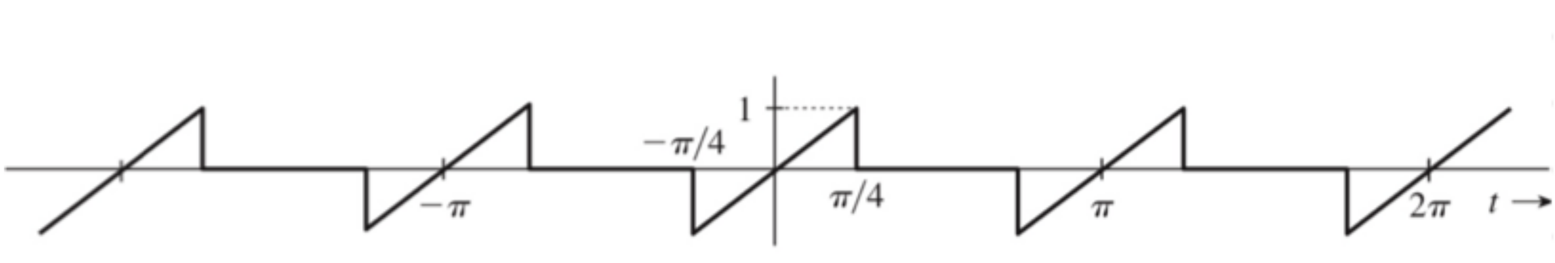


(a)

(b)



(c)



(d)

A. $$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

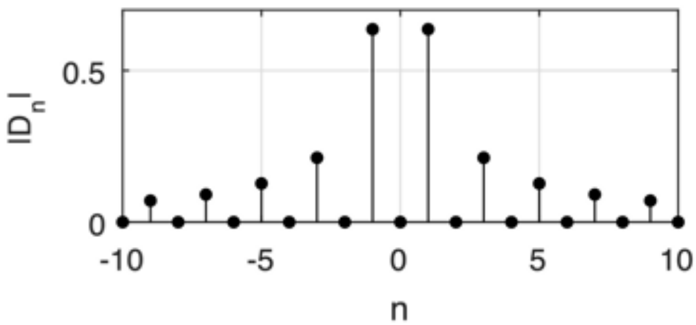
where $$D_n = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

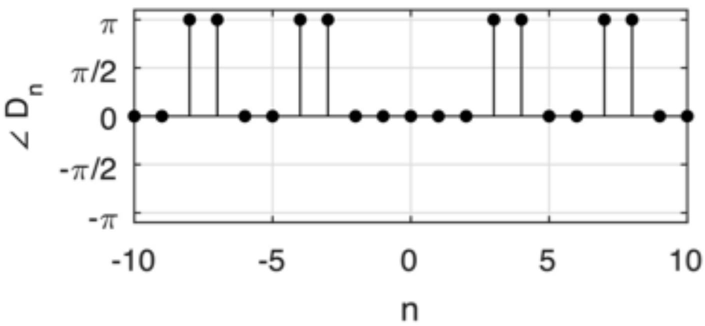
(a). Here, $T_0 = 4$, $\omega_0 = \frac{\pi}{2}$

$$D_0 = \frac{1}{4}\int_{-1}^{3} x(t)\, dt = 0$$

For $|n| \geq 1$, $D_n = \frac{1}{2\pi}\int_{-1}^{1} e^{-j(n\pi/2)t}\, dt - \int_{1}^{3} e^{-j(n\pi/2)t}\, dt$

$$\Rightarrow D_n = \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right)$$



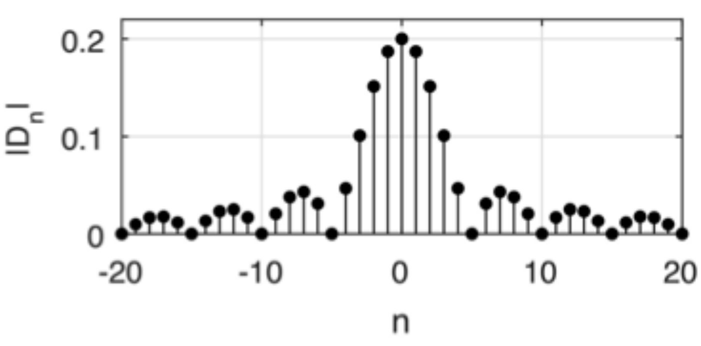


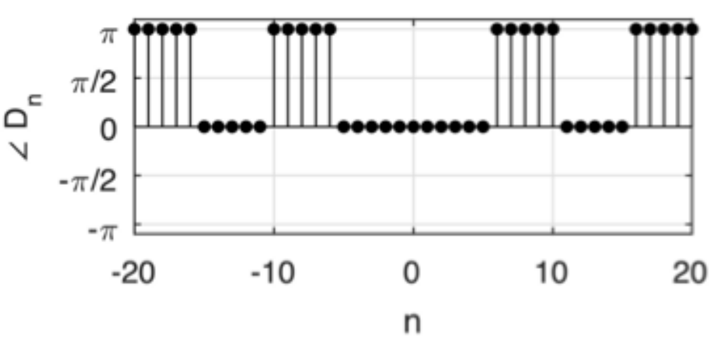
(b). $T_0 = 10\pi$, $\omega_0 = \frac{1}{5}$

$$D_0 = \frac{1}{10\pi}\int_{-\pi}^{\pi} x(t)\, dt = \frac{2\pi}{10\pi} = \frac{1}{5}$$

For $|n| \geq 1$, $D_n = \frac{1}{10\pi}\int_{-\pi}^{\pi} e^{-j\frac{n}{5}t}\, dt = \frac{j}{2\pi n}\left(-2j \sin\frac{n\pi}{5}\right)$

$$\Rightarrow D_n = \frac{1}{\pi n}\sin\left(\frac{n\pi}{5}\right)$$



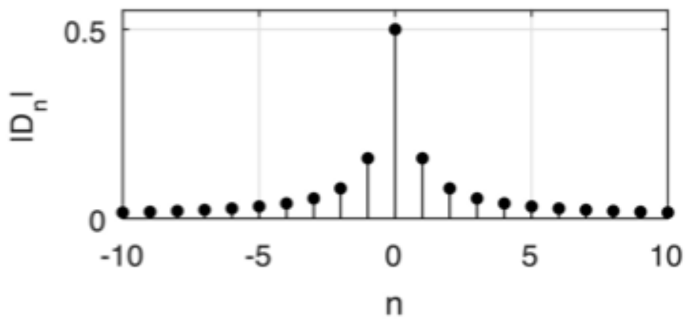


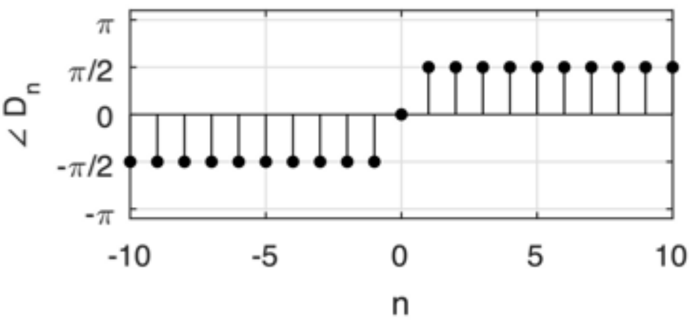
(c). $T_0 = 2\pi, \quad \omega_0 = 1$

$$D_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\,dt = \frac{1}{2}$$

For $|n| \geq 1$, $\quad D_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi} e^{-jnt}\,dt = \frac{j}{2\pi n}$

$$\Rightarrow \quad |D_n| = \frac{1}{2\pi n} \quad \text{and} \quad \angle D_n = \begin{cases} \pi/2 & n > 0 \\ -\pi/2 & n < 0 \end{cases}$$



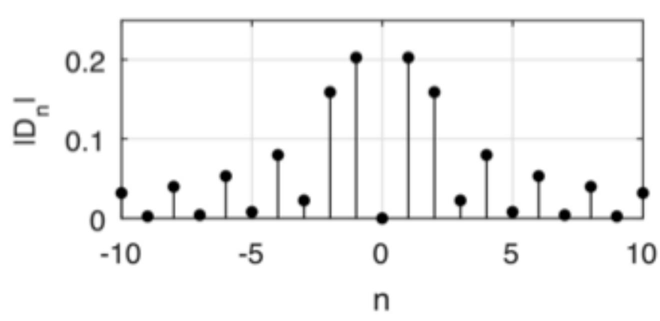


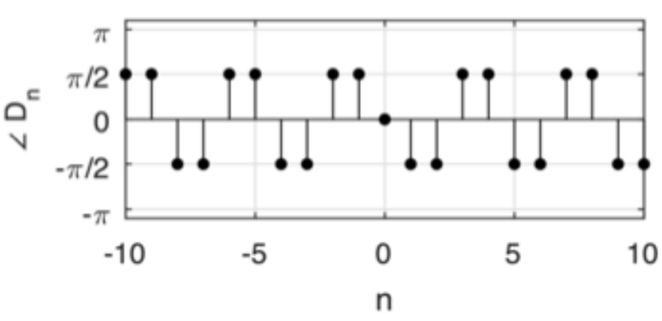
(d). $T_0 = \pi, \quad \omega_0 = 2$

$$D_0 = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4t}{\pi}\,dt = 0$$

For $|n| \geq 1$, $\quad D_n = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{4t}{\pi} e^{-j2nt}\,dt$

$$\Rightarrow \quad D_n = \frac{-j}{\pi n}\left(\frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right)$$





Q. Suppose a periodic signal $x(t)$ has an exponential Fourier series spectrum D_n. Then prove the following:

(a). If $x(t)$ has even symmetry, then D_n also has even symmetry ($D_n = D_{-n}$).

(b). If $x(t)$ has odd symmetry, then D_n also has odd symmetry ($D_n = -D_{-n}$).

(c). If $x(t)$ is real, then D_n is conjugate symmetric ($D_n = D_{-n}^*$).

(d). If $x(t)$ is imaginary, then D_n is conjugate anti symmetric ($D_n = -D_{-n}^*$).

A. (a). Given $x(t) = x(-t)$

Then, $$D_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(-t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} x(t)\, e^{jn\omega_0 t}\, dt = D_{-n}$$

Hence, $D_n = D_{-n}$

(b). Given $x(t) = -x(-t)$, then

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{-1}{T_0}\int_{T_0} x(-t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{-1}{T_0}\int_{T_0} x(t)\, e^{jn\omega_0 t}\, dt = -D_{-n}$$

Hence, $D_n = -D_{-n}$

(c). Given $x(t) = x^*(t)$, then

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} x^*(t)\left(e^{jn\omega_0 t}\right)^* dt$$

$$= D_{-n}^*$$

Hence, $D_n = D_{-n}^*$

(d). Given $x(t) = -x^*(t)$, then

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{-1}{T_0}\int_{T_0} x^*(t)\left(e^{jn\omega_0 t}\right)^* dt$$

$$= -D_{-n}^*$$

Hence, $D_n = -D_{-n}^*$