Parseval's Theorem:

We know that $x(t) = \sum_{n=1}^{\infty} c_n x_n(t)$ → orthogonal signals

$$\Rightarrow \quad x^2(t) = \left(c_1 x_1(t) + c_2 x_2(t) + \dots\right) \cdot \left(c_1 x_1(t) + c_2 x_2(t) + \dots\right)$$

integrating on both sides from t_1 to t_2, we get

$$\int_{t_1}^{t_2} x^2(t)\, dt = \int_{t_1}^{t_2} \left(c_1^2 x_1^2(t) + c_2^2 x_2^2(t) + \dots\right) dt$$

(note that the orthogonal terms will integrate to zero)

$$\Rightarrow \quad E_x = \sum_{n=1}^{\infty} c_n^2 E_n$$

Energy of the signal $x(t)$ over $[t_1, t_2]$

Energy of the orthogonal components $c_n x_n(t)$

Hence, the energy of the signal $x(t)$ is equal to the sum of the energies of the orthogonal components. This equation is called the Parseval's theorem.

Parseval's Theorem for Fourier Series:

The power of $x(t)$ is equal to the sum of the power of all the orthogonal components.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

Thus, $$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = a_0^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

$$= \underbrace{C_0^2}_{\text{power of the DC term}} + \underbrace{\frac{1}{2} \sum_{n=1}^{\infty} C_n^2}_{\text{power of the sinusoids/cosines}}$$

Similarly, in terms of exponential Fourier series:

We have $$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

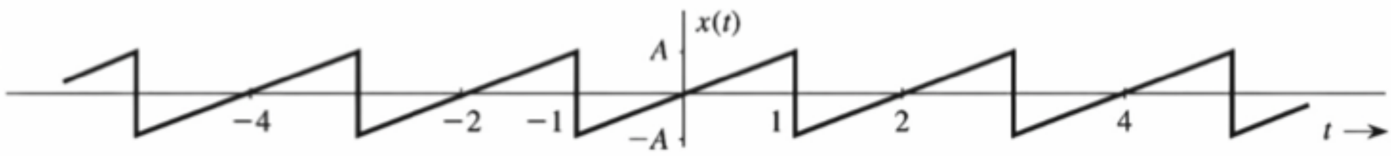
Then, $$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |D_n|^2$$

Furthermore, if $x(t)$ is real, then $|D_{-n}| = |D_n|$

$$\Rightarrow P_x = D_0^2 + 2 \sum_{n=1}^{\infty} |D_n|^2$$

Q. Verify Parseval's theorem for the following:

(a).



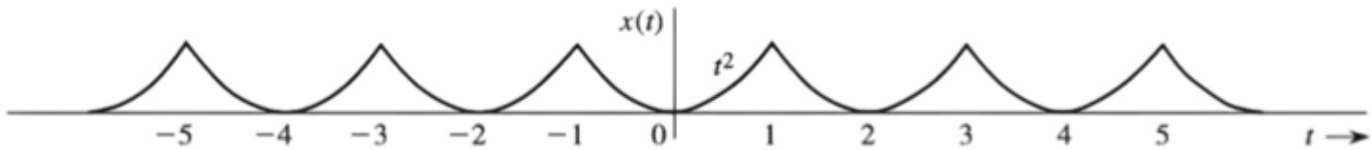
First, find the power of $x(t)$.

Now, the trignometric Fourier series of $x(t)$ is:

$$x(t) = \sum_{n=1}^{\infty} \frac{2A}{\pi} (-1)^{n+1} \frac{1}{n} \sin(n\pi t) , \quad -1 \leq t \leq 1$$

Given $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, show that Parseval's theorem holds for $x(t)$.

(b).



First, find the power of $x(t)$.

Now, the trignometric Fourier series of $x(t)$ is:

$$x(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2} \frac{(-1)^n}{n^2} \cos(n\pi t) , \quad -1 \leq t \leq 1$$

Given $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, show that Parseval's theorem holds for $x(t)$.

A. (a). The power of $x(t)$ is:

$$P_x = \frac{1}{2}\int_{-1}^{1} (At)^2\,dt = \frac{A^2}{3}$$

Now, $x(t) = \sum_{n=1}^{\infty} \frac{2A}{\pi}(-1)^{n+1}\frac{1}{n}\sin(n\pi t), \quad -\pi \le t \le \pi$

From Parseval's theorem,

$$P_x = a_0^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2A}{\pi n}\right)^2 (-1)^{2n+2}$$

$$= \frac{2A^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{A^2}{3}$$

Hence, Parseval's theorem holds.

(b). The power of $x(t)$ is: $P_x = \frac{1}{2}\int_{-1}^{1} (t^2)^2\,dt = \frac{1}{5}$

Now, $x(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2}\frac{(-1)^n}{n^2}\cos(n\pi t), \quad -1 \le t \le 1$

From Parseval's theorem,

$$P_x = C_0^2 + \sum_{n=1}^{\infty}\frac{C_n^2}{2} = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{\pi^2 n^2}\right)^2$$

$$= \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{9} + \frac{8}{90} = \frac{1}{5}$$

Hence, Parseval's theorem holds.

Fourier Series Properties:

If $x(t)$ has Fourier series spectrum D_n, that is, $x(t) \Longleftrightarrow D_n$, then

1. Scalar multiplication : $kx \Longleftrightarrow kD_n$

2. Addition : $x_1 + x_2 \Longleftrightarrow D_{n_1} + D_{n_2}$
   (same ω_0)

3. Time Shifting : $x(t-t_0) \Longleftrightarrow D_n e^{-jn\omega_0 t_0}$

4. Reversal : $x(-t) \Longleftrightarrow D_{-n}$

5. Frequency Shifting : $x(t)\, e^{jn_0\omega_0 t} \Longleftrightarrow D_{n-n_0}$

6. Conjugation : $x^*(t) \Longleftrightarrow D^*_{-n}$

7. Frequency convolution : $x_1(t)\, x_2(t) \Longleftrightarrow D_{n_1} * D_{n_2}$
   (same ω_0)

8. Time differentiation : $\dfrac{d^k x(t)}{dt^k} \Longleftrightarrow (jn\omega_0)^k D_n$

Q. A periodic signal $x(t)$ with $T_0 = \pi$ has Fourier series spectrum D_n. Find the Fourier series spectrum of the following transformed signals:

(a). $\frac{1}{3} x(-t-5)$ (b). $x(t) \cos(10t)$

(c). $x(-t) - 3x(t+2)$

A. (a).

$$x(t) \Longleftrightarrow D_n$$

$$x(t-5) \Longleftrightarrow D_n e^{-jn \frac{2\pi}{\pi} \cdot 5} = D_n e^{-10jn}$$

$$x(-t-5) \Longleftrightarrow D_{-n} e^{10jn}$$

$$\frac{1}{3} x(-t-5) \Longleftrightarrow \frac{1}{3} D_{-n} e^{j10n}$$

(b).

$$x(t) \Longleftrightarrow D_n$$

$$x(t) \cdot \frac{1}{2} e^{j10t} \Longleftrightarrow \frac{1}{2} D_{n-5}$$

$$x(t) \frac{1}{2} \left[e^{j10t} + e^{-j10t} \right] \Longleftrightarrow \frac{1}{2} \left[D_{n-5} + D_{n+5} \right]$$

(c).

$$x(t) \Longleftrightarrow D_n$$

$$x(-t) - 3x(t+2) \Longleftrightarrow D_{-n} - 3e^{j4n} D_n$$

Q. If a periodic signal $x(t)$ with a period T_0 is half-wave symmetric, then

$$x\left(t \pm \frac{T_0}{2}\right) = -x(t)$$

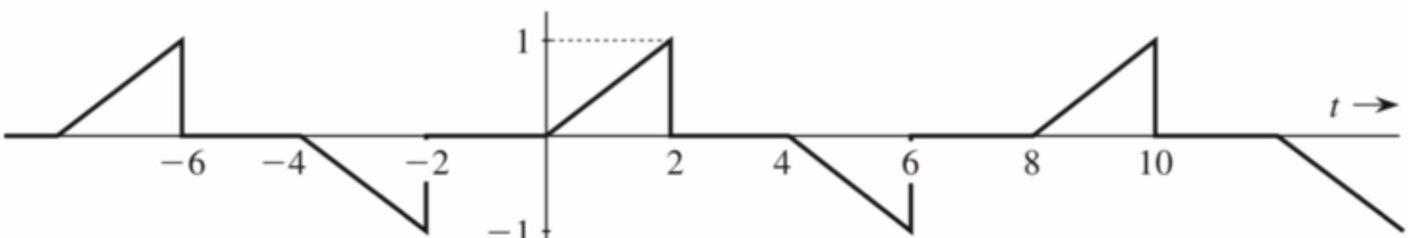
In this case, show that all the even-numbered harmonics vanish, and the odd-numbered harmonic coefficients are given by:

$$a_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} x(t) \cos(n\omega_0 t)\, dt \qquad (n \text{ odd})$$

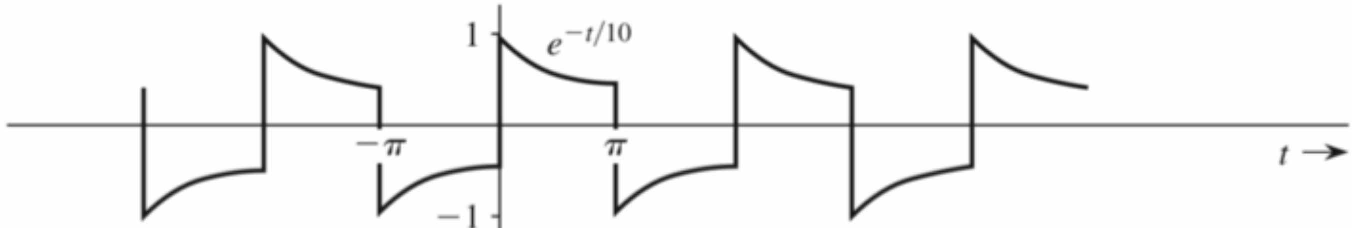
and

$$b_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} x(t) \sin(n\omega_0 t)\, dt \qquad (n \text{ odd})$$

Using these results, find the Fourier Series for the following periodic signals:



(a)



(b)

A. Given $x\left(t \pm \frac{T_0}{2}\right) = -x(t)$

$$a_n = \frac{2}{T_0}\left[\int_0^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)\,dt + \int_{\frac{T_0}{2}}^{T_0} x(t)\cos(n\omega_0 t)\,dt\right]$$

substituting $\tau = t - \frac{T_0}{2}$

$$= \frac{2}{T_0}\left[\int_0^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)\,dt + \int_0^{\frac{T_0}{2}} \underbrace{x\left(\tau+\frac{T_0}{2}\right)}_{-x(\tau)}\ \underbrace{\cos n\omega_0\left(\tau+\frac{T_0}{2}\right)}_{\substack{\cos n\omega_0\tau \ (\text{for even } n) \\ -\cos n\omega_0\tau \ (\text{for odd } n)}}\,d\tau\right]$$

Hence, if n is even, $a_n = 0$

For odd n,

$$a_n = \frac{2}{T_0}\left[\int_0^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)\,dt + \int_0^{\frac{T_0}{2}} \left(-x(\tau)\right)\left(-\cos n\omega_0\tau\right)d\tau\right]$$

Hence, $$a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)\,dt \quad (n \text{ odd})$$

Similarly, $$b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} x(t)\sin(n\omega_0 t)\,dt \quad (n \text{ odd})$$

(a). Now, for the first waveform,

$$T_0 = 8 , \quad \omega_0 = \frac{\pi}{4}$$

$$a_0 = \frac{1}{8} \int_{T_0} x(t)\, dt = 0 , \quad a_n = b_n = 0 \ (n \text{ even})$$

$$a_n = \frac{4}{8} \int_0^4 x(t) \cos\left(\frac{n\pi}{4} t\right) dt = \frac{4}{n^2\pi^2} \left(\frac{n\pi}{2} \sin\left(\frac{n\pi}{2}\right) - 1\right)$$

$$(n \text{ odd})$$

$$b_n = \frac{4}{8} \int_0^4 x(t) \sin\left(\frac{n\pi}{4} t\right) dt = \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \quad (n \text{ odd})$$

(b). $T_0 = 2\pi \Rightarrow \omega_0 = 1$

$$a_0 = \frac{1}{2\pi} \int_{T_0} x(t)\, dt = 0 , \quad a_n = b_n = 0 \ (n \text{ even})$$

$$a_n = \frac{4}{2\pi} \int_0^{\pi} e^{\frac{-t}{10}} \cos(nt)\, dt = \frac{\left(e^{-\pi/10} - 1\right)}{5\pi\left(n^2 + 0.01\right)} \quad (n \text{ odd})$$

$$b_n = \frac{4}{2\pi} \int_0^{\pi} e^{\frac{-t}{10}} \sin(nt)\, dt = \frac{2n\left(e^{-\pi/10} - 1\right)}{\left(n^2 + 0.01\right)} \quad (n \text{ odd})$$

Q. Consider a $T_0 = 1$ periodic signal $x(t)$ defined as :

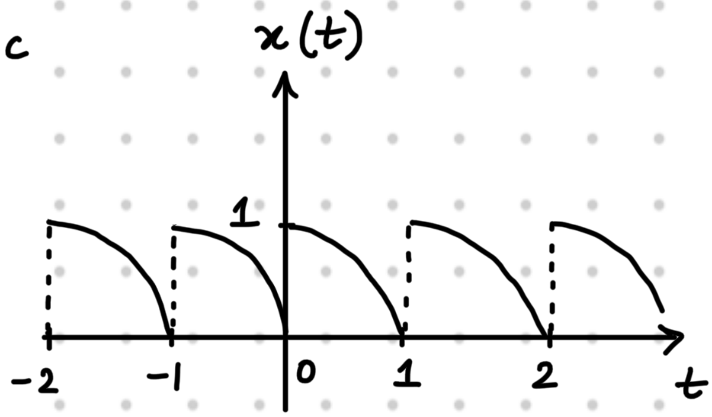
$$x(t) = \begin{cases} 1 - t^2 & 0 < t \leq 1 \\ x(t+1) & \forall\, t \end{cases}$$

(a). Sketch $x(t)$ for $-2 \leq t \leq 2$

(b). Determine the exponential Fourier Series spectrum of $x(t)$

(c). If $x(t)$ is applied to an ideal band pass filter with a 1 Hz passband centered at 3 Hz, determine the output $y(t)$.

A. (a). $x(t)$ is periodic with $T_0 = 1$



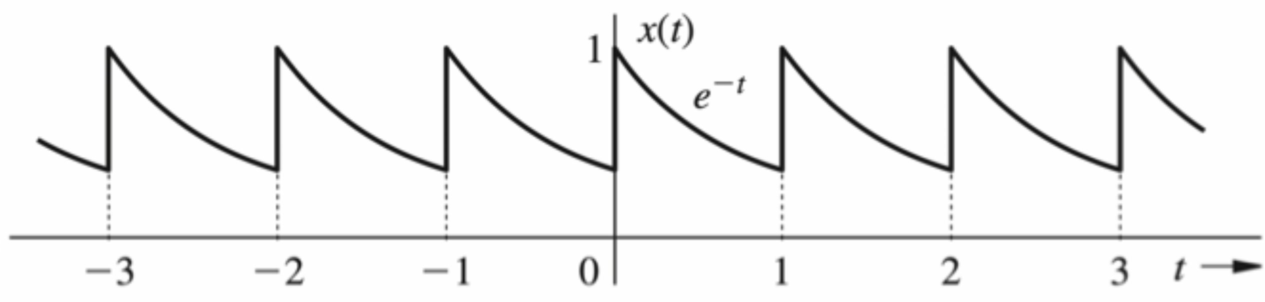
(b). Using $D_n = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$

$$D_n = \frac{1}{2\pi j n} - \frac{1}{2n^2\pi^2} \quad (n \neq 0), \quad D_0 = \frac{2}{3}$$

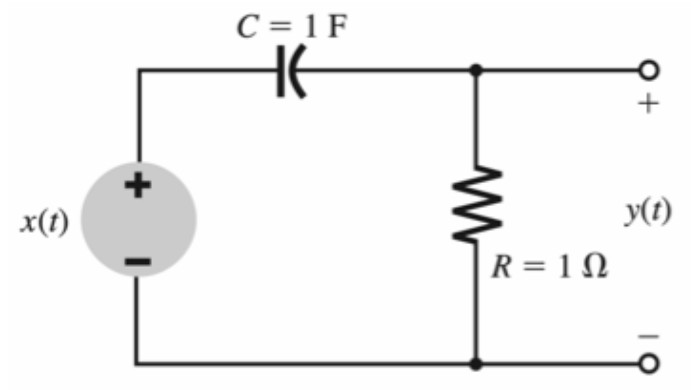
(c). Only the 3 Hz component will pass through,

$$\Rightarrow y(t) = \frac{\sin 6\pi t}{3\pi} - \frac{\cos 6\pi t}{9\pi^2}$$

Q. Find the exponential Fourier Series coefficients for the signal $x(t)$ shown below.



If $x(t)$ is applied at the input of an LTIC system shown, find the zero-state output if the unit impulse response is $h(t) = \delta(t) - \frac{e^{-t/RC}}{RC} u(t)$



A. $$D_n = \int_0^1 e^{-t} e^{-jn2\pi t} \, dt = \frac{1 - 1/e}{1 + 2\pi j n}$$

For an LTIC system, if $x(t) = \sum_n D_n e^{s_n t}$

then $y(t) = \sum_n D_n H(s_n) e^{s_n t}$

where $H(s_n) = \int_{-\infty}^{\infty} h(\tau) e^{-s_n \tau} d\tau$

substituting $s_n = jn(2\pi)$ and solving, $H(s_n) = \frac{j 2\pi n}{1 + j 2\pi n}$

Hence, $$y(t) = \sum_{n=-\infty}^{\infty} \frac{(1 - 1/e)}{(1 + 2\pi j n)^2} (2\pi j n) e^{j 2\pi n t}$$