Q. Compute the following:

(a). $\displaystyle\int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt$ (b). $\displaystyle\int_{-\infty}^{\infty} \text{sinc}^2(t-2)\,dt$

A. (a). $x(t) = \text{sinc}(kt) \iff X(\omega) = \frac{\pi}{k}\,\text{rect}\left(\frac{\omega}{2k}\right)$

Using Parseval's theorem,

$$E_x = \int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi^2}{k^2}\left[\text{rect}\left(\frac{\omega}{2k}\right)\right]^2 d\omega$$

$$= \frac{\pi}{2k^2}\int_{-k}^{k} d\omega = \frac{\pi}{k}$$

(b). $x(t) = \text{sinc}(t-2) \iff X(\omega) = e^{-j2\omega}\,\pi\,\text{rect}\left(\frac{\omega}{2}\right)$

Using Parseval's theorem,

$$E_x = \int_{-\infty}^{\infty} \text{sinc}^2(t-2)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi^2\left[\text{rect}\left(\frac{\omega}{2}\right)\right]^2 d\omega$$

$$= \frac{\pi}{2}\int_{-1}^{1} d\omega = \pi$$

Note: This can simply be also computed as

$$E_x = \int_{-\infty}^{\infty} \text{sinc}^2 t\,dt = \pi \quad \left(\text{as time shift does not alter } E_x\right)$$

Q. For the signal $x(t) = e^{-at}u(t)$, find the frequency ω_0 so that the energy contributed by all the spectral components below ω_0 is 95% of the signal energy.

A. Using Parseval's theorem,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$x(t) = e^{-at}u(t) \iff \frac{1}{a+j\omega}$$

To find ω_0, we compute

$$\frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \frac{1}{a^2+\omega^2} d\omega = \frac{95}{100} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2} d\omega$$

$$2 \operatorname{Tan}^{-1}\left(\frac{\omega_0}{a}\right) = (0.95)\pi$$

$$\Rightarrow \quad \omega_0 \simeq 12.71\, a \text{ rad/s}$$

Q. Prove that :

(a). $$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(-\omega)\, X_2(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2(-\omega)\, d\omega$$

(b). $$\int_{-\infty}^{\infty} \text{sinc}(\tau t - m\pi)\, \text{sinc}(\tau t - n\pi)\, dt$$

$$= \begin{cases} 0 & m \neq n \\ \dfrac{\pi}{\tau} & m = n \end{cases}$$

A.

(a). $$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(t)\left[\int_{-\infty}^{\infty} X_2(\omega)\, e^{j\omega t}\, d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)\left[\int_{-\infty}^{\infty} x_1(t)\, e^{j\omega t}\, dt\right] d\omega = \frac{1}{2\pi}\int X_1(-\omega)\, X_2(\omega)\, d\omega$$

Switch $x_1(t)$ and $x_2(t)$ for the other expression.

(b). $$\int_{-\infty}^{\infty} \text{sinc}(\tau t - m\pi)\, \text{sinc}(\tau t - n\pi)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\pi}{\tau}\right)^2 \left[\text{rect}\left(\frac{\omega}{2\tau}\right)\right]^2 e^{\frac{j(n-m)\pi\omega}{\tau}}\, d\omega$$

$$= \frac{\pi}{\tau} \;(\text{if } m=n) \quad \text{and} \quad 0 \;(\text{if } m \neq n)$$

Q. Verify Parseval's theorem for a Gaussian pulse $x(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-t^2}{2\sigma^2}} \Longleftrightarrow e^{-\sigma^2\omega^2/2}$

Given $\int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} dt = \sqrt{2\pi}$

A. $E_x = \int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{\frac{-t^2}{\sigma^2}} dt$

substituting $\frac{t}{\sigma} = \frac{\lambda}{\sqrt{2}}$, $E_x = \frac{1}{2\pi\sigma^2} \cdot \frac{\sigma}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\lambda^2/2} d\lambda$

using $\int_{-\infty}^{\infty} e^{-\lambda^2/2} d\lambda = \sqrt{2\pi}$, $E_x = \frac{1}{2\sigma\sqrt{\pi}}$

We also have $\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-t^2}{2\sigma^2}} \Longleftrightarrow e^{-\sigma^2\omega^2/2}$

Using Parseval's theorem, $E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

$$\Rightarrow E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sigma^2\omega^2} d\omega = \frac{1}{2\pi} \frac{1}{\sigma\sqrt{2}} \int_{-\infty}^{\infty} e^{-\lambda^2/2} d\lambda$$

$$= \frac{\sqrt{2\pi}}{2\pi\sigma\sqrt{2}} = \frac{1}{2\sigma\sqrt{\pi}}$$