Q. Are the following signals periodic or aperiodic?

For periodic signals, find the period and state which of the harmonics are present in the series.

(a). $3 \sin t + 2 \sin 3t$

(b). $3 \cos \sqrt{2} t + 5 \cos 2t$

(c). $\sin \frac{5t}{2} + 3 \cos \frac{6t}{5} + 3 \sin\left(\frac{t}{7} + 30^\circ\right)$

(d). $2 \sin 3t + 7 \cos \pi t$

(e). $(3 \sin 2t + \sin 5t)^2$

A. A signal is periodic if $x(t+T) = x(t) \ \forall t$

If the above property does not hold, the signal is aperiodic.

Further, every frequency in a periodic signal is an integral multiple of the fundamental frequency ω_0. Therefore, the ratio of any two frequencies is a rational number and they are said to be harmonically related. The largest positive number of which all the frequencies are integral multiples is the fundamental frequency.

(a). The frequencies in the spectrum are: 1 and 3.

$\Rightarrow$ The ratio of these frequencies is rational.

Hence, the two frequencies are harmonically related and the signal is periodic with $\omega_0 = 1 \Rightarrow T = 2\pi$

(b). The frequencies in the spectrum are: $\sqrt{2}$ and 2.

$\Rightarrow$ The ratio of these frequencies is irrational.

Hence, the signal is aperiodic

(c). The frequencies in the spectrum are:

$\frac{5}{2}$, $\frac{6}{5}$, and $\frac{1}{7}$.

$\Rightarrow$ The ratio of these frequencies is rational.

Hence, these frequencies are harmonically related and the signal is periodic with $\omega_0 = \frac{1}{70} \Rightarrow T = 140\pi$

(d). The frequencies in the spectrum are: 3 and π.

$\Rightarrow$ The ratio of these frequencies is irrational.

Hence, the signal is aperiodic

(e). The frequencies in the spectrum are: 4, 10, 2, 5.

$\Rightarrow$ The ratio of these frequencies is rational.

Hence, these frequencies are harmonically related and the signal is periodic with $\omega_0 = 1 \Rightarrow T = 2\pi$

Q. Determine the trignometric Fourier series coefficients a_n and b_n for the following signals. In each case, also determine the signals' fundamental radian frequency ω_0.

(a). $\cos(3\pi t)$

(b). $2 + 4\cos(3\pi t) - 2j\sin(7\pi t)$

(c). $\sin(3\pi t + 1) + 2\cos(7\pi t - 2)$

A. $$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

(a). $$a_n = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}, \quad b_n = 0 \text{ and } \omega_0 = 3\pi$$

(b). $$a_n = \begin{cases} 2 & n = 0 \\ 4 & n = 3 \\ 0 & \text{otherwise} \end{cases}, \quad b_n = \begin{cases} -2j & n = 7 \\ 0 & \text{otherwise} \end{cases}$$

, and $\omega_0 = \pi$.

(c). $$\sin(3\pi t)\cos(1) + \cos(3\pi t)\sin(1) + 2\cos(7\pi t)\cos(2) + 2\sin(7\pi t)\sin(2)$$

$$a_n = \begin{cases} \sin(1) & n = 3 \\ 2\cos(2) & n = 7 \\ 0 & \text{otherwise} \end{cases}, \quad b_n = \begin{cases} \cos(1) & n = 3 \\ 2\sin(2) & n = 7 \\ 0 & \text{otherwise} \end{cases}$$

, and $\omega_0 = \pi$.

Exponential Fourier Series:

An orthogonal set of exponential signals can also be used for generalized Fourier series. The set of exponentials e^{njw_0t} $(n = 0, \pm 1, \pm 2, \dots)$ is orthogonal over any interval of duration $T_0 = \frac{2\pi}{w_0}$.

That is,

$$\int_{T_0} e^{jmw_0t} \cdot \left(e^{jnw_0t}\right)^* dt = \int_{T_0} e^{j(m-n)w_0t} dt = \begin{cases} 0 & m \neq n \\ T_0 & m = n \end{cases}$$

Hence, a signal $x(t)$ can be expressed as:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jnw_0t}$$

→ just another form of trignometric Fourier series

where $$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jnw_0t} dt$$

substituting $e^{-jnw_0t} = \cos(nw_0t) - j \sin(nw_0t)$

gives us $D_n = \frac{1}{2}(a_n - j b_n)$

Trignometric $\longleftrightarrow$ Compact Trignometric $\longleftrightarrow$ Exponential

The trignometric, compact trignometric, and exponential Fourier series expansions can be converted from one form to another.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$t_1 \le t \le t_1 + T_0$$

where $T_0 = \dfrac{2\pi}{\omega_0}$

Using $a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = C_n \cos(n\omega_0 t + \theta_n)$

where $C_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \tan^{-1}\left(\dfrac{-b_n}{a_n}\right)$

Substituting $C_0 = a_0$,

we can write $x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$

$$t_1 \le t \le t_1 + T_0$$

This is called as the compact trignometric series

Now, using

$$C_n \cos(n\omega_0 t + \theta_n) = \frac{C_n}{2}\left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right]$$

$$= \left(\frac{C_n}{2} e^{j\theta_n}\right) e^{jn\omega_0 t} + \left(\frac{C_n}{2} e^{-j\theta_n}\right) e^{-jn\omega_0 t}$$

substituting $C_0 = D_0$, $\frac{C_n}{2} e^{j\theta_n} = D_n$, $\frac{C_n}{2} e^{-j\theta_n} = D_{-n}$

we can write $x(t) = D_0 + \sum_{n=1}^{\infty} \left(D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t} \right)$

$$\Rightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

Exponential Fourier Series

where $D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} \, dt$

Note:

1. $D_0 = a_0 = C_0$

2. $|D_n| = |D_{-n}| = \frac{1}{2} C_n = \frac{1}{2} \sqrt{a_n^2 + b_n^2}$, $n \neq 0$

$\angle D_n = \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$ and $\angle D_{-n} = -\theta_n = -\tan^{-1}\left(\frac{-b_n}{a_n}\right)$

Thus, $D_n = |D_n| e^{j\theta_n}$ and $D_{-n} = |D_n| e^{-j\theta_n}$