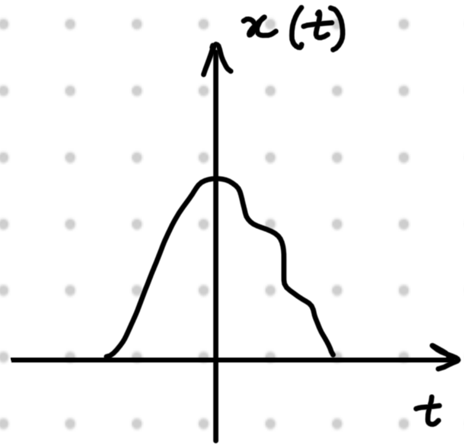
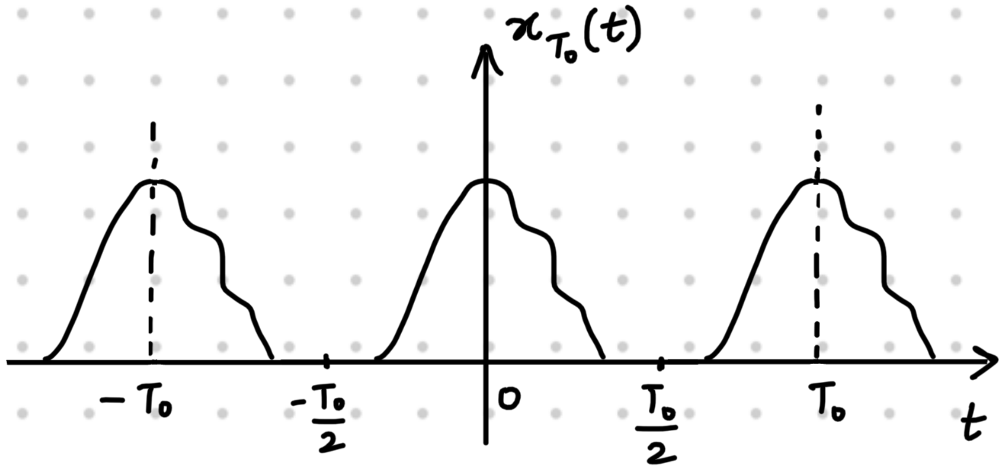
The Fourier Transform

Although Fourier Series helped us represent a periodic signal as a linear combination of sinusoids (or exponentials), it has two limitations. Firstly, it cannot be used for aperiodic signals and secondly, it cannot be used for marginally stable or unstable systems. The Fourier Transform helps us address the first limitation and the Laplace Transform helps us address the second limitation.

Suppose we have an aperiodic signal $x(t)$ as shown here.



Then, we can construct a new periodic signal $x_{T_0}(t)$ by repeating the signal $x(t)$ at intervals of T_0 seconds.



(T_0 is chosen long enough to avoid overlap between the pulses.)

Now, we know that the periodic signal $x_{T_0}(t)$ can be represented using an exponential Fourier series. If we let $T_0 \to \infty$, then $x_{T_0}(t) \to x(t)$.

That is, $\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$

Thus, the Fourier series representing $x_{T_0}(t)$ will also represent $x(t)$ as $T_0 \to \infty$.

If $x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$ with $\omega_0 = \frac{2\pi}{T_0}$

and Fourier coefficients $D_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_{T_0}(t)\, e^{-jn\omega_0 t}\, dt$

Integrating $x_{T_0}(t)$ over $\left(-\frac{T_0}{2}, \frac{T_0}{2}\right)$ is the same as integrating $x(t)$ over $(-\infty, \infty)$.

Hence, $D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t)\, e^{-jn\omega_0 t}\, dt$

Now,

let $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Rightarrow D_n = \frac{1}{T_0} X(n\omega_0)$

Substituting D_n in $x_{T_0}(t)$ gives,

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{X(n\omega_0)}{T_0} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{X(n\omega_0)}{2\pi} \omega_0 e^{jn\omega_0 t}$$

As $T_0 \to \infty$, $\omega_0 \to 0$ and $x_{T_0}(t) \to x(t)$

$$\Rightarrow \lim_{T_0 \to \infty} x_{T_0}(t) = x(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \underbrace{\sum_{n=-\infty}^{\infty} X(n\omega_0)\, \omega_0\, e^{jn\omega_0 t}}_{\text{area under the function } X(\omega) e^{j\omega t}}$$

Therefore,

$$x(t) = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega}_{\text{Fourier Integral}}$$

Basically a Fourier series in the limit $\omega_0 \to 0$

$X(\omega)$ is called the direct Fourier transform of $x(t)$, and $x(t)$ is called the inverse Fourier transform of $X(\omega)$.

Hence, $X(\omega) = \mathcal{F}(x(t))$ and $x(t) = \mathcal{F}^{-1}(X(\omega))$

or $$x(t) \Longleftrightarrow X(\omega)$$

Dirichlet conditions: The Fourier transform is assured for any $x(t)$ satisfying the Dirichlet conditions.

Since $|X(\omega)| \leq \int_{-\infty}^{\infty} |x(t)|\, dt$

$\Rightarrow$ If $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$, the existence of the Fourier transform is assured.

Note: Although this condition is sufficient, it is not necessary. Example: The signal $\frac{\sin at}{t}$ does not obey the property, but does have a Fourier transform.

The remaining Dirichlet conditions are:
In any finite interval, $x(t)$ may have only a finite number of maxima and minima and a finite number of finite discontinuities.

Any signal that can be generated in practice satisfies the Dirichlet conditions and therefore has a Fourier transform. Thus, the physical existence of a signal is a sufficient condition for the existence of its transform.

Fourier Transforms of Some Useful Functions:

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	