The Laplace Transform

The Fourier transform helps us represent signals as continuous sum of exponentials of the form $e^{j\omega t}$, whose frequencies are restricted to the imaginary axis in the complex plane $(s = j\omega)$. However, Fourier transform exists only for a restricted class of signals and cannot be used easily to analyze unstable or even marginally stable systems.

These problems with Fourier transform can be resolved if we use $s = \sigma + j\omega$ as the complex frequency (instead of $s = j\omega$). For example, $x(t) = e^{2t} u(t)$ does not have a Fourier transform, but $\phi(t) = x(t)\, e^{-\sigma t}$ can be made Fourier transformable if we choose $\sigma > 2$. This extension is known as the bilateral Laplace transform.

We know, $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Now, the Fourier transform of $x(t)e^{-\sigma t}$ (σ real) is:

$$\mathcal{F}\left(x(t)e^{-\sigma t}\right) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

$$\Rightarrow \mathcal{F}\left(x(t)e^{-\sigma t}\right) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt = X(\sigma+j\omega)$$

Applying inverse Fourier transform to the above equation gives,

$$\mathcal{F}^{-1}\mathcal{F}\left(x(t)e^{-\sigma t}\right) = \mathcal{F}^{-1}\left(X(\sigma+j\omega)\right)$$

$$\Rightarrow x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{j\omega t}\,d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)e^{(\sigma+j\omega)t}\,d\omega$$

substituting $s = \sigma + j\omega$ gives us,

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

and

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Bilateral Laplace Transform pair

Region of Convergence

The region of convergence (ROC) of $X(s)$ is the set of values of s (the region in the complex plane) for which the integral defining the Laplace transform $X(s)$ converges.

That is, set of s for which

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt < \infty$$

Unilateral Laplace Transform

For a given $X(s)$, there may be more than one inverse transform, depending on the ROC. Hence, unless the ROC is specified, there is no one-to-one correspondence between $X(s)$ and $x(t)$. This complexity is the result of trying to handle causal as well as noncausal signals. If we restrict all our signals to the causal type, such an ambiguity does not arise.

The unilateral Laplace transform is a special case of the bilateral Laplace transform, where all signals are restricted to being causal. Therefore, the unilateral Laplace transform $X(s)$ of a signal $x(t)$ is defined as:

$$X(s) \equiv \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

The price for this simplification is that we cannot analyze noncausal systems or use noncausal inputs. However, in most practical problems this is of little consequence. Hence, in practice, the term Laplace transform means the unilateral Laplace transform.

We choose 0^- (instead of 0^+) to ensure inclusion of an impulse function at $t=0$ and allow us to use initial conditions at 0^- (rather than 0^+) in the solution of differential equations via the Laplace transform.

Laplace Transforms of Some Useful Functions:

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at}\cos(bt+\theta)u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)u(t)$ $r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$ $\theta=\tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right)$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$
10d	$e^{-at}\left[A\cos bt+\frac{B-Aa}{b}\sin bt\right]u(t)$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$

Existence of the Laplace Transform

$$X(s) = \int_{0^-}^{\infty} \left(x(t)\, e^{-\sigma t}\right) e^{-j\omega t}\, dt$$

This integral converges if $\int_{0^-}^{\infty} \left| x(t)\, e^{-\sigma t} \right| dt < \infty$

Hence the existence of the Laplace transform is guaranteed if the integral is finite for some value of σ. Thus, if for some M and σ_0,

$$|x(t)| \leq M e^{\sigma_0 t}$$

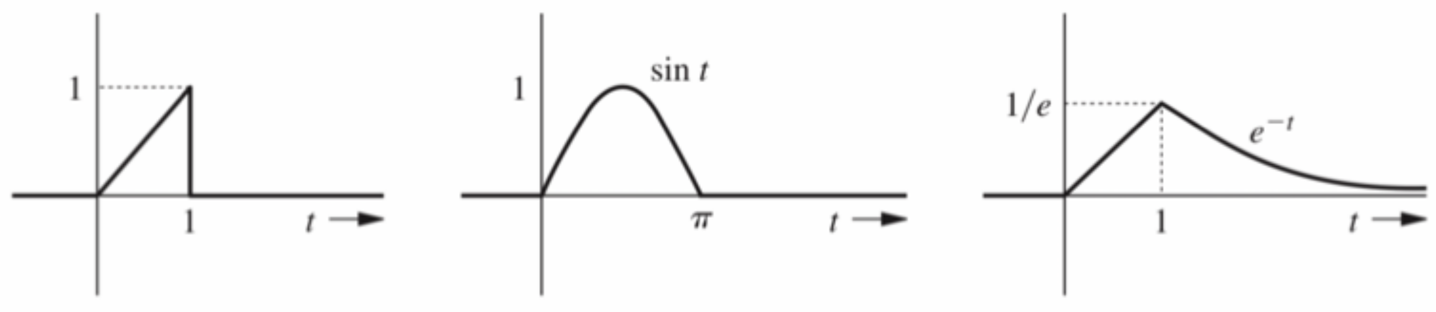
we can choose $\sigma > \sigma_0$ to guarantee the existence of the Laplace transform.

Finding the Inverse Transform

Finding the inverse Laplace transform using the conventional formula is highly complex.

Since most of the transforms of practical interest are rational functions, we can find the inverse transforms by expressing them as a sum of simpler functions by using partial fraction expansion.

Q. Find the Laplace transforms of the signals shown below:



(a) (b) (c)

A. (a). $$X(s) = \int_0^1 t\, e^{-st}\, dt = \frac{e^{-st}}{s}(-st-1)\Big|_0^1$$

$$= \frac{1}{s^2}\left(1 - e^{-s} - s e^{-s}\right) \qquad \left[\text{ROC: entire s-plane}\right]$$

(b). $$X(s) = \int_0^{\pi} (\sin t)\, e^{-st}\, dt = \frac{e^{-st}}{s^2+1}\left(-s(\sin t) - \cos t\right)\Big|_0^{\pi}$$

$$= \frac{1 + e^{-\pi s}}{s^2+1} \qquad \left[\text{ROC: entire s-plane}\right]$$

(c). $$X(s) = \int_0^1 \frac{t}{e}\, e^{-st}\, dt + \int_1^{\infty} e^{-t} e^{-st}\, dt$$

$$= \frac{e^{-st}}{es}(-st-1)\Big|_0^1 - \frac{1}{s+1} e^{-(s+1)}\Big|_1^{\infty}$$

$$= \frac{1}{es^2}\left(1 - e^{-s} - s e^{-s}\right) + \frac{1}{s+1} e^{-(s+1)} \qquad \left[\text{ROC: Re}(s) > -1\right]$$