Some Properties of FT:

1. Linearity: If $x_1(t) \Longleftrightarrow X_1(\omega)$ and

$x_2(t) \Longleftrightarrow X_2(\omega)$ then

$$a_1 x_1(t) + a_2 x_2(t) \Longleftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$$

2. Time-Shifting: If $x(t) \Longleftrightarrow X(\omega)$

then $x(t-t_0) \Longleftrightarrow e^{-j\omega t_0} X(\omega)$

3. Frequency-Shifting: If $x(t) \Longleftrightarrow X(\omega)$

then $e^{j\omega_0 t} x(t) \Longleftrightarrow X(\omega-\omega_0)$

4. Scaling: If $x(t) \Longleftrightarrow X(\omega)$

then $x(at) \Longleftrightarrow \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

5. Symmetry / Duality: If $x(t) \Longleftrightarrow X(\omega)$

then $X(t) \Longleftrightarrow 2\pi\, x(-\omega)$

6. Time Differentiation: If $x(t) \Longleftrightarrow X(\omega)$

then $\dfrac{d^n x}{dt^n} \Longleftrightarrow (j\omega)^n X(\omega)$

7. Time Integration: If $x(t) \Longleftrightarrow X(\omega)$

then $\displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau \Longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$

8. Conjugation: If $x(t) \Longleftrightarrow X(\omega)$

then $x^*(t) \Longleftrightarrow X^*(-\omega)$

9. Convolution: $x_1(t) * x_2(t) \Longleftrightarrow X_1(\omega)\cdot X_2(\omega)$

and $x_1(t)\,x_2(t) \Longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

LTIC System Response:

We know the LTIC zero-state response is:

$$
\begin{array}{ccccc}
y(t) & = & x(t) & * & h(t) \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\text{Then, } Y(\omega) & = & X(\omega) & \cdot & H(\omega)
\end{array}
$$

Parseval's theorem:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} x(t)\,x^*(t)\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\,e^{-j\omega t}\,d\omega\right)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

energy spectral density

Q. Prove the following using duality property:

(a). $\frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right] \Longleftrightarrow u(\omega)$

(b). $\delta(t+T) + \delta(t-T) \Longleftrightarrow 2\cos(T\omega)$

(c). $\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j\sin(T\omega)$

A. (a). We know, $u(t) \Longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$

Applying the duality property,

$$\pi\delta(t) + \frac{1}{jt} \Longleftrightarrow 2\pi u(-\omega)$$

$$\Rightarrow \quad \frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \Longleftrightarrow u(-\omega)$$

using $x(-t) \Longleftrightarrow X(-\omega)$,

$$\frac{1}{2}\left[\delta(-t) - \frac{1}{j\pi t}\right] \Longleftrightarrow u(\omega)$$

since $\delta(-t) = \delta(t)$,

$$\Rightarrow \quad \frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right] \Longleftrightarrow u(\omega)$$

(b). We know,

$$\cos(\omega_0 t) \Longleftrightarrow \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

Applying the duality property,

$$\pi\left[\delta(t+\omega_0) + \delta(t-\omega_0)\right] \Longleftrightarrow 2\pi\cos(-\omega_0\omega)$$

$$= 2\pi\cos(\omega_0\omega)$$

setting $\omega_0 = T$,

$$\delta(t+T) + \delta(t-T) \Longleftrightarrow 2\cos(T\omega)$$

(c). We know,

$$\sin(\omega_0 t) \Longleftrightarrow j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

Applying the duality property,

$$j\pi\left[\delta(t+\omega_0) - \delta(t-\omega_0)\right] \Longleftrightarrow 2\pi\sin(-\omega_0\omega)$$

$$= -2\pi\sin(\omega_0\omega)$$

Setting $\omega_0 = T$,

$$\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j\sin(T\omega)$$

Q. Prove the frequency differentiation property:

$$-jt\,x(t) \Longleftrightarrow \frac{d}{d\omega} X(\omega)$$

Using this property, determine the Fourier transform of $t e^{-at} u(t)$.

A. $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \frac{d}{d\omega} X(\omega) = \frac{d}{d\omega} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[-jt\, e^{-j\omega t}\right] dt = \int_{-\infty}^{\infty} \left(-jt\, x(t)\right) e^{-j\omega t}\, dt$$

Hence, $-jt\,x(t) \Longleftrightarrow \frac{d}{d\omega} X(\omega)$

Now, $$e^{-at} u(t) \Longleftrightarrow \frac{1}{j\omega + a}$$

$$\Rightarrow -jt\, e^{-at} u(t) \Longleftrightarrow (-j)\cdot \frac{1}{(j\omega+a)^2}\cdot j = \frac{1}{(j\omega+a)^2}$$

$$t\, e^{-at} u(t) \Longleftrightarrow \frac{1}{(j\omega+a)^2}$$

Q. A signal $x(t)$ has Fourier transform $X(\omega)$.

Determine the Fourier transform $Y(\omega)$ in terms of $X(\omega)$ for the following signals $y(t)$:

(a). $y(t) = \frac{1}{5} x(-2t+3)$

(b). $y(t) = e^{j2t} x^*(-3t-6)$

A. (a). $x(t) \Longleftrightarrow X(\omega) \Rightarrow x(-2t) \Longleftrightarrow \frac{1}{2} X\left(\frac{-\omega}{2}\right)$

$$x(-2(t-3/2)) \Longleftrightarrow \frac{e^{-j3\omega/2}}{2} X\left(\frac{-\omega}{2}\right)$$

$$\Rightarrow \frac{1}{5} x(-2(t-3/2)) \Longleftrightarrow \frac{e^{-j3\omega/2}}{10} X\left(\frac{-\omega}{2}\right)$$

Hence, $Y(\omega) = \frac{e^{-j3\omega/2}}{10} X\left(\frac{-\omega}{2}\right)$

(b). $x(t) \Longleftrightarrow X(\omega) \Rightarrow x(-3t) \Longleftrightarrow \frac{1}{3} X\left(\frac{-\omega}{3}\right)$

$$x^*(-3t) \Longleftrightarrow \frac{1}{3} X^*\left(\frac{\omega}{3}\right) \Rightarrow x^*(-3(t+2)) \Longleftrightarrow \frac{e^{j2\omega}}{3} X^*\left(\frac{\omega}{3}\right)$$

$$\Rightarrow e^{j2t} x^*(-3(t+2)) \Longleftrightarrow \frac{e^{j2(\omega-2)}}{3} X^*\left(\frac{\omega-2}{3}\right)$$

Hence, $Y(\omega) = \frac{e^{j2(\omega-2)}}{3} X^*\left(\frac{\omega-2}{3}\right)$

Q. A signal $x(t)$ has Fourier transform $X(\omega)$.

Determine the inverse Fourier transform $y(t)$ in terms of $x(t)$ for the following spectra $Y(\omega)$:

(a). $Y(\omega) = \frac{4}{3} e^{-j\frac{2\omega}{3}} X\left(\frac{-\omega}{3}\right)$

(b). $Y(\omega) = \frac{1}{3} e^{j2(\omega-2)} X^*\left(\frac{\omega-2}{3}\right)$

A. (a). $x(t) \Longleftrightarrow X(\omega) \Rightarrow x(-3t) \Longleftrightarrow \frac{1}{3} X\left(\frac{-\omega}{3}\right)$

$$x(-3(t-2/3)) \Longleftrightarrow \frac{e^{-j2\omega/3}}{3} X\left(\frac{-\omega}{3}\right)$$

$$\Rightarrow \quad 4x(-3(t-2/3)) \Longleftrightarrow \frac{4e^{-j2\omega/3}}{3} X\left(\frac{-\omega}{3}\right)$$

Hence, $y(t) = 4x(-3t+2)$

(b). $x(t) \Longleftrightarrow X(\omega) \Rightarrow x(-3t) \Longleftrightarrow \frac{1}{3} X\left(\frac{-\omega}{3}\right)$

$$x^*(-3t) \Longleftrightarrow \frac{1}{3} X^*\left(\frac{\omega}{3}\right) \Rightarrow x^*(-3(t+2)) \Longleftrightarrow \frac{e^{j2\omega}}{3} X^*\left(\frac{\omega}{3}\right)$$

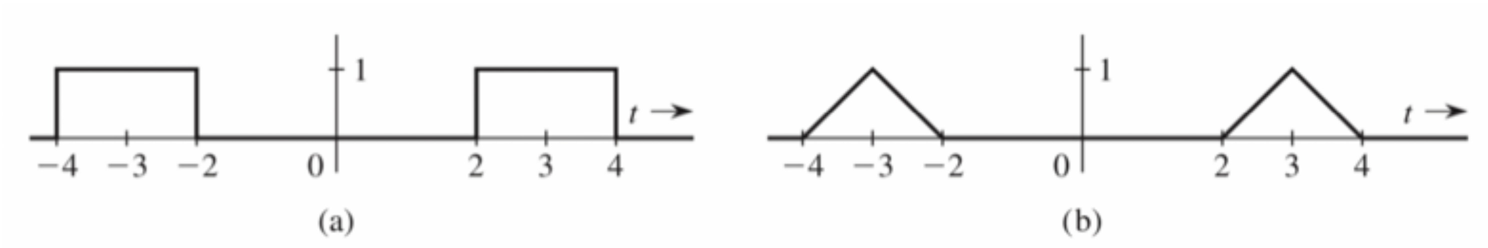
$$\Rightarrow \quad e^{j2t} x^*(-3(t+2)) \Longleftrightarrow \frac{e^{j2(\omega-2)}}{3} X^*\left(\frac{\omega-2}{3}\right)$$

Hence, $y(t) = e^{j2t} x^*(-3t-6)$

Q. Show that if $x(t) \Longleftrightarrow X(\omega)$, then

$$x(t+T) + x(t-T) \Longleftrightarrow 2X(\omega)\cos(T\omega)$$

Using this result, find the Fourier transforms of the following signals:



A. $x(t \pm T) \Longleftrightarrow X(\omega)\, e^{\pm j\omega T}$

$$\Rightarrow \quad x(t+T) + x(t-T) \Longleftrightarrow X(\omega) \cdot 2\cos(\omega T)$$

(a). Let $x(t) = \text{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\,\text{sinc}(\omega)$

Then, $x(t+3) + x(t-3) \Longleftrightarrow 4\,\text{sinc}(\omega)\cos(3\omega)$

(b). Let $x(t) = \Delta\left(\frac{t}{2}\right) \Longleftrightarrow \text{sinc}^2\left(\frac{\omega}{2}\right)$

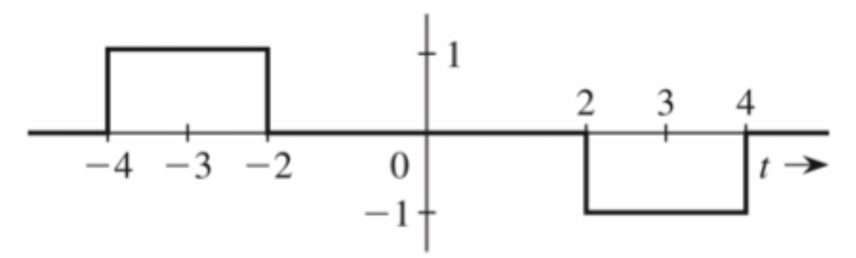
Then, $x(t+3) + x(t-3) \Longleftrightarrow 2\,\text{sinc}^2\left(\frac{\omega}{2}\right)\cos(3\omega)$

Q. Prove the following:

$$x(t)\sin\omega_0 t \Longleftrightarrow \frac{1}{2j}\left[X(\omega-\omega_0) - X(\omega+\omega_0)\right]$$

$$\frac{1}{2j}\left[x(t+T) - x(t-T)\right] \Longleftrightarrow X(\omega)\sin(T\omega)$$

Using this, find the Fourier transform of the following signal:



A. We know, $x(t)\, e^{\pm j\omega_0 t} \Longleftrightarrow X(\omega \mp \omega_0)$

Now, $x(t)\sin\omega_0 t = \frac{1}{2j}\left[x(t)\, e^{j\omega_0 t} - x(t)\, e^{-j\omega_0 t}\right]$

$$\Rightarrow \quad x(t)\sin\omega_0 t \Longleftrightarrow \frac{1}{2j}\left[X(\omega-\omega_0) - X(\omega+\omega_0)\right]$$

Furthermore, $x(t \pm T) \Longleftrightarrow X(\omega)\, e^{\pm j\omega T}$

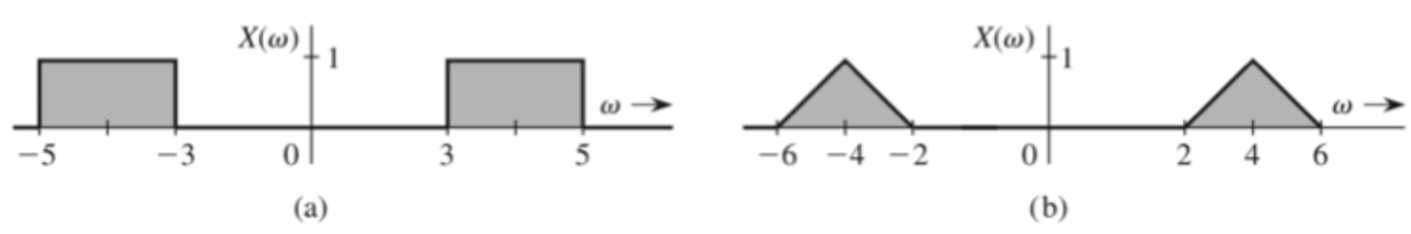
$$\Rightarrow \quad x(t+T) - x(t-T) \Longleftrightarrow X(\omega)\cdot 2j\sin(\omega T)$$

$$\Rightarrow \quad \frac{1}{2j}\left[x(t+T) - x(t-T)\right] \Longleftrightarrow X(\omega)\sin(\omega T)$$

Let $x(t) = \text{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\,\text{sinc}(\omega)$

then $x(t+3) - x(t-3) \Longleftrightarrow 4j\,\text{sinc}(\omega)\cdot\sin(3\omega)$

Q. Using the frequency-shifting property, find the inverse Fourier transform of the following spectra:



A. (a). $$x(\omega) = rect\left(\frac{\omega-4}{2}\right) + rect\left(\frac{\omega+4}{2}\right)$$

Using duality, $$\frac{1}{\pi} \operatorname{sinc}(t) \Longleftrightarrow rect\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \quad x(\omega) = rect\left(\frac{\omega-4}{2}\right) + rect\left(\frac{\omega+4}{2}\right)$$

$$\Longleftrightarrow \quad \frac{2}{\pi} \operatorname{sinc}(t) \cos(4t) = x(t)$$

(b). $$x(\omega) = \Delta\left(\frac{\omega+4}{4}\right) + \Delta\left(\frac{\omega-4}{4}\right)$$

Using duality, $$\frac{1}{\pi} \operatorname{sinc}^2(t) \Longleftrightarrow \Delta\left(\frac{\omega}{4}\right)$$

$$\Rightarrow \quad x(\omega) = \Delta\left(\frac{\omega+4}{4}\right) + \Delta\left(\frac{\omega-4}{4}\right)$$

$$\Longleftrightarrow \quad \frac{2}{\pi} \operatorname{sinc}^2(t) \cos(4t)$$