Total Response

The total response of a linear system can be expressed as the sum of its zero-input and zero-state components:

$$\text{Total Response} = \underbrace{\begin{matrix}\text{zero-input}\\ \text{component}\end{matrix}}_{\sum_{j=1}^{n} c_j e^{\lambda_j t}} + \underbrace{\begin{matrix}\text{zero-state}\\ \text{component}\end{matrix}}_{x(t) * h(t)}$$

(modify this appropriately for repeated roots)

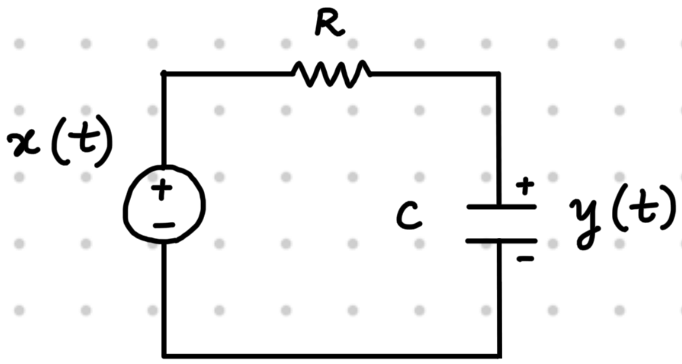
We can lump together all the characteristic mode terms in the total response, giving us a component known as the natural response.

The remainder, consisting entirely of noncharacteristic mode terms, is known as the forced response.

$$\Rightarrow \text{Total Response} = \underbrace{\begin{matrix}\text{Natural}\\ \text{Response}\end{matrix}}_{\text{characteristic mode terms}} + \underbrace{\begin{matrix}\text{Forced}\\ \text{Response}\end{matrix}}_{\text{noncharacteristic mode terms}}$$

Q. Consider the simple RC circuit shown below.

(a). Write the differential equation that relates the input $x(t)$ to output $y(t)$. Is this system LTI?



(b). Find the system's characteristic roots.

(c). Find the system's total response to a unit step input with the initial capacitor voltage $y(0^-) = 2V$. The system's unit impulse response is given as $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$.

A. (a). Using KVL,

$$x(t) = iR + y(t) = RC \frac{dy}{dt} + y(t)$$

$$\Rightarrow \quad \frac{dy}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

Yes, the system is both linear and time-invariant.

(b). $$\left(D + \frac{1}{RC}\right) y(t) = \frac{1}{RC} x(t)$$

$$Q(D) = D + \frac{1}{RC} \quad \Rightarrow \quad Q(\lambda) = \lambda + \frac{1}{RC}$$

Hence, the system's characteristic root is $-\frac{1}{RC}$.

(c). Total response = Zero-input response + Zero-state response

$$y_{zi}(t) = c\, e^{\frac{-t}{RC}}$$

using $y(0^-) = 2V$, $\quad y_{zi} = 2\, e^{\frac{-t}{RC}}$

$$y_{zs}(t) = \underbrace{x(t)}_{u(t)} * \underbrace{h(t)}_{\frac{1}{RC} e^{\frac{-t}{RC}} u(t)}$$

(this can also be derived using $[P(D)\, y_n(t)]\, u(t)$)

$$\Rightarrow \quad y_{zs}(t) = u(t) * \frac{1}{RC} e^{\frac{-t}{RC}} u(t)$$

$$= \left(1 - e^{\frac{-t}{RC}}\right) u(t)$$

Hence, the total response is :

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

$$\Rightarrow \quad y(t) = \left(1 + e^{\frac{-t}{RC}}\right) u(t)$$

System Stability
- Zero-state stability (External)
- Zero-input stability (Internal)

External (BIBO) Stability: The zero-state response is $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

If $x(t)$ is bounded (i.e. $|x(t)| < \infty$), then $y(t)$ is also bounded $\iff \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$ (BIBO stability criterion)

Hence, the BIBO stability criterion is that the impulse response $h(t)$ be absolutely integrable.

Internal (Asymptotic) Stability

Suppose we have an LTI system in equilibrium state (zero state, in which all initial conditions are zero) and we change this state by creating some nonzero initial conditions. Then, if the system returns to zero state (or the system's output due to the non zero initial conditions $\to 0$ as $t \to \infty$), we say that the system is <u>asymptotically stable</u>.

If a system has n distinct characteristic roots $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then we know the system's zero-output is made up of its characteristic modes:

$$y_0(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}$$

Then, $$\lim_{t \to \infty} y_0(t) = \lim_{t \to \infty} \sum_{i=1}^{n} c_i e^{\lambda_i t}$$

$$\Rightarrow \lim_{t \to \infty} y_0(t) \to 0 \iff \lim_{t \to \infty} e^{\lambda_i t} \to 0$$

Note that

$$\lim_{t \to \infty} e^{\lambda_i t} = \begin{cases} 0 & \text{if } Re(\lambda_i) < 0 \\ \text{bounded} & \text{if } Re(\lambda_i) = 0 \\ \infty & \text{if } Re(\lambda_i) > 0 \end{cases}$$

If we have r repeated roots, $y_0(t) = \sum_{i=1}^{r} c_i t^{i-1} e^{\lambda t}$

Then, $\lim_{t \to \infty} y_0(t) \to 0 \iff \lim_{t \to \infty} t^{i-1} e^{\lambda t} \to 0$

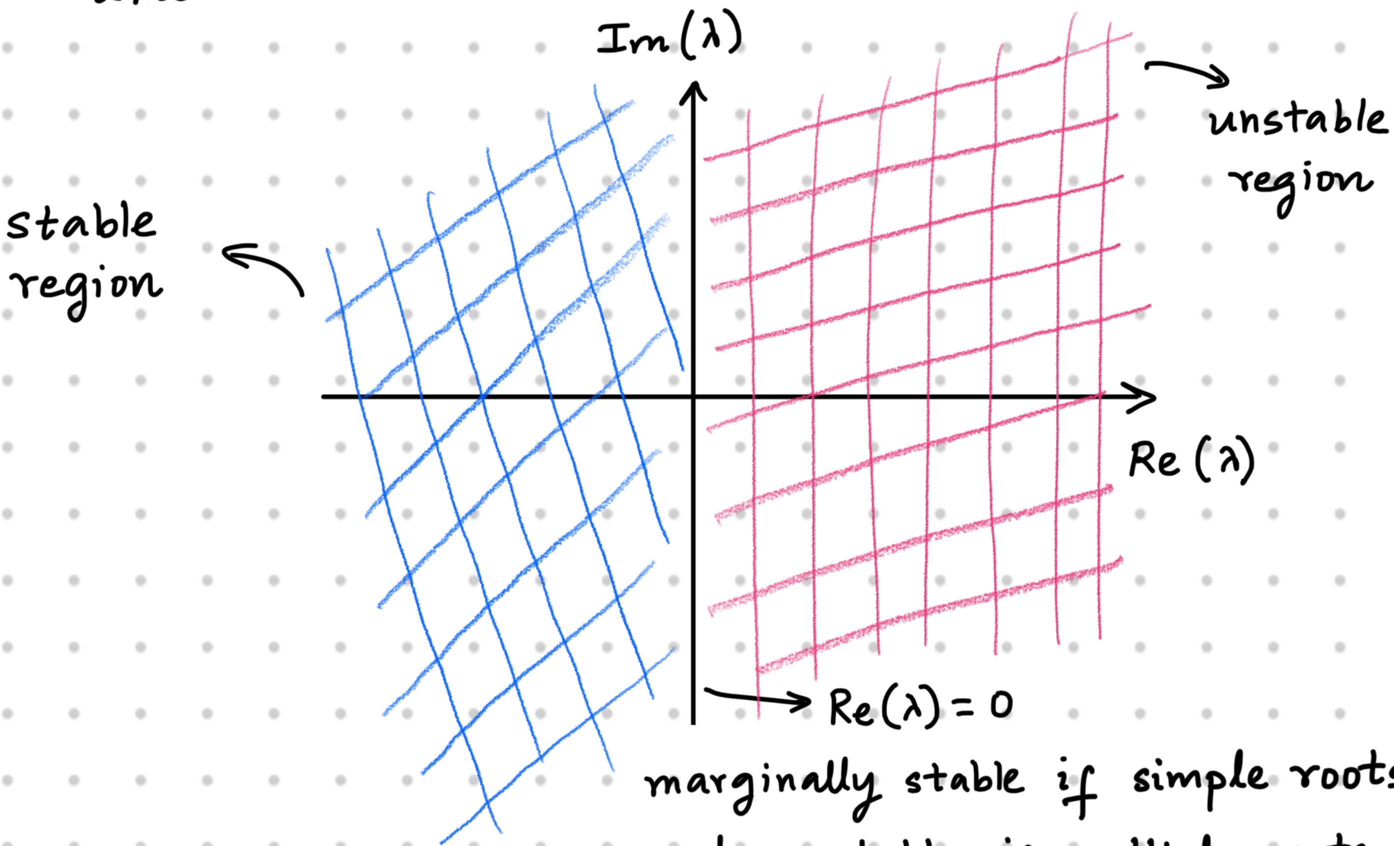
Note that $$\lim_{t \to \infty} t^{i-1} e^{\lambda t} = \begin{cases} 0 & \text{if } Re(\lambda) < 0 \\ \infty & \text{if } Re(\lambda) \geq 0 \end{cases}$$

Hence, we can summarize:

1. An LTIC system is <u>asymptotically stable</u> ($y_0(t) \to 0$ as $t \to \infty$) $\iff$ all the characteristic roots (distinct or repeated) are in the LHP.

2. An LTIC system is <u>marginally stable / bounded</u> ($|y_0(t)| < k$ as $t \to \infty$ $\iff$ there are no roots in the RHP and there are some unrepeated roots on the imaginary axis.

3. An LTIC system is <u>unstable</u> ($y_0(t) \to \infty$ as $t \to \infty$) $\iff$ at least one root is in the RHP or there are repeated roots on the imaginary axis.



marginally stable if simple roots and unstable if multiple roots

Q. Investigate the asymptotic (internal) and the BIBO (external) stability of the following LTIC systems:

(a). $(D^2 + 8D + 12)\, y(t) = (D - 1)\, x(t)$

(b). $D(D^2 + 3D + 2)\, y(t) = (D + 5)\, x(t)$

(c). $(D + 1)(D^2 + 2D + 5)^2\, y(t) = x(t)$

(d). $D^2 (D^2 + 2)\, y(t) = x(t)$

(e). $(D^2 + 1)(D^2 + 4)(D^2 + 9)\, y(t) = 3D\, x(t)$

(f). $(D + 1)(D^2 - 6D + 5)\, y(t) = (3D + 1)\, x(t)$

A. (a). The roots of the characteristic equation are : -2 and -6.

All roots in LHP $\Rightarrow$ Asymptotically stable.

Also, $h(t)$ will be a linear combination of the characteristic modes (e^{-2t} and e^{-6t}). Hence, $h(t)$ is absolutely integrable $\Rightarrow$ BIBO Stable.

(b). The roots of the characteristic equation are : $0, -1, -2$.

One root on the imaginary axis $\Rightarrow$ Marginally stable

However,

$h(t)$ is not absolutely integrable $\Rightarrow$ BIBO unstable.

(c). The roots of the characteristic equation are : -1, $-1 \pm 2j$ (repeated twice).

All roots in LHP $\Rightarrow$ Asymptotically stable and BIBO stable.

(d). The roots of the characteristic equation are : 0 (repeated twice), $\pm\sqrt{2}j$.

Repeated roots on the imaginary axis $\Rightarrow$ Asymptotically unstable and BIBO unstable.

(e). The roots of the characteristic equation are : $\pm j$, $\pm 2j$, and $\pm 3j$.

Unrepeated roots on the imaginary axis $\Rightarrow$ Marginally stable and BIBO unstable.

(f). The roots of the characteristic equation are : -1, 1, and 5. Roots exist in RHP

$\Rightarrow$ Asymptotically and BIBO unstable.

Q. Consider the following LTIC systems with their unit impulse responses. Are these systems BIBO stable?

(a). $h(t) = u(t)$

(b). $h(t) = e^{t}\left[\frac{2}{3}\cos\left(\frac{3t}{2}\right) + \frac{1}{3}\sin(\pi t)\right] u(123-t)$

(c). $h(t) = \frac{1}{t}\, u(t-T)$

(d). $h(t) = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i} \delta(t-i)$

A. For the system to be BIBO stable, $h(t)$ should be absolutely integrable.

(a). $\int_{-\infty}^{\infty} |h(t)|\, dt \to \infty$. Hence, not BIBO stable.

(b). $\int_{-\infty}^{\infty} |h(t)|\, dt < e^{123} < \infty$. Hence, BIBO stable.

(c). $\int_{-\infty}^{\infty} |h(t)|\, dt \to \infty$. Hence, not BIBO stable.

(d). $\int_{-\infty}^{\infty} |h(t)|\, dt = 2 < \infty$. Hence, BIBO stable.