Solutions

A1. (a). $y(t) = 3\,x(t)\,\sin(t)$

Causal — ✓

Linear — ✓ $\quad 0.25 \times 4 = 1$

Time-invariant — ✗

Invertible — ✗

(b). $y(t) = x(t) + 2$

Causal — ✓

Linear — ✗ $\quad 0.25 \times 4 = 1$

Time-invariant — ✓

Invertible — ✓

(c). $y(t) = x\left(\cos(t)\right)$

Causal — ✗

Linear — ✓ $\quad 0.25 \times 4 = 1$

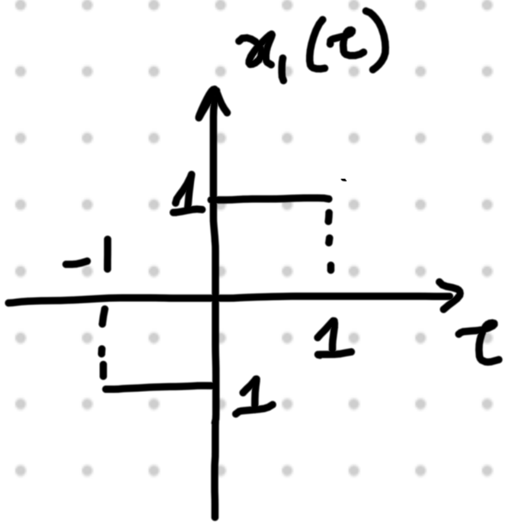
Time-invariant — ✗

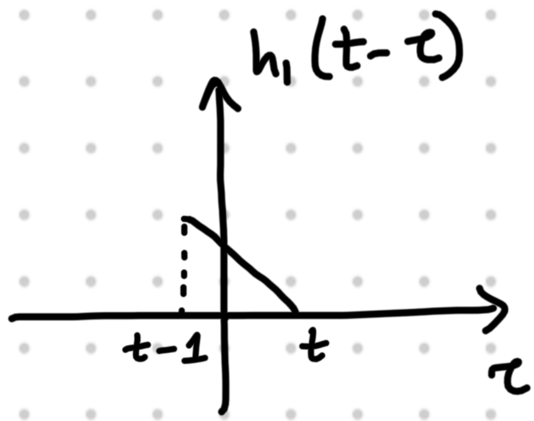
Invertible — ✗

A2. $y_1(t) = x(t) * h(t)$

$$x(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

$$h_1(t) = \begin{cases} t & 0 < t < 1 \\ 0 & \text{else} \end{cases}$$





If $t < -1$: $y_1(t) = 0$ 1

If $-1 < t < 0$:

$$y_1(t) = \int_{-1}^{t} (-1)(t-\tau)\,d\tau$$

$$= -\frac{(1+t+t^2)}{2} = -\frac{(t+1)^2}{2}$$ 1

If $0 < t < 1$: $\quad y_1(t) = \int_{t-1}^{0} (-1)(t-\tau)\,d\tau + \int_{0}^{t} 1\,(t-\tau)\,d\tau$

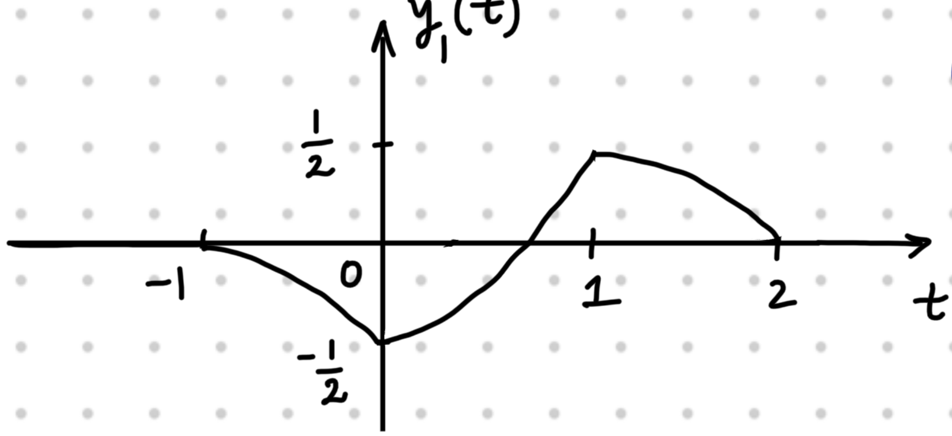
$$y_1(t) = t^2 - \frac{1}{2}$$ 1

If $1 < t < 2$: $\quad y_1(t) = \int_{t-1}^{1} (t-\tau)\,d\tau = t - \frac{t^2}{2}$ 1

If $t \geqslant 2$: $\quad y_1(t) = 0$ 1

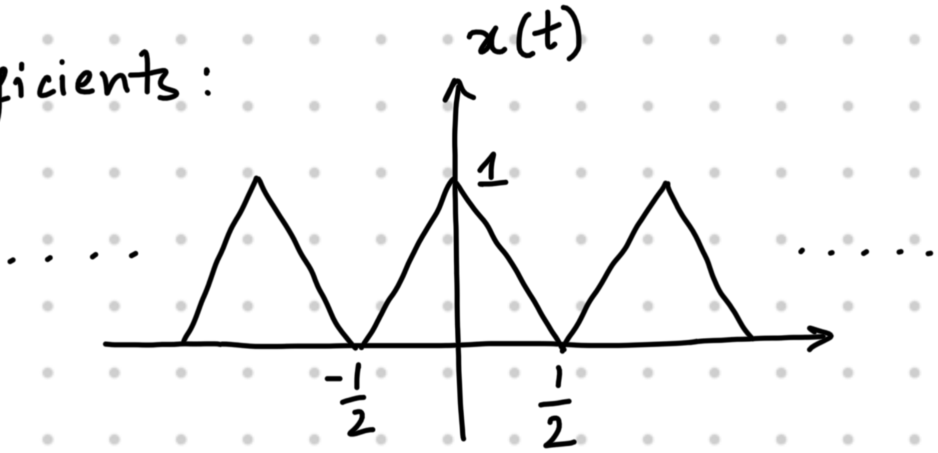
Hence,

$$y_1(t) = \begin{cases} 0 & t \le -1 \\ -\frac{(t+1)^2}{2} & -1 \le t \le 0 \\ t^2 - \frac{1}{2} & 0 \le t \le 1 \\ t - \frac{t^2}{2} & 1 \le t \le 2 \\ 0 & 2 \le t \end{cases}$$



1

A3(a). D_n coefficients :



$$T_0 = 1$$

$$\omega_0 = 2\pi$$

$$D_n = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$D_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x(t)\, dt = \frac{1}{2}\cdot 1 \cdot 1 = \frac{1}{2} \qquad 1$$

For $n \neq 0$:

$$D_n = \int_{-\frac{1}{2}}^{0} (2t+1)\, e^{-jn2\pi t}\, dt + \int_{0}^{\frac{1}{2}} (-2t+1)\, e^{-jn2\pi t}\, dt$$

$$= \left. \frac{2\,t\,e^{-jn2\pi t}}{-jn2\pi} \right|_{-\frac{1}{2}}^{0} + \int_{-\frac{1}{2}}^{0} \frac{2\,e^{-jn2\pi t}}{jn2\pi}\, dt + \left. \frac{e^{-jn2\pi t}}{-jn2\pi} \right|_{-\frac{1}{2}}^{0}$$

$$+ \left. \frac{2\,t\,e^{-jn2\pi t}}{jn2\pi} \right|_{0}^{\frac{1}{2}} - \int_{0}^{\frac{1}{2}} \frac{2\,e^{-jn2\pi t}}{jn2\pi}\, dt - \left. \frac{e^{-jn2\pi t}}{jn2\pi} \right|_{0}^{\frac{1}{2}}$$

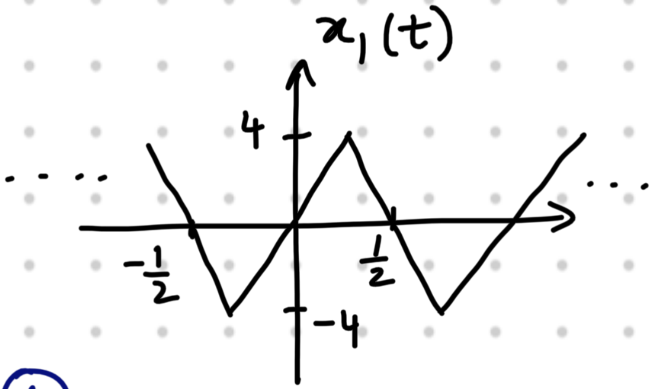
$$= \frac{-e^{jn\pi}}{jn2\pi} + \frac{1-e^{jn\pi}}{2n^2\pi^2} + \frac{e^{jn\pi}-1}{jn2\pi} + \frac{e^{-jn\pi}}{jn2\pi}$$

$$+ \frac{1-e^{-jn\pi}}{2n^2\pi^2} + \frac{1-e^{-jn\pi}}{jn2\pi} = \frac{1-(-1)^n}{n^2\pi^2} \quad 3$$

Hence, $$D_n = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{1-(-1)^n}{n^2\pi^2} & n \neq 0 \end{cases}$$

Further, $D_1 + D_2 + D_3 + D_4 + D_5 = \frac{518}{225} \cdot \frac{1}{\pi^2}$ 1

A 3(b).



$x_1(t) = 8\,x\left(t-\frac{1}{4}\right) - 4$ 1

$\Rightarrow \quad D_0^1 = 8D_0 - 4 = 0$ 1

$D_n^1 = 8\,D_n\,e^{-jn\frac{\pi}{2}} = -8j\,D_n \sin\left(\frac{n\pi}{2}\right)$ 1

Hence, $$D_n^1 = \begin{cases} 0 & n = 0 \\ -8j\left(\frac{1-(-1)^n}{n^2\pi^2}\right)\sin\left(\frac{n\pi}{2}\right) & n \neq 0 \end{cases}$$

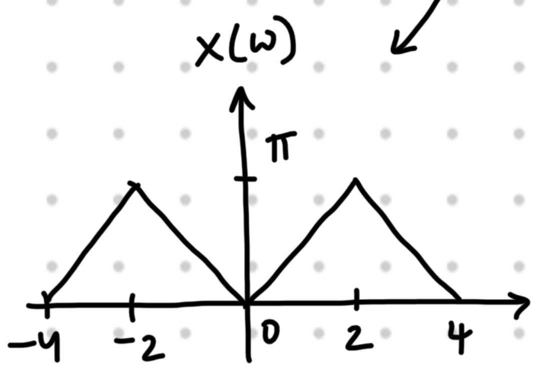
A 4.

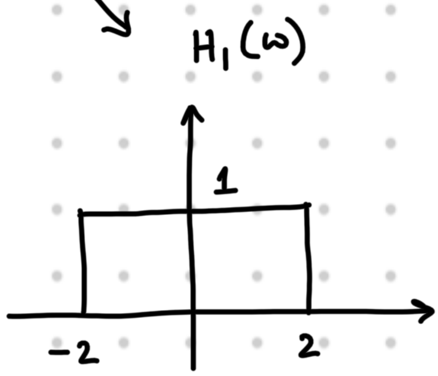
(a). Plot $Y_1(\omega)$

Given
$$x(t) = 2\ \text{sinc}^2(t)\cos 2t$$
$$h_1(t) = \frac{\sin 2t}{\pi t}$$

$$y_1(t) = x(t) * h_1(t)$$

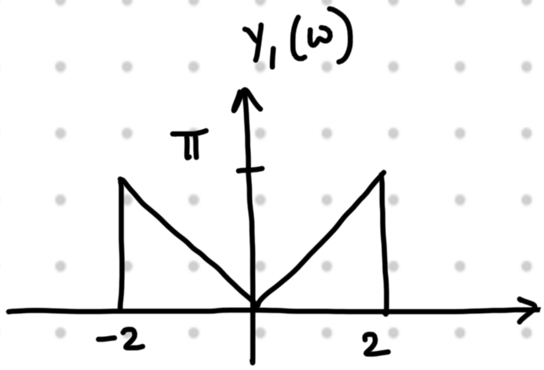
$$\Rightarrow \quad Y_1(\omega) = X(\omega) \cdot H_1(\omega)$$





1

$\Rightarrow$ Plot of $Y_1(\omega)$:

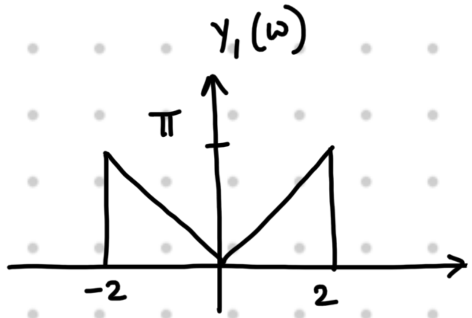


1

(b). <u>Plot $y_1(t) \cdot z_1(t)$</u>

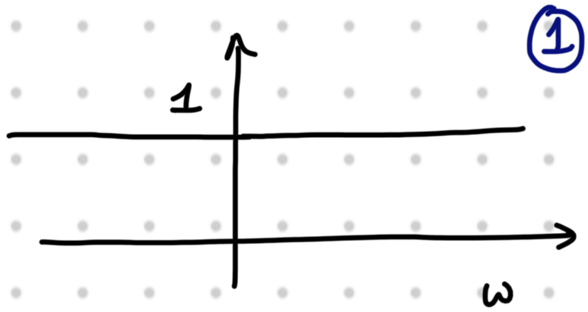
Given $z_1(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\pi)$

$$y_1(t) \cdot z_1(t) \Longleftrightarrow \frac{1}{2\pi} \, Y_1(\omega) * Z_1(\omega)$$



$*$ $\quad 2 \sum_{n=-\infty}^{\infty} \delta(\omega - 2n)$

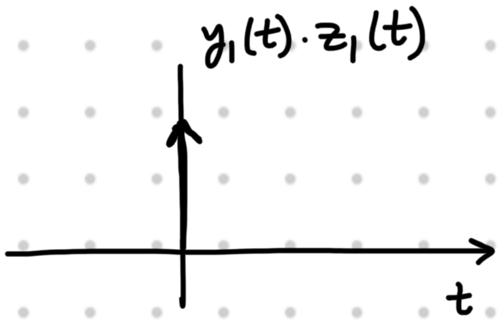
$$y_1(t) \cdot z_1(t) \Longleftrightarrow \frac{1}{\pi} \sum_{n=-\infty}^{\infty} Y_1(\omega - 2n)$$



1

$$\Rightarrow \quad y_1(t) \cdot z_1(t) = \delta(t)$$

Hence,



1

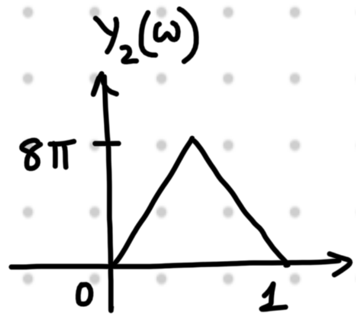
(c). Compute $y_2(\pi)$ and $Y_2(\pi)$

Given $h_2(t) = 2\,\text{sinc}^2\left(\frac{t}{4}\right) e^{\frac{jt}{2}}$

$$y_2(t) = \underbrace{\left(y_1(t) \cdot z_1(t)\right)}_{\delta(t)} * h_2(t) = h_2(t)$$

$$\Rightarrow \quad y_2(\pi) = h_2(\pi) = \frac{16j}{\pi^2} \quad 1$$

Also, $Y_2(\omega) = H_2(\omega)$



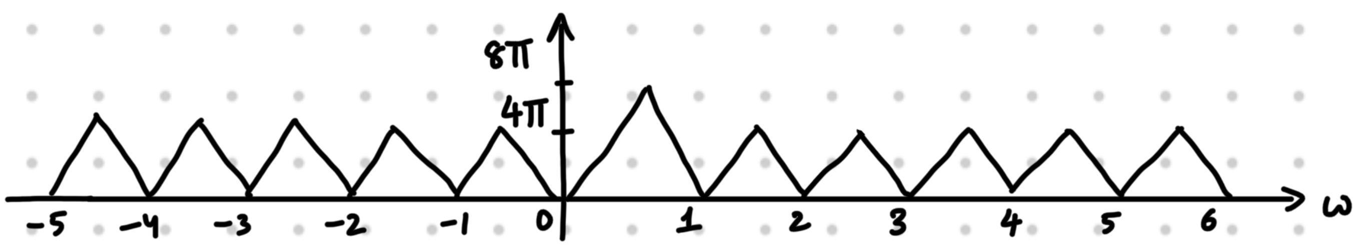
$$\Rightarrow \quad Y_2(\omega) = 8\pi\,\Delta\left(\omega - \frac{1}{2}\right)$$

$$\Rightarrow \quad Y_2(\pi) = 0 \quad 1$$

(d). Plot the F.T. of $y_2(t)\, z_2(t)$

Given $z_2(t) = \sum_{k=0}^{5} \cos kt$

$$y_2(t) \cdot z_2(t) \iff \frac{1}{2\pi} Y_2(\omega) * Z_2(\omega)$$



2

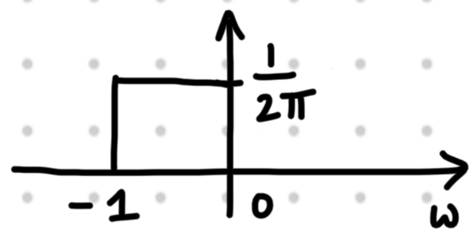
(e). Compute $\displaystyle\int_{-\infty}^{\infty} |y(t)|^2 \, dt$

Given $\displaystyle h_3(t) = \frac{1}{4\pi^2} \operatorname{sinc}\left(\frac{t}{2}\right) e^{-jt}$

$$y(t) = \left(y_2(t) \cdot z_2(t)\right) * h_3(t)$$

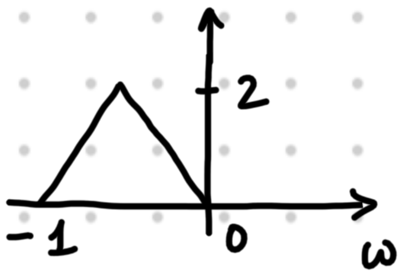
$$\Rightarrow \quad Y(\omega) = \left(\frac{1}{2\pi} Y_2(\omega) * Z_2(\omega)\right) \cdot H_3(\omega)$$

$$H_3(\omega) = \frac{1}{2\pi} \operatorname{rect}(\omega + 1)$$



1

$\Rightarrow$ $Y(\omega)$

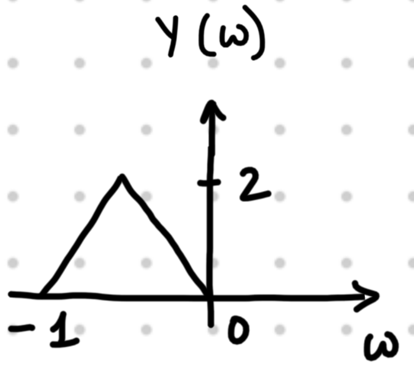


Hence, $$\int_{-\infty}^{\infty} |y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \left(\int_{-1}^{-\frac{1}{2}} \left(4(1+\omega)\right)^2 d\omega + \int_{-\frac{1}{2}}^{0} (-4\omega)^2 d\omega \right)$$

$$= \frac{1}{2\pi} \left(\frac{2}{3} + \frac{2}{3} \right) = \frac{2}{3\pi}$$ 1

(f). <u>Plot $y(t)\, e^{jt}$</u>

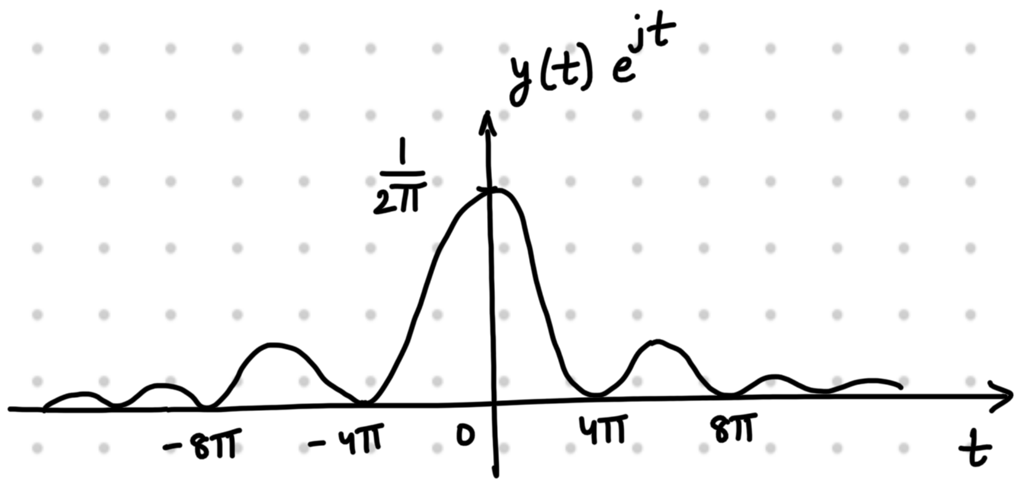


$$= \quad 2\,\Delta(\omega+1)$$

$$2\,\Delta(\omega+1) \iff \frac{1}{2\pi}\,\mathrm{sinc}^2\left(\frac{t}{4}\right) e^{-jt}$$

Hence, $$y(t) = \frac{1}{2\pi}\,\mathrm{sinc}^2\left(\frac{t}{4}\right) e^{-jt}$$

$$\Rightarrow \quad y(t)\, e^{jt} = \frac{1}{2\pi}\,\mathrm{sinc}^2\left(\frac{t}{4}\right)$$ 1



1

A5.

$$\int_{-\infty}^{\infty} \operatorname{sinc} t \; e^{-j\omega t} \, dt = X(\omega)$$

where $X(\omega) = \mathcal{F}\{\operatorname{sinc} t\} \Longleftrightarrow \pi \operatorname{rect}\left(\frac{\omega}{2}\right)$

Hence, $$\int_{-\infty}^{\infty} \operatorname{sinc} t \, dt = \underset{\omega \to 0}{Lt} X(\omega) = \pi$$ 0.5

$$\Rightarrow \int_{0}^{\infty} \operatorname{sinc}(2t-1) \, dt = \frac{\pi}{4}$$ 1

Next, $$x(t) = \int_{-\infty}^{\infty} X(\omega) \, e^{j\omega t} \, d\omega$$

using $\Delta\left(\frac{t}{\tau}\right) \Longleftrightarrow \frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$

$\Rightarrow \frac{1}{2} \Delta\left(\frac{t}{4}\right) \Longleftrightarrow \operatorname{sinc}^2(\omega)$ 0.5

Hence, $$\int_{-\infty}^{\infty} 4 \operatorname{sinc}^2(\omega) \, e^{j\omega} \, d\omega = 4\,x(1) = 4 \, \frac{1}{2} \Delta\left(\frac{1}{4}\right)$$

$$= 1$$ 1

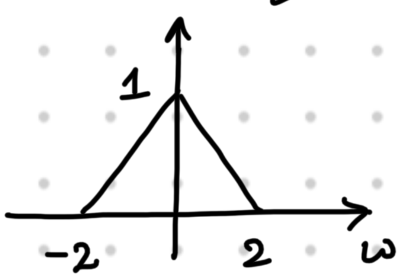
Extra Credit

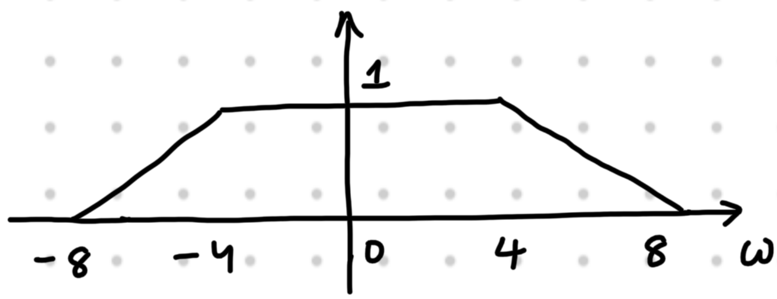
A6. $$\int_{-\infty}^{\infty} \left| \underbrace{\left(\frac{\sin c^2 t}{\pi}\right)}_{x(t)} * \underbrace{\left(\frac{\pi}{2} \, \frac{\sin 2t}{\pi t} \, \frac{\sin 6t}{\pi t}\right)}_{h(t)} \right|^2 dt$$

Using $$\int_{-\infty}^{\infty} |y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

1





1

Hence, $Y(\omega) = X(\omega)$

$$\int_{-\infty}^{\infty} |y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{4}{3} = \frac{2}{3\pi}$$ 1