Q. (a). If $x(t) \Longleftrightarrow X(s)$, then prove that $-t\,x(t) \Longleftrightarrow \frac{d}{ds}X(s)$.

(b). Using $e^{-2t}u(t) \Longleftrightarrow \frac{1}{s+2}$, determine the unilateral Laplace Transform of :

$$x(t) = t\,e^{-2(t-3)}u(t-2)$$

A. (a).
$$X(s) = \int_{0^-}^{\infty} x(t)\,e^{-st}\,dt$$

$$\Rightarrow \quad \frac{dX(s)}{ds} = \int_{0^-}^{\infty} \left(-t\,x(t)\right) e^{-st}\,dt \Longleftrightarrow -t\,x(t)$$

(b).
$$e^{-2t}u(t) \Longleftrightarrow \frac{1}{s+2}$$

$$e^{-2(t-2)}u(t-2) \Longleftrightarrow e^{-2s}\frac{1}{s+2}$$

$$e^{-2(t-3)}u(t-2) \Longleftrightarrow e^{-2(s-1)}\frac{1}{s+2}$$

$$t\,e^{-2(t-3)}u(t-2) \Longleftrightarrow -\frac{d}{ds}\left(e^{-2(s-1)}\cdot\frac{1}{s+2}\right)$$

$$t\,e^{-2(t-3)}u(t-2) \Longleftrightarrow \frac{2s+5}{(s+2)^2}\,e^{-2(s-1)}$$

Q. Using initial and final value theorems, find the initial and final values of the zero-state response of a system with :

(a). $H(s) = \dfrac{6s^2 + 3s + 10}{2s^2 + 6s + 5}$, $x(t) = e^{-t} u(t)$

(b). $Y(s) = \dfrac{s^2 + 5s + 6}{s^2 + 3s + 2}$

A. (a). $Y(s) = H(s) \cdot X(s) = \dfrac{6s^2 + 3s + 10}{(2s^2 + 6s + 5)(s+1)}$

$$y(0^+) = \lim_{s \to \infty} s\,Y(s) = \lim_{s \to \infty} \frac{6s^3 + 3s^2 + 10s}{2s^3 + 8s^2 + 11s + 5} = 3$$

$$y(\infty) = \lim_{s \to 0} s\,Y(s) = 0$$

(b). $Y(s) = \dfrac{s^2 + 5s + 6}{s^2 + 3s + 2} = 1 + \dfrac{(2s + 4)}{s^2 + 3s + 2}$

$$y(0^+) = \lim_{s \to \infty} \frac{2s^2 + 4s}{s^2 + 3s + 2} = 2$$

$$y(\infty) = \lim_{s \to 0} \frac{2s^2 + 4s}{s^2 + 3s + 2} = 0$$

Q. An LTI system produces output

$y(t) = e^{-t}u(t) - e^{-2t}u(t)$ for a unit step input.

Determine the output of this system for a new input $x(t) = \delta(t-\pi) - 3u(t)$.

A. For a unit step input, $X(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}$$

$$\Rightarrow \quad H(s) = \frac{s}{(s+1)(s+2)} = \frac{2}{s+2} - \frac{1}{s+1}$$

For the new input,

$$X(s) = e^{-s\pi} - \frac{3}{s}$$

$$\Rightarrow \quad Y(s) = \frac{2e^{-s\pi}}{s+2} - \frac{e^{-s\pi}}{s+1} - \frac{3}{(s+1)(s+2)}$$

Hence,

$$y(t) = \left[2e^{-2(t-\pi)} - e^{-(t-\pi)}\right]u(t-\pi) + 3\left(e^{-2t} - e^{-t}\right)u(t)$$

Q. For a system with transfer function

$$H(s) = \frac{s+5}{s^2+5s+6}$$

find the zero-state response if the input is :

(a). $e^{-3t}\,u(t)$

(b). $e^{-4(t-5)}\,u(t-5)$

(c). $e^{-4(t-5)}\,u(t)$

A. (a). $e^{-3t}\,u(t) \Longleftrightarrow \frac{1}{s+3} = X(s)$

$$\Rightarrow \quad Y(s) = H(s)\,X(s) = \frac{s+5}{(s+3)(s+2)} \cdot \frac{1}{(s+3)}$$

$$\Rightarrow \quad Y(s) = \frac{k_1}{s+2} + \frac{k_2}{s+3} + \frac{k_3}{(s+3)^2}$$

$$\Rightarrow \quad Y(s) = \frac{3}{s+2} - \frac{3}{s+3} - \frac{2}{(s+3)^2}$$

$$\Rightarrow \quad y(t) = \left(3\,e^{-2t} - 3\,e^{-3t} - 2t\,e^{-3t}\right)u(t)$$

(b). First, $x_1(t) = e^{-4t}\,u(t) \Longleftrightarrow \dfrac{1}{s+4} = X_1(s)$

$$\Rightarrow \quad Y_1(s) = H(s)\,X_1(s) = \frac{s+5}{(s+2)(s+3)(s+4)}$$

$$= \frac{3}{2}\,\frac{1}{(s+2)} - \frac{2}{(s+3)} + \frac{1}{2(s+4)}$$

$$\Rightarrow \quad y_1(t) = \left(\frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}\right)u(t)$$

Now, $x(t) = e^{-4(t-5)}\,u(t-5) = x_1(t-5)$

$$\Rightarrow \quad X(s) = X_1(s)\,e^{-5s} \Rightarrow Y(s) = Y_1(s)\cdot e^{-5s}$$

$$\Rightarrow \quad y(t) = \left[\frac{3}{2}e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2}e^{-4(t-5)}\right]u(t-5)$$

(c). $x(t) = e^{-4(t-5)}\,u(t) = e^{20}\cdot x_1(t)$

$$\Rightarrow \quad X(s) = e^{20}\,X_1(s) \Rightarrow Y(s) = e^{20}\,Y_1(s)$$

$$\Rightarrow \quad Y(s) = e^{20}\left(\frac{3}{2}\,\frac{1}{s+2} - \frac{2}{s+3} + \frac{1}{2(s+4)}\right)$$

$$\Rightarrow \quad y(t) = e^{20}\left(\frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}\right)u(t)$$