Fourier Series Representation

We saw earlier that an arbitrary signal $x(t)$ can be expressed as a sum of its impulse components. This decomposition helped in obtaining the zero-state response of an LTIC system by simply computing the convolution of the input and the impulse response.

That is, $y(t) = x(t) * h(t)$

zero-state component

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Suppose we have $x(t) = e^{st}$ (complex exponential)

then $$y(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)}\, h(\tau)\, d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} e^{-s\tau}\, h(\tau)\, d\tau}_{H(s)}$$

$$\Rightarrow \quad y(t) = x(t)\, H(s)$$

Hence, we see that the output of $x(t) = e^{st}$ is $H(s)$ (a constant) times the input. Hence, the complex exponential is an eigenfunction of the system and $H(s)$ is the corresponding eigenvalue.

More generally, if the input to an LTIC system is represented as a linear combination of complex exponentials (or sinusoids), then the output is also a linear combination of the same complex exponentials.

That is, $\sum_k a_k e^{s_k t} \longrightarrow \boxed{S} \longrightarrow \sum_k a_k H(s_k) e^{s_k t}$

Hence, representing signals as complex exponentials (or sinusoids) will help us in analyzing LTIC systems.

It so happens that there are infinite possible ways of expressing an input $x(t)$ in terms of other signals. We will now see how to express $x(t)$ as the sum of a general set of orthogonal signals.

Orthogonal Signal Set

Suppose we have N signals $x_1(t), x_2(t), \cdots, x_N(t)$ that are mutually orthogonal over an interval $[t_1, t_2]$.

Then,
$$\int_{t_1}^{t_2} x_m(t) \cdot x_n(t)\, dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

where E_n is the energy of the signal $x_n(t)$

Now, let us approximate a signal $x(t)$ over the interval $[t_1, t_2]$ using the N signals as:

$$x(t) \approx c_1 x_1(t) + c_2 x_2(t) + \cdots\cdots + c_N x_N(t)$$

$$\Rightarrow x(t) \approx \sum_{n=1}^{N} c_n x_n(t)$$

The error $e(t)$ is then given by:

$$e(t) = x(t) - \sum_{n=1}^{N} c_n x_n(t)$$

We can minimize the error signal $e(t)$ by minimizing its energy E_e over the interval $[t_1, t_2]$.

That is, solve $\dfrac{dE_e}{dc_n} = 0$

Substituting $E_e = \displaystyle\int_{t_1}^{t_2} \left(x(t) - \sum_{n=1}^{N} c_n x_n(t)\right)^2 dt$ and

solving the equation gives us:

$$c_n = \frac{\displaystyle\int_{t_1}^{t_2} x(t)\, x_n(t)\, dt}{\displaystyle\int_{t_1}^{t_2} x_n^2(t)\, dt} = \frac{1}{E_n} \int_{t_1}^{t_2} x(t)\, x_n(t)\, dt$$

For the above choice of c_n, the error signal energy is given by :

$$E_e = \int_{t_1}^{t_2} x^2(t)\, dt - \sum_{n=1}^{N} c_n^2 E_n$$

Clearly, an N increases, E_e decreases. Hence, it is possible that $E_e \to 0$ as $N \to \infty$. When this happens, the orthogonal set is said to be complete.

Hence, $x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots\cdots$

$$\Rightarrow \quad x(t) = \sum_{n=1}^{\infty} c_n x_n(t)$$

This is called as the generalized Fourier series of $x(t)$ with respect to the orthogonal set $\{x_n(t)\}$, which is also called a set of basis signals.

<u>Note</u>: Extending the analysis for complex signals, suppose we have N mutually orthogonal functions,

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

Then, $x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots\cdots = \sum_{n=1}^{\infty} c_n x_n(t)$

where $$c_n = \frac{1}{E_n} \int_{t_1}^{t_2} x(t)\, x_n^*(t)\, dt$$