Q. An LTIC system is specified by the equation:

$$(D^2 + 5D + 6)\, y(t) = (D + 1)\, x(t)$$

(a). Find the characteristic polynomial, characteristic equation, characteristic roots, and characteristic modes of this system.

(b). Find $y_0(t)$, the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial conditions are $y_0(0^-) = 2$ and $\dot{y}_0(0^-) = -1$.

A. (a). The characteristic polynomial is $\lambda^2 + 5\lambda + 6$.

The characteristic equation is $\lambda^2 + 5\lambda + 6 = 0$

The characteristic roots are $\lambda_1 = -2$ and $\lambda_2 = -3$.

The characteristic modes are e^{-2t} and e^{-3t}.

(b).

$$y_0(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$\Rightarrow \quad \dot{y}_0(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

substituting $y_0(0^-) = 2$ and $\dot{y}_0(0^-) = -1$ and solving the equation, we get

$$y_0(t) = 5e^{-2t} - 3e^{-3t}.$$

Q. An LTIC system is specified by the equation:

$$(D^2 + 4D + 4)\, y(t) = D\, x(t)$$

(a). Find the characteristic polynomial, characteristic equation, characteristic roots, and characteristic modes of this system.

(b). Find $y_0(t)$, the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial conditions are $y_0(0^-) = 3$ and $\dot{y}_0(0^-) = -4$.

A. (a). The characteristic polynomial is $\lambda^2 + 4\lambda + 4$.

The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$

The characteristic roots are $\lambda_1 = \lambda_2 = -2$

The characteristic modes are e^{-2t} and te^{-2t}.

(b).
$$y_0(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$\Rightarrow \quad \dot{y}_0(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

substituting $y_0(0^-) = 3$ and $\dot{y}_0(0^-) = -4$ and solving the equation, we get

$$y_0(t) = (3 + 2t)\, e^{-2t}.$$

Q. An LTIC system is specified by the equation: $(D^2 + 9)\, y(t) = (3D + 2)\, x(t)$

(a). Find the characteristic polynomial, characteristic equation, characteristic roots, and characteristic modes of this system.

(b). Find $y_0(t)$, the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial conditions are $y_0(0^-) = 0$ and $\dot{y}_0(0^-) = 6$.

A. (a). The characteristic polynomial is $\lambda^2 + 9$.

The characteristic equation is $\lambda^2 + 9 = 0$

The characteristic roots are $\lambda_1 = 3j$, $\lambda_2 = -3j$

The characteristic modes are e^{3jt} and e^{-3jt}.

(b). $$y_0(t) = c \cos(3t + \theta)$$

$$\Rightarrow \dot{y}_0(t) = -3c \sin(3t + \theta)$$

substituting $y_0(0^-) = 0$ and $\dot{y}_0(0^-) = 6$ and solving the equation, we get

$$y_0(t) = 2 \cos(3t - \pi/2) = 2 \sin 3t.$$

Q. An LTIC system is specified by the equation:

$$D^2(D+1)\,y(t) = (D^2+2)\,x(t)$$

(a). Find the characteristic polynomial, characteristic equation, characteristic roots, and characteristic modes of this system.

(b). Find $y_0(t)$, the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial conditions are $y_0(0^-) = 4$, $\dot{y}_0(0^-) = 3$, and $\ddot{y}_0(0^-) = -1$.

A. (a). The characteristic polynomial is $\lambda^2(\lambda+1)$.

The characteristic equation is $\lambda^2(\lambda+1) = 0$

The characteristic roots are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -1$

The characteristic modes are 1, t, and e^{-t}.

(b). $$y_0(t) = c_1 + c_2 t + c_3 e^{-t}$$

$$\Rightarrow \quad \dot{y}_0(t) = c_2 - c_3 e^{-t}, \quad \ddot{y}_0(t) = c_3 e^{-t}$$

substituting $y_0(0^-) = 4$, $\dot{y}_0(0^-) = 3$, and $\ddot{y}_0(0^-) = -1$, and solving the equation, we get

$$y_0(t) = 5 + 2t - e^{-t}.$$

Zero - State Response :

This is the system response $y(t)$ to an input $x(t)$ when the system is in zero state (i.e. all initial conditions are zero).

But before studying the zero-state response for an arbitrary input $x(t)$, we should be familiar be the unit impulse response $h(t)$.

The unit impulse response $h(t)$ of a system is the system's response to an impulse input applied at $t=0$ with all initial conditions zero at $t = 0^-$

$$\delta(t) \longrightarrow \boxed{\;S\;} \longrightarrow h(t)$$

For an LTIC system described by the n-th order differential equation with unit impulse input,

$$Q(D)\, h(t) = P(D)\, \delta(t)$$

The impulse $\delta(t)$ appears momentarily at $t=0$ and disappears forever. But in that moment, it generates energy storages ; that is it creates non zero initial conditions in the system at $t=0^+$ and then vanishes.

Hence, the impulse response $h(t)$ must consist of the system's characteristic modes for $t \geq 0^+$.

$$h(t) = \text{characteristic mode terms} \quad \text{for } t \geq 0^+$$

At $t=0$, there can at most be an impulse for $m \leq n$ (practical systems).

Hence, $$h(t) = A_0\,\delta(t) + \text{characteristic mode terms} \quad t \geq 0$$

For $m=n$,

$$h(t) = \underbrace{b_n}_{\text{coefficient of the } n^{th} \text{ order term in } P(D)}\,\delta(t) + \underbrace{\big(P(D)\,y_n(t)\big)}_{\text{linear combination of the characteristic modes of the system subject to the following initial conditions:}}u(t)$$

coefficient of the n^{th} order term in $P(D)$

linear combination of the characteristic modes of the system subject to the following initial conditions:

$$y_n^{(n-1)}(0) = 1 \quad \text{and} \quad y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = \cdots\cdots = y_n^{(n-2)}(0) = 0$$

For $m < n$, $b_n = 0$

$$\Rightarrow \quad h(t) = \big(P(D)\,y_n(t)\big)\,u(t)$$

Q. Find the unit impulse response of a system specified by the equation:

$$\left(D^2 + 4D + 3\right) y(t) = (D+5)\, x(t)$$

A. The characteristic equation is :

$$\lambda^2 + 4\lambda + 3 = 0$$

$$\Rightarrow \quad (\lambda+1)(\lambda+3) = 0 \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = -3$$

$$y_n(t) = c_1 e^{-t} + c_2 e^{-3t}$$

$$\dot{y}_n(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$$

substituting $y(0) = 0$ and $\dot{y}(0) = 1$, we get

$$y_n(t) = \frac{e^{-t} - e^{-3t}}{2}$$

Now, $$h(t) = \left[P(D)\, y_n(t)\right] u(t) = \left[(D+5)\, \frac{e^{-t} - e^{-3t}}{2}\right] u(t)$$

$$\Rightarrow \quad h(t) = \frac{-e^{-t} + 3e^{-3t} + 5e^{-t} - 5e^{-3t}}{2} \cdot u(t)$$

Hence, $$h(t) = \left(2e^{-t} - e^{-3t}\right) u(t)$$

Q. Find the unit impulse response of a system specified by the equation:

$$\left(D^2 + 6D + 9\right) y(t) = \left(2D + 9\right) x(t)$$

A. The characteristic equation is:

$$\lambda^2 + 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda + 3)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = -3$$

$$y_n(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

$$\dot{y}_n(t) = -3c_1 e^{-3t} - 3c_2 t e^{-3t} + c_2 e^{-3t}$$

substituting $y(0) = 0$ and $\dot{y}(0) = 1$, we get

$$y_n(t) = t e^{-3t}$$

Now, $h(t) = \left[P(D)\, y_n(t)\right] u(t) = \left[(2D+9)\, t e^{-3t}\right] u(t)$

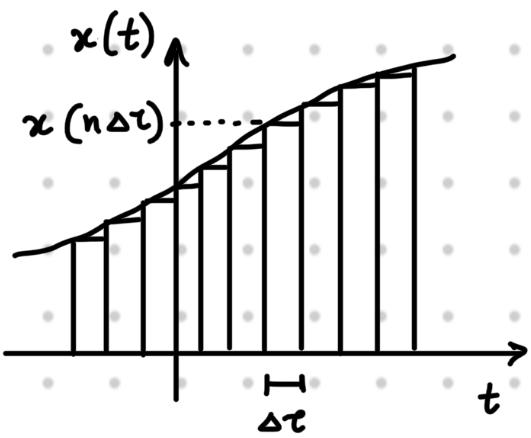
$$\Rightarrow \quad h(t) = \left(2e^{-3t} - 6te^{-3t} + 9te^{-3t}\right) \cdot u(t)$$

Hence, $h(t) = (2 + 3t)\, e^{-3t} \cdot u(t)$

Having understood the unit impulse response $h(t)$, we can now derive the zero-state response $y(t)$ to an arbitrary input $x(t)$ with no initial conditions.

First, using the superposition principle, if we express $x(t)$ in terms of impulses, then the total system response is the sum of the system's response to each impulse component.

Hence, if we simply know the impulse response of the system, we can determine the system's response to any arbitrary input $x(t)$.



In the figure, $x(t)$ is shown as a sum of rectangular pulses, each of width $\Delta\tau$.
In the limit as $\Delta\tau \to 0$, each pulse approaches an impulse having a strength equal to the area under the pulse.

As $\Delta\tau \to 0$, the shaded rectangular pulse located at $t = n\Delta\tau$ can be represented as an impulse:

$$x(n\Delta\tau) \cdot \Delta\tau \cdot \delta(t - n\Delta\tau)$$

Hence, for an LTIC system, if

$$\delta(t) \longrightarrow \boxed{S} \longrightarrow h(t)$$

then,

$$\delta(t - n\Delta\tau) \longrightarrow \boxed{S} \longrightarrow h(t - n\Delta\tau)$$

$$x(n\Delta\tau)\,\Delta\tau\;\delta(t - n\Delta\tau) \longrightarrow \boxed{S} \longrightarrow x(n\Delta\tau)\cdot\Delta\tau\cdot h(t - n\Delta\tau)$$

$$\lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau)\cdot\Delta\tau\cdot\delta(t - n\Delta\tau) \longrightarrow \boxed{S} \longrightarrow \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau)\cdot\Delta\tau\cdot h(t - n\Delta\tau)$$

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}_{x(t)} \longrightarrow \boxed{S} \longrightarrow \underbrace{\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}_{y(t)}$$

Therefore, if we know the impulse response $h(t)$ of an LTIC system, we can obtain the system response to any input $x(t)$ as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

convolution integral