Q. Fill in the blanks below:

(a). $\delta(t) \Longleftrightarrow$ ______ (b). $\delta(\omega) \Longleftrightarrow$ ______

(c). $\delta(\omega-\omega_0) \Longleftrightarrow$ ______ (d). $\cos(\omega_0 t) \Longleftrightarrow$ ______

(e). $\sum_{n=-\infty}^{\infty} \delta(t-nT_0) \Longleftrightarrow$ ______ (f). $u(t) \Longleftrightarrow$ ______

Infinite impulse train

(g). $\operatorname{sgn}(t) = \begin{cases} 1 & t>0 \\ -1 & t<0 \\ 0 & t=0 \end{cases} \Longleftrightarrow$ ______

signum function

A. (a). $\int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$. Hence, $\delta(t) \Longleftrightarrow 1$.

(b). $\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}$. Hence, $\delta(\omega) \Longleftrightarrow \frac{1}{2\pi}$

and $1 \Longleftrightarrow 2\pi\,\delta(\omega)$

(c). $\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$.

Hence, $\delta(\omega-\omega_0) \Longleftrightarrow \frac{1}{2\pi} e^{j\omega_0 t}$ and $e^{j\omega_0 t} \Longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$

(d). $\cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

using $e^{j\omega_0 t} \Longleftrightarrow 2\pi\, \delta(\omega - \omega_0)$,

$$\cos \omega_0 t \Longleftrightarrow \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

(e). This is a periodic signal with $T_0 = \frac{2\pi}{\omega_0}$.

Hence, $x(t) = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\omega_0 t}$ with $D_n = \frac{1}{T_0} = \frac{\omega_0}{2\pi}$

Then, $X(\omega) = \sum_{n=-\infty}^{\infty} \frac{\omega_0}{2\pi} \cdot 2\pi\, \delta(\omega - \omega_0) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0)$

(f). The Fourier Transform of $u(t)$ cannot be computed using direct integration. Hence, we write $u(t) = \lim_{a \to 0} e^{-at} u(t)$

Then, $\mathcal{F}\{u(t)\} = \lim_{a \to 0} \mathcal{F}\{e^{-at} u(t)\}$

$$= \lim_{a \to 0} \frac{1}{a + j\omega} = \lim_{a \to 0} \left\{\frac{a}{a^2 + \omega^2} - \frac{j\omega}{a^2 + \omega^2}\right\}$$

Hence, $\mathcal{F}\{u(t)\} = \pi\, \delta(\omega) + \frac{1}{j\omega}$

(g). $\text{sgn}(t) = u(t) - u(-t) = 2u(t) - 1$

Hence, $\mathcal{F}\{\text{sgn}(t)\} = 2\pi\, \delta(\omega) + \frac{2}{j\omega} - 2\pi\, \delta(\omega) = \frac{2}{j\omega}$

Q. Find the Fourier Transforms of :

(a). $$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0 & \text{else} \end{cases}$$

(b). $$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{2|t|}{\tau} & |t| < \frac{\tau}{2} \\ 0 & \text{else} \end{cases}$$

A. (a). $$X(\omega) = \int_{-\tau/2}^{\tau/2} 1 \cdot e^{-j\omega t}\, d\omega = \frac{2 \sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

$$= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

Hence, $\text{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$

(b). $$X(\omega) = \int_{-\frac{\tau}{2}}^{0} \left(1 + \frac{2t}{\tau}\right) e^{-j\omega t}\, dt + \int_{0}^{\frac{\tau}{2}} \left(1 - \frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$= \frac{8}{\tau\omega^2} \sin^2\left(\frac{\omega\tau}{4}\right) = \frac{\tau}{2} \frac{\sin^2\left(\frac{\omega\tau}{4}\right)}{\left(\frac{\omega\tau}{4}\right)^2} = \frac{\tau}{2} \,\text{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

Hence, $\Delta\left(\frac{t}{\tau}\right) \Longleftrightarrow \frac{\tau}{2} \,\text{sinc}^2\left(\frac{\omega\tau}{4}\right)$