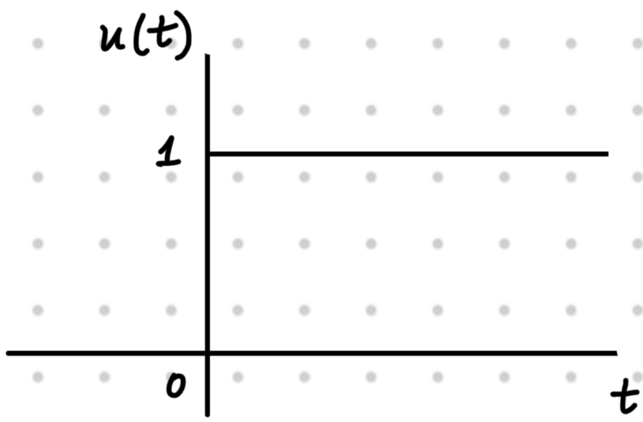
Useful Functions :

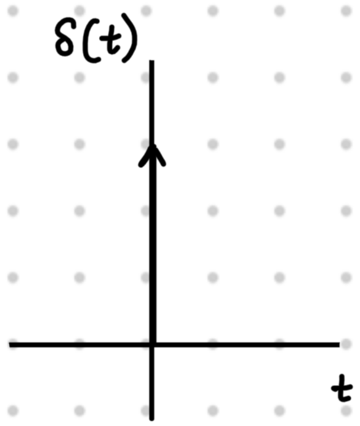
1. Unit Step Function $u(t)$:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



2. The Unit Impulse Function $\delta(t)$: A tall, rectangular pulse of unit area

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad , \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



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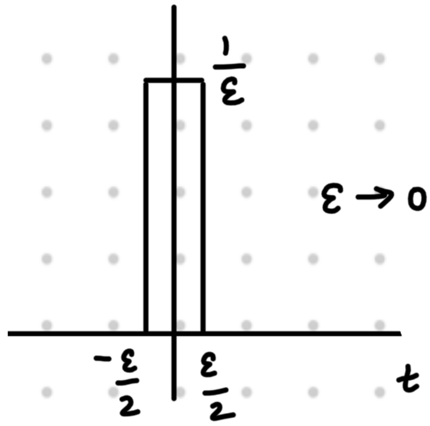

Some notes on the Impulse function:

(a) $\phi(t)\,\delta(t) = \phi(0)\,\delta(t)$

$$\Rightarrow \int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \phi(0)$$ → sampling property of the unit impulse

Similarly, $\phi(t)\,\delta(t-T) = \phi(T)\,\delta(t-T)$

$$\Rightarrow \int_{-\infty}^{\infty} \phi(t)\,\delta(t-T)\,dt = \phi(T)$$

(b) Our earlier definition of $\delta(t)$ is not mathematically rigorous. To circumvent this problem, we define the impulse as a generalized function (by its effect on other functions instead of by its value).

Hence, we define a unit impulse as a function for which the area under its product with a function $\phi(t)$ is equal to the value of the function $\phi(t)$ at the instant where the impulse is located (assuming $\phi(t)$ is continuous at the location of the impulse).

(c.) $\dfrac{du}{dt} = \delta(t)$

Proof:

$$\int_{-\infty}^{\infty} \frac{du}{dt}\,\phi(t)\,dt = \phi(t)\cdot u(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi'(t)\,u(t)\,dt$$

$$= \phi(\infty) - \int_{0}^{\infty} \phi'(t)\,dt = \phi(0)$$

Hence, $\dfrac{du}{dt}$ satisfies the sampling property of $\delta(t)$.

Therefore, it is an impulse $\delta(t)$ in the generalized sense. That is, $\dfrac{du}{dt} = \delta(t)$

$$\Rightarrow \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

3. The exponential function e^{st}:

$s = \sigma + j\omega$ is the complex frequency

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

$$e^{s^*t} = e^{(\sigma-j\omega)t} = e^{\sigma t}(\cos\omega t - j\sin\omega t)$$

Q. Simplify the following expressions:

(a). $\left(\frac{\sin t}{t^2+2}\right)\delta(t)$

(b). $\left(\frac{j\omega+2}{\omega^2+9}\right)\delta(\omega)$

(c). $\left[e^{-t}\cos(3t-60^\circ)\right]\delta(t)$

(d). $\frac{\sin\left[\frac{\pi}{2}(t-2)\right]}{t^2+4}\cdot\delta(1-t)$

(e). $\left(\frac{1}{j\omega+2}\right)\delta(\omega+3)$

(f). $\left(\frac{\sin k\omega}{\omega}\right)\delta(\omega)$

A. We know $f(t)\,\delta(t) = f(0)\,\delta(t)$. Hence,

(a). $\left(\frac{\sin t}{t^2+2}\right)\delta(t) = 0$

(b). $\left(\frac{j\omega+2}{\omega^2+9}\right)\delta(\omega) = \frac{2}{9}\delta(\omega)$

(c). $\left[e^{-t}\cos(3t-60^\circ)\right]\delta(t) = \frac{1}{2}\delta(t)$

(d). $\frac{\sin\left[\frac{\pi}{2}(t-2)\right]}{t^2+4}\cdot\delta(1-t) = \frac{-1}{5}\delta(1-t)$

(e). $\left(\frac{1}{j\omega+2}\right)\delta(\omega+3) = \frac{1}{2-3j}\delta(\omega+3)$

(f). $\left(\frac{\sin k\omega}{\omega}\right)\delta(\omega) = k\,\delta(\omega)$

Q. Evaluate the following integrals :

(a). $\int_{-\infty}^{\infty} \delta(\tau)\, x(t-\tau)\, d\tau$

(b). $\int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$

(c). $\int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$

(d). $\int_{-\infty}^{\infty} \delta(2t-3)\, \sin \pi t\, dt$

(e). $\int_{-\infty}^{\infty} \delta(t+3)\, e^{-t}\, dt$

(f). $\int_{-\infty}^{\infty} (t^3+4)\, \delta(1-t)\, dt$

(g). $\int_{-\infty}^{\infty} x(2-t)\, \delta(3-t)\, dt$

(h). $\int_{-\infty}^{\infty} e^{(x-1)} \cos\left(\frac{\pi}{2}(x-5)\right) \delta(x-3)\, dx$

A. (a). $x(t)$

(b). $x(t)$

(c). 1

(d). $\frac{-1}{2}$

(e). e^{3}

(f). 5

(g). $x(-1)$

(h). $-e^{2}$

Q. What are the frequencies of the following sinusoids in the complex plane?

(a) $\cos 3t$

(b) $e^{-3t} \cos 3t$

(c) $e^{2t} \cos 3t$

(d) e^{-2t}

(e) e^{2t}

(f) 5

A. Complex frequency $s = \alpha + j\beta$

(a) $$\cos 3t = \frac{e^{j3t} + e^{-j3t}}{2}$$

Hence, the complex frequencies are $s = \pm 3j$

(b). $s = -3 \pm 3j$

(c). $s = 2 \pm 3j$

(d). $s = -2$

(e). $s = 2$

(f). $s = 0$