Q. Find the Laplace transforms and the region of convergence of the following functions:

(a). $u(t) - u(t-1)$

(b). $t\,e^{-t}\,u(t)$

(c). $t\cos(\omega_0 t)\,u(t)$

(d). $\left(e^{2t} - 2e^{-t}\right)u(t)$

(e). $\cos(\omega_1 t)\cos(\omega_2 t)\,u(t)$

(f). $\cosh(at)\,u(t)$

(g). $\sinh(at)\,u(t)$

(h). $e^{-2t}\cos(5t+\theta)\,u(t)$

A. (a). $$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt = \int_{0}^{1} e^{-st}\,dt$$

$$= -\frac{e^{-st}}{s}\Bigg|_{0}^{1} = \frac{-1}{s}\left[e^{-s} - 1\right] = \frac{1}{s}\left[1 - e^{-s}\right]$$

$X(s)$ is valid for all values of $s$. Hence, the region of convergence is the entire s-plane.

(b). $$X(s) = \int_{0}^{\infty} t\,e^{-t}\,e^{-st}\,dt = \int_{0}^{\infty} t\,e^{-(s+1)}\,dt$$

$$= -\frac{e^{-(s+1)t}}{(s+1)^2}\left[-(s+1)t - 1\right]\Bigg|_{0}^{\infty} = \frac{1}{(s+1)^2}$$

provided that $e^{-(s+1)t} \to 0$ as $t \to 0$

$\Rightarrow\ \mathrm{Re}(s+1) > 0 \quad \Rightarrow\ \mathrm{ROC}:\ \mathrm{Re}(s) > -1$

(c). $X(s) = \int_0^\infty t \cos(\omega_0 t)\, e^{-st}\, dt$

$$= \frac{1}{2}\left\{ \int_0^\infty \left(t\, e^{(j\omega_0 - s)t} + t\, e^{-(j\omega_0 + s)t}\right) dt \right\}$$

$$= \frac{1}{2}\left[ \frac{1}{(s - j\omega_0)^2} + \frac{1}{(s + j\omega_0)^2} \right], \quad \text{ROC} : \operatorname{Re}(s) > 0$$

(d). $X(s) = \int_0^\infty \left(e^{2t} - 2e^{-t}\right) e^{-st}\, dt$

$$= \int_0^\infty e^{-(s-2)t}\, dt - 2\int_0^\infty e^{-(s+1)t}\, dt = \frac{1}{s-2} - \frac{2}{s+1}$$

$$\text{ROC} = \{\operatorname{Re}(s) > 2\} \cap \{\operatorname{Re}(s) > -1\}$$

$$\Rightarrow \quad \text{ROC} : \operatorname{Re}(s) > 2$$

(e). $x(t) = \cos(\omega_1 t)\cos(\omega_2 t)\, u(t)$

$$= \frac{1}{2}\left[\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t\right] u(t)$$

$$X(s) = \frac{1}{2}\int_0^\infty \left[\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t\right] e^{-st}\, dt$$

$$= \frac{1}{2}\left[ \frac{s}{s^2 + (\omega_1 + \omega_2)^2} + \frac{s}{s^2 + (\omega_1 - \omega_2)^2} \right]$$

$$\text{ROC} : \operatorname{Re}(s) > 0$$

(f). $$X(s) = \frac{1}{2}\left[\int_0^{\infty} e^{at} e^{-st}\,dt + \int_0^{\infty} e^{-at} e^{-st}\,dt\right]$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-(s-a)t}\,dt + \int_0^{\infty} e^{-(s+a)t}\,dt\right]$$

$$= \frac{s}{s^2 - a^2}, \quad \text{ROC}: \ \mathrm{Re}(s) > |a|$$

(g). $$X(s) = \frac{1}{2}\left[\int_0^{\infty} e^{-(s-a)t}\,dt - \int_0^{\infty} e^{-(s+a)t}\,dt\right]$$

$$= \frac{a}{s^2 - a^2}, \quad \text{ROC}: \ \mathrm{Re}(s) > |a|$$

(h). $$x(t) = e^{-2t}\cos(5t+\theta)\,u(t)$$

$$= \frac{1}{2}\left[e^{-2t+j(5t+\theta)} + e^{-2t-j(5t+\theta)}\right]$$

$$\Rightarrow \quad X(s) = \frac{1}{2}e^{j\theta}\left(\frac{1}{s+2-j5}\right) + \frac{1}{2}e^{-j\theta}\left(\frac{1}{s+2+j5}\right)$$

provided that $\mathrm{Re}(s) > -2 \Rightarrow$ ROC : $\mathrm{Re}(s) > -2$

Hence,

$$X(s) = \frac{(s+2)\cos\theta - 5\sin\theta}{s^2 + 4s + 29}; \quad \text{ROC}: \ \mathrm{Re}(s) > -2$$

Q. Find the inverse (unilateral) Laplace transforms of the following functions:

(a). $\dfrac{7s-6}{s^2-s-6}$

(b). $\dfrac{2s+5}{s^2+5s+6}$

(c). $\dfrac{2s^2+5}{s^2+3s+2}$

(d). $\dfrac{6(s+34)}{s(s^2+10s+34)}$

(e). $\dfrac{8s+10}{(s+1)(s+2)^3}$

A. (a). $$X(s) = \frac{7s-6}{(s-3)(s+2)} = \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

$$\Rightarrow \quad 7s-6 = k_1(s-3) + k_2(s+2)$$

For $k_1$ and $k_2$, substitute $s=-2$ and $s=3$ respectively

$$k_1 = \left.\frac{7s-6}{s-3}\right|_{s=-2} = 4, \quad k_2 = \left.\frac{7s-6}{s+2}\right|_{s=3} = 3$$

$$\Rightarrow \quad X(s) = \frac{4}{s+2} + \frac{3}{s-3}$$

$$\Rightarrow \quad x(t) = 4e^{-2t}u(t) + 3e^{3t}u(t)$$

Hence, $x(t) = \left(4e^{-2t} + 3e^{3t}\right)u(t)$

(b). $$X(s) = \frac{2s+5}{s^2+5s+6} = \frac{1}{s+2} + \frac{1}{s+3}$$

$$\Rightarrow \quad x(t) = \left(e^{-2t} + e^{-3t}\right) u(t)$$

(c). $$X(s) = \frac{2s^2+5}{s^2+3s+2} = \frac{2s^2+5}{(s+1)(s+2)}$$

$X(s)$ is an improper fraction with $m=n$

Hence, $$X(s) = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$\Rightarrow \quad 2s^2+5 = 2(s+1)(s+2) + k_1(s+2) + k_2(s+1)$$

$$k_1 = \left.\frac{2s^2+5}{s+2}\right|_{s=-1} = 7, \qquad k_2 = \left.\frac{2s^2+5}{s+1}\right|_{s=-2} = -13$$

$$\Rightarrow \quad X(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2}$$

Hence, $$x(t) = 2\,\delta(t) + \left(7e^{-t} - 13e^{-2t}\right) u(t)$$

(d). $$X(s) = \frac{6(s+34)}{s(s^2+10s+34)} = \frac{6(s+34)}{s(s+5-3j)(s+5+3j)}$$

$$\Rightarrow \quad X(s) = \frac{k_1}{s} + \frac{k_2}{s+5-3j} + \frac{k_3}{s+5+3j}$$

$$\Rightarrow \quad 6(s+34) = k_1(s+5-3j)(s+5+3j) + k_2(s)(s+5+3j) + k_3(s)(s+5-3j)$$

$$\Rightarrow \quad k_1 = \left.\frac{6(s+34)}{(s+5-3j)(s+5+3j)}\right|_{s=0} = 6$$

$$k_2 = \left.\frac{6(s+34)}{s(s+5+3j)}\right|_{s=-5+3j} = -3+4j = 5e^{j(126.9^\circ)}$$

$$k_3 = \left.\frac{6(s+34)}{s(s+5-3j)}\right|_{s=-5-3j} = -3-4j = 5e^{-j(126.9^\circ)}$$

$$\Rightarrow \quad X(s) = \frac{6}{s} + \frac{5e^{j(126.9^\circ)}}{s+5-3j} + \frac{5e^{-j(126.9^\circ)}}{s+5+3j}$$

$$\Rightarrow \quad x(t) = \left[6 + 10e^{-5t}\cos(3t+126.9^\circ)\right]u(t)$$

(e). $$X(s) = \frac{8s+10}{(s+1)(s+2)^3} = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^3} + \frac{k_3}{(s+2)^2} + \frac{k_4}{(s+2)}$$

$$\Rightarrow \quad 8s+10 = k_1(s+2)^3 + k_2(s+1) + k_3(s+1)(s+2) + k_4(s+1)(s+2)^2$$

$$\Rightarrow \quad k_1 = \left.\frac{8s+10}{(s+2)^3}\right|_{s=-1} = 2 \ , \quad k_2 = \left.\frac{8s+10}{s+1}\right|_{s=-2} = 6$$

For $k_3$, take derivative of the equation w.r.t. s and substitute $s = -2$,

$$8 = k_1\, 3(s+2)^2 + k_2 + k_3(s+2+s+1)$$
$$+ k_4\left((s+2)^2 + 2(s+1)(s+2)\right)$$

$$\Rightarrow \quad 8 = 6 + k_3(-1) + 0 \quad \Rightarrow \quad k_3 = -2$$

For $k_4$, take double derivative of the equation with respect to s and substitute $s = -2$,

$$0 = 6\, k_1(s+2) + 0 + k_3(2) + k_4\left(2(s+2) + 2(s+2)\right.$$
$$\left. + 2(s+1)\right)$$

$$\Rightarrow \quad 0 = 2(-2) + 2k_4(-1) \quad \Rightarrow \quad k_4 = -2$$

Hence, $$X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^3} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)}$$

$$\Rightarrow \quad x(t) = \left(2e^{-t} + (3t^2 - 2t - 2)e^{-2t}\right) u(t)$$