Convolution Integral

For two functions $x_1(t)$ and $x_2(t)$, the convolution integral is defined as :

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Some properties of the convolution integral:

1. $x_1(t) * x_2(t) = x_2(t) * x_1(t)$

2. $x_1(t) * \big(x_2(t) + x_3(t)\big) = x_1(t) * x_2(t) + x_1(t) * x_3(t)$

3. $x_1(t) * \big(x_2(t) * x_3(t)\big) = \big(x_1(t) * x_2(t)\big) * x_3(t)$

4. If $x_1(t) * x_2(t) = c(t)$, then

$$x_1(t) * x_2(t-T_0) = c(t-T_0)$$

5. $x(t) * \delta(t) = x(t)$

6. If $x(t)$ is a causal signal, then the zero-state response can be simplified as :

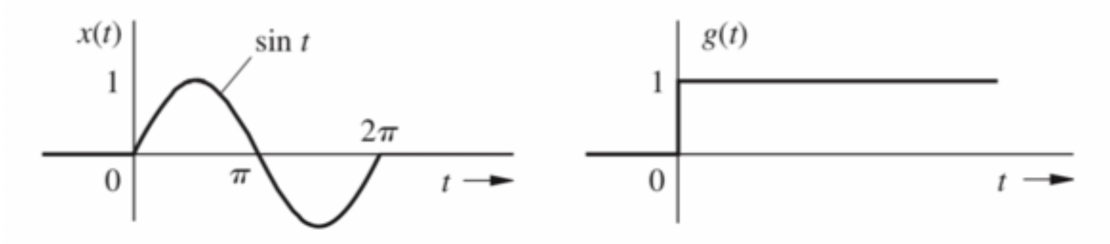
<u>for $t \geq 0$:</u>

$$y(t) = x(t) * h(t) = \int_{0^-}^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{t} x(t-\tau)\, h(\tau)\, d\tau$$

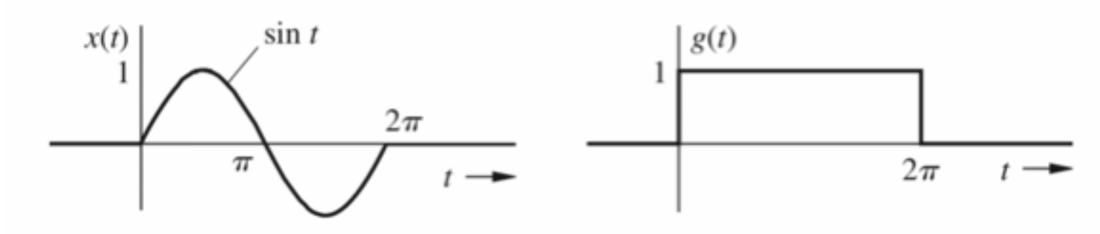
<u>for $t < 0$:</u> $\quad y(t) = 0$

Q. Find and sketch $c(t) = f(t) * g(t)$ for the functions depicted below :

(a).



(b).



A. (a). $c(t) = \int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau$

For $t \leq 0$: $\int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau = 0$

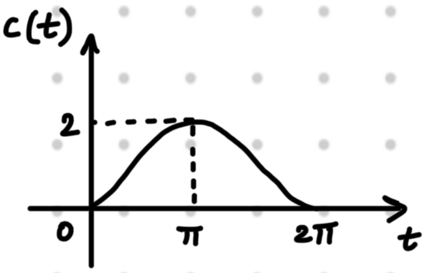
For $0 \leq t \leq 2\pi$: $\int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau = \int_{0}^{t} (\sin\tau)(1)\, d\tau$

$= 1 - \cos t$

For $t \geq 2\pi$: $\int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau = \int_{0}^{2\pi} (\sin\tau)(1)\, d\tau = 0$

Hence,

$$c(t) = \begin{cases} 0 & t \leq 0 \\ 1-\cos t & 0 \leq t \leq 2\pi \\ 0 & t \geq 2\pi \end{cases}$$



(b). $$c(t) = \int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau$$

For $t \leq 0$: $$\int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau = 0$$

For $0 \leq t \leq 2\pi$: $$\int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau = \int_{0}^{t} (\sin\tau)(1)\, d\tau$$

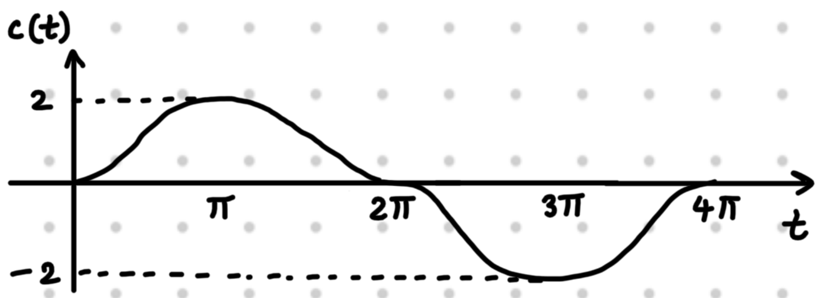
$$= 1 - \cos t$$

For $2\pi \leq t \leq 4\pi$: $$\int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau = \int_{t-2\pi}^{2\pi} (\sin\tau)(1)\, d\tau$$

$$= -1 + \cos t$$

For $t \geq 4\pi$: $$\int_{-\infty}^{\infty} x(\tau)\, g(t-\tau)\, d\tau = 0$$

Hence, $$c(t) = \begin{cases} 0 & t \leq 0 \\ 1 - \cos t & 0 \leq t \leq 2\pi \\ -1 + \cos t & 2\pi \leq t \leq 4\pi \\ 0 & t \geq 4\pi \end{cases}$$



Q. Calculate:

(a). $u(t) * u(t)$

(b). $t\, u(t) * u(t)$

(c). $e^{-at} u(t) * e^{-bt} u(t)$

(d). $(\sin t)\, u(t) * u(t)$

A. (a). $u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau$

For $t < 0$: $\int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = 0$

For $t \geqslant 0$: $\int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = \int_{0}^{t} (1)(1)\, d\tau = t$

Hence, $u(t) * u(t) = t\, u(t)$

(b). $t\, u(t) * u(t) = \int_{-\infty}^{\infty} \tau\, u(\tau)\, u(t-\tau)\, d\tau$

For $t < 0$: $\int_{-\infty}^{\infty} \tau\, u(\tau)\, u(t-\tau)\, d\tau = 0$

For $t \geqslant 0$: $\int_{-\infty}^{\infty} \tau\, u(\tau)\, u(t-\tau)\, d\tau = \int_{0}^{t} \tau\, (1)(1)\, d\tau = \frac{t^2}{2}$

Hence, $u(t) * t\, u(t) = \frac{t^2}{2} u(t)$

(c). $e^{-at}u(t) * e^{-bt}u(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-b(t-\tau)} u(t-\tau)\, d\tau$

For $t < 0$: $\int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-b(t-\tau)} u(t-\tau)\, d\tau = 0$

For $t \geqslant 0$: $\int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-b(t-\tau)} u(t-\tau)\, d\tau = \int_{0}^{t} e^{-a\tau} \cdot e^{-b(t-\tau)} d\tau$

$$= \frac{e^{-bt} - e^{-at}}{a-b} = \frac{e^{-at} - e^{-bt}}{b-a}$$

Hence, $e^{-at}u(t) * e^{-bt}u(t) = \frac{e^{-at} - e^{-bt}}{b-a} \cdot u(t)$

(d). $(\sin t)\, u(t) * u(t) = \int_{-\infty}^{\infty} (\sin \tau)\, u(\tau) \cdot u(t-\tau)\, d\tau$

For $t < 0$: $\int_{-\infty}^{\infty} (\sin \tau)\, u(\tau) \cdot u(t-\tau)\, d\tau = 0$

For $t \geqslant 0$: $\int_{-\infty}^{\infty} (\sin \tau)\, u(\tau) \cdot u(t-\tau)\, d\tau = \int_{0}^{t} (\sin \tau)\, d\tau$

$$= 1 - \cos t$$

Hence, $(\sin t)\, u(t) * u(t) = (1 - \cos t)\, u(t)$

Q. The unit impulse response of an LTIC system is :

$$h(t) = \left[2e^{-3t} - e^{-2t}\right] u(t)$$

Find this system's zero-state response $y(t)$ if the input $x(t)$ is :

(a). $u(t)$ (b). $e^{-2t} u(t)$

A. (a). $y(t) = h(t) * x(t) = \left[2e^{-3t} - e^{-2t}\right] u(t) * u(t)$

We know, $e^{-at} u(t) * e^{-bt} u(t) = \dfrac{e^{-at} - e^{-bt}}{b-a} \cdot u(t)$

Hence, $y(t) = 2\left(\dfrac{e^{-3t} - 1}{-3}\right) u(t) - \left(\dfrac{e^{-2t} - 1}{-2}\right) u(t)$

$$\Rightarrow \quad y(t) = \left(\frac{1}{6} - \frac{2}{3}e^{-3t} + \frac{1}{2}e^{-2t}\right) u(t)$$

(b). $y(t) = h(t) * x(t) = \left[2e^{-3t} - e^{-2t}\right] u(t) * e^{-2t} u(t)$

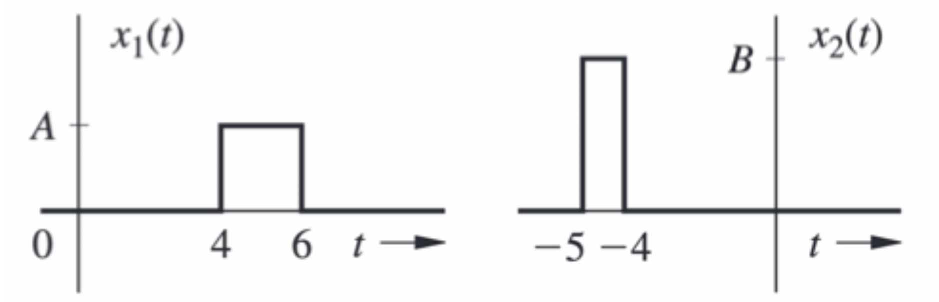
$$\Rightarrow \quad y(t) = 2\left(\frac{e^{-3t} - e^{-2t}}{2-3}\right) u(t) - \left(e^{-2t} u(t) * e^{-2t} u(t)\right)$$

$$= 2e^{-2t} u(t) - 2e^{-3t} u(t) - t e^{-2t} u(t)$$

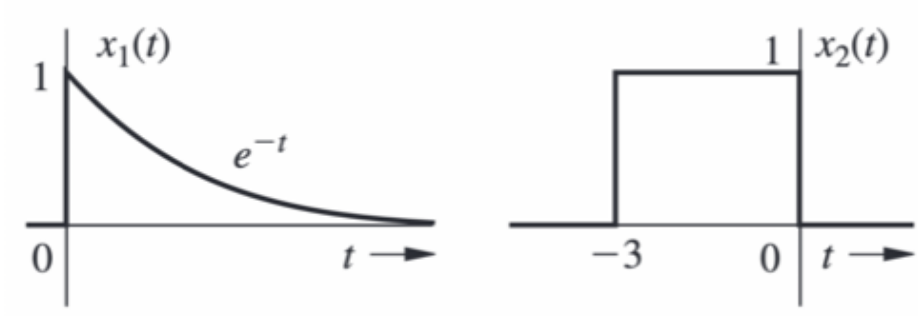
$$\Rightarrow \quad y(t) = \left(2e^{-2t} - 2e^{-3t} - t e^{-2t}\right) u(t)$$

Q. Find and sketch $c(t) = x_1(t) * x_2(t)$ for the functions depicted below :

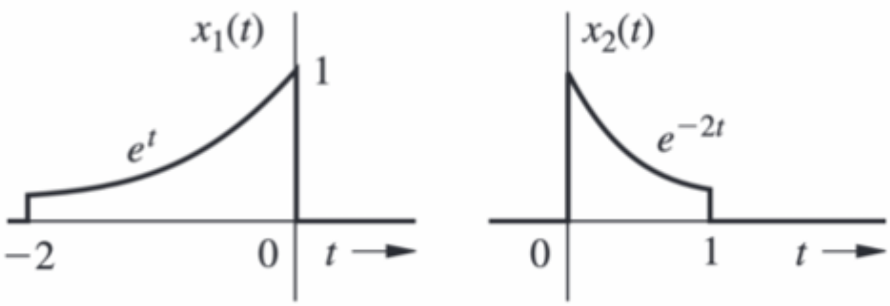
(a).



(b).



(c).



A. (a). $c(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau$

For $t \leq -1$: $\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = 0$

For $-1 \leq t \leq 0$: $\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = \int_{-5}^{-4+t} (A)(B)\, d\tau$

$= AB(1+t)$

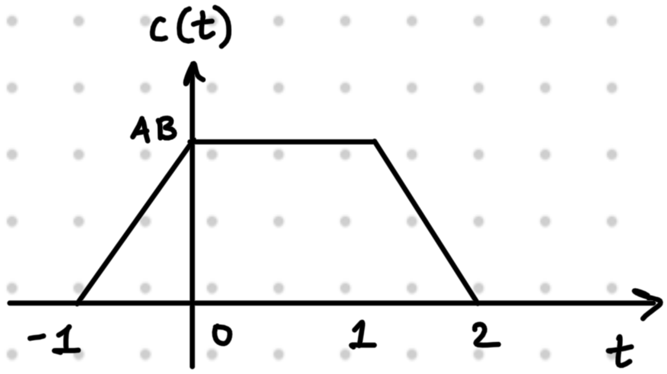
For $0 \le t \le 1$: $\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = \int_{-5}^{-4} (A)(B)\, d\tau = AB$

For $1 \le t \le 2$: $\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = \int_{-6+t}^{-4} (A)(B)\, d\tau$

$$= AB(2-t)$$

For $t \geqslant 2$: $\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = 0$

Hence,



(b). $c(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau$

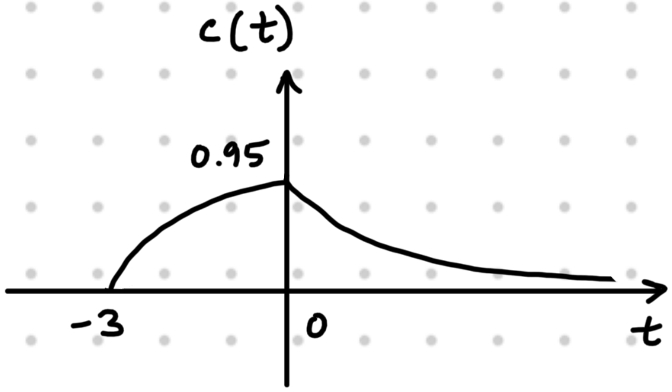
For $t \le -3$: $\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = 0$

For $-3 \le t \le 0$: $\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = \int_{-3}^{t} e^{\tau - t}\,(1)\, d\tau$

$$= 1 - e^{-(t+3)}$$

For $t \geq 0$: $$\int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = \int_{-3}^{0} e^{\tau - t}\,(1)\, d\tau$$
$$= e^{-t}\left(1 - e^{-3}\right)$$

Hence,



(c). For $t \leq -2$: $$c(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\, x_2(\tau)\, d\tau = 0$$

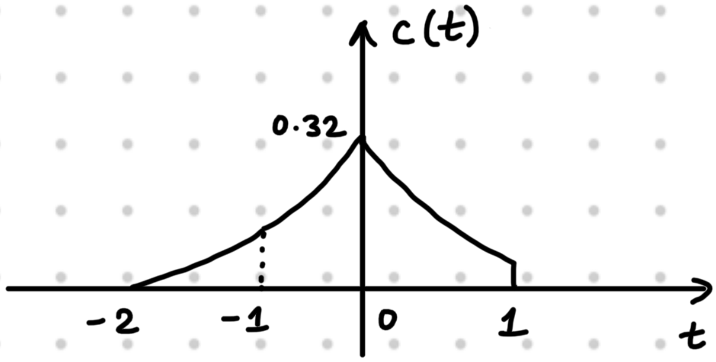
For $-2 \leq t \leq -1$: $$c(t) = \int_{-2}^{t} e^{2(\tau - t)}\, e^{\tau}\, d\tau = \frac{e^{t} - e^{-2(t+3)}}{3}$$

For $-1 \leq t \leq 0$: $$c(t) = \int_{-1+t}^{t} e^{2(\tau - t)}\, e^{\tau}\, d\tau = \frac{e^{t} - e^{-(3-t)}}{3}$$

For $0 \leq t \leq 1$: $$c(t) = \int_{-1+t}^{0} e^{2(\tau - t)}\, e^{\tau}\, d\tau = \frac{e^{-2t} - e^{-3+t}}{3}$$

For $t \geq 1$: $c(t) = 0$

Hence,



Q. If $f(t) * g(t) = c(t)$, then show that

(a). $f(at) * g(at) = \left|\frac{1}{a}\right| c(at)$.

(b). $\dot{f}(t) * g(t) = f(t) * \dot{g}(t) = \dot{c}(t)$

A. (a). $c(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$

Now, $f(at) * g(at) = \int_{-\infty}^{\infty} f(a\tau)\, g(a(t-\tau))\, d\tau$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} f(\theta)\, g(at-\theta)\, d\theta = \frac{1}{|a|} c(at)$$

(b). $c(t) = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$

$$\Rightarrow \dot{c}(t) = \frac{d}{dt} \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau = \frac{d}{dt} \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$$\Rightarrow \dot{c}(t) = \int_{-\infty}^{\infty} \dot{f}(t-\tau)\, g(\tau)\, d\tau = \int_{-\infty}^{\infty} f(\tau)\, \dot{g}(t-\tau)\, d\tau$$

Hence, $\dot{c}(t) = \dot{f}(t) * g(t) = f(t) * \dot{g}(t)$