Q. A stable LTIC system is specified by the frequency response $H(\omega) = \frac{1}{2-j\omega}$. Find the zero-state response of this system if the input $x(t)$ is :

(a). $e^{-t}u(t)$ (b). $e^{t}u(-t)$

A. (a). $x(t) = e^{-t}u(t) \Rightarrow X(\omega) = \frac{1}{j\omega+1}$

$$Y(\omega) = X(\omega)\cdot H(\omega) = \frac{1}{j\omega+1}\cdot\frac{1}{2-j\omega}$$

$$\Rightarrow Y(\omega) = \frac{1}{3}\left[\frac{1}{j\omega+1} + \frac{1}{2-j\omega}\right]$$

Hence, $y(t) = \frac{1}{3}\left[e^{-t}u(t) + e^{2t}u(-t)\right]$

(b). $x(t) = e^{t}u(-t) \Rightarrow X(\omega) = \frac{1}{1-j\omega}$

$$Y(\omega) = \frac{1}{1-j\omega}\cdot\frac{1}{2-j\omega} = \frac{1}{1-j\omega} - \frac{1}{2-j\omega}$$

Hence, $y(t) = \left(e^{t} - e^{2t}\right)u(-t)$

Q. A stable LTIC system is specified by the impulse response $h(t) = e^{-t} u(t)$. Find the zero-state response of this system if the input $x(t)$ is:

(a). $e^{-2t} u(t)$ (b). $u(t)$

A. $H(\omega) = \dfrac{1}{1+j\omega}$

(a). $X(\omega) = \dfrac{1}{2+j\omega} \Rightarrow Y(\omega) = \dfrac{1}{(1+j\omega)(2+j\omega)}$

$$\Rightarrow Y(\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$$\Rightarrow y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$

(b). $X(\omega) = \pi\,\delta(\omega) + \dfrac{1}{j\omega}$

$$\Rightarrow Y(\omega) = \frac{1}{1+j\omega} \cdot \pi\,\delta(\omega) + \frac{1}{j\omega}\left(\frac{1}{1+j\omega}\right)$$

$$\Rightarrow y(t) = u(t)\left[1 - e^{-t}\right]$$

Q. Consider an LTI system whose response to the input $x(t) = \left(e^{-t} + e^{-3t}\right) u(t)$ is given as $y(t) = \left[2 e^{-t} - 2 e^{-4t}\right] u(t)$. What is the system's impulse response ?

A. $$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}$$

Expanding $H(\omega)$ into partial fractions and taking inverse fourier transform,

$$h(t) = \frac{3}{2}\left[e^{-4t} + e^{-2t}\right] u(t)$$

Q. A causal and stable LTI system has the frequency response $H(\omega) = \dfrac{j\omega + 4}{6 - \omega^2 + 5j\omega}$.

What is the output of this system when the input is $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$?

A. $$H(\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} = \frac{2}{2+j\omega} - \frac{1}{3+j\omega}$$

$$X(\omega) = \frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2}$$

Therefore, $Y(\omega) = X(\omega)\, H(\omega)$

$$Y(\omega) = \frac{1}{(4+j\omega)(2+j\omega)}$$

Expanding $Y(\omega)$ into partial fractions and taking inverse fourier transform,

$$y(t) = \frac{1}{2}\left[e^{-2t} - e^{-4t}\right]u(t)$$

Q. Suppose $X(\omega) = \text{rect}(\omega)$. Then,

(a). Compute $y(t)$, where $y(t) = x(t) * x(t)$.

(b). Compute $y(t)$, where $y(t) = x(t) * x\left(\frac{t}{2}\right)$.

(c). Compute $y(t)$, where $y(t) = 1 - x^2(t)$.

A. (a). $y(t) = x(t) * x(t)$

$$\Rightarrow \quad Y(\omega) = X(\omega) \cdot X(\omega) = \text{rect}^2(\omega) = \text{rect}(\omega)$$

$$\Rightarrow \quad y(t) = \frac{1}{2\pi} \text{sinc}\left(\frac{t}{2}\right)$$

(b). $y(t) = x(t) * x\left(\frac{t}{2}\right)$

$$\Rightarrow \quad Y(\omega) = X(\omega) \cdot 2\,X(2\omega) = 2\,\text{rect}(2\omega)$$

$$\Rightarrow \quad y(t) = \frac{1}{2\pi} \text{sinc}\left(\frac{t}{4}\right)$$

(c). $y(t) = 1 - x^2(t)$

$$\Rightarrow \quad Y(\omega) = 2\pi\,\delta(\omega) - \frac{1}{2\pi} X(\omega) * X(\omega)$$

$$= 2\pi\,\delta(\omega) - \frac{1}{2\pi} \Delta\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \quad y(t) = 1 - \frac{1}{4\pi^2} \text{sinc}^2\left(\frac{t}{2}\right)$$

Q. Consider an LTI system with impulse response $h(t) = \dfrac{\sin 4(t-1)}{\pi(t-1)}$. Determine the zero-state output for the following inputs:

(a). $\cos\left(6\left(t+\frac{1}{12}\right)\right)$ (b). $\dfrac{\sin(4(t+1))}{\pi(t+1)}$ (c). $\left(\dfrac{\sin 2t}{\pi t}\right)^2$

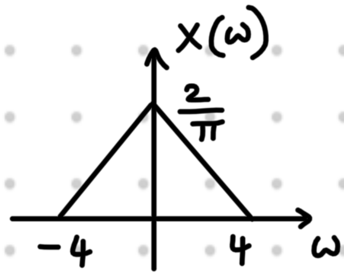
A. $H(\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 4 \\ 0, & \text{else} \end{cases}$

(a). $X(\omega) = \pi e^{j\frac{\omega}{12}}\left[\delta(\omega-6) + \delta(\omega+6)\right]$

$\Rightarrow Y(\omega) = X(\omega)\cdot H(\omega) = 0 \Rightarrow y(t) = 0$

(b). $X(\omega) = \begin{cases} e^{j\omega}, & |\omega| < 4 \\ 0, & \text{else} \end{cases} \Rightarrow Y(\omega) = \begin{cases} 1, & |\omega| < 4 \\ 0, & \text{else} \end{cases}$

$\Rightarrow y(t) = \dfrac{\sin 4t}{\pi t}$



(c). $X(\omega) = \dfrac{2}{\pi}\Delta\left(\dfrac{\omega}{8}\right)$

$Y(\omega) = e^{-j\omega}\dfrac{2}{\pi}\Delta\left(\dfrac{\omega}{8}\right), \; |\omega| < 4$

$\Rightarrow y(t) = \left(\dfrac{\sin 2(t-1)}{\pi(t-1)}\right)^2$

Q. An input $x(t) = 1 + 2\cos(5\pi t) + 3\sin(8\pi t)$ is applied to an LTIC system with unit impulse response $h(t) = 8\,\text{sinc}(4\pi t)\cos(2\pi t)$. Determine the zero-state output $y(t)$.

A. $$x(t) = 1 + e^{j5\pi t} + e^{-j5\pi t} + \frac{3}{2j}\left(e^{j8\pi t} - e^{-j8\pi t}\right)$$

$$\Rightarrow X(\omega) = 2\pi\,\delta(\omega) + \delta(\omega - 5\pi) + \delta(\omega + 5\pi) + \frac{3}{2j}\left(\delta(\omega - 8\pi) - \delta(\omega + 8\pi)\right)$$

$$h(t) = 8\,\text{sinc}(4\pi t)\cos(2\pi t)$$

$$= \underbrace{4\,\text{sinc}(4\pi t)}_{\Updownarrow\ \text{rect}\left(\frac{\omega}{8\pi}\right)} \cdot \underbrace{2\cos(2\pi t)}_{\Updownarrow\ \left[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\right]}$$

$$\Rightarrow H(\omega) = \text{rect}\left(\frac{\omega}{8\pi}\right) * \left[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\right]$$

$$= \left[\text{rect}\left(\frac{\omega - 2\pi}{8\pi}\right) + \text{rect}\left(\frac{\omega + 2\pi}{8\pi}\right)\right]$$

Then, $$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$= 4\pi\,\delta(\omega) + \delta(\omega - 5\pi) + \delta(\omega + 5\pi)$$

Hence, $$y(t) = 2 + 2\cos(5\pi t)$$

Q. A periodic delta train $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - \pi n)$ is applied to an LTIC system with impulse response $h(t) = \sin(3t) \cdot \text{sinc}^2(t)$. Find the zero-state output $y(t) = x(t) * h(t)$.

A. $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - \pi n)$, $\omega_0 = \frac{2\pi}{T_0} = 2$

$$\Rightarrow \quad X(\omega) = 2 \sum_{n=-\infty}^{\infty} \delta(\omega - 2n)$$

$$h(t) = \sin(3t)\, \text{sinc}^2 t \iff \frac{1}{2\pi}\left[j\pi\,\delta(\omega+3) - j\pi\,\delta(\omega-3)\right] * \pi\,\Delta\left(\frac{\omega}{4}\right)$$

$$\Rightarrow \quad H(\omega) = \frac{j\pi}{2}\,\Delta\left(\frac{\omega+3}{4}\right) - \frac{j\pi}{2}\,\Delta\left(\frac{\omega-3}{4}\right)$$

$$Y(\omega) = X(\omega)\cdot H(\omega) = 2\,\delta(\omega+4)\,\frac{j\pi}{4} + 2\,\delta(\omega+2)\,\frac{j\pi}{4} + 2\,\delta(\omega-2)\,\frac{-j\pi}{4} + 2\,\delta(\omega-4)\,\frac{-j\pi}{4}$$

Hence,

$$y(t) = \frac{1}{2\pi}\left(\frac{j\pi}{2}e^{-j4t} + \frac{j\pi}{2}e^{-j2t} - \frac{j\pi}{2}e^{j2t} - \frac{j\pi}{2}e^{j4t}\right)$$

$$\Rightarrow \quad y(t) = \frac{1}{2}\left(\sin 2t + \sin 4t\right)$$