Introduction to Graph Theory

Notes for MATH 4171

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Contents

0 Preliminaries

We assume that the reader is familiar with the rudiments of set theory, including the notations $x \in X$ to indicate that x is an element of the set X and $x \notin X$ to indicate that it is not, and the use of expressions such as $\{x \in \mathbb{R} \mid x > 1\}$, which denotes the set of all real numbers greater than 1. We recall here some (but not all) of the ideas the reader should have encountered. Two sets are equal iff they have exactly the same elements. There is a unique set with no elements, the *empty set* $\emptyset$. We use the notation $X \subseteq Y$ to indicate that X is a subset of Y; that is, that $x \in X$ implies $x \in Y$. We have $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$. We use $X \subset Y$ to indicate that X is a proper subset of Y; that is, $X \subseteq Y$ and $X \neq Y$. The intersection, union and difference (or relative complement) of sets X and Y are defined by

$$\begin{aligned} X \cap Y &= \{x \mid x \in X \text{ and } x \in Y\}, \\ X \cup Y &= \{x \mid x \in X \text{ or } x \in Y\}, \\ \text{and} \quad X - Y &= \{x \mid x \in X \text{ and } x \notin Y\}, \end{aligned}$$

respectively. (We follow the standard mathematical convention that "or" is used inclusively, so that $X \cap Y \subseteq X \cup Y$.) The Cartesian product $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. We deal mostly with finite sets. The number of elements, or *cardinality* of a finite set X will be denoted by $|X|$.

We also assume familiarity with the idea of a function, and a few related notions. The notation $f\colon X \to Y$ means that f is a function from the set X to the set Y; that is, to each element x of X is associated a unique element $f(x)$ of Y. Functions $f\colon X \to Y$ and $f'\colon X' \to Y'$ are equal iff $X = X'$, $Y = Y'$ and $f(x) = f'(x)$ for all $x \in X$. For any set X, the identity function on X is the function id_X (or just id) from X to X defined by $\mathrm{id}_X(x) = x$ for all $x \in X$. If $f\colon X \to Y$ and $A \subseteq X$, the image of A under f is the subset $f(A) = \{f(a) \mid a \in A\}$ of Y. The function f is *injective* (or *an injection*) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for x_1, $x_2 \in X$. It is *surjective* (or *a surjection*) if $f(X) = Y$, and it is *bijective* (or *a bijection*) if it is both injective and surjective. If f is a bijection from X to Y, there is an inverse function $f^{-1}\colon Y \to X$ defined by $f^{-1}(y) = x$ iff $y = f(x)$ for $x \in X$ and $y \in Y$.

If $f\colon X \to Y$ and $g\colon Y \to Z$, the composite $g \circ f\colon X \to Z$ is defined by $(g \circ f)(x) = g(f(x))$. We also use gf to denote the composite. If $f\colon X \to Y$ is a bijection we have $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$.

The rest of this section discusses equivalence relations and partitions, which the reader may not have met before. A *binary relation* on a set X is a subset R of $X \times X$. If $(x, y) \in R$, we say that x is related to y by R, and write $x\,R\,y$. Typically, specific binary relations are denoted by symbols such as $=$, $\leq$, $>$ and $\sim$ rather than letters. Thus, formally, the relation $>$ on the set $\{1, 2, 3\}$ is the set $\{(2, 1), (3, 1), (3, 2)\}$ of ordered pairs. Let R be a binary relation on X.

(1) R is *reflexive* if $x\,R\,x$ for all $x \in X$.

(2) R is *symmetric* if $x\,R\,y$ implies $y\,R\,x$ for x, $y \in X$.

(3) R is *transitive* if $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$ for x, y and $z \in X$.

An *equivalence relation* is a relation that is reflexive, symmetric and transitive. If R is an equivalence relation on X and $x \in X$, the *equivalence class* of x is the set $[x]$ of all $y \in X$ such that $x\,R\,y$.

Example 0.1. The equality relation on any set is an equivalence relation, and $[x] = \{x\}$.

Example 0.2. On the set $\mathbb{R}$ of real numbers, the relation $\leq$ is reflexive and transitive, but not symmetric.

Example 0.3. On the set $\mathbb{R}$, the relation $<$ is transitive, but neither reflexive nor symmetric.

Example 0.4. If n is a positive integer, the relation of congruence modulo n on the set $\mathbb{Z}$ of all integers is defined by $a \equiv b \pmod{n}$ if n divides $a - b$. It is an equivalence relation, and $[a] = \{qn + a \mid q \in \mathbb{Z}\}$. The distinct equivalence classes are $[0], [1], \dots, [n-1]$, which we shall sometimes denote just by $0, 1, \dots, n-1$. The set of equivalence classes will be denoted by $\mathbb{Z}_n$ (a notation that annoys some algebraists of my acquaintance, who prefer $\mathbb{Z}/n$ or $\mathbb{Z}/n\mathbb{Z}$). We define operations of addition and multiplication on $\mathbb{Z}_n$ by $[i] + [j] = [i + j]$ and $[i][j] = [ij]$. It is left to the reader to verify that these are well-defined operations. They satisfy many of the usual laws of arithmetic (commutativity, associativity, distributivity, ...); those who have met the concept will know that they give $\mathbb{Z}_n$ the structure of a ring. At one point we shall need the operation of addition on $\mathbb{Z}_2 \times \mathbb{Z}_2$, defined in the obvious way: $(a, b) + (c, d) = (a + c, b + d)$.

Proposition 0.5. *Let R be an equivalence relation on a set X, and x, $y \in X$. Then $x \in [x]$, and $[x] = [y]$ iff $x \, R \, y$.*

Proof. First, $x \in [x]$ because R is reflexive. Second, if $[x] = [y]$ then $y \in [x]$, so $x \, R y$. Finally, suppose $x \, R y$. If $z \in [y]$ then $y \, R \, z$, so $x \, R \, z$ by transitivity. This gives $z \in [x]$, and so $[y] \subseteq [x]$. By symmetry of R, $y \, R \, x$, so also $[x] \subseteq [y]$; that is, $[x] = [y]$. □

A *partition* of a set X is a collection of disjoint, non-empty subsets of X whose union is X.

Proposition 0.6. *Let X be a set.*

(1) *The collection of equivalence classes under any equivalence relation on X is a partition of X.*

(2) *Every partition of X is the collection of equivalence classes for a unique equivalence relation on X.*

Proof. (1) Let R be an equivalence relation on X, and $\mathcal{P}$ the set of equivalence classes under R. The relation $x \in [x]$ of Proposition 0.5 shows that the elements of $\mathcal{P}$ are non-empty and have union X. Suppose $[x]$ and $[y]$ are elements of $\mathcal{P}$ that are not disjoint. Let $z \in [x] \cap [y]$. Then $x \, R \, z$ and $y \, R \, z$, so by Propsition 0.5, $[x] = [z] = [y]$. Thus $\mathcal{P}$ is a partition of X.

(2) Let $\mathcal{P}$ be a partition of X, and for $x \in X$ let $\langle x \rangle$ be the unique element of $\mathcal{P}$ containing X. By Proposition 0.5, if $\mathcal{P}$ is the set of equivalence

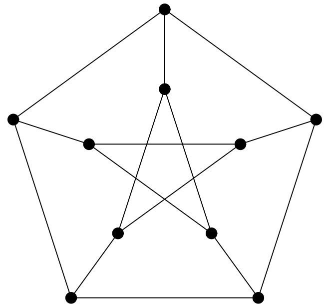


Figure 1: The Petersen graph

classes for an equivalence relation R then $x\,R\,y$ iff $\langle x\rangle = \langle y\rangle$, which proves the uniqueness part. To prove existence, define a relation R by $x\,R\,y$ if $\langle x\rangle = \langle y\rangle$. It is trivial to check that R is an equivalence relation. We must show that the equivalence class $[x]$ is equal to $\langle x\rangle$. Suppose $y \in [x]$. Then $\langle x\rangle = \langle y\rangle$, and since $y \in \langle y\rangle$ by definition, $y \in \langle x\rangle$. Conversely, if $y \in \langle x\rangle$, then since also $y \in \langle y\rangle$, $\langle x\rangle$ and $\langle y\rangle$ are not disjoint. Hence $\langle x\rangle = \langle y\rangle$, so $x\,R\,y$ and $y \in [x]$. $\square$

1 Graphs

Definition 1.1. A *graph* G is a pair (V, E), where V is a non-empty finite set, and E is a set of unordered pairs of elements of V. The elements of V are called the *vertices* of G, and the elements of E are the *edges* of G. We always assume that $V \cap E = \emptyset$.

We can represent a graph pictorially as in Figure 1. The dots represent the vertices, and each edge $\{u, v\}$ is represented by an arc connecting the dots corresponding to the vertices u and v. We will usually write uv instead of $\{u, v\}$; of course, $uv = vu$. The set of vertices of a graph G may be written $V(G)$; the number $|V(G)|$ of vertices is the *order* of G, denoted by $n(G)$ or just n. The set of edges may be written $E(G)$; the number $|E(G)|$ of edges is the *size* of G, denoted by $m(G)$ or just m. By definition, $n > 0$, but $m = 0$ is allowed. If $n = 1$ (and so $m = 0$), G is *trivial.* In general, $0 \le m \le \binom{n}{2} = n(n-1)/2$. If $m = 0$ we have an *empty graph*, while if $m = \binom{n}{2}$ we have a *complete graph*; see Figure 2 for an example. Vertices u and v of a graph G are *adjacent* if $uv \in E(G)$; an edge $e = uv$ is *incident* with the vertices u and v, and distinct edges e and f are *adjacent* if there is a vertex incident with both of them. Vertices u and v are *independent* if

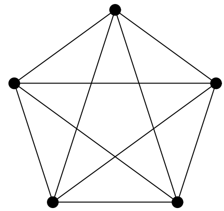


Figure 2: The complete graph on 5 vertices

$u \neq v$ and $uv \notin E(G)$. If u and v are adjacent we shall also say that u and v are *neighbors*.

Definition 1.2. Let G_1 and G_2 be graphs. An *isomorphism* from G_1 to G_2 is a bijection $\phi\colon V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ iff $\phi(uv) \in E(G_2)$. (Here we are using the standard notation for the image of a set under a function: $\phi(uv) = \phi(\{u, v\}) = \{\phi(u), \phi(v)\} = \phi(u)\phi(v)$.) If there exists an isomorphism $G_1 \to G_2$, G_1 is *isomorphic to* G_2, written $G_1 \cong G_2$.

Note that any graph is isomorphic to itself; if $G_1 \cong G_2$ then $G_2 \cong G_1$; and if $G_1 \cong G_2$ and $G_2 \cong G_3$ then $G_1 \cong G_3$. (One is tempted to say that isomorphism is an equivalence relation on the set of graphs. However, there is no such thing as the set of all graphs. Isomorphism *is* an equivalence relation on any set of graphs.) Isomorphic graphs are "essentially the same"; they differ only in the names of their vertices. Some authors use $G_1 = G_2$ to mean that G_1 and G_2 are isomorphic, but I prefer to reserve the equal sign for, well, equality. Nevertheless, I do follow the universal practice of referring to *the* Petersen graph, or *the* complete graph on n vertices, meaning some one of a collection of isomorphic graphs. The complete graph on n vertices will be denoted by K_n.

Sometimes we wish to determine the number of graphs in a given collection, and the question arises as to whether we are counting the graphs up to identity or up to isomorphism. We adopt a convention that should be clear from the following example. If X is a given set of n elements, "the number of graphs with vertex set X" (or "on X") means precisely what it says: the number of collections of unordered pairs of elements of X, which is $2^{\binom{n}{2}}$. However, "the number of graphs of order n" (or "on n vertices") means the maximum number in a collection of pairwise non-isomorphic graphs of order n, or equivalently the number of isomorphism classes in the set of all graphs on a fixed set of size n. This number has no known formula; the reader may care to verify that the numbers of graphs of orders 1, 2, 3, 4 and 5 are 1, 2,

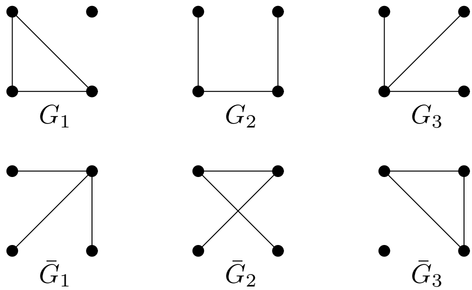


Figure 3: The graphs of order 4 and size 3

4, 11 and 34, respectively. (For order 5, particularly, you may want to use the following definition to reduce the tedium.)

Definition 1.3. The *complement* of a graph G is the graph $\bar{G}$ with $V(\bar{G}) = V(G)$ in which, for distinct vertices u and v, $uv \in E(\bar{G})$ iff $uv \notin E(G)$. Put another way, if K is the complete graph with vertex set $V(G)$, $E(\bar{G}) = E(K) - E(G)$.

For example, the complement $\bar{K}_n$ of the complete graph on n vertices is the empty graph on n vertices. If G has order n and size m then $\bar{G}$ has size $\binom{n}{2} - m$. A graph is *self-complementary* if $G \cong \bar{G}$. If this is so we must have $2m = \binom{n}{2}$, so $\binom{n}{2}$ is even, which happens iff $n \equiv 0$ or $1 \pmod 4$. In fact, for any such n there exist self-complementary graphs of order n. For $n = 4$, the size must be 3. In Figure 3, the top row shows all graphs of order 4 and size 3, while the bottom row shows their complements. We see that G_1 and G_3 are (up to isomorphism) complements of each other, while G_2 is self-complementary.

The *degree* of a vertex v of a graph G is the number of vertices adjacent to v. It is denoted by $\deg_G v$ or just $\deg v$. With this definition we can finally state a theorem.

Theorem 1.4. *Let G be a graph of size m. Then $\sum_{v \in V(G)} \deg v = 2m$.*

Proof. We can count the number p of pairs (v, e) where v is a vertex of the edge e in two ways. First, there are m edges, each with two vertices, so $p = 2m$. Second, the vertex v is incident to $\deg v$ edges, so $p = \sum_{v \in V(G)} \deg v$. □

A vertex is called *even* or *odd* according as its degree is even or odd.

Corollary 1.5. *Any graph contains an even number of odd vertices.* □

The minmum degree of the vertices of a graph is denoted by $\delta(G)$, and the maximum by $\Delta(G)$. Clearly $0 \leq \delta(G) \leq \Delta(G) \leq n(G) - 1$. A vertex of degree 0 is an *isolated vertex*, while one of degree 1 is an *end-vertex*. If $\delta(G) = \Delta(G)$ then G is *regular*; if the common value is d, G is *d-regular* or *regular of degree d*. A 3-regular graph is also called a *cubic graph*. The complete graph K_n is $(n-1)$-regular, and the Petersen graph of Figure 1 is cubic. For a regular graph, any two of the order n, the size m and the degree d determine the third by the equation $2m = nd$.

A *subgraph* of a graph G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$. If U is any proper subset of $V(G)$, $G - U$ is the subgraph obtained by deleting all vertices of U and all edges with at least one vertex in U. When $U = \{u\}$, we abbreviate $G - \{u\}$ to $G - u$. Similarly, if F is a subset of $E(G)$, $G - F$ is the subgraph obtained by deleting the edges of F, and $G - \{e\}$ is written as $G - e$. (This notation is logically indefensible, since an edge is a set of vertices. Nevertheless, for an edge $e = uv = \{u, v\}$, we adopt the convention that $G - e$ and $G - uv$ denote G with just an edge deleted, and $G - \{u, v\}$ denotes G with two vertices and all incident edges deleted.) Note that $V(G - F) = V(G)$, and $F = E(G)$ is allowed, with $G - E(G)$ being the empty graph on $V(G)$. A *spanning subgraph* of G is a subgraph of the same order (i.e., containing all the vertices of G). These are precisely the subgraphs of the form $G - F$ for $F \subseteq E(G)$. If u and v are independent vertices of G and $f = uv$, the graph $G + f$ has vertex set $V(G)$ and edge set $E(G) \cup \{f\}$.

If U is a non-empty set of vertices of the graph G, the subgraph $\langle U \rangle$ *induced* by U is the subgraph with vertex set U and edge set consisting of all edges of G with both vertices in U. A *vertex-induced subgraph* (or just an *induced subgraph*) is a subgraph of this form. If F is a non-empty set of edges of G, the subgraph $\langle F \rangle$ *induced* by F is the subgraph with vertex set all vertices of edges of F and edge set F. An *edge-induced subgraph* is a subgraph of this form.

Theorem 1.6 (König [22]). *Let G be a graph and d an integer with $d \geq \Delta(G)$. Then there is a d-regular graph containing G as an induced subgraph.*

Proof. The proof is by induction on $d - \delta(G)$. If this is 0, G is d-regular and there is nothing to do. If $d > \delta(G)$, let G' be another copy of G and form H by adding edges joining each vertex of G of degree $\delta(G)$ to the corresponding vertex of G'. Then H contains G as an induced subgraph, and has $\delta(H) = \delta(G) + 1$ and $\Delta(H) \leq d$. The result follows. $\square$

The graph constructed in this proof is generally not of minimal order;

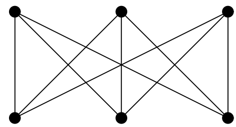
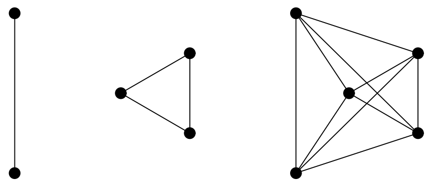


Figure 4: The complete bipartite graph $K_{3,3}$

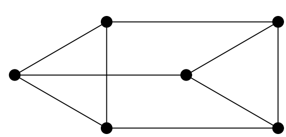


G_1 G_2 $G_1 * G_2$

Figure 5: The join of two graphs

see Exercise 1.7.

A graph G is *bipartite* if there is a partition of $V(G)$ into two sets V_1 and V_2, called *partite sets*, such that every edge has one vertex in V_1 and one in V_2. It is a *complete bipartite graph* if uv is an edge for all $u \in V_1$ and $v \in V_2$. The complete bipartite graph with r vertices in one partite set and s in the other is denoted by $K_{r,s}$; see Figure 4 for an example. The complete bipartite graph $K_{1,n}$ is called a *star*. More generally, G is *k-partite* if there is a partition of $V(G)$ into k subsets ($k \geq 2$) such that the vertices of any edge are in different sets of the partition, and *complete k-partite* if every pair consisting of two vertices from different sets is an edge. The complete k-partite graph with partite sets of sizes $n_1, n_2, \ldots, n_k$ is denoted by $K_{n_1,n_2,\ldots,n_k}$. A graph is *(complete) multipartite* if it is (complete) k-partite for some $k \geq 2$. (We exclude $k = 1$ because the only 1-partite graphs would be empty, and we would have a conflict in the use of K_n. At the other extreme, any graph of order n is n-partite, with the partite sets being the one-vertex sets, and $K_{1,1,\ldots,1}$ is a superfluous notation for K_n.)



$G_1 \times G_2$

Figure 6: The product of two graphs

Let G_1 and G_2 be graphs. Their *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Their *disjoint union* $G_1 \sqcup G_2$ is any graph of the form $G_1' \cup G_2$ where $G_1' \cong G_1$ and $V(G_1')$ is disjoint from $V(G_2)$; it is well-defined up to isomorphism. The disjoint union of k copies of a graph G is denoted kG. The *join* of G_1 and G_2 is the graph $G_1 * G_2$ obtained from $G_1 \sqcup G_2$ by adding an edge $v_1 v_2$ for each vertex v_1 of G_1 and v_2 of G_2; see Figure 5. Note that $\bar{G}_1 * \bar{G}_2 \cong \overline{G_1 \sqcup G_2}$. The complete bipartite graph $K_{r,s}$ can be described as either $\bar{K}_r * \bar{K}_s$ or $\overline{K_r \sqcup K_s}$. The *Cartesian product* $G_1 \times G_2$ has vertex set $V(G_1) \times V(G_2)$, and there is an edge with vertices (u_1, u_2) and (v_1, v_2) iff either $u_1 v_1 \in E(G_1)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$. Figure 6 shows the product of the graphs G_1 and G_2 of Figure 5.

Two classes of graphs we shall meet frequently are the paths and cycles. The path P_n $(n \geq 1)$ of order n has vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$ for $1 \leq i < n$. Of course, $P_1 \cong K_1$ and $P_2 \cong K_2$. For $n \geq 3$, the cycle C_n is obtained from P_n by adding the edge $v_n v_1$. It can be represented as a regular n-gon. For small values of n we refer to C_n as a triangle, quadrilateral, pentagon, ... Another special class of graphs consists of the cubes. The n-cube Q_n $(n \geq 0)$ is defined recursively by setting $Q_0 = K_1$ and $Q_{n+1} = Q_n \times K_2$. It may also be defined as the graph with vertices all sequences of zeroes and ones of length n, with two sequences being adjacent iff they differ in exactly one place.

Remark. The definition of a graph is mutable. A variant is that a graph G consists of disjoint sets $V(G)$ of vertices and edges $E(G)$, together with a relation of incidence between vertices and edges such that every edge is incident with either one or two vertices. (Some authors also allow edges incident with no vertices; well, I have, at times.) Edges incident with only one vertex are *loops*, while distinct edges incident with the same vertices are *parallel*. We shall refer to such objects as *multigraphs*, and hardly ever mention them again. Obviously every graph determines a multigraph. Further, every multigraph may be converted to a graph by subdividing its edges (see §15) while preserving most of its interesting properties.

Exercises for §1

1.1. Determine all non-isomorphic graphs of order at most 5.

1.2. Show that in any non-trivial graph there are two distinct vertices with the same degree.

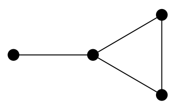
1.3. Let G be a self-complementary graph, and let G' be obtained from $G \sqcup P_4$ by adding edges from every vertex of G to the two end-vertices of P_4. Show that G' is self-complementary, and deduce that there are self complementary graphs of order n for every positive integer n with $n \equiv 0$ or $1 \pmod 4$.

1.4. Let G be a self-complementary graph of order $n \equiv 1 \pmod 4$. Prove that G has an odd number of vertices of degree $(n-1)/2$.

1.5. Give an example of two non-isomorphic regular graphs of order 6 and degree 3. [Hint: examine the figures in this section.]

1.6. For any integer $k \geq 2$, show that there are k non-isomorphic regular graphs, all of the same order and degree. [Hint: the disjoint union of regular graphs of the same degree is regular.]

1.7. Show that the minimum order of a 3-regular graph containing the graph G below as an induced subgraph is 6. Also draw the 3-regular graph constructed from G as in the proof of Theorem 1.6.



2 Degree sequences

Let G be a graph with vertices $v_1, \ldots, v_n$, and set $d_i = \deg v_i$. The sequence $(d_1, \ldots, d_n)$ is called a *degree sequence* of G. A sequence $(d_1, \ldots, d_n)$ of integers is *graphical* if it is a degree sequence of some graph. Necessary conditions for the sequence to be graphical are that $0 \leq d_i \leq n-1$ for all i and (by Theorem 1.4) that $d_1 + \cdots + d_n$ is even. These are not sufficient; it is easy to see that $(3, 3, 3, 1)$ is not graphical.

Theorem 2.1 (Havel [17], Hakimi [15]). *Let $d = (d_1, \ldots, d_n)$ be a sequence of integers with $n \geq 2$, $d_1 \geq \cdots \geq d_n \geq 0$ and $d_1 > 0$. Then d is graphical iff $d_1 \leq n-1$ and the sequence*

$$d' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$$

is graphical.

Evidently this theorem gives an algorithm for determining whether a sequence d of integers is graphical. The only graphical sequence of length 1 is (0), any sequence consisting entirely of zeroes is graphical, and any sequence with negative terms is not graphical. For any other sequence, arrange it in descending order. If $d_1 \geq n$, it is not graphical. Otherwise, form the sequence d' as above, and repeat the process until a decision is reached. For $d = (3, 3, 3, 1)$, we get $d' = (2, 2, 0)$ and then $d'' = (1, -1)$, so indeed d is not graphical.

Proof. Sufficiency is easy. Suppose that $d_1 \leq n - 1$ and d' is graphical. Let G' be a graph with vertices $v_2, \dots, v_n$ such that $\deg_{G'} v_i = d_i - 1$ for $2 \leq i \leq d_1 + 1$ and $\deg_{G'} v_i = d_i$ for $d_1 + 2 \leq i \leq n$. Form G by adding to G' a new vertex v_1 and edges $v_1 v_i$ for $2 \leq i \leq d_1 + 1$. Then $\deg_G v_i = d_i$ for $1 \leq i \leq n$.

Conversely, suppose that d is graphical. Certainly $d_1 \leq n - 1$. Consider all graphs G of order n and labellings of their vertices as $v_1, \dots, v_n$ for which $\deg v_i = d_i$, $1 \leq i \leq n$. Amongst all such, choose one for which the sum of the degrees of the vertices adjacent to v_1 is a maximum. We claim that in this G, the degrees of the vertices adjacent to v_1 are $d_2, \dots, d_{d_1+1}$, from which it will follow that $G - v_1$ has d' as a degree sequence, completing the proof.

Suppose the claim is false. Then there exist integers r and s with $2 \leq r < s \leq n$, $d_r > d_s$, and v_r not adjacent to v_1 and v_s adjacent to v_1. Since $\deg v_r > \deg v_s$, there is a vertex v_t adjacent to v_r and independent of v_s; v_t is automatically distinct from v_1 and v_r. Deleting the edges $v_1 v_s$ and $v_r v_t$ and adding edges $v_1 v_r$ and $v_s v_t$ gives a graph G' with $\deg_{G'} v_i = d_i$ for $1 \leq i \leq n$, but the sum of the degrees of the vertices adjacent to v_1 in G' is greater than the sum for G by $d_r - d_s$, a contradiction. □

Theorem 2.2 (Erdös-Gallai [11]). *Let $d = (d_1, \dots, d_n)$ be a sequence of integers with $n \geq 2$ and $d_1 \geq \cdots \geq d_n \geq 0$. Then d is graphical iff $\sum_{i=1}^n d_i$ is even and, for $1 \leq k \leq n - 1$,*

$$\text{(EG)} \qquad \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}.$$

Proof. We first show the necessity of the conditions. That $\sum_{i=1}^n d_i$ must be even has already been remarked. Let G be a graph of order n with vertices $v_1, \dots, v_n$ for which $\deg v_i = d_i$, $1 \leq i \leq n$, and let $1 \leq k \leq n - 1$. Consider all ordered pairs (i, j) with $1 \leq i \leq k$ and $1 \leq j \leq n$ for which $v_i v_j \in E(G)$.

For $1 \leq i \leq k$ there are d_i such pairs, so the total number of pairs is $\sum_{i=1}^k d_i$. The number of such pairs with $j \leq k$ is at most $k(k-1)$, and for each $j > k$ the number of such pairs is at most $\min\{d_j, k\}$, so (EG) holds.

For the converse, we prove the stronger result that if $\sum_{i=1}^n d_i$ is even and (EG) holds for all k with $1 \leq k \leq d_k$ then G is graphical. (It is easy to see that these cases of (EG) imply the remainder, but we do not need this.) If $n = 2$, the only such sequences are $(0,0)$ and $(1,1)$, which are graphical. Suppose then that $n > 2$ and the result holds for sequences of length $n-1$. If $d_1 = 0$ then d is the degree sequence of an empty graph. Suppose that $d_1 > 0$. The case $k = 1$ of (EG) is $d_1 \leq \sum_{i=2}^n \min\{d_i, 1\}$, and the right-hand side of this inequality is the number of $i \geq 2$ with $d_i \neq 0$. Thus $\hat{d} = d_{d_1+1}$ is positive. Let a and c be the least and greatest elements of the set of those $i \in \{2, \dots, n\}$ with $d_i = \hat{d}$. We have $a \leq d_1 + 1 \leq c$, so setting $b = a + c - d_1 - 1$ we have $a \leq b \leq c$. Partition the set $\{2, \dots, n\}$ into $R = \{i \mid 2 \leq i < a \text{ or } b \leq i \leq c\}$ and $S = \{i \mid a \leq i < b \text{ or } c < i \leq n\}$, and note that $|R| = d_1$. For $2 \leq i \leq n$, let $d'_i = d_i - 1$ if $i \in R$ and $d'_i = d_i$ if $i \in S$. The sequence $d' = (d'_2, \dots, d'_n)$ has $d'_2 \geq \cdots \geq d'_n \geq 0$ and $\sum_{i=2}^n d'_i = \sum_{i=1}^n d_i - 2d_1$. (It is just the sequence of Theorem 2.1 arranged in descending order.) We show that it satisfies the inequalities corresponding to (EG), namely

$$\sum_{i=2}^{k} d'_i \leq (k-1)(k-2) + \sum_{i=k+1}^{n} \min\{d'_i, k-1\},$$

for $1 \leq k-1 \leq d'_k$. Once this has been done, it will follow from the inductive hypothesis that d' is graphical, and then that d is graphical as in the earlier proof. For $i \geq c$ we have

$$d'_i \leq d'_c = \hat{d} - 1 \leq d_1 - 1 \leq c - 2 \leq i - 2,$$

so $k - 1 \leq d'_k$ implies $k < c$. Suppose first that $k < a$. If $k \leq \hat{d}$, then $\min\{d'_i, k-1\} = k - 1 = \min\{d_i, k-1\}$ for $i \in R$, while clearly $\min\{d'_i, k-1\} = \min\{d_i, k-1\}$ for $i \in S$. Hence

$$\begin{aligned}
\sum_{i=2}^{k} d'_i &= \sum_{i=2}^{k} d_i - k + 1 \leq \sum_{i=1}^{k-1} d_i - k + 1 \\
&\leq (k-1)(k-2) + \sum_{i=k}^{n} \min\{d_i, k-1\} - k + 1 \\
&= (k-1)(k-2) + \sum_{i=k+1}^{n} \min\{d'_i, k-1\}.
\end{aligned}$$

If $k \geq \hat{d}+1$, then $\min\{d'_i, k-1\} = d_i = \min\{d_i, k\}$ for $i \in S$, and of course $\min\{d'_i, k-1\} = \min\{d_i, k\} - 1$ for $i \in R$. Therefore

$$\begin{aligned}
\sum_{i=2}^{k} d'_i &= \sum_{i=1}^{k} d_i - d_1 - k + 1 \\
&\leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} - d_1 - k + 1 \\
&= k(k-1) + \sum_{i=k+1}^{n} \min\{d'_i, k-1\} + (d_1 - k + 1) - d_1 - k + 1 \\
&= (k-1)(k-2) + \sum_{i=k+1}^{n} \min\{d'_i, k-1\}.
\end{aligned}$$

For the rest of the proof, we assume that $a \leq k$. We distinguish three cases.

Case 1. $a \leq k < b$ and $k \leq \hat{d}$.

Case 2. $a \leq k < b$ and $k = \hat{d}+1$.

Case 3. $b \leq k \leq \hat{d}$.

We have

$$\begin{aligned}
\sum_{i=2}^{k} d'_i &\leq (a-2)(d_1-1) + \sum_{i=a}^{k} d'_i \\
&= (a-2)(d_1-1) + (k-a+1)\hat{d} - x_k,
\end{aligned}$$

where x_k is 0 in cases 1 and 2, and $k-b+1$ in case 3. Also,

$$\begin{aligned}
&(k-1)(k-2) + \sum_{i=k+1}^{n} \min\{d'_i, k-1\} \\
&\quad \geq (k-1)(k-2) + \sum_{i=k+1}^{c} \min\{d'_i, k-1\} \\
&\quad = (k-1)(k-2) + (c-k)(k-1) - y_k \\
&\quad = (k-1)(c-2) - y_k,
\end{aligned}$$

where y_k is 0 in cases 1 and 3, and $c-b+1$ in case 2. It is therefore enough to prove that $z_k \geq 0$, where

$$\begin{aligned} z_k &= (k-1)(c-2) - y_k - ((a-2)(d_1-1) + (k-a+1)\hat{d} - x_k) \\ &= (a-2)(c-d_1-1) + (k-a+1)(c-\hat{d}-2) + x_k - y_k. \end{aligned}$$

Case 1. Here $z_k = (a-2)(c-d_1-1) + (k-a+1)(c-\hat{d}-2)$, and since $a < b$ we have $\hat{d} \leq d_1 \leq c-2$, so $z_k \geq 0$.

Case 2. Here

$$\begin{aligned} z_k &= (a-2)(c-d_1-1) + (k-a+1)(c-\hat{d}-2) - (c-b+1) \\ &= (a-2)(c-d_1-1) + (k-a)(c-\hat{d}-2) + b - \hat{d} - 3. \end{aligned}$$

As in the previous case, $\hat{d} \leq d_1 \leq c-2$, and also $b \geq k+1 = \hat{d}+2$, so $z_k \geq 0$ except perhaps if $a = 2$ and $b = \hat{d}+2$. In this case $c = \hat{d} + d_1 + 1$ and $z_k = (\hat{d}-1)(d_1-1) - 1$. If $\hat{d} = 1$ then $\sum_{i=1}^n d_i = d_1 + c - 1 = 2d_1 + 1$, which is impossible. Hence $1 < \hat{d} \leq d_1$ and $z_k \geq 0$.

Case 3. Here

$$z_k = (a-2)(c-d_1-1) + (k-a+1)(c-\hat{d}-2) + k - b + 1,$$

and since $\hat{d} \leq d_1 \leq c-1$, $z_k > 0$ unless $\hat{d} = d_1 = c-1$, and in that case $b = a$ so $z_k = 0$. □

Exercises for §2

2.1. Let n and d be integers with $0 \leq d < n$ and dn even. Prove that there is a d-regular graph of order n.

2.2. Come up with 5 different sequences $(d_1, \ldots, d_7)$ 0f integers with $d_1 \geq \cdots \geq d_7 > 0$, $d_1 + \cdots + d_7$ even and $d_1 \leq 6$, of which 2 are graphical and 3 are not. Prove that those that aren't graphical aren't, and give drawings showing that the other two are.

3 Connectedness

Definition 3.1. A *walk* in a graph G is a sequence $W = (u_0, u_1, \ldots, u_l)$ $(l \geq 0)$ of vertices of G such that u_{i-1} and u_i are adjacent for $1 \leq i \leq l$. The *length* of W is l (the number of edges on W rather than the number of vertices). We say that W is a u_0-u_l walk, or a walk from u_0 to u_l. If

$u_0 = u_l$, W is a *closed* walk; otherwise it is *open*. A *trail* is a walk with no repeated edges ($u_iu_{i+1} \neq u_ju_{j+1}$ for $i \neq j$). A *path* is a walk with no repeated vertices ($u_i \neq u_j$ for $i \neq j$). Every path is a trail, and for any vertex u the trivial u-u walk (u) is a path. A *circuit* is a non-trivial closed trail, and a *cycle* is a closed walk $(u_0, u_1, \ldots, u_l)$ of length $l \geq 3$ such that $u_i \neq u_j$ for $1 \leq i < j \leq l$. Every cycle is a circuit, and every circuit has length at least 3.

Note that the subgraph induced by the edges of a path in G of length l is P_{l+1}, and that induced by the edges of a cycle of length l is C_l. We shall often confuse a walk (trail, path, circuit, cycle) with the subgraph induced by its edges. If W is a u-v walk, we refer to u and v as the *endpoints* of W, and when W is a path any other vertices on W are called *internal vertices* of W. We say that a walk $(u_0, u_1, \ldots, u_l)$ contains a walk $(v_0, v_1, \ldots, v_k)$ if there are integers $0 \leq i_0 < i_1 < \cdots < i_k \leq l$ such that $v_j = u_{i_j}$ for $0 \leq j \leq k$.

Theorem 3.2. *Any u-v walk in a graph G contains a u-v path.*

Proof. The proof is by induction on the length. Suppose

$$W = (u = u_0, u_1, \ldots, u_l = v)$$

is a u-v walk, and assume inductively that every u-v walk (if any) of length less than l contains a u-v path. If W is a path, there is nothing to do. Otherwise, there exist i and j with $0 \leq i < j \leq l$ and $u_i = u_j$. Now W contains the u-v walk $(u_0, u_1, \ldots, u_i, u_{j+1}, \ldots, u_l)$ of length less than l, which in turn contains a u-v path. □

Definition 3.3. We say that vertices u and v of a graph G are *connected* if there is a u-v path in G, and in this case we write $u \sim v$. The graph G is *connected* if any two of its vertices are connected.

Note that by Theorem 3.2, u and v are connected iff there is a u-v walk. If $W = (u_0, u_1, \ldots, u_l)$ is a walk, we denote by W^r the reverse walk $(u_l, u_{l-1}, \ldots, u_0)$. If $W' = (v_0, v_1, \ldots, v_k)$ is another walk with $v_0 = u_l$ we denote by $W \cdot W'$ the walk $(u_0, u_1, \ldots, u_l, v_1, \ldots, v_k)$.

Lemma 3.4. *Let G be a graph. The relation of Definition 3.3 is an equivalence relation on $V(G)$.*

Proof. Let u, v and w be vertices of G. The trivial u-u path shows that $u \sim u$. If $u \sim v$, let P be a u-v path. Then P^r is a v-u path, so $v \sim u$. If also $v \sim w$, let Q be a v-w path. Then $P \cdot Q$ is a u-w walk, so $u \sim w$. □

The subgraphs of G induced by the equivalence classes for the relation $\sim$ are called the *components* of G. Each component is connected, and every connected subgraph of G is contained in a component. The number of components is denoted by $k(G)$. Thus $k(G) = 1$ iff G is connected.

Definition 3.5. Let u and v be vertices of a connected graph G. The *distance* from u to v, $d(u, v) = d_G(u, v)$, is the minimum of the lengths of all paths from u to v in G. A u-v path of length $d(u, v)$ is a u-v *geodesic*.

By Theorem 3.2, $d(u, v)$ is also the minimum of the lengths of all walks from u to v. In a disconnected graph, we may define $d(u, v)$ as above for vertices in the same component, and set it to ∞ for vertices in different components.

Theorem 3.6. *Let u, v and w be vertices of a connected graph G.*

(1) $d(u, u) = 0$ *and* $d(u, v) > 0$ *if* $u \neq v$.

(2) $d(u, v) = d(v, u)$.

(3) $d(u, w) \leq d(u, v) + d(v, w)$.

Proof. (1) is obvious. The other parts follow by examining the lengths of the walks used to show symmetry and transitivity in the proof of Lemma 3.4. □

A cycle in a graph is called *odd* or *even* according as its length is odd or even.

Theorem 3.7. *A non-trivial graph G is bipartite iff it has no odd cycles.*

Proof. We may assume that G is connected, since a non-trivial graph is bipartite iff all its non-trivial components are bipartite. Suppose first that G is bipartite, with partite sets V_1 and V_2, and let $(u_0, u_1, \ldots, u_k)$ be a cycle in G. Without loss of generality, $u_0 \in V_1$. It follows by an easy induction that $u_i \in V_1$ if i is even and $u_i \in V_2$ if i is odd. Since $u_k = u_0 \in V_1$, k is even.

Suppose conversely that G has no odd cycles, and pick a vertex w. Let V_1 be the set of those $u \in V(G)$ with $d(u, w)$ even, and V_2 the set of those with $d(u, w)$ odd. We have $w \in V_1$, and since G is connected and non-trivial, there is a vertex adjacent to w, and therefore in V_2. Thus $\{V_1, V_2\}$ is a partition of $V(G)$. We show that G is bipartitite with respect to this partition. Suppose for a contradiction that u and v are adjacent vertices in the same set of the partition; that is, with $d(u, w) \equiv d(v, w) \pmod 2$. Let $(w = u_0, u_1, \ldots, u_k = u)$ and $(w = v_0, v_1, \ldots, v_l = v)$ be geodesics, so that $k \equiv l \pmod 2$. For $0 \le i \le j \le k$, $d(u_i, u_j) = j - i$, since otherwise we could replace the subpath $(u_i, \ldots, u_j)$ by a shorter path to obtain a shorter w-u walk. Similarly, $d(v_i, v_j) = j - i$ for $1 \le i \le j \le l$. Hence if $u_i = v_j$ then $i = d(w, u_i) = d(w, v_j) = j$. Let i be the greatest integer for which $u_i = v_i$. Since $u \ne v$, $(k-i)+(l-i) \ge 1$, and since $k \equiv l \pmod 2$, $(k-i)+(l-i) \ge 2$. Now $(u_i, \ldots, u_{k-1}, u, v, v_{l-1}, \ldots, v_i)$ is a cycle of odd length $k+l-2i+1$. □

Definition 3.8. Let G be a connected graph. The *eccentricity* $e_G(u) = e(u)$ of a vertex u of G is defined by

$$e(u) = \max\{d(u, v) \mid v \in V(G)\}.$$

The *radius* $\operatorname{rad} G$ of G is the minimum of $e(u)$ over all vertices u, while the *diameter* $\operatorname{diam} G$ is the maximum of $e(u)$. We also have

$$\operatorname{diam} G = \max\{d(u, v) \mid u, v \in V(G)\}.$$

A vertex u is *central* if $e(u) = \operatorname{rad} G$ and *peripheral* if $e(u) = \operatorname{diam} G$, and the *center* $\operatorname{Cen} G$ and *periphery* $\operatorname{Per} G$ of G are the subgraphs induced by the central and peripheral vertices, respectively.

Example 3.9. $\operatorname{rad} K_n = \operatorname{diam} K_n = 1$ and $\operatorname{Cen} K_n = \operatorname{Per} K_n = K_n$ for $n \ge 2$.

Example 3.10. $\operatorname{rad} C_n = \operatorname{diam} C_n = \lfloor n/2 \rfloor$ and $\operatorname{Cen} C_n = \operatorname{Per} C_n = C_n$.

Example 3.11. Let P_n have vertices $v_1, \ldots, v_n$ in that order, where $n \ge 3$. We have $\operatorname{diam} P_n = n - 1$ and $\operatorname{Per} P_n = \langle v_1, v_n \rangle \cong \bar{K}_2$. If $n = 2k + 1$ then $\operatorname{rad} P_n = k$ and $\operatorname{Cen} P_n = \langle v_{k+1} \rangle \cong K_1$, while if $n = 2k$ then $\operatorname{rad} P_n = k$ and $\operatorname{Cen} P_n = \langle v_k, v_{k+1} \rangle \cong K_2$.

Theorem 3.12. *For any connected graph G, $\operatorname{rad} G \le \operatorname{diam} G \le 2 \operatorname{rad} G$.*

Proof. That $\operatorname{rad} G \le \operatorname{diam} G$ is immediate from the definitions. For the other inequality we must show that $d(u, v) \le 2 \operatorname{rad} G$ for any vertices u and v. If w is a central vertex we have

$$d(u, v) \le d(u, w) + d(w, v) \le 2e(w) = 2 \operatorname{rad} G.$$ □

Theorem 3.13 (Hedetniemi; see [5]). *Any graph G is the center of some connected graph.*

Proof. Form a graph H by adding to G four new vertices v_1, v_2, w_1 and w_2, and edges uv_i and v_iw_i for $u \in V(G)$ and $i = 1$ or 2. Then H is connected, $e_H(u) = 2$ for $u \in V(G)$, $e_H(v_1) = e_H(v_2) = 3$ and $e_H(w_1) = e_H(w_2) = 4$, so $G = \operatorname{Cen} H$. □

Theorem 3.14 (Bielak and Syslo [3]). *A graph G is the periphery of some connected graph iff G is complete or no vertex of G has eccentricity 1.*

If G is not connected, the eccentricity of any vertex is infinite, so the theorem asserts that G is the periphery of some connected graph.

Proof. A complete graph is its own periphery, so suppose G is not complete. Suppose first that there is a connected graph H with $G = \operatorname{Per} H$. Then H is also not complete, so $\operatorname{diam} H \geq 2$. For $u \in V(G)$, $e_G(u) \geq e_H(u) = \operatorname{diam} H$, so no vertex of G has eccentricity 1. Conversely, if no vertex of G has eccentricity 1, set $H = K_1 * G$, with new vertex v. For $u \in V(G)$, $e_H(u) = 2$, while $e_H(v) = 1$, so $G = \operatorname{Per} H$. □

Definition 3.15. If u is a vertex of the connected graph G, the *total distance* $\operatorname{td}(u) = \operatorname{td}_G(u)$ is defined by $\operatorname{td}(u) = \sum_{v \in V(G)} d(u, v)$. A *median vertex* of G is a vertex with minimum total distance, and the *median* of G is the subgraph $\operatorname{Med} G$ induced by the median vertices.

Theorem 3.16 (Hendry [18]). *For any graphs G_1 and G_2, there is a connected graph H with* $\operatorname{Cen} H \cong G_1$ *and* $\operatorname{Med} H \cong G_2$.

Proof. We may assume that $V(G_1)$ and $V(G_2)$ are disjoint. For $i = 1$ or 2, set $n_i = n(G_i)$ and $\delta_i = \delta(G_i)$. Let a and b be positive integers with $a > (n_2 - n_1 - \delta_2 - 3)/2$ and $b > 2a + n_1 + n_2 - \delta_2$. Form H by adding to $G_1 \cup G_2$ vertices x_i for $1 \leq i \leq a + 2$, y_i for $1 \leq i \leq a$ and z_i for $1 \leq i \leq b$, and edges

$$
\begin{array}{rl}
x_i x_{i+1} & \text{for } 1 \leq i < a + 2; \\
y_i y_{i+1} & \text{for } 1 \leq i < a; \\
z_i z_j & \text{for } 1 \leq i < j \leq b; \\
u_1 x_1 \text{ and } u_1 y_1 & \text{for } u_1 \in V(G_1); \text{ and} \\
u_2 y_a \text{ and } u_2 z_i & \text{for } u_2 \in V(G_2) \text{ and } 1 \leq i \leq b - \deg_{G_2} u_2 + \delta_2.
\end{array}
$$

In what follows, d, e and td denote distance, eccentricity and total distance in H. For distinct vertices u_i and u_i' of G_i ($i = 1$ or 2) we have $d(u_i, u_i') = 1$ if $u_i u_i' \in E(G_i)$ and $d(u_i, u_i') = 2$ otherwise. Let u_2 be a vertex of G_2.

Since $b - \deg_{G_2} u_2 + \delta_2 > 0$, u_2 is adjacent to z_1, and so $d(u_2, z_i) \leq 2$ for $1 \leq i \leq b$. Since $b - \deg_{G_2} u_2 + \delta_2 \leq b$, the number of z_i with $d(u_2, z_i) = 1$ is $b - \deg_{G_2} u_2 + \delta_2$. The remaining distances in H are given below; for those involving a vertex z_i, note that z_i is adjacent to at least one vertex of G_2, namely a vertex of minimum degree in G_2. We have

$$\begin{aligned}
d(x_i, x_j) &= j - i && \text{for } 1 \leq i < j \leq a+2; \\
d(x_i, u_1) &= i && \text{for } 1 \leq i \leq a+2 \text{ and } u_1 \in V(G_1); \\
d(x_i, y_j) &= i + j && \text{for } 1 \leq i \leq a+2 \text{ and } 1 \leq j \leq a; \\
d(x_i, u_2) &= i + a + 1 && \text{for } 1 \leq i \leq a+2 \text{ and } u_2 \in V(G_2); \\
d(x_i, z_j) &= i + a + 2 && \text{for } 1 \leq i \leq a+2 \text{ and } 1 \leq j \leq b; \\
d(u_1, y_i) &= i && \text{for } u_1 \in V(G_1) \text{ and } 1 \leq i \leq a; \\
d(u_1, u_2) &= a + 1 && \text{for } u_1 \in V(G_1) \text{ and } u_2 \in V(G_2); \\
d(u_1, z_i) &= a + 2 && \text{for } u_1 \in V(G_1) \text{ and } 1 \leq i \leq b; \\
d(y_i, y_j) &= j - i && \text{for } 1 \leq i < j \leq a; \\
d(y_i, u_2) &= a - i + 1 && \text{for } 1 \leq i \leq a \text{ and } u_2 \in V(G_2); \\
d(y_i, z_j) &= a - i + 2 && \text{for } 1 \leq i \leq a \text{ and } 1 \leq j \leq b; \text{ and} \\
d(z_i, z_j) &= 1 && \text{for } 1 \leq i < j \leq b.
\end{aligned}$$

Hence

$$\begin{aligned}
e(x_i) &= i + a + 2 && \text{for } 1 \leq i \leq a+2; \\
e(u_1) &= a + 2 && \text{for } u_1 \in V(G_1); \\
e(y_i) &= i + a + 2 && \text{for } 1 \leq i \leq a; \\
e(u_2) &= 2a + 3 && \text{for } u_2 \in V(G_2); \text{ and} \\
e(z_i) &= 2a + 4 && \text{for } 1 \leq i \leq b,
\end{aligned}$$

so $\operatorname{Cen} H = G_1$. Further,

$$\operatorname{td}(x_i) = \sum_{j=1}^{i-1} j + \sum_{j=1}^{a-i+2} j + i n_1 + \sum_{j=i+1}^{i+a} j + (i + a + 1) n_2 + (i + a + 2) b$$

for $1 \leq i \leq a+2$;

$$\operatorname{td}(u_1) \geq \sum_{j=1}^{a+2} j + n_1 - 1 + \sum_{j=1}^{a} j + (a+1) n_2 + (a+2) b$$

for $u_1 \in V(G_1)$;

$$\mathrm{td}(y_i) = \sum_{j=i+1}^{i+a+2} j + in_1 + \sum_{j=1}^{i-1} j + \sum_{j=1}^{a-i} j + (a-i+1)n_2 + (a-i+2)b$$

for $1 \le i \le a$;

$$\mathrm{td}(u_2) = \sum_{j=a+2}^{2a+3} j + (a+1)n_1 + \sum_{j=1}^{a} j + 2n_2 + b - \delta_2 - 2$$

for $u_2 \in V(G_2)$; and

$$\mathrm{td}(z_i) \ge \sum_{j=a+3}^{2a+4} j + (a+2)n_1 + \sum_{j=2}^{a+1} j + n_2 + b - 1$$

for $1 \le i \le b$. For $1 \le i < a+2$

$$\mathrm{td}(x_{i+1}) - \mathrm{td}(x_i) = 2i + n_1 + n_2 + b - 2 > 0,$$

so the minimum value of $\mathrm{td}(x_i)$ is

$$\mathrm{td}(x_1) = \sum_{j=1}^{a+1} j + n_1 + \sum_{j=2}^{a+1} j + (a+2)n_2 + (a+3)b,$$

while for $1 \le i < a$

$$\mathrm{td}(y_i) - \mathrm{td}(y_{i+1}) = -2i - n_1 + n_2 + b - 2 \ge -2a - n_1 + n_2 + b > 0,$$

so the maximum and minimum values of $\mathrm{td}(y_i)$ are

$$\mathrm{td}(y_1) = \sum_{j=2}^{a+3} j + n_1 + \sum_{j=1}^{a-1} j + an_2 + (a+1)b$$

$$\text{and} \qquad \mathrm{td}(y_a) = \sum_{j=a+1}^{2a+2} j + an_1 + \sum_{j=1}^{a-1} j + n_2 + 2b,$$

respectively. We have, for $u_1 \in V(G_1)$, $u_2 \in V(G_2)$ and $1 \le i \le b$,

$$\begin{aligned}
\mathrm{td}(x_1) - \mathrm{td}(y_1) &= 2n_2 + 2b - 4 > 0; \\
\mathrm{td}(u_1) - \mathrm{td}(y_1) &\ge n_2 + b - 3 > 0; \\
\mathrm{td}(y_a) - \mathrm{td}(u_2) &= -2a - n_1 - n_2 + b + \delta_2 > 0; \text{ and} \\
\mathrm{td}(z_i) - \mathrm{td}(u_2) &\ge 2a + n_1 - n_2 + \delta_2 + 3 > 0,
\end{aligned}$$

so $\mathrm{Med}\, H = G_2$. □

Exercises for §3

3.1. Prove that any graph G contains a path of length $\delta(G)$.

4 Graphs and matrices

Definition 4.1. Let G be a graph of order n and size m, with vertices $v_1, \ldots, v_n$ and edges $e_1, \ldots, e_m$. The *adjacency matrix* of G is then $n \times n$ matrix $A = A(G)$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G); \\ 0 & \text{if } v_i v_j \notin E(G). \end{cases}$$

The *degree matrix* of G is the $n \times n$ diagonal matrix $D = D(G)$ with diagonal entries $d_{ii} = \deg v_i$, and the *Lagrangian matrix* is $L = L(G) = D - A$. Finally, if $m > 0$, the *incidence matrix* of G is the $m \times n$ matrix $B = B(G)$ with entries

$$b_{ij} = \begin{cases} 1 & \text{if } e_i \text{ is incident to } v_j; \\ 0 & \text{if not.} \end{cases}$$

Evidently an $n \times n$ 0-1 matrix (a matrix with all entries 0 or 1) is the adjacency matrix of some graph iff it is symmetric and has all diagonal entries 0, and an $m \times n$ 0-1 matrix is the incidence matrix of some graph iff every row has exactly two non-zero entries and different rows are unequal. Either the adjacency or the incidence matrix determines the graph up to isomorphism.

Theorem 4.2. *Let G be a graph of order n with vertices $v_1, \ldots, v_n$ and adjacency matrix $A = [a_{ij}]$. For any integer $l \geq 0$, let $A^l = [a_{ij}^{(l)}]$. Then $a_{ij}^{(l)}$ is the number of v_i-v_j walks of length l.*

Proof. This is by induction on l, the case $l = 0$ being obvious. Suppose the result holds for some l. Then

$$a_{ij}^{(l+1)} = \sum_{k=1}^{n} a_{ik}^{(l)} a_{kj} = \sum_{k: v_k v_j \in E(G)} a_{ik}^{(l)},$$

which is the number of walks of length l from v_i to a vertex adjacent to v_j. This is clearly the number of walks of length $l+1$ from v_i to v_j. $\square$

The adjacency matrix A of a graph G of order n, being a real symmetric matrix, is diagonalizable over $\mathbb{R}$; that is, there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Theorem 4.3. *Let G be a graph of order n with vertices $v_1, \ldots, v_n$ and adjacency matrix A. If G is d-regular, A has $\mathbf{x}_1 = (1, 1, \ldots, 1)$ as an eigenvector of eigenvalue d, and for any eigenvalue λ, $|\lambda| \leq d$. If also G is connected, the multiplicity of the eigenvalue d is 1.*

Proof. For any graph G, $A\mathbf{x}_1 = (d_1, d_2, \ldots, d_n)$ where $d_i = \deg v_i$, giving the first assertion. Suppose that G is d-regular, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be an eigenvector of eigenvalue λ. Then, for $1 \leq i \leq n$,

$$x_{j(i,1)} + x_{j(i,2)} + \cdots + x_{j(i,d)} = \lambda x_i,$$

where $v_{j(i,1)}, v_{j(i,2)}, \ldots, v_{j(i,d)}$ are the vertices adjacent to v_i. Take i so that $|x_i|$ is a maximum. Then $|\lambda x_i| \leq \sum_{k=1}^{d} |x_{j(i,k)}| \leq d|x_i|$, so $|\lambda| \leq d$. If $\lambda = d$, we must have

$$x_{j(i,1)} = x_{j(i,2)} = \cdots = x_{j(i,d)} = x_i.$$

It follows that $x_j = x_k$ if v_j and v_k are in the same component, completing the proof. $\square$

Theorem 4.4. *Let A be the adjacency matrix of a connected graph G. If A has at most two eigenvalues then G is complete.*

Proof. Let G have order n. If $n = 1$ then $G \cong K_1$, so suppose $n \geq 2$. A diagonalizable matrix with only one eigenvalue is a scalar multiple of the identity, so A has at least two eigenvalues. Suppose it has exactly two. Then its minimal polynomial has degree 2, so $A^2 = aA + bI$ for some a and b, and every diagonal entry of A^2 is b. The diagonal entry of A^2 corresponding to a vertex v is the number of v-v walks of length 2, which is $\deg v$. Therefore G is b-regular, so one eigenvalue is b and the other is -1. The minimal poynomial is thus $(x - b)(x + 1)$, so $A(A + I) = b(A + I)$. That is, every column of $A + I$ is an eigenvector of A with eigenvalue b, and since every diagonal entry of $A+I$ is 1, it follows from Theorem 4.3 that $A+I = J$, the $n \times n$ matrix with all entries 1. That is, $A = J - I$, the adjacency matrix of K_n. $\square$

Exercises for §4

4.1. Let G be a connected d-regular graph with adjacency matrix A. Show that $-d$ is an eigenvalue of A iff G is bipartite, in which case it has multiplicity 1.

5 Blocks

Definition 5.1. A vertex v of a graph G is a *cut-vertex* if G is non-trivial and $k(G-v) > k(G)$. A graph is *non-separable* if it is connected and has no cut-vertex; otherwise it is *separable.*

Note that an end-vertex is never a cut-vertex.

Theorem 5.2. *A vertex u of a non-trivial graph G is a cut-vertex iff there exist vertices v and w, different from u, that belong to the same component of G, and for which every v-w path contains u.*

Proof. Since every component of $G-u$ is contained in a component of G, u is a cut-vertex iff some component of G contains more than one component of $G-u$. Since vertices v and w of $G-u$ lie in different components of $G-u$ iff every v-w path in G contains u, the result follows. □

Theorem 5.3. *Let G be a non-trivial connected graph. No peripheral vertex of G is a cut-vertex, and in particular G has at least two vertices that are not cut-vertices.*

Proof. Let u be a peripheral vertex of G, and v a vertex with $d(u,v) = \operatorname{diam} G$. Suppose that u is a cut-vertex, and let w be a vertex of a component of $G-u$ not containing v. Then every v-w path, and in particular every v-w geodesic, contains u, so $d(v,w) > d(u,v) = \operatorname{diam} G$, a contradiction. □

Definition 5.4. An edge e of a graph G is a *bridge* if $k(G-e) > k(G)$.

The proof of the following theorem is similar to that of Theorem 5.2, and is left as an exercise.

Theorem 5.5. *An edge e of a graph G is a bridge iff there exist vertices v and w that belong to the same component of G, and for which every v-w path contains e.* □

It follows that if e is a bridge then $k(G-e) = k(G)+1$ and the vertices of e belong to different components of $G-e$. A vertex of a bridge is either a cut-vertex or an end-vertex, and so the only connected graph with a bridge but no cut-vertices is K_2.

Theorem 5.6. *An edge e of a graph G is a bridge iff it lies on no cycle in G.*

Proof. Let $e = uv$. Suppose that e lies on a cycle C. For any two vertices w and x in a single component of G, take a w-x path in G. If it contains the edge e, replace e by the other u-v or v-u path in C to get a w-x path in $G - e$. Hence e is not a bridge.

Now suppose e is not a bridge. Then there is a u-v path P in $G - e$, and $P \cdot (v, u)$ is a cycle in G containing e. □

Definition 5.7. A graph is *acyclic* if it contains no cycles. A connected acyclic graph is a *tree*, and an acyclic graph (whose components are trees) is also called a *forest*.

In a forest, every edge is a bridge, so every vertex is either a cut-vertex or an end-vertex. A non-separable graph of order at least three has no bridges, so it is not acyclic. The following result is an immediate consequence of Theorem 5.3.

Theorem 5.8. *A non-trivial tree has at least two end-vertices.* □

Theorem 5.9. *Let G be a graph of order $n \geq 3$. Then the following are equivalent.*

(1) *G has no isolated vertices and any two edges of G lie on a common cycle.*

(2) *G is non-empty and any vertex and edge of G lie on a common cycle.*

(3) *Any two vertics of G lie on a common cycle.*

(4) *G is non-separable.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial. Suppose that (3) holds. Certainly G is connected. Let u be any vertex of G, and v and w vertices distinct from u. There is a cycle C containing v and w, and at least one of the two v-w paths in C does not contain u, so u is not a cut-vertex by Theorem 5.2. Thus G is non-separable.

Suppose that G is non-separable, and let e_1 and e_2 be edges of G. Let k be the minimum distance from a vertex of e_1 to a vertex of e_2. We prove by induction on k that there is a cycle containing e_1 and e_2. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$, where $d(u_1, u_2) = k$. We show first that there is a cycle C containing e_1 and u_2. If $k = 0$, this is because G has no bridges, so there is a cycle containing e_1. If $k > 0$, there is a vertex w adjacent to u_2 with $d(u_1, w) = k - 1$, and by inductive hypothesis there is a cycle containing e_1 and the edge u_2w.

If e_2 is on C, we are done, so suppose it is not. Since u_2 is not a cut-vertex, there is a path P in $G - u_2$ from v_2 to a vertex x of C. Taking P to have minimum length, x is the only vertex of P on C. Let Q be the x-u_2 path in C containing e_1. Then $P \cdot Q \cdot (u_2, v_2)$ is the desired cycle. $\square$

Definition 5.10. Paths P and Q in a graph G are *internally disjoint* if no internal vertex of either one is on the other.

The next result is just a reformulation of the equivalence (3) $\Leftrightarrow$ (4) of the previous one, and is left as an exercise.

Corollary 5.11. *A graph G of order $n \geq 3$ is non-separable iff, for $u \neq v \in V(G)$, there are at least two internally disjoint u-v paths.*

Definition 5.12. A *block* of a graph G is a maximal non-separable subgraph of G. (That is, it is a non-separable subgraph H such that $H \subseteq H' \subseteq G$ and H' non-separable imply that $H = H'$.)

Every block of G is contained in a component of G, and every edge is contained in a block, so the only trivial blocks of G are isolated vertices. The subgraph induced by a bridge is a block of G, called an *acyclic block of G*. If G is non-separable, it is its own unique block.

Lemma 5.13. *Let G be a connected graph.*

(1) *Two distinct blocks of G meet in at most a single vertex, which must be a cut-vertex of G.*

(2) *Every cut-vertex of G is in at least two blocks.*

(3) *If v and w are vertices such that no block of G contains both v and w, there is a cut-vertex u such that v and w are in different components of $G - u$.*

Proof. The proof is by induction on the order of G. If G has no cut-vertices, (1)–(3) are vacuously true, so suppose G has a cut-vertex u. Let the components of $G - u$ be $H_1, \ldots, H_k$ ($k \geq 2$), and set $G_i = \langle V(H_i) \cup \{u\} \rangle$ for $1 \leq i \leq k$. Each G_i is a connected graph of smaller order than G, so by induction the result holds for G_i. If v and w belong to different components of $G - u$ then any connected subgraph of G containing v and w contains u as a cut-vertex, so there is no block of G containing v and w. That is, every block of G is contained in some G_i, and so the blocks of G are just the blocks of all the G_i's. Also, a vertex of G_i is a cut-vertex of G_i iff it is a cut-vertex of G different from u. Consider two distinct blocks of G. If they lie in the same G_i, they meet in at most a single vertex which is a cut-vertex of G_i, and if not they meet in at most u. Every cut-vertex of G other than u is in at least two blocks of some G_i, and u is in $k \geq 2$ blocks of G, one from each G_i. Finally, suppose v and w are vertices contained in no block of G. If they lie in a single G_i, there is a cut-vertex u' of G_i such that v and w lie in different components of $G_i - u'$, and hence of $G - u'$. Otherwise, v and w lie in different components of $G - u$. □

Theorem 5.14 (Harary and Norman [16]). *Let G be a connected graph. Then* $\operatorname{Cen} G$ *is contained in a single block of G.*

Proof. Let u and v be vertices such that no block of G contains both u and v, and let w be the cut-vertex provided by Lemma 5.13. Let x be a vertex with $d(w,x) = e(w)$. At least one of u and v is in a different component of $G - w$ from x; say u is. Then $e(u) \geq d(u,x) = d(u,w) + d(w,x) > e(w)$, so u is not a central vertex. The result follows. □

Corollary 5.15. *Let T be a tree. Then* $\operatorname{Cen} T$ *is isomorphic to K_1 or K_2.*

Proof. If T is non-trivial, every block of T is isomorphic to K_2, and the only induced subgraphs of K_2 are K_1 and K_2. □

If a connected graph G has a cut-vertex, every block of G contains a cut-vertex. A block of G containing exactly one cut-vertex is called an *end-block*.

Definition 5.16. The *block graph* $\operatorname{Blk} G$ of a graph G is a bipartite graph with partite sets V_1 and V_2, where V_1 is the set of all blocks of G, and V_2 is the set of all cut-vertices. A block B and a cut-vertex v are adjacent in $\operatorname{Blk} G$ iff B contains v.

The block graph of G is the disjoint union of the block graphs of its components, so it is enough to consider $\operatorname{Blk} G$ for G connected. By (2) of Lemma 5.13, the end-vertices of $\operatorname{Blk} G$ are precisely the end-blocks of G.

Theorem 5.17. *The block graph of a connected graph G is a tree.*

Proof. If G is non-separable then $\operatorname{Blk} G$ is trivial, so suppose G has a cut-vertex. Since every block contains a cut-vertex, to show that $\operatorname{Blk} G$ is connected it is enough to show that any cut-vertices u and v of G are connected in $\operatorname{Blk} G$. Let $(u_0, u_1, \ldots, u_r)$ be a u-v path in G, and let the cut-vertices on this path be $u_{i_0}, u_{i_1}, \ldots, u_{i_s}$ where $0 = i_0 < i_1 < \cdots < i_s = r$. Since any edge or non-cut-vertex of G is contained in a unique block, each sub-path from $u_{i_{j-1}}$ to u_{i_j} is contained in a block B_j. Now

$$(u, B_1, u_{i_1}, B_2, \dots, u_{i_{s-1}}, B_s, v)$$

is a u-v path in $\operatorname{Blk} G$.

It remains to prove that $\operatorname{Blk} G$ is acyclic. If not, it contains a cycle

$$(B_0, u_1, B_1, \dots, u_k, B_k = B_0)$$

where each B_i is a block and each u_i is a cut-vertex. Let $H = B_1 \cup \cdots \cup B_k$, which is clearly connected. Let v be a vertex of H, and let B_i' be $B_i - v$ if v is in B_i, and B_i otherwise. Each B_i' is connected, and if $v \neq u_i$ then B_{i-1}' and B_i' have a common vertex, and hence lie in a single component of $H - v$. Since v is equal to at most one u_i, $H - v = B_1' \cup \cdots \cup B_k'$ is connected, so H is non-separable. This contradicts the maximality of the blocks B_i. □

The next result is immediate from the previous one and Theorem 5.8.

Theorem 5.18. *A connected graph with a cut-vertex contains at least two end-blocks.* □

Theorem 5.19. *Let G be a connected graph with a cut-vertex. Then there is a cut-vertex u such that at most one block of G containing u is not an end-block.*

Proof. Let T be the result of deleting from $\operatorname{Blk} G$ all its end-vertices. Then T is also a tree. If T is trivial, then G has a unique cut-vertex u and all blocks of G are end-blocks, so u has the desired property. Otherwise, T has an end-vertex, which must be a cut-vertex u of G since blocks are not adjacent in $\operatorname{Blk} G$. Now every block of G containing u is an end-block, except for the unique vertex of T adjacent to u. □

The next lemma will be used in §17.

Lemma 5.20. *Let G be a non-trivial connected graph and U a set of non-cut-vertices of G containing at most one vertex from each block. Then $G-U$ is connected.*

Proof. This is by induction on the number of blocks, the case of a single block being trivial. Suppose G has more than one block, let G_1 be an end-block, and let G_2 be the union of the other blocks. Then G_1 and G_2 meet in a single cut-vertex u of G. For $i = 1$ or 2, let $U_i = U \cap V(G_i)$. By induction, $G_1 - U_1$ and $G_2 - U_2$ are connected, and since they have u in common, so is their union $G - U$. □

Exercises for §5

5.1. Prove Theorem 5.5.

5.2. Prove Corollary 5.11.

5.3. Prove that a graph with only even vertices has no bridge.

5.4. Let T be a tree of order $n \geq 3$, and let T' be the tree obtained by deleting all end-vertices of T. Prove that for every vertex u of T', $e_{T'}(u) = e_T(u) - 1$. Use this to give an alternative proof of Theorem 5.14.

6 Connectivity

Definition 6.1. A *vertex-cut* of a graph G is a proper subset U of $V(G)$ such that $G - U$ is disconnected.

If G is not complete, let u and v be independent vertices. Then $V(G) - \{u, v\}$ is a vertex-cut. The *connectivity* of G, $\kappa(G)$, is defined to be the minimum number of vertices in a vertex-cut. If G has order n, we have $0 \leq \kappa(G) \leq n-2$. A complete graph has no vertex-cut; we define $\kappa(K_n) = n-1$. In all cases, $\kappa(G)$ is the minimum cardinality of a set U of vertices such that $G - U$ is disconnected or trivial. We have $\kappa(G) = 0$ iff G is disconnected or trivial. Also $\kappa(G) = 1$ iff G is connected and non-trivial, and either $G \cong K_2$ or G has a cut-vertex. Thus $\kappa(G) \geq 2$ iff G is non-separable of order at least 3.

Definition 6.2. An *edge-cut* of a graph G is a subset X of $E(G)$ such that $G - X$ is disconnected.

If G is non-trivial, G has an edge-cut (for instance, $E(G)$), and the *edge-connectivity* $\kappa_1(G)$ is the minimum cardinality of an edge-cut. We also set $\kappa_1(K_1) = 0$. We have $\kappa_1(G) = 0$ iff G is disconnected or trivial and $\kappa_1(G) = 1$ iff G is connected, non-trivial, and has a bridge. If X is a minimal edge cut of a connected graph G then $G - X$ has exactly two components, and each edge of X has one vertex in each.

Theorem 6.3 (Whitney [31])**.** *For any graph G, $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$.*

Proof. We may assume that G is connected and non-trivial. Suppose that X is an edge-cut with $|X| = \kappa_1(G)$, and let the components of $G - X$ be G_1 and G_2. Suppose first that every vertex of G_1 is adjacent to every vertex of G_2.

Let the orders of G, G_1 and G_2 be n, n_1 and n_2. Then $(n_1 - 1)(n_2 - 1) \geq 0$, so $|X| = n_1 n_2 \geq n_1 + n_2 - 1 = n - 1 \geq \kappa(G)$. Otherwise, let $u \in V(G_1)$ and $v \in V(G_2)$ be non-adjacent. For every edge e of X, pick a vertex $f(e)$ of e different from u and v, and set $U = \{f(e) \mid e \in X\}$, so $|U| \leq |X|$. Then U is a vertex-cut, because u and v are vertices of $G - U$, and every u-v path in G must contain an edge e of X, and therefore the vertex $f(e)$ of U. We have proved that $\kappa(G) \leq \kappa_1(G)$ in all cases. The set of edges incident with any vertex is an edge-cut, which gives $\kappa_1(G) \leq \delta(G)$. □

Theorem 6.4. *Let G be a graph of diameter 2. Then $\kappa_1(G) = \delta(G)$.*

Proof. Let $k = \kappa_1(G)$, and let X be an edge-cut with $|X| = k$. The graph $G - X$ has two components H_1 and H_2. We show first that either every vertex of H_1 is adjacent to a vertex of H_2, or every vertex of H_2 is adjacent to a vertex of H_1. Suppose there is a vertex u of H_1 adjacent to no vertex of H_2. If v is any vertex of H_2, $d(u, v) = 2$ and there is a path (u, w, v). Since w is adjacent to u, w is in H_1. Thus every vertex of H_2 is adjacent to a vertex of H_1.

Let n_1 and n_2 be the orders of H_1 and H_2, and choose the numbering so that $n_1 \leq n_2$. In the first case above $k \geq n_1$ and in the second $k \geq n_2$, so in any case $k \geq n_1$. For a vertex u of H_1, let $d_1(u)$ and $d_2(u)$ be the numbers of vertices of H_1 and H_2 adjacent to u. Then

$$n_1 \leq k \leq \delta(G) \leq \deg u = d_1(u) + d_2(u) \leq n_1 - 1 + d_2(u).$$

Hence $d_2(u) \geq 1$; that is, every vertex of H_1 is adjacent to a vertex of H_2. For any vertex u of H_1,

$$k = \sum_{v \in V(G_1)} d_2(v) \geq d_2(u) + n_1 - 1 \geq \delta(G) \geq k,$$

so $k = \delta(G)$. □

Let u and v be vertices of a graph G. We say that a set W of vertices of G *separates u and v* if u and v belong to different components of $G - W$; such a set is of course a vertex-cut.

Theorem 6.5 (Menger [24])**.** *Let u and v be independent vertices of a graph G. Then the minimum number of vertices that separate u and v is equal to the maximum number of internally disjoint u-v paths.*

Proof. For independent vertices u and v of a graph G let $s_G(u,v)$ be the minimum number of vertices that separate u and v (which exists because the set of all other vertices separates), and let $p_G(u,v)$ be the maximum number of internally disjoint u-v paths. If P is any such set of paths and W is a set of vertices separating u and v, W must contain at least one internal vertex of every path in P, so $|P| \leq |W|$ and hence $p_G(u,v) \leq s_G(u,v)$. If $p_G(u,v) = 0$, then u and v are in different components of G, and so $s_G(u,v) = 0$. It follows that if $s_G(u,v) = 1$ then $p_G(u,v) = 1$, so if $p_G(u,v) < s_G(u,v)$ then $s_G(u,v) \geq 2$. Suppose there exist independent vertices u and v of a graph G with $p_G(u,v) < s_G(u,v)$. Let t be the smallest integer for which there exist such u, v and G with $s_G(u,v) = t$, and amongst such u, v and G let G have minimum size. We make the following claims about G.

(1) There is no vertex adjacent to both u and v.

(2) If w_1 and w_2 are adjacent vertices different from u and v, there is a set W of $t-1$ vertices such that $W \cup \{w_1\}$ and $W \cup \{w_2\}$ both separate u and v.

(3) If W is a set of t vertices separating u and v, either every vertex of W is adjacent to u, or every vertex of W is adjacent to v.

For (1), suppose that w is adjacent to u and v, and let $H = G - w$. A set W of vertices separates u and v in H iff $W \cup \{w\}$ separates them in G, so $s_H(u,v) = t$ or $t-1$. If $s_H(u,v) = t$, then since H has smaller size than G, $p_H(u,v) = t$. But t internally disjoint u-v paths in H are also paths in G, a contradiction. If $s_H(u,v) = t-1$, then $p_H(u,v) = t-1$. However, adding to $t-1$ internally disjoint u-v paths in H the path (u,w,v) gives t such paths in G, again a contradiction.

As for (2), let H be obtained by deleting the edge w_1w_2 of G. If W is a set of vertices separating u and v in H, then $W \cup \{w_i\}$ separates them in G for $i = 1$ or 2, so $s_H(u,v) \geq t-1$. On the other hand, any set of vertices separating u and v in G also separates in H, so $s_H(u,v) \leq t$. If $s_H(u,v) = t$, then since H has smaller size than G, $p_H(u,v) = t$. As before, t internally disjoint u-v paths in H are also paths in G, a contradiction. Hence $s_H(u,v) = t-1$, and a set of $t-1$ vertices separating u and v in H is the set required by the claim.

Finally, suppose, contrary to (3), that there is a set W of t vertices separating u and v in G that contains a vertex not adjacent to u, and one not adjacent to v. Let K_u be the component of $G - W$ containing u, and H_u the subgraph of G induced by all edges with at least one vertex in K_u.

By minimality of W, every vertex of W is in K_u, and since W has a vertex not adjacent to u, H_u has at least $t+1$ edges. Define K_v and H_v similarly. Let G_u be the graph obtained from H_u by adding the vertex v and edges vw for all w in W. Since H_v has at least $t+1$ edges, G_u has smaller size than G. Clearly W separates u from v in G_u; let W' be any set of vertices separating u from v in G_u. Every u-v path P in G gives a u-v path P' in G_u, namely the initial subpath in P from u to some $w \in W$, followed by (w, v). Since P' contains a vertex of W', so does P. It follows that $s_{G_u}(u, v) = t$, and so $p_{G_u}(u, v) = t$. Now t internally disjoint u-v paths in G_u give internally disjoint paths in H_u from u to each vertex of W. Similarly, there are internally disjoint paths in H_v from each vertex of W to v. Putting these together with the paths in H_u gives t internally disjoint u-v paths in G, a contradiction.

At this point, (1)–(3) have been established, and we now derive a final contradiction. Let $(w_0, w_1, \ldots, w_k)$ be a u-v geodesic in G. By (1), $k \geq 3$. By (2), there is a set W of $t-1$ vertices such that $W \cup \{w_1\}$ and $W \cup \{w_2\}$ both separate u and v in G. Since w_1 is not adjacent to v, (3) implies that every vertex of W is adjacent to v, and since w_2 is not adjacent to u, it implies that every vertex of W is adjacent to v. Since $t \geq 2$, W is non-empty and we have a contradiction to (1). □

Definition 6.6. Let k be a positive integer. A graph G is *k-connected* if $\kappa(G) \geq k$, and *edge k-connected* if $\kappa_1(G) \geq k$.

G is 2-connected iff G is non-separable of order at least 3, so Corollary 5.11 can be restated as saying that a non-trivial graph is 2-connected iff every two vertices are connected by two internally disjoint paths. This is generalized in Theorem 6.8 below.

Lemma 6.7. *Let e be an edge of a k-connected graph G, where $k \geq 2$. Then $G - e$ is $(k-1)$-connected.*

Proof. Let $e = uv$, and let W be a set of vertices such that $G - e - W$ is disconnected or trivial. If u or v is in W, $G - e - W = G - W$, so $|W| \geq k$. Otherwise, u and v are vertices of $G - e - W$, which is therefore disconnected. If $G - u - W$ is disconnected, we have $|W| \geq k-1$. Otherwise, $G - e - W$ has u as an isolated vertex, and just one other component. Hence $G - v - W$ is trivial or disconnected, so again $|W| \geq k-1$. □

Theorem 6.8 (Whitney [31]). *Let G be a non-trivial graph. Then G is k-connected ($k \geq 1$) iff for all $u \neq v \in V(G)$ there are k internally disjoint u-v paths.*

Proof. Suppose first that such paths exist. Take distinct vertices of G. Of a set of k internally disjoint paths connecting them, at most one has no internal vertices, so G has order at least $k+1$. Thus it suffices to show that any vertex-cut U has at least k elements. Let P be a set of k internally disjoint paths joining vertices of different components of $G-U$. Then U contains an internal vertex of every path in P, so indeed $|U| \geq k$.

Suppose conversely that G is k-connected, and let $u \neq v \in V(G)$. If $k=1$ the result is trivial, so suppose $k \geq 2$. If u and v are not adjacent, any set of vertices separating u and v has at least k elements, so Menger's Theorem gives at least k internally disjoint u-v paths. Otherwise, $G-uv$ is $(k-1)$-connected by Lemma 6.7, so Menger's Theorem gives at least $k-1$ internally disjoint u-v paths in $G-uv$, and adding the path (u,v) gives at least k internally disjoint u-v paths in G. □

Lemma 6.9. *Let $u_1, \ldots, u_k$ be distinct vertices of a k-connected graph G, and form H by adding a new vertex v and edges $u_i v$ for $1 \leq i \leq k$. Then H is k-connected.*

Proof. Let U be a set of fewer than k vertices of H. If $v \notin U$ then $G-U$ is connected and contains at least one u_i, so $H-U$ is connected and non-trivial. If $v \in U$ then $H-U = G-(U-\{v\})$, which is connected and non-trivial. □

Theorem 6.10. *Let $u, v_1, \ldots, v_k$ be distinct vertices of a k-connected graph G. Then there are internally disjoint u-v_i paths for $1 \leq i \leq k$.*

Proof. Form H by adding a new vertex w and edges $v_i w$ for $1 \leq i \leq k$. By Lemma 6.9, H is k-connected, so there are k internally disjoint u-w paths in H by Theorem 6.8. Deleting their last edges gives the required paths in G. □

Theorem 6.11. *Let G be k-connected, where $k \geq 2$. Then any k vertices of G lie on a common cycle.*

Proof. Let U be a set of k vertices of G, and let C be a cycle containing the maximum number, say l, of vertices of U. Certainly $l > 0$. (In fact, Theorem 5.9 implies that $l \geq 2$, but we do not need this.) We suppose, for a contradiction, that $l < k$. Let u be a vertex of U not on C. Suppose first that C has length l. By Theorem 6.10, there are internally disjoint paths from u to every vertex of C. Replacing an edge of C by two of these paths gives a cycle containing at least $l+1$ vertices of U, a contradiction. Otherwise, let V be a set of $l+1$ vertices of C. Since $l+1 \leq k$, Theorem

6.10 gives internally disjoint paths from u to every vertex of V. Replacing these paths by their shortest initial segments ending on C, and V by the set of terminal points of the new paths, we may assume that each path in the set meets C only in its terminal point. Now there are vertices v_1 and v_2 of V such that one of the v_1-v_2 paths in C has no element of V or U as an internal vertex. Replacing this path by a path through u gives a cycle containing at least $l+1$ vertices of U, completing the proof. □

Theorem 6.12. *Let G be a graph of order $n \geq 2$, and let k be an integer with $1 \leq k \leq n-1$. If $\delta(G) \geq \frac{1}{2}(n+k-2)$ then G is k-connected.*

Proof. Suppose that G is not k-connected. Then G has a vertex-cut U with $|U| = l \leq k-1$. Let G_1 be a component of $G - U$ of minimum order n_1. Then $n_1 \leq (n-l)/2$. If v is a vertex of G_1 then v is adjacent only to vertices of U and other vertices of G_1, so

$$\delta(G) \leq \deg v \leq l + n_1 - 1 \leq (n+l-2)/2 \leq (n+k-3)/2. \qquad \square$$

Exercises for §6

6.1. (a) Let G_1 and G_2 be disjoint copies of K_n, $n \geq 2$, and form a graph G by adding to $G_1 \cup G_2$ k edges, $0 < k < n$, which must have one vertex in G_1 and one in G_2. Suppose that n_1 vertices of G_1 and n_2 of G_2 are incident with one or more of these k edges. Determine $\kappa(G)$, $\kappa_1(G)$ and $\delta(G)$.

(b) Show that for any positive integers a, b and c with $a \leq b \leq c$, there is a graph G with $\kappa(G) = a$, $\kappa_1(G) = b$ and $\delta(G) = c$.

6.2. Determine the connectivity and edge-connectivity of any complete multipartite graph.

7 Trees

Recall that a spanning subgraph of a graph G is a subgraph containing all the vertices of G.

Definition 7.1. A *spanning tree* of a graph is a spanning subgraph that is a tree, and a *spanning forest* is a spanning subgraph that is a forest.

We shall be concerned with spanning trees more than spanning forests. Evidently a graph with a spanning tree is connected.

Theorem 7.2. *Any connected graph G has a spanning tree.*

Proof. The proof is by induction on the size m of G. If G is itself a tree, there is nothing to do. Otherwise, G has an edge e that is not a bridge, and then $G - e$ is a connected spanning subgraph of G of size $m - 1$, which has a spanning tree T by induction. Now T is also a spanning tree of G. □

We have seen that any non-trivial tree T has end-vertices. Deleting an end-vertex from T leaves a tree. Conversely, adding to a tree a single vertex v and an edge with v as a vertex yields a tree (with v as an end-vertex).

Theorem 7.3. *For a graph G of order n and size m, any two of the following conditions imply the third.*

(1) *G is connected.*

(2) *G is acyclic.*

(3) $m = n - 1$.

Thus to show that a graph is a tree, it is enough to verify any two of these conditions.

Proof. (a) We show that (1) and (2) imply (3) by induction on n. If $n = 1$ then $m = 0$ and (3) holds. If $n > 1$ then as remarked above we may delete an end-vertex, leaving a tree of order $n-1$ and size $m-1$, giving the inductive step.

(b) Suppose that (2) and (3) hold and G has k components. Applying part (a) to each component and adding, we have $m = n - k$, so $k = 1$, as required.

(c) Suppose that (1) and (3) hold. Then G has a spanning tree T. By part (a), T has size m, so $T = G$, giving (2). □

Corollary 7.4. *Let G be an acyclic graph of order n and size m with k components. Then $m = n - k$.* □

Lemma 7.5. *Let u and v be vertices of a graph G, and suppose there are distinct u-v paths P and Q of lengths k and l. Then there is a cycle of length at most $k + l$ in G, and if $P \cdot Q^r$ is not a cycle, there is one of length less than $k + l$.*

Proof. Let $P = (u_0, u_1, \ldots, u_k)$. There is some i, $0 \leq i < k$, for which u_i is on Q and the edge $u_i u_{i+1}$ is not. Let j be the smallest integer greater than i for which u_j is on Q. Let P_1 and Q_1 be the u_i-u_j subpaths of P and Q or

Q^r, respectively. Then $P_1 \cdot Q_1^r$ is a cycle in G of length at most $k+l$, with equality only if $P = P_1$ and $Q = Q_1$. □

Theorem 7.6. *A graph G is a tree iff there is a unique u-v path for all u, $v \in V(G)$.*

Proof. Suppose that any two vertices of G are connected by a unique path. Certainly G is connected. If C is a cycle in G and u and v are any two distinct vertices of C, there are two distinct u-v paths in C, a contradiction. Thus G is a tree. The converse is immediate from the preceding lemma. □

Theorem 7.7. *Let $d = (d_1, \dots, d_n)$ be a sequence of positive integers, where $n \geq 2$. Then d is a degree sequence of a tree iff $\sum_{i=1}^n d_i = 2n - 2$.*

Proof. If d is a degree sequence of a tree of size m then $\sum_{i=1}^n d_i = 2m = 2n-2$. For the converse, if $n = 2$ then $d = (1,1)$, the degree sequence of the tree K_2. Suppose then that $n > 2$ and the result holds for sequences of length $n-1$. We may assume $d_1 \geq \cdots \geq d_n$. Then $d_1 > 1$, since otherwise $\sum_{i=1}^n d_i = n < 2n-2$, and $d_n = 1$, since otherwise $\sum_{i=1}^n d_i \geq 2n$. The sequence $(d_1 - 1, d_2, \dots, d_{n-1})$ of positive integers has sum $2n-4$, so it is a degree sequence of a tree T of order $n-1$. Adding to T a vertex u and an edge uv where $\deg_T v = d_1 - 1$ gives a tree with degree sequence d. □

Theorem 7.8. *Let T be a tree of order n. If G is a graph with $\delta(G) \geq n-1$ then G has a subgraph isomorphic to T.*

Proof. This is by induction on n, the case $n = 1$ being trivial. Suppose that $n > 1$, and let u be an end-vertex of T and v the vertex of T adjacent to u. By inductive hypothesis, there is an isomorphism ϕ from $T' = T - u$ to a subgraph H of G. Since $\deg_T v \leq n-1$, $\deg_{T'} v \leq n-2$. Since $\deg_G \phi(v) \geq n-1$, there is a vertex u' of G adjacent to $\phi(v)$ and not in H. Then ϕ may be extended to an isomorphism from T to a subgraph of G by sending u to u'. □

8 Counting trees

There is no known formula for the number of trees of order n (up to isomorphism). For small values of n one can enumerate the trees of order n by considering all ways of adding a vertex and edge to a tree of order $n-1$. The unique trees of orders 1, 2 and 3 are K_1, K_2 and P_3. For $n = 4$, there are two trees, the path P_4 and the star $K_{1,3}$. Of course, any path or star is a tree. Other special classes of trees are the *double stars*, which have

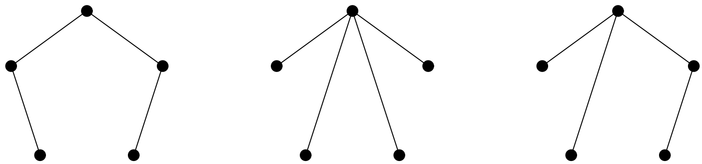


Figure 7: The trees of order 5

exactly two (necessarily adjacent) vertices of degree greater than 1, and the *caterpillars*, the trees for which deleting the end-vertices leaves a path (the *spine* of the caterpillar). There are three trees of order 5, shown in Figure 7, a path, a star, and a double star. There are 6 of order 6, whose enumeration is left as an exercise.

For the number of trees on a given set of size n (up to equality), sometimes known as the number of labelled trees of order n, there is the following theorem of Cayley [6]. The proof below is due to Prüfer [26].

Theorem 8.1. *Let $X = \{1, \ldots, n\}$. The number of trees with vertex set X is n^{n-2}.*

Proof. The cases $n = 1$ and 2 are obvious, so assume $n \geq 3$. Let T be a tree with $V(T) = X$. We define trees T_i for $0 \leq i \leq n-2$ and integers $s_i \in X$ for $1 \leq i \leq n-2$ such that $V(T_i) \subseteq X$ and $|V(T_i)| = n - i$ as follows. We set $T_0 = T$. Suppose that $1 \leq i \leq n-2$ and T_{i-1} has been constructed. Let v_i be the least end-vertex of T_{i-1}, let s_i be the vertex of T_{i-1} adjacent to v_i and set $T_i = T_{i-1} - v_i$. The sequence $(s_1, \ldots, s_{n-2})$ so constructed is the *Prüfer sequence* of T. For later use, we show that, for $0 \leq i \leq n-2$,

$$\begin{aligned} &\text{a vertex of } T_i \text{ is an end-vertex} \\ &\Leftrightarrow \text{it is not in the set } \{s_{i+1}, \ldots, s_{n-2}\}. \end{aligned} \tag{8.1}$$

For $i = n-2$ this is clear: T_{n-2} has two vertices, which are end-vertices, and the set in question is empty. Suppose that (8.1) holds for some i with $1 \leq i \leq n-2$. T_{i-1} has one more vertex than T_i, namely v_i, which is an end-vertex of T_{i-1} and is not in the set $\{s_i, \ldots, s_{n-2}\}$. Further, a vertex of T_i is an end-vertex of T_{i-1} iff it is an end vertex of T_i and not equal to s_i, which is the case iff it is not in $\{s_i, \ldots, s_{n-2}\}$. This establishes (8.1) for $i-1$.

Now let $(s_1, \ldots, s_{n-2})$ be any sequence of elements of X. We construct graphs G_i and sets $X_i \subseteq X$ for $0 \leq i \leq n-2$ such that $V(G_i) = X$, $|E(G_i)| = i$, $|X_i| = n - i$, $\{s_{i+1}, \ldots, s_{n-2}\} \subseteq X_i$, and each component of

G_i has exactly one vertex in X_i. We take G_0 to be the empty graph on X and $X_0 = X$. Suppose that $1 \le i \le n-2$ and G_{i-1} and X_{i-1} have been constructed. Then at least two elements of X_{i-1} are not in the set $\{s_i, \dots, s_{n-2}\}$; let v_i be the least such element. Then v_i and s_i are in distinct components of G_{i-1}. Setting $G_i = G_{i-1} + v_i s_i$ and $X_i = X_{i-1} - \{v_i\}$, G_i and X_i have the desired properties. The set X_{n-2} has two elements v_{n-1} and v_n, and the graph G_{n-2} has $n-2$ edges and two components, one containing v_{n-1} and the other containing v_n. Hence $T = G_{n-2} + v_{n-1}v_n$ is a tree. For $0 \le i \le n-2$, let T_i be the subgraph of T induced by the vertices X_i. We show that T_i is a tree whose end vertices are the elements of $X_i - \{s_{i+1}, \dots, s_{n-2}\}$. For $i = n-2$, $X_{n-2} = \{v_{n-1}, v_n\}$ and $v_{n-1}v_n$ is an edge of T, so the result holds in this case. Suppose it holds for some i, $1 \le i \le n-2$. Then $X_{i-1} = X_i \cup \{v_i\}$, $s_i \in X_i$, and $v_i s_i$ is an edge of G_i, and hence of T. Thus $T_{i-1} = T_i + v_i s_i$, which is a tree, and its set of end-vertices is obtained from that of T_i by deleting s_i (if present) and adding v_i. This proves the result for $i-1$. Note also that v_i is the least end vertex of T_{i-1}. It follows that the trees T_i are also the trees used in constructing the Prüfer sequence of T, and that this sequence is equal to $(s_1, \dots, s_{n-2})$.

Now, starting with a tree T, construct the trees $T_0, \dots, T_{n-2}$ and the Prüfer sequence $(s_1, \dots, s_{n-2})$ as in the first paragraph, and then the graphs $G_0, \dots, G_{n-2}$ and the sets $X_0, \dots, X_{n-2}$ from this sequence. We show that, for $0 \le i \le n-2$,

$$X_i = V(T_i) \text{ and } E(G_i) = E(T) - E(T_i), \tag{8.2}$$

the case $i = 0$ being obvious. Suppose that $1 \le i \le n-2$, and (8.2) holds for $i-1$. Let v_i be the least end-vertex of T_{i-1}, so that s_i is the vertex of T_{i-1} adjacent to v_i and $T_i = T_{i-1} - v_i$. By (8.1), v_i is also the least element of $V(T_{i-1}) = X_{i-1}$ not in $\{s_i, \dots, s_{n-2}\}$, so $G_i = G_{i-1} + v_i s_i$ and $X_i = X_{i-1} - \{v_i\}$, proving (8.2) for i. If $X_{n-2} = \{v_{n-1}, v_n\}$, the case $i = n-2$ of (8.2) shows that $G_{n-2} + v_{n-1}v_n = T$. Thus sending a tree on X to its Prüfer sequence gives a bijection from the set of such trees to X^{n-2}, proving the theorem. □

The following result of Kirchoff [21] is known as the Matrix-Tree Theorem.

Theorem 8.2. *Let G be a graph. The number of spanning trees of G is equal to any cofactor of the Lagrangian matrix $L(G)$.*

The number of trees on a given set is the same as the number of spanning trees in the complete graph on that set, so consistency with Cayley's

Theorem requires that any cofactor of $L(K_n)$ is equal to n^{n-2}; this is left as an exercise.

Proof. The sum of each row or column of $L(G)$ is zero, from which it follows easily that all cofactors of $L(G)$ are equal. For any non-empty proper subset U of $V = V(G)$, let $L_U(G)$ be the matrix obtained from $L(G)$ by deleting the rows and columns corresponding to elements of U. Define a U-forest in G to be a spanning forest in which every component contains exactly one element of U. When U has a single element, a U-forest is just a spanning tree, so the result will follow if we show that the number of U-forests is equal to $\det L_U(G)$. This we do by induction on $|V - U|$. Suppose this is 1, and let v be the element of $V - U$. Then $\det L_U(G) = \deg v$, while the U-forests are the spanning subgraphs with just one edge, incident to v. The result holds in this case. Now suppose that $|V - U| > 1$, and the result holds for larger subsets of V. Here we use induction on the number k of edges with one vertex in U and one in $V - U$. If $k = 0$, there are no U-forests, while $L_U(G)$ is just the Lagrangian of the subgraph induced by $V - U$, and so has determinant zero. Suppose then that $k > 0$, and let $e = uv$ be an edge with $u \in U$ and $v \in V - U$. Set $H = G - e$. The matrix $L_U(G)$ is obtained from $L_U(H)$ by adding 1 to the diagonal entry corresponding to v, so $\det L_U(G) = \det L_U(H) + \det L_{U'}(H)$, where $U' = U \cup \{v\}$. By the induction on k, $\det L_U(H)$ is the number of U-forests in H, and by the induction on $|V - U|$, $\det L_{U'}(H)$ is the number of U'-forests in H. The U-forests in H are precisely the U-forests in G that do not contain e. Also, there is a one-to-one correspondence between the U'-forests in H and the U-forests in G that do contain e, given by adding e to a U'-forest in H. The result follows. □

Exercises for §8

8.1. Draw all trees of order 6.

8.2. Let A be a square matrix in which every row and column sums to zero. Show that all cofactors of A are equal.

8.3. Show that any cofactor of $L(K_n)$ is equal to n^{n-2}.

9 Permutation groups

In what follows, we need some knowledge of groups. However, the general theory is not really needed, and a few easy facts about permutation groups will suffice.

Definition 9.1. Let X be a finite set. A *permutation* of X is a bijection $\pi\colon X \to X$. The set of all permutations of X will be denoted by $S(X)$. When $X = \{1, \ldots, n\}$, we write S_n for $S(X)$, and we will often identify $S(X)$ with S_n when X has n elements.

For any set X we have the identity permutation $\mathrm{id} = \mathrm{id}_X \in S(X)$, and for any $\pi \in S(X)$ the inverse π^{-1} is in $S(X)$. Finally, for any π and σ in $S(X)$, the composite $\pi\sigma$ is in $S(X)$. We have:

(1) $\mathrm{id}\,\pi = \pi = \pi\,\mathrm{id}$ for all $\pi \in S(X)$;

(2) $\pi\pi^{-1} = \mathrm{id} = \pi^{-1}\pi$ for all $\pi \in S(X)$;

(3) $\pi(\sigma\tau) = (\pi\sigma)\tau$ for all π, σ, $\tau \in S(X)$.

It follows easily that $\pi\sigma = \pi\tau$ implies $\sigma = \tau$, as does $\sigma\pi = \tau\pi$. The set $S(X)$ together with the operation of composition is called the *symmetric group* of X. Clearly the number of elements (or *order*) of S_n is $n!$.

Definition 9.2. Let $x_1, \ldots, x_k$ be distinct elements of X. The *permutation k-cycle* $(x_1 \ldots x_k)$ is the permutation π of X defined by $\pi(x_i) = x_{i+1}$ for $1 \le i < k$, $\pi(x_k) = x_1$, and $\pi(x) = x$ if $x \notin \{x_1, \ldots, x_k\}$. Permutation cycles $(x_1 \ldots x_k)$ and $(y_1 \ldots y_l)$ are *disjoint* if $x_i \neq y_j$ for all i and j.

Of course, the same cycle may be written in different ways: $(1\,2\,3) = (2\,3\,1) = (3\,1\,2)$. Any 1-cycle (x) is equal to the identity id. If $\pi = (x_1 \ldots x_k)$ and $\sigma = (y_1 \ldots y_l)$ are disjoint then π and σ commute ($\pi\sigma = \sigma\pi$). It is easy to see that any permutation can be written as a product of disjoint cycles, and that this representation is unique except for the order of the factors and the presence or absence of 1-cycles.

Example 9.3. $S_3 = \{\mathrm{id}, (1\,2), (1\,3), (2\,3), (1\,2\,3), (3\,2\,1)\}$.

Definition 9.4. A *permutation group* on the finite set X is a subset A of $S(X)$ such that:

(1) $\mathrm{id} \in A$;

(2) if $\alpha \in A$ then $\alpha^{-1} \in A$;

(3) if $\alpha \in A$ and $\beta \in A$ then $\alpha\beta \in A$.

We also call A a *subgroup* of $S(X)$.

Example 9.5. Let G be a graph. An *automorphism* of G is an isomorphism $\alpha\colon G \to G$. The set $\mathrm{Aut}(G)$ of all automorphisms of G is a permutation group on $V(G)$.

Example 9.6. It is well-known that any permutation $\pi \in S(X)$ can be written as a product of (not necessarily disjoint) 2-cycles, or *transpositions*, and that if this is done in two ways, the numbers of factors are either both even or both odd; π is called an *even* or *odd* permutation accordingly. The *sign* $\mathrm{sgn}(\pi)$ of π is 1 if π is even and -1 if π is odd. Clearly $\mathrm{sgn}(\mathrm{id}) = 1$, $\mathrm{sgn}(\pi^{-1}) = \mathrm{sgn}(\pi)$ and $\mathrm{sgn}(\pi\sigma) = \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma)$, and it follows that the set $\mathrm{Alt}(X)$ of even permutations is a subgroup of $S(X)$, called the *alternating group* on X. When $X = \{1, \ldots, n\}$, we write A_n for $\mathrm{Alt}(X)$.

Proposition 9.7. *A non-empty subset A of $S(X)$ is a subgroup iff, for any α and $\beta \in A$, $\alpha\beta^{-1} \in A$.*

Proof. That any subgroup has this property follows from (2) and (3) in Definition 9.4. Suppose, conversely, that A is a non-empty subset of $S(X)$ with the given property. Take some $\alpha \in A$; then $\mathrm{id} = \alpha\alpha^{-1} \in A$. Hence, for $\alpha \in A$, $\alpha^{-1} = \mathrm{id}\,\alpha^{-1} \in A$. Finally, for α and $\beta \in A$, $\alpha\beta = \alpha(\beta^{-1})^{-1} \in A$. □

Theorem 9.8. *Let A be a subgroup of S_n. Then the order of A divides $n!$.*

Proof. We define a relation $\sim$ on $S(X)$ by $\pi \sim \sigma$ if $\pi\sigma^{-1} \in A$, and show that this is an equivalence relation. For $\pi \in S(X)$, $\pi \sim \pi$ since $\pi\pi^{-1} = \mathrm{id} \in A$. If $\pi \sim \sigma$ then $\pi\sigma^{-1} \in A$, so $\sigma\pi^{-1} = (\pi\sigma^{-1})^{-1} \in A$, and $\sigma \sim \pi$. Finally, if $\pi \sim \sigma$ and $\sigma \sim \tau$ then $\pi\sigma^{-1} \in A$ and $\sigma\tau^{-1} \in A$, so $\sigma\tau^{-1} = (\pi\sigma^{-1})(\sigma\tau^{-1}) \in A$, and $\pi \sim \tau$.

Next we show that σ is in the equivalence class of π iff $\sigma = \alpha\pi$ for some $\alpha \in A$. If $\sigma = \alpha\pi$ then $\sigma\pi^{-1} = \alpha \in A$, so $\sigma \sim \pi$, and conversely if $\sigma \sim \pi$ then defining $\alpha = \sigma\pi^{-1}$ we have $\alpha \in A$ and $\sigma = \alpha\pi$.

Since $\alpha\pi = \beta\pi$ implies $\alpha = \beta$, it follows that every equivalence class has size $|A|$, and hence that the size $n!$ of S_n is the product of $|A|$ and the number of equivalence classes. □

Example 9.9. Consider the subgroup A_n of S_n, and suppose $n > 1$ so that there exist odd permutations and $A_n \neq S_n$. Clearly $\pi\sigma^{-1}$ is even if and only if π and σ have the same parity, so there are just two equivalence classes under the relation in the above proof, namely A_n and the set of odd permutations. Hence A_n has order $n!/2$.

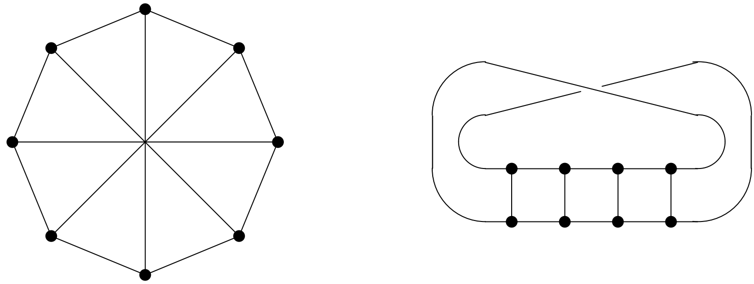


Figure 8: Two pictures of M_4

Definition 9.10. Let A and B be permutation groups (possibly on sets of different sizes). An *isomorphism* from A to B is a bijection $f\colon A \to B$ such that $f(\alpha\beta) = f(\alpha)f(\beta)$ for all α, $\beta \in A$. (It follows that $f(\mathrm{id}) = \mathrm{id}$ and $f(\alpha^{-1}) = f(\alpha)^{-1}$.) If there exists an isomorphism from A to B, A and B are *isomorphic*, written $A \cong B$. It is easy to see that isomorphism is an equivalence relation on any set of permutation groups.

Example 9.11. If $|X| = n$ then $S(X) \cong S_n$.

10 Automorphism groups of graphs

Example 10.1. For any graph G, $\mathrm{Aut}(\bar{G}) = \mathrm{Aut}(G)$.

Example 10.2. $\mathrm{Aut}\, K_n = \mathrm{Aut}\, \bar{K}_n \cong S_n$, of order $n!$. For any other graph G of order n, $|\mathrm{Aut}\, G|$ is less than $n!$, and divides $n!$ by Theorem 9.8.

Example 10.3. $\mathrm{Aut}\, P_n$ has order 2 for any $n > 1$.

Example 10.4. $\mathrm{Aut}\, C_n$ has order $2n$ for any $n \geq 3$. If C_n is drawn as a regular n-gon, the automorphisms are n rotations (counting the identity as a rotation), and n reflections. $\mathrm{Aut}\, C_n$ is known as the *dihedral group of order* $2n$.

Example 10.5. The *n-rung Möbius ladder* M_n ($n \geq 2$) is the graph obtained from a cycle C_{2n} (called the rim) by adding edges (called the rungs) joining each pair of opposite vertices (i.e., vertices at distance n in C_{2n}). The case $n = 4$ is shown in Figure 8. Clearly $\mathrm{Aut}\, M_n$ contains the dihedral group $\mathrm{Aut}\, C_{2n}$ of order $4n$. For $n = 2$, $M_2 = K_4$, so $\mathrm{Aut}\, M_2 \cong S_4$, of order 24. For $n = 3$, $M_3 = K_{3,3}$, so $\mathrm{Aut}\, M_3$ has order 72. Let $n \geq 4$. We shall show that $\mathrm{Aut}\, M_n = \mathrm{Aut}\, C_{2n}$. In M_n, there is a unique cycle containing no

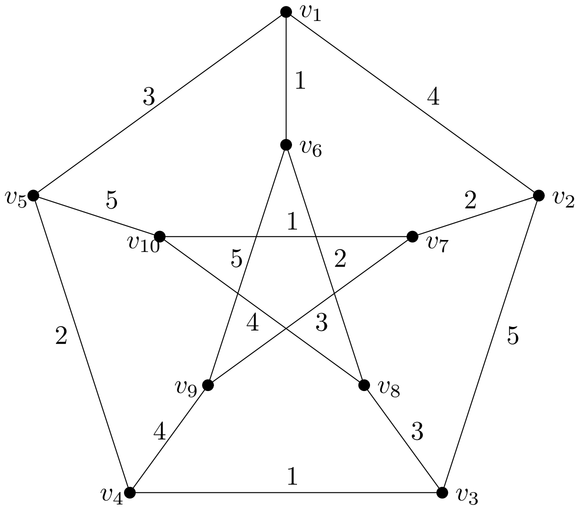


Figure 9: Labelling the Petersen graph

rungs, namely the rim, of length $2n \geq 8$, and any cycle containing a single rung has length $n+1 \geq 5$. Therefore a 4-cycle must contain at least two rungs, and hence consists of a pair of opposite edges of the rim together with the two adjacent rungs. This shows that each edge of the rim lies on just one 4-cycle, while each rung lies on two. It follows that any automorphism of M_n takes rim edges to rim edges (and rungs to rungs), and therefore lies in $\operatorname{Aut} C_{2n}$.

Theorem 10.6. *Let G be the Petersen graph. Then* $\operatorname{Aut}(G) \cong S_5$.

Proof. Label the edges of G with the numbers $1, \ldots, 5$ as in Figure 9. You can check that two distinct edges e and f get different labels iff they are adjacent, or there is a third edge adjacent to both. Hence, if $\alpha \in \operatorname{Aut}(G)$ and e and f are edges of G, then e and f have the same label iff $\alpha(e)$ and $\alpha(f)$ have the same label, so α induces an element $F(\alpha)$ of S_5 which takes i to the label of $\alpha(e)$, where e is any edge labelled i. Clearly $F(\alpha\beta) = F(\alpha)F(\beta)$. Now label each vertex v of G with the pair $\{i, j\}$ where i and j are the numbers that do *not* occur as labels of the edges at v. (For instance, v_1 gets the label $\{2, 5\}$.) You can check that each two-element subset $\{i, j\}$ of $\{1, \ldots, 5\}$ occurs exactly once, and that vertices u and v are adjacent iff their labels are disjoint (in which case the edge uv is labelled with the number not occurring in either vertex label). It follows that for each $\pi \in S_5$ there is a unique $\alpha \in \operatorname{Aut}(G)$ with $F(\alpha) = \pi$, sending a vertex labelled $\{i, j\}$ to the vertex labelled $\{\pi(i), \pi(j)\}$. $\square$

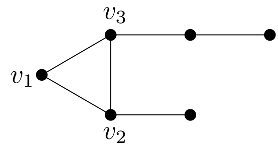


Figure 10: The case $a = 1$, $b = 2$, $c = 3$ of Theorem 10.8

Example 10.7. With the notation of Theorem 10.6,

$$\alpha = (v_1\, v_4\, v_7\, v_8)(v_2\, v_3)(v_5\, v_9\, v_{10}\, v_6)$$

is an automorphism of the Petersen graph with $F(\alpha) = (1\,2\,3\,4) \in S_5$.

Vertices u and v of a graph G are *similar* if there is an automorphism α of G with $\alpha(u) = v$. This is an equivalence relation on $V(G)$, and the equivalence classes are the *orbits* of G. A graph with a single orbit is *vertex transitive*. A vertex transitive graph is regular, but not conversely (Exercise 10.2). At the other extreme, a graph is *asymmetric* if its only automorphism is the identity; equivalently, its orbits are all singletons.

Theorem 10.8. *There is an asymmetric graph of order $n > 1$ iff $n \geq 6$.*

Proof. Let a, b and c be distinct positive integers. Take the disjoint union H of P_a, P_b and P_c, and let v_1, v_2 and v_3 be endpoints of the three components. Form G by adding edges v_1v_2, v_1v_3 and v_2v_3 (see Figure 10). Any automorphism α of G must take the unique cycle induced by $\{v_1, v_2, v_3\}$ to itself, and so induce an automorphism of H. Since the components of H have different orders, each must be taken to itself, and it follows that $\alpha = \mathrm{id}$. Thus G is an asymmetric graph of order $a + b + c$, which can be any integer $n \geq 6$.

It remains to show that there is no asymmetric graph of order n for $2 \leq n \leq 5$. A reader who has done Exercise 1.1 can simply examine all the graphs of these orders. To avoid examining all of them, we can note the following. Since $\mathrm{Aut}(G) = \mathrm{Aut}(\bar{G})$ it is enough to consider graphs of size $m \leq \frac{1}{2}\binom{n}{2}$. Any disconnected graph is either empty or has a non-trivial component of smaller order, so if the result has already been verified for smaller values of n it is enough to consider connected graphs. There is no graph satisfying these conditions of order 2 or 3, and for order 4 such a graph must be a tree, and therefore either P_4 or $K_{1,3}$, neither of which is asymmetric. For order 5, such a graph is either a tree or has size 5. The three trees of order 5 are shown in Figure 7, and are not asymmetric. A connected graph of order 5 and size 5 contains a unique cycle, of length 3, 4 or 5, so there are five such graphs; it is left to the reader to verify that they are not asymmetric. □

A reader who has done Exercise 8.1 can verify that there are no asymmetric trees of order 6.

Exercises for §10

10.1. Find an automorphism α of the Petersen graph such that, in the notation of Theorem 10.6, $F(\alpha) = (1\,2\,3)(4\,5) \in S_5$.

10.2. Find a connected regular graph that is not vertex transitive.

10.3. Show that there is an asymmetric tree of any order $n \geq 7$.

10.4. Show that for $n \geq 2$ there is a graph of order n with vertices of $n-1$ different degrees.

10.5. Show that for integers n and k with $1 \leq k \leq n-1$ there is a graph of order n with k orbits.

11 Eulerian graphs

The first paper in graph theory was Euler's 1736 work on the problem of the Königsberg bridges, in which he studied what are now known as Eulerian circuits and trails. An English translation is in Biggs, Lloyd and Wilson [4], which contains much historical detail, as well as all or part of several important papers, and references to others.

Definition 11.1. A *Eulerian circuit* in a graph G is a circuit containing every edge and vertex of G, and G is *Eulerian* if it contains a Eulerian circuit. An *Eulerian trail* is an open trail containing every edge and vertex of G.

The essential part of the definition of an Eulerian circuit or trail is that it contains every edge; that it contains every vertex is automatic unless G has isolated vertices. Recall that a trail has no repeated edges, so an Eulerian trail or circuit contains every edge exactly once. A graph with an Eulerian trail or circuit is connected and non-trivial.

Remark 11.2. We make repeated use of the following obvious facts. If C is a circuit in a graph G, every vertex of G is incident with an even number of edges of C, while if T is a u-v trail with $u \neq v$ then u and v are incident with an odd number of edges of T, and every other vertex is incident with an even number.

Theorem 11.3 (Euler)**.** *A non-trivial connected graph G is Eulerian iff every vertex of G is even.*

Proof. The "only if" part follows from the preceding remark. Suppose that every vertex of G is even, and let T be a trail of maximum length. Suppose T is a u-v trail. If $u \neq v$, v is incident with an odd number of edges of T, and since $\deg v$ is even there is an edge vw not on T. But then $T \cdot (v, w)$ is a longer trail, which is impossible. Thus T is a circuit. If T is not an Eulerian circuit, there is an edge xy that is not on T but has at least the vertex x on T. There is an x-x trail T' with the same edges as T, and now $T' \cdot (x, y)$ is a longer trail than T. Thus T is indeed an Eulerian circuit. □

Theorem 11.4 (Euler)**.** *A connected graph G has an Eulerian trail iff there are exactly two odd vertices in G.*

Proof. The "only if" part again follows from Remark 11.2, so suppose G has just two odd vertices u and v. Form a graph G' by adding to G a vertex w and edges uw and vw. By the previous theorem, G' is Eulerian. We may take an Eulerian circuit in G' of the form $T \cdot (v, w, u)$, and then T is an Eulerian trail in G. □

When there are more than two odd vertices, something can still be said.

Theorem 11.5. *Let G be a connected graph with $2k$ odd vertices, $k \geq 2$. There are k trails in G whose edge-sets partition $E(G)$, and of which at most one is of odd length.*

Proof. Let the odd vertices of G be $u_1, v_1, \ldots, u_k, v_k$. Form a graph G' by adding to G k new vertices $w_1, \ldots, w_k$ and edges $u_i w_i$ and $v_i w_i$ for $1 \leq i \leq k$. Then G' is Eulerian by Theorem 11.3. Pick an Eulerian circuit C of G'. There are k sub-trails of C that start at some w_i, end at some w_j, but do not otherwise pass through any w_k, and their edge-sets partition $E(G')$. Deleting their first and last edges gives k trails in G whose edge-sets partition $E(G)$.

We shall call a partition $\mathcal{P}$ of $E(G)$ into k subsets, each the edge set of a trail, a *trail partition.* For any trail partition $\mathcal{P}$ let $\nu(\mathcal{P})$ be the number of elements X of $\mathcal{P}$ with $|X|$ odd, and suppose for a contradiction that $\nu(\mathcal{P}) > 1$ for any trail partition. We associate a graph $H(\mathcal{P})$ to each trail partition. The vertices are the elements of $\mathcal{P}$, and $X \neq Y \in \mathcal{P}$ are adjacent iff they contain adjacent edges. We show that $H(\mathcal{P})$ is connected. Let X and $Y \in \mathcal{P}$, and let u and v be vertices of edges of X and Y respectively. Let $u = u_0, u_1, \ldots, u_r = v$ be a path in G. For $1 \leq i \leq r$, the edge $u_{i-1}u_i$ of G belongs to some $Z_i \in \mathcal{P}$. In the sequence $X, Z_1, \ldots, Z_r, Y$ of vertices of $H(\mathcal{P})$, any two consecutive terms are equal or adjacent, so X and Y are connected in $H(\mathcal{P})$. Hence we may define $\mu(\mathcal{P})$ to be the minimum distance in $H(\mathcal{P})$ between distinct elements X and Y of $\mathcal{P}$ with $|X|$ and $|Y|$ odd.

Let $\mathcal{P}$ be a trail partition such that for every other trail partition $\mathcal{P}'$, either $\nu(\mathcal{P}) < \nu(\mathcal{P}')$, or $\nu(\mathcal{P}) = \nu(\mathcal{P}')$ and $\mu(\mathcal{P}) \leq \mu(\mathcal{P}')$. We consider two cases, depending on whether or not $\mu(\mathcal{P}) = 1$. If $\mu(\mathcal{P}) = 1$, there are adjacent vertices X and Y of $H(\mathcal{P})$ with $|X|$ and $|Y|$ odd. These are the edge sets of trails $u_0, u_1, \ldots, u_r$ and $v_0, v_1, \ldots v_s$ in G, where r and s are odd, and $u_i = v_j$ for some i and j. By reversing one trail if necessary, we may assume that $i \not\equiv j \pmod 2$. We may form a new trail partition $\mathcal{P}'$ by replacing X and Y by the edge sets X' and Y' of the trails $u_0, u_1, \ldots, u_i, v_{j+1}, \ldots, v_s$ and $v_0, v_1, \ldots, v_j, u_{i+1}, \ldots, u_r$. Since $|X'|$ and $|Y'|$ are even, $\nu(\mathcal{P}') = \nu(\mathcal{P}) - 2$, a contradiction.

Now suppose that $\mu(\mathcal{P}) > 1$. Then there are vertices X, Y, Z and W of $H(\mathcal{P})$ with $|X|$ and $|W|$ odd, $|Y|$ even, Y adjacent to both X and Z, and $d_{H(\mathcal{P})}(Z, W) = \mu(\mathcal{P}) - 2$. Then X and Y are the edge sets of trails $u_0, u_1, \ldots, u_r$ and $v_0, v_1, \ldots v_s$ in G, where r is odd, s is even, $u_i = v_j$ for some i and j, and some v_k is a vertex of an edge of Z. Note that $j \neq k$, since otherwise X and Z would be equal or adjacent. By reversing the trails if necessary, we may assume that $j < k$, and then that $i \not\equiv j \pmod 2$. We may form a new trail partition $\mathcal{P}'$ by replacing X and Y by the edge sets X' and Y' of the trails $u_0, u_1, \ldots, u_i, v_{j+1}, \ldots, v_s$ and $v_0, v_1, \ldots, v_j, u_{i+1}, \ldots, u_r$. Since $|X'|$ is odd and $|Y'|$ is even, $\nu(\mathcal{P}') = \nu(\mathcal{P})$. Since X' and Z have adjacent edges, $\mu(\mathcal{P}') \leq d_{H(\mathcal{P}')}(X', W) \leq \mu(\mathcal{P}) - 1$. This contradiction completes the proof. □

Theorem 11.6. *A non-trivial connected graph G is Eulerian iff every edge of G is on an odd number of cycles.*

In the statement of this theorem, "cycle" means a subgraph isomorphic to some C_k rather than a walk inducing such a subgraph.

Proof. Suppose first that G is Eulerian, so that every vertex is even, and let $e = uv$ be any edge. Let $\mathcal{T}$ be the set of all trails in $G - e$ that start at u

and either do not contain v, or contain it only as the terminal vertex. For $k \geq 0$, let $\mathcal{T}_k$ be the set of trails in $\mathcal{T}$ of length at most k that either have v as terminal vertex or are of length exactly k. For $k = 0$, $\mathcal{T}_0$ contains only the trivial u-u path, while for sufficiently large k, $\mathcal{T}_k$ is the set $\mathcal{T}'$ of all trails in $\mathcal{T}$ that end at v. Suppose that $T \in \mathcal{T}_k$ is a u-x trail where $x \neq v$. If $x \neq u$ then x has even degree in $G - e$ and is incident with an odd number of edges of T, while if $x = u$ then x has odd degree in $G - e$ and is incident with an even number of edges of T. In either case there are an odd number of edges xy of $G - e$ that are not on T, and for each one $T \cdot (x, y)$ is in $\mathcal{T}_{k+1}$. Every element of $\mathcal{T}_{k+1}$ is either of this form, or is an element of $\mathcal{T}_k$ ending at v. It follows by induction that $|\mathcal{T}_k|$ is odd for all k, so $|\mathcal{T}'|$ is odd. Now the set $\mathcal{P}$ of all u-v paths in $G - e$ is a subset of $\mathcal{T}'$, and is in one-to-one correspondence with the set of all cycles in G containing e. Suppose $T = (u = u_0, u_1, \ldots, u_k = v)$ is an element of $\mathcal{T}' - \mathcal{P}$. Let j be the smallest integer for which there exists $i < j$ with $u_i = u_j$. Then

$$\alpha(T) = (u_0, u_1, \ldots, u_i, u_{j-1}, \ldots, u_{i+1}, u_j, u_{j+1}, \ldots, u_k)$$

is a different element of $\mathcal{T}' - \mathcal{P}$, and $\alpha^2(T) = T$, so $|\mathcal{T}' - \mathcal{P}|$ is even and $|\mathcal{P}|$ is odd, as required.

Now suppose that every edge of G lies on an odd number of cycles. Let u be a vertex of G, and consider the set P of all pairs (e, C), where e is an edge incident with u and C is a cycle containing e. Because there are an odd number of such pairs for each edge incident with u, $|P| \equiv \deg u \pmod 2$. Because there are two such pairs for each cycle containing u, $|P|$ is even. Hence $\deg u$ is even, and G is Eulerian by Theorem 11.3. □

12 Hamiltonian graphs

Definition 12.1. A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G, and G is *Hamiltonian* if it contains a Hamiltonian cycle.

A Hamiltonian graph has order at least 3 and is connected, and indeed, by Theorem 5.9, 2-connected. Not every 2-connected graph is Hamiltonian, however; the Petersen graph, for instance, is not Hamiltonian. The graph of the dodecahedron, shown in Figure 11, is Hamiltonian; the association of Hamilton's name with this idea comes from his invention of a game based on this fact; see [4].

Theorem 12.2. *Let G be a graph of order n and u and v independent vertices of G with $\deg u + \deg v \geq n$. Then G is Hamiltonian iff $G + uv$ is Hamiltonian.*

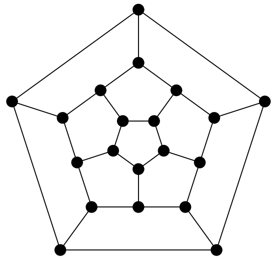


Figure 11: The graph of the dodecahedron

Proof. The "only if" part is trivial. Suppose that $G' = G + uv$ is Hamiltonian. A Hamiltonian cycle in G' not containing the edge uv is a Hamiltonian cycle in G, so suppose that G' has a Hamiltonian cycle

$$(u = u_1, u_2, \dots, u_n = v, u).$$

The set of integers i with $2 \leq i \leq n$ such that u_i is adjacent to u has cardinality $\deg u$, and the set of those such that u_{i-1} is adjacent to v has cardinality $\deg v$, so there is some such i such that u_i is adjacent to u and u_{i-1} is adjacent to v. Now

$$(u_1, u_i, u_{i+1} \dots, u_n, u_{i-1}, u_{i-2}, \dots, u_1)$$

is a Hamiltonian cycle in G. □

Given a graph G of order n, we may form a sequence $G = G_0, G_1, \dots, G_r$ of graphs where, for $0 < i \leq r$, $G_i = G_{i-1} + u_i v_i$ for independent vertices u_i and v_i of G_{i-1} with $\deg_{G_{i-1}} u_i + \deg_{G_{i-1}} v_i \geq n$, such that there are no independent vertices u and v of G_r with $\deg_{G_r} u + \deg_{G_r} v \geq n$. We call G_r a *closure* of G.

Lemma 12.3. *Any graph has a unique closure.*

Proof. Let G be a graph of order n, and let $G = G_0, G_1, \dots, G_r$, $u_1, \dots u_r$ and $v_1, \dots, v_r$ be as above. Let G' be another closure of G. We show, by induction on i, that each edge $u_i v_i$ added to G in forming G_r is an edge of G'. Suppose then that, for some i, $1 \leq i \leq r$, $u_j v_j \in E(G')$ for $1 \leq j < i$. Then

$$\deg_{G'} u_i + \deg_{G'} v_i \geq \deg_{G_{i-1}} u_i + \deg_{G_{i-1}} v_i \geq n,$$

so u_i and v_i are not independent in G', and $u_i v_i \in E(G')$, proving the inductive step. By symmetry, every edge added to G in forming G' is an edge of G_r, so $G_r = G'$. □

We may therefore speak of *the* closure of G, denoted $C(G)$. The next three results are immediate consequences of Theorem 12.2.

Corollary 12.4. *Let G be a graph of order $n \geq 3$. If $C(G) \cong K_n$ then G is Hamiltonian.* □

Corollary 12.5. *Let G be a graph of order $n \geq 3$. If $\deg u + \deg v \geq n$ for any independent vertices u and v of G then G is Hamiltonian.* □

Corollary 12.6. *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$ then G is Hamiltonian.* □

The following consequence of Theorem 12.2 takes a bit more work.

Theorem 12.7. *Let G be a graph of order $n \geq 3$ and $(d_1, \ldots, d_n)$ the degree sequence of G with $d_1 \leq \cdots \leq d_n$. If there is no k with $1 \leq k < n/2$ such that $d_k \leq k$ and $d_{n-k} \leq n-k-1$ then G is Hamiltonian.*

Proof. Let $H = C(G)$. Note that G, and hence H, has no isolated vertices. We claim that $H \cong K_n$. Suppose not, and let u and v be independent vertices of H such that $\deg_H u + \deg_H v$ is a maximum. By the definition of closure, $\deg_H u + \deg_H v \leq n-1$. Suppose that $\deg_H u \geq \deg_H v$ and set $k = \deg_H v$. Then $1 \leq k \leq (n-1)/2 < n/2$. There are $n-1-\deg_H u \geq k$ vertices of H independent of u, and for each such vertex w, $\deg_G w \leq \deg_H w \leq \deg_H v = k$. That is, $d_k \leq k$. There are $n-1-\deg_H v = n-1-k$ vertices independent of v in H, and for each such vertex w, $\deg_G w \leq \deg_H w \leq \deg_H u \leq n-1-k$. Since also $\deg_G v \leq k \leq n-1-k$, there are at least $n-k$ vertices of degree at most $n-k-1$ in G, and $d_{n-k} \leq n-k-1$. □

Definition 12.8. A set U of vertices of a graph G is *independent* if any two distinct elements of U are independent. The *independence number* of G, $\beta(G)$, is the maximum number of vertices in an independent set.

For example, $\beta(G) = 1$ iff G is complete.

Theorem 12.9. *Let G be a graph of order $n \geq 3$. If $\kappa(G) \geq \beta(G)$ then G is Hamiltonian.*

Proof. Let $k = \kappa(G)$. Then $k \geq 2$, since otherwise $\beta(G) = 1$ and $G \cong K_n$, contradicting $k = 1$. Hence G contains a cycle. Let C be a cycle of maximum length l. By Theorem 6.11, $l \geq k$. Suppose, for a contradiction, that there is a vertex u not on C. By Theorem 6.10, we may find k internally disjoint paths $Q_1, \ldots, Q_k$ from u to distinct vertices of C. Let v_i be the first vertex of Q_i that is on C and P_i the u-v_i subpath of Q_i. If C contains an edge v_iv_j, we may replace it by $P_i^r \cdot P_j$ to get a longer cycle, which is impossible. Let w_i be the vertex following v_i in one cyclic ordering of C. If some w_i is adjacent to u, we may replace the edge v_iw_i by $P_i^r \cdot (u, w_i)$ to get a longer cycle, which is again impossible. Now $|\{u, w_1, \ldots, w_k\}| = k + 1 > \beta(G)$, so this set is not independent and G contains an edge w_iw_j (which is of course not in C). Replacing the edges v_iw_i and v_jw_j by $P_i^r \cdot P_j$ and (w_i, w_j) gives a longer cycle, and this contradiction completes the proof. □

Exercises for §12

12.1. Prove that if G is Hamiltonian of order n then $\beta(G) \leq n/2$.

12.2. Let G be a graph of order $n \geq 3$ such that $\deg u + \deg v \geq n$ for any independent vertices u and v of G. Let $(d_1, \ldots, d_n)$ be the degree sequence of G with $d_1 \leq \cdots \leq d_n$. Prove that there is no k with $1 \leq k < n/2$ such that $d_k \leq k$ and $d_{n-k} \leq n - k - 1$.

12.3. Show that the complete k-partite graph $K_{r_1,\ldots,r_k}$, where $r_1 \leq \cdots \leq r_k$, is Hamiltonian iff $r_k \leq \sum_{i=1}^{k-1} r_i$.

13 Elements of plane topology

We review briefly some definitions from MATH 2057 and a few related ideas. The *length* or *norm* of $x = (x_1, x_2) \in \mathbb{R}^2$ is $\|x\| = \sqrt{x_1^2 + x_2^2}$, the *Euclidean distance* between x and $y \in \mathbb{R}^2$ is $\|x - y\|$, and the *(open) ball* of radius $r > 0$ centered at $x \in \mathbb{R}^2$ is $B(x, r) = \{y \in \mathbb{R}^2 \mid \|x - y\| < r\}$. A subset A of $\mathbb{R}^2$ is *open* if, for every $x \in A$, there is some $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$, and A is *closed* if $\mathbb{R}^2 - A$ is open. The *closure* of A is the set $\bar{A}$ of all $x \in \mathbb{R}^2$ such that $B(x, \epsilon)$ meets A for all $\epsilon > 0$; it is the smallest closed set containing A. The *interior* of A is the set A° of all $x \in \mathbb{R}^2$ such that there is some $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$; it is the largest open set contained in A. The *boundary* of A is $\bar{A} - A^\circ$; alternatively, it is the set of all $x \in \mathbb{R}^2$ such that $B(x, \epsilon)$ meets both A and $\mathbb{R}^2 - A$ for all $\epsilon > 0$.

A *curve* in $\mathbb{R}^2$ is a continuous function $f\colon [0, 1] \to \mathbb{R}^2$. An *arc* in $\mathbb{R}^2$ is the image of an injective curve, and its *endpoints* are the images of 0 and 1. It can be shown that this is well-defined; that is, if f and g are injective curves with the same image then $f(\{0, 1\}) = g(\{0, 1\})$. (This can be proved using two theorems usually stated but not proved in a first calculus course, the Intermediate and Extreme Value Theorems.) A subset A of $\mathbb{R}^2$ is *path-connected* if, for any x, $y \in A$, there is a curve f in A with $f(0) = x$

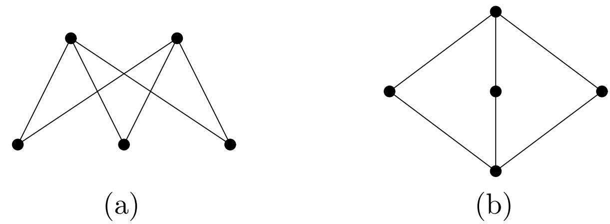


Figure 12: The planar graph $K_{2,3}$

and $f(1) = y$. The *path-components* of A are the maximal path-connected subsets of A; they form a partition of A and the path-components of an open set are open. A *simple closed curve* is the image of a curve f such that, for $0 \le s < t \le 1$, $f(s) = f(t)$ iff $s = 0$ and $t = 1$. If C is a simple closed curve in $\mathbb{R}^2$, $\mathbb{R}^2 - C$ has exactly two path-components, one bounded and one unbounded, each with boundary C. This is the celebrated Jordan Curve Theorem, and will be taken on trust.

Suppose that A, B and C are arcs in $\mathbb{R}^2$ such that any two meet only in their endpoints x and y. If X is one of A, B or C, then the union of the other two is a simple closed curve, whose complement has path-components U_X disjoint from X, and V_X containing $X - \{x, y\}$. The path-components of $\mathbb{R}^2 - (A \cup B \cup C)$ are U_A, U_B and U_C.

14 Planar graphs

Definition 14.1. A *plane embedding* of a graph G consists of the following. First, a bijection α from $V(G)$ to a subset of $\mathbb{R}^2$. Second, a bijection β from $E(G)$ to a set of arcs in $\mathbb{R}^2$. These are required to satisfy the following conditions. First, if e is the edge uv, then the endpoints of $\beta(e)$ are $\alpha(u)$ and $\alpha(v)$. Second, if e and f are distinct edges, then $\beta(e)$ and $\beta(f)$ intersect only if e and f are adjacent, and then only in $\alpha(u)$ where u is the common vertex of e and f.

Less formally, a plane embedding of G is a picture of G in which no edge crosses itself or any other edge. A graph is *planar* if it has a plane embedding. A *plane graph* is a graph G together with a plane embedding of G. Given a plane graph we usually identify each vertex v with the point $\alpha(v)$ in $\mathbb{R}^2$, each edge e with the arc $\beta(e)$, and the graph G with the union of these subsets of $\mathbb{R}^2$. Figure 12 shows two pictures of $K_{2,3}$. That in (a) is not a plane embedding, while the one in (b) is, and shows that $K_{2,3}$ is planar.

If G is a plane graph, the path-components of the open set $\mathbb{R}^2 - G$ are called the *regions* of G. There is one unbounded region, the *exterior* region. The boundary of a region R is a subgraph of G denoted by ∂R; we say that a region is *adjacent* to the vertices and edges in its boundary. A plane graph G may be re-embedded so that any given vertex or edge is adjacent to the exterior region as follows. Regard $\mathbb{R}^2$ as the plane of the first two coordinates in $\mathbb{R}^3$, let S^2 be the unit sphere centered at the origin in $\mathbb{R}^3$, and let $N = (0, 0, 1)$, the "north pole" of S^2. There is a bijection $\phi\colon S^2 - \{N\} \to \mathbb{R}^2$, called stereographic projection. For $x \in S^2 - \{N\}$, $\phi(x)$ is the point in which the line through N and x meets $\mathbb{R}^2$. Let x be a point of some region adjacent to the given vertex or edge, and take the image of G under the composite of ϕ^{-1}, followed by a rotation of S^2 taking $\phi^{-1}(x)$ to N, followed by ϕ.

Clearly a graph is planar iff each component is planar. Let G be a non-trivial, connected plane graph. A *arrow* of a graph is an ordered pair (u, v) of adjacent vertices (i.e., it is a path of length 1). For each arrow (u, v) of G, there is a region R_+ to its left and a region R_- to its right. If uv is a bridge then $R_+ = R_-$; otherwise $R_+ \neq R_-$. A *side* of a region R is an arrow having R to its left. The number of sides of R is thus twice the number of bridges plus the number of other edges of ∂R. We may assume that at each vertex v there is a circle centered at v meeting each edge incident to v just once. Traversing this circle either clockwise or counterclockwise gives a cyclic ordering of the vertices adjacent to v. For any region R of G we construct a closed walk in its boundary as follows. Start with a side (u_0, u_1) of R. Suppose we have constructed $(u_0, u_1, \ldots, u_k)$ so that (u_i, u_{i+1}) is a side of R for $0 \leq i < k$. Let v be the vertex adjacent to u_k that follows u_{k-1} in the clockwise ordering, or u_{k-1} if u_k has degree 1. Then (u_k, v) is a side of R. If $u_k = u_0$ and $v = u_1$, stop; otherwise extend the walk by taking $u_{k+1} = v$. The result is a walk containing each side of R just once. If there are no bridges in ∂R the walk is a circuit, and if there are also no cut-vertices it is a cycle. In particular, if G is 2-connected, the boundary of any region is a cycle.

Theorem 14.2 (Euler's formula)**.** *Let G be a connected plane graph of order n and size m having r regions. Then $n - m + r = 2$.*

Proof. The proof is by induction on m. If G is a tree then $m = n - 1$, and since every edge is a bridge, $r = 1$, so the result holds. Otherwise, let e be an edge that lies on a cycle. Then $G - e$ is a connected plane graph of order n and size $m - 1$ having $r - 1$ regions (the two regions of G adjacent to e lying in a single region of $G - e$). The result follows. ☐

Corollary 14.3. *Let G be a plane graph of order n and size m having r regions and k components. Then $n - m + r = k + 1$.* □

Theorem 14.4. *If G is a planar graph of order $n \geq 3$ and size m then $m \leq 3n - 6$.*

Proof. We may assume that G is connected, since otherwise we may add edges to get a connected planar graph of order n and size greater than m. Embed G in the plane and let the number of regions be r. The number of arrows in G is $2m$. Now each arrow is a side of a unique region, and each region has at least three sides, so $2m \geq 3r$, and by Euler's formula $6 = 3n - 3m + 3r \leq 3n - m$. □

Example 14.5. For K_5 we have $m = 10$ and $3n - 6 = 9$, so K_5 is not planar.

Corollary 14.6. *Every planar graph G has $\delta(G) \leq 5$.*

Proof. Let G have order n and size m. If $n < 3$ the result is trivial, and otherwise $m \leq 3n - 6$. Now $2m = \sum_{v \in V(G)} \deg v \geq \delta(G)n$, so $\delta(G)n \leq 6n - 12$. □

A *maximal planar graph* is a planar graph G of order $n \geq 3$ such that, for any independent vertices u and v, $G + uv$ is non-planar.

Theorem 14.7. *Let G be a maximal planar graph of order n and size m. Then $m = 3n - 6$, and in any plane embedding of G, the boundary of any region is a triangle.*

Proof. The second statement implies the first, because it gives $2m = 3r$, where r is the number of regions, and together with Euler's formula this gives $m = 3n - 6$. Clearly G is connected; in fact, it is 2-connected, because if u were a cut vertex, there would be vertices v and w adjacent to u, with w following v in the counterclockwise order of the neighbors of u, and in different components of $G - u$. But then we could embed $G + vw$ by putting the edge vw in the region to the left of (u, v). Thus the boundary of every region is a cycle. Suppose that the boundary of some region R has length greater than 3, and let u_1, u_2, u_3 and u_4 be consecutive vertices around ∂R. Not both of u_1u_3 and u_2u_4 can be edges of G, and for one that is not we can embed a corresponding arc in R to contradict maximal planarity of G. □

Theorem 14.8. *A graph is planar iff each of its blocks is planar.*

Proof. One direction is trivial: every subgraph of a planar graph is planar. Suppose that G is a graph all of whose blocks are planar. We may assume that G is connected. The proof is by induction on the number k of blocks, the case $k = 1$ being obvious. Suppose that $k > 1$. By Theorem 5.18, G has an end-block B. Let v be the cut-vertex in B, and H the union of all other blocks of G, so that $G = B \cup H$ and B and H have only v in common. By induction, H is planar. We may embed both B and H in $\mathbb{R}^2$ so that v is adjacent to the exterior region. Further, we may choose these embeddings so that $v = (0, 0)$ in both cases, every other point (x_1, x_2) of B has $x_1 < 0$, and every other point (x_1, x_2) of H has $x_1 > 0$. This gives a plane embedding of G. □

Definition 14.9. The *girth* of a graph that contains a cycle is the minimum length of a cycle.

Theorem 14.10. *Let G be a 2-connected planar graph of order n, size m and girth g. Then $m \leq \frac{g}{g-2}(n-2)$.*

Proof. Embed G in the plane and let the number of regions be r. Since G is 2-connected, the boundary of any region is a cycle, so it has at least g sides. Thus $2m \geq gr$, so $2g = gn - gm + gr \leq gn - (g-2)m$. □

Example 14.11. For $K_{3,3}$, $m = 9$, $n = 6$, $g = 4$ and $\frac{g}{g-2}(n-2) = 8$, so $K_{3,3}$ is non-planar.

Example 14.12. For the Petersen graph, $m = 15$, $n = 10$, $g = 5$ and $\frac{g}{g-2}(n-2) = \frac{40}{3}$, so it is non-planar.

Theorem 14.13. *Let G be a connected plane graph with $\delta(G) \geq 3$. Then there is a region with at most 5 sides.*

Proof. Let the order, size and number of regions be n, m and r. We have $2m \geq 3n$ and if every region has at least 6 sides $2m \geq 6r$, or $m \geq 3r$. Hence $6 = 3n - 3m + 3r \leq 0$, a contradiction. □

Theorem 14.14. *Let G be a connected plane graph with $\delta(G) \geq 3$. Let n_k be the number of vertices of degree k and r_k the number of regions with k sides. Then $n_3 + r_3 \geq 8$.*

Proof. Let the order, size and number of regions be n, m and r. Note that $n_k = r_k = 0$ for $k < 3$. We have $n = \sum_{k\geq 3} n_k$, $r = \sum_{k\geq 3} r_k$, $2m = \sum_{k\geq 3} kn_k$, and $2m = \sum_{k\geq 3} kr_k$. Hence

$$8 = (4n - 2m) + (4r - 2m) = \sum_{k\geq 3}(4-k)(n_k + r_k) \leq n_3 + r_3. \qquad \square$$

The following theorem is due independently to Fáry [13] and Wagner [30]. The proof below is based on Wood [32].

Theorem 14.15. *Any planar graph has a plane embedding in which each edge is a straight line segment.*

Proof. It is enough to prove this for a maximal planar graph G, which we do by induction on the order n. In fact we prove that for any embedding of G there is a line-segment embedding in which the counterclockwise order on the neighbors of any vertex is unchanged. When $n = 3$, G is isomorphic to K_3 and the result is clear, so suppose $n \geq 4$. Recall that all regions of the embedding are bounded by triangles. If u and v are adjacent vertices, the edge uv is in the boundary of two regions, whose other vertices are common neighbors of u and v. We show that u and v may be chosen so that these are their only common neighbors. If there are three common neighbors, uv is in three triangles of G, at least one of which has a vertex in its interior and one in its exterior. Amongst all triangles of G which have a vertex in both their interior and their exterior, choose an innermost one, choose a vertex u in its interior, and let v be any neighbor of u. Then indeed u and v have just two common neighbors p and q.

With a suitable choice of notation, the neighbors of u in counterclockwise order are $v, p, x_1, \ldots, x_r, q$, and those of v are $u, q, y_1, \ldots, y_s, p$. Form a graph G' by deleting the vertex u and adding edges vx_i for $1 \leq i \leq r$. (This operation is called *contracting the edge uv*, and will appear in the next section.) We may extend the embedding of $G - u$ to an embedding of G', in which the neighbbors of v in counterclockwise order are $p, x_1, \ldots, x_r, q, y_1, \ldots, y_s$. By inductive hypothesis we may re-embed G' so the edges are line segments, preserving orderings of neighbors. Suppose the angle θ between vp and vq, measured counterclockwise, is at most π. Consider the cycle $C = (v, p, x_1, \ldots, x_r, q, v)$. Its interior contains the edges vx_i, except for their endpoints. Delete these edges, take u to be in the interior of C, and add the line segments from u to v, p, q and each x_i. If u is taken sufficiently close to v, these segments are in the interior of C, except for their endpoints on C, and we have the desired embedding of G.

If $\theta > \pi$, rename v as u, and similarly situate v in the interior of $(u, q, y_1, \ldots, y_s, p, u)$. □

15 Kuratowski's Theorem

Definition 15.1. An *elementary subdivision* of a graph G is the operation of deleting an edge uv and adding a new vertex w and edges uw and vw.

A *subdivision* of G is a graph isomorphic to one obtained by zero or more elementary subdivisions.

Example 15.2. The Petersen graph has a subgraph that is a subdivision of $K_{3,3}$, obtained by deleting two edges with the same label in Figure 9.

Clearly a sudivision of G is planar iff G is planar, and as already remarked, a subgraph of a planar graph is planar. Together with Examples 14.5 and 14.11, these observations prove the "only if" part of the following theorem.

Theorem 15.3 (Kuratowski [23]). *A graph G is planar iff no subgraph of G is a subdivision of K_5 or $K_{3,3}$.*

Lemma 15.4. *Let G be a 2-connected graph of order $n \geq 4$. Then G has either a vertex v such that $G - v$ is 2-connected, or a vertex of degree 2.*

Proof. Suppose that G has no vertex v such that $G - v$ is 2-connected. Then, for evey vertex x of G there is a vertex $y \neq x$ such that $G - \{x, y\}$ is disconnected. Let u and v be such a pair of vertices for which some component of $G - \{u, v\}$, say G_1, has minimum order. Note that each of u and v is adjacent to some vertex in every component of $G - \{u, v\}$. Let G_2 be the union of the components of $G - \{u, v\}$ other than G_1, and for $i = 1$ or 2, let $H_i = \langle V(G_i) \cup \{u, v\}\rangle$. Let w_1 be a vertex of G_1. Then there is a vertex $w_2 \neq w_1$ such that $G - \{w_1, w_2\}$ is disconnected. If w_2 is in G_1, then because H_2 is connected, some component of $G - \{w_1, w_2\}$ is contained in $G_1 - \{w_1, w_2\}$, a contradiction. If $w_2 = u$, then because $\langle V(G_2) \cup \{v\}\rangle$ is connected, some component of $G - \{w_1, w_2\}$ is contained in $G_1 - w_1$, again a contradiction. A similar contradiction arises if $w_2 = v$, so w_2 is in G_2.

For $i = 1$ or 2, every vertex of $H_i - w_i$ is connected to either u or v in $H_i - w_i$, because it is connected to, say, u in $G - w_i$ and a path can only leave H_i at u or v. It follows that $H_i - w_i$ has exactly two components, one containing u and the other containing v. Let the components of $H_1 - w_1$ be K_u and K_v. If K_u is non-trivial, then $K_u - u$ is a union of components of $G - \{w_1, u\}$, contradicting the minimality of the order of G_1. Thus K_u is trivial, and similarly K_v is trivial. This means that G_1 has only the single vertex w_1, which is therefore adjacent to only u and v, and of degree 2. □

Lemma 15.5. *Let G be a 2-connected graph of order $n \geq 4$. Then G has either an edge e such that $G - e$ is 2-connected, or a vertex of degree 2.*

Proof. By the previous lemma, we may assume that G has a vertex v such that $G - v$ is 2-connected. Let e be an edge incident with v. If $G - e$ is

2-connected, we are done. Otherwise there is a vertex u such that $G-e-u$ is disconnected; u is not a vertex of e since otherwise $G-e-u=G-u$. Now e is a bridge of $G-u$ but its vertex v is not a cut-vertex of $G-u$. Hence v has degree 1 in $G-u$, and therefore degree 2 in G. □

Let G be a graph and Γ a cycle of G. We assume that an orientation of Γ (that is, a cyclic ordering of its vertices) has been chosen. For distinct vertices u and v of Γ, we let $[u,v]$ be the u-v path in Γ determined by the orientation. (The other one is the reverse of $[v,u]$.) We let $(u,v]$, $[u,v)$ and (u,v) be the sets obtained from the set of vertices of $[u,v]$ by removing u, v, or both. (The "open interval" (u,v) may be empty.) We also allow the degenerate closed interval $[u,u]$, the trivial u-u path. By a *chord* of Γ we mean a path C in G having only its end vertices in common with Γ. We define *pieces* of G relative to Γ of two kinds. For each component K of $G-V(\Gamma)$ there is a piece, which is the subgraph of G induced by all edges with at least one vertex in K. Also, the subgraph induced by an edge not on Γ, but with both vertices on Γ, is a piece. The edge sets of Γ and all pieces partition $E(G)$. If G is connected, each piece has at least one vertex on Γ, and if G is 2-connected, at least two. If u and v are distinct vertices of some piece P that lie on Γ, P contains a u-v chord of Γ. We say that two subgraphs H_1 and H_2 of G, each of which is either a chord or a piece, *cross* if there are vertices u_i and v_i of H_i on Γ ($i=1$ or 2), such that u_1, u_2, v_1 and v_2 are distinct and appear in that order around Γ.

Suppose that G has been embedded in the plane. For definiteness, we shall assume that Γ has the counterclockwise orientation. The plane is divided by Γ into two regions, the interior and the exterior. Each vertex and edge not on Γ, as well as each chord or piece, and each region of G, may be classified as interior or exterior. Note that if two subgraphs (each a chord or piece) cross, one is interior and the other exterior.

Lemma 15.6. *Let G be a 2-connected plane graph, and Γ a cycle of G. Suppose that u and v are vertices of Γ that are not adjacent to any single interior region of G. Then there is an interior chord of Γ with one endpoint in (u,v) and the other in (v,u).*

Remark. The assumption that G is 2-connected is not essential, but simplifies the proof, and is satisfied in our applications.

Proof. The sets (u,v) and (v,u) are non-empty; let $w\in(u,v)$ and $w'\in(v,u)$ be adjacent to u. We first define a w-w' walk with all edges (and hence vertices) on or interior to Γ. Set $w_0=w$. Let R_1 be the interior region of G adjacent to uw_0, and take the path going counterclockwise around the boundary of R_1 from w_0 to its other vertex w_1 adjacent to u. If uw_1 is interior, let R_2 be the other (necessarily interior) region of G adjacent to uw_1, and take the path going counterclockwise around the boundary of R_2 from w_1 to its other vertex w_2 adjacent to u. Continue in this fashion, stopping when uw_k is on Γ, and therefore $w_k = w'$. Stringing these paths together gives the desired walk W. Note that neither u nor v is on W.

Let W be $w = x_0, x_1, \ldots, x_r = w'$. There exist i and j with $0 \le i < j \le r$ such that x_i is on (u, v), x_j is on (v, u), and x_k is interior to Γ for $i < k < j$. An x_i-x_j path contained in the walk $x_i, x_{i+1}, \ldots, x_j$ is the required chord. □

If a graph G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, we shall say that G *has a forbidden subgraph.* The next lemma is the heart of the proof of Kuratowski's Theorem.

Lemma 15.7. *Suppose that G is a non-planar graph that has an edge e such that $G - e$ is 2-connected and planar. Then G has a forbidden subgraph.*

Proof. Let $e = u_0u_2$. (The reason for the peculiar indexing will emerge later.) Set $H = G - e$. Since H is 2-connected, it has a cycle Γ containing u_0 and u_2. The pieces of G relative to Γ are the pieces of H together with the chord C_0 induced by e. For any plane embedding of H, there must be an interior piece of H crossing C_0, since otherwise Lemma 15.6 would give an interior region adjacent to u_0 and u_2, and hence a plane embedding of G.

Now fix some embedding of H, and choose Γ to have the maximum number of interior regions amongst all cycles of H containing u_0 and u_2. Let P be an exterior piece of H. Since H is 2-connected, P has at least two vertices on Γ. Suppose it has two such v and w for which (v, w) contains neither u_0 nor u_2. Then the union of the path $[w, v]$ and a v-w chord in P is a cycle containing u_0 and u_2 with more interior regions than Γ, contrary to hypothesis. It follows that P has exactly two vertices on Γ, with one, v_P, on (u_0, u_2), and the other, w_P, on (u_2, u_0). Amongst the regions of the subgraph $\Gamma \cup P$ of H, there are three whose boundaries contain edges of Γ. One is the interior of Γ. Of the others, one, R_P^-, has boundary meeting Γ in $[w_P, v_P]$, and the otherR_P^+, has boundary meeting Γ in $[v_P, w_P]$. The boundary of any other region has at most the vertices v_P and w_P in common with Γ, and cannot have both. (Suppose it does. Then its boundary is the union of two v_P-w_P chords of Γ, both of which must contain exterior vertices. However, any path in $\Gamma \cup P$ from an exterior vertex of one chord to an exterior vertex of the other must pass through v_P or w_P, contradicting the fact that the

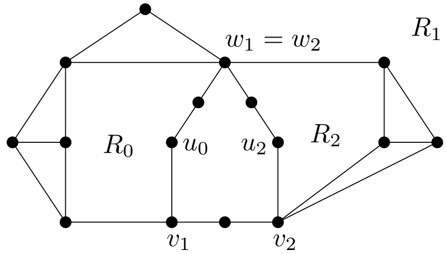


Figure 13: Exterior pieces and the regions R_i

exterior vertices of P belong to a single component of $H - V(\Gamma)$.) If P' is any other exterior piece of H, its edges and exterior vertices lie in some region of $\Gamma \cup P$, and since it has vertices in both (u_0, u_2) and (u_2, u_0), this must be R_P^- or R_P^+. In the first case we say that P follows P', and in the second that P precedes P'. Then P precedes P' iff P' follows P, and if P precedes P' and P' precedes P'' then P precedes P''. Hence we may number the exterior pieces of H as $P_1, \ldots, P_k$ so that P_i precedes P_{i+1} for $1 \leq i < k$. We set $v_i = v_{P_i}$ and $w_i = w_{P_i}$. For $1 \leq i < k$, the intersection of $R_{P_i}^+$ and $R_{P_{i+1}}^-$ is a region R_i of H whose boundary meets Γ in the paths $[v_i, v_{i+1}]$ and $[w_{i+1}, w_i]$. We also set $R_0 = R_{P_1}^-$ and $R_k = R_{P_k}^+$. (See Figure 13, in which the interior pieces of H are not shown.)

Now consider an interior piece Q of H. If Q does not cross some P_i, all its vertices on Γ lie on the boundary of some R_i. Thus we may re-embed Q in R_i. We may do this simultaneously for all such interior pieces, since they do not cross one another. In this way, we obtain a new embedding of H in which every interior piece crosses some P_i. As remarked above, in this embedding there must be some interior piece Q crossing C_0. Suppose that Q also crosses P_j. Set $u_1 = v_j$ and $u_3 = w_j$, so that u_0, u_1, u_2 and u_3 appear in that order around Γ. We shall take the subscripts on these vertices modulo 4. Choose a u_1-u_3 chord C_1 in P_j, and note that the union of Γ, C_0 and C_1 is a subdivision of K_4. The piece Q has a vertex in each of the four open intervals (u_i, u_{i+2}). There are two cases, (each with two subcases) depending on whether or not it has a vertex in some (u_i, u_{i+1}).

Suppose first that Q has a vertex x_1 in (u_i, u_{i+1}). If it also has a vertex x_2 in (u_{i+2}, u_{i+3}), the union of Γ, C_0, C_1 and an x_1-x_2 chord in Q is a subdivision of $K_{3,3}$. (See Figure 14(a).) Otherwise, Q has vertices x_2 in $(u_{i+1}, u_{i+2}]$ and x_3 in $[u_{i+3}, u_i)$. Take an x_2-x_3 chord D_1 in Q. Then D_1 has interior vertices, and each one may be joined to x_1 by a path in Q whose only vertex on Γ is x_1. Taking a shortest such path, we obtain an x_1-x_4

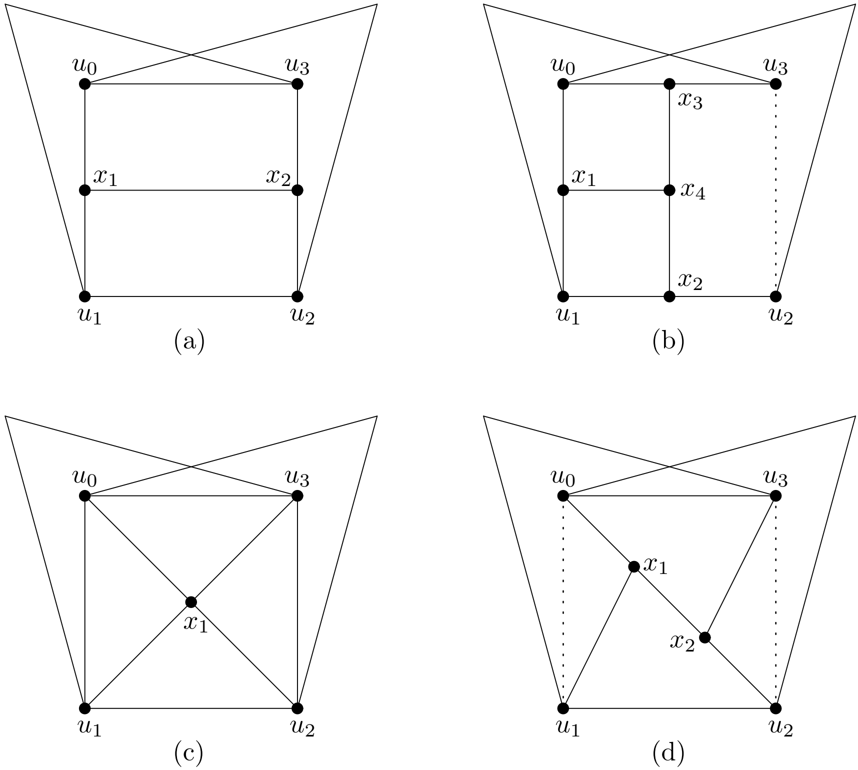


Figure 14: Forbidden subgraphs

path D_2 in Q whose only vertex on Γ is x_1 and whose only vertex on D_1 is the interior vertex x_4. The union of $[u_{i+3}, u_{i+2}]$, C_0, C_1, D_1 and D_2 is a subdivision of $K_{3,3}$. (See Figure 14(b).) The proof is thus complete in this case.

Now suppose that Q has no vertex in any (u_i, u_{i+1}). Then its only possible vertex in (u_{i-1}, u_{i+1}) is u_i, so every u_i is a vertex of Q. Take a u_0-u_2 chord D_1 in Q. As above, we may find a u_1-x_1 path D_2 and a u_3-x_2 path D_3 in Q where the only vertex of D_2 on Γ is u_1, the only vertex of D_2 on D_1 is the interior vertex x_1, the only vertex of D_3 on Γ is u_3, and the only vertex of D_3 on D_1 is the interior vertex x_2. Of the two regions into which D_1 divides the interior of Γ, the internal vertices of D_2 lie in one, and those of D_3 lie in the other, so D_2 and D_3 are internally disjoint. If $x_1 = x_2$, the union of Γ, C_0, C_1, D_1, D_2 and D_3 is a subdivision of K_5. (See Figure 14(c).) Otherwise, one of x_1 and x_2 precedes the other as we traverse D_1 from u_0 to u_2. Suppose it is x_1. Then the union of $[u_1, u_2]$, $[u_3, u_0]$, C_0, C_1,

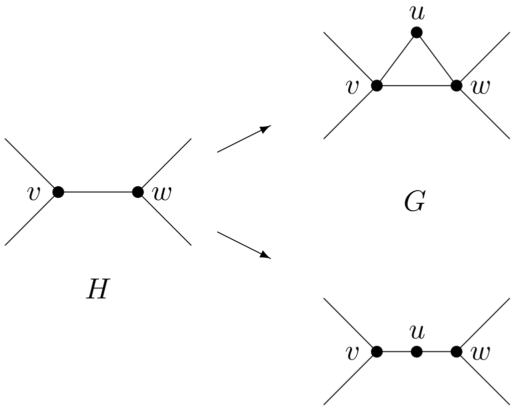


Figure 15: The moves for the proof of Theorem 15.3

D_1 D_2 and D_3 is a subdivision of $K_{3,3}$. (See Figure 14(d).) The remaining case is similar, and we are done. □

Proof of Theorem 15.3. What remains to be proved is that every non-planar graph has a forbidden subgraph. Note that if H has a forbidden subgraph and either H is a subgraph of G or G is a subdivision of H, then G has a forbidden subgraph. Let us say that a graph G_1 is *smaller* than a graph G_2 if either $n(G_1) < n(G_2)$, or $n(G_1) = n(G_2)$ and $m(G_1) < m(G_2)$. It suffices to prove that if G is a non-planar graph such that every smaller non-planar graph has a forbidden subgraph, then G has a forbidden subgraph. If G is not 2-connected, then every block of G is smaller than G, and by Theorem 14.8, G has a non-planar block, so the result follows in this case. Suppose then that G is 2-connected. If $\delta(G) < 3$ then G has a vertex u of degree 2. Let the two vertices adjacent to u be v and w. Form H by deleting the vertex u and, if v and w are not adjacent in G, adding an edge vw (in which case G is a subdivision of H). Then H is smaller than G and is non-planar, since a plane embedding of H would give rise to one of G by one of the two moves in Figure 15. Therefore H, and hence G, has a forbidden subgraph. Suppose further that $\delta(G) \geq 3$. By Lemma 15.5, G has an edge e such that $G - e$ is 2-connected. If $G - e$ is non-planar then, being smaller than G, it has a forbidden subgraph, and hence so does G. Finally, if $G - e$ is planar then G has a forbidden subgraph by Lemma 15.7. □

Let $e = uv$ be an edge of a graph G. The operation of *contracting* e is defined as follows. To $G - \{u, v\}$ add a new vertex w together with edges wx for all x adjacent to either u or v.

Definition 15.8. A *minor* of a graph G is a graph isomorphic to one obtained from G by zero or more applications of the operations of deleting or contracting an edge, or deleting a vertex. A minor obtained using only contractions is a *contraction* of G.

Example 15.9. The Petersen graph has K_5 as a contraction.

Clearly any subgraph of G is a minor of G, and if H is a minor of G and K is a minor of H then K is a minor of G.

Lemma 15.10. *If G is a subdivision of H then H is a minor of G.*

Proof. It suffices to consider the case where G is obtained from H by an elementary subdivision. If the edge of H involved is uv and the new vertex in G is w, contracting the edge vw of G gives a graph isomorphic to H. □

Theorem 15.11. *A graph G is planar iff neither K_5 nor $K_{3,3}$ is a minor of G.*

Proof. It is obvious that if G is planar, so is any graph obtained by contracting an edge, and hence any minor of G, which proves the "only if" part. The converse is given by Kuratowski's Theorem and the preceding lemma. □

Exercises for §15

15.1. Show that any minor of G is a contraction of a subgraph of G.

15.2. Show that H is a contraction of G iff there is a surjection $\phi\colon V(G) \to V(H)$ such that, for vertices u and v of H, the subgraph of G induced by $\phi^{-1}(u)$ is connected, and u and v are adjacent iff there is an edge of G with one vertex in $\phi^{-1}(u)$ and the other in $\phi^{-1}(v)$.

15.3. Show, without using Kuratowski's Theorem, that if G has K_5 or $K_{3,3}$ as a minor then it has a subgraph that is a subdivision of K_5 or $K_{3,3}$.

16 Crossing number

In this section we need to be more precise about what we mean by a drawing of a graph.

Definition 16.1. A *(plane) drawing* of a graph G consists of the following. First, a bijection α from $V(G)$ to a subset of $\mathbb{R}^2$. Second, a bijection β from $E(G)$ to a set of arcs in $\mathbb{R}^2$. These are required to satisfy the following conditions. First, if e is the edge uv, then the endpoints of $\beta(e)$ are $\alpha(u)$ and $\alpha(v)$. Second, if e and f are adjacent edges, $\beta(e)$ and $\beta(f)$ intersect only in $\alpha(u)$ where u is the common vertex of e and f. Thirdly, if e and f are distinct non-adjacent edges then $\beta(e)$ and $\beta(f)$ intersect in at most one point, at which $\beta(e)$ and $\beta(f)$ cross, and which lies on the image of no other edge. The *crossing number* of G is the minimum number $\nu(G)$ of crossings in any drawing of G.

Thus $\nu(G) = 0$ iff G is planar. As with embeddings, we identify a vertex with its image under α and an edge with its image under β. The preceding definition could be relaxed by allowing an edge to cross itself, or adjacent edges to cross, or edges to cross more than once. However, it is easy to see that such crossings could be eliminated while decreasing the total number of crossings, so this would not change the crossing number. If we allow more than two edges to cross at a point, we would have to replace "number of crossings" in the definition of $\nu(G)$ by "number of unordered pairs of edges that cross". (For instance, if three edges have a common point at which they mutually cross, we can perturb one slightly to produce three crossings on two edges each.) It will sometimes be more convenient to describe drawings in this generalized sense.

Definition 16.2. The *rectilinear crossing number* $\bar{\nu}(G)$ of a graph G is the minimum number of crossings in a drawing of G in which all edges are straight line segments.

Clearly $\nu(G) \leq \bar{\nu}(G)$. By the Fáry-Wagner Theorem (14.15), if $\nu(G) = 0$ then $\bar{\nu}(G) = 0$, but equality does not hold in general. It is true that allowing more than two edges to cross at a single point does not change $\bar{\nu}(G)$ provided one counts crossings properly; in this case, we need to perturb the vertices.

Computing crossing numbers seems to be hard; mostly what is known is upper bounds, conjectured to be exact.

Theorem 16.3. *The complete graph K_n has*

$$\nu(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Proof. If $n < 5$, $\nu(K_n) = 0$ and the result holds, so suppose $n \geq 5$. Suppose first that $n = 2k$, $k \geq 3$. In the strip $\{(r, s) \mid 1 \leq r \leq 2\}$ of $\mathbb{R}^2$, consider the set of all straight line segments from $(1, i)$ to $(2, i+a)$ for i, $a \in \mathbb{Z}$ and $0 \leq a < k$. Under the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(r, s) = (r\cos\frac{2\pi s}{k}, r\sin\frac{2\pi s}{k})$, the endpoints of these segments map to $2k$ points $u_i = f(1, i)$ and $v_i = f(2, i)$

for $0 \le i < k$, and the segments to k^2 arcs which are the edges of a drawing of $K_{k,k}$. To obtain a drawing of K_n we first add edges between distinct u_i and u_j that are straight line segments. These do not cross any edges of the first type, and for any four of the u_i there is just one crossing on the edges they induce. Thus the number of crossings on edges of this type is $\binom{k}{4}$. We may similarly add edges between distinct v_i and v_j lying in the set of points at distance at least 2 from the origin, having a further $\binom{k}{4}$ crossings. We now count, for a vertex w, the number of crossings on edges of the first type incident to w. This is independent of w, so we take $w = u_0$. For given a and b with $0 \le a, b < k$, we determine the number of crossings of $u_0 v_a$ with edges $u_i v_{i+b}$. If $a = b$, there are none. If $a \ne b$ the corresponding line segments cross if either $i < 0$ and $i + b > a$, or $i > 0$ and $i + b < a$; that is, if $a - b < i < 0$ or $0 < i < a - b$. At most one of these ranges is non-empty, and different values of i in the range give different edges, so the number of crossing edges is $|a - b| - 1$. Hence the number of crossings on edges of the first type incident with w is

$$2\sum_{a=3}^{k}\sum_{b=1}^{a-2}(a-b-1) = 2\sum_{a=3}^{k}\binom{a-1}{2} = 2\binom{k}{3}.$$

There are $2k$ vertices, and each crossing is on edges incident with four vertices, so the total number of crossings on edges of the first type is $2k \cdot 2\binom{k}{3}/4 = k\binom{k}{3}$, and the number of all crossings in the diagram is

$$k\binom{k}{3} + 2\binom{k}{4} = \tfrac{k(k-1)^2(k-2)}{4} = \tfrac{1}{4}\left\lfloor \tfrac{n}{2}\right\rfloor \left\lfloor \tfrac{n-1}{2}\right\rfloor \left\lfloor \tfrac{n-2}{2}\right\rfloor \left\lfloor \tfrac{n-3}{2}\right\rfloor.$$

Now suppose $n = 2k - 1$, $k \ge 3$. In the drawing of K_{2k} above, the number of crossings on the edges incident with any vertex is $2\binom{k}{3} + \binom{k-1}{3} = (k-1)^2(k-2)/2$, so deleting a vertex gives a drawing of K_n with

$$\begin{aligned}\tfrac{k(k-1)^2(k-2)}{4} - \tfrac{(k-1)^2(k-2)}{2} &= \tfrac{(k-1)^2(k-2)^2}{4}\\ &= \tfrac{1}{4}\left\lfloor \tfrac{n}{2}\right\rfloor \left\lfloor \tfrac{n-1}{2}\right\rfloor \left\lfloor \tfrac{n-2}{2}\right\rfloor \left\lfloor \tfrac{n-3}{2}\right\rfloor\end{aligned}$$

crossings. □

The bound in this theorem is known to be exact for $n \le 10$. It gives 0 for $n \le 4$ and 1 for $n = 5$, which are clearly exact. Let us verify that the bound $\nu(K_6) \le 3$ is also exact. Suppose we have a drawing of K_6 with c crossings. Replacing each crossing by a vertex gives a plane graph of order $c + 6$ and size $2c + 15$, so by Theorem 14.4, $2c + 15 \le 3(c + 6) - 6$, or $c \ge 3$. The known case $n = 8$ gives $\nu(K_8) = 18$; it is also known that $\bar{\nu}(K_8) = 19$.

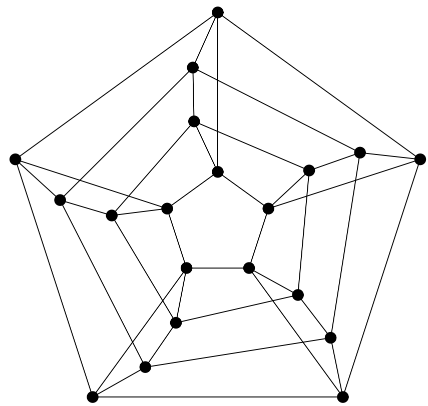


Figure 16: $C_4 \times C_5$ with 10 crossings

Theorem 16.4. *For the complete bipartite graph $K_{r,s}$,*

$$\nu(K_{r,s}) \le \bar{\nu}(K_{r,s}) \le \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor .$$

Proof. As the vertices of one partite set we take $r_+ = \left\lfloor \frac{r}{2} \right\rfloor$ points on the positive x-axis and $r_- = \left\lceil \frac{r}{2} \right\rceil$ points on the negative x-axis, and for the other we take $s_+ = \left\lfloor \frac{s}{2} \right\rfloor$ points on the positive y-axis and $s_- = \left\lceil \frac{s}{2} \right\rceil$ points on the negative y-axis. The edges are represented by line segments. Each crossing determines two points on one half of the x-axis and two on one half of the y-axis, and conversely any such points determine one crossing. Hence the number of crossings is

$$\begin{aligned}\binom{r_+}{2}\binom{s_+}{2} + \binom{r_+}{2}\binom{s_-}{2} + \binom{r_-}{2}\binom{s_+}{2} + \binom{r_-}{2}\binom{s_-}{2} &= \left(\binom{r_+}{2} + \binom{r_-}{2}\right)\left(\binom{s_+}{2} + \binom{s_-}{2}\right) \\ &= \left\lfloor \tfrac{r}{2} \right\rfloor \left\lfloor \tfrac{r-1}{2} \right\rfloor \left\lfloor \tfrac{s}{2} \right\rfloor \left\lfloor \tfrac{s-1}{2} \right\rfloor . \qquad \square\end{aligned}$$

It is conjectured that both bounds in this Theorem are exact, and this is known for $r \le 6$ and any s, and $r = 7$ and $s \le 10$.

Figure 16 shows a straight-line drawing of $C_4 \times C_5$ with $(4-2) \cdot 5 = 10$ crossings, from which the general case of $C_r \times C_s$ given in the next result should be clear.

Theorem 16.5. $\nu(C_r \times C_s) \le \bar{\nu}(C_r \times C_s) \le (r-2)s.$ $\square$

It is conjectured that, for $r \le s$, both these bounds are exact, and this is known for $r = 3$ and 4. We shall prove it for $r = 3$.

Lemma 16.6. $\nu(C_3 \times C_3) = 3.$

Proof. What remains to be proved is that any drawing of $G = C_3 \times C_3$ has at least three crossings. Consider a drawing with c crossings. Certainly G is not planar (see below), so $c > 0$. Pick an edge containing a crossing, and then a 4-cycle containing this edge. Deleting the edges of this cycle leaves a subdivision of K_5, so there is at least one crossing among the remaining edges. If there are two or more then $c \geq 3$, so suppose there is just one. This means that our original drawing can be obtained from a one-crossing drawing of K_5 by picking points subdividing the edges of a 4-cycle in K_5, and adding an edge between the subdivision points on each adjacent pair of edges. There is an essentially unique one-crossing diagram of K_5; replacing the crossing by a vertex gives the plane embedding of the graph of the octahedron. We must consider all ways of choosing a 4-cycle in the drawing of K_5, and for any edge on the cycle through the crossing, choosing which side of the crossing to put the subdivision point.

We consider the drawing as lying on the unit sphere in $\mathbb{R}^3$, with vertices $u_\pm = (\pm 1, 0, 0)$, $v_\pm = (0, \pm 1, 0)$ and $w = (0, 0, 1)$, and with edges lying in the coordinate planes. The edges u_+u_- and v_+v_- are semicircles crossing at $(0, 0, -1)$, and the other edges are quarter-circles. Rotation through a multiple of $\frac{\pi}{2}$ about the z-axis induces an automorphism of the K_5, as does reflection in any of the vertical planes $x = 0$, $y = 0$, $x = y$ and $x = -y$. This means we may assume that our 4-cycle is one of the four below. In each case, we denote the subdivision points on the edges by a_1, a_2, a_3 and a_4 in the order in which the edges appear in the cycle as written.

(a) $(u_+, v_+, u_-, v_-, u_+)$. In this case every added edge must have a crossing, so $c \geq 5$.

(b) $(u_+, v_+, v_-, u_-, u_+)$. In this case we may further assume that a_4 has positive x-coordinate and that a_2 has positive y-coordinate. Then each edge added at a_3 must have a crossing, so $c \geq 3$.

(c) (u_+, v_+, u_-, w, u_+). Here each of a_1a_2 and a_3a_4 must have a crossing, so $c \geq 3$.

(d) (u_+, u_-, v_+, w, u_+). The edge a_1a_4 must have one crossing if a_1 has positive x-coordinate, and two if it has negative x-coordinate, and if a_1 has positive x-coordinate the edge a_1a_2 must also have a crossing, so again $c \geq 3$. □

Theorem 16.7. *For any $s \geq 3$, $\nu(C_3 \times C_s) = s$.*

Proof. This is by induction on s, the case $s = 3$ being the previous lemma. Suppose then that $s > 3$ and the result is true for $s - 1$. We let the vertices of $G = C_3 \times C_s$ be u_i, v_i and w_i for $i \in \mathbb{Z}_s$, where the triangles are $T_i =$

$\langle u_i, v_i, w_i \rangle$ and the s-cycles are induced by the sets $U = \{u_i \mid i \in \mathbb{Z}_s\}$, $V = \{v_i \mid i \in \mathbb{Z}_s\}$ and $W = \{w_i \mid i \in \mathbb{Z}_s\}$, with edges u_iu_{i+1}, v_iv_{i+1} and w_iw_{i+1}. We must prove that, given a drawing of G with c crossings, $c \geq s$. If any edge of some T_i contains a crossing, deleting the edges of T_i gives a drawing of a subdivision of $C_3 \times C_{s-1}$ with at most $c-1$ crossings, so by inductive hypothesis $c - 1 \geq s - 1$.

Suppose then that there are no crossings on the edges of any T_i. Let c_e be the number of crossings on the edge e and note that $\sum_{e \in E(G)} c_e = 2e$. For $i \in \mathbb{Z}_s$, let X_i be the set of edges $\{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}\}$. We show that $\sum_{e \in X_i} c_e \geq 2$, which will complete the proof. If two edges of X_i cross each other this is clear, so suppose they do not. Then our drawing contains a plane embedding of the subgraph H_i induced by the vertices of the elements of X_i, which is isomorphic to $C_3 \times K_2$. It has five regions, two bounded by triangles and three by 4-cycles. The triangle T_{i+2} must lie in one of these regions, say R. If R were bounded by T_i, the cycle induced by U would meet T_i in a point other than u_i, which would be a crossing on an edge of T_i. Thus R is not bounded by T_i, and similarly not by T_{i+1}. Without loss of generality, we may assume that R is bounded by the 4-cycle $(u_i, v_i, v_{i+1}, u_{i+1}, u_i)$. Now the cycle induced by W must meet this 4-cycle in two points, which are crossings on edges of X_i. □

17 Vertex colorings

A *vertex coloring*, or just a *coloring*, of a graph G from a set C of colors is a function $\gamma \colon V(G) \to C$ such that for adjacent vertices u and v, $\gamma(u) \neq \gamma(v)$. It is a k-coloring if the image of γ has k elements, and the *chromatic index* $\chi(G)$ is the minimum k for which G has a k-coloring. G is *k-colorable* if $\chi(G) \leq G$. It is clear that $\chi(G)$ is the maximum of $\chi(H)$ over the components H of G.

Lemma 17.1. *For any graph G, $\chi(G)$ is the maximum of $\chi(B)$ over all blocks B of G.*

Proof. We may assume that G is connected. Let k be the maximum over blocks; clearly $k \leq \chi(G)$. We prove the reverse inequality by induction on the number of blocks, the case of a single block being trivial. Suppose that G has more than one block. Let B be an end-block, and let G' be the union of the other blocks. Then B and G' meet in a single cut-vertex u of G. By induction, B and G' are k-colorable. We may color them both from the same set of k colors, so that u receives the same color from B and G', and this gives a k-coloring of G. □

A graph has chromatic index 1 iff it is empty, and chromatic index 2 iff it is non-empty and bipartite. More generally, a k-coloring of G partitions $V(G)$ into k *color classes*, the sets of vertices of each color, and G is k-partite with respect to these sets. Conversely, if G is k-partite with partite sets $U_1, \ldots, U_k$, we may color G from $\{1, \ldots, k\}$ by setting $\gamma(u) = i$ for u in U_i. Some classes of graphs have easily determined chromatic indices: $\chi(K_n) = n$, $\chi(P_n) = 2$ for $n > 1$, and $\chi(C_n) = 2$ if n is even and 3 if n is odd. In general, though, determining $\chi(G)$ is hard.

A graph G is *k-chromatic* if $\chi(G) = k$, and, for $k \geq 2$, *critically k-chromatic* if in addition $\chi(G-u) < k$ (and hence $\chi(G-u) = k-1$) for every vertex u. By Lemma 17.1, a critically k-chromatic graph is non-separable. The only critically 2-chromatic graph is K_2. An odd cycle is critically 3-chromatic. Conversely, if G is critically 3-chromatic, it is not bipartite, and so contains an odd cycle. Let C be a shortest odd cycle in G. If vertices of C are not adjacent in C, they are not adjacent in G, since otherwise we could find two shorter cycles, one of them odd. If G had a vertex not on C, then $G - u$, containing an odd cycle, would be 3-chromatic, a contradiction. Thus G is an odd cycle.

Theorem 17.2. *Let G be a critically k-chromatic graph, $k \geq 2$. Then G is edge $(k-1)$-connected.*

Proof. Certainly G is connected. Let X be an edge-cut of cardinality $\kappa_1(G)$, and suppose, for a contradiction that $|X| < k - 1$. Then $G - X$ has two components G_1 and G_2, which are both $(k-1)$-colorable. Choose colorings of G_1 and G_2 from some set C of $k-1$ colors. Let the color classes of G_1 that contain a vertex adjacent to some vertex of G_2 be $U_1, \ldots, U_r$, and let the number of edges with one vertex in U_i and one in G_2 be k_i. Then $k_1 + \cdots + k_r \leq |X|$. We re-color G_1 from C, keeping the same color classes, so as to obtain a coloring of G from C, which is the desired contradiction. The vertices of U_1 are adjacent to vertices of G_2 having at most k_1 different colors, and since $k_1 < k-1$ we may choose some other color for the vertices of U_1. Suppose we have assigned colors to the vertices of U_j for $j < i$ so that all adjacent vertices so far colored have different colors. The number of different colors appearing at vertices of all U_j for $j < i$, or at vertices of G_2 adjacent to a vertex of U_i, is at most $i - 1 + k_i \leq k_1 + \cdots + k_i < k - 1$, so we may choose a different color for vertices of U_i.

In this way, all vertices of $U_1 \cup \cdots \cup U_r$ are colored, and we may assign colors to the remaining color classes of G_1 subject only to the requirement that all classes of G_1 get different colors. □

From the inequality $\kappa_1(G) \leq \delta(G)$ of Theorem 6.3 we have the following corollary.

Corollary 17.3. *Let G be a critically k-chromatic graph, $k \geq 2$. Then $\delta(G) \geq k-1$.* □

Theorem 17.4. *Let G be a connected graph that is neither complete nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.*

Of course, if G is complete or an odd cycle then $\chi(G) = \Delta(G) + 1$.

Proof. Let $k = \chi(G)$. Then G contains a critically k-chromatic subgraph H. If H is complete or an odd cycle then $H \neq G$. Since G is connected and H is regular, $\Delta(H) < \Delta(G)$, so $k = \Delta(H) + 1 \leq \Delta(G)$. Suppose then that H is neither complete nor an odd cycle, which implies that $k \geq 4$, and so, by the preceding corollary, $\delta(H) \geq 3$. Let n be the order of H. We shall show below that H contains vertices u and v with $d_H(u,v) = 2$ such that $H - \{u,v\}$ is connected. Given this, we may number the remaining vertices as $w_1, \ldots, w_{n-2}$, where w_1 is adjacent to u and v, and, for $1 < i \leq n-2$, w_i is adjacent to some w_j with $j < i$. We may now color H with the colors $\{1, \ldots, \Delta(H)\}$, showing $k \leq \Delta(H) \leq \Delta(G)$, as follows. We first give u and v the color 1. Then, for $i = n-2, \ldots, 2$ in turn we give w_i a color different from any color already assigned to some vertex adjacent to w_i, which is possible since w_i is adjacent to at most $\Delta(H)$ vertices, of which at least one has not yet been colored. Finally, since w_1 is adjacent to at most $\Delta(H)$ vertices, two of which have the same color, it may be assigned a color to complete the coloring.

Since H is not complete (and is connected), it has vertices u and v with $d_H(u,v) = 2$, and if $H - \{u,v\}$ is connected we are done. Otherwise, $H - u$ is connected (since H is 2-connected) and has v as a cut-vertex. Let B_1 and B_2 be distinct end-blocks of $H - u$ containing cut-vertices w_1 and w_2 respectively. For $i = 1$ or 2, since $H - w_i$ is connected, there is a vertex x_i of $B_i - w_i$ adjacent to u. Then $d_H(x_1, x_2) = 2$. By Lemma 5.20, $H - u - \{x_1, x_2\}$ is connected, and since $\deg_H u \geq 3$, so is $H - \{x_1, x_2\}$. □

Theorem 17.5. *For any graph G, $\chi(G) \leq \max \delta(G') + 1$, where the maximum is over all subgraphs G' of G.*

Proof. Let $k = \chi(G)$. If $k = 1$ the result is clear, and otherwise G has a subgraph G' that is critically k-chromatic, and $k \leq \delta(G') + 1$ by Corollary 17.3. □

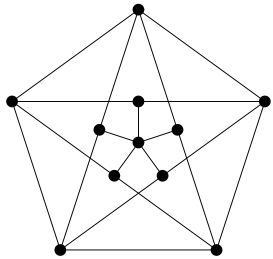


Figure 17: The Grötzsch graph

Theorem 17.6. *For any graph G, let $l(G)$ be the maximum length of a path in G. Then $\chi(G) \le l(G) + 1$.*

Proof. By the previous theorem there is a subgraph G' of G with $\chi(G) \le \delta(G') + 1$, and by Exercise 3.1 or Theorem 7.8, $l(G) \ge l(G') \ge \delta(G')$. □

A graph is *triangle-free* if, well, it contains no triangles; that is, it is either acyclic or of girth at least 4.

Theorem 17.7. *For any positive integer k there is a triangle free, k-chromatic graph.*

Much more is true; there are graphs of any chromatic index of arbitrarily high girth, but we do not prove this.

Proof. If G is a graph of order n with vertices $u_1, \ldots, u_n$, form a graph G' of order $2n+1$ by adding new vertices $v_1, \ldots v_n$ and w, and edges $v_i u_j$ whenever u_i and u_j are adjacent, and $v_i w$ for $1 \le i \le n$. If G is triangle-free, so is G'. Given a k-coloring of G, we get a $(k+1)$-coloring of G' by giving each v_i the same color as u_i, and w a $(k+1)$'st color. On the other hand, given a $(k+1)$-coloring of G', let c be the color of w. Color G by giving u_i its color in G' if this is not c, and the color of v_i if it is. This produces a coloring of G with at most k colors. Thus $\chi(G') = \chi(G) + 1$, and the result follows by induction on k. □

If the construction of the above proof is applied to K_1 (a triangle-free 1-chromatic graph), the result is $K_1 \sqcup K_2$, which is indeed triangle-free and 2-chromatic, but so is K_2. If the construction is applied to K_2 the result is C_5, which is clearly the unique triangle-free 3-chromatic graph of minimum order. If the construction is applied to C_5, the result is the Grötzsch graph of order 11, shown in Figure 17. We verify that this is the unique triangle-free 4-chromatic graph of minimum order.

Proposition 17.8. *Let G be a triangle-free graph of order $n \leq 11$ with $\chi(G) \geq 4$. Then G is isomorphic to the Grötzsch graph.*

Proof. We assume that n is minimal, and that amongst triangle free graphs of order n with chromatic index at least 4, G is maximal. If we prove that G is isomorphic to the Grötzsch graph, the result will follow since deleting any edge of the Grötzsch graph leaves a 3-chromatic graph. Since adding edges does not decrease the chromatic index, the maximality requirement means that G has diameter 2.

By Theorem 17.4, $\Delta(G) \geq 4$. Let w be a vertex of maximum degree, V the set of its neighbors, and U the set of all other vertices of G. Then $|U| \leq 6$. The subgraph induced by U is not bipartite, since otherwise we could 2-color it, give the vertices of V a third color, and w one of the original 2 colors to obtain a 3-coloring of G. Hence there is a 5-cycle C with vertices in U, and at most one vertex x of U not on C. Any vertex $v \in V$ is a adjacent to at most two vertices of C. Any vertex u of C not adjacent to v has a common neighbor with v, which must be one of its neighbors in C, or x. It follows that v is adjacent to two vertices of C. Moreover, distinct v_1 and v_2 in V cannot be adjacent to the same two vertices of C, since then the set of neigbors of one of them, say v_1, would be a subset of the neighbors of the other. A 3-coloring of $G - v_1$ could then be extended to a 3-coloring of G by giving v_1 the same color as v_2.

If $\Delta(G) = 5$, it follows that G is isomorphic to the Grötzsch graph. Otherwise, let G' be $G - x$ if x exists, and G if not. Then G' is isomorphic to the Grötzsch graph with some v_i (in the notation of the proof of Theorem 17.7) deleted. Hence there is a 3-coloring of G' in which some color appears only at two vertices of degree 4 in G'. Since x, if it exists, cannot be adjacent to these vertices, it can be given the same color to 3-color G, a contradiction. □

18 The Four-color Theorem

Theorem 18.1. *Every planar graph is 4-colorable.*

The original statement of this theorem was in terms of coloring regions of a plane graph. A *region coloring* of a plane graph G is a function γ from the set of regions of G to a set C of colors such that, for each arrow (u, v) of G, the regions to the left and right of (u, v) get different colors. (This form of the definition means that if G has a bridge then it has no region colorings.) It is a region k-coloring if the image of γ has k elements, and the *region chromatic index* $\chi^*(G)$ is the minimum k for which G has a region k-coloring. G is *region k-colorable* if $\chi^*(G) \leq k$.

Proposition 18.2. *Let k be a positive integer. The following are equivalent.*

(1) *Every bridgeless cubic plane graph is region k-colorable.*

(2) *Every planar graph is k-colorable.*

(3) *Every bridgeless plane graph is region k-colorable.*

The basic idea here is that a coloring of the vertices of a plane graph corresponds to a coloring of the regions of its dual. The *dual* of a plane graph G is obtained by taking a vertex in every region and taking, for each edge e of G, an edge in the dual adjacent to the vertices in the adjoining regions and crossing e at one point. However, the dual is generally a multigraph, and we are avoiding multigraphs in our formal treatment. (Loops in the dual correspond to bridges in G, and parallel edges arise when regions of G have more than one edge in common.) Therefore our proof will avoid taking the dual of a general plane graph.

Proof. Suppose first that every bridgeless cubic planar graph is region k-colorable. It is enough to prove that every maximal planar graph G is k-colorable. Take a plane embedding of G. Then the dual G^* of G is a graph, is bridgeless and cubic, and has dual G, so a region coloring of G^* gives a vertex coloring of G using the same colors.

Suppose now that every planar graph is k-colorable, and let G be a bridgeless plane graph, which we may assume is connected. Form a plane graph H as follows. Take a vertex in every region of G. For any regions of G with an edge in common, pick one such, e, and take an edge in H whose vertices are those in the regions and which crosses e at one point. A coloring of the vertices of H gives a region-coloring of G using the same colors. (Here the dual of G has no loops since G is bridgeless, and H is obtained by dropping all but one of each set of parallel edges.) □

For the early history of the four-color theorem, see [4]. A fallacious proof was published by Kempe in 1879. The error was only pointed out by Heawood in 1890. The four-color theorem was then reduced to the status of a conjecture, only becoming a theorem again in 1976 (Appel, Haken and Koch [1]). Even the simplified proof of Robertson, Sanders, Seymour and Thomas [27] relies heavily on the use of a computer, and no more will be said about it here. We give Kempe's incorrect proof, Heawood's observation

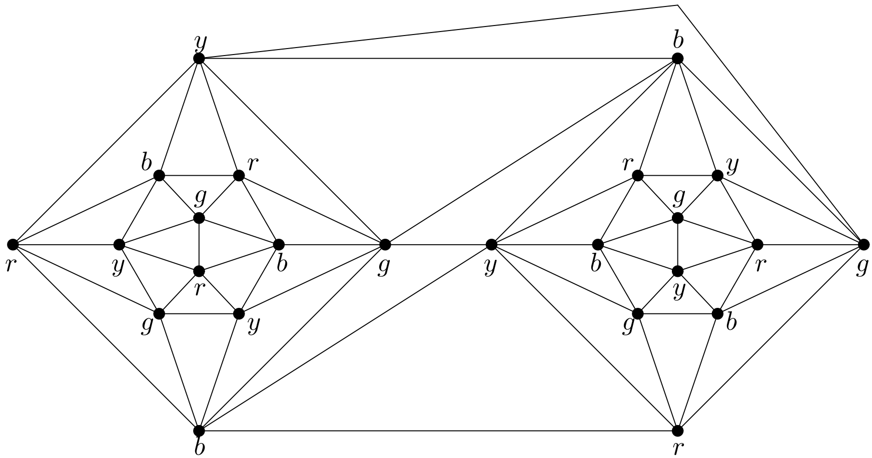


Figure 18: Heawood's counterexample

of the flaw, and his modification of Kempe's method to prove a weaker theorem. We shall use the vertex-coloring version, rather than the dual region-colorings of the originals.

Failed proof of Theorem 18.1. We attempt to use induction on the order n of a planar graph G, the cases $n \leq 4$ being obvious. Suppose then that $n \geq 5$ and the result is true for graphs of order less than n. We may assume that G is maximal planar, and we take a plane embedding of G. By Corollary 14.6, G has a vertex u of degree at most 5. The planar graph $G - u$ may be colored using (at most) four colors, which we take to be red, green, blue and yellow (r, g, b and y). If some color does not appear at any neighbor of u, we get a 4-coloring of G by giving u this color. One possibility is that u has degree 4, with neigbors v_1, v_2, v_3 and v_4 in counterclockwise order, having colors r, g, b and y respectively. For any two colors, we may consider the subgraph of $G - u$ induced by the vertices of those colors. A component of such a graph is a *Kempe chain*, and an rg-chain if the colors r and g were used. We may switch the two colors at the vertices of any Kempe chain to produce another coloring of $G - u$. If the rb-chain containing v_1 does not contain v_3, then switching red and blue on this chain gives a coloring of $G - u$ in which red does not appear at a neighbor of u, and hence a 4-coloring of G. If this chain does contain v_3, it contains a v_1-v_3 path, which together with (v_3, u, v_1) gives a cycle in G separating v_2 and v_4. Hence the gy-chain containing v_2 does not contain v_4, and we may use this chain to get a 4-coloring of G.

It remains to consider the case where u has degree 5, with neighbors $v_1, \ldots, v_5$ in counterclockwise order. By maximality, $(v_1, \ldots, v_5, v_1)$ is a cycle, and so without loss of generality the colors of $v_1, \ldots, v_5$ are b, r, y, g, r respectively. If the by-chain contaijning v_1 does not contain v_3, we are done as before, so we assume it does. Hence the rg-chain containing v_2 does not contain v_4. Similarly we may assume that the bg-chain containing v_1 contains v_4, and so the ry-chain containing v_5 does not contain v_3. Kempe thought that these two chains could be used to get a coloring of $G - u$ in which red does not appear at a neighbor of u, completing the proof. Heawood pointed out that switching colors on one chain can alter the other, so that the new chain contains two neighbors of u, and gave the example shown in Figure 18. Here $G - u$ and its coloring are shown; u is in the unbounded region. It will be seen that the relevant by- and bg-chains have the properties stated, and changing colors on the rg- and ry-chains, in either order, leaves a coloring with all four colors at neighbors of u. □

Theorem 18.3 (Heawood). *Every planar graph is 5-colorable.*

Proof. We use induction on the order n of a planar graph G, the cases $n \leq 5$ being obvious. Suppose then that $n \geq 6$ and the result is true for graphs of order less than n. Take a plane embedding of G. By Corollary 14.6, G has a vertex u of degree at most 5. The planar graph $G - u$ may be colored using (at most) five colors, which we take to be red, green, blue, yellow and taupe (r, g, b, y and t). If some color does not appear at any neighbor of u, we get a 5-coloring of G by giving u this color. Otherwise u has degree 5, with neigbors v_1, v_2, v_3, v_4 and v_5 in counterclockwise order, having colors, say, r, g, b, y and t respectively. Either the rb-chain contaijning v_1 does not contain v_3, or the by-chain containing v_2 does not contain v_4, and in either case we get a coloring of $G - u$ with a missing color at the neighbors of u, and hence a coloring of G. □

19 Edge colorings

An *edge coloring* of a graph G from a set C of colors is a function $\gamma\colon E(G) \to C$ such that for adjacent edges e and f, $\gamma(e) \neq \gamma(f)$. It is an edge k-coloring if the image of γ has k elements, and the *edge chromatic index* $\chi_1(G)$ is the minimum k for which G has an edge k-coloring. G is *edge k-colorable* if $\chi_1(G) \leq G$. Clearly $\Delta(G) \leq \chi_1(G)$.

Theorem 19.1 (Vizing). *For any graph G, $\chi_1(G) \leq \Delta(G) + 1$.*

Proof. This is by induction on the size, the case of an empty graph being obvious. Suppose that G is non-empty and the result holds for any graph with fewer edges. Let C be some fixed set of $\Delta(G)+1$ colors. In this proof, an edge coloring from C will be called just a *coloring.* Let $e = uv_0$ be an edge of G, and set $H = G - e$. Since $\Delta(H) \leq \Delta(G)$, H has a coloring. For any coloring γ of H and vertex x of G, we say that a color $c \in C$ *appears at* x if there is an edge f of H incident with x such that $\gamma(f) = c$. At any vertex, there is at least one color that does not appear. Suppose that, for some coloring γ of H and integer $k \geq 0$, there are distinct vertices $v_0, v_1, \ldots, v_k$ adjacent to u such that

(1) for $1 \leq i \leq k$, the color $a_i = \gamma(uv_i)$ does not appear at v_{i-1};

(2) there is a color b that appears at neither u nor v_k.

Then we obtain a coloring of G by giving uv_{i-1} the color a_i for $1 \leq i \leq k$, uv_k the color b, and every other edge its color under γ.

Now let γ be an arbitrary coloring of H, and suppose we cannot find vertices as above. Let k be the maximum integer for which there exist distinct vertices $v_0, v_1, \ldots, v_k$ adjacent to u satisfying (1). (There is such an integer, because (1) is vacuously true if $k = 0$.) There is some color not appearing at v_k, which must appear at u, and by maximality of k it must be a_j for some j with $0 < j < k$. Let b be a color not appearing at u, which must appear at every v_i for $0 \leq i \leq k$. Consider the subgraph K of H induced by the edges colored a_j or b. Every component is either a cycle or a path, and the endpoints of the paths are the vertices at which just one of a_j and b appears. One such vertex is v_k; let the component of K containing it be P, and let w be the other endpoint of P. Define a new coloring γ' of H by switching a_j and b on the edges of P. Note that at every vertex other than v_k and w the same colors appear in γ and γ'. We claim that there is some k', $0 \leq k' \leq k$, such that, with respect to γ',

(1′) for $1 \leq i \leq k'$, the color $a'_i = \gamma'(uv_i)$ does not appear at v_{i-1};

(2′) there is a color b' that appears at neither u nor $v_{k'}$,

which will complete the proof. Note that $a'_i = a_i$ except perhaps for $i = j$. Suppose first that $w = u$, so the vertex adjacent to w in P is v_j. Then we may take $k' = j - 1$ and $b' = a_j$. If $w \neq u$ then we also have $a'_j = a_j$. Suppose that $w = v_l$, where $0 \leq l < k$ and $l \neq j$. Since b appears at v_l in γ, it does not in γ' and we may take $k' = l$ and $b' = b$. The only remaining possibility is that w is different from u and all v_i, $0 \leq i \leq k$, and then we may take $k' = k$ and $b' = b$. □

A graph G is of *class 1* if $\kappa_1(G) = \Delta(G)$, and of *class 2* if $\kappa_1(G) = \Delta(G) + 1$.

Theorem 19.2. *A regular graph of class 1 has even order.*

Proof. Suppose G is d-regular of class 1. In any edge d-coloring of G, every color appears at every vertex. Thus the edges with any given color span G. Since they are pairwise non-adjacent, the result follows. □

The converse is false; see Exercise 19.1. The existence of edge 3-colorings of cubic graphs is related to the four-color theorem by the following result.

Theorem 19.3 (Tait)**.** *Let G be a connected, bridgeless, cubic plane graph. Then G is edge 3-colorable iff it is region 4-colorable.*

Proof. We color regions using $\mathbb{Z}_2 \times \mathbb{Z}_2$ and edges using the non-zero elements. Given a region coloring, we color each edge with the sum of the colors of the adjacent regions, which is non-zero because adjacent regions have different colors. Since the three regions around a vertex have different colors, so do the three edges.

Conversely, given an edge coloring, give one region the color $(0, 0)$. Then the color of every other region is determined by the rule that the sum of the colors of adjacent regions is the color of their common edge. This is well-defined because at every vertex the colors of the edges are the three non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, whose sum is zero. □

By Proposition 18.2, it follows that the four-color theorem is equivalent to the edge 3-colorability of all bridgeless, cubic, planar graphs. Martin Gardner [14] gave the name *snark* to a connected, bridgeless cubic graph that has no edge 3-coloring (i.e., is of class 2), and called a planar snark a *boojum*. (The names are taken from Lewis Carroll's "The Hunting of the Snark".) In these terms, the four-color theorem is equivalent to the non-existence of a boojum. Actually, snarks are required to satisfy extra conditions, such as being edge 4-connected and of girth at least 5, to rule out examples that are easily constructed from simpler ones. The first snark discovered was the Petersen graph (Exercise 19.1). Another is shown in Figure 19.

Exercises for §19

19.1. Show that the Petersen graph is a snark.

19.2. Let n be odd. Show that in any edge n-coloring of K_n, each color appears $(n-1)/2$ times.

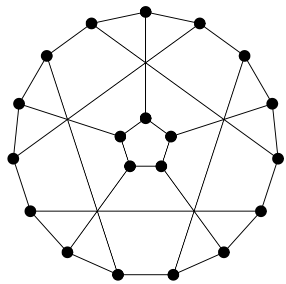


Figure 19: The flower snark

19.3. Show that K_n is of class 1 if n is even.

19.4. Show that a Hamiltonian cubic graph is not a snark.

20 Strongly regular graphs

Definition 20.1. A graph G is *strongly regular* with parameters (n, d, λ, μ) if it is connected of order n, is d-regular where $d < n - 1$, any two adjacent vertices have λ common neighbors, and any two independent vertices have μ common neighbors.

The condition that $d < n - 1$ just rules out complete graphs, and the requirement that G be connected rules out graphs whose components are all complete of the same order. We have $0 \leq \lambda < d$, $0 < \mu \leq d$ and $\operatorname{diam} G = 2$.

Theorem 20.2. *Let G be strongly regular with parameters (n, d, λ, μ). Then $d(d - \lambda - 1) = (n - d - 1)\mu$.*

Proof. For any vertex v, the number of walks of length 2 starting at v is d^2. Of these, d end at v, and $d\lambda$ end at a vertex adjacent to v, so $d(d - \lambda - 1)$ end at a vertex independent of v. On the other hand, there are $n - d - 1$ vertices independent of v, and μ walks of length 2 from v to each of them. The result follows. □

Lemma 20.3. *Let G be strongly regular with parameters (n, d, λ, μ) and adjacency matrix A. Then $A^2 + (\mu - \lambda)A - (d - \mu)I = \mu J$, where I is the $n \times n$ identity matrix, and J is the $n \times n$ matrix with all entries 1.*

Proof. By Theorem 4.2, A^2 counts walks of length 2 in G. The number of u-v walks of length 2 is d if $u = v$, λ if u and v are adjacent, and μ otherwise, so $A^2 = dI + \lambda A + \mu(J - I - A)$, as required. □

Theorem 20.4. *Let G be strongly regular with parameters (n, d, λ, μ). Then*

$$\tfrac{1}{2}\left(n-1 \pm \frac{(n-1)(\mu-\lambda)-2d}{\sqrt{(\mu-\lambda)^2+4(d-\mu)}}\right)$$

are positive integers.

Proof. By Lemma 20.3, every eigenvector of A is an eigenvector of J. Now J has a 1-dimensional eigenspace with eigenvalue n, generated by $\mathbf{x}_1 = (1, 1, \ldots, 1)$, and an $(n-1)$-dimensional eigenspace with eigenvalue 0. The vector $\mathbf{x}_1$ is an eigenvector of A, with eigenvalue d. The eigenvalue of any independent eigenvector of A must be a root of $x^2 + (\mu-\lambda)x - (d-\mu)$. The roots of this quadratic are

$$r_\pm = \tfrac{1}{2}\left(-\mu+\lambda \pm \sqrt{(\mu-\lambda)^2+4(d-\mu)}\right).$$

Let the multiplicity of $r_\pm$ as an eigenvalue of A be $a_\pm$. By Theorem 4.4, A must have three distinct eigenvalues, so a_+ and a_- are positive integers. Then $a_+ + a_- = n-1$ and, since A has trace 0, $d + r_+a_+ + r_-a_- = 0$. These equations give

$$a_\pm = \tfrac{1}{2}\left(n-1 \pm \frac{(n-1)(\mu-\lambda)-2d}{\sqrt{(\mu-\lambda)^2+4(d-\mu)}}\right). \qquad \square$$

Theorem 20.5. *Let G be strongly regular with parameters (n, d, λ, μ). Then either $(\mu-\lambda)^2 + 4(d-\mu)$ is a perfect square, or $\lambda = \mu - 1$, $d = 2\mu$ and $n = 4\mu + 1$.*

Proof. By Theorem 20.4, if $(\mu-\lambda)^2 + 4(d-\mu)$ is not a perfect square then $(n-1)(\mu-\lambda) = 2d$. Since $0 < d < n-1$, $\mu - \lambda = 1$ and $n = 2d+1$. Combining these equations with Theorem 20.2 gives the result. $\square$

Theorem 20.6. *Let G be a connected d-regular graph of order n whose adjacency matrix A has exactly three distinct eigenvalues. Then G is strongly regular.*

Proof. By Theorem 4.3, A has d as an eigenvalue of multiplicity 1. Let the sum and product of the other two eigenvalues be a and b. Then the minimal polynomial of A is $(x-d)q(x)$, where $q(x) = x^2 - ax + b$. Now $Aq(A) = dq(A)$, so every column of $q(A)$ is an eigenvector of A with eigenvalue d, and all its entries are equal. Since each diagonal entry of A^2, A or I is d, 0 or 1, respectively, every diagonal entry of $q(A)$ is $d + b$, and so $q(A) = (d+b)J$. That is, $A^2 = aA - bI + (d+b)J$, so G is strongly regular with parameters $(n, d, d+a+b, d+b)$. $\square$

Exercises for §20

20.1. Let G be strongly regular with parameters (n, d, λ, μ) where $\mu < d$. Show that $\bar{G}$ is strongly regular with parameters

$$(n, \bar{d}, \bar{d} - d + \mu - 1, \bar{d} - d + \lambda + 1),$$

where $\bar{d} = n - d - 1$. (The strongly regular graphs with $\mu = d$ are complete multipartite graphs of the form $K_{r,r,\ldots,r}$, and have disconnected complements.)

20.2. Find a strongly regular graph with parameters $(9, 4, 1, 2)$.

21 Moore graphs

Theorem 21.1. *Let G be a connected graph of order n, diameter k and maximum degree Δ. Then $n \le 1 + \Delta \sum_{i=0}^{k-1} (\Delta - 1)^i$.*

Proof. Fix a vertex v_0, and for $0 \le r \le k$ let S_r be the set of vertices v with $d(v_0, v) = r$, so that $V(G) = S_0 \cup S_1 \cup \cdots \cup S_k$. We have $S_0 = \{v_0\}$ and $|S_1| \le \Delta$. For $0 < r < k$, every vertex of S_r is adjacent to a vertex of S_{r-1} and is therefore adjacent to at most $\Delta - 1$ vertices of S_{r+1}, so $|S_{r+1}| \le (\Delta - 1)|S_r|$. Hence $|S_r| \le \Delta(\Delta - 1)^{r-1}$ for $0 < r \le k$, and the result follows. □

Definition 21.2. A *Moore graph* is a graph for which equality holds in the inequality of Theorem 21.1.

This is the original definition of a Moore graph. The term has since been extended to a larger class of graphs, as will be discussed later. If G is a Moore graph, every inequality in the above proof must be an equality. This implies that G is regular, $\operatorname{rad} G = \operatorname{diam} G$, and that for every vertex v_0 the subgraph induced by the edges of all geodesics starting at v_0 is a tree T, and the vertices of an edge not in T are end-vertices of T. The graphs of diameter less than 2 are the complete graphs, and these are Moore graphs. The 2-regular Moore graphs are the odd cycles. Recall that the girth of a graph that is not a forest is the length of a shortest cycle. Clearly the girth of a Moore graph of diameter k is $2k + 1$.

Theorem 21.3. *Let G be a connected graph of order n, girth $2k + 1$ and minimum degree δ. Then $n \ge 1 + \delta \sum_{i=0}^{k-1} (\delta - 1)^i$.*

Proof. As before, fix a vertex v_0 and let S_r be the set of vertices v with $d(v_0, v) = r$. Then $S_0 = \{v_0\}$ and $|S_1| \geq \delta$. For $0 < r \leq k$, every vertex of S_r is adjacent to only one vertex of S_{r-1}, since otherwise there would be a cycle of length at most $2r$ (by Lemma 7.5). If there are adjacent vertices of S_r then there is a cycle of length at most $2r + 1$, so $r \geq k$. It follows that $|S_r| \geq \delta(\delta - 1)^{r-1}$ for $0 < r \leq k$. □

Theorem 21.4. *Let G be a connected graph of order n, girth $2k$ and minimum degree δ. Then $n \geq 2\sum_{i=0}^{k-1}(\delta - 1)^i$.*

Proof. This time, fix adjacent vertices v_0 and v_1, and let S_r be the set of vertices v for which the minimum of $d(v_0, v)$ and $d(v_1, v)$ is r, so $S_0 = \{v_0, v_1\}$. For $0 < r < k$, every vertex of S_r is adjacent to only one vertex of S_{r-1}, since otherwise there would be a cycle of length at most $2r + 1$. If there are adjacent vertices of S_r then there is a cycle of length at most $2r + 2$, so $r \geq k - 1$. It follows that $|S_r| \geq 2(\delta - 1)^r$ for $0 \leq r < k$, giving the result. □

If equality holds in Theorem 21.3, then G is regular of diameter k, and therefore a Moore graph. If equality holds in Theorem 21.4 then G is also regular of diameter k. Some authors use the term Moore graph to mean a graph for which equality holds in Theorem 21.3 or 21.4, as appropriate. We shall keep the original meaning, and use *generalized Moore graph* for the broader class. The generalized Moore graphs of girth 4 are the complete bipartitite graphs $K_{r,r}$, and the 2-regular generalized Moore graphs of even girth are the even cycles. It can be shown that, for any integers $d \geq 2$ and $g \geq 3$, there is a d-regular graph of girth g. Such a graph of minimum order is called a (d, g)*-cage*. Thus every generalized Moore graph is a cage; however, there are many pairs (d, g) for which a (d, g)-cage is not a generalized Moore graph.

If G is d-regular of diameter k and girth $2k+1$, then Theorems 21.1 and 21.3 give $1+d\sum_{i=0}^{k-1}(d-1)^i \leq n \leq 1+d\sum_{i=0}^{k-1}(d-1)^i$, so G is a Moore graph. The hypothesis of regularity is superfluous, as was shown by Singleton [28].

Theorem 21.5. *Suppose that G is a graph of diameter k and girth $2k + 1$. Then G is regular.*

Proof. By Lemma 7.5, for vertices u and v there is a unique u-v path of length $d(u, v)$, and any other u-v path has length at least $2k + 1 - d(u, v)$, with equality only if the two paths give a cycle of length $2k+1$. We say that u and v are *antipodes* if $d(u, v) = k$. Suppose this is so, and let P be the u-v path of length k. Any neighbor w of u not on P must also have distance

k from v, and P, a v-w path of length k and the edge wu give a $(2k+1)$-cycle. Hence the number of $(2k+1)$-cycles through u and v is $\deg u - 1$, and by symmetry in u and v, $\deg u = \deg v$. Suppose $C = (v_0, v_1, \ldots, v_{2k}, v_0)$ is a $(2k+1)$-cycle, and take the subscripts modulo $2k+1$. The sequence $(v_0, v_k, v_{2k}, \ldots, v_{(2k)k})$ contains all vertices of C, and consecutive terms are antipodes, so all vertices of C have the same degree.

Now fix a $(2k+1)$-cycle C. For any vertex v, we may construct a path of length k starting at v by taking a shortest path from v to a vertex of C, followed by a path in C. Hence v has an antipode. It follows that no vertex has degree 1, since its neighbor would not have an antipode. Hence any path of length $l < 2k$ can be extended to a path of length $l+1$. This implies that, for any vertices u and v there is some path of length k containing u and v, and this is contained in a $(2k+1)$-cycle. Thus $\deg u = \deg v$. $\square$

Theorem 21.6 (Hoffman-Singleton [19])**.** *If there exists a d-regular Moore graph of diameter 2, then $d = 2$, 3, 7 or 57.*

As for the existence of Moore graphs of diameter 2, the following is known. Obviously the only one of degree 2 is the pentagon. The Petersen graph is an example of degree 3, and it is not hard to see that it is unique. For degree 7, there is also a unique example, as we show below; this was also proved in [19]. Whether there is an example of degree 57 is unknown.

Proof. If G is such a graph, it is strongly regular with parameters $(d^2+1, d, 0, 1)$. By Theorem 20.5, either $4d-3$ is a perfect square or $d = 2$. Suppose then that $s = \sqrt{4d-3}$ is an integer. By Theorem 20.4, s divides $d(d-2) = \frac{1}{16}(s^2+3)(s^2-5)$. That is, for some integer a, $s^4 - 2s^2 - 16as - 15 = 0$, so s is a divisor of 15. If $s = 1$ then $d = 1$, which is impossible, while if $s = 3$, 5 or 15 then $d = 3$, 7 or 57. $\square$

Example 21.7 (The Hoffman-Singleton graph)**.** This is a 7-regular Moore graph G of diameter 2, and hence order 50. The vertices are $x_{i,j}$ and $y_{i,j}$ for i, $j \in \mathbb{Z}_5$. The edges are $x_{i,j}x_{i,j+1}$, $y_{i,j}y_{i,j+2}$ and $x_{i,k}y_{j,ij+k}$ for i, j and $k \in \mathbb{Z}_5$. It is easy to check that G is 7-regular of diameter 2, as required. This graph is shown Figure 20. (This drawing can be constructed as follows. The vertices lie on a circle centered at the origin. To determine their angular coordinates, let $a = (0, 2, 24, 28, 6)$ and $b = (1, 33, 45, 49, 37)$. The angle for $x_{i,j}$ is $\pi(a_i + 20j)/25$, and that for $y_{i,j}$ is $\pi(b_i + 20j)/25$.)

Theorem 21.8. *Let G be a Moore graph of degree 7 and diameter 2. Then G is isomorphic to the graph of Example 21.7.*

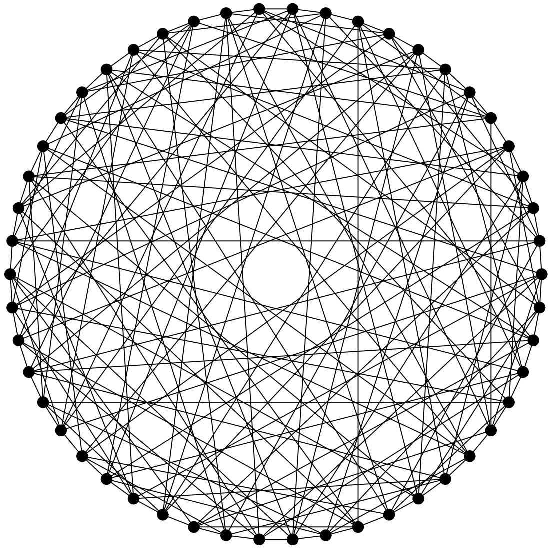


Figure 20: The Hoffman-Singleton graph

The following proof is taken from Elkies [10].

Proof. In this proof we shall distinguish between a 5-cycle, which is a sequence of vertices, and a pentagon, the underlying graph of a 5-cycle. It is clear that in any graph the number of 5-cycles is 10 times the number of pentagons. All subscripts will be in $\mathbb{Z}_5$.

As remarked in the previous proof, G is strongly regular with parameters $(50, 7, 0, 1)$. If $P = (u_0, u_1, u_2, u_3)$ is a path of length 3 then u_0 and u_3 are non-adjacent, so they have a unique common neighbor u_4, and there is a unique 5-cycle $(u_0, u_1, u_2, u_3, u_4, u_0)$ starting with P. There are thus $50 \cdot 7 \cdot 6 \cdot 6 = 12600$ 5-cycles, and hence 1260 pentagons, in G.

Let $C = (u_0, u_1, u_2, u_3, u_4, u_0)$ be a 5-cycle, and let A_i be the set of vertices adjacent to u_i other than $u_{i\pm1}$. If $i \neq j \in \mathbb{Z}_5$ then either u_i and u_j have no common neighbors or they have a common neighbor in C, so A_i and A_j are disjoint. Setting $A = A_0 \cup \cdots \cup A_4$ and $B = V(G) - A$, $|A| = |B| = 25$. A vertex of A_i can be adjacent to a vertex of A_j only if $i = j \pm 2$, and any vertex not on C can be adjacent to at most one vertex in each A_i. If $v \in A_i$ and $j = i \pm 2$, v and u_j have a common neighbor, which must be in A_j. It follows that each vertex of A is adjacent to 2 vertices of A and 5 of B. There are 125 edges from a vertex of A to a vertex of B, and since each vertex v of B is adjacent to at most 5 vertices of A (those of A_i if $v = u_i$ and at most one from each A_i if v is not on C), v is adjacent to 5 vertices of A and 2 of B.

Now consider a 5-cycle $(v_0, v_1, v_2, v_3, v_4, v_0)$. There are $25 \cdot 5 \cdot 4 \cdot 4 = 2000$ such cycles with $v_0,\ v_2 \in A$ and $v_1,\ v_3 \in B$, and for each one there is just one other cycle of the same form giving the same pentagon, namely $(v_4, v_3, v_2, v_1, v_0, v_4)$ if $v_4 \in A$ and $(v_2, v_1, v_0, v_4, v_3, v_2)$ if $v_4 \in B$, so these cycles account for 1000 of the pentagons. There are $25 \cdot 2 \cdot 5 \cdot 2 = 500$ cycles with $v_0,\ v_1 \in A$ and $v_2,\ v_3 \in B$, and again for each one there is one other cycle of the same form giving the same pentagon, namely $(v_0, v_4, v_3, v_2, v_1, v_0)$ if $v_4 \in A$ and $(v_1, v_0, v_4, v_3, v_3, v_1)$ if $v_4 \in B$. These cycles give a further 250 pentagons, so there are 10 pentagons not yet accounted for. These are given by cycles in which v_0, v_1, v_2 and v_3 belong to the same one of A and B, and there are $50 \cdot 2 \cdot 1 \cdot 1 = 100$ of these. If v_4 is not in the same set, there is only one other cycle of this form giving the same pentagon, $(v_3, v_2, v_1, v_0, v_4, v_3)$, which gives too many pentagons. Hence A is the disjoint union of pentagons $P_0, \ldots, P_4$, and B of pentagons $Q_0, \ldots, Q_4$.

Each vertex of P_i is adjacent to one vertex of every Q_j, and vice-versa. We shall label the vertices of P_i as $x_{i,j}$ and those of Q_i as $y_{i,j}$ $(i,\ j \in \mathbb{Z}_5)$, in such a way that the edges are $x_{i,j}x_{i,j+1}$ and $y_{i,j}y_{i,j+2}$. However this is done, there are, for $i,\ j \in \mathbb{Z}_5$, elements $a(i,j)$ and $b(i,j) = \pm 1$ of $\mathbb{Z}_5$ such that $x_{i,k}$ is adjacent to $y_{j,a(i,j)+b(i,j)k}$. We may choose the labelling of the $x_{0,k}$ arbitrarily, then that of the $y_{j,l}$ so that $a(0,j) = 0$ and $b(0,j) = 1$, then that of the remaining $x_{i,k}$ so that $a(i,0) = 0$ and $b(i,0) = 1$. Suppose that there exist i and j with $b(i,j) = -1$. Then for any $k \in \mathbb{Z}_5$ there is a trail $(x_{i,k}, y_{j,a(i,j)-k}, x_{0,a(i,j)-k}, y_{0,a(i,j)-k}, x_{i,a(i,j)-k)})$ of length 4. However, if $k = 3a(i,j)$ this is a cycle, which is impossible. Thus $x_{i,k}$ is adjacent to $y_{j,a(i,j)+k}$. Since there are no 4-cycles, for $i_1 \neq i_2$ and $j_1 \neq j_2$ we have

$$a(i_1, j_1) - a(i_2, j_1) + a(i_2, j_2) - a(i_1, j_2) \neq 0. \tag{21.1}$$

In particular,

$$\begin{aligned} a(i, j_1) &\neq a(i, j_2) \quad \text{for } i \neq 0 \text{ and } j_1 \neq j_2; \\ a(i_1, j) &\neq a(i_2, j) \quad \text{for } i_1 \neq i_2 \text{ and } j \neq 0. \end{aligned} \tag{21.2}$$

We may re-number the P_i and Q_j so that, in addition to $a(i,0) = a(0,j) = 0$, we have $a(i,1) = i$ and $a(1,j) = j$. All values of $a(i,j)$ are now determined. We have $a(2,3) \neq 0$, 2 or 3 by (21.2), and taking $i_1 = j_1 = 1$, $i_2 = 2$ and $j_2 = 3$ in (21.1) gives $a(2,3) \neq 4$, so $a(2,3) = 1$. Similarly $a(3,2) = 1$, and now (21.2) determines the remaining values. We find $a(i,j) = ij$, so G is isomorphic to the Hoffman-Singleton graph. □

Following [19], we shall show that the only Moore graph of diameter 3 is the heptagon C_7. (It was subsequently proved by Bannai and Ito [2] and Damerell [8] that all Moore graphs of diameter greater than 2 are odd cycles, so we have expended a lot of effort determining the properties of the graphs in a small, and, with the exceptions of the Petersen and Hoffman-Singleton graphs, boring class. Such is life.)

For integers $d \geq 2$ and $r \geq 0$, define polynomials $F_{d,r} \in \mathbb{Z}[x]$ by

$$F_{d,r} = \begin{cases} 1 & \text{if } r = 0; \\ x+1 & \text{if } r = 1; \\ xF_{d,r-1} - (d-1)F_{d,r-2} & \text{if } r \geq 2. \end{cases}$$

$F_{d,r}$ is monic of degree r.

Lemma 21.9. *If G is a d-regular Moore graph of diameter k with adjacency matrix A then $F_{d,k}(A) = J$.*

Proof. The discussion after Definition 21.2 shows that if u and v are vertices with $d(u,v) = r > 0$ then u has one neighbor at distance $r-1$ from v, and if $r < k$ its other neighbors are at distance $r+1$ from v. We show by induction that, for $0 \leq r \leq k$, the entry of $F_{d,r}(A)$ corresponding to vertices u and v is 1 if $d(u,v) \leq r$ and 0 otherwise, the cases $r = 0$ and 1 being obvious, and the case $r = k$ giving the lemma. Suppose that $r \geq 2$ and the result is true for smaller values. The entry of $AF_{d,r-1}(A)$ corresponding to u and v is the number of neighbors of u whose distance from v is at most $r-1$, which is d if $d(u,v) \leq r-2$; 1 if $d(u,v) = r-1$ or r; and 0 if $d(u,v) > r$. The result follows. □

Theorem 21.10. *For integers $d \geq 3$ and $k \geq 2$, if the polynomial $F_{d,k}$ is irreducible (over $\mathbb{Z}$, or equivalently over $\mathbb{Q}$), there is no d-regular Moore graph of diameter k.*

Proof. Suppose there is such a Moore graph G with adjacency matrix A. Then A has d as an eigenvalue of multiplicity 1, and every other eigenvalue is a root of $F_{d,k}$. The minimal polynomial of A has the form $(x-d)f$, where $f \in \mathbb{Z}[x]$ divides $F_{d,k}$. If $F_{d,k}$ is irreducible, $f = F_{d,k}$ and the characteristic polynomial of A is $(x-d)F_{d,k}^a$ for some positive integer a. The coefficient of x^{k-1} in $F_{d,k}$ is 1, so the sum of the roots of $F_{d,k}$ is -1. Since A has trace zero the sum of its eigenvalues (with multiplicities) is zero, so $a = d$. Therefore $1 + kd$ is the order of G, which is $1 + d\sum_{i=0}^{k-1}(d-1)^i$, and this is impossible for $d \geq 3$ and $k \geq 2$. □

Corollary 21.11. *There is no d-regular Moore graph of diameter 3 for $d \geq 3$.*

Proof. We have $F_{d,3} = x^3 + x^2 - 2(d-1)x - (d-1)$. If this polynomial is reducible, it has an integral root r. But then $d - 1 = \frac{r^2(r+1)}{2r+1}$, and since $2r+1$ is coprime to r and $r+1$, this implies $2r+1 = \pm 1$ and $d = 1$, a contradiction. □

Exercises for §21

21.1. Show that the automorphism group of the Hoffman-Singleton graph has order 252000.

22 Kneser graphs

Let n and r be integers with $0 < r < n$ and let X be the set $\{1, \dots, n\}$. We denote the collection of all r-element subsets of X by $\binom{X}{r}$. The *Kneser graph* $\mathrm{Kn}(n, r)$ has vertex set $\binom{X}{r}$, and two vertices A and B are adjacent iff they are disjoint. In the proof of Theorem 10.6 we showed that the Petersen graph is isomorphic to $\mathrm{Kn}(5, 2)$. If s is an integer with $0 \leq s < r$, the *generalized Kneser graph* $\mathrm{Kn}(n, r, s)$ also has vertex set $\binom{X}{r}$, and two vertices A and B are adjacent iff $|A \cap B| \leq s$. Thus $\mathrm{Kn}(n, r) = \mathrm{Kn}(n, r, 0)$. If $n < 2r - s$ then $\mathrm{Kn}(n, r, s)$ is empty, while if $s = 0$ and $n = 2r$ then every component is a K_2, so for the rest of this section we exclude these cases. We shall always use t to denote $n - 2(r - s)$; writing t as $(n - (2r - s)) + s$ makes it clear that our restrictions imply that $t > 0$. Sending $A \in \binom{X}{r}$ to $X - A$ gives an isomorphism from $\mathrm{Kn}(n, r, s)$ to $\mathrm{Kn}(n, n - r, n - 2r + s)$, so we could restrict attention to the cases $n \geq 2r$, but this does not seem to simplify any arguments significantly.

Lemma 22.1. *Let A, $B \in \binom{X}{r}$. Then A and B have a common neighbor in $\mathrm{Kn}(n, r, s)$ iff $|A \cap B| \geq r - t$.*

Proof. Let $k = |A \cap B|$. Suppose first that there is a common neighbor C of A and B. Then $|(A \cup B) \cap C| \leq 2s$, so

$$\begin{aligned} n - 2r + k &= |X - (A \cup B)| \\ &\geq |C - (A \cup B)| \\ &\geq r - 2s, \end{aligned}$$

or $k \geq 3r - n - 2s = r - t$. Now suppose that $k \geq r - t$. It suffices to prove that there is a set $Y \subseteq X$ with $|Y| \geq r$, $|A \cap Y| \leq s$ and $|B \cap Y| \leq s$. If

$s \leq r-k$, we may take sets $A' \subseteq A-B$ and $B' \subseteq B-A$ with $|A'| = |B'| = s$, and set $Y = A' \cup B' \cup (X - (A \cup B))$. We have $|A \cap Y| = |B \cap Y| = s$ and $|Y| = 2s + n - 2r + k \geq r$. If $s > r-k$, take $Z \subseteq A \cap B$ with $|Z| = r-s$ and set $Y = X - Z$. Then $|A \cap Y| = |B \cap Y| = s$ and $|Y| = n - (r-s) \geq r$. $\square$

Lemma 22.2. *Let A, $B \in \binom{X}{r}$ and let $p \geq 0$ be an integer. There is an A-B walk of length $2p$ in $\mathrm{Kn}(n,r,s)$ iff $|A \cap B| \geq r - pt$.*

Proof. The case $p = 0$ is trivial. For $p \geq 1$ we proceed by induction on p, the case $p = 1$ being Lemma 22.1. Suppose there is an A-B walk of length $2(p+1)$. Then there is a vertex C and A-C and B-C walks of lengths $2p$ and 2, respectively. By induction, $|A \cap C| \geq r - pt$ and $|B \cap C| \geq r - t$. Now

$$\begin{aligned}
|A \cap B| &\geq |A \cap B \cap C| \\
&= |A \cap C| + |B \cap C| - |(A \cap C) \cup (B \cap C)| \\
&\geq (r - pt) + (r - t) - r \\
&= r - (p+1)t.
\end{aligned}$$

Suppose conversely that $|A \cap B| \geq r - (p+1)t$. We show there is a vertex C with $|A \cap C| \geq r - pt$ and $|B \cap C| \geq r - t$; it will follow by induction that there are A-C and B-C walks of lengths $2p$ and 2, and hence an A-B walk of length $2(p+1)$. If $|A \cap B| \geq r - pt$ we may take $C = B$, so suppose $|A \cap B| < r - pt$. It suffices to show that there is a set $Y \subseteq X$ with $|Y| \leq r$, $|A \cap Y| \geq r - pt$ and $|B \cap Y| \geq r - t$. Let $A' \subseteq A - B$ and $B' \subseteq B - A$ have $|A'| = r - pt - |A \cap B|$ and $|B'| = r - t - |A \cap B|$, and set $Y = A' \cup B' \cup (A \cap B)$. Then $|A \cap Y| = r - pt$, $|B \cap Y| = r - t$, and $|Y| = 2r - (p+1)t - |A \cap B| \leq r$. $\square$

Lemma 22.3. *Let A, $B \in \binom{X}{r}$ and let $p \geq 0$ be an integer. There is an A-B walk of length $2p+1$ in $\mathrm{Kn}(n,r,s)$ iff $|A \cap B| \leq pt + s$.*

Proof. By Lemma 22.2, there is an A-B walk of length $2p+1$ iff there exists $C \in \binom{X}{r}$ with $|A \cap C| \geq r - pt$ and $|B \cap C| \leq s$. Suppose such a C exists. Then $|A \cap B| \leq |A - C| + |B \cap C| \leq pt + s$. Suppose conversely that $|A \cap B| \leq pt + s$. If $|A \cap B| \leq s$ we may take $C = A$. Otherwise, we may take $C_1 \subseteq A \cap B$ with $|C_1| = s$ and, since $|A - B| < r - s \leq n - r = |X - B|$, C_2 with $A - B \subseteq C_2 \subseteq X - B$ and $|C_2| = r - s$. Setting $C = C_1 \cup C_2$, we have $|C| = r$, $|A \cap C| = |A - B| + s \geq r - pt$, and $|B \cap C| = s$. $\square$

Definition 22.4. The *odd girth* of a graph G is the minimum length of an odd cycle (assuming there is one; that is, G is not bipartite).

The odd girth of $\mathrm{Kn}(n, r)$ was determined by Poljak and Tuza [25], and this was extended to $\mathrm{Kn}(n, r, s)$ under mild restrictions on n, r and s by Denley [9].

Theorem 22.5. *The odd girth of* $\mathrm{Kn}(n, r, s)$ *is* $2\left\lceil \frac{r-s}{t} \right\rceil + 1$.

Proof. For any graph, the odd girth is the minimal length of an odd closed walk. By Lemma 22.3, there is a closed walk in $\mathrm{Kn}(n, r, s)$ of length $2p+1$ iff $r \le pt + s$, and the result follows. $\square$

Proposition 22.6. *Let A, $B \in \binom{X}{r}$ and set $k = |A \cap B|$. The distance $d(A, B)$ from A to B in* $\mathrm{Kn}(n, r, s)$ *is given by*

$$d(A, B) = \min\left\{2\left\lceil \frac{r-k}{t} \right\rceil, 2\left\lceil \frac{k-s}{t} \right\rceil + 1\right\}.$$

Proof. By Lemma 22.2, there is an A-B walk of length $2p$ iff $p \ge \left\lceil \frac{r-k}{t} \right\rceil$. By Lemma 22.3, there is an A-B walk of length $2p+1$ iff $p \ge \left\lceil \frac{k-s}{t} \right\rceil$, provided that $k - s > -t$. Of course, if $k < s$ then $d(A, B) = 1$, but for completeness we verify that in fact $k - s > -t$ and our formula is valid in all cases. Now $k - s \le -t$ iff $n - 2r + s \le -k$, which can happen only if $n = 2r - s$ and $k = 0$. But we assumed that if $n = 2r - s$ then $s > 0$, and in this case there do not exist disjoint elements of $\binom{X}{r}$. $\square$

The diameter of $\mathrm{Kn}(n, r)$ was determined by Valencia-Pabon and Vera [29].

Theorem 22.7. *The diameter of* $\mathrm{Kn}(n, r, s)$ *is* $\left\lceil \frac{r-s-1}{t} \right\rceil + 1$.

Proof. Set $a = \left\lceil \frac{r-s-1}{t} \right\rceil + 1$, so that $r - s - 1 \le t(a-1) \le r - s + t - 2$. The first of these inequalities may be written as $r - s - 1 \le t\left(\left\lfloor \frac{a-1}{2} \right\rfloor + \left\lfloor \frac{a}{2} \right\rfloor\right)$, or $r - t\left\lfloor \frac{a}{2} \right\rfloor \le s + t\left\lfloor \frac{a-1}{2} \right\rfloor + 1$. Hence, for any integer k, either $k \ge r - t\left\lfloor \frac{a}{2} \right\rfloor$ or $k \le s + t\left\lfloor \frac{a-1}{2} \right\rfloor$. In the first case, $2\left\lceil \frac{r-k}{t} \right\rceil \le a$, and in the second $2\left\lceil \frac{k-s}{t} \right\rceil + 1 \le a$, and it follows by Proposition 22.6 that $\operatorname{diam} \mathrm{Kn}(n, r, s) \le a$.

The inequality $t(a-1) \le r - s + t - 2$ may be similarly rewritten as $s + t\left\lfloor \frac{a}{2} \right\rfloor - t + 1 \le r - t\left\lfloor \frac{a-1}{2} \right\rfloor - 1$. If k is an integer with

$$s + t\left\lfloor \frac{a}{2} \right\rfloor - t + 1 \le k \le r - t\left\lfloor \frac{a-1}{2} \right\rfloor - 1 \tag{22.1}$$

then $2\left\lceil \frac{r-k}{t} \right\rceil \ge 2\left(\left\lfloor \frac{a-1}{2} \right\rfloor + 1\right) \ge a$ and $2\left\lceil \frac{k-s}{t} \right\rceil + 1 \ge 2\left\lfloor \frac{a}{2} \right\rfloor + 1 \ge a$. Therefore if we can find A and B in $\binom{X}{r}$ with $k = |A \cap B|$ satisfying (22.1) we will have $\operatorname{diam}\operatorname{Kn}(n,r,s) \geq a$, completing the proof. Certainly there exist A and B with $|A \cap B| = k$ provided $s \leq k \leq r$, so it is enough to prove that $s \leq r - t\left\lfloor \frac{a-1}{2} \right\rfloor - 1$ and $s + t\left\lfloor \frac{a}{2} \right\rfloor - t + 1 \leq r$. The first inequality is equivalent to $t\left\lfloor \frac{a-1}{2} \right\rfloor \leq r - s - 1$ and (since $\left\lfloor \frac{a}{2} \right\rfloor \leq \left\lfloor \frac{a-1}{2} \right\rfloor + 1$) implies the second. Now $t\left\lfloor \frac{a-1}{2} \right\rfloor \leq \frac{1}{2}(r-s+t-2)$, which is at most $r-s-1$ provided $t \leq r-s$. On the other hand, if $t > r-s$ then $a = \left\lceil \frac{r-s-1}{t} \right\rceil + 1 \leq 2$, and so $t\left\lfloor \frac{a-1}{2} \right\rfloor = 0 \leq r-s-1$. ☐

In the rest of this section we determine the automorphism group of $\operatorname{Kn}(n,r,s)$. For $\operatorname{Kn}(n,r)$, this can be determined using the Erdös-Ko-Rado Theorem [12, 20], but the argument below does work in this case. If $s = r-1$ then $\operatorname{Kn}(n,r,s)$ is the complete graph on $\binom{n}{r}$ vertices, so from now on we further assume $s \leq r-2$, and hence $2 \leq r \leq n-2$. The symmetric group S_n acts naturally on $\binom{X}{r}$ as a group of automorphisms of $\operatorname{Kn}(n,r,s)$. Further, if $n = 2r$ there is an automorphism χ of $\operatorname{Kn}(n,r,s)$ given by $\chi(A) = X - A$, which is not induced by an element of S_n, and commutes with those that are. We show that these automorphisms generate the whole group.

For $A \in \binom{X}{r}$ we let $N(A)$ be the set of neighbors of A in $\operatorname{Kn}(n,r,s)$, and for distinct vertices A and B we let $N(A,B) = N(A) \cap N(B)$. Clearly the cardinality of $N(A,B)$ depends only on that of $A \cap B$. We let $f(k) = |N(A,B)|$ if there exist A and B with $|A \cap B| = k$, and 0 otherwise.

Lemma 22.8. *For $0 < k < r$, $f(k) \geq f(k-1)$, and $f(r-1) > f(r-2)$.*

Proof. Suppose there exist A, $B \in \binom{X}{r}$ with $|A \cap B| = k-1$. (Otherwise there is nothing to do.) Choose $b \in B - A$ and $c \in A - B$ and set $C = B - \{b\} \cup \{c\}$, so that $|A \cap C| = k$. For $D \in \binom{X}{r}$, we have $D \in N(B) - N(C)$ iff $|B \cap C \cap D| = s$, $b \notin D$ and $c \in D$, and similarly with the rôles of B and C reversed. Hence there is a bijection $\phi \colon N(B) - N(C) \to N(C) - N(B)$ given by $\phi(D) = D \cup \{b\} - \{c\}$. Since $|A \cap \phi(D)| = |A \cap D| - 1$, we have $f(k) \geq f(k-1)$, and the inequality is strict if there exists $E \in N(C) - N(B)$ with $|A \cap E| = s$. Suppose then that $k = r-1$, and let x and y be the unique elements of $A - C$ and $C - A$. If $s > 0$ we may take $E_1 \subseteq A \cap B$ with $|E_1| = s-1$ and $E_2 \subseteq X - (A \cup C)$ with $b \in E_2$ and $|E_2| = r-s-1$, and set $E = E_1 \cup E_2 \cup \{x,y\}$; while if $n > 2r-s$ we may take $E_1 \subseteq A \cap B$ with $|E_1| = s$ and $E_2 \subseteq X - (A \cup C)$ with $b \in E_2$ and $|E_2| = r-s$ and set $E = E_1 \cup E_2$. At least one of these constructions is possible, and we are done. ☐

Remark. We need not have $f(k) > f(k-1)$, even if $f(k) > 0$. For example, with $n = 8$, $r = 4$ and $s = 2$ we have $f(1) = f(0) = 36$.

We define a new graph with vertex set $\binom{X}{r}$, the *maximal overlap graph* $M(n,r)$, in which A and B are adjacent iff $|A \cap B| = r-1$. (It is the complement of $\mathrm{Kn}(n,r,r-2)$.) An immediate consequence of the previous lemma is:

Corollary 22.9. *If $s \le r-2$, every automorphism of $\mathrm{Kn}(n,r,s)$ is an automorphism of $M(n,r)$.* $\square$

Let $T = \{A, B, C\}$ be a triangle in $M(n,r)$. Then $|A \cap B \cap C|$ is equal to $r-1$ or $r-2$, and we say that T is of the first or second kind accordingly. Since $2 \le r \le n-2$, triangles of both kinds exist. If T is of the first kind, there are $n-r-2$ vertices adjacent to A, B and C in $M(n,r)$ (the sets $(A \cap B \cap C) \cup \{x\}$ for $x \notin A \cup B \cup C$), while if T is of the second kind, there are $r-2$ (the sets $(A \cup B \cup C) - \{x\}$ for $x \in A \cap B \cap C$). Hence if $n \neq 2r$, any automorphism of $M(n,r)$ takes each triangle to one of the same kind. When $n = 2r$ this is not the case; the complementation automorphism χ interchanges the two kinds of triangles, so the following result is non-trivial in this case.

Lemma 22.10. *Let T and T' be triangles in $M(n,r)$, and α an automorphism of $M(n,r)$. Then T and T' are of the same kind iff $\alpha(T)$ and $\alpha(T')$ are of the same kind.*

Proof. Let $\mathcal{T}$ be the graph whose vertices are the triangles in $M(n,r)$, with two triangles being adjacent in $\mathcal{T}$ if they share an edge. If $T = \{A, B, C\}$ and $T' = \{A, B, C'\}$ are adjacent triangles, then C and C' are adjacent in $M(n,r)$ iff T and T' are of the same kind, so the result holds in this case. It therefore suffices to prove that $\mathcal{T}$ is connected. Any triangle of the second kind is adjacent to one of the first, so it is enough to prove that any two triangles $T_1 = \{A_1, B_1, C_1\}$ and $T_2 = \{A_2, B_2, C_2\}$ of the first kind are connected. Let $Y_i = A_i \cap B_i \cap C_i$, so that $A_i = Y_i \cup \{a_i\}$, $B_i = Y_i \cup \{b_i\}$ and $C_i = Y_i \cup \{c_i\}$ for distinct a_i, b_i and $c_i \in X - Y_i$. The proof is by induction on $k = |Y_1 - Y_2|$, the case $k = 0$ being obvious. Suppose that $k > 0$. We may assume that $a_1 \in Y_2$, using the case $k = 0$ if necessary. Let $y \in Y_1 - Y_2$, and set $Y' = Y_1 - \{y\} \cup \{a_1\}$, $A' = A_1$, $B' = B_1 - \{y\} \cup \{a_1\}$ and $C' = C_1 - \{y\} \cup \{a_1\}$. Then $T' = \{A', B', C'\}$ is a triangle of the first kind with $A' \cap B' \cap C' = Y'$, and since $Y' - Y_2 = Y_1 - Y_2 - \{y\}$, T' is connected to T_2 by induction. Since $\{A_1, B_1, B'\}$ is a triangle (of the second kind) adjacent to both T_1 and T', T_1 and T_2 are connected. $\square$

Lemma 22.11. *Let α be an automorphism of $M(n,r)$ that takes each triangle to one of the same kind. Then α is induced by an element of S_n.*

Proof. Let A, B and C be elements of $\binom{X}{r}$ with A adjacent to both B and C in $M(n, r)$. We claim that:

(1) if $A - B = A - C$ then $\alpha(A) - \alpha(B) = \alpha(A) - \alpha(C)$;

(2) if $B - A = C - A$ then $\alpha(B) - \alpha(A) = \alpha(C) - \alpha(A)$.

If $B = C$ this is trivial. Otherwise, $A - B = A - C$ iff $\{A, B, C\}$ is a triangle of the first kind, and $B - A = C - A$ iff it is a triangle of the second kind, and the claim follows.

We now show that if A_1B_1 and A_2B_2 are edges of $M(n, r)$ with $A_1 - B_1 = A_2 - B_2$, then $\alpha(A_1) - \alpha(B_1) = \alpha(A_2) - \alpha(B_2)$. The proof is by induction on $k = |A_1 - A_2|$, the case $k = 0$ being (1). Suppose that $k > 0$ and pick $a_1 \in A_1 - A_2$ and $a_2 \in A_2 - A_1$. Note that $A_1 - B_1 = A_2 - B_2$ implies that $a_1 \in B_1$. Let $B_1 - A_1 = \{b_1\}$, and set

$$\begin{aligned} A' &= A_1 - \{a_1\} \cup \{a_2\}; \\ B' &= B_1 - \{b_1\} \cup \{a_2\}. \end{aligned}$$

(It may be that $B' = B_1$.) We have $A_1 \cap B' = A_1 \cap B_1$, so A_1 and B' are adjacent and $A_1 - B' = A_1 - B_1$. Thus $\alpha(A_1) - \alpha(B') = \alpha(A_1) - \alpha(B_1)$ by (1). Also $A' - B' = A_1 - B'$ (since a_1 and a_2 are both in B'), which implies that A' and B' are adjacent and $\alpha(A') - \alpha(B') = \alpha(A_1) - \alpha(B')$ by (2). Now $|A' - A_2| = k - 1$, so $\alpha(A') - \alpha(B') = \alpha(A_2) - \alpha(B_2)$ by induction, and we are done.

It follows that there is a well-defined function $\pi \colon X \to X$ such that if AB is an edge of $M(n, r)$ and $A - B = \{a\}$ then $\alpha(A) - \alpha(B) = \{\pi(a)\}$. If $a \neq b \in X$, we can pick an edge AB with $A - B = \{a\}$ and $B - A = \{b\}$. Hence $\pi(a) \neq \pi(b)$, and so $\pi \in S_n$. We must show that $\pi(A) = \alpha(A)$ for $A \in \binom{X}{r}$. If $a \in A$ then $\pi(a) \in \alpha(A)$ by the definition of π. That is, $\pi(A) \subseteq \alpha(A)$, and so $\pi(A) = \alpha(A)$. □

Theorem 22.12. *If $s \leq r - 2$, the automorphism group of* $\mathrm{Kn}(n, r, s)$ *is isomorphic to S_n if $n \neq 2r$, and to $S_n \times \mathbb{Z}_2$ if $n = 2r$.*

Proof. Let α be an automorphism of $\mathrm{Kn}(n, r, s)$. We must show that either α is induced by an element of S_n, or $n = 2r$ and $\chi\alpha$ is so induced. By Corollary 22.9, α is an automorphism of $M(n, r)$. By the discussion before Lemma 22.10, if $n \neq 2r$ then α takes each triangle of $M(n, r)$ to one of the same kind, while by that lemma, if $n = 2r$ either α or $\chi\alpha$ has this property. Lemma 22.11 completes the proof. □

A More topology

Here I fill in a few details from section 13 that my inner pedant insists must be covered. I have tried, perhaps quixotically, to limit the prerequisites to things that should have at least been stated in 1550 – 2057. The first lemma is left as an exercise.

Lemma A.1. *Let A and B be subsets of $\mathbb{R}^2$. If A and B are open (resp. closed), then $A \cup B$ and $A \cap B$ are open (resp. closed).* □

Lemma A.2. *Let $f\colon [0,1] \to [0,1]$ be a continuous bijection. Then $f(\{0,1\}) = \{0,1\}$.*

Proof. Suppose, for a contradiction, that $0 < f(0) < 1$. Since f is surjective, there exist a, $b \in [0,1]$ with $f(a) = 0$ and $f(b) = 1$. By the Intermediate Value Theorem, there is some c between a and b with $f(c) = f(0)$, contradicting the injectivity of f. A similar contradiction arises if $0 < f(1) < 1$, completing the proof. □

Lemma A.3. *Let $f\colon [a,b] \to \mathbb{R}^2$ be continuous. The image of f is a closed and bounded subset of $\mathbb{R}^2$.*

Proof. Let $A = f([a,b])$. We show first that A is closed. Let $x \in \mathbb{R}^2 - A$, and define $g\colon [a,b] \to \mathbb{R}$ by $g(s) = \|f(s) - x\|$. By the Extreme Value Theorem, g has a minimum value ϵ, and $\epsilon > 0$. If $y \in B(x,\epsilon)$ then $y \notin A$, so $\mathbb{R}^2 - A$ is open and A is closed.

Now define $h\colon [a,b] \to \mathbb{R}$ by $h(s) = \|f(s) - \mathbf{0}\|$. If M is the maximum value of h then $A \subseteq B(\mathbf{0}, M)$. □

Lemma A.4. *Let f and g be injective paths in $\mathbb{R}^2$ with the same image. Then the composite $h = g^{-1} \circ f\colon [0,1] \to [0,1]$ is continuous.*

Proof. Let $x_0 \in [0,1]$, $y_0 = h(x_0)$. and $\epsilon > 0$. We must show that there exists $\delta > 0$ such that $x \in [0,1]$ and $|x - x_0| < \delta$ imply that $|h(x) - y_0| < \epsilon$. Let $Y = \{y \in [0,1] \mid |y - y_0| \geq \epsilon\}$. Replacing ϵ by a smaller number, if necessary, we may assume that Y is a closed interval or a disjoint union of two closed intervals. By Lemmas A.1 and A.3, $g(Y)$ is a closed subset of $\mathbb{R}^2$. Hence there exists $\gamma > 0$ such that $B(f(x_0),\gamma) \subseteq \mathbb{R}^2 - g(Y)$, and then $\delta > 0$ such that $x \in [0,1]$ and $|x - x_0| < \delta$ imply $f(x) \in B(f(x_0),\gamma)$, and therefore $h(x) \notin Y$ and $|h(x) - y_0| < \epsilon$, as required. □

The next result follows immediately from Lemmas A.2 and A.4.

Theorem A.5. *Let f and g be injective paths in $\mathbb{R}^2$ with the same image. Then $f(\{0,1\}) = g(\{0,1\})$.* □

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List of errata

Problem 6.1 in the notes cannot be answered as stated. See the corrected version in the separate collection of exercises.

Page	Line	reads	should read
15	-9	0f	of
16	13	$(u_0, u_1, \ldots, u_l)$	$W_1 = (u_0, u_1, \ldots, u_l)$
16	13	$(v_0, v_1, \ldots, v_k)$	$W_2 = (v_0, v_1, \ldots, v_k)$
16	14	$0 \leq j \leq k$	$0 \leq j \leq k$ and every edge of W_2 is an edge of W_1
29	7 – 8	Theorem 5.14	Corollary 5.15
77	3	regular graph	non-empty regular graph