Three-dimensional orthogonal graph drawing with direction constrained edges

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ABSTRACT

Graph drawing studies the problem of producing layouts of relational structures that can be represented by combinatorial graphs. An orthogonal drawing is one whose edges are drawn as polygonal lines, with constituent segments drawn parallel to the coordinate axes. Orthogonal drawings arise in applications in diverse fields such as information visualization and VLSI circuit layout. One of the most successful methodologies for generating 2D orthogonal layouts of graphs is the so-called Topology-Shape-Metrics approach, where the task of defining the combinatorial shape of the drawing is separated from the task of determining the geometric coordinates of the vertices in the final drawing.

In contrast to its 2D counterpart, however, the aforementioned Topology-Shape-Metrics approach has not been exploited in 3D. The first step toward achieving this goal is due to Di Battista *et al.* [10, 11], who give combinatorial characterizations of paths and cycles with given shape such that they admit simple (i.e. non-self-intersecting) orthogonal 3D drawings. In particular, [10] studies the following problem: given a cycle with an axis-parallel direction label on each edge, can we compute a simple orthogonal drawing of the cycle? However, the necessity part of the proof for the characterization in [10] was later discovered by its authors to be incomplete. The aim of this thesis, therefore, is to complete the proof of the characterization given by Di Battista *et al.*, and to discuss further results that arise as consequences of this now complete characterization.

Résumé

Le dessin de graphe étudie le problème de produire des plans de structures relationnelles pouvant être représentées par des graphes combinatoires. Un dessin orthogonal est un graphe dont les arêtes sont des lignes polygonales parallèles aux axes de coordonnées. Les dessins orthogonaux sont utiles dans plusieurs applications de divers champs comme la visualisation d'information et la fabrication de plan pour l'intégration de circuits à très grande échelle (very large scale integration - VLSI). Une des meilleures méthodes pour générer des plans orthogonaux bidimensionnels de graphes est l'approche dite de «Topology-Shape-Metrics» [Topologie-Forme-Métrique], où la tâche de définir les formes combinatoires du dessin est séparée de celle de déterminer les coordonnées géométriques des sommets dans le dessin final.

Par opposition à son équivalent bidimensionnel, la méthode de «Topology-Shape-Metric» mentionnée précédemment n'a pas encore été exploitée en trois dimensions. La première étape afin d'atteindre ce but est énoncée par Di Battista et autres [10, 11] lorsqu'il donne les caractéristiques combinatoires des chemins et cycles d'une forme donnée tels qu'ils admettent des dessins 3D simples, (c'est-à-dire: sans intersections). En particulier, [10] étudie le problème suivant: étant donné un cycle avec une étiquette associant chaque arête à son axe parallèle, peut-on obtenir un dessin orthogonal simple du cycle? La preuve de la condition nécessaire pour la caractérisation dans [10] s'est néanmoins révélée comme étant incomplète par les auteurs. Le but de ce mémoire est donc de compléter la preuve de la caractérisation donnée par Di Battista et autres et aussi de discuter les résultats futurs résultant des conséquences de la complétion de la caractérisation.

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Chapter 1

Introduction

Graphs are abstract mathematical objects to model relationships between entities. The relational structure that graphs can model comes from many different areas in social science, engineering, biology, chemistry, mathematics, and, of course, computer science. For example, road networks, the world wide web, the structure of protein molecules, and digital circuits are all graphs in disguise. Due to this abundance in real world applications, algorithmic graph theory has now become a well-established area in combinatorics.

In many applications, it is also of interest to be able to automatically generate *drawings* of graphs, which has given rise to the field of graph drawing. Problems studied in graph drawing often assert specific aesthetic criteria for the drawings. These aesthetic criteria decide whether a drawing is good or bad, and which criterion should be considered varies from one application to another. Examples of aesthetic criteria include area of the drawing, readability as measured by the number of bends on the edges, etc. The objective of this chapter is to give, in Section 1.1, a brief but concrete list of examples where graph drawing finds applications in real life. Moreover, in Section 1.2, we will begin discussing the principal topic of this thesis by presenting the problem statement.

1.1 Applications of Graph Drawing

In this section, we introduce areas in various disciplines where graph drawing plays an important role. It should be noted, however, that this list is by no means an extensive survey on applications of graph drawing. Rather, these concrete examples are provided to help the reader understand the types of problems studied in graph drawing. Many comprehensive surveys are available on results in graph drawing. See, for example, [8, 19, 31, 34].

Software Engineering In software engineering, visualizing information effectively is an important tool for developing software systems. One of the first graph drawing algorithms for this purpose is an algorithm due to Knuth [21], for drawing flowcharts. Since the appearance of Knuth's algorithm, a variety of algorithms have been designed to draw diagrams such as entity-relationship diagrams [2], class diagrams [37], etc. Figure 1.1 shows an example of a UML diagram[1] drawn using a visualization tool from Tom Sawyer Software (http://www.tomsawyer.com). Here the graph is drawn as an *orthogonal drawing*, meaning that the edges are drawn as polygonal lines whose constituent segments are parallel to coordinate axes.

Biology Structural biologists study protein structures that consist of linear chains of amino acids. Due to limitations in experimental techniques, researchers must predict the geometric shape of protein molecules, also known as the *tertiary structure*, from sequences of amino acids. A protein molecule can easily be modeled as a graph-

[1]UML stands for Unified Modeling Language. Various UML diagrams such as class diagrams and use case diagrams let developers view a software system from different perspectives.

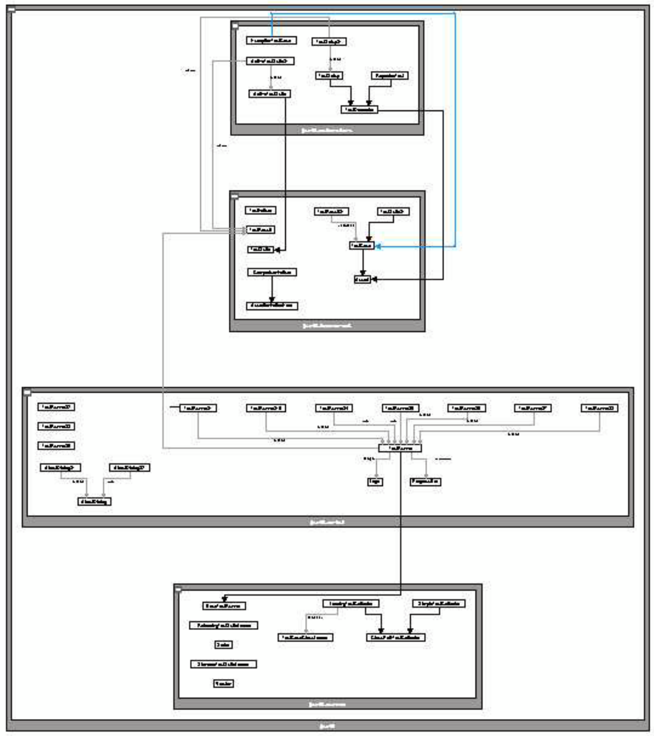

Figure 1.1: An orthogonal graph drawing illustrating a nested UML diagram (courtesy of Tom Sawyer Software)

theoretic path, where each vertex represents an amino acid, and a peptide bond between two amino acids is represented by an edge. Figure 1.2 shows a visualization of a protein molecule as a chain of amino acids. An abundance of literature in mathematics and theoretical computer science studies the protein folding problem from the perspective of graph drawing. The goal here is not to construct visually pleasing drawings, but rather to find the correct structure dictated by the rules from biology. See, for example, [3, 7, 14, 24].

Engineering Another example is in designing VLSI circuits. It is a well-known fact that every boolean circuit may be expressed as a combinatorial graph. The technology of building digital circuits has created a tremendous amount of demand

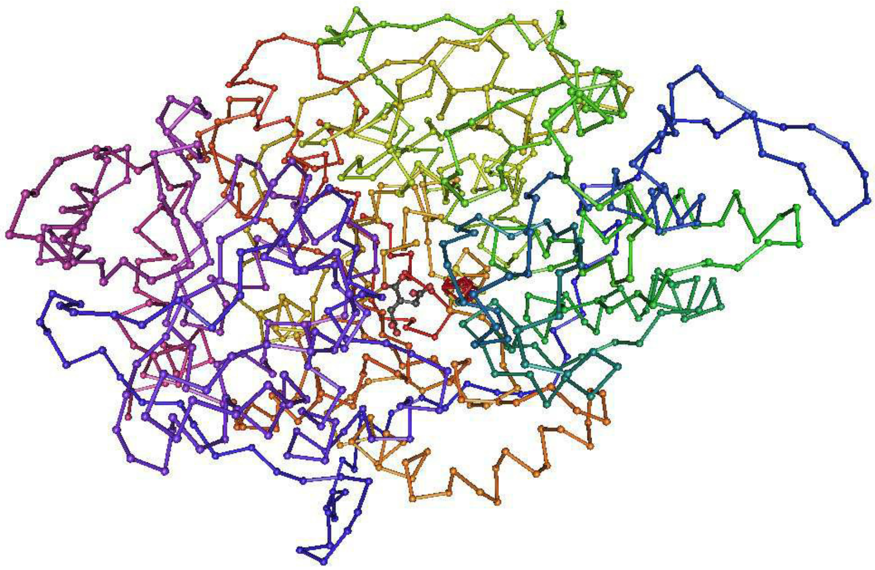

Figure 1.2: A visualization of a protein molecule as a chain of amino acids; courtesy of Structure Database at National Center for Biotechnology Information

for producing graph drawings that meet the design constraints imposed by VLSI manufacturers. Graph drawing problems for VLSI layouts have been studied both in 2D [4, 23] and in 3D [22, 29].

As such, graph drawing is the study of these problems from a purely mathematical and algorithmic point of view.

1.2 Problem Statement

In this thesis, we focus on graph drawing with a particular drawing style called *orthogonal drawing*. In an orthogonal drawing, all edges in the drawing must be parallel to the X-, Y-, or Z-axis, while vertices are generally drawn as points. In particular, we study the problem of drawing graphs whose edges have preassigned directions.

The main topic of this thesis can be stated as follows.

Problem 1.1. Orthogonal Cycle Drawing in 3D *Given a directed cycle with n vertices, where each edge in the cycle is labeled with a direction* North, South, East, West, Up, *or* Down, *our problem is to compute a drawing of the cycle in 3D by assigning a length to each edge, so that the drawing respects the direction labels while no two edges intersect except at a shared endpoint.*

Besides its theoretical interest, this problem arises in the famous algorithmic framework called *Topology-Shape-Metrics Approach*, which will be discussed in more detail in Chapter 2.

Cycle whose edges are given direction labels are called *shape cycles*. An example of a shape cycle and its orthogonal drawing is shown in Figure 1.3. This thesis studies a combinatorial characterization of graph-theoretic cycles that admit such drawings.

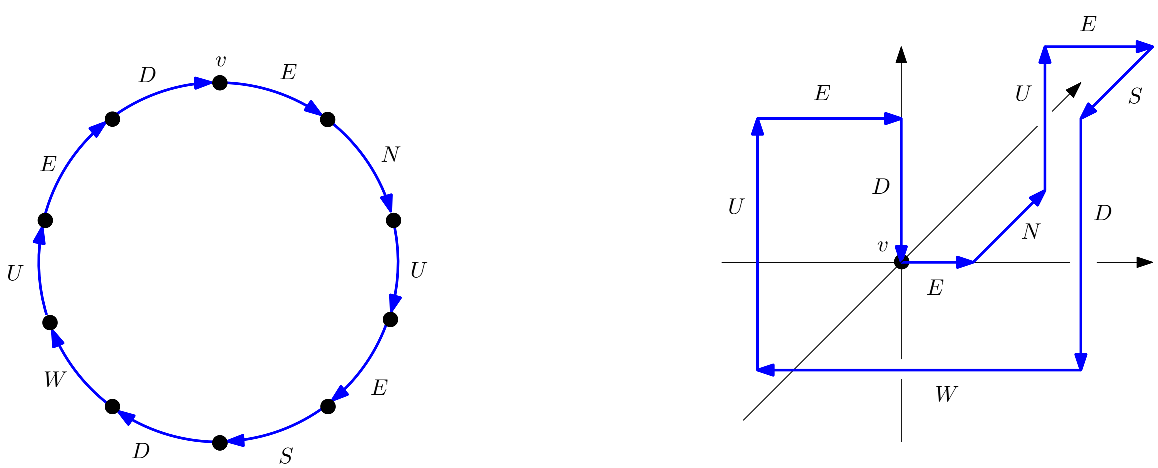

Figure 1.3: An orthogonal drawing of a shape cycle, with vertex v at the origin

1.3 Organization and Contributions

The rest of this thesis is organized as follows.

In **Chapter 2**, we give a background on orthogonal graph drawing, focusing on previous work related to the Topology-Shape-Metrics approach and graph drawing with direction-constrained edges, thereby providing motivations and implication of the work in this thesis.

In **Chapter 3**, we introduce definitions and notation used throughout the thesis.

In **Chapter 4**, we study rudimentary properties of shape cycles. These properties will provide a toolkit for use in later chapters. We introduce, with references where applicable, the results from previous work that will be used to prove the main result of this thesis. Finally, we end the chapter by discussing the proof techniques presented in Di Battista *et al.* [10], and their shortcoming. This result, and its shortcomings, is the main topic of study in this thesis.

In **Chapter 5**, based on the results from Chapter 4, we develop a necessary condition for the characterization of shape cycles. The necessary condition constructed in this chapter is essentially a completion, and shortening, of earlier work in [9, 10].

In **Chapter 6**, we complete the proof of our principal result, by using results discussed in Chapters 3, 4, and 5.

Finally, we conclude and present a list of open problems in **Chapter 7** by discussing consequences of the results studied in this thesis.

1.4 Statement of Originality

This thesis contains no material which has been accepted in whole, or in part, for any other degree or diploma. Except for results whose authors are mentioned, materials presented in Chapters 4, 5, 6, and 7 of this thesis is an original contribution to knowledge. Contributions on the results presented in Chapter 4, in particular, is clearly stated in Sections 4.4.2 and 4.5.

Chapter 2

Related Work & Motivations

In this chapter, we will build some background on direction constrained graph drawing by introducing various related topics in graph drawing. While doing so, we aim to understand where the principal topic of this thesis stands among the related work, and find out the implications of this thesis.

2.1 Orthogonal Graph Drawing

In automatic graph drawing, there are often optimization problems that are defined by maximizing or minimizing some quantitative aesthetic value for the drawing. Here, *aesthetic* need not mean visually pleasing, but rather, simply desirable in some technical way. For example, for visualization purposes, it is undesirable to draw edges too close to one another because such edges make the drawing less legible. Thus, one may wish to produce drawings of graphs with good angular resolution by maximizing the minimum angle between adjacent edges. Other examples of aesthetic properties are the area of a drawing whose vertices are placed on a grid, the total length of its edges, etc. These objective functions often conflict. For example, good angular resolution may conflict with area minimization. Thus, there may be trade-off issues between competing criterion.

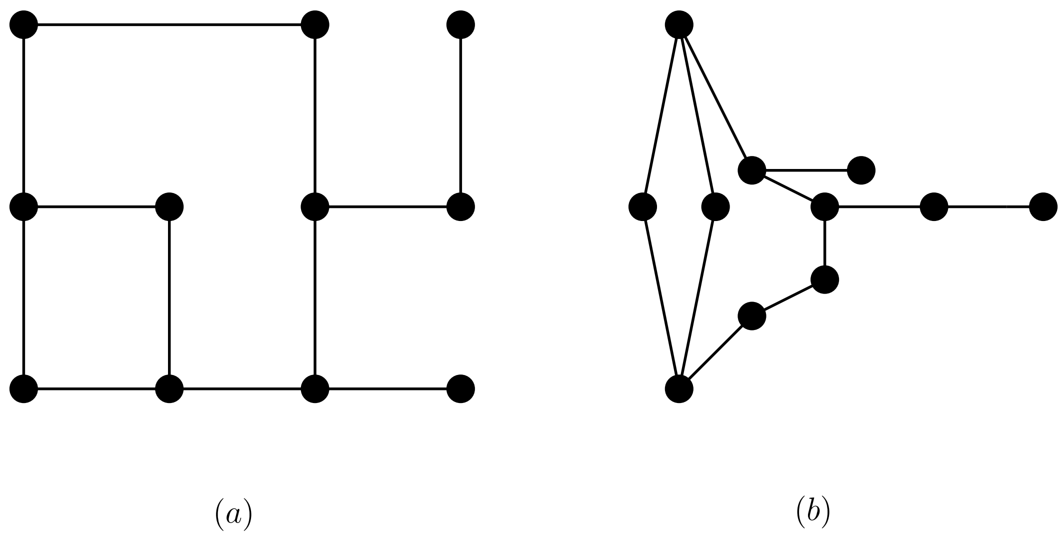

Figure 2.1: An orthogonal drawing of a graph and a drawing with bad angular resolution; these two graphs are isomorphic.

One way to achieve good angular resolution is to enforce all angles between adjacent edges to be multiples of $\frac{\pi}{2}$. In other words, all edges must be parallel to an axis. This style of drawing is called *orthogonal drawing*. See Figure 2.1 for an example of an orthogonal drawing of a graph in comparison with a drawing with bad angular resolution.

However, good angular resolution of orthogonal drawings comes with a few trade-offs. Because orthogonal drawing requires axis-parallel edges, inevitably, *bends* occur on the edges. Unfortunately, it is NP-hard to decide whether a planar graph with maximum degree 4 has an orthogonal drawing, as shown by Formann *et al* [17]. In most cases, however, the topological embedding of the graph is given, and thus we can then hope to optimize the number of bends in the drawing efficiently. Indeed, Tamassia [33] gives an elegant solution to the bend minimization problem in orthogonal drawings by reducing the problem to a network flow problem.

Obviously, another restriction imposed by the orthogonal drawing style is on the degree of vertices in the graphs. If the vertices are drawn as points, vertices with degree greater than 4 cannot be drawn without edges overlapping. One of the ways to resolve this is by drawing vertices as boxes in lieu of points. Much literature studies orthogonal box drawings; see chapters 6 and 7 of Kaufmann and Wagner [19] for a survey on orthogonal box drawings.

Many 2D graph drawing styles can be extended to 3D, including the orthogonal drawing style. Although angular resolution is no longer a good measure of readability of drawings in 3D, there are certain advantages of drawing graphs orthogonally in 3D. For example, 3D orthogonal drawings can handle graphs with maximum degree 6, as opposed to 4. Moreover, edge crossings can always be avoided due to the flexibility given by the extra dimension. Three dimensional orthogonal drawings have been studied extensively; see, for example, [5, 6, 16, 38, 39].

2.2 The Topology-Shape-Metrics Approach

Originally proposed in [1, 33, 35], and explained in detail in [8], the Topology-Shape-Metrics approach is an algorithmic paradigm to produce drawings of graphs in the orthogonal style. The overall approach can be divided into three phases:

1. *Topology*: Two drawings are said to be topologically equivalent, if the sequences of edges around corresponding faces are the same. To encode the topology of an embedding, we give a cyclic ordering of the edges to the adjacent vertices in the embedding, for each vertex.

2. *Shape*: Two drawings are said to have the same shape, if they have the same topology, and the corresponding angles formed by two adjacent edges are the same. In other words, two drawings have the same shape if one can be obtained from the other by changing lengths of the constituent segments of the edges.

3. *Metrics*: Two drawings have the same metrics if one can be obtained from the other by a translation and/or a rotation.

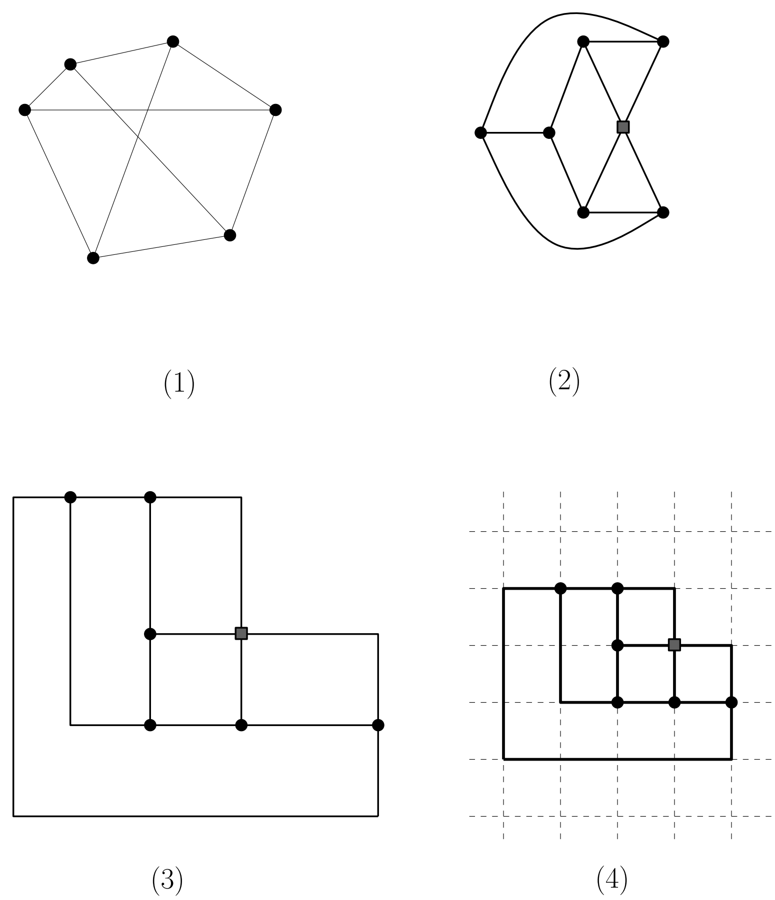

Figure 2.2: Topology-Shape-Metric approach: (1) A combinatorial graph is given without embedding; (2) Planarization gives an embedding of the graph. The light-shaded square marks the dummy vertex caused by a crossing; (3) Orthogonalization decides the shape of the embedding; (4) Compaction algorithm decides the metric of the embedding on a grid.

We now describe how the Topology-Shape-Metrics approach can be used to produce orthogonal drawings of graphs. Figure 2.2 shows an overview of the steps below.

Given a graph, we can first construct the topology of the embedding. This can be done via a planarization step, which is a well-understood problem. A graph is *planar* if there exists an embedding of the graph onto a plane, so that no two edges intersect except at a shared endpoint. A planar graph with a planar embedding is called a *plane graph*. Chapter 2 of Kaufmann *et al.* [19] and Nishizeki *et al.* [25]

provide extensive surveys on drawing planar graphs as plane graphs. Of course, not all graphs can be drawn as a plane graph. In such cases, one is generally interested in a drawing where the number of edge crossings is minimized. The problem of crossing minimization is another intensively studied problem both in graph drawing and in topological graph theory. For example, see [18, 15, 26] and the survey on crossing number by Szekely [32].

Once the topology of an embedding is specified, we then wish to specify the shape of the embedding. In the case of orthogonal drawings, a seminal work of Tamassia [33] gives an orthogonalization algorithm as mentioned in the previous section.

Given the shape of an embedding, we are now ready to determine the metric of the embedding by giving the coordinates of the vertices. In the case of grid drawings, where every vertex must be located at a grid point, we can produce a compact drawing of the graph by a compaction algorithm, as shown in Figure 2.2(3) and (4). Compactness of a drawing can be measured in various ways. For example, we can produce a compact layout by minimizing (1) area of the drawing, (2) total sum of the edge lengths, or (3) the maximum length of an edge. Unfortunately, given the shape of an embedding, it is NP-hard to produce a compact drawing defined by each of these three quantitative measures, as shown by Patrignani [27]. Nonetheless, many branch-and-bound approaches have been proposed to produce compact layouts of orthogonal drawings. See, for example, [20, 30], and a survey on compaction algorithms in Chapter 6 of Kaufmann and Wagner [19].

Observe that different objective functions can be formulated for each step depending on the desired property of the drawing. Minimizing the number of crossings, minimizing the number of edge bends, and minimizing the area of the final embedding are only a few examples of such objective functions. Regarding each phase of this approach as an independent problem, the main result in this thesis addresses the question of feasibility in the third phase of this approach: given a shape of an embedding, does there exist an assignment of coordinates to vertices and bends that is consistent with the shape? If this question can be answered positively, we can then try to optimize over an objective function, such as producing a compact grid embedding with a bounding box of small area or volume.

2.3 Direction Constrained Graph Drawing

Efforts in solving the feasibility problem have already begun. Vijayan and Wigderson [36] studied the problem of finding 2D orthogonal drawings of general graphs with given shape. They gave an $O(n)$ time recognition algorithm, together with an $O(n^2)$ time embedding algorithm for drawing general graphs in 2D. In particular, 2D cycles can be drawn in a simple orthogonal manner if and only if the cycle can be simplified to a rectangle by a series of edge contractions and vertex removals. In other words, a 2D cycle admits a simple orthogonal drawing if and only if it contains exactly 4 more right turns than left turns, or vice versa.

In contrast to its 2D counterpart, the 3D extension of this problem, or the Topology-Shape-Metrics approach in general, is not well understood. The first step towards this goal was taken by Di Battista *et al.* [11], where the path reachability problem was studied. In this work, the problem of finding an embedding of a path is studied, where a path must be drawn in such a way that the first vertex is drawn at the origin, and the last vertex is placed at a prescribed point in 3D. Rectilinear paths that admit such simple orthogonal drawings are characterized, which in turn yields a linear time recognition algorithm as well as a linear time drawing algorithm. In this work, this characterization was also generalized to higher dimensions.

The second attempt to extend the problem to 3D was by another work of the same authors [10], where the problem of drawing cycles is studied. Using similar characterization tools to the path drawing result, 3D cycles are characterized to yield linear time recognition and drawing algorithms. However, it was later discovered by its authors that the proof of the characterization is incomplete. Indeed, the topic of this thesis is to complete the proof of their characterization.

On another note, the characterization given by Di Battista *et al.* does not extend to even seemingly simple graphs. Di Giacomo *et al.* [12] discovered a graph, consisting of only 3 cycles, such that every simple cycle induced by its vertices admits a simple orthogonal drawing, but the graph itself as a whole cannot be drawn in a simple orthogonal manner.

Finally, the latest result on orthogonal shape graphs is by Patrignani [28], where it is shown that finding a drawing of a shape of a graph is, in general, NP-hard. This is particularly bad news for the Topology-Shape-Metrics approach, since it is now no longer plausible to separate the task of defining the shape of the drawing from the task of determining the metrics of the drawing. However, finding a nice characterization of shapes that admit simple orthogonal drawings still remains open. While recognizing any characterization of shape graphs would still be NP-hard, there may exist algorithms that are useful in practice. Furthermore, finding nontrivial classes of graphs that admit efficient recognition and drawing algorithms may be of interest. Independent of its practical applications, understanding which shapes can be drawn without intersection remains a fascinating, basic question. In Chapter 7, we give a survey of open problems that are closely related to the work in this thesis.

Application of Direction Constrained Drawings As discussed in previous sections, direction constrained drawings can be regarded as the final phase of the Topology-Shape-Metrics framework. Considering each phase of this framework as independent problem, Problem 1.1 is a feasibility problem where one wishes to check if the given shape indeed admits a feasible solution.

One of the major applications of direction constrained drawing is in VLSI layouts [4, 23]. In particular, the compaction problem where one wishes to minimize the size of a circuit layout while preserving its shape has been one of the challenging tasks in VLSI research. Various other constraints are often added into formulation of problems, such as the vertices being represented as squares or rectangles, and such complex set of constraints lead to problems that are difficult to formulate. Previous results solve the compaction problem using techniques from combinatorial optimization such as branch-and-bound or branch-and-cut [20].

With the recent development of VLSI research, it is now of interest to design circuit layouts in 2.5 dimensions(i.e. multiple layers of two-dimensional circuit layouts), or even full three-dimensions [22, 29]. Therefore, the compaction problem in 3D has been the subject of a renewed research interest. In the spirit of extending the Topology-Shape-Metrics approach to 3D, the feasibility problem of three-dimensional shapes is an essential, fundamental problem to consider.

Chapter 3

Preliminaries

In this chapter, we introduce definitions and the notation used throughout the thesis.

3.1 Graphs and Shapes

Throughout this thesis, we denote a graph by $G = (V, E)$, where V denotes the set of vertices and E denotes the set of edges. Unless otherwise stated, the number of vertices and edges of G are denoted by $n = |V|$ and $m = |E|$, respectively. We say a graph is *directed* if the edges in the graph are assigned directions. A *path* is a connected graph with two vertices of degree 1, and the other $n - 2$ vertices of degree 2. A *cycle* is a connected graph such that every vertex has degree 2[1]. For other terms in graph theory, see Diestel [13].

Let P be a graph theoretic path, where each edge is labeled with an axis-parallel direction in 3D: *North*, *South*, *East*, *West*, *Up*, or *Down*. If we label the edges of P

[1]These definitions are sufficient since we do not consider paths or cycles as subgraphs of other graphs.

with these directions we call this path a *shape path*. Here, labels on adjacent edges must be distinct but not oppositely directed. In other words, labels on adjacent edges must indicate orthogonal directions. Let C be a graph theoretic cycle with direction labels on its edges, again such that adjacent labels give orthogonal directions. We call such an edge labeled cycle a *shape cycle*. Shape paths and shape cycles are often denoted by σ. Furthermore, we sometimes make distinction between shape paths or cycles in different dimension. A shape path(cycle) is called a r-D shape path(cycle) if the direction labels on its edges contain r distinct axis-parallel directions that are mutually orthogonal. For example, a shape path labeled as $NESU$ is a 3D shape path while a shape cycle labeled as $NESW$ is a 2D shape cycle.

A *drawing* of a shape path or cycle σ is a geometric representation of σ in real space such that the vertices are positioned at real coordinates, and the edges are drawn as straight-line segments between the endpoints. Therefore, given a shape path or cycle σ, we can represent a drawing of σ by giving coordinates to the vertices in σ. We say a drawing of a shape cycle or a shape path is *simple* if no two edges intersect except at a shared endpoint. Furthermore, a drawing is said to be *orthogonal* if every edge is drawn parallel to an axis. Throughout the rest of this thesis, $\Gamma(\sigma)$ refers to a simple orthogonal drawing of σ such that the drawing respects the direction labels on the edges. We say a shape path or a shape cycle σ is *drawable* if σ admits such a drawing. Furthermore, we often use $\overline{uv}$ to denote the line segment that represents an edge uv in $\Gamma(\sigma)$. See Figure 1.3 for an example of a simple orthogonal drawing of a shape cycle. When we refer to a particular element, i.e., an edge, in a shape cycle or a shape path σ, we often denote it by σ_i, where i is an index for the edges of σ. Also, when we refer to a subpath from vertex a to vertex b, we often denote it by σ_{ab}. We typically describe a shape path or a shape cycle σ by giving the sequence of the direction labels on its edges in the order that they appear in σ. For example, $\sigma = NESUNDWU$.

3.2 Flats and Canonical Sequences

In this section, we first define the notions of *flats* and *canonical sequences*. Then we show how to decompose a shape cycle or its drawing into a collection of objects combinatorially defined by the notion of flats.

Definition 3.1. [10, 11] *Let σ be a shape cycle or a shape path. A* flat *of σ is a consecutive subsequence of σ that is maximal with respect to the property that its labels come either from the set $\{N, S, E, W\}$, or from the set $\{N, S, U, D\}$, or from the set $\{E, W, U, D\}$. We say a flat is* heavy *if it contains 3 or more elements, and* light *otherwise.*

In any simple orthogonal drawing of σ that respects the direction labels, the set of edges within a flat is contained in an axis-perpendicular plane. It is easy to see that every three-dimensional shape cycle or shape path that contains at least 3 mutually orthogonal labels consists of two or more flats. Furthermore, observe that the flats in a shape cycle or a shape path are not disjoint; the first edge of a flat is the last edge of the previous flat, and the last edge of a flat is the first edge of the next flat. We call such edges that belong to two consecutive flats *transition elements*. For example, consider the following example.

Example 3.2. *Let $\sigma = NESUNDWU$ be a shape cycle. Then, there are four flats in σ, namely $F_1 = NES$; $F_2 = SUND$; $F_3 = DWU$; and $F_4 = UN$. The flats F_1, F_2, and F_3 are heavy, but F_4 is a light flat. The transition elements in σ are $\sigma_1 = N$, $\sigma_3 = S$, $\sigma_6 = D$, and $\sigma_8 = U$.*

Throughout this thesis, we often use a combinatorial diagram to analyze the structure of a shape cycle. Figure 3.1 shows a diagram for the σ of Example 3.2.

We now define the notion of *canonical sequence* for sequences of direction labels on shape cycles. This definition will later help us design and analyze the simple

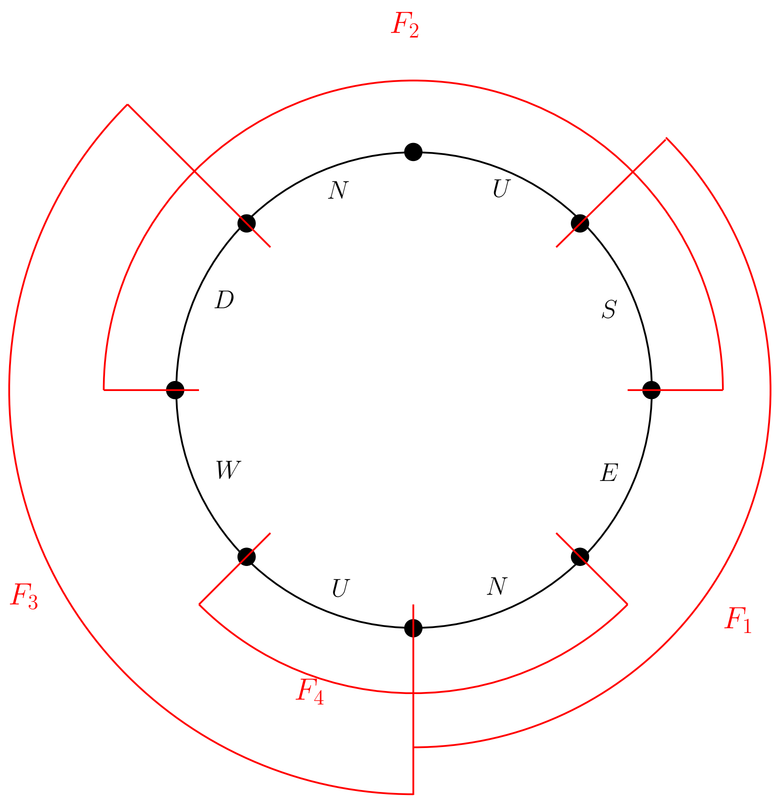

Figure 3.1: A combinatorial diagram of a shape cycle $\sigma = NESUNDWU$ with its flats

orthogonal drawings of shape cycles.

Definition 3.3. [10] *Let τ be a not necessarily consecutive subsequence of σ. We say τ is a* canonical sequence *if the following conditions are met.*

(1) The labels in τ are distinct;

(2) no flat of σ contains more than three labels of τ; and

(3) if a flat F of σ contains two or more labels of τ, then $\tau \cap F$ is a consecutive subsequence of σ.

To give some examples, $\sigma_1\sigma_2\sigma_5 = NEN$ is not a canonical sequence of σ of Example 3.2 since the label N is repeated; $\sigma_3\sigma_4\sigma_5\sigma_6 = SUND$ is not a canonical sequence since all four labels come from the same flat F_2; $\sigma_1\sigma_3 = NS$ is not canonical, because the two labels are from the same flat, but not consecutive in σ. Observe that the longest canonical sequence in this example is $\sigma_1\sigma_2\sigma_3\sigma_7\sigma_8 = NESWU$, which is of length 5.

We often call the direction labels of a canonical sequence *canonical labels*, and use special notation on these labels such as $\bar{}, \hat{}, \tilde{}$, etc. For example, when we say $\tau = \bar{N}\ \bar{E}\ \bar{U}$, each of $\bar{N}, \bar{E}, \bar{U}$ refers to a particular occurrence of that label in the shape cycle.

When we speak of canonical sequences, we sometimes disregard the ordering of the canonical labels in the sequence by using the set notation $\{\}$. For example, observe that $\tau = \{\bar{U}, \bar{N}, \bar{E}\}$ completely describes a canonical sequence. When we consider canonical sequences as sets of direction labels, we can thus use any conventional set operation on them. Consider the following example.

Example 3.4. *Let $\tau_1 = \{\bar{U}, \bar{N}\}$ and $\tau_2 = \{\hat{U}, \hat{D}\}$ be two individual canonical sequences for a shape cycle σ. If $\bar{U} = \hat{U}$, then $\tau_1 \cap \tau_2 = \{\bar{U} = \hat{U}\}$ and $\tau_1 \cup \tau_2 = \{\bar{U} = \hat{U}, \bar{N}, \hat{D}\}$. If $\bar{U} \neq \hat{U}$, then $\tau_1 \cap \tau_2 = \emptyset$ and $\tau_1 \cup \tau_2 = \{\bar{U}, \bar{N}, \hat{U}, \hat{D}\}$. Observe that even if τ_1 and τ_2 are two individual canonical sequences for σ, $\tau_1 \cup \tau_2$ is not necessarily canonical for σ.*

Finally, we define the *type* of a canonical sequence to be the set of its direction labels by disregarding the ordering of the canonical labels in the sequence. For example, $\tau = \{\bar{E}, \bar{N}, \bar{U}\}$ is a canonical sequence of type $\{U, N, E\}$.

3.3 Relationship between vertices, transition elements, and more

Here we study the relationship between vertices and transition elements of shape cycles and their drawings. The relationship between entities in a drawing $\Gamma(\sigma)$ allows us to analyze the drawing in a combinatorial manner, rather than treating it as a geometric object. We first look at the relationship between two vertices of $\Gamma(\sigma)$.

Suppose vertex u is located at the origin. Then, if vertex v shares exactly two of the three coordinates with u, v lies on an axis; if it shares exactly one coordinate with u, it lies on a quadrant; if it does not share any coordinate with u, it lies in an octant. This motivates the following definition.

Definition 3.5. *Let σ be a 3D shape cycle or path, and let $\Gamma(\sigma)$ be a drawing of σ. If two vertices u, v in $\Gamma(\sigma)$ share two of their three coordinates, we say they are in an* axis-relation*. If they share exactly one coordinate, we say they are in a* quadrant-relation*. Finally, if they do not share any coordinates, we say they are in an* octant-relation*.*

We now consider the relationship between two transition elements of σ in terms of flats containing the vertices of the transition elements.

Definition 3.6. *Let $e_1 = (a, a')$ and $e_2 = (b, b')$ be two transition elements in σ. Then, e_1 and e_2 are said to be* well-related *if no flat in σ contains a vertex from $\{a, a'\}$ and a vertex from $\{b, b'\}$.*

Finally, we consider the relationship between an edge and a vertex.

Definition 3.7. *Let $e_1 = (a, a')$ be an edge, and let $b \notin \{a, a'\}$ be a vertex in $\Gamma(\sigma)$. Suppose we place the origin at vertex b. Then, if a and a' are in the same octant, or the same quadrant, or the same axis, we say a and a' are* equivalent with respect to

b, and we write it as $a \sim_b a'$. Otherwise, we say they are not equivalent with respect to b*, and we write it as $a \not\sim_b a'$.*

3.4 Characterization of Shape Paths and Cycles

The first result on characterizing the drawability of 2D shape cycles is due to Vijayan and Widgerson [36] in terms of editing operations on the shape cycle. These editing operations can be regarded as taking shortcuts at u-turns in a rectilinear polygon such that the shortcuts do not intersect other edges of the polygon. Their characterization states the following: a shape cycle σ is drawable if and only if, after applying these editing operations repeatedly until no further operations can be done, σ becomes a rectangular shape, i.e., a sequence of four direction labels with consecutive labels orthogonal.

The next result on characterizing the drawability of 2D shape paths is due to Di Battista *et al.* [11]. In this work, one wishes to compute a simple orthogonal drawing of the given shape path so that the path starts at the origin and ends at a prescribed point p. The characterization of shape paths that admit such drawings can be summarized as follows.

Theorem 3.8. [11] *Let σ be a 2D shape path, and let p be any point of the UN quadrant. Then σ admits a simple orthogonal drawing $\Gamma(\sigma)$ that starts at the origin and terminates at p if and only if σ contains a canonical sequence of type $\{U, N\}$.*

Furthermore, this result has been extended to 3D and even higher dimensions. Here we state their characterization for the drawability of 3D shape paths.

Theorem 3.9. [11] *Let σ be a shape path and let p be a point in the UNE octant. Then there exists a simple, orthogonal, polygonal curve of shape σ that starts at the* *origin and that terminates at p if and only if σ contains a canonical sequence of type $\{U, N, E\}$.*

It should be noted, however, that the prescribed point p must be within an octant, i.e., p cannot be a point in a quadrant or an axis. Characterizations of drawability for such degenerate cases remain open. One example of a degenerate case is when the prescribed point p is located at the origin. In other words, a shape path is to be drawn as a cycle by placing the initial and the final vertex at the same location. Drawability of 3D shape cycles is studied in [10], and are characterized as follows.

Theorem 3.10. [10] *A 3D shape cycle σ admits a simple orthogonal drawing if and only if it contains a canonical sequence of length six.*

In other words, shape cycles that admit simple orthogonal drawings are those with a canonical sequence of type $\{U, N, E, D, S, W\}$. However, the original proof for the necessary condition in [10] was discovered by its authors to be incomplete. Hence, this thesis mainly focuses on the necessary condition to complete the proof of Theorem 3.10.

Chapter 4

Fundamental Properties of Shape Cycles

In this chapter, we study various fundamental questions regarding shape cycles and their drawings. Many results presented in this chapter are not new, but will be used as a toolkit for discussions in later chapters. While ideas from earlier work are clearly cited, we include proofs for propositions that were rewritten and have not been in any published material. This chapter is organized as follows. We will first introduce the notion of drawings in general position. Then, we introduce a special type of shape path drawing: a shape path drawing that reaches an octant and then reaches back to a quadrant. Certain properties of such drawings will be used extensively in many results in this thesis. Finally, we discuss the proof techniques from [10] that were used to prove Theorem 3.10, and point out the shortcomings of the techniques presented. For the discussions in this chapter, recall that when we speak of a drawing $\Gamma(\sigma)$ of a shape path or a shape cycle σ, we assume the drawing $\Gamma(\sigma)$ is simple and orthogonal.

4.1 Drawings in General Position

Given a simple orthogonal drawing of a shape cycle or a shape path, we show that the drawing can be perturbed slightly to satisfy a certain general position property as defined below.

Definition 4.1. *Let u, v be distinct, nonadjacent vertices of a drawing Γ of a shape path or cycle σ. Let E_u and E_v denote the sets of edges of Γ having u and v, respectively, as endpoints. Suppose that no flat of σ contains both an element of E_u and an element of E_v. Then Γ is in* general position *if, for each such pair u, v of vertices of Γ, no axis-parallel plane contains both u and v.*

In other words, Γ is in general position if, for every pair u, v of vertices that share a coordinate, there is a flat in σ that contains both u and v. The following lemma tells us that every simple orthogonal drawing can be perturbed to be in general position.

Lemma 4.2. (General Position Lemma) *A drawing Γ of a shape cycle or path σ can always be perturbed to be in general position.*

Proof. Let $\mathcal{P}$ be a NSUD plane that sweeps Γ from W to E. Each time there is more than one vertex of Γ on $\mathcal{P}$, we *stretch* Γ to obtain Γ' as follows. We partition the vertices in Γ into three subsets: S_1 the set of vertices that lie west of $\mathcal{P}$; S_2 the set of vertices that lie on $\mathcal{P}$; and S_3 the set of vertices that lie east of $\mathcal{P}$. First, we move all the vertices in S_3 east by unit distance. Then, we choose an arbitrary vertex u in S_2. For every other vertex v in S_2, we move v east by unit distance if v is not in the same flat as u. Otherwise, we do nothing. Finally, for vertices in S_1, we do nothing.

Observe that there are two invariants for this sweeping and stretching process. In particular, after the vertices are processed with respect to $\mathcal{P}$ at a given position, the following two properties hold for Γ':

(1) if a NSUD plane contains two vertices that are west of $\mathcal{P}$ then they belong to a flat.

(2) no two edges intersect except at their endpoints.

Furthermore, Γ' is also an orthogonal drawing. To see this, suppose that there exists an edge uv that is not parallel to an axis. Since uv is an edge, u and v both belong to a flat. Therefore, both u and v must have stayed or moved together during the stretching process described above. This is a contradiction.

Now, we repeat this sweeping and stretching process with $\mathcal{P}$ in the other two directions to obtain the final drawing in general position. □

Now, we may make the following assumption.

Assumption 4.3. *As a consequence of Lemma 4.2, we assume all drawings of shape paths and shape cycles discussed in this thesis are in general position.*

4.2 Manipulating Canonical Sequences

In this section, we present some basic properties on canonical sequences. Given a canonical sequence τ, it is sometimes useful to be able to select only a subsequence of τ. Furthermore, given a pair of canonical sequences, we may wish to combine the labels in the canonical sequences to construct a longer canonical sequence. The following set of tools will help us manipulate canonical sequences in such manner. Note that some of these results apply to only shape paths, or only shape cycles.

Property 4.4. [9] *Let τ be a canonical sequence for* a shape path or a shape cycle *σ. If τ contains three elements σ_i, σ_j, and σ_k that are consecutive on the same flat of σ, then σ_i and σ_k are oppositely directed.*

Property 4.5. [9] *Let* τ *be a canonical sequence for* a shape path σ. *If we remove from* τ *its first or last element, the resulting sequence is still canonical for the shape path* σ.

Property 4.6. [9] *Let* τ *be a canonical sequence for* a shape path or a shape cycle σ. *If* τ *consists of three elements that are pairwise orthogonal, then any subsequence of* τ *is also canonical.*

Lemma 4.7. [9] *Let* τ_1 *and* τ_2 *be two canonical sequences for* a shape path or a shape cycle σ *such that (1)* τ_1 *and* τ_2 *have no direction labels in common, and (2) no flat of* σ *contains an element from* τ_1 *and an element from* τ_2. *Then* $\tau_1 \cup \tau_2$ *is canonical for* σ.

Lemma 4.8. [9] *Let* $\sigma = \sigma_1 \ldots \sigma_i \sigma_{i+1} \ldots \sigma_n$ *be* a shape path *such that there is a light flat* $F_l = \sigma_i \sigma_{i+1}$. *If* τ_1 *and* τ_2 *are two canonical sequences for* σ *such that* $\tau_1 \subseteq \sigma_1 \ldots \sigma_i$ *and* $\tau_2 \subseteq \sigma_{i+1} \ldots \sigma_n$, *and* τ_1 *and* τ_2 *have no direction labels in common, then* $\tau_1 \cup \tau_2$ *is canonical for* σ.

Similarly, let $\sigma' = \sigma'_1 \ldots \sigma'_i \sigma'_{i+1} \ldots \sigma'_j \sigma'_{j+1} \ldots \sigma'_n$ *be* a shape cycle *(so* σ_1 *follows* σ_n*) such that there are two light flats* $F_1 = \sigma'_i \sigma'_{i+1}$ *and* $F_2 = \sigma'_j \sigma'_{j+1}$. *If* τ'_1 *and* τ'_2 *are two canonical sequences for* σ' *such that* $\tau'_1 \subseteq \sigma'_{j+1} \ldots \sigma'_i$ *and* $\tau'_2 \subseteq \sigma'_{i+1} \ldots \sigma'_j$, *and* τ'_1 *and* τ'_2 *have no direction labels in common, then* $\tau'_1 \cup \tau'_2$ *is canonical for* σ'.

Lemma 4.9. [9] *Let* $\sigma = \sigma_1 \ldots \sigma_k \ldots \sigma_n$ *be* a shape path *where* σ_k *is a transition element for* σ. *Suppose* τ_1 *and* τ_2 *are canonical sequences for* σ *such that:*

(1) $\tau_1 \subseteq \sigma_1 \ldots \sigma_k$ *and* $\tau_2 \subseteq \sigma_k \ldots \sigma_n$, *or vice versa;*

(2) The direction label of σ_k *must appear in both* τ_1 *and* τ_2 [1] *; and*

(3) Every subsequence of τ_2, *not necessarily consecutive, is also canonical for* σ.

[1] By direction label, we mean the direction of the element of σ, and not the element itself. Thus, σ_k might or might not belong to τ_1 or τ_2.

Then, there exists a canonical sequence of length $|\tau_1| + |\tau_2| - 1$ whose set of direction labels is equal to the union of direction labels in τ_1 and direction labels in τ_2.

Proof. We construct a new canonical sequence τ based on distinct cases as follows.

1. $\sigma_k \in \tau_1 \cap \tau_2$: Remove from τ_2 the elements whose direction labels appear in τ_1, and call it τ_2'. By assumption, τ_2' is canonical. Then, let $\tau = \tau_1 \cup \tau_2'$.

2. $\sigma_k \in \tau_1$, and $\sigma_k \notin \tau_2$: Let $\tau_1' = \tau_1 \setminus \sigma_k$. By Property 4.5, τ_1' is canonical. Remove from τ_2 the element whose direction labels appear in τ_1', and call it τ_2'. By assumption, τ_2' is canonical. Then, let $\tau = \tau_1' \cup \tau_2'$.

3. $\sigma_k \notin \tau_1$, and $\sigma_k \in \tau_2$: Remove from τ_2 the elements whose direction labels appear in τ_1, and call it τ_2'. By assumption, τ_2' is canonical. Then, let $\tau = \tau_1 \cup \tau_2'$.

4. $\sigma_k \notin \tau_1 \cup \tau_2$: Remove from τ_2 the elements whose direction labels appear in τ_1, and call it τ_2'. By assumption, τ_2' is canonical. Then, let $\tau = \tau_1 \cup \tau_2'$.

We now show that the constructed sequence τ is canonical for σ. By assumption and construction, the direction labels in τ are distinct. If a flat F contains more than one element of τ, then either $F \cap \tau$ is contained in τ_1 or τ_2 because σ_k is a transition element. Since τ_1 and τ_2 are canonical, the elements in $F \cap \tau$ are consecutive. Hence τ is canonical for σ. ☐

Observe that the merging operations done by Lemma 4.7 and 4.8 preserve the ordering of canonical labels in τ_1 and τ_2. In other words, the new canonical sequence constructed by these two merging operations is simply a concatenation of the two canonical sequences. On the other hand, the merging operation of Lemma 4.9 does not necessarily preserve such orderings. Nonetheless, the constructive nature of the proof of Lemma 4.9 dictates how to merge the two canonical sequences, removing, if necessary, some elements to avoid repetition of labels in τ.

4.3 The Shootback Lemma

In this section, we discuss the properties of shape paths that are drawn in such a way that the vertex at one end is drawn at the origin, and a later edge reaches a quadrant or an axis orthogonally(i.e. the path *shoots back*). We say an edge uv *intersects a quadrant* Q *orthogonally* if uv is perpendicular to Q, and an endpoint of uv lies in Q, or u and v lie in the two distinct octants adjacent to Q. Similarly, we say an edge uv *intersects an axis* A *orthogonally* if uv is perpendicular to A, and an endpoint of uv lies in A, or u and v lie in two distinct quadrants adjacent to A. We first present a result on 2D shape paths.

Lemma 4.10. (2D Shootback Lemma [9]) *Let* $\Gamma(\sigma)$ *be a simple drawing of a 2D shape path* σ *that starts at vertex* a *located at the origin* O*, and has a later edge* uv *that intersects an axis orthogonally. Let* X *be the direction label that denotes the axis that* uv *intersects, and let* Y, Y' *be the direction labels that denote the direction of* uv *and its opposite direction. Then* σ_{av} *contains a canonical sequence of type* $\{X, Y, Y'\}$*.*

In other words, if a drawing of a 2D shape path starts at the origin and intersects an axis orthogonally, there is a canonical sequence that consists of exactly those 3 labels needed to make the u-turn. Intuitively, one may expect a similar statement for 3D shape paths. However, the corresponding necessary condition for 3D shape paths is slightly weaker than its 2D counterpart.

Lemma 4.11. (3D Shootback Lemma [9]) *Let* $\Gamma(\sigma)$ *be a simple drawing of a 3D shape path* σ *that starts at vertex* a *located at the origin* O*, and has a later edge* uv *that intersects a quadrant orthogonally. Let* X, Y *be the direction labels that denote the quadrant that* uv *intersects, and let* Z, Z' *be the direction labels that denote the direction of* uv *and its opposite direction. Then* σ_{av} *contains a canonical sequence whose set of direction labels has* $\{X, Y, Z, Z'\}$ *as a subset.*

For conciseness, we write such canonical sequences as $\tau = \{X, Y, Z, Z'^{+}\}$ with a

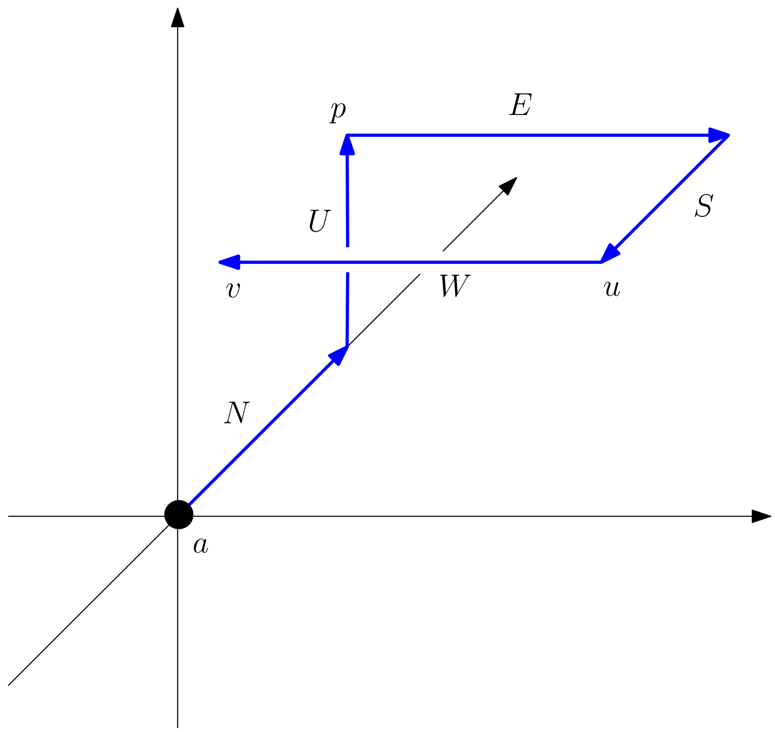

Figure 4.1: A drawing of $\sigma = NUESW$ that intersects the UN quadrant orthogonally

$+$ sign denoting that τ is a set of direction labels that contains $\{X, Y, Z, Z'\}$ as a subset. This result tells us that if a drawing of a 3D shape path intersects a quadrant orthogonally, there is a canonical sequence that contains those 4 direction labels as a subset. Unfortunately, we cannot strengthen this result to have a canonical sequence consisting of exactly those 4 direction labels and nothing more. To see this, consider the following counterexample.

Counterexample 4.12. *The shape path $\sigma = NUESW$ does not contain a canonical sequence of type $\{N, U, E, W\}$, yet σ admits a drawing that starts at the origin, and has a later edge intersects the UN quadrant orthogonally.*

This shape path can be drawn as in Figure 4.1. The final edge in the shape path intersects the UN quadrant, but $NUEW$ is not canonical for σ because S separates E and W while $\{E, S, W\}$ all belong to the same flat. On the other hand, $NUESW$ is canonical, and contains the desired 4 direction labels as the Lemma 4.11 claims.

On another note, observe that Lemma 4.11 also handles drawings where the edge uv simply *touches* a quadrant orthogonally. In such cases, the final vertex v lies in a quadrant when the initial vertex a is located at the origin. Because the vertex u is in an octant, the two vertices a and v do not belong to the same flat. Although the general position assumption allows us to safely ignore such drawings, if the shape path under consideration is part of a shape cycle, such cases may occur. For example, if the shape cycle that contains the shape path in discussion has a flat that contains v and a, the edge uv in the drawing may touch a quadrant without violating the general position assumption. By taking such cases into consideration, the statement of Lemma 4.11 allows us to analyze shape paths that are parts of a shape cycle.

In what follows, we give a proof for Lemma 4.11.

4.3.1 Proof of Lemma 4.11

We use induction on the length of σ, denoted by $|\sigma|$. When $|\sigma| = 4$, the statement holds trivially. Suppose the statement holds for all values $4 \leq |\sigma| \leq k$, and consider the case where $|\sigma| = k+1$. If uv is not the last edge of σ, we are done by the induction hypothesis. If there exists an edge other than uv that intersects the XY quadrant orthogonally, we are again done by the induction hypothesis. So, assume uv is the last edge of σ, and no other edge intersects the XY quadrant orthogonally.

Let F_{pv} denote the flat that contains uv, and consider the first vertex p of F_{pv}. We assume, without loss of generality, that u and v belong to two distinct octants adjacent to the XY quadrant (i.e. uv crosses the XY quadrant.). Otherwise, we can stretch the edge uv slightly to cross the quadrant without otherwise changing the drawing. Then, we assume, without loss of generality, that u is in the UNE octant, v is in the UNW octant, and thus uv is crossing the UN quadrant. Further, we assume F_{pv} is a NSEW flat. Other cases can be handled by applying the same arguments after permuting the direction labels. In the following subsections, we distinguish different cases depending on the location of p.

4.3.1.1 p as a point of an octant

We first consider cases where p lies in an octant. Observe that no edge in F_{pv} crosses or touches the U axis. To see this, observe that if an edge crosses or touches the U axis, by the general position assumption, there is a flat that contains a and the both endpoints of this edge. This edge must be either the edge pp', or the edge uv. In both cases, this is a contradiction since each of p and u is in an octant-relation with a, by assumption.

Case 1. p is in the UNE octant Consider the subpath σ_{ap}. Since p is in UNE octant, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{N}, \bar{E}\}$. We have that $uv = \hat{W}$. By Lemma 4.7, $\tau = \tau_1 \cup \{\hat{W}\}$ is canonical.

Now, in cases 2-4 below, we suppose p is in some octant other than UNE. Because p is not in the UNE octant, there must exist an edge qq' in σ_{pv} that enters the UNE octant for the last time. (Note that this edge qq' may be the edge pp'.) Observe that F_{pv} is a NSEW flat, and thus $\overline{qq'}$ must intersect either the UN quadrant or the UE quadrant. If $\overline{qq'}$ intersects UN quadrant, we are done by the induction hypothesis. So assume $\overline{qq'}$ intersects the UE quadrant orthogonally, and has direction N.

Case 2. p is in the USE octant By Theorem 3.9, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{S}, \bar{E}\}$. By Theorem 3.8, σ_{qv} contains $\tau_2 = \{\hat{N}, \hat{W}\}$. If $p \neq q$, we can merge τ_1 with $\{W\}$ by Lemma 4.7. So assume $p = q$. Then $\sigma_{ap'}$ contains $\tau_3 = \{\tilde{U}, \tilde{S}, \tilde{E}, \tilde{N}^{+}\}$, by the induction hypothesis. By Lemma 4.9, we can merge τ_2 and τ_3 to give a canonical sequence that contains the desired labels.

Case 3. p **is in the UNW octant** By Theorem 3.9, σ_{ap} has $\tau_1 = \{\bar{U}, \bar{N}, \bar{W}\}$. Observe that $p \neq q$, because q is in the USE octant. Consider σ_{pq}. Because p is in the UNW octant and q is in the USE octant, σ_{pq} must contain an east label $\hat{E}$. If $\overline{pp'}$ is the only east label in σ_{pq} then $\overline{pp'}$ must intersect the UN quadrant to reach q in USE octant. Then we are done by the induction hypothesis on $\sigma_{ap'}$. So assume that $\hat{E} \neq \overline{pp'}$. By Lemma 4.7, $\tau = \tau_1 \cup \hat{E}$ is canonical.

Case 4. p **is in the USW octant** By Theorem 3.9, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{S}, \bar{W}\}$. Because p is in the USW octant and u is in the UNE octant, σ_{pu} contains $\tau_2 = \{\hat{N}, \hat{E}\}$. There are three cases to consider:

- $\overline{pp'} \notin \tau_2$: By Lemma 4.7, $\tau_1 \cup \tau_2$ is canonical.

- $\overline{pp'} = \hat{N}$: To reach q in the USE octant from p, σ_{pq} must intersect either the UN quadrant or the US quadrant orthogonally since no edge in F_{pv} intersects the U axis. If σ_{pq} intersects the UN quadrant orthogonally, we are done by the induction hypothesis. So assume σ_{pq} intersects the US quadrant, and let $\overline{rr'}$ denote an edge in σ_{pq} that enters the USE octant by intersecting the US quadrant orthogonally. Observe that $\overline{pp'} = \hat{N}$ and $\overline{rr'} = E$, so $p \neq r$. Then u is NE of r, so σ_{ru} contains $\{\tilde{N}, \tilde{E}\}$. By Property 4.6 applied to τ_1, $\{\bar{U}, \bar{W}\}$ is canonical. Finally, by Lemma 4.7, $\{\tilde{N}, \tilde{E}, \bar{U}, \bar{W}\}$ is canonical.

- $\overline{pp'} = \hat{E}$: Because σ_{pv} constitutes only one flat, $\overline{pp'} = \hat{E}$ implies that the succeeding edge $\overline{p'p''} = \hat{N}$. Consider the subpath $\sigma_{ap'}$. If $\overline{pp'}$ intersects the US quadrant orthogonally, $\sigma_{ap'}$ has $\tau_3 = \{\bar{U}, \bar{S}, \bar{E}, \bar{W}^{+}\}$, by the induction hypothesis. Then, we can merge τ_2 and τ_3 by Lemma 4.9 to obtain a canonical sequence that contains the desired labels. If $\overline{pp'}$ does not intersect the US quadrant, then $\sigma_{p'u}$ has $\tau_4 = \{\tilde{N}, \tilde{E}\}$. By Lemma 4.7, $\tau = \tau_1 \cup \tau_4$ is canonical.

4.3.1.2 p as a point of a quadrant

Here we consider cases where p is in some quadrant. By the general position assumption, a and p must lie in the same flat. Because p is the first vertex of the flat containing $\overline{uv}$, $\overline{pp'}$ is a transition element, and there are two cases: (i) There is a flat F_{ap} that starts at a and ends at p. Then there is a light flat between F_{ap} and F_{pv}; (ii) There is a flat $F_{ap'}$ that starts at a and ends at p'.

Now in cases 5-8, we consider the 4 quadrants in which p could lie.

Case 5. p is in the UE quadrant Because p is in UE quadrant, by Theorem 3.9, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{E}\}$, and σ_{pv} contains $\tau_2 = \{\hat{N}, \hat{W}\}$. If σ is case (i), $\tau_1 \cup \tau_2$ is canonical, by Lemma 4.8. So, assume case (ii) holds. Then, $\overline{pp'}$ can be either E or W.

- $\sigma_{pp'} \notin \tau_2$: By Lemma 4.7, $\tau_1 \cup \tau_2$ is canonical.
- $\sigma_{pp'} \in \tau_2$: Then $pp' = \hat{W}$. If $\overline{pp'}$ does not intersect the U axis, $\sigma_{p'v}$ contains $\tau_3 = \{\tilde{N}, \tilde{W}\}$, and $\tau_1 \cup \tau_3$ is canonical by Lemma 4.7. If $\overline{pp'}$ does intersect the U axis, $\sigma_{ap'}$ contains $\tau_4 = \{\check{U}, \check{E}, \check{W}\}$ by Lemma 4.10. Then, we can merge τ_2 and τ_4 by Lemma 4.9 to obtain a canonical sequence that contains the desired direction labels.

Case 6. p is in the UW quadrant Because p is in UW quadrant, by Theorem 3.9, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{W}\}$, and σ_{pu} contains $\tau_2 = \{\hat{N}, \hat{E}\}$. If σ is case (i), $\tau_1 \cup \tau_2$ is canonical, by Lemma 4.8. So, assume case (ii) holds. Then, $\overline{pp'}$ can be either E or W.

- $\sigma_{pp'} \notin \tau_2$: By Lemma 4.7, $\tau_1 \cup \tau_2$ is canonical.

- $\sigma_{pp'} \in \tau_2$: Then $pp' = \hat{E}$. If $\overline{pp'}$ does not intersect the U axis, $\sigma_{p'u}$ contains $\tau_3 = \{\tilde{N}, \tilde{E}\}$, and $\tau_1 \cup \tau_3$ is canonical by Lemma 4.7. If $\overline{pp'}$ does intersect the U axis, $\sigma_{ap'}$ contains $\tau_4 = \{\check{U}, \check{E}, \check{W}\}$ by Lemma 4.10. Then, we can merge τ_2 and τ_4 by Lemma 4.9 to obtain a canonical sequence that contains the desired direction labels.

Case 7. p is in the US quadrant By Theorem 3.9, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{S}\}$. Furthermore, σ_{pv} contains $\tau_2 = \{\hat{E}, \hat{N}, \hat{W}\}$ by Lemma 4.10. If σ is in case (i), $\tau_1 \cup \tau_2$ is canonical, by Lemma 4.8. So, assume case (ii) holds. Then, $\overline{pp'}$ is either N or S. Therefore, $\overline{pp'}$ is neither $\hat{E}$ nor $\hat{W}$. By Lemma 4.7, $\tau_1 \cup \tau_2$ is canonical.

Case 8. p is in the UN quadrant By Theorem 3.9, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{N}\}$. Depending on where $\overline{uv}$ intersects the UN quadrant, there are two cases:

1. p is south of $\overline{uv}$: By Lemma 4.10, σ_{pv} contains $\tau_2 = \{\hat{E}, \hat{N}, \hat{W}\}$. If σ is in case (i), $\{\bar{U}\} \cup \tau_2$ is canonical by Lemma 4.8. If σ is in case (ii), then $\overline{pp'}$ is either N or S. Thus, $\overline{pp'}$ is neither $\hat{E}$, nor $\hat{W}$. Furthermore, since τ_2 belongs to the flat F_{pv}, $\hat{N}$ must appear between $\hat{E}$ and $\hat{W}$, so $\overline{pp'}$ cannot be $\hat{N}$. Hence, $\overline{pp'} \notin \tau_2$. By Lemma 4.7, $\{\bar{U}\} \cup \tau_2$ is canonical.

2. p is north of $\overline{uv}$: By Lemma 4.10, σ_{pv} contains $\tau_3 = \{\tilde{E}, \tilde{S}, \tilde{W}\}$. If σ is in case (i), then $\tau_1 \cup \tau_3$ is canonical by Lemma 4.8. If σ is in case (ii), then $\overline{pp'}$ is either N or S. Thus, $\overline{pp'}$ is neither $\tilde{E}$ nor $\tilde{W}$. Furthermore, since τ_3 belongs to the flat F_{pv} it follows that $\tilde{S}$ must appear between $\tilde{E}$ and $\tilde{W}$, so $\overline{pp'}$ cannot be $\tilde{S}$. Hence, $\overline{pp'} \notin \tau_3$. By Lemma 4.7, $\tau_1 \cup \tau_3$ is canonical.

4.3.1.3 p as a point in an axis

The following case 9 completes the proof.

Case 9. p is in the U axis σ_{ap} contains $\tau_1 = \{\bar{U}\}$. By Lemma 4.10, σ_{pv} contains $\tau_2 = \{\hat{N}, \hat{E}, \hat{W}\}$. Recall that p is the first vertex of the flat F_{pv}. If there is a flat F_{ap} that starts at a and ends at p, then there must be a light flat consisting of the last element of F_{ap} and the first element of F_{pv}. Then, $\tau_1 \cup \tau_2$ is canonical, by Lemma 4.8. So, assume that the first flat of σ starts at a and ends at p' ($F_{ap'}$); and the next flat F_{pv} follows. We then have the following subcases.

- $\overline{pp'} \notin \tau_2$: By Lemma 4.7, $\tau_1 \cup \tau_2$ is canonical.
- $\overline{pp'} = \hat{N}$: Since τ_2 belongs to a flat F_{pv}, $\hat{N}$ must appear between $\hat{E}$ and $\hat{W}$, and so $\overline{pp'}$ cannot be $\hat{N}$.
- $\overline{pp'} = \hat{E}$: Because $\overline{pp'}$ is in the same flat as a, σ_{ap} contains $\tau_3 = \{\tilde{U}, \tilde{E}\}$ by Theorem 3.8. Then, by Lemma 4.9, we can merge τ_2 and τ_3 to obtain a canonical sequence that contains the desired direction labels.
- $\overline{pp'} = \hat{W}$: Because $\overline{pp'}$ is in the same flat as a, σ_{ap} contains $\tau_4 = \{\tilde{U}, \tilde{W}\}$ by Theorem 3.8. By Lemma 4.9, we can merge τ_2 and τ_4 to obtain a canonical sequence that contains the desired direction labels.

□

4.3.1.4 Consequence

The following result is a direct consequence of Lemmas 3.8, 4.8, and 4.11.

Lemma 4.13. *Let σ be a shape cycle with three consecutive flats F^-, F, and F^+ such that F^- and F^+ are both light. If the first and the last vertex of F are in a quadrant-relation, then σ contains a canonical sequence of length 6.*

Proof. Let a denote the first vertex of F, and let b denote the last vertex of F. Since a and b are in a quadrant-relation, assume, without loss of generality, that σ_{ab} contains $\tau_1 = \{\bar{N}, \bar{E}\}$ by Theorem 3.8. Then σ_{ba} must contain $\tau_2 = \{\hat{D}, \hat{S}, \hat{W}, \hat{U}^+\}$ by Lemma 4.11. If τ_2 contains all 6 labels, we are done. Otherwise, τ_2 together with the element(s) of τ_1 having the remaining direction label(s) is canonical by Lemma 4.8. $\square$

4.4 Shortcomings of Previous Work

In this section, we describe the proof technique used to prove Theorem 3.10 in [9], and point out the shortcomings of the approach. Theorem 3.10 first appeared in [10], and its proof was given in an unpublished manuscript [9]. It was later discovered by the original authors, however, that the necessity part of the proof was incomplete. In what follows, we sketch the necessity proof for Theorem 3.10, as presented in [9]. Then, we describe why the proof fails to show the necessity of Theorem 3.10 for shape cycles with 4-10 flats.

4.4.1 Proof Sketch as given by Di Battista *et al.*

Suppose Γ is a simple orthogonal drawing of a given shape cycle σ. One wishes to show that σ contains a canonical sequence of length six. For cases where σ contains fewer than four flats, a straightforward case analysis can be used to prove the necessity. Now, consider the case where σ contains four or more flats. Because there are only three different orientations of flats (e.g. NSEW,NSUD, EWUD), by the pigeon hole principle, there exists at least two flats F_a and F_b that are parallel.

Note that Γ may be perturbed so that F_a and F_b do not belong to the same isothetic plane. Hence one can assume that F_a and F_b are drawn on two distinct parallel planes. Now, consider the first edge $\overline{aa'}$ of F_a and the first edge $\overline{bb'}$ of F_b.

Note that these two edges are transition elements. Using the relations defined in Chapter 3, one can analyze the relationship between the two transition elements $\overline{aa'}$ and $\overline{bb'}$. The authors distinguished four cases as follows.

Proposition 4.14. [9] *If $a, a', b,$ and b' are such that $a \sim_b a'$, and $b \sim_a b'$, then a canonical sequence of length six exists for σ.*

Proposition 4.15. [9] *If $a, a', b,$ and b' are such that $a \sim_b a'$, and $b \not\sim_a b'$, then a canonical sequence of length six exists for σ.*

Proposition 4.16. [9] *If $a, a', b,$ and b' are such that $a \not\sim_b a'$, and $b \sim_a b'$, then a canonical sequence of length six exists for σ.*

Proposition 4.17. [9] *If $a, a', b,$ and b' are such that $a \not\sim_b a'$, and $b \not\sim_a b'$, then a canonical sequence of length six exists for σ.*

We omit repeating the proofs for the above propositions. As a consequence of the above propositions, the authors concluded that every shape cycle σ with four or more flats contains a canonical sequence of length six.

4.4.2 Historical Notes & Contribution Statements

When using the relations between entities as defined in Chapter 3, one must be careful not to make unsafe assumptions. Recall the definition of equivalence between two vertices a, a' with respect to a third vertex b, as stated in Definition 3.7. Observe that when a and a' are not equivalent with respect to b, it does not necessarily imply that a and a' are in different octant-relations with respect to b. For example, suppose b is located at the origin, and a is in the UNE octant. If a' is in some octant other than the UNE octant, we say that a and a' are not equivalent with respect to b. Also, if a' is in some quadrant, or in some axis, we still say that a and a' are not equivalent with respect to b. In other words, when two vertices a, a' are not equivalent with

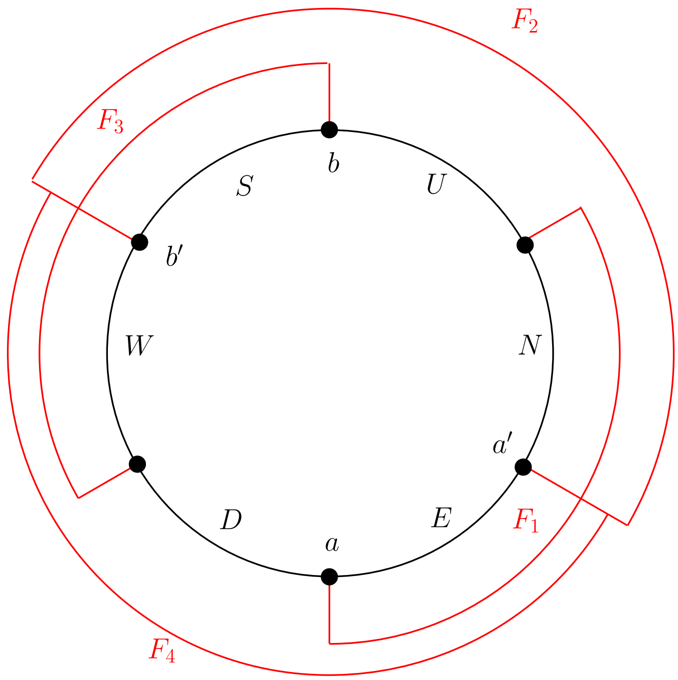

Figure 4.2: A counterexample for the assumption made in the proof of Propositions 4.14-4.17

respect to another vertex b , it simply means they do not have the *same relations* with respect to b. This is indeed what was overlooked by the original authors, as they assumed that each of a and a' is in an octant-relation with each of b and b'. Their proofs for Propositions 4.14-4.17 assume that the two transition elements $\overline{aa'}$ and $\overline{bb'}$ are well-related, and ignore the cases where there is a flat containing a vertex from $\{a, a'\}$ and a vertex from $\{b, b'\}$. They eventually discovered this and found an example to show this possibility must be considered. Here, we give our own example.

Example 4.18. *The shape cycle $\sigma = ENUSWD$ contains 4 flats: $F_1 = EN$, $F_2 = NUS$, $F_3 = SW$, and $F_4 = WDE$. F_1 and F_3 are the only parallel flats. Then the first element of F_1 and the first element of F_3 are not well-related.*

See Figure 4.2 for an illustration of σ. The two flats F_1 and F_3 are the only parallel flats. Let the two transition elements $(a, a') = \sigma_1 = E$, and let $(b, b') = \sigma_4 = S$. Then, F_2 contains both a' and b, and therefore a' and b are not in an octant-relation.

Moreover, the oversight in the original manuscript was particularly problematical in the case of the proof of Proposition 4.17. Not only the first edges from F_a and F_b were assumed to be well-related, but also each of the first *and the last* edges of F_a and each of the first and the last edges of F_b were assumed to be well-related. This is false if the two parallel flats are separated by only one flat. The fallacy of this assumption can also be seen in Example 4.18.

In order to fix these problems, the original authors looked for a condition in which there are two parallel flats such that the transition elements in those flats are pairwise well-related. In particular, they looked for 4 parallel flats. Suppose a shape cycle σ contains four parallel flats F_a, F_b, F_c, and F_d, appearing in σ in that order. Observe that F_a and F_c are separated by another parallel flat in both ways. Then, each of the two transition elements of F_a and each of the two transition elements of F_c are well-related, as required. (This fact can be generalized further. See Lemma 5.2 in Chapter 5.) By choosing F_a and F_c, one can use Propositions 4.14-4.17 with the original assumption that each of a and a' is in an octant-relation with each of b and b'. Hence, if a shape cycle with a simple orthogonal drawing contains 4 or more parallel flats, there is a canonical sequence of length 6. As the original authors noted, because there are 3 different orientations of a flat, by the pigeon hole principle, a shape cycle with 10 or more flats must contain 4 parallel flats. However, shape cycles with fewer than 10 flats, on the other hand, are not guaranteed to contain 4 parallel flats. Hence the gap from 4 to 9 flats remains open.

The authors began attempts to fill this gap via a rigorous case analysis based on the concept of well-related transition elements from two parallel flats. At this point, the author of this thesis began his research by carrying out a lengthy case analysis on shape cycles with 4 flats. Later, it was conjectured that it is unnecessary to have the condition that the two transition edges aa' and bb' come from two parallel flats. In the next chapter, we study the properties of well-related transition elements, and how they can help us prove the necessity of Theorem 3.10, in all cases.

4.5 Bibliographic Notes

In this chapter, we studied fundamental properties of shape paths and cycles. While many results in this chapter were initially stated and proved by Di Battista *et al.* [9, 10, 11], here we gave proofs that are more concise. For example, Lemma 4.2 first appeared and used extensively in [9]. The proof of the lemma, however, did not appear in [10] nor in the unpublished manuscript [9]. We note that the proof given in Section 4.1 is an independent effort by the author.

Lemma 4.9 was first proposed and proved in [9], but the original version of the lemma only handled the cases where $\tau_1 \cap \tau_2 = \sigma_k$. The statement in Lemma 4.9 is now more general, and handles the cases where σ_k is not necessarily in τ_1 or τ_2, and furthermore, there may be direction labels appearing in both τ_1 and τ_2 that are different from that of σ_k. This new, more general version of this lemma effectively combines the original version with other lemmas. While the statement of the new version becomes more complex, the new lemma gives a powerful tool to provide succinct proof of Lemma 4.11 and many other results presented in subsequent chapters.

Properties 4.4-4.6 and Lemmas 4.7-4.8 first appeared and proved in [9]. Lemmas 4.10 and 4.11 first appeared in [10]. However, the statement of Lemma 4.11 in [10] is different from Lemma 4.11 in this chapter. In [10], it was claimed that every shape path that intersects the XY-quadrant orthogonally contains a canonical sequence of length 4 with the direction labels $\{X, Y, Z, Z'\}$. This was later discovered by the original authors to be false, as discussed in Section 4.3. Then the correct statement, as stated in this chapter, was given and proved in [9]. The proof presented in Section 4.3 has been rewritten for conciseness and simplification, while the overall framework of the proof is heavily borrowed from the original proof.

Chapter 5

Well-related Transition Elements

In this chapter, we develop a necessary condition for shape cycles to have a simple orthogonal drawing. Recall from the previous chapter that the initial attempt to prove the necessity of Theorem 3.10 was proceeded by looking for two parallel flats whose transition elements are well-related. As it turned out, such parallel flats can only be guaranteed to exist in shape cycles with 10 or more flats, leaving the smaller cases uncovered. We will remedy this problem by studying well-related transition elements, without the requirement that they must belong to two parallel flats.

5.1 Properties of Well-related Transition Elements

We first study some basic properties of shape cycles that contain well-related transition elements. Recall the definition of well-related transition elements: two transition elements aa' and bb' are said to be *well-related* if no flat contains a vertex from $\{a, a'\}$ and a vertex from $\{b, b'\}$. Our first result is a characterization of well-related transition elements in terms of the number of flats between them.

Definition 5.1. *We say a flat F is a* separating flat from aa' to bb' *if all the elements* *in F occur in $\sigma_{ab'}$.*

Observe that the number of separating flats from aa' to bb' may be different from the number of separating flats from bb' to aa'. Moreover, notice that, for any pair of transition elements aa' and bb', the sum of the number of separating flats from aa' to bb' and the number of separating flats from bb' to aa' is equal to the total number of flats in the shape cycle. Now we are ready to characterize well-related transition elements.

Lemma 5.2. (Spacing Property) *Let aa' and bb' be two transition elements in a shape cycle σ. Let $\mathcal{F}$ be the sequence of separating flats from aa' to bb', and let $\mathcal{F}'$ be the sequence of separating flats from bb' to aa'. Then, aa' and bb' are well-related if and only if, for each of the sequences $\mathcal{F}$ and $\mathcal{F}'$, one of the following conditions holds:*

(1) the sequence consists of 2 flats F_1 and F_2 such that they are both heavy, or

(2) the sequence consists of 3 flats F_1, F_2 and F_3 such that either F_1 or F_3 is heavy, or

(3) the sequence consists of at least 4 flats.

Proof. Due to symmetry, it suffices to show that aa' and bb' are well-related if and only if no flat in $\mathcal{F}$ contains both a' and b. See Figure 5.1.

(1) There are two flats F_1 and F_2 in $\mathcal{F}$: If one of the two flats is light, the other flat contains both a' and b. On the other hand, if F_1 and F_2 are both heavy, neither flat can contain both a' and b.

(2) There are three flats F_1, F_2 and F_3 in $\mathcal{F}$: If the two flats F_1 and F_3 are both light, the flat F_2 must contain both a' and b. On the other hand, if either F_1 or F_3 is heavy, none of the three flat can contain both a' and b.

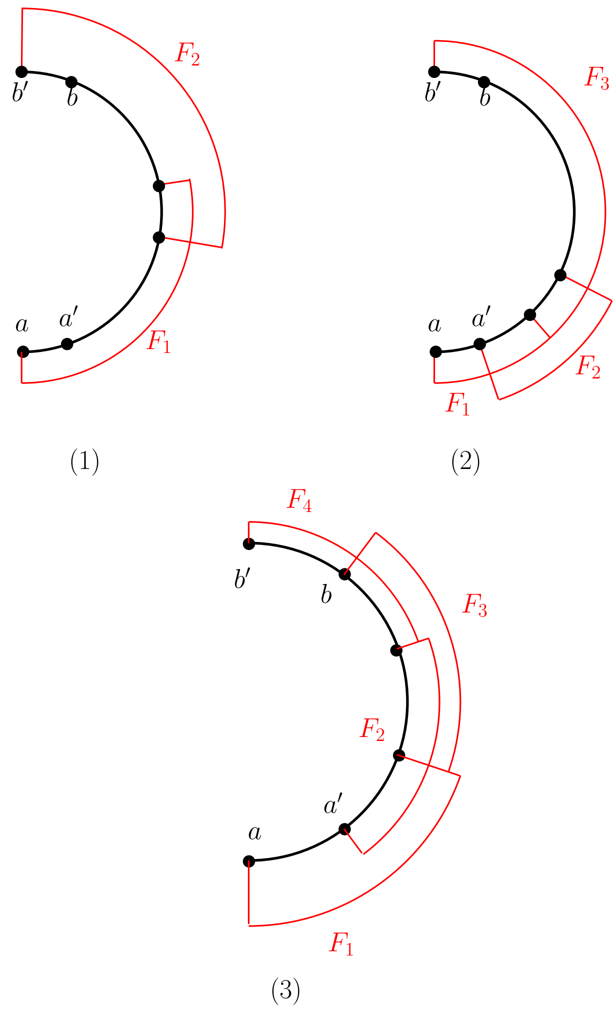

Figure 5.1: Separating flats from aa' to bb': (1) Two heavy flats; (2) Three flats with one heavy extreme flat; (3) Four or more flats

(3) There are four or more flats in $\mathcal{F}$: Suppose that there exists a flat containing both a' and b. Then, at most two other separating flats can exist from aa' to bb'. This implies that there are at most three flats in $\mathcal{F}$. This is a contradiction.

$\square$

Using this characterization, we can establish bounds on the number of flats in a shape cycle with well-related transition elements. If a shape cycle has fewer than 4 flats, it is impossible to find two transition elements that are well-related. On the other hand, if a shape cycle contains more than or equal to 8 flats, there must exist a pair of transition elements aa' and bb' such that there are at least 4 separating flats from aa' to bb', and similarly from bb' to aa'. Therefore, there is a pair of well-related transition elements. It should be noted, however, that these bounds are not sharp. A more careful analysis will be done in Chapter 6.

It is also useful to know if there exist any other pairs of well-related transition elements in the given shape cycle. In the following lemma, we show that if there exists a pair of well-related transition elements, there is at least one other pair of well-related transition elements.

Lemma 5.3. *Let aa', bb' be two well-related transition elements in σ. Then there exists another pair of transition elements pp', qq' that are well-related to each other. In particular, these two pairs of transition elements appear in alternating order within σ (i.e. aa', pp', bb', qq').*

Proof. For illustrations of the following cases, see Figure 5.2.

1. Suppose that there are 2 heavy separating flats F_1 and F_2 from aa' to bb'. Then, we choose the shared transition element between F_1 and F_2, and call it pp'. There are the following subcases to consider:

   (a) If there are 2 heavy separating flats F_3 and F_4 from bb' to aa', choose the shared transition element between F_3 and F_4, and call it qq'. Then pp' and qq' are well-related.

   (b) Suppose that there are 3 separating flats F_3, F_4, and F_5 from bb' to aa', and either F_3 or F_5 is a heavy flat. Then, choose the transition element in the heavy flat that is different from aa' and bb', and call it qq'. Then, pp' and qq' are well-related.

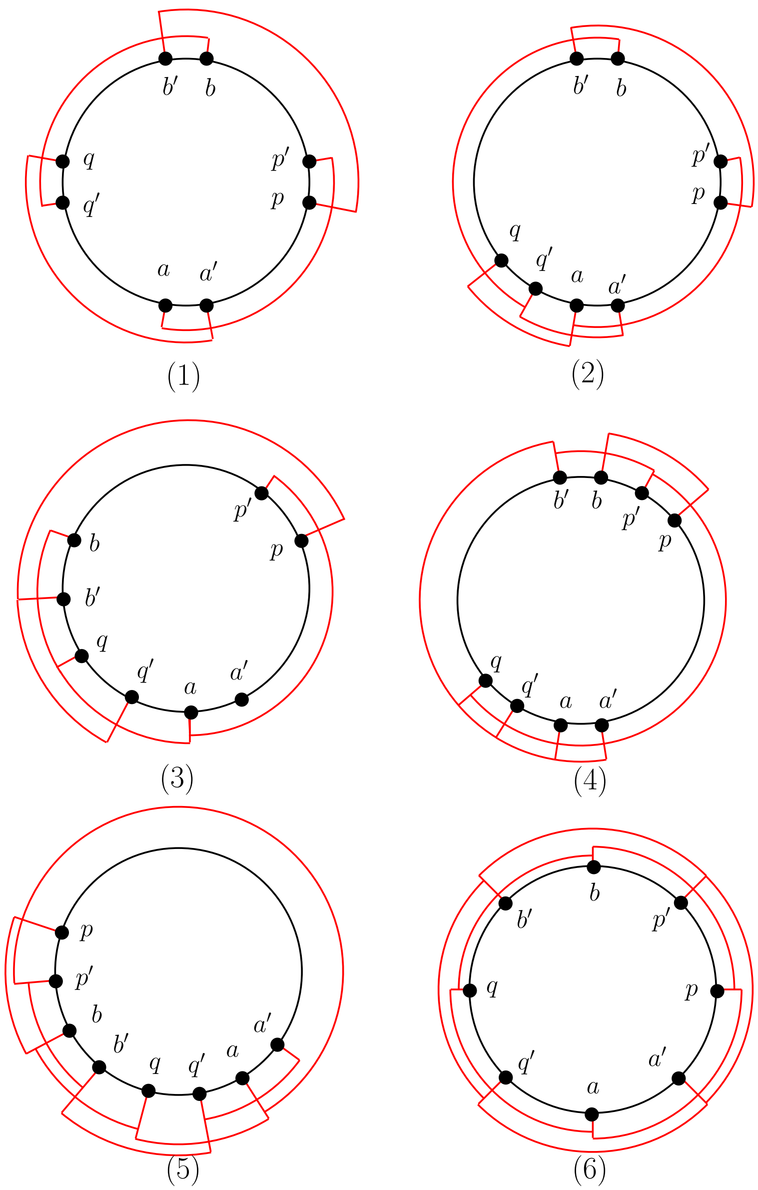

Figure 5.2: Finding another pair of well-related transition elements: $(aa', bb'), (pp', qq')$ are well-related pairs. (1) 2 heavy separating flats from aa' to bb' and 2 heavy separating flats from bb' to aa'; (2) 2 heavy separating flats from aa' to bb' and 3 separating flats (with a heavy extreme flat) from bb' to aa'; (3) 2 heavy separating flats from aa' to bb' and 4 separating flats from bb' to aa'; (4) 3 separating flats (with a heavy extreme flat) from aa' to bb' and 3 separating flats (with a heavy extreme flat) from bb' to aa'; (5) 3 separating flats (with a heavy extreme flat) from aa' to bb' and 4 separating flats from bb' to aa'; (5) 4 separating flats from aa' to bb' and 4 separating flats from bb' to aa'

(c) If there are 4 or more separating flats from bb' to aa', choose the transition element qq' in $\sigma_{ba'}$ such that there are 2 separating flats from bb' to qq'. Then, pp' and qq' are well-related.

2. Suppose that there are 3 separating flats F_1, F_2 and F_3 from aa' to bb' such that either F_1 or F_3 is heavy. Then, choose the transition element of the heavy flat that is different from aa' and bb', and call it pp'. There are the following subcases to consider:

   (a) Suppose that there are 3 separating flats F_4, F_5, and F_6 from bb' to aa' such that either F_4 or F_6 is heavy. Then, choose the transition element in the heavy flat that is different from aa' and bb', and call it qq'. Then pp' and qq' are well-related.

   (b) If there are 4 or more separating flats from bb' to aa', choose the transition element qq' in $\sigma_{ba'}$ such that there are 2 separating flats from bb' to qq'. Then pp' and qq' are well-related.

3. Finally, suppose that there are 4 or more separating flats from aa' to bb', and 4 or more separating flats from bb' to aa'. Choose the transition element pp' in $\sigma_{ab'}$ such that there are two separating flats from aa' to pp'. Similarly, choose the transition element qq' in $\sigma_{ba'}$ such that there are two separating flats from bb' to qq'. Then pp' and qq' are well-related.

$\square$

5.2 A Necessary Condition

In this section, we study properties of shape cycles that admit simple orthogonal drawings and contain two well-related transition elements. In particular, we develop a necessary condition for such shape cycles to contain a canonical sequence of length six. We will distinguish various cases based on the relationship between the vertices of the well-related transition elements in the drawing.

Lemma 5.4. *Let σ be a shape cycle with a simple orthogonal drawing Γ, and let aa', bb' be two well-related transition elements of σ. If $a \sim_b a'$, or $a \sim_{b'} a'$, or $b \sim_a b'$, or $b \sim_{a'} b'$, then σ contains a canonical sequence of length 6.*

Proof. Suppose a and a' are equivalent with respect to b (i.e. $a \sim_b a'$). All other cases are symmetric, and can be shown by the same arguments after exchanging a's and b's.

Without loss of generality, assume b is UNE of a. This implies that b is UNE of a' as well, so there exists $\tau_1 = \{\bar{U}, \bar{N}, \bar{E}\}$ in $\sigma_{a'b}$ and $\tau_2 = \{\hat{D}, \hat{S}, \hat{W}\}$ in σ_{ba}. If τ_2 is contained in $\sigma_{b'a}$, we are done by applying Lemma 4.7 on τ_1 and τ_2. So, it must be that $bb' \in \tau_2$, and thus the label X on bb' is in $\{D, S, W\}$. If $b \sim_a b'$, then $\sigma_{b'a}$ must contain a canonical sequence of type $\{D, S, W\}$, and we are again done by Lemma 4.7. So we assume that $b \not\sim_a b'$. Then, by Lemma 4.11, $\sigma_{ab'}$ contains a canonical sequence $\tau_3 = \{\tilde{U}, \tilde{N}, \tilde{E}, \tilde{X}^+\}$, where X belongs to $\{D, S, W\}$. Now, there are two cases to consider.

Case 1. $b \not\sim_{a'} b'$**:** Since b is UNE of a', $\sigma_{a'b'}$ must contain $\tau_4 = \{\check{U}, \check{N}, \check{E}, \check{X}^+\}$ by Lemma 4.11. Then we can merge τ_2 and τ_4 by Lemma 4.9.

Case 2. $b \sim_{a'} b'$**:** Then the relations between vertices can be summarized as follows.

- b is UNE of a.
- b is UNE of a'.
- b' is *not* UNE of a.

- b' is UNE of a'.

Since b' is UNE of a', but not UNE of a, aa' must be one of $\{D, S, W\}$, like bb'. Furthermore, the direction label for aa' must be the same direction label X as that of bb'; otherwise it would be that $a \sim_{b'} a'$, which contradicts the assumption. Therefore, only one of aa' and bb' can belong to τ_3. Now, there are the following three subcases.

1. $aa' \notin \tau_3$ and $bb' \notin \tau_3$: Remove from τ_2 the direction labels appearing in τ_3 and call the result τ_2'. By Property 4.6, τ_2' is canonical. Then, by Lemma 4.7, $\tau_2' \cup \tau_3$ is canonical.

2. $aa' \in \tau_3$ and $bb' \notin \tau_3$: Since b' is UNE of a', $\sigma_{b'a'}$ contains $\tau_5 = \{\dot{D}, \dot{S}, \dot{W}\}$. Then we can merge τ_3 and τ_5 by Lemma 4.9.

3. $aa' \notin \tau_3$ and $bb' \in \tau_3$: Then we can merge τ_2 and τ_3 by Lemma 4.9.

This completes the proof. □

Observe that Lemma 5.4 considers all cases where, given a pair of well-related transition elements, two vertices in one transition element are equivalent with respect to a vertex in the other transition element. Next, we study the case where no such equivalence exists for any pair of well-related transition elements in the shape cycle. By proving Lemma 5.4 before Lemma 5.5, we can safely disregard situations where such equivalence does exist.

Lemma 5.5. *Let σ be a shape cycle with a simple orthogonal drawing Γ such that each and every pair of transition element aa' and bb' satisfies the following: $a \not\sim_b a'$, and $a \not\sim_{b'} a'$, and $b \not\sim_a b'$, and $b \not\sim_{a'} b'$. Then, there exists a canonical sequence of length 6.*

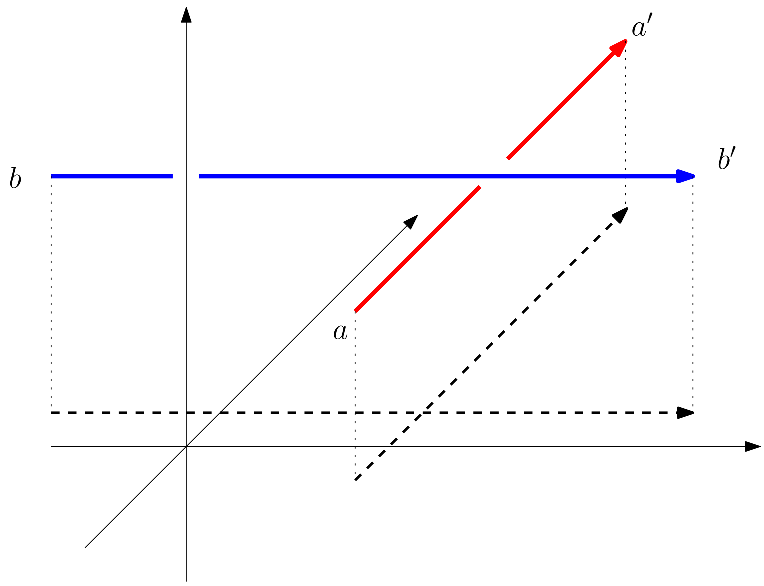

Figure 5.3: Drawing of two well-related transition elements aa' and bb' considered in Lemma 5.5; in order to satisfy the conditions in the lemma, the two edges must cross perpendicularly when projected onto the plane parallel to $\overline{aa'}$ and $\overline{bb'}$.

Proof. Geometrically, these conditions assert that the line segments $\overline{aa'}$ and $\overline{bb'}$ are perpendicular to each other, and their projections onto an axis-perpendicular plane cross each other. For example, see Figure 5.3.

Assume, without loss of generality, that b is UNE of a. Since $a \not\sim_b a'$, $aa' \in \{U, N, E\}$. Similarly, since $b \not\sim_a b'$, we have $bb' \in \{D, S, W\}$.

By Lemma 5.3, there is at least one other pair of well-related transition elements. Let pp', qq' denote the other pair of well-related transition elements. By assumption, this new pair of well-related transition elements must also satisfy the conditions stated in this lemma. Since there are three different axes to which an edge could be parallel (i.e. NS, EW, UD), two of these four transition elements must be parallel. Because aa' cannot be parallel to bb' by assumption, we assume, without loss of generality, that $\overline{aa'}$ and $\overline{pp'}$ are parallel. If $\overline{aa'}$ and $\overline{pp'}$ are well-related, we are done by Lemma 5.4. So assume that $\overline{aa'}$ and $\overline{pp'}$ are not well-related, i.e., some flat contains a vertex from $\{a, a'\}$, and a vertex from $\{p, p'\}$. Since $\overline{aa'}$ and $\overline{pp'}$ are parallel, there must exist a

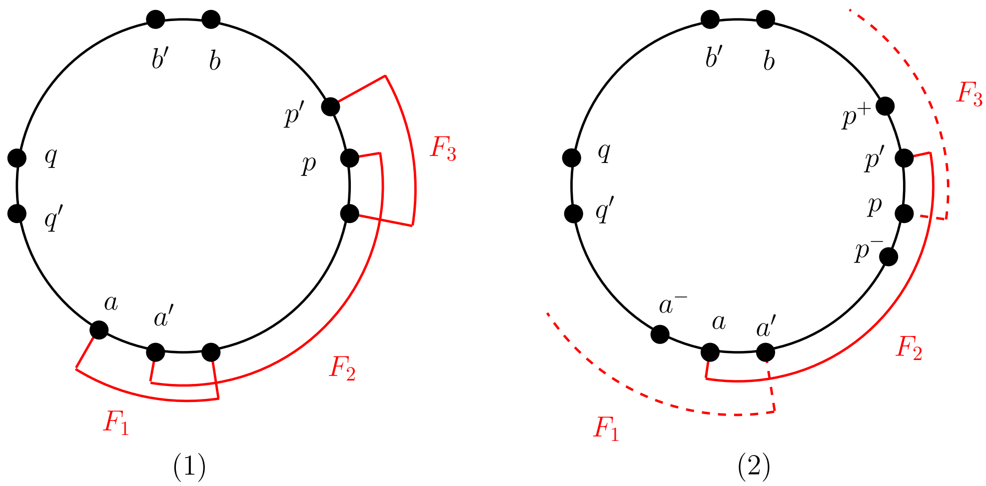

Figure 5.4: Two types of flats between $\overline{aa'}$ and $\overline{pp'}$. (1) Type I: F_2 starts at a' and ends at p. (2) Type II: F_2 starts at a and ends at p'.

flat that either (i) starts at a' and ends at p (Type I), or (ii) starts at a and ends at p' (Type II). Let F_2 denote this flat, and let F_1 and F_3 denote the preceding and succeeding flats, respectively. See Figure 5.4 for these two cases.

Suppose σ is Type I. By Lemma 4.2, a and p' are in an octant-relation. Since $\overline{aa'}$ is parallel to $\overline{pp'}$, a' is in a quadrant-relation with p. Then we are done by Lemma 4.13.

Now, suppose σ is Type II. See Figure 5.4(2) for a combinatorial illustration. First, observe that F_2 must be heavy, since the two transition elements $\overline{aa'}$ and $\overline{pp'}$ are parallel. Assume, without loss of generality, that F_2 is a NSEW flat, and $\overline{aa'}$ and $\overline{pp'}$ both run parallel to the NS axis. Then, it follows that F_1 and F_3 are both NSUD flats. Observe that the two segments $\overline{p'p^+}$ and $\overline{a^-a}$ are both perpendicular to the plane of the flat F_2, so they must be either U or D. Now, consider the vertex a and p' of F_2. Suppose a and p' are in an axis-relation. Then, p', p^+, a^-, a must all lie on an axis-perpendicular plane. By the general position assumption, there must then exist a flat that contains all of p', p^+, a^-, and a. This implies that $bb' = p'p^+$ and $qq' = a^-a$. This is impossible because aa' and bb' are well-related, and pp' and qq' are well-related.

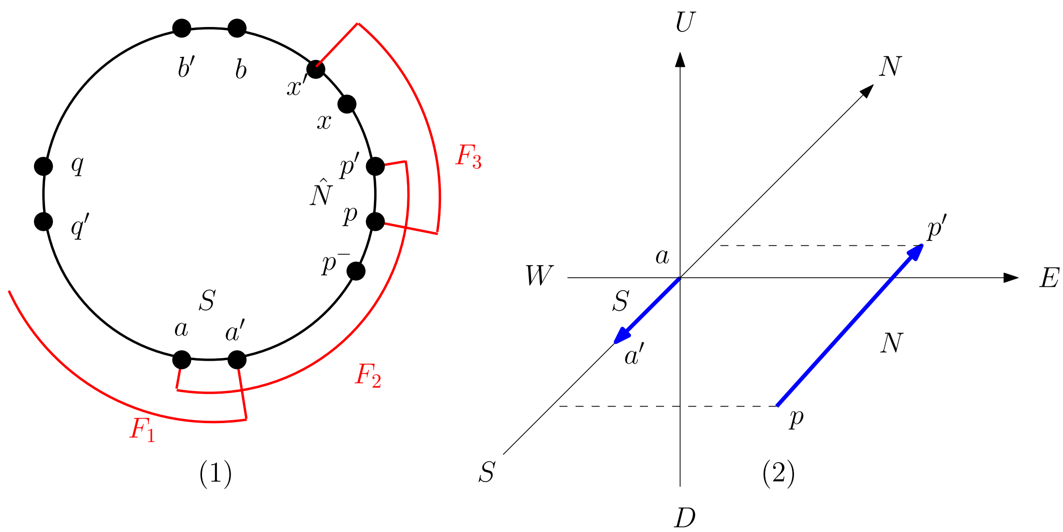

Figure 5.5: (1) Shape cycle σ with $aa' = S, pp' = N$; p' is NE of a, but p is not NE of a'. (2) This implies that the projection of $\overline{pp'}$ onto a NSUD plane completely contains the projection of $\overline{aa'}$.

So, a and p' must be in a quadrant-relation. Assume, without loss of generality, that p' is NE of a. By Lemma 4.11, $\sigma_{p'a}$ contains $\tau_3 = \{\bar{D}, \bar{S}, \bar{W}, \bar{U}^+\}$. Furthermore, by Theorem 3.8, $\sigma_{ap'}$ contains $\tau_4 = \{\hat{N}, \hat{E}\}$. If τ_4 is contained in $\sigma_{a'p}$, then $\tau_3 \cup \tau_4$ is canonical by Lemma 4.7. By similar reasoning, we can assume that there is no adjacent NE or EN in $\sigma_{a'p}$. Further, $\hat{N}$ of τ_4 must be either at aa' or pp'. Now, we consider various cases based on direction labels for aa' and pp'.

Case 1: $aa' = pp' = S$. This is impossible since $\hat{N}$ of τ_4 must be either aa' or pp'.

Case 2: $aa' = S$, $pp' = N$. Because there is no adjacent NE or EN in $\sigma_{a'p}$, p is not NE of a'. This implies that, in the drawing Γ of σ, the projection of $\overline{pp'}$ onto an NSUD plane $\mathcal{P}$ must contain the projection of $\overline{aa'}$ on $\mathcal{P}$ completely. See Figure 5.5.

Observe that τ_4 consists of p^-p and pp', and so τ_4 is contained in $\sigma_{a'p'}$. If F_3 is light, $\tau_3 \cup \tau_4$ is canonical by Lemma 4.8. So, we assume that F_3 is heavy. Now, consider the transition element $\overline{xx'}$ of F_3 that is different from $\overline{pp'}$. Because F_2 and

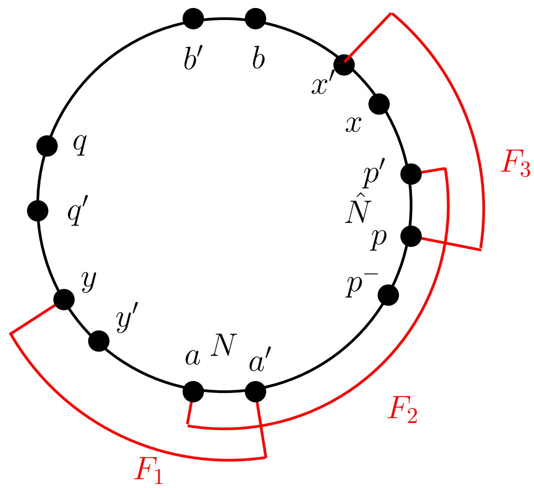

Figure 5.6: Shape cycle σ with $aa' = N, pp' = N$

F_3 are both heavy, $\overline{xx'}$ and $\overline{aa'}$ are well-related. Furthermore, by the conditions in this lemma, $\overline{xx'}$ and $\overline{aa'}$ must also cross in projection onto $\mathcal{P}$. Then, since $\overline{xx'}$ and $\overline{pp'}$ both belong to the same NSUD flat and the projection of $\overline{pp'}$ onto $\mathcal{P}$ contains the projection of $\overline{aa'}$ completely, $\overline{xx'}$ would have to intersect $\overline{pp'}$. This contradicts the assumption that Γ is simple.

Case 3: $aa' = N$, $pp' = S$. This case is symmetric to Case 2.

Case 4: $aa' = pp' = N$. See Figure 5.6. Then, $\hat{N}$ of τ_4 must be at aa' or at pp'. Assume, without loss of generality, that $\hat{N} = pp'$ and $\hat{E} = p^-p$. (The case that $aa' = \hat{N}$ is symmetric.) Then, observe that τ_4 is contained in $\sigma_{a'p'}$. If F_3 is light, $\tau_3 \cup \tau_4$ is canonical by Lemma 4.8. So assume F_3 is heavy. This implies that xx' is well-related to aa', and by the conditions in this lemma, the projection of $\overline{xx'}$ and $\overline{aa'}$ cross perpendicularly. Furthermore, because F_3 is a NSUD flat, xx' must be either U or D.

1. Suppose F_1 is light. Then $y' = a$, and yy' is U or D. So , $\overline{yy'}$ is parallel to $\overline{xx'}$.

If xx' and yy' are well-related, we're done by Lemma 5.4. Hence we assume otherwise. Then we have the following two subcases.

(i) There is a flat F_4 containing $\{x', y\}$, but not $\{x, y'\}$. By Lemma 4.2, x and y' are in an octant-relation, and so x' and y are in a quadrant-relation. Then we are done by Lemma 4.13.

(ii) There is a flat F_4 containing all of $\{x, x', y, y'\}$. Then it must be the case that $yy' = qq'$. Then, the flat F_2 contains q' and p, contradicting that pp' and qq' are well-related.

2. Suppose F_1 is heavy. Then yy' is well-related to pp', and $\overline{yy'}$ must be crossing $\overline{pp'}$ in projection. This implies that yy' is either U or D, and so $\overline{yy'}$ is parallel to $\overline{xx'}$. If $\overline{xx'}$ and $\overline{yy'}$ are well-related, we're done by Lemma 5.4. Hence we assume otherwise. Then we have the following two cases.

   (i) There is a flat F_4 containing $\{x', y\}$, but not $\{x, y'\}$. By Lemma 4.2, x and y' are in an octant-relation, and so x' and y are in a quadrant-relation. Then we are done by Lemma 4.13.

   (ii) There is a flat F_4 containing all of $\{x, x', y, y'\}$. Observe that F_1 and F_3 are two distinct parallel NSUD planes. Therefore, F_4 must be a EWUD plane in order to contain all of $\{x, x', y, y'\}$. Then, it must be that $x \sim_a x'$ and $y \sim_p y'$. This is a contradiction.

□

The above two lemmas can be summarized as follows.

Theorem 5.6. *If a shape cycle σ with a simple orthogonal drawing Γ contains two well-related transition elements, then σ contains a canonical sequence of length 6.*

Proof. This follows immediately from Lemma 5.4 and Lemma 5.5. □

5.3 Chapter Summary

In this chapter, we studied some rudimentary properties of well-related transition elements. Then, we established a necessary condition for drawable shape cycles with a pair of well-related transition elements. In the next chapter, we will complete the proof of necessity that a 3D shape cycle must contain a canonical sequence of length 6 in order to admit a simple orthogonal drawing.

Chapter 6

Proof of Necessity

In this chapter, we complete the proof of Theorem 3.10. For ease of reference, we state below the necessity of the condition in Theorem 3.10 that characterizes shape cycles admitting a simple orthogonal drawing.

Theorem 6.1. *If a 3D shape cycle σ admits a simple orthogonal drawing, then σ contains a canonical sequence of length six.*

We prove Theorem 6.1 by distinguishing different cases based on the number of flats in the shape cycle. For discussions in this chapter, we let A, B, and C denote the three different orientations of flats, i.e., NSEW, NSUD, EWUD. Furthermore, given a shape cycle σ with n flats, we let F_i denote the i^{th} flat in a cyclic ordering of flats in σ.

The following properties provide useful tools for the proof of Theorem 6.1.

Property 6.2. *Let F_i be a flat in a shape cycle σ with a drawing Γ. If F_i lies in between two flats F_{i-1} and F_{i+1} that are parallel in Γ, then F_i must be heavy.*

Proof. Suppose F_i is light. Then, the two elements of F_i are transition elements for

F_i. Since F_{i-1} and F_{i+1} are parallel, it follows that F_i is also parallel to F_{i-1} and F_{i+1}. This is a contradiction. □

Lemma 6.3. *Let F_a and F_b be two light flats in a shape cycle σ with a drawing Γ such that $F_a = a^-aa^+$ and $F_b = b^-bb^+$. If there are at least 2 separating flats from aa^+ to b^-b and at least 2 separating flats from bb^+ to a^-a, then σ contains a canonical sequence of length 6.*

Proof. By Lemma 4.2, a and b are in an octant-relation in Γ. Assume, without loss of generality, that b is UNE of a. Then , σ_{ab} contains $\tau_1 = \{\overline{U}, \overline{N}, \overline{E}\}$, and σ_{ba} contains $\tau_2 = \{\hat{D}, \hat{S}, \hat{W}\}$. By Lemma 4.8, $\tau = \tau_1 \cup \tau_2$ is canonical in σ. □

This result will help us handle several cases while looking at different permutations of flat orientations in the shape cycle. Throughout the rest of this chapter, we assume that σ denotes a 3D shape cycle that admits a simple orthogonal drawing, and Γ denotes a drawing of σ in general position (By Lemma 4.2, σ admits such a drawing.). Now we begin the proof of Theorem 6.1.

6.1 Shape Cycles with 8 or more flats

Let σ be a drawable shape cycle with 8 or more flats. Then, σ must contain 2 transition elements aa' and bb' such that there are at least 4 separating flats from aa' to bb', and there are at least 4 separating flats from bb' to aa'. Then, By Lemma 5.2, aa' and bb' are well-related. By Theorem 5.6, σ contains a canonical sequence of length 6.

6.2 Shape Cycles with 7 flats

Let σ be a drawable shape cycle with exactly 7 flats $F_1, \ldots, F_7$. Suppose all 7 flats in σ are light. Then by applying Lemma 6.3 to, say, F_1 and F_4, there exists a canonical sequence of length 6. Now, assume without loss of generality that F_1 is heavy. Let aa' denote the shared transition element between F_1 and F_2, and let bb' denote the shared transition element between F_5 and F_6. By Lemma 5.2, aa' and bb' are well-related, and thus there exists a canonical sequence of length 6, by Theorem 5.6.

6.3 Shape Cycles with 6 flats

Let σ be a drawable shape cycle with exactly 6 flats $F_1, \ldots, F_6$. If all 6 flats in σ are light, we are done by applying Lemma 6.3 to F_1 and F_4. So we assume F_1 is heavy. Suppose F_2 is also heavy. Then consider the shared transition element aa' between F_6 and F_1, and the shared transition element bb' between F_2 and F_3. By Lemma 5.2, aa' and bb' are well-related. Then, by Theorem 5.6, σ contains a canonical sequence of length 6.

So, assume F_2 is light. By a similar reasoning, F_6 must also be light. If F_3 is light, we are done by applying Lemma 6.3 to F_3 and F_6. So we assume F_3 is heavy. Now, consider the shared transition element bb' between F_2 and F_3, and the shared transition element cc' between F_5 and F_6. (Observe that $c' = a$, since F_6 is light.) By the general position assumption, b' and c must be in an octant-relation. This implies that the first and the last vertices of F_1 (a and b) are in a quadrant-relation. Then, we are done by applying Lemma 4.13 to F_1.

6.4 Shape Cycles with 5 flats

Let σ be a drawable shape cycle with 5 flats $F_1, \ldots, F_5$. Because there are 5 flats, there must be $\lceil \frac{5}{3} \rceil = 2$ flats in the same orientation, say A. Moreover, we cannot have 3 flats in the same orientation, because there must be a flat in another orientation between each pair of parallel flats. This gives rise to the following two patterns of flat orientations:

Case 1. $(F_1, F_2, F_3, F_4, F_5) = (A, B, C, A, B)$**:** Since each of F_1 and F_5 is between two parallel flats, they must be heavy. Suppose F_2 is heavy. Then consider the shared transition element aa' between F_2 and F_3, and the shared transition element bb' between F_1 and F_5. By Lemma 5.2, aa' and bb' are well-related, so we are done by Theorem 5.6. By a similar reasoning, if F_4 is heavy, we are done. So assume F_2 and F_4 are both light. This implies F_3 is between two light flats. Since the first and the last vertices of F_3 are in a quadrant-relation in Γ, we are done by applying Lemma 4.13 to F_3.

Case 2. $(F_1, F_2, F_3, F_4, F_5) = (A, B, C, A, C)$**:** Since each of F_4 and F_5 is between two parallel flats, they must be heavy. As in the previous case, if F_1 or F_3 is heavy, the shared transition element between F_3 and F_4 and the shared transition element between F_5 and F_1 are well-related transition elements. So assume F_1 and F_3 are light. Then, F_2 is between two light flats. Furthermore, the first and the last vertices of F_3 are in a quadrant-relation in Γ, so we are done by applying Lemma 4.13 to F_2.

6.5 Shape Cycles with 4 flats

Let σ be a drawable shape cycle with 4 flats $F_1, \ldots, F_4$. Then, there are $\lceil \frac{4}{2} \rceil = 2$ flats of some orientation A. This gives rise to the following patterns of flat orientations:

Case 1. $(F_1, F_2, F_3, F_4) = (A, B, A, B)$**:** Since each flat lies between two parallel flats, every flat is heavy. Then, by Lemma 5.2, the shared transition element between F_1 and F_2, and the shared transition element between F_3 and F_4 are well-related. By Theorem 5.6, σ contains a canonical sequence of length 6.

Case 2. $(F_1, F_2, F_3, F_4) = (A, B, A, C)$**:** F_2 and F_4 must be heavy since they lie between two parallel flats. For reasons of concreteness, we assume $A = NSEW, B = NSUD, C = EWUD$ without loss of generality. There are following subcases to consider.

1. *F_1 and F_3 are both heavy.* Then, as in the case above, there exists a pair of well-related transition elements.

2. *F_1 and F_3 are both light.* Let F_1 and F_3 consist of three vertices each, so that $F_1 = aa'a''$ and $F_3 = bb'b''$, respectively. Then, it must be the case that b' is in an axis-relation with a' in Γ. Assume, without loss of generality, that b' is U of a'. By Lemma 4.10, $\sigma_{a'b'}$ has $\tau_1 = \{\bar{U}, \bar{N}, \bar{S}\}$, and $\sigma_{b'a'}$ has $\tau_2 = \{\hat{D}, \hat{E}, \hat{W}\}$. Then, by Lemma 4.8, $\tau_1 \cup \tau_2$ is canonical.

3. *F_1 is heavy, and F_3 is light.* See Figure 6.1. Let aa' denote the shared transition element between F_4 and F_1, and let bb' denote the shared transition element between F_2 and F_3. By the general position assumption, a and b must be in an octant-relation in Γ. Assume, without loss of generality, that b is UNE of a. Due to the orientations of F_2 and F_3, the direction label of bb' is either N or S.

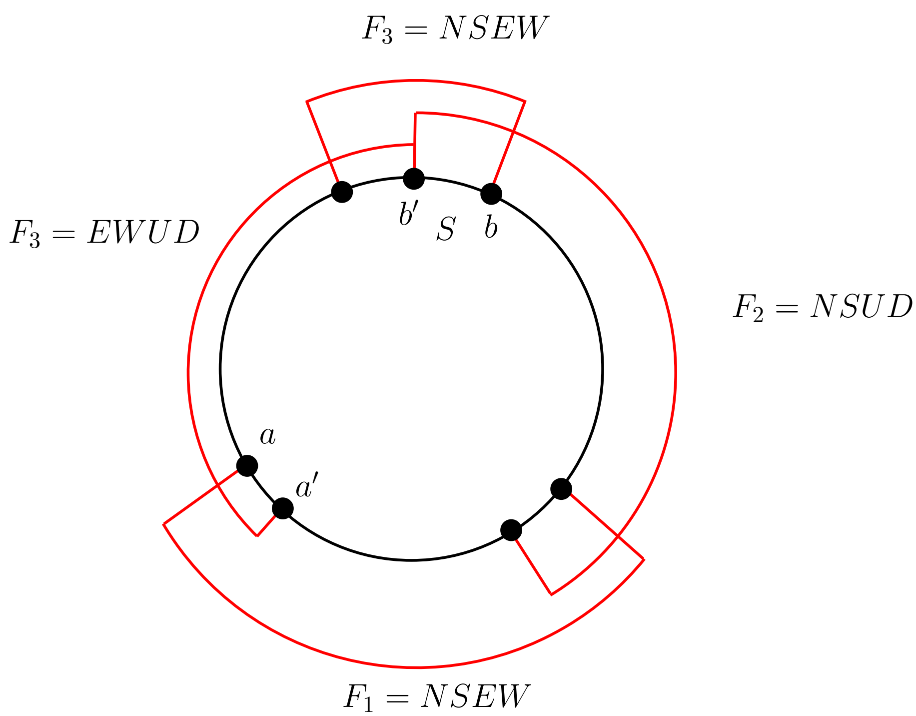

Figure 6.1: A shape cycle with flat pattern $(F_1, F_2, F_3, F_4) = (A, B, A, C)$; here, it is assumed, without loss of generality, that $A = NSEW$, $B = NSUD$, and $C = EWUD$.

Observe that there is a flat that contains b' and a, while b is UNE of a. This implies that bb' is an S label.

First, suppose $a \sim_b a'$. This implies that b is UNE of a'. Since b' and a' belong to the same flat, by Lemma 4.11, $\sigma_{a'b'}$ contains $\tau_3 = \{\bar{U}, \bar{N}, \bar{E}, \bar{S}^+\}$, and σ_{ba} contains $\tau_4 = \{\hat{D}, \hat{S}, \hat{W}\}$. Then, we can merge τ_3 and τ_4 by Lemma 4.9.

So, assume $a \not\sim_b a'$. Since no flat contains both b and a', they must lie in an octant-relation. This implies that b is UNW of a', and $aa' = E$. If we place the origin at a, then b is UNE of a, and the edge $\overline{bb'}$ intersects the UE quadrant orthgonally. Hence, by Lemma 4.11, $\sigma_{ab'}$ has $\tau_5 = \{\tilde{U}, \tilde{N}, \tilde{E}, \tilde{S}^+\}$. Furthermore, b is UNW of a', so $\sigma_{ba'}$ has $\tau_6 = \{\check{D}, \check{S}, \check{E}\}$. There are the following cases to consider.

(a) $bb' \in \tau_5 \cap \tau_6$:

- $aa' \in \tau_5 \cap \tau_6$: If $\tilde{D} \in \tau_5$, remove $\check{D}$ from τ_6, and call it τ_6'. Otherwise, let $\tau_6' = \tau_6$. By Property 4.6, τ_6' is canonical. Then, observe that $\tau = \tau_5 \cup \tau_6'$ contains all six labels. If a flat F contains 2 or more elements of τ, $F \cap \tau$ is either $F \cap \tau_5$ or $F \cap \tau_6'$. Thus, τ is canonical.

- $aa' \in \tau_5 \setminus \tau_6$: Remove aa' from τ_5, and call it τ_5'. By Property 4.5, τ_5' is canonical. Then, we can merge τ_5' and τ_6 by Lemma 4.9.
- $aa' \in \tau_6 \setminus \tau_5$: Remove aa' from τ_6, and call it τ_6'. By Property 4.5, τ_6' is canonical. Then, we can merge τ_5 and τ_6' by Lemma 4.9.
- $aa' \notin \tau_5 \cup \tau_6$: Then, we can merge τ_5 and τ_6 by Lemma 4.9.

(b) $bb' \in \tau_5 \setminus \tau_6$: Remove bb' from τ_5, and call it τ_5'. By Property 4.5, τ_5' is canonical. Then, we can merge τ_5' and τ_6 by Lemma 4.9.

(c) $bb' \in \tau_6 \setminus \tau_5$: Remove bb' from τ_6, and call it τ_6'. By Property 4.5, τ_6' is canonical. Then, we can merge τ_5 and τ_6' by Lemma 4.9.

(d) $bb' \notin \tau_5 \cup \tau_6$: Then, we can merge τ_5 and τ_6 by Lemma 4.9.

4. *F_1 is light, and F_3 is heavy.* This case is identical to the above, after interchanging B with C.

6.6 Shape Cycles with 3 or fewer flats

A straightforward case analysis handles shape cycles with fewer than four flats, as shown in [9, 10].

This completes the proof of Theorem 6.1. $\square$

Chapter 7

Concluding Remarks and Open Problems

The main contribution of this thesis is the completion of the proof of Theorem 3.10. Furthermore, due to the completion of this characterization, it is now confirmed that the recognition and the drawing problem for shape cycles can be solved in linear time, as presented in [10]. However, a series of problems remain unexplored. In this concluding chapter, we discuss the consequences of the results presented in this thesis, and present a survey of some open problems related to this work.

7.1 Degenerate cases of Shape Paths

As discussed in previous chapters, Di Battista *et al.* [11] discuss the reachability problem: given a shape path σ and a point p in an octant, does σ admit a simple orthogonal drawing such that the initial vertex of σ is located at the origin while the final vertex of σ is located at p? This question can be answered in linear time by checking for the characterization of shape paths presented in [11]. However, this characterization only handles the case where p is a point in an octant. When the final vertex is to be drawn at some point in a quadrant or on an axis with respect to the position of the initial vertex, this characterization does not apply. In this spirit, shape cycles can be regarded as a degenerate case of a shape path, where the position of the final vertex coincides with the position of the initial vertex. The reachability problems for the remaining degenerate cases are still open.

Open Problem 7.1. *Let σ be a 3D shape path with initial vertex u and final vertex v. Let p be a point in a quadrant or on an axis. Does there exist an efficient algorithm to check if σ admits a simple orthogonal drawing such that u is located at the origin, and v is located at p? Furthermore, does there exist an efficient algorithm to produce such a drawing?*

We present some initial exploration to show that our Lemma 4.11 can provide a useful tool. We also present some examples to show that handling degenerate cases may be a complex issue for further research.

Some partial results can be obtained as consequences of Theorem 3.10 and Lemma 4.11. The following result provides a useful tool for shape paths that intersects an axis orthogonally.

Lemma 7.2. *Let $\Gamma(\sigma)$ be a simple orthogonal drawing of a 3D shape path σ that starts at the origin, and has a later edge uv intersecting an axis orthogonally. Let X denote the direction defined by the axis that uv intersects orthogonally. Then, σ has a canonical sequence that contains $\{X, Y, Y', Z, Z'\}$ as a subset.*

Proof. We use induction on $|\sigma|$. When $|\sigma| = 5$, the statement holds trivially. Suppose the statement holds for all values $5 \leq |\sigma| \leq k$, and consider the case where $|\sigma| = k+1$. Let a denote the first vertex of σ, and let uv denote an edge that intersects the X axis orthogonally. If uv is not the last edge of σ, we are done by the induction hypothesis. If there exists an edge that appears before uv such that, in the drawing $\Gamma(\sigma)$, it intersects the X axis orthogonally, we are again done by induction. So, assume uv is the last edge of σ, and that no other edge intersects the X axis orthogonally. If $\overline{uv}$ crosses the X axis so that u and v lie on two distinct quadrants adjacent to the X axis, shrink the edge $\overline{uv}$ so that v lies on the X axis.

Now, consider the X axis between v and a. Then, imagine walking along the X axis from v to a. Let p denote the first vertex that is encountered when walking along the X axis from v to a. Then, it must be that $p = a$, or p is the second vertex of σ in the case where the first direction label in σ is X. Then, add a dummy edge from v to p, and assign X' as the direction for the dummy edge. Let σ' be a shape consisting of the shape path σ_{pv} and the final edge vp. Then, σ' is a shape cycle. Before adding the edge vp, there was no other edge that intersects the X axis between v and p. Therefore, the drawing of the shape path $\Gamma(\sigma_{pv})$ and the newly created dummy edge $\overline{vp}$ form a simple orthogonal drawing of σ'. By Theorem 3.10, σ' contains a canonical sequence τ of length six. Now, if $vp \in \tau$, remove vp from τ and call the result τ'. Otherwise, let $\tau' = \tau$. In either case, the resulting sequence τ' contains $\{X, Y, Y', Z, Z'\}$ as a subset, and is canonical for σ. □

Notice that this result is analogous to Lemma 4.11. Similarly to Lemma 4.11, Lemma 7.2 cannot be strengthened so that σ contains exactly those 5 labels. Consider the following counterexample.

Counterexample 7.3. [1] *The shape path $\sigma = NUSEDW$ can be drawn so that $\Gamma(\sigma)$ starts at the origin and intersects the U axis orthogonally. See Figure 7.1 for an example of $\Gamma(\sigma)$. However, σ does not contain a canonical sequence of type $\{U, N, E, S, W\}$.*

Note that Lemma 7.2 assumes that the last edge of σ is orthogonally intersecting the X axis. The following result is more general, and analogous to the necessity direction of the characterization in Theorem 3.9.

[1]In this counterexample, $\Gamma(\sigma)$ does not contain a vertex in an octant-relation with the origin. It would be interesting to know a counterexample for which the drawing does contain such a vertex.

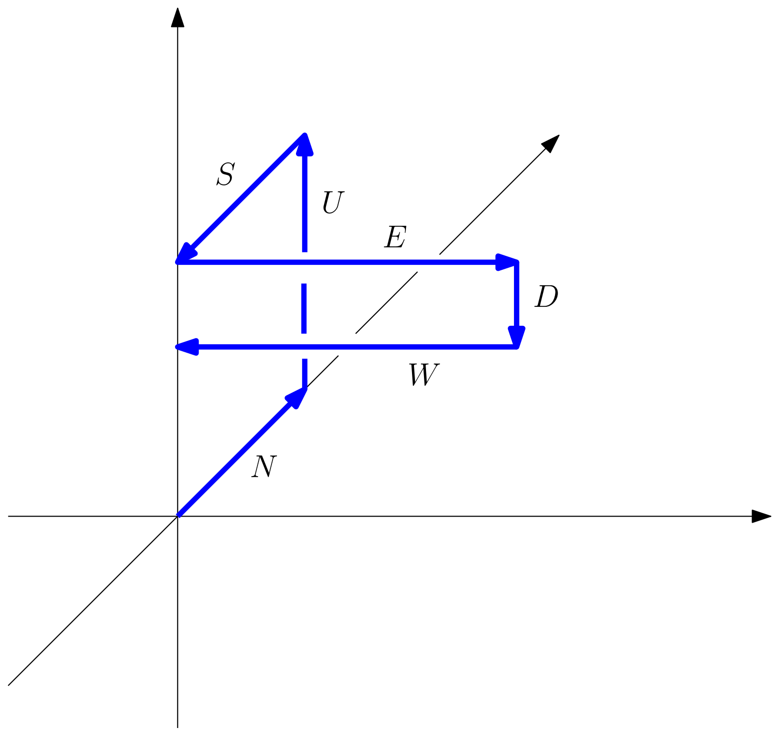

Figure 7.1: A drawing of a shape path $\sigma = NUSEDW$ that intersects the U axis orthogonally; but σ itself does not contain a canonical sequence of type $\{U, N, E, S, W\}$.

Lemma 7.4. *Let $\Gamma(\sigma)$ be a simple orthogonal drawing of a 3D shape path σ that start at the origin, and end at some point p in the X axis. Then, σ has a canonical sequence that contains $\{X, Y, Y', Z, Z'\}$ as a subset.*

Proof. Let a be the first vertex of σ, and let uv the last edge of σ. By assumption, v lies in the X axis. If uv is orthogonal to the X axis, we are done by Lemma 7.2. Otherwise, because σ is a 3D shape path, there must exist an edge bc before uv such that $\overline{bc}$ touches the axis orthogonally, and σ_{ac} is a 3D shape path. Then, σ_{ac} contains the desired canonical sequence by Lemma 7.2. $\square$

Similarly, we can develop a necessary condition for shape paths that starts at the origin, and ends at some point in a quadrant.

Lemma 7.5. *Let $\Gamma(\sigma)$ be a drawing of a 3D shape path σ that starts at the origin, and ends at some point in the XY quadrant. Then, σ has a canonical sequence that contains $\{X, Y, Z, Z'\}$ as a subset.*

Proof. Let a denote the first vertex of σ, and let uv be the last edge of σ. If $\overline{uv}$ is orthogonal to the XY quadrant, we are done by Lemma 4.11. By the same reasoning, if there exists an edge such that, in the drawing $\Gamma(\sigma)$, it intersects the XY quadrant orthogonally, we are done. Otherwise, $\Gamma(\sigma)$ must reach the XY quadrant either through the X axis or through the Y axis. Let pq denote the last edge of σ that enters the XY quadrant (i.e. q lies in the XY quadrant while p does not.). Then, pq must intersect either the X axis or the Y axis orthogonally. Furthermore σ_{aq} is a 3D path. If pq intersects the X axis, by Lemma 7.2, there is a canonical sequence in σ_{aq} that contains $\{X, Y, Y', Z, Z'\}$ as a subset. If pq intersects the Y axis, by Lemma 7.2, there is a canonical sequence in σ_{aq} that contains $\{X, X', Y, Z, Z'\}$ as a subset. □

Observe that Lemmas 7.4 and 7.5 may serve as necessary conditions for characterizing shape paths that start at the origin and end at some point in a quadrant or an axis. If the converse of these two lemmas is true, we would have characterizations of such shape paths that can be checked in linear time, using the algorithm of Di Battista *et al.* [10, 11]. Unfortunately, the converse of these two lemmas does not hold. We offer the following counterexamples.

Counterexample 7.6. *Let σ be a shape path such that $\sigma = UNESDW$. Then, $\tau = UNESW$ is a canonical sequence for σ. However, σ does not admit a simple orthogonal drawing Γ such that Γ starts at the origin, and ends at some point in the U axis.*

Counterexample 7.7. *Let σ be a shape path such that $\sigma = NEDSWU$. Then $\tau = NEWU$ is a canonical sequence for σ. However, σ does not admit a simple orthogonal drawing Γ such that Γ starts at the origin, and ends at some point in the UN quadrant.*

Therefore, a different combinatorial characterization must be found in order to solve Open Problem 7.1.

7.2 Drawing in an integer grid

As is the case with many other orthogonal drawing results, one may be interested in drawing shape cycles in the grid model, i.e. with all vertices are drawn at integer coordinates. The drawing Γ generated by the algorithm in [10] can easily be converted to a grid model.

Corollary 7.8. *Let σ be a shape cycle. Then there exists a $O(n \log n)$ time algorithm to draw σ in an integer grid, in $O(n) \times O(n) \times O(n)$ volume.*

Proof. The drawing algorithm presented in Di Bittista *et al.* [10] produces a drawing Γ of σ in R^3 in $O(n)$ time. Now we scale Γ to construct Γ' as follows. Sort the vertices by their x-coordinates, and simply modify their x-coordinates to 1 through n, in the sorted order. Similarly, modify the y- and z-coordinates. Observe that the modified drawing Γ' remains simple during this scaling process. Then, Γ' contains at most n vertices in each dimension, and hence the volume of the drawing is $O(n) \times O(n) \times O(n)$. □

In fact, this scaling technique can be applied to convert any orthogonal drawings in R^3 to the grid model. However, in the case of drawing shape paths where the positions of the initial and the final vertices are fixed, this technique does not work in general. We recall the following problem, which has remained unsolved since the appearance of [11].

Open Problem 7.9. [11] *Let σ be a shape path with n vertices, and let $p = (p_x, p_y, p_z)$ be a grid point in an integer grid. Do there exist a recognition and a drawing algorithm for the grid model such that σ starts at the origin and ends at p?*

If all of p_x, p_y, and p_z are greater than or equal to n, the scaling technique will convert the drawing Γ to the grid model Γ' such that the initial and the final vertices are located at the desired grid points. However, if any of the coordinates of p is less than n, the orderings of the vertices in x, y, and z directions defined by Γ may be such that the scaled drawing Γ' cannot fit in the rectilinear box enclosing the origin and p. Thus, a new drawing algorithm must be designed to solve this problem.

On another note, the complexity results discussed in Patrignani [28] still hold for the integer grid model, since grid drawings are special cases of drawings in R^3. Hence, given a shape σ for a general graph, it is NP-hard to check if there exists a grid drawing of σ.

7.3 Extension to other classes of graphs

Although the feasibility problem for shape graphs is NP-hard in general [28], it may be of interest to consider restricted classes of graphs that admit efficient recognition and drawing algorithms. A characterization is unknown even for simple classes of graphs such as trees or planar graphs.

Open Problem 7.10. [12, 28] *Are there any nontrivial classes of shape graphs that admit efficient recognition and drawing algorithms?*

As discussed in Chapter 2, Di Giacomo *et al.* [12] exhibited a a shape graph whose induced cycles are independently realizable, but the graph itself cannot be realized. This shows that simply applying the techniques discussed in this thesis together with cycle decomposition of graphs does not work in general.

7.4 Extension to higher dimensions

In [11], the characterization for drawing shape paths has been extended to arbitrary dimensions. The characterization for drawing shape cycles, however, has only been shown to hold in 3D.

Open Problem 7.11. *Can we extend Theorem 3.10 to higher dimensions?*

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