On Bus Graph Realizability

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Problem Definition

Instance: A bipartite graph $G = (\mathcal{B}, \mathcal{C}; \mathcal{E})$ such that $\forall c \in \mathcal{C}$, $\deg(c) \leq 4$.

Question: Can G be drawn onto a grid so that the following properties hold?

1. Each vertex $B \in \mathcal{B}$ is drawn as a closed line segment along a grid line.
2. Each vertex $c \in \mathcal{C}$ is drawn as a point at a grid point.
3. Each edge $(B, c) \in \mathcal{E}$ is drawn as a closed line segment between B and c, that is perpendicular to B, and contains no other connectors or buses apart from B and c; an edge can, however, cross other edges. An edge may connect to a bus at an endpoint of the bus.
4. No buses or connectors may intersect.

Summary

• The BGR problem is NP-complete, when the maximum degree of $\mathcal{C}$-vertices is $2, 3$, or 4.

• The BGR problem with the lengths of buses given as input is NP-Hard

• Partition by Orientation (an easier sub-problem) is also NP-Complete

Definitions and Preliminaries

• An instance of BGR is given by $G = (\mathcal{B}, \mathcal{C}; \mathcal{E})$, where $\mathcal{B}$ denotes the bus vertices, and $\mathcal{C}$ denotes the connector vertices.

• The Γ function is used to denote an embedding of a combinatorial bus graph G.

Membership in NP

Lemma 1. *Bus graph realizability is in NP.*

Given an embedding $\Gamma(G)$ of a bus graph G, one can construct a *compact* embedding $\Gamma'(G)$, such that the size of the embedding is linear in any dimension.

Gadgets

Definition 1. *An (A, B)-perp is a bus graph component as shown in the Figure 1.*

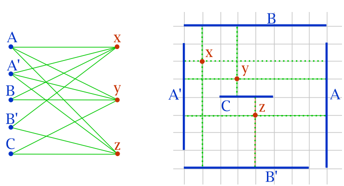

FIGURE 1: A combinatorial graph of an (A, B)-perp and its embedding.

Property 1. *In any embedding Γ of an (A, B)-perp,*

1. $\Gamma(B)$ and $\Gamma(B')$ are parallel.

2. $\Gamma(A)$ and $\Gamma(B)$ are perpendicular.

Definition 2. *A (B, o)-flipper is a bus graph component as shown in Figure 2.*

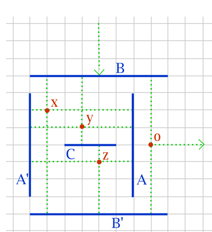

FIGURE 2: Example embedding of a (B, o)-flipper.

Let i be a $\mathcal{C}$-vertex joined with B, and let O be a $\mathcal{B}$-vertex joined with o. Then, we have the following property.

Property 2. *In any embedding Γ of a (B, o)-flipper,*

1. $\Gamma((B, i))$ and $\Gamma((O, o))$ are perpendicular.

2. $\Gamma(B)$ and $\Gamma(O)$ are perpendicular.

Definition 3. *An (A, k, B, l)-variable-box is a bus graph component as shown in Figure 3.*

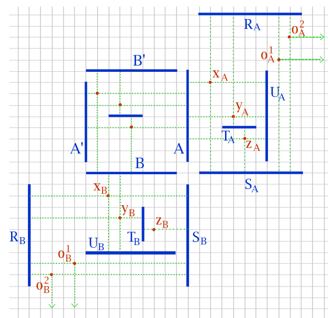

FIGURE 3: Embedding of an $(A, 2, B, 2)$-variable-box.

Property 3. *Let (O_A^i, o_A^i) for $i = 1, 2, \ldots, k$ and (O_B^j, o_B^j) for $j = 1, 2, \ldots, l$ be edges joined with an (A, k, B, l)-variable-box. Then in any embedding Γ,*

1. $\Gamma((O_A^i, o_A^i))$ and $\Gamma(A)$ are perpendicular for any $i = 1, 2, \ldots, k$.

2. $\Gamma((O_B^i, o_B^i))$ and $\Gamma(B)$ are perpendicular for any $i = 1, 2, \ldots, l$.

Definition 4. *An (I, O)-chain is a bus graph component consisting of four "connected" flippers as shown in Figure 4.*

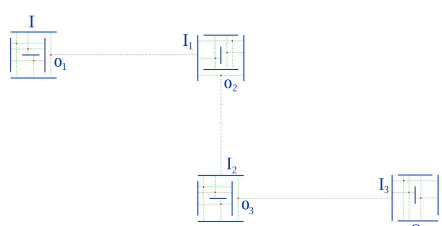

FIGURE 4: Embedding of an (I, O)-chain.

Property 4. *In any embedding Γ of an (I, O)-chain, $\Gamma(I)$ and $\Gamma(O)$ are parallel.*

Reduction from NAE-3SAT

Let ϕ be an instance of NAE-3SAT, consisting of boolean variables $x_1, \ldots, x_n$, and clauses $C_1, \ldots, C_m$. Construct a bus graph G as follows.

1. For each boolean variable x_i, create a $(X_i, t_i, \overline{X}_i, f_i)$-variable-box, where t_i and f_i are the numbers of distinct occurrences of the literals x_i and $\overline{x_i}$, respectively, in ϕ.
2. For each clause $C_q = (x_i^* \vee x_j^* \vee x_k^*)$, where x^* is either x or $\overline{x}$, create

(a) a $\mathcal{C}$-vertex c_q,

(b) an $(I_{q,1}, O_{q,1})$-chain, an $(I_{q,2}, O_{q,2})$-chain, and an $(I_{q,3}, O_{q,3})$-chain,

(c) edges $(O_{q,1}, c_q)$, $(O_{q,2}, c_q)$ and $(O_{q,3}, c_q)$,

(d) edges $(I_{q,1}, p_i)$, $(I_{q,2}, p_j)$ and $(I_{q,1}, p_k)$, where $p_i = o^r_{X_i}$ if $x_i^* = x_i$ and $p_i = o^r_{\overline{X}_i}$ if $x_i^* = \overline{x_i}$, and it is the rth occurrence of x_i^* being considered.

Lemma 2. *$\phi \in$ NAE-3SAT if and only if $G \in$ BGR.*

• Suppose $\phi \in$ NAE-3SAT. An embedding Γ of G can be constructed as demonstrated in the Figure 5.

• Conversely, suppose $G \in$ BGR, and take an embedding Γ of G. Assign each variable x_i to be true if $\Gamma(X_i)$ is vertical and false otherwise; this is a satisfying assignment.

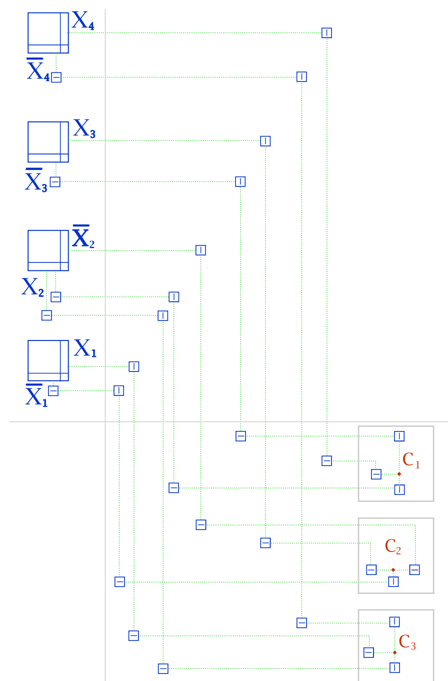

FIGURE 5: A schematic embedding of G, where ϕ consists of the clauses $C_1 = (x_2 \vee \overline{x_3} \vee x_4)$, $C_2 = (\overline{x_1} \vee \overline{x_2} \vee x_3)$,and $C_3 = (x_1 \vee x_2 \vee \overline{x_4})$ with a satisfying truth assignment $x_1 = x_3 = x_4 =$ true and $x_2 =$false.

BGR with maximum degree two or three

It is possible to design perps, flippers and variable boxes using at most 2 or 3 degree per each $\mathcal{C}$-vertex. This allows us to construct similar proofs for BGR instances with $\mathcal{C}$ having maximum degree of 2 or 3. Note, however, the problem is trivial when the $\mathcal{C}$-vertices have maximum degree of 1, or strictly greater than 4.

Theorem 1. *Bus Graph Realizability is NP-complete if and only of the maximum degree of $\mathcal{C}$-vertices is $2, 3$, or 4.*

Related Problems

Partition by Orientation.

Instance: A bipartite graph $G = (\mathcal{B}, \mathcal{C}; \mathcal{E})$ such that $\forall c \in \mathcal{C}, deg(c) \leq 4$.

Question: Can B be partitioned into two disjoint sets $\mathcal{B}_{\mathcal{H}}$ and $\mathcal{B}_{\mathcal{V}}$ such that $\forall c \in C, c$ is adjacent to no more than two vertices in $\mathcal{B}_{\mathcal{H}}$ and no more than two vertices in $\mathcal{B}_{\mathcal{V}}$?

This problem is NP-complete, by a similar reduction from NAE-3SAT. Although this problem is a subproblem of the general BGR problem, a proper bipartition by orientation does not guarantee that the bus graph is realizable.

BGR with Given Bus Lengths.

In this version of the problem, the lengths of the buses are given as input. This problem is NP-hard by a reduction from PARTITION. Given an instance $\langle A, s \rangle$ of PARTITION, where A is a set of elements and $s : A \rightarrow Z^+$ is a size function, construct a bus graph G as follows. Create a $\mathcal{B}$-vertex B^* of length $\frac{1}{2}\sum_{a \in A} s(a) - 1$. Then, for each $a \in A$, create a $\mathcal{B}$-vertex B_a of length $s(a) - 1$. Now, for each element $a \in A$, create exactly $s(a)$ copies of $\mathcal{C}$-vertex, and join them to both B_a and B^*. Now, it can be shown that G is realizable if and only if $\langle A, s \rangle \in$ PARTITION.

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