Math 221 Sec 10M : W1 2023 | Homework 7
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**Question 1.**

*Solution.*

a) Is $D$ a linear transformation? Explain your answer. If yes, write down its standard matrix.
The transformation D: $\mathbb{R}^4 \to \mathbb{R}^3$ describing the derivative of p(x) for some vector $\vec{c}$ which represents the constants of the polynomial $p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$. The derivative of $p'(x) = 0 + c_1 + 2c_2x + 3c_3x^2$ with $\vec{c'} = \begin{pmatrix} c_1 \\ 2c_2 \\ 3c_3 \end{pmatrix}$. We can see that this transformation removes the first entry in $\vec{c}$, therefore the standard matrix can be written as

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

For $D$ to be a linear transformation $D(\vec{u} + \vec{c}) = D(\vec{u}) + D(\vec{c})$ and $D(c\vec{u}) = cD(\vec{u})$. Check for closure along addition, given $\vec{u}, \vec{c} \in V$,

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \vec{u} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$D(\vec{u} + \vec{c}) = D \begin{pmatrix} u_0 + c_0 \\ u_1 + c_1 \\ u_2 + c_2 \\ u_3 + c_3 \end{pmatrix} = \begin{pmatrix} u_1 + c_1 \\ 2(u_2 + c_2) \\ 3(u_3 + c_3) \end{pmatrix}$$

Check $D(\vec{u}) + D(\vec{c})$

$$D(\vec{u}) = D \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ 2u_2 \\ 3u_3 \end{pmatrix}$$

$$D(\vec{c}) = D \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2c_2 \\ 3c_3 \end{pmatrix}$$

$$D(\vec{u}) + D(\vec{c}) = \begin{pmatrix} u_1 + c_1 \\ 2(u_2 + c_2) \\ 3(u_3 + c_3) \end{pmatrix} = D(u \vec{+} c)$$

Check for closure along multiplication

$$D(c\vec{u}) = D\begin{pmatrix} cu_0 \\ cu_1 \\ cu_2 \\ cu_3 \end{pmatrix} = \begin{pmatrix} cu_1 \\ 2cu_2 \\ 3cu_3 \end{pmatrix}$$

$$cD(\vec{u}) = D\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = c\begin{pmatrix} u_1 \\ 2u_2 \\ 3u_3 \end{pmatrix} = \begin{pmatrix} cu_1 \\ 2cu_2 \\ 3cu_3 \end{pmatrix} = D(c\vec{u})$$

The transformation D is a linear transformation since it follows both principles.

b) To check if D is one-to-one we reduce D to RREF.

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$Dred = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

D represents the matrix transformation from $\mathbb{R}^4 \to \mathbb{R}^3$. We can see that there is not a pivot in every column, and therefore, the transformation is not one-to-one and implies $Ax = 0$ has non-trivial solutions.

Given the vectors $\mathbf{u}, \mathbf{v}$, where $\mathbf{u} = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Computing the transform of both of these vectors,

$$T(\mathbf{u}) = D\begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(\mathbf{v}) = D\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(\mathbf{u}) = T(\mathbf{v})$$

c) We can see again by looking at the reduced matrix of D, there is a pivot in every row, therefore the transformation is onto.

$$Dred = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\square$