Math 221 section 10M, W1 2023

Homework 5 Solutions

Due: 10pm on Thursday, 5 October 2023

- Make sure to bring a copy of your completed homework to class on Friday the 6th of October as we will be discussing one of the problems from there.
- Upload your homework to Canvas with plenty of time to deal with technical issues. Late homework will not be accepted.
- Unless otherwise specified, you are expected to carefully explain all of your responses.

1. Consider the following subsets of $\mathbb{R}^3$ and $\mathbb{R}^2$. For each of the subsets, determine if it is a subspace of $\mathbb{R}^3$ (or $\mathbb{R}^2$) by checking the conditions in the definition. If it is a subspace, describe it as a span.

   (a) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = y + z \right\}$

   (b) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y + 1 \right\}$

   (c) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y^2 \right\}$

   Answer.

   (a) V is a subspace:

   - $\mathbf{0}$ is in V, since $0 = 0 + 0$ (taking components of the zero vector).
   - If $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ are in V, then $x_i = y_i + z_i$ for $i = 1, 2$ and $x_1 + x_2 = y_1 + z_1 + y_2 + z_2 = (y_1 + y_2) + (z_1 + z_2)$, so the sum $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is in V as well.
   - Let $c \in \mathbb{R}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$, then $cx = c(y + z) = cy + cz$ so $c \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ is in V

   (b) W is not a subspace: it does not contain the zero vector: if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $x = 0 \neq y+1 = 0+1 = 1$.

   (c) W is not a subspace: for example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ satisfies the condition $x = y^2$, but the scalar multiple $2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ does not: $x = 2$ but $y^2 = 4$. Thus, W is not closed under scalar multiplication, so it cannot be a subspace.

2. **True or False** Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ be vectors in $\mathbb{R}^4$. Suppose that the set $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. Determine if the following statements are true or false.

(a) Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$ and $\{\mathbf{w}, \mathbf{x}\}$ are linearly independent sets. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly independent.

(b) Suppose that $u_1 = v_1 = w_1 = 0$ and $x_1 \neq 0$. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly independent.

Answer.

(a) FALSE: suppose $\mathbf{x} = \mathbf{w} + \mathbf{v}$ for example, Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$ is linearly independent and $\{\mathbf{w}, \mathbf{x}\}$ is linearly independent, but $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is not because $\mathbf{x}$ is in the span of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. As a specific counter-example: let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Then $\{\mathbf{u}, \mathbf{v}, \mathbf{x}\}$ is linearly independent: if c_1, c_2, c_3 are such that

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{x} = \mathbf{0}$$

then in components:

$$\begin{bmatrix} c_1 \\ c_2 + c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the first and third components tell us that $c_1 = 0$ and $c_3 = 0$, which in turn forces c_2 to be 0.
We also see that $\{\mathbf{w}, \mathbf{x}\}$ is linearly independent, since the vectors are not scalar multiples of each other.
However, the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly dependent, because $\mathbf{x} = \mathbf{v} + \mathbf{w}$.

(b) TRUE: suppose that c_1, c_2, c_3, c_4 are scalars such that $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_4\mathbf{x} = \mathbf{0}$. Then, if we consider the first component of the vector equation, we get

$$c_1u_1 + c_2v_1 + c_3w_1 + c_4x_1 = 0$$

But we know that $u_1 = v_1 = w_1 = 0$, so this equation becomes $c_4x_1 = 0$. Since $x_1 \neq 0$, $c_4 = 0$. Then the vector equation becomes

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$$

But we know that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set, so $c_1 = c_2 = c_3 = 0$. Thus, the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is linearly independent.

3. Kawhi and Pascal are discussing the set $S = \left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ -4 \end{bmatrix} \right\}$, and want to find out if it is linearly independent:

KAWHI: We saw that a set of vectors $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent if they aren't scalar multiples of each other.
PASCAL: Ok, so we check if the first two vectors are scalar multiples of each other: if the first vector is a scalar multiple of the second, then there's a scalar k such that

$$\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

or in components:

$$\left\{ \begin{array}{rcr} -1 & = & 2k \\ 3 & = & k \\ 2 & = & -k \end{array} \right.$$

This clearly isn't possible, so the first two vectors are linearly independent.
KAWHI: Right, so let's check if the second and third vectors are also linearly independent: if the third vector were a scalar multiple of the second vector, then

$$\begin{bmatrix} -1 \\ -11 \\ -4 \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

It's pretty clear by looking at components that these are linearly independent as well.
PASCAL: Ok, we can't forget to check if the first and third vetors are linearly independent. If not, then

$$\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = k \begin{bmatrix} -1 \\ -11 \\ -4 \end{bmatrix}$$

It's clear looking at the components that there isn't a value of k that makes this system of equations consistent, so the first vector isn't a scalar multiple of the third vector, and they're linearly independent too.
KAWHI: All of the vectors are linearly independent with each of the others, so the set S is linearly independent.

(a) Do you agree with Kawhi and Pascal that the set S is linearly independent? Use our matrix method from class to check if the set is linearly independent. If it is dependent, find an explicit linear dependence relation between the vectors.

(b) If you do not agree with their conclusion, go through each line of the argument and identify any errors they make.

Answer.

(a) Kawhi and Pascal are incorrect in their conclusion: the set is linearly dependent. We see this as follows:
Let $A = \begin{bmatrix} -1 & 2 & -1 \\ 3 & 1 & -11 \\ 2 & -1 & -4 \end{bmatrix}$ be the matrix whose columns are the vectors in S. Using Python, we

row reduce A to find its RREF:

$$RREF(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns of a matrix are linearly independent if it has pivots in every column, and we see that the third column of A is not a pivot column. In fact, the RREF of A gives us a linear dependence relation between the columns of A:

$$\begin{bmatrix} -1 \\ -11 \\ -4 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

(b) The main problem with the argument is that while it is certainly true that a set of two vectors is linearly independent if and only if the vectors are scalar multiples of each other, this is not the case for sets of three or more vectors. To really illustrate this: consider the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then we see that any pair of the vectors is linearly independent: none of the vectors are scalar multiples of each other, but the full set of three vectors is linearly dependent, since $\mathbf{v} = \mathbf{u} + \mathbf{w}$. This is because linear dependence is about considering linear combinations of the vectors, and linear combinations of two vectors give direct relations between the two vectors, but linear combinations of three vectors give relations between one of the vectors and the two other vectors. Essentially, the only mistake in this argument is in the conclusion: just because the vectors in the set are pairwise linearly independent, it is not enough to conclude that the whole set is linearly independent.

4. Emmaford Prefect is travelling in the Heart of Gold, a spaceship that travels in hyperspace (this can reliably, so I'm told, be modelled by $\mathbb{R}^4$). Her spaceship has a few quirks, due to its improbability drive. Firstly, she can only travel in directions given by the following vectors:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -1 \\ -13 \\ -10 \\ -8 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} -1 \\ -3 \\ 4 \\ -2 \end{bmatrix}$$

and she can travel along them in both the positive and negative directions. Furthermore, each trip can only involve each of these direction vectors once, though she can travel in each direction for as long as she likes.

(a) Emmaford takes off from her home at the Bay at the End of the Universe in her spaceship. Is she able to do a round trip just using the spaceship? (in other words, is she able to return home after travelling away from home all with the spaceship?). If she is able to, describe the set of all trips she is able to make that start and end at her home. Your description should involve vectors.

(b) Can Emmaford reach any other point in hyperspace using her spaceship? If not, describe the subset of $\mathbb{R}^4$ that she can reach with her spaceship. Your description should involve vectors.

(c) Show that the sets from the previous parts are subspaces by describing each of them as a column space or a null space of a matrix.

(d) Emma's grandfather Allenod Beeblebrox is visiting the planet Krikkit, located at the coordinates $\begin{bmatrix} 3 \\ 5 \\ 6 \\ 6 \end{bmatrix}$ away from Emmaford's home. Can she reach him using her spaceship? Desribe the set of ways she can reach Allenod. Your description should involve vectors.

(e) Is the set in the previous part a subspace? Explain your answer.

Answer. The point of this question is to understand what the mathematical concepts we have been learning mean in a practical sense. Therefore, we need to be able to translate all of these questions into purely mathematical ones.

(a) Under our assumptions about trips, this question comes down to: what linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ and $\mathbf{v_4}$ are equal to the zero vector? We see this because a trip is a linear combination of the vectors, and a trip that starts and ends at the same place has a total displacement of $\mathbf{0}$. To answer the question, we can make a matrix A whose columns are the vectors $\mathbf{v_i}$:

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & -13 & -2 & -3 \\ 3 & -10 & 2 & 4 \\ 1 & -8 & 0 & -2 \end{bmatrix}$$

The question translates to: what is the solution set of the matrix equation $A\mathbf{x} = \mathbf{0}$? We know how to find these solutions: we row reduce the matrix A using Python and obtain:

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There is a pivot in every column, so we see that the columns of A are linearly independent, and thus the solution set of $A\mathbf{x} = \mathbf{0}$ is trivial: it contains only the zero vector. Thus, the only way Emmaford can "return" home is to do a trivial trip where she only travels by a 0 amount.

(b) The question of whether Emmaford can reach any other point in hyperspace with the spaceship translates to: is the span of the vectors $\mathbf{v_i}$ equal to the whole of $\mathbb{R}^4$? We see this as follows: a point in hyperspace is a vector in $\mathbb{R}^4$. The set of all such vectors is all of $\mathbb{R}^4$. A trip is a linear combination of the vectors $\mathbf{v_i}$, so asking if she can reach all points in $\mathbb{R}^4$ is asking whether every point is a linear combination of the vectors $\mathbf{v_i}$. The set of all linear combinations of the $\mathbf{v_i}$ is by definition the span of the vectors. To answer our question, we look back at the RREF of the matrix A from the previous part: we have seen in class that a set of vectors in $\mathbb{R}^m$ span $\mathbb{R}^m$ if the matrix whose columns are those vectors has a pivot in every row (think about why this is!). The RREF of A is the identity matrix of size 4×4 so indeed it has a pivot in every row. Thus, the columns of A (which are the vectors $\mathbf{v_i}$) span $\mathbb{R}^4$, or equivalently $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\} = \mathbb{R}^4$ and Emmaford can reach every other point in $\mathbb{R}^4$.

(c) Having translated our previous questions to mathematical ones, we now see how the sets we computed, namely, the $\{\mathbf{0}\}$ in part (a) and $\mathbb{R}^4$ in part (b) are subspaces:

- We found $\{\mathbf{0}\}$ as the solution set to the equation $A\mathbf{x} = \mathbf{0}$. This is by definition the null space of the matrix A. Thus $\{\mathbf{0}\} = \text{Nul}(A)$ is a subspace.
- We found $\mathbb{R}^4$ as the span of the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$, which are the columns of the matrix A. This is by definition the column span of A. Thus $\mathbb{R}^4 = \text{Col}(A)$ is a subspace.

(d) Emmaford can indeed reach Allenod by our results from part (a), which told us that she can reach any other point in hyperspace using the spaceship. Since the RREF of A is the identity matrix, we know that there will be a unique trip that will allow her to reach Allenod, but to determine what this solution is, we need to reduce the augmented matrix (using Python) corresponding to the matrix equation $A\mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 6 \end{bmatrix}$:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 3 \\ 0 & -13 & -2 & -3 & 5 \\ 3 & -10 & 2 & 4 & 6 \\ 1 & -8 & 0 & -2 & 6 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

The solution to the matrix equation is $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix}$, so Emmaford needs to travel 4 times along $\mathbf{v_1}$, -1 time along $\mathbf{v_3}$ and -1 time along $\mathbf{v_4}$.

(e) The set from part (c) is $\left\{ \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$. This is not a subspace: a subspace must contain the zero vector, which is clearly not in this set. More generally, a solution set to an inhomogeneous equation, as we have seen, is never a subspace: it is not a span (rather, it is always a translated span), and spans are equivalent to subspaces.