## ASSIGNMENT 1

### Contributors


- #75173526 Ethan **Wong**
- #83313395 Mariana **Cardoso**


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### Reflection question

*Reflection questions encourage you to think about how mathematics is done. This is an important ingredient of success. Reflection questions contribute to your* ***engagement grade****.*

1. To work productively as a team, it is helpful to have shared expectations. This question consists of prompts that form a "team contract", a document that guides how you will work together. Before answering the prompts, read the document "Group resources" that is posted along with the assignment on the Canvas course page.
   (a) What are your overarching "ground rules"? Come up with 4-6 specific expectations regarding communication (including how often and what medium), meetings (how often, how long, and where), preparation and attendance.
   (b) What actions will your team take if a member does not follow the ground rules? Be specific.
   (c) What happens if a team member does not fulfill an agreed-upon task for an assignment? Consider both how the team will handle any dropped tasks, as well as actions you will take as a team.

   **Answers**

   (a) 1. Meetings should be planned beforehand to avoid absences
   2. Communication through Instagram DMs
   3. Assignment parts will be equally divided between the two students
   4. Meeting place will vary from meeting to meeting, but should also be decided beforehand.
   5. If there is an emergency, the student needs to provide an explanation and reschedule the meeting as soon as possible. Absences without warning or a few minutes before the meeting should not be tolerated.
   6. Communication should be clear and respectful at all times

   (b) First receive a verbal warning and then report or be removed from the assignment.

   (c) Since the team consists of only two people, the other party must complete the assignment alone, remove the other party from the assignment and report their behavior.

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### Assignment questions

*The questions in this section contribute to your* ***assignment grade****. Starred questions are at the level of Part 2 exam questions.*

2. (★★☆☆)

We are going to consider the classical gravitational force associated with the planet Earth. Forces are vector quantities, which have magnitudes and directions, but we will focus on the scalar functions associated with these forces.

Let's assume Earth is a perfect sphere. Let's also assume the material making up Earth is uniformly distributed throughout this sphere, meaning the density of the Earth is uniform. (This is not true in reality as the core of the Earth is denser than the crust, for example.)

Suppose Earth has a radius $r_E$ and that we measure distances $r$ to any object from the centre of Earth, which we say is $r = 0$. Since we are interested in understanding the gravitational force of Earth alone, it is useful to imagine this object to be small and to have small mass since we will ignore the gravitational force produced by this object itself.

Physics tells us the gravitational force due to the mass of Earth on our small object will only depend on the distance $r$ this object is from the centre of Earth. We would like to find a function $F$ to describe the gravitational force as a function of the distance $r$.

For distances $r$ that are greater than $r = r_E$, Newton's law tell us the gravitational force of Earth on our small object is proportional to the inverse of the squared distance $r$; that is, $F(r) = \alpha/r^2$.

If we imagine drilling a hole from the surface of Earth to its centre and placing our object at some depth in this hole a distance $r$ from the centre, then an interesting application of Gauss's law tells us the gravitational force on our small object is proportional to its distance $r$ from the centre of Earth; that is, $F(r) = \beta r$.

(a) Find the explicit expression of the function $F(r)$ that describes the gravitational force associated with Earth, where $r \geq 0$.

(b) Sketch a graph of $F(r)$ and write a few observations about this function and its graph. What do you know about the gravitational force at the surface of Earth and what does this tell you about the function $F(r)$ when $r = r_E$? What do you know about the relationship between the constants of proportionality $\alpha$ and $\beta$?

*Hint:* The function $F(r)$ will be a piecewise-defined function; it has a different form for distances that place the object inside Earth than for objects outside Earth's surface.

### Answers

(a) We know that the surface of the Earth is exactly $r_E$ distance from its center. The gravitational force acted on an object from the surface of the earth can be represented by the function

$$F(r) = \frac{a}{r^2}$$

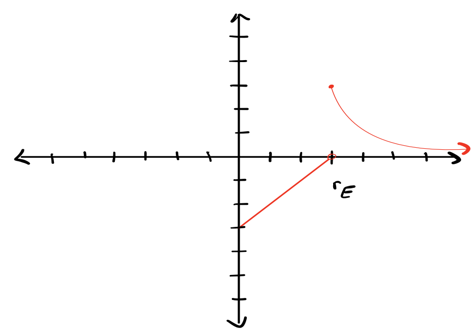
The gravitational force acted on an object when they are below the surface of the earth is represented by the function

$$F(r) = \beta \cdot r$$

We can then define a piecewise function that represents the gravitational force of the Earth as:

$$F(r) = \left\{ \begin{array}{ll} \frac{a}{r^2} & \text{if } r \geq r_E \\ \beta \cdot r & \text{if } r < r_E \end{array} \right\}$$

(b) At the surface, there is a jump discontinuity since the function changes from $\frac{\alpha}{r^2}$ to $\beta \cdot r$. Where $\alpha$ determines how steep the function is when $r \geq r_E$. $\beta$ determines the steepness of the slope of the function when $r < r_E$.




3. (★★★☆)

Let us consider now a slightly more realistic model of the internal structure of Earth. In particular, imagine the material inside Earth is still distributed with spherical symmetry, meaning the density only depends on the distance from the centre of Earth, but the density of the material varies with this distance. Suppose there is a dense central core of radius equal to $0.4r_E$ of uniform (constant) high density, where $r_E$ is the radius of Earth. This core is surrounded by a mantle of uniform (constant) density that is 60% of the density of the core.

Let's find the gravitational force on a small object placed at a point inside this Earth.

Since we are only looking inside the planet, we will measure distances from the centre of Earth as a fraction of Earth's radius. Call this fractional distance $R$ and so $R$ goes from 0 (the centre of Earth) to 1 (the surface of Earth). In terms of the usual distance $r$, we have $R = r/r_E$.

The gravitational force on a small object in the core is $F_c(r) = 1.6r$.

The expression for the gravitational force on a small object in the mantle is slightly more complicated: $F_m(r) = \dfrac{0.04}{r^2} + r$.

(a) Write down the function $F(r)$ describing the gravitational force an object will experience anywhere inside Earth given this model of Earth's internal structure.

(b) Sketch a graph of this function.

(c) If you drilled a narrow hole from the surface straight to the centre of Earth and dropped a small object into this hole, describe carefully how the object would accelerate as it fell towards the centre. (Remember Newton's Second Law says $F = ma$.)

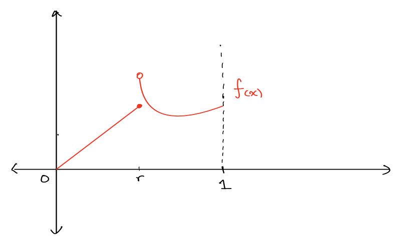
(d) Some approximations were made to produce the numerical constants in the expressions for the gravitational forces in the core and in the mantle. How can you tell these constants are not given exactly based on properties of the function $F(r)$?

**Answers**

(a) Again we can describe the force anywhere inside the earth as a piecewise function

$$F(r) = \left\{ \begin{array}{ll} \dfrac{0.04}{r^2} + r & \text{if } 0.4r_E < x \leq 1 \\ \\ 1.6r & \text{if } 0 \leq x \leq 0.4r_E \end{array} \right\}$$

(b) Sketched graph:




(c) We can first describe the acceleration of the object using the functions described above and using Newton's Second Equation. Solving for acceleration:

$$a = \frac{F}{m}$$

We can see that the acceleration is directly related to the gravitational force acting on it divided by its mass. From the surface of the earth all the way to $0.4r_E$ the force acting on the object due to gravity can be expressed as

$$F(R) = \frac{0.04}{r^2} + r$$

where $r$ is the distance from the center of the Earth. This function of the force only occurs in the mantle of the Earth. Passing the mantle (i.e. $r \leq 0.4r_E$), the force acting on the object becomes the function

$$F(R) = 1.6r$$

To find the acceleration of the object at any distance from the center, we first determine if the object is in the mantle or if it's in the core, find the force of gravity that would act on an object at that distance, and then divide by the object's mass.

(d) The constant given for the function of gravity when we are in the core of the Earth is a simple $1.6r$ implying that gravity is linear in the core where in reality there are other factors that would affect gravity. On top of that, we assumed that the Earth is uniformly formed and has a constant density when in reality this is not the case as some areas of the Earth are more dense than others, this is specifically true in the mantle of the Earth.

4. (★★★☆)

In 1970, science fiction author Larry Niven wrote *Ringworld*, in which there is a large artificial world constructed as a ring that has a diameter approximately the same size as Earth's orbit around the Sun, which means the circumference of Ringworld is about 1 billion kilometers. The total mass of this world is about $2 \times 10^{27}$ kilograms (about 333 times the mass of Earth), which means it has a large gravitational field. For the purposes of this question, we will consider this mass to be uniformly distributed around the ring.

While it is tricky to calculate exactly this gravitational field at a general point in space around this world, the symmetry of Ringworld about an axis running through the centre of the ring and perpendicular to the plane of the ring makes it possible to calculate exactly the gravitational force a small object placed anywhere on this axis will experience.

The easiest place to calculate the gravitational force is at the centre of the ring: it must be exactly 0 there because of the symmetry since each bit of the ring has a corresponding bit of the ring of equal mass diametrically opposite to it on the ring. These antipodal bits of the ring exert equal and opposite gravitational forces on a small object at the centre, meaning the net force there is 0 (technically, the gravitational force is a vector, so this is the 0-vector, but we will focus on the magnitude of it, which is the scalar 0). This is true for all such bits of the ring and so their total contributions add up to 0 at the centre of the ring.

Now consider the axis that passes through the centre of the ring, and that is perpendicular to the plane of the ring. Think of this axis as a straight line acting as a coordinate line where the origin of the coordinate is at the centre of the ring. Let's call this coordinate $z$ and say $z = 0$ is the centre. If $z > 0$, then we are looking at points in one direction along this axis, and when $z < 0$, we are considering points in the other direction. (The choice of positive direction is arbitrary.)

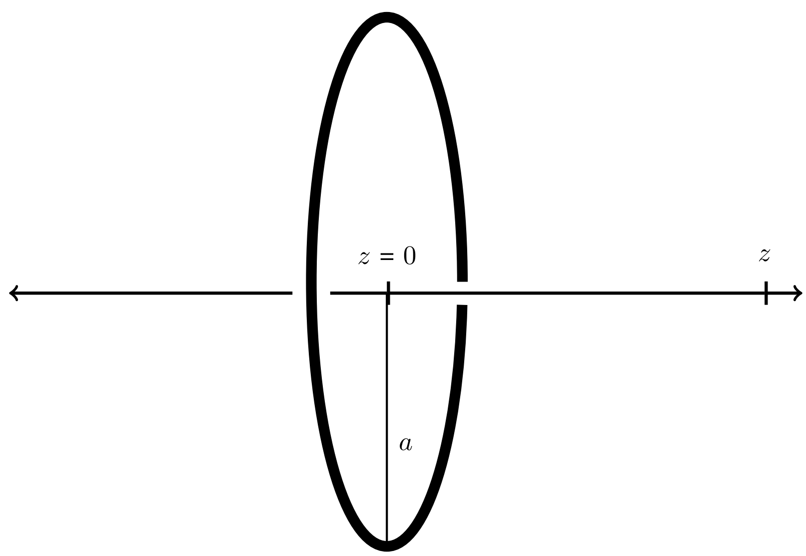



Figure 1: Ringworld and its Central Axis

To find the gravitational force at a point $z$ on this axis requires adding up all the contributions of bits of the ring. This requires knowing how to integrate, which is a topic covered in a later course, so we will just use the final result of this calculation: the scalar function associated to the gravitational force at a point $z$ on this central axis is given as

$$F(z) = \frac{Kz}{(a^2 + z^2)^{3/2}},$$

where $K > 0$ is a constant, and $a$ is the radius of the ring (≈150 million kilometers). This force points along the axis towards the centre of the ring. (Why?) Note this result includes the fact $F(0) = 0$, which says the net gravitational force at the centre of the ring is 0.

(a) If you are very close to the centre of Ringworld along this central axis, what does the gravitational force look like? That is, consider the behaviour of the function $F(z)$ for small $z$ – what simpler function does $F(z)$ look like in this case?

(b) If you are very far away from Ringworld along this central axis, what does the gravitational force look like? This is, consider the behaviour of the function $F(z)$ for large $z$ – what simpler function does F(z) look like in this case? Comment on the reasonableness of using this simpler function to study the gravitational field at large distances from Ringworld.

(c) Even though you don't have an expression for the function describing the gravitational force at a general point in space around Ringworld, what do you expect this function to look like when you consider points at large distances from Ringworld in any direction? What is the reason for your expectation?

**Answers**

(a) At a very small $z$ value, the $a^2$ dominates the numerator. This leaves us with

$$F(z) = \frac{Kz}{(a^2)^{3/2}} \longrightarrow F(z) = \frac{Kz}{a^3}$$

If we take the limit of this equation as $z \to 0$ we get:

$$\lim_{z \to 0} \frac{Kz}{a^3} = 0$$

This implies that the gravitational force as we get closer to $z = 0$ becomes weaker and weaker.

(b) At very large $z$ values the $z^2$ dominates the numerator leaving

$$F(z) = \frac{Kz}{z^3} \longrightarrow F(z) = \frac{K}{z^2}$$

If we take the limit of this equation as $z \to \infty$ we get:

$$\lim_{z \to \infty} \frac{K}{z^2} = 0$$

This again implies that the gravitational forces become weaker as our $z$ values increase to infinity.

(c) It is expected that at large distances away from the Ringworld will behave similarly to how they do if they were on the axis. Meaning that at larger and larger distances from the Ringworld the distance from the Ringworld will dominate the function and the gravitational force will be directly related to how for an object is from the Ringworld.