9
Optimization: What Is the Best…?

This final chapter is about achieving the *best result*, obtaining the *maximum gain*, finding the *optimal outcome*. Thus, this chapter is about *optimization*—an especially interesting subject because finding an optimum result may be difficult, and at times even impossible. Our experience with finding maxima and minima in calculus suggests that we can often find a point where the derivative of a function vanishes and an extreme value exists. But in engineering design and in life generally, we often have to "satisfice," that is, in the word of Herbert A. Simon, be satisfied with an acceptable outcome, rather than an optimal one. Here, however, we will focus on modeling the ways we seek optimal solutions. In so doing, we will see that the formulation of an optimization problem depends strongly on how we express the *objective function* whose extreme values we want and the *constraints* that limit the values that our variables may assume.

Why?
How?

Much of the work on finding optimal results derives from an interest in making good decisions. Many of the ideas about formulating optimization problems emerged during and after World War II, when a compelling need to make the very best use of scarce military and economic resources translated in turn into a need to be able to formulate and make the *best decisions* about using those resources. Thus, with improved decision making as the theme, we will also present (in Section 9.4) a method of choosing the best of an available set of alternatives that can be used in a variety of settings.

We will close with a miscellany of interesting, "practical" optimization problems.

9.1 Continuous Optimization Modeling

We start with a basic minimization problem whose solution is found using

Find? elementary calculus. Suppose that we want to find the minimum values of the *objective function*

$$U(x) = \frac{x^2}{2} - x, \tag{9.1}$$

which we have drawn in Figure 9.1. That picture of the objective function $U(x)$—so called because we set our objective as finding its extreme value—is a parabolic function of x, as the algebraic form of eq. (9.1) confirms.

How? Thus, it has only a single minimum value, called the *global* minimum. The value of x at which this global minimum is found is determined by setting the first derivative of $U(x)$ to zero:

$$\frac{dU(x)}{dx} = x - 1 = 0, \tag{9.2}$$

from which it follows that the minimum value of $U(x)$ occurs when $x_{\min} = 1$ and is

$$U_{\min} = U(x_{\min}) = -\frac{1}{2}. \tag{9.3}$$

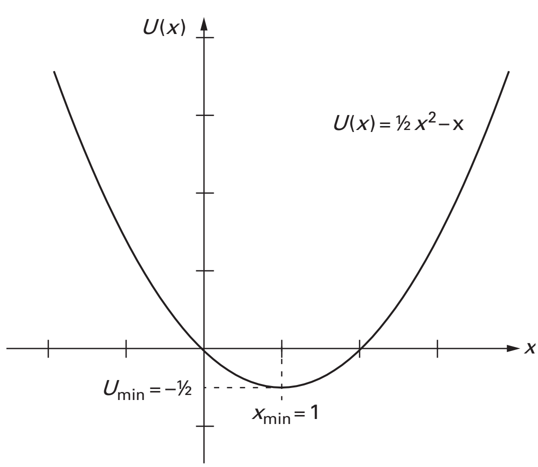

Figure 9.1 The objective function $U(x) = x^2/2 - x$ plotted over the unrestricted range of $-\infty \leq x \leq +\infty$. The minimum value of the objective function, $U_{\min} = -1/2$, occurs at $x_{\min} = 1$.

We also note from eq. (9.2) that the slope of $U(x)$ increases monotonically as x goes from $-\infty$ to $+\infty$, which means that $U(x)$ itself can have only one flat spot. We can confirm this by calculating the rate of change or derivative of the slope,

$$\frac{d^2U(x)}{dx^2} = 1, \tag{9.4}$$

which is always positive. Thus, there is only one minimum, and it is a global minimum. In fact, we can go a step further and identify the minimum value of eq. (9.3) as an *unconstrained minimum* because we did not constrain or limit the values that the variable x could assume.

Suppose we did impose a constraint, say of the form $x \leq x_0$, which requires the independent variable, x, to always be less than or equal to a given constant, x_0. This means that search for the minimum of $U(x)$ is limited to the *admissible values* of x: $x \leq x_0$.We can visualize a procedure for implementing this constraint as putting a line on the same graph as the curve, $U(x)$, and then "moving" this line to different values of x_0, as shown in Figure 9.2. The constraint then shows as the set of lines, $x_{01} < x_{02} < x_{03}$, so we can now briefly consider the three problems of determining the minimum values of $U(x)$ with $x \leq x_{0i}$, $i = 1, 2, 3$. In the first case, $i = 1$, the admissible range of x is so restricted that the constrained minimum value

Assume?

How?

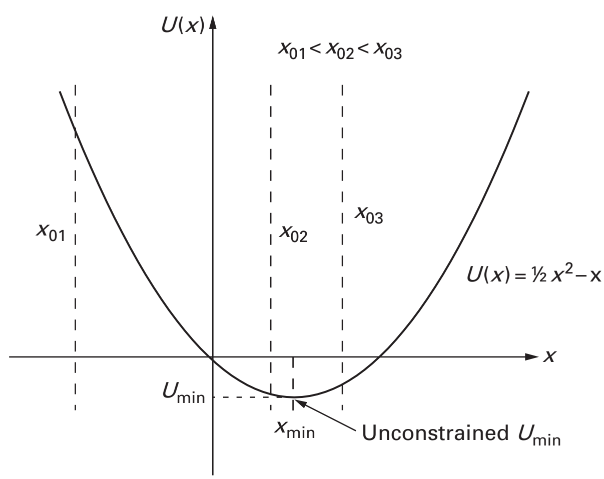

Figure 9.2 The objective function $U(x) = x^2/2 - x$ plotted together with three constraints that restrict the range of admissible values: the set of lines, $x_{01} < x_{02} < x_{03}$. These lines allow us to consider the three problems of determining the minimum values of $U(x)$ with $x \leq x_{0i}$, $i = 1, 2, 3$.

of $U(x)$ is apparently significantly greater than the unconstrained minimum of eq. (9.3). For example, if $x_{01} = -3$, the corresponding constrained minimum is $U(-3) = 7.5$. As the constraint "moves" further to the right $(i = 2, 3)$, we approach and then go through the unconstrained minimum. Thus, the range of *feasible solutions* for the minimum of $U(x)$ may include the unconstrained minimum, $U_{\min}$—or it may not—depending on just where the constraint boundaries happen to be.

The constraints so far imposed are *inequality constraints*, $x \leq x_0$, that bound the range of feasible values at the upper end by the equality, $x = x_0$, and include the interior region, $x < x_0$. We might have posed only a simple *equality constraint*, $x = x_0$, in which case we would have found a (highly) constrained minimum $U(x_0)$.

If our objective function were only slightly more complicated, the search for extreme points would become significantly more complicated. Consider the objective function

$$U(x) = \sin x, \tag{9.5}$$

This elementary function could have, depending on the limits placed on the range of admissible values of x, an infinite number of maxima and of minima, or a constrained extremum somewhere between the two (see Problem 9.1). The point of this seemingly trivial example is simple. Characterizing and finding the extrema can be complicated even when the objective function is well known and its properties well understood.

The objective functions (9.1) and (9.5) have only a single variable.

Why? However, *multi-dimensional optimization problems* are almost always the norm in engineering practice because engineered devices and processes rarely, if ever, depend only on a single variable. One simple example can be found at the local post office, where postal regulations typically stipulate that the rectangular package shown in Figure 9.3 can be mailed only if the sum of its girth $(2x + 2y)$ and length (z) do not exceed 84 in (2.14 m).

Find? What is the largest volume that such a rectangular package can enclose?

The objective function is the package's volume,

$$V(x, y, z) = xyz, \tag{9.6}$$

where x and y are the two smaller dimensions whose sum comprises the

Assume? package's girth, and the length, z, is its longest dimension. We assume that these three dimensions are positive real numbers (i.e., $x > 0$, $y > 0$, $z > 0$).

The constraint on the package dimensions stemming from the postal regulations can be written as:

$$\underbrace{2x + 2y}_{\text{girth}} + \underbrace{z}_{\text{length}} \leq 84, \tag{9.7}$$

Since we seek the largest possible volume, this inequality constraint on the package dimensions can be expressed as an equality constraint: **Assume?**

$$2x + 2y + z = 84. \tag{9.8}$$

Thus, the volume maximization problem is expressed as the objective function (9.6) to be maximized, subject to the equality constraint (9.8). Although the problem is formulated in three dimensions, we can use the equality constraint to eliminate one variable, say the length, z, so that the objective function becomes: **How?**

$$V(x, y) = xy(84 - 2x - 2y) = 84xy - 2x^2y - 2xy^2. \tag{9.9}$$

Now we want to find the maximum value of $V(x, y)$ as a function of x and y. As we recall from calculus, the necessary condition that $V(x, y)$ takes on an extreme value is:

$$\frac{\partial V(x, y)}{\partial x} = 84y - 4xy - 2y^2 = 2y(42 - 2x - y) = 0, \tag{9.10a}$$

and

$$\frac{\partial V(x, y)}{\partial y} = 84x - 2x^2 - 4xy = 2x(42 - x - 2y) = 0. \tag{9.10b}$$

Equations (9.10a–b) can be reduced to a pair of linear algebraic equations whose non-trivial solution can be found ($x = y = 14$ in) to determine the corresponding package volume, $V = 5488$ in^3. This volume can be confirmed to be a maximum (see Problems 9.4, 9.5).

The package problem, albeit multi-dimensional, was still relatively simple because its inequality constraint could logically and appropriately be reduced to an equality constraint that could, in turn, be used to reduce the dimensionality of the problem. Then we found the maximum volume of the package by applying standard calculus tools and seemingly without any further reference to constraints (see Problem 9.6). Consider for a moment the problem of finding the minimum of the following objective function:

$$U(x, y) = x^2 + 2(x - y)^2 + 3y^2 - 11y. \tag{9.11}$$

We show a three-dimensional rendering of this parabolic surface in Figure 9.3. It has an unconstrained minimum at the point ($x = 1, y = 1.5$), where $U_{\min} = -8.25$ (see Problem 9.7). What happens if an equality constraint is imposed? That is, in the style and terminology of the field of operations research, suppose that we want to find the **Find?**

$$\begin{array}{ll} \text{minimum of} & U(x, y) = x^2 + 2(x - y)^2 + 3y^2 - 11y, \\ \text{subject to} & x + y = 3. \end{array} \tag{9.12}$$

We could again use standard calculus techniques to show that the constrained minimum occurs at the point ($x = 31/24$, $y = 41/24$), where $U_{\min} = -385/48$ (see Problem 9.8). Note that the minimum is located on the boundary plane where the constraint intersects $U(x, y)$, that is, at a point such that $x + y = 31/24 + 41/24 = 72/24 = 3$. If the equality constraint of eq. (9.12) was replaced with the (strict) inequality constraint $x + y < 3$, we would find that the minimum sought lies inside the intersecting boundary plane.

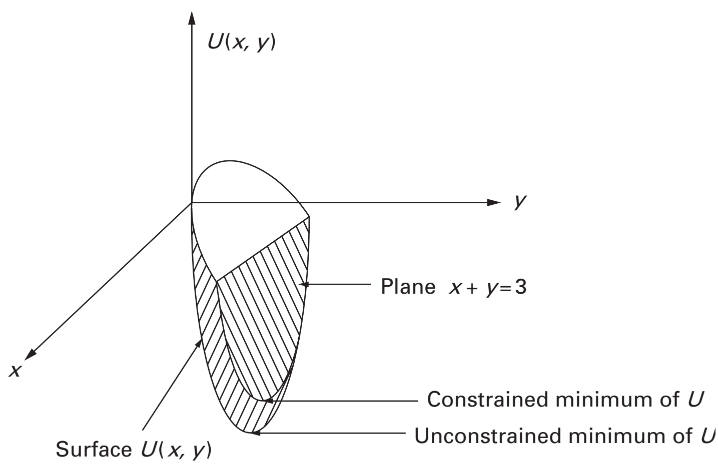

Figure 9.3 The objective function $U(x, y) = x^2 + 2(x - y)^2 + 3y^2 - 11y$ "plotted" in three dimensions, along with the plane $x + y = 3$ that could form the boundary of an equality constraint or of a corresponding inequality constraint.

Problem 9.1. Determine the maxima and minima of the elementary function, $U(x) = \sin x$, when the range of admissible values is:

(a) unconstrained;
(b) $0 < x < \pi/2$;
(c) $0 \leq x \leq \pi/2$; and
(d) $3\pi/4 \leq x \leq 9\pi/4$.

Problem 9.2. Assume an equality constraint is applied to the minimization of the objective function (9.1).

(a) Determine the corresponding value of $U_{\min}$.

(b) How would you characterize that extreme value (e.g., it is an _____ minimum)?

Problem 9.3. Solve the linear algebraic equations (9.10) and determine the three corresponding package dimensions and the package's volume, V.

Problem 9.4. If eqs. (9.10a–b) are the *necessary* conditions to find the maximum value of the function, V, of eq. (9.9), what additional requirements are needed to have *sufficient* conditions to obtain the maximum of V?

Problem 9.5. Apply the sufficient condition(s) found in Problem 9.4 to the package volume problem to confirm that the result calculated in Problem 9.3 is, in fact, a maximum.

Problem 9.6. Are there any "invisible" or implicit constraints in the package maximization problem? (*Hint*: Start with the fact that x, y, and z represent real physical quantities that can never be negative.)

Problem 9.7. Determine the location and value of the minimum of the parabolic surface given by eq. (9.11).

Problem 9.8. Use standard calculus methods to determine the constrained minimum defined by eq. (9.12). (*Hint*: Eliminate a variable.)

9.2 Optimization with Linear Programming

The section just completed showed that the search for an optimum or extreme value of a function subject to an inequality constraint requires a search over the interior of the region defined by the constraint boundary. Thus, as shown in Figure 9.2, we must search for all values of $x \leq x_{0i}$. This is true more generally because an objective function may fluctuate in value, perhaps like the sinusoid of eq. (9.5). Consider, for example, the sketch of a generic objective function in Figure 9.4. The good news is that the standard methods of calculus are usually adequate for searches where the objective functions are relatively tractable. The bad news is that, in such cases, we generally need to search the entire domain, $x_{04} \leq x \leq x_{05}$ to find global optima. However, there is a very important class of problems where a search of the interior region is not required because the optimum point must occur on one of the constraint boundaries. This class of problems is made up of objective functions that are linear functions of the independent variables, and their optimization searches are known as *linear programming* (LP).

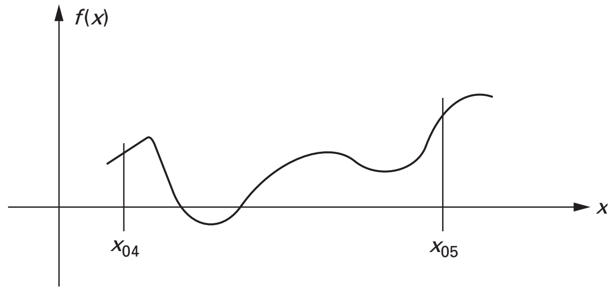

Figure 9.4 A generic sketch of an objective function that shows some variation or fluctuation, with peaks and valleys in the domain of interest. The bad news is that here we do need to search the entire domain, $x_{04} \leq x \leq x_{05}$, to find a global optimum. The good news is that the standard methods of calculus are usually adequate for searches if the objective functions are relatively straightforward.

Suppose we want to find (see Figure 9.5) the

$$\begin{array}{ll} \text{minimum of} & U(x) = mx + b, \\ \text{subject to} & x_1 \leq x \leq x_2. \end{array} \tag{9.13}$$

Now, the minimum of $U(x)$ must lie within the admissible range of values of x, defined by the two inequality constraints just given. Geometry, however, tells us that the optimal values of the linear objective function,

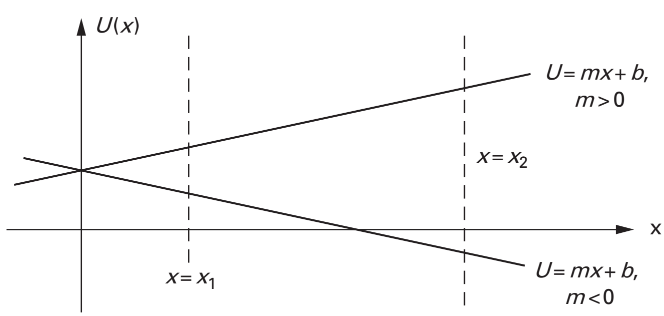

Figure 9.5 A generic *linear programming* problem which is characterized by an objective function that is a linear function of the variable x. Note that the optimal values, both maxima and minima, for $m > 0$ or $m < 0$, occur at points where the objective function intersects the constraint boundaries, that is, on the constraint boundaries themselves.

$U(x) = mx + b$, occur at points where $U(x)$ intersects one of the two constraint boundaries. For $m > 0$, $U_{\min}$ must occur at $x = x_1$ and $U_{\max}$ must occur at $x = x_2$. Thus, for this linear programming problem, we can find the optima of $U(x)$ without searching the interior region defined by the constraint boundaries: We know *a priori* that the optima must occur on the constraint boundaries. In fact, it can be shown that the optimum solutions for LP problems are found by searching only at the boundary intersections or *vertices.* The search problem is thus "reduced" to solving for a set of intersection points defined by various linear equations.

Is requiring an objective function to be linear too much of a simplification? Are LP problems useful, or a cute mathematical artifact? In fact, LP is extremely important and useful, and is one of the cornerstones of the field of *operations research.* The field of operations research (OR)—pronounced "oh r"—developed first in Britain and then in the United States during World War II when there was a compelling interest in optimizing scarce military and economic resources. Since that time, OR has been applied to both military and civil problems, including in the latter a wide variety of commercial enterprises, allocating medical resources, managing traffic, and modeling the criminal justice system. The hallmark of LP is the determination of optimal results for *single* objectives: *minimizing* transportation costs, *optimizing* the product mix, *maximizing* hospital bed availability, *minimizing* the number of highway toll attendants when traffic is slack, or *minimizing* drivers' waiting times when traffic is heavy.

9.2.1 Maximizing Profit in the Furniture Business

Suppose that we are in the furniture business and making desks and tables that are made of oak and maple. Desks and tables consume different amounts of lumber: a desk requires 6 board-feet (bft) each of oak and maple, while a table requires 3 bft of oak and 9 bft of maple. The local lumber mill will supply up to 1200 bft of oak at \$6.00/bft and up to 1800 bft of maple at \$4.00/bft. The market for desks and tables is such that they can be sold for, respectively, \$90.00 and \$84.00. How many desks and how many tables should we make to maximize our profit?

Why?
Given?
Assume?
Find?

We will soon find out (see eq. (9.16)) that under the conditions assigned here, the profits earned by selling a desk are the same as the profits earned by selling a table, namely, \$30.00 each. Suppose that this was not the case and that the profit in selling a table was only \$18.00. Then it might seem reasonable to first make only desks to maximize profit—except that we will run out of oak after only 200 desks are made and have an excess, unusable supply of maple left over. It will also turn out that the profit earned in this case, \$6000, is not the maximum profit possible. This problem is interesting because the constraints supply *limits* on the available materials, which means in turn that we must make *trade-offs* between desks and tables to maximize our overall profit.

How? We formulate this profit optimization problem as an LP problem, meaning that we build an objective function—the difference between sales income and cost of manufacture—and the relevant operating constraints. If x_1 is the number of desks made, and x_2 the number of tables, the income derived by selling desks and tables is:

$$\begin{aligned}\text{income} &= (\$/\text{desk})x_1 + (\$/\text{table})x_2 \\ &= \$(90x_1 + 84x_2).\end{aligned} \tag{9.14}$$

The cost of manufacture is reckoned in terms of the quantity of lumber required for each product and the unit costs of that lumber:

$$\begin{aligned}\text{cost} =&(\$/\text{oak})[(\text{oak}/\text{desk})x_1 + (\text{oak}/\text{table})x_2] \\ &+ (\$/\text{maple})[(\text{maple}/\text{desk})x_1 + (\text{maple}/\text{table})x_2] \\ =&(\$6.00)(6x_1 + 3x_2) + (\$4.00)(6x_1 + 9x_2) \\ =&\$(60x_1 + 54x_2).\end{aligned} \tag{9.15}$$

The profit to be maximized, the objective function, is the difference between the income (eq. (9.14)) and the cost (eq. (9.15)):

$$\begin{aligned}\text{profit} &= \$(90x_1 + 84x_2) - \$(60x_1 + 54x_2) \\ &= \$(30x_1 + 30x_2).\end{aligned} \tag{9.16}$$

Note that the income, cost, and profit are all expressed in a common unit of currency, in this case U.S. dollars ($).

There are three constraints for this linear programming problem, deriving from the limitations of the wood's availability from the lumber mill and the fact that wood is a real physical object. On availability, the manufacturer can't use any more wood than the lumber mills can make available, so that

$$\begin{aligned}\text{amount of oak used} &= (\text{oak}/\text{desk})x_1 + (\text{oak}/\text{table})x_2 \\ &= 6x_1 + 3x_2 \leq 1200 \text{ (bft)},\end{aligned} \tag{9.17a}$$

and

$$\begin{aligned}\text{amount of maple used} &= (\text{maple}/\text{desk})x_1 + (\text{maple}/\text{table})x_2 \\ &= 6x_1 + 9x_2 \leq 1800 \text{ (bft)}.\end{aligned} \tag{9.17b}$$

Note that both constraints are expressed in terms of a common dimension, board-feet (see Problems 9.9 and 9.10).

The third constraint follows from the simple fact that the numbers of tables and desk must be real, positive numbers. Thus, there is a *non-negativity constraint*:

$$x_1, x_2 \geq 0. \tag{9.18}$$

So, to sum up our furniture LP problem, then, we want to find the

$$\begin{array}{ll} \text{maximum of} & \$(30x_1 + 30x_2), \\ \text{subject to} & \begin{cases} 6x_1 + 3x_2 \leq 1200(\text{bft}), \\ 6x_1 + 9x_2 \leq 1800(\text{bft}), \\ x_1, x_2 \geq 0. \end{cases} \end{array} \tag{9.19}$$

Note that the non-negativity constraint is almost always a part of LP formulations, largely because the variables involved in LP or optimization problems are real physical variables that by their very nature are greater than (or sometimes equal to) zero. In addition, it is fairly easy to convert "negative" variables to "positive" variables by suitable sign changes in the objective function and in the constraints. Finally, and likely most importantly, there is a significant computational advantage when all of the variables are positive because the non-negativity constraint limits the admissible space of the variables substantially. **How?**

The solution to the LP problem posed in eq. (9.19) will be found graphically. In Figure 9.6 we show the admissible space as the first quadrant in the (x_1, x_2) plane; the objective function (9.16) as a series of dotted lines; and the two inequality constraints (9.17) as (labeled) solid lines. The feasible region, wherein *all* (in this case *both*) constraints are satisfied, is shaded. The objective function can be thought of as a series of parallel dotted lines that can take on different, *increasing* values as the variables x_1 and x_2 increase (see Problem 9.11). We observe that the objective function will reach its largest value when it reaches the point (150, 100) because it is the last point on the constraint boundary that is within the feasible space—here at the intersection of the two inequality constraints (9.17). The value attained by the objective function at that point is \$6300. **How?**

We might have spotted this solution immediately on the basis of our previous observation that LP optima must lie on the constraint boundary. In the present case, the objective function, $30(x_1 + x_2)$, must reach its maximum at one of the vertices formed by the intersection of the constraint boundary with the feasible region ((200, 0), (0, 200)) or with each other (150, 100). Clearly, the point among these three that produces the largest objective function is that at the intersection of the two inequality constraints. So, again, the furniture maker will reap the most profit by making (and selling) 150 desks and 100 tables, which will yield a profit of \$6300. **Valid?** **Use?**

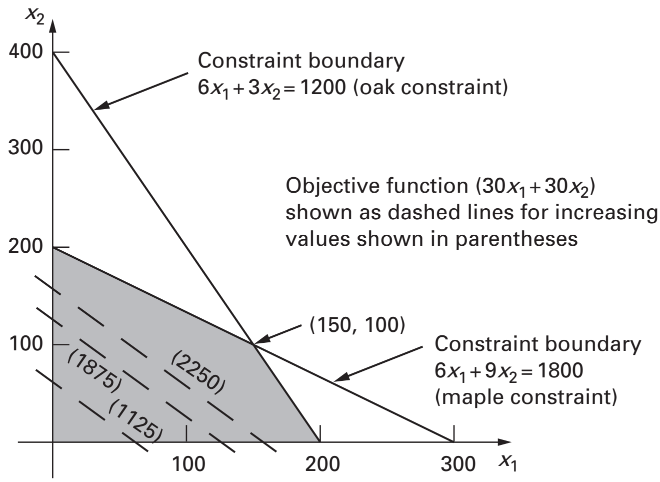

Figure 9.6 The graphical solution to the LP problem posed in eq. (9.19). Note that the admissible space is the first quadrant in the (x_1, x_2) plane; the objective function (9.16) is portrayed in a series of dotted lines; and the two inequality constraints (9.17) are the (labeled) solid lines. The region of feasible solutions, wherein *all* (in this case *both*) constraints are satisfied, is shaded.

9.2.2 On Linear Programming and Extensions

The example presented just above is rather simple because it has only two variables, which made it amenable to graphical solution. LP problems often have hundreds or even thousands of variables and so cannot be handled graphically, but they can be solved straightforwardly with a variety of standard computational approaches. All of these approaches to LP problems, the most notable of which is the *simplex* method, work by identifying the boundary vertices at which optima must lie in ways that are analogous to our graphical solution. There are also many other classical OR/LP problems, including the *feed-mix* and *product-mix* problems that occur repeatedly in industry, and the *transportation* problem that we will discuss in the next section.

There are other so-called programming problems and methodologies that are used to solve more complicated optimization problems. For example, *nonlinear programming* (NLP) refers to the set of techniques used when the objective functions are nonlinear. *Dynamic programming* refers to the class of problems that require sequential, hierarchical decisions, that is, problems where the solution to one problem serves as input to or a starting point for another problem. For example, in an extension of the furniture problem just solved, the lumber prices might vary over time because of external supply factors, in which case different production decisions might be made. Similarly, *integer programming* is designed to deal with those problems in which variables must be treated as integers, rather than as continuous variables. For example, we treated the numbers of desks and tables as continuous variables in the furniture optimization problems, but it is hard to imagine that we would make 150.7 desks or 99.6 tables. The results could, of course, be rounded off, but then we lose our guarantee of optimality. More importantly, however, integer programming is used for problems that have binary variables (for example, zero or one). For example, *scheduling* problems, such as when an election agency locates polling places within particular zip codes, are the kind of "go or no go" situations where integer programming is of most use.

9.2.3 On Defining and Assessing Optima

The optimal or best solution may well depend on the perspective of the person conducting or sponsoring the study. Someone has to say what he or she means by "the best." Consider the three cities that are spaced as in Figure 9.7. New connecting highways are to be built between the three cities, and the taxpayers clearly want the best result. The question is: What is the best result? If the best result is defined as the shortest travel time between two adjacent cities, then the configuration shown in Figure 9.7(a) will be the best. If the best result is defined as the least amount of road construction, then the configuration shown in Figure 9.7(b) will be the best. Thus, as the old saying goes, "Where you stand depends on where you sit." Optimizing travel time (the commuter's perspective) may be the best, but minimizing road construction (the taxpayer's perspective) may also be the best.

Further, the choices are rarely as simple as that. The reduction of commuter travel time may lead to tangible economic benefits, perhaps from more rapid delivery of goods, perhaps from a reduced pollution burden, or it may prompt more travel that increases noise and air pollution. A careful calculation of such benefits might be used to decrease the net cost of the first highway configuration, which might change the assessment of which highway pattern is truly the best. Clearly, doing such calculations requires the assignment of economic values to waiting time, delivery time, inventory costs, pollution burdens, and to other aspects of "reality" that may be relevant. Even the choice between scenic and direct routes can be modeled as an economic choice because it reflects a value judgment about whether

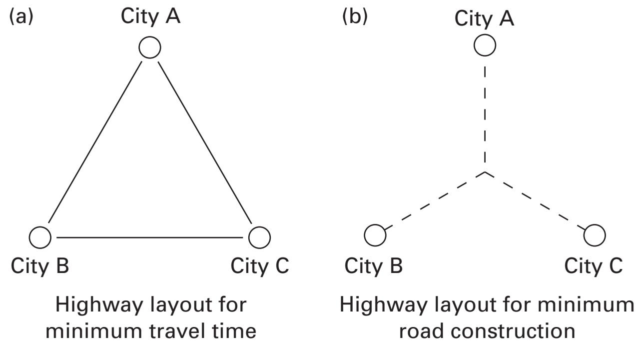

Figure 9.7 Alternate highway configurations for connecting three cities. (a) The first configuration minimizes travel time between adjacent cities. (b) The second configuration minimizes the amount of road construction needed.

it is more important to enjoy the landscape or to get to the destination as quickly as possible. Thus, three important points are:

- With LP and other OR techniques, we are modeling *decisions*, rather than physical behavior or the like.
- When making such decisions, we are making *trade-offs* between *costs* and *benefits.*
- When formulating and modeling such decisions, we are using cost-benefit analysis to make explicit our values and preferences.

Problem 9.9. Verify the dimensions of eqs. (9.14–9.16).

Problem 9.10. Verify the dimensions of eqs. (9.17a–b).

Problem 9.11. Determine the slope of the dotted lines in Figure 9.6 that represents the objective function (9.16) of the furniture LP problem. How does it compare with the slopes of the two constraint boundaries?

9.3 The Transportation Problem

Why? Having decided in Section 9.2 how many desks and tables we must make in order to maximize our profit from making furniture, we now turn to selling our desks through a *distribution network* of furniture outlets. The stores are at varying distances from the furniture maker's two plants—we have been doing well and have expanded our operations!—and each store has its own demand, based on its own marketing analyses. Thus, we have the logistical problem of deciding how to allocate the desks among the stores. This class of OR problems is called the *transportation problem.*

Three furniture stores have ordered desks: Mary's Furniture Emporium wants 30, Lori's Custom Furniture wants 50, and Jenn's Furniture Bazaar wants 45. We have made 70 desks at Plant 1 and another 80 at Plant 2. The distances between the two plants and the three stores are given in Table 9.1, and the shipping cost is \$1.50 per mile per desk. We want to minimize the shipping costs of filling the three orders. Since the cost of shipping a desk is easily calculated (see Problem 9.12), we have to calculate how many desks go from a specified plant (of two) to a particular store (of three).

Given?

What?

How?

Table 9.1 The distances (in miles) between Plants 1 and 2, where the desks are made, to Mary's Furniture Emporium, Lori's Custom Furniture, and Jenn's Furniture Bazaar, where the desks will be sold.

	Product		
Plant	**(1) Mary's**	**(2) Lori's**	**(3) Jenn's**
1	10	5	30
2	7	20	5

Is an optimal solution available easily? Doesn't it seem reasonable to ship first along the shortest (and cheapest) routes? Here we would send 50 desks from Plant 1 to Lori's, 45 desks from Plant 2 to Jenn's, and so on; and this approach might yield the optimal solution in this case (see Problem 9.13). However, this is a simple problem that has only two plants and three stores, and thus only six plant-store combinations to consider. In addition, the supply of desks is distributed so that the demand for all three stores can be met with only the shortest routes (see Problem 9.14). Thus, this simple problem does not need the kind of trade-off among alternatives that was needed to maximize profit for making furniture. However, it is a useful template for more complex problems, so let us solve it in a formal way.

This transportation problem of shipping desks from two plants to three stores can be represented as an elementary *network problem*, as depicted in Figure 9.8. The circles represent *nodes* at which desks are either supplied (the plants) or consumed (the stores). In more elaborate problems the nodes may be points to which material is supplied and from which material is distributed. The directed line segments (the arrows) are *links*

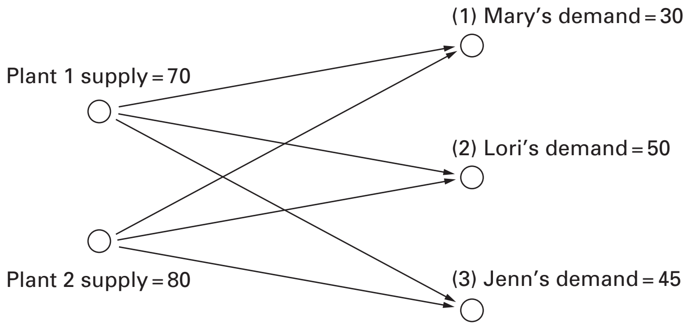

Figure 9.8 A *network representation* of the elementary transportation problem of shipping desks from two plants to three stores. The circles represent *nodes* at which desks are either supplied (the plants) or consumed (the stores). The directed line segments (the arrows) are *links* that represent the routes along which the desks could be shipped.

that represent the routes along which the desks could be shipped, and in more elaborate problems, these directed line segments may thus signify two-way or bi-directional links.

One possible—although *sub-optimal*—solution to this transportation problem is shown in Figure 9.9. We can calculate the shipping cost for this solution as \$1387.50 (see Problem 9.15), which is substantially higher than the optimal solution (see Problem 9.13 again).

The transportation problem can be formulated as an LP problem, for which some additional notation will be useful. Thus, we now identify x_{ij} as the number of desks shipped from Plant $i = 1, 2$ to Store $j = 1, 2, 3$. (Note

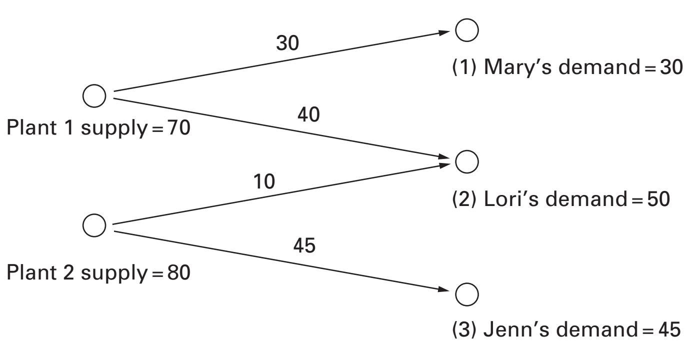

Figure 9.9 One possible solution to the elementary transportation problem of shipping desks from Plants 1 and 2 to Mary's Furniture Emporium, Lori's Custom Furniture, and Jenn's Furniture Bazaar.

that the two plants were numbered from the beginning, and the stores were assigned numbers in Table 9.1.) Then we have

x_{11} = number of desks from Plant 1 to Store 1 (Mary's),
x_{12} = number of desks from Plant 1 to Store 2 (Lori's),
x_{13} = number of desks from Plant 1 to Store 3 (Jenn's).
x_{21} = number of desks from Plant 2 to Store 1 (Mary's),
x_{22} = number of desks from Plant 2 to Store 2 (Lori's), and
x_{23} = number of desks from Plant 2 to Store 3 (Jenn's).

Since the unit cost of shipping is a constant ($1.50 per desk per mile), we can use the data in Table 1 to establish an objective function that is equal to the shipping cost:

$$\text{shipping cost } = (\$1.50)(10x_{11} + 5x_{12} + 30x_{13} + 7x_{21} + 20x_{22} + 5x_{23}). \tag{9.20}$$

The constraints for this problem arise from the supply of desks produced by the two plants and the demand for the desks by the three stores. The two plants cannot exceed their capacities for producing desks:

$$\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 70, \\ x_{21} + x_{22} + x_{23} &\leq 80. \end{aligned} \tag{9.21}$$

The stores, in turn, must have enough desks shipped to them to meet their demands:

$$\begin{aligned} x_{11} + x_{21} &\geq 30, \\ x_{12} + x_{22} &\geq 50, \\ x_{13} + x_{23} &\geq 45. \end{aligned} \tag{9.22}$$

Finally, the numbers of desks must satisfy a non-negativity constraint because the desks are real, so that:

$$x_{ij} \geq 0. \tag{9.23}$$

Thus, to sum up the formulation of our shipping problem as an LP problem, we want to find the

$$\begin{aligned} &\text{minimum of} \quad (\$1.50)(10x_{11} + 5x_{12} + 30x_{13} + 7x_{21} + 20x_{22} + 5x_{23}), \\ &\text{subject to} \quad \begin{cases} x_{11} + x_{12} + x_{13} \leq 70, \\ x_{21} + x_{22} + x_{23} \leq 80, \\ x_{11} + x_{21} \geq 30, \\ x_{12} + x_{22} \geq 50, \\ x_{13} + x_{23} \geq 45, \\ x_{ij} \geq 0. \end{cases} \end{aligned} \tag{9.24}$$

The shipping problem posed thus far has supply exceeding demand. A more restricted version, the *classical transportation problem*, sets the total supply equal to the total demand. The five inequality constraints (9.21) and (9.22) become simple equality constraints, which reduces the number of independent constraints by one. Suppose that Plant 1 produces only 45 desks (instead of 70). This reduces the (total) supply to a level that equals the demand level of 125 desks. Thus, the supply constraints (9.21) become

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= 45, \\ x_{21} + x_{22} + x_{23} &= 80, \end{aligned} \tag{9.25}$$

whose sum, a constraint on the total supply, can then be found to be:

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = 125. \tag{9.26}$$

Similarly, the demand constraints (9.22) become

$$\begin{aligned} x_{11} + x_{21} &= 30, \\ x_{12} + x_{22} &= 50, \\ x_{13} + x_{23} &= 45, \end{aligned} \tag{9.27}$$

and their sum, a constraint on the total demand, adds up to the same result as for the total supply [eq. (9.26)]. Since the total demand and the total supply equations are the same, the set of constraints (9.25) and (9.27) represent only four—not five—independent equations.

This reduction in the number of independent constraints produces some real computational benefits in solving classical transportation problems. One of the benefits is that all of the variables turn out to be integers when the constraints are expressed as integers. Further, the "extra" constraint produced by equating supply to demand results in comparatively straightforward and efficient computations of the optimum.

If the demand exceeds the supply, the LP model cannot even get started because it is impossible to get into the feasible region—from which the solutions derive. This is clear from summing the supply constraint inequalities (9.21),

$$\sum_{i,j} x_{ij} \leq \text{supply}, \tag{9.28}$$

and comparing it to the sum of the demand inequalities (9.22),

$$\sum_{i,j} x_{ij} \geq \text{demand}. \tag{9.29}$$

Of course, if supply exceeds demand, so that there is a net, positive surplus, an LP solution can proceed in a straightforward fashion.

Problem 9.12. Formulate and verify the dimensions of the equation needed to calculate the cost of shipping a desk from either plant to any of the three stores.

Problem 9.13. Complete the easily available optimal solution to the desk shipping problem and determine the minimum shipping cost.

Problem 9.14. Which of the six available plant-store routes were used in achieving the optimal solution of Problem 9.13?

Problem 9.15. Find the actual shipping cost of the transportation solution shown in Figure 9.9.

Problem 9.16. Confirm the objective function (9.20). (*Hint*: Use the result of Problem 9.12.)

Problem 9.17. Verify the constraints (9.21) and (9.22).

Problem 9.18. Verify that the sums of the supply (9.25) and demand (9.28) equality constraints add to the same sum as eq. (9.26).

9.4 Choosing the Best Alternative

Why?

People choose among alternatives all of the time: voters rank candidates; designers rank objectives; and students rank colleges. In each of these circumstances, the *voter* or *decision maker* is charged with choosing among the alternatives.

9.4.1 Rankings and Pairwise Comparisons

In recent years, questions have been raised about *how* voters establish rankings of alternatives. Further, since people seem to compare objects in a list on a pairwise basis before rank ordering the entire list, there is a special focus on how pairwise comparisons are performed as a means of assembling information for doing rank orderings. In *pairwise comparisons*, the elements in a set (i.e., the candidates, design objectives, or colleges) are ranked two at a time, on a pair-by-pair basis, until all of the permutations have been exhausted. Points are awarded to the winner of each comparison. Then the points awarded to each element in the set are summed, and the rankings are obtained by ordering the elements according to points accumulated. However, it is worth noting that as both described here and practiced, the number of points awarded in such pairwise comparisons is often non-uniform and arbitrarily weighted. But, as we will note below, it is quite important that the points awarded be measured in fixed increments.

The pairwise comparison methodology has been criticized particularly because it violates the famous *Arrow impossibility theorem* for which Kenneth J. Arrow was awarded the 1972 Nobel Prize in Economics. In that theorem, Arrow proved that a perfect or fair voting procedure cannot be developed whenever there are more than two candidates or alternatives that are to be chosen. He started by analyzing the properties that would typify a *fair* election system, and stated (mathematically) that a voting procedure can be characterized as fair *only* if four axioms are obeyed:

1. *Unrestricted*: All conceivable rankings registered by individual voters are actually possible.
2. *No Dictator*: The system does not allow one voter to impose his/her ranking as the group's aggregate ranking.
3. *Pareto Condition*: If every individual ranks *A* over *B*, the societal ranking has *A* ranked above *B*.
4. *Independence of Irrelevant Alternatives* (IIA): If the aggregate ranking would choose *A* over *B* when *C* is not considered, then it will not choose *B* over *A* when *C* is considered.

Arrow proved that at *least one of these properties must be violated* for problems of reasonable size (at least three voters expressing only ordinal preferences among more than two alternatives). It is worth noting that a consistent social choice (voting) procedure can be achieved by violating any one of the four conditions. Further, some *voting procedures* based on pairwise comparisons are faulty in that they can produce ranking results that offend our intuitive sense of a reasonable outcome—and quite often a desired final ranking can be arrived at by specifying a voting procedure.

Among pairwise comparison procedures, the *Borda count* (which we describe below, in Section 9.4.2) most "respects the data" in that it avoids the counter-intuitive results that can arise with other methods. As D. G. Saari notes, the Borda count "*never elects the candidate which loses all pairwise elections … always ranks a candidate who wins all pairwise comparisons above the candidate who loses all such comparisons*."

The Borda count does violate Arrow's final axiom, the *independence of irrelevant alternatives* (IIA). What does it mean that IIA is violated? And, is that important? The meaning depends to some extent on the domain and whether or not there are meaningful alternatives or options that are being excluded. In an election with a finite number of candidates, the IIA axiom is likely not an issue. In conceptual design, where the possible space of design choice is large or even infinite, IIA could be a problem. However, rational designers must find a way to limit their set of design alternatives to a finite, relatively small set of options. Thus, options that don't meet some criteria or are otherwise seen as poor designs may be eliminated. It is unlikely that IIA matters much if it is violated for one of these two reasons—and there is some evidence to support this—unless it is shown that promising designs were wrongly removed early in the process.

The violation of IIA leads to the possibility of *rank reversals*, that is, changes in order among n alternatives that may occur when one alternative is dropped from a once-ranked set before a second ranking of the remaining n–1 alternatives (as we will soon see below). The elimination of designs or candidates *can* change the tabulated rankings of those designs or candidates that remain under consideration. The determination of which design is "best" or which candidate is "preferred most" may well be sensitive to the set of designs considered.

Rank reversals occur when there are *Condorcet cycles* in the voting patterns: $[A \succ B \succ C, B \succ C \succ A, C \succ A \succ B]$. When aggregated over all voters and alternatives, these cycles cancel each other out because each option has the same Borda count. When one of the alternatives is removed, this cycle no longer cancels. Thus, removing C from the above cycle unbalances the Borda count between A and B, resulting in a unit gain for A that is propagated to the final ranking results. Thus, the rank reversals symbolize a loss of information that occurs when an alternative is dropped or removed from the once-ranked set.

We now describe a way to use pairwise comparisons in a structured approach that parallels the role of the Borda count in voting procedures and, in fact, produces results that are identical to the accepted vote-counting standard, the Borda count. The method is a structured extension of pairwise comparisons to a *pairwise comparison chart* (PCC) or matrix. The PCC produces consistent results quickly and efficiently, and these results are identical with results produced by a Borda count.

9.4.2 Borda Counts and Pairwise Comparisons

We begin with an example that highlights some of the problems of (non-Borda count) pairwise comparison procedures. It also suggests the equivalence of the Borda count with a structured pairwise comparison chart (PCC).

Twelve (12) voters are asked to rank order three candidates: A, B, and C. In doing so, the twelve voters have, collectively, produced the following sets of orderings:

$$\begin{array}{ll} 1 \text{ preferred } A \succ B \succ C, & 4 \text{ preferred } B \succ C \succ A, \\ 4 \text{ preferred } A \succ C \succ B, & 3 \text{ preferred } C \succ B \succ A. \end{array} \tag{9.30}$$

Pairwise comparisons other than the Borda count can lead to inconsistent results for this case. For example, in a widely used plurality voting process called *the best of the best*, A gets 5 first-place votes, while B and C each get 4 and 3, respectively. Thus, A is a clear winner. On the other hand, in an "antiplurality" procedure characterized as *avoid the worst of the worst*, C gets only 1 last-place vote, while A and B get 7 and 4, respectively. Thus, under these rules, C could be regarded as the winner. In an iterative process based on *the best of the best*, if C were eliminated for coming in last, then a comparison of the remaining pair A and B quickly shows that B is the winner:

$$\begin{array}{ll} 1 \text{ preferred } A \succ B, & 4 \text{ preferred } B \succ A, \\ 4 \text{ preferred } A \succ B, & 3 \text{ preferred} B \succ A. \end{array} \tag{9.31}$$

On the other hand, a Borda count produces a clear result. The Borda count procedure assigns numerical ratings separated by a common constant to each element in the list. Thus, sets such as (3, 2, 1), (2, 1, 0) and (10, 5, 0) could be used to rank a three-element list. If we use (2, 1, 0) for the rankings presented in eq. (9.30), we find total vote counts of (A: $2+8+0+0 = 10$), (B: $1+0+8+3 = 12$) and (C: $0+4+4+6 = 14$), which clearly shows that C is the winner. Furthermore, if A is eliminated and C is compared only to B in a second Borda count,

$$\begin{array}{ll} 1 \text{ preferred } B \succ C, & 4 \text{ preferred } B \succ C, \\ 4 \text{ preferred } C \succ B, & 3 \text{ preferred } C \succ B. \end{array} \tag{9.32}$$

C remains the winner, as it also would here by a simple vote count. It must be remarked that this consistency cannot be guaranteed, as the Borda count violates the IIA axiom.

We now make the same comparisons in a PCC matrix, as illustrated in Table 9.2. As noted above, a point is awarded to the winner in each pairwise comparison, and then the points earned by each alternative are summed. In the PCC of Table 9.2, points are awarded row-by-row, proceeding along

Table 9.2 A pairwise comparison chart (PCC) for the ballots cast by twelve (12) voters choosing among the candidates A, B and C (see eq. (9.30)).

Win/Lose	A	B	C	Sum/Win
A	——	$1+4+0+0$	$1+4+0+0$	10
B	$0+0+4+3$	——	$1+0+4+0$	12
C	$0+0+4+3$	$0+4+0+3$	——	14
Sum/Lose	14	12	10	——

each row while comparing the row element to each column alternative in an individual pairwise comparison. This PCC result shows that the rank ordering of preferred candidates is entirely consistent with the Borda results just obtained:

$$C \succ B \succ A. \tag{9.33}$$

Note that the PCC matrix exhibits a special kind of symmetry, as does the ordering in the "Win" column (largest number of points) and the "Lose" row (smallest number of points): the sum of corresponding off-diagonal elements, $X_{ij} + X_{ji}$, is a constant equal to the number of comparison sets.

We have noted that a principal complaint about some pairwise comparisons is that they lead to rank reversals when the field of candidate elements is reduced by removing the lowest-ranked element between orderings. (Strictly speaking, rank reversal can occur when any alternative is removed. In fact, and as we note further in Section 9.4.3, examples can be constructed to achieve a specific rank reversal outcome. Such examples usually include a dominated option that is not the worst. Also, rank reversals are possible if new alternatives are *added*.) Practical experience suggests that the PCC generally preserves the original rankings if one alternative is dropped. If element A is removed above and a two-element runoff is conducted for B and C, we find the results given in Table 9.3. Hence, once again we find

$$C \succ B. \tag{9.34}$$

The results in inequality (9.34) clearly preserve the ordering of inequality (9.33), that is, no rank reversal is obtained as a result of applying the PCC approach. In those instances where some rank reversal does occur, it is often among lower-ranked elements where the information is strongly influenced by the removed element (see Section 9.4.3).

Table 9.3 A reduced pairwise comparison chart (PCC) for the problem in Table 9.2 wherein the "loser" A in the first ranking is removed from consideration.

Win/Lose	B	C	**Sum/Win**
B	——	$1+0+4+0$	5
C	$0+4+0+3$	——	7
Sum/Lose	7	5	——

9.4.3 Pairwise Comparisons and Rank Reversals

Rank reversals do sometimes occur when alternatives are dropped and the PCC procedure is repeated. We now show how such an example can be constructed.

Thirty (30) designers (or consumers) are asked to rank order five designs, A, B, C, D, and E, as a result of which they produce the following sets of orderings:

$$\begin{aligned} &10 \text{ preferred} \quad A \succ B \succ C \succ D \succ E, \\ &10 \text{ preferred} \quad B \succ C \succ D \succ E \succ A, \\ &10 \text{ preferred} \quad C \succ D \succ E \succ A \succ B. \end{aligned} \tag{9.35}$$

Here too, the procedure chosen to rank order these five designs can decidedly influence or alter the results. For example, all of the designers ranked C and D ahead of E in the above tally. Nonetheless, if the following sequence of pairwise comparisons is undertaken, an inconsistent result obtains:

$$C \text{ vs } D \Rightarrow C; \quad C \text{ vs } B \Rightarrow B; \quad B \text{ vs } A \Rightarrow A; \quad A \text{ vs } E \Rightarrow E. \tag{9.36}$$

Table 9.4 shows the PCC matrix for this five-design example, and the results clearly indicate the order of preferred designs to be:

$$C \succ B \succ D \succ A \succ E. \tag{9.37}$$

If the same data are subjected to a Borda count, using the weights (4, 3, 2, 1, 0) for the place rankings, we then find the results displayed in Table 9.5. When we compare these results to the PCC results shown in Table 9.4, we see that the PCC has achieved the same Borda count results, albeit in a slightly different fashion.

Table 9.4 A collective pairwise comparison chart (PCC) for a case before alternatives are dropped and the PCC is repeated.

Win/Lose	A	B	C	D	E	**Sum/Win**
A	——	10 + 0 + 10	10 + 0 + 0	10 + 0 + 0	10 + 0 + 0	50
B	0 + 10 + 0	——	10 + 10 + 0	10 + 10 + 0	10 + 10 + 0	70
C	0 + 10 + 10	0 + 0 + 10	——	10 + 10 + 10	10 + 10 + 10	90
D	0 + 10 + 10	0 + 0 + 10	0 + 0 + 0	——	10 + 10 + 10	60
E	0 + 10 + 10	0 + 0 + 10	0 + 0 + 0	0 + 0 + 0	——	30
Sum/Lose	70	50	30	60	90	——

What happens if we drop the lowest-ranked design and redo our assessment of alternatives? Here design E is least preferred, and we find the results shown in Table 9.5 if it is dropped. These results show a rank ordering of

$$C \succ B \succ A \succ D. \tag{9.38}$$

Rank order is preserved here for the two top designs, C and B, while the last two change places. Why does this happen? Quite simply, because of the relative narrowness of the gap between A and D when compared to the gap between A and E, the two lowest ranked in the first application of the PCC in this example.

Table 9.5 The Borda count with weights (4, 3, 2, 1) for the case where alternative E is dropped and the PCC is repeated. Compare these results with those in Table 9.4.

Element	Points
A	$40 + 10 + 20 = 70$
B	$30 + 40 + 10 = 80$
C	$20 + 30 + 40 = 90$
D	$10 + 20 + 30 = 60$

It is also useful to "reverse engineer" this example. Evidently, it was constructed by taking a Condorcet cycle [$A \succ B \succ C$, $B \succ C \succ A$, $C \succ A \succ B$] and replacing C with an ordered set ($C \succ D \succ E$) that introduces two dominated (by C) options that are irrelevant by inspection. Removing only E produces a minor rank reversal of the last two alternatives, A and D. Removing only D, the third best option, produces the same result among A, B, and C as removing E, although without creating a rank reversal. Removing both D and E produces a tie among A, B, and C.

9.4.4 Pay Attention to All of the Data

We now present an example that shows how pairwise ranking that does not consider other alternatives can lead to a result exactly opposite to a Borda count, which does consider other alternatives. It also indicates that attempting to select a single best alternative may be the wrong approach.

One hundred (100) customers are "surveyed on their preferences" with respect to five mutually exclusive design alternatives, A, B, C, D, and E. The survey reports that "45 customers prefer A, 25 prefer B, 17 prefer C, 13 prefer D, and no one prefers E." These data suggest that A is the preferred choice, and that E is entirely "off the table."

However, as reported, these results assume either that the customers are asked to list only one choice or, if asked to rank order all five designs, that only their first choices are abstracted from their rank orderings. Suppose that the 100 customers were asked for rankings and that those rankings are:

$$\begin{aligned} &45 \text{ preferred } A \succ E \succ D \succ C \succ B, \\ &25 \text{ preferred } B \succ E \succ D \succ C \succ A, \\ &17 \text{ preferred } C \succ E \succ D \succ B \succ A, \\ &13 \text{ preferred } D \succ E \succ C \succ B \succ A. \end{aligned} \tag{9.39}$$

Again, the procedure used to choose among the rank orderings of these five designs can decidedly influence or alter the results. For example, if A and B are compared as a (single) pair, B beats A by a margin of 55 to 45. And, continuing a sequence of pairwise comparisons, we can find that (see Problem 9.23):

$$A \text{ vs } B \Rightarrow B; \quad B \text{ vs } C \Rightarrow C; \quad C \text{ vs } D \Rightarrow D; \quad D \text{ vs } E \Rightarrow E. \tag{9.40}$$

Proposition (9.40) provides an entirely different outcome, one that is not at all apparent from the vote count originally reported. How do we sort out this apparent conflict?

We resolve this dilemma by constructing a PCC matrix for this five-product example, and the results clearly indicate the order of preferred designs to be (see Problem 9.24):

$$E(300) \succ D(226) \succ A(180) \succ C(164) \succ B(130). \tag{9.41}$$

A Borda count of the same data (of eqs. (9.39)), using the weights (4, 3, 2, 1, 0) for the place rankings, confirms the PCC results, with the Borda count numbers being identical to those in eq. (9.41) (see Problem 9.25). In this case, removing B and re-voting generates a relatively unimportant rank reversal between A and C, thus demonstrating the meaning of IIA and showing that dropping information can have consequences.

This example is one where the "best option" as revealed by the PCC/Borda count is not the one most preferred by anyone. Is the PCC lying to us? In a real market situation, where all five options are available, none of the surveyed customers would buy E. Perhaps this data was collected across too broad a spectrum of customers in a very segmented market in which design E provided a "common denominator," while the other four designs responded better to their separate market "niches." There is really no "best design" under these circumstances. It is also possible that these designs were extremely close to each other in performance, so that small variations in performance have translated into large differences in the PCC. Both of the above explanations point to the need to treat PCC results with caution because there are cases where more detailed selection procedures might be more appropriate.

9.4.5 On Pairwise Comparisons and Making Decisions

The structured PCC—an implementation of the Borda count—can support consistent decision making and choice, notwithstanding concerns raised about pairwise comparisons and violations of Arrow's theorem. Rank reversals and other infelicities do result when "losing" alternatives are dropped from further consideration. But simulation suggests that such reversals are limited to alternatives that are nearly indistinguishable. Pairwise comparisons that are properly aggregated in a pairwise comparison chart (PCC) produce results that are identical to the Borda count, which in Saari's words is a "unique positional procedure which should be trusted."

Practicing designers use the PCC and similar methods very early in the design process where rough ordinal rankings are used to bound the scope of further design work. The PCC is more of a discussion tool than a device intended to aggregate individual orderings of design team members into a "group" decision. Indeed, design students are routinely cautioned against over-interpreting or relying too heavily on small numerical differences. In political voting, we usually end up with only one winner, and any winner *must* be one of the entrants in the contest. In early design, it is perfectly fine to keep two or more winners around, and the ultimate winner often does not appear on the initial ballot. Indeed, it is often suggested that designers look at *all* of the design alternatives and try to incorporate the good points of each to create an improved, composite design. In this framework, the PCC is a useful aid for understanding the strengths and weaknesses of individual design alternatives. Still, pairwise comparison charts should be applied carefully and with restraint. As noted above, it is important to cluster similar choices and to perform the evaluations at comparable levels of detail.

In addition, given the subjective nature of these rankings, when we use such a ranking tool, we should ask *whose* values are being assessed. Marketing values are easily included in different rankings, as in product design, for example, where a design team might need to know whether it's "better" for a product to be cheaper or lighter. On the other hand, there might be deeper issues involved that, in some cases, may touch upon the fundamental values of both clients and designers. For example, suppose two competing companies, GRAFT and BJIC, are trying to rank order design objectives for a new beverage container. We show the PCCs for the GRAFT- and BJIC-based design teams in Tables 9.6(a) and (b), respectively. It is clear from these two charts and the scores in their right-hand columns that the GRAFT designers were far more interested in a container that would generate a strong brand identity and be easy to distribute than in it being environmentally benign or having appeal for parents. At BJIC, on the other hand, the environment and taste preservation ranked more highly, thus demonstrating that subjective values show up in PCCs and, eventually, in the marketplace!

Table 9.6 Using PCCs to rank order design objectives at two different companies designing new beverage containers (Dym and Little, 2003).

Goals	Environ. Benign	Easy to Distribute	Preserve Taste	Appeals to Parents	Market Flexibility	Brand ID	Score
(a) GRAFT's weighted objectives							
Environ. Benign	•••	0	0	0	0	0	0
Easy to Distribute	1	•••	1	1	1	0	4
Preserve Taste	1	0	•••	0	0	0	1
Appeals to Parents	1	0	1	•••	0	0	2
Market Flexibility	1	0	1	1	•••	0	3
Brand ID	1	1	1	1	1	•••	5
(b) BJIC's weighted objectives							
Environ. Benign	•••	1	1	1	1	1	5
Easy to Distribute	0	•••	0	0	1	0	1
Preserve Taste	0	1	•••	1	1	1	4
Appeals to Parents	0	1	0	•••	1	1	3
Market Flexibility	0	0	0	0	•••	0	0
Brand ID	0	1	0	0	1	•••	2

It is also tempting to take our *ranked* or ordered objectives and put them on a *scale* so that we can manipulate the rankings in order to attach relative weights to goals or to do some other calculation. It would be nice to be able to answer questions such as: *How much more* important is portability than cost in a ladder? Or, in the case of a beverage container, *How much more* important is environmental friendliness than durability? A little more? A lot more? Ten times more? We can easily think of cases where one of the objectives is substantially more important than any of the others, such as safety compared to attractiveness or to cost in an air traffic control system, and other cases where the objectives are essentially very close to one another. However, and sadly, there is *no mathematical foundation* for normalizing the rankings obtained with tools such as the PCC. The numbers obtained with a PCC are *approximate*, *subjective* views or judgments about relative importance. We must *not* inflate their importance by doing further calculations with them or by giving them unwarranted precision.

Valid?

Use?

Problem 9.19. Would you find election procedures that violated Arrow's third axiom offensive? Explain your answer.

Problem 9.20. Would you find election procedures that violated the Pareto condition, Arrow's fourth axiom, offensive? Explain your answer.

Problem 9.21. Engineering designers often use quantified performance rankings to compare alternatives on the basis of measurable criteria. If this comparison were done on a pairwise basis, would it violate Arrow's fourth axiom? Explain your answer.

Problem 9.22. Defend or refute the proposition that ranking criteria that are of the less-is-better, more-is-better, or nominal-is-best varieties will violate Arrow's first axiom. (*Hint*: Are all theoretically possible orders admissible in practice?)

Problem 9.23. Verify the ordering of the five alternatives displayed in eq. (9.40) by performing the appropriate individual pair-by-pair comparisons.

Problem 9.24. Construct a PCC of the data presented in eq. (9.39) and confirm the Borda count results given in eq. (9.41).

Problem 9.25. Using the weights (4, 3, 2, 1, 0), perform a Borda count of the preferences expressed in eq. (9.39) and confirm the results obtained in eq. (9.41) and in the previous problem.

9.5 A Miscellany of Optimization Problems

In this section we present some simple yet interesting optimization and "Can we do better?" problems. Their interest derives more from their subject matter than from the optimization technique applied. As a result, some elementary models are introduced and described in just enough detail to make the search for optimum behavior meaningful. These optimization problems include forming nuclei in solids, maximizing the range of planes or birds, and reducing the weight of a cantilever beam. Along the way we will introduce some further wrinkles in the modeling of searches for a good—if not globally optimum—method of searching for an optimum result.

9.5.1 Is There Enough Energy to Create a Sphere?

Nucleation refers to the formation of tiny, even submicroscopic, particles. Such particles or *nuclei* initiate the phase transformations in which the microstructures of materials are changed during various materials processes. For example, steel alloys come in various forms (e.g., cementite and ferrite) that have substantially different properties (e.g., ferrite is softer than cementite, but it is also less brittle). How do such nuclei form?

Why?

Given?

The nucleation process occurs in a solution that has, for example, a small number of β atoms relative to a much larger number of α atoms. The β atoms diffuse together, form a small volume and then re-arrange into a crystal structure that is enclosed in a volume, V, with an interfacial (with the surrounding α atoms) area, A. This process can occur only if an *activation energy barrier* is overcome. In the simplest formulation, wherein the distribution of the interfacial energy is *isotropic* (or independent of direction within the solution), the total *free energy exchange* ΔG needed to bring about this change is given by

$$\Delta G = -(\Delta G_{\mathrm{V}} - \Delta G_{\mathrm{S}})V + \gamma A, \tag{9.42}$$

where $(\Delta G_{\mathrm{V}} - \Delta G_{\mathrm{S}})$ is the (positive) difference between the volume free energy and the misfit strain energy, and γ is the surface free energy per unit area. Further, the misfit strain energy reduces the free energy exchange because it is subtracted from the volume free energy. Note that all of the energy terms are *specific*, expressed as they are in terms of energy per unit volume of β.

Notwithstanding this meager, skeletal introduction to some of the language of thermodynamics, the important point is that the free energy exchange needed to allow creation of a volume, V, is the sum of a term that decreases with V but increases with its area, A. Is there a point below which the free energy exchange cannot happen? If so, what is the value of that free energy exchange barrier?

Predict?

Assume?

To answer these questions,we assume that the volume will form, at least initially, a sphere of radius r. With appropriate substitutions for the sphere's surface area and volume, eq. (9.42) becomes

$$\Delta G = -(\Delta G_V - \Delta G_S)\left(\frac{4}{3}\pi r^3\right) + \gamma\left(\pi r^2\right). \tag{9.43}$$

We can now employ the standard techniques of calculus to show that there is a minimum radius, r^*, below which there is not enough free energy to overcome the free energy exchange barrier, ΔG^* (see Problems 9.25 and 9.26):

$$r^* = \frac{2\gamma}{(\Delta G_V - \Delta G_S)}, \tag{9.44}$$

and

$$\Delta G^* = \frac{16\pi\gamma^3}{3(\Delta G_V - \Delta G_S)^2}. \tag{9.45}$$

The free energy exchange is plotted on Figure 9.10 and it shows the barrier that needs to be overcome quite clearly. It also shows how the behavior of the free energy exchange depends differently on r, depending on whether the sphere is smaller or larger than that with the minimum radius, r^* (see Problems 9.27 and 9.28).

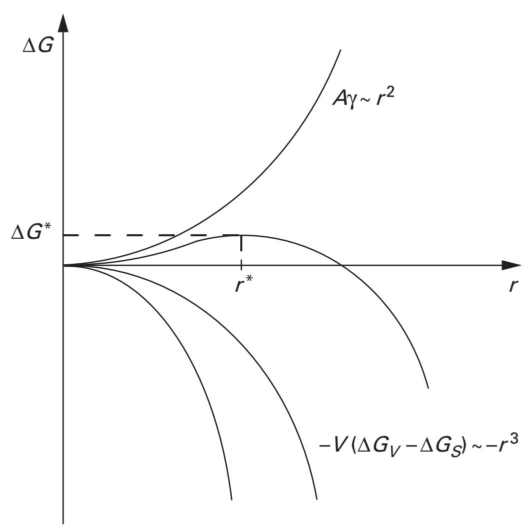

Figure 9.10 The variation of the free energy exchange, ΔG, with the radius, r, of a nucleating sphere (Porter and Easterling, 1992). We see that there is an activation energy barrier, ΔG^*, that must be overcome, and that the free energy exchange decreases for values of $r < r^*$, while it increases for $r > r^*$.

Problem 9.25. Verify eqs. (9.44) and (9.45) by performing the appropriate calculus.

Problem 9.26. Demonstrate that eqs. (9.44) and (9.45) have the correct dimensions.

Problem 9.27. Write eq. (9.43) in terms of a dimensionless coordinate, $\rho = r/r^*$, and expand it in a power series valid for *small* values of ρ. What part of the curves in Figure 9.10 does that result portray?

Problem 9.28. Expand eq. (9.43) in a power series valid for *large* values of the dimensionless coordinate $\rho = r/r^*$. What part of the curves in Figure 9.10 does that result portray?

Problem 9.29. Is the surface area-to-volume ratio of a cylinder of radius, R, and length, L, smaller or larger than that of a sphere of radius R? (Hint: Write a ratio of the ratios as a function of R/L.)

9.5.2 Maximizing the Range of Planes and Birds

Why? Airplane pilots share a challenge with flying birds: How far can they go—what is their *range*—for a fixed amount of fuel? Still better, can they maximize their range? It turns out that for a given amount of fuel, the speed that maximizes the range is the one that maximizes the aerodynamic quantity, called the *lift-to-drag ratio*, or, conversely, minimizes its inverse, the *drag-to-lift ratio*.

Given? We show a typical jet in Figure 9.11 with a free-body diagram (FBD) superposed. The plane is climbing at an angle, α, at a speed, V, relative to the ground. The climb or flight direction angle, α, is zero for level flight, and positive for ascending flight and negative for descending flight. The FBD shows the forces that act to support the plane and move it forward, as described in the aerodynamic literature. The plane's weight, W, is supported by a *lift* (force), L, that is perpendicular to the flight path. The engines provide a thrust, T, that moves the plane along the flight path by overcoming the *drag* (force), D, that also acts along the flight path, albeit it in a direction that retards flight. Due largely to preceding experimental work and subsequent confirming analysis, aerodynamicists have known since the end of the 17th century that the lift and drag forces on a flying body can be expressed in terms of the density of the surrounding air, ρ, the wing or *lifting surface* area, S, and the body's speed, V, as,

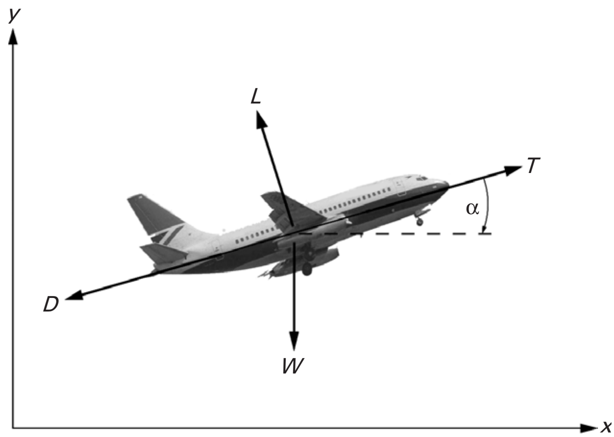

Figure 9.11 A typical jet with a superposed free-body diagram showing the aerodynamic forces acting. The plane is climbing at an angle, α, at a speed, V, relative to the earth below. The plane's weight, W, is supported by a *lift* (force), L, that is perpendicular to the flight path. The jet engines provide a thrust, T, that moves the plane along the flight path by overcoming the *drag* (force), D, that also acts along the flight path, although in a direction that opposes flight. The plane's wing has a surface area, S, and span, b.

respectively,

$$L = \frac{1}{2}\rho S V^2 C_{\mathrm{L}}, \tag{9.46a}$$

and

$$D = \frac{1}{2}\rho S V^2 C_{\mathrm{D}}, \tag{9.46b}$$

where C_{L} and C_{D} are the corresponding *lift* and *drag coefficients.* (We should note that the drag-velocity relation is more complicated when planes fly closer to the speed of sound, due to drag produced by compressibility effects either on rapidly rotating propellers or on the wings of jet aircraft.) The makeup of the C_{L} and C_{D} coefficients and their relationship provide the complexity we will see in our search for an optimum flight speed. But first we need to do a little equilibrium analysis because taken superficially, eqs. (9.46a–b) suggest that the drag-to-lift ratio L/D is independent of the speed V, so how could it be minimized with respect to V?

We sum the forces superposed on the plane in Figure 9.11 in the x and y directions:

$$\sum F_x = -T\cos\alpha + L\sin\alpha + D\cos\alpha = 0, \tag{9.47a}$$

and

$$\sum F_y = T\sin\alpha + L\cos\alpha - W - D\sin\alpha = 0. \tag{9.47b}$$

If the climb angle, α, is assumed to be small, along the lines of the approximations introduced in Section 4.1.2, eqs. (9.47a–b) can be simplified and solved to show that the lift L is, in fact, a constant (see Problem 9.30),

$$L \cong \frac{W}{1+\alpha^2} \cong W, \tag{9.48a}$$

which means that the *drag-to-lift ratio* is simply

$$D/L \cong D/W. \tag{9.48b}$$

Equation (9.48a) clearly shows that the lift force supports the plane's weight, while eq. (9.48b) provides a speed-dependent ratio of the drag force to the weight.

Now we return to the drag coefficients because that is the logical step for casting the D/L ratio in terms of the plane's speed, V. It turns out that the drag coefficient is expressed as a sum of two terms,

$$C_{\mathrm{D}} = C_{\mathrm{D}_0} + \frac{kSC_{\mathrm{L}}^2}{\pi b^2}. \tag{9.49}$$

The first term represents the *parasite* or *friction drag* caused by shear stresses resulting from the air speeding over and separating from the wing. The second term is the *induced drag*: it is independent of the air viscosity and is created by wings of finite span (i.e., real wings!) because of momentum changes needed to produce lift, according to Newton's second law. Note that the induced drag is proportional to the square of the lift coefficient, C_{L}^2.

Now we can combine eqs. (9.46b) and (9.48b) to write the drag-to-lift ratio as

$$\frac{D}{L} = \frac{\rho SV^2 C_{\mathrm{D}}}{2W}, \tag{9.50}$$

after which we can further combine eqs. (9.46a), (9.48a), and (9.49) to rewrite eq. (9.50) as (see Problem 9.31):

$$\frac{D}{L} = C_{01}V^2 + C_{02}V^{-2}, \tag{9.51}$$

with the *constants* C_{01} and C_{02} defined as:

$$C_{01} = \frac{\rho S C_{D_0}}{2W}, \quad C_{02} = \frac{2kW}{\pi\rho b^2}. \tag{9.52}$$

Thus, the objective function or *cost* for this optimization problem is defined in eq. (9.51), and its coefficients as presented in eq. (9.52) are simply constants reflecting the values of the problem's physical parameters: ρ, S, W, the wing span, b, the parasite drag coefficient, C_{D_0}, and a dimensionless shape constant, k (see Problem 9.32).

The extreme value of this unconstrained optimization problem is then found by the standard calculus approach, that is,

$$\frac{d}{dV}\left(\frac{D}{L}\right) = 2C_{01}V - 2C_{02}V^{-3} = 0, \tag{9.53}$$

which has the following extreme value:

$$\left(\frac{D}{L}\right)_{\min} = 2\sqrt{C_{01}C_{02}} \quad \text{at} \quad V_{\min} = \left(\frac{C_{02}}{C_{01}}\right)^{1/4}. \tag{9.54}$$

With the aid of eq. (9.52), the minimum drag-to-lift ratio can then be written in its final form (see Problem 9.33):

$$\left(\frac{D}{L}\right)_{\min} = 2\sqrt{\frac{kSC_{D_0}}{\pi \mathrm{b}^2}}. \tag{9.55}$$

This is a classical result in aerodynamics. Further, it is also easily demonstrated that this minimum D/L ratio occurs only when the parasite drag and the induced drag are equal and, consequently, independent of the plane weight W (see Problem 9.34). In the next section we will obtain this result again by introducing still another method of searching for optimal results.

Problem 9.30. Solve eqs. (9.47a–b) for $(T - D)$ as a function of L and W and confirm that eqs. (9.48a–b) are correct while identifying any additional needed approximations.

Problem 9.31. Combine eqs. (9.46a), (9.48a), and (9.49) and develop eq. (9.51).

Problem 9.32. Show that the constants C_{01} and C_{02} have the correct physical dimensions. (*Hints*: What are their physical dimensions according to eq. (9.51)? Do they have those dimensions?)

Problem 9.33. Use the standard calculus test to confirm that the value of D/L given in eqs. (9.54) and (9.55) is a minimum.

Problem 9.34. Show that the induced drag equals the parasite drag at the minimum D/L ratio, and that both are independent of the plane weight, W.

Problem 9.35. The minimum of eq. (9.51) can also be seen "by inspection." Inspect eq. (9.51) and explain why that minimum can be so determined.

9.5.3 Geometric Programming for a Plane's Optimum Speed

The objective function (9.51) for the plane range problem considered just above is a member of the class of functions called *posynomials*, polynomials whose coefficients are always positive. Clarence Zener, inventor of the Zener diode, noted that if the objective functions whose minima were being sought were posynomials, then each *term* in such an objective function could be considered an independent variable whose contribution to the overall minimum sum could be established. Zener proposed doing that by constructing a dual function that would be maximized. The mathematician Richard J. Duffin recognized that a posynomial cost function could be viewed as a *weighted arithmetic mean*, and Zener's dual function as its *weighted geometric mean*. Cauchy's inequality—the arithmetic mean is always greater than or equal to its geometric mean—could be brought to bear, and thus Zener's optimization invention became known as *geometric programming*.

How?

Consider a rectangle bounded by lines of length a and b. Geometrically, then, the rectangle's perimeter is $P = 2(a + b)$ and its area is $A = ab$. The Greeks asked, What is the smallest perimeter of a rectangle of given area? Well, the answer to that equation is not hard to find. The perimeter can be written as

$$P = 2(a + b) = 4\left[\left(\frac{a}{2} + \frac{b}{2}\right)^2\right]^{1/2}, \tag{9.56}$$

from which it follows that (see Problem 9.36):

$$P = 4\left[ab + \left(\frac{a-b}{2}\right)^2\right]^{1/2} \geq 4\sqrt{ab}. \tag{9.57}$$

Thus, in terms of the rectangle's perimeter and area, eqs. (9.56) and (9.57) tell us that (see Problem 9.37):

$$P \geq 4\sqrt{A}. \tag{9.58}$$

Equation (9.58) also tells us something else. For any two numbers a and b, we can define their *arithmetic mean*, $\overline{a}_{\text{arith}} = \frac{1}{2}(a+b)$, and their *geometric mean*, $\overline{a}_{\text{geom}} = \sqrt{ab}$. Then eq. (9.58) turns out to be a very simple expression of the *Cauchy inequality*:

$$\overline{a}_{\text{arith}} \geq \overline{a}_{\text{geom}}. \tag{9.59}$$

For a collection of numbers or functions, U_i, the Cauchy inequality becomes

$$\overline{U}_{\text{arith}} = \frac{1}{N}\sum_{i=1}^{N} U_i \geq \prod_{i=1}^{N} U_i^{1/N} = \overline{U}_{\text{geom}}. \tag{9.60}$$

Equation (9.60) can be generalized still further. Consider that each object in the sum that is the arithmetic mean is weighted by a positive constant, w_i. Then the extended Cauchy inequality is:

$$\overline{U}_{\text{arith}} = \sum_{i=1}^{N} w_i U_i \geq \prod_{i=1}^{N} U_i^{w_i} = \overline{U}_{\text{geom}}. \tag{9.61}$$

Finally, if we define a set of modified numbers or functions, $V_i = w_i U_i$, we can write the central inequality of eq. (9.61) as

$$\sum_{i=1}^{N} V_i \geq \prod_{i=1}^{N}\left(\frac{V_i}{w_i}\right)^{w_i}, \tag{9.62a}$$

or, written *in extenso*,

$$V_1 + V_2 + \cdots V_N \geq \left(\frac{V_1}{w_1}\right)^{w_1}\left(\frac{V_2}{w_2}\right)^{w_2}\cdots\left(\frac{V_N}{w_N}\right)^{w_N}. \tag{9.62b}$$

The weights, w_i, are restricted in two ways that reflect their roots in geometry. The first is that they must satisfy a *normality condition*, that is, their values must sum to one:

$$w_1 + w_2 + \cdots w_N = 1. \tag{9.63}$$

The second restriction is an *orthogonality condition*, which requires that the geometric mean terms on the right-hand sides of eqs. (9.62a–b) must be free of—or dimensionless in—the independent variables that make up the functions, V_i. The weights that satisfy both the normality and orthogonality conditions then *maximize the geometric mean* and, consequently, *minimize the arithmetic mean.* This lovely piece of geometry brings us back to our optimization problem.

We start with an objective or cost function that is written as a sum of posynomials, $V_i(x)$, each of which is a function of some or all of a set of independent design variables, $x = (x_1, x_2, \ldots x_k)$:

$$V(x) = V(x)_1 + V_2(x) + \cdots V_N(x) = \sum_{i=1}^{N} V_i(x). \tag{9.64}$$

We then define the following weighted product as the *dual* to the cost function, $U(x)$:

$$d(w) = \left(\frac{V_1}{w_1}\right)^{w_1} \left(\frac{V_2}{w_2}\right)^{w_2} \cdots \left(\frac{V_N}{w_N}\right)^{w_N} = \prod_{i=1}^{N} \left(\frac{V_j}{w_j}\right)^{w_j}, \tag{9.65}$$

where $w = (w_1, w_2, \ldots w_k)$ is the set of weights that satisfy the appropriate normality and orthogonality conditions. By the geometric analysis culminating in eqs. (9.62a–b), we can then say that:

$$\min V(x) = \max d(w). \tag{9.66}$$

To illustrate the application of geometric programming (GP), consider once again the determination of the optimum D/L ratio for maximizing a plane's range. The objective function is the D/L ratio given in eq. (9.51), and it is clearly a sum of two posynomials: $V_1 = C_{01}V^2$ and $V_2 = C_{01}V^{-2}$. The corresponding dual function can then be constructed as defined by eq. (9.65):

$$d_\gamma(w) = \left(\frac{C_{01}V^2}{w_1}\right)^{w_1} \left(\frac{C_{02}V^{-2}}{w_2}\right)^{w_2} = \left(\frac{C_{01}}{w_1}\right)^{w_1} \left(\frac{C_{02}}{w_2}\right)^{w_2} \left(V^{2(w_1-w_2)}\right) \tag{9.67}$$

The orthogonality condition that renders eq. (9.67) dimensionless with respect to the independent variable V is:

$$2w_1 - 2w_2 = 0. \tag{9.68}$$

In conjunction with the appropriate ($N = 2$) version of the normality condition (9.63), eq. (9.68) produces the weights $w_1 = w_2 = 1/2$, which can then immediately be substituted into the dual function (9.67) to yield the minimum D/L ratio:

$$d_\gamma(w) = \left(\frac{C_{01}}{1/2}\right)^{1/2}\left(\frac{C_{02}}{1/2}\right)^{1/2} = 2\sqrt{C_{01}C_{02}} \tag{9.69}$$

This result is, of course, exactly the same as the one we obtained before (see eq. (9.54)).

Two features of this solution are worth special note. The first is that the solution proceeded quite directly to the sought minimum D/L ratio. This contrasts with the calculus solution, which yielded first the velocity at which the minimum ratio occurs, with the ratio itself being determined after its additional calculation from the critical value of the velocity. The second point is that this solution was remarkably simple. In principle, and by extension to more complicated cases (see Problem 9.40), all we had to do was solve a set of linear equations—the normality and orthogonality conditions—to immediately obtain the optimum we were after. The typical calculus approach, for such evidently nonlinear cost functions, clearly requires much more work.

Finally on GP, we note that we have presented GP in its simplest form. We have not dealt with any constraints, whether equality or inequality. The principles applied to more complicated, more "real" problems are similar—but they will require more work. Given the role of computers in our lives, techniques such as GP are not invoked much any more. However, GP still offers a neat and direct approach to an interesting class of (posynomial) problems.

Problem 9.36. Construct the steps that get one from eq. (9.56) to eq. (9.57).

Problem 9.37. When does the soft inequality in eq. (9.58) become a simple equality?

Problem 9.38. Use the principle of induction to prove the general statement (9.60) of Cauchy's inequality.

Problem 9.39. Can eq. (9.43)—in the discussion of nucleation in Section 9.5.1—be cast into a form suitable for solution by geometric programming? Explain your answer.

Problem 9.40. Minimize the objective function $2x_1^2x_2^{-1} + 4x_2^3x_3^{1/2} + x_1^{-2}x_2x_3^{-1/2} + 2x_2^{-3}$ using geometric programming.

Problem 9.41. Confirm the results of Problem 9.40 using the standard calculus approach of determining extrema.

9.5.4 The Lightest Diving Board (or Cantilever Beam)

Cantilever beams are ubiquitous in life, appearing as diving boards, trees, tall slender buildings, freeway signs, and the arms of grandparents picking up their grandchildren. Their optimal design will vary with their situational circumstance. We present here an analysis that leads toward significantly improved designs that may or may not be optimal. The question answered by such analysis is, therefore, much less like "What is the best ...?" and much more like "Can we do better than ...?".

Find? Our "doing better" problem is simple. We want to determine the profile or shape of a tip-loaded cantilever beam that weighs significantly less while **Given?** yielding the same tip deflection, δ, for a given tip load, P. We will use the classic elementary model of beam bending, the origin of which can be traced to Galileo Galilei (1564–1642), to model the cantilever beam shown in Figure 9.12. This widely applicable model assumes that the beam is long and slender, meaning that its thickness, h, and width, b, are both small compared to its length, L, and that its response to an applied load is almost entirely due to bending, meaning that the stress through the thickness is distributed as

$$\sigma_{xx}(x,z) = \frac{M(x)z}{I}, \tag{9.70}$$

where $\sigma_{xx}(x,z)$ is the axial stress that occurs when the beam is bent, $M(x)$ is the moment that forces the beam to bend, z the vertical coordinate measured positive downward from the centerline of the beam's cross-sectional area, and I is the second moment of that area (see Figure 9.12).

Assume? We will assume that the cross-sectional area is rectangular, with constant width but thickness that may vary with the axial coordinate, x. In this case the second moment, I, is

$$I(x) = \frac{bh^3(x)}{12}. \tag{9.71}$$

Finally, the bending theory of beams states that the moment produced by a force, P, at the cantilever tip ($x = 0$) is $M(x) = -Px$, and that the resulting deflection at the tip is:

$$\delta(x=0) = \int_0^I \frac{Px^2\,dx}{EI(x)}. \tag{9.72}$$

Our base model for comparison is the case of a uniform cantilever of constant thickness, h_0, and length, L_0. For this case, the second moment of the area is the constant

$$I_0 = \frac{bh_0^3}{12}, \tag{9.73}$$

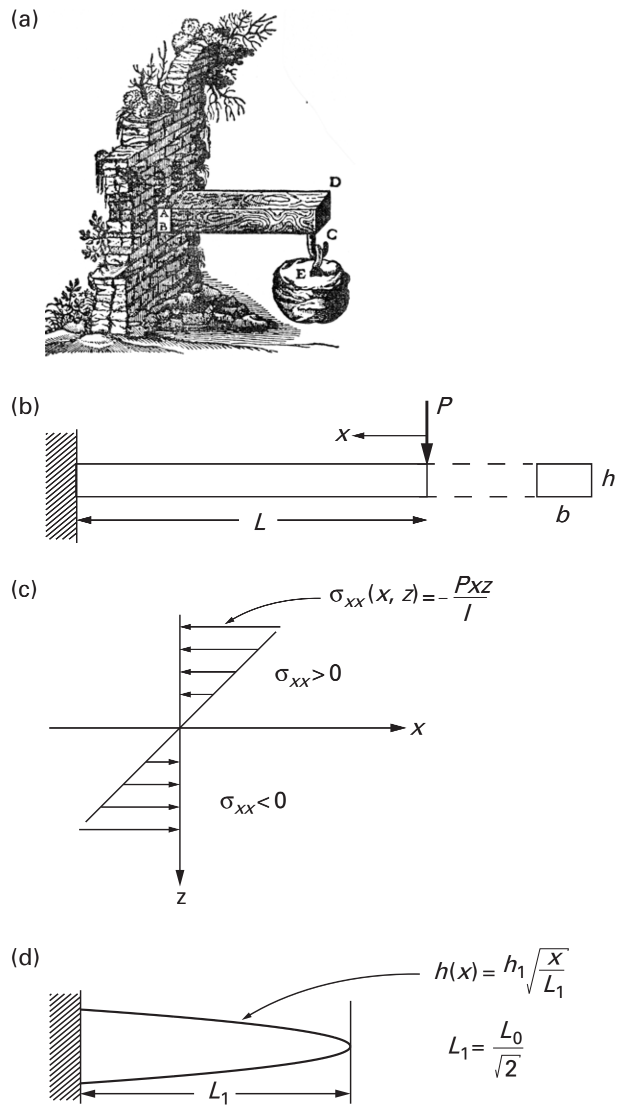

Figure 9.12 The classic cantilever beam: (a) Galileo Galilei's famous picture; (b) a modern incarnation; (c) the distribution of stress through the thickness, as indicated by eq. (9.70); and (d) the optimal shape, including a significantly shorter length, that produces a 67% reduction in the total volume (and thus the beam's weight) (after Bejan, 2000).

and its volume is

$$V_0 = bh_0L_0, \tag{9.74}$$

while its tip deflection is found by integrating eq. (9.72):

$$\delta_0 = \frac{PL_0^3}{3EI_0} = \frac{4PL_0^3}{Ebh_0^3}. \tag{9.75}$$

The maximum stress in the beam will occur at the root or support, $x = L_0$, at the beam's outer edges, $z = h_0/2$, and is determined by eq. (9.70) to be:

$$\sigma_{\max} = \sigma_{xx}(L_0, h_0/2) = \frac{6PL_0}{bh_0^2}. \tag{9.76}$$

Therefore, if a maximum stress that must not be exceeded is given as a design constraint, it follows that the minimum thickness required for a uniform beam to support a load, P, is:

$$h_0 = \sqrt{\frac{6PL_0}{b\sigma_{\max}}}. \tag{9.77}$$

Finally, for the base case, and in view of eqs. (9.75) and (9.76), we can rewrite the volume (9.74) in a form that is independent of the beam's geometrical parameters:

$$V_0 = 9\frac{EP\delta_0}{\sigma_{\max}^2}. \tag{9.78}$$

Consider now a second case where the cantilever has variable thickness, $h(x)$, and length, L_1. The variation of the thickness is determined by the requirement that, again, a given maximum stress not be exceeded. In this instance, eq. (9.70) states that (see Problem 9.42):

$$h(x) = \sqrt{\frac{6PL_1}{b\sigma_{\max}}}\sqrt{\frac{x}{L_1}} \equiv h_1\sqrt{\frac{x}{L_1}}, \tag{9.79}$$

where h_1 is the maximum thickness of a parabolic profile that begins at the tip ($x = 0$) and reaches its maximum at the support ($x = L_1$).

The volume for this beam of varying thickness is

$$V_1 = \int_0^{L_1} bh(x)dx = \frac{2}{3}bh_1L_1. \tag{9.80}$$

The corresponding tip deflection is found by substituting eq. (9.79) into eq. (9.72) and then performing the indicated integration:

$$\delta_1 = \frac{8PL_1^3}{Ebh_1^3}. \tag{9.81}$$

Once again we can cast the volume (9.80) of the beam with parabolically-varying thickness in a form independent of the beam's geometrical parameters, now in view of eqs. (9.79) and (9.81):

$$V_1 = 3\frac{EP\delta_1}{\sigma_{\max}^2}. \tag{9.82}$$

How are we to compare these two cases? Which is the better beam? What do we mean by the "better beam"? There are (at least) two bases for comparison. In the first, we ask: How do the volumes compare if we require each beam to have the same tip deflection while supporting the same load, P? This question is easily answered by comparing the volumes given by eqs. (9.78) and (9.82):

$$\frac{V_1}{V_0} = \frac{1}{3}\frac{\delta_1}{\delta_0} = \frac{1}{3}. \tag{9.83}$$

Thus, we have the astounding result that we can reduce the volume by 2/3 or 67%! An amazing improvement. By equating the formulas (9.77) and (9.81) for the respective deflections, and by rewriting the volumes in terms of their geometries, we can find that (see Problem 9.43):

$$\frac{h_1}{h_0} = \frac{L_1}{L_0} = \frac{1}{\sqrt{2}}. \tag{9.84}$$

Our volume savings come at a price of a beam that is not only thinner, but almost 30% shorter. This length shortening may or may not matter; it depends on the context in which this beam will be used.

Suppose we looked for a different "better beam." Suppose we require that the beams carry the same load, P, have the same maximum thickness, $h_0 = h_1$, and the same length, $L_0 = L_1 = L$. It is then easy enough to show that (see Problems 9.44 and 9.45):

$$\frac{V_1}{V_0} = \frac{2}{3} \tag{9.85}$$

and

$$\frac{\delta_1}{\delta_0} = 2. \tag{9.86}$$

In this case we still have a substantial volume reduction of 33% , but at the price of doubling the deflection. We have maintained the original length (and maximum thickness), but we now pay a different price for a different saving.

There are other ways to improve the behavior of a beam. While we have focused on the more visible external structure of the beam (i.e., its thickness and its length), we could also change the inner structure. For example, we might consider the volume (and material) saving that results from taking a beam of a solid circular cross-section, and then hollowing it out to make it a tube (see Problems 9.57 and 9.58). Further, we might also combine changes in the internal and external structure to see what costs are reduced (see Problems 9.59 and 9.60). In fact, we can see such examples in nature any time we choose to look. Tree trunks are thicker at their bases, and branches thicker at their initial branching points, thus exemplifying external structuring. And the internal structuring of tubes shows up in bamboo and various reeds. Thus, nature seems to be paying attention to the search for optimal behavior.

Problem 9.42. Show that a beam with varying thickness, $h(x)$, that is required to have a maximum stress $\leq \sigma_{\max}$, will have the thickness distribution given in eq. (9.79). (*Hint*: Where does the maximum stress occur for varying $h(x)$?)

Problem 9.43. Show that eq. (9.84) is correct for a comparison of beams required to have the same tip deflection, δ, when under the same load, P.

Problem 9.44. Confirm that the volume ratio (9.85) is correct when beams whose lengths and maximum thickness are required to support the same load, P.

Problem 9.45. Confirm that the tip deflection ratio (9.86) is correct when beams whose lengths and maximum thickness are required to support the same load, P.

9.6 Summary

This final chapter has been devoted to optimization, the search for the optimum or best outcome to a problem. We have briefly reviewed several well-founded techniques, including calculus, linear programming (LP), and geometric programming (GP). We also talked about making the best decision when voting for candidates and choosing among alternatives. Our emphasis throughout has been less on the intricacies of the particular techniques, and more on framing the question. In this context, it is particularly important to recognize that any search for an optimum solution is to some extent “biased” or influenced by the way the question is framed. This was most evident in the discussion of voting and the expression of preferences, but it is also the case in the more “rigorous” calculus- and programming-based approaches. When we ask which is the cheapest design or product, we are choosing money as our metric, not the design's esthetics or the product's effects on the environment. To be sure, such externals can be taken into account, but the means for so doing are neither rigorous nor entirely objective.

Having said that, we also note that we have only scratched the surface of tools that support the making of optimal decisions. For example, while linear programming is a very valuable tool and an important part of operations research, there are many other optimization techniques, including nonlinear, integer, dynamic, and geometric programming. Operations research also includes queueing theory, game theory, and simulation (particularly Monte Carlo simulation). These approaches are concerned with such issues as assessing the costs of having too few or too many service lines at a service facility, rationalizing economic and strategic decisions in the face of uncertainty, and performing simulations of problems that are analytically intractable or experimentally too expensive. There is also a vast body of literature on and experience with what might be called "continuous optimization" techniques, and their digital implementations are often used with finite element methods (FEM) and other numerical programs to seek the best designs of large complex designs, such as aircraft. All in all, the foregoing discussion is only an appetizer; more than a few full meals remain.

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9.8 Problems

9.46. (a) Find the extreme values of the function $y = \sin x$ for $0 \leq x \leq \pi$. Are the extreme values maxima or minima?

(b) What are maxima and minima of y=sin x in the interval $0 \leq x \leq 2\pi$. Are the extreme values maxima or minima?

9.47. (a) What are the extreme values of the function $y = x$ in the interval $0 \leq x \leq 2\pi$?

(b) What are the extreme values of the function $y = x - x^3/3!$ in the interval $0 \leq x \leq 2\pi$?

(c) How do the answers to parts (a) and (b) of this question relate to the answers to Problem 9.46?

9.48. A string of length, l, can be used to outline many simple geometrical figures, such as an equilateral triangle with sides $l/3$, a square with sides $l/4$, a pentagon with sides $l/5$, and a circle of circumference, l. For the figures mentioned:

(a) calculate their areas and show how they vary with the number of sides; and

(b) guess (and explain!) the maximum area that can be enclosed by a string of given length l.

9.49. Determine the maximum area of a triangle that can be inscribed in the shown semicircle of diameter, d. (*Hint*: Show that the area of the triangle is $bc/2$ and that the height, c, can be expressed [and eliminated] through a relationship between the triangle's sides and the semicircle's diameter.)

9.50. Graphically solve the following linear programming problem cast in terms of two nonnegative variables, x_1 and x_2.

$$\text{Maximize} \quad z = 5x_1 + 3x_2$$
$$\text{subject to} \quad \begin{cases} 3x_1 + 5x_2 \leq 15 \\ 5x_1 + 2x_2 \leq 10 \end{cases}$$

9.51. Graphically solve the following linear programming problem cast in terms of two nonnegative variables, x_1 and x_2.

$$\text{Maximize} \quad z = 2x_1 + x_2$$
$$\text{subject to} \quad \begin{cases} 4x_1 + 3x_2 \leq 24 \\ 3x_1 + 5x_2 \leq 15 \end{cases}$$

9.52. Graphically solve the following linear programming problem cast in terms of two nonnegative variables, x_1 and x_2.

$$\text{Maximize} \quad z = 2x_1 + x_2$$
$$\text{subject to} \quad \begin{cases} x_1 + x_2 \leq 4 \\ 3x_1 + x_2 \leq 10 \end{cases}$$

9.53. A manufacturing company regularly produces three products that are sold at unit prices of, respectively, \$6, \$11, and \$22. These prices seem to be independent of the firm's output, that is, the market seems able to absorb any amount of product without any adverse effect on their price. Four input factors are needed to make these three products, with the specific amounts, costs, and available supplies shown in the table below. Assuming that no other restrictions are placed on the company's manufacturing decisions, how much of each product should be made to maximize the company's profits? (*Hint*: Formulate the problem as a linear programming problem.)

		Product			
Input	**Unit cost (\$)**	**1**	**2**	**3**	**Supply of input**
1	2.0	0	1	2	150
2	1.0	1	2	1	200
3	0.5	4	6	10	400
4	2.0	0	0	2	100

9.54. A vendor customarily produces three washers whose materials and *unit* (per washer) costs are, respectively, brass at \$0.60, steel at \$1.20, and aluminum at \$1.00. The washers are sold in two collections of mixed types, as shown in the table below, as is the supply of raw materials. Mixture A sells at \$1.50/lb and mixture B sells at \$1.80/lb. The vendor would like to know, how much of each mixture should she make? (*Hint*: Formulate the problem as a linear programming problem.)

Mixture	Brass	Steel	Aluminum
A	0.25	0.50	0.25
B	0.00	0.50	0.50
Supply (lb)	1,000	400	400

9.55. A trucking firm has received an order to move 3000 tons of miscellaneous goods. The firm has fleets of 150 15-ton trucks and 100 10-ton trucks whose operating costs per ton are, respectively, \$30.00 and \$40.00. The firm also has a policy of retaining in reserve at least one 150-ton truck with every two 10-ton trucks. How many of each fleet should be dispatched to move the goods at minimal operating cost? (*Hint*: Formulate the problem as a linear programming problem.)

9.56. Ingredients A and B are mixed in varying proportions to make massage oil and machine oil, each of which is sold at the (same) wholesale price of \$3.00 per quart. The cost of massage oil is \$1.50 per quart, while machine oil costs \$2.00 per quart. While there is no fixed formula or algorithm for mixing A and B to obtain a specific type of oil, two rules are generally followed: (1) Massage oil may contain no less than 25% of A and no less than 50% of B; and (2) machine oil may contain no more than 75% of A. If 30 quarts of A and 20 quarts of B are available for mixing, how much of each oil should be made to maximize profit? (*Hint*: Formulate the problem as a linear programming problem.)

9.57. Determine the volume and tip deflection of a tip-loaded cantilever beam of length, L, and circular cross-section of constant radius, R.

9.58. What savings of volume (or material) could be made for the beam of Problem 9.57 if the beam cross-section were a hollow tube of constant mean radius, R, and tube wall thickness, t?

9.59. What savings of volume (or material) if a beam of constant rectangular cross-section $b \times h$ were replaced with a beam of the same length whose cross-section is an idealized I-beam that has two, symmetrically placed small rectangles of thickness $t < h$ and area $A = b \times t$ that are separated by the beam's height, h? (*Hint*: Remember that I is the second moment of the area, $I = \int_A z^2 dA$.)

9.60. What savings would be made if the radius of the solid circular beam varied along the axis, $R = R(x)$, and was restricted to have the same deflection under the same load?

9.61. What savings would be made if (only) the radius of the tubular circular beam varied along the axis, $R = R(x)$, and was restricted to have the same deflection under the same load?

9.62. Develop a model for the *glide angle* γ of a glider. (*Hints*: Reconsider the small plane model developed in Section 9.5.2 in the absence of thrust. How does the climb angle, α, relate to the glide angle, γ?)

9.63. In the light of Problem 9.62, what is the optimum glide angle for a glider?

9.64. Show that the power $P = D \times V$ of a small, propeller-driven plane for equilibrium flight, during which the plane's acceleration is zero, can be modeled as:

$$P = C'_{01} V^3 + C'_{02} V^{-1}.$$

How do the constants, C'_{01} and C'_{02}, relate to those given for the small plane model of Section 9.5.2?

9.65. Determine the optimum speed that minimizes the power consumption for the plane model developed in Problem 9.64 using the standard calculus approach.

9.66. Determine the optimum speed that minimizes the power consumption for the plane model developed in Problem 9.64 using geometric programming (GP). Does this answer agree with that obtained in Problem 9.65?