# Monitoring Structural Breaks in Dynamic Regression Models with Bayesian Sequential Probability Test

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**Abstract**

Structural breaks are pervasive among macroeconomic and financial time series; consequently, forecasts may lose accuracy out of sample, which renders monitoring structural breaks a critical practice. We develop a structural break monitoring schema, Bayesian Sequential Probability Test (BSPT), for dynamic regression models, which consists of two components: the probabilistic detecting statistics of a structural break, and a sequential stopping procedure. We demonstrate the finite sample property and effectiveness of BSPT by comparing its performance with that of CUSUM under a variety of DGPs and in a few economic applications.

*Keywords:* Structural Breaks; Bayesian Decision Theory; Optimal Stopping; Change Point Detection

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# 1. Introduction

## 1.1. Motivation

The economy is a complicated, dynamic, and evolving system. Relations among economic variables are unstable due to changes in laws, policies, technologies, and consumer preferences. Stock and Watson (1996) documented substantial evidence of parameter instability in their models using 76 representative US monthly time series. However, practitioners of econometrics usually rely on relative parameter stability to build statistical models for forecasts and inferences; when structural breaks happen unacknowledged, forecasts lose accuracy, and inferences may be erroneous. Therefore it is of practical importance to detect the structural breaks promptly.

Data arrives sequentially in reality, which raises a concern: is the data generating process of the newly arrived data different from the one of the historical data? More specifically, practitioners need to update inference about the occurrence of a structural break and decide whether to conclude or no sufficient evidence to suggest so. Such decisions will be made sequentially and incur two risks: a false alarm and delayed detection. Furthermore, the risks cannot be reduced simultaneously: one risk can only be reduced at the cost of increasing the other. In order to make optimal decisions, practitioners need to strike a balance between these two risks.

To achieve this goal, we need a sequential statistical decision framework that consists of at least two components: the detecting statistics and a decision rule. The detecting statistics should be updated with newly arrived data to convey uncertainty about the occurrence of a structural break; ideally, the proper language to communicate uncertainty is probability; the decision rule should minimize the risk function.

Bayesian statistics are naturally sequential and probabilistic: the notion of updating belief in the wake of new information is fundamental to the Bayesian approach (West, 1986). Bayesian decision theory based on the Bayesian statistics updated with the latest information, combined with the decision maker's objective, will deliver the optimal decision rule. It is not always possible to make statistical inference concerning the use of such information, as represented by a risk function. However, such practice should be encouraged in general (Berger, 1985, p. 1-2). For an early study to incorporate a risk function into statistical analysis, see Wald (1949).

In this paper, we develop a Bayesian Sequential Probability Test(BSPT), as a residual-based structural break monitoring schema for dynamic regression models. We will demonstrate the finite sample property of the schema and the effectiveness through numerical studies.

*1.2. Literature Review*

There are extensive studies on structural breaks, most of which deal with tests and identifications in fixed-length data. Some classical references include Andrews (1993), Andrews and Ploberger (1994), Kuan and Hornik (1995), Bai and Perron (1998) and Bai (1994, 1995, 1996, 1997). Such methods are retrospective and cannot be applied repeatedly for monitoring purposes, since the law of iterated logarithm implies that, the empirical size will approach one with a constant boundary function as the sample size grows and the test repeatedly applied (Robbins, 1970). Hence, researchers resort to functional central limit theorem (FCLT) to approximate the boundary-crossing probabilities of the CUSUM type detecting statistics by those of standard Brownian motions.

Chu et al. (1996) broke the ground of monitoring structural breaks in linear regression models and proposed two monitoring schemas: the CUSUM and FL monitoring. The CUSUM detector is a partial sum of OLS recursive residuals as the detecting statistics, which will converge to standard Brownian motion under the null of no structural breaks. The detecting statistics of FL monitoring are derived on the deviation of updated parameter estimates from the historical estimates. These statistics will converge to a vector of Brownian bridges under the null. Whenever one element of the detecting statistics exceeds the boundary, the structural break is concluded.

Subsequently, Zeileis et al. (2005) proposed a unified framework for monitoring structural breaks; in particular, they used the OLS-residuals based CUSUM test, where the regression coefficients are estimated only once, and residuals in the monitoring period, are computed with the estimated coefficients. If there were structural breaks in the monitoring period, the residuals would deviate from zero consistently[1]. The authors also proposed moving window OLS estimates as another monitoring schema. The monitoring statistics consist of the differences between moving window parameter estimates and the historical estimates.

[1] In this paper, we adopt a similar approach, but we will focus on the finite sample property.

Both Chu et al. (1996) and Zeileis et al. (2005) pointed out that the boundary functions in their monitoring schema are determined for mathematical convenience rather than optimality.

Besides the literature in econometrics, there is a long history of studies on sequential tests in statistics. Early work dates back at least to Shewhart (1931). Wald (1947)'s sequential probability ratio test (SPRT) is a seminal work of sequential tests. Since its inception, the subject has been extensively studied and applied in quality control (Page, 1955) and clinical trials (Armitage, 1975). Please refer to Lai (2001) for a comprehensive review.

Shiryaev (1963) adopted the Bayesian approach to solve the change detection problem with detecting statistics derived through Bayes' theorem and the optimal stopping time obtained by minimizing a Bayes risk function, which is the sum of the probability of a false alarm and expected length of delayed detection[1]. However, the alternative distribution in Shiryaev (1963)'s problem is assumed to be known; such an assumption is too restrictive for economic applications. Li et al. (forthcoming 2020) extended Shiryaev's framework to composite parameter to monitor US recessions. In this paper, We will extend Shiryaev (1963)'s original work to cases with unknown alternative distribution in monitoring structural breaks in dynamic regression models.

## 2. The Model

### *2.1. Setup of the Problem*

Consider a dynamic regression model

$$y_t = x_t'\beta_t + \epsilon_t, \tag{2.1}$$

where $x_t$ is a $k$ dimensional vector of regressors and $\beta_t$ is a $k \times 1$ vector of regression coefficients, and the error terms $\epsilon_t$ is IID $\mathcal{N}(0, \sigma^2)$. Assume that in the history of $t \in \{1, ..., n\}$, the coefficients in Equation (2.1) are constant and equal to $\beta_0$. We want to monitor new data from $n+1$ onward to test if there is any structural break happening since. For some unknown time $\kappa > n$, the null

1 Barry and Hartigan (1993), another notable Bayesian approach, devised block particitations to model changes in parameters, which is mainly for restrospective analysis.

and alternative of the hypothesis can be specified as

$$\begin{aligned} H_0 : y_t &= x_t'\beta_0 + \epsilon_t \text{ for } t = 0, 1, 2, ..., \kappa - 1, \\ H_1 : y_t &= x_t'\beta_1 + \epsilon_t \text{ for } t = \kappa, \kappa + 1, ... \end{aligned}$$

Furthermore, we can show the error process evolves as

$$\begin{aligned} H_0 : \varepsilon_t &= \epsilon_t \text{ for } t = 0, 1, 2, ..., \kappa - 1, \\ H_1 : \varepsilon_t &= x_t'(\beta_1 - \beta_0) + \epsilon_t \text{ for } t = \kappa, \kappa + 1, ... \end{aligned}$$

We take the unconditional mean of the error process in the alternative ($E(\varepsilon_t) = \mu$) to arrive at the simplified stochastic process of the error terms:

$$\begin{aligned} H_0 : \zeta_t &= \epsilon_t \text{ for } t = 0, 1, 2, ..., \kappa - 1, \\ H_1 : \zeta_t &= \mu + \epsilon_t \text{ for } t = \kappa, \kappa + 1, ... \end{aligned}$$

The purpose of the above procedure is to derive a stochastic process that can capture the statistical property of a structural break of a dynamic regression model, which is, under the alternative, the error terms will deviate form zero consistently. Finally, we have arrived at the stochastic process:

$$\zeta_t \sim \begin{cases} H_0 : f_0 = \mathcal{N}(0, \sigma^2) & \text{for } t = 1, 2, ..., \kappa - 1 \\ H_1 : f_1 = \mathcal{N}(\mu, \sigma^2) & \text{for } t = \kappa, \kappa + 1, ... \end{cases} \tag{2.2}$$

### *2.2. The Bayesian Sequential Probability Test*

The Bayesian Sequential Probability Test is a Bayesian sequential statistical decision framework for monitoring structural breaks in dynamic regression models. It consists of two components: the detecting statistics of a structural break and a sequential stopping procedure, which is a decision rule derived by minimizing the practitioner's risk function. Please see Figure(7) for an overview of the framework.

*2.2.1. The Detecting Statistics of a Structural Break*

Consider a random sequence of errors $\zeta_t$ defined by Equation (2.2) with $t = 1$ as the starting time of monitoring period. Assume that conditional on the unobservable parameter $\kappa$, the structural break time, the sequence $\{\zeta_1, \zeta_2, ..., \zeta_{\kappa-1}\}$ being IID with marginal distribution $f_0$, and $\{\zeta_\kappa, \zeta_{\kappa+1}, ...\}$ being IID with marginal distribution $f_1(\theta)$. The null distribution $f_0$ is assumed to be known, and the alternative distribution $f_1(\theta)$ is a parameterized distribution with $\theta \in \Theta$ as the parameters.

Assume a geometric prior distribution for the structural break time $\kappa$:

$$P(\kappa = k) = \begin{cases} \pi & \text{if } k = 0 \\ (1-\pi)\rho(1-\rho)^{k-1} & \text{if } k = 1, 2, ... \end{cases} \tag{2.3}$$

Let $\Pi_0(\theta)$ be the prior distribution over the parameter space $\Theta$, and $\Pi_t(\theta)$ be the posterior distribution given information set $\mathcal{I}_t$.

Let $\zeta^t = \{\zeta_1, ..., \zeta_t\}$ be the history of random variable $\zeta_t$, and $\zeta_t \in Z_t$, and $Z^t = Z_0 \times Z_1 \times, ..., \times Z_t$. Define the set $A \in Z^t$ as $A = \{\omega \colon \zeta_1 \leq x_1, ..., \zeta_t \leq x_t\}$. Given $\theta \in \Theta$, the probability of the event $A$ is defined as

$$\begin{aligned} P_\theta(A) \quad = \quad & \pi P_\theta^1(A) + (1-\pi)\sum_{i=0}^{t-1} \rho(1-\rho)^i P^0\{\zeta_1 \leq x_1, ..., \zeta_i \leq x_i\} \\ & \times P_\theta^1\{\zeta_{i+1} \leq x_{i+1}, ..., \zeta_t \leq x_t\} + (1-\pi)(1-\rho)^t P^0(A), \end{aligned}$$

where $P^0$ and $P_\theta^1$ are the probability measures underlying the null and alternative distribution given $\theta \in \Theta$ respectively. Furthermore, the probability measure given the distribution $\Pi(\theta)$ over the parameter space $\Theta$ is defined as

$$P^\Pi(A) = \int_\Theta P_\theta(A) d\Pi(\theta) \tag{2.4}$$

The expectation $E^\Pi$ in this paper is defined with probability measure defined above.

**Definition 2.1.** *Define the detecting statistics as the posterior probability of a structural break* *happened before time t:*

$$\pi_t = P^{\Pi}(\kappa \leq t \mid \mathcal{I}_t), \textit{ for } t = 1, 2, 3, ..., \tag{2.5}$$

*where $\mathcal{I}_t$ denotes the information available at time $t$.*

We will describe the evolution of the probability of a structural break $\pi_t$. Let $\pi_t(\theta)$ be the probability of a structural break given $\theta \in \Theta$, and $\pi_t(\theta)$ evolves as

$$\pi_t(\theta) = \frac{(\rho(1-\pi_{t-1}) + \pi_{t-1})\, f_1(\zeta_t \mid \theta)}{f(\zeta_t \mid \theta)}, \tag{2.6}$$

given $\pi_{t-1}$ and $\zeta_t$, where

$$f(\zeta_t \mid \theta) \quad = \quad (1-\rho)(1-\pi_{t-1}) f_0(\zeta_t) + (\pi_{t-1} + \rho(1-\pi_{t-1}))\, f_1(\zeta_t \mid \theta).$$

Given the likelihood at time $t$, we can update the posterior at $t-1$:

$$\Pi_t(\theta) = \frac{f(\zeta_t \mid \theta)\Pi_{t-1}(\theta)}{\int f(\zeta_t \mid \theta) d\Pi_{t-1}(\theta)}. \tag{2.7}$$

Lastly, $\pi_t$ is updated as

$$\pi_t = \int_{\Theta} \pi_t(\theta) d\Pi_t(\theta). \tag{2.8}$$

Since the parameter $\theta$ is unknown, the idea is to average it out with available information. It is trivial to show that given the posterior distribution $\Pi_t$, the expectation of $\pi_t$ only depends on $\pi_{t-1}$, that makes value function iteration a possibility, which is key to solve the practitioner's optimization problem.

### *2.2.2. The Objective Function*

The practitioner's objective is to detect the structural break as soon as possible, with minimum probability of a false alarm. This is a sequential decision problem, in which the practitioner will examine a sequence of observations one at a time, and decide after each observation whether to conclude the break or continue sampling.

The practitioner's decision is represented as stopping time $\tau$, which is the time to conclude the structural break. It is a random function of information available at the time. In this paper, we will focus on the set of stopping time with finite expected value:

$$\Gamma \equiv \{\tau : E^{\Pi_0}(\tau) < \infty\}. \tag{2.9}$$

The risk function of stopping time $\tau$ given parameter $\theta$ is

$$L(\theta, \tau) = E_\theta \left\{ 1(\tau < \kappa) + c(\tau - \kappa)^+ \right\}. \tag{2.10}$$

The first component is the probability of a false alarm, and the second component is the expected length of delayed detection multiplied by a controlling factor $c$, which reflects the practitioner's penalty on delay relative to a false alarm[1].

The Bayes risk of stopping time $\tau$ with prior $\Pi_0$ is

$$r^{\Pi_0}(\pi, \tau) = E^{\Pi_0} L(\theta, \tau). \tag{2.11}$$

The Bayes risk of the problem with prior $\Pi_0$ is

$$r^{\Pi_0}(\pi) = \inf_{\tau \in \Gamma} E^{\Pi_0} L(\theta, \tau), \tag{2.12}$$

where the expectation is taken over the parameter space $\Theta$.

The optimization problem in Equation(2.12) is very complicated, and to solve it with Shiryaev (1963) type of stopping time at time 0 is intractable, mainly because of the prior distribution $\Pi_t$ that defines the expectation, is evolving. This issue is not present in Shiryaev (1963)'s problem with known alternative distribution. We delineate the solution in two steps: first, we solve the optimization problem with fixed prior $\Pi_t$ at any time $t$; second, we take the solution to sequential stopping procedure with evolving prior $\Pi_t$.

[1] This risk function with known alternative distribution was first proposed and solved by Shiryaev (1963)

*2.2.3. The Solution to the Problem with Fixed Prior*

In this section, we will solve the Bayes risk problem with prior fixed at any time $t$:

$$r^{\Pi_t}(\pi) = \inf_{\tau \in \Gamma} E^{\Pi_t} L(\theta, \tau). \tag{2.13}$$

To solve this problem we need to transform the objective function in Equation (2.13) into more tractable form.

**Lemma 2.1.** *For any generic prior distribution $\Pi_t$ over the parameter space, and the space of stopping time $\Gamma$ defined in Equation(2.9). Suppose $E^{\Pi_t}(\kappa) < \infty$ and the sequence $\pi_t$ defined in Definition (2.1), then for $\tau \in \Gamma$, we can write*

$$E^{\Pi_t}\left\{1\left(\tau < \kappa\right) + c\left(\tau - \kappa\right)^{+}\right\} = E^{\Pi_t}(1 - \pi_\tau + c\sum_{i=0}^{\tau-1} \pi_i). \tag{2.14}$$

We are now ready to characterize the optimal stopping time with the following lemma.

**Lemma 2.2.** *Given $\pi_t$ defined in Definition 2.1, for some appropriately chosen constant $\pi^*$, the stopping time*

$$\tau_t = \begin{cases} 1 & \textit{if } \pi_t \geq \pi^* \\ 0 & \textit{if otherwise} \end{cases} \tag{2.15}$$

*is Bayes optimal to the objective function (2.14) with prior distribution $\Pi_t$, if $c \geq 1$ then $\pi^* = 0$, where the subscript of $\tau_t$ is the suppressed index for prior $\Pi_t$.*

*2.2.4. The Solution to the Problem with Evolving Prior*

So far, we have solved the Bayes risk problem for any generic prior $\Pi_t$, which is held fixed. In this section, we will consider the case that the prior $\Pi_t$ is updated with newly arrived information. We formulate the solution based on the general framework laid down by Berger (1985)[1] for Bayesian sequential decision problems: after each new observation arrives, we should consider a new sequential problem with updated prior, i.e., the posterior as the new prior, and minimize the

[1] Chapter 7 of Berger (1985)

Bayes risk function looking forward with available information[1]. To be more specific, consider $\zeta_t$ is observed, then we obtain the posterior $\Pi_t$ given $\zeta_t$ and the previous history already observed. We can consider a new sequential problem with $\Pi_t$ as the prior and $\{\zeta_{t+1},\ \zeta_{t+2},...\}$ as possible future observations. We solve the optimization problem every time there is new data arrives, with the optimal stopping time reflecting the latest information available, represented by the posterior distribution at the time; that will result in a sequence of optimal stopping time indexed by the posterior distribution $\Pi_t$. We present the solution more formally in the flowing.

Let $\Gamma$ be defined as in Equation(2.9), and define a sequential stopping procedure $T$ as:

$$T = \min\{t \geq 0 : \tau_t(\zeta^t) = 1\}, \tag{2.16}$$

where $\tau_t(\zeta^t) \in \Gamma$. The sequential stopping procedure is also a random function of information available at time $t$. The Bayes risk of a sequential stopping procedure $T$ is given by

$$r^{\Pi_0}(\pi, T) = E^{\Pi_0} L(\theta, T), \tag{2.17}$$

and the prior $\Pi_t$ will evolve according to Equation (2.7). The Bayes risk of the sequential decision problem is

$$r^{\Pi_0}(\pi) = \inf_{T \in \Gamma^\infty} E^{\Pi_0} L(\theta, T), \tag{2.18}$$

where $\Gamma^\infty \equiv \Gamma \times \Gamma \times \ldots$. At any time $t$, after each new observation is taken, the practitioner is considering a new sequential decision problem starts at that point looking forward, and the Bayes risk of the sequential decision problem is

$$r^{\Pi_t}(\pi) = \inf_{T \in \Gamma^\infty} E^{\Pi_t} L(\theta, T). \tag{2.19}$$

**Lemma 2.3.** *Given the Bayes risk of sequential stopping procedure $T$, The optimal sequential*

[1] The problem, in general, is very complicated and challenging; other approaches in change detection literature include Lai and Xing (2010). The authors proposed a solution which is asymptotically Bayes in the sense of $\rho \to 0$.

*stopping procedure*

$$T^* = \min\{t \geq 0 : \tau_t^* = 1\} \tag{2.20}$$

*solves the sequential decision problem*

$$r^{\Pi_0}(\pi) = \inf_{T \in \Gamma^\infty} E^{\Pi_0} L(\theta, T),$$

*where $\tau_t^*$ is defined in Equation (2.15).*

## 3. Simulation

We present some Monte Carlo studies in this section. Following Chu et al. (1996) and Zeileis et al. (2005), the data is generated from IID $\mathcal{N}(0,1)$ and $\mathcal{N}(0.8,1)$ for null and alternative distribution respectively. The monitoring period starts at 100 and the break times are $\kappa \in \{110, 150, 200, 300\}$. The number of simulation for each monitoring exercise is 10,000. The mean of the alternative distribution is unknown to the practitioner. The results are summarized in Table(1). Figure(1) provides an illustration of the BSPT. From Table (1), one can see that with higher value of the controlling factor $c$, the expected delay is shorter and probability of false alarm is higher. This is due to the trade-off between the two risks that is weighted by the controlling factor.

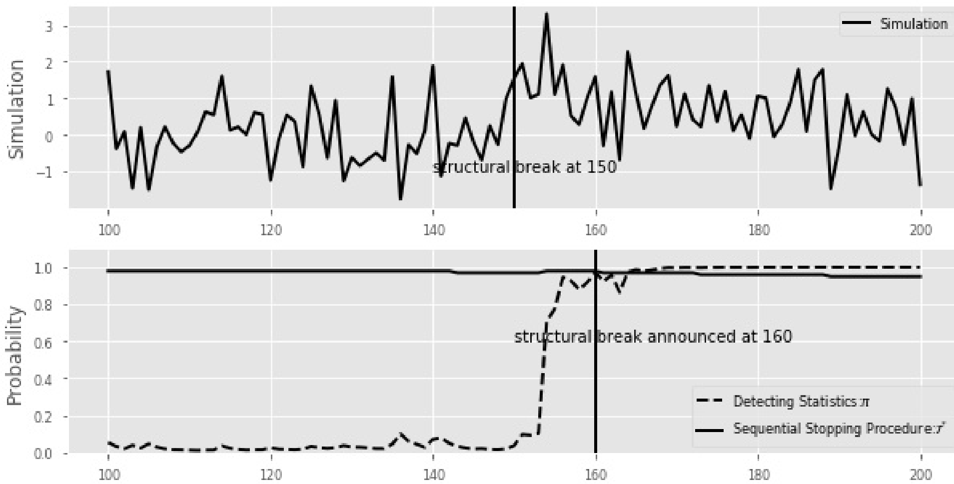


Figure 1: A demonstration of the BSPT with Monte Carlo simulation. The mean of the simulated process changes from 0 to 0.8 at 150. When the detecting statistics exceed the decision threshold: $\pi_t \geq \tau^*_{160}$ at 160, then the structural break is concluded.

| Control Factor | Structure Break Time | Expected Delay(sd) | Prob of False Alarm |
|---|---|---|---|
| $c = 0.005$ | 110 | 27.64 (14.70) | 0.28% |
| | 150 | 29.27 (21.43) | 0.52% |
| | 200 | 30.68 (27.75) | 1.09% |
| | 300 | 33.75 (38.77) | 2.61% |
| $c = 0.008$ | 110 | 25.65 (14.24) | 0.40% |
| | 150 | 27.31 (20.37) | 0.81% |
| | 200 | 28.57 (25.84) | 1.71% |
| | 300 | 31.02 (35.49) | 4.19% |
| $c = 0.01$ | 110 | 24.88 (13.97) | 0.48% |
| | 150 | 26.41 (20.04) | 1.04% |
| | 200 | 27.56 (24.63) | 2.16% |
| | 300 | 29.49 (32.63) | 5.28% |

Table 1: Summary of Simulation Results with Structural Break in the Mean

We also compare the performance of the BSPT with that of the CUSUM. The results are presented in Table (3). The alternative process is simulated with the following equation

$$y_t = \alpha + \gamma y_{t-1} + \epsilon_t, \tag{3.21}$$

where $\epsilon_t$ is white noise. This exercise also exams the BSPT's ability to deal with intertemporally dependent alternative processes. In general, the BSPT can detect the break with much less average delay under similar empirical size. More importantly, the BSPT is much more robust against early and late structural breaks. It is well known that the CUSUM type of detecting statistics is powerful for the early break and loss its power gradually as the monitoring process continues.

| | | CUSUM | | | BSPT | | |
|---|---|---|---|---|---|---|---|
| | Break Time | Exp Delay(sd) | type I | type II | Exp Delay(sd) | type I | type II |
| $\alpha = 0.8$ | 110 | 30(19) | 0.55% | 0% | 26(14) | 0.36% | 0% |
| | 150 | 58(34) | 1.90% | 0% | 27(20) | 0.86% | 0% |
| $\gamma = 0$ | 200 | 90(52) | 2.52% | 0% | 28(25) | 1.57% | 0% |
| | 300 | 151(84) | 2.71% | 0.12% | 31(35) | 4.40% | 0% |
| $\alpha = 0.3$ | 110 | 61(74) | 0.48% | 0.18% | 34(32) | 0.36% | 0% |
| | 150 | 108(99) | 1.99% | 0.91% | 37(35) | 0.86% | 0% |
| $\gamma = 0.5$ | 200 | 153(120) | 2.64% | 2.29% | 38(38) | 1.57% | 0% |
| | 300 | 223(134) | 3.41% | 8.50% | 41(45) | 4.40% | 0% |
| $\alpha = 0.1$ | 110 | 86(129) | 0.55% | 8.76% | 33(36) | 0.36% | 0% |
| | 150 | 152(162) | 1.92% | 19.66% | 34(35) | 0.86% | 0% |
| $\gamma = 0.7$ | 200 | 199(168) | 2.64% | 30.52% | 34(34) | 1.57% | 0% |
| | 300 | 240(158) | 3.16% | 48.43% | 35(36) | 4.40% | 0% |
| $\alpha = 0.0$ | 110 | 48(80) | 0.57% | 6.80% | 22(23) | 0.36% | 0% |
| | 150 | 96(119) | 1.89% | 20.39% | 23(24) | 0.86% | 0% |
| $\gamma = 0.8$ | 200 | 134(138) | 2.45% | 36.58% | 24(23) | 1.57% | 0% |
| | 300 | 172(139) | 2.99% | 59.24% | 24(24) | 4.40% | 0% |

Table 2: We compare the performance of the CUSUM and BSPT under the same alternative process: $y_t = \alpha + \gamma y_{t-1} + \epsilon_t$, where $\epsilon_t$ is white noise. The results of the CUSUM are obtained with the OLS-CUSUM method in the R package strucchange by Zeileis et al. (2015) with 10,000 simulations for each monitoring exercise. The control factor in the BSPT is set at 0.008, and the prior is presented in Table (3). Type II error refers to the break is not detected before 900 periods.

This result can be attributed to a few advantages of the BSPT: First, the sequential stopping procedure consists of a sequence of optimal stopping time that reflect dynamic information in the data arriving sequentially. Second, it is well known in the literature that a significant deviation from the null is evidence in favor of the alternative; but the increase in the frequency of such deviations is also crucial for detection. The detecting statistics of the BSPT is keen to incorporate the increase in the frequency and convert it to the posterior probability through Bayesian updating. In contrast to the CUSUM type of statistics, the decision rule usually is a deterministic function of time.

## 4. Some Applications of the BSPT

We demonstrate the ability of the BSPT to monitor structural breaks with a few economic applications, including German M1 money demand, US labor productivity, and US HPI. We use the same set of prior as in Table (3), we also set the initial value of $\pi = 0.5$ and $\rho = 0.01$.

| Prior Distribution in Simulation | |
|---|---|
| $\mu$ | $[-2, -0.6] \cup [0.6, 2]$ |

Table 3: Prior Distribution For the Parameters

### *4.1. The Boom-Bust Cycles and Forecast Performance of the House Price Models*

The recent financial crisis in 2008 has revived research interest in housing price forecast, in particular, the challenge posed by the boom-bust cycles of the housing price: When the housing price gets into a boom cycle, the model tends underestimate the housing price; while in a bust cycle, the model tends overestimate it. The boom-bust cycles present multiple structural breaks, BSPT handles multiple structural breaks by dealing with each structural break piecewise.

Following Miles (2008), we use the AR(2) model for one period ahead forecast for five states with diverse housing price dynamics in the US: California, Florida, Massachusetts, Ohio, and Texas. We use FHFA HPI quarterly purchase-only indexes and focus on the boom-bust cycle in the 2000s.

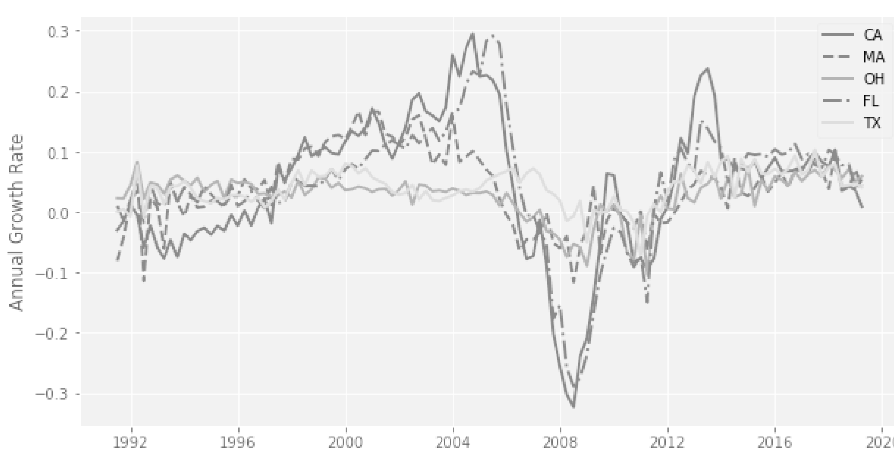


Figure 2: The Annual Growth Rate of HPI for Five States

We use data from 1991 Q2 to 2000 Q4 as the historical sample to estimate the AR(2) model for the five states, and start monitoring the one period ahead forecast errors from 2000 Q1.

For FL, we first identify a structural break in the forecast errors with a positive mean (actual HPA higher than estimates) in 2004 Q1. We then update the model estimation with data from 1991 to 2005 and then start monitoring from 2005. We identify forecast errors with a negative mean (actual HPA lower than estimates) in 2006 Q1. Please see Figure(3,4) for the graphical presentation of the exercise.

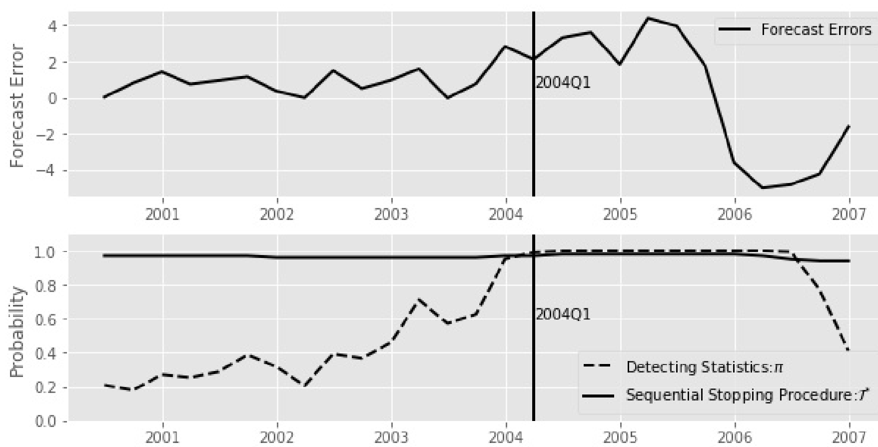


Figure 3: Monitoring the Boom of Housing Price in FL

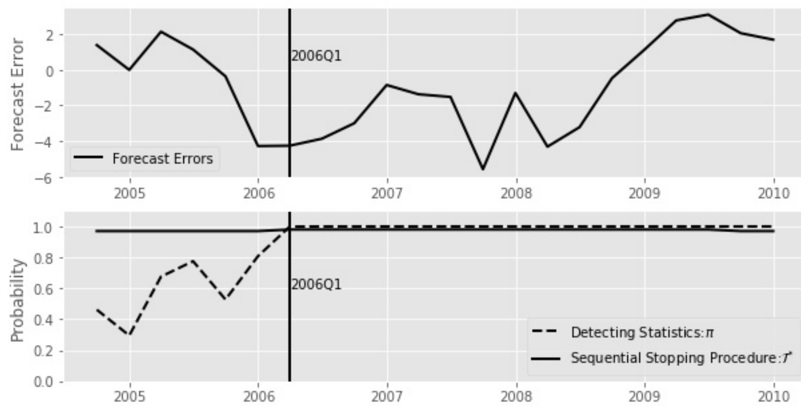


Figure 4: Monitoring the Bust of Housing Price in FL

As for CA, we detect a structural break in the forecast errors with negative mean in 2005 Q3. Similar to CA, we also identify bust in 2006 Q1 for OH and 2010 Q2 for TX. For MA, we did not find sufficient evidence to reject the null with zero mean forecast errors. Please see Figure(5,7,6 and 8).

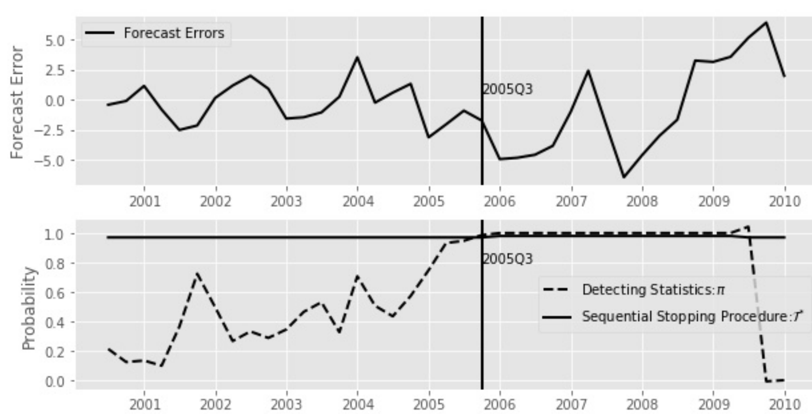


Figure 5: Monitoring the Bust of Housing Price in CA

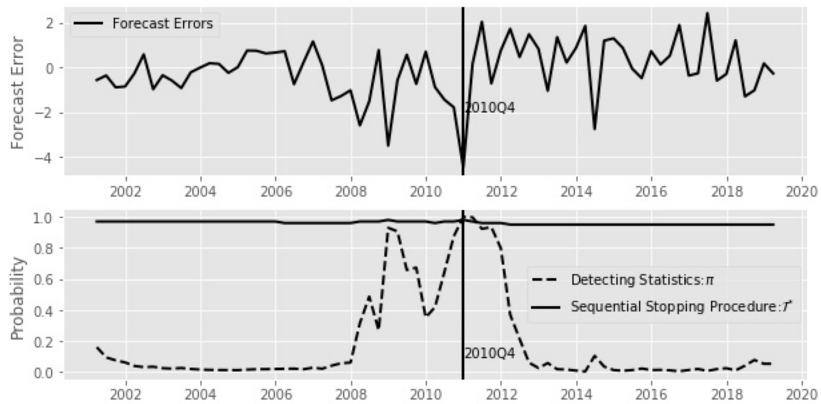


Figure 6: Monitoring the Bust of Housing Price in TX

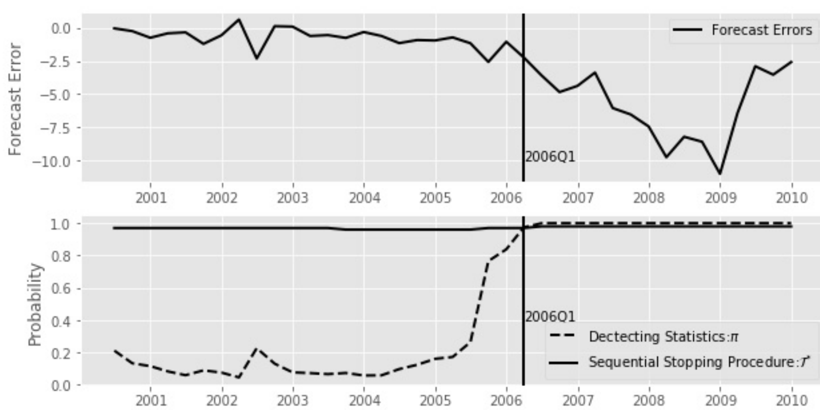


Figure 7: Monitoring the Bust of Housing Price in OH

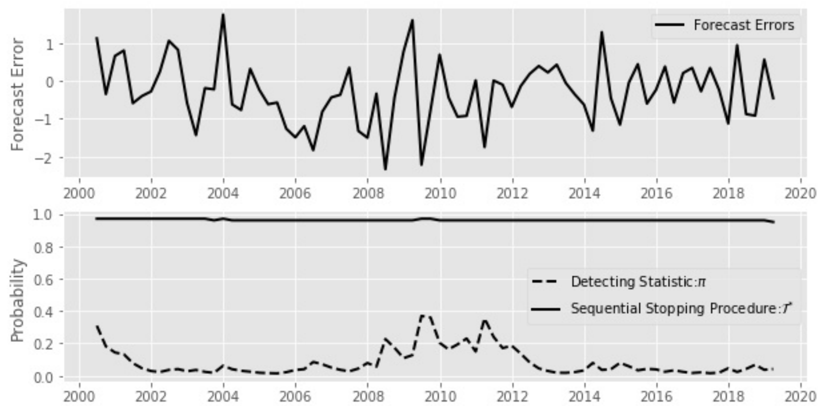


Figure 8: Monitoring the Housing Price Forecast Errors in MA

*4.2. US Labor Productivity*

It has been asserted that the US labor productivity has experienced a speedup in the second half of the 1990s. Hansen (2001) investigated the US labor productivity in the manufacturing/durable sector, and found strong evidence of a structural break sometime between 1992 and 1996, and weaker evidence of structural breaks in the 1960s and early 1980s. We use data ranging from Jan 1964 to Dec 1980 as the historical sample to estimate an AR(1) model, and starting monitoring the standardized forecast errors from Jan 1980 to Dec 1997. We detect a structural break in Aug 1997, and present the result in Figure(9). The BSPT yields a little delayed result than Hansen (2001). This is because Hansen (2001)'s test statistics are derived from the whole sample, while the stopping procedure in the BSPT only utilizes the information available at the time of the decision.

As shown in Figure (9), the forecast errors start to deviate from zero and become positive since early 1992. The detecting statistics start to pick up, but, as the information favoring the alternative is not strong, the detecting statistics stay way below the threshold before 1994. After such evidence, favoring the alternative keep flowing in, the information is picked up through likelihood function and update the detecting statistics. This illustrates an essential aspect of the BSPT: not only the magnitude of the information favoring the alternative is crucial to detection, but also the increase in the frequency of such information. Such intuition is reflected in Equation (2.6).

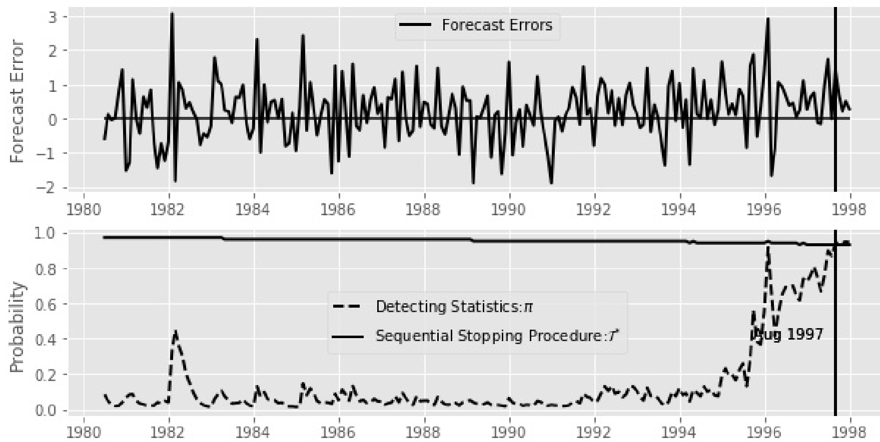


Figure 9: The upper panel is the forecast errors of the US labor productivity. The lower one consists of the detecting statistics $\pi_t$ and the sequential stopping procedure $T^*$. The forecast errors deviate from zero since 1992, and the probability starts to pick up as more evidence favoring the alternative flows in. Eventually, it crosses the threshold in Aug 1997.

### *4.3. German M1 Money Demand*

Lütkepohl et al. (1999) studied the stability and linearity of the German M1 money demand function and found clear evidence of structural instability after the monetary unification of German since 1990 Q3. We use the same linear model and data as in Zeileis et al. (2005) and detect the structural break in 1990 Q4 from a monitoring perspective. Figure (10) demonstrates the monitoring process.

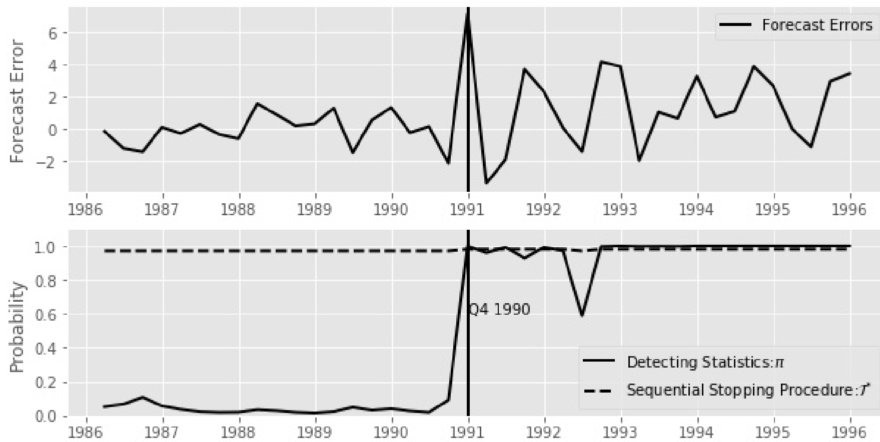


Figure 10: Monitoring Structural Breaks in German M1 Money Demand

## 5. Discussion

Monitoring schemas of structural breaks suffers from huge potential alternatives. To make the matter more complicated, since delay is costly, the practitioner has to make decisions with limited post-change data. Andrews (2003) studied one-time end-of-sample instability test, and proposed the S test that relies on the fact that under the null, the errors are centered around zero, while the post-change errors will deviate from zero, and large deviations from zero are evidence against the null. We rely on the same notion that under the alternative, the errors will deviate from zero consistently. Also, the primary interest of monitoring is the structural break time $\kappa$. All other parameters are of secondary concern.

We choose the prior for the mean of the alternative distribution to be the uniform distribution over the set $[-2, -0.6] \cap [0.6, 2]$ since the uniform prior is the most uninformative prior. The monitoring process is essentially a sequential hypothesis test, and the prior distribution also serves as the specification of the composite alternative in the hypothesis. The prior chosen is used throughout the exercises in this paper. This also raises another interesting intuition: the purpose of monitoring is to identify the break as soon as possible with minimum risk of false alarm. If the practitioner has knowledge about the alternative, that should be incorporated into the prior. The better the alternative is known, the quicker the break can be identified[1]. The special case is when the alternative is known. Following Shiryaev (1963), we also assume geometric prior for the breaking time $\kappa$ in Equation (2.3), a higher value of $\rho$ means an earlier break in the detecting period. This challenge to specify $\rho$ is not unique to the Bayesian approach that the decision about the likely timing of the future break. It is well known that the boundary in Chu et al. (1996) performs well for early breaks, but gets increasingly insensitive to late breaks. Zeileis et al. (2005) proposed different boundaries to distribute the empirical size more evenly. The authors also acknowledge that it is desirable to have a more flexible and less heuristic instrument to select boundaries according to a specified prior distribution of the timing of the break. If there is information regarding the likely timing of the future break, it can be incorporated as the prior. We choose $\rho = 0.01$ in all our exercises. The simulation results of other specifications can be provided per request.

The controlling factor $c$ is the cost of one period of delay. A higher value of the controlling factor suggests costlier delay. We tabulated different values of the controlling factor and the corresponding probability of a false alarm and expected length of delay in Table(1). More tabulation can be provided by request.

We can have a better understanding of how the sequential stopping procedure depends on the evolution of newly arrived information, through the posterior distribution $\Pi_t$, by examining Figure(1) and (11). We can see from Figure(1), the decision threshold is decreasing since 150. This is because the information favoring the alternative starts to accumulate after 150. the evolution of the posterior distribution reflects this: the probability mass starts to accumulate in the positive region(see Figure(11)). As such information accumulates, there is less probability of false alarm; so the threshold can be lower to minimize the delay. Hence the threshold reflects the inflow of the information dynamically.

[1] In certain situations, such prior knowledge is applicable. For example, the growth rate of GDP is positive when the economy is booming and will change from positive to negative in recessions.

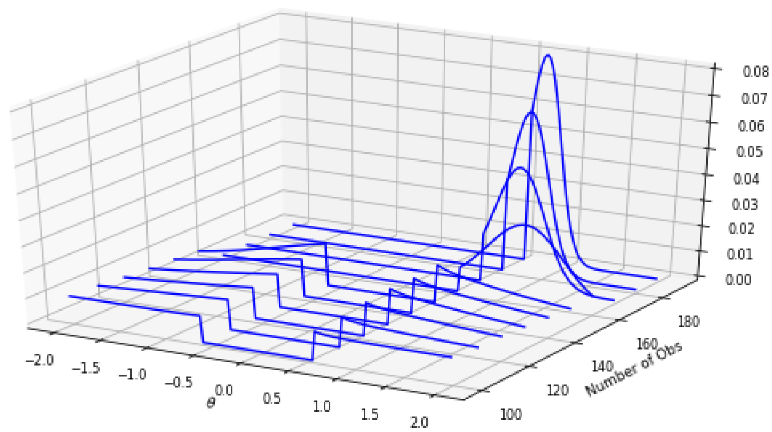


Figure 11: This figure shows the evolution of posterior distribution $\Pi_t$. The structural break happens at 150, and is detected at 160. As more information supporting the alternative accumulates, the posterior distribution amasses around 0.8.

## 6. Concluding Remarks

Monitoring structural breaks is essentially a process of sequential statistical decision making. To solve such a problem, we start with formulating the practitioner's objective function as a weighted sum of two risks: the probability of a false alarm and the expected length of delayed detection, and then derive the decision rule by optimizing the objective function sequentially, that results in a sequence of optimal stopping time. It is important to note that the magnitude of deviation from the null is sure evidence favoring the alternative, but the increase in the frequency of such signals is also crucial for detecting purpose. We demonstrate the effectiveness of the framework with Monte Carlo simulations and economic applications.

Bayesian statistics is inherently sequential, and we believe this is a promising line of research. Future research might include a broader range of unknown parameters, recursively estimated OLS parameters, etc.

## 7. Appendix

*Proof of Lemma 2.1.*

**Remark.** *This surprising result was first approved by Shiryaev (1963) for known distribution, and Poor and Hadjiliadis (2009) presented it with modern notations. We extend the proof to the unknown alternative distribution.*

For stopping time $\tau \in \Gamma$, we can show that

$$\begin{aligned} P(\tau < \kappa) &= E^{\Pi}(1(\tau < \kappa)) \\ &= E^{\Pi}(1 - \pi_\tau). \end{aligned}$$

The second equality follows from the definition of $\pi_t$. Now we show

$$E^{\Pi}(\tau - \kappa)^+ = E^{\Pi}\left(\sum_{t=0}^{\tau-1} \pi_t\right). \tag{7.22}$$

For any $t > 0$,

$$\begin{aligned} E^{\Pi}(t - \kappa)^+ &= \sum_{k=0}^{t}(t-k)P(\kappa = k \mid \mathcal{I}_t) \\ &= \sum_{k=0}^{t-1} P(\kappa \le k \mid \mathcal{I}_t) \\ &= \sum_{k=0}^{t-1}[P(\kappa \le k \mid \mathcal{I}_t) - P(\kappa \le k \mid \mathcal{I}_k)] + \sum_{k=0}^{t-1} P(\kappa \le k \mid \mathcal{I}_k) \\ &= \sum_{k=0}^{t-1}[P(\kappa \le k \mid \mathcal{I}_t) - P(\kappa \le k \mid \mathcal{I}_k)] + \sum_{k=0}^{t-1} \pi_k \end{aligned}$$

Define

$$\begin{aligned} M_t &= \sum_{k=0}^{t-1}[P(\kappa \le k \mid \mathcal{I}_t) - P(\kappa \le k \mid \mathcal{I}_k)] \\ &= -\sum_{k=0}^{t-1}[P(\kappa > k \mid \mathcal{I}_t) - P(\kappa > k \mid \mathcal{I}_k)] \end{aligned}$$

We can show that $M_t$ is a regular martingale:

$$E^{\Pi}(M_{t+1} \mid \mathcal{I}_t) = M_t. \tag{7.23}$$

Define

$$|M| \leq \lim_{t\to\infty} \sum_{k=0}^{\infty} [P(\kappa > k \mid \mathcal{I}_t) + \sum_{k=0}^{\infty} P(\kappa > k \mid \mathcal{I}_k)].$$

And also by assumption, we have

$$\begin{aligned} E^{\Pi} \sum_{k=0}^{\infty} [P(\kappa > k \mid \mathcal{I}_t) &= \sum_{k=0}^{\infty} P(\kappa > k) \\ &= E^{\Pi}(\kappa) < \infty \end{aligned}$$

and

$$E^{\Pi} \sum_{k=0}^{\infty} P(\kappa > k \mid \mathcal{I}_k) = E^{\Pi}(\kappa) < \infty.$$

Hence, $M$ is integrable. Now we want to show $M$ is regular:

$$\begin{aligned} E^{\Pi}(M \mid \mathcal{I}_m) &= \sum_{k=0}^{\infty} E^{\Pi}[P(\kappa \leq k \mid \mathcal{I}_t) - P(\kappa \leq k \mid \mathcal{I}_k) \mid \mathcal{I}_m] \\ &= \sum_{k=0}^{m-1} [P(\kappa \leq k \mid \mathcal{I}_m) - P(\kappa \leq k \mid \mathcal{I}_k)] \\ &= M_m \end{aligned}$$

The second equation follows since for any $m < k$, we have $I_m \subset I_k$ and the first term will be canceled by the second term in the equation due to iteration property of expectation. Hence $M_t$ is regular. By optional sampling, it implies

$$\begin{aligned} E^{\Pi}(M_T) &= E^{\Pi}(M_0) \\ &= 0, \end{aligned}$$

then the Lemma follows. $\square$

*Proof of Lemma 2.2.*

**Remark.** *The proof with known alternative distribution can be found in Shiryaev (1963, 1978) and Poor and Hadjiliadis (2009). The proof we did here is for problem with unknown alternative distribution.*

At $t = 0$, we have the following problem:

$$v^{\Pi_0}(\pi) = \inf_{\tau \in \Gamma} E^{\Pi_0}(1 - \pi_\tau + c\sum_{k=0}^{\tau-1} \pi_k). \tag{7.24}$$

The practitioner's problem is to decide either to stop or continue sampling given information available. If the decision is to stop, the loss will be $1 - \pi$, which is the probability of false alarm; otherwise, and the experiment continues, it will yield one period of expected delay plus the value of continuation from $t = 1$ on, which is

$$v^{\Pi_1}(\pi) = \inf_{\tau \in \Gamma} E^{\Pi_1}(1 - \pi_\tau + c\sum_{k=1}^{\tau-1} \pi_k) \tag{7.25}$$

The value function at $t = 0$ can be written in recursive form as

$$\begin{aligned} v^{\Pi_0}(\pi) &= \min\{1 - \pi, c\pi + E^{\Pi_0} v^{\Pi_1}(\pi' \mid \pi)\} \\ v^{\Pi_0}(\pi) &= \min\{1 - \pi, c\pi + v^{\Pi_0}(\pi' \mid \pi)\} \end{aligned}$$

With the same argument, we can derive the recursive presentation of the problem for any time $t$ with prior distribution $\Pi_t$, where we suppress the subscript:

$$v^{\Pi}(\pi) = \min\{1 - \pi, c\pi + v^{\Pi}(\pi' \mid \pi)\} \tag{7.26}$$

Next we will show that the recursive formulation for the valuation in Equation (7.26) is convergent.

One can define a finite time problem that $t \in \{0, 1, 2, ..., n\}$, by backward induction, we can define $v^{\Pi}_{n-1,n}(\pi)$ given prior distribution $\Pi$ as

$$v^{\Pi}_{n-1,n}(\pi) = \min\{1-\pi, c\pi + v^{\Pi}_{n}\left(\pi' \mid \pi\right)\},$$

Furthermore,

$$v^{\Pi}_{n-2,n}(\pi) = \min\{1-\pi, c\pi + v^{\Pi}_{n-1,n}\left(\pi' \mid \pi\right)\}.$$

We can define in general for any time $k < n$

$$v^{\Pi}_{k,n}(\pi) = \min\{1-\pi, c\pi + v^{\Pi}_{k+1,n}\left(\pi' \mid \pi\right)\},$$

It is evident that $v^{\Pi}_{k,n} \leq v^{\Pi}_{k+1,n}$, since the option of stopping has more value when there are more time left. By the same argument, we can also define a sequence in the same way: $v^{\Pi}_{k,n}, v^{\Pi}_{k,n+1}, \ldots$ with the property that $v^{\Pi}_{k,n} \geq v^{\Pi}_{k,n+1}$. In this way, we can construct a monotone non-increasing sequence, and the limit of which is well defined as

$$v^{\Pi}_{k} = \lim_{n\to\infty} v^{\Pi}_{k,n}.$$

The value function is bounded from below by $\min\{1-\pi, c\pi\}$. We can solve for $v^{\Pi}$ through value function iteration.

Define operator $Q$ as

$$(Qv^{\Pi})(\pi) = \min\{1-\pi, c\pi + v^{\Pi}(\pi' \mid \pi)\}$$

The value function

$$v^{\Pi}(\pi) = \lim_{n\to\infty} (Q^n v^{\Pi})(\pi). \tag{7.27}$$

and the stopping time for any generic prior distribution $\Pi$

$$\tau = \begin{cases} 1 & \text{if } v^{\Pi}(\pi) \geq 1-\pi \\ 0 & \text{if otherwise} \end{cases} \tag{7.28}$$

When the value function of continuation evaluated at $\pi$ equal the expected loss of stopping right now $(1-\pi)$ then it is time to stop. Further more, the above equation can be converted to

$$\tau = \begin{cases} 1 & \text{if } \pi \geq \pi^* \\ 0 & \text{if otherwise,} \end{cases} \tag{7.29}$$

where $v^{\Pi}(\pi^*) = 1 - \pi^*$. $\square$

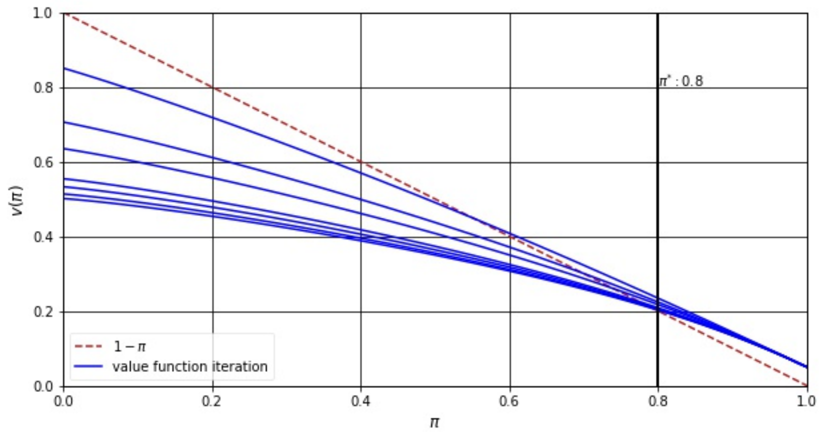


Figure 12: Iteration of value function until it converges. The value function crosses the line $1-\pi$ at $\pi^*$, where the continuation region and stopping region are defined.

*Proof of Lemma 2.3.* Let $\{N = t\}$ denote the event that $\tau_t = 1$. We can extend the sequential decision problem as

$$\begin{aligned} r^{\Pi_{t=0}}(\pi) \quad = \quad & P(N=0)E^{\Pi_{t=0}}(L(\theta, \tau_0 = 1)) + \\ & \sum_{t=1}^{\infty} \int_{\Theta} \int_{\{N=t\}} L(\theta, \tau_t = 1) dF(\zeta^t \mid \theta) dF^{\Pi_t}(\theta). \end{aligned}$$

The optimal sequential stopping procedure $T^* = \{\tau_0^*, \tau_1^*, \dots\}$ consist of a collection of stopping times that is optimal for Bayes risk function at any time $t$; hence the optimal sequential stopping procedure $T^*$ solves the practitioner's problem at time 0. $\square$

Figure 13: The Skeleton of Bayesian Sequential Probability Test

Statement of Problem

$$y_t = x_t' \beta_t + \zeta_t$$

$$\zeta_t \sim \begin{cases} \mathcal{N}(0, \sigma^2) & \text{for } t = 1, 2, ..., \kappa - 1 \\ \mathcal{N}(\mu, \sigma^2) & \text{for } t = \kappa, \kappa + 1, ... \end{cases}$$

**The Detecting Statistics:**

Definition of detecting statistics:

$$\pi_t = P(\kappa \leq t \mid \mathcal{I}_t)$$

.

The probability is updated with newly arrived data according to Bayesian theorem given parameter:

$$\pi_t(\theta) = \frac{(\rho(1 - \pi_{t-1}) + \pi_{t-1}) f_1(\zeta_t \mid \theta)}{f(\zeta_t \mid \theta)}.$$

The prior distribution of parameter $\theta$ evolves as:

$$\Pi_t(\theta) = \frac{f(\zeta_t \mid \theta) \, \Pi_{t-1}(\theta)}{\int f(\zeta_t \mid \theta) d\Pi_{t-1}(\theta)}.$$

The detecting statistics becomes:

$$\pi_t = \int_\Theta \pi_t(\theta) d\Pi_t(\theta).$$

For any time $t$, the optimal stopping time is

$$\tau_t^* = \{1 \text{ if } \pi_t \geq \pi^*, 0 \text{ otherwise}\},$$

and the optimal sequential stopping procedure is

$$T^* = \min\{t \geq 0 : \tau_t^* = 1\}.$$

**The Sequential Stopping Procedure:**

The objective function:

$$v^{\Pi_0}(\pi) = \inf_{T \in \Gamma^\infty} E^{\Pi_0}\{1(\tau < \kappa) + c(\tau - \kappa)^+\}.$$

The objective function given $\Pi_t$ can be converted to

$$E^{\Pi_t}\{1(\tau < \kappa) + c(\tau - \kappa)^+\} = E^{\Pi_t}(1 - \pi_\tau + c \sum_{i=0}^{\tau-1} \pi_i).$$

Then the value function given state variable $\pi$ and prior distribution $\Pi_t$ is

$$v^{\Pi_t}(\pi) = \min\{1 - \pi, c\pi + v^{\Pi_t}(\pi' \mid \pi)\}.$$

Define operator $Q$ as

$$(Qv^{\Pi_t})(\pi) = \min\{1 - \pi, c\pi + v^{\Pi_t}(\pi' \mid \pi)\}.$$

The value function can be obtained through iteration:

$$v^{\Pi_t}(\pi) = \lim_{n \to \infty} (Q^n v^{\Pi_t})(\pi),$$

and the stopping time for any generic prior distribution $\Pi_t$ is

$$\tau^* = \{1 \text{ if } v^{\Pi_t}(\pi) \geq 1 - \pi, 0 \text{ otherwise}\}.$$

Furthermore, we have

$$\tau^* = \{1 \text{ if } \pi \geq \pi^*, 0 \text{ otherwise}\},$$

where $v^{\Pi_t}(\pi^*) = 1 - \pi^*$.

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